CONJUGACY CLASSES OF SOME PROJECTIVE

LINEAR GROUPS

by

ERNEST ATHUR DIETRICH

Submitted in partial fulfilment of the requirements for the degree

MASTER OF SCIENCE

IN

MATHEMATICS

UNIVERSITY OF THE WESTERN CAPE

PROMOTOR : PROF. T.G. SCHULTZ

CONTENTS

ACKNOWLEDGEMENTS

First and foremost I wish to express my sincere gratitude towards my supervisor prof. T.G. Schultz for his continued guidance in often trying circumstances. His assistance has contributed to my education in no insignificant way.

I would also like to thank the University of the Western Cape for the financial assistance granted towards the completion of my studies.

My thanks also to Mrs A. Booth for her excellent typing of this manuscript, and to prof. G. Booth for his technical assistance in this regard.

Lastly, I wish to thank my parents, Ernest and Sylvia Dietrich for their continued support and encouragement.

ABSTRACT

Given a finite set X of distinct symbols the symmetric group S_X and the alternating group A_X are obtained without further constructions. More interesting groups are contrived, however, by imposing a certain structure on the set X and observing the subgroups formed by those elements of S_X that preserve this structure.

In this thesis we concern ourselves with one such imposition viz. that defining the notion of a finite projective plane. We look at the different subgroups of S_X arising in this manner, with particular emphasis on the projective linear groups and their action on the projective plane.

We conclude this work with a detailed study of the structure of the projective linear groups of orders 168 and 5616, respectively. Of particular interest to us are the distinct conjugacy classes of these groups, and the manner in which they relate to one another, within each particular group.

PREFACE

In Chapter 1 we present results relevant to the topic of this thesis. Section 1.1 deals with the concept of group representations and group characters. We also see how the structure constants of a finite group are evident of the interaction of the group's distinct conjugacy classes. The latter becomes a useful tool in our study of the two projective linear groups investigated in Chapter 3.

In 1.2 we present results pertaining to projective geometries and their associated groups. Emphasis is placed on those projective geometries of (projective) dimension two. We show that the latter comply with the abstract definition of a finite projective plane and look at the action of the projective linear groups on the projective plane.

Section 1.3 is devoted to the relationships between projective geometries and other areas in Mathematics.

In Chapter 2 we proceed to construct a number of finite projective planes and show the existence of a bijection between projective planes of the same order.

Chapter 3 concludes this work with a detailed study of the structure of the projective linear groups PGL(3,2) and PGL(3,3).

NOTATION

G, a finite group

Z(G), the centre of G

|G|, the order of G

X_i, group characters

K_i, conjugacy classes

C(K), centralizer of K

F_q, a finite field with q elements, q a prime power

ρ, an F–representation of G

λ, an element of F

$F^* := F \setminus \{0\}$

$\mathbb{Z}_m$, the integers modulo m

V = V(n,q), an n–dimensional vector space over the field F with q elements

GL(V) = GL(n,q), the general linear group of non–singular transformations of V = V(n,q)

SL(V) = SL(n,q), the special linear group of V = V(n,q)

PGL(V) = PGL(n,q), the projective general linear group over V = V(n,q)

PSL(V) = PSL(n,q), the projective special linear group over V = V(n,q)

PG(n–1,q), projective geometry of (projective) dimension n–1

~, equivalent to

[x], the equivalence class of x

$x^{\perp}$, the orthogonal complement of x

<x>, the subspace generated by x

$c_{ij\ell}$ structure constants

$A_{ij\ell}$ square matrix with $c_{ij\ell}$ as ij–th entry

S, a finite set

card (S), the cardinality of S

$\mathcal{L}$, a family of subsets of S

θ, a permutation

CHAPTER 1

PRELIMINARIES

Two distinct areas relevant to the topic of this thesis are dealt with in this chapter, the first being that of group representations, whilst the second consists of topics from finite geometries and their associated groups. Results to be referred to in subsequent chapters are stated, often without proof. At the same time we will present results relevant to our theme in order to enhance understanding and to identify its relations with other areas in Mathematics.

1.1 GROUP REPRESENTATIONS

Let G be a finite group and $\{K_1, \ldots, K_n\}$ the set of its distinct conjugacy classes (we say n is the class number of G). If F is any field and V is an m–dimensional vector space over F, then a group homomorphism $\rho: G \to GL(V)$, where $GL(V)$ is the set of all nonsingular F–linear transformations of V, is called an F–representation of G. In this case m is called the degree of the representation ρ. If ρ is injective it is called faithful.

Given a basis B of V, we have that $\rho(g)$, for all $g \in G$, has an associated matrix $[\rho(g)]_B$, so that the map $X_\rho : G \to F$ defined by $g \to \text{trace}([\rho(g)]_B)$, is well defined. We call X_ρ the character of the representation ρ. Now, if B' is any other basis for V, then there exists a nonsingular mxm matrix P such that $P^{-1}[\rho(g)]_B P = [\rho(g)]_{B'}$, and since $\text{trace}(XAX^{-1}) = \text{trace}(A)$ for any square matrix A and nonsingular matrix X, we have that $X_\rho(g) = \text{trace}([\rho(g)]_{B'}) = \text{trace}([\rho(g)]_B)$. Hence we have;

1.1.1 **Lemma**: The character X_ρ of the representation ρ is independent of the choice of basis B of V and clearly, if $g = hah^{-1}$, for $g,h,a \in G$, then $X_\rho(a) = X_\rho(g)$ i.e. X_ρ is constant valued on the conjugancy classes K_i of G. □

Now, let FG denote the set of all formal sums $\sum_{x \in G} f_x x$, $f_x \in F$ and, with a finite number of exceptions, $f_x = 0$. With addition and multiplication defined by

$$(\Sigma f_x x) + (\Sigma f'_x x) := \Sigma_x (f_x + f'_x)x \text{ and}$$

$$(\Sigma f_x x)(\Sigma f'_x x) := \Sigma_x \left(\sum_{yz=x} f_y f_z \right)x, \text{ respectively,}$$

we call FG the group ring of G over F. Since F is a field, FG has a natural F–module structure given by $f(\Sigma f_x x) = \Sigma_x (f f_x)x$, where $f \in F$. This makes FG into an F–vector space. Furthermore, the centre of FG, Z(FG), has the natural basis $\{1, x_1, \ldots, x_n\}$, where $x_i := \sum_{x \in K_i} x$ ([11], [20], [26]). Its dimension therefore equals the class number of G.

Now, if $\rho: G \to GL(V)$ is an F–representation of G with degree n, then V can be turned into a right FG–module by means of the rule

$$a\left(\sum_{x \in G} f_x x \right) := \sum_{x \in G} f_x (a[\rho(x)]), \text{ where } a \in V.$$

Conversely, if V is a right FG–module with F–dimension n, there exists a corresponding F–representation $\rho: G \to GL(V)$ of degree n given by $a[\rho(g)] = ag$, where $a \in V$. Thus, what we have is none other than a bijection between the F–representations of G with degree n, and the right FG–modules with F–dimension n.

We call two F–representations ρ_V and ρ_{V^1} of a group G equivalent if they arise from isomorphic right FG–modules V and V^1, respectively. From the aforementioned bijection it follows that equivalent representations have the same degree and character.

An F–representation ρ of G is called reducible if the right FG–module V from which it arises has a proper nonzero submodule. An F–representation is called irreducible if its associated right FG–module V has no proper nonzero submodules (if V itself is nontrivial).

1.1.2 **Proposition** ([26]): Every character is a sum of irreducible characters (where the sum X + Y of characters X and Y is defined by $(X + Y)(g) = X(g) + Y(g)$, $g \in G$). □

The following result, due to Frobenius and also stated without proof, provides a fundamental relation between the irreducible characters of a group G.

1.1.3 **Proposition** ([26]): Let G be a finite group and F a field. Let X and Y be distinct irreducible characters of F–representations of G. Then;

(i) $\sum_{x \in G} X(x)Y(x^{-1}) = 0$

(ii) If F is algebraically closed and its characteristic does not divide the order of G, then

$$\sum_{x \in G} X(x)X(x^{-1}) = |G|$$

(iii) If F has characteristic 0, then

$$(|G|)^{-1} \sum_{x \in G} X(x)X(x^{-1}) \text{ is always a positive integer.}$$

□

1.1.3 (i) and (ii) are called the orthogonality relations.

In view of the fact that the number of irreducible characters equals the class number (see[11]), it is convenient to display the character values of a group G in a table which we call the character table of G. In particular, if G is a finite group with $\{K_1, \ldots, K_n\}$, the set of its distinct conjugacy classes, F an algebraically closed field whose characteristic does not divide the order of G, and $X_1, \ldots, X_n$ are the irreducible F–characters of G, then we write;

	K_1	...	K_n
X_1	$X_1^{(1)}$	...	$X_1^{(n)}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
X_n	$X_n^{(1)}$	...	$X_n^{(n)}$

Here $X_i^{(j)}$ denotes the value of the character X_i on the conjucacy class K_j.

The orthogonality properties of the characters may now be translated into row and column orthogonality of the character table in the following manner:

By 1.1.3 we can write $\sum_{x \in G} X_i(x)X_j(x^{-1}) = m\delta_{ij}$, where $m = |G|$ and δ_{ij} is the Kronecker delta function. Writing $\ell_i = |K_i|$ and $K_{i^*} = (K_i)^{-1}$, this becomes

$$\sum_{r=1}^{n} \ell_r X_i^{(r)} X_j^{(r^*)} = m\delta_{ij} \qquad (1)$$

which expresses the orthogonality of rows of the character table, whilst

$$\sum_{i=1}^{n} X_i^{(r^*)} X_i^{(s)} = \frac{m}{\ell_r}\delta_{rs} \qquad (2)$$

expresses the orthogonality of columns of the character table.

When $F = \mathbb{C}$, the field of complex numbers, equations 1 and 2 become

$$\sum_{r=1}^{n} \ell_r X_i^{(r)} \overline{X}_j^{(r)} = m\delta_{ij}$$

and

$$\sum_{i=1}^{n} \overline{X}_i^{(r)} X_i^{(s)} = \frac{m}{\ell_r}\delta_{rs},$$

respectively, where $\overline{X}_i : G \to \mathbb{C}$ is given by $\overline{X}_i(g) = \overline{X_i(g)}$ and is called the complex conjugate of X_i.

If K_i, K_j and K_ℓ are distinct conjugacy classes of a finite group G, then the number of solutions (x,y) of xy = z, where $x \in K_i$, $y \in K_j$ and $z \in K_\ell$, is called the structure constant $c_{ij\ell}$ of G. Thus, if $K_iK_j := \{xy : x \in K_i, y \in K_j\}$, then each element of K_ℓ occurs $c_{ij\ell}$ times in K_iK_j. Thus the structure constants connect the conjugacy classes of G with respect to product formation. The following formula for computing the structure constants of a group was established by Burnside ([8]).

1.1.4 **Proposition** : Let n be the number of conjugacy classes of the finite group G, and let $K_1, \ldots, K_n$ be its distinct conjugacy classes. Let x_i be a representative of the elements in K_i, $1 \leq i \leq n$. Then

$K_iK_j = \sum_{\ell=1}^{n} c_{ij\ell}K_\ell$, where

$$c_{ij\ell} = \frac{|K_i| \cdot |K_j|}{|G|} \sum_{X} \frac{X(x_i)X(x_j)\overline{X}(x_\ell)}{X^{(1)}},$$

where $|K_i|$, $|K_j|$ are the orders of the conjugacy classes K_i and K_j, respectively, $X(x_i)$, $X(x_j)$ are the values of the character X on the conjugacy classes K_i and K_j, respectively, $X^{(1)}$ is the value of the character X on the identity, and $|G|$ is the order of G.

From [3] we obtain a generalization of this product to that of any finite number of conjugacy classes viz.

$$K_{i_1} \cdots K_{i_m} = \frac{|K_{i_1}| \ldots |K_{i_m}|}{|G|} \sum_{X} \frac{X(x_{i_1}) \ldots X(x_{i_m}) \overline{X}(x_{\ell})}{X(1)^{(m-1)}}$$

□

Burnside's formula therefore allows us to compute the structure constants of a group G when the irreducible characters of G are known. Conversely, the character table of a group G may be determined if its structure constants are known ([11]).

The usefulness of representation theory, and in particular that of group characters, in the study of finite groups is well known. A number of results on abstract groups have been proved through the use of group characters, the best known being that of Burnside which states that a finite group whose order has at most two distinct prime divisors, must be solvable. Basic references [11], [20], [21] and [26] may be consulted.

1.2 FINITE GEOMETRIES AND THEIR ASSOCIATED GROUPS

Throughout this section, let $V = V(n,q)$ denote an n–dimensional vector space over the field F with $q = p^r$ elements, where p is a prime. In section 1.1 we defined the general linear group $GL(V)$ to be the set of all linear automorphisms of V over F. The set of all linear automorphisms of V of determinant 1 is called the special linear group $SL(V)$. It is easy to see that the inverse ℓ^{-1} of a linear automorphism ℓ of V is also a linear automorphism, and since $(\det \ell)(\det \ell^{-1}) = \det(\ell.\ell^{-1}) = \det(1) = 1$ the linear automorphisms have nonzero determinant. If δ is the determinant map, then it is clear that δ maps $GL(V)$ into $F^* = F \setminus \{0\}$. Furthermore, $\delta(v.w) = \det(v.w) = \det(v).\det(w)$ for all $v,w \in GL(V)$, so that δ is in fact a homomorphism of GL(V) onto the multiplicative group F^*. Since SL(V), by definition, is the kernel of this map, we have, by the first isomorphism theorem for groups, that;

1.2.1 **Lemma** : SL(V) is a normal subgroup of GL(V) and
$[GL(V) : SL(V)] = |F^*| = q-1$.

For each pair of ordered bases $\{v_1, \ldots, v_n\}, \{w_1, \ldots, w_n\}$ of V there exists a unique linear automorphism ℓ of V such that $\ell(v_i) = w_i$, $1 \le i \le n$ and conversely, for each linear automorphism ℓ, $\{\ell(v_1),\ldots,\ell(v_n)\}$ is an ordered basis. Thus the order of GL(V) is equal to the number of ordered bases of V. Since $|V| = q^n$ and the first member of an ordered basis may be any nonzero element of V it can be chosen in q^n-1 ways. The second member, linearly independent on the first, may be chosen in q^n-q ways, the third in q^n-q^2 ways, and so on. Continuing in this fashion we find that:

1.2.2 **Lemma** : The orders of GL(V) and SL(V) are given by :

(i) $$|GL(V)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i-1)$$

(ii) $$|SL(V)| = q^{n(n-1)/2} \prod_{i=2}^{n} (q^i-1),$$ from Lemma 1.2.1. □

From Biggs, et al ([7]) we have the following result concerning the centres of GL(V) and SL(V).

1.2.3 **Lemma** : The centre of GL(V) consists of the q–1 scalar transformations $g \to \lambda g$, $(\lambda \in F^*)$, whilst the centre of SL(V) consists of those scalar transformations for which $\lambda^n = 1$.

□

If $V = V(n,q)$ is a vector space over a field F, then the equation $x = \lambda y$, where $x,y, \in V^* = V \setminus \{0\}$, $\lambda \in F^*$, defines an equivalence relation on V^*. We shall denote by [x] the equivalence class of $x \in V^*$. The set of equivalence classes $PG(V) = \{[x] : x \in V^*\}$ is called a projective geometry (vector space) of projective dimension n–1. When necessary, to avoid ambiguity we shall write PG(n–1,q) for PG(V).

Let $\varphi : V^* \to PG(V)$ be the natural map $x \to [x]$. A subset S of PG(V) of the form $\varphi(W^*)$, for some (m+1)–dimensional subspace W of V, is called a projective m–subspace or a projective geometry of (projective) dimension m. If m = 0, 1 or 2, S is called a projective point, a projective line, or a projective plane, respectively.

1.2.4 **Example** : Let V(n,2) be any n–dimensional vector space over the field $F = \mathbb{Z}_2$, then $|V| = 2^n$. Since F^* has only one element, 1, PG(n–1,2) has 2^n-1 points.

□

1.2.5 **Lemma** : If V has dimension $n \geq 2$, then;

(i) for $x,y \in V^*$, $[x] \neq [y]$ if and only if $\{x,y\}$ is linearly independent.

(ii) every two distinct points in PG(n–1, q) lie on a unique line.

Proof: (i) If $[x] \neq [y]$, then $x \neq \lambda y$, for any $\lambda \in F^*$, and $\{x,y\}$ is linearly independent. Conversely, if $\{x,y\}$ is linearly independent, then $x \neq \lambda y$, for all $\lambda \in F^*$, and hence $[x] \neq [y]$.

(ii) Let $[x] \neq [y]$ be points in PG(n–1,q) with $x,y \in V^*$ representing each. A projective line L containing both $[x]$ and $[y]$ is of the form $\varphi(W^*)$, where W^* is a 2–dimensional subspace of V containing x and y. By (i) above $\{x,y\}$ is linearly independent, so that $\langle x,y\rangle = W$. This proves both the existence and uniqueness of L.

□

1.2.6 **Proposition** : The points and lines of PG(2,q) satisfy the following:

(i) every pair of distinct points lie on a unique common line,

(ii) every pair of distinct lines intersect at a unique common point,

(iii) PG(2,q) contains a set of four points with the property that no three of them lie on a common line.

Proof: (i) was proved in 1.2.5 (ii)

(ii) If L_1 and L_2 are distinct lines in PG(2,q), then there exist 2–dimensional vector spaces V_i, i = 1,2, such that $L_i = \{[x] \mid x \in V_i\}$. Since $\dim(V_1 \cap V_2) = 1$, $\{[x] \mid x \in V_1 \cap V_2\}$ represents a unique point on L_1 and L_2.

(iii) We show that the points of the subspaces generated by the vectors {(1,0,0), (0,1,0), (0,0,1), (1,1,1)} exhibit the required property. Suppose that the first three vectors in our set generate subspaces which, as points, lie on a common line. This implies that there is a 2–dimensional subspace containing the three vectors. However, our three vectors are linearly independent, which leads to a contradiction. Since any subset of three vectors from our set is linearly independent the same will result when considering all such subsets.

□

1.2.7 **Example** : Consider PG(2,2) of example 1.2.4. A 1–dimensional subspace (line) in PG(2,2) is the image of a 2–dimensional subspace W of V(3,2). [W] in PG(2,2) contains 3 points, since W contains 4 points, one being the origin. Choosing coordinates (x_1,x_2,x_3) for a point x in V and denoting by $[x_1,x_2,x_3]$ the point [x] in PG(2,2) we obtain;

The subspace W with equation $x_1+x_2+x_3 = 0$ gives rise to a line [W] in PG(2,2) containing the points [1,0,1], [1,1,0] and [0,1,1]. The subspace with equation $x_1+x_2 = 0$ gives rise to a line in PG(2,2) containing [1,1,0], [0,0,1] and [1,1,1]. Continuing in this fashion we obtain seven lines, the other five being {[1,0,0,], [1,1,1], [0,1,1]}, {[0,1,0], [1,1,1], [1,0,1]}, {[1,0,0,], [1,0,1], [0,0,1]}, {[0,1,0], [0,1,1,], [0,0,1]} and {[1,0,0], [1,1,0], [0,1,0]}. With the points [1,0,0], [0,1,0], [0,0,1], [1,1,1] satisfying the third property of 1.2.6 we see that PG(2,2) exhibits all three of the properties given. □

1.2.8 **Proposition** ([27]): (i) For every n > 0 and every prime power q, the number of points in PG(n–1,q) is $(q^{n-1})/(q-1)$. In particular, every projective line has exactly q+1 points.

(ii) The number of projective lines in PG(2,q) equals the number of points in PG(2,q) viz. q^2+q+1. We say q is the order of PG(2,q). □

As an example, consider PG(2,2) of 1.2.7. We see that PG(2,2) contains $2^2+2+1 = 7$ points, 7 lines, and each line contains 3 points.

Given $g \in GL(V)$, we define a permutation $\hat{g}$ of PG(V) by the rule $\hat{g}[x] = [g(x)]$, for $x \in V^*$. This definition is independent of the chosen representative of [x] since, if $[x] = [x']$, then $x = \lambda x'$, for some $\lambda \in F^*$, and $g(x) = g(\lambda x') = \lambda g(x')$, so that $[g(x)] = [g(x')]$. However, $g \to \hat{g}$ is not faithful, since some non–identity automorphisms may well induce the identity on PG(V). In fact, we have:

1.2.9 **Lemma** ([7]): For $g \in GL(V)$, the induced permutation $\hat{g}$ is the identity on PG(V) if and only if g is a scalar transformation. □

Since we are not concerned with the action of the scalar transformations on PG(V), we eliminate them by 'collapsing' the general and special linear groups onto their respective centres. To this effect we obtain;

1.2.10 The projective general linear group PGL(V) and the projective special linear group PSL(V) are respectively defined as follows;

$$PGL(V) := {}^{GL(V)}/_{Z(GL(V))},$$

$$PSL(V) := {}^{SL(V)}/_{Z(SL(V))},$$

and their orders are;

$$|PGL(V)| = q^{n(n-1)/2} \prod_{i=2}^{n} (q^i-1),$$

$$|PSL(V)| = [\gcd(q-1,n)]^{-1}|PGL(V)|,$$

where gcd (,) denotes the highest common factor and V has dimension $n \geq 2$ ([7]).

□

A group G of permutations is said to be transitive if, given a pair of letters a, b (which need not be distinct), there is at least one permutation $\beta \in G$ which transforms a into b. G is said to be k–transitive if it contains at least one permutation ξ which changes any ordered set of k distinct objects $a_1, a_2, \ldots, a_k$ into any other such set $b_1, b_2, \ldots, b_k$ (the two sets may have elements in common).

From [7] we have;

1.2.11 **Proposition** : Both PGL(V) and PSL(V) act 2–transitively on the points of PG(n–1,q). That they are not 3 –transitive on PG(2,q) is clear from 1.2.6. □

1.2.12 **Proposition** : PSL(V) is simple, provided that $n \geq 2$ and $(n,q) \neq (2,2)$ or $(2,3)$.

□

A permutation θ on the points of PG(V) that takes lines to lines in PG(V) is called a colineation. Thus, the set of colineations of PG(V) is a subgroup of the symmetric group on PG(V).

If $[x]$ and $[y]$ are distinct points on a line in $PG(V)$, then $\dim\langle x,y\rangle = 2$. Thus for each $g \in GL(V)$, $\dim (g\langle x,y\rangle) = \dim\langle g(x),g(y)\rangle = 2$ and hence $[g(x)]$ and $g[y]$ are distinct points on a line in $PG(V)$. Therefore $\hat{g}$ is a colineation, where $\hat{g}[x] = [g(x)]$, for all $x \in V$.

If V is a vector space over a field F, then a permutation ℓ of V, such that

$\ell(x+y) = \ell(x)+\ell(y)$,

$\ell(\lambda x) = \alpha(\lambda)\ell(x)$, where $x,y \in V$, $\lambda \in F$, and α is a fixed automorphism for each ℓ, is called a semilinear automorphism of V. It is obvious that if α is the identity automorphism, then ℓ is linear.

For example, let V be a vector space over the field F with $q = 2^2$. If α is the automorphism $\alpha{:}\lambda \to \lambda^2$ and ℓ is a permutation of V with $\ell(x+y) = \ell(x)+\ell(y)$, and $\ell(\lambda x) = \alpha(\lambda)\ell(x) = \lambda^2\ell(x)$, where $x,y \in V$, $\lambda \in F$, then ℓ is a semilinear automorphism of V.

Each semilinear automorphism ℓ of V induces a permutation $\hat{\ell}$ of $PG(V)$ in the same way as for linear automorphisms so that $\hat{\ell}$ is also a colineation of $PG(V)$.
The group of all colineations of $PG(V)$ is denoted by $\Gamma L(V)$. $GL(V)$ and $SL(V)$ are subgroups of $\Gamma L(V)$ whilst $PGL(V)$ and $PSL(V)$ are quotient groups of $\Gamma L(V)$ ([7]).
If $|F| = p^r$, then $|Aut(F)| = r$, and $[\Gamma L(V) : GL(V)] = r$.

We earlier defined a projective geometry $PG(V)$ of (projective) dimension two, to be a projective plane. However, since there exist other structures, not quite the same as a $PG(V)$, that exhibit the properties of $PG(2,q)$ (see [16]), we need to give a more general definition;

1.2.13 A finite projective plane or geometry is a finite set S, together with a family $\mathcal{L}$ of subsets of S, satisfying:

PP_1: Each pair of distinct elements of S belongs to only one set ϱ in $\mathcal{L}$

PP_2: The intersection of each pair of distinct sets in $\mathcal{L}$ is a single element of S.

PP_3: At least four elements of S have the property that no three of them occur in a single set ϱ of $\mathcal{L}$.

The analogy with 1.2.6 is clear when we consider the points of PG(2,q) to be the elements of S with the lines of PG(2,q) being the elements of $\mathcal{L}$.

1.2.14 **Example:** Numbering the points of PG(2,2) constructed in example 1.2.7 from 1 to 7 we obtain:

$S = \{1,2,3,4,5,6,7\}$ and

$\mathcal{L} = \{1,2,4\}, \{2,3,5\}, \{3,4,6\}, \{4,5,7\}, \{5,6,1\}, \{6,7,2\}, \{7,1,3\}$.

Thus far all known projective planes have order a prime power, the most well known unsolved problem in the study of projective planes being the question on the existence of a projective plane of order other than a prime power. The only result showing the general non–existence of any finite projective plane of given order is the Bruck–Ryser theorem which states that if $q \equiv 1$ or 2 (mod 4), then there cannot be a projective plane of order q unless q can be expressed as a sum of two integral squares. For a proof, see [16], pp 87 – 89.

1.3 RELATIONSHIPS WITH OTHER AREAS

A projective plane PG(2,q) may also be described in terms of a (0,1)–matrix A in the following manner;

Let $P_1, ..., P_m$ and $L_1, ..., L_m$ denote the points and lines of PG(2,q), respectively. Then the matrix A of zeros and ones will have as its ij–th entry the number 1 if and only if P_i is on L_j. We call A the incidence matrix of PG(2,q).

1.3.1 **Example:** From Example 1.2.14, with the given order, we have that:

$$A = \begin{bmatrix} 1&0&0&0&1&0&1 \\ 1&1&0&0&0&1&0 \\ 0&1&1&0&0&0&1 \\ 1&0&1&1&0&0&0 \\ 0&1&0&1&1&0&0 \\ 0&0&1&0&1&1&0 \\ 0&0&0&1&0&1&1 \end{bmatrix}$$ □

From [19] we also have:

1.3.2 **Proposition** : For $m > 3$ an mxm (0,1)–matrix A defines a projective plane if and only if $A^TA = AA^T = kI+J$ where J is the kxk matrix, all of whose entries are 1.

A balanced incomplete block design (BIBD) ([15]) of type (b,ν,r,k,λ) consists of a family B_i, i = 1 to b, of subsets of a set V with ν elements such that (i) $|B_i| = k < \nu$ for all i, (ii) $|\{i|x \in B_i\}| = r$ for all $x \in V$, (iii) $|\{i|x \in B_i \text{ and } y \in B_i\}| = \lambda$ for all $x \neq y$ in V.

BIBD's are used in statistics in the design of experiments in which it is not convenient to test every value of a factor α against every value of a factor β, for varying values of a third factor γ. The block B_i is normally considered to be the i–th set of experiments and V is considered to be the set of varieties to be tested. We show that a projective plane of order q is a BIBD of type $(q^2+q+1, q^2+q+1, q+1, q+1, 1)$:

Let PG(2,q) be the set of q^2+q+1 points of the projective plane of order q, and let B_i, for $1 \leq i \leq q^2+q+1$, be the lines of the plane. By 1.2.6 and 1.2.8 each line contains q+1 points and each point occurs on q+1 lines. Therefore

(i) $|B_i| = q+1 < q^2+q+1$ for all i,

(ii) $|\{i \mid x \in B_i\}| = q+1$ for all $x \in PG(2,q)$,

(iii) $|\{i \mid x \in B_i \text{ and } y \in B_i\}| = 1$ for all $x \neq y \in PG(2,q)$.

A further interesting application of finite geometries lies in its interaction with Abelian groups. Every projective geometry PG(n–1,q), where $q = p^r$ (p a prime), can be represented by an Abelian group G of order p^{nr} and of type (1, ..., 1). This implies that every abstract theorem pertaining to the geometry PG(n–1, q) may be translated into a corresponding result relating to the Abelian group G. Conversely, some of the results concerning the group G may well be translated into results pertaining to PG(n–1,q). A detailed exposé of this relation may be found in [9], pages 328 – 344.

CHAPTER 2

SOME FINITE PROJECTIVE PLANES

2.1 INTRODUCTION

In this section we explain the methodology that will be followed for the construction of some finite projective planes in the subsequent sections.

Let V be a 3–dimensional euclidean space over the finite field F. The construction of finite projective planes is based on the following fact regarding the 2–dimensional subspaces of V.

If U and W are 2–dimensional subspaces of V,
then $U = W$ or $\dim(U \cap W) = 1$.

For each $x \in V \smallsetminus \{0\}$, let $[x]^{\perp} := \{[u] \mid u \in x^{\perp}\}$, where $x^{\perp}$ is the orthogonal complement of x in V. $[x]^{\perp}$ is a line in $PG(V)$. Now, for distinct $[x], [y] \in PG(V)$, we have that $x^{\perp} = y^{\perp}: \Rightarrow [x] = [y]$ – a contradiction. So we must have $\dim(x^{\perp} \cap y^{\perp}) = 1$; hence $[x]^{\perp}$ intersects $[y]^{\perp}$ in the point $\{[z] \mid z \in x^{\perp} \cap y^{\perp}\}$.

Thus, if we consider the set $PG(V)$ together with the lines $\mathcal{L} := \{[x]^{\perp} \mid x \in V \smallsetminus \{0\}\}$, then :

(a) For each distinct pair $[x], [y] \in PG(V)$, $[x], [y] \in [z]^{\perp}$, where $\langle z \rangle := \langle x,y \rangle^{\perp}$.

(b) From the above we have that either $[x]^{\perp} = [y]^{\perp}$ or $[x]^{\perp}$ intersects $[y]^{\perp}$ in a unique point.

(c) The elements $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ and $[1,1,1]$ cannot lie in any one element of $\mathcal{L}$.

Thus, the pair (PG(V), $\mathcal{L}$) is a projective plane.

In the sequel we will use this method to construct finite projective planes associated with sets PG(2,3), PG(2,4), PG(2,5) and PG(2,8). The chapter is concluded by providing an algorithm to construct an isomorphism between planes which are obtained by renumbering the elements of PG(V).

2.2 A PROJECTIVE PLANE FOR PG(2,3)

Let V(3,3) be a vector space over the field $F = \mathbb{Z}_3$ with addition and multiplication defined by

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

and

$\cdot$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{1}$

,

respectively.

Choosing representatives for the equivalence classes of V^* we obtain;

[1,0,0], [0,1,0], [1,1,0], [1,2,0,], [0,0,1], [1,0,1], [1,0,2], [0,1,1], [0,1,2], [1,1,1], [1,1,2], [1,2,1], and [1,2,2].

Numbering the equivalence classes from 1 to 13 and determining the orthogonal complement of each, we have;

i	$[x]_i$	$[x]_i^\perp$
1	[1,0,0]	{2,5,8,9}
2	[0,1,0]	{1,5,6,7}
3	[1,1,0]	{4,5,12,13}
4	[1,2,0]	{3,5,10,11}
5	[0,0,1]	{1,2,3,4}
6	[1,0,1]	{2,7,11,13}
7	[1,0,2]	{2,6,10,12}
8	[0,1,1]	{1,9,11,12}
9	[0,1,2]	{1,8,10,13}
10	[1,1,1]	{4,7,9,10}
11	[1,1,2]	{4,6,8,11}
12	[1,2,1]	{3,7,8,12}
13	[1,2,2]	{3,6,9,13}

With the third axiom of 1.2.13 satisfied by the set {1,2,5,10}, it is easily verified that we have a projective plane of order 3.

2.3 A PROJECTIVE PLANE FOR PG(2,4)

Let F be the field F = {0, 1, x, y}, where y = x+1, with addition and multiplication defined by

+	0	1	x	y
0	0	1	x	y
1	1	0	y	x
x	x	y	0	1
y	y	x	1	0

and

.	0	1	x	y
0	0	0	0	0
1	0	1	x	y
x	0	x	y	1
y	0	y	1	x

,

respectively.

In this case V^* has 63 elements and, since F has 3 nonzero scalars, we have 21 equivalence classes. Again choosing representatives for the distinct classes, numbering them from 1 to 21, and determining their orthogonal complements we obtain:

i	$[x]_i$	$[x]_i^{\perp}$
1	[1,0,0]	{2,3,6,20,21}
2	[0,1,0]	{1,3,5,10,11}
3	[0,0,1]	{1,2,4,8,9}
4	[1,1,0]	{3,4,7,14,18}
5	[1,0,1]	{2,5,7,13,17}
6	[0,1,1]	{1,6,7,12,19}
7	[1,1,1]	{4,5,6,15,16}
8	[x,1,0]	{3,9,15,17,19}
9	[1,x,0]	{3,8,12,13,16}
10	[x,0,1]	{2,11,16,18,19}
11	[1,0,x]	{2,10,12,14,15}
12	[x,1,1]	{6,9,11,13,14}
13	[x,1,x,]	{5,9,12,18,21}
14	[1,1,x+1]	{4,11,12,17,20}
15	[x,x+1,1]	{7,8,11,15,21}
16	[x,1,x+1]	{7,9,10,16,20}
17	[1,x,1]	{5,8,14,19,20}
18	[1,1,x]	{4,10,13,19,21}
19	[x+1,1,1]	{6,8,10,17,18}
20	[0,1,x]	{1,14,16,17,21}
21	[0,1,x+1]	{1,13,15,18,20}

With {1,2,3,7} satisfying PP_3 of 1.2.13 we again have a projective plane, this time of order 4.

2.4 A PROJECTIVE PLANE FOR PG(2,5)

Let F be the field $F = \mathbb{Z}_5$ with addition and multiplication defined by

+	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$

and

.	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

respectively.

Numbering the 31 equivalence classes of V^* and determining their orthogonal complements, we obtain:

i	$[x]_i$	$[x]_i^{\perp}$
1	[1,0,0]	{2,3,11,19,22,24}
2	[0,1,0]	{1,3,4,8,16,17}
3	[0,0,1]	{1,2,5,7,9,18}
4	[1,0,1]	{2,17,21,25,27,28}
5	[1,1,0]	{3,18,20,26,27,31}
6	[1,1,1]	{10,13,15,17,18,19}
7	[2,1,0]	{3,7,13,23,25,29}
8	[2,0,1]	{2,8,15,23,26,30}
9	[3,1,0]	{3,9,10,12,28,30}
10	[3,1,1]	{6,9,16,19,25,26}
11	[0,2,1]	{1,11,12,15,25,31}
12	[1,2,1]	{9,11,14,17,20,23}
13	[1,3,1]	{6,7,17,24,30,31}
14	[1,1,2]	{12,16,18,21,23,24}
15	[1,1,3]	{6,8,11,18,28,29}
16	[3,0,1]	{2,10,14,16,29,31}
17	[4,0,1]	{2,4,6,12,13,20}
18	[1,4,0]	{3,5,6,14,15,21}
19	[0,4,1]	{1,6,10,22,23,27}
20	[1,4,1]	{5,12,17,22,26,29}
21	[1,1,4]	{4,14,18,22,25,27}
22	[0,1,1]	{1,19,20,21,29,30}

i	$[x]_i$	$[x]_i^{\perp}$
23	[2,1,1]	{7,8,12,14,19,27}
24	[0,3,1]	{1,13,14,24,26,28}
25	[2,1,3]	{4,7,10,11,21,26}
26	[2,3,1]	{5,8,10,20,24,25}
27	[4,1,1]	{4,5,19,23,28,31}
28	[1,2,4]	{4,9,15,24,27,29}
29	[3,4,1]	{7,15,16,20,22,28}
30	[1,2,3]	{8,9,13,21,22,31}
31	[2,3,4]	{5,11,13,16,27,30}

Here the set {1,2,3,6} satisfies PP_3 of 1.2.13 and we have a projective plane of order 5.

2.5 A PROJECTIVE PLANE FOR PG(2,7)

Let F be the field $F = \mathbb{Z}_7$ with addition and multiplication defined by

+	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{5}$	$\bar{5}$	$\bar{6}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{6}$	$\bar{6}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$

and

$\cdot$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{6}$	$\bar{1}$	$\bar{3}$	$\bar{5}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{6}$	$\bar{2}$	$\bar{5}$	$\bar{1}$	$\bar{4}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{1}$	$\bar{5}$	$\bar{2}$	$\bar{6}$	$\bar{3}$
$\bar{5}$	$\bar{0}$	$\bar{5}$	$\bar{3}$	$\bar{1}$	$\bar{6}$	$\bar{4}$	$\bar{2}$
$\bar{6}$	$\bar{0}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

,

respectively.

Following the same procedure as in the preceding sections, we obtain:

i	$[x]_i$	$[x]_i^{\perp}$
1	[1,0,0]	{2,4,5,49,50,51,52,53}
2	[0,1,0]	{1,5,6,44,45,46,47,48}
3	[1,1,0]	{5,18,19,23,36,40,42,55}
4	[0,1,1]	{1,14,20,34,37,40,53,57}
5	[0,0,1]	{1,2,3,9,10,12,17,19}
6	[1,0,1]	{2,18,21,26,29,34,43,48}
7	[1,1,1]	{16,19,24,28,33,38,48,53}
8	[1,2,1]	{13,17,22,27,32,40,48,52}
9	[1,2,0]	{5,16,17,25,37,39,43,56}

i	$[x]_i$	$[x]_i^{\perp}$
10	[2,1,0]	{5,11,12,20,21,22,38,54}
11	[3,1,1]	{10,13,21,31,36,39,46,53}
12	[3,1,0]	{5,10,14,24,27,29,35,41}
13	[4,1,1]	{8,11,17,23,29,30,45,53}
14	[6,3,4]	{4,12,25,28,29,31,40,44}
15	[1,2,3]	{15,17,34,35,36,38,44,51}
16	[5,1,1]	{7,9,27,42,43,44,53,54}
17	[5,1,0]	{5,8,9,13,15,26,28,57}
18	[6,1,1]	{3,6,22,25,26,35,53,55}
19	[6,1,0]	{3,5,7,30,31,32,33,34}
20	[2,3,4]	{4,10,32,38,43,45,55,57}
21	[2,3,5]	{6,10,11,28,34,42,52,56}
22	[2,3,6]	{8,10,18,22,33,37,44,50}
23	[3,4,5]	{3,13,29,37,38,42,47,49}
24	[3,5,6]	{7,12,24,26,36,37,45,52}
25	[4,5,6]	{9,14,18,30,38,46,52,56}
26	[5,3,2]	{6,17,18,24,31,49,54,57}
27	[6,3,2]	{8,12,16,27,34,46,49,55}
28	[6,5,3]	{7,14,17,21,28,47,50,55}
29	[4,2,3]	{6,12,13,14,23,33,43,51}
30	[1,1,2]	{13,19,25,34,41,45,50,54}
31	[1,1,3]	{11,14,19,26,32,39,44,49}
32	[1,1,4]	{8,19,20,31,35,43,47,52}
33	[1,1,5]	{7,19,22,29,46,51,56,57}
34	[1,1,6]	{4,6,15,19,21,27,30,37}

i	$[x]_i$	$[x]_i^{\perp}$
35	[2,1,1]	{12,15,18,32,41,47,53,56}
36	[3,4,1]	{3,11,15,24,40,43,46,50}
37	[4,5,2]	{4,9,22,23,24,34,39,47}
38	[3,1,3]	{7,10,15,20,23,25,48,49}
39	[5,1,5]	{9,11,31,37,41,48,51,55}
40	[6,1,6]	{3,4,8,14,36,48,54,56}
41	[1,4,1]	{12,30,35,39,42,48,50,57}
42	[6,1,4]	{3,16,21,23,41,44,52,57}
43	[5,1,2]	{6,9,16,20,29,32,36,50}
44	[1,0,2]	{2,14,15,16,22,31,42,45}
45	[1,0,3]	{2,13,20,24,30,44,55,56}
46	[1,0,4]	{2,11,25,27,33,36,47,57}
47	[1,0,5]	{2,23,28,32,35,37,46,54}
48	[1,0,6]	{2,6,7,8,38,39,40,41}
49	[0,1,2]	{1,23,26,27,31,38,50,56}
50	[0,1,3]	{1,22,28,30,36,41,43,49}
51	[0,1,4]	{1,15,29,33,39,52,54,55}
52	[0,1,5]	{1,8,21,24,25,32,42,51}
53	[0,1,6]	{1,4,7,11,13,16,18,35}
54	[6,2,3]	{10,16,26,30,40,47,51,54}
55	[6,1,5]	{3,18,20,27,28,39,45,51}
56	[4,5,1]	{9,21,25,33,35,40,45,49}
57	[1,2,5]	{4,17,20,26,33,41,42,46}

Here the set {1,2,5,7} satisfies PP_3 of 1.2.13 and we have a projective plane of order 7.

2.6 A PROJECTIVE PLANE FOR PG(2,8)

Let F be the field $F = \{0, 1, x, y_1, y_2, y_3, y_4, y_5\}$, where $y_1 = x^2$, $y_2 = x+1$, $y_3 = x^2+x$, $y_4 = x^2+x+1$, $y_5 = x^2+1$, with addition and multiplication defined by:

+	0	1	x	y_1	y_2	y_3	y_4	y_5
0	0	1	x	y_1	y_2	y_3	y_4	y_5
1	1	0	y_2	y_5	x	y_4	y_3	y_1
x	x	y_2	0	y_3	1	y_1	y_5	y_4
y_1	y_1	y_5	y_3	0	y_4	x	y_2	1
y_2	y_2	x	1	y_4	0	y_5	y_1	y_3
y_3	y_3	y_4	y_1	x	y_5	0	1	y_2
y_4	y_4	y_3	y_5	y_2	y_1	1	0	x
y_5	y_5	y_1	y_4	1	y_3	y_2	x	0

and

$\cdot$	0	1	x	y_1	y_2	y_3	y_4	y_5
0	0	0	0	0	0	0	0	0
1	0	1	x	y_1	y_2	y_3	y_4	y_5
x	0	x	y_1	y_2	y_3	y_4	y_5	1
y_1	0	y_1	y_2	y_3	y_4	y_5	1	x
y_2	0	y_2	y_3	y_4	y_5	1	x	y_1
y_3	0	y_3	y_4	y_5	1	x	y_1	y_2
y_4	0	y_4	y_5	1	x	y_1	y_2	y_3
y_5	0	y_5	1	x	y_1	y_2	y_3	y_4,

respectively.

Again we find the orthogonal complements of the 73 equivalence classes to obtain:

i	$[x]_i$	$[x]_i^{\perp}$
1	[1,0,0]	{4,5,7,22,23,24,25,26,27}
2	[1,1,0]	{2,3,5,39,40,41,42,43,44}
3	[1,1,1]	{2,4,6,14,18,59,63,67,69}
4	[0,1,1]	{1,3,4,34,35,36,37,38,45}
5	[0,0,1]	{1,2,7,8,9,10,11,12,73}
6	[1,0,1]	{3,6,7,46,47,48,49,50,51}
7	[0,1,0]	{1,5,6,28,29,30,31,32,33}
8	$[x,x^2,0]$	{5,45,51,68,69,70,71,72,73}
9	$[x,x+1,0]$	{5,12,21,38,50,64,65,66,67}
10	$[x,x^2+x,0]$	{5,11,20,37,49,60,61,62,63}
11	$[x,x^2+x+1,0]$	{5,10,19,36,48,56,57,58,59}
12	$[x,x^2+1,0]$	{5,9,18,34,47,52,53,54,55}
13	$[x,x^2,x+1]$	{15,18,27,32,36,39,50,61,73}
14	$[x,x^2,x^2+x]$	{3,14,21,26,31,55,56,60,73}
15	$[x,x^2,x^2+x+])$	{13,25,30,38,44,49,54,59,73}
16	$[x,x^2,x^2+1]$	{20,24,29,35,43,48,53,67,73}
17	$[x,x^2,1]$	{19,23,28,34,42,46,63,66,73}
18	$[x,x+1,1]$	{3,13,12,18,24,28,58,62,71}
19	$[x,x^2+x,1]$	{11,17,25,28,36,40,51,55,67}
20	$[x,x^2+x+1,1]$	{10,16,26,28,37,44,50,53,69}
21	$[x,x^2+1,1]$	{9,14,27,28,43,45,49,57,65}
22	$[0,x,x^2]$	{1,27,44,46,52,56,62,67,72}
23	$[0,x,x+1]$	{1,17,26,43,47,59,61,66,71}
24	$[0,x,x^2+x]$	{1,16,18,25,42,48,60,65,70}

i	$[x]_i$	$[x]_i^{\perp}$
25	$[0,x,x^2+x+1]$	{1,15,19,24,41,49,55,64,69}
26	$[0,x,x^2+1]$	{1,14,20,23,40,50,54,58,68}
27	$[0,x,1]$	{1,13,21,22,39,51,53,57,63}
28	$[x,0,x^2]$	{7,17,18,19,20,21,33,44,45}
29	$[x,0,x+1]$	{7,16,32,38,43,55,58,63,72}
30	$[x,0,x^2+x]$	{7,15,31,37,42,54,57,67,71}
31	$[x,0,x^2+x+1]$	{7,14,30,36,41,53,62,66,70}
32	$[x,0,x^2+1]$	{7,13,29,34,40,56,61,65,69}
33	$[x,0,1]$	{7,28,35,39,52,59,60,64,68}
34	$[x,x+1,x+1]$	{4,12,17,32,42,49,53,56,68}
35	$[x,x^2,x^2]$	{4,16,33,40,47,57,62,64,73}
36	$[x,x^2+x,x^2+1]$	{4,11,13,19,31,43,50,52,70}
37	$[x,x^2+x+1,x^2+x+1]$	{4,10,20,30,39,46,55,65,71}
38	$[x,x^2+1,x^2+1]$	{4,9,15,29,44,51,58,60,66}
39	$[x,x,x^2]$	{2,13,27,33,37,48,55,66,68}
40	$[x,x,x+1]$	{2,19,26,32,35,51,54,62,65}
41	$[x,x,x^2+x]$	{2,25,31,45,46,53,58,61,64}
42	$[x,x,x^2+x+1]$	{2,17,24,30,34,50,57,60,72}
43	$[x,x,x^2+1]$	{2,16,21,23,29,36,49,52,71}
44	$[x,x,1]$	{2,15,20,22,28,38,47,56,70}
45	$[x,1,1]$	{4,8,21,28,41,48,54,61,72}
46	$[x,x^2,x]$	{6,17,22,37,41,52,58,65,73}
47	$[x,x+1,x]$	{6,12,23,35,44,55,57,61,70}
48	$[x,x^2+x,x]$	{6,11,16,24,39,45,54,56,66}
49	$[x,x^2+x+1,x]$	{6,10,15,21,25,34,43,62,68}

i	$[x]_i$	$[x]_i^{\perp}$
50	$[x,x^2+1,x]$	{6,9,13,20,26,36,42,64,72}
51	$[x,1,x]$	{6,8,19,27,38,40,53,60,71}
52	$[x,x+1,x^2]$	{12,22,33,36,43,46,54,60,69}
53	$[x,x+1,x^2+x]$	{12,16,20,27,31,34,41,51,59}
54	$[x,x+1,x^2+x+1]$	{12,15,26,30,40,45,48,52,63}
55	$[x,x+1,x^2+1]$	{12,14,19,25,29,37,39,47,72}
56	$[x,x^2+x,x+1]$	{11,14,22,32,34,44,48,64,71}
57	$[x,x^2+x,x^2+x+1]$	{11,21,27,30,35,42,47,58,69}
58	$[x,x^2+x,x^2+1]$	{11,18,26,29,38,41,46,57,68}
59	$[x,x^2+x,x^2]$	{3,11,15,23,33,53,59,65,72}
60	$[x,x^2+x+1,x^2]$	{10,14,24,33,38,42,51,52,61}
61	$[x,x^2+x+1,x+1]$	{10,13,23,32,41,45,47,60,67}
62	$[x,x^2+x+1,x^2+x]$	{10,18,22,31,35,40,49,66,72}
63	$[x,x^2+x+1,x^2+1]$	{3,10,17,27,29,54,63,64,70}
64	$[x,x^2+1,x^2]$	{9,25,33,35,41,50,56,63,71}
65	$[x,x^2+1,x+1]$	{9,21,24,32,37,40,46,59,70}
66	$[x,x^2+1,x^2+x]$	{9,17,23,31,38,39,48,62,69}
67	$[x,x^2+1,x^2+x+1]$	{3,9,16,19,22,30,61,67,68}
68	$[x,1,x^2]$	{8,26,33,34,39,49,58,67,70}
69	$[x,1,x+1]$	{3,8,20,25,32,52,57,66}
70	$[x,1,x^2+x]$	{8,24,31,36,44,47,63,65,68}
71	$[x,1,x^2+x+1]$	{8,18,23,30,37,43,51,56,64}
72	$[x,1,x^2+1]$	{8,22,29,42,45,50,55,59,62}
73	$[x,1,0]$	{5,8,13,14,15,16,17,35,46}

With the set {1,3,5,7} satisfying PP_3 of 1.2.13 we have a projective plane of order 8.

2.7 ISOMORPHIC PROJECTIVE PLANES

A permutation θ on PG(V) is an isomorphism from $(PG(V),\mathcal{L})$ to $(PG(V),\mathcal{L}')$ if $\theta(\ell) \in \mathcal{L}'$, for all $\ell \in \mathcal{L}$

It is easily seen that a renumbering of the equivalence classes will in each case result in a different projective plane. This renumbering would necessarily define a bijection which preserves the orthogonality relation between vectors, i.e. if θ is the bijection, then $x.y = 0$ implies $\theta(x).\theta(y) = 0$. In concluding this chapter we provide a general procedure, which depends solely on the manipulation of the associated incidence matrices, to establish the required bijection between two projective planes of the same order.

An (r,s)–interchange operation on an nxn incidence matrix A is the operation which interchanges rows r and s of A to obtain A', followed by the interchange of columns r and s of A' to obtain A''. The sequence of operation on the rows and columns in this definition is immaterial since:

if E(r,s) is the matrix obtained from the nxn identity matrix by interchanging rows r and s, then A' = E(r,s).A (effecting the row operation on A), and A'' = A'.E(r,s) (effecting the column operation on A'). Hence A'' = [E(r,s).A].E(r,s). But, since matrix multiplication is associative, we have that A'' = E(r,s).[A.E(r,s)] i.e. the column operation may be executed before the row operation.

2.6.1 **Example**

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Executing the (2,3)–interchange operation we obtain:

$$A' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ after row operation on A,}$$

$$\text{and } A'' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ after column operation on A'.}$$

On the other hand, reversing the order of operations we obtain:

$$A' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ after column operation on A,}$$

$$\text{and } A'' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ after row operation on A'.}$$

□

Two incidence matrices are said to be equivalent if one can be obtained from the other by a sequence of (r,s)–interchanges, for a finite sequence of r and s values.

2.6.2 Algorithm: For any projective geometry $(S,\mathcal{L})$ (as defined in 1.2.13) such that card $(S) =$ card $(\mathcal{L})$ there exists an incidence matrix $I(S,\mathcal{L})$.

Without loss of generality, assume that $S = \{1,2,3,\ldots,n\}$.

Define the bijection $f:S \rightarrow \mathcal{L}$ recursively as follows:

consistently apply the rule that;

if s is in $f(r)$, then r must be in $f(s)$.

Step 1:

Let $f(1)$ be an arbitrary element of $\mathcal{L}$

and let $I(S,\mathcal{L})(k,1) = I(S,\mathcal{L})(1,k)$

$= 1$, if $k \in f(1)$

$= 0$, otherwise (where $I(S,\mathcal{L})(i,j)$ denotes the ij–th entry of $I(S,\mathcal{L})$).

Step 2:

For each $k \in f(1)$ select $f(k)$ such that

(i) $1 \in f(k)$

(ii) $f(k) \neq f(k')$ if $k \neq k'$

(iii) if $I(S,\mathcal{L})(k,s) = 1$, then s must be in $f(k)$

(iv) if $I(S,\mathcal{L})(s,k) = 1$, then k must be in $f(s)$.

Now fill in row and column k of $I(S,\mathcal{L})$ as prescribed in step 1.

Step 3:

For each $s \in f(k)$, where $k \in f(1)$, proceed as in step 2.

□

2.6.3 Example:

Let $\mathcal{L}_1 = \{(1,2,4), (2,3,5), (3,4,6), (4,5,7), (5,6,1), (6,7,2), (7,1,3)\}$ and suppose $f(1) = (2,3,5)$.

Then, by Step 1, above,

$$I = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & & & & & & \\ 1 & & & & & & \\ 0 & & & & & & \\ 1 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \end{bmatrix},$$

with the remaining entries still to be ascertained in steps 2 and 3.

Now if $k = 2, 3,$ or 5, then $f(k) = (1,2,4), (5,6,1),$ or $(7,3,1)$. Choosing $f(2) = (5,6,1)$ we have, by the definition of f above, that $f(5) = (1,2,4)$. Therefore $f(3) = (7,3,1)$. Thus

$$I = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a_1 & 1 & a_2 & a_3 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_1 & 0 & b_2 & b_3 \\ 0 & 0 & 1 & c_1 & 0 & c_2 & c_3 \end{bmatrix}$$

Since f(5) = (1,2,4) it follows that f(4) must contain 5. Therefore f(4) = (2,3,5), (4,5,7), or (5,6,1). But f(4) ≠ (2,3,5) = f(1) and f(4) ≠ (5,6,1) = f(2). Hence f(4) = (4,5,7), so that $a_1 = 1$, $b_1 = a_2 = 0$, and $c_1 = a_3 = 1$. Furthermore, f(2) = (5,6,1) ⇒ f(6) must contain 2. Therefore f(6) = (1,2,4), (2,3,5), or (6,7,2). However, f(5) = (1,2,4) and f(1) = (2,3,5), so that f(6) = (6,7,2). Thus $b_2 = c_2 = b_3 = 1$.

Finally, since f(7) must contain 3 and 6 we have that f(7) = (3,4,6), so that

$$I = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

□

2.6.4 **Algorithm**: Let P_1 and P_2 be projective planes on the sets S_1 and S_2 of the same order q. If the associated incidence matrices are equivalent, then P_1 and P_2 are isomorphic.

Let $(s_1,r_1), (s_2,r_2),\ldots, (s_k,r_k)$ be the interchanges required to establish the equivalence between the incidence matrices I_1 and I_2 of P_1 and P_2, respectively. Execute the permutations $(s_1,r_1), (s_2,r_2),\ldots, (s_k,r_k)$ on the ordered set $\{1,2,3,\ldots,n\}$, $n \geq q$, to obtain $\{p_1,p_2,\ldots,p_n\}$.

For example, if the interchanges were $(2,3)$, $(5,3)$ and $(4,2)$, then the permutations on the ordered set $\{1,2,3,4,5,6\}$ will sequentially be:

$\{1,2,3,4,5,6\} \xrightarrow{(2,3)} \{1,3,2,4,5,6\} \xrightarrow{(5,3)} \{1,3,5,4,2,6\} \xrightarrow{(4,2)} \{1,4,5,3,2,6\}$.

Define now the map $\theta : S_2 \to S_1$, $\theta(i) = p_i$. In the above example θ will be the permutation $(2,4,3,5)$. It is easily seen that θ is a bijection between the points and lines of P_1 and P_2.

□

2.6.5 **Example**: For the numbering

$1: = (1,0,0)$, $2: = (0,1,0)$,

$3: = (1,1,0)$, $4: = (0,0,1)$,

$5: = (1,0,1)$, $6: = (0,1,1)$,

$7: = (1,1,1)$, i.e. for

$\mathcal{L}_2 = \{(1,2,3), (1,4,5), (1,6,7), (2,4,6), (3,4,7), (2,5,7), (3,5,6)\}$, we have the incidence matrix

$$J = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Executing the interchanges (4,5), (5,6) and (3,6) on I we obtain:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{after the (4,5) interchange,}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$ after the (5,6) interchange,

and $$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$ = J, after the (3,6) interchange.

Sequentially executing the permutations (4,5)–, (5,6)– and (3,6)– on the ordered set {1,2,3,4,5,6,7} we obtain:

$\{1,2,3,4,5,6,7\} \xrightarrow{(4,5)} \{1,2,3,5,4,6,7\} \xrightarrow{(5,6)} \{1,2,3,5,6,4,7\} \xrightarrow{(3,6)} \{1,2,4,5,6,3,7\}$.

Hence the required bijection will be the permutation (3,4,5,6}. In executing this permutation $\theta = (3,4,5,6)$ on $\mathcal{L}_2$ we obtain:

$(2,4,6) = (2,5,3)$

$\theta(1,4,5) = (1,5,6)$

$\theta(3,4,7) = (4,5,7)$

$\theta(1,2,3) = (1,2,4)$

$\theta(2,5,7) = (2,6,7)$

$\theta(1,6,7) = (1,3,7)$

$\theta(3,5,6) = (4,6,3)$

i.e. $\theta : \mathcal{L}_2 \to \mathcal{L}_1$. Similarly, for $\theta^{-1} = (6,5,4,3)$ we have $\theta^{-1} : \mathcal{L}_1 \to \mathcal{L}_2$. Hence θ is an isomorphism between the two projective planes P_1 and P_2 represented by the incidence matrices I and J, repectively.

□

CHAPTER 3

TWO PROJECTIVE GENERAL LINEAR GROUPS

In this chapter we calculate the structural constants of the centres of the group rings PGL(3,2) and PGL(3,3). This information is part of a collection required for an investigation into the group–theoretic nature and implications of the structural constants.

Secondly, no text on group rings explicitly provided this information, although the centre reflects many of the intrinsic group theoretic properties of the groups themselves. This is a major consideration in the works of Arad et al on simple groups (see [1], [2], [3], [4]). The work therefore complements the existing texts for the aforementioned purpose.

If F is a field, G a finite group, and $\{x_1, ..., x_t\}$ a basis for the centre of the group ring, Z(FG), as a vectorspace over F, then t equals the class number and each x_i is completely described by the conjugacy classes of G. Furthermore, since the product maps

$$f_{x_i} : Z(FG) \to Z(FG), \; y \to x_i y$$

are F–linear maps, there exists a txt matrix A_i such that

$$[x_i y] = A_i [y]$$

where [v] denotes the co–ordinate vector of v with respect to the ordered basis $\{x_1, ..., x_t\}$ found on page 2. The entries of the matrices A_i, which are called the structural constants of the group, are solely dependent on the properties of the group and there exists a well established formula, using the characters of the group, to determine them.

In this chapter all the necessary detail required to establish these matrix entries are provided, and the entries are calculated for the groups PGL(3,2) and PGL(3,3) (the first two groups within the family PGL(V) which conforms to the construction methods). Due to the large orders of these groups computer algorithms were used to determine some of this information.

3.1 PGL(3,2)

Let V be a 3–dimensional vector space over the field $F = \mathbb{Z}_2$. In chapter one (see example 1.2.7) we constructed the projective plane PG(2,2) consisting of the finite set $S = \{1,2,3,4,5,6,7\}$, with the family

$\mathcal{L} = [\{1,2,4\}, \{2,3,5\}, \{3,4,6\}, \{4,5,7\}, \{5,6,1\}, \{6,7,2\}, \{7,1,3\}]$ of subsets of S. Since F has no automorphisms other than the identity, we have $[\Gamma L(V) : GL(V)] = 1$. Therefore the general linear group GL(V) contains all the colineations of PG(2,2). Furthermore, since the identity is the only nonzero scalar in F, we have

$|Z(GL(V)| = 1$, so that $|GL(V)| = |PGL(V)|$

i.e. $GL(V) = PGL(V)$. In this case the projective general linear group PGL(V) therefore contains all colineations of the plane PG(2,2).

By 1.2.10 $|PGL(V)| = 2^{3(3-1)/2} \prod_{i=1}^{3} (2^i - 1)$

$= 168$

Moreover, since $\gcd(q-1,n) = \gcd(1,3) = 1$, we have, by 1.2.10, that $|PSL(V)| = |PGL(V)|$ i.e. $PGL(V) = PSL(V)$. Thus, by 1.2.12 $PGL(V)$ is simple.

$PGL(V)$ is 2–transitive on $PG(2,2)$, by 1.2.11. That it is not 3–transitive is readily seen from the lines of $PG(2,2)$. For example, there is no colineation that can take the subset (line) $\{2,3,5\}$ to $\{2,3,7\}$. This also follows readily from the definition of the finite projective plane. For suppose that G, the group of permissible permutations on a projective geometry $(S,\mathcal{L})$ is 3–transitive and $\ell \in \mathcal{L}$ is such that $\{a,b,c\} \subsetneq \ell$. If $d \neq c$, then there exists a $\sigma \in G$ such that

$$(\sigma(a),\sigma(b),\sigma(c)) = (a,b,d)$$

But then there exists an $\ell' \in \mathcal{L}$ such that $\{a,b,d\} \subsetneq \ell'$ and hence $|\ell' \cap \ell| \geq 2$, violating PP_2 of 1.2.13.

In determining the conjugacy classes of $PGL(V)$ we obtain six distinct classes. The order, together with a representative of each class, is presented below:

I	Class	Class Respresentative	Class order
1.	K_1	identity	1
2.	K_2	(2,4)(5,6)	21
3.	K_3	(2,7,6)(4,3,5)	56
4.	K_4	(2,3,4,7)(5,6)	42
5.	K_5	(1,2,3,4,5,6,7)	24
6.	K_6	(1,7,6,5,4,3,2)	24

From Arad, et al ([3]) we obtain the character table of PGL(V):

I	1	2	3	4	5	6
Class	K_1	K_2	K_3	K_4	K_5	K_6
$\|C(I)\|$	168	8	3	4	7	7
X_1	1	1	1	1	1	1
X_2	3	–1	0	1	ω	$\bar{\omega}$
X_3	3	–1	0	1	$\bar{\omega}$	ω
X_4	6	2	0	0	–1	–1
X_5	7	–1	1	–1	0	0
X_6	8	0	–1	0	1	1

where C(I) is the centralizer of the class numbered I and $\omega = \frac{-1+\sqrt{7}\,i}{2}$

Before proceeding to determine the structure constants of PGL(3,2), consider the following argument;

Let G be a finite group with $K_1, \ldots, K_n$ its distinct conjugacy classes. For a fixed $\ell \in K_\ell$ say, there are n^2 structure constants $c_{ij\ell}$, since we have n choices for i and n choices for j. This must be done for each of the n conjugacy classes, so that we have a total of $n^2.n = n^3$ structure constants. Of course, not all of the structure constants are distinct, particularly since multiplication of conjugacy classes is commutative

i.e. $c_{ij\ell} = c_{ji\ell}$

Hence, by the above argument, PGL(3,2), consisting of 6 distinct conjugacy classes, will have a total of $6^3 = 216$ structure constants. We will use Burnside's formula (1.1.4) to calculate the constants. For example.

$$c_{4,3,2} = \frac{|K_4| \cdot |K_3|}{|PGL(3,2)|} \cdot \sum_{X} \frac{X(K_4)X(K_3)\overline{X(K_2)}}{X(1)},$$

where $|K_4|$, $|K_3|$ are the orders of the conjugacy classes numbered 4 and 3 respectively, $X(K_4)$, $X(K_3)$ are the values of the character X on the classes K_4 and K_3, respectively, $\overline{X(K_2)}$ is the complex conjugate of the value of X on K_2, and $X(1)$ is the value of X on the identity i.e. on K_1.

Hence we obtain;

$$c_{4,3,2} = \frac{42 \cdot 56}{168} \cdot \left[\frac{1 \cdot 1 \cdot 1}{1} + \frac{1 \cdot 0 \cdot (-1)}{3} + \frac{1 \cdot 0 \cdot (-1)}{3} + \frac{0 \cdot 0 \cdot 2}{6} + \frac{(-1) \cdot 1 \cdot (-1)}{7} + \frac{0 \cdot (-1) \cdot 0}{8}\right]$$

$= 16 = c_{3,4,2}$, i.e. each element of K_2 occurs 16 times in K_4K_3.

$$c_{5,4,3} = \frac{24 \cdot 42}{168} \cdot \left[\frac{1 \cdot 1 \cdot 1}{1} + \frac{\omega \cdot 1 \cdot 0}{3} + \frac{\overline{\omega} \cdot 1 \cdot 0}{3} + \frac{(-1) \cdot 0 \cdot 0}{6} + \frac{0 \cdot (-1) \cdot 1}{7} + \frac{1 \cdot 0 \cdot (-1)}{8}\right]$$

$= 6 = c_{4,5,3}$, etc.

Finally, we present the structure constants of PGL(3,2) in matrix form, as follows ; the matrix $A_{ij\ell}$ will have as its ij–th entry, the structure constant $c_{ij\ell}$;

$$A_{ij1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 21 & 0 & 0 & 0 & 0 \\ 0 & 0 & 56 & 0 & 0 & 0 \\ 0 & 0 & 0 & 42 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 24 & 0 \end{bmatrix}$$

For example, the entry in the third row and fourth column of the above matrix is the structure constant $c_{3,4,1}$ which is of course equal to $c_{4,3,1}$, etc.

From A_{ij1} we see that the conjugacy classes K_1, K_2, K_3 and K_4 all contain their own inverses, whilst K_5 consists of the inverses of K_6.

$$A_{ij2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 8 & 8 & 0 & 0 \\ 0 & 8 & 16 & 16 & 8 & 8 \\ 0 & 8 & 16 & 2 & 8 & 8 \\ 0 & 0 & 8 & 8 & 8 & 0 \\ 0 & 0 & 8 & 8 & 0 & 8 \end{bmatrix}$$

$$A_{ij3} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 6 & 6 & 3 & 3 \\ 1 & 6 & 19 & 12 & 9 & 9 \\ 0 & 6 & 12 & 12 & 6 & 6 \\ 0 & 3 & 9 & 6 & 3 & 3 \\ 0 & 3 & 9 & 6 & 3 & 3 \end{bmatrix}$$

$$A_{ij4} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 8 & 1 & 4 & 4 \\ 0 & 8 & 16 & 16 & 8 & 8 \\ 1 & 1 & 16 & 16 & 4 & 4 \\ 0 & 4 & 8 & 4 & 0 & 8 \\ 0 & 4 & 8 & 4 & 8 & 0 \end{bmatrix}$$

$$A_{ij5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 7 & 0 & 7 \\ 0 & 7 & 21 & 14 & 7 & 7 \\ 0 & 7 & 14 & 7 & 14 & 0 \\ 1 & 0 & 7 & 14 & 1 & 1 \\ 0 & 7 & 7 & 0 & 1 & 9 \end{bmatrix}$$

$$A_{ij6} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 7 & 7 & 7 & 0 \\ 0 & 7 & 21 & 14 & 7 & 7 \\ 0 & 7 & 14 & 7 & 0 & 14 \\ 0 & 7 & 7 & 0 & 9 & 1 \\ 1 & 0 & 7 & 14 & 1 & 1 \end{bmatrix}$$

3.2 PGL(3,3)

In 2.1 we constructed a projective plane of order 3, consisting of the set $S = \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$ and the family of subsets $\mathcal{L} = [\{2,5,8,9\}, \{1,5,6,7\}, \{4,5,12,13\}, \{3,5,10,11\}, \{1,2,3,4\}, \{2,7,11,13\}, \{2,6,10,12\}, \{1,9,11,12\}, \{1,8,10,13\}, \{4,7,9,10\}, \{4,6,8,11\}, \{3,7,8,12\}, \{3,6,9,13\}]$ over the field $F = \mathbb{Z}_3$.

Since the identity is the only field automorphism on $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ we again have that $[\Gamma L(V) : GL(V)] = 1$ i.e. the general linear group $GL(V)$ is the group of all colineations of $PG(2,3)$. By 1.2.2 the order of $GL(V)$ is

$$|GLV(3,3)| = 3^{3(3-1)/2} \prod_{i=1}^{3} (3^i-1) = 3^3(3-1)(3^2-1)(3^3-1) = 11\,232$$

and, since the field $\mathbb{Z}_3$ contains two nonzero scalars, the order of the projective general linear group, $PGL(V)$, is;

$$|PGL(3,3)| = 3^{3(3-1)/2} \prod_{i=2}^{3} (3^i-1)$$

$$= 3^3(3^2-1)(3^3-1)$$

$$= 5616$$

$$= |GLV(3,3)|/2$$

Again we find that the projective general linear group is the same as the projective special linear group, PSL(3,3), since $|PSL(3,3)| = [\gcd(2,3)]^{-1}|PGL(3,3)|$

$$= 1 \cdot |PGL(3,3)|.$$

Hence, by 1.2.12 PGL(3,3) is simple. It is 2–transitive by 1.2.11. That it is not 3–transitive is evident. For example, no colineation can take the line {2,5,8,9} to {2,5,10,11}.

PGL(3,3) has 12 distinct conjugacy classes and a representative of each class, together with the order of each class are given below:

I	Class	Class Representative	Class order
1.	K_1	identity	1
2.	K_2	(6,7)(8,9)(10,11)(12,13)	117
3.	K_3	(2,8,9)(3,10,11)(4,13,12)	104
4.	K_4	(1,11,13)(2,8,9)(3,4,6)(7,10,12)	624
5.	K_5	(2,5)(8,9)(3,7,4,6)(10,11,13,12)	702
6.	K_6	(8,9)(1,7,6)(3,13,10,4,11,12)	936
7.	K_7	(2,9,8,5)(3,12,13,7,4,11,10,6)	702
8.	K_8	(2,5,8,9)(3,6,10,11,4,7,13,12)	702
9.	K_9	(1,5,2,6,8,10,11,13,3,7,9,12,4)	432
10.	K_{10}	(1,5,8,11,12,13,10,3,7,2,6,9,4)	432
11.	K_{11}	(1,4,12,9,7,3,13,11,10,8,6,2,5)	432
12.	K_{12}	(1,4,9,6,2,7,3,10,13,12,11,8,5)	432

The character table of PGL(3,3) is given below:

I	1	2	3	4	5	6	7	8	9	10	11	12
Class	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}
|C(I)|	5616	48	54	9	8	6	8	8	13	13	13	13
X_1	1	1	1	1	1	1	1	1	1	1	1	1
X_2	12	4	3	0	0	1	0	0	–1	–1	–1	–1
X_3	13	–3	4	1	1	0	–1	–1	0	0	0	0
X_4	16	0	–2	1	0	0	0	0	ω_1	ω_2	ω_3	ω_4
X_5	16	0	–2	1	0	0	0	0	ω_4	ω_1	ω_2	ω_3
X_6	16	0	–2	1	0	0	0	0	ω_3	ω_4	ω_1	ω_2
X_7	16	0	–2	1	0	0	0	0	ω_2	ω_3	ω_4	ω_1
X_8	26	2	–1	–1	2	–1	0	0	0	0	0	0
X_9	26	–2	–1	–1	0	1	v	$\bar{v}$	0	0	0	0
X_{10}	26	–2	–1	–1	0	1	$\bar{v}$	v	0	0	0	0
X_{11}	27	3	0	0	–1	0	–1	–1	1	1	1	1
X_{12}	39	–1	3	0	–1	–1	1	1	0	0	0	0

where C(I) is the centralizer of the class numbered I, $\omega_1 = \alpha+\alpha^3+\alpha^9$, $\omega_2 = \alpha^2+\alpha^5+\alpha^6$, $\omega_3 = \alpha^4+\alpha^{10}+\alpha^{12}$, $\omega_4 = \alpha^7+\alpha^8+\alpha^{11}$, $\alpha = e^{2\pi i/13}$, $v = \beta+\beta^3$, and $\beta = e^{2\pi i/8}$.

PGL(3,3) has $12^3 = 1728$ structure constants. In concluding this chapter we again use Burnside's formula (1.1.4) to calculate the constants and present them in matrix form as before i.e. the matrix $A_{ij\ell}$ shall have as its ij–th entry the structure constant $c_{ij\ell}$.

$$A_{ij1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 117 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 104 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 624 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 702 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 936 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 702 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 702 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 432 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 432 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 432 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 432 & 0 & 0 \end{bmatrix}$$

Again the structure constants given in A_{ij1} above show that the conjugacy classes K_1, K_2, K_3, K_4, K_5 and K_6 all contain their own inverses, whilst K_7 consists of the inverses of K_8, K_9 of the inverses of K_{11}, and K_{10} of the inverses of K_{12}.

$$A_{ij2} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 12 & 16 & 16 & 24 & 48 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 40 & 24 & 24 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 48 & 48 & 128 & 96 & 96 & 48 & 48 & 48 & 48 \\ 0 & 24 & 0 & 48 & 102 & 96 & 120 & 120 & 48 & 48 & 48 & 48 \\ 0 & 48 & 40 & 128 & 96 & 192 & 120 & 120 & 48 & 48 & 48 & 48 \\ 0 & 0 & 24 & 96 & 120 & 120 & 102 & 48 & 48 & 48 & 48 & 48 \\ 0 & 0 & 24 & 96 & 120 & 120 & 48 & 102 & 48 & 48 & 48 & 48 \\ 0 & 0 & 0 & 48 & 48 & 48 & 48 & 48 & 48 & 48 & 48 & 48 \\ 0 & 0 & 0 & 48 & 48 & 48 & 48 & 48 & 48 & 48 & 48 & 48 \\ 0 & 0 & 0 & 48 & 48 & 48 & 48 & 48 & 48 & 48 & 48 & 48 \\ 0 & 0 & 0 & 48 & 48 & 48 & 48 & 48 & 48 & 48 & 48 & 48 \end{bmatrix}$$

$$A_{ij3} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 18 & 0 & 0 & 0 & 45 & 27 & 27 & 0 & 0 & 0 & 0 \\ 1 & 0 & 13 & 36 & 27 & 27 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 36 & 48 & 108 & 108 & 54 & 54 & 54 & 54 & 54 & 54 \\ 0 & 0 & 27 & 108 & 108 & 135 & 54 & 54 & 54 & 54 & 54 & 54 \\ 0 & 45 & 27 & 108 & 135 & 189 & 108 & 108 & 54 & 54 & 54 & 54 \\ 0 & 27 & 0 & 54 & 54 & 108 & 135 & 108 & 54 & 54 & 54 & 54 \\ 0 & 27 & 0 & 54 & 54 & 108 & 108 & 135 & 54 & 54 & 54 & 54 \\ 0 & 0 & 0 & 54 & 54 & 54 & 54 & 54 & 54 & 54 & 0 & 54 \\ 0 & 0 & 0 & 54 & 54 & 54 & 54 & 54 & 54 & 54 & 54 & 0 \\ 0 & 0 & 0 & 54 & 54 & 54 & 54 & 54 & 0 & 54 & 54 & 54 \\ 0 & 0 & 0 & 54 & 54 & 54 & 54 & 54 & 54 & 0 & 54 & 54 \end{bmatrix}$$

$$
A_{ij4} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 9 & 9 & 24 & 18 & 18 & 9 & 9 & 9 & 9 \\
0 & 0 & 6 & 8 & 18 & 18 & 9 & 9 & 9 & 9 & 9 & 9 \\
1 & 9 & 8 & 84 & 90 & 108 & 72 & 72 & 45 & 45 & 45 & 45 \\
0 & 9 & 18 & 90 & 81 & 126 & 81 & 81 & 54 & 54 & 54 & 54 \\
0 & 24 & 18 & 108 & 126 & 138 & 117 & 117 & 72 & 72 & 72 & 72 \\
0 & 18 & 9 & 72 & 81 & 117 & 108 & 81 & 54 & 54 & 54 & 54 \\
0 & 18 & 9 & 72 & 81 & 117 & 81 & 108 & 54 & 54 & 54 & 54 \\
0 & 9 & 9 & 45 & 54 & 72 & 54 & 54 & 27 & 27 & 54 & 27 \\
0 & 9 & 9 & 45 & 54 & 72 & 54 & 54 & 27 & 27 & 27 & 54 \\
0 & 9 & 9 & 45 & 54 & 72 & 54 & 54 & 54 & 27 & 27 & 27 \\
0 & 9 & 9 & 45 & 54 & 72 & 54 & 54 & 27 & 54 & 27 & 27
\end{bmatrix}
$$

$$
A_{ij5} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 8 & 17 & 16 & 20 & 20 & 8 & 8 & 8 & 8 \\
0 & 0 & 4 & 16 & 16 & 20 & 8 & 8 & 8 & 8 & 8 & 8 \\
0 & 8 & 16 & 80 & 72 & 112 & 72 & 72 & 48 & 48 & 48 & 48 \\
1 & 17 & 16 & 72 & 116 & 96 & 80 & 80 & 56 & 56 & 56 & 56 \\
0 & 16 & 20 & 112 & 96 & 164 & 120 & 120 & 72 & 72 & 72 & 72 \\
0 & 20 & 8 & 72 & 80 & 120 & 89 & 89 & 56 & 56 & 56 & 56 \\
0 & 20 & 8 & 72 & 80 & 120 & 89 & 89 & 56 & 56 & 56 & 56 \\
0 & 8 & 8 & 48 & 56 & 72 & 56 & 56 & 32 & 32 & 32 & 32 \\
0 & 8 & 8 & 48 & 56 & 72 & 56 & 56 & 32 & 32 & 32 & 32 \\
0 & 8 & 8 & 48 & 56 & 72 & 56 & 56 & 32 & 32 & 32 & 32 \\
0 & 8 & 8 & 48 & 56 & 72 & 56 & 56 & 32 & 32 & 32 & 32
\end{bmatrix}
$$

$$
A_{ij6} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 5 & 16 & 12 & 24 & 15 & 15 & 6 & 6 & 6 & 6 \\
0 & 5 & 3 & 12 & 15 & 21 & 12 & 12 & 6 & 6 & 6 & 6 \\
0 & 16 & 12 & 72 & 84 & 92 & 78 & 78 & 48 & 48 & 48 & 48 \\
0 & 12 & 15 & 84 & 72 & 123 & 90 & 90 & 54 & 54 & 54 & 54 \\
1 & 24 & 21 & 92 & 123 & 171 & 120 & 120 & 66 & 66 & 66 & 66 \\
0 & 15 & 12 & 78 & 90 & 120 & 72 & 99 & 54 & 54 & 54 & 54 \\
0 & 15 & 12 & 78 & 90 & 120 & 99 & 72 & 54 & 54 & 54 & 54 \\
0 & 6 & 6 & 48 & 54 & 66 & 54 & 54 & 36 & 36 & 36 & 36 \\
0 & 6 & 6 & 48 & 54 & 66 & 54 & 54 & 36 & 36 & 36 & 36 \\
0 & 6 & 6 & 48 & 54 & 66 & 54 & 54 & 36 & 36 & 36 & 36 \\
0 & 6 & 6 & 48 & 54 & 66 & 54 & 54 & 36 & 36 & 36 & 36
\end{bmatrix}
$$

$$
A_{ij7} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 16 & 20 & 20 & 8 & 17 & 8 & 8 & 8 & 8 \\
0 & 4 & 0 & 8 & 8 & 16 & 16 & 20 & 8 & 8 & 8 & 8 \\
0 & 16 & 8 & 64 & 72 & 104 & 72 & 96 & 48 & 48 & 48 & 48 \\
0 & 20 & 8 & 72 & 80 & 120 & 89 & 89 & 56 & 56 & 56 & 56 \\
0 & 20 & 16 & 104 & 120 & 160 & 132 & 96 & 72 & 72 & 72 & 72 \\
1 & 8 & 16 & 72 & 89 & 132 & 80 & 80 & 56 & 56 & 56 & 56 \\
0 & 17 & 20 & 96 & 89 & 96 & 80 & 80 & 56 & 56 & 56 & 56 \\
0 & 8 & 8 & 48 & 56 & 72 & 56 & 56 & 32 & 32 & 32 & 32 \\
0 & 8 & 8 & 48 & 56 & 72 & 56 & 56 & 32 & 32 & 32 & 32 \\
0 & 8 & 8 & 48 & 56 & 72 & 56 & 56 & 32 & 32 & 32 & 32 \\
0 & 8 & 8 & 48 & 56 & 72 & 56 & 56 & 32 & 32 & 32 & 32
\end{bmatrix}
$$

$A_{ij8} =$

0	0	0	0	0	0	0	1	0	0	0	0
0	0	4	16	20	20	17	8	8	8	8	8
0	4	0	8	8	16	20	16	8	8	8	8
0	16	8	64	72	104	96	72	48	48	48	48
0	20	8	72	80	120	89	89	56	56	56	56
0	20	16	104	120	160	96	132	72	72	72	72
0	17	20	96	89	96	80	80	56	56	56	56
1	8	16	72	89	132	80	80	56	56	56	56
0	8	8	48	56	72	56	56	32	32	32	32
0	8	8	48	56	72	56	56	32	32	32	32
0	8	8	48	56	72	56	56	32	32	32	32
0	8	8	48	56	72	56	56	32	32	32	32

$A_{ij9} =$

0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	13	13	13	13	13	13	13	13	13
0	0	0	13	13	13	13	13	0	13	13	13
0	13	13	65	78	104	78	78	78	39	39	39
0	13	13	78	91	117	91	91	52	52	52	52
0	13	13	104	117	143	117	117	78	78	78	78
0	13	13	78	91	117	91	91	52	52	52	52
0	13	13	78	91	117	91	91	52	52	52	52
1	13	0	78	52	78	52	52	13	40	13	40
0	13	13	39	52	78	52	52	40	13	40	40
0	13	13	39	52	78	52	52	13	40	67	13
0	13	13	39	52	78	52	52	40	40	13	40

$$A_{ij10} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \\
0 & 0 & 0 & 13 & 13 & 13 & 13 & 13 & 13 & 0 & 13 & 13 \\
0 & 13 & 13 & 65 & 78 & 104 & 78 & 78 & 39 & 78 & 39 & 39 \\
0 & 13 & 13 & 78 & 91 & 117 & 91 & 91 & 52 & 52 & 52 & 52 \\
0 & 13 & 13 & 104 & 117 & 143 & 117 & 117 & 78 & 78 & 78 & 78 \\
0 & 13 & 13 & 78 & 91 & 117 & 91 & 91 & 52 & 52 & 52 & 52 \\
0 & 13 & 13 & 78 & 91 & 117 & 91 & 91 & 52 & 52 & 52 & 52 \\
0 & 13 & 13 & 39 & 52 & 78 & 52 & 52 & 40 & 40 & 40 & 13 \\
1 & 13 & 0 & 78 & 52 & 78 & 52 & 52 & 40 & 13 & 40 & 13 \\
0 & 13 & 13 & 39 & 52 & 78 & 52 & 52 & 40 & 40 & 13 & 40 \\
0 & 13 & 13 & 39 & 52 & 78 & 52 & 52 & 13 & 13 & 40 & 67
\end{bmatrix}$$

$$A_{ij11} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \\
0 & 0 & 0 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 0 & 13 \\
0 & 13 & 13 & 65 & 78 & 104 & 78 & 78 & 39 & 39 & 78 & 39 \\
0 & 13 & 13 & 78 & 91 & 117 & 91 & 91 & 52 & 52 & 52 & 52 \\
0 & 13 & 13 & 104 & 117 & 143 & 117 & 117 & 78 & 78 & 78 & 78 \\
0 & 13 & 13 & 78 & 91 & 117 & 91 & 91 & 52 & 52 & 52 & 52 \\
0 & 13 & 13 & 78 & 91 & 117 & 91 & 91 & 52 & 52 & 52 & 52 \\
0 & 13 & 13 & 39 & 52 & 78 & 52 & 52 & 67 & 13 & 13 & 40 \\
0 & 13 & 13 & 39 & 52 & 78 & 52 & 52 & 13 & 40 & 40 & 40 \\
1 & 13 & 0 & 78 & 52 & 78 & 52 & 52 & 13 & 40 & 13 & 40 \\
0 & 13 & 13 & 39 & 52 & 78 & 52 & 52 & 40 & 40 & 40 & 13
\end{bmatrix}$$

$$A_{ij12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \\ 0 & 0 & 0 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 0 \\ 0 & 13 & 13 & 65 & 78 & 104 & 78 & 78 & 39 & 39 & 39 & 78 \\ 0 & 13 & 13 & 78 & 91 & 117 & 91 & 91 & 52 & 52 & 52 & 52 \\ 0 & 13 & 13 & 104 & 117 & 143 & 117 & 117 & 78 & 78 & 78 & 78 \\ 0 & 13 & 13 & 78 & 91 & 117 & 91 & 91 & 52 & 52 & 52 & 52 \\ 0 & 13 & 13 & 78 & 91 & 117 & 91 & 91 & 52 & 52 & 52 & 52 \\ 0 & 13 & 13 & 39 & 52 & 78 & 52 & 52 & 13 & 40 & 40 & 40 \\ 0 & 13 & 13 & 39 & 52 & 78 & 52 & 52 & 40 & 67 & 13 & 13 \\ 0 & 13 & 13 & 39 & 52 & 78 & 52 & 52 & 40 & 13 & 40 & 40 \\ 1 & 13 & 0 & 78 & 52 & 78 & 52 & 52 & 40 & 13 & 40 & 13 \end{bmatrix}$$

REFERENCES

[1] **Arad Z,** On products of conjugacy classes and irreducible characters in finite groups, *Proc. Symp. Pure Math.* 47 No.2 (1987), 3–9.

[2] ________ Survey on table algebras and applications to finite group theory, Ring Theory 1989, 96–110 *Israel Math. Conf. Proc.*, 1, 1989.

[3] ________ **and Herzog M,** Lecture notes in Mathematics, Volume 1112, Products of conjugacy classes in groups, Springer–Verlag, 1985.

[4] ________ **and Lipman–Gutweter, H,** On products of characters in finite groups. *Houston J. Math.* 15 (1989), no.3, 305–326.

[5] **Aschbacher Michael,** The Finite Simple Groups and their classification, Yale Mathematical Monographs 7, Yale University Press, 1980.

[6] **Atkins P.W., Child M.S. and Phillips C.S.G,** Tables for Group Theory, Oxford University Press, 1970.

[7] **Biggs N.L. and White A.T,** Permutation groups and Combinational structures, Cambridge University Press, 1979.

[8] **Burnside W,** The Theory of Groups of Finite Order, second edition, Cambridge University Press, 1911.

[9] **Carmichael Robert D,** Introduction to the Theory of Finite Groups, Dover Publications, 1956.

[10] **Chillag D. and Herzog M,** On the length of the conjugacy classes of finite groups, *J. Algebra* 131 (1990) 110–125.

[11] **Curtis Charles W. and Reiner Irving,** Representation theory of Finite Groups and Associated Algebras, John Wiley & Sons, 1962.

[12] **Dembowski P**, Finite Geometries, Springer–Verlag, 1968.

[13] **Greub Werner**, Linear Algebra, fourth edition, Springer–Verlag, 1975.

[14] **Green J.H**, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* 80 (1955), 402–447.

[15] **Gupta S.C. and Kapoor P.K.**, Fundamentals of Applied Statistics, 3rd Edition, S.Chand and Sons, 1987.

[16] **Hughes D.R. and Piper F.C**, Projective Planes, Springer–Verlag, 1973.

[17] **Hungerford Thomas W**, Algebra, Springer–Verlag, 1974.

[18] **James G.D**, The irreducible representations of the finite linear groups, *Proc. London Math. Soc.* (3) 52 (1986), 236–268.

[19] **Ki Hang Kim and Roush Fred W**, Applied Algebra, Ellis Horwood, 1983.

[20] **Lang Serge**, Algebra, Addison–Wesley, 1977.

[21] **Lederman Walter**, Introduction to Group Characters, Cambridge University Press, 1977.

[22] ______________________, Introduction to the Theory of Finite Groups, Oliver and Boyd, 1961.

[23] **Mackey George W**, Unitary Group Representations in Physics, Probability, and Number Theory, Benjamin–Cummings, 1978.

[24] **Noble B. and Daniel W**, Applied Linear Algebra, Prentice Hall, 1977.

[25] **Passman D.S**, Permutation groups, Benjamin, 1968.

[26] **Robinson D.J.S**, A Course in the Theory of Groups, Springer–Verlag, 1982.

[27] **Rotman J.J**, An Introduction to the Theory of Groups, third edition, Wm. C. Brown, 1988.

[28] **Winter David J**, The Structure of Fields, Springer–Verlag, 1974.