VARIANCES OF PURITY WITHIN TORSION-FREE ABELIAN GROUPS

by

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PREFACE

In chapter 0 we give the notation used, pertinent definitions and concepts used throughout, and state fundamental results.

Chapter 1 is a historical overview of the development of the variances of purity. Here we state the concept, the originator of the concept and approximate date when the concept was first introduced by means of a paper. We then state the motivation for the new concept if it is apparent in the literature.

From chapter 2 onwards, we limit our discussion to variances of purity applicable only to torsion-free abelian groups.

In chapter 2 we compare variances of purity. We show which definition imply others and show examples where possible, of a subgroup which satisfies a weaker but not a stronger definition and thus providing the weaker definition with a raison d'être.

Chapter 3 is a collection of theorems which deal with groups with special conditions. These conditions are interesting in that when a group has these conditions, a certain class of subgroup is guaranteed to have stronger properties.

We have included appendices A.1 and A.2 since certain concepts discussed in this dissertation, can only be fully understood once these sections have been perused. However, despite its importance, we felt that these explanations interrupted the flow of the main document. To this end, A.1 discusses the concepts of primitivity, valuated coproduct, *-valuated coproducts and *-pure subgroups; A.2 discusses the concept of k-groups.

ABSTRACT

In 1921, Prüfer introduced the concept of a pure subgroup of an abelian group. This concept, which is applicable only to abelian groups, proved to be a very useful one. Subsequently, this concept has sparked off numerous definitions of subgroups of abelian groups which are either generalizations or refinements of the pure subgroup.

We look firstly, at how these ideas have developed since Prüfer's time. This picture has been gleaned by the perusal of the Mathematical Reviews to see which papers have been published regarding this topic and then, where available, by studying these papers to try to understand the rationale of the author.

Secondly, we group certain concepts which are comparable and then study the interrelation between these concepts.

In chapter 3.1, it is shown that, for a torsion-free abelian groupG, the following conditions are equivalent:

(i) G is a finite rank completely decomposable group,

(ii) all pure subgroups of G are summands,

(iii) all pure subgroups of G are balanced in G.

One of the interesting results of section 3.2 is the theorem that states that a subgroup of a finite rank completely decomposable group is *-purely generated if and only if it is strongly regular pure and that of 3.3 is that any finite rank *-pure subgroup of a separable group is a completely decomposable summand.

Section 3.4 uses for a basis, the theorem proved by P. Hill and C. Megibben which states that a Σ-pure subgroup of a k-group is itself a k-group. What is so interesting about this theorem is that one of its corollaries states that a Σ-pure subgroup of a separable group is also strongly pure.

The last section of the dissertation discusses the relationship between knice subgroups and balanced subgroups. A pure subgroup is knice if and only if it is balanced and its quotient group is a k-group. This result looks as though it could be helpful when trying to look at alternative definitions of balancedness.

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CHAPTER 0 - PRELIMINARIES

Throughout this dissertation the word **group**, unless otherwise stated, refers to an additively written abelian group.

The **standard references** are [Fu1] and [Fu2] and in this chapter, we introduce the notation, definitions, basic concepts and preliminary results frequently used in this dissertation.

§ 0.1 NOTATION

$\boldsymbol{Z}$: the set of integers.

$\boldsymbol{N}$: the set of non-negative integers.

$\boldsymbol{Q}$: the set of rational numbers.

$\boldsymbol{P}$: the set of prime numbers.

$\aleph_0$ or ω : the first infinite ordinal.

$\aleph_1$ or ω_1 : the second infinite ordinal, etc.

$\subseteq$, $\subset$: is contained, properly contained in.

$\langle S \rangle$, $\langle S \rangle_*$: the subgroup, pure subgroup generated by S.

T(G): the torsion subgroup of G.

G/H : the quotient group G modulo H.

A\B : the elements of A which are not in B.

A+B, ΣA_i : the subgroup generated by A and B, the A_i.

$\varnothing$: the empty set.

$\rightarrow$: mapping between sets.

1_A : the identity map of A.

$\alpha|_A$: the restriction of α to A.

ker α : the kernel of the homomorphism α.

Im α : the image of the homomorphism α.

$\oplus_I$, $\oplus_E$, $\oplus$: the internal, external, and (where there is no ambiguity) the direct sum of...

$h_p^G(x)$, $h_p(x)$: the p-height (p prime) of x in G, and (where there is no ambiguity) the p-height of x.

$\chi_G(x)$, $\chi(x)$: the height sequence of x in the torsion free group G, and (where there is no ambiguity) the height sequence of x.

$type_G(x)$, type(x) : the type of x in the torsion free group G, and (where there is no ambiguity) the type of x.

T(G) : the typeset of the torsion free group G.

$\mathscr{E}(G)$: The critical typeset of the torsion free group G.

$\mathfrak{C}$: The set of characteristic sequences.

T : The set of types.

§ 0.2 DEFINITIONS AND BASIC CONCEPTS

We start by mentioning concepts fundamental in the study of arbitrary abelian groups.

0.2.1 Internal direct sums

Let A and B be subgroups of G. We say that G is the internal direct sum of A and B (written $G = A \oplus_I B$) if $G = A + B$ (i.e. every element of G can be written as a sum of elements in A and B) **and** $A \cap B = \{ 0 \}$. We say that A is a direct summand of G if $G = A \oplus_I B$ for some B a subgroup of G. This notion can be generalized to an arbitrary collection of subgroups in G viz. $\{A_i : i \in I\}$. We say that $G = \oplus_I A_i$ if the following two conditions are satisfied :

a) $G = \sum_{i \in I} A_i$, and

b) $A_i \bigcap \sum_{\substack{j \in I \\ j \neq i}} A_j = \{ 0 \}$, for all $i \in I$.

Each A_i is a direct summand of G.

An abelian group is called indecomposable if it has only trivial direct summands. Q is known to be indecomposable.

0.2.2 Direct Products and External direct sums.

Let $(A_i : i \in I)$ be a family of groups indexed by I.

The direct product, $G = \prod_{i \in I} A_i$

is defined as follows:

$f \in G$ if f is a function with domain I and range $\bigcup_{i \in I} A_i$

and $f(i) \in A_i$ for every $i \in I$.

$f + g: I \to \bigcup_{i \in I} A_i$ is defined to be $(f+g)(i) = f(i) + g(i)$

and thus $(f + g)(i) \in A_i$. It can be shown that G becomes an abelian group under the binary operation defined above.

For every $i \in I$, define

(a) $\pi_i : G \to A_i$ by $\pi_i(f) = f(i)$ and call this the **i-th projection**;

(b) $\rho_i : A_i \to \prod_{j \in I} A_j$ as follows:

for any $x \in A_i$, $\rho_i(x) = \rho_{i,x}$

$$\text{where } \rho_{i,x}(j) = \begin{cases} x, & \text{if } i=j \\ 0, & \text{if } i \neq j. \end{cases}$$

For every $f \in G$, we define

supp(f) = { i ∈ I : f(i) ≠ 0 } and call this subset of I the support of f. If we put $F = \{ f \in G : \text{supp}(f) \text{ is finite} \}$, then since $\text{supp}(f+g) \subseteq \text{supp}(f) \cup \text{supp}(g)$ for every $f, g \in G$, it is easy to see that F is a subgroup of $G = \prod_{i \in I} A_i$. F is called <u>the external direct sum of the A_i's</u> and is denoted by $\oplus_E A_i$. It can be shown that :

(1) $\oplus_E A_i = \oplus_I \rho_i(A_i)$ and that

(2) for any $f \in \bigoplus_{E\, i \in I} A_i$, $f = \sum_{i \in supp(f)} \rho_i \pi_i f$.

0.2.3: Free and divisible abelian Groups

An abelian group F is **free on a set X** if there exists a function $i : X \to F$ such that for any function $f : X \to G$, G an abelian group, there exists a group homomorphism $\theta : F \to G$ such that $\theta i = f$. It can be shown that for any non-empty set X there exists a free abelian group F satisfying the definition above. This implies that every abelian group is an epimorphic image of a free abelian group (see [Fu1: p.74]) i.e. for every abelian group G, there exists a free abelian group F, and an epimorphism $\theta : F \to G$. The free group F together with θ, is called a <u>free resolution of G</u>.

An abelian group P is said to be **projective** if, for any abelian groups A and B, if $\alpha : A \to B$ and $\beta : P \to B$ are homomorphisms with α onto, there exists a homomorphism $\theta : P \to A$ such that $\alpha\theta = \beta$.

An abelian group D is :

a) **divisible** if for every non-zero integer n and $x \in D$, there exists $y \in D$ such that $ny = x$.

b) **injective** if, for any abelian groups A and B, if $\alpha : A \to B$ and $\beta : A \to D$ are homomorphisms with α monic, there exists a homomorphism $\theta : B \to D$ such that $\theta\alpha = \beta$.

0.2.4 Linear independence and rank

A subset $L = \{x_i : i \in I\}$ of an arbitrary abelian group, G, is called *Z*-independent (or just independent if there is no ambiguity) if for any finite sum $\Sigma n_i x_i = 0$, with $n_i \in Z$, then $n_i x_i = 0$ for all i. It can be shown [Fu1:p.85] that a system L is independent if and only if the subgroup generated by L is the direct sum of cyclic groups $\langle x_i \rangle$, $i \in I$. An element g of G depends on L if there exist integers $n, n_1, n_2, \ldots, n_k$ such that $0 \neq ng = n_1x_1 + n_2x_2 + \ldots + n_kx_k$ with $x_i \in L$. An independent system L is maximal if there is no independent system in G containing L properly. By Zorn's lemma, every independent system in G can be extended to a maximal one and, if the initial independent system contained only elements of infinite or prime power orders, then the same can be assumed of the maximal one. The rank of G, denoted r(G) is the cardinal number of a maximal independent system in G. It can be shown that r(G) depends only on G see [Fu1: Theorem 16.3, p.85]. Note that if $r(G) = 1$, then any two elements in G depend on each other.

0.2.5 Height/characteristic sequences and types

We now introduce the concepts of height, height sequence and type which will play a fundamental role throughout this dissertation.

A sequence $s = (s_p : p \in \boldsymbol{P})$ is called a **characteristic sequence** if s_p is either a non-negative integer or the symbol ∞. (The reason that we index the sequence by $\boldsymbol{P}$ will be apparent later on.)

We define a relation $\sim$ on the set $\mathfrak{C}$ of characteristic sequences, as follows:

$s \sim t$ if (i) $s_p = \infty$ if and only if $t_p = \infty$ and

(ii) $\{p : s_p \neq t_p\}$ is finite.

It can be shown that this defines an equivalence relation on $\mathfrak{C}$ and an equivalence class with respect to this relation is called a type.

We can also define an order relation $\leq$ on $\mathfrak{C}$ by :

$s \leq t$ if and only if $s_p \leq t_p$ for all p in $\boldsymbol{P}$. If neither $s \leq t$ nor $t \leq s$ then s and t are said to be incomparable and we write : $s \parallel t$.

With respect to this order relation, $\mathfrak{C}$ forms a complete distributive lattice with meet and join operations defined by:

$(s \wedge t)_p = \min\{ s_p, t_p \}$ and $(s \vee t)_p = \max\{ s_p, t_p \}$ respectively.

Let $n = \prod_{p \in P} p^{r_p} \in N$, where $r_p \in \boldsymbol{N}$, and let s be any characteristic sequence. (Note that $r_p = 0$ if and only if $(n,p) = 1$ in this product. We find it convenient to define the "product" of s and n to be the characteristic sequence $ns = (t_p)$ where $t_p = s_p + r_p$.

The order relation and meet and join operations defined on $\mathfrak{C}$ induce corresponding notions on the set T of types as follows:

for τ_1 and τ_2 in T,

a) $\tau_1 \leq \tau_2$ if and only if there exist $s_i \in \tau_i$, where $i = 1, 2$ such that $s_1 \leq s_2$;

b) $\tau = \tau_1 \wedge \tau_2$ if and only if τ contains $s_1 \wedge s_2$ with $s_i \in \tau_i$ where $i = 1,2$; $\tau_1 \vee \tau_2$ is defined in a similar manner.

If neither $\tau_1 \leq \tau_2$ nor $\tau_2 \leq \tau_1$, then τ_1 and τ_2 are incomparable (written $\tau_1 \parallel \tau_2$). For convenience, if $\tau_1 \leq \tau_2$ but $\tau_1 \neq \tau_2$, then we say that $\tau_1 < \tau_2$.

Let G be a torsion free group and p a fixed prime. We obtain the following sequence of subgroups of G:

$$G = p^0G \supseteq pG \supseteq p^2G \supseteq \ldots \supseteq p^nG \supseteq \ldots \qquad n \in \mathbb{N}.$$

If $x \in G$, the **p-height of x in G**, denoted by $h_p^G(x)$, is defined to be n if $x \in p^nG \setminus p^{n+1}G$, otherwise it is ∞. The characteristic sequence $(h_p^G(x) : p \in \boldsymbol{P})$ is called the **height sequence** of x in G and will be denoted by $\chi_G(x)$. The equivalence class containing the height sequence of x in G will be called the **type of x in G** and will be denoted by $\text{type}_G(x)$. The **typeset of the group G**, T(G), is the set of types of all the non-zero elements in G. Note that for any $n \in \boldsymbol{Z}$, $\chi_G(x) \sim \chi_G(nx)$ and thus $\text{type}_G(x) = \text{type}_G(nx)$. If $|T(G)| = 1$ (i.e. all the elements in G have the same type) then G is said to be homogeneous.

Every height sequence $s \in \mathfrak{C}$ determines a fully invariant subgroup $G(s) = \{x \in G: \chi_G(x) \geq s\}$ of G which contains the fully invariant subgroups $G(s^*)$ and $G(s^*,p)$, where $G(s^*)$ is generated by all the x's in G(s) with $\chi_G(x)$ not equivalent with s, while $G(s^*,p) = G(s^*) + pG(s)$. Note that :

(a) if $s \leq t$, then $G(t) \subseteq G(s)$, $G(t^*) \subseteq G(s^*)$, and $G(t^*,p) \subseteq G(s^*,p)$; and

(b) if $t = ns$, then $G(t) = nG(s)$, $G(t^*) = nG(s^*)$, and $G(t^*,p) = nG(s^*,p)$.

For every type τ we also have the following fully invariant subgroups of G:

$$\mathbf{G(\tau)} = \sum_{s \in \tau} G(s) = \bigcup_{s \in \tau} G(s) \text{ and } \mathbf{G(\tau^*)} = \sum_{s \in \tau} G(s^*) = \bigcup_{s \in \tau} G(s^*).$$

Remark: The subgroup $G(\tau)$ of G is simply the collection of all elements x of G such that $type_G(x) \geq \tau$ and is, in general, a pure subgroup of G. However, the collection of elements in G(s) with height sequences not equivalent to s and the collection of elements of type $> \tau$ do not necessarily form a group. Hence we have to define $G(s^*)$ and $G(\tau^*)$ as the subgroups generated by these respective elements in G.

Clearly $G(\tau^*) \subseteq G(\tau)$ for every type, τ, and if $G(\tau^*) \subset G(\tau)$ then $\tau \in \mathrm{T}$ is said to be a <u>critical type</u> of G. The set of critical types in T(G) is called the <u>critical typeset</u> of G and will be denoted by $\mathcal{T}(G)$.

<u>0.2.6 Completely decomposable and separable torsion free groups</u>

A torsion-free group is said to be <u>completely decomposable</u> if it is a direct sum of rank 1 groups and is called <u>almost completely decomposable</u> if it contains a completely decomposable subgroup of finite index. If every finite subset of a torsion-free group G can be embedded in a completely decomposable finite rank direct summand of G, then G is said to be **separable.** Trivially, every completely decomposable group is separable but there are examples of separable groups that are not completely decomposable.

§ 0.3 SOME PRELIMINARY RESULTS

0.3.1: Let F be a free group and H a subgroup of F. Then

a) F is a direct sum of infinite cyclic groups;

b) H is also free;

c) if rank(F) = n, a positive integer, then there exists a basis $\{x_1, x_2, \ldots, x_n\}$ of F and integers r, k_1, k_2, ..., k_r, where $1 \leq r \leq n$ and $k_i | k_{i+1}$, $i=1,2,\ldots,r-1$ such that $\{k_1x_1, \ldots, k_rx_r\}$ is a basis of H. (**Stacked Basis Theorem for finite rank free groups**).

0.3.2: A group is free if and only if it has the projective property.

0.3.3: A group is divisible if and only if it has the injective property.

0.3.4: A divisible subgroup D of an abelian group, G, is a direct summand of G and we can write $G = D \oplus C$ where C contains no divisible subgroups other than 0. C is called a **reduced** group.

0.3.5: If s and t are height sequences, then $s \sim t$ if and only if there exist $n, m \in \mathbf{Z}$ such that $ns = mt$. **Thus if G is a rank 1 group, then G is homogeneous.**

0.3.6: If $G = \oplus G_i$ then:

$G(s) = \oplus G_i(s)$ and $G(s^*) = \oplus G_i(s^*)$;

$G(\tau) = \oplus G_i(\tau)$ and $G(\tau^*) = \oplus G_i(\tau^*)$.

0.3.7: If G is completely decomposable, then $\tau \in \mathscr{T}(G)$ if and only if there is at least one summand G_α of G such that $T(G_\alpha) = \{\tau\}$. Let G_τ be the direct sum of all rank one summands of G of type τ then $G = \oplus \{G_\tau : \tau \in \mathscr{T}(G)$.

0.3.8: If $G = H \oplus K$ then $\mathscr{T}(H) \subseteq \mathscr{T}(G)$.

CHAPTER 1 - HISTORICAL OVERVIEW

In this chapter the various notions of purity that appear in the research papers reviewed in Mathematical Reviews vol 21 to vol 92g will be introduced. As far as possible, the names of the people who first introduced the different concepts of purity, as well as the approximate date that the idea was introduced, will be given. We will also endeavour, where possible, to supply the observations that led to some of these definitions.

In 1921, Prüfer [Pr] introduced the notion of a "pure subgroup" which has turned out to be one of the most useful and powerful concepts in the study of abelian groups. A subgroup H of an arbitrary abelian group, G, is said to be **pure in G** if $nH = nG \cap H$ for any $n \in \mathbb{Z}$. Note that, in general, $nH \subseteq nG \cap H$ for all integers n and, in order to check whether a subgroup is pure or not, we need only verify the other containment.

The notion of pure subgroups is intermediate between subgroups and direct summands and it reflects a way in which a subgroup is embedded in the whole group. Specifically, it can easily be shown that all direct summands are pure subgroups. However, since a subgroup H is pure in a group G whenever G/H is torsion-free, if we take a free resolution of the group of rationals $\mathbb{Q}$, the kernel is pure but not a summand otherwise its complement would be isomorphic to Q. On the other hand, $2Z$ is a subgroup of Z which is not pure. Thus, being a direct summand is stronger than being pure, which in turn, is stronger than being an ordinary subgroup.

One can appreciate the significant role played by pure subgroups when one considers, for example, basic subgroups. These play a crucial part in the study of p-groups and are, by definition, pure.

It is also a well-known fact [Fu1: p.106] that every abelian group can be embedded as a subgroup of a divisible group - called the **divisible hull** of the group. The structure of divisible groups is described completely by Theorem 23.1 [Fu1: p.104] and, thus, one of the approaches to describing the structure of all abelian groups is to study how a subgroup is embedded in its divisible hull. We can also be interested in when certain properties of the larger group are inherited by a particular class of its subgroups. It can be easily demonstrated that pure subgroups of divisible groups inherit the property of being divisible. Thus, a group which is not divisible cannot be a pure subgroup of its divisible hull. Hence one way of getting closer to understanding the structure of abelian groups, is to study subgroups with properties that are weaker than being pure.

According to the famous Baer-Kaplansky-Kulikov Theorem (see [Fu2: p.114]), direct summands of completely decomposable groups are themselves completely decomposable. Bican [Bi2] gave an example of a pure subgroup of a completely decomposable group which is not completely decomposable. The question can thus be asked whether a variance of purity (stronger than ordinary purity, yet weaker than being a summand) exists so that this class of subgroups of a completely decomposable group is guaranteed to be completely decomposable. To date, there are no examples in literature which give a negative or affirmative answer to this, except in restricted cases (e.g. homogeneous pure subgroups of completely decomposable group G are completely decomposable provided that $\mathcal{E}$(G) is countable) [No3].

Since the introduction of the concept of purity, many variances of purity were introduced and studied. The question posed in the previous paragraph shows that the pursuit of finding new variances of purity is still an interesting and worthwhile activity. What follows in this chapter is a collection of the definitions of the variances of purity that exist in literature.

The torsion part of an abelian group is a direct sum of its p-components and thus the study of the torsion part can be reduced to the study of p-groups. The observation that every p-subgroup H of a group G has the property that $q^nH = H \cap q^nG$ for all $n \in \mathbf{N}$, $q \in \mathbf{P}$ and $q \neq p$ leads to the definition of **p-purity**. A subgroup H of a group G is p-pure in G if $p^nH = H \cap p^nG$ for any $n \in \mathbf{N}$. From the remark above, a p-subgroup H of G is q-pure for all $q \neq p$ and H will therefore be pure in G if and only if it is p-pure in G.

If, in the definition of purity, we restrict the integers to be only prime numbers, then we get the definition of neatness as introduced by K. Honda [Ho]. In other words, H is **neat** in G if $pH = H \cap pG$ for all $p \in \mathbf{P}$. A pure subgroup is trivially neat and in torsion-free groups, neatness and purity coincide.

If $pH = H \cap pG$ for a **fixed** prime $p \in \mathbf{P}$, then H is said to be **p-neat** in G. For any prime p, a p-pure subgroup of G is necessarily p-neat.

A pure subgroup H of a group G has the property that if $0 \neq ng \in H$, where $g \in G$, then there is an element $h \in H$ such that $nh = ng$. In torsion-free groups this implies that $g \in H$. In mixed or torsion groups, however, this is not necessarily the case, for if $ng \in H$, there might exist an element $h \in H$, **distinct** from g, with the property that $ng = nh$. A. Abian and D. Rinehart [AR], in 1963, called a subgroup H of an arbitrary group G **honest** if, for any $n \in \mathbf{N}$, $0 \neq ng \in H$ implies that $g \in H$. Thus an honest subgroup will necessarily be a pure subgroup and, if G is torsion-free, then H would be honest in G if and only if H is pure in G. It is proved in [AR] that if H is honest in G, then either $H \subset T(G)$ or $T(G) \subseteq H$, where T(G) is the torsion subgroup of G.

Suppose that H is a proper subgroup of the torsion subgroup of G. Then H is honest in G if and only if the following three conditions are satisfied :

(i) T(G) is a p-group for some prime p;

(ii) H is a direct sum of cyclic subgroups of order p; and

(iii) H is a direct summand of G.

On the other hand, if $T(G) \subseteq H$, then H is honest if and only if G/H is torsion-free.

It can be seen, in view of condition (iii) above, that for torsion and mixed groups where H is properly contained in T(G), honesty is stronger than being a direct summand.

In 1964 K. M. Rangaswamy defined H to be **absorbing** in G if, whenever $nx \in H$ for some non-zero integer n, then $x \in H$ [Ra1]. This definition is stronger than A. Abian and D. Rinehart's definition of honesty in that absorbing subgroups are necessarily honest. As Rangaswamy did not specify that nx be non-zero, absorbing subgroups contain all elements in G of finite order whereas this is not the case with honest subgroups. In fact, an honest subgroup

H of G is absorbing if and only if $T(G) \subseteq H$. Rangaswamy, in the same paper, defined a subgroup H of an arbitrary abelian group G to be **full** in G if $G[p] \subseteq H$ where p is a fixed prime number. By the remark above, every absorbing subgroup is full. Rangaswamy proved that every full and neat subgroup is absorbing.

For every non-negative integer n, put $G_n = p^nG$ and recall that we can get a decreasing chain of subgroups : $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n \supseteq \ldots$ We extend the definition of G_n to $G_\alpha = p^\alpha G$ where α is any ordinal as follows:

$$G_\alpha = p^\alpha G = \bigcap_{\beta<\alpha} G_\beta, \qquad \text{if } \alpha \text{ is a limit ordinal;}$$

$$G_\alpha = p^\alpha G = pG_{\alpha-1} \qquad \text{if } \alpha \text{ is a successor ordinal.}$$

In the same way we extend the definition of p-heights as follows : an element g in G is said to have <u>generalized p-height in G</u> of α (i.e. $h^*_{G,p}(g) = \alpha$) if $g \in G_\alpha$ but $g \notin G_{\alpha+1}$. Note that in torsion-free groups, $p^\alpha G = p^\omega G$ for any ordinal $\alpha > \omega$. Thus, in torsion-free groups, the generalized p-height is the same as the p-height.

In 1952, Kulikov [Ku] generalized the definition of p-purity as follows: a subgroup H of a p-group G is **isotype** in G if $p^\alpha H = p^\alpha G \cap H$ for every ordinal α. An isotype subgroup is necessarily pure. If G is torsion-free, then G is isotype if and only if G is pure and, if G is a p-group containing no elements of infinite height, a subgroup H of G is isotype if and only if it is pure in G.

Let G be a group, B and H subgroups of G. The subgroup H is said to be **B-high** in G if $H \cap B = \{0\}$ **and** if $H \subset H' \subseteq G$ implies that $H' \cap B \neq \{0\}$. It was shown by Irwin and Walker (see [Fu2: Theorem 80.1, p.76]) that, for any ordinal α, any $p^{\alpha}G$-high subgroup of a p-group G is isotype. This result gives a simple way of constructing non-trivial isotype subgroups of a p-group that contains elements of infinite p-height.

In a paper published in 1966 [deR], E. de Robert introduced the concepts of p^{α}-pure and α-pure. If α is any ordinal and H is a subgroup of an arbitrary abelian group G, then H is said to be **p^{α}-pure** in G if $p^{\beta}G \cap H = p^{\beta}H$ for all ordinals $0 \leq \beta \leq \alpha$. H is said to be **α-pure** in G if H is p^{α}-pure for all primes p. If G is a p-group, then a subgroup H is α-pure in G if and only if H is p^{α}-pure in G. A subgroup is p^{ω}-pure if and only if it is p-pure and ω-purity coincides with ordinary purity.

Let $S \subseteq \boldsymbol{P}$ and define a subgroup H of a group G to be **S-pure** if H is p-pure in G for all p in S. This notion was first introduced by J.M. Maranda in 1960 ([Ma]). If $S = \boldsymbol{P}$, then S-purity coincides with ordinary purity and, if $S = \{p\}$, p a prime number, then S-purity becomes p-purity.

This concept was generalized by T.J. van Dyk, [vD] in 1979 as follows: instead of restricting S to be a subset of the set of prime numbers, let S be a non-empty subset of integers with the property that whenever $0 \neq n \in S$, then every positive divisor of n is also in S . Clearly, the following sets of integers satisfy the property stated above:

a) $\mathbf{Z}$;

b) $\mathbf{P} \cup \{1\}$;

c) the union of {1} with any **subset** of primes;

d) $S^{(p)} = \{p^k : p \text{ is fixed and } k = 0, 1, 2, \ldots\}$, and

e) $S(\alpha) = \{p^\beta : p \text{ prime and } 0 \leq \beta \leq \alpha\}$.

Thus H is

i) $\mathbf{Z}$-pure in G if and only if it is pure in G;

ii) $(\mathbf{P} \cup \{1\})$-pure if and only if it is neat in G;

iii) $S^{(p)}$-pure if and only if it is p-pure in G; and

iv) $S(\alpha)$-pure if and only if H is p^α-pure in G.

Let G be an abelian group and H a subgroup of G. The natural epimorphism $\psi : G \rightarrow G/H$ need not preserve the properties of elements of G. For example, if G is torsion-free and rank(G) = rank(H), then ψ maps elements of infinite order into elements of finite order. However, by [Fu1: Theorem 28.1], H is pure in G if and only if ψ preserves the order of at least one element in every coset of G modulo H. Also, ψ need not preserve the (generalised) p-heights of elements of G (e.g. in the free resolution of $\mathbb{Q}$, elements of finite p-height are mapped to elements of infinite p-height for every prime p).

An element x in G\H is said to be **(p-)proper with respect to H** if the generalized p-height of x in G equals the generalized p-height of the coset x + H in G/H (i.e. ψ preserves the generalised p-height of x).

A close analysis of the Kaplansky-Mackey proof of Ulm's Theorem led P Hill to the discovery of a significant type of subgroup which embodies the properties of finite subgroups relevant to the proof ([Fu2: paragraph 77]). He called a subgroup, N of a p-group G, **nice** (see [Fu2: paragraph 79]) if every non-zero coset of G mod N contains an element which is p-proper with respect to N. A subgroup B of a p-group is **balanced** if it is both nice and isotype. This idea was introduced by L. Fuchs in [Fu2: p77].

This concept can easily be extended to torsion-free abelian groups as follows: if H is a pure subgroup of a torsion free group G, then G/H is torsion free and vice versa; a pure subgroup H is said to be **balanced** in G if the natural epimorphism preserves the height sequence of at least one element in every coset of G modulo H.

The definitions of balancedness given above apply **only** to p-groups (or **only** to torsion-free groups). The extension of this concept to abelian groups in general (i.e. mixed groups), which yields the above definitions for p-groups and torsion free groups, was carried through by Hunter ([Hu]) in 1976. In order to introduce this concept, we need to define the height matrices.

Let an extended characteristic sequence, (s_p) be a sequence of **ordinals** and symbols ∞. Let (s_p) and (t_p) be two extended characteristic sequences. We say that $(s_p) \leq (t_p)$ if $s_p \leq t_p$ for all $p \in P$.

A height matrix, **M** is defined to be an $\omega \times \omega$ matrix $[\sigma_{pk}]$ where $p \in P$ and $k \in N$ and whose entries, σ_{pk}, are ordinals and symbols ∞. Given a height matrix, $\mathbf{M} = [\sigma_{pk}]$, p a prime, we define p**M** to be the matrix with p-th row $(\sigma_{p1}, \sigma_{p2}, \ldots)$ (i.e. drop σ_{p0} and shift all other entries one place to the left) and all other rows are identical to the corresponding rows in **M**. Multiplication of height matrices by a power of a prime p and therefore by an arbitrary integer is defined in the obvious way. Note that for arbitrary positive integers n and k, $(nk)\mathbf{M} = n(k\mathbf{M})$ and the definitions here give a scalar multiplication of height matrices by positive integers.

Two height matrices **M** and **N** are said to be **equivalent** if there are positive integers m and n such that $m\mathbf{M} = n\mathbf{N}$. As in the case for torsion-free abelian groups, this obviously defines an equivalence relation on the set of height matrices, but we have been unable to find in literature, an investigation of this equivalence relation parallel to the one for torsion free groups.

Let $\mathbf{M} = [\sigma_{pk}]$ and $\mathbf{N} = [\rho_{pk}]$ be height matrices. We say that $\mathbf{M} \le \mathbf{N}$ if $\sigma_{pk} \le \rho_{pk}$ for all $p \in P$ and $k \in \mathbf{N}$. We denote the pth-row of the height matrix $\mathbf{M}$ by $\mathbf{M}_p$.

Let $x \in G$. We define the <u>extended height sequence of x</u>, (written $\chi^*_G(x)$), to be (s_p) where (s_p) is an extended characteristic sequence with $s_p = h^*_{G,p}(x)$. A <u>height matrix, $\mathbf{H}_G(x)$, of an element x in G</u> is defined to be the height matrix $\mathbf{M} = [\sigma_{pk}]$ where $\sigma_{pk} = h^*_{G,p}(p^k x)$, the generalized p-height of $p^k x$ in G.

For each group G and height matrix $\mathbf{M}$, define

$$G(\mathbf{M}) = \{g \in G : \mathbf{H}_G(a) \ge \mathbf{M}\} = \bigcap_{p \in P} G(\mathbf{M}_p).$$

Now let H be a subgroup of G. An element g in G\H is **<u>H-proper with respect to H</u>** if $\mathbf{H}_G(g) = \mathbf{H}_{G/H}(g+H)$ and H is **<u>H-nice</u>** in G if every coset g+H contains an element which is **H**-proper with respect to H.

An exact sequence :

$$0 \to A \to^{\alpha} B \to^{\beta} C \to 0$$

is said to be **<u>balanced</u>** if the induced sequence :

there are positive integers m and n such that $m\mathbf{M} = n\mathbf{N}$. As in the case for torsion-free abelian groups, this obviously defines an equivalence relation on the set of height matrices, but we have been unable to find in literature, an investigation of this equivalence relation parallel to the one for torsion free groups.

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For each group G and height matrix $\mathbf{M}$, define

$$G(\mathbf{M}) = \{g \in G : \mathbf{H}_G(a) \ge \mathbf{M}\} = \bigcap_{p \in P} G(\mathbf{M}_p).$$

Now let H be a subgroup of G. An element g in G\H is **<u>H-proper with respect to H</u>** if $\mathbf{H}_G(g) = \mathbf{H}_{G/H}(g+H)$ and H is **<u>H-nice</u>** in G if every coset g+H contains an element which is **H**-proper with respect to H.

An exact sequence :

$$0 \to A \to^{\alpha} B \to^{\beta} C \to 0$$

is said to be **<u>balanced</u>** if the induced sequence :

$$0 \to A(M) \to^{\alpha} B(M) \to^{\beta} C(M) \to 0 \quad .$$

is exact for every height matrix **M**. **A subgroup H of G is said to be balanced in G** if the exact sequence above is balanced when we replace A, B, C, α, and β with H, G, G/H, the inclusion map and the canonical map respectively. Hunter also proved in [Hu] that if H is a subgroup of G, then the following are equivalent :

(a) H is balanced in G;

(b) H is both isotype and **H**-nice in G;

(c) to each c in G/H, there is an element g in G such that $\psi(g) = c$ (where ψ is the canonical map), $\mathbf{H}(g) = \mathbf{H}(c)$ and $o(g) = o(c)$; and

(d) the sequences $0 \to H/H(\mathbf{M}) \to G/G(\mathbf{M}) \to C/C(\mathbf{M}) \to 0$ (where C = G/H) is exact for all height matrices **M**.

Hunter also defined an exact sequence $0 \to A \to B \to C \to 0$ to be **H-Balanced** if $0 \to A(K) \to B(K) \to C(K) \to 0$ is exact for all extended height sequences K. If G is a p-group, then the rows of the height matrix consist of ∞'s except for a single row and the original definition of balanced coincides with Hunter's one ([Fu2: Exercise 6, p.93]).

If G is a torsion-free group, then the first column of the height matrix of an element is the (ordinary) height sequence of that element which gives the same amount of information as the entire height matrix in view of the fact that for every $x \in G$, p prime and n a non-negative integer, $h_p(p^n x) = n + h_p(x)$. Thus if the first column of the height matrix is s, then $x \in G(s)$ if and only if $x \in G(\mathbf{H}(x))$ and by (b) above, if H is balanced in G according to Hunter, H is balanced in G according to Fuchs's definition.

In 1977, C.J.Boshoff [Bo] defined a subgroup H of a **p-group** G to be **peaked** in G if for every non-zero coset g+H in G/H, there is an $h \in H$ such that $h_p^G(g+h) = h_p^{G/H}(g+H)$. She defined a **fine** subgroup to be one that is peaked and pure. Nice subgroups are by definition peaked and thus a pure subgroup which is also nice would be a fine subgroup.

Note that if H is a subgroup of a group G, then H is isotype in G if and only if the height matrix of an element in H, evaluated in H, is the same as its height sequence evaluated in G. If G is torsion-free, H is **regular** in G if $\text{type}_H(h) = \text{type}_G(h)$ for all $h \in H$.

Let H be a subgroup of an abelian p-group G without elements of infinite height. It was observed by S. Janakiraman and K. M. Rangaswamy in 1975 [JR] that H is pure in G if and only if for every $a \in H$, there exists a homomorphism $\alpha : G \to H$ satisfying $\alpha(a) = a$. This observation led to their defininition of strongly pure subgroups: a subgroup H of a group G is **strongly pure** if, for every $h \in H$, there exists a homomorphism $\alpha : G \to H$ such that $\alpha(h) = h$. Strongly pure subgroups are pure, but the converse is not true as will be shown in Chapter 2.

The definition of a strongly pure subgroup can be generalised in an obvious manner as follows: H is **strongly regular** in G if there exists, for every $h \in H$, a homomorphism $\alpha : G \to H$ and a non-zero integer n such that $\alpha(h) = nh$. This generalisation was undertaken by L. Nongxa [No2] in 1987; it will be shown in Chapter 2 (Lemma 2.3.4) that for torsion-free groups, strongly regular subgroups are regular - hence the terminology.

In 1979, Rangaswamy [Ra2] dualised the concept of strong purity as follows : H is **strongly balanced** in G if, for every $a \in G$ there exists a homomorphism $\phi: G/H \to G$ such that $\psi(a) = \psi\phi\psi(a)$, where $\psi : G \to G/H$ is the natural epimorphism. Strongly balanced subgroups are balanced in the sense defined above (see Lemma 2.3.3).

Recall that a balanced subgroup w.r.t. p-groups is a subgroup that is both isotype and nice. H. Bowman and K.M.Rangaswamy in [BR] defined a short exact sequence :

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \text{ ....(**)}$$

to be **strongly isotype** if, to each $a \in A$, there is a homomorphism $f: B \to A$ such that $f\alpha(a) = a$;

to be **strongly nice** if, to each $c \in C$, there is a homomorphism $g : C \to B$ s.t. $\beta g(c) = c$; and to be **strongly balanced** if it is both strongly isotype and strongly nice. In our view, the definition of strongly isotype coincides with the definition of strongly pure and that of strongly nice is equivalent to strongly balanced. In the paper referred to above, subgroups which are both strongly isotype and strongly nice/balanced were not investigated in any detail.

Let m be any cardinal. Rangaswamy [Ra3] called the short exact sequence (**) :

a) **strongly m-isotype** if, to each subset X of cardinality less than m in A, there exists a homomorphism $f : B \to A$ such that $f\alpha|_X$ = identity on X;

b) **strongly m-nice** if, to each subset Y of cardinality less than m in C, there exists a homomorphism $g : C \to B$ such that $bg|_Y$ = identity on Y; and

c) **strongly m-balanced** if it is both strongly m-isotype and strongly m-nice.

A subgroup H of G is strongly m-balanced if the short exact sequence $0 \to H \to G \to G/H \to 0$ is strongly m-balanced. Strongly ω-balanced is strongly balanced.

In their paper [BR], Bowman and Rangaswamy also defined a ***-balanced** subgroup to be a balanced subgroup H of a torsion-free group G such that for each type τ

$$H \cap \langle G(\tau^*) \rangle_* = \langle H(\tau^*) \rangle_*.$$

It is easy to see that the assumption that H be balanced in G is an unnecessary restriction. This restriction was lifted in a definition introduced by L.Nongxa in [No2]. He thus defined a pure subgroup H of a torsion free group G to be ***-pure in G** if

$\langle H(\tau^*)\rangle_* = H \cap \langle G(\tau^*)\rangle_*$ for every type τ.

A torsion-free group G is called a **Butler group** if G is a pure subgroup of a finite rank completely decomposable group. The following were found to be equivalent:

(1) G is a pure subgroup of a finite rank completely decomposable group;

(2) G is an epimorphic image of a (finite rank) completely decomposable group;

(3) T(G) is finite and closed under infimums and, for every $\tau \in T(G)$,

(i) $G(\tau) = G_\tau \oplus \langle G(\tau^*)\rangle_*$, where G_τ is a homogeneous (possibly 0) completely decomposable group,

(ii) $\langle G(\tau^*)\rangle_*/G(\tau^*)$ is a finite group; and

(4) $G = \sum_{i=1}^{n} G_i$ where G_i is a pure, rank one subgroup of G

The equivalence of (1),(2) and (3) was established by M.C.R. Butler in 1965 [Bu] and the equivalence of (2) and

(4) was established by L. Bican in 1970 [Bi1].

In 1985, U. Albrecht and P. Hill [AH], defined a decent subgroup in order to get a characterization of a particular generalization of Butler groups, to groups having infinite rank viz. the class of B_2-groups. (A torsion-free group G is called a B_2-group if G is the union of a smooth chain $0 = G_0 \subseteq G_1 \subseteq \ldots G_\alpha \subseteq \ldots$ of pure subgroups G_α such that, for each α, $G_{\alpha+1} = G_\alpha + B_\alpha$ where B_α is a Butler group (of finite rank).) They called a subgroup H of the torsion-free group G **decent** if for any finite subset S of G there exists a finite number of rank 1 pure subgroups A_i, $1 \leq i \leq n$, of G such that $H + \sum_{1 \leq i \leq n} A_i$ is pure in G and contains S. U. Albrecht and P. Hill proved that a group G is a B_2-group if and only if G satisfies the third axiom of countability with respect to decent subgroups.

M. Dugas and K.M. Rangaswamy [DR2] defined a group G to satisfy the **torsion extension property** (for short, T.E.P.) over a pure subgroup H, if every homomorphism $f : H \to T$, where T is any torsion group, extends to a homomorphism $g : G \to T$. They went on to prove that if G is a Butler group of rank ω and H a pure subgroup, then H is decent in G if and only if G satisfies the T.E.P. over H.

If G is a torsion-free group and H a pure subgroup of G, it was shown by L.S. Ljapin that, for every $x \in G\backslash H$

$$\chi_{G/H}(x + H) = \sup\{\chi_G(x + h) : h \in H\}.$$

It can be concluded that the definition of balanced subgroups implies that there exists $h \in H$ such that $\chi_G(x + h)$ is the supremum of the set on the right-hand side.

Generalising this observation, F. Richman in [Ri] defined a pure subgroup H of a torsion-free group G, to be **semi-balanced** in G if for every $g \in G\backslash H$ there is a finite subset $\{h_1, \ldots, h_n\} \subseteq H$ such that

$$\chi_{G/H}(g+H) = \sup \{ \chi_G(g+h_i) : 1 \le i \le n \}$$

In the case that n is equal to 1, H is balanced in G. This concept was studied by L. Fuchs and G. Viljoen in [FV] who called these subgroups **prebalanced.**

REMARK: Lemma 1 in [FV] asserts that the equation above is satisfied if and only if

$$\langle H,g\rangle_* = H + \langle g+h_1\rangle_* + \ldots + \langle g+h_n\rangle_*$$

However, although the former equation implies the latter, the reviewer of this paper, William J. Wickless noted that in general the two equations are not equivalent see [Mathematical Review 91a:20062]. In [FM], L. Fuchs and C. Metelli redefined a **prebalanced** subgroup as follows : H is prebalanced in G if and only if for every $g \in G$, there is a non-zero integer m and a finite subset $\{h_1, h_2, \ldots, h_n\}$ of H such that $\chi_{G/H}(mg+H) = \sup \{ \chi_G(mg+h_i) : 1 \le i \le n \}$. A torsion-free group G, is called **a locally Butler group** if every finite subset of G can be embedded in a pure subgroup of G which is a finite rank Butler group.

L. Fuchs and G. Viljoen proved that a pure subgroup H of a torsion-free group is decent if and only if H is prebalanced in G and G/H is a locally Butler group.

For any type τ, let X_τ be a rank 1 torsion free group of type τ. It is easy to see that $G(\tau)$ is the subgroup of G generated by $\{f(X_\tau) : f \in Hom(X_\tau, G)\}$. D. Arnold and C. Vinsonhaler in [AV] called this subgroup the **τ-socle** of G and "dualised" it as follows: the subgroup of G called the **τ-radical of G**, denoted by $G[\tau]$, is defined to be $\bigcap_{f \in Hom(G, X_\tau)} \ker f$, and $G^*[\tau] = \bigcap_{\sigma < \tau} G[\sigma]$. They defined a pure subgroup H of G to be **co-balanced** if $0 \to H/H[\tau] \to G/G[\tau]$ is pure exact for every type τ.

A. J. Giovannitti and K. M. Rangaswamy [GR] "dualised" the notion of a prebalanced subgroup by calling a subgroup H of G to be **precobalanced** if for any subgroup K in H with $H/K \simeq R$, a rank one group, there are subgroups $K_1, K_2, K_3, \ldots, K_n$ of G satisfying:

(1) each $G/K_i \simeq R_i$, a rank one group;

(2) $K = \cap \{K_i \cap H : 1 \leq i \leq n\}$; and

(3) for each $h \in H$, $\chi_{H/K}(h+K) = \bigwedge_{1 \leq i \leq n} \chi_{G/K_i}(h + K_i)$. The authors showed that exact sequences of Butler groups are always prebalanced and precobalanced. More generally,

they proved that if the exact sequence

$$0 \to A \to B \to C \to 0$$

is a sequence of torsion-free groups, then:

(a) if B is a Butler group and A is precobalanced in B, then A is prebalanced in B;

(b) if C is a Butler group and A is prebalanced, then A is precobalanced.

In [DHR], M. Dugas, P. Hill and K.M. Rangaswamy defined a pure subgroup H, of a torsion-free group G, to be **hyperbalanced** if for each $g \in G$ and each countable subset C of H there is some $h \in H$ with $g_C^\# = h_C^\#$ where $a_C^\#$ is defined to be a map from C into the set of all height sequences by $a_C^\#(x) = \chi_G(a+x)$ for all $x \in C$. They proved that if H is hyperbalanced in G, then H is also balanced in G (see Lemma 2.5.20).

P. Hill in [Hi] defined a subgroup H of a torsion-free group G to be **separable** if for each $g \in G$ there is a countable subset $\{h_n : n<\omega\}$ of H satisfying the following condition: for $h \in H$, there is a corresponding $n < \omega$ such that $\chi_G(g+h) \leq \chi_G(g+h_n)$. (This concept is now called **separative**). If H is balanced in G, then the countable subset $\{h_n\}$ is a set containing only one element and this set satisfies the condition above. Thus all balanced subgroups are necessarily separable. Trivially all countable subsets of G are also separable. He went on to define a subgroup to be **absolutely separable** if it is separable in any torsion-free group in which it appears as a pure subgroup.

P. Hill and C. Megibben [HM1] introduced the notion of kniceness which they claimed generalized niceness to torsion-free groups. To define knice subgroups however, requires the definitions of primitive elements, valuated coproducts, *-valuated coproducts, and free *-valuated subgroups. These concepts will be dealt with in great detail in the Appendix A.1. A subgroup N of the torsion-free group G, is said to be a **knice** subgroup if for each finite subset S of G, there are primitive elements y_1, y_2, ..., y_m such that $N' = N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \ldots \oplus \langle y_m \rangle$ with N′ a *-valuated coproduct and with $\langle S,N' \rangle / N'$ finite.

In [HM1], P. Hill and C. Megibben also defined a pure subgroup H of G to be *-pure in G if $H \cap G(s^*) = H(s^*)$ **and** $H \cap G(s^*,p) = H(s^*,p)$. Direct summands and rank one pure subgroups generated by primitive elements, are necessarily *-pure. The ascending union of *-pure subgroups is also *-pure and the *-valuated coproduct (see A.1.3 in the appendix) $H = \oplus H_i$ is a *-pure subgroup if each H_i is *-pure. In this paper Hill and Megibben also proved that pure and knice subgroups are necessarily *-pure. This definition was published about the same time as, and independent of L. Nongxa's definition of a *-pure subgroup. The relationship between these two different types of *-pure subgroups will be demonstrated in Chapter 2.

In 1987 P. Hill and C. Megibben [HM2] defined a subgroup H of an arbitrary abelian group G to be **Σ-pure** in G if, whenever $h = g_1 + g_2 + \ldots + g_n$ with $h \in H$ and $g_i \in G(s_i)$, then $h = h_1 + h_2 + \ldots + h_n$ where $h_i \in H(s_i)$. Hill and Megibben showed in [HM2] that pure knice subgroups are necessarily Σ-pure and that Σ-pure subgroups are necessarily *-pure. This variance of purity will also be discussed in greater detail in Chapter 2. The discovery of Σ-pure subgroups led Hill and Megibben to answer - in the negative - the question posed by L. Nongxa in [No1]: "Are all strongly pure subgroups of completely decomposable groups also completely decomposable?"

CHAPTER 2

Note that from henceforth, the word "group", unless otherwise stated, will mean a "**torsion-free** abelian group".

§ 2.1 **Introduction:** In this chapter we compare the relative strengths of some of the notions of purity restricted to torsion-free abelian groups which appear in Chapter 1. In particular, we will compare the concepts of, "being a direct summand", ordinary purity, strong purity, *-purity, Σ-purity, **s*-purity**, **t*-purity** (to be introduced in this chapter), regularity, strong (pure) regularity, kniceness, balancedness, *-balancedness, strong balancedness, $\mathcal{Z}$-strong balancedness (to be introduced in this chapter), semibalancedness, pre-balancedness, decency, and hyperbalancedness.

§ 2.2 Strong purity, direct summands and purity:

Recall that a subgroup H of G is said to be strongly pure if, for every $h \in H$, there exists a homomorphism $\psi : G \to H$ such that $\psi(h) = h$.

If G is any abelian group, and H **a summand** of G, then the projection $\pi : G \to H$ satisfies $\pi(h) = h$ for all $h \in H$. Thus:

2.2.1 Remark: **Any summand is a strongly pure subgroup** of G (thus "being a summand" implies strong purity).

In order to show that the implication sign above is not reversible, we need the following three remarks:

2.2.2 Remark: Let $F = \bigoplus_{i=1}^{n} \langle x_i \rangle$ be any **finite rank** free group and let H be a pure subgroup of F. By the Stacked Basis Theorem, there exist k, m_1, ..., m_k such that $H = \bigoplus_{i=1}^{k} \langle m_i x_i \rangle$. As H is pure in F, $H = \bigoplus_{i=1}^{k} \langle x_i \rangle$ which is a summand of G. Thus **each pure subgroup of a finite rank free group is a summand.**

2.2.3 Remark: Any pure subgroup of a free group is strongly pure.

Proof: Let $F = \bigoplus_{i \in I} \langle x_i \rangle$ be a free group and let H be a pure subgroup of F. For any $h \in H$, $h = \sum_{i \in I_0} r_i x_i$, where I_0 is a finite subset of I. Thus $h \in \bigoplus_{i \in I_0} \langle x_i \rangle$ which is a finite rank (free) summand of F. Now $\langle h \rangle_* \subseteq H$ and $\langle h \rangle_* \subseteq \bigoplus_{i \in I_0} \langle x_i \rangle$. By 2.2.2, $\langle h \rangle_*$ is a summand of $\bigoplus_{i \in I_0} \langle x_i \rangle$ and thus also a summand of F. The projection from F to $\langle h \rangle_*$ is a map from F to H which fixes h implying that H is strongly pure. ■

2.2.4 Remark: The converse of 2.2.1 does not necessarily hold; for, if we take a free resolution of $\mathbb{Q}$, then the kernel, which is pure, is, by 2.2.3, also strongly pure but is not a summand, otherwise, its complement, which is free would be isomorphic to $\mathbb{Q}$, a contradiction.

Suppose H is strongly pure in G and let $x \in H \cap p^kG$ for some $p \in \mathbf{P}$ and some $k \in \mathbf{N}$. Then $x = p^kg$ for some $g \in G$ and there exists $\phi : G \to H$ such that $x = \phi(x) = \phi(p^kg) = p^k\phi(g) \in p^kH$ and this implies that H is pure in G. Hence:

2.2.5 Remark [JR] : **Strongly pure subgroups are necessarily pure.**

In order to show that the converse of Remark 2.2.5 is not necessarily true, we require the following result, obtained by S. Janakiraman and K. Rangaswamy in [JR] and its corollary.

2.2.6 Lemma [JR] : If H is strongly pure in G, then for any **finite** subset $S = \{h_1, h_2, \ldots h_n\}$ of H, there exists $\phi : G \to H$ such that $\phi(x) = x$ for all $x \in S$.

Proof: This is by induction on n and the statement is true for n=1 by definition of strong purity.
Assume this is true for n-1 and let $\psi_{n-1} : G \to H$ be such that $\psi_{n-1}(h_i) = h_i$ for all $i = 1, 2, \ldots, n-1$. Since $h_n - \psi_{n-1}(h_n) \in H$, there is $\psi_n : G \to H$ such that

$\psi_n(h_n - \psi_{n-1}(h_n)) \quad = \quad h_n - \psi_{n-1}(h_n).$ (*)

Let $\phi = \psi_n + \psi_{n-1} - \psi_n\psi_{n-1}$. Then, for $1 \le i \le n-1$,

$$\begin{aligned}\phi(h_i) &= \psi_n(h_i) + \psi_{n-1}(h_i) - \psi_n(\psi_{n-1}(h_i)) \\ &= \psi_n(h_i) + h_i - \psi_n(h_i) = h_i, \text{ and}\end{aligned}$$

$$\begin{aligned}\phi(h_n) &= (\psi_n + \psi_{n-1} - \psi_n\psi_{n-1})(h_n) \\ &= \psi_n(h_n) + \psi_{n-1}(h_n) - \psi_n\psi_{n-1}(h_n) \\ &= h_n - \psi_{n-1}(h_n) + \psi_{n-1}(h_n) \quad \text{by } (*) \\ &= h_n\end{aligned}$$

Thus $\phi: G \to H$ fixes the whole of S. ■

<u>2.2.7 Corollary [JR]</u> : Finite rank strongly pure subgroups of torsion-free abelian groups are summands.

<u>Proof</u>: Let G be torsion-free and H a finite rank strongly pure subgroup of G, then $H = \langle S \rangle_*$ where $S = \{h_1, h_2, ..., h_n\}$ is finite. Let $\phi : G \to H$ be such that ϕ fixes the whole of S. If $x \in \langle S \rangle_*$, then $nx = \Sigma\, n_i h_i$, for some n, $n_i \in \mathbb{Z}$ which implies that $\phi(nx) = n\phi(x) = \Sigma\, n_i\, \phi(h_i) = \Sigma\, n_i h_i = nx$ and, by torsion-freeness, $\phi(x) = x$. This implies that ϕ is a projection and thus $H = \langle S \rangle_*$ is a summand of G. ■

Let G be any indecomposable torsion-free group of finite rank (see for example [Fu2: Example 5, p.125]), and let S be a proper non-zero pure subgroup of G. Then S cannot be strongly pure otherwise S would have to be a summand of G by Corollary 2.2.7, contradicting the assumption that G is indecomposable. Thus pure subgroups **need not**

be strongly pure.

§ 2.3 Regularity, strong regularity

Note that if H is a pure subgroup of G, then $\chi_G(h) = \chi_H(h)$ for any h ϵ H. Recall that a subgroup H of a torsion-free group G is said to be regular in G if, for every h ϵ H, $type_H(h) = type_G(h)$.

2.3.1 Remark: Since, from the definition of purity, the p-height of an element in a pure subgroup is the same as its p-height in the main group, **pure subgroups are regular**. However, $2\mathbb{Z}$ is not pure in $\mathbb{Z}$ but the type of any element in $2\mathbb{Z}$ is the same as its type in $\mathbb{Z}$ since both groups are cyclic. Hence $2\mathbb{Z}$ is regular in $\mathbb{Z}$ but not pure in $\mathbb{Z}$.

Recall that a subgroup H of G is said to be strongly regular in G if for every h ϵ H, there exists a homomorphism $\psi : G \rightarrow H$ such that $\psi(h) = n_h.h$ for some $n_h \epsilon \mathbb{Z}$. As pointed out in Chapter 1, this is an obvious generalisation of the definition of strong purity thus:

2.3.2 Remark: **Strongly pure subgroups are strongly regular.**

The group of even integers, $2\mathbb{Z}$, is in fact strongly regular in $\mathbb{Z}$ since multiplication by 2 gives a homomorphism satisfying the definition of strong regularity. We therefore also conclude that **strongly regular subgroups need not be pure** and therefore **not necessarily strongly pure**.

An analogue of the observation made by S. Janakiraman and K. Rangaswamy (Lemma 2.2.6) also holds for strongly regular subgroups, namely:

<u>2.3.3 Lemma</u> : If H is a strongly regular subgroup of G then, for any **finite** subset S of H, there exist an integer m and a homomorphism $\phi : G \rightarrow H$ such that $\phi(x) = mx$ for all $x \in S$.

<u>Proof</u>: Let $S = \{x_1, x_2, \ldots, x_n\}$ be any finite subset of H. The proof will be by induction on n and the case n = 1 follows immediately from the definition of strong regularity.

Suppose that there exist an integer r and a homomorphism $\psi_1 : G \rightarrow H$ such that $\psi_1(x_i) = rx_i$, $1 \le i \le n-1$. Then there exist an integer s and a homomorphism $\psi_2 : G \rightarrow H$ such that

$$\psi_2(rx_n - \psi_1(x_n)) = s(rx_n - \psi_1(x_n))$$
$$= rsx_n - s\psi_1(x_n)$$

If we define $\phi : G \rightarrow H$ by:

$\phi = s\psi_1 + r\psi_2 - \psi_2\psi_1$, then, for $1 \le i \le n-1$ we have

$$\phi(x_i) = s\psi_1(x_i) + r\psi_2(x_i) - \psi_2\psi_1(x_i)$$

$$= rsx_i + r\psi_2(x_i) - r\psi_2(x_i)$$

$$= rsx_i \text{ and}$$

$$\phi(x_n) = s\psi_1(x_n) + r\psi_2(x_n) - \psi_2\psi_1(x_n)$$

$$= s\psi_1(x_n) + \psi_2(rx_n - \psi_1(x_n))$$

$$= s\psi_1(x_n) + srx_n - s\psi_1(x_n) = srx_n \text{ thus } m = rs.$$ ■

The name given to this concept was motivated by the simple observation that:

2.3.4 Lemma: **Strongly regular subgroups are regular.**

Proof: Suppose H is strongly regular in G. Let $h \in H$. Then there exists $\psi : G \to H$ a homomorphism and $n_h \in \mathbb{Z}$ such that $\psi(h) = n_h h$. Thus

$$\text{type}_H(h) \leq \text{type}_G(h) \leq \text{type}_H(\psi(h)) = \text{type}_H(n_h h)$$

$= \text{type}_H(h)$ and thus $\text{type}_H(h) = \text{type}_G(h)$ and H is regular in G. ■

§ 2.4 **s*-purity, t*-purity, *-purity, *-purely generated and Σ-purity.**

Recall that, in [No2], a pure subgroup H of a torsion-free group is called ***-pure** if

$$\langle H(\tau^*)_* = H \cap \langle G(\tau^*)\rangle_* \text{ for every type } \tau.$$

In [HM1], a pure subgroup H of a torsion-free group G is also called ***-pure** if $H \cap G(s^*) = H(s^*)$ and $H \cap G(s^*,p) = H(s^*,p)$ for all height sequences s and all primes p. These two definitions were introduced almost simultaneously and independent of each other. It will be shown that they are not equivalent and, the names possibly coincided since in the one case use is made of the fully invariant subgroup $G(\tau^*)$ and in the other, use is made of $G(s^*)$. We will attempt to resolve this situation as follows:

2.4.1 Definition: Let H be a pure subgroup of a torsion-free group G; then H is:

a) ***-pure** in G if it satisfies the definition of P.Hill and C. Megibben;

b) **s^*-pure** in G if $H(s^*) = H \cap G(s^*)$ for all height sequences s;

c) **t^*-pure** in G if $H(\tau^*) = H \cap G(\tau^*)$ for all types τ; and

d) ***-purely generated** in G if it satisfies the definition of L. Nongxa.

2.4.2 Remark: It follows immediately from the definition that *-purity implies s*-purity. However, the converse is not true as we see in the following example.

<u>2.4.3 Example:</u> Let $G = G_1 \oplus G_2$, where G_1 is infinite cyclic and $G_2 = \{m/2^k : m \text{ and } k \text{ integers}\}$. Let us choose any prime $p \neq 2$, x_1 in G_1 whose height sequence is (0,0, ..., 0), x_2 in G_2 of height sequence

(∞, 0,0, ..., 0). Let $x = px_1 + x_2$ and $H = \langle x \rangle_*$. Then H is s^*-pure in G since, for all height sequences s, $H(s^*) = \{0\} = H \cap G_2$ and $G(s^*) = G_2$. However, $x \notin pH(s^*) = H(s^*,p)$ whereas $x \in G(s^*,p) \cap H$. Thus H is not *-pure in G.

<u>2.4.4 Remark</u>: From the observation that, for every type τ, and any torsion-free group g, $G(\tau^*) = \bigcup_{s \in \tau} G(s^*)$ we can deduce that every **s^*-pure subgroup is t^*-pure** since

$$H \cap G(\tau^*) = H \cap (\bigcup_{s \in \tau} G(s^*)) = \bigcup_{s \in \tau} (H \cap G(s^*)) = \bigcup_{s \in \tau} H(s^*)$$

$$= H(\tau^*).$$

<u>2.4.5 Remark</u>: It can also be seen that the notion of **t*-purity introduced above implies *-purely generated,** since, if $H(\tau^*) = H \cap G(\tau^*)$ for any type τ, with H pure in G, then $\langle H(\tau^*) \rangle_* \subseteq H \cap \langle G(\tau^*) \rangle_*$. Also, if

$h \in H \cap \langle G(\tau^*) \rangle_*$, then $nh \in G(s^*)$ for some nonzero integer n and $s \in \tau$. This implies that

$nh \in H \cap G(s^*) = H(s^*) \subseteq H(\tau^*) \subseteq \langle H(\tau^*) \rangle_*$ and therefore

$h \in \langle H(\tau^*)\rangle_*$. However, the converse is not true as can be illustrated by the following example:

<u>2.4.6 Example</u>: Let $G = G_1 \oplus G_2 \oplus G_3$ be a torsion-free group where G_i is of rank one, $T(G_i) = \tau_i$, G_i is reduced for $1 \leq i \leq 3$, $\tau_1 \parallel \tau_2$, and $\tau_3 \geq \max\{\tau_1, \tau_2\}$. Let p be a prime for which G_3 - and thus G_1 and G_2 - is not p-divisible and let $g_i \in G_i$ with $h_p^G(g_i) = 0$. Let $H = \langle g_3+pg_1, g_3+pg_2\rangle_*$. H is $*$-purely generated as $\langle H(\tau_i^*)\rangle_* = \{0\} = H \cap G = H \cap G(\tau_i^*)$ for $i = 1,2$ and $H = \langle H(\tau_0^*)\rangle_* = H \cap G = H \cap G(\tau_0^*)$ but $H(\tau_0^*) = H(\tau_1) \oplus H(\tau_2) \neq H$ as by Lemma 1 in [Bi2], H is not completely decomposable. Hence H is not t^*-pure in G.

<u>2.4.7 Lemma</u> : If H is a pure, strongly regular subgroup of G, then H is $*$-purely generated.

<u>Proof</u>: Let H be a pure, strongly regular subset of G and let τ be any type. For any $h \in H \cap \langle G(\tau^*)\rangle_*$, h in H implies that there exists a homomorphism $\psi : G \to H$ so that $\psi(h) = n_h.h$; and h in $\langle G(\tau^*)\rangle_*$ implies that there is an integer m_h such that $m_h.h \in G(\tau^*)$. Thus $m_h.h = \Sigma_i g_i$ such that $\text{type}(g_i) > \tau$, where $g_i \in G$.
Now $m_h n_h h = \psi(m_h h) = \Sigma_i \psi(g_i) \in H(\tau^*)$ and this implies that $h \in \langle H(\tau^*)\rangle_*$. Since $H(\tau^*) \subseteq H \cap G(\tau^*)$, purity of H in G implies that $\langle H(\tau^*)\rangle_* \subseteq H \cap \langle G(\tau^*)\rangle_*$ and hence we have equality. ■

Recall that in [HM2] a subgroup H of a torsion-free group

G is **Σ-pure** if, for any finite set of height sequences $\{s_1, s_2, \ldots, s_n\}$, $\Sigma_i H(s_i) = H \cap \Sigma_i G(s_i)$.

2.4.8 Proposition [HM2] :

Strongly pure ⇒ Σ-pure ⇒ *-pure ⇒ pure.

Proof:

1. **Strongly pure implies Σ-pure.**

Let H be a strongly pure subgroup of G and suppose that $g_1 + \ldots + g_n = h \in H$, with $g_i \in G(s_i)$. There is a homomorphism, $\phi_h : G \to H$ that leaves h fixed. Thus $h = \Sigma_i h_i$ where $h_i = \phi_h(g_i) \in H$, and

$\chi_H(h_i) = \chi_H(\phi_h(g_i)) \geq \chi_G(g_i) \geq s_i$ for $1 \leq i \leq n$. Thus $h \in \Sigma_i H(s_i)$ which implies that H is Σ-pure in G.

2. **Σ-pure implies pure.**

Suppose H is Σ-pure in G. Let $h = ng \in H \cap nG$ and put $s = \chi_G(g)$. Then $h \in H \cap G(ns) = H(ns) = nH(s)$. Thus $h = nh'$ where $h' \in H(s)$ and torsion-freeness implies that $g = h'$ and $h \in nH$.

3. **Σ-pure implies *-pure.**

Suppose H is Σ-pure in G. By 2, H is pure in G and thus for any $h \in H$, $\chi_G(h) = \chi_H(h)$. Let $h \in H \cap G(s^*)$. Then $h = g_1 + \ldots + g_n$ with $g_i \in G(s)$ and $\chi_G(g_i) \not\geq s$. By Σ-purity, $h = h_1 + \ldots + h_n$ with each $h_i \in H(\chi_G(g_i))$ and thus for each $i = 1, 2, \ldots, n$, $\chi_H(h_i) \not\geq s$ which means that $h \in H(s^*)$. Let $h \in H \cap G(s^*, p)$ for any $p \in \boldsymbol{P}$. Then

$h = g_1 + \ldots + g_n + g$ where $g \in G(ps)$ and for each $i=1,2,\ldots,n$, $g_i \in G(s)$ and $\chi_G(g_i) \nsucceq s$. By Σ-purity, $h = h_1 + \ldots + h_n + h'$ where $h' \in H(\chi_G(g)) \subseteq H(ps)$ and for each $i=1,2,\ldots,n$, $h_i \in H(\chi_G(g_i))$ and thus $\chi_H(h_i) \geq s$ and $\chi_H(h_i) \nsucceq s$. Thus $h \in H(s^*,p)$ and H is $*$-pure in G.

4. By definition, *-purity implies purity. ■

2.4.9 Remark: 2.4.3 is an example of a pure subgroup which is not *-pure. 3.4.7 in Chapter 3 is an example of a *-pure set which is not Σ-pure and 2.6.5 is an example of a Σ-pure subgroup which is not strongly pure. Thus we see that none of the implications of proposition 2.4.8 is reversible.

§2.5 VARIANCES OF BALANCEDNESS

In this section we compare balancedness with being a direct summand, strongly balanced, *-balanced, semi-balanced, prebalanced, hyperbalanced and Z-strongly balanced (a concept which will be introduced in this section).

We quote the following result which is especially useful when we quote examples.

2.5.0 Lemma (See [Fu2]) :

For any abelian group G, there exist a completely decomposabale group A and an epimorphism $\gamma : A \to G$ such that ker γ is balanced in G. $\gamma : A \to G$ is called the **balanced resolution of G.**

2.5.1 Remark: **All direct summands of a group G are also balanced in G.** This follows from the fact that if $H \oplus K$ is a decomposition of G and $g = h + k$ is an arbitrary element of G then $\chi_G(h+k) = \chi_G(h) \wedge \chi_G(k) \leq \chi(k)$ for any $h \in H$ and $k \in K$ and hence $\chi_{G/H}(g+H) = \chi_{G/H}(k+H) = \chi_G(k)$.

2.5.2 Remark: Recall that a *-balanced subgroup H of G is a subgroup that is balanced and which satisfies $\langle H(\tau^*)\rangle_* = H \cap \langle G(\tau^*)\rangle_*$. Thus by definition, *-balanced subgroups are necessarily balanced.

Recall that a subgroup H of G is said to be strongly balanced in G if given any coset g+H, there is a homomorphism $\phi:G/H \to G$ such that g+H is fixed by $\psi\phi$ where ψ is the canonical map from G to G/H. The motivation for the name 'strongly balanced' is not apparent from the definition. However, H strongly balanced in G implies that H is balanced in G as is proved in the following lemma.

2.5.3 Lemma: All strongly balanced subgroups are balanced.

Proof: Suppose H is strongly balanced in G. Then for any $g + H \in G/H$, there is $\sigma : G/H \to G$ such that $\psi\sigma\psi(g) = \psi(g)$. If we put $g' = \sigma(g + H)$, then $\psi(g) = g+H = \psi\sigma(\psi(g)) = \psi(g') = g'+H$. Now $\chi_G(g') \leq \chi_{G/H}(g'+H) = \chi_{G/H}(g+H) \leq \chi_G(\sigma(g+H)) = \chi_G(g')$. Thus $g' = \sigma(g+H)$ is proper in g+H. ■

Examples of strongly balanced subgroups can be found easily in the light of the following result.

2.5.4 Theorem: Suppose H is balanced in G and G/H is separable then H is strongly balanced in G.

Proof: If $g \in G \backslash H$ then $0 \neq g+H \in G/H$ and by separability, there exists some $C/H \subseteq G/H$ such that C/H is a finite rank completely decomposable summand of G/H containing g+H. As completely decomposable groups are balanced projective, we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & C/H & & \\ & & & & & \theta \swarrow & \pi \uparrow\downarrow i & & \\ 0 & \to & H & \to & G & \xrightarrow{\psi} & G/H & \to & 0 \end{array}$$

where i is the inclusion map, ψ is the canonical map, π is the projection map, and θ is the induced map so that $\psi\theta = i$. Now $\theta\pi : G/H \to G$ with

$$\begin{aligned} \psi\theta\pi\psi(g) &= \psi\theta\pi(g+H) \\ &= \psi\theta(g+H) \quad (\text{as } g+H \in C/H) \\ &= g+H \quad (\text{as } \psi\theta = i) \end{aligned}$$

$= \psi(g)$.

Thus H is strongly balanced in G. ■

2.5.5 Remark: Strongly pure and strongly regular subgroups of G are defined in terms of a single element in G. However, if we have a finite subset S in G, we can find a homomorphism from G to H which fixes S (see Lemma 2.2.6 in the case where H is strongly pure) or multiplies the elements in S by integers (see Lemma 2.3.3 in the case that H is strongly regular). In exactly the same way, if S is a finite set in G/H and H is strongly balanced in G, we can find a homomorphism θ from G/H to G so that $\psi\theta$ fixes all the elements of S (where ψ is the canonical homomorphism from G to G/H).

2.5.6 Theorem [Ra3]: Suppose H is strongly balanced in G where G/H is countable, then H is a direct summand of G.

Proof: Let $G/H = \{g_1+H, g_2+H, g_3+H, \ldots\}$. For each $n=1,2,3,\ldots$ let $G_n/H = \langle g_1+H, g_2+H, \ldots, g_n+H\rangle$. By 2.5.5, for each $i \geq 1$, we can find $f_i:G/H \to G$ such that ψf_i restricted to G_i/H is the identity (here again, ψ is the canonical epimorphism from G to G/H). As $g_{i+1}+H - \psi f_i(g_{i+1}+H) \in G/H$, there is a homomorphism $\gamma:G/H \to G$ such that $\psi\gamma$ fixes $((g_{i+1}+H) - \psi f_i(g_{i+1}+H))$. Now define $f_{i+1} = \gamma + f_i - \gamma\psi f_i$ and note that $f_{i+1}:G/H \to G$, f_{i+1} restricted to G_i/H is f_i, and ψf_{i+1} restricted to G_{i+1}/H is the identity map. Now define $f: G/H \rightarrow G$ by $f(x+H) = f_n(x+H)$ if n is the least positive integer such that $x+H \in G_n/H$. Thus ker f = H is a summand of G. ■

2.5.7 Remark: Let G be a countable indecomposable group and let $A \rightarrow^{\gamma} G$ be a balanced resolution of G. This means that A is completely decomposable, γ is an epimorphism and ker γ is balanced in G. Thus we have that $G \simeq A/\ker \gamma$. If ker γ were strongly balanced in G, then by 2.5.6 ker γ would be a direct summand of A and thus by the Baer-Kaplansky-Kulikov Theorem, A/ker γ is completely decomposable, a contradiction. Hence balancedness does not imply strongly balancedness.

We introduce now a dual of the concept of strongly regular.

2.5.8 Definition: A pure subgroup H of G is said to be ***Z*-strongly balanced** if for every $g \in G$ there is a homomorphism $\sigma: G/H \rightarrow G$ and an integer n_g such that $\psi\sigma\psi(g) = n_g\psi(g)$.

The name *Z*-strongly **balanced** was chosen as this notion is related to being strongly balanced. However, we have not investigated the properties of *Z*-strongly balanced subgroups and we have not ascertained whether such groups are in fact **balanced**.

Subgroups of this nature occur naturally as will be demonstrated in the next result.

<u>2.5.9 Theorem</u>: Suppose H is balanced in G and G/H is almost completely decomposable, then H is $\mathbb{Z}$-strongly balanced in G.

<u>Proof</u>: As G/H is almost completely decomposable there is a completely decomposable subgroup C/H of G/H such that $|(G/H)/(C/H)| = n$, say and thus, for any $g \in G$, $n(g+H) \in C/H$. Hence define the homomorphism $\nu: G/H \to C/H$ by $\nu(g+H) = ng+H$. As C/H is completely decomposable and hence balanced projective, there is a $\theta: C/H \to G$ such that the following is a commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & C/H & & \\ & & & & & \theta \swarrow & \nu \uparrow \downarrow i & & \\ 0 & \to & H & \to & G & \xrightarrow{\psi} & G/H & \to & 0 \end{array}$$

where i is the inclusion map, ψ is the canonical map. The homomorphism $\theta\nu: G/H \to G$ satisfies

$$\begin{aligned} \psi\theta\nu\psi(g) &= \psi\theta\nu(g+H) \\ &= \psi\theta(ng+H) \\ &= ng+H \qquad (\text{as } \theta\psi = i) \\ &= n(\psi(g)). \end{aligned}$$

Thus H is $\mathbb{Z}$-strongly balanced in G. ■

<u>2.5.10 Remark</u> It is clear that all strongly balanced

subgroups are Z-strongly balanced for in this case, $n_g = 1$ for all g in G.

Recall that a pure subgroup H of G is said to be <u>semi-balanced</u> if for every $g \in G$, there exists $\{h_1, h_2, \ldots, h_n\} \subseteq H$ such that $\chi_{G/H}(g+H) = \chi_G(g+h_1) \vee \chi_G(g+h_2) \vee \ldots \vee \chi_G(g+h_n)$. Recall also that a pure subgroup H of G is said to be <u>prebalanced</u> in G if for every $g \in G$, there exist $h_1, h_2, \ldots, h_n$ in H such that $\langle H,g\rangle_* = H + \langle g+h_1\rangle_* + \langle g+h_2\rangle_* + \ldots + \langle g+h_n\rangle_*$.

<u>2.5.11 Remark</u>: All balanced subgroups are semi-balanced for then n would just be 1.

<u>2.5.12 Lemma:</u> Let G be a torsion-free group whose typeset is a chain. A pure subgroup of G is semi-balanced if and only if it is balanced.

<u>Proof:</u> By 2.5.11, we need only show that if H is semi-balanced in G, then H is balanced in G. Let $g \in G \setminus H$ and let $\{h_1, h_2, \ldots, h_n\}$ be such that $\chi_{G/H}(g+H) = \bigvee_{i=1}^{n} \chi_G(g+h_i)$. Since the typeset of G is a chain, we can assume that $\text{type}_{G/H}(g+H) = \bigvee_{i=1}^{n} \text{type}_G(g+h_i) = \text{type}_G(g+h_n)$. The coset g+H contains an element of the same type as the coset and by Baer's Lemma [Fu2: Theorem 86.4] g+H contains an element of the same height sequence as the height

sequence of the coset. As g+H was an arbitrary coset, H is balanced in G. ■

<u>2.5.13 Lemma [FV]</u> : Semi-balanced subgroups are necessarily prebalanced.

<u>Proof</u>: Let g be any element in G. By our assumption, $\chi_{G/H}(g+H) = \chi_G(g+h_1) \vee \chi_G(g+h_2) \vee \ldots \vee \chi_G(g+h_n)$ where $h_i \in H$, $1 \leq i \leq n$. Since for all the i's, $h_i \in H$, we obviously have that

$$H + \langle g+h_1 \rangle_* + \langle g+h_2 \rangle_* + \ldots + \langle g+h_n \rangle_* \subseteq \langle H,g \rangle_*.$$

Now let $x \in \langle H,g \rangle_*$. There are integers n and m such that $nx = m(h+g)$ for some $h \in H$ and, as G is torsion-free, we can assume that $(n,m)=1$. For any prime p and integer k satisfying $p^k \mid n$, we have that $p^k \nmid m$ which implies that $p^k \mid (h+g)$. Let n_k and g_k be such that $n = p^k n_k$ and $h+g = p^k g_k$. Since $p^k \mid (h+g)$, $p^k \mid g+H$ which by our assumption, implies that $p^k \mid g+h_i$ for at least one i. Thus let g_i be such that $g+h_i = p^k g_i$. Now $p^k g_k - p^k g_i = g+h - (g+h_i) = h - h_i \in H$. The torsion-freeness of G and the purity of H means that $g_k - g_i \in H$. Thus $g_k \in H + \langle g+h_i \rangle_*$ and

$$n_k x = m g_k \in H + \langle g+h_i \rangle_*.$$

Repeating this process with all the prime power factors of n, we conclude that $x \in H + \sum_{i=1}^{n} \langle g+h_i \rangle_*$. ■

We have the following characterization of a prebalanced subgroup.

2.5.14 Lemma [FV] : H is prebalanced in G if and only if any rank one pure subgroup K/H of G/H is the sum of the images of pure, rank one subgroups X_1, X_2, ... , X_n of G under the canonical map.

Proof: Suppose H is prebalanced in G. Let K/H be a rank one subgroup of G/H which means that $K/H = \langle g+H \rangle_*$ for some $g \in G$. By our assumption, there exist h_1, h_2, ..., h_n all in H such that $\langle H,g \rangle_* = H + \langle g+h_1 \rangle_* + \langle g+h_2 \rangle_* + \ldots + \langle g+h_n \rangle_*$. Let ψ be the canonical map. $\psi(\langle H,g \rangle_*) = \langle g+H \rangle_*$ and thus $\langle g+H \rangle_* = \Sigma\psi(\langle g+h_i \rangle_*)$.

Conversely, suppose the rank one, pure subgroup, $\langle g+H \rangle_* = K/H = \langle H,g \rangle_*/H = \Sigma\psi(X_i)$ where the X_i's are all rank one pure subgroups in G. This implies that $\langle H,g \rangle_* = H + \Sigma_i X_i$. For $1 \leq i \leq n$, let $X_i = \langle x_i \rangle_*$. We now have that $x_i \in \langle H,g \rangle_*$ and thus there are integers n, m with $(n,m)=1$ such that $nx_i = mg + h_i$ for some h_i in H which in turn implies that $\psi(h_i + mg) = \psi(nx_i)$. Hence $\langle x_i \rangle_* + H = \langle g+h_i \rangle_* + H$ and we have found the desired set of h_i's. ■

2.5.15 Theorem [FV] : Let H be a pure subgroup of G where G is of finite rank. G is a Butler group if and only if :

(1) both H and G/H are Butler groups; and

(2) H is prebalanced in G.

Proof: Suppose G is a Butler group. Since G/H is an epimorphic image of G and H is pure in G, G/H and H are also Butler groups. Let $g \in G\backslash H$ and $H' = \langle H,g\rangle_*$. H' is pure in G and thus H' is also a Butler group. Thus

$$H' = G_1 + G_2 + \ldots + G_n = H + G_1 + G_2 + \ldots + G_n$$

where each G_i is of rank 1 and is pure in G. By Lemma 2.5.14, H is prebalanced in G.

Conversely, suppose (1) and (2) hold. As G/H is Butler, $G/H = C_1 + C_2 + \ldots + C_n$ where each C_i is rank one and pure in G/H. As H is prebalanced in G, each $C_i = \psi(D_i)$ where $D_i = H + H_{i1} + H_{i2} + \ldots + H_{i,m(i)}$ and each H_{ij} is rank one. Hence $G = H + \sum_{i=1}^{n} (H_{i1} + H_{i2} + \ldots H_{im(i)})$. Since H is itself a Butler group, G is Butler. ■

<u>2.5.16 Corollary [FV]</u> : A group of finite rank is a Butler group if and only if all of its pure subgroups are prebalanced.

Proof: Let G have finite rank. If G is a Butler group, then the result follows from Theorem 2.5.15. Conversely, suppose all pure subgroups H of G are prebalanced. Let $\{g_1, g_2, \ldots, g_n\}$ be a maximal independent set in G and define $G_i = \langle g_1, \ldots, g_i\rangle_*$. As each G_i is pure in G, G_i is prebalanced in G. G_1 and each G_i/G_{i-1} are rank 1 and thus Butler. Thus by induction and theorem 2.5.15, each G_i is

Butler. Thus $G = G_n$ is Butler. ■

2.5.17 Example: Let $G = G_1 \oplus G_2$, where G_1 is infinite cyclic and $G_2 = \{m/2^k : m \text{ and } k \text{ are integers}\}$. Let p be any prime not equal to 2, $x_1 \in G_1$ whose height sequence is $(0,0,\ldots,0,\ldots)$, $x_2 \in G_2$ whose height sequence is $(\infty,0,\ldots,0,\ldots)$, $x = px_1 + x_2$ and $H = \langle x\rangle_*$. Let $\{\tau_1\} = T(G_1)$ and $\{\tau_2\} = T(G_2)$ and note that $\tau_1 < \tau_2$ and thus the typeset of G is a chain. By 2.5.16, H is prebalanced. Suppose H is semi-balanced. Then by 2.5.12, H is balanced in G. Now G/H is rank one and thus is balanced projective, making H a direct summand of G. Now if x_2 has non-zero component in H, then $\text{type}_G(x_2) = \tau_1$ thus $1 = h_p^G(px_1) = h_p^G(px_1+x_2-x_2) = h_p^G(px_1+x_2) \wedge h_p^G(x_2) = 0$, which gives a contradiction. Thus a semi-balanced subgroup is necessarily prebalanced but prebalancedness does not necessarily imply semi-balancedness.

We now compare decent, balanced and prebalanced.

2.5.18 Lemma [FV] : If a pure subgroup H of G is decent, then it is prebalanced.

Proof: Let $g \in G\backslash H$ and let H be decent in G. There are pure rank one subgroups, $G_1, G_2, \ldots, G_n$ such that $g \in H + C$ where $C = G_1 + G_2 + \ldots + G_n$ and $H + C$ is pure in G and thus $\langle H,g\rangle_* \subseteq H + C$. By the modular law we have that $\langle H,g\rangle_* = \langle H,g\rangle_* \cap (H + C) = H + (C \cap \langle H,g\rangle_*)$. C is a Butler group and thus $C \cap \langle H,g\rangle_*$, which is a pure subgroup of C, is a Butler group which in turn implies that $\langle H,g\rangle_* = H + \Sigma C_i$ where each C_i is rank one and pure in G. ■

2.5.19 Example: Let A be a finite rank homogeneous, indecomposable group and let $G \to^{\gamma} A$ be a balanced resolution of A. G is completely decomposable and as $H = \ker \gamma$ is balanced in G, H is prebalanced. Suppose H is decent. Let $T = \{a_1, a_2, \ldots, a_n\}$ be a maximal independent set in A and let $S = \{g_1, g_2, \ldots, g_n\}$ be such that $\gamma(g_i) = a_i$ for each $i = 1, 2, \ldots, n$. As H is decent, $S \subseteq G' = H + G_1 + G_2 + \ldots + G_m$, where each G_i is a pure, rank one subgroup of G, and G′ is also pure in G. The Butler group, G′/H is pure in $G/H \simeq A$ and G′/H contains the whole of T and thus G′/H = A. Thus A is a homogeneous, finite rank Butler group and M. C. R. Butler proved in [Bu] that homogeneous, finite rank Butler groups are completely decomposable providing the contradiction. Thus a prebalanced subgroup of G is necessarily decent in G, but the converse is not true.

We will now look at the relationship between balanced and hyperbalanced.

2.5.20 Lemma [DHR] : Any hyperbalanced subgroup H of G is also balanced in G.

Proof : Suppose H is hyperbalanced in G i.e. for any $g \in G$ and any countable set C in H, there is some $h \in H$ with $g_C^{\#} = h_C^{\#}$ where $k_C^{\#}: C \rightarrow$ the set of height sequences, k any element in G, is defined as $k_C^{\#}(x) = \chi(k+x)$ for all $x \in C$. Assume that H is **not** balanced in G. Then there is at least one coset, g+H, say without proper element. Let g_0 be any element in g+H and then suppose that g_γ has been defined for all $\gamma < \alpha$ where $\alpha < \omega_1$. Note that $C = \{g_\gamma - g : \gamma < \alpha\}$ is countable. Since H is hyperbalanced, there is some $h_1 \in H$ with $\chi_G(g+(g_\gamma-g)) = \chi_G(h_1+g_\gamma-g)$ for all $\gamma < \alpha$. Thus $\chi_G(g-h_1) \geq \chi_G(g_\gamma)$ for each γ. As g+H has no proper element, there is some $h_2 \in H$ with $\chi_G(g-h_1) < \chi_G(g-h_2)$. Now define $C' = C \cup \{-h_1; h_2\}$ which is still a countable set in H. Thus there is $h_3 \in H$ with $\chi_G(g+(g_\gamma-g)) = \chi_G(h_3+g_\gamma-g)$. As above, $\chi_G(g-h_3) \geq \chi_G(g_\gamma)$, $\chi_G(g-h_3) \geq \chi_G(h_2)$, and $\chi_G(g-h_3) \geq \chi_G(h_1)$. Note that for all $\gamma < \alpha$, $\chi_G(g_\gamma) < \chi_G(g-h_3)$ for else

$$\chi_G(g+h_2) \leq \chi_G(g+h_3) = \chi_G(g_\gamma) \leq \chi_G(g-h_1) < \chi_G(g-h_2)$$

which is a contradiction. Set $g_\alpha = g+h_3$. We have now constructed an uncountable sequence of elements in g+H in such a way that the height sequences of the elements of the sequence are strictly increasing. This contradicts the fact that each sequence of strictly increasing height sequences can have only countable many elements. ■

§2.6 **KNICE SUBGROUPS**

The concept of knice subgroups requires the introduction of primitive elements, valuated coproducts, *-valuated coproducts and free *-valuated subgroups as introduced by P. Hill and C. Meggiben in [HM1]. We have omitted a discussion of these concepts in this chapter and left it to appendix A.1 in order not to interrupt our exposition of the different notions of purity.

2.6.1 Definition: A subgroup N of the torsion-free group G, is said to be a knice subgroup if for each finite subset S of G, there are primitive elements y_1, $y_2, \ldots, y_m$ such that $N' = N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \ldots \oplus \langle y_m \rangle$ with N′ a *-valuated coproduct with $\langle S, N' \rangle / N'$ finite.

2.6.2 Lemma [HM1] : If N is both pure and knice in G, then the y_i's can be chosen so that $S \subseteq N'$.

Proof:

Suppose $S = \{x\}$ and suppose that $N' = N \oplus \langle y_1 \rangle \oplus \ldots \oplus \langle y_n \rangle$, then as $(N' + \langle x \rangle)/N'$ is finite, there is a $k \in \mathbb{N}$ so that $kx \in N'$. Thus $kx = n + \sum_{i=1}^{n} l_i y_i$ where $l_i \in \mathbb{Z}$ for each i. If $s = \chi(x)$, then $kx \in G(ks)$. However, N′ is a *-valuated coproduct and thus $l_i y_i \in G(ks) = kG(s)$. Thus, for each i, there exists $(y')_i \in G(s)$ such that $l_i y_i = k(y')_i$. Now

$n = kx - \sum_{i=1}^{n} ky_i' = k(x - \sum_{i=1}^{n} y_i')$ and thus, by purity of N, $x - \sum_{i=1}^{n} y_i' \in N$ and $x \in N + \sum_{i=1}^{n} y_i'$.

Furthermore, as the y_i's and the $(y')_i$'s are linearly dependent, the $(y')_i$'s generate their own direct sum, $N \oplus \bigoplus_{i=1}^{n} \langle y_i' \rangle$ is a *-valuated coproduct and $N'' = N \oplus \bigoplus_{i=1}^{n} \langle y_i' \rangle$ is such that $(N'' + S)/N''$ is finite. Thus N" satisfies the required conditions. Now suppose that $S = \{x_1; x_2\}$. In the same way as above, we can find k_1, k_2, l_i, m_i integers and y'_i, and y''_i such that $k_1x_1 = n_1 + \sum_{i=1}^{n} l_i y_i$; $k_2x_2 = n_2 + \sum_{i=1}^{n} m_i y_i$;

$x_1 + \sum_{i=1}^{n} y_i' \in N$ and $x_2 + \sum_{i=1}^{n} y_i'' \in N$. For each i, $m_i k_1 y'_i = l_i k_2 y''_i$ and thus $\chi(y'_i) \sim \chi(y''_i)$. Thus $A = \{p : h_p(y'_i) \neq h_p(y''_i) < \infty\}$ is finite. Construct each z_i as follows: for each $p \in A$, $h_p(z) = 0$ and for each $p \notin A$, $h_p(z) = h_p(y'_i) = h_p(y''_i)$ and at the same time, $n_p = \prod_{p \in A} [h_p(y'_i) \vee h_p(y''_i)]$ is such that hz is both a multiple of y'_i and y''_i. Then $x_i \in N \oplus \langle z_1 \rangle \oplus \ldots \oplus \langle z_n \rangle$ for each i. Continuing in this way, we can show that if $|S|$ = an arbitrary m, say, then we can find suitable y_i's so that S is completely contained in $N \oplus \langle y_1 \rangle \oplus \ldots \oplus \langle y_n \rangle$. ■

<u>2.6.3 Proposition [HM1]</u> : If $N' = N \oplus \langle x_1 \rangle \oplus \ldots . \oplus \langle x_n \rangle$

is a $*$-valuated coproduct in G with N a knice subgroup of G and each x_i primitive in G, then N′ is a knice subgroup of G.

Proof: By induction, it suffices to consider case n=1. Assume then that $N' = N \oplus \langle x \rangle$ is a $*$-valuated coproduct in G with N knice and x primitive. Let S be a finite subset of G. Let $S' = S \cup \{x\}$. Since N is knice, there is a $*$-valuated coproduct $F = N \oplus \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \ldots \oplus \langle y_m \rangle$, where the y_i's are primitive and $\langle S', F \rangle / F$ is finite. Thus there is a multiple of x, say x′, such that $x' \in F$. Thus $x' = z + t_1 y_1 + \ldots + t_m y_m$, where $t_i \in \mathbf{N}$. Since $N \oplus \langle x' \rangle$ is a $*$-valuated coproduct, if all the $t_i y_i$'s had type greater than the type of x′, the primitivity of x′ would be contradicted. Then $x' = z + y + g$ where the primitive element y is the sum of the $t_i y_i$'s having the same type as x′. ■

2.6.4 Proposition [HM2] : A subgroup which is pure and knice is Σ-pure.

Proof: Suppose $h = g_1 + g_2 + \ldots + g_n$, where $h \in H$ and each g_i is in $G(s_i)$ and let $S = \{g_1, g_2, \ldots, g_n\}$ then, by kniceness, there is a $*$-valuated coproduct $H \oplus A$ in G which contains s. Writing $g_i = h_i + a_i$, $a_i \in A$ and $h_i \in H$, we see that $\chi(g_i) \leq \chi(h_i)$. Thus $h_i \in H(s_i)$ for all the i's and, as
$h - (h_1 + h_2 + \ldots + h_n) = a_1 + \ldots + a_n$ and is thus in

$H \cap A = \{0\}$. Thus $h = h_1 + h_2 + \ldots + h_n$ and H is Σ-pure in G. ■

The following example shows that the above implication is not reversible.

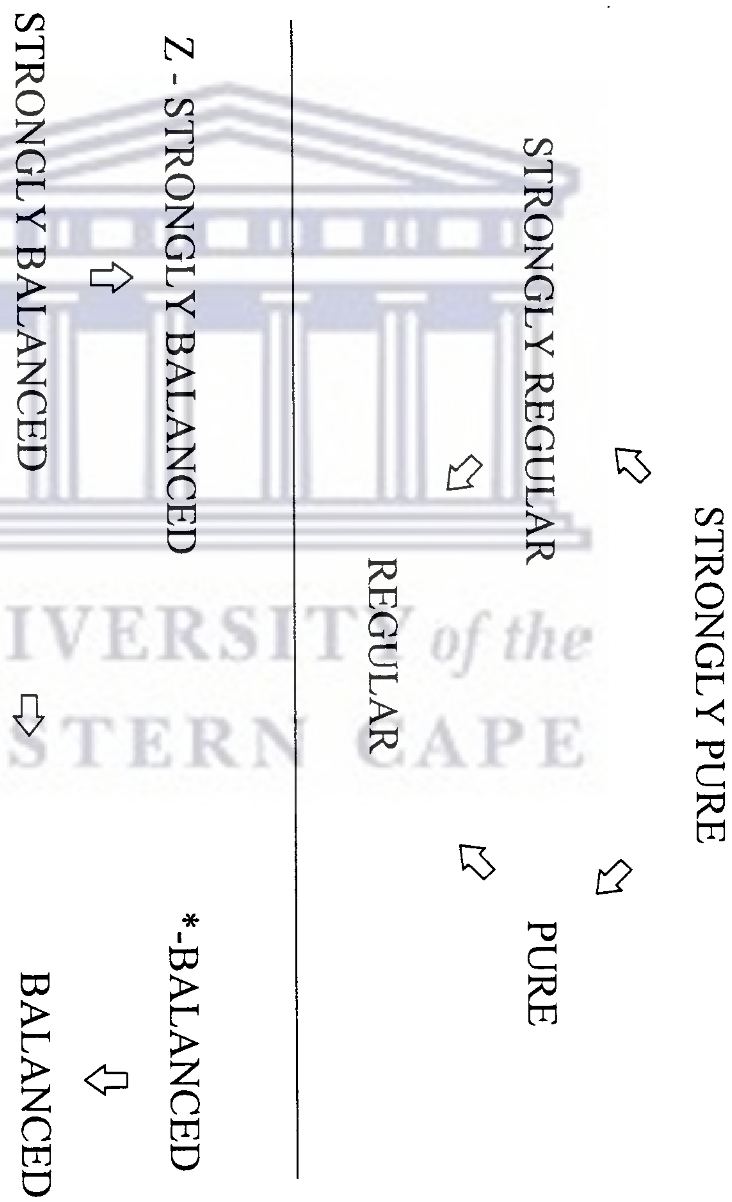
2.6.5 Example: Let $G = \langle \oplus_\omega Z,\ 2\Pi_\omega Z \rangle$. All countable sets in G are free and thus every non-zero element in G is primitive in G and if $F = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_n \rangle$ is pure in G, then F is a *-valuated subgroup of G. Let $x = (2,2,\ldots,2,\ldots)$ then x is not contained in any finite rank summand of G but $\langle x \rangle_*$ is pure and knice in G. By 2.6.4, $\langle x \rangle_*$ is Σ-pure in G. However, as $\langle x \rangle_*$ fails to be a summand of G, $\langle x \rangle_*$ is not strongly pure in G and thus, by 2.4.8, $\langle x \rangle_*$ is not knice. Thus a subgroup of G which is knice and pure is necessarily Σ-pure but $\langle x \rangle_*$ above is pure and Σ-pure but not knice in G.

2.6.6 Proposition [HM1] : A pure and knice subgroup H of G is balanced in G.

Proof: Let H be pure and knice in G. Let $x \in G\backslash H$. Then there are $y_1, y_2, \ldots, y_n$ such that $x \in H \oplus \langle y_1 \rangle \oplus \ldots \oplus \langle y_n \rangle$ which is a *-valuated coproduct in G. Thus there exist $z \in H$ and integers $m_1, m_2, \ldots, m_n$ such that $x = z + m_1y_1 + m_2y_2 + \ldots + m_ny_n$. By A.1.1.6, each m_iy_i is primitive and $H \oplus \langle m_1y_1 \rangle \oplus \ldots \oplus \langle m_ny_n \rangle$ is a valuated coproduct. $y = m_1y_1 + \ldots + m_ny_n$ is such that $H \oplus \langle y \rangle$ is also a valuated coproduct and $x = z + y$. For any $h \in H$,

$\chi(x+h) = \chi(z+h) \min \chi(y) \leq \chi(y)$. Thus $\chi(y)$ is an upper bound for $\{\chi(x+h) : h \in H\}$. However, $\chi(x-z) = \chi(y)$ and thus $\chi(y) = \sup\{\chi(x+h) : h \in H\} = \chi_{G/H}(x+H)$ and H is balanced in G. ■

We represent the above schematically by the following:



DECENT

⇩

PREBALANCED ⇦ SEMI-BALANCED

⇧

KNICE + PURE ⇨ BALANCED

⇩

DIRECT SUMMAND ⇨ STRONGLY PURE ⇨ Σ - PURE ⇨ *-PURE ⇨ PURE

PURE + STRONGLY REGULAR

⇩

-PURE ⇨ s-PURE ⇨ t*-PURE ⇨ *-PURELY GENERATED

CHAPTER 3

In this chapter we will see under which conditions, if any, the different kinds of purity discussed in Chapter 2, coincide.

§3.1 In this section we will prove that all pure subgroups of a group G are direct summands of G if and only if G is a homogeneous, finite rank, separable group. We will also prove that all pure subgroups are balanced in G if and only if G is a homogeneous, finite rank, separable group. Furthermore we show that pure subgroups are strongly pure if and only if G is homogeneous and separable.

The following theorem was initially proved by Baer in [Ba] but the proof given is an alternate one given by Hill and Megibben in [HM1] on page 741.

3.1.1 Theorem : Any finite rank pure subgroup of a homogeneous separable group is a direct summand.

Proof: Let H be a non-zero finite rank, pure subgroup of a homogeneous separable group, G. Let $T(G) = \{\tau\}$, say and hence $G(s^*) = 0$ for any s in τ. Thus each element in G is primitive in G. Let $\{y_1, y_2, \ldots, y_n\}$ be a maximal independent set in H and let $N = \langle y_1\rangle \oplus \langle y_2\rangle \oplus \ldots \oplus \langle y_n\rangle$.

As G is separable, $\{y_i: i = 1, \ldots, n\} \subseteq \bigoplus_{j=1}^{N} G_j$ where each G_j is a rank one subgroup and $\bigoplus_{j=1}^{N} G_j$ is a summand of G. (N = n as N > n would lead to a contradiction to the independence of the y_i's and N < n would lead to a contradiction of the fact that $\{y_i: i = 1, \ldots, n\}$ is a **maximal** independent set.) Each $y_i = \sum_{j=1}^{n} g_{ij}$ where each $g_{ij} \in G_j$. As G_j is a rank one group, $\langle g_{1j}, \ldots, g_{nj}\rangle = \langle x_j\rangle$ for some $x_j \in G_j$. Thus $N \subseteq F = \langle x_1\rangle \oplus \ldots \oplus \langle x_n\rangle$. We show that N is a *-valuated subgroup by invoking A.1.3.9 as follows:

F is a *-valuated coproduct and G is homogeneous and thus by A.1.3.9, we can find $z_{12}, z_{13}, \ldots, z_{1n}$ in F such that $F_1 = \langle y_1\rangle \oplus \langle z_{12}\rangle \oplus \ldots \oplus \langle z_{1n}\rangle$ is a *-valuated coproduct and F/F_1 is finite. Now $\langle z_{12}\rangle \oplus \ldots \oplus \langle z_{1n}\rangle$ is also a *-valuated coproduct ensuring that there exist $z_{23}, z_{24}, \ldots, z_{2n}$ in F such that

$F_2 = \langle y_1\rangle \oplus \langle y_2\rangle \oplus \langle z_{23}\rangle \oplus \ldots \oplus \langle z_{2n}\rangle$ is a *-valued coproduct with F/F_2 finite. We continue like this until we get $N = F_n = \langle y_1\rangle \oplus \ldots \oplus \langle y_n\rangle$ is a *-valuated coproduct with F/N finite. By A.1.3.13, the pure closure of N (which is H) is a summand of G. ■

L. Fuchs, A. Kertesz and T Szele in [FKS] proved that the converse of this theorem is also true. We state the theorem in its strongest form but prove only its converse.

3.1.2 Theorem [FKS]: Every pure subgroup of a group G is a summand if and only if G is separable, of finite rank and homogeneous.

Proof: Suppose $G = \{g_\alpha : \alpha < \mu\}$ is such that every pure subgroup of G is a direct summand. Define $G_\alpha = \langle\{g_\beta : \beta < \alpha\}\rangle_*$. By our assumption, G_α is a summand of G and thus also of $G_{\alpha+1}$. Let $C_\alpha = G_{\alpha+1}/G_\alpha$ then for each α, C_α is either zero or of rank 1. It follows that **$G = \bigoplus_{\alpha<\mu} C_\alpha$ is completely decomposable**. If τ_1 and τ_2 are **distinct** elements in $\mathscr{E}(G)$, then G has a summand $C = C_1 \oplus C_2$ with each C_i of rank 1 and type$(C_i) = \tau_i$, i=1,2. If $\tau_1 \| \tau_2$, then let $0 \neq g_i \in C_i$, i=1,2 and observe that $H = \langle g_1 + g_2\rangle_*$ is a summand of G which is impossible. If $\tau_1 < \tau_2$, say then, as without loss of generality we can assume that G is reduced, let p be a prime such that $pG \neq G$. Choose $0 \neq g_2 \in C_2$ such that $h_p^G(g_2) = 0$. For any $0 \neq g_1 \in C_1$, $h_p^G(pg_1 + g_2) = 0$ and $H = \langle pg_1 + g_2\rangle_*$ cannot be a summand of G otherwise, as all the elements in H would be of type τ_1, $C = H \oplus C_2$ and
$1 \leq h_p^G(pg_1) = h_p^G(pg_1 + g_2) \wedge h_p^G(-g_2) = 0$ which is a contradiction and **G is homogeneous**. If rank(G) is infinite, then let K be any finite rank indecomposable homogeneous group of the same type as G. Observe that there is an epimorphism $\theta : G \to K$. As $\ker\theta$ is pure in G, by our initial assumption, $\ker\theta$ is a summand of G. By the Baer - Kaplansky - Kulikov Theorem, $G/\ker\theta \simeq K$ is completely decomposable which provides the desired

contradiction. We conclude thus that G is completely decomposable, homogeneous and of finite rank. ■

3.1.3 Theorem: All pure subgroups of a group G are balanced if and only if G is a homogeneous, finite rank, separable group.

Proof: Suppose G is homogeneous, finite rank, and separable. Then by 3.1.2, every pure subgroup is a direct summand and is thus balanced, by 2.5.1.
Conversely, suppose $G = \{g_\alpha : \alpha < \mu\}$ is such that every pure subgroup of G is balanced. Define $G_\alpha = \langle\{g_\beta : \beta < \alpha\}\rangle_*$ and observe that $G = \bigcup_{\alpha<\mu} G_\alpha$ and $G_\alpha \subseteq G_{\alpha+1}$ provided $\alpha+1 < \mu$. Since G_α is balanced in G, it is balanced in $G_{\alpha+1}$ and $G_{\alpha+1}/G_\alpha$ is either zero or rank one and hence balanced projective. Thus $G_{\alpha+1} = G_\alpha \oplus C_\alpha$ and $G = \bigoplus_{\alpha<\mu} C_\alpha$ and is thus **completely decomposable**. If τ_1 and τ_2 are **distinct** elements in $\mathscr{E}(G)$, then G has a summand $C = C_1 \oplus C_2$ with each C_i of rank 1 and type$(C_i) = \tau_i$, i=1,2. If $\tau_1 \| \tau_2$, then let $0 \neq g_i \in C_i$, i=1,2. By assumption $H = \langle g_1 + g_2\rangle_*$ is balanced in G and thus in C. $C/H = C_1/H \oplus C_2/H$ and is thus completely decomposable and hence balanced projective. Thus H is a summand of C which is impossible as $T(H) = \{\tau_1 \wedge \tau_2\}$. If $\tau_1 < \tau_2$, say, then, as without loss of generality we can assume that G is reduced, let p be a prime such that $pG \neq G$. Choose $0 \neq g_2 \in C_2$ such that $h_p^G(g_2) = 0$. For any $0 \neq g_1 \in C_1$.

$h_p^G(pg_1 + g_2) = 0$ and $H = \langle pg_1 + g_2 \rangle_*$ cannot be balanced in G for else it would be a summand of G and as in 3.1.2, we would get a contradiction. We conclude that **G is homogeneous**. Let rank(G) be infinite and let K be a rank one group of the same type as G. Note that as K is rank one it is also balanced projective. Since K and G are of the same type, we can find an epimorphism $\theta : G \to K$. Ker θ is pure in G and thus by our assumption, ker θ is balanced in G. $G/\ker \theta \simeq K$ and thus ker θ is a summand of G. By the Baer-Kaplansky-Kulikov Theorem, $K \simeq G/\ker \theta$ is completely decomposable which provides the desired contradiction. We conclude thus that G is completely decomposable, homogeneous and of **finite rank**. ■

3.1.4 Theorem: All pure subgroups H of G are strongly pure if and only if G is homogeneous and separable.

Proof: Suppose G is homogeneous and separable. Let H be pure in G. For any $0 \neq h \in H$, $\langle h \rangle_*$ is a summand of G by 3.1.1. Thus the projection $\theta : G \to \langle h \rangle_*$ followed by the inclusion map $i : \langle h \rangle_* \to H$ is a homomorphism from G to H which fixes h and H is strongly pure in G. Conversely suppose every pure subgroup is strongly pure in G. Let $S = \{x_1, \ldots, x_n\}$ be a finite set in G. As $\langle S \rangle_*$ is strongly pure, there is a homomorphism θ which takes G to H and which fixes the whole of S and thus the whole of $\langle S \rangle_*$. Thus $\theta^2 = \theta$ and is a projection from G to $\langle S \rangle_*$ which means that $\langle S \rangle_*$ is a summand of G. Every pure subgroup in

$\langle S \rangle_*$ is strongly pure and therefore a summand. By 3.1.2, $\langle S \rangle_*$ is completely decomposable and homogeneous. We conclude that G is separable and homogeneous. ■

<u>3.1.5 Remark</u>: Every element of a free group F can be embedded in a finite rank free summand of F. The Stacked Basis Theorem implies that every pure subgroup of F is a summand of F.

<u>§3.2</u> In this section we prove that a pure subgroup of a finite rank completely decomposable group G is *-purely generated if and only if it is strongly regular pure. Throughout this section, G will denote a completely decomposable group. Recall that $\mathscr{T}(G) = \{\tau \in T(G) : \langle G(\tau^*) \rangle_* \subset G(\tau)\}$ and $\mathscr{T}(G)$ is called the **critical typeset of G.** Note that in completely decomposable groups, a type would be an element of the critical typeset if and only if the group has a rank one summand of that type. Thus we let $G = \bigoplus \{G_\tau : \tau \in \mathscr{T}(G)\}$ be a homogeneous decomposition of G (i.e. $G_\tau \cong G(\tau)/G(\tau^*)$) and $\pi_\tau : G \to G_\tau$ is the projection such that $\ker \pi_\tau = \bigoplus_{\tau' \neq \tau} G_{\tau'}$ for every $\tau \in \mathscr{T}(G)$.

<u>3.2.1 Lemma</u>: Let H be *-purely generated in G. For every $\tau \in \mathscr{T}(H)$, $H(\tau) = H_\tau \oplus \langle H(\tau^*) \rangle_*$ where H_τ is a homogeneous completely decomposable group.

<u>Proof</u>: $H(\tau)/\langle H(\tau^*)\rangle_* = (H \cap G(\tau))/(H \cap G(\tau^*)$

$$= (H \cap G(\tau))/(H \cap G(\tau) \cap G(\tau^*)$$

$$\simeq \langle H \cap G(\tau),\ G(\tau^*)\rangle/G(\tau^*)$$

$$\leq G(\tau)/G(\tau^*)$$

which is a homogeneous completely decomposable group by [Fu2: Theorem 86.6]. By Baer's Lemma, [Fu2: Theorem 86.4], $\langle H(\tau^*)\rangle_*$ is balanced in $H(\tau)$ and since completely decomposable groups are balanced projective, $\langle H(\tau^*)\rangle_*$ is a summand of $H(\tau)$ with complement H_τ, and as $\tau \in \mathscr{E}(H)$, H_τ is non-zero. ■

<u>3.2.2 Lemma</u>: Let H and G be as above. Then $\mathscr{E}(H) \subseteq \mathscr{E}(G)$.

<u>Proof</u>: Let $\tau \in \mathscr{E}(H)\backslash\mathscr{E}(G)$. Then $G(\tau^*) = G(\tau)$ which implies that $\langle H(\tau^*)\rangle_* = H \cap G(\tau^*) = H \cap G(\tau) = H(\tau)$ contradicting the notion of a critical type set. ■

<u>3.2.3 Lemma</u>: Let G, H and H_τ be as above. For every $\tau \in \mathscr{E}(H)$ and $0 \neq h \in H_\tau$, $\pi_\tau(h) \neq 0$ where π_τ is the projection from G to the homogeneous summand of G of type τ.

<u>Proof</u>: Let $g = \oplus\{G_\tau : \tau \in \mathscr{E}(G)\}$ be a decomposition of G into maximal homogeneous components and for every $\tau \in \mathscr{E}G)$, let $\pi_\tau : G \to G_\tau$ be the corresponding projection. If $0 \neq h \in H_\Pi$, where $H(\tau) = H_\tau \oplus \langle H(\tau^*)\rangle_*$, then $h = \Sigma\{\pi_\sigma(h) : \sigma$ is an element of a finite subset I of $\mathscr{E}(G)\}$ and $\tau = \inf\{\sigma : \sigma \in I\} \leq \sigma$ for every $\sigma \in I$. If $\tau \notin I$ or if $\tau \in I$ but $\pi_\tau(h) = 0$, then

$h \in G(\tau^*) \cap H \cap H_\tau = \langle H(\tau^*)\rangle_* \cap H_\tau = 0$, a contradiction. Thus $\pi_\tau(h) \neq 0$. ■

3.2.4 Proposition: Let G, H and H_τ be as above. H_τ is strongly regular in G.

Proof: Let $\tau \in \mathscr{E}(H)$ and let $0 \neq h \in H_\tau$. $\langle \pi_\tau(h) \rangle_*$ is a summand of $G_\tau = \pi_\tau(G)$ as G_τ is homogeneous and completely decomposable group. Let θ be the projection from G to $\langle \pi_\tau(h) \rangle_*$. Let $A = \{p_1, p_2, \ldots, p_k\}$ be the set of primes such that

$$h_{p_i}^G(h) = h_{p_i}^H(h) < h_{p_i}^G(\pi_\tau(h)) \quad i=1,2,\ldots,k.$$

Let $n = \prod_{i=1}^{k} p_i^{n(i)}$ where $n(i) = h_{p_i}^G(\pi_\tau(h)) - h_{p_i}^G(h)$, and thus $n \mid \pi_\tau(h)$ in $\langle \pi_\tau(h) \rangle_*$. Let $\pi_\tau(h) = ng'$. For any $p \notin A$,

$$h_p^H(h) = h_p^G(\pi_\tau(h)) = h_p^G(g')$$

and for any $p \in A$, $h_p^G(g') = h_p^H(h)$. Thus $\chi_G(g') = \chi_G(h)$ and there is an isomorphism $\psi' : \langle g' \rangle_* \to \langle h \rangle_*$ with $\psi'(g') = h$. We then have
$\psi'\theta\pi_\tau(h) = \psi'\theta(ng') = n\psi'(g') = nh$ and $\psi = \psi'\theta\pi_\tau$ is the desired homomorphism. ■

3.2.5 Theorem [No2]: A subgroup H of finite rank completely decomposable group G is *-purely generated if and only if it is strongly regular pure.

Proof: By 2.4.7, strongly regular pure implies *-purely generated. By the proposition above, H_τ is strongly regular for every τ in $\mathscr{E}(H)$. By [No2], the H_τ's generate their direct sum. Let $0 \neq h \in \oplus \{H_\tau : \tau \in \mathscr{E}(G)\}$ i.e. $h = \sum_{i=1}^{n} h_i$ *where each* $h_i \in H_{\tau_i}$ and each $\tau_i \in \mathscr{E}(H)$. For each $i=1,2,\ldots,n$, there is a $\psi_i : G \to H_\tau$ and an integer n_i such that $\psi_i(h_i) = n_i h_i$. Thus $\psi_i(h) = \sum_{j=1, j\neq i}^{n} \psi_i(h_j)$ and as each $\psi_i(h_j) \in H_{\tau_i}$, $\Sigma\psi_i(h_j) \in H_{\tau_i}$. As H_{τ_i} is rank one which contains h_i, $H_{\tau_i} = \langle h_i \rangle_*$. This implies that there are integers k_i and l_i such that $k_i h_i = l_i \Sigma\psi_i(h_j)$. Let $\Pi k_j = k$ and $\Pi_{j\neq i} k_j = k_i'$. Then

$$\begin{aligned}\psi_i(l_i h) &= l_i n_i h_i + l_i \Sigma_j \psi_i(h_j)\\ &= l_i n_i h_i + k_i h_i\\ &= (l_i n_i + k_i) h_i.\end{aligned}$$

Let $r_i = l_i n_i + k_i$ and let $r = \Pi_i r_i$ and $r_i' = \Pi_{j\neq i} r_j$. Then $rh = r\Sigma_i(h_i) = \Sigma_i r_i' r_i h_i = \Sigma_i r_i' \psi_i(l_i h) = \Sigma_i r_i' \psi_i(h)$. Thus $\oplus \{H_\tau : \tau \in \mathscr{E}(G)\}$ is strongly regular in G. By [No2], $\oplus \{H_\tau : \tau \in \mathscr{E}(G)\}$ has finite index in H. Thus for every $g \in H$ there is an integer n_g such that $n_g g \in \oplus \{H_\tau : \tau \in \mathscr{E}(G)\}$ and thus $rn_g g = \Sigma_i r_i' \psi_i n_g(g)$. ∎

<u>§3.3</u> In this section we will study under which conditions $*$-pure subgroups are summands.

<u>3.3.1 Lemma [DR1]</u> : $y \in G$ is primitive if and only if the pure subgroup $\langle y \rangle_*$ is $*$-pure in G.

<u>Proof</u> : Suppose $H = \langle y \rangle_*$ is $*$-pure in G. If $s \sim \chi(y)$ and $p \in P$ such that $s_p \neq \infty$ and $s_p = h_p^G(y)$ but $y \in G(s^*,p)$,

then, by our assumption, $y \in H(s^*,p)$. Thus $y = \sum_{i=1}^{n+1} h_i$ with $h_{n+1} \in H(ps)$ and for each $i=1,\ldots,n$, $h_i \in G(s)$ but $\chi(h_i) \nsim s$. As h_i depends on y, $\chi(y) \sim \chi(h_i) \sim s$ and thus for each $i=1,\ldots,n$, $h_i = 0$. We thus have that $y \in G(ps)$ which is a contradiction to the choice of p and y is primitive. Conversely, suppose y is primitive. Let $x \in H = \langle y \rangle_*$. Then there are n,m, such that $(n,m) = 1$ and $nx = my$. But my, and hence nx and x are primitive by A.1.1.6. Let $x \in H \cap G(s^*,p)$. By A.1.1.3, either $\chi(x) \nsim s$ or $x \in H \cap G(ps)$. This implies that either $x \in H(s^*)$ or $x \in H(ps)$ (as H is pure in G) which in turn implies that $x \in H(s^*,p)$. Therefore $H \cap G(s^*,p) \subseteq H(s^*,p)$

and $H \cap G(s^*,p) = H(s^*,p)$

$H \cap G(s^*) = H(s^*)$ as H is pure in G. Thus H is $*$-pure in G. ■

The following lemma is useful when proving 3.3.3.

<u>3.3.2 Lemma:</u> Let y be primitive in G and let

$y = \sum_{i=1}^{n} y_i$ where each y_i is also primitive in G. There exists at least one y_j, $1 \le j \le n$ such that $\text{type}_G(y) = \text{type}_G(y_j)$.

<u>Proof</u>: Let $s = \chi_G(y)$ and let $\tau = \text{type}_G(y)$. If all the y_i's are such that $\text{type}_G(y_i) > \tau$, then $y \in G(s^*,p)$ for any $p \in \mathbf{P}$ which contradicts the primitivity of y in G. ■

3.3.3 Theorem [DR1] : If G is separable, then any finite rank *-pure subgroup H is a completely decomposable summand.

Proof: The proof is by induction on n, the rank of H. Let n = 1. Then H = $\langle y\rangle_*$ and as G is separable, H $\subseteq$ K, a completely decomposable summand of G. Thus $K = \langle y_1\rangle_* \oplus \langle y_2\rangle_* \oplus \ldots \oplus \langle y_m\rangle_*$. There exist k, l_1, l_2, ..., l_m such that $ky = \Sigma_i l_i y_i$. But $\langle l_i y_i\rangle_* = \langle y_i\rangle_*$ and $\langle ky\rangle_* = \langle y\rangle_*$. So without loss of generality, we may assume that $y = \Sigma_i y_i$. $\chi(y) = \bigwedge_{i=1}^{m} \chi(y_i) \leq \chi(y_i)$ for each i. By our assumption, H is *-pure in G and thus by Lemma 3.3.1, y is primitive in G. By A.1.1.9, each y_i is primitive and Lemma 3.3.2 implies that at least one of the y_i's is of the same type as y and rearranging the y_i's if necessary, write $y = y' + b$ where $y' = y_1 + \ldots + y_k$ with type(y) = type(y_i) for each i=1,2,...,k and $b = y_{k+1} + \ldots + y_n$. $\langle y'\rangle_*$ which is pure in the homogeneous completely decomposable group $\langle y_1\rangle_* \oplus \ldots \oplus \langle y_k\rangle_*$ is, by 3.1.1, a summand of K and thus of G and thus $G = \langle y'\rangle_* \oplus N$, say with $b \in N$. We thus conclude that $G = \langle y\rangle_* \oplus N$.

Now suppose H is of rank n > 1 and the result holds for *-pure subgroups of rank less than n. Let $y \in H$ be an element of maximal type τ and let $\chi(y) = s$. $H(s^*) = 0$ and for each prime p for which $s_p = h_p(y) \neq \infty$, $y \notin H(ps) = H(s^*,p) = H \cap G(s^*,p)$. This is true for any $t \in \tau$ and any prime p with $t_p = h_p(y_p)$ and thus y is primitive in G. By Lemma 3.3.1, $\langle y\rangle_*$ is *-pure and thus by our assumption, $\langle y \rangle_*$ is a summand of G and thus of H. Let $G = \langle y \rangle_* \oplus K$ and $H = \langle y \rangle_* \oplus L$. Let $x \in L \cap G(s^*)$. Since H is *-pure in G, $x \in H(s^*)$ and $x = \Sigma h_i$ with each $h_i \in G(s)$ but $\chi(h_i) \not\to s$. Each $h_i = k_i + l_i$ with $k_i \in \langle y \rangle_*$ and $l_i \in L$ and $l_i \in G(s)$ but $\chi(l_i) \not\to s$. Now $x - \Sigma l_i = \Sigma k_i = 0$ by properties of direct sums and thus $x = \Sigma l_i$ and $x \in L(s^*)$ which implies that $L \cap G(s^*) = L(s^*)$. Similarly, $L \cap G(s^*,p) = L(s^*,p)$ and hence L is *-pure of rank n-1 and which by the inductive hypothesis is a completely decomposable summand of G. ■

§3.4 In this section, we use results that are dependent on the theory of k-groups. A k-group, which is a generalization of a separable group, was first introduced by P. Hill and C. Megibben in [HM1] and later studied by M. Dugas and K.M. Rangaswamy in [DR1]. The definition and properties of k-groups can be found in the appendix A.2.

We quote the following result of P. Hill and C. Megibben.

3.4.1 Theorem: [HM2: Theorem 4.1]
A Σ-pure subgroup of a torsion-free k-group is again a k-group.

3.4.2 Corollary [HM2]: A strongly-pure subgroup of a k-group is again a k-group.

Proof: A strongly-pure subgroup is, by 2.4.8, necessarily a Σ-pure subgroup. ■

3.4.3 Corollary [HM1, Fu2]: A direct summand of a k-group is again a k-group and a direct summand of a separable group is again a separable group.

Proof: This follows directly from 2.2.1 viz. that direct summands are strongly pure subgroups and thus, by 2.4.8, Σ-pure subgroups as well. ■

3.4.4 Corollary [No1]: A strongly pure subgroup of a separable group is again a separable group.

Proof: Let H be a strongly pure subgroup of the separable group G. By 3.4.2, H is a k-group. Let $A = \langle y_1 \rangle \oplus \ldots \oplus \langle y_n \rangle$ be a finite rank, free *-valuated subgroup of H. As a consequence of 2.2.6, there is a homomorphism $\phi: G \rightarrow H$ which fixes the whole of A. Thus $\langle A \rangle_*$ is a summand of G and hence of H as well. By A.2.3, H is separable. ■

3.4.5 Corollary [HM1, DR]: A pure knice subgroup of a k-group is again a k-group.

Proof: From 2.6.4, a pure and knice subgroup is also a Σ-pure subgroup. Thus the result follows from Theorem

3.4.1. ■

<u>3.4.6 Corollary [HM2]</u>: **A Σ-pure subgroup of a separable group is strongly pure.**

<u>Proof</u>: Let G be a separable group and let H be a Σ-pure subgroup of G. Let h ∈ H. By 3.4.1, H is a k-group and thus we can find a finite rank, free *-valuated subgroup, N, in H such that h ∈ N. By A.2.3, $\langle N\rangle_*$ is a summand of G and thus $\langle N\rangle_*$ is a summand of H. Thus the projection map from G to $\langle N\rangle_*$ is a map from G to H which fixes h. ■

Recall (2.4.8) that Σ-pure subgroups are necessarily *-pure. The following is an example quoted by P. Hill and C. Megibben that illustrates that *-pure subgroups do not coincide with Σ-pure subgroups.

<u>3.4.7 Example</u>: [HM2] Let $\mathcal{F}$ denote the countable collection of all finite subsets of the set $\boldsymbol{P}$. Let $\{F_n\}_{n<\omega}$ be a sequence such that $F_n \in \mathcal{F}$ and for each $F \in \mathcal{F}$, there are infinitely many n for which $F_n = F$. It is well known that there exists a sequence $\{E_n\}_{n<\omega}$ that satisfies the following conditions whenever $i,j < \omega$:

(1) $E_i \subseteq \boldsymbol{P}$.

(2) E_i is infinite and $E_i \supseteq F_i$.

(3) $E_i \cap E_j$ is finite when $i \neq j$.

(4) $E_k \cap E_j$ properly contains $E_i \cap E_j$ when $i \neq j$ and $k > \max\{i,j\}$.

Select the sequence $\{E_n\}_{n<\omega}$ which satisfies the conditions (1) -(4) above. Let s_n be the height sequence consisting of 0's and ∞'s such that s_n assumes the value of ∞ if and only if $p \in E_n$. Choose a rank one group $\langle g_n \rangle_* = G_n \subseteq Q$ so that the height sequence of g_n in G_n is precisely s_n. Let $G = \bigoplus_{n<\omega} G_n$. Let ϕ_n be the map from G_n to Q that maps g_n to 1 and let $\phi: G \to Q$ be the induced map. Hill and Megibben proved in [HM2: Theorem 6.2] that $\ker\phi = H$ is *-pure in G but H is not completely decomposable and in fact does not contain primitive elements. G is completely decomposable and H is countable. If H were Σ-pure in G, then as G is a k-group, H would be a k-group by 3.4.1. However, this contradicts A.2.5 which states that all countable k-groups are completely decomposable. Thus we have a ***-pure subgroup which is not Σ-pure**.

<u>§3.5</u> In this section, we will look at the connection between balancedness and kniceness and under which conditions kniceness implies being a summand.

<u>3.5.1 Lemma [HM1]</u>: If H is balanced in G and s is any height sequence, then $(G/H)(s) = (G(s) + H)/H$, $(G/H)(s^*) = (G(s^*) + H)/H$, and $(G/H)(s^*,p) = (G(s^*,p) + H)/H$.

<u>Proof</u>: We only look at the last equation as the verifications of the other two equations are very similar. Suppose $x+H \in (G/H)(s^*,p)$ then

$x+H = y_1+H + \ldots + y_n+H + y+H$ where each y_i+H is such that $\chi(y_i+H) \geq s$ but $\chi(y_i+H) \not> s$ and $\chi(y+H) \geq ps$. As H is balanced, each coset y_i+H contains an element z_i such that $\chi(z_i) = \chi(y_i+H) \geq s$ and thus $\chi(z_i) \not> s$ and there is a $z \in y+H$ such that $\chi(z) = \chi(y+H)$.

$$\begin{aligned} x+H &= z_1+H + \ldots + z_n+H + z+H \\ &= (z_1+z_2+\ldots+z_n+z)+H \\ &\in (G(s^*,p) + H)/H. \end{aligned}$$

Now suppose $x+H \in (G(s^*,p) + H)/H$. Then $x = y + h$ where $y \in G(s^*,p)$ and $h \in H$. $y = y_1 + \ldots + y_k + y'$ where $\chi(y') \geq ps$ and $\chi(y_i) \geq s$ but $\chi(y_i) \not> s$. Thus $\chi(y'+H) \geq \chi(y') \geq ps$ and $\chi(y_i+H) \geq \chi(y_i) \geq s$ and $\chi(y_i+H) \not> s$. Now

$$x+H = y+H = y_1+H + \ldots + y_n+H + y'+H$$

which implies that $x+H \in (G(s^*,p) + H)/H$. ■

<u>3.5.2 Theorem [HM1]</u>: A pure subgroup H of G is a knice subgroup if and only if H is balanced in G and G/H is a k-group.

<u>Proof</u>: Suppose H is pure and knice in G. By 2.6.6, H is balanced in G. Let $S = \{x_1+H, x_2+H, \ldots, x_n+H\}$ be contained in G/H then, as H is pure and knice in G, $S' = \{x_1, \ldots, x_n\}$ is contained in $H + N$ where $N = \langle y_1\rangle \oplus \langle y_2\rangle \oplus \ldots \oplus \langle y_k\rangle$ and $H + N$ is a $*$-valuated coproduct in G and all the y_i's are primitive in G. Let $s_i \sim \chi^{G/H}(y_i+H)$ and let $p \in \boldsymbol{P}$ be such that $h_p^{G/H}(y_i+H) = (s_i)_p$. If $y_i+H \in (G/H)((s_i)^*,p)$, then $y_i+H \in (G((s_i)^*,p) + H)/H$ by

lemma 3.5.1. Thus $y_i+h \in G((s_i)^*,p)$ for some $h \in H$. As $H \oplus N$ is a *-valuated coproduct, $y_i \in G((s_i)^*,p)$. As H is balanced in G, there is an $h \in H$ such that $\chi^G(y_i+h) = \chi^{G/H}(y_i+H)$ and as $H \oplus N$ is a valuated coproduct, $\chi(y_i) \leq \chi(y_i+h) \sim s_i$ which contradicts the primitivity of y_i. Thus each y_i+H is primitive in G/H. In a similar way we can show that $K = \langle y_1+H \rangle \oplus \ldots \oplus \langle y_k+H \rangle$ is a *-valuated coproduct in G/H and thus K is the free, *-valuated subgroup of G/H containing S and G/H is a k-group. Conversely, suppose H is balanced in G and G/H is a k-group. Let $S = \{x_1, x_2, \ldots, x_n\}$ be in H then $S' = \{x_1+H, \ldots, x_n+H\}$ is a finite set in G/H and thus S′ in $\langle y_1+H \rangle \oplus \langle y_2+H \rangle \oplus \ldots \oplus \langle y_m+H \rangle$ which is a *-valuated coproduct and each y_i+H is primitive in G/H. Suppose that the y_i's have been chosen so that $\chi_G(y_i) = \chi_{G/H}(y_i+H)$ then the primitivity of y_i+H guarantees the primitivity of each y_i. Let $x = h + t_1y_1 + t_2y_2 + \ldots + t_my_m$ where $h \in H$. $x+H = t_1y_1+H + \ldots + t_my_m+H$ and $\langle t_1y_1+H \rangle \oplus \ldots \oplus \langle t_my_m+H \rangle$ is also a valuated coproduct. Now suppose $x \in G(s)$ for some height sequence s. $\chi(x+H) \geq \chi(x) \geq s$ and thus for each i, $t_iy_i+H \in (G/H)(s)$. Since $\chi(t_iy_i) = \chi(t_iy_i+H)$, $t_iy_i \in G(s)$. Thus $H \oplus \langle y_1 \rangle \oplus \ldots \oplus \langle y_m \rangle$ is a valuated coproduct. Now suppose that $x \in G(s^*,p)$. As before, each $t_iy_i+H \in (G/H)(s^*,p)$ but as t_iy_i+H is primitive, either $t_iy_i+H \in (G/H)(ps)$ or $\chi(t_iy_i+H) \not\sim s$. In either case, $\chi(t_iy_i) = \chi(t_iy_i+H)$ implies that $t_iy_i \in G(s^*,p)$. ■

<u>3.5.3 Corollary [HM1]</u>: If H is a pure knice subgroup of

G and if G/H is countable, then H is a summand of G.

Proof: As H is pure and knice, H is balanced and G/H is a k-group. G/H is countable and thus by A.2.5, G/H is completely decomposable and is thus balanced projective which immediately gives that H is a summand of G. ■

3.5.4 Corollary [DR1]: G is a k-group if and only if G = C/B where C is completely decomposable and B is knice in C.

Proof: Let $0 \to H \to C \to^{\theta} G \to 0$ be a balanced resolution of G. Suppose G is a k-group. Then G ≅ C/H where C is completely decomposable and H is balanced in C. By 3.5.2, H is knice in C. Conversely, suppose H is knice in C then by 3.5.2, H is balanced and C/H, which is isomorphic to G, is a k-group. ■

3.5.5 Corollary [DR1]: A countable knice subgroup of a completely decomposable group is a summand.

Proof: Let H be a countable knice subgroup of G, a completely decomposable group. By 3.5.2, G/H is a k-group. We can assume that G is countable and thus G/H is a countable k-group which by A.2.5, is completely decomposable and thus balanced projective. Hence H is a summand of G. ■

APPENDIX A.1

§A.1.1 **Primitive Elements**.

PRIMITIVE ACCORDING TO BAER

A.1.1.1 Definition: Baer, in [Ba], defined an element x in G, a torsion-free group, to be primitive of type τ if $x \in G(\tau)\backslash G(\tau^*)$ and x is proper with respect to $G(\tau^*)$ (i.e. $\chi^G(x) \geq \chi^G(x+g)$ for all $g \in G(\tau^*)$).

PRIMITIVE ACCORDING TO HILL AND MEGIBBEN

A.1.1.2 Definition: P. Hill and C. Meggibben [HM1] defined primitive elements in a torsion-free group as follows :- Let $x \in G$ and let $(p,s)_x$ be a pair with p a prime and s a height sequence equivalent to $\chi(x)$ satisfying:

(i) $h_p(x) \neq \infty$

(ii) $s_p = h_p(x)$.

If $x \notin G(s^*,p)$ for all pairs $(p,s)_x$ as above, then x is said to be primitive in G.

A.1.1.3 Remark [HM1]: Suppose x is primitive (Hill and Megibben) in G but $x \in G(s^*,p)$ for some prime p and some height sequence s, then either $\chi(x) \sim s$ or $\chi(x) \nsim s$. In the event that $\chi(x) \nsim s$, $h_p(x) > s_p$; while in the event that $\chi(x) \sim s$ either $h_p(x) = \infty$ or $h_p(x) > s_p$. Thus in both cases, either

$\Sigma_{p\in P} |h_p(x) - s_p| = \infty$ or $x \in G(ps)$.

A.1.1.4 Lemma: Primitive according to Baer implies primitive according to Hill and Megibben.

Proof: Let $x \in G$ be primitive (Baer) of type τ for some type τ. Then $x \in G(\tau)\backslash G(\tau^*)$ which implies that $\text{type}(x) = \tau$ and for any height sequence $s \in \tau$, $s \sim \chi(x)$. Let $p \in P$ be such that $s_p = h_p(x) \neq \infty$. Suppose $x \in G(s^*,p)$ then $x = x_1 + x_2$ where $x_1 \in G(s^*)$ - which implies that $x_1 = y_1 + y_2 + \ldots + y_n$ where $\chi(y_i) \geq s$ and $\chi(y_i) \nsim s$ - and $x_2 \in G(ps)$. Each $y_i \in G(\tau^*)$ and thus $x_1 \in G(\tau^*)$ which implies that $x - x_1 = x_2 \in x+G(\tau^*)$. Primitivity (Baer) implies that $\chi(x) \geq \chi(x-x_1) = \chi(x_2)$ and recalling that $h_p(x) = s_p$ and $h_p(x_2) > s_p$, we get the desired contradiction. Hence x is primitive (Hill and Megibben).

From now on we will refer to primitive (Hill and Megibben) merely as **primitive**.

A.1.1.5 Note that if H is *-pure in G then x is primitive in H if and only if x is primitive in G.

A.1.1.6 Lemma [HM1] : Let $x \in G$ and suppose n is a non-zero integer. Then x is primitive in G if and only if nx is primitive in G.

Proof: Suppose x is not primitive in G. Then for some pair $(p,s)_x$ with (i) $s \sim \chi(x)$; (ii) $h_p(x) \neq \infty$; and (iii) $s_p = h_p(x)$, $x \in G(s^*,p)$. By (i), $ns \sim \chi(nx)$; by (ii), $h_p(nx) \neq \infty$; and by (iii), $ns_p = h_p(nx)$. $x \in G(s^*,p)$ implies that $nx \in nG(s^*,p)$ which in turn is equal to $G((ns)^*,p)$. Thus nx is not primitive in G.

Conversely, suppose that nx is not primitive. Then there is a pair $(p,s)_{nx}$ such that (i) $s \sim \chi(nx)$;
(ii) $h_p(nx) \neq \infty$; (iii) $s_p = h_p(nx)$; and $x \in G((ns)^*,p)$.
Let k be such that $(n/p^k,p^k) = 1$. Let $t=(t)_p$ be such that $t_q = s_q$ for $q \neq p$ and $t_p = s_p - k$. Then by (i), $t \sim \chi(x)$; by (ii) $h_p(x) \neq \infty$; and by (iii), $t_p = h_p(x)$ while torsion-freeness of G implies that $x \in G(s^*,p)$. Thus x is not primitive in G. ■

A.1.1.7 Lemma [HM1]: If x is primitive in G with $s = \chi(x)$, then each element of the coset $x + G(s^*)$ is primitive with s as its height sequence.

Proof : Let x be primitive and $s = \chi(x)$. Let $y = x + z$, where $z \in G(s^*)$. Suppose y is not primitive i.e. there is a pair $(t,p)_y$ such that $t \sim \chi(y)$ and $t_p = h_p(y) \neq \infty$. $t_p = h_p(y) \geq \min\{h_p(x); h_p(z)\} = s_p$ as $z \in G(s^*)$.

But $x = y - z \in G(t^*,p) + G(s^*) = G(t^*) + G(pt) + G(s^*)$ and $t \wedge s \leq t$ and $t \wedge s \leq s$.

Therefore $G(t) \subseteq G(t \wedge s)$ and $G(t^*) \subseteq G((t \wedge s)^*)$

and $G(s) \subseteq G(t \wedge s)$ and $G(s^*) \subseteq G((t \wedge s)^*)$

and $G(pt) \subseteq G(p(t \wedge s))$.

Thus $x \in G((t \wedge s)^*, p)$;

$\chi(y) \geq \chi(x) \wedge \chi(z) = s$ and therefore $t \wedge s = s$; and $\infty \neq t_p > (t \wedge s)_p = s_p$ which contradicts the primitivity of x. ■

<u>A.1.1.8 Remark</u>: If $y \in \langle x \rangle_*$ and $\chi(x) = s$, then since y depends on x, $\chi(y) \sim s$.

<u>A.1.1.9 Lemma [HM1]</u>: If $\langle x \rangle_*$ is a direct summand of G, then x is primitive in G.

<u>Proof</u>: Let $\chi(x) = t$ and let $H = \langle x \rangle_*$. Suppose $G = H \oplus K$ for some $K \leq G$ and suppose x is not primitive in G There must be some height sequence s and prime p satisfying $s \sim t$, $t_p = h_p(x) = s_p \neq \infty$, such that $x \in G(s^*, p)$. Thus $x = x^* + y$ where $x^* \in G(s^*)$,

$y \in G(ps)$. By properties of direct sums and by A.1.1.8, $G(s^*) \leq K$ and thus $x - x^*$ is a decomposition of y into H and K. Hence $\chi(y) = \chi(x) \wedge \chi(x^*)$ and in particular, $s_p < h_p(y) = h_p(x) \wedge h_p(x^*) = s_p$. Thus x is primitive in G. ■

<u>A.1.1.10 Lemma</u>: Suppose $x = x_1 + x_2 + \ldots + x_n$ with the property that if $\chi(x_i) \geq \chi(x) \geq s$ for each i and for some height sequence s. If $\chi(x_i) \sim s$ for **any** i, then $\chi(x) \sim s$.

Proof: Assume that $\chi(x_i) \sim s$. Since $\chi(x_i) \geq \chi(x) \geq s$ and since $h_p(x_i) = \infty$ if and only if $s_p = \infty$, $h_p(x) = \infty$ if and only if $s_p = \infty$. If $h_p(x) > s_p$, then $h_p(x_i) > s_p$ which only occurs finitely many times. Thus $\chi(x) \sim s$. ■

A.1.1.11 Lemma: If $x \in G$, a separable group, and $\chi(x) = s$, $\text{type}(x) = \sigma$, then x is primitive if and only if:

(1) $\text{type}(x) \in \mathscr{E}(G)$ and

(2) $\chi(\pi_\sigma(x)) = \chi(x)$.

Proof: Without loss of generality, we can assume that G is completely decomposable and thus $G = \oplus_{\tau\in\mathscr{E}(G)} G(\tau)$. If π_τ is the projection from G to G_τ, then $s = \chi(x) \leq \chi(\pi_\tau(x))$ and $\sigma = \text{type}(x) \leq \text{type}(\pi_\tau(x))$. We can write $x = \Sigma\{\pi_\tau(x) : \tau \in \mathscr{E}(G)\}$ where $\mathscr{E}(x) = \{\tau \in \mathscr{E}(G) : \pi_\tau(x) \neq 0\}$. Now suppose x is primitive but $\sigma \notin \mathscr{E}(A)$. Then for any $\tau \in \mathscr{E}(x)$, $\sigma < \tau = \text{type}(\pi_\tau(x))$. Thus $x \in G(s^*) \subseteq G(s^*,p)$ for any $p \in P$ contradicting the primitivity of x. Suppose $\chi(x) < \chi(\pi_\sigma(x))$ then for some $p \in P$, $h_p(x) < h_p(\pi_\sigma(x))$ and $x \in G(ps) \subseteq G(s^*,p)$ and x cannot be primitive.

Conversely, suppose conditions (1) and (2) hold but x is not primitive in G. There are thus a height sequence t such that $t \sim s$ and $p \in P$ such that $h_p(x) = t_p < \infty$ and $x \in G(t^*,p)$. Thus $x = \sum_{i=1}^{n+1} x_i$ where $x_{n+1} \in G(pt)$ and for each $i = 1,\ldots,n$, $\chi(x_i) \geq t$ and $\chi(x_i) \not> t$. As G is a direct sum, $x_i = \Sigma\{\pi_\tau(x_i) : \tau \in \mathscr{E}(G)\}$ for each $i = 1,\ldots,n, n+1$ and thus $x = \sum_{i=1}^{n+1} \Sigma\{\pi_\tau(x_i) : \tau \in \mathscr{E}(G)\}$. By Lemma A.1.1.10, $\chi(\pi_\sigma(x_i)) \not> t$ for any $i = 1,\ldots,n$ which forces $\pi_\sigma(x_i) = 0$. We now have that $\pi_\sigma(x) = \pi_\sigma(x_{n+1})$ and thus, by (2), $t_p = h_p(x) = h_p(\pi_\sigma(x)) = h_p(\pi_\sigma(x_{n+1})) > t_p$, a contradiction. Thus, under these conditions, x will be primitive. ■

§A.1.2 **Valuated coproducts**

A.1.2.1 Definition [HM1]: A direct sum $A_1 \oplus A_2$ of independent subgroups of G is said to be a valuated coproduct in G if $\chi(a_1 + a_2) = \chi(a_1) \wedge \chi(a_2)$ for all $a_1 \in A_1$, and $a_2 \in A_2$.

A.1.2.2 Lemma [HM1]: $A_1 \oplus A_2$ is a valuated coproduct if and only if for any height sequence s, whenever $a_1 + a_2 \in G(s)$, then a_1 and a_2 are elements of G(s) too.

Proof : Let s be any height sequence. Suppose $A_1 \oplus A_2$ is a valuated coproduct in G. Let $a_i \in A_i$ for i = 1 or 2. Then $(a_1 + a_2) \in G(s)$ implies that $\chi(a_1) \wedge \chi(a_2) = \chi(a_1 + a_2) \geq s$ and since $\chi(a_i) \geq \chi(a_1) \wedge \chi(a_2)$ for each i = 1 or 2, $a_i \in G(s)$. Suppose now that for some $a_1 \in A_1$, $a_2 \in A_2$,

$t = \chi(a_1 + a_2) > \chi(a_1) \wedge \chi(a_2)$ while at the same time, for any height sequence s, $a_1 + a_2 \in G(s)$ implies that $a_i \in G(s)$, i = 1 **and** 2. Thus $\chi(a_1) \geq t$ and $\chi(a_2) \geq t$ and hence $\chi(a_1) \wedge \chi(a_2) \geq t$ which is a contradiction. We can thus conclude that $\chi(a_1 + a_2) = \chi(a_1) \wedge \chi(a_2)$. ■

A.1.2.3 Definition [HM1] : We can extend the definition of a valuated coproduct to an arbitrary sum of independent subgroups of G as follows: $\bigoplus_{i \in I} A_i$ is a valuated coproduct if, for any height sequence s, whenever $\chi(\Sigma_i a_i) \in G(s)$, then $a_i \in G(s)$ for each i.

Alternatively, $\bigoplus_{i \in I} A_i$ is a valuated coproduct if, for any

$a = \sum_{i \in I_0} a_i$, where I_0 is a finite subset of I, then

$\chi(a) = \bigwedge_{i \in I_0} \{\chi(a_i)\}$. As in A.1.2.2, we can show that these definitions are equivalent.

A.1.2.4 Lemma [HM1]: If B_i/A_i is torsion for all $i \in I$,

then $B = \bigoplus_{i \in I} B_i$ is a valuated coproduct if and only if

$A = \bigoplus_{i \in I} A_i$ is a valuated coproduct.

Proof: Suppose B is a valuated coproduct. Let s be an arbitrary height sequence. Suppose

$$a_{i_1} + a_{i_2} + a_{i_3} + \ldots + a_{i_n} \in G(s), \qquad a_{i_j} \in A_{i_j}.$$

Now $A_{i_j} \subseteq B_{i_j}$ for each j = 1, ..., n and as B is a valuated coproduct, $a_{i_j} \in G(s)$ for each j. Thus A is also a valuated coproduct. Conversely, suppose that A is a valuated coproduct and let s be an arbitrary height sequence. Suppose

$$b = b_{i_1} + b_{i_2} + \ldots + b_{i_m} \in G(s)$$
$$\text{with } b_{i_j} \in B_{i_j} \text{ for all } j=1,2,\ldots,m$$

There is an n_{i_j} such that $n_{i_j}b_{i_j} \in A_{i_j}$ and thus $n = (\prod_{j=1}^{m} n_{i_j})$ is such that nb $\in$ A. Now b $\in$ G(s) implies that nb $\in$ nG(s) = G(ns). $nb = \sum_{j=1}^{m} b_{i_j}$ and as A is a valuated coproduct, $nb_{i_j} \in G(ns)$ and thus $b \in G(s)$ as G is torsion-free. Thus B is a valuated coproduct. ■

<u>A.1.2.5 Remark</u>: If $G = \bigoplus_{i \in I} G_i$ then $\chi(\Sigma_i g_i) = \wedge_i(\chi(g_i))$ where $g_i \in G_i$ and thus G is a valuated coproduct.

<u>A.1.2.6 Lemma [HM1]</u>: The valuated coproduct, $A = \bigoplus_{i \in I} A_i$ in G is pure in G if and only if each A_i is pure in G.

<u>Proof</u>: As each element in G can be represented as a sum of finitely many elements in the A_i's, the proof reduces (using induction if necessary) to the case when the

valuated coproduct $A = A_1 \oplus A_2$. Suppose A_1 and A_2 are pure in G. Let $ng \in A$. Now $ng = a_1 + a_2$ where $a_i \in A_i$, $i = 1$ or 2. $\chi(ng) = \min\{\chi(a_1); \chi(a_2)\}$. Thus $n\chi(g) \leq \chi(a_i)$ for $i = 1,2$ and therefore $n|a_1$ and $n|a_2$. Thus there are b_1 and b_2 elements in A_1 and A_2 respectively such that :

$a_1 = nb_1$ and $a_2 = nb_2$ and $ng = nb_1 + nb_2 = n(b_1 + b_2)$

and A is pure in G.

Now suppose that A is pure in G. For any $ng \in A_1$, ng is also in A and by the purity of A, $g \in A$ and thus

$g = b_1 + b_2$ with $b_i \in A_i$. Thus $ng = nb_1 + nb_2$ and

$ng - nb_1 = nb_2 \in A_1 \cap A_2 = \{0\}$. Hence $ng = nb_1$ and by torsion-freeness, $g = b_1 \in A_1$ and A_1 is pure in G. Similarly, A_2 is pure in G. ■

§A.1.3 **<u>*-Valuated Coproducts</u>**

<u>A.1.3.1 Definition [HM1]</u>: Let $A = \bigoplus_{i \in I} A_i$ be a valuated coproduct in G and represent each $a \in A$ as a finite sum $a = \sum_{i \in I_0} a_i$, where each $a_i \in A_i$ and I_0 is a finite subset of I. If for each prime p and each height sequence s it is the case that $a \in G(s^*)$ implies that $a_i \in G(s^*)$ for all the i's <u>and</u> $a \in G(s^*,p)$ implies that $a_i \in G(s^*,p)$ for all the i's then we say that $A = \oplus A_i$ is a <u>*-valuated coproduct</u>.

<u>A.1.3.2 Lemma</u>: If $H = \langle H_1 \rangle_* \oplus \langle H_2 \rangle_* \oplus \ldots \oplus \langle H_n \rangle_*$ and $N = H_1 \oplus \ldots \oplus H_n$ then H is a *-valuated coproduct if and only if N is a *-valuated coproduct.

<u>Proof</u>: Suppose H is a *-valuated coproduct in G. Each element in N is a sum of elements in the H_i's which are contained in the $\langle H_i \rangle_*$'s. Thus N is also a *-valuated coproduct in G. Conversely, suppose N is a *-valuated coproduct in G. Let $h = x_1 + x_2 + \ldots + x_n$ where each $x_i \in \langle H_i \rangle_*$ and let $h \in G(s^*,p)$ for some height sequence s. Then there are integers k_i, $i = 1,2,\ldots,n$ such that $k_i x_i \in H_i$. If $k = \Pi_i k_i$ then for each i, $kx_i \in H_i$ and thus $kh \in N$. As N is a *-valuated coproduct and $kh \in G((ks)^*,p)$, $kx_i \in G((ks)^*,p)$ for each i and thus by torsion-freeness, each $x_i \in G(s^*,p)$ and H is a *-valuated coproduct. ■

<u>A.1.3.3 Lemma</u>: If $G = \bigoplus_{i \in I} G_i$, I some index set, then G is a *-valuated coproduct in G.

<u>Proof</u>: By A.1.2.5, G is at least a valuated coproduct. Let $g \in G$ then $g = g_1 + \ldots + g_n$ with each $g_i \in G_{ji}$ for some $j \in I$. Suppose $g \in G(s^*,p)$ for some height sequence s and some prime p. Thus $g = x_1 + x_2 + \ldots + x_m + y$ where $\chi(y) \geq ps$, $\chi(x_j) \geq s$ and $\chi(x_j) \not\geq s$ for each $j=1,2,\ldots,m$.

Each $x_j = x_{1j} + x_{2j} + \ldots + x_{nj}$ and $y = y_1 + \ldots + y_n$ where each y_i and each x_{ij} are contained in G_i. Since G is a direct sum and by A.1.1.10, all the x_{ij}'s are such that $\chi(x_{ij}) \geq s$ and $\chi(x_{ij}) \not\sim s$ and $\chi(y_i) \geq ps$.

$g = \Sigma_j\Sigma_i x_{ij} + \Sigma_i y_i = \Sigma_i(\Sigma_j x_{ij} + y_i)$ where each $\Sigma_j x_{ij} + y_i \in G_i$. Thus each $g_i = \Sigma_j x_{ij} + y_i$ and hence $g_i \in G(s^*,p)$. Similarly, if $g \in G(s^*)$, then each $g_i \in G(s^*)$. ■

<u>A.1.3.4 Lemma [HM1]</u>: Let $A = A_1 \oplus A_2$ be a valuated coproduct where $A_2 = \langle x \rangle$ with x primitive in G. If $a = a_1 + a_2 \in G(s^*,p)$ implies that $a_i \in G(s^*,p)$ for all the i's then A is a *-valuated coproduct.

<u>Proof</u>: Let $a \in G(s^*)$. As $G(s^*) \subseteq G(s^*,p)$, $a \in G(s^*,p)$ and by our hypothesis, $a_1 \in G(s^*,p)$ and, for some n, $nx = a_2 \in G(s^*,p)$. But nx is primitive and thus either (i) $\chi(nx) \not\sim s$ or (ii) $\chi(nx) \sim s$ but at the same time, $h_p(nx) > s_p$. If (i) is true then $nx \in G(s^*)$ and $a_1 = a - a_2$ is also in $G(s^*)$ and A is a *-valuated coproduct.

Suppose (ii) is true. As A is a valuated coproduct, A.1.1.10 implies that $\chi(a) \sim s$ and $a \in G(s^*)$ implies that $a = x_1 + x_2 + \ldots + x_n$ where for each i, $\chi(x_i) \geq s$ and $\chi(x_i) \not\sim s$. By A.1.1.8, each $x_i \in A_1$ and thus $a_1 = a$ and $n = 0$ which trivially implies that $a_1 \in G(s^*)$ and $nx \in G(s^*)$ and hence that A is a *-valuated coproduct. ■

<u>A.1.3.5 Lemma [HM1]</u>: If $N \oplus \langle x \rangle$ is a *-valuated

coproduct in G with x primitive and if $y = x + z$ where $z \in N$ and $\chi(y) = \chi(x)$, then y is primitive and $N \oplus \langle y \rangle$ is a *-valuated coproduct in G.

<u>Proof</u>: Let $(s,p)_x$ be any pair such that s is a height sequence equivalent to $\chi(y) = \chi(x)$ and p is a prime such that $s_p = h_p(y) = h_p(x) \neq \infty$. If $y \in G(s^*,p)$ then as $N \oplus \langle x \rangle$ is a *-valuated coproduct, $x \in G(s^*,p)$ contradicting the primitivity of x. Thus y is primitive too. $N + \langle y \rangle \subseteq N + \langle x \rangle$ by definition of y.
$a \in N \oplus \langle x \rangle$ implies that $a = n + kx$ for some $n \in N$, and $k \in \mathbb{Z}$. Thus $a = n - kz + ky \in N + \langle y \rangle$ and hence $N \oplus \langle x \rangle = N + \langle y \rangle$ and since, $N \cap \langle y \rangle = \{0\}$ by properties of direct sums, $N \oplus \langle x \rangle = N \oplus \langle y \rangle$. We now have to prove that $N \oplus \langle y \rangle$ is a *-valuated coproduct. Let $w \in N$ and $k \in \mathbb{Z}$ then $\chi(w+ky) = \chi(w + kz + kx)$

$$= \chi(w + kz) \wedge \chi(kx) \leq \chi(kx) = \chi(ky).$$

Suppose that $w + ky \in G(s)$ then
$s \leq \chi(w + ky) \leq \chi(ky)$. Thus $ky \in G(s)$ and $w \in G(s)$ as G(s) is a subgroup of G. Thus we have that $N \oplus \langle y \rangle$ is a valuated coproduct in G. Now suppose that
$w + ky \in G(s^*,p)$. This implies that kx and $w + kz$ are in $G(s^*,p)$ as $N \oplus \langle x \rangle$ is a *-valuated coproduct. The primitivity of x implies, thus, that either $kx \not\geq s$ (in which case $ky \not\geq s$ and $ky \in G(s^*) \subseteq G(s^*,p)$) <u>OR</u> that $h_p(kx) = h_p(ky) > s_p$ (in which case $ky \in pG(s) \subseteq G(s^*,p)$). In either case, $ky \in G(s^*,p)$. Because $G(s^*,p)$ is a subgroup of G, $kz = ky - kx \in G(s^*,p)$ and hence

$w = w + ny - ny \in G(s^*,p)$. By A.1.3.4, $N \oplus \langle y \rangle$ is a $*$-valuated coproduct. ■

A.1.3.6 Remark: If $H \subseteq P \subseteq G$ and G is torsion-free but G/H is torsion and P is pure in G, then $P = G$. This is proved as follows: Let $g \in G$ then, by torsion freeness of G/H, there is an $n \in \mathbb{Z}$ such that $ng \in H$ and thus $ng \in P$. Purity of P implies that there is an $h \in P$ such that $ng = nh$ and torsion-freeness of G implies that $g = h$. Thus $G = P$.

A.1.3.7 Corollary [HM1]: If $G = \langle x \rangle_* \oplus K$ and $\chi(x) = s$, and if $y = x + z$ with $z \in G(s^*)$, then $G = \langle y \rangle_* \oplus K$.

Proof: $N = \langle x \rangle \oplus K$ is a $*$-valuated coproduct in G by A.1.3.2. As $z \in G(s^*)$, $\chi(y) = s$ and by A.1.1.8, $z \in K$; by A.1.1.9, x is primitive in G and thus by A.1.3.5, $N = \langle y \rangle \oplus K$ is a $*$-valuated coproduct which in turn implies (A.1.3.2) that $\langle y \rangle_* \oplus K$ is also a $*$-valuated coproduct. Now by A.1.2.6, $\langle y \rangle_* \oplus K$ is pure in G and G/N is torsion. Thus, by A.1.3.6, $G = \langle y \rangle_* \oplus K$. ■

A.1.3.8 Lemma [HM1]: Suppose $N = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \ldots \oplus \langle x_n \rangle$ is a $*$-valuated coproduct in G, where $x_1, x_2, \ldots, x_n$ are all primitive elements of the same type. Then every element of N is primitive in G. Moreover, if $y = x_1 + x_2 + \ldots + x_n$, then there exist elements $y_2, \ldots, y_n$ in N such that $N = \langle y_1 \rangle \oplus \langle y_2 \rangle \oplus \ldots \oplus \langle y_n \rangle$ is a *-valuated coproduct in G.

Proof: The proof is by induction on n.

Let $N = \langle x_1 \rangle \oplus \langle x_2 \rangle$ with x_1, x_2 both primitive. Let $\chi(x_1) = t_1$, and $\chi(x_2) = t_2$, say. By our assumption, t_1 and t_2 belong to the same type, τ, say. Let $y = n_1x_1 + n_2x_2$ and let $\chi(y) = t$. N is a *-valuated coproduct and therefore $t \in \tau$. Suppose s is a height sequence such that $s \in \tau$ and p is such that $s_p = t_p = h_p(y) \neq \infty$. If $y \in G(s^*,p)$ and if without loss of generality, $h_p(n_1x_1) \leq h_p(n_2x_2)$, then $n_1x_1 \in G(s^*,p)$ as N is a *-valuated coproduct which contradicts the primitivity of n_1x_1. Thus y is primitive and my is primitive for any $m \in \mathbb{Z}$ and hence every element in N is primitive.

Let $y_1 = x_1 + x_2$.

Since $t_1 \sim t_2$, the sets $A_1 = \{p \in P: h_p(x_1) < h_p(x_2) \neq \infty\}$ and $A_2 = \{p \in P: h_p(x_2) < h_p(x_1) \neq \infty\}$ are finite. Let $|A_1| = n'$, and $|A_2| = m'$. Let n_i be such that $h_{p_i}(p_i^{n_i} x_1) = h_{p_i}(x_2)$ for all $p_i \in A_1$ and all $i = 1, \ldots, n'$ and let m_j be such that $h_{p_j}(p_j^{m_j} x_2) = h_{p_j}(x_1)$ for all $p_j \in A_2$ and all $j = 1, \ldots, m'$.

Let $n = p_1^{n_1}p_2^{n_2}\ldots p_{n'}^{n_{n'}}$ where each $p_i \in A_1$

and $m = p_1^{m_1}p_2^{m_2}\ldots p_{m'}^{m_{m'}}$ where each $p_j \in A_2$. $(n,m) = 1$ as $A_1 \cap A_2 = \varnothing$. Thus $h_{p_i}(nx_1) = h_{p_i}(x_2)$ for all $p_i \in A_1$ and $h_{p_j}(x_1) = h_{p_j}(mx_2)$ for all $p_j \in A_2$. Let $k,l \in \mathbb{Z}$ such that $1 = kn + lm$ and let $y_2 = -knx_1 + lmx_2$. If $p \in A_1$, then $h_p(y_1) = h_p(x_1) \wedge h_p(x_2) = h_p(x_1)$ (*)

and $h_p(y_2) = h_p(-knx_1) \wedge h_p(lmx_2)$

$= h_p(-kx_2) \wedge h_p(lx_2)$ (as $(p,m)=1$)

$= h_p(x_2)$ as $(k,l) = 1$. (**)

Similarly, if $p \in A_2$, then $h_p(y_1) = h_p(x_2)$ and $h_p(y_2) = h_p(x_1)$ and for $p \in P\setminus(A_1 \cup A_2)$, $h_p(y_1) = h_p(y_2) = h_p(x_1) = h_p(x_2)$.

Now

$$T = \begin{bmatrix} 1 & 1 \\ -km & ln \end{bmatrix} \text{ is such that } T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

$\det(T) = 1$ and $T^{-1} = \begin{bmatrix} ln & -1 \\ km & 1 \end{bmatrix}$.

Let $x \in N$ then $x = n_1x_1 + n_2x_2$ for some $n_1, n_2 \in \mathbf{Z}$.

$$= [n_1\ n_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [n_1\ n_2]\,T^{-1}T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [n_1\ n_2]\,T^{-1}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [n_1\ n_2]\begin{bmatrix} ln & -1 \\ km & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= (n_1ln + n_2km)y_1 + (n_2 - n_1)y_2.$$

Therefore $N = \langle y_1 \rangle + \langle y_2 \rangle$.

Let $g \in \langle y_1 \rangle \cap \langle y_2 \rangle$ then $g = m_1y_1 = m_2y_2$ for some $m_1, m_2 \in \mathbf{Z}$.

Thus $m_1(x_1 + x_2) = m_2(-kmx_1 + lnx_2)$ and

$(m_1 + m_2km)x_1 = (m_2ln - m_1)x_2$. As $\langle x_1 \rangle \oplus \langle x_2 \rangle$ is a direct sum, $m_1 + m_2km = 0$ <u>and</u> $m_2ln - m_1 = 0$. We thus have the following system of equations:

$$\begin{bmatrix} 1 & km \\ -1 & ln \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Bearing in mind that $km + ln = 1$, we get that $m_1 = 0$ and $m_2 = 0$. Thus $N = \langle y_1 \rangle \oplus \langle y_2 \rangle$.

Claim: N is a valuated coproduct.

Proof: Let $y = n_1y_1 + n_2y_2$. Note that

$$y = [n_1 \quad n_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [n_1 \quad n_2]\ T\ T^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [n_1 \quad n_2] \begin{bmatrix} 1 & 1 \\ -km & ln \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [\ n_1 - n_2km \quad n_1 + n_2ln\ ] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus $y = n_1y_1 + n_2y_2 = m_1x_1 + m_2x_2$ where $m_1 = n_1 - n_2km$ and $m_2 = n_1 + n_2ln$. We need to show that $h_p(n_1y_1 + n_2y_2) = h_p(n_1y_1) \wedge h_p(n_2y_2)$ for all n_1, $n_2 \in \boldsymbol{Z}$ and $p \in \boldsymbol{P}$. We need only check the case when $h_p(n_1y_1) = h_p(n_2y_2)$. G is torsion-free and therefore common p-power factors can be cancelled and hence at least one of n_1 and n_2 is coprime to p. If $p \in A_1 \cup A_2$ then $h_p(y_1) < h_p(y_2)$ and $h_p(n_1y_1) = h_p(n_2y_2)$ implies that $p \mid n_1$ but $p \nmid n_2$. On the other hand, if $p \notin A_1 \cup A_2$, then $p \nmid n_1$ and $p \nmid n_2$.

If $p \in A_1$, then $h_p(n_1x_1) = h_p(n_1y_1)$ by (*)

$= h_p(n_2y_2)$ by assumption

$= h_p(y_2)$ as $p \nmid n_2$

$= h_p(x_2)$ by definition of y_2.

Now $h_p(n_2kmx_1) = h_p(kmx_1)$ as $p \nmid n_2$

$= h_p(kx_2)$ by definition of m.

Therefore, by definition of A_1, $p \mid n_2km$ and thus, $p \mid m_1$. Note that $m_2 - m_1 = n_2$ and $p \nmid n_2$ implies that $p \nmid m_2$. Hence, $h_p(m_2x_2) = h_p(n_2y_2)$ as $p \nmid m_2$, $p \nmid n_2$, and (**)

$= h_p(n_1y_1)$ by assumption

$\leq h_p(m_1x_1)$.

Thus $h_p(n_1y_1 + n_2y_2) = h_p(m_1x_1) \wedge h_p(m_2x_2) = h_p(m_2x_2) = h_p(x_2)$

And $h_p(n_1y_1) \wedge h_p(n_2y_2) = h_p(n_1y_1)$ by assumption

$= h_p(x_2)$ by above.

We thus have the desired result that :

$h_p(n_1y_1 + n_2y_2) = h_p(n_1y_1) \wedge h_p(n_2y_2)$.

If $p \in A_2$, then $h_p(x_1) > h_p(x_2)$, $h_p(y_1) = h_p(x_2)$, and $h_p(y_2) = h_p(x_1)$. Thus :

$h_p(n_1x_2) = h_p(n_1y_1) = h_p(n_2y_2) = h_p(y_2) = h_p(x_1)$;

$h_p(lnn_2x_2) = h_p(lnx_2)$ as $p \nmid n_2$

$= h_p(lx_1)$ by definition of n; and

$h_p(m_2x_2) = h_p(n_1x_2 + lnn_2x_2) \geq h_p(x_1) \wedge h_p(lx_1) = h_p(x_1)$.

As $p \in A_2$, $p \mid m$ and thus $p \mid m_2$ and again, as $m_2 - m_1 = n_2$, $p \nmid m_1$. Hence $h_p(m_1x_1) = h_p(x_1) \leq h_p(m_2x_2)$ and so $h_p(n_1y_1 + n_2y_2) = h_p(m_1x_1) \wedge h_p(m_2x_2) = h_p(m_1x_1) = h_p(x_1)$ and $h_p(n_1y_1) \wedge h_p(n_2y_2) = h_p(n_1y_1) = h_p(x_1)$. Thus we have the desired result that $h_p(n_1y_1 + n_2y_2) = h_p(n_1y_1) \wedge h_p(n_2y_2)$. On the other hand, if $p \notin A_1 \cup A_2$, the equation $m_2 - m_1 = n_2$ ensures that p divides at most one of the integers m_1 and m_2 and consequently, in this case, we have that $h_p(m_1x_1) \wedge h_p(m_2x_2) = h_p(x_1) = h_p(x_2) = h_p(y_1) = h_p(y_2)$ and $h_p(n_1y_1) \wedge h_p(n_2y_2) = h_p(y_1) = h_p(y_2)$ as $p \nmid n_i$, i=1,2. Thus, in each case, $h_p(n_1y_1 + n_2y_2) = h_p(n_1y_1) \wedge h_p(n_2y_2)$ and **N is a valuated coproduct.** To show that N is a *-valuated coproduct, let $0 \neq y = n_1y_1 + n_2y_2 \in G(s^*,p)$. y is primitive and therefore either :

(1) $\chi(y) \not\sim s$ or

(2) $\chi(y) \sim s$ but $y \in G(ps)$. In (1), $\chi(n_1y_1) \not\sim s$ and $\chi(n_2y_2) \not\sim s$ as $n_1y_1, n_2y_2 \in G(s)$ and thus $n_1y_1 \in G(s^*)$ and $n_2y_2 \in G(s^*) \subseteq G(s^*,p)$. In (2), as N is already a valuated coproduct, $y \in G(ps)$ implies that

n_1y_1 and $n_2y_2 \in G(ps) \subseteq G(s^*,p)$. Thus **N is a *-valuated coproduct.** Assume that if $N = \langle x_1 \rangle \oplus \ldots \oplus \langle x_{n-1} \rangle$ is a *-valuated coproduct in G; $x_1, x_2, \ldots, x_k, x_{n-1}$ are all primitive elements of the same type; then every element in N is primitive in G; and if $y_1 = \langle x_1 \rangle + \ldots + \langle x_{n-1} \rangle$, then there exist $y_2, \ldots, y_k$ in N such that $N = \langle y_1 \rangle \oplus \ldots \oplus \langle y_{n-1} \rangle$ is a *-valuated coproduct in G.

Now let $N = \langle x_1 \rangle \oplus \ldots \oplus \langle x_{n-1} \rangle \oplus \langle x_n \rangle$ and let $y \in N$. For simplicity, let $N_{n-1} = \langle x_1 \rangle \oplus \ldots \oplus \langle x_{n-1} \rangle$. Then

$y = a_{n-1} + k_nx_n$ where $a_{n-1} \in N_{n-1}$, and $k_n \in Z$. By the inductive hypothesis, a_{n-1} is primitive and thus, by exactly the same argument as in the case when n = 2, y is primitive. In particular, if $y_1 = x_1 + x_2 + \ldots + x_{n-1} + x_n$ then, for simplicity, let $x_1 + \ldots + x_k = b_1$. By the inductive hypothesis, there are $y_2, y_3, \ldots, y_{n-1}$ such that $N_{n-1} = \langle b_1 \rangle \oplus \langle y_2 \rangle \oplus \ldots \oplus \langle y_{n-1} \rangle$ is a *-valuated coproduct.

Note that $\text{type}(b_1) = \bigwedge_{i=1}^{n-1} \{type(x_i)\} = \text{type}(x_i)$ for every i thus, as in the case of n=2, choose y_n such that

$\langle b_1\rangle \oplus \langle x_n\rangle = \langle y_1\rangle \oplus \langle y_n\rangle$ is a *-valued coproduct. Now

$N = \langle x_1\rangle \oplus \ldots \oplus \langle x_{n-1}\rangle \oplus \langle x_n\rangle$

$= \langle b_1\rangle \oplus \langle y_2\rangle \oplus \ldots \oplus \langle y_{n-1}\rangle \oplus \langle x_n\rangle$

$= \langle y_1\rangle \oplus \langle y_2\rangle \oplus \ldots \oplus \langle y_{n-1}\rangle \oplus \langle y_n\rangle$ which is also a *-valued coproduct. ■

<u>A.1.3.9 Theorem [HM1]</u> : Suppose that $N = \langle x_1\rangle \oplus \langle x_2\rangle \oplus \ldots \oplus \langle x_m\rangle$ is a *-valued coproduct in G where each of the x_i's are primitive. If $y_1 \neq 0$ is a primitive element contained in N, then there exist primitive elements $y_2, \ldots, y_m$ such that $N' = \langle y_1\rangle \oplus \langle y_2\rangle \oplus \ldots \oplus \langle y_m\rangle$ is a *-valued coproduct with N/N' finite.

<u>Proof</u>: $y_1 \in N$ implies that we can write $y_1 = n_1x_1 + n_2x_2 + \ldots n_mx_m$. Note that:

1. in the event that n_i is zero, we can, for the sake of this part of the proof, rewrite N as $N = \langle x_1\rangle \oplus \ldots \oplus \langle x_{i-1}\rangle \oplus \langle x_{i+1}\rangle \oplus \ldots \oplus \langle x_m\rangle$ and add $\langle x_i\rangle$ to N and N' in the end for the result to hold; and
2. $N' = \langle n_1x_1\rangle \oplus \langle n_2x_2\rangle \oplus \ldots \oplus \langle n_mx_m\rangle$ is a *-valued coproduct and $y_1 \in N'$ with N/N' finite.

By the above, we can consider y_1 to be $y_1 = x_1 + x_2 + \ldots x_m$.

Now $\chi(y_1) = \bigwedge_{i=1}^{m} \chi(x_i) \leq \chi(x_i)$, *for each i*.

Rearrange the x_i's so that the first k x_i's are of the same type as y_1; let $y = x_1 + \ldots + x_k$ and

$g = x_{k+1} + \ldots + x_m$. $y \subseteq 0$ for else $y_1 \in G((\chi(y_1))^*) \subseteq G((\chi(y_1))^*, p)$ for any $p \in P$ which contradicts the primitivity of y_1. As N and N′ are $*$-valuated coproducts, $\chi(y_1) = \chi(y+g) \leq \chi(y)$ which implies that $\chi(y) = \chi(y_1)$ for else $y_1 \in G((\chi(y_1))^*)$ which again contradicts the primitivity of y_1. By lemma A.1.3.8 there are primitive elements $y_2, \ldots, y_k$ that $N' = \langle y \rangle \oplus \langle y_2 \rangle \oplus \ldots \oplus \langle y_k \rangle \oplus \langle x_{k+1} \rangle \oplus \ldots \oplus \langle x_m \rangle$ and by lemma we can replace y by y_1. ■

<u>A.1.3.10 Corollary</u> : If x is a primitive element in the separable group G, then $\langle x \rangle_*$ is a direct summand of G.

<u>Proof</u>: By separability of G, x is contained in a direct summand $A = A_1 \oplus A_2 \oplus \ldots \oplus A_m$ where each A_i is a rank one subgroup of G. Thus $x = x_1 + \ldots + x_m$ where each $x_i \in A_i$ and $x \in N = \langle x_1 \rangle \oplus \ldots \oplus \langle x_m \rangle$. Bearing in mind that $A_i = \langle x_i \rangle_*$, A/N is torsion and each $\langle x_i \rangle_*$ is a direct summand of G which implies that each x_i is primitive in G. The previous theorem yields $N' = \langle x \rangle \oplus \langle y_2 \rangle \oplus \ldots \oplus \langle y_m \rangle$ where each y_i is primitive and N/N′ is finite. $B = \langle x \rangle_* \oplus \langle y_2 \rangle_* \oplus \ldots \oplus \langle y_m \rangle_*$ is pure in G and thus pure in A with A/N′ torsion. Thus by A.1.3.6, B = A and $\langle x \rangle_*$ is a summand of A and thus of G too. ■

<u>A.1.3.11 Definition [HM1]</u> : Any subgroup, F, of G which can be represented as a $*$-valuated coproduct

$F = \bigoplus_{i\in I} \langle x_i \rangle$, where the x_i's are non-zero primitive elements of G, is called a free *-valuated subgroup of G. Under these circumstances, the x_i's are said to form a set of free generators of F.

A.1.3.12 Theorem [HM1] : If F and N are free *-valuated subgroups of G, where N has finite rank and $N \subseteq F$, then there is a *-valuated coproduct $F' = N \oplus M$, where F/F' is finite and M is also a free *-valuated subgroup of G.

Proof: Let $y_1, y_2, \ldots, y_m$ be a set of free generators of N. We proceed by induction on the number of y_i's that are also free generators of F'.

Let $M = F$ and let $F_0 = \langle 0 \rangle$. Then $F = F_0 \oplus M$ is such that F/F is finite and M is a free *-valuated subgroup of G. Assume that $F_{n-1} = \langle y_1 \rangle \oplus \ldots \oplus \langle y_{n-1} \rangle \oplus M_{n-1}$ is such that F/F_{n-1} is finite and M_{n-1} is a free -valuated subgroup of G. The finiteness of F/F_{n-1} guarantees that there is a multiple y_n' of y_n so that $y_n' \in F_{n-1}$. Thus we can write $y_n' = -y_1' - y_2' - \ldots - y_{n-1}' + m_{n-1}$ where $m_{n-1} \in M_{n-1}$ and for each $i = 1,\ldots,n-1$, y_i' is a multiple of y_i. Let $\chi(y_n') = s$. As F_{n-1} is a *-valuated coproduct, $s = \chi(y_n') \leq \chi(m_{n-1})$ and, for all $i=1,\ldots,n-1$, $\chi(y_n') \leq \chi(y_i')$. Rearrange the y_i's so that $y_1, y_2, \ldots, y_k$ all have the same types as type(y_n') and note that $k = 0$ makes no difference to the proof and that y_{k+1}',

..., y_{n-1}' are all contained in $G(s^*)$. By the first part of lemma A.1.3.8, $y_n' + y_1' + y_2' + \ldots + y_k'$ is primitive in G. $\chi(y_n' + y_1' + \ldots + y_k') = \wedge\{\chi(y_n'), \chi(y_1'), \ldots \chi(y_k')\} = \chi(y_n') = s$. Because M_{n-1} is also a free *-valuated subgroup, we can, as above, write $m_{n-1} = y + g$ where y is the sum of primitive elements in M_{n-1} of the same type as $\text{type}(y_n') = \tau$, say and $g \in G(s^*) \cap M_{n-1}$. Here again, by the first part of lemma A.1.3.8, y is primitive in G and, as $G(\tau)$ is a subgroup and type(y) ≤ type of each of its summands, type(y) = τ. Note that if y = 0, then $a = y_n' + y_1' + \ldots + y_k' \in G(s^*)$ a contradiction to the primitivity of a. Suppose $\chi(y_n') < \chi(y)$, then for at least one $p \in \boldsymbol{P}$, $h_p(y_n') < h_p(y)$ and $y \in G(ps)$ which means that $y_n' \in G(s^*,p)$ which contradicts the primitivity of y_n'. Thus $\chi(y_n') = \chi(y)$. By Theorem A.1.3.9, we can find $M_{n-1}' = \langle y\rangle \oplus M_n$ so that M_{n-1}/M_{n-1}' is finite and by Lemma A.1.3.5, $M'_{n-1} = \langle y_n'\rangle \oplus M_n$. Thus we have $F'_{n-1} = \langle y_1\rangle \oplus \ldots \oplus \langle y_{n-1}\rangle \oplus \langle y_n'\rangle \oplus M_n$ with F_{n-1}/F'_{n-1} finite and thus F/F'_{n-1} is finite. Since $\langle y_n\rangle/\langle y_n'\rangle$ has finite order, $F'_n = \langle y_1\rangle \oplus \ldots \oplus \langle y_{n-1}\rangle \oplus \langle y_n\rangle \oplus M_n$ satisfies the conditions of the theorem. ■

<u>A.1.3.13 Corollary [HM1]</u> : If N is a finite rank, free *-valuated subgroup of the separable group G, then the pure closure of N is a direct summand of G.

<u>Proof</u>: Let $N = \langle y_1\rangle \oplus \ldots \oplus \langle y_n\rangle$, where the y_i's are free generators of N. Since N is separable, N is contained in

a direct summand $A = A_1 \oplus A_2 \oplus \ldots \oplus A_m$ of G, where each A_i is a rank one subgroup. Each $y_i = \sum_{j=1}^{m} n_j a_{ij}$, $a_{ij} \in A_j$. Each A_i is locally cyclic and therefore we have nonzero x_j's such that $\langle x_j \rangle = \langle a_{1j}, a_{2j}, \ldots, a_{nj} \rangle \subseteq A_j$ and hence $F = \langle x_1 \rangle \oplus \ldots \oplus \langle x_m \rangle$ contains N. By A.1.3.12, we have a *-valuated coproduct $F' = N \oplus M$, where F/F' is finite and M is a free *-valuated subgroup of G. A/F is torsion as $A = \langle x_1 \rangle_* \oplus \ldots \oplus \langle x_m \rangle_*$ and F/F' is finite. Hence A/F' is torsion and by A.1.3.6, A is equal to the pure closure of F'. ■

APPENDIX A.2

In this appendix we introduce the concept of a k-group and show that any finite rank, free *-valuated subgroup of a k-group G, is knice in G.

k - Groups were first introduced by Hill and Megibben in [HM1:p 741]. They were also studied by M. Dugas and K.M. Rangaswamy in [DR1].

Definition [HM1]: A torsion-free group, G, is called a k-group if each finite subset can be imbedded in a finite rank, free *-valuated subgroup.

A.2.1 Lemma [HM1] : Any separable group is a k-group.

Proof : Any finite subset, S say, of a separable group, G is contained in a finite rank, completely decomposable summand, A, say, of G. In the same way as in the first part 3.1.1, we can find a free *-valuated subgroup, N which contains S and which in turn is contained in A. ■

A.2.2 Proposition [DR1] : A finite rank summand of a k-group is completely decomposable.

Proof: Let A be a finite rank summand of the k-group G. Then there is a finitely generated subgroup, F, of A with A/F torsion. Since G is a k-group, F is contained in a free *-valued subgroup N of G. The pure closure, B, of N is a completely decomposable group containing A and A is a direct summand of B. Thus, as direct summands of completely decomposable groups are completely decomposable too, the proof is complete. ■

<u>A.2.3 Theorem [HM1]</u> : G is separable if and only if G is a k-group with property that the pure closure of each finite rank, free *-valued subgroup is a direct summand.

<u>Proof</u>: Lemma A.2.1 shows that separable groups are k-groups and 3.1.1 proves that separable groups have the property that the pure closure of a finite rank, free *-valued subgroup of a separable group is a summand of that group. Now suppose that G is a k-group with the property stated above. Let S be any finite subset of G then, as G is a k-group, S can be embedded in a finite rank, free *-valued subgroup, N, say, of G. By the assumption, the pure closure, B, of N is a finite rank summand of G. By A.2.2, B is completely decomposable and thus G is separable. ■

<u>A.2.4 Lemma [HM1:Lemma 3.4]</u> : Any finite rank, free *-valued subgroup of a k-group G, is a knice subgroup of G.

<u>Proof</u>: Let x_1, x_2, ..., x_n be a set of free generators of

N and let S be any finite set in G. Take $S' = S \cup \{x_1, \ldots, x_n\}$. Since G is a k-group, we can select a finite rank, free *-valuated subgroup F containing S'. Then $N \subseteq F$ and by theorem A.1.3.12, we have a *-valuated coproduct $N' = N \oplus \langle y_1 \rangle \oplus \ldots \oplus \langle y_m \rangle$, where the y_i's are primitive and F/N' is finite. Since $S \subseteq F$, the proof is complete. ■

<u>A.2.5 Theorem [HM1]</u> : A countable k-group is completely decomposable.

<u>Proof</u>: Suppose $x_1, x_2, \ldots, x_n, \ldots$ is an enumeration of the elements of G and let $X_n = \{x_i : i < n\}$ for each $n < \aleph$. As G is a k-group, $\{x_1\}$ is contained in a finitely generated free *-valuated subgroup,

$F_1 = \langle y_{11} \rangle \oplus \langle y_{12} \rangle \oplus \ldots \oplus \langle y_{1,n_1} \rangle$. By 3.2.4, F_1 is knice in G and therefore there are primitive elements y_{21}, y_{22}, $\ldots$, y_{2,n_2} such that $F_2 = F_1 \oplus \langle y_{21} \rangle \oplus \ldots \oplus \langle y_{2,n_2} \rangle$ and $\langle X_2, F_2 \rangle / F_2$ is finite. Let $S_m = \bigcup_{i=1}^{m} \{y_{i1}, y_{i2}, \ldots, y_{i,n_i}\}$

and $F = \bigcup_{n<\omega} F_n$. Then T is a free *-valuated subgroup of G. Suppose $g \in G \backslash F$ then $g = x_N$ for some $N < \aleph$. Thus there is an $n_g \in \mathbb{Z}$ such that $n_g g \in F_N$ and G/F is torsion and G is the pure closure of F. $F = \bigoplus_{i<\omega} \bigoplus_{j=1}^{n_i} \langle y_{ij} \rangle$ and

hence $G = \bigoplus_{i<\omega} \bigoplus_{j=1}^{n_i} \langle y_{ij} \rangle_*$ which implies that G is completely decomposable. ■

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