SINGULAR INTEGRAL EQUATIONS AND REALIZATION: A SURVEY OF THE STATE SPACE METHOD

by

YOLLANDA VUYOKAZI GANTSHO

submitted in partial fulfilment of the requirements
for the degree of Master of Science
in the Department of Mathematics and Applied Mathematics,
University of the Western Cape.

Supervisor: Dr G.J. Groenewald

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Chapter 0

INTRODUCTION

Different methods for solving singular integral equations exist. One of the most recent methods is the so-called *state space* method. This method is based on the fact that a rational matrix function $W(\lambda)$ which is analytic and invertible at infinity can be represented by

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \tag{0.1}$$

where A is a square matrix whose order may be larger than that of $W(\lambda)$, and B, C and D are matrices of appropriate sizes. The representation (0.1) allows one to reduce analytic problems about rational matrix functions to linear algebra ones involving constant matrices, and often it provides explicit and readily computable formulas for the solutions. In the last fifteen years the state space approach has proved to be effective in solving various problems of mathematical analysis (see the survey paper [BGK3]).

In this mini-thesis we employ the state space method to solve singular integral equations. These equations serve as a tool to solve problems in numerous fields of application. For the general theory and examples of applications (see, for instance, [GKr], [M] and [V]).

We will consider equations with a rational matrix symbol and which are of the form:

$$A(\lambda)\phi(\lambda)+B(\lambda)\left(\frac{1}{\pi i}\int_\Gamma \frac{\phi(\mu)}{\mu-\lambda}d\mu\right)=f(\lambda),\quad \lambda\in\Gamma. \tag{0.2}$$

Here the contour Γ consists of a finite number of disjoint smooth simple Jordan curves, $A(\cdot)$ and $B(\cdot)$ are given $m\times m$ rational matrix functions, which have no poles on Γ, and f is a given function from $L_2^m(\Gamma)$, the space of all $\mathbb{C}^m$-valued functions that are square integrable on Γ. The matrix function $W(\lambda)$,

$$W(\lambda):=[A(\lambda)-B(\lambda)]^{-1}[A(\lambda)+B(\lambda)],$$

which plays an important part in the analysis of (0.2), is, in general, not proper, and so we use a modification of the representation (0.1), namely, (see [GK1])

$$W(\lambda)=D+(\lambda-\alpha)C(\lambda G-A)^{-1}B. \tag{0.3}$$

Here A, B and C are as in (0.1), G is a square matrix of the same order as A, D denotes an invertible $m\times m$ matrix and α is a nonzero complex number which is neither a pole nor a zero of $W(\lambda)$.
In fact, we follow a similar program as in [GK3], but with a different representation. We use (0.3) instead of

$$W(\lambda)=I+C(\lambda G-A)^{-1}B.$$

The aim is to give necessary and sufficient conditions for the inversion of the equation (0.2) and an explicit formula for its solution in terms of the matrices A, G, B, C and D. In addition, the Fredholm characteristics of equation (0.2) will be described in terms of these five matrices.
This mini-thesis consists of two chapters. In chapter 1, we discuss the coupling method, (see [BGK1]). Firstly, the basic properties of this method are described. Next, the coupling method is applied to the class of singular integral operators. This method reduces various classes of integral operators to simpler ones, which are often just finite matrices. In particular, finding the inverse, generalized

inverse, kernel and image of an integral operator is reduced to the corresponding problem for finite matrices. Chapter 2 concerns the state space method for solving singular integral equations with rational symbol. Here we provide explicit formulae for invertibility and Fredholm characteristics of the singular integral equation with rational symbol using the representation (0.3) which is different from the one used in [GK3]. The main idea of the proof is to reduce the inversion problem to one for input/output systems.

Finally, we give a review of the factorization method. This last section may be seen as an extension of the theory developed in [BGK2], [GK2] which concerns Wiener-Hopf integral operators, infinite block Toeplitz matrices and singular integral operators with proper rational symbols.

Chapter 1

THE COUPLING METHOD FOR SOLVING SINGULAR INTEGRAL EQUATIONS

We follow the paper [BGK1] in our treatment of Sections 1–3 of this chapter. In Section 4, we follow the paper [GK2], see also the paper [Gr2].

1.1 Matricial coupling and indicator

Throughout this section and the next one all spaces are assumed to be complex Banach spaces and all operators are bounded and linear. The identity operator on a Banach space X is denoted by I_X or I.

In this section, the method of reducing operators of various classes to simpler ones is introduced. It is based on the notion of matricial coupling of operators, which is defined as follows:

Let $T : X_1 \to Z_1$ and $S : Z_2 \to X_2$ be bounded linear operators acting between Banach spaces. We call T and S *matricially coupled* if S and T are related in the following way:

$$\begin{pmatrix} T & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & S \end{pmatrix}. \tag{1.1}$$

This means that one can construct an invertible 2×2 operator matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} : X_1 \oplus X_2 \to Z_1 \oplus Z_2 \tag{1.2}$$

with $A_{11} = T$, whose inverse is given by

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} : Z_1 \oplus Z_2 \to X_1 \oplus X_2 \tag{1.3}$$

with $B_{22} = S$. We shall refer to (1.1) as the *coupling relation* and to (1.2) and (1.3) as the *coupling matrices.* If T and S are matricially coupled operators, then we say that S is an *indicator* of T (and conversely, T is an indicator of S). This notion is of particular interest if S is simpler than T.

Example: Let $A : X \to Y$ and $B : Y \to X$ be given operators, and let D and K be invertible operators acting on the spaces X and Y, respectively. Then the operators $D - BK^{-1}A$ and $K - AD^{-1}B$ are matricially coupled operators. Indeed,

$$\begin{pmatrix} D - BK^{-1}A & -BK^{-1} \\ K^{-1}A & K^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} D^{-1} & D^{-1}B \\ -AD^{-1} & K - AD^{-1}B \end{pmatrix}. \tag{1.4}$$

Theorem 1.1 *Assume $T : X_1 \to Z_1$ and $S : Z_2 \to X_2$ are matricially coupled operators, and let the coupling relation be given by*

$$\begin{pmatrix} T & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & S \end{pmatrix}.$$

Then

$$\begin{pmatrix} T & 0 \\ 0 & I_{X_2} \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_{Z_1} \end{pmatrix} F, \tag{1.5}$$

where E and F are invertible 2×2 operator matrices

$$E = \begin{pmatrix} -A_{12} & TB_{11} \\ I_{X_2} & B_{21} \end{pmatrix}, \quad F = \begin{pmatrix} A_{21} & A_{22} \\ T & A_{12} \end{pmatrix},$$

with inverses

$$E^{-1} = \begin{pmatrix} -B_{21} & SA_{22} \\ I_{Z_1} & A_{12} \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} B_{12} & B_{11} \\ S & B_{21} \end{pmatrix}.$$

Proof. By direct computation, using (1.1). □

From the definition of equivalent matrices one notes that (1.5) says that after a simple extension the operators T and S are equivalent.

Theorem 1.1 is of particular interest when the operators T and S depend on a parameter. For example, if the entries of the coupling matrix (1.2) depend analytically on a parameter λ, for λ in some open subset of $\mathbb{C}$, then the same is true for the entries in its inverse (assuming it exists) and in this case the operators E and F appearing in Theorem 1.1 also depend analytically on λ.

1.2 Invertibility of matricially coupled operators

In this section we describe Fredholm properties of matricially coupled operators.

Theorem 2.1 *Let T and S be matricially coupled operators, and let the coupling relation be given by*

$$\begin{pmatrix} T & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & S \end{pmatrix}. \tag{2.1}$$

Then

$$Ker\ T = B_{12}(Ker\ S) \quad ; \quad Ker\ S = A_{21}(Ker\ T). \tag{2.2}$$

$$Im\ T = B_{21}^{-1}(Im\ S) \quad ; \quad Im\ S = A_{12}^{-1}(Im\ T). \tag{2.3}$$

Furthermore, T has a generalized inverse (resp., right, two-sided inverse) if and only if S has a generalized inverse (resp., right, two-sided inverse). If S^+ is a generalized inverse of S, then

$$T^+ = B_{11} - B_{12}S^+B_{21} \tag{2.4}$$

is a generalized inverse of T, conversely, if T^+ is a generalized inverse of T, then

$$S^+ = A_{22} - A_{21}T^+A_{12} \tag{2.5}$$

is a generalized inverse of S. Also T is a (semi-) Fredholm operator if and only if S is a (semi-) Fredholm operator, and in this case ind T = ind S.

Proof. Since the first matrix in (2.1) is the inverse of the second matrix in (2.1), we know that $B_{21}T + SA_{21} = 0$. This shows that $\text{Im}\ T \subset B_{21}^{-1}(\text{Im}\ S)$. Now assume that $B_{21}y = Sz$. Then

$$\begin{aligned} y &= TB_{11}y + A_{12}B_{21}y \\ &= TB_{11}y + A_{12}Sz \\ &= TB_{11}y - TB_{12}z \in \text{Im}\ T. \end{aligned}$$

We have proved the first identity in (2.3). Similarly with the other identities. All other statements in the theorem are straightforward consequences of the equivalence relation in (1.5). $\square$

Example: The usual method of reducing the inversion of an operator $I - F$, where F has finite rank, to that of a matrix is to be understood and made precise in the context of matricially coupled operators. To see this, assume $F : X \to X$ is given by

$$F = \sum_{j=1}^{n} \langle \cdot \, , \, \phi_j^* \rangle \psi_j \quad ,$$

where $\psi_1, \ldots, \psi_n$ are given vectors in the Banach space X and $\phi_1^*, \ldots, \phi_n^*$ are continuous linear functionals on X.

Define $A : X \to \mathbb{C}^n$ and $B : \mathbb{C}^n \to X$ by setting

$$Ax = \operatorname{col}(\langle x, \phi_i^* \rangle)_{i=1}^{n} \quad , \quad x \in X,$$

$$B \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \sum_{j=1}^{n} \alpha_j \psi_j \quad .$$

Note that $G = AB$ acts on $\mathbb{C}^n$, and its matrix with respect to the standard basis of $\mathbb{C}^n$, is given by

$$\text{mat } (G) = (\langle \psi_j, \phi_i^* \rangle)_{i,\, j=1}^{n} . \tag{2.6}$$

Since $F = BA$, the operators $I_X - \mu F$ and $I_n - \mu G$ are matricially coupled; in fact

$$\begin{pmatrix} I_X - \mu F & \mu B \\ A & I_n \end{pmatrix}^{-1} = \begin{pmatrix} I_X & \mu B \\ -A & I_n - \mu G \end{pmatrix} . \tag{2.7}$$

From (2.7) it follows (cf., formula (2.4)) that $(I_X - \mu F)^{-1} = I_X + \mu B(I_n - \mu G)^{-1} A$ whenever $\det\, (I_n - \mu G) \neq 0$.

1.3 Singular integral equations

In the remaining sections of this chapter, we apply the principle of matricial coupling to the class of singular integral operators with analytical symbol.

Consider the singular integral equation

$$A(\lambda)\phi(\lambda) + B(\lambda)(\frac{1}{\pi i}\int_\Gamma \frac{\phi(\mu)}{\mu - \lambda}\, d\mu) = f(\lambda) \quad , \quad \lambda \in \Gamma. \tag{3.1}$$

Here Γ consists of a finite number of disjoint smooth simple Jordan curves, A and B are given continuous $n \times n$ matrix functions defined on Γ, and the given function f and the unknown function ϕ are vector functions. As usual in the theory of singular integral equations, it is assumed that the inner domain Δ_+ of Γ is connected and contains 0, while the outer domain Δ_- of Γ contains ∞.

The problem is to find ϕ such that (3.1) is satisfied. For ϕ a rational function without poles on Γ, we put

$$(S_\Gamma \phi)(\lambda) = \frac{1}{\pi i}\int_\Gamma \frac{\phi(\mu)}{\mu - \lambda}\, d\mu \quad , \quad \lambda \in \Gamma,$$

where the integral is taken in the sense of the Cauchy principle value.

The operator S_Γ defined in this way can be extended by continuity to a bounded linear operator, again denoted by S_Γ, on a suitable space E. Equation (3.1) can now be written as

$$(M_A + M_B S_\Gamma)\phi = f, \tag{3.2}$$

where M_A and M_B are the operators of multiplication by A and B respectively, and S_Γ is the *basic singular integral operator* and S_Γ enjoys the property that $S_\Gamma^2 = I$ (see [CG]). The operators M_A and M_B are also assumed to be bounded linear operators on E. In what follows E will be the space $L_2^n(\Gamma)$ of square integrable functions from Γ into a Banach space $Y = \mathbb{C}^n$.

Put $P_\Gamma = \frac{1}{2}(I + S_\Gamma)$ and $Q_\Gamma = \frac{1}{2}(I - S_\Gamma)$, where $I = I_E$ is the identity operator on E. The operators P_Γ and Q_Γ are complementary projections, which can be used to rewrite the operator $M_A + M_B S_\Gamma$.

Assume that the values of the function $A - B$ are invertible operators on Y and that the operator $M_{(A-B)^{-1}}$ of multiplication by $[A(\lambda) - B(\lambda)]^{-1}$ is a well-defined bounded linear operator on E. Then

$$M_A + M_B S_\Gamma = M_{A-B}(P_\Gamma M_W P_\Gamma + Q_\Gamma)(I + Q_\Gamma M_W P_\Gamma),$$

where M_W is the operator of multiplication by $W(\lambda)$, with $W(\lambda) = [A(\lambda) - B(\lambda)]^{-1}[A(\lambda) + B(\lambda)]$. Observe that M_{A-B} and $I + Q_\Gamma M_W P_\Gamma$ are both invertible operators. It follows that the invertibility properties of $M_A + M_B S_\Gamma$ are completely determined by those of the operator

$$T_W = P_\Gamma M_W P_\Gamma \ : \ E^+ \ \to \ E^+.$$

Here $E^+ = \operatorname{Im} P_\Gamma$ is the space consisting of all functions in E that have an extension which is analytic on the inner domain Δ_+ of Γ and continuous on the closure $\Delta_+ \cup \Gamma$ of Δ_+. Similarly, the image of Q_Γ is the subspace of all functions in $L_2^m(\Gamma)$ that admit an analytic continuation into Δ_- and vanish at ∞. We shall write E^- for $\operatorname{Im} Q_\Gamma$, and thus $E = E^+ \oplus E^-$. The operator T_W is called the *Toeplitz operator with symbol* W. The action of T_W on E^+ is given by

$$(T_W\phi)(\lambda) = \frac{1}{2}W(\lambda)\phi(\lambda) + \frac{1}{2\pi i}\int_\Gamma \frac{W(\mu)\phi(\mu)}{\mu - \lambda}d\mu \quad , \quad \lambda \in \Gamma. \tag{3.3}$$

Note that the symbol W is a function on Γ whose values are in $\mathcal{L}(Y)$, the space of all bounded linear operators on the Banach space Y.

1.4 Indicator

Continuing the discussion of Section 3 we now assume that the symbol $W : \Gamma \to \mathcal{L}(\mathbb{C}^m)$ is regular. The following result (taken from [GK1]) will play a fundamental role in the analysis of the Toeplitz operator T_W associated with the singular integral operator.

Theorem 4.1 *Let W be a regular $m \times m$ rational matrix function, and let $\alpha \neq 0$ be such that α is neither a pole nor a zero of W. Then W admits a representation*

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B \quad , \quad \lambda \in \Gamma, \tag{4.1}$$

where G and A are square matrices of the same order, B, C and D are matrices of appropriate sizes with D invertible and the pencil $\lambda G - A$ is Γ-regular.

We shall refer to the right hand side of (4.1) as a *realization* of W. In the sequel the next two lemmas will be useful, see [Gr1].

Lemma 4.2 *Let W be as in (4.1), where $\lambda G - A$ is Γ-regular. Set $G^{\times} = G + BD^{-1}C$ and $A^{\times} = A + \alpha BD^{-1}C$. Then det $W(\lambda) \neq 0$ for each $\lambda \in \Gamma$ if and only if the pencil $\lambda G^{\times} - A^{\times}$ is Γ-regular, and in this case*

$$W(\lambda)^{-1} = D^{-1} - (\lambda - \alpha)D^{-1}C(\lambda G^{\times} - A^{\times})^{-1}BD^{-1} \;, \quad \lambda \in \Gamma. \tag{4.2}$$

Lemma 4.3 *Let W be as in (4.1), where $\lambda G - A$ is Γ-regular. Assume that det $W(\lambda) \neq 0$ for each $\lambda \in \Gamma$, and set $G^{\times} = G + BD^{-1}C$ and $A^{\times} = A + \alpha BD^{-1}C$. Then for $\lambda \in \Gamma$,*

$$\begin{aligned}
W(\lambda)^{-1}C(\lambda G - A)^{-1} &= D^{-1}C(\lambda G^{\times} - A^{\times})^{-1}, \\
(\lambda G - A)^{-1}BW(\lambda)^{-1} &= (\lambda G^{\times} - A^{\times})^{-1}BD^{-1}, \\
(\lambda G^{\times} - A^{\times})^{-1} &= (\lambda G - A)^{-1} - (\lambda - \alpha)(\lambda G - A)^{-1}BW(\lambda)^{-1}C(\lambda G - A)^{-1}.
\end{aligned}$$

Next, we turn to operator pencils. Let X be a complex Banach space, and let G and A be bounded linear operators on X. The expression $\lambda G - A$, where λ is a complex parameter, will be called a *(linear) pencil of operators on X*. Given a non-empty subset Δ of the Riemann sphere $\mathbb{C}_{\infty}$, we say that $\lambda G - A$ is Δ-*regular* if $\lambda G - A$ (or just G if $\lambda = \infty$) is invertible for each λ in Δ.

Now we recall the very useful spectral decomposition theorem for linear (matrix) pencils, (see [GK2] Theorem 2.1).

Theorem 4.4 *Let Γ be a Cauchy contour with Δ_+ and Δ_- as inner and outer domain respectively, and let $\lambda G - A$ be a Γ-regular pencil of operators on the Banach space X. Then there exists a projection P and an invertible operator E, both acting on X, such that relative to the decomposition $X = Ker\ P \oplus\ Im\ P$ the following partitioning holds:*

$$(\lambda G - A)E = \begin{pmatrix} \lambda\Omega_1 - I_1 & 0 \\ 0 & \lambda I_2 - \Omega_2 \end{pmatrix} : Ker\ P \oplus\ Im\ P \to Ker\ P \oplus\ Im\ P, \tag{4.3}$$

where I_1 (resp. I_2) denotes the identity operator on $Ker\ P$ (resp. $Im\ P$), the pencil $\lambda\Omega_1 - I_1$ is $\bar{\Delta}_+$-regular and $\lambda I_2 - \Omega_2$ is $\bar{\Delta}_-$ -regular. Furthermore, P and E (and hence also the operators Ω_1 and Ω_2) are uniquely determined. In fact,

$$P = \frac{1}{2\pi i}\int_\Gamma G(\lambda G - A)^{-1} d\lambda, \tag{4.4}$$

$$E = \frac{1}{2\pi i}\int_\Gamma (1 - \lambda^{-1})(\lambda G - A)^{-1} d\lambda, \tag{4.5}$$

$$\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix} = \frac{1}{2\pi i}\int_\Gamma (\lambda - \lambda^{-1})G(\lambda G - A)^{-1} d\lambda. \tag{4.6}$$

The 2×2 operator matrix in (4.3) is called the $\Gamma-$*spectral decomposition* of the pencil $\lambda G - A$, and the operator Ω in (4.6) will be referred to as the *associated operator* corresponding to $\lambda G - A$ and Γ. We will refer to the projection P and the operator E in Theorem 4.4 as the *separating projection* and the *right equivalence operator*, respectively.

For the proof of Theorem 4.4, we refer to [GK2], see also [GGK], Chapter 4. Here we give a few essential steps of the proof. Put

$$Q = \frac{1}{2\pi i}\int_\Gamma (\lambda G - A)^{-1} G d\lambda. \tag{4.7}$$

It can be shown that

$$PG = GQ \quad , \quad PA = AQ,$$

and hence the pencil $\lambda G - A$ admits the following partitioning.

$$\lambda G - A = \begin{pmatrix} \lambda G_1 - A_1 & 0 \\ 0 & \lambda G_2 - A_2 \end{pmatrix} : Ker\ Q \oplus\ Im\ Q \to Ker\ P \oplus\ Im\ P. \tag{4.8}$$

The next step is to show that the pencil $\lambda\Omega_1 - I_1$ is $\bar{\Delta}_+$-regular and $\lambda I_2 - \Omega_2$ is $\bar{\Delta}_-$-regular. Since $0 \in \Delta_+$ and $\infty \in \Delta_-$ it follows that A_1 and G_2 are invertible. Thus we may set

$$E = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & G_2^{-1} \end{pmatrix} : Ker\ P \oplus Im\ P \to Ker\ Q \oplus Im\ Q, \tag{4.9}$$

and $\Omega_1 = G_1 A_1^{-1}$ and $\Omega_2 = A_2 G_2^{-1}$. Then (4.3) holds and it also follows that the pencils $\lambda\Omega_1 - I_1$ and $\lambda I_2 - \Omega_2$ are $\bar{\Delta}_+$-regular and $\bar{\Delta}_-$-regular, respectively. Now we can prove that E is also given by (4.5) and Ω by (4.6).

Next, the realization (4.1) can be employed to compute the Fourier coefficients of W. This leads to the following proposition, see [J] and [Gr2].

Proposition 4.5 *Let W be a rational $m \times m$ matrix function without poles on the unit circle $\mathbb{T}$, and let*

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B\ ,\quad \lambda \in \mathbb{T},$$

be a realization of W. Then the k-th Fourier coeffficient W_k of W admits the following representation:

$$W_k = \begin{cases} -CE(\Omega^{k-1} - \alpha\Omega^k)(I - P)B & , \quad k > 0 \\ D + \alpha CE(I - P)B + CEPB & , \quad k = 0 \\ CE(\Omega^{-k} - \alpha\Omega^{-k-1})PB & , \quad k < 0. \end{cases}$$

Here P, E and Ω are, respectively, the separating projection, the right equivalence operator and the associated operator corresponding to the pencil $\lambda G - A$ and $\mathbb{T}$, that is, P, E and Ω are given by (4.4)-(4.6). In particular, Ω has all its eigenvalues in the open unit disc and Ω commutes with P.

Finally, we give the main result of this section.

Theorem 4.6 *Let T be a block Toeplitz operator on ℓ_p^m with a rational symbol*

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B\ , \quad \lambda \in \mathbb{T},$$

given in realized form. Set $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$. Let P and $P^\times$ be the generalized Riesz projections given by

$$P = \frac{1}{2\pi i}\int_{\mathbb{T}} G(\lambda G - A)^{-1} d\lambda \quad , \quad P^\times = \frac{1}{2\pi i}\int_{\mathbb{T}} G^\times(\lambda G^\times - A^\times)^{-1} d\lambda.$$

Then the operator

$$J^\times = P^\times|_{Im\ P}\ :\ Im\ P \to Im\ P^\times \tag{4.10}$$

is an indicator for the Toeplitz operator T. More precisely, the following coupling relation holds:

$$\begin{pmatrix} T & U \\ R & J \end{pmatrix}^{-1} = \begin{pmatrix} T^\times & U^\times \\ R^\times & J^\times \end{pmatrix}, \tag{4.11}$$

where

$$\begin{array}{lcllcllcl}
U & : & Im\ P^\times & \to \ell_p^m & , & (Ux)_j & = & -CE\Omega^j(I-P)x & , \ x \in Im\ P^\times, \\
U^\times & : & Im\ P & \to \ell_p^m & , & (U^\times x)_j & = & -D^{-1}CE^\times(\Omega^\times)^j(I-P^\times)x & , \ x \in Im\ P, \\
R & : & \ell_p^m & \to Im\ P & , & R\eta & = & \sum_{j=0}^\infty P\Omega^j B\phi_j & , \ \eta = (\phi_0, \phi_1 \ldots) \in \ell_p^m, \\
R^\times & : & \ell_p^m & \to Im\ P^\times & , & R^\times\eta & = & -\sum_{j=0}^\infty P^\times(\Omega^\times)^j BD^{-1}\phi_j & , \ \eta = (\phi_0, \phi_1 \ldots) \in \ell_p^m, \\
J & : & Im\ P^\times & \to Im\ P & , & Jx & = & Px & , \ x \in Im\ P^\times.
\end{array}$$

Here E and Ω are the right equivalence operator and the associate operator corresponding to $\lambda G - A$ and $\mathbb{T}$. The operator $T^\times$ is the block Toeplitz operator on ℓ_p^m with symbol $W(\cdot)^{-1}$.

Proof. In the sequel we assume that the pencil $\lambda G^\times - A^\times$ is $\mathbb{T}$-regular. To establish the coupling relation (4.11) we employ the method of matricial coupling (see [BGK1]). Introduce the following operators:

$$\begin{pmatrix} T & U \\ R & J \end{pmatrix} : \ell_p^m \oplus Im\ P^\times \quad \to \quad \ell_p^m \oplus Im\ P,$$
$$\begin{pmatrix} T^\times & U^\times \\ R^\times & J^\times \end{pmatrix} : \ell_p^m \oplus Im\ P \quad \to \quad \ell_p^m \oplus Im\ P^\times,$$

$$\begin{aligned}
(Ux)_j &= -CE\Omega^j(I-P)x\ , \quad x \in Im\ P^\times, \\
(U^\times x)_j &= -D^{-1}CE^\times(\Omega^\times)^j(I-P^\times)x\ , \quad x \in Im\ P, \\
R\eta &= \sum_{j=0}^{\infty} P\Omega^j B\phi_j\ , \quad \eta = (\phi_0, \phi_1, \ldots) \in \ell_p^m, \\
R^\times\eta &= -\sum_{j=0}^{\infty} P^\times(\Omega^\times)^j BD^{-1}\phi_j\ , \quad \eta = (\phi_0, \phi_1, \ldots) \in \ell_p^m, \\
Jx &= Px \ \ (x \in Im\ P^\times), \ \ J^\times x = P^\times x \ \ (x \in Im\ P).
\end{aligned}$$

Here E and Ω are the right equivalence operator and associate operator corresponding to $\lambda G - A$ and $\mathbb{T}$. The operator $T^\times$ is the block Toeplitz operator on ℓ_p^m with symbol $W(\cdot)^{-1}$. Note that $J^\times$ is the operator defined by (4.10). Since Ω and $\Omega^\times$ have their eigenvalues in the open unit disc (Proposition 4.5), the operators U, $U^\times$, R and $R^\times$ are well-defined. We will prove that (4.11) holds. In fact, proving (4.11) boils down to verifying eight identities. Here we will establish four of them, namely

$$TT^\times + UR^\times = I_{\ell_p^m}, \tag{4.12}$$

$$RT^\times + JR^\times = 0, \tag{4.13}$$

$$TU^\times + UJ^\times = 0, \tag{4.14}$$

$$RU^\times + JJ^\times = I_{ImP}. \tag{4.15}$$

The other four identities can be obtained similarly or by interchanging the roles of $W(\cdot)$ and $W(\cdot)^{-1}$.

We first consider the case $p = 2$. Let $L_2^m(\mathbb{T})$ be the Hilbert space of all $\mathbb{C}^m$-valued square integrable functions on $\mathbb{T}$, and let $H_2^m(\mathbb{T})$ be the subspace consisting of all $\phi \in L_2^m(\mathbb{T})$ with Fourier coefficients $c_n = 0$ for $n = -1, -2, \ldots$. The orthogonal projection of $L_2^m(\mathbb{T})$ onto $H_2^m(\mathbb{T})$ will be denoted by $\mathbb{P}$. If $g \in L_2^m(\mathbb{T})$, then $\mathbb{P}g$ has a natural extension to an analytic function on $\mathbb{D}_+$ (also denoted by $\mathbb{P}g$), and we will use the fact that

$$(\mathbb{P}g)(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g(\mu)}{\mu - \zeta} d\mu\ , \quad |\zeta| < 1. \tag{4.16}$$

It will be convenient to use the Fourier transform

$$F\ :\ H_2^m(\mathbb{T}) \to \ell_2^m\ ,\quad F\phi = (c_j)_{j=0}^{\infty},$$

where c_j is the j-th Fourier coefficient of ϕ. Set

$$\begin{aligned} S_W &= F^{-1}TF\ , & V &= F^{-1}U\ , & N &= RF, \\ S_{W^{-1}} &= F^{-1}T^\times F\ , & V^\times &= F^{-1}U^\times\ , & N^\times &= R^\times F. \end{aligned}$$

Then

$$\begin{aligned} (Vx)(\zeta) &= C(\zeta G - A)^{-1}(I-P)x\ ,\quad x \in Im\ P^\times,\quad \zeta \in \mathbb{T}, \\ (V^\times x)(\zeta) &= D^{-1}C(\zeta G^\times - A^\times)^{-1}(I - P^\times)x\ ,\quad x \in Im\ P,\quad \zeta \in \mathbb{T}, \\ N\phi &= \frac{1}{2\pi i}\int_{\mathbb{T}}(\zeta - \alpha)PG(\zeta G - A)^{-1}B\phi(\zeta)d\zeta\ ,\quad \phi \in H_2^m(\mathbb{T}), \\ N^\times\phi &= -\frac{1}{2\pi i}\int_{\mathbb{T}}(\zeta-\alpha)P^\times G^\times(\zeta G^\times - A^\times)^{-1}BD^{-1}\phi(\zeta)d\zeta\ ,\quad \phi \in H_2^m(\mathbb{T}), \\ S_W\phi &= \mathbb{P}M_W\phi\ ,\quad S_{W^{-1}} = \mathbb{P}M_{W^{-1}}\phi\ ,\quad \phi \in H_2^m(\mathbb{T}), \end{aligned}$$

where $\mathbb{P}$ is the orthogonal projection of $L_2^m(\mathbb{T})$ onto $H_2^m(\mathbb{T})$ and M_W (resp. $M_{W^{-1}}$) is the operator of multiplication by W (resp. W^{-1}). We have to prove the following identities:

$$S_W S_{W^{-1}} + VN^\times = I_{H_2^m(\mathbb{T})}, \tag{4.17}$$

$$NS_{W^{-1}} + JN^{\times} = 0, \tag{4.18}$$

$$S_W V^{\times} + VJ^{\times} = 0, \tag{4.19}$$

$$NV^{\times} + JJ^{\times} = I_{ImP}. \tag{4.20}$$

First we compute $S_W S_{W^{-1}}$. Note that

$$(\zeta - \alpha)BD^{-1}C = (\mu G^{\times} - A^{\times}) - (\zeta G - A) - (\mu - \zeta)G^{\times}. \tag{4.21}$$

Thus

$$\begin{aligned}
W(\zeta)W(\mu)^{-1} &= \left\{D + (\zeta - \alpha)C(\zeta G - A)^{-1}B\right\}\left\{D^{-1} - (\mu - \alpha)D^{-1}C(\mu G^{\times} - A^{\times})^{-1}BD^{-1}\right\} \\
&= I - (\mu - \alpha)C(\mu G^{\times} - A^{\times})^{-1}BD^{-1} + (\zeta - \alpha)C(\zeta G - A)^{-1}BD^{-1} \\
&\quad -(\mu - \alpha)C(\zeta G - A)^{-1}(\zeta - \alpha)BD^{-1}C(\mu G^{\times} - A^{\times})^{-1}BD^{-1} \\
&= I - (\mu - \zeta)C(\zeta G - A)^{-1}BD^{-1} \\
&\quad +(\mu - \alpha)(\mu - \zeta)C(\zeta G - A)^{-1}G^{\times}(\mu G^{\times} - A^{\times})^{-1}BD^{-1}.
\end{aligned}$$

Let $g \in H_2^m(\mathbb{T})$, and suppose that g is a polynomial. Then, by formula (4.16),

$$(S_{W^{-1}}g)(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{W(\mu)^{-1}g(\mu)}{\mu - \zeta}d\mu, \quad |\zeta| < 1.$$

It follows that for $|\zeta| < 1$,

$$\begin{aligned}
(M_W S_{W^{-1}}g)(\zeta) &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{W(\zeta)W(\mu)^{-1}g(\mu)}{\mu - \zeta}d\mu \\
&= g(\zeta) + C(\zeta G - A)^{-1}\left(\frac{1}{2\pi i}\int_{\mathbb{T}}(\mu - \alpha)G^{\times}(\mu G^{\times} - A^{\times})^{-1}BD^{-1}g(\mu)d\mu\right).
\end{aligned}$$

Now use the $\mathbb{T}$-spectral decomposition of the pencil $\mu G^{\times} - A^{\times}$ (Theorem 4.4). It follows that $(I - P^{\times})G^{\times}(\mu G^{\times} - A^{\times})^{-1}$ is analytic on $\mathbb{D}_+$. Since $g \in H_2^m(\mathbb{T})$, we conclude that

$$\frac{1}{2\pi i}\int_{\mathbb{T}}(\mu - \alpha)(I - P^{\times})G^{\times}(\mu G^{\times} - A^{\times})^{-1}BD^{-1}g(\mu)d\mu = 0. \tag{4.22}$$

Thus

$$(M_W S_{W^{-1}} g)(\zeta) = g(\zeta) - C(\zeta G - A)^{-1} N^\times g\ , \quad |\zeta| < 1.$$

The $\mathbb{T}$-spectral decomposition of $\zeta G - A$ implies that $C(\zeta G - A)^{-1} P$ is analytic on $\mathbb{D}_-$ and $C(\zeta G - A)^{-1}(I - P)$ is analytic on $\mathbb{D}_+$. Note that all functions involved are rational. Thus $S_W S_{W^{-1}} = g - VN^\times g$ for each polynomial g in $H_2^m(\mathbb{T})$. But the polynomials are dense in $H_2^m(\mathbb{T})$, so the identity (4.17) is proved.

Again, let $g \in H_2^m(\mathbb{T})$ be a polynomial. Then

$$\begin{aligned}
(JN^\times g)(\zeta) &= (PN^\times g)(\zeta) \\
&= P\left(-\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta - \alpha) P^\times G^\times (\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta\right) \\
&= -\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta - \alpha) PG(\zeta G - A)^{-1} BD^{-1} g(\zeta) d\zeta \\
&\quad -P\left(\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta - \alpha) BD^{-1} C(\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta\right) \\
&= -\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta - \alpha) PG(\zeta G - A)^{-1} BW(\zeta)^{-1} g(\zeta) d\zeta \\
&\quad -P\left(\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta - \alpha) BD^{-1} C(\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta\right),
\end{aligned}$$

by an application of Lemma 4.3. Since $\mathbb{P}\ W(\zeta)^{-1} g(\zeta) = W(\zeta)^{-1} g(\zeta)$, we get that

$$\begin{aligned}
(JN^\times g)(\zeta) &+ (NS_{W^{-1}} g)(\zeta) \\
&= -P\left(\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta - \alpha) BD^{-1} C(\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta\right) = 0.
\end{aligned}$$

Indeed,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta - \alpha) BD^{-1} C(\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta \\
= &\frac{1}{2\pi i}\int_{\mathbb{T}} \left[(\zeta G^\times - A^\times) - (\zeta G - A)\right] (\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta \\
= &\frac{1}{2\pi i}\int_{\mathbb{T}} BD^{-1} g(\zeta) d\zeta - \frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta G - A)(\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta \\
= &\frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta G - A)(\zeta G - A)^{-1} BW(\zeta)^{-1} g(\zeta) d\zeta = 0.
\end{aligned}$$

Here we applied Lemma 4.3 and Cauchy's theorem (twice). Since the polynomials are dense in $H_2^m(\mathbb{T})$, the formula (4.18) is proved.

Next, we take $x \in Im\ P$. Note that $(I-P)(I-P^\times)x = -(I-P)P^\times x$. Thus, using Lemma 4.3,

$$\begin{aligned}
(M_W V^\times x)(\zeta) &= W(\zeta)D^{-1}C(\zeta G^\times - A^\times)^{-1}(I-P^\times)x \\
&= C(\zeta G - A)^{-1}(I-P^\times)x \\
&= C(\zeta G - A)^{-1}P(I-P^\times)x + C(\zeta G - A)^{-1}(I-P)(I-P^\times)x \\
&= C(\zeta G - A)^{-1}P(I-P^\times)x - C(\zeta G - A)^{-1}(I-P)P^\times x \\
&= C(\zeta G - A)^{-1}P(I-P^\times)x - C(\zeta G - A)^{-1}(I-P)J^\times x \\
&= C(\zeta G - A)^{-1}P(I-P^\times)x - (VJ^\times x)(\zeta).
\end{aligned}$$

Now use the fact that $(\zeta G - A)^{-1}P$ is analytic on $\mathbb{D}_-$, it follows that $S_W V^\times = -VJ^\times$, and (4.19) is proved.

Formula (4.21) (with $\mu = \zeta$) implies that

$$(\zeta G - A)^{-1}(\zeta - \alpha)BD^{-1}C(\zeta G^\times - A^\times)^{-1} = (\zeta G - A)^{-1} - (\zeta G^\times - A^\times)^{-1}.$$

For $x \in Im\ P$,

$$\begin{aligned}
NV^\times x &= \frac{1}{2\pi i}\int_{\mathbb{T}}(\zeta-\alpha)PG(\zeta G - A)^{-1}B(V^\times x)(\zeta)d\zeta \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} PG\left[(\zeta G - A)^{-1} - (\zeta G^\times - A^\times)^{-1}\right](I-P^\times)x d\zeta \\
&= P(I-P^\times)x - PP^\times(I-P^\times)x + P\left(\frac{1}{2\pi i}\int_{\mathbb{T}} BD^{-1}C(\zeta G^\times - A^\times)^{-1}(I-P^\times)x d\zeta\right) \\
&= x - JJ^\times x + P\left(\frac{1}{2\pi i}\int_{\mathbb{T}} BD^{-1}C(\zeta G^\times - A^\times)^{-1}(I-P^\times)x d\zeta\right) \\
&= x - JJ^\times x,
\end{aligned}$$

since $P\left(\frac{1}{2\pi i}\int_{\mathbb{T}} BD^{-1}C(\zeta G^{\times}-A^{\times})^{-1}(I-P^{\times})xd\zeta\right)=0.$
Indeed,

$$P\left(\frac{1}{2\pi i}\int_{\mathbb{T}} BD^{-1}C(\zeta G^{\times}-A^{\times})^{-1}(I-P^{\times})xd\zeta\right)=PB\left(\frac{1}{2\pi i}\int_{\mathbb{T}} D^{-1}C(\zeta G^{\times}-A^{\times})^{-1}zd\zeta\right),$$

where $z\in Ker\ P^{\times}$. Now, recall that

$$\phi_{+}(\zeta)=D^{-1}C(\zeta G^{\times}-A^{\times})^{-1}z\ ,\quad z\in Ker\ P^{\times},$$

has an analytic continuation to $\mathbb{D}_{+}$, whence

$$\frac{1}{2\pi i}\int_{\mathbb{T}} D^{-1}C(\zeta G^{\times}-A^{\times})^{-1}zd\zeta=0\ ,$$

by Cauchy's theorem. Hence (4.20) is established.

We have now proved the identities (4.12)–(4.15) for $p=2$. Next, take an arbitrary p, $1\leq p\leq\infty$. Since T and $T^{\times}$ are block Toeplitz operators with symbols from the Wiener class, the operator $TT^{\times}$ on ℓ_p^m has a matrix representation, that is,

$$(TT^{\times}x)_k=\sum_{j=0}^{\infty}M_{kj}x_j\ ,\quad k=0,1,2,\ldots,$$

for each $x=(x_0,x_1,x_2,\ldots)$ in ℓ_p^m. The same is true for $UR^{\times}$. So to check (4.12) it suffices to show that $(TT^{\times}+UR^{\times})x=x$ for all sequences $x=(x_k)_{k=0}^{\infty}$ with a finite number of non-zero elements . But the latter sequences are all in ℓ_2^m, and hence (4.12) holds for any $1\leq p\leq\infty$.

A similar argument proves that (4.13) holds for any $1\leq p\leq\infty$. The identities (4.14) and (4.15) do not depend on p. $\square$

Chapter 2

THE STATE SPACE METHOD FOR SOLVING SINGULAR INTEGRAL EQUATIONS

In [GK3] the state space method was used to give explicit formulas for the solutions of singular integral equations with rational symbol of the form

$$W(\lambda) = I + C(\lambda G - A)^{-1}B. \tag{0.1}$$

Here A is a square matrix whose order n may be much larger than the size of $W(\lambda)$, and B and C are matrices of appropriate sizes. G is a square matrix of the same order as A, and I stands for the $m \times m$ identity matrix.

In this chapter we carry out a similar program as in [GK3] but with a different representation of the rational symbol, namely

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B. \tag{0.2}$$

Here α is a non-zero complex number which is neither a pole nor a zero of W. A, G, B and C are as in (0.1) and D is an invertible matrix. The main ideas from [GK3] are extended to the case considered here, i.e., explicit formulas for the solutions of singular integral equations with the above representation of the symbol are given.

2.1 Preliminaries about matrix pencils and realization

Throughout this chapter Γ is a contour consisting of a finite number of disjoint smooth simple Jordan curves. The inner domain of Γ will still be denoted by Δ_+ and its outer domain by Δ_-. In what follows we assume that $\infty \in \Delta_-$.

1(a). Realization

This subsection concerns the special representation (0.2).

Proposition 1.1 *A rational $m \times m$ matrix function W without poles on the contour Γ admits the following representation:*

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B \quad , \quad \lambda \in \Gamma, \tag{1.1}$$

where $\alpha \neq 0$ and α is neither a pole nor a zero of W. Here G and A are square matrices of the same size; $n \times n$ say, the pencil $\lambda G - A$ is Γ-regular, and B, C and D are matrices of sizes $n \times m$, $m \times n$ and $m \times m$ respectively.

The representation (1.1) may be derived from classical realization results by applying the Möbius transformation

$$\phi(\lambda) = \alpha\frac{2\lambda - 1}{2\lambda + 1} \quad , \quad \phi^{-1}(z) = -\frac{1}{2}\frac{z + \alpha}{z - \alpha}.$$

Indeed, since $W(\lambda)$ can be written in the form

$$W(\lambda) = D + C(\lambda - A)^{-1}B \quad , \quad \lambda \in \Gamma.$$

Put $W'(\lambda) = W(\phi^{-1}(\lambda))$. Then

$$\begin{aligned}
W(\phi^{-1}(\lambda)) &= D + C(\phi^{-1}(\lambda) - A)^{-1}B \\
&= D + C\left[-\frac{1}{2}\frac{\lambda+\alpha}{\lambda-\alpha} - A\right]^{-1} B \\
&= D + C\left[\frac{-\frac{1}{2}(\lambda+\alpha) - (\lambda-\alpha)A}{\lambda-\alpha}\right]^{-1} B \\
&= D + (\lambda-\alpha)C\left[\lambda(-\frac{1}{2} - A) - \alpha(\frac{1}{2} - A)\right]^{-1} B \\
&= D' + (\lambda-\alpha)C'(\lambda G' - A')^{-1}B'.
\end{aligned}$$

Here $A' = \alpha(\frac{1}{2} - A)$, $G' = -\frac{1}{2} - A$, $B' = B$, $C' = C$ and $D' = D$. Then (1.1) holds.
If W is as in (1.1), then we shall say that W is in *realized form*, and we shall call the right-hand side of (1.1) a *realization of W*. The following proposition will be used in Section 3; its proof can be found in [Gr1], Section 1.2.

Proposition 1.2 *Let $W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B$, $\lambda \in \Gamma$, be a given realization, where $\lambda G - A$ is Γ -regular. Set $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$. Then $\det W(\lambda) \neq 0$ for each $\lambda \in \Gamma$ if and only if the pencil $\lambda G^\times - A^\times$ is Γ -regular, and in this case we have the following identities:*

$$W(\lambda)^{-1} = D^{-1} - (\lambda - \alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}BD^{-1} \quad , \quad \lambda \in \Gamma, \tag{1.2}$$

$$(\lambda G^\times - A^\times)^{-1} = (\lambda G - A)^{-1} - (\lambda - \alpha)(\lambda G - A)^{-1}BW(\lambda)^{-1}C(\lambda G - A)^{-1}, \ \lambda \in \Gamma. \tag{1.3}$$

1(b). Matrix pencils

Let A and G be $n \times n$ complex matrices. The expression $\lambda G - A$, where λ is a complex parameter, is called a *(linear matrix) pencil*. We say that the pencil $\lambda G - A$ is Γ-*regular* if det $(\lambda G - A) \neq 0$ for each λ on the contour Γ. In this case one can define the following matrices:

$$
\begin{aligned}
P &= \frac{1}{2\pi i}\int_\Gamma G(\zeta G - A)^{-1} d\zeta\,, \qquad & Q &= \frac{1}{2\pi i}\int_\Gamma (\zeta G - A)^{-1} G d\zeta, \\
P^\times &= \frac{1}{2\pi i}\int_\Gamma G^\times(\zeta G^\times - A^\times)^{-1} d\zeta\,, \qquad & Q^\times &= \frac{1}{2\pi i}\int_\Gamma (\zeta G^\times - A^\times)^{-1} G^\times d\zeta.
\end{aligned}
\tag{1.4}
$$

We shall need the following spectral decomposition result. For its proof we refer to [GK2], Section 2.

Proposition 1.3 *Let $\lambda G - A$ be Γ-regular, and let the matrices P and Q be defined by* (1.4). *Then P and Q are projections which have the following properties:*

(1) *$PG = GQ$ and $PA = AQ$;*

(2) *$(\lambda G - A)^{-1} P = Q(\lambda G - A)^{-1}$ on Γ and this function has an analytic continuation on Δ_- which vanishes at ∞;*

(3) *$(\lambda G - A)^{-1}(I - P) = (I - Q)(\lambda G - A)^{-1}$ on Γ and this function has an analytic continuation on Δ_+.*

Note that the above proposition also holds for the associate pencil $\lambda G^\times - A^\times$ and the corresponding separating projection $P^\times$.

2.2 Reduction of the inversion problem for singular integral equations

This section consists of a proposition that summarizes one of the main steps in the proofs of theorems that will be dealt with in the next section. Its central idea is the reduction of the inversion problem to a problem for input/output systems.

We shall refer to the integral equation introduced in Section 1.3, namely,

$$A(\lambda)\phi(\lambda) + B(\lambda)\left(\frac{1}{\pi i}\int_{\Gamma}\frac{\phi(\mu)}{\mu-\lambda}d\mu\right) = f(\lambda) \quad ,\lambda \in \Gamma. \tag{2.1}$$

As before, the contour Γ consists of a finite number of disjoint smooth simple Jordan curves and the coefficients $A(\cdot)$ and $B(\cdot)$ are $m \times m$ rational matrix functions, which have no poles on Γ. Q_Γ, P_Γ and S_Γ are as before.

Assume now that det $(A(\lambda) - B(\lambda)) \neq 0$ for $\lambda \in \Gamma$. Then equation (2.1) may be rewritten in the form

$$(M_W P_\Gamma + Q_\Gamma)\phi = g, \tag{2.2}$$

where M_W is the operator of multiplication by the $m \times m$ matrix function

$$W(\lambda) = [A(\lambda) - B(\lambda)]^{-1}[A(\lambda) + B(\lambda)], \quad \lambda \in \Gamma, \tag{2.3}$$

and the right-hand side g is given by

$$g(\lambda) = [A(\lambda) - B(\lambda)]^{-1} f(\lambda) \quad , \quad \lambda \in \Gamma. \tag{2.4}$$

We shall refer to $M_W P_\Gamma + Q_\Gamma$ as the *singular integral operator* with symbol W. In fact, the symbol is the diagonal matrix $W(\cdot) \oplus I_\Gamma$, where I_Γ denotes the function which is identically equal on Γ to the $m \times m$ identity matrix; in the sequel we will omit this second function.

Proposition 2.1 *Let $M_W P_\Gamma + Q_\Gamma$ be the singular integral operator on $L_2^m(\Gamma)$ with symbol* (1.1), *and let $g \in L_2^m(\Gamma)$. Put $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$. Assume that $\lambda G^\times - A^\times$ is Γ -regular, and let P and $P^\times$ be the projections defined by* (1.4). *Then the equation*

$$(M_W P_\Gamma + Q_\Gamma)\phi = g \tag{2.5}$$

has a solution $\phi \in L_2^m(\Gamma)$ if and only if

$$\int_\Gamma (\zeta - \alpha) P^\times G^\times (\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta \in P^\times[Im\ P], \tag{2.6}$$

and in this case the general solution of (2.5) is given by

$$D\phi_+(\lambda) + \phi_-(\lambda) = [g(\lambda) - C(\lambda G - A)^{-1}y + C(\lambda G^\times - A^\times)^{-1}y - (\lambda - \alpha)C(\lambda G^\times - A^\times)^{-1}BD^{-1}(P_\Gamma g)(\lambda)], \tag{2.7}$$

where y is an arbitrary vector in $Im\ P$ such that

$$P^\times y = \frac{1}{2\pi i}\int_\Gamma (\zeta - \alpha) P^\times G^\times (\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta. \tag{2.8}$$

Here $\phi_+(\lambda) = (P_\Gamma \phi)(\lambda)$ and $\phi_-(\lambda) = (Q_\Gamma \phi)(\lambda)$ for $\phi \in L_2^m(\Gamma)$.

Proof. We follow the same line of reasoning as in the proof of Proposition 3.1 in [GK3]. See also Proposition 2.3 in [Gr1], Section II.2.

For $\phi \in L_2^m(\Gamma)$, put

$$\phi_+(\lambda) := (P_\Gamma \phi)(\lambda) = \frac{1}{2}\phi(\lambda) + \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\zeta)d\zeta}{\zeta - \lambda}\ , \quad \lambda \in \Gamma, \tag{2.9}$$

$$\phi_-(\lambda) := (Q_\Gamma \phi)(\lambda) = \frac{1}{2}\phi(\lambda) - \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\zeta)d\zeta}{\zeta - \lambda}\ , \quad \lambda \in \Gamma. \tag{2.10}$$

Assume now that $\phi \in L_2^m(\Gamma)$ is a solution of (2.5). We shall now show that in this case g satisfies (2.6) and that ϕ is given by (2.7). First, we introduce the auxiliary function

$$\rho(\lambda) = (\lambda - \alpha)(\lambda G - A)^{-1}B\phi_+(\lambda), \quad \lambda \in \Gamma.$$

Next, from the representation (1.1) for W it follows that the connection between ϕ and g in (2.5) is described by the following input/output system:

$$\begin{cases} \lambda G\rho(\lambda) = A\rho(\lambda) + (\lambda - \alpha)B\phi_+(\lambda) \quad , \quad \lambda \in \Gamma, \\ g(\lambda) = C\rho(\lambda) + D\phi_+(\lambda) + \phi_-(\lambda). \end{cases} \tag{2.11}$$

Note that $\rho \in L_2^n(\Gamma)$. The first identity in (2.11) implies that the function $(\lambda G - A)\rho(\lambda) = (\lambda - \alpha)B\phi_+(\lambda) \in \mathrm{Im}\ P_\Gamma$ (where P_Γ is now considered onto $L_2^n(\Gamma)$). Since $P_\Gamma = \frac{1}{2}(I + S_\Gamma)$ and $(\lambda G - A)\rho(\lambda) \in \mathrm{Im}\ P_\Gamma$ it follows from (2.9) that

$$(\lambda G - A)\rho(\lambda) = \frac{1}{2}(\lambda G - A)\rho(\lambda) + \frac{1}{2\pi i}\int_\Gamma \frac{(\zeta G - A)}{\zeta - \lambda}\rho(\zeta)d\zeta.$$

Hence,

$$\begin{aligned} \frac{1}{2}(\lambda G - A)\rho(\lambda) &= \frac{1}{2\pi i}\int_\Gamma \frac{(\zeta G - A)}{\zeta - \lambda}\rho(\zeta)d\zeta \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta - \lambda}[(\zeta - \lambda)G + (\lambda G - A)]\rho(\zeta)d\zeta \\ &= Gx + (\lambda G - A)\left(\frac{1}{2\pi i}\int_\Gamma \frac{\rho(\zeta)}{\zeta - \lambda}d\zeta\right) \quad , \quad \lambda \in \Gamma, \end{aligned}$$

where

$$x = \frac{1}{2\pi i}\int_\Gamma \rho(\zeta)d\zeta \in \mathbb{C}^n.$$

But then we see (use (2.10)) that

$$\frac{1}{2}\rho(\lambda) = (\lambda G - A)^{-1}Gx + \frac{1}{2\pi i}\int_\Gamma \frac{\rho(\zeta)}{\zeta - \lambda}d\zeta \ , \quad \lambda \in \Gamma,$$

i.e.,

$$(\lambda G - A)^{-1}Gx = \frac{1}{2}\rho(\lambda) - \frac{1}{2\pi i}\int_\Gamma \frac{\rho(\zeta)}{\zeta - \lambda}d\zeta = (Q_\Gamma\rho)(\lambda) \ , \quad \lambda \in \Gamma.$$

Hence,

$$\rho_-(\lambda) = (\lambda G - A)^{-1}Gx \quad , \quad \lambda \in \Gamma. \tag{2.12}$$

Since $PGx \in \text{Im } P$, we may apply Proposition 1.3(2) to show that $(\lambda G - A)^{-1}PGx$ extends to an analytic function on Δ_- which vanishes at infinity. The function ρ_- has the same properties. Thus, by (2.12), also the function $(\lambda G - A)^{-1}(I - P)Gx$ may be extended to an analytic function on Δ_- which vanishes at ∞. On the other hand, by Proposition 1.3(3), the function $(\lambda G - A)^{-1}(I - P)Gx$ is analytic on $\Delta_+ \cup \Gamma$. Thus this function is an entire function which is zero at infinity. Therefore, by Liouville's theorem, $(\lambda G - A)^{-1}(I - P)Gx$ is identically zero, which implies that $Gx = PGx \in \text{Im } P$. From (2.12) it follows that the first identity in (2.11) can be written as:

$$\lambda G\rho_+(\lambda) = A\rho_+(\lambda) - Gx + (\lambda - \alpha)B\phi_+(\lambda), \quad \lambda \in \Gamma. \tag{2.13}$$

By applying P_Γ to the second identity in (2.11) we get

$$g_+(\lambda) = C\rho_+(\lambda) + D\phi_+(\lambda), \quad \lambda \in \Gamma. \tag{2.14}$$

Now multiply (2.14) from the left by $(\lambda - \alpha)BD^{-1}$ and subtract the resulting identity from (2.13). This yields

$$\lambda G^\times \rho_+(\lambda) = A^\times \rho_+(\lambda) - Gx + (\lambda - \alpha)BD^{-1}g_+(\lambda), \quad \lambda \in \Gamma, \tag{2.15}$$

and thus

$$(\lambda - \alpha)(\lambda G^\times - A^\times)^{-1}BD^{-1}g_+(\lambda) = \rho_+(\lambda) + (\lambda G^\times - A^\times)^{-1}Gx \quad , \quad \lambda \in \Gamma. \tag{2.16}$$

From Proposition 1.3(3) (with $(\lambda G^\times - A^\times)$ instead of $(\lambda G - A)$) we know that the function $(\lambda G^\times - A^\times)^{-1}(I - P^\times)Gx$ extends to a function which is analytic at each point of $\Delta_+ \cup \Gamma$, and thus the function $(\lambda G^\times - A^\times)^{-1}(I - P^\times)Gx$ belongs to $\text{Im } P_\Gamma = \text{Ker } Q_\Gamma$. Also $\rho_+ \in \text{Ker } Q_\Gamma$. Therefore Q_Γ applied to (2.16) yields:

$$\begin{aligned} &\frac{1}{2}(\lambda - \alpha)(\lambda G^\times - A^\times)^{-1}BD^{-1}g_+(\lambda) - \frac{1}{2\pi i}\int_\Gamma \frac{(\zeta - \alpha)}{(\zeta - \lambda)}(\zeta G^\times - A^\times)^{-1}BD^{-1}g_+(\zeta)d\zeta \\ &= (\lambda G^\times - A^\times)^{-1}P^\times Gx \quad , \quad \lambda \in \Gamma, \end{aligned} \tag{2.17}$$

and so

$$\begin{aligned}
P^\times Gx &= \frac{1}{2}(\lambda-\alpha)BD^{-1}g_+(\lambda) - \frac{1}{2\pi i}\int_\Gamma \frac{(\zeta-\alpha)}{(\zeta-\lambda)}[(\lambda-\zeta)G^\times + (\zeta G^\times - A^\times)] \\
&\quad \cdot(\zeta G^\times - A^\times)^{-1}BD^{-1}g_+(\zeta)d\zeta \\
&= \frac{1}{2}(\lambda-\alpha)BD^{-1}g_+(\lambda) - \frac{1}{2\pi i}\int_\Gamma \frac{(\zeta-\alpha)}{(\zeta-\lambda)}BD^{-1}g_+(\zeta)d\zeta \\
&+ \frac{1}{2\pi i}\int_\Gamma (\zeta-\alpha)G^\times(\zeta G^\times - A^\times)^{-1}BD^{-1}g_+(\zeta)d\zeta \\
&= \frac{1}{2\pi i}\int_\Gamma (\zeta-\alpha)G^\times(\zeta G^\times - A^\times)^{-1}BD^{-1}g_+(\zeta)d\zeta.
\end{aligned} \tag{2.18}$$

Proposition 1.3(1) and 1.3(2) imply that the last integral does not change if in the integrand $G^\times$ is replaced by $P^\times G^\times$. But $P^\times G^\times(\zeta G^\times - A^\times)^{-1}BD^{-1}$ is analytic on Δ_- and vanishes at ∞. Therefore,

$$\frac{1}{2\pi i}\int_\Gamma (\zeta-\alpha)P^\times G^\times(\zeta G^\times - A^\times)^{-1}BD^{-1}g_-(\zeta)d\zeta = 0.$$

Thus

$$P^\times Gx = \frac{1}{2\pi i}\int_\Gamma (\zeta-\alpha)P^\times G^\times(\zeta G^\times - A^\times)^{-1}BD^{-1}g_+(\zeta)d\zeta$$

which shows that (2.6) is satisfied.

Put $y = Gx$. Then (2.8) holds. Furthermore, by the second identity in (2.11), and formulas (2.12) and (2.16) we have

$$\begin{aligned}
D\phi_+(\lambda) + \phi_-(\lambda) &= g(\lambda) - C\rho_+(\lambda) - C\rho_-(\lambda) \\
&= g(\lambda) - C(\lambda G - A)^{-1}y + C(\lambda G^\times - A^\times)^{-1}y \\
&\quad -(\lambda-\alpha)C(\lambda G^\times - A^\times)^{-1}BD^{-1}(P_\Gamma g)(\lambda)
\end{aligned}$$

which proves (2.7).

Next, we prove the converse statement. So, we assume that ϕ is given implicitly by (2.7), with y a vector in Im P satisfying (2.8).

Put

$$\rho_1(\lambda) = (\lambda G - A)^{-1} y, \quad \rho_2(\lambda) = -(\lambda G^\times - A^\times)^{-1} y,$$

$$\rho_3(\lambda) = (\lambda - \alpha)(\lambda G^\times - A^\times)^{-1} B D^{-1} g_+(\lambda),$$

where $\lambda \in \Gamma$. From $y \in \operatorname{Im} P$ and Proposition 1.3(2) it follows that $\rho_1 \in \operatorname{Im} Q_\Gamma$.

Further, note that

$$\rho_2(\lambda) = -(\lambda G^\times - A^\times)^{-1} P^\times y - (\lambda G^\times - A^\times)^{-1}(I - P^\times) y.$$

Now, applying Proposition 1.3 and Q_Γ to the above equation yields

$$(Q_\Gamma \rho_2)(\lambda) = -(\lambda G^\times - A^\times)^{-1} P^\times y \ , \quad \lambda \in \Gamma.$$

Furthermore,

$$\begin{aligned} P^\times y &= \frac{1}{2\pi i} \int_\Gamma (\zeta - \alpha) P^\times G^\times (\zeta G^\times - A^\times)^{-1} B D^{-1} g_+(\zeta) d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma (\zeta - \alpha) G^\times (\zeta G^\times - A^\times)^{-1} B D^{-1} g_+(\zeta) d\zeta \\ &= \frac{1}{2}(\lambda - \alpha) B D^{-1} g_+(\lambda) - (\lambda G^\times - A^\times) \left(\frac{1}{2\pi i} \int_\Gamma \frac{(\zeta - \alpha)}{(\zeta - \lambda)} (\zeta G^\times - A^\times)^{-1} B D^{-1} g_+(\zeta) d\zeta \right). \end{aligned}$$

To prove the last equality one uses the same type of reasoning as in (2.18).
From the above calculation it follows that (2.17) holds with y instead of Gx, i.e.,

$$(\lambda G^\times - A^\times)^{-1} P^\times y = \frac{1}{2}(\lambda - \alpha)(\lambda G^\times - A^\times)^{-1} B D^{-1} g_+(\lambda) - \frac{1}{2\pi i} \int_\Gamma \frac{(\zeta - \alpha)}{(\zeta - \lambda)} (\zeta G^\times - A^\times)^{-1} B D^{-1} g_+(\zeta) d\zeta$$

which shows that

$$(Q_\Gamma \rho_3)(\lambda) = (\lambda G^\times - A^\times)^{-1} P^\times y \ , \quad \lambda \in \Gamma.$$

Thus $\rho_2 + \rho_3 \in \operatorname{Ker} Q_\Gamma = \operatorname{Im} P_\Gamma$. As $D\phi_+ + \phi_- = g - C\rho_1 - C(\rho_2 + \rho_3)$, we conclude that

$$\phi_- = g_- - C\rho_1 \ , \quad D\phi_+ = g_+ - C(\rho_2 + \rho_3).$$

From the definitions of ρ_2 and ρ_3 we have

$$\begin{aligned}(\lambda G - A)(\rho_2(\lambda) + \rho_3(\lambda)) &= (\lambda G^\times - A^\times)(\rho_2(\lambda) + \rho_3(\lambda)) - (\lambda - \alpha)BD^{-1}C(\rho_2(\lambda) + \rho_3(\lambda)) \\ &= -y + (\lambda - \alpha)BD^{-1}g_+(\lambda) - (\lambda - \alpha)BD^{-1}C(\rho_2(\lambda) + \rho_3(\lambda)) \\ &= -y + (\lambda - \alpha)BD^{-1}\left[g_+(\lambda) - C(\rho_2(\lambda) + \rho_3(\lambda))\right] \\ &= -y + (\lambda - \alpha)B\phi_+(\lambda) \\ &= -(\lambda G - A)\rho_1(\lambda) + (\lambda - \alpha)B\phi_+(\lambda).\end{aligned}$$

It follows that with $\rho = \rho_1 + \rho_2 + \rho_3$ the identities in (2.11) hold. But this implies that

$$\begin{aligned}W(\lambda)\phi_+(\lambda) + \phi_-(\lambda) &= [D + (\lambda - \alpha)C(\lambda G - A)^{-1}B]\phi_+(\lambda) + \phi_-(\lambda) \\ &= D\phi_+(\lambda) + C\rho(\lambda) + \phi_-(\lambda) \\ &= C\rho(\lambda) + D\phi_+(\lambda) + \phi_-(\lambda) \\ &= g(\lambda), \quad \lambda \in \Gamma,\end{aligned}$$

and thus $D\phi_+ + \phi_-$ is a solution of (2.5). $\square$

2.3 Inversion and Fredholm properties

Equation (2.1) has a unique solution $\phi \in L_2^m(\Gamma)$ for each choice of $f \in L_2^m(\Gamma)$ if and only if the singular integral operator $M_W P_\Gamma + Q_\Gamma$ is invertible, and in this case the solution ϕ is given by

$$\phi = (M_W P_\Gamma + Q_\Gamma)^{-1} g,$$

where g is defined by (2.4). In this section we give a necessary and sufficient condition for the invertibility of $M_W P_\Gamma + Q_\Gamma$ and an explicit formula for its inverse. Also we shall describe the Fredholm properties of the operator $M_W P_\Gamma + Q_\Gamma$.

Since the coefficients $A(\cdot)$ and $B(\cdot)$ in (2.1) are rational and have no poles on Γ, we see from (2.3) that the same is true for W. It follows (see Section 1(a)) that W admits a realization of the form

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1} B, \quad \lambda \in \Gamma, \tag{3.1}$$

where $\lambda G - A$ is a Γ-regular matrix pencil.

Recall (see [TL]) that an operator T on $L_2^m(\Gamma)$ is *Fredholm* if $\operatorname{Im} T$ is closed and

$$\dim \operatorname{Ker} T < \infty, \quad \operatorname{codim} \operatorname{Im} T = \dim \left(\frac{L_2^m(\Gamma)}{\operatorname{Im} T} \right) < \infty.$$

If T is Fredholm, then its *index* is the integer

$$\operatorname{ind} T := \dim \operatorname{Ker} T - \operatorname{codim} \operatorname{Im} T.$$

We say that T^+ is a *generalized inverse* (in a weak sense) of T if $TT^+T = T$.

We now have the following theorems.

Theorem 3.1 *Let $T = M_W P_\Gamma + Q_\Gamma$ be the singular integral operator on $L_2^m(\Gamma)$ with symbol* (3.1). *Put $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$. Then T is a Fredholm operator if and only if the pencil $\lambda G^\times - A^\times$ is Γ - regular, and in this case the following equalities hold:*

$$Ker\ T = \{\phi \mid D\phi_+(\lambda) + \phi_-(\lambda) = -C(\lambda G - A)^{-1}y + C(\lambda G^\times - A^\times)^{-1}y\ ,\ \ y \in Im\ P \cap Ker\ P^\times\}, \tag{3.2}$$

$$Im\ T = \{g \in L_2^m(\Gamma) \mid \int_\Gamma (\zeta - \alpha)P^\times G^\times(\zeta G^\times - A^\times)^{-1} BD^{-1} g(\zeta) d\zeta \in Im\ P + Ker\ P^\times\} \tag{3.3}$$

$$dim\ Ker\ T = dim(Im\ P \cap Ker\ P^\times)\ ,\ codim\ Im\ T = dim \frac{\mathbb{C}^n}{Im\ P + Ker\ P^\times}\ , \tag{3.4}$$

$$ind\ T = rank\ P - rank\ P^\times. \tag{3.5}$$

Here P and $P^\times$ are as before, and n is the order of the matrices A and G. Furthermore, a generalized inverse T^+ of T is given by:

$$\begin{aligned}(T^+g)(\lambda) &= g(\lambda)-(\lambda-\alpha)C(\lambda G^\times-A^\times)^{-1}BD^{-1}(P_\Gamma g)(\lambda)\\ &\quad+\{C(\lambda G^\times-A^\times)^{-1}-C(\lambda G-A)^{-1}\}\\ &\quad\cdot J^+\left(\frac{1}{2\pi i}\int_\Gamma(\zeta-\alpha)P^\times G^\times(\zeta G^\times-A^\times)^{-1}BD^{-1}g(\zeta)d\zeta\right),\quad \lambda\in\Gamma,\end{aligned}\tag{3.6}$$

where $J^+ : Im\ P^\times \to Im\ P$ is a generalized inverse of the linear transformation

$$J = P^\times \mid Im\ P\ :\ Im\ P \to Im\ P^\times. \tag{3.7}$$

Proof. From the general theory of singular integral equations (see [G], also [CG]) it is known that $T = M_W P_\Gamma + Q_\Gamma$ is Fredholm if and only if $\det W(\lambda) \neq 0$, $\lambda \in \Gamma$. But by Proposition 1.2, T is Fredholm if and only if $\det\,(\lambda G^\times - A^\times) \neq 0$ for each $\lambda \in \Gamma$.

Assume that the latter condition holds. An immediate application of Proposition 2.1 (with $g = 0$) gives (3.2). Also (3.3) follows directly from Propostion 2.1; one only has to note that for $x \in \text{Im}\ P^\times$:

$$x \in P^\times[\text{Im}\ P] \Leftrightarrow x \in \text{Im}\ P + \text{Ker}\ P^\times.$$

To prove the first identity in (3.4) it suffices to show that for $y \in \text{Im}\ P \cap \text{Ker}\ P^\times$ the identity

$$C(\lambda G - A)^{-1}y = C(\lambda G^\times - A^\times)^{-1}y\ \ ,\ \ \lambda \in \Gamma \tag{3.8}$$

implies $y = 0$. Since $y \in \text{Im}\ P$, the left-hand side of (3.8) extends to an analytic function on Δ_- which vanishes at ∞. From $y \in \text{Ker}\ P^\times$ it follows that the right hand-side of (3.8) has an analytic continuation on Δ_+. So, by Liouville's theorem, both functions are identically zero on Γ. But then we can apply the identity (1.3) to show that

$$(\lambda G^\times - A^\times)^{-1}y = (\lambda G - A)^{-1}y\ \ ,\ \ \lambda \in \Gamma. \tag{3.9}$$

Apply $G^\times$ to both sides of (3.9) and integrate over the contour Γ. One sees that $y = Py = P^\times y = 0$ and thus the first identity in (3.4) is proved. In an analogous way one proves the second identity in (3.4).

From (3.4) it follows that

$$\begin{aligned} \text{ind } T &= \dim(\text{Im } P \cap \text{Ker } P^\times) - \dim \frac{\mathbb{C}^n}{\text{Im } P + \text{Ker } P^\times} \\ &= \dim \text{Im } P - \dim \frac{\text{Im } P}{\text{Im } P \cap \text{Ker } P^\times} - \dim \frac{\mathbb{C}^n}{\text{Im } P + \text{Ker } P^\times} \\ &= \dim \text{Im } P - \dim \frac{\text{Im } P + \text{Ker } P^\times}{\text{Ker } P^\times} - \dim \frac{\mathbb{C}^n}{\text{Im } P + \text{Ker } P^\times} \\ &= \dim \text{Im } P - \dim \text{Im } P^\times \end{aligned}$$

which proves (3.5).

Finally, let us show that the operator T^+ defined by (3.6) is a generalized inverse of T. Take an arbitrary $\phi \in L_2^m(\Gamma)$, and put $g = T\phi$. Then (2.6) holds, that is,

$$z := \int_\Gamma (\zeta - \alpha) P^\times G^\times (\zeta G^\times - A^\times)^{-1} B D^{-1} g(\zeta) d\zeta \in \text{Im } J \tag{3.10}$$

where J is defined by (3.7). Put $y = J^+ z$. Since J^+ is a generalized inverse of J, the map JJ^+ acts as the identity operator on Im J, and therefore $P^\times y = JJ^+ z = z$. It follows that (2.8) holds. Also $y \in$ Im P. Thus Proposition 2.1 implies that $T^+ g$ is a solution of (2.5). But then

$$T\phi = g = T(T^+ g) = TT^+ T\phi.$$

Since ϕ is arbitrary, we have proved that T^+ is a generalized inverse of T. $\square$

Theorem 3.2 *Let $M_W P_\Gamma + Q_\Gamma$ be the singular integral operator on $L_2^m(\Gamma)$ with symbol* (3.1)*. Put $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$. Then $M_W P_\Gamma + Q_\Gamma$ is invertible if and only if the following two conditions are satisfied:*

(α) the pencil $\lambda G^\times - A^\times$ is Γ - regular,

(β) $\mathbb{C}^n = \ Im\ P \oplus Ker\ P^\times$,

where n is the order of the matrices G and A, and

$$P = \frac{1}{2\pi i}\int_\Gamma G(\lambda G - A)^{-1} d\lambda \quad , \quad P^\times = \frac{1}{2\pi i}\int_\Gamma G^\times(\lambda G^\times - A^\times)^{-1} d\lambda. \tag{3.11}$$

In this case,

$$\begin{aligned}(M_W P_\Gamma + Q_\Gamma)^{-1} g(\lambda) \ &= \ g(\lambda) - (\lambda - \alpha) C(\lambda G^\times - A^\times)^{-1} B D^{-1} (P_\Gamma g)(\lambda) \\ &\quad + \{C(\lambda G^\times - A^\times)^{-1} - C(\lambda G - A)^{-1}\}(I - \pi) \\ &\quad \cdot \left(\frac{1}{2\pi i}\int_\Gamma (\zeta - \alpha) P^\times G^\times (\zeta G^\times - A^\times)^{-1} B D^{-1} g(\zeta) d\zeta\right) \quad , \quad \lambda \in \Gamma,\end{aligned}$$

where π is the projection of $\mathbb{C}^n$ onto $Ker\ P^\times$ along $Im\ P$.

Proof. Assume that $T := M_W P_\Gamma + Q_\Gamma$ is invertible. Then T is Fredholm, and thus, by Theorem 3.1 condition (α) is fulfilled. Furthermore, since

$$\dim \operatorname{Ker} T = 0 \quad , \quad \operatorname{codim} \operatorname{Im} T = 0 \tag{3.12}$$

formula (3.4) shows that condition (β) is fulfilled.

Conversely, assume that (α) and (β) hold. Then, by Theorem 3.1, the operator T is Fredholm and (3.12) holds. But, this means that T is invertible.

To compute T^{-1}, let π be the projection of $\mathbb{C}^n$ onto Ker $P^\times$ along Im P, and define $J^+ \ : \ \operatorname{Im} P^\times \to \operatorname{Im} P$, by setting,

$$J^+ x = (I - \pi)x \quad , \quad x \in \operatorname{Im} P^\times. \tag{3.13}$$

Let J be the map defined by (3.7). From

$$J J^+ J z = P^\times (I - \pi) J z = P^\times J z \quad , \quad z \in \operatorname{Im} P,$$

it follows that J^+ is a generalized inverse of J. Now, let T^+ be the operator defined by (3.6) with J^+ given by (3.13). Then T^+ is a generalized inverse of T. But T is invertible, and thus $T^+ = T^{-1}$, which proves the formula for T^{-1}. $\square$

2.4 The factorization method

The classical way to invert the singular integral operator $M_W P_\Gamma + Q_\Gamma$ is based on the idea of factorization. First, one looks for a so-called *right canonical factorization* of the symbol W relative to the contour Γ, that is, a factorization of the form

$$W(\lambda) = W_-(\lambda)W_+(\lambda) \quad , \quad \lambda \in \Gamma, \tag{4.1}$$

where, for $\nu = +, -$, the matrix function W_ν is continuous on $\Delta_\nu \cup \Gamma$ and analytic on Δ_ν, and $\det W_\nu(\lambda) \neq 0$ for each $\lambda \in \Delta_\nu \cup \Gamma$. In particular, the factor W_- is analytic at ∞ and $\det W_-(\infty) \neq 0$.

As in the previous sections, let us assume that the symbol W is rational. Then it is well-known (see e.g. [CG], Theorem I.3.1) that the singular integral operator $M_W P_\Gamma + Q_\Gamma$ is invertible if and only if its symbol admits a right canonical factorization, and in this case

$$(M_W P_\Gamma + Q_\Gamma)^{-1} g)(\lambda) = W_+(\lambda)^{-1}(P_\Gamma(W_-^{-1}g))(\lambda) W_-(\lambda)(Q_\Gamma(W_-^{-1}g))(\lambda) \quad , \quad \lambda \in \Gamma. \tag{4.2}$$

where W_- and W_+ are factors in a right canonical factorization of W relative to Γ. By definition, $W_-^{-1}g$ is the function $W_-(\cdot)^{-1}g(\cdot)$. To apply this method in an effective way one needs necessary and sufficient conditions that guarantee the existence of the canonical factorization and one needs explicit formulas for the factors in the factorization (and also for their inverses). The representation of the symbol (3.1) allows one to find such conditions and to derive the factors and their inverses explicitly. The following theorem holds; its proof may be found in [Gr1, Theorem I.3.1].

Theorem 4.1 *Let W be a rational $m \times m$ matrix function without poles on the contour Γ, and let W be given in realized form:*

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B \quad , \quad \lambda \in \Gamma.$$

Put $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$. Then W admits a right canonical factorization relative to Γ if and only if the following two conditions hold:

(i) the pencil $\lambda G^\times - A^\times$ is Γ - regular,

(ii) $\mathbb{C}^n = Im\ P \oplus Ker\ P^\times$ and $\mathbb{C}^n = Im\ Q \oplus Ker\ Q^\times$.

Here n is the order of the matrices G and A, and

$$\begin{aligned} P &= \frac{1}{2\pi i}\int_\Gamma G(\lambda G - A)^{-1}d\lambda \quad , \quad P^\times = \frac{1}{2\pi i}\int_\Gamma G^\times(\lambda G^\times - A^\times)^{-1}d\lambda, \\ Q &= \frac{1}{2\pi i}\int_\Gamma (\lambda G - A)^{-1}G d\lambda \quad , \quad Q^\times = \frac{1}{2\pi i}\int_\Gamma (\lambda G^\times - A^\times)^{-1}G^\times d\lambda. \end{aligned}$$

In this case a right canonical factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$ of W relative to Γ is obtained by taking:

$$\begin{aligned} W_-(\lambda) &= D + (\lambda - \alpha)C(\lambda G - A)^{-1}(I - \pi)B \quad , \quad \lambda \in \Gamma \cup \Delta_-, \\ W_+(\lambda) &= I + (\lambda - \alpha)D^{-1}C\tau(\lambda G - A)^{-1}B \quad , \quad \lambda \in \Gamma \cup \Delta_+, \\ W_-(\lambda)^{-1} &= D^{-1} - (\lambda - \alpha)D^{-1}C(I - \tau)(\lambda G^\times - A^\times)^{-1}BD^{-1} \quad , \quad \lambda \in \Gamma \cup \Delta_-, \\ W_+(\lambda)^{-1} &= I - (\lambda - \alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}\pi B \quad , \quad \lambda \in \Gamma \cup \Delta_+. \end{aligned}$$

Here τ is the projection of $\mathbb{C}^n$ onto $Ker\ Q^\times$ along $Im\ Q$ and π is the projection of $\mathbb{C}^n$ onto $Ker\ P^\times$ along $Im\ P$. Furthermore, the two equalities in (ii) are equivalent.

Let $M_W P_\Gamma + Q_\Gamma$ be the singular operator with symbol W. Assume that W is rational and given in the realized form (3.1). Theorem 4.1 and the general theory of singular integral operators reviewed in the first two paragraphs of this section imply that $M_W P_\Gamma + Q_\Gamma$ is invertible if and only if conditions (i) and (ii) in Theorem 4.1 are fulfilled. Since the two conditions in Theorem 4.1(ii) are equivalent, we reprove in this way the first part of Theorem 3.2.

The formula for $(M_W P_\Gamma + Q_\Gamma)^{-1}$ appearing in Theorem 3.2, may also be obtained from Theorem 4.1 and the general theory referred to above. For this purpose we use formula (4.2), and we insert in this expression the explicit formulas for the factors W_-, W_-^{-1} and W_+^{-1} appearing in Theorem 4.1. We first rewrite (4.2) in the following form:

$$\begin{aligned}((M_W P_\Gamma + Q_\Gamma)^{-1} g)(\lambda) &= \frac{1}{2} g(\lambda) + \frac{1}{2} W(\lambda)^{-1} g(\lambda) \\ &+ \frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta - \lambda} \{W_+(\lambda)^{-1} - W_-(\lambda)\} W_-(\zeta)^{-1} g(\zeta) d\zeta \quad , \quad \lambda \in \Gamma.\end{aligned} \tag{4.3}$$

Next, observe that, by Theorem 4.1,

$$\begin{aligned}\{W_+(\lambda)^{-1} - W_-(\lambda)\} W_-(\zeta)^{-1} &= -(\lambda - \alpha) C(\lambda G^\times - A^\times)^{-1} \pi B \\ &\quad -(\lambda - \alpha) C(\lambda G - A)^{-1} (I - \pi) B \\ &\quad +(\lambda - \alpha) C(\lambda G^\times - A^\times)^{-1} (\zeta - \alpha) \\ &\quad \cdot \pi B C(I - \tau)(\zeta G^\times - A^\times)^{-1} B \\ &\quad +(\zeta - \alpha) C(\lambda G - A)^{-1} (\lambda - \alpha)(I - \pi) \\ &\quad \cdot B C(I - \tau)(\zeta G^\times - A^\times)^{-1} B.\end{aligned} \tag{4.4}$$

The latter formulas can be simplified further. Indeed, note that

$$\begin{aligned}\pi A(I - \tau) = 0 \quad &, \quad \pi G(I - \tau) = 0, \\ (I - \pi) A^\times \tau = 0 \quad &, \quad (I - \pi) G^\times \tau = 0.\end{aligned}$$

Since $(\zeta-\alpha)BC=(\zeta G^{\times}-A^{\times})-(\zeta G-A)$, it follows that

$$\begin{aligned}(\zeta-\alpha)\pi BC(I-\tau) &= \zeta\pi G^{\times}(I-\tau)-\pi A^{\times}(I-\tau)-\zeta\pi G(I-\tau)+\pi A^{(}I-\tau) \\ &= \zeta\pi G^{\times}-\zeta G^{\times}\tau-\pi A^{\times}+A^{\times}\tau \\ &= (A^{\times}-\lambda G^{\times})\tau-\pi(A^{\times}-\zeta G^{\times})-(\zeta-\lambda)G^{\times}\tau.\end{aligned}$$

Also, note that

$$\begin{aligned}(\zeta-\alpha)(I-\pi)BC(I-\tau) &= (I-\pi)(\zeta G^{\times}-A^{\times})-(\zeta G-A)(I-\tau) \\ &= (A-\lambda G)(I-\tau)-(I-\pi)(A^{\times}-\zeta G^{\times})-(\zeta-\lambda)G(I-\tau).\end{aligned}$$

But $G=G^{\times}-BC$ and $G(I-\tau)=(I-\pi)G(I-\pi)$ imply that

$$\begin{aligned}(\zeta-\alpha)(I-\pi)BC(I-\tau) &= (A-\lambda G)(I-\tau)-(I-\pi)(A^{\times}-\zeta G^{\times})-(\zeta-\lambda)(I-\pi)G^{\times}(I-\tau) \\ &\quad +(\zeta-\lambda)(I-\pi)BC(I-\tau)\end{aligned}$$

whence

$$(\lambda-\alpha)(I-\pi)BC(I-\tau)=(A-\lambda G)(I-\tau)-(I-\pi)(A^{\times}-\zeta G^{\times})-(\zeta-\lambda)(I-\pi)G^{\times}(I-\tau).$$

Inserting these expressions into (4.4) yields

$$\begin{aligned}\left\{W_{+}(\lambda)^{-1}-W_{-}(\lambda)\right\}W_{-}(\zeta)^{-1} &= -(\zeta-\alpha)C(\zeta G^{\times}-A^{\times})^{-1}B+(\zeta-\lambda) \\ &\quad \cdot C\tau(\zeta G^{\times}-A^{\times})^{-1}B+(\zeta-\lambda)C(\lambda G-A)^{-1}(I-\pi)B \\ &\quad -(\zeta-\lambda)(\lambda-\alpha)C(\lambda G^{\times}-A^{\times})^{-1}\pi G^{\times}\tau(\zeta G^{\times}-A^{\times})^{-1}B \\ &\quad -(\zeta-\lambda)(\zeta-\alpha)C(\lambda G-A)^{-1} \\ &\quad \cdot(I-\pi)G^{\times}(I-\tau)(\zeta G^{\times}-A^{\times})^{-1}B.\end{aligned} \tag{4.5}$$

Next, use that

$$\begin{aligned}&(\zeta-\lambda)C(\lambda G^{\times}-A^{\times})^{-1}G^{\times}(\zeta G^{\times}-A^{\times})^{-1}B \\ &= C(\lambda G^{\times}-A^{\times})^{-1}\left\{(\zeta G^{\times}-A^{\times})-(\lambda G^{\times}-A^{\times})\right\}(\zeta G^{\times}-A^{\times})^{-1}B \\ &= C(\lambda G^{\times}-A^{\times})^{-1}B-C(\zeta G^{\times}-A^{\times})^{-1}B\end{aligned}$$

and thus rewrite (4.5) as

$$
\begin{aligned}
\left\{W_+(\lambda)^{-1} - W_-(\lambda)\right\} W_-(\zeta)^{-1} &= -(\zeta-\alpha)C(\zeta G^\times - A^\times)^{-1}B + (\zeta-\lambda)C\tau(\zeta G^\times - A^\times)^{-1}B \\
&\quad +(\zeta-\lambda)C(\lambda G - A)^{-1}(I-\pi)B - (\lambda-\alpha)C(\lambda G^\times - A^\times)^{-1}B \\
&\quad +(\lambda-\alpha)C(\zeta G^\times - A^\times)^{-1}B \\
&\quad +(\zeta-\lambda)(\lambda-\alpha)C(\lambda G^\times - A^\times)^{-1}(G^\times - \pi G^\times\tau)(\zeta G^\times - A^\times)^{-1}B \\
&\quad -(\zeta-\lambda)(\zeta-\alpha)C(\lambda G - A)^{-1}(I-\pi)G^\times(I-\tau)(\zeta G^\times - A^\times)^{-1}B \\
&= -(\zeta-\lambda)C(I-\tau)(\zeta G^\times - A^\times)^{-1}B + (\zeta-\lambda)C(\lambda G - A)^{-1}(I-\pi)B \\
&\quad -(\lambda-\alpha)C(\lambda G^\times - A^\times)^{-1}B \\
&\quad +(\zeta-\lambda)(\lambda-\alpha)C(\lambda G^\times - A^\times)^{-1}G^\times(I-\tau)(\zeta G^\times - A^\times)^{-1}B \\
&\quad -(\zeta-\lambda)(\zeta-\alpha)C(\lambda G^\times - A^\times)^{-1}(I-\pi)G^\times(I-\tau)(\zeta G^\times - A^\times)^{-1}B \\
&\quad +(\zeta-\lambda)(\zeta-\alpha)C(\lambda G^\times - A^\times)^{-1}(I-\pi)G^\times(I-\tau)(\zeta G^\times - A^\times)^{-1}B \\
&\quad -(\zeta-\lambda)(\zeta-\alpha)C(\lambda G - A)^{-1}(I-\pi)G^\times(I-\tau)(\zeta G^\times - A^\times)^{-1}B.
\end{aligned}
$$

So,

$$
\begin{aligned}
\{W_+(\lambda)^{-1} - W_-(\lambda)\} W_-(\zeta)^{-1} &= -(\lambda-\alpha)C(\lambda G^\times - A^\times)^{-1}B \\
&\quad -(\zeta-\lambda)C(I-\tau)(\zeta G^\times - A^\times)^{-1}B \\
&\quad +(\zeta-\lambda)C(\lambda G - A)^{-1}(I-\pi)B \\
&\quad -(\zeta-\lambda)^2C(\lambda G^\times - A^\times)^{-1}G^\times(I-\tau)(\zeta G^\times - A^\times)^{-1}B \\
&\quad +(\zeta-\lambda)(\zeta-\alpha)C(\lambda G^\times - A^\times)^{-1}\pi G^\times \\
&\quad \cdot(I-\tau)(\zeta G^\times - A^\times)^{-1}B \\
&\quad +(\zeta-\lambda)\left\{C(\lambda G^\times - A^\times)^{-1} - C(\lambda G - A)^{-1}\right\} \\
&\quad \cdot(\zeta-\alpha)(I-\pi)P^\times G^\times(\zeta G^\times - A^\times)^{-1}B.
\end{aligned} \tag{4.6}
$$

By inserting (4.6) and (1.3) in (4.3) we obtain

$$\begin{aligned}
(M_W P_\Gamma + Q_\Gamma)^{-1} g(\lambda) &= g(\lambda) - \frac{1}{2}(\lambda-\alpha)C(\lambda G^\times - A^\times)^{-1}Bg(\lambda) \\
&\quad -(\lambda-\alpha)C(\lambda G^\times - A^\times)^{-1}B\left(\frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta-\lambda}g(\zeta)d\zeta\right) \\
&\quad +\left\{C(\lambda G^\times - A^\times)^{-1} - C(\lambda G - A)^{-1}\right\}(I-\pi) \\
&\quad \cdot\left(\frac{1}{2\pi i}\int_\Gamma (\zeta-\alpha)P^\times G^\times(\zeta G^\times - A^\times)^{-1}Bg(\zeta)d\zeta\right) \\
&\quad -\frac{1}{2\pi i}\int_\Gamma C(I-\tau)(\zeta G^\times - A^\times)^{-1}Bg(\zeta)d\zeta \\
&\quad +\frac{1}{2\pi i}\int_\Gamma C(\lambda G - A)^{-1}(I-\pi)Bg(\zeta)d\zeta \\
&\quad -\frac{1}{2\pi i}\int_\Gamma (\zeta-\lambda)C(\lambda G^\times - A^\times)^{-1}G^\times(I-\tau)(\zeta G^\times - A^\times)^{-1}Bg(\zeta)d\zeta \\
&\quad +\frac{1}{2\pi i}\int_\Gamma (\zeta-\alpha)C(\lambda G^\times - A^\times)^{-1}\pi G^\times(I-\tau)(\zeta G^\times - A^\times)^{-1}Bg(\zeta)d\zeta.
\end{aligned}$$

Since $(I-\pi)P^\times = I-\pi$ and P_Γ is given by (2.9), we have found the expression for $(M_W P_\Gamma + Q_\Gamma)^{-1}$ appearing in Theorem 3.2 and four additional terms. It remains to show that these surplus terms are equal to zero.

Indeed, it follows from the spectral decomposition theorem (see [GK2], Theorem 2.1) that the third term,

$$\frac{1}{2\pi i}\int_\Gamma C(I-\tau)(\zeta G^\times - A^\times)^{-1}Bg(\zeta)d\zeta$$

$$= \begin{pmatrix} \frac{1}{2\pi i}\int_\Gamma C(\zeta G_1^\times - A_1^\times)^{-1}Bg(\zeta)d\zeta & 0 \\ 0 & 0 \end{pmatrix}$$

Since $g_+(\cdot)$ and $\zeta\Omega_1^\times - I_1^\times$ are analytic on Δ_+ and

$$\frac{1}{2\pi i}\int_\Gamma C(\zeta G_1^\times - A_1^\times)^{-1}Bg_+(\zeta)d\zeta = \frac{1}{2\pi i}\int_\Gamma C(A_1^\times)^{-1}(\zeta\Omega_1^\times - I_1^\times)^{-1}Bg_+(\zeta)d\zeta,$$

it follows by Cauchy's theorem that

$$\frac{1}{2\pi i}\int_\Gamma C(I-\tau)(\zeta G^\times - A^\times)^{-1}Bg_+(\zeta)d\zeta = 0.$$

On the other hand, since $I-\tau = (I-\tau)Q^\times$ we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_\Gamma C(I-\tau)(\zeta G^\times - A^\times)^{-1}Bg_-(\zeta)d\zeta\\ =&\frac{1}{2\pi i}\int_\Gamma C(I-\tau)Q^\times(\zeta G^\times - A^\times)^{-1}Bg_-(\zeta)d\zeta.\end{aligned}$$

But $Q^\times(\zeta G^\times - A^\times)^{-1}B$ is analytic on Δ_- and vanishes at ∞. Hence,

$$\frac{1}{2\pi i}\int_\Gamma C(I-\tau)(\zeta G^\times - A^\times)^{-1}Bg_-(\zeta)d\zeta = 0,$$

so the third term is zero.

Next, since $g_+(\cdot)$ is analytic on Δ_+, it follows by Cauchy's theorem that

$$\frac{1}{2\pi i}\int_\Gamma C(\lambda G - A)^{-1}(I-\pi)Bg_+(\zeta)d\zeta = 0.$$

On the other hand, note the resolvent identity

$$(\lambda G - A)^{-1} = (\zeta G - A)^{-1} + (\zeta - \lambda)(\lambda G - A)^{-1}G(\zeta G - A)^{-1}.$$

Clearly, since $(I-\pi)Bg_-(\cdot) \in$ Im P, we have that

$$\begin{aligned}&\frac{1}{2\pi i}\int_\Gamma C(\lambda G - A)^{-1}(I-\pi)Bg_-(\zeta)d\zeta\\ =&\frac{1}{2\pi i}\int_\Gamma C(\zeta G - A)^{-1}P(I-\pi)Bg_-(\zeta)d\zeta\\ &+C(\lambda G - A)^{-1}G\left(\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)(\zeta G - A)^{-1}P(I-\pi)Bg_-(\zeta d\zeta\right)\end{aligned}$$

But, $(\zeta G - A)^{-1}P$ is analytic on Δ_- and vanishes at ∞. Therefore,

$$\frac{1}{2\pi i}\int_\Gamma C(\zeta G - A)^{-1}P(I-\pi)Bg_-(\zeta)d\zeta = 0$$

and

$$\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)(\zeta G-A)^{-1}P(I-\pi)Bg_-(\zeta)d\zeta=0,$$

whence the fourth term is zero.

Furthermore, the fifth term equals

$$C(\lambda G^\times-A^\times)^{-1}G^\times\left(\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)(I-\tau)(\zeta G^\times-A^\times)^{-1}Bg(\zeta)d\zeta\right).$$

Then, it follows once again from the spectral decomposition theorem (see [GK2], Theorem 2.1) that

$$\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)(I-\tau)(\zeta G^\times-A^\times)^{-1}Bg(\zeta)d\zeta$$

$$=\begin{pmatrix}\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)(\zeta G_1^\times-A_1^\times)^{-1}Bg(\zeta)d\zeta & 0\\ 0 & 0\end{pmatrix}$$

Since $g_+(\cdot)$ and $\zeta\Omega_1^\times-I_1^\times$ are analytic on Δ_+ and

$$\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)(\zeta G_1^\times-A_1^\times)^{-1}Bg_+(\zeta)d\zeta=\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)(A_1^\times)^{-1}(\zeta\Omega_1^\times-I_1^\times)^{-1}Bg_+(\zeta)d\zeta,$$

it follows by Cauchy's theorem that

$$\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)(I-\tau)(\zeta G^\times-A^\times)^{-1}Bg_+(\zeta)d\zeta=0.$$

On the other hand, since $I-\tau=(I-\tau)Q^\times$ we have

$$\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)C(\lambda G^\times-A^\times)^{-1}G^\times(I-\tau)Q^\times(\zeta G^\times-A^\times)^{-1}Bg_-(\zeta)d\zeta$$

$$=C(\lambda G^\times-A^\times)^{-1}G^\times(I-\tau)\left(\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)Q^\times(\zeta G^\times-A^\times)^{-1}Bg_-(\zeta)d\zeta\right).$$

But, $Q^\times(\zeta G^\times-A^\times)^{-1}B$ is analytic on Δ_- and vanishes at ∞. Thus,

$$\frac{1}{2\pi i}\int_\Gamma(\zeta-\lambda)Q^\times(\zeta G^\times-A^\times)^{-1}Bg_-(\zeta)d\zeta=0,$$

so the fifth term is zero.

Analogously, one can show that the sixth term is zero.

The Fredholm properties of $M_W P_\Gamma + Q_\Gamma$ may also be derived via the factorization method. This one can do by constructing a non-canonical factorization via the state space method (see [BGK2], [GKR] and [Gr1]). However, the formulas are much more complicated than those in Theorem 4.1, and hence for the Fredholm case the approach employed in Section 3 via input/output systems is more direct.

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LIST OF SYMBOLS

$\subset$	subset
$\mathbb{C}$	set of complex numbers
Γ	Cauchy contour in $\mathbb{C}$
Δ_+	inner domain of Γ
Δ_-	outer domain of Γ
Ker T	kernel (nullspace) of the operator T
Im T	image (range) of the operator T
T^{-1}	inverse of the operator T
T^+	generalized inverse of the operator T
$T\mid_X$	restriction of the operator T to the set X
ind T	index of the operator T
$I_X,\ I_m$	identity operator on X, $m \times m$ identity matrix
$X \oplus Y$	direct sum of the linear spaces X and Y
$\mathbb{C}^n$	Unitary space of dimension n over the field $\mathbb{C}$
$\mathcal{L}(X)$	class of bounded linear operators on a space X
$L_2^m(\Gamma)$	space of $\mathbb{C}^m$ -valued square-integrable functions on Γ
$\langle x, y\rangle$	inner product of x and y

SUMMARY

In this dissertation we studied the state space method for solving singular integral equations explicitly from the representation of a matrix function in realization form. A rational matrix function W, which is analytic and invertible at infinity, may be represented in the form

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \tag{1}$$

where A is a $n \times n$ square matrix, say, B and C are $n \times m$ and $m \times n$ matrices, respectively, and D is an invertible $m \times m$ matrix. The process of constructing explicit formulas for the generalized inverse (resp., inverse) of a singular integral operator with rational symbol is well-known for rational matrix functions in the form (1). However, in our work, we have concentrated on the case where W does not have these properties at infinity and has a realization of the form

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \tag{2}$$

where A, B, C, and D are as above and G is of the same order as A. In the main results in Chapter 2, we give necessary and sufficient conditions for the existence of an inverse (resp., generalized inverse) of a singular integral operator with rational symbol. In addition, we have shown that the explicit formulas (in realized form (2)) for the generalized inverse (resp., inverse) may be written in terms of the matrices A, G, B, C and D and various other matrices derived from them. In this chapter, we made extensive use of the Riesz theory associated with the decomposition of the spectrum of the pencil $\lambda G - A$. Finally, we review the factorization method.