A description of entry level tertiary students' mathematical achievement

(Towards an analysis of student texts)

Mark Solomon Jacobs

University of Western Cape

A thesis submitted in fulfilment of the requirements for the degree of Ph. D in the School of Mathematics and Science Education, University of the Western Cape.

November 2006

Keywords

Mathematics

Cognition

Achievement

Procedural knowledge

Conceptual knowledge

Assessment

Semiotics

Taxonomy

Text

Profile

ABSTRACT

A DESCRIPTION OF ENTRY LEVEL TERTIARY STUDENTS' MATHEMATICAL ACHIEVEMENT

(Towards an analysis of student texts)

M.S. Jacobs

PhD Thesis, School of Mathematics and Science Education, University of the Western Cape.

This research provides insights into the mathematical achievement of a cohort of tertiary mathematics students. The context for the study is an entry level mathematics course, set in an engineering programme at a tertiary institution, the Cape Peninsula University of Technology (CPUT). The participant members are first year, first semester students. The materials for the inquiry are student produced test scores and examination scripts taken from their entry level course.

The characteristics of the mathematical achievement of the cohort concern the understanding of procedural and conceptual knowledge and problem solving abilities in mathematics. The facility with mathematics is another central concern of this study as it forms the dominant aspect of mathematical achievement which is accessible to research in the materials employed for the study. This research also develops a mathematical achievement profile for individual members of the cohort. The methodology makes use of content - and textual analytic methods for profiling the students. When viewed across the different kinds of profiling techniques adopted, this study suggests that these techniques complement one another: the profiles developed provide a cohesive and complementary overview of the achievement of the cohort.

This study challenges perceptions that responses to constructed response questions offer little information about the mathematical knowledge of students. This study investigates the possibilities of providing a bridge between the assessment of students by means of tests scores and a taxonomy of mathematical objectives, on the one hand, and the critical analysis of student produced texts, on the other. Findings suggest that diagnostic uses of paper and pencil tests can be revealing about the achievement of students. The wide range of responses to test items revealed a distribution of incompleteness in terms of employing algorithmic techniques. This research revealed that even in cases of wrong solutions, participant members' responses were reasonable, meaningful, clear and logical. The participants responded in many ways as predicted by the research literature. Evidence could be found for the use of child methods; poor use of reflective abstraction for coordination; accessing the wrong cognitive frames; not seeing the underlying structure of the mathematics and treating letters as objects.

Findings suggest that the use of a textual analytic method, which led the creation of critical indicators as a way of sign-posting events, enhanced the achievement profile of the students.

DECLARATION

I declare that *A description of entry level tertiary students' mathematical achievement (Towards an analysis of student texts)* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Mark Solomon Jacobs November 2006

Signed …………………………………………..

DEDICATION

This thesis is dedicated to the memory of my mother, Olive Anne Jacobs, who believed in the goodness of people.

It is also dedicated to my English teacher, René Kapp, who opened my eyes to the world through her love for learning.

Finally, this thesis is dedicated to all my students, from whom I have learnt much.

ACKNOWLEDGEMENTS

I wish to acknowledge the many colleagues, friends and family who have contributed to the completion of this thesis.

Professor Cyril Julie encouraged me over many years to further my educational qualifications and proposed me as a candidate for the Grassmate Project (Norway/ South Africa) based at the University of The Western Cape, South Africa. He has been an inspiration and I thank him for the support that he has provided. He was a coordinator along with Dr. Oyvind Mikalsen of the Grassmate Project and I am grateful to them for the privilege to have been a student on that project.

I would also like to acknowledge the supervising team assigned to me by the project, namely Professor Trygve Breiteig, Associate Professor Ole Einar Torkildsen and Professor Julie.

The other supervisors and the students helped to create an atmosphere of collegiality, mutual support and intellectual stimulation which contributed to my development and the writing of this thesis and I would like to thank all of them.

Special word of thanks goes to Dr Lorna Holtman (UWC and Grassmate) for dedicated professional and administrative support and also to her staff at PET who helped me through crucial technical aspects of the thesis.

A personal friend and colleague with impeccable credentials, Arthur (Maseo) from New Jersey, USA, has assisted me with detailed critique and analysis of my thoughts and work over the length of the dissertation and I salute him here.

I wish to acknowledge the Dean of Engineering, Dr O. Franks and senior management at my institution, the Cape Peninsula University of Technology, for giving me the time needed to complete my thesis and for providing me with encouragement and support.

I thank my sister, Isobel, and all her family for their encouragement and support.

Finally, a warm thank you to my children Julia, Rebecca and Anna, and my wife Jane Coombe for providing me with love and support to embark on and complete this thesis.

Contents

1. Chapter 1: Introduction

This study is an epistemological and textual- analytic inquiry into the mathematical achievement of entry level tertiary mathematics students. It is situated in the discipline of mathematics education. Mathematical achievement concerns the extent of knowledge about, facility with and use of mathematics, within a given context, in terms of stated measuring instruments. Knowledge about mathematics in this study will mean mainly conceptual and procedural knowledge, but also knowledge used in solving problems. These terms will be elaborated on in the literature review in chapter two. The facility with mathematics is another central concern of this study as it forms the dominant aspect of mathematical achievement which is available in the materials employed for the study. By facility I mean the ease of use of mathematical symbols and expressions and employing mathematical language tools in order to engage in and communicate with mathematics.

The techniques required to equip students with good facility in mathematics are a central aspect of teaching and learning in South African mathematics classrooms. This is largely the case since it is the dominant aspect of the teaching and learning of mathematics which is practiced and assessed in South African schools (Taylor and Vinjevold, 1999; Long, 1998, p. 18). These schools form the major feeder for the course under study and therefore the central features of these schools, in relation to mathematics, inform much of the context for the study. This is elaborated on below.

1.1 Mathematics in the South African school context

The experiences of students at tertiary level, particularly during the first year, are significantly informed by their experiences at school. At the heart of learning mathematics in the context of South African schools is the interplay between procedural knowledge and conceptual knowledge, the way in which they inform and reinforce each other, and the debate about the order in which they happen. What is referred to as school mathematics informs the study integrally. This is mainly because what the participants have learnt in school mathematics is often revealed through the texts under review in this study. It is also the case that school mathematics forms part of the learnt content which is embedded in the tertiary course under review. Though I do not focus on "school" as a notion in this thesis, some of its properties have informed characteristics expected in the members of the cohort. Some properties are:

- Goal direction
- Knowledge transmission
- Organised form of work
- Use of technology (Romberg, 2003)

It is probably true, though, that the phrase "school mathematics" conjures up notions of "an absolutist social form". The subject school mathematics, bound by a specific interpretation of the discipline, mathematics, and set within a structured environment known as a school, is geared to function in certain "absolutist" ways. These ways include: control, organisation, routine, drill and practice; hard work, exercising the mind; rule following procedures, doing the mathematics that others have done, and so on. For the learner, the pattern is: a passive absorber of knowledge, in small manageable bits to be stored for later retrieval. For the teacher, the pattern is: to teach, transmit knowledge, interpret the syllabus to fit into the classroom, adhere to time constrains and examination requirements maintain order and control and manage the production line (Taylor and Vinjevold, 1999; Julie, 1992). The awareness of the existence of these expected relations within the subject, its students and teacher forms a necessary backdrop to the textual analysis: it is part of the inference-rich context of the text

Built on a thin foundation of procedural presentation (as opposed to development), in standard textbook mode, through a predominantly exposition-type teaching style, the stereotypical South African classroom is a preparatory camp for testing in examinations, including the final school leaving examinations. In this way, a whole system is geared to channel masses of new recruits through algorithmic-dominant mathematics. For many students this form of learning leads to a stale and ultimately devoid-of- meaning kind of mathematics. This is a typical picture of the kind of preparation many students entering tertiary institutions have had. And what is often forgotten is that many of these students are the *survivors* of the system; these students have come through, not because of, but in spite of their experiences (Julie, 1992; Taylor and Vinjevold, 1999; Khuzwayo, 2005).

In a similar vein, the TIMMS[1] reports have highlighted conditions in South African schools pertaining to achievement in mathematics. The 1996 report drew attention to the conditions under which pupils are expected to learn mathematics and other subjects in schools and the resultant effects. The HSRC's media response, which quotes a university researcher, highlighted the plight of South African school mathematics education:

> *The number of pupils failing mathematics at school is unacceptably high in South Africa… The completely unacceptable situation regarding mathematics that currently exists in South Africa can only be solved if the study situation of pupils improves dramatically. Pupils' emotions, habits and attitudes in mathematics, as well as the way in which they perceive the subject, their teachers and the teaching of the*

[1] Third International Mathematics and Science Study, implemented by the international Association for the Evaluation of Educational Achievement

subject, the class atmosphere and their home circumstances, play a significant role in their eventual achievement in mathematics. Due to their limited experiences pupils from non-stimulating environments are frequently at a disadvantage, they struggle in the subject and they learn more slowly. Language problems caused by being taught in a language other that the pupil's mother tongue contribute to anxiety in mathematics and undermine achievement in the subject. Pupils experience mathematics anxiety when they do not master the limited technical language and are afraid to ask questions or discuss their problems with their teachers (HSRC, 1997).

Not much has changed as revealed in the follow up TIMSS study. The HSRC report (Reddy, 2006) on South Africa's performance at TIMSS[2] 2003 has relevance for this study (see Chapter Four) since the HSRC also gave the same test to the grade nine students of that year. The entry level students would mainly have been in grade nine the year of the TIMMS/HSRC study. The report is instructive in its findings about the students, the teachers as well as the system:

Internationally, most teachers had at least a four year degree qualification. The comparison with the international cadre of TIMSS teachers illustrates the SA mathematics teachers are among the least qualified (p. 10)

It found that in the test areas: content domain and the cognitive domain, less than 30% of the tested grade eights scored correctly, and found overall that the knowledge and understanding of mathematics was less than 30%. The authors express their "disappointment" that the grade nine learners who were given the same test managed to beat the grade eights by only 20 points on average.

[2] For a discussion of the philosophy of TIMSS see chapter two

The emphasis on procedural knowledge and on developing good algorithmic skill is entrenched in the South African school system (Adler, 1998; Engelbrecht, et al., 2005). Students entering tertiary institutions usually represent a mixture of abilities in terms of mathematical achievement. There are those students who have a good, but, still procedurally dominant, preparation in mathematics and those who have not. Of the ones who have been prepared well to take the final examination, which is the dominant focus, some have lent credence to notions that expositionary approaches nonetheless can produce students with good quality mathematical achievement. In other words, if the context is affirming and the classroom situation is supportive, an emphasis on procedural knowledge and a development of procedural proficiency may have intended and unintended positive consequences for conceptual development and understanding. This aspect is discussed in chapter two (see Sfard, 1991). Vast numbers of schools which are well resourced, including having teachers who are well qualified and who, in the main have been themselves exposed to this same manner of teaching at school and university, practice in this way. They are often driven by expectations of output in school leaving examinations. Here we might have some tangible examples of the notion that the *templates-driven* approach paves the way for a deeper *object-mediated* understanding later on (Sfard, 2000; see Chapter Two of this thesis). The effective use of textbooks in this regard may act as a support to the mathematical development of students:

> *Textbooks provide a limited content expertise for a topic, plus a logical sequencing, and a variety of pedagogical supports: activities, test items, and sometimes summaries of expected student difficulties and misconceptions. They may help make the curriculum content oriented and comprehensive while allowing more effective use of teacher time than might occur on average without them. In the third world textbooks are significantly more important and are often the only books that students encounter in their studies (Venezky, 1992, p. 442).*

Thus the preference for procedural approaches to mathematics is entrenched in the South African school system. The implication for tertiary institutions is that they have to find ways of bridging the various procedural and conceptual lags that the students may exhibit to the kind of skills that they require for their programmes. These are often done through what has become known as “access” policies. These programmes are mainly curriculum responses geared to induct students perceived to be in need of programme support into tertiary study.

The next section looks at the notion of a mathematical achievement profile. The idea is to home in on the individual students in order to understand their individual mathematical achievement in the context of the background sketched in this section.

1.2 Mathematical achievement profile

Linked to the study of the mathematical achievement of tertiary students is the notion of a mathematical achievement profile of a student. By a profile is meant “the extent to which a person exhibits various tested characteristics” (Collins, 2001). The tested characteristics which will be discussed in this study concern knowledge and understanding of mathematical concepts, procedures and limited forms of problem solving abilities. These are the aspects of mathematical achievement which I chose to interrogate. Methods are introduced which will be used to analyse the chosen evidence of the mathematical achievement of students in order to point the way to determining their group and individual profiles.

This study investigates the possibilities of providing a bridge between the assessment of students by means of tests scores and a taxonomy of mathematical objectives, on the one hand, and the critical analysis of student produced texts, on the other.

1.3 Goals of the study and research questions

In deciding on the goals of the thesis I was mindful of a number of things. I was keen to describe the mathematical achievement of entry level tertiary students at my institution, the Cape Peninsula University of Technology. One of the practical benefits intended is that these results would inform the institution's access policies as raised in section 1.1.

I was aware that the mathematics knowledge that the students had on arrival was learnt mainly at our South African schools. In light of this knowledge, I anticipated that the character of their mathematics would pervade the entry-level course, would influence the flow of the course, their engagement with it and their results. Largely on account of the reality that the didactics at tertiary entry level is so similar to what is in use in the old system[3] in South African schools, I opted to assess their mathematical achievement within the confines of the standard examination format used at my tertiary institution.

So it is in the analysis of what they write and specifically how they write in these tests and examinations that I explored questions about their mathematical achievement. I had done a number of pre-trials with the examination scripts of earlier students in which I interrogated their scripts using semiotics and other sign systems. I found that the use of these methods enhanced my findings. On that account I decided to make use of semiotics to do a textual analysis of the student scripts in addition to analysing their test scores and performing a standard content analysis of their scripts. I have called this three-in-one method *triangulation*.

This is how I derived the goals of this thesis. The goals of the thesis are:

[3] South Africa has embarked on an outcomes – based education since 2001, away from a more traditional drill-and-practice approach. Current (2006) tertiary students still came through the traditional schooling system and curriculum.

- To analyse student responses to constructed-response questions at tertiary entry level.
- To describe the mathematical achievement profile of a cohort of tertiary students.
- To develop a language of interpretation of student produced texts which will extend current practices of analysing such texts.

In an attempt to achieve these goals I pursue a number of research questions as follows:

- How do tertiary entry level students respond to constructed response questions?
- What is the mathematical achievement profile of such students?
- What additional details does a textual analysis of the student produced scripts reveal about such students?

1.4 The structure of the thesis

The thesis has been structured into chapters as follows:

Chapter One is the introduction. This deals with the main focus of the study, which is to describe the mathematical achievement of tertiary students. The notion of a mathematical achievement profile is introduced as a novel concept. The chapter outlines the goals of the thesis and the research questions.

Chapter Two provides a review of literature related to learning and mathematics. The review situates the description of tertiary students' mathematical achievement within an epistemological, psychological and sociological context and examines what it means to learn and know mathematics. In the review it is argued that learners possess the intellectual faculty to learn mathematics. An overview of the nature of mathematics is given in the context of the interplay between conceptual and procedural knowledge. This is followed by a review of the cognitive and pedagogic properties which inform the derivation of taxonomies of educational objectives, such as that of Bloom, et al. (1956). An overview is then given of assessment and its place in determining the achievement of students. In conclusion I consider the contribution of this study to discussions about mathematical achievement, specifically at tertiary level.

In Chapter Three I describe the creation of a theoretical framework. This framework is used to analyse students' tests and scripts. These texts are interrogated, using the framework, to determine students' understanding of concepts and procedures in mathematics. Firstly, a content analysis framework for the study is derived. This provides the rationale for making use of a taxonomy for classifying the cognitive tasks the cohort performed in the assessments and for analysing the detailed breakdown of the test items. Secondly, a textual analytic framework is created in order to do a semiotic analysis of individual scripts of selected participant members of the cohort.

In Chapter Four I describe the research methodology, design and data collection. Firstly, a taxonomy is designed based on the literature review of Chapter Two and theoretical discussions in the previous chapter. Secondly, I outline the particular characteristics of the participants and also of the different data. Thirdly, I give a detailed breakdown of the test items of the various tests. Fourthly, I follow this up with the construction of the tables for the different parts of the analysis, namely, the taxonomy table for test scores and test items, the content analysis table for test items and the textual analysis table for individual scripts. Lastly, I end the chapter with the creation of a competence scale.

Chapter Five describes the data interpretation and analysis. I interpret the data by using the theoretical frameworks outlined in chapter three. Every test item is analysed in great detail and general trends are described. My use of textual analysis complements my other methods. As a result, I am able to realise the creation of a mathematics achievement profile.

Chapter Six provides concluding remarks about the study. Based on my analysis of the data, I elaborate responses to the study's research questions and indicate the limitations of the investigation. Possibilities for future research are outlined as well as implications for pedagogy. A summary is provided at the end.

2. Chapter 2: Literature Review

2.1 Introduction

This review will situate the description of tertiary students' mathematical achievement within an epistemological, psychological and textual-analytic context. In the review it will be argued that while learners acquire the intellectual knowledge to learn mathematics at an early age, the advent of school mathematics often leads to a loss or submergence of that knowledge, depending on the experience of the learners. This result appears to reside in part in the nature of school mathematics. Key terms in mathematical achievement which are reviewed in this study are factual, conceptual and procedural knowledge. So, too, are the terms assessment and symbolic language. A review of taxonomies and their rationale is included. This acts as a lead in to the taxonomy created for the mathematical content of the cohort in Chapter Four. It is intended that this review will indicate the contribution of this study to discussions about assessment of mathematical achievement in the research community, specifically at tertiary level. This review will also indicate the intended contribution of this study to discussions about the role of textual analysis in assessment in the mathematics education research community.

2.2 Conceptual and procedural knowledge in mathematics

This section reviews the input of researchers into ideas about conceptual and procedural knowledge. Procedural and conceptual knowledge is knowledge about procedures and concepts. The derivation of these procedures and concepts in mathematics is at the outset a rich area for research in the field of psychology and is explored here in some depth. Also, the nature of beginning mathematics is explored in its relation to the acquisition of these concepts and procedures. The use of symbols as a means of communication and expression of ideas in mathematics is an additional aspect in the development of procedural and conceptual knowledge in mathematics.

It has been argued that concepts are gradually formed through the engagement of thoughts and actions on an external world by the child. The child, it is argued, builds or organises ‘schemas’ in the mind as a result of these thoughts and actions. Piaget (1973) states that

> *before all language, at the purely sensorimotor level, actions are susceptible to repetition and then to generalization thus building up what could be called assimilation schemes” (pp. 79-80)*

These schemes may inter-lead and connect in increasingly complex ways and become interconnected structures through the mind’s tendency to organise (Shuell, 2001). Through two key terms, assimilation and accommodation, Piagetian psychologists, following Piaget, describe how new experiences are incorporated into the mind’s structures: where an experience can fit into previously existing structures it is assimilated; where it cannot be assimilated because the structures are incomplete, it is accommodated. Thus the accommodation aspect of learning leads to extensions of cognitive structures. This is how, according to this school of thought, the child adapts to her environment: “Where a new idea does not fit into the assimilation framework of that child’s mind, an accommodation is made” (Gelman and Gallistel, 1978).

Thus, it is argued, very young children at play develop intellectually and assimilate and accommodate various experiences, building up their knowledge of the world around them (Piaget, 1960). Freudenthal (1991) considers, in this vein, the manner in which children appear to acquire number and the means to express it, as an “astonishing feature”, and especially so since this link is seen as the first “algorithm of mathematical character” (p. 6).

His approach is more tied to a notion of common sense, that is, what has developed over time within a community (or all of humanity) and is accepted without the need for reasons, outside of theoretical considerations. The theoretical considerations follow. He suggests that Piaget's work on number stems less from common sense notions about number than from a theoretical sense of number which became largely devoid of common sense once it became formalised. He claims that children spontaneously develop common sense notions about number in their interaction with their reality. These two opposing views are played out in pedagogical terrains and this study is an example of the effects of the first view on conceptual development.

Children's development of conceptual thinking, it is claimed, begins at an early age, even involving that which is not visible: "children's concepts incorporate non-perceptual elements from a young age" (Gelman, 2006). Elsewhere it is claimed that cognitive competencies relevant to children's learning of mathematics develop from infancy onwards (Lipton and Spelke, 2003; Xu and Spelke, 2000).

Thus, researchers argue, as children grow up, interact with their surroundings, play and learn to talk; they acquire the mental means to communicate, think about and act on their surroundings. This kind of mental development is often referred to as cognition, that is, mental processes such as abstraction and generalizations. For my purposes, cognition is seen to underpin the learning of mathematics, within which I will situate the formation of conceptual and procedural knowledge.

Mathematics is seen as a field or discipline where "a peculiar mental activity can be exercised most adequately and most efficiently…creating certainty (it would seem) beyond compare" (Freudenthal, 1991, p. 2). Perhaps more specifically, mathematics can be thought of as "…the science of order, patterns, structure and logical relationships" (Devlin, 2004, p.11). That is, before the use of symbols and notations, mathematics is about thinking about patterns: patterns in nature, purely abstract patterns, patterns of reasoning, patterns of shape, real or imagined, and so on. In this sense, it is argued, children, who develop the skills to make patterns, structure and order relationships and so on, develop the mental constructs to learn mathematics. For Gattegno (1970) the many activities of children are algebraic in essence because they involve the organisation of mental operations.

In other words, when mathematics is viewed as being broader than what is contained in the standard school textbook, we can find examples of mathematical thinking in children from an early age. Moreover, it can be said that most children from an early age engage their minds in the act of mathematising[4] . Researchers claim that mathematics is involved Even for language acquisition and use: "the child's mastery of grammar can only be adequately described in terms of mathematical operations, and this mastery is not derived by imitation" (Wheeler, 1982, pp.45-46).

There is considered speculation that "the feature of our brain that enables us to use language is the same feature that makes it possible for us to do mathematics" (Devlin, 2004, p.70). This is consistent with Gattegno's (1970) findings. It is also of significance that distinctions be made between concepts such as number and processes such as reasoning. Similarly, distinctions can be made between factual recall and procedural fluency. This is because conflating these concepts may blur attempts to understand what it is that went wrong when things go wrong. Thus the claims of Butterworth (1999) inform this study and are considered in this review:

> *Common sense and some expert opinion suggest that our numerical abilities are closely bound up with our linguistic abilities, with our ability to memorize…and with*

[4] that is, engaging in thought processes and activities which can be described as mathematical in nature

our ability to reason. What these patients show is the exact opposite. In the brain, these abilities are separable (p.180).

He shows, experimentally, through a process of *double dissociation*[5], in which the independence of mental processes is determined, that, among other details, arithmetic facts and arithmetic procedures occupy different circuits in the brain. He is not alone in this. Others have claimed similarly that a system necessary for the acquisition of higher level mathematical skills develops after the number representation system. Some posit that this second more sophisticated system may be dependent on the acquisition of language skills (Ansari et al., 2003; Wynn, 1990; Dehaene, et al. 1999).

The notion of abstractness is another factor essential to ideas about mathematics. Mathematics can be seen as a process of thinking which involves the building and application of abstract, logically connected, networks of ideas (Gelman, 2006). The need to move from the concrete object to an abstraction of it in the mind, then to an expressed symbolic form, for example, in diagrammatic, verbal or written text, is at the core of what has been called "mathematization" (Freudenthal, 1991). This is linked to the formation of conceptual knowledge. A typical example would be when children are exploring relationships in a known context and then go on to find mathematical ways of describing what they experience. The "pizza toppings" task is one such example: children are asked to find out how many different pizzas they can make if they have four different toppings available. A mathematical goal would be for them to grapple with ideas about combinatorics. There will also be instances when children may be required to explore relationships between abstract things, without reference to concrete entities (Gelman, 2006; Devlin, 2004).

[5] A neuropsychological term: good at X but bad at Y vs good at Y but bad at X

There are suggestions, then, that notions of abstraction begin to be present in the early child as evidenced by the child's number sense and use of language grammar. Now as the child engages with his/her world in increasingly complex levels, these thinking processes develop in increasing complexity. Associated with this, in line with Butterworth's (1999) observations, it would appear that children's ability to abstract in a mathematical sense increases in complexity much as their ability to use language does. There is thus also the possibility (not explored in this study) that processes which contribute positively to the child's increasingly complex use of language may have parallels in mathematical development. Mathematics involves thinking about abstract objects. Thinking in this way increases in difficulty the wider the gap between these objects in the mind and any obvious relation to real objects. In Devlin's (2004) hierarchy this widening gap between real and visualised objects represents the highest level of abstraction.

The use of symbolism in mathematical expression is a necessary development in the growth and use of mathematics for the child, in particular for her growth in procedural and conceptual knowledge in this domain. This symbolic language is at first accommodated in the child's growing experiences as she adapts to the awareness of number symbols in her world. Schooling accelerates this acquisition of mathematical symbolism. Standard mathematical expression (for example, writing solutions to problems) uses this symbolic language, which is one of its key features. The slow but definite introduction of symbolic mathematical expressions into children's schooling is the beginning of this induction programme which increases in complexity the higher the mathematics they study. The linking of number symbols to pre-concepts about number, through language, is by then already in place. So, too are many of the operations which flow from this, such as addition and subtraction. There is general acceptance of this growth in mathematical expression:

> *The complexity and abstraction of most mathematical patterns make anything other than symbolic notation prohibitively cumbersome to use. And so the development of mathematics has involved a steady increase in the use of abstract notation (Devlin, 2004, p. 12)*

Now, the derivation of *concepts,* which arise through the development of mathematical ideas, many of them informed initially by the manipulation of objects in the real world, finds expression through mathematical symbolism. The requirement to conform means that such knowledge is assimilated (when there is cognitive conflict) or accommodated within the schemas of the learner and mediated by social pressure, in the developmental framework of Piaget (1960) as discussed earlier.

Freudenthal (1991) asserts that before the formation of concepts there is the formation of mental objects and sites the history of the development of the function concept as an example. Mental objects are first formed in reference to objects in reality (or later, existing concepts). The use of mental operations lead to the concept formation variously, depending on context and individuals involved. For evidence that this concept formation is happening Freudenthal suggest looking at use and expression of the object/concept. This is the sense in which Hiebert and Lefevre (1986) talk about conceptual and procedural knowledge. Hiebert and Lefevre (1986) characterize conceptual knowledge as “knowledge that is rich in relationships” (p. 3). Conceptual knowledge is developed through building relationships between new and existing pieces of knowledge and among the existing pieces of knowledge. These relate to the use of schemas and structures of Piaget (1963, 1970). Developing conceptual knowledge means building those relationships and having the awareness of those processes, that is, of how the concepts are formed (Skemp, 1987). Having conceptual knowledge is therefore not a static fact but comes about as a result of a dynamic interactive process of engaging with existing concepts and relationships and assimilating and accommodating new ideas and new thoughts in a constant flow. These mental processes involve the use of reasoning and the development of logical thinking. These processes and growing forms of mental engagement come together in notions such as "understanding". Thus to have conceptual understanding involves having conceptual knowledge, engaging in logical processes which involve reasoning, and building relationships. In this sense knowing what the concepts are is only one part of this conceptual knowledge; there is also what they mean and how they work and particularly how they relate to other concepts (Skemp, 1987).

By procedural knowledge is usually meant knowledge about definitive and specific procedures which are required in the solution of questions and problems. These procedures tend to involve the use of algorithms or rules and usually are expressed through the symbolic or formal representation system of mathematics, as discussed above (Hiebert and Lefevre, 1986, p. 6). As with conceptual understanding, having procedural understanding is a dynamic process. It involves knowing what procedures are required, how they are required and where they are required. It also involves making adaptations to a procedure if needed and making choices between procedures when confronted with a question which requires a solution in the form of a procedure or when solving a problem.

Star (2004) challenges the comfort which exists in the research community literature around the clear distinction between conceptual and procedural knowledge. He notes concern that the definitions of Hiebert and Lefevre (1986) are built up from studies conducted in elementary school, implying that the framework works on account of that context. Also, he notes that the assessment of these two kinds of knowledge is carried out in different ways. The implications are that conceptual knowledge is more complex and multi-faceted. However, Star suggests that the treatment of procedural knowledge is inadequate. Star's use of the expression "'conceptual' knowledge about procedures" (p. 6) underscores concerns that a rigid division between concepts and procedures does not explain his findings that students can be differentiated in terms of their flexibility with and knowledge about various procedures to solve a linear equation.

Similar discomfort has been expressed in the work of Kilpatrick, et al. (2001, p. 122):

> *Procedural fluency and conceptual understanding are often seen as competing for attention in school mathematics. But pitting skill against understanding creates a false dichotomy. As we noted earlier, the two are interwoven. Understanding makes learning skills easier, less susceptible to common errors, and less prone to forgetting. By the same token, a certain level of skill is required to learn many mathematical concepts with understanding, and using procedures can help strengthen and develop that understanding. For example, it is difficult for students to understand multidigit calculations if they have not attained some reasonable level of skill in single-digit calculations. On the other hand, once students have learned procedures without understanding, it can be difficult to get them to engage in activities to help them understand the reasons underlying the procedure.*

"Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121).

The interconnectedness between procedural proficiency and conceptual understanding, that knowledge of and skill in the one impinges on the other in a constantly forward and backward shifting manner, is an example of the interwoven nature of the framework of Kilpatrick, et al. (2001). This theme of the link between concept knowledge and procedural knowledge has been taken up elsewhere (for example, Sfard, 1991). In some ways the "mutual constitution" of a symbol and its object in mathematics as a form of concept formation is a pre-cursor to conceptual understanding (Sfard, 2000, p.47). The crux of the dilemma in acquiring procedural proficiency, both as a result of conceptual understanding and as a lead in towards it, is the manner in which it is acquired, the uses to which these procedures are put, and the environment in which these procedures are developed. Much of the concern around this relates *directly* to the central focus of this study. That focus is: what is the profile of the mathematical achievement of a cohort of students. This kind of mathematical achievement is often attained *in the context of assessment that emphasises procedural fluency over conceptual understanding.* In this study, the recognition that there is a dynamic interplay between these two concepts and that the one does not have primacy over the other is an important factor in the analysis of the cohort's mathematical achievement. Sfard's (2000) description of a *template- driven* approach to mathematics to one which is object- mediated is consistent with this, where the templates driven phase is akin to the notion of "pre-concept" use (pp. 57-58). The templates-driven phase is seen as a phase when the outward, structural features of a concept or procedure are adopted, often similar to existing concepts or procedures. At that stage there is no expectation that the recipients, such as a class of learners, will imbue the new concept or procedure with meaning and understanding. Sfard talks about creating the "semantic space" which will be filled over time (p. 58). In other words, as the learners become proficient and gain understanding about the place of the procedure or concept in their body of knowledge about procedures and concepts, they change in their use of the procedure or concept. They begin to adopt an object-mediated approach whenever using the new concept or procedure. The object- mediated phase involves the use of real or mental objects through which mathematical engagement is conducted. Where the "object" is a mental one, its creation in order to focus on it leads to the creation of "name and symbol" – in this sense they are mutually constitutive (p. 48). Thus the notion of concept formation for the individual, in an active sense entails a process which starts with concept naming, and what has been termed pre-conceptual uses, and builds to eventual conceptual maturity as envisaged in the use of the concept by the individual.

The power of mathematics to abstract is at the same time a source for much anxiety, uncertainty and unknowing within a pedagogic context. The link between the thought (the mental object) and the object in reality is mediated through the use of *symbols* (diagram, gestures, written text) (Chandler, 1995). The process of *generating* one's own system of symbol construction as a result of constructing knowledge and experience has a unique place in building and cementing individual mathematical thought. This is largely the constructivist and interactionist view points (Sfard, 2000). The reality for school learners, especially once they begin algebra, is that a symbolic system already exist for them and is imposed through the implementation of the curriculum: "the structural interpretation of symbols is imposed by the way in which the symbols are used…in most cases the act of introduction is performed by a teacher (p. 55).

One of the key questions, then, is: how do learners understand meaning in this context? An example would be the mathematical expression

$$y = x^2 + 2$$

This symbolic collection contains a number of meanings for students of mathematics; possibly the more meanings, the more mathematics the reader of the text knows. Possibly the first use of this expression that many school learners encounter is under the heading: substitution. Here learners learn to determine values for the *y* symbol or letter when values for the *x* symbol are known. Better still they may derive this formula from a table of given and derived values written on a worksheet or blackboard: a kind of "introduction to algebra" lesson. To move from those beginnings (the templates) to having spatial patterns in the mind in which graphs are moving about a plane (the graph as an object), for example, is a development of conceptual understanding.

Tall (2004), in casting his eye over initial theories about mathematical development from babies to maturity talks about the transition early learners have to make when switching from whole numbers to fractions, for example:

> *For instance the transition from whole numbers to fractions is highly complex; the embodied representation of a number as a physical collection of counters must be replaced by a sharing of an object or a collection of objects into equal parts and selected a number of them…In experiencing whole numbers, the child will encounter the idea that each number has a next number and there are none in between. This "met-before" can cause confusion with fractions wherein there is no next fraction, and two fractions always have many others in between (p. 286).*

It is thus, when making the transition to algebra and the use of letters in place of number-symbols, that an arithmetic inspired "bridge" is constructed. This process renders the acquisition of this new symbol system dependent, to start with, on the rules, regulations and *discourse* of the other system. In fact, algebra is usually referred to as generalised arithmetic. Then, at some point along this trajectory, another shift is required – when rules, regulations and the discourse of arithmetic obstructs further conceptual development and understanding of other topics which incorporate some but not all the ideas of algebra (see examples below). As Booth (1981, p. 30) puts it:

> *The successful use of this approach on the easy items within each topic may itself prevent the child from moving on to the more advanced kind, since the child sees no need for a formal strategy.*

Eisenhart, et al. (1993) considers that learning to teach conceptual knowledge on the back of patterns of teaching could lead to problematic situations. Tall (2004) refers to the expression: $2+3x$ as something that is unknown, unless the value of x is known. This is an example of a "sum" whose answer cannot be known if the unknown is not known, something new for a child with an arithmetic background to learn and accept.

Learners employ a number of strategies to deal with such hurdles. In algebra, for instance, students have been known to employ strategies which rely on arithmetic familiarity in order to deal with algebraic concepts, often leading to wrong choices. Also, in the students' responses to questions about logarithms and trigonometry it is common to see interpretations of algebraic rules employed. For example:

1. $\sin(a+b) = \sin a + \sin b$

and

2. $\log(a+b) = \log a + \log b$

are interpreted as extensions of the arithmetic and algebraic distributive rule:

$$a(b+c) = ab + bc$$

There is common acceptance that these sorts of examples are illustrative of a lack of understanding of the underlying concepts. There are different theoretical positions which seek to describe and explain these situations. For example, Davis, et al.(1978) use the metaphor of the *frame*, to describe "what went wrong": the student is said to have retrieved the *wrong frame* from memory (p. 97). Their theoretical preference is drawn from cognitive science and computer programming. For Sfard and Linchevski (1994), students make use of a "pseudo structural" conception. That is, they take the representation of the object for the object itself. The claim is that for the students the form of the structure is all it is. Küchemann (1981, 1983) and Booth (1981, 1984), on the other hand, have emphasised children's use of strategies to cope with conceptual demands in secondary mathematics, some with limited success. Booth (1981) makes the claim that "children attempt to solve mathematics within a 'human sense' framework". This is consistent with Freudenthal's (1991) "mathematics as common sense" (p. 6). Booth (1981) claimed that for some children in schools there exists two systems of mathematics: one the formal taught mathematics and one a child methods mathematics which uses "building up" strategies such as "counting" and "adding up" (p.29). The limitations of using these build-up strategies are ultimately exposed: "each such method has a limit in applicability which is reached very quickly" (Booth, 1981, p. 30). Even the more advanced children in the study[6] who dealt with very difficult problems very well with sophisticated build up strategies found that their strategies "eventually broke down when the problem became either generalised or too complex" (p. 31). Similarly, this is the case when referring to templates- driven strategies[7] (Sfard, 2000).

In summary, conceptual and procedural knowledge in mathematics is premised on extensive build-up strategies and cognitive development, on abstraction in algebra and concept formation and on learning to be proficient in the use of the symbolic language of mathematics.

2.2.1 Theories of learning in the design of taxonomies in classifying educational objectives

In this section I perform a literature review of the theoretical debates which have influenced the design of a number of key taxonomies and educational frameworks. This is done as a backdrop to deriving a framework for this study to capture the cognitive tasks the students perform in their tests and examination in an organised way. The framework will be designed in Chapter Four. The use of this kind of educational framework is usually underpinned by one or more theories of learning. These will be reflected in the discussion here. I will begin

---

[6] Concepts in Secondary Mathematics and Science Project, 1980, Chelsea College, University of London, UK.

[7] Despite the time gap: Booth 1981 – Sfard 2000 and their use of alternative theoretical insights these researchers nonetheless make very similar points.

with a review of Bloom's taxonomy as the original and most influential classification of educational objectives and set a few key subsequent developments in the area of taxonomies against that. Differences between factual knowledge and conceptual knowledge will also be outlined.

Since the publication of Bloom's et al. Taxonomy of Educational Objectives (1956), there have been many changes in the educational landscape. The taxonomy was set within a pre-dominantly behaviourist paradigm. This paradigm was in part premised on conditioned responses, was narrowly focussed, and claimed to be based on empirical, scientific methods. It has been challenged increasingly by more modern understandings of teaching and learning in education (NCTM, 1989). These later versions have been influenced by new ideas in psychology and sociology, as will be shown below. Nonetheless, the esteem with which the taxonomy has been held over the years, as shown by Krathwohl (2004), bears testimony to the fact that many of the original objectives for writing the handbook were successfully met and that the essential aspects of the taxonomy have endured.

The standard taxonomy is a content- by- cognitive behaviour grid, also sometimes known as an item specification table. Achievement of test items or pieces of curriculum is assigned to individual cells. Each item is categorised as representing exactly one level of cognitive behaviour and one element of content (Robitaille, et al., 1993, p. 41). The original (Bloom's et al. 1956) taxonomy levels were: knowledge; comprehension; application; analysis; synthesis; evaluation, in a strictly hierarchical sense. But, as the condensed version in the handbook shows, the sublevels for each category flesh out in greater detail what the group of examiners and educators intended who drew up the document. Thus *knowing,* for example has 32 sub categories, ranging from simple to complex, as the authors outline in the handbook. The authors also showed awareness of the limited ways in which the taxonomy could end up being used. For example, one could end up far from the objective with which one started as a result of fragmenting the objective to fit into the taxonomy. This action thus unknowingly pre – empted later criticism that the taxonomy leads to a fragmented curriculum and that it fails on interrelatedness of content and cognitive behaviours:

> *Although this was recognised as a very real danger, one solution for this problem appeared to be setting the taxonomy at a level of generality where the loss by fragmentation would not be too great (Bloom, et al., 1956, p. 6)*

Romberg (1992) and Robitaille, et al. (1993, p. 42) also expressed similar doubts. Cognitive psychology also informed the design of the taxonomy. The arguments put forward for the cognitive rationale which provided a basis for the taxonomy are less persuasive, more tentative:

> *we reviewed theories of personality and learning but were unable to find a single view which, in our opinion, accounts for the varieties of behaviours represented in the educational objectives we attempted to classify" (Bloom, et al.,p.17)*

This rationale seemed to centre on a correspondence between the performance of students on tests and a notion of educational behaviours, ranging from simple to complex. The suggestion that there might be a "scale of consciousness" and that learners become more conscious as problems become more complex is of interest, especially in light of subsequent developments in that field[8]. This idea has some parallels with the notion of "adaptive reasoning" (Kilpatrick, 2001) as discussed below.

Freudenthal (1991) is quite scathing of the claims made by Bloom about the value of achievement testing. According to Freudenthal, tests become ends in themselves, creating "objects" on which tests are conducted, without regard to the usefulness of those objects in

---

[8] I'm thinking about the research area of neuro-science and learning; see below

explaining phenomena. He posits that such tests are "condemned to sterility" should they fail to be properly controlled.

But developments in knowledge production, especially cognitive science (for our purposes) and an increasing awareness of the importance of applied knowledge and the ability to think in complex ways in a modern technological world have meant that many other attempts have been made to produce a more modern "knowledge taxonomy" ala Bloom, et al. The authors of the *revised* Bloom's Taxonomy took account of changes in knowledge and cognition since the original publication and have the advantage of continuity through David R. Krathwohl, who served on the original committee. They made a significant shift from the behaviourism of the fifties, with its potential for conditioning and stimulus response associations, to the modern, preferred theories about cognitive processes. These processes regard learners as "active agents in their own learning (who) …select the information to which they will attend and construct their own meaning" (Anderson, 2001, p.38).

The major departure from Bloom's et al. taxonomy is the emphasis on different kinds of knowledge. In particular, reference is made to *factual* knowledge, which seems to echo the list in Bloom, et al., and conceptual knowledge, for which there is no obvious parallel in Bloom, et al. (1956). In Bloom, et al., there are terms such as "knowledge of abstractions" and "generalizations" which have some parallel. These changes reflect as much changes in ways of thinking about learning and thinking as they reflect a new language which accompanies that. For example, "abstractions" appear where we would perhaps put "concepts" today. Anderson, et al. (2001) makes a significant distinction between factual knowledge and conceptual knowledge. Factual knowledge is defined to be knowledge referring to specific content elements such as terms or facts. *Conceptual* knowledge is knowledge of larger, organised bodies of knowledge (concepts, principles, models, or theories). They indicate a qualitative difference between those students who make the connections between facts (factual knowledge connections) and the "larger system of ideas" and those who don't. Another addition is *meta-cognitive* knowledge. Meta cognitive knowledge, subset strategic knowledge, refers to knowing when to use a strategy when solving certain problems, even routine algorithms, and requires a form of thinking which it is expected will lead students to correct solutions. This form of knowledge is considered one of the highest forms of cognition. It is claimed that this knowledge will often underpin the learning of new ideas. Though this kind of knowledge may fall into the second tier (conceptual knowledge), repeated use of this strategy may have the overall effect of making certain students skilled in using it across sectors and even across disciplines.

The following example is intended to illustrate how what counts as conceptual knowledge in a school context is often assumed to be factual knowledge in a tertiary setting. The example acts as a caution against interpreting the narrow framework above too rigidly. It is common teaching practice, after having defined the trigonometric functions, to manipulate those definitions and produce identities. This is standard practice in schools and is reflected in standard textbooks (Gonin, et al., 1973, De Jager, et al. 1992, Schreiber, et al. 1993). It is possible that this has become standard practice because of the power of textbooks in our context. Students are encouraged to explore the definitions in order to derive the identities from them: this is a worthwhile pedagogical task as it has a potential empowering effect. This could be the case even though such learning could be constrained by the teacher's objectives for the lesson. Thus, the identity

$$\cos^2 x + \sin^2 x = 1$$

may start off being "apply conceptual knowledge" but very soon becomes "remember factual knowledge".

Another change from the original Bloom's et al. taxonomy, but in contrast to the revised version, comes from the TIMSS authors. The TIMSS curriculum framework for mathematics provides a window into both past practices and assessment but also modern ones (Robitaille, et al., 1993). One of the ambitions of TIMSS is to incorporate into its analysis the societal context in which the school is situated, the background of the participants in the school system and the actual school arrangements. As in the case of the revised Bloom's taxonomy, TIMSS' earlier "organizing structures" used for the construction of achievement measures were "content -by –cognitive- behaviour grids". Bloom's et al. influence is noted in the discussion document (p.41). Also, there were reasons for changing the curriculum framework based on Bloom, et al. to one which could include "flexibility in accommodating different theoretical notions about how students learn" (p. 42). The performance expectations in the TIMSS framework are presented as a "re-conceptualization of the former cognitive behaviour dimension" (p. 44). Thus a test item, for example, is not confined in this framework to a single cell in a strictly hierarchical sense, as we saw above in the revised Bloom's taxonomy, but may be related to all or any of the three aspects listed or any or all of the categories which are subsets of the three aspects, and so on. An advantage of this framework is that it is much less reductionist. This is so despite the list of performance expectations which reads in parts like the earlier taxonomies: knowing; using routine procedures; investigating and problem solving; mathematical reasoning; proportionality and communication. The TIMSS framework is critical of the limitations of the theory of learning underpinning Bloom's et al. taxonomy which is seen as a 'simplistic classification scheme that distorts and impoverishes the student experience. TIMSS's framework is seen as more flexible in accommodating different theoretical notions about how students learn: representing, formulating, relating representations and describing and discussing (Robitaille, et al., 1993).

In summary, changes in learning theories and how these influence pedagogy have led to adaptations and changes to the original Bloom's et al. taxonomy of educational objectives. The use of a taxonomy to classify cognitive achievements is underpinned by one or more theories of learning.

2.3 Assessment

The use of constructed response questions is an integral part of the assessment of the kind of tertiary students under review in this study. This section is a literature review of assessment in mathematics, specifically the use of tests and examinations. When one considers assessment in mathematics, the awareness of the interlinking of symbolic representation and conceptual understanding is made evident in the text. To make sense of mathematics at tertiary level our students have to make links between their experience and knowledge on the one hand and the new ideas and especially ways of expressing these ideas, on the other. They have to learn the symbolic language of any new mathematics and learn how to function with any new mathematics. They have to learn about understanding the rules and regulations, the nuances and shortcuts which the symbolic language makes possible; they have to adjust and expand much of their knowledge of mathematics. Much of our awareness of how students adopt this new discourse and how comfortable they are within it comes from our assessment of their work, oral and written.

Assessments in typical paper- and –pencil tests are limited in the tasks that they can achieve, for a number of reasons. Questions tend to be short and closed as there is a time limit involved. More complicated mathematics which requires a deeper understanding is best assessed through a variety of methods, namely interviews, assignments, report writing and essays (Lingefjärd, 2000; Taylor, 1999). Assessments in mathematics have the potential to provide huge insight into:

..student achievements…student learning…specific misunderstandings…quality of teaching…quality of learning…what it means students can do (Ramsden, 1992, p. 182).

However, the use of controlled paper and pencil tests is not usually recommended as a good form of assessment if one wants to assess in the manner as described in the quotation above. Paper and pencil tests are often considered too narrow, focussing on testing skills and procedures (Romagnano, 2001). Freudenthal (1991) is disdainful of this sort of testing. Nonetheless, written tests continue to be a dominant means to assess students (Tomás Ferreira, 2004). The fact is that such tests

- Allow the screening of the whole class (ibid).
- Are valued by students, parents, and the general society (ibid).
- Are at least objective measures of skills and procedures (Romagnano, 2001, p. 31).
- Are easy to set up
- can easily be duplicated
- can be standardised
- use relatively few resources
- are easily graded
- can be used as a benchmark and a filter.

There are other reasons why these sorts of test will continue to be a mainstay of educational practice:

Pencil and paper tests are relatively independent measures of student performances, and hence give data on what students know and are able to do which is relatively more valid and reliable than information from other sources. This is not because written tests do not involve a subjective element, but because their interpretive elements are subjected to two forms of control. Firstly standardised tests *represent a consensus: they are a distillation of a range of expert judgements. Secondly, piloting tests against a target population is a measure of the extent to which they fulfil the prediction of experts, allowing for adjustments to provide for a better fit (Taylor, 1999, p.198)*

These features make the use of such forms of assessment attractive for those individuals and institutions which are geared towards greater control and conformity, but also greater economy and accountability. That pencil and paper tests are 'relatively more reliable than information from other sources' is open to debate and some of my analysis unravels that confident tone (see Chapter Five). Much would appear to depend on the depth of assessment of these tests, especially the extent to which one is able to interrogate the individual response in the sort of depth that is envisaged in this thesis.

Assessments can also be viewed within the semiotic world as earlier stated (O'Shaughnessy and Stadler, 2002). Examiners are the audience of much of what the participant members produce. Examiners' expectations are shaped by their interpretation of what mathematical produced texts should look like. They are shaped by their own training and by authors of mathematical textbooks. They do not, as a rule, make use of notes of other mathematicians nor do they present their students with opportunities for individual explorations of solutions to problems set in these tests and examinations. The notes to students, the notebooks and textbooks they refer to have an implicitly and explicitly formalised presentation and they expect the students to produce the same. In this way, the students are inducted into the formalised discipline of mathematics. Thus the assessment methods emphasise this formal, written text as evidence that the student is progressing within the subject. In assessing these texts the examiners grade according to set grade schemes which emphasise uniformity and formality over individuality and informality. As a consequence, the texts are said to contain meanings according to the criteria built into the grade schemes. The grade schemes are built up via the formalised approaches, notes and so on, as mentioned above. In other words, the reading of the lecturer is the dominant reading for the student produced texts.

If it is assumed that objectivity is a worthwhile educational goal to strive for in assessments, then tests which maximise this would be more desirable. The issue, however, is not so straightforward. Objectivity remains elusive. Considering test results from the point of view of consistency and reliability may be more achievable (Romagnano, 2001). This is one of the reasons why a number of different tests are considered for the same cohort in this study. Romagnano makes use of a form of textual analysis when considering a student's response to a "most common and traditional type" quiz question:

$$\textit{Solve} \quad x^2 + x - 6 = 0 \qquad \textit{(p. 32)}$$

In his analysis he raises many questions about the reading that it is possible to make about the student's answer. For example,

What does the student know about solving quadratic equations? (p. 32)

But his conclusion favours the view that the scorers are subjective, the scores assigned are inconsistent and consequently these types of questions "may not offer much information about the mathematical knowledge of the student" (Romagnano, 2001, p. 32). The paper – and –pencil test is a dominant form of assessment in South African schools and also in the case of this study. That South Africa is by no means the exception is backed up by research by Tomás Ferreira (2004). She found that student teachers "had low expectations for their students and complained about student underachievement, misbehaviour and lack of motivation" and also that "parents overvalued typical written tests" (p. 32). She also found that most of the student teachers saw no obvious reason why they should abandon testing as the main form of assessment. One of her groups saw the use of quizzes as a means of revising lesson plans and classroom teaching in order to accommodate students' needs. In another case the students were given their tests back with comments on how to improve them. They were expected to improve on their tests and return the new versions after some time. This was considered a development of the method of the paper-and-pencil test.

These examples illustrate again the pervasiveness of written tests in school systems and also reiterate the existence of a cycle of didactic teaching and learning within which new teachers find themselves and the consequent difficulties of changing established practices. Although there is clear evidence that these tests don't reveal enough about the mathematical conceptual development of students (Webb and Coxford, 1993; Romagnano, 2001), the view that it is still possible to interrogate them for possible conceptual and other mathematical properties, still holds sway among many educators. This is the case despite the limited nature of these tests, as described.

Because this study has a focus on multiple choice type tests, some comments about those kinds of tests will be made here. Multiple choice tests have particular characteristics compared to other tests:

Readability levels: the need to ensure that students are not going to be penalised because they could not understand the question (need for clarity and unambiguity).

Plausibility of options: the need to ensure that there are no obvious wrong and right answers, and also that a right answer won't be chosen for the wrong reasons (besides guessing).

Language control: there is a need to ensure that language use does not lead the student on or away from the solution.

Item difficulty: this refers to a percentage of the cohort, for example, who would get the correct answer for an item.

Item discrimination: the need for answers to test items to reflect correctly the differences between students, with respect to their mathematical faculty. (Cohen, et al., 2003)

These features of multiple choice questions can lead to disagreements about difficulty levels assigned to test items. The panel which discussed the PISA[9] 2003 assessment, with specific reference to mathematical literacy, raised some points about the characteristics of test items which inform my discussions and analysis in chapter five. On the issue of cultural influence on item difficulty for respondents, Shimizu (2005) in referring to an item in which Japanese students faired better than their counterparts in other countries, asked:

> *Does the result suggest that those (Japanese) students have a cultural practice with number cubes, or Origami, inside and outside schools? (p.78)*

This observation seems to point towards the *specificity* of learning. By that I mean focussing your teaching and learning on a specific problem type, for example. Neubrand (2005) expressed concerns about item difficulty in another way: "not the same features make a problem difficult in any of the three types of mathematical activities[10] (p. 81).

Kieran (2005), in her presentation, disagrees with the difficulty level assigned to item 4 by the PISA panel (PISA 2003) on the grounds that, comparatively, item 5, which is assigned the same difficulty level, is more difficult. She quotes other studies to support her arguments. My interest in referring to her presentation lies in the doubt she creates around the weighting of item difficulty. By pointing to the debate which she raises I am drawing attention to the existence of differences among the research community on the question of test item difficulty. My own sense is that test item difficulty is specific, as stated before.

---

[9] Programme for International Student Assessment, OECD

[10] viz., employing only techniques; modelling and problem solving activities using mathematical tools and procedures; modelling and problem solving activities calling for connections using mathematical conceptions.

Student performances in a test have to be assessed against more factors than the test itself. This much is not new knowledge, but my system of triangulation as outlined in 4.2.3 below, is intended to support such attempts. My own arguments, put forward in section 4.2.2 when discussing individual test items, reside, in the main, in an assessment of conceptual and procedural difficulty, as discussed in this chapter. This focus is not unlike that as discussed by the PISA panel.

2.4 Intended Contribution of the study

Despite the criticism that written tests with constructed response questions are too narrow and focus on skills and procedures at the cost of problem solving skills and conceptual depth, these types of tests continue to dominate the mathematical educational landscape. Constructed response questions are questions which, in an empiricist framework, require a written production or reproduction of solutions which follow certain norms. These norms are usually set by the lecturers and are also often reproduced in standard textbooks. Detailed analyses of responses to test items, such as in TIMSS (Reddy, 2006) provide extensive and very useful feedback, but part of their limitations lie in the nature of the tests. While such analyses are able to give detailed group insight of the content and cognitive domain strengths and weaknesses of a cohort, there is no mechanism for exploring the mathematical achievement of individuals. Group testing at best profiles individuals against group performances. The literature has focussed on this aspect of testing.

While Romagnano's (2001) view that the solutions to response type questions may offer very little information about the mathematical knowledge of the student, it is my intention to show that this is a limited view. Taking the allegedly incomplete and ambiguous evidence furnished by the student together with evidence provided by other results, I posit that I can say something stronger and more consistent about the mathematical achievement of that student. Furthermore, by making a closer analysis of the student response, by looking for signs and clues in the text itself, I posit that I will be able to furnish more information which will add to the general picture I am trying to get about the student. So, this is the first contribution I intend to make to the analysis of student responses to pencil- and- paper tests.

On the other end of the scale, assessment of what has become known in the mathematics education world as “standards-based teaching” (NCTM, 1989), is about assessing competencies such as problem solving abilities in settings other than paper-and –pencil tests. Since the publication of the standards much has been written about its impact on teaching, learning and assessment (Ross, et al., 2003). There is a dirth in the literature about the use of paper-and–pencil type tests for the kind of individual diagnostic purposes usually associated with “standards-based teaching”, for example group work, discussions, modelling, building conjectures and so on. This study is intended to contribute to bridging that gap by offering diagnostic methods for paper and pencil tests.

At the same time, interest in semiotics as a method of analysis for research in mathematics education has opened up new avenues in the discipline (Vile, 1998; Sfard, 2000; Radford 2000, 2002, 2004).) Semiotics is discussed as a theoretical framework in Chapter Three. For example, Radford (2004), commenting on meaning and the structural properties of mathematics, makes a claim for the use of a semiotics lens as a way of seeing ‘beyond certainty’ in mathematics:

> *The idea that mathematical meaning goes beyond the production of mathematical structures and the claim that meaning is produced in the crossroads of diverse semiotic systems (mathematical and non- mathematical) is certainly one of the cornerstones of non-structural approaches to mathematical thinking (Research Forum 2, p. 163)*

His overview of researchers grappling with the problem of structure and symbolic manipulation in algebra with, or in absence of, meaning is done through a semiotic lens (in looking at signs, their referents and interpretents). In this way he uses semiotics both to argue his views and to show the advantages of using a semiotic form of analysis. One advantage is that through semiotics use one is able to critique the view that logic and formal symbolic manipulations are examples of certainty in mathematics by raising the possibility of other interpretations. Thus, secondly, this study is intended to make a contribution to the field of semiotics in mathematics education in the following way. I show how a new category of sign-posting events can be employed in the analysis of a student produced text in an attempt to enhance an achievement profile of the student.

3. Chapter 3: Theoretical Framework

3.1 Introduction

This chapter is concerned with the theoretical framework of this study. The framework has two sections: the first deals with content analysis and the second with textual analysis. The content analysis framework serves two purposes in the design of this study: one, to provide the theoretical underpinning for the use of a taxonomy for the cognitive tasks of the cohort, and, two, to provide the theoretical rationale for the content analysis of the tests data and test items in Chapter Four. The textual analysis framework is based within the theory of semiotics for mathematics education and is used to analyse the examination texts data. This chapter will elaborate on these various sections and situate them in the context of my study.

3.2 Theoretical framework for the content analysis

The theory which informs the content analysis performed in Chapter Five is discussed in this section. By content analysis I mean the coding and categorization of my data. This theory is underpinned by discussions in Chapter Two about the role of factual, conceptual and procedural knowledge. I make the assumption that a philosophy of teaching and learning informs any content analysis. Robitaille, et al. (1993) introduces the notion of the intended, implemented and attained curricula in the conceptual framework for TIMMS. The intended curriculum "is the mathematics content as defined at the national or educational system level", the implemented curriculum "is the mathematics content as it is interpreted by teachers and made available to students", and the attained curriculum "consists of the outcomes of schooling – the concepts, processes, attitudes towards mathematics that the students have acquired in the course of their schooling years" (p. 27-29). In light of these definitions I assume, further, that the school - based and tertiary-based 'intended curricula' and 'implemented curricula' that the tertiary students were exposed to will contain certain common features. While it is not possible to determine those patterns with great accuracy from the limited vantage point that the study affords, the study may reveal aspects of them through my analysis. Furthermore, I make another assumption, namely, that the 'attained curriculum' is what will be revealed through the content analysis done in Chapter Five.

In light of this, Bloom's et al. (1956) taxonomy, as the original taxonomy for educational objective, had a huge influence in mathematical assessment and built upon the behaviouristic notions prevalent then. These included conditioning and stimulus response associations, the so called empty vessel approach to teaching in which knowledge has to be *inserted* into the learner's brain. The "chalk and talk" approach to teaching was the prevalent didactics methodology (Atherton, 2005). This was followed by the developmental psychology phase of Piaget (1960, 1973) and the notion of learners *constructing* their own knowledge, of being active learners with choices. Current shifts include notions of mental representations in the brain, of the interconnectedness and organisation of pieces of information, of the role of the affective in learning and of the role of meta-cognition in learning, retention and retrieval (Butterworth, 1999; Robitaille, et al., 1993; Devlin, 2004).

This linear process, however, does not necessarily reflect current practices in teaching, although the case for adaptation is made quite strongly on the basis of what the most recent understandings are about how people learn. As demonstrated, changes in schooling have repeated phases we have seen before. There is the switch from an emphasis on computational skills and procedures, to one focussing on the structure of mathematics and its unifying ideas (reform in the 1960s Europe and United States of America). Then there is the "back to basics" of computation and tables and procedures and so on. Then this moves on to yet another reform movement involving problem solving, communication and use of modern technology. Finally to the present, where there is a kind of reaction to reform, where memorization and facility in computation comes to the fore again (Kilpatrick, et al., 2001, pp. 114-6).

What tends to happen is that these phases of policy reform, firstly, take a long time to become the norm. In South Africa the education authorities began to implement outcomes based education in grade 1 in 2000 and by 2006 grade 10 began this new educational process (DOE, 2003). Secondly, a core ingredient in education, teachers, also take a long time to change from the old to the new and they may act as restrainers and undermine the new movement in the classroom (Taylor and Vinjevold, 1999, pp. 137- 144). Thus, what you have when you analyse trends is often a hodgepodge of practice.

The focus within the tertiary mathematics course on content in terms of concepts, procedures and limited problem solving forms the basis for deriving a set of cognitive areas which tend to get assessed in this course. It provides the rationale for analysing the test scores and test items. Given the history of South African education, the nature of our classroom practices, and my own personal experience as a student and teacher in high school, it is so that most of our practices and especially assessment practices remain locked into a *procedural, computational framework,* including surface level problem-solving (Reddy, 2006; Adler, 1998). This is one reason why the results of a diagnostic test are included: the test contains a number of non routine problems which students would normally find unusual and unfamiliar. On account of that assumption, the test was used to assess students' non routine skills.

The cognitive areas which will be emphasised in the assessment analysis are given below:

Recall basic facts, procedures and concepts

This refers to knowing what the basic facts, procedures and concepts in the requisite and pre-requisite course are. Examples include knowledge about the basic operations, knowing the number system, knowing basic definitions within topics such as trigonometry and logarithms.

Perform routine calculations

These refer to calculations as used with basic arithmetic, substitution of values in formulas and solving basic algebraic equations.

Apply routine calculations

These refer to routine calculations which are applied to situations in which the mathematics is embedded, but at a very routine level. For example, the problem which requires learners to determine the number of bricks in a double layer wall of bricks with some bricks missing.

Perform routine procedures

These are procedures or algorithms which underpin much of the content learnt, such as determining derivatives, using differentiation rules, determining a basic limit and solving a trigonometric equation. Use of the term *procedural breakdown* is employed in this context when a participant member fails to complete a procedural process. Usually there is evidence in the text of this kind of breakdown. The wider context of a specific text may also be considered, for example, by looking for patterns.

Apply conceptual knowledge

This refers to knowing what the conceptual demands of a problem are and applying that knowledge, usually in conjunction with procedural knowledge and /or problem solving strategies.

Problem solving

This relates to solving problems requiring mathematics for their solution wherein the mathematics in the problem is embedded. It could also relate to mathematical problems where the mathematical strategy to be used for solving the question requires more than a routine procedure. This type of problem may involve a string of procedures which are interwoven.

The test items in the examination scripts of the cohort provide opportunities for explorations about the variety of responses which the cohort might make. Set against this is the reality of a myriad of student responses which are normally assessed against these expected norms. For example, the examiner has a constructed response in mind for the following question:

Determine the derivative of $y = e^{-x^2}$ *with respect to x*

Moreover, similar examples may have been reproduced in the classroom and/or in the notes. In the assessment of this solution there is thus an expectation contained in the memorandum of the test that the student will reproduce the format used. Methods which are radically different and have wrong answers are at risk of being judged to be deviant, error-prone or simply wrong. This is the case despite the existence of a wide variety of student responses, many of which do lead to the correct solution. In that sense, then, there are expectations that the examiner will draw up questions covering certain topics and conceptual areas and in return that the student will respond to those questions in certain standardised ways. There is an added assumption that both will play their parts in this relationship; that for the participant student there are definite returns for solutions provided as required.

Thus, in this study, when I seek to interpret the student texts it is done with the assumption of the standard practice as outlined. This involves the curriculum, the course, the classroom, the notes, the humans involved and so on.

In an attempt to characterise *differences* among the participant members of the cohort in terms of their mathematical practices, and to compare or contrast these findings with the data tables derived from the cognitive areas outlined above, a further analysis of the test items will also be done. The aim here is to look for patterns across scripts as possible clues to the mathematical achievement of the cohort. I wanted to see whether the cohort had internalised earlier mathematics sufficiently to recall and apply as needed; whether the cohort has understood the mathematics taught on the course, enough to respond well to the questions set and whether the cohort have attempted the difficult questions.

On the basis of a pre-trial, the categories that I initially included were:

- *percentage who attempted a question*

Was there anything about the question which caused some students not to bother to even try it? Why? What does this say about the question, the content it refers to or the skill or conceptual understanding of the participants, and so on?

- *percentage answered correctly, using the prescribed or recognisable procedure*

If the percentage is too high or too low we can say the question did not discriminate enough, for example. Here I am not considering individual cases but trends.

- *numbers making the same mistake/s in a question*

Here we may get clues as to the cause of some students getting wrong answers. If there is a general pattern it should alert us to possible causes or even to problems with the question itself. As always, here I look for trends specific to an item.

- *None of the above*

There are many variations of those who do not fit the pattern as outlined above. The textual analysis of the third section is set-up to analyse these scripts.

3.3 Theoretical framework for the textual analysis

Introduction

By qualitative textual analysis is usually meant the use of theoretical tools with given procedures used to analyse texts. Text in this theoretical world includes speech, gestures, visuals, written text and much more. In the case of this study, textual analysis means the analysis of the written assessment which students do at the end of the tertiary mathematics entry level course in terms of one or more of the theories of textual analysis. I have chosen to review the field of semiotics in mathematics education as one of the emerging theoretical fields in the discipline. I have chosen to include the use of semiotics as a tool for doing the textual analysis in my theoretical framework. To begin with, I will give a synopsis of the main points for which semiotics, as it is used in the field of mathematics education, is known.

I have already referred to the introduction of symbols and names in order to constitute new mathematical objects, in Chapter Two, section 2.2 (Sfard, 2000). Sfard introduces these notions on the back of the language of the semiotics of Saussure. Saussure's lecture notes, captured by some of his students, are the resources from which researchers have drawn key facets of his ideas. Chandler (1995), for example, outlines some of these key facets, as discussed next.

Semiotics is a system of signs and the meanings that are conveyed through it. A sign can be an object, an utterance, a gesture which conveys meaning that has been put into it by the convention of a culture or a micro-culture. So for example, marks on a page can be read by a person and understood as conveying meaning about literature, psychology or mathematics. Signs are seen to represent the unity of two components, a signifier and a signified. An example is the word, quadratic and the symbol x^2: the symbol connects the word to the mathematical idea, the concept of quadratic.

In mathematics, as the example illustrates, the symbol is often itself a component in another sign-system, namely language. But symbols, as one type of sign, play a highly significant role in mathematics. This is so, particularly in light of the fact that mathematical objects are mental constructs with little or no concrete referents. Symbols are signs which link concepts with things. These things may themselves be abstract. Signs are seen to be arbitrary but within certain sign-systems signs acquire fixed meanings and interpretations through convention. One such system is language itself, another is mathematics. In mathematics conventions allow that symbols have specific meanings, mostly different from their meaning in other sign-system worlds. For example the use of "and" and "or" mean different things in mathematics than in English. Signs are said to be either syntagmatic or paradigmatic. For example, in mathematics an identity in trigonometry such as $\sin^2 x + \cos^2 x = 1$ can be said to be syntagmatic: the terms are connected in that statement (equation). An example of a paradigmatic situation is in Euclidean geometry where, for example, equal angles can be exchanged in an attempt to arrive at a proof: the process of interchanging those angles is said to be paradigmatic.

The key semiotic focus in this study concern written signs, that is, signs which are produced on paper, in tests and examinations. These written signs are mathematical terms, expressions, and statements. They are often written in sequence and connected by logical, deductive steps. Often, in the context of the study, these texts conform or seek to conform to routinised algorithms: they seek to imitate patterned solution responses. The students in our context are required to conform or seek to conform to a set of expectations about what written mathematics, as required in a tertiary setting, is all about. In this sense students seek to adhere to a system outside of their normal, everyday experience, in the way that schools are conceived of as not normal places. This view is consistent with that which Booth (1981) found for learners at secondary school.

The text can be viewed as an act of performance by the student and received as such by the lecturer. In this regard, the performance is like an utterance within a system where "performance", the ideal thing, is a concept. Judgements made of the performance by the student relate to this joint collusion by student and lecturer. In this regard, the respondent's answer book is structured according to an agreed pattern (performance), and the grading is the judgement (Prior, 2004). The recipient of the answer book is the lecturer and the responses given are couched in mathematical phrases and terms which the student expects and can understand. In the same way the question paper is structured with the students as readers in mind. In this context, evaluation is also seen as a substantial part of the didactical contract that is being established between student and teacher (Brousseau, 1997).

The expectation of this conventional form of the mathematical text is premised on students having been inducted over a period of time into the writing of mathematics. A mathematical argument follows a certain internal logic, a way of connecting lines, statements and of making inferences. This includes various conventions such as how to answer a question paper, how to write down your solutions and how to label objects. The language involves mathematical expressions, equations and statements and also includes the medium of instruction. The relationships outside the text relate to properties, rules, theorems, other knowledge that is often implied within the text at hand, but also the context which gives rise to the text, for example the notes and classroom activities.

Using the language of semiotics, the text of the scripts can be viewed as a system of *signs,* with distinctive *form,* distinctive uses of *language and conventions.* There are assumptions about authorship and readership, and having *relationships* outside of the text itself (Atkinson and Coffey, 2004). Alphabetic letters, numerals and other symbols in this discipline have very specific meanings different from their meaning when they are used elsewhere (and some are only used here): as such, they are part of the mathematical *code*. There is also a range of uses for the same symbol, its position in the statement indicating its property. For example, the number 2 has a different use and meaning in each of the following:

$$x^2\ ;\ 2x;\ \sin\frac{x}{2}$$

In the same way, the position of the *2* in the following means each expression has a different meaning even though the same letters are used:

$$\sin^2 x;\ \sin x^2; 2\sin x; \sin 2x$$

The use of the *2* in each case has a meaning in its context, which the student has learnt (or is expected to) and will apply as and when required. This is taken as further evidence for the notion of the arbitrariness of the sign. In mathematics, each sign (symbol and placement) becomes associated with a meaning which is attributed to it through convention.

Mathematics is a highly developed symbolic system, with some of its symbols having been in use for so long that their symbolic status is often overlooked. This is true, for example, with our use of the Hindu numerals: after some time of becoming familiar with them we simply use the numerals in daily life in an ordinary sense, without a conscious awareness of their symbolic meaning or past history. One could say these numerals have been inducted into the code of ordinary language. Our number sense and our symbols for numbers have fused.

O'Shaughnessy and Stadler (2002), using the language of semiotics, reflect on the notion of a text in much the same vein. For them, to make sense of the texts produced by the students it is necessary that one be aware of the context, the participants and their intended audience as well as how they use and make sense of the text. For them, texts produce meanings by using pre-existing codes of representation. This way of reading texts I found helpful when I analysed the assessment of the student-produced scripts.

In the requisite mathematics for the cohort, the system of mathematics is well defined and the individual utterances, as long as they follow the rules, are allowed and understood within this system. For example, there are acceptable ways to determine the value of an unknown in an equation, say, even if those ways are finite. This system fits the general description of structuralism, as outlined by Saussure. But this system has its dangers. Applied to mathematics the distributive rule in algebra: $a(b+c) = ab + ac$ creates a *structure*; this structure can accept many versions, such as $2(a+4) = 2a + 8$ but not every version which contains mathematics. So, for example, $\sin(a+b) \neq \sin a + \sin b$ for all values of a and b.

Structural semiotics has been criticised for focussing on the internal only:

> *One of the weaknesses of structuralist semiotics is the tendency to treat individual texts as discrete, closed-off entities and to focus exclusively on internal structures. Even where texts are studied as a 'corpus' (a unified collection), the overall generic structures tend themselves to be treated as strictly bounded (Chandler, 1995).*

Others have suggested that the relationship between signifier and signified is not only arbitrary but unstable, even in cases where that relationship has given the impression of stability (Selden, 1989, p. 71). For example, in mathematics the use of the numeral 2 to mean different things in a number of ways, as shown above, can lead to a sense of instability in the learner who has always linked the symbol 2 with the meaning of quantity. Radford (2000) refers to this change from structural semiotics as a "theoretical shift from what signs *represent* to what they *enable* us to do" (p. 241).

As a consequence, there have been many developments in the field of semiotics and some of these have been echoed by mathematics education researchers. This shift has relevance for my own focus in terms of my attempts to investigate the student produced texts beyond a formalistic, structuralist view. For example, Sfard's (2000) notion that a mathematical concept and the naming of that concept "mutually constitute each other into being" (p.47) is an example of symbol use in mathematics which is part of mathematical activity "*in which students come to participate*" (Sfard, 2000, p. 19). Thus the active social and cultural contextual aspects are drawn into semiotic activity: they are an integral part. So much so that a student's use of symbols as a proxy for concepts or ideas does not take place in isolation but is set with social and cultural norms, for example, of the classroom.

One of the key issues is whether the user of the symbols is culturally at ease with that use. The test for that, as with a child learning to speak words in the company of others, lies with the norms of the culture, or for that matter, micro culture. In other words, does the micro culture, say, provide the "space" for the student's elaboration of her meanings through her symbol use? A mathematical culture which is closed-bounded, that is, which has fixed rules about mathematical conduct and symbol use, may stifle and reject symbol- use which is outside of those norms. An example is precisely the required solutions to constructed response questions of the kind which is part of the data for this study. These solutions belong to a mould outside of which students are seldom expected to wander.

Further, understanding the whole system is not dependent on seeing all the bits, or, in the case of mathematics, understanding all the meanings of all the symbols. On the other hand, understanding the bits is a necessary step towards understanding the system. In practice this translates into learning the use of concepts, as represented by symbols, in activities, such as problem solving, the repetition of algorithms and so on. Understanding the system comes about with time. Sfard's (2000) sense of the mutual constitution between symbol and object also contains her claim that the object with which the symbol is constituted comes into being "retroactively" (p.49). Examples of this are students who learn to use symbols before understanding the meaning they are meant to convey – the understanding follows from the use of the symbol, from engaging in a socio-cultural mathematical world, say, a classroom, where the symbol has a place. Berger (2006), working from theories of Vygotsky (1986), among others, reinforces the ideas of Sfard presented here. In her study she shows that her subject, a student at tertiary level, adopts the symbol of an improper integral with an infinite limit, makes use of the operations in a template- driven format, without understanding, initially. In this sense, she says, her student (John) "is using the mathematical signs in mathematical activities" (incoherent as they are to an outsider) (p. 19). But over time he develops understanding, with feedback drawn from a textbook, which she sees as representing the social influence as discussed by Vygotsky. Radford (2000) emphasises that this socio-cultural effect frames the development of an awareness that the student necessarily has to undergo in her induction into the "cultural symbolic system":

> *Signs with which the individual acts and in which the individual thinks belong to cultural symbolic systems (like mathematics – my insertion) which transcend individual qua individual. Signs hence have a double life. On the one hand they function as tools allowing the individuals to engage in cognitive praxis. On the other hand they are part of those systems transcending the individual and through which a social reality is objectified. The sign –tools with which the individual thinks appear then as framed by social meanings and rules of use and provide the individual with social means of semiotic objectification (p.241).*

Elsewhere Radford (2002) explains semiotic objectification as a process aimed at bringing something in front of someone's attention to persuade the person to focus on it. He asserts that the means used to do that are considered semiotic means such as signs. Thus the use of a sign functions on two levels, as an induction into the object-sign duality and as an entry into the cultural symbolic world of, say, the requisite mathematics of a tertiary course. Radford's notion of “semiotic means of objectification” provides an avenue for my own investigation of the cohort’s written scripts in the following sense. Radford extends the historical view that mathematical objects ought to be mediated through *written* symbols only, by including “sensible experience” (p.15). While this is generally taken to mean words, gestures, graphics and artefacts (p.15), I seek for signs comparative to “sensible experience” *in the text*, *through and beyond* the formalised mathematical expression. My use of such *indicators* in the texts of the cohort is calculated to extend the perceptions derived from other more traditional means of analysis, as suggested in sections 3.2 and 3.3 above.

As can be observed from these accounts, a semiotic analysis of an event provides tools and a theoretical framework which is coherent and consistent. More so, the use of semiotics makes possible a form of positioning relative to the event (or text) under discussion which is refreshing. By stepping outside the reality as commonly perceived and using a different lens to reinterpret that reality, semiotics makes possible new and different findings. In the words of Vile (1998, p. 25):

> *As an interpretive framework with focus on meaning rather than knowledge and with a body of pre-existing theoretical and practical ideas, semiotics has potential to provide insights into the ways in which students make meaning in the mathematics classroom*

Drawing on the language of semiotic analysis I approach the texts from a defined perspective which will allow me to be open to what the texts may indicate. To deal with the available evidence in the text I have created a new set of criteria which I call *critical indicators*.

Critical indicators (the theoretical framework for the textual analysis)

By critical indicators I mean those indications in the text which assist the reader/analyst (in this case myself) to build a profile of the author of the text (the participant). I am aware that by introducing markers into the text such as these, I am in danger of placing the text into theoretical space. This is because notions such as "reader", "author" and "text" conjure up debates about signs and meanings which became somewhat derailed with the advent of the deconstruction movement (Selden, 1989). In that environment, 'the written sign can break its "real context" and can be read in a different context regardless of what the writer intended' (p. 90)

While I do "read into" the text and seek meanings which are not visible in the "real context" of the text (that is, the content matter dealing with mathematics), I am constrained by the social context within which the text is created. This refers to the cohort set within a tertiary environment. That is, my search for indications in the text of a mix of cognitive and affective features of the author (the participant member) is guided by an emphasis on social factors which inform the production of the text. The text stands supported by at least one clear context, the context of the examinations. Other influencing contexts can at best only be inferred, and with not very good evidence in the text to support such inferences[11].

Thus critical indicators were derived from the data in a constant backwards and forwards movement between the analysis and the reflection on it. The use of semiotics as a method provided the entrée. As a result, the indicators were more visible to me. As indicated below, these indicators may speak directly to the text, and in that way they are not surprising. Others are induced, analytically, by the in-depth research reported in Chapter Five.

The critical indicators are:

[11] I'm thinking of a situation where a participant was not feeling in good health, for example

Procedural competence

This refers to a direct interpretation of the procedural competence of the participant as revealed by the in-depth analysis and as indicated on the body of knowledge schema.

Conceptual competence

The indicator refers to signs that the participant had understood the conceptual basis of the problem.

Logic

Irrespective of whether the answer was correct or not, the logic indicator refers to logically executed steps in the response.

Reflection

This was used when it was evident that the participant reflected on his work. Evidence for this was looked for in the text. An example could be "scratching out" work.

Confidence

This indicator referred to evidence in the text of forthright solutions, preparedness to follow through, with little or no hesitancy.

Dealing with crisis

This indicator indicates how the participants dealt with test items which they struggled with.

Creativity

Irrespective of the correctness of the mathematics used to solve the problem, this indicator labels behaviour in the text which shows that the participant ventured outside of the norm in dealing with the problem. The norm would be that which I have described as constructed response solutions.

In light of the above, then, I can say that, although there is general awareness and expectation that other realities exist for the students all the time, it is the reality of the classroom and the discipline of mathematics which is the dominant reality and context in this study. Part of the implicit curriculum is to gear the students to excise, minimise, co-opt, suspend, and subsume all other realities. Thus I accept at the outset two main realities with which students contend. One reality has to do with their textual production as an expression of mathematical thought in response to questions set, the other not. How the first reality is revealed in the text is the subject of this part of this study.

4. Chapter 4: Research Methodology and Data Collection

4.1 Introduction

This research has been designed to accommodate a three-prong approach to describing the mathematical achievement of the entry level tertiary students. The choice of instruments and nature of the data sampling is deliberate. The data has been collected as part of the ongoing formal activities of a cohort of entry level students, without intrusion and obstruction, specifically to minimize external influences, especially with regard to the role and visibility of the researcher.

The use of quantitative and qualitative methods in this research project is done with the awareness of criticisms that this kind of methodology has attracted. The central argument against using such an approach relates to the inherent assumptions about epistemology and ontology that each method purports to have as its basis. It is commonly understood that a positivist epistemology underpins the use of quantitative methods which are considered scientific, objective, and deductive. An interpretive epistemology on the other hand usually underpins qualitative research. Such research is considered constructivist, subjective and inductive. While the former is often associated with theory testing the latter is often associated with theory building (Spicer, 2004).

It is, however, particularly in social settings such as educational environments where assumptions about what it is we know are so clearly fraught. This is largely because of a lack of agreement about the issues involved and how we critically define and study them. All research run risks and makes assumptions which are susceptible to dispute and contradiction and hence, no methodology is without its dangers. However, it is generally assumed that the use of quantitative methods is best where the data is largely clear and undisputed– and this usually involves numbers, like test scores. There will always be reasons to critique even something as neutral sounding as test scores, simply because there is always a context to everything and that context requires interpretation which is open to dispute. A content analysis of data, for example, which uses quantitative methods, has the risk that the data so chosen requires interpretation and thus involves some kind of hidden value judgement which usually resides within a qualitative framework. Charmaz (2005) suggests that we share in constructing what we define as data.

On the other hand, choosing to analyse textual data for meaning making, for example, is best done qualitatively. Making clear the definition of meaning, making choices about how to partition the data or select from it, and so on, is all about using qualitative methods.

The challenge, then, will be to assess whether the different findings, resulting from the different approaches, have coherence and so can be presented together or not. Approaching this research from these different angles, however, is deliberate because of the complexity of the issues involved.

This research, then, uses a sequencing of methods in a structured way. For the analysis of test scores a scientific instrumentation approach is used, but mainly as a premise for a deeper interpretive analysis further on. A quantitative research method is then the first part, set within a post - positivist framework since it is assumed that this is the most practical. Also, it tends to meet typical scientific standards: the tests can be repeated under similar conditions and the comparisons likewise. This is done against the background of an objectivist viewpoint which accepts the achievements of the cohort as a reality: the scores stand for something. Also, within the general understanding of a world which is ordered and makes sense, the scores, and the processes which led to their generation, are viewed against certain norms and standards, established over time within this same scientific framework. There is also strong emphasis on norm-referenced forms of testing. The individual is judged against the group in terms of set standards; individual differences are smoothed out to create a uniformity and those who don't or can't are considered anomalies.

The context for the study is an entry level mathematics course, set in an engineering programme at a tertiary institution, the Cape Peninsula University of Technology (CPUT). The participant members are a cohort of tertiary entry level students. The majority had recently passed secondary school and thus their experience of mathematics, by the time of the study, is still deeply embedded in their school experiences. It is for this reason that the South African school system informs the study. It is a cohort study in that results of the same group of individuals at different points over a short period of time are interrogated, connected and compared.

The materials for the inquiry are student produced test scores and examination scripts taken from their entry level course. These materials have been chosen because they represent a commonality familiar to tertiary entry level courses in mathematics throughout the institution and others like it elsewhere in the country. This is so because the students whose scripts they are form a representative sample for tertiary entry level students elsewhere in the institution and in the country. The materials are thus a reasonable sample for the inquiry and the nature of the inquiry is seen to be of value in that regard.

This study takes on board the dynamics generated by group testing, but also dissects the individual responses in an examination set-up. There are a number of assumptions:

- that the test scores are the result of something the cohort did

- that the test scores contain information about the cohort, hidden as it were inside the scores, especially in terms of patterns which are discernable
- that the test scores fit into a body of knowledge about test scores, how they are derived and what they are meant to impart
- that the test scores have predictability built in because the process of generating them is repeatable with the same or similar groups.

The schooling of the cohort, in the main, conforms also to this uniformity, even if different styles of schooling are possible within it: the order, the goals, the assumptions about knowledge and the world are similar. For the students, the standard examination format is the dominant form to which they have been exposed. Our national school examination system, with which most of the cohort is familiar, took place just 6 months prior to my analysis. This examination system is premised, in mathematics, on technique-based examination questions, also referred to as constructed response questions (Reddy, 2006). These examinations have the central feature that students communicate their mathematical knowledge and their attitude about it through writing down solutions to questions. For the purposes of summative assessment, there is no dialogue, no group work, no verbal feedback, no inter- human exchange other than writing solutions to constructed response questions on a page.

The desire to posit generalizations about the cohort based on the findings is underlined by the empiricist nature of the research: the test scores act as verifiable evidence. But criticism of positivism has supported the search for alternatives:

> *Where positivism is less successful, however, is in its application to the study of human behaviour, where the immense complexity of human nature and the elusive*

and intangible quality of social phenomena contrast strikingly with the order and regularity of the natural world. This point is nowhere more apparent than in the contexts of classroom and school where the problems of teaching, learning and human interaction present the positivistic researcher with a mammoth challenge. (Cohen, et al., 2003)

As a consequence, the use of an interpretive framework is considered for the textual analysis which follows the qualitative analysis. The interpretive framework is premised on the notions of the individual and group actions of the students as well as on attempts to make statements about what they do. The actions of the cohort are seen as being

- individualistic
- coming from within
- possibly resisting attempts from outside (the system) to conform

(Cohen, et al, 2003)

An important assumption made here is that the contradictions inherent in posing a research approach with theoretically opposing paradigms are reflective of the phenomena under scrutiny. The power of such an approach is that the combined strengths of each can give rise to meaningful perspectives and provide means for action if desired. The system within which the cohort have placed themselves requires the kind of uniformity and attention to detail and rules which are characterised by the nature of the assessment practices through which the cohort have to go. The cohort, at the same time, brings to the system that which the system recognises, as well as that which is necessarily hidden, both the personal and collective. For example, the group may develop their own dynamics in order to deal with the challenges of the course: they may all refuse to engage with difficult material and reflect this attitude in a test. The study of these details at the micro and individual level, reminiscent of an ethno-methodological approach, is therefore meant to support findings from the qualitative study and vice versa. This is intentional: the focus is on the student produced text; all else which may or may not impact is assumed but not considered in meaningful detail. This is mainly to keep the research balanced, as it is anticipated that the quantitative analyses will highlight many relationships between the different tests and the textual analysis is intended to direct at least some of those findings. In the section on recommendations I return to the contextual concerns raised here.

In contrast to the positivist analysis of the test scores, I use a subjectivist viewpoint when I consider the individuals who make up the cohort. I closely analyse their written texts for clues to their individuality or conformity. Since this viewpoint accepts that individuals may choose to construe reality differently and may reflect this in their practices, I am mindful of the individualist nature of the participants. In so far as they exhibit their individual characteristics within a context where the cost to them in material terms might be high[12], I believe it is valuable to analyse their behaviour. It is my contention that it is likely that they may make other kinds of statements by such behaviour.[13] By choosing this kind of analysis, I am expecting to find that there will be room for detailed probing by me in my quest to get a fuller description of their mathematical achievement profile.

This study intentionally reflects on the system which produces the kinds of students which make up the cohort. To the extent that the study does so it is also a systemic study. Neubrand (2005), in discussing the general purpose of PISA, contrasts its focus on the systemic as opposed to the individual in mathematics education:

---

[12] for example, they could fail the course

[13] for example, that the course holds no meaning for them

> *The key question therefore is on the system level: What do we know about the mathematical achievement and its conditions in an educational system compared to what one can observe in an international overview? (p.80)*

The contrast with the individual is that studies focussed on the individual tend to be concerned about her "thoughts, difficulties, sources, and strategies when learning mathematics" (p. 79). For Neubrand, "the …question towards a system's efficiency in mathematics teaching and learning is not less challenging (p. 80). He further posits that:

> *Different didactical traditions and ways of teaching lead to different "inner structures" of mathematical achievement, made visible by different performance in the types of mathematical activities[14] (p. 82).*

It is this sense, of a system which produces a particular mathematical achievement which is "made visible" by certain performances, as evidenced by participant members of a cohort, which acts as the *fundamental rationale* for this study.

4.2 The Research Design

This section outlines the design for capturing the data in the different ways described in the theoretical framework:

- Data as test scores, examination results and content
- Data as text

The first part of the design features a taxonomy for capturing the cognitive skills performed by the cohort which were derived out of categories discussed in Chapter Three, section 3.2.

[14] viz., employing only techniques; modelling and problem solving activities using mathematical tools and procedures; modelling and problem solving activities calling for connections using mathematical conceptions.

This is followed by an outline of the characteristics of the participants. This leads into the specific context for the study, in particular the South African final school examination setting and the tertiary examination setting. I then present the characteristics of the data. This is a significant part of the design since the data is analysed in three ways:

- by means of a taxonomy of cognitive tasks,
- by an analysis of the individual items, and
- by a textual analysis of the tertiary examination scripts of the cohort (see table 4).

These analyses are presented in Chapter Five. The detail of individual test items is given. This has significance for understanding how the participants responded to the separate test items, as well as to the test items as a whole. Finally the schemas derived from the theoretical framework are presented with the data features.

This next section derives the taxonomy for the content analysis of the study. By classifying "knowledge" into four categories, partly suggested by Bloom, et al., (1956) and changing the nouns to verbs, the authors of the revised Bloom's et al. taxonomy (Anderson, 2005) have developed a two dimensional taxonomy table or grid (table 1).

It is instructive to note that a subset of this taxonomy and especially the cells which have been marked (table 2), together with instruction and assessment guidelines, describe almost the complete *attained curriculum* (Robitaille, 1993) in use in many South African schools (Adler, 1998, p. 17).

The Cognitive Process Dimension						
The knowledge dimension	1 remember	2 understan d	3 apply	4 Analyse	5 evaluate	6 Create
a. factual knowledge						
b. conceptual knowledge						
c. procedural knowledge						
Meta-cognitive knowledge						

Table 1: revised Bloom, et al. taxonomy (Anderson, 2005, p. 105)

	1 remember	2 understand	3 apply
a. factual knowledge	*		
b. conceptual knowledge		*	
c. procedural knowledge			*

Table 2: subset of revised Bloom, et al. taxonomy, (Anderson, 2005, p. 105), my additions.

It would appear that South African school textbooks (pre- outcomes –based education) follow this pattern quite closely, too, with "applications" added at the end of chapters (Gonin, et al., 1973).

Examples of the cognitive phases are:

- *to recall factual knowledge*: knowledge of definitions, for example, for the sine function in the *xy-plane;*
- *to understand conceptual knowledge*: knowledge about mathematical classes, for example, straight line functions; and,
- *to apply procedural knowledge*: routine algorithms to factorise trinomials

These are some of the most basic operations and procedures which students are familiarised with.

Against the background of the review of taxonomies outlined in Chapter Two, section 2.3 and the classification discussed in Chapter Three, section 3.2, I have created a framework in this section which will appropriately analyse certain standardised tests given to the cohort, with minimal extraneous influences. So when I analyse test items for the multiple choice test and tertiary examination, the grouping of those items into the categories listed in table 3 will reflect the reality of that class at that point in time and space. The framework is a taxonomy of content in terms of concepts, procedures and problem solving. It provides the rationale for analysing the test scores and test items. Given the history of South African education, the nature of our classroom practices, and my own personal experience as a student and teacher in high school, it is so that most of our practices and especially assessment practices remain locked into a procedural, computational framework, including surface level problem-solving (Reddy, 2006; Adler, 1998). This is one reason why the result of the diagnostic test which the cohort wrote is included: it contains a number of non routine problems which I assume the members of the cohort would find unusual. On account of that assumption, the test was used to assess their non routine skills.

Therefore I have constructed a revised framework taking those facts into account and placed them in a hierarchical order (table 3).

Taxonomy for analysing test scores and test items
Recall basic facts, procedures and concepts
Perform routine calculations
Problem solving level 1 with routine calculations)
perform routine procedures
Apply conceptual knowledge
Problem solving level 2 (with routine problems)
Problem solving level 3 (non routine problems)

Table 3: Taxonomy for analysis of test scores and test items

This is the framework which I will first use to analyse part of the data in Chapter Five.

4.2.1 The characteristics of the participants

The 129 participants of this study are first year, first semester students enrolled in an electrical engineering programme. As noted in the introductory chapter, they are mostly recent matriculants, that is, they have recently passed the final school leaving examinations in a South African school or college. Some of them are foreigners who have equivalent school leaving certificates. For the purposes of this study their relevant characteristics are as follows:

- they have all written the final examinations of the senior certificate course in mathematics or an equivalent examination.

- they have all written the computer-based diagnostic test hosted by the tertiary institution, CPUT, in February 2006.
- they have all written the final examination of the tertiary entry level mathematics course that they were enrolled for in June 2006.
- they all enrolled in the tertiary entry level mathematics course in electrical engineering for the first time in 2006, semester 1.

These are the common characteristics of the cohort which I have isolated because they represent a form of coherence and thus make reliability checks possible.

There are bound to be a number of additional, affective and non-academic differences within the cohort. Here I am thinking of cultural differences, for example, such as:

- language
- class and other socio economic differences
- schooling background

Although it may be possible to pick up some of these additional differences when analysing the text, for this researcher's purposes it is intended that the analysis of the data will be the key focus. The analysis of the data may show trends about student preparedness for further study in more depth than is possible by simply taking the results of the school leaving examination into account.

It is necessary to understand the immediate context within which these exam and test texts are produced. Students are taken through the course material via lectures and tutorials. The assessment for the course includes a minimal amount of coursework, short tutorial – type quizzes and three class tests. The students typically follow a routine pattern of questions, and would have seen examples of the test questions in class and also on revision sheets. The test questions are thus of a fairly standard type (see Appendix A). They sit apart, are given a formula sheet, and have to write these examinations alone (i.e. not in groups or pairs). No discussion is allowed at any time and no exchange of or reference to material is permitted. An invigilator monitors the test or examination and calls a halt when the time is up at which point the scripts are collected.

On the scripts, students are allowed to scribble, scratch out, and use sections of the examination book to work out their strategy or solutions, and so on. It is generally assumed that they will indicate clearly the part which is for the examiner to grade.

4.2.2 The characteristics of the data

The data that has been collected are the student results of the following achievement tests:

- Results from the Senior Certificate final examinations of the Department of Education, South Africa in Mathematics (scores only)
- Tertiary diagnostic multiple choice test in problem solving (basic and complex non routine questions) (scores only)
- Final examination in tertiary entry level mathematics course (scores and scripts)
- Final mark for tertiary entry level mathematics course. (scores only)

The data is described in the following sections. Since there are four types of scores, for ease of reference the following will be used:

School leaving examination marks are referred to as: *examination results (school).*

Tertiary entry level examination results are referred to as: *examination results (tertiary).*

Tertiary entry level final course marks are referred to as: *final results (tertiary).*

Multiple choice diagnostic test results are referred to as: *test scores (multiple choice).*

The examination scripts from the entry level tertiary course will be referred to as : *examination scripts (tertiary).*

Also, the source of the various test items are either the examination (tertiary) or test scores (multiple choice) and so will be referred to as follows:

Item 2 from the examination (tertiary) will become *item 2 (tertiary)*

Item 2 from the multiple choice test will become *item 2 (multiple choice)*

A schematic version of the data analysis is given in table 4.

Data analysis		
Choice of instrument	Nature of data	Research method
Statistical analysis	examination results (school), examination results (tertiary), final results (tertiary)	quantitative
taxonomy of cognitive skills	test scores and test items (multiple choice) and examination results (tertiary)	quantitative
Content analysis in terms of procedural and conceptual knowledge and limited problem solving skills	test items (multiple choice and tertiary)	qualitative
Semiotic analysis	examination scripts (tertiary)	qualitative

Table 4: Data analysis outline for the study

Examination results of the Senior Certificate, Department of Education, South Africa

These examinations are parametric: that is, they are standardised and are used to make statements about the whole community of students who write these examinations across the country. The examinations are norm referenced: they compare students according to set norms, established over many years. The data was collected from the university student enrolment forms, which include a copy of their official matriculation examination results. These examinations evaluate blind; that is, they do not consider socio-political contextual factors. Since these are national results from one examination body a number of assumptions are made. I assume that the results are valid and reliable, that is, that an A symbol for Mathematics is the same wherever the symbol was obtained in the country, and that a similar examination would yield similar results. These assumptions are made, mindful of the many threats to validity and reliability of national testing of this order. In particular, there are a number of factors which would influence the reliability or validity of such an examination:

- Individual factors (for example, language differences)
- Situational factors (for example, where the examinations were held)
- Instrument variables (for example, some examinations are harder or easier than the norm for that subject over the years)
- Assessment variables (although examinations are moderated, differences between examiners are bound to affect individual results)

This is one reason why, to enhance the reliability of the results, this research uses the method of *triangulation:* results from one set of data are cross checked with results from another.

Diagnostic multiple-choice test in problem solving

This is a non-parametric, computer based multiple – choice test. The computer-based test is situated in a *webct* programme run by CPUT, the tertiary institution from where the cohort is drawn. The original test was drawn up by Allan Walton of Highline Community College in the USA, in line with standardised diagnostic tests for two year, post school colleges in that country (personal communication, January 2006). The level of the test is pre-calculus. The test is norm-referenced since we can get some idea of the norm for the cohort's understanding of problem solving (non routine questions), and criterion – referenced since individual responses can be checked against those individual performances on other tests. The test has a fixed time limit of 30 minutes. Students were informed that they would have to take the test in a specific computer laboratory; that they had one day in which to take the test; and, that they could only have one attempt. Assistants were around to answer questions related to setting up and getting started. Students were not allowed to use calculators but they were allowed to use paper and pencil. The test items were mostly unfamiliar and very visual and are thus considered non-routine. There were a number of biases built into the test:

- The language was not the first language of the entire cohort which meant that ESL students were disadvantaged since a few questions required close reading. Phrases not in common English-as-a-Second-language (ESL) use include "adjacent faces", "tie", "helix".
- Cultural bias included use of culturally biased phrases and words such as "piggy-bank" and "locker padlock"

Characteristics of the test items of the multiple choice test

The following is a subset of the test items which the students were given in the multiple choice diagnostic computer based test. The subset was chosen to represent the full range of cognitive skills which were tested. A classification in terms of the cognitive skills taxonomy developed in section 3.2.5 is done for each test item:

item 1 (multiple choice): How many bricks…? This is an *apply routine calculations* problem since the participant has to count the rows and deduct for the bricks which were not there. It is routine in so far as it deals with counting in rows, even though the sketch is three dimensional.

item 2 (multiple choice): The next picture …? This is a *problem solving level 3* (non-routine) problem. The image of the corner which has been cut off as viewed from above was considered non-routine for the cohort by me.

item 3 (multiple choice): What fraction of the …? This is an *apply routine calculations (problem solving level 1)* problem. The two dimensional nature of the sketch and the familiarity of this sort of problem from secondary school makes this level 1.

item 4 (multiple choice): All sides of this figure …? This is an *apply routine calculations* problem because, after making the adjustment for the length and breadth the participant simply has to double to calculate the total score on the sketch.

item 5 (multiple choice): The numbers in this …? This is an *apply routine calculations* problem. The participant has to multiply by 3: that is the part to solve. The rest is to multiply and determine the correct answer.

item 6 (multiple choice): The mass of a container …? The word sum is a *level 1* problem: the language will test their understanding even though the problem itself is straightforward.

item 7 (multiple choice): A piggy bank holds…? This required an algebraic procedure, but the presence of words makes this a *level 2* problem.

item 8 (multiple choice): A pile of 50 sheets…? This is a straight *application of routine calculations* problem.

item 9 (multiple choice): Four children play…? This is a *level two* problem since the story is quite clear and can be worked out from scratch.

item 10 (multiple choice): Gina has a 4-digit…? This is a *level 3* problem for the cohort. They have not done combinatorics before.

item 11 (multiple choice): A boy and a girl run…? This is a *level 3* problem because of the word order involved in the calculations.

item12 (multiple choice): Half of 10^{-8} *is…?* This is a *routine calculation*.

item 13 (multiple choice): The diagonals on two…? This is a *level 3* problem. Understanding that the diagonals in the question meet a third diagonal to form an equilateral triangle is the key, but I expect that most will be distracted by the surface angles.

item 14 (multiple choice): A string is wound…? This is a *level 3* problem. The participant has to "see" that in the triangle with height 4 (circumference) and length 3 ($\frac{1}{4}$ of 12) the string has length 5 and that there are four such pieces.

item 15 (multiple choice): Two amoebas are placed…? This is a *level 3* problem as it is non routine. The task itself is closer to *level 2*.

Test items are included in Appendix D and the results are in Appendices E, F, G and H. A summary of the performance of the group is detailed in the next chapter (data analysis).

Final examination in tertiary entry level mathematics course

This examination forms the main source of the data. The examination is available in Appendix A. This was a non-parametric, summative examination. The assessment covered the second term's work, which is made up of basic calculus. I elaborate extensively on the test items as these items are subjected, in the study, to deep analysis. The test items are characterised as follows:

- they are consistent with test items covered in similar tests over a period of years
- they cover the content of the syllabus done in class by the students
- students are able to demonstrate their performance in terms of the various objectives of the test
- the test instructions are clearly marked
- independence of items are ensured: no one item is dependent on results from other items
- the test items cover the most basic aspects of each section of the syllabus: the level of difficulty is assured
- there are no known distracters in the test
- the items are all intended to discriminate between the cohort

Characteristics of the individual test items of the tertiary entry level examination

Item 1 (tertiary): Determine $\frac{dy}{dx}$ *and simplify where possible: a.* $y = \ln x^{\pi}$

This is a "perform routine procedure" item: the solution makes use of the derivative for $\ln x$ and also either the power law for natural logarithms, which is the preferred method or the chain rule for the composite function of natural logarithm with x^{π}.

The solution we want is: $\frac{dy}{dx} = \frac{\pi}{x}$.

Some students might apply the product rule for derivatives once they have $y = \pi \ln x$. In the item analysis we are only interested in the solution. The incomplete solution:

$$\frac{dy}{dx} = \frac{1}{x}\pi x^{\pi - 1}$$

was also accepted.

Item 2 (tertiary): Determine $\frac{dy}{dx}$ *and simplify where possible: b.* $y = \frac{1+x^2}{\sqrt{1-x^2}}$

This is another "perform routine procedure" item, even though the procedure is much more involved than item 1. The students may make use of the formula sheet and apply the correct formula (quotient rule). The simplification involves working with a square root in the denominator, something which they have been exposed to.

Item 3 (tertiary): Determine $\frac{dy}{dx}$ *and simplify where possible: c.* $y = (\sin 2x)^x$

This is similar to item 2 in that it is a routine procedure, except that the students have to recall it from their preparations or when it was done during exercises or previous tests. It involves the chain rule in a similar way to item 1.

Item 4 (tertiary): Determine the equation of the tangent line to the curve

$$x^2 - 2xy + 3y^2 = 4$$

at the point $(-2;0)$.

This item falls in the category "apply conceptual knowledge" since knowledge of what a tangent line to a curve means is the key to solving this question. The procedure required is not trivial at this level, but the completion of the question requires understanding of the concept mentioned. So there is a procedural aspect, involving implicit differentiation and the product rule, for which students have much practice, as well as rules for determining the equation of a straight line and these two procedures are connected conceptually.

Item 5 (tertiary): Determine $f''(0)$ *if* $f(x) = e^{-x^2}$

This item requires routine procedure. In terms of difficulty it involves the chain rule for the first derivative and second derivative, the product rule as well as functional substitution.

Item 6 (tertiary): Determine the coordinates and the nature of the stationary points of

$$y = x \ln x$$

This is placed in the "apply conceptual knowledge" cell because it involves some of the core ideas about differential calculus and stationary points, even though the procedure can be learnt. A student who shows a perfect solution would have had to combine knowledge of derivatives, what it means for the derivative of a function to equal zero and also make use of the second derivative to ascertain the nature of the stationary points.

Item 7 (tertiary): Calculate the maximum area that a piece of wire of length 40cm can enclose if it is bent in the form of a sector of a circle

This may be one of those problems where students can respond perfectly without understanding the core idea and only really move beyond the templates driven understanding further along in their studies, if at all. This is a procedural type question in the course but is used here as a 'apply conceptual knowledge" question because of the conceptual understanding that students must bring to this type of question in order for them to treat it successfully as a procedural problem. It would be the norm, given the number of concepts that have to be carried in the problem, for most of the cohort to find this a difficult problem.

Item 8 (tertiary): Determine the following integrals: a. $\int \left(\frac{3}{x} - 1\right)^2 dx$

Item 9 (tertiary): Determine the following integrals: b. $\int \frac{\sin^2 x}{\cos x}\, dx$

Item 10 (tertiary): Determine the following integrals: c. $\int_{-1}^{0} \left(x^2 + 2x + 1\right) dx$

These three items deal with basic integration methods. They are considered together since they test routine procedures. Item 8 requires a routine procedure, preceded by the algebraic manipulation of the square. Item 9 has an added difficulty in that sorting out the trigonometric fraction in the problem in such a way that the procedure for solving it is clearer may prove to be a difficulty. This is so especially given that the cohort did not have much time to consolidate this section of their work and given that trigonometry has mainly been taught in school. Item 10 is a routine procedure problem.

4.2.3 The tables of the data analysis

Taxonomy table

By using the taxonomy outlined in section 4.2 and the discussions of the test items in section 4.2.2 in terms of their procedural and conceptual content, a classification of the test items is done individually (table 5).

Taxonomy for content analysis (test scores)		
	MC non routine	AT Calculus
Recall basic facts, procedures and concepts		
Perform routine calculations	12	
Apply routine calculations (problem solving level 1)	1,3,4,5,6,8	
perform routine procedures		1,2,3, 8, 10
Apply conceptual knowledge		4, 5,6
problem solving level 2 (with routine problems)	7,9	7, 9
problem solving level 3 (non routine problems)	2,10,11,13,14,15	

Table 5: Taxonomy for content analysis of test scores

Content analysis table

The original framework for the content analysis of test items is given in table 6.

Content Analytic Framework (test items)	
Attempted question:	
Solution correct	
Same mistake	
None of above	

Table 6: Content analytic framework

It turned out that the results were much more complex and caused me to extend the categorization (see Chapter 6). These characteristics, however, were a good starting point for the analysis.

Textual analysis table (critical indicators)

The critical indicators derived from the textual analysis, as discussed in section 3.4.2 are reproduced in tabular form (table 7):

Critical indicators (textual analysis)
Procedural competence
Conceptual competence
Logic
Reflection
Confidence
Dealing with crisis
Creativity

Table 7: Critical indicators for textual analysis

The competence scale

In order to distinguish the degree of competence between participants, but at the level of individual test items, I have set up a competence scale (table 8). This scale will be used in reference to the main cognitive areas as outlined in the taxonomy for the analysis of the content: procedural and conceptual competence, knowledge of facts, procedures and concepts, and problem solving (levels 1, 2 and 3).

Competence scale				
Not competent	Poor competence	competent	Good competence	Very good competence

Table 8: Scale of competence of cohort

5. Chapter 5: Data Interpretation and Analysis

5.1 Introduction

In this chapter I describe the various analyses which I engaged in to arrive at the mathematical achievement of the cohort. Firstly, I show correlations between the examination results (school) and the examination and course results (tertiary). This is done using the statistical package SPSS, version 14. Certain conclusions are drawn from this section, some of which lead to further exploration. Then a statistical analysis is done for the test scores (multiple choice). This is followed up by a statistical analysis of individual test items (multiple choice). The use of the taxonomy of cognitive skills set up in Chapter 4, section 4.2.4, to complete the classification of the cohort's test items (multiple choice and tertiary) follows.

The next section deals with the content analysis of the test items (tertiary). These are placed on the content analysis table (table 6) created in Chapter Four, section 4.2.4. Possible trends emerging from the various analyses are considered at this stage. Suggestive outcomes from the statistical analysis of the examination results (tertiary) are followed up using further statistical analysis. In particular, the correlation between item 2 (tertiary) and final results (tertiary) is investigated using SPSS, version 14. The semiotic analysis of a sample group of participant members of the cohort is conducted at this stage, including three members who performed in a curious way on item 10 (multiple choice). The Chapter ends with a discussion on the building of an integrated profile of individual members of the cohort.

A schematic version of the outline of the data analysis is given in table 9.

Data analysis		
Choice of instrument	Nature of data	Research method
Statistical analysis	Examination results (school), examination results (tertiary), final results (tertiary)	Quantitative
Statistical analysis	Test scores (multiple choice)	Quantitative
Taxonomy of cognitive skills	Test scores, examination results and test items (multiple choice and tertiary)	Quantitative
Content analysis in terms of procedural and conceptual knowledge and limited problem solving skills	Test items (tertiary)	Qualitative
Statistical analysis	Item 2 (tertiary) vs. final results (tertiary) Item 3 vs. item 5 (tertiary)	Quantitative
Semiotic analysis	Examination scripts (tertiary)	Qualitative

Table 9: data analysis

5.2 Comparison between examination results (school) and examination results (tertiary)

In this section, I show the correlations between the participants' examination results (school), the examination results (tertiary) and the final results (tertiary). The data was analysed using SPSS, version 14, and summarized in tables and charts, to substantiate the claims that I make.

To begin with, I was interested to know what relationship, if any, existed between the participants' examination results (school) and their examination results (tertiary). To get a first picture, I drew a scatterplot (figure 1). The statistical result was clear but the implications were inconclusive. Although low scores in the school examination related to low scores in the tertiary examination, and high scores in the school examination related to high scores in tertiary maths, there were many outliers. In fact there was no obvious trend. Some participants scoring between 40% and 60% on the school examination scored less than 20% on the tertiary examination, while a few scoring around 30% on the school examination scored above 40% on the tertiary examination. The scatter plot indicated that the correlation was low.

A matrix scatterplot (figure 2) for the variables—examination results (school), examination mark (tertiary) and the final results (tertiary)—revealed a definite relationship between the latter two. This is not entirely unexpected as the examination mark (tertiary) is an integral part of the final results (tertiary). As for the case of the examination mark (tertiary) with the examination results (school), the relationship between the examination results (school) and the final results (tertiary) was statistically clear but the implications for my purposes were inconclusive. These results do not appear to tally with the descriptive statistics (table 10a). The means of the final results (tertiary) and the examination results (school) are very close: 53.2 and 54.1, respectively, but this is not relevant for an inquiry about correlation. The difference in the standard deviations (4.7) explains only partially the differences between the scores, but does not touch upon the outliers as revealed by the scatter plot. The lower mean for the tertiary examination (38.6) relative to the final results (tertiary) can be seen on the scatterplot diagram as the final results (tertiary) are higher than the examination results (tertiary).

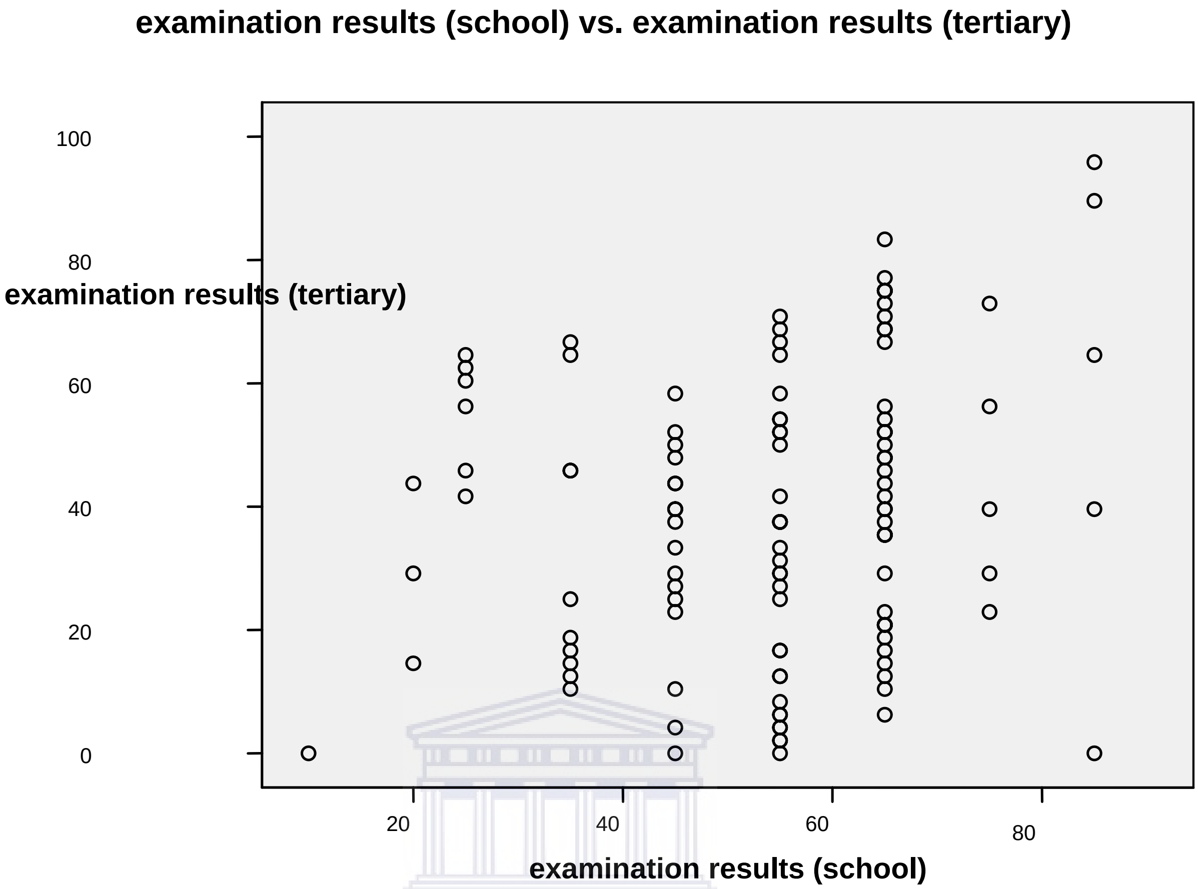


Figure 1: scatterplot school mathematics (matriculation) vs. tertiary examination (percentage exam)

scatter plot analysis

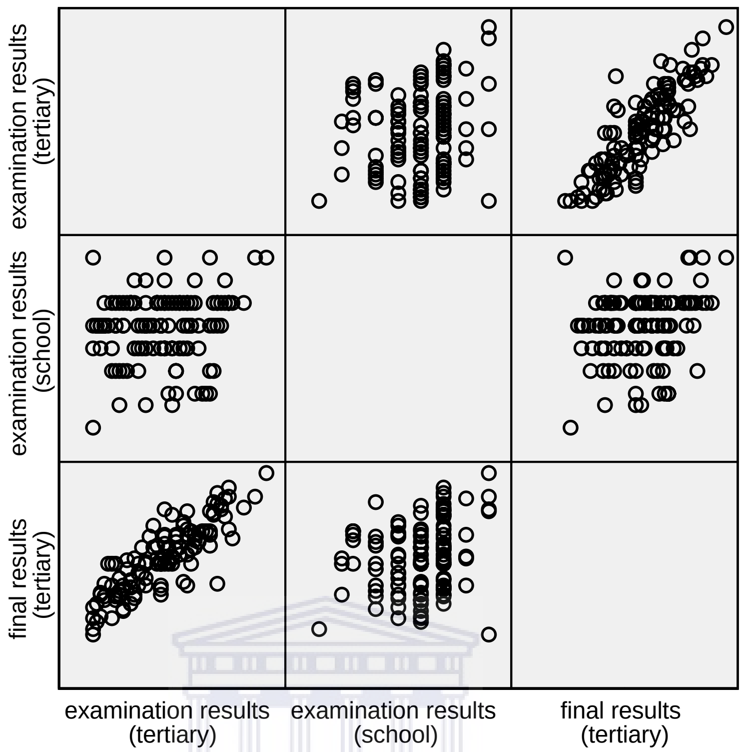


Figure 2: matrix scatterplot for examination results (school), examination results (tertiary) and final results (tertiary).

Descriptive Statistics

	Mean	Std. Deviation	N
Final results (tertiary)	53.2	20.1	135
Examination results (school) results	54.1	15.4	120
Examination results (tertiary)	38.6	23.2	130

Table 10a: descriptive statistics for final mark (tertiary), matriculation results, and examination results (tertiary)

To conduct the bivariate correlation between any two out of the three variables under discussion, I used Pearson's product moment correlation coefficient. The results are in table 10b. The correlation between the examination results (school) and the final results (tertiary) is highly significant ($p < 0.001$) and the relationship is positive (an increase in one matches an increase in the other). But the strength is low ($r = 0.243$). This matches the scatterplot image of a less than cohesive relationship. The correlation between examination results (school) and the examination results (tertiary) was significant at the 0.05 level ($p < 0.05$), positive, but not strong ($r = 0.182$). As expected, the correlation between the examination results (tertiary) and the final results (tertiary) was very high ($r = 0.836$), positive and highly significant ($p < 0.01$).

Bivariate correlations between final results (tertiary), examination results (school) and examination results (tertiary)

		Final results (tertiary)	Examination results (school)	Examination results (tertiary)
Final mark	Pearson Correlation	1	.243(**)	.836(**)
	Sig. (2-tailed)		.008	.000
	N	135	119	130
Matriculation results	Pearson Correlation	.243(**)	1	.182(*)
	Sig. (2-tailed)	.008		.048
	N	119	120	118
Percentage exam	Pearson Correlation	.836(**)	.182(*)	1
	Sig. (2-tailed)	.000	.048	
	N	130	118	130

** Correlation is significant at the 0.01 level (2-tailed).
* Correlation is significant at the 0.05 level (2-tailed).

Table 10b: Bivariate correlations between final results (tertiary), examination results (school) and examination results (tertiary).

In summary, the statistics reveal positive relationships between the examination results (school) and the tertiary results, although those relationships did not appear to be very strong. The data does not suggest that good results on the school leaving examination correlate positively with the tertiary entry level examination. Even if it did, inferring causality from correlation is not correct.

5.3 Analyses of the multiple choice diagnostic test

The multiple choice test is in Appendix C and the results are in Appendices D and E. Individual scores per test item were used as part of my attempt to highlight the performance of the sample cohort subgroup[15]. Appendix E gives the performance of the cohort as a whole for each test item. This is what I will focus on below.

Performance of cohort on multiple choice test: total score statistical analysis

Table 11 contains descriptive statistics of the cohort's performance on the multiple choice test items. It is clear looking at the table that the measures of central tendency are all clustered around 8. The deviation from the mean is 2.87, indicating that, assuming a normal distribution, all those with scores between 5 and 11 would lie within approximately 68% clustered around the mean. In light of a small standard error, I can conclude that the data is fairly accurately reflected. The range of 13 is perhaps expected as the problems are of a non-routine kind and the differences between the students would show up this dramatically, with the highest ones at 14 and the lowest ones at 1. The value of the skewness and kurtosis converted to *z*- scores to standardise them (using the standard errors), are respectively -0.774 and -1.24. This showed that the distribution was flat and that there was a pile up to the immediate right of the mean. Thus the distribution was not perfectly normal but the low scores close to zero indicate that it is reasonably normal. The histogram and normal overlay illustrates these points (see figure 3). Thus I can conclude that the performance of the students on the multiple choice test indicated that there were wide differences among the cohort. Also, that the spread of scores for the individual participants was close to normally distributed.

[15] See section 5.4

Multiple Choice descriptive statistics

N	Valid	125
Mean		7.87
Std. Error of Mean		.25
Median		8.0
Mode		8.0
Std. Deviation		2.88
Variance		8.33
Skewness		-.16
Std. Error of Skewness		.21
Kurtosis		-.53
Std. Error of Kurtosis		.43
Range		13.0
Minimum		1.0
Maximum		14.0
Sum		984.0
Percentiles	25	6.0
	50	8.0
	75	10.0

Table 11: Descriptive statistics of the multiple choice test

Performance of cohort on multiple choice test: individual test item statistical analysis

A description of the table in Appendix E follows. The two columns in the worksheet, column E and F refer to the upper and lower 25%. This reference is to the 75th and 25th percentiles of the cohort, that is, the upper 25% and the lower 25% of the cohort if they were ranked from best overall performer to lowest. The discrimination refers to the difference between these two groups and is a statistic that is used to decide how well a test item discriminates among the cohort.

item 1 (multiple choice): How many bricks…? As indicated in D7, 93.94% of the cohort answered this question correctly. As an "apply routine calculations" problem, the cohort showed that the problem was within their mathematical faculty.

MC item 6(multiple choice): The mass of a container…? This item was answered correctly by 48.75% of the cohort, that is, about half of them (D12). Of the top performing 25% of the cohort overall, 85.71% got this answer correct (E12). This percentage suggests that the problem was a good discriminator. Of the bottom 25% of the cohort 12.5% got the correct answer (F12). The value of the discrimination (73.21, G12) confirms that the problem was a good discriminator among the cohort.

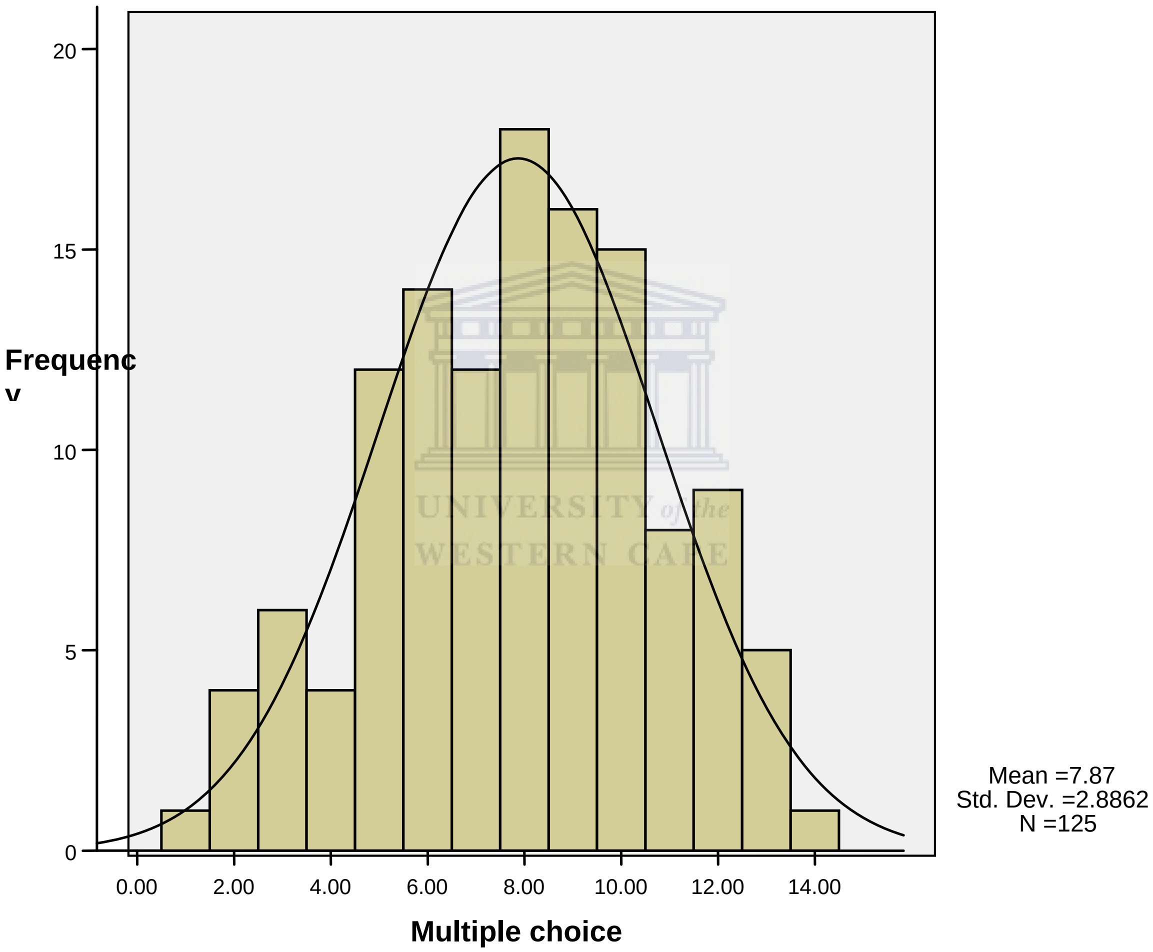


Figure 3: histogram and normal curve overlay for the multiple choice test

MC item 10 (multiple choice): Gina has four digits…? This item was correctly answered by only 28.57% of the cohort (D16). This is a very low percentage. I classified this item as a level 3 problem (section 4.2.2), the highest level of cognitive demand. And I predicted that the cohort would not manage this level of problem very easily, given my assumptions about their schooling background. Of the top 25% overall, only 52.63% got the correct answer (E16), while of the bottom 25% overall, 19.05% got the correct answer (F16). Considering that for a number of other test items none of the bottom 25% got the answers correct (items 4, 11, 12, 13 and 14), I considered this particular group worthy of further investigation. I thought this could be done even if only to ascertain that they got the correct answer by chance. To do this I first ranked the cohort in terms of their overall scores for the multiple choice test. Then I took the bottom quarter and looked for participants who fitted this profile. I then found their total scores for the assessment examination and course. The participants who fitted this profile and whom I ended up investigating further were participants number 75, 96 and 100 (table 12):

Participants who answered item 10 (multiple choice) correctly but were from the bottom 25%		
Participants Number	Total score for multiple choice test Out of 15	Total score for tertiary examination Final result (tertiary) in brackets
No. 75	5 (33%)	29% (50%)
No. 96	6 (40%)	52% (66%)
No. 100	6 (40%)	29% (50%)

Table 12: participants who answered item 10 (multiple choice) correctly but were drawn from the bottom 25% overall.

As can be seen from the descriptive statistics shown in table 11, the mean for the cohort was 7.87 and all three participants were below that figure. The median and mode were both 8. The standard deviation of 2.87 meant that the three participants were part of 68% approximately who were within one standard deviation of the mean. From this I expected that the three participants were near the top end of the lowest group. This was indeed the case as the ranking shows (table 13). All three participants obtained a mark higher than 50% and thus passed the course although only number 96 passed the tertiary examination.

Ranking of participants in lowest quartile	
Participan t Number	Rank position of selected students from lowest to highest of 34 Participants
75	21st
96	31st
100	33rd

Table 13: ranking of participants in lowest quartile

I decided to make one more follow-up with these participants: I wondered which problems they answered correctly or nearly correctly. My interest was mainly to assess whether they were part of the 63% of the cohort who answered the multiple choice test item 10 correctly, by chance or not. I was interested in finding some indication of the manner in which they answered the tertiary examination test items which could indicate that there was more involved than mere chance. I was aware that my answer would remain speculative but I was interested to explore it nonetheless. The results of this investigation followed that of the item analysis of the tertiary examination which is recorded in section 5.4. This was a direct example of the method of triangulation mentioned in section 4.2.3. By cross- referencing the data with the different kinds of analysis my attention was drawn, in this case, to these three participants. The results of my further investigations of their achievement are detailed in section 5.7 after the analyses of the test items of the tertiary examination in section 5.4 following this discussion.

5.4 Analyses of the test items of the tertiary examination

Introduction

Using the framework developed in section 4.2.4, the analysis of the test items (tertiary) which follows are to be viewed in light of the discussions in section 2.2, particularly. There is no attempt in this section to discuss in detail the classification of the test items beyond providing the labels discussed in section 2.2. In other words, terminology such as templates-driven methods, the use of child methods, the interpretations of variables, as discussed in section 2.2, will here simply be referred to where applicable.

It will be shown that specific patterns of response to some of the items by members of the cohort do emerge. It would appear that these patterns have not been characterised in the research literature in any great depth before in the way it is done here.

Analysing Item 1 (tertiary): Determine $\frac{dy}{dx}$ and simplify where possible: a.

$y = \ln x^{\pi}$

By using the content analytic framework I sorted the responses to item 1. The results are given in table 14.

Content Analytic Framework: item 1	
Content analysis	Number of participant members
Attempted question:	126
Solution correct	69
Same mistake	57
None of above	3

Table 14: The content analytic breakdown of test item 1 (tertiary)

Discussion: There were four common mistakes which the 57 participant members made:

a. applying the natural log rule for differentiation as if the composite function was a simple function: $y' = \frac{1}{\pi^x}$

b. taking natural logs both sides of the equation but using the log laws incorrectly on the right hand side: $\ln y = \ln x^\pi - \ln x^\pi + \ln 2^{x \log_2 3}$

c. taking the derivative of x^π as if the natural log was not present: $y' = \pi \ln x^{\pi-1}$

d. simply inverting the right hand side: $y' = \frac{1}{\ln x^\pi}$

The face-value difficulties which the 57 participant members demonstrated were:

- Not recognising composite functions and hence when to use the chain rule (Dubinsky, 1991)
- Not seeing the structure of the equation (Dreyfus and Hoch, 2004)
- Ignoring natural log (ignoring letters as objects)
- Applying the structure of the derivative of $\ln x$ inappropriately (that is, assuming the answer must be 1 over something) (child methods, over generalisation)

Dubinsky (1991), in commenting on students' difficulties with the formation of the composition of two functions, places this difficulty in the category *coordination*: the composition of two or more processes to construct a new one. A student would first have to understand each function as an object; then understand that each object came about as a result of a certain process; then coordinate the two processes to form a new one. This new process would eventually lead to the formation of a new object, the composite function. The participant would then have to link this new object with the notion of derivative. As became evident, members of the cohort had great difficulty in dealing with composite functions, especially as applied to the chain rule.

Analysing Item 2 (tertiary): Determine $\frac{dy}{dx}$ and simplify where possible: b.

$$y = \frac{1+x^2}{\sqrt{1-x^2}}$$

The initial breakdown in terms of the textual analytic framework (fig 4.2.4) produced the results depicted in table 15.

Content Analytic Framework: item 2 (tertiary)	
	Number of participant members
Attempted question:	114
Solution correct	22
Same mistakes	92
None of above	15

Table 15: The content analytic breakdown of test item 2 (tertiary)

Discussion: At first the difference between those who produced the correct answer and those who did not, seemed much too wide. Given that the solution to the question required a number of procedures, further analysis was considered. This led to an expansion of the content analytic framework which worked for the rest of the test items as well. It also led to further speculation: could the classification of the students in terms of their responses to item 2 have significant effects when compared to other results? In other words, was item 2 a good discriminator for overall performance? Would a student who is able to hold the multiplicity of procedures together as required by this problem demonstrate thereby the necessary cognitive maturity to do well overall? These latter questions were investigated using statistical analysis with data from item 2 and also the final examination marks for the cohort. The results are discussed in section 5.5.

To uncover the extended classification I measured the student responses against a constructed response. A typical algorithm for solving the problem is given as follows:

Step 1: Recognise that the problem is to differentiate a quotient and make use of the quotient rule; alternatively take natural logs and make use of log differentiation. A total of 114 participant members attempted to do this (that is, groups 1, 2 and 3, table 16). Of them, 58 carried out this procedure correctly (groups 1 and 2, table 16), 56 did not (group 3, table 16).

The quotient rule on the formula sheet is given by: $\frac{dy}{dx} = \frac{f'g - fg'}{g^2}$ `

The thinking and skill involved include

- Recognising the form for the quotient rule and applying it to the problem
- Performing the routine procedures for differentiation
- Squaring g

From the analysis, it appeared that unsuccessful students experienced the following difficulties:

- Getting the order of the difference in the numerator wrong (writing $fg' - f'g$ [16])
- Not being able to fully differentiate $\sqrt{(1-x^2)}$ which requires the use of the chain rule, a differentiation rule of complexity as discussed under item 1 above.

Step 2: Simplify the result and get the correct answer: 22 did this correctly, 36 did not.

In simplifying, participant members had to deal with:

$$2x\sqrt{1-x^2} - (1+x^2).\frac{-2x}{2\sqrt{1-x^2}} \quad \text{or}$$

$$\left(\frac{1+x^2}{\sqrt{1-x^2}}\right)\left(\frac{2x}{1+x^2} + \frac{-2x}{2\sqrt{1-x^2}}\right).$$

Some of the difficulties exposed included being unable to work with the $\sqrt{\ }$ sign in the denominator and in some cases getting the sign for the second term wrong. There were some who did not make use of the quotient rule but who made some attempt to answer the question (group 4 on table 16). The main approach seemed to be to determine the derivative of the numerator and denominator separately, and the denominator was the difficult one. This subgroup was an example of a group which was united in terms of what they did not do as opposed to what they did. An analysis of what each individual did would have been valid within the framework of my analysis. In section 5.5 I do exactly this with a sample of the students.

The summary is given in table 16.

[16] - often confused with the product rule order: $fg' + f'g$)

Content analytic framework (revised): Item 2 (tertiary)		
Stage completed	Participant members	number
Group 1: Produced correct solution	3,12,13,26,**27,29,50**,54,58,62,85,86,91,**93,95,109**, **116**,118,**120,127**,131,**135**	22
Group 2: Succeeded in first step only	1,2,4,5,7,20,23,24,28,31,33,39,40,44,48,52,53,57,61,66,68,76,81,87,89,94,101,104,106,110,113,115,122,126,132,138	36
Group 3: Knew the correct rule to use in first step but could not proceed	10,15,16,17,18,19,21,22,25,30,34,37,38,41,42,43,45,47,51,55,56,63,64,65,67,69,71,72,73,75,77,78,80, 82,83,88,90,96,97,99,100,102,103,105,107,108,111, 112,117,121,123,124,128,130,133,137	56
Group 4: Followed a path leading elsewhere (did not use the rule)	6,8,11,14,35,36,70,74,79,84,98,114,125,129,136,	15

Table 16: The content analytic breakdown of test item 2 (tertiary)

The participants marked in bold text in group 1 of table 16, who produced the correct solution for item 2, also produced the correct solution for item 3. These two items (item 2 and item 3) were placed in the same cell on the taxonomy framework in section 4.2.4 (table 5), namely, ‘perform routine procedures’. Though the routine procedures were not identical, these two items were compared for consistency. The participants in the bold text in table 16 and also the participants in group 4, table 16 (the ones who did not fit any pattern listed here, for both items) performed consistently across the two items. This kind of tracking across items proved useful for determining consistency of procedural and conceptual behaviour. Where the behaviour was not consistent, it was desirable to look for other factors.

Analysing Item 3 (tertiary): Determine $\frac{dy}{dx}$ and simplify where possible: c.

$y = (\sin 2x)^x$

A constructed-response solution to item 3 is given as follows:

*Step 1: T*ake natural logs of both sides of the equation, and drop the power *x* on the right hand side: $\ln y = x \ln(\sin 2x)$

Step 2: Determine the derivatives:

2.1 Implicit differentiation for: $\ln x$

2.2 The product rule for: $x \ln \sin 2x$ and

2.3 The chain rule for: $\ln \sin 2x$

to get the equation: $\frac{1}{y}\frac{dy}{dx} = \ln(\sin 2x) + x.\frac{2\cos 2x}{\sin 2x}$

Step 3: Simplify and get: $\frac{dy}{dx} = (\sin 2x)^x (\ln(\sin 2x) + 2x \cot 2x$

Those participant members who made use of the natural log strategy as outlined above did not divide themselves easily. The simplest initial division was between those who produced the correct solution and those who did not. Of the types of differentiation outlined in step 2 above it turned out that there was no obvious hierarchy in terms of how the participants responded. This was the case even though the derivative for $\ln x$ was included on the formula sheet and could therefore have been copied down, whereas the derivative for $\ln \sin 2x$ had to be determined by the participants.

From my analysis, it turned out that there were two significant routes, other than the one leading to the correct answer, which a number of participant members preferred to take:

- Some saw the function as a composite exponent and used the rules for differentiating such functions.
- Some saw the function as a polynomial and used the power rule, namely $\frac{dy}{dx} = nx^{n-1}$, to find the answer.

Both these routes produced the wrong answer. It could be argued that the problem was not with the procedural or even necessarily conceptual difficulties, but with *perceptual* difficulties: the participant members did not recognise what sort of function they were dealing with. Such a viewpoint would fit the analysis of frames*:* they accessed the wrong frame for solving the problem (Davis, et al., 1978). It is conceivable that some of these participant members would have faired well had they started off correctly, that is, taken the natural log route, since many did demonstrate their ability to differentiate the composite functions. Thus their procedural proficiency was not necessarily the issue. It is possible that a lack of understanding about functions in general was more the issue.

These three groups, that is, the ones who chose the standard route, the ones who used the composite exponent route and the ones who treated the function as a polynomial, stand out from the rest of the participants. The rest of the participants either did not make any attempt or their attempts defied description in terms of my stated criteria. Of significance is that the same group who were excluded in item 2 were again excluded here, but there were a number of additional participant members. I decided not to investigate the group from item 2 further, unless I discovered other relevant details during the analysis of the other items. The division of the cohort for this item was an indication that the question, including its interpretation, was of a greater difficulty than item 2. The results are given in table 17.

Content Analytic framework (revised): Item 3 (tertiary)		
Stage completed	participant members	Number of students
Take logs, determine $\frac{d(\ln x)}{dx}$	3,20,39,99,105,128,132,133	8
Take logs, determine $\frac{d(\ln x)}{dx}, \frac{d(\ln \sin 2x)}{dx}$ (not correct)	18,38,75,88,131	5
Take logs, show use of the product rule and do chain rule correct	28,94,106,118,130	5
Take logs, show use of product rule and correct $\frac{d(\ln x)}{dx}$	2,13,42,62,63,72,77,91,107	
Correct solution	16,23,**27,29**,47,**50**,52,57,76,80,87,90,**93,95**,100,**109**,112,**116,120,127,135**,137	22
Common error: treated $(\sin 2x)^x$ as an exponent	4,10,12,15,21,24,30,31,33,34,40,43,53,58,65,66,68,71,83,85,89, 96,101,104,111,113,115,122,138	29
Common error: treated $(\sin 2x)^x$ as a polynomial	1,5,19,25,45,48,55,64,110,117,121,123,124	13
None of above	6,8,11,14,35,36,70,74,79,84,98,114,125,129,136,7,17,22,26,37,41,44,51,54,56,61,67,69,73,78,81,82,86,97,102,103,108,126	

Table 17: The content analytic breakdown of test item 3 (tertiary)

Analysing Item 4 (tertiary): Determine the equation of the tangent line to the curve $x^2 - 2xy + 3y^2 = 4$

The solution of this test item can be broken down into a number of components:

Step 1: Determine the derivative of the equation

Step 2: Determine the value of the derivative at the point (-2; 0)

Step 3: Substitute this value and the point (-2; 0) into a standard equation for the straight line and determine the value of the constant c. Write out this equation of a straight line.

Most of the work required is procedural but the participant members must know that the link between the tangent line and the curve is that *the two gradients are equal at the tangent point*. The aim is for the participant members to determine the value of this gradient. That is the conceptual requirement.

The breakdown of the responses of the cohort revealed that most attempted to determine the derivative, but not all determined the equation of the line. This difference would appear to be a good dividing feature, notwithstanding that determining the derivative proved somewhat difficult for some. Thus I distinguished between those who attempted to determine an equation for the line and those who did not. Again their scores within these chosen groups is not the main issue here: I am prioritising the conceptual divide, but also the range of procedural routes, most of which were incomplete. The summary is given in table 18.

Discussion: Some 36% seemed to know what the links are between the derivative of the equation and the equation of the straight line. Twenty one percent made some attempt but could not complete due to procedural difficulties. 36% did not approach the question in a way which indicated that they knew that they were meant to introduce the equation of a straight line. The range of categories which results from this kind of breakdown of responses presents a challenge for how one is to understand the participants who produced these results. 80% of the candidates knew that they had to determine the derivative.

Content Analytic Framework (revised): Item 4 (tertiary)		
Stage completed	no	Participant members
No attempt	4	35,79,107,114
Solution correct	38	1,4,5,11,12,13,16,23,27,29,31,33,34,50,52,57,58,62, 77,80,83,86,91,93,95,104,109,113,115,116,118,120, 124,128,130,132,135
Sign wrong, process correct	9	17,18,38,48,68,82,96,127,137
Variations: coding for this table: y', line r – right, w-wrong		
a. w, w	4	14,36,87,138
b. w, r	24	3,10,19,21,45,55,56,67,69,72,73,74,81,84,85,88,90, 97,99,105,106,111,122,133
c. r, w	2	44,61
No line: y', r	9	7,15,28,37,43,63,94,100,101
y' w	17	2,20,22,39,41,42,47,54,65,71,78,89,103,108,110,126, 131,102
Other methods, unrelated to the problem		
1. Substitution	9	8,25,64,70,75,76,112,125,129
Make equation = 0	5	6,24,40,51,53
Other	7	30,66,98,117,121,123,136

Table 18: The content analytic breakdown of test item 4 (tertiary)

60% of the candidates knew that they had to work with the straight line. These are overarching figures which give insight into the achievement of the cohort. Of course, these figures also indicate what the pedagogical challenges are. By cross checking and identifying common themes across items I made assertions about the cohort with more confidence.

While the ideal is to analyse the items individually, this raises the problem of practicality, as the group is large. I decided that I would have to make a choice in terms of grouping the overall responses in a meaningful way. By determining the classification for all the items I was able to assess better the direction of the analysis (section 5.5 below).

The fact that the responses of the students for item 4 were so divergent appeared to reinforce claims that there can be no understanding without conceptual understanding, irrespective of the order of the procedural and conceptual knowledge development. Sfard's (1991) idea that the former precedes the latter is one strand of that debate. The system to which many of the participants were exposed prior to doing tertiary mathematics (South African context) privileges the procedural at the expense of the conceptual at the secondary level.[17] This may well be a major source of the issues being explored here.

Analysing Item 5 (tertiary): Determine $f''(0)$ if $f(x) = e^{-x^2}$

The results of this test item posed an initial problem: 48 of the participant members could not differentiate the function correctly. This contrasted dramatically with other items which all required differentiation as well but had better results. Since the question requires almost entirely a procedural approach (and understanding of functional notation at the end) most attempted the two derivatives. The substitution, $f''(0)$, posed problems for some of those who successfully determined the derivatives. I would argue that this was another case for one of Dubinsky's (1991) construction methods in reflective abstraction known as *coordination.* This was discussed in the analysis of test item 1 above. The participant members have to recognise the composite functions as a coming together of the exponential function and the polynomial square function; this has then to be combined with the chain

[17] See Mathematics in the South African school context, Chapter One

rule method of determining the derivative. The summary of the breakdown is given in table 19.

Content Analytic Framework (revised): Item 5 (tertiary)		
Stage completed	no	Participant members
No attempt	1	114
Solution correct	26	2,4,7,20,26,31,40,44,47,57,62,75,77,84,89,91,94,95,104,109,118,120,124,127,135,138
y' (right) y'' (wrong)	42	3,5,10,12,13,14,15,16,19,23,25,29,33,37,41,43,48,52,55,58,1,63,65,68,72,80 81,85,90,100,105,106,110,111,113,115,121,122,126,133,137
y' (wrong) y'' (wrong)	47	6,8,11,17,21,22,24,27,30,35,36,38,39,42,45,51,53,54,64,66,67,69,70,71,73,74,78,79,82,88,93,97,98,99,101,102,108,112,116,117,123,125,128,129,130,132
$f''(0)$ (wrong)	13	1,18,28,34,50,56,83,86,87,96,107

Table 19: The content analytic breakdown of test item 5 (tertiary)

Discussion: The central issue appeared to be the interpretation of the function. Seeing it as a composite exponential function would require participants to make use of the chain rule and consult the formula sheet for the different derivatives. Making use of logarithmic differentiation, however, would have been an option, assuming the derivatives were then determined correctly. The evidence indicated that the participants did not know how to interpret the composite exponential function. This prevented many from demonstrating the procedural fluency that they have shown elsewhere.

Analysing Item 6 (tertiary): Determine the coordinates and the nature of the stationary points of $y = x\ln x$

I had to consider what would be involved in the algorithmic solution to this problem for the participants. On the face of it the algorithmic solution should be straightforward enough for the average student who has studied school calculus, where learning an algorithm is part of the syllabus. A standard algorithm is as follows:

Step 1: Determine the derivative of y: $y' = 1 + \ln x$ *(procedural)*

Step 2: Determine the value/s of x for which the derivative is zero: $x = \frac{1}{e}$ *(conceptual)*

*Step 3: Determine the y-component of this **critical / stationary** point:* $y = -\frac{1}{e}$ *(procedural)*

Step 4: Determine the second derivative: $y'' = \frac{1}{x}$ *(procedural)*

Step 5: Determine the sign of the second derivative for $x = \frac{1}{e}$ *: positive (conceptual)*

*Step 6: Deduce the nature of the stationary point**: minimum** turning point (conceptual)*

In attempting a classification of how participant members responded to the question, I had to make choices. One approach was to assume that that which requires conceptual application is more difficult than that which requires a procedure. Of course, it is possible that one could encounter a simple conceptual problem versus a rather difficult procedure, but this case I did not consider beforehand. I started by taking as undisputed the following classification:

a) those who did not attempt the question
b) those who went through the whole process (showing conceptual understanding if not always correct calculation).

That left me with a large number of participants who fell into a very broad category made up of:

c) those who know to determine y'
d) those who know to determine y''
e) those who know to seek values for x for which y' equals zero.
f) those who know to determine the coordinates of the stationary points.

g) those who know to find the sign of the second derivative at the point determined in e), and
h) a combination of d to g above.

The choice that I made was to distinguish between those who employ the concepts and those who don't, irrespective of correct steps taken along the way or not. Thus the most important factor I considered was whether the participant member understood what needed to be done. To place such a person onto a correct path would involve relatively straight forward algorithmic practice. This is based on an assumption that the procedural difficulty is not held up by a conceptual problem from mathematics done earlier in the participant's school career. Those who did not proceed further because they encountered a procedural problem leave me with a dilemma: would they have continued had they not encountered that difficulty? In other words, was it a procedural hold up which caused the breakdown or would the participant member have failed to proceed anyway on account of a conceptual block? The resultant analysis is given in table 20.

Content Analytic Framework (revised): Item 6 (tertiary)		
Stage completed	no	participant members
Did not attempt the question:	12	36,61,71,79,82,105,110,112,114,125,126,136
Solution correct	22	5,10,17,21,23,29,33,40,50,52,53,58,84,86,87,90,91,111,115,120,127,137
Procedural only: y' wrong	30	8,14,18,19,25,30,35,37,41,42,48,55,56,34,63,67,70,74,78,98,100,101,102, 108,113,117,122,123,129,133
Procedural only: y' right	17	7,28,44,45,47,54,64,65,66,68,81,83,97,99,103,124,132
Procedural only: y' and y''	16	2,3,12,15,27,31,38,73,85,88,94,104,109,116,107,138
Procedural only	63	Total
Conceptual $y' = 0$	14	1,4,11,13,43,57,62,69,75,89,95,118,121,130
Conceptual $y' = 0$ and y''	18	6,16,20,22,24,26,39,51,72,76,77,80,93,96,106,128,131,135
Conceptual	54	(including the correct responses) total

Table 20: The content analytic breakdown of test item 6 (tertiary)

My bias when choosing how to classify the participant members

What was my bias when choosing how to classify the responses of the participant members of the cohort? Some choices seemed clear: a participant member either could determine the first derivative or not. One who did not determine the derivative correctly may have done a host of other interesting things worthy of analysis, but not for the purposes of this discussion (again a choice). But what to make of the rest who stopped short of making use of the first and second derivatives to determine the nature of the stationary point, possibly because that conceptual understanding was absent? Is there a hierarchy among this group of responses for this question? Is it better to determine the first derivative and then the value of the independent variable for which the derivative is zero or is it better to determine the first and second derivatives? Ideally of course it is better to determine everything that is required to answer the question. The first case shows the participant member is on track to determine the stationary point. The second case indicates the participant member is on course to answer the question about the nature of the stationary point. Both stop midway and so that does not leave any clearer sign that they knew the end goal of their pursuit. I made the assumption that both fell short of knowing fully what is expected. Again, a participant member may have stopped midway because she experienced difficulties with the symbolic manipulation or the calculation, not the conceptual issues – perhaps with easier calculation she would easily have completed the solution. So I chose to group and label in particular ways, in anticipation that I may be able to arrive at certain useful conclusions. But I am aware that this is but one way to interpret the incomplete procedural actions.

Analysing Item 7(tertiary): Calculate the maximum area that a piece of wire of length 40 cm can enclose if it is bent in the form of a sector of a circle

This test item represents the typical maximum/minimum calculus problem from the point of view of the curriculum. The conceptual demand is similar to test item 6 above but the word-problem situation adds another dimension. The typical expected student behaviour is that most of them will not complete the question and a significant number will ignore the question. This expectation of their possible responses derives from the historical school context of (not) doing word problems in classrooms. And yet, it is this kind of question which is most telling in predicting student conceptual development. The information available in the problem is as follows:

- there is a piece of wire with a given length
- the wire can be enclosed in the form of a sector of a circle in a number of ways[18]
- This abstract sector of a circle has an area which can be calculated (further abstraction)
- Of all the different ways of bending the piece of wire into a sector of a circle, one way will provide a maximum area – this is what is to be determined.

The standard approach which the lecturers on the course use to solving the problem is as follows:

Part one

Step 1: Set up an equation for the area of the sector

Step 2: Set up an equation for the perimeter of the sector

Step 3: Make a substitution, using the equation of the perimeter, to get an equation for the area of the sector in one independent variable

Part two

Step 4: Determine the derivative of the equation of the area; determine where it is zero

[18] this point represents an abstraction, a fact that may escape a number of participant members

Step 5: Determine the sign of the second derivative of the area at the value determined in step 4. This is the value we need to use to calculate the area.

Then finally: *Calculate the area using the value determined in step 5.*

The results of the breakdown are given in table 21.

Content Analytic Framework (revised): Item 7 (tertiary)		
Stage completed	no	participant members
No attempt	39	15,22,26,27,38,41,43,45,47,50,51,54,61,64,65,69,70,71,73,74,78,79,87, 90,94,96,102,109,112,114,115,124,126,129,130,131,132,135,136
Attempted (no score)	49	1,3,5,7,8,11,14,17,18,21,24,28,31,34,36,37,42,48,55,56,57,63,66,67,75, 80,81,82,86,88,89,93,97,99,100,105,106,108,110,111,113,117,118,121, 122,123,125,133,137
Attempted (some score)	26	2,6,10,19,20,25,29,33,35,40,44,52,62,72,76,77,84,95,98,101,103,104,10 7,116,128,138
Complete solution	15	4,12,13,16,23,30,39,53,58,68,83,85,91,120,127

Table 21: The content analytic breakdown of test item 7 (tertiary)

The number of participant members who made a successful or even partially successful approach is only 41, i.e. 31,7% of the cohort. This was partly expected as this type of test item usually has a low success rate. In addition, the context is that this work was done with the cohort within the three weeks leading up to the assessment and was followed by the basic integration section which was also assessed. Thus it could be argued that the cohort did not have sufficient time to consolidate the ideas contained in this section, nor did they have enough time to practice solving problems of this type. A factor which should also be considered is that the formula for the area of a sector of a circle was not provided on the formula sheet. On the other hand, 37% of the cohort did offer some sign that they knew what to do, and 11,6% had a complete solution. Of interest is *how* the cohort responded, given the context sketched above. In other words, given that the routinised, practised procedures could not easily be called upon, what strategies did the participants adopt to deal with this problem? To answer this problem I did a textual analysis of a sample of participants. This is dealt with in section 5.7 below.

Analysing: Item 8 (tertiary): Determine the following integrals: a. $\int\left(\frac{3}{x}-1\right)^2 dx$

Analysing: Item 9 (tertiary): Determine the following integrals: b. $\int \frac{\sin^2 x}{\cos x}\, dx$

Analysing: Item 10 (tertiary): Determine the following integrals: c.

$\int_{-1}^{0}\left(x^2+2x+1\right)dx$

As indicated above, these three items are taken together as they constitute the assessment for the basic integration section in the course and the type is procedural. It turned out that item 9 was the least well answered. This is partly explained by the requirement to manipulate the trigonometric fraction. The way the responses were sorted is given in table 22.

Content Analytic Framework (revised): Item 8, 9 and 10 (tertiary)		
Stage completed	no	participant members
Some attempt	21	8,14,19,22,29,35,38,40,41,43,45,51,64,66,68,70,78,79,104,114,130 (final scores (%): 0%,0%, 8%, 4%,14%, 12%,16%,16%,2%,12%,12%,8%,0%,6%,**42%,34%,34%, 26%,30%,62%,58%)**
Item 8 and/or 10 correct	67	1,2,3,4,6,10,11,12,13,16,17,20,21,24,27,30,31,33,36,39,42,47,48,50,54,55,56,58,63,67,69,72,73,75,77,82,84,85,87,89,94,95,96,97,99, 100,101,102,105,106,107,108,110,111,112,115,117,118,121,122,123,124,128,132,133,135,137
Item 9 correct	9	18,25,28,52,61,93,109,125,127
All correct	28	5,7,15,23,26,34,37,44,53,57,62,74,76,80,81,83,86,88,90,91,103,113 116,120126,131,136,138
No attempt	4	65, 71, 98,129 (final scores: 7%, 5%, 2%, and 0%)

Table 22: The content analytic breakdown of test items 8, 9, and 10 (tertiary)

In the "no attempt" row all the participant members scored very low final marks (below 10 %). Similarly, in the "some attempt" row, with the exception of 7 participant members (marked in bold) who scored above 30 %, the rest scored very low marks. I read this in a number of ways. Firstly, the fact that the cohort was first exposed to these concepts near the end of the course is a good reason for them not to have done well. This is so especially in light of their difficulties with differentiation. Secondly, even if they had more time, the procedural emphasis on teaching the concepts implies that memorization about techniques plays a very important role in mastering the techniques. The techniques for differentiation are the reverse of those for integration, in most cases. This can potentially lead to much confusion. Nonetheless, these results (item 8, 9 and 10) perhaps reflect more on the attitude of the participants, rather than their understanding. This is inferred on account of the presence of the participant members marked in bold in table 22 in those two categories who scored better in the assessment as a whole. In other words, their presence acts as a caution against saying that the rest got things wrong because they were procedurally and conceptually poor. Given that this section had been taught at the end of the course, so close to the final assessment, and given the pressures of the other engineering courses, this may well be a case of “one too many new things to learn”.

5.5 Trends arising from the item analysis (tertiary)

The myriad categories which arose as a result of doing the content analysis required further interrogation and the search for possible trends. There is, firstly, the analysis of questions according to how the cohort responded, in some cases leading to the creation of categories I had not considered beforehand. For example, when analysing responses to item 4 (tertiary), I was able to make up 11 categories. These different categories relate in part to the difference between conceptual demands as opposed to procedural facility. This seemed, in light of the literature review and theoretical framework analysis, to be the clearest way to distinguish among the participants’ responses to various items. At the same time, the range of procedural routes and strategies adopted was in some cases quite rich. It was possible to delineate the responses to item 4 (tertiary), for example, in even more categories than eleven, but then the purpose of grouping participants would have less analytical power the smaller the groups became. An individual textual analysis makes this more possible (section 5.7). In the end, I returned to the divide between conceptual understanding and procedural fluency as the main theoretical distinction. Where a participant member has shown that she understands the conceptual issues involved she has been given the benefit of the doubt as far as her procedural facility is concerned. The argument is that the procedural facility can easily be taught and practiced once there is conceptual understanding.

Also, it was clear that the participants were not consistent in their performances. This was probably related more to the nature of each question, as perceived by each participant, rather than an indication of what the participants are capable of overall. It was also probably related to the conditions under which individual participants learnt the different topics. For example, someone may not have been present, or someone else may have had another project uppermost in her mind at the time. This lack of consistency does indicate that one should be cautious about over generalisation about what students are capable of. In other words, detailing the specific responses to specific questions highlights the dangers and difficulties of generalising, either for individuals or whole groups.

Following on from the argument about procedural versus conceptual learning difficulties, I next took a selection of the cohort and put together a picture for each, tracking their performance as outlined under the various items above. This was the beginning of building a profile for each individual. To begin with, I selected some of the cohort from the "no attempt" group from items 8, 9 and 10 (tertiary). I wanted to know more about those who made no attempt to answer questions done at the end of the course. I also selected one from each of the other categories. To have a good spread in terms of final scores I chose the students listed in table 23 (scores included).

Cohort selection: individual profile

Participant member	Final score
8	0%
15	42%
16	70%
19	12%
34	56%
40	34%
93	78%
130	62%

Table 23: Cohort selection for determining individual profiles

The results are given in table 24. Written in brackets, in the column ‘task achieved’ in the table is the percentage of the group who responded similarly.

Content analytic framework and profile of participant member no. 8		
Item	Task achieved	Profile
2	Group 4: Followed a path leading elsewhere (10, 6%)	Failed to identify the rules required
3	None of above (see item 3) (29%)	Insufficient information
4	Substitution (0.06%)	Made use of a different method
5	y' wrong, y'' wrong (36%)	Knew what to use, did not use it correctly
6	*Procedural only:* y' wrong (23%)	Knew where to start, got it wrong
7	Attempted (no score) (37%)	Attempted (no score)
8,9,10	Some attempt (16%)	Some attempt

Table 24: Content analytic framework and profile of participant member no. 8

The overview of participant number 8's profile based on the summary in table 24 suggests that this participant is not performing to the required standard. This participant was placed in the cell for those whose solutions were wrong and not easily categorised for most of the test items.

Content analytic framework and profile of participant member no. 15		
Item	Task achieved	Profile
2	Group 3: Knew the correct rule to use but could not proceed (43%)	Procedural difficulty
3	Common error: treated $(\sin 2x)^x$ as an exponent (22%)	Conceptual visualisation, procedural competence
4	No line: y' right (0.06%)	Conceptual difficulty
5	y' right, y'' wrong (32%)	Partial procedural competence
6	*Procedural only:* y' and y'' (12%)	Procedural competence, conceptual difficulty
7	No attempt (30%)	Conceptual difficulty
8,9,10	All correct (29%)	Conceptual and procedural competence

Table 25: Content analytic framework and profile of participant member no. 15

The overview of participant number 15 suggests that this participant has procedural competence but conceptual difficulties.

Content analytic framework and profile of participant member no. 16		
Item	Task achieved	Profile
2	Group 3: Knew the correct rule to use but could not proceed (42%)	Algebraic procedural difficulty
3	Correct solution (16%)	Conceptual and procedural competence
4	Solution correct (29%)	Conceptual and procedural competence
5	y' right, y'' wrong (32%)	Some procedural competence
6	Conceptual $y' = 0$ and y'' (14%)	Conceptual and procedural competence
7	Complete solution (10%)	Conceptual and procedural competence
8,9,10	Item 8 and/or 10 correct (51%)	Some procedural competence

Table 26: Content analytic framework and profile of participant member no. 16

The overview of participant number 16 suggests that this participant has good conceptual and procedural competence.

Content analytic framework and profile of participant member no. 19		
Item	Task achieved	Profile
2	Group 3: Knew the correct rule to use but could not proceed (42%)	lacked procedural facility
3	Common error: treated $(\sin 2x)^x$ as a polynomial (22%)	Wrong procedure, conceptual visualisation
4	Variations: y' wrong, line right (18%)	Procedural problem
5	y' right, y'' wrong (32%)	Procedural difficulty
6	*Procedural only:* y' wrong (22%)	Procedural difficulty
7	Attempted (some score) (19%)	Conceptual difficulty
8,9,10	Some attempt (16%)	Conceptual difficulty

Table 27: Content analytic framework and profile of participant member no. 19

The overview of participant number 19 suggests that the participant has conceptual difficulties with the content and also procedural difficulties, although there is evidence of attempts to function in a procedurally competent way.

Content analytic framework and profile of participant member no. 34		
Item	Task achieved	Profile
2	All correct (21%)	procedural competence
3	Common error: treated $(\sin 2x)^x$ as an exponent (22%)	Conceptual visualisation difficulty, procedural competence
4	Solution correct (29%)	Procedural and conceptual competence
5	$f''(0)$ wrong10%)	Procedural competence, conceptual difficulty
6	*Procedural only:* y' wrong (23%)	Procedural and conceptual difficulty
7	Attempted (no score) (37%)	Conceptual difficulty
8,9,10	All correct (21%)	Procedural and conceptual competence

Table 28: Content analytic framework and profile of participant member no. 34

The overview of participant number 34 suggests that this participant has good procedural competence and good conceptual competence.

Content analytic framework of participant member no. 40		
Item	Task achieved	Profile
2	Group 2: Succeeded in first level only (27%)	Algebraic manipulation problem
3	Common error: treated $(\sin 2x)^x$ as an exponent (22%)	Conceptual visualisation problem
4	Other methods, unrelated to the problem: Make equation equal to zero (0.03%)	Conceptual difficulty
5	Solution correct (20%)	Procedurally competent
6	Solution correct (16%)	Procedurally and conceptually competent
7	Attempted (some score (19%)	Conceptual difficulty
8,9,10	Some attempt (16%)	Procedural and conceptual difficulty

Table 29: Content analytic framework and profile of participant member no. 40

The overview for participant number 40 suggests that this participant is procedurally competent but has conceptually poor competence with the content. The participant was in a small group of those who had correct responses to two questions, which is a highly positive sign for the participant's further development.

Content analytic framework and profile of participant member no. 93		
Item	Task achieved	Profile
2	Group 1:Got correct answer (16%)	Procedural competence
3	Group 1: Got correct answer (16%)	Procedural competence
4	Solution correct (29%)	Procedural and conceptual competence
5	y' wrong, y'' wrong (36%)	Procedural difficulty
6	Conceptual $y' = 0$ and y'' (13%)	Conceptual and procedural competence
7	Attempted (no score) (37%)	Conceptual difficulty
8,9,10	Item 9 correct (0.06%)	Procedural competence

Table 30: Content analytic framework and profile of participant member no. 93

The overview of participant number 93's profile suggests that this participant has good procedural competence and poor conceptual competence.

Content analytic framework and profile of participant member no. 130		
Item	Task achieved	Profile
2	Group 3: Knew the correct rule to use but could not proceed (42%)	Algebraic Procedural difficulty
3	Take logs, show use of the product rule and does chain rule correct (0.03%)	Procedural competence
4	Solution correct (29%)	Procedural and conceptual competence
5	y' wrong, y'' wrong (36%)	Procedural difficulty
6	Conceptual $y' = 0$ (10%)	Procedural competence, conceptual difficulty
7	No attempt (29%)	Conceptual difficulty
8,9,10	Some attempt (16%)	Procedural and conceptual difficulty

Table 31: Content analytic framework and profile of participant member no. 130

The overview of participant number 130 suggests that this participant has good competence procedurally but has conceptual difficulties.

5.5.1 Building the profile

In this section I made use of the taxonomy table framework of chapter four (table 5) and completed a profile for individual participants in terms of their test scores. I then compared those tables with the content analysis tables above and proceed to extend the profile up to this point for a selected subgroup of the cohort, namely, the group as listed in table 23 above.

Taxonomy of conceptual, procedural and problem solving skills of participant no. 8		
Cognitive areas	Tests	
	Multiple choice test	Tertiary examination
Recall basic facts, procedures and concepts		
Perform routine calculations	100%	
Apply routine calculations (problem solving level 1)	50%	
perform routine procedures		0
Apply conceptual knowledge		0
problem solving level 2 (with routine problems)	50%	0
problem solving level 3 (non routine problems)	66%	

Table 32: Taxonomy of conceptual, procedural and problem solving skills of participant no. 8

This taxonomy suggests that participant number 8 is competent in problem solving for multiple choice items. The participant displays no strength for written solutions.

Taxonomy of conceptual, procedural and problem solving skills of participant no. 15		
Cognitive areas	Tests	
	Multiple choice test	Tertiary examination
Recall basic facts, procedures and concepts		
Perform routine calculations	0%	
Apply routine calculations (problem solving level 1)	100%	
perform routine procedures		55%
Apply conceptual knowledge		50%
problem solving level 2 (with routine problems)	50%	30%
problem solving level 3 (non routine problems)	50%	

Table 33: Taxonomy of conceptual, procedural and problem solving skills of participant no. 34

This taxonomy suggests that participant number 15 performed very similarly to participant number 34. Number 15 could just cope with the demands of all types of problems but showed a further weakness in the area of problem solving level 2.

Taxonomy of conceptual, procedural and problem solving skills of participant no. 16		
Cognitive areas	Tests	
	Multiple choice test	Tertiary examination
Recall basic facts, procedures and concepts		
Perform routine calculations	0%	
Apply routine calculations (problem solving level 1)	83%	
perform routine procedures		75%
Apply conceptual knowledge		66,7%
problem solving level 2 (with routine problems)	50%	70%
problem solving level 3 (non routine problems)	50%	

Table 34: Taxonomy of conceptual, procedural and problem solving skills of participant no. 16

This taxonomy suggests that participant number 16 has good competence in almost all areas.

Taxonomy of conceptual, procedural and problem solving skills of participant no. 19		
Cognitive areas	Tests	
	Multiple choice test	Tertiary examination
Recall basic facts, procedures and concepts		
Perform routine calculations	0%	
Apply routine calculations (problem solving level 1)	66%	
perform routine procedures		0
Apply conceptual knowledge		22,2%
problem solving level 2 (with routine problems)	100%	10%
problem solving level 3 (non routine problems)	33%	

Table 35: Taxonomy of conceptual, procedural and problem solving skills of participant no.19

This taxonomy suggests that participant number 19 has difficulties in almost all cognitive areas, but is competent in applying routine calculations and has very good competence in problem solving level 2.

Taxonomy of conceptual, procedural and problem solving skills of participant no. 34		
Cognitive areas	Tests	
	Multiple choice test	Tertiary examination
Recall basic facts, procedures and concepts		
Perform routine calculations	0%	
Apply routine calculations (problem solving no. 1)	83%	
perform routine procedures		60%
Apply conceptual knowledge		55,6%
problem solving level 2 (with routine problems)	50%	30%
problem solving level 3 (non routine problems)	50%	

Table 36: Taxonomy of conceptual, procedural and problem solving skills of participant no. 34

This taxonomy suggests that participant number 34 could just cope with all types of problems and performed consistently overall. The exception appears to be the area of problem solving level 2 in which the participant fared more poorly.

Taxonomy of conceptual, procedural and problem solving skills of participant no. 40		
Cognitive areas	Tests	
	Multiple choice test	Tertiary examination
Recall basic facts, procedures and concepts		
Perform routine calculations	0%	
Apply routine calculations (problem solving level 1)	66%	
perform routine procedures		25%
Apply conceptual knowledge		61%
problem solving level 2 (with routine problems)	100%	20%
problem solving level 3 (non routine problems)	66%	

Table 37: Taxonomy of conceptual, procedural and problem solving skills of participant no. 40

This taxonomy suggests that participant number 40 has good competence in problem solving (non-routine and routine) and is competent in applying conceptual knowledge.

Taxonomy of conceptual, procedural and problem solving skills of participant no. 93		
Cognitive areas	Tests	
	Multiple choice test	Tertiary examination
Recall basic facts, procedures and concepts		
Perform routine calculations	0%	
Apply routine calculations (problem solving level 1)	66%	
perform routine procedures		90%
Apply conceptual knowledge		61%
problem solving level 2 (with routine problems)	100%	30%
problem solving level 3 (non routine problems)	16%	

Table 38: Taxonomy of conceptual, procedural and problem solving skills of participant no. 93

This taxonomy suggests that participant number 93 has good competence in applying conceptual knowledge and applying routine calculations. This participant has very good competence in problem solving level 2 (non-routine).

Taxonomy of conceptual, procedural and problem solving skills of participant no. 130		
Cognitive areas	Tests	
	Multiple choice test	Tertiary examination
Recall basic facts, procedures and concepts		
Perform routine calculations	0%	
Apply routine calculations (problem solving level 1)	33%	
perform routine procedures		45%
Apply conceptual knowledge		55,6%
problem solving level 2 (with routine problems)	0%	0%
problem solving level 3 (non routine problems)	16%	

Table 39: Taxonomy of conceptual, procedural and problem solving skills of participant no. 130

This taxonomy suggests that participant number 130 has poor competence in most of the areas, except possibly in applying conceptual knowledge.

The results for the individual participant members, produced here (table 40) for reference.

Multiple Choice non routine results summary: individual participants															
Problems:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Participant No. 8	X	√	X	√	√	√	X	X	√	X	√	√	√	X	√
Participant No. 19	X	√	√	X	√	√	√	√	√	X	X	X	X	X	√
Participant No. 40	√	√	√	X	√	X	√	√	√	√	√	X	X	X	√
Participant No. 130	X	X	√	X	X	X	X	√	X	X	X	X	√	X	X
Participant No. 16	√	√	√	√	√	X	X	√	√	√	X	X	√	X	X
Participant No. 93	√	X	√	X	√	X	√	√	√	X	X	X	√	X	X
Participant No. 34	√	√	√	√	√	√	X	X	√	X	√	X	X	√	X
Participant no. 15	√	√	√	√	√	√	√	√	X	X	X	X	√	X	√

Table 40: multiple choice test scores summary for individual participants

As discussed in chapter two, Neubrand (2005) expressed a particular concern about item difficulty: “not the same features make a problem difficult in any of the three types of mathematical activities[19] (1, p. 81). This point resonates with what I found in my analysis of the tests items (tertiary) (Chapter 5, section 5.4). But it also resonates with my analysis of the test items (multiple choice) in a different way (Chapter 5, section 5.3). It was not easy to see why two participants got the same item wrong yet showed different strengths with items classified by me into the same cell in the content analysis framework. That is, items given the same conceptual and procedural difficulty level. My sense is that to explain the differences one would have to do the kind of triangulation analysis which I propose (and

[19] viz., employing only techniques; modelling and problem solving activities using mathematical tools and procedures; modelling and problem solving activities calling for connections using mathematical conceptions.

possibly more) and also test the three types of mathematical activities as outlined by Neubrand (2005).

5.6 Statistical overview of test items (tertiary)

I explored two relationships that stood out when I delved into the details of the test items (tertiary). The idea for one of the relationships that I explored arose when I analysed the participants' responses to item 2 (tertiary). The question that arose at the end of that process and that led me to this statistical exploration was the following:

Was item 2 (tertiary) a good discriminator for overall performance?

In other words, I wanted to know if a participant member who did well in item 2 (tertiary) would also do well in the examination (tertiary). I based this query on my analysis that suggested that the procedural and conceptual requirements of item 2 (tertiary) were of an order sufficiently intricate to discriminate among the cohort. The data was analysed using the software package Statistical Package for Social Sciences (SPSS, version 14).

Noting that the total score for item 2 (tertiary) was 6 points, I initially chose to set up two groups: (1) those with 4 points and above and (2) those below 4 points. Because of the missing data of candidates who were not part of the study but appeared on the sheet, SPSS reconfigures the different percentages as these are labelled "Valid Percent". The cumulative percent has the normal meaning of cumulative. The split is shown in table 41.

Item2

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Low marks	73	52.9	56.6	56.6
	High marks	56	40.6	43.4	100.0
	Total	129	93.5	100.0	
Missing	System	9	6.5		
Total		138	100.0		

Table 41*:* division of cohort into two groups for item 2: those with 4 -6 points written as high marks and those with 0 -3 points written as low marks

From table 41 it can be seen that there were 73 participants in the low mark group and 56 participants in the high mark group.

The next step was to divide the group of participants in terms of their final test score (written as a percentage). I was keen to divide them into a 0% - 59% low group and 60%-100% high group. I chose that division because it is my experience that 60%+ is a reasonable standard to attain in such a test and because it is a probable pointer to the individual's capability to succeed at the next course level. Unfortunately this split of the cohort was too extreme, as shown (table 42). Consequently I chose a 50% cut – off: those with 50% and above in one group (high) and those below 50% (low) (table 40). I then had 66,9% in the low scoring group and 33,1% in the high scoring group.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Low percentage group	112	81.2	81.2	81.2
	High percentage group	26	18.8	18.8	100.0
	Total	138	100.0	100.0	

Percentage: total examination mark (per TM)

Table 42: Cohort split into low percentage group and high percentage group for total examination mark

Percentage: total examination mark (per TM)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Low percentage group	87	63.0	66.9	66.9
	High percentage group	43	31.2	33.1	100.0
	Total	130	94.2	100.0	
Missing	System	8	5.8		
Total		138	100.0		

Table 43: Cohort split into low percentage group and high percentage group for total examination mark

By doing a cross tabulation and placing the information in a contingency table, I was able to determine the nature of the relationship between the two sets of data (table 44). The data confirmed that a low mark for item 2 (tertiary) indicated a low percentage for the overall score of the examination. Also, the reverse seemed true: a high mark for item 2 (tertiary) indicated a high percentage for the total score of the examination.

Item2 (tertiary) * examination results (tertiary) Cross tabulation

			Examination results (tertiary)		Total
			Low percentage	High percentage	
Item2	Low marks	Count	65	8	73
		% within Item2	89.0%	11.0%	100.0%
	High marks	Count	21	35	56
		% within Item2	37.5%	62.5%	100.0%
Total		Count	86	43	129
		% within Item2	66.7%	33.3%	100.0%

Table 44: Cross tabulation for item 2 and examination results (tertiary)

To test for *significance* of these relationships I used the Pearson chi-square test (table 45). I wanted to see if there was a difference between my perception as evidenced in the cross tabulation and a statistic, and the Pearson chi-square statistic is ideal for that purpose. The results showed that since the expected minimum count was 18.67 which exceeds 5 this meant that the assumption for the chi square had been met. The statistic for the chi-square was 37.88 which is larger than the 18.67 minimum. This meant that the relationship was significant. In fact this relationship is highly significant ($p < 0.001$).

Chi-Square Tests

	Value	df	Asymp. Sig. (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-Square	37.88(b)	1	.000		
N of Valid Cases	129				

a. Computed only for a 2x2 table
b. 0 cells (.0%) have expected count less than 5. The minimum expected count is 18.67.

Table 45:Chi-Square test for item 2 (tertiary) versus total examination results (tertiary)

It remained to test for the *strength* of the relationship between scores in item 2 (tertiary) and examination results (tertiary). I used the Pearson's R statistic to do this correlation (table 43). This is a standard correlation to use for these purposes: to investigate the tendency of two sets of data to vary consistently. The correlation statistic of $r = 0.542$, significant at $p < 0.001$ showed that there was a positive relationship. The strength of the relationship was strong at .542, but perhaps not very strong.

Symmetric Measures

		Value	Asymp. Std. Error(a)	Approx. T(b)	Approx. Sig.
Interval by Interval	Pearson's R	.542	.074	7.266	.000(c)
N of Valid Cases		129			

a Not assuming the null hypothesis.
b Using the asymptotic standard error assuming the null hypothesis.
c Based on normal approximation.

Table 46: Pearson's correlation coefficient r for item 2 (tertiary) versus examination results (tertiary)

The second relationship I wanted to test was between items 3 and item 5 (tertiary) since both involved composite functions and the chain rule and both had total scores of 5. The question I wanted to pursue was:

Was there a strong relationship between the results for items 3 and item 5 (tertiary)?

I made use of the correlation coefficient Spearman's Rho after having first ranked the data (scores) in an ordinal way (table 47). This statistic is useful for measuring the degree of association between two ordinal variables, such as the scores for items 3 and 5. The correlation was two tailed since the direction of the relationship (positive of negative) was open. As table 47 shows, the relationship was positive although not strong (.298) and it was significant at the 0.01 level.

Correlation of items 3 and 5 (tertiary)

			ITEM 3	ITEM 5
Spearman's rho	ITEM 3	Correlation Coefficient	1.000	.298(**)
		Sig. (2-tailed)	.	.001
		N	129	129
	ITEM 5	Correlation Coefficient	.298(**)	1.000
		Sig. (2-tailed)	.001	.
		N	129	129

** Correlation is significant at the 0.01 level (2-tailed).

Table 47: Spearman's rho correlation between scores for item 3 and item 5

5.7 Textual analysis of the examination scripts (tertiary) of the cohort

I will now examine the texts of the selected group (table 23) in terms of the semiotic analysis set up in 3.3. The complete set of examination scripts (tertiary) of the selected group (table 23) is in Appendix B.

Participant No.8

This participant scored no points in the examination. Closer scrutiny revealed some surprising behaviour. The first item is scratched out but there is an apology enclosed in a frame and a sad face (lines 1 – 3)! Item two indicates to the examiner, however, that all is not well: a perfectly good derivative of the numerator is given but the denominator is manipulated to produce a solution of 1 (line 6). The existence of a quotient is not acknowledged at the start, so the quotient rule, provided on the formula sheet, is not employed. Item 3 sees the use of a trigonometric identity (line 9). Then the derivative of each of the trigonometric functions is correctly applied, ignoring the product and also the power (line 10). The deleted '$2\sin$' may be nothing or perhaps it indicates participant number 8 started off repeating line 9 but then changed his mind. The next line sees a jump to $\cos^2 x$: I found this to be unexpected (line 12). Still the power is ignored. Then in the next line the power is simply dropped (line 12). It would appear that there is another derivative applied in line 13 as if the $\cos x$ is a polynomial.

The participant's solution of this item is filled with steps which to my interpretation do not relate to each other. Unlike with participant number 40, say, who took wrong turns but then followed through logically, participant number 8 appears to act differently at each step. One could say this solution lacks coherence. Also, the dropping of the power without any attempt to engage with it could indicate a lack of conviction. Item 4 is also done without any indication that no.8 knows what is expected and how to get there. There is a strong showing of how to get intercepts with the axes, and both are done correctly. This could be a throwback to earlier mathematics, perhaps when the student had more confidence and knew better what to do. Item 5 contains much of the same sort of abandonment that has been shown before. Each line (lines 35 – 38) follows the previous without a pattern: they are text on a page, defying my interpretation. The only correct (or for that matter, visibly logical step) is the division (line 39-40). Notably, however, there is no obvious indication of disregard or disaffection with the examination. Item 6 contains one correct derivative but no use of the product rule and no $\frac{dy}{dx}$. This makes it possible to replace y with 1 (line 45) and "solve" for x (line 46). The employment of the derivative of $\ln x$, again without any $\frac{dy}{dx}$

leads to the equation in line 48 and the logical line 49. So here there is a series of logical steps somehow set inside repeated replacements for $\ln x$ with its derivative. The manipulations are very elementary, however, and probably learnt well at school. Line 50 again defies any explanation I can offer and then participant number 8 has derived the stationary points. In Item 7 the participant interprets the length of a sector to refer only to the arc, a common mistake, and adds an unknown entity, *x*, to complete the circle. All this is provided with the aid of a sketch. Then again there is a jump without an obvious logical step when the radius is assumed to be $40-x$ (line 56). The use of the school value 3, 14 in place of π again betrays a reliance on school learnt content. This type of behaviour is typical of the assimilation schemas of Piaget, as discussed in section 2.1. Then follows manipulation which, again illogically to me, comes full circle: participant number 8 multiplies with 3,14, correctly (lines 57-8) and then divides by 3,14 (lines 59 -60). Then follows a factorization (without the square) and a solution for x (the assumption being that the equation $x-40=0$ had to be solved) (line 62). Thus from line 55, beginning with the area of a circle formula, he determines a value for x. This value is then substituted into the derived formula for the perimeter of the circle (line 52) to get a value for the area of the circle (lines 63-5). There is in fact in this question a continuation of the performance of logical steps unrelated to other logical steps. The substitution of the area of the circle for the perimeter of the circle (line 63) is consistent with the dropping of the power in item 3 (line 12): instead of engagement with difficulty there seems to be escape from it. This could be interpreted as an example off cognitive conflict not resulting in accommodation. In item 8 there is the correct product, but without the integral sign (line 67), a rearranging of the product (line 68 -9) and then the derivative (correctly) of each term. The manipulation of the quadratic is good and appears to be well learnt in school and the derivative of each term is good. So again we have a kind of denial of the problem at hand, possibly because the problem is difficult, and the solution, perfect in this case, of a problem within the capabilities of the participant. Item 9 is similar, with no.8 converting the quotient to one which is equal to $\tan x$ (lines72-3) not engaging with the integral. And item 10 is handled similarly, where the integrand is factored (line 75), the integral dropped (repeat of the pattern), then x "solved" and, bringing back a memory from a past lesson, -1 is substituted for the complex j^2 (line 78).

Critical indicators: participant no.8 has not engaged with the course in any significantly meaningful way. Besides knowing to make use of a few derivatives, available from the formula sheet as well and possibly learnt at school level, no.8 has not made the shift to tertiary level mathematics. The presence of correct procedures which relate almost entirely to school level topics is another indicator that no.8 has not adjusted to tertiary level mathematics. The dominant indicators are: no engagement, no reflection, limited creativity[20] and a "child method" response to crisis.

Participant No.15

No.15 discards the first attempt to do item 1, which is actually done correctly. There are some scratch marks and lines drawn across the solution to indicate the work is not to be considered, but no notes (lines 1-5 left hand side). The revised version is incorrect (lines 1-7, right hand side). Item two has two versions again. The version to be considered for points is on the left hand side (lines 8-14) and the version in pencil on the right hand side is reconstructed by me and placed at the end (lines 63-70). Apart from leaving out the derivative expression on the right hand side (a common error), No.15 has worked well up to line 12, with one mistake, namely the denominator remains a square root despite being squared (line 11). Line 13 is again erased work. This work is more manipulation of the equation and is correctly done. The final answer is, however, incomplete (line14): the

[20] limited to downgrading the question

available evidence indicates that the participant knew that there was more to the solution than is given as the final answer. Perhaps the answer is left incomplete, but there is no indication.

In contrast, another version is given (lines 63-70). This version contains the expression for the derivative on the left hand side as is required (line 64). The denominator problem remains (line 67 forward) and is left off a few times (lines 65-6). There is a sign change in line 66 which was only applied to the first sign-sign product but not the second one, but good manipulation in the numerator in line 68. This is simplified to give the answer in lines 69-70. The pencil work next to the work in ink (which is the one for grading by the examiner) is work done subsequent to the first one. This is a second attempt at the question. This shows that, possibly, No.15 was not convinced about the first solution. This solution is also more complete up to the final answer. Why did No.15 not replace the first solution? Was it time or lack of conviction? So thus far for two questions there have been four solutions and much indication of doubt and hesitancy. Item 3 sees a repeat process: there are two versions, one in ink for grading and one in pencil. The ink version (lines15 – 18) is scratched out. The approach taken is to view the expression on the right hand side as an exponent, as was the case for 22% of the cohort. Again the base, $2\sin x\cos x$, is taken to be the power as well, and the derivative of the product is done as if the terms are separate (line 17). Then the derivative is multiplied with the argument of the natural logarithm (line 18). In the pencil version (lines 18a-18e, reconstructed) $\sin 2x$ is used as the base[21] and the power, although the derivative of $\sin 2x$ is incomplete. Then there follows a replacement of $\sin 2x$ and $\cos 2x$ with formulae meant for $\sin^2 x$ *and* $\cos^2 x$ and quite an intense period of manipulation. The power, *x*, is dropped, then brought back and has no effect on the

[21] I am having the formulae in mind: $y = a^x$ *and* $y = a^u$. The common error is to replace *a* with *u*.

manipulations; it is simply carried over to the next line. Item 4 is dealt with efficiently up to the result for the gradient (lines 19 – 25) but no further. The equation of the tangent line is not dealt with: there is no engagement with the conceptual demands of the problem. This is repeated elsewhere. Item 5 is dealt with equally efficiently except that the second derivative is wrong; again the product rule was not employed. Apart from that flaw the solution is properly presented. Item 6 is dealt with equally efficiently but again only up to a point (and with a flaw). The derivative is handled well with No.15 making good use of the product rule, indicating that he *can* use the product rule even though he failed to do so on two earlier occasions[22]. The equation written in line 38 but not part of the solution indicates a thought process: "since $\ln e = 1$ and I must solve $\ln x = -1$, I just need to multiply the first equation (line 38) by -1 both sides. Therefore the value for the unknown is "-.... e". This is also a common error (personal experience). But there is no engagement with the conceptual issues contained in the problem: the coordinates and nature of the stationary points. Item 7 is not done apart from writing down a piece of the information provided (line 42). The pattern that emerges is that No.15 appears to be quite prepared to engage with procedures and manipulation but appears to have difficulties when dealing with the conceptual issues. It is clear from considering his responses to items 8-10 that No.15 had prepared well for those. For these questions, on the most recent work, he is methodical, sets out his solutions in an orderly way, using a number of additional steps (for example lines 46-7 and 53-4) and gained full points (part of 29% of the cohort).

Critical indicators: apart from some glaring errors, No.15 is comfortable with manipulative strategies and is procedurally fair. Within that brief he has shown himself to be logical. He has not engaged with the conceptual demands of the questions. I interpreted his conduct as

---

[22] One might conclude that the problem is with the perception and recognition of expressions, not exercising the rule

showing a lack of confidence: attempting more than one version of a solution on a few occasions. He also did not attempt the question with the biggest conceptual demand. It would appear that he prefers to do only that which he knows. This could be a strategic position. His response to crisis in the examination is to attempt multiple versions of the solution, to ignore problems (the power of x was an example), to stop when his confidence is high (for example item 4) or simply not to start. There was no obvious showing of creativity in seeking alternative solutions although reflection was indicated in the duplication of solutions.

Participant No.16

In item 1 participant No.16 scratched out her third line and replaced it with a new answer. This action set the tone for one pattern throughout the script: checking, substituting, often for a better solution. This form of behaviour would fit the pattern of assimilation (section 2.1). Item 2 was handled very well. The side working showed care in planning (lines 8 – 10) and a correction was made to the substitution for u. There is one error: the derivative of $\sqrt{1-x^2}$ lacks the negative sign (line 8, coming from the side working in line 9). This error is carried through to the end of the solution, which is otherwise well done. Item 3 starts off with No. 16 deciding, like other participants, that the expression $(\sin 2x)^x$ is an exponent and also taking the base for the power in the exponent formula. The derivative of the sine function is done correctly. But then No.16 changes his mind and attempts the problem by introducing logarithms, the standard way to do it (line 13). To save space[23] he starts back on the first line, next to the erased work, but soon moves back under it in line 15. From that point No.16 proceeds to work very well and determines the correct solution. Item 4 is handled with equal ease, an incorrect step quickly erased in line 20. He alternates between

[23] a needless, but common, phenomenon

spreading out his quotients on two lines or inside one line. In item 5 he finds only the second derivative for one part of the product in line 35 (line 36) and though the subsequent substitution is right, the solution is wrong. In item 6 participant No.16 misses the concept: the nature of the stationary points. He does, however, perform the procedural tasks properly, up to determining the x component where the derivative is zero (line 50). Again in this problem we see No.16 pursuing a line of argument, only to abandon it and attempt something different. In this case, he starts by turning $x \ln x$ into an exponent and then determines the derivative (which is not clearly visible) and also the second derivative (equal to $\frac{1}{-x}$) (line42 – 44, left hand side). This is then discarded. A second attempt, again using the exponent form, is made (incorrectly, as before) (lines 43-44, in the centre). This, too, is abandoned. A third attempt is made, this time correct (lines 44 – 45). There are more signs of further struggle, this time with the idea of an intercept or value at which y' equals zero. Left hand side lines 46-7 there is work which probably relates to graph sketching and intercepts. This value of x (zero) is included next to $x = 0{,}37$, but then discarded (line 51). There is the existence of $x = 0{,}37$, but what to do with it? Substitute it into y'? No (discarded line 54). Substitute it into y''? Yes (line 55). But then the interpretation betrays No.16. The result of the substitution (line 57) becomes the $y-$ component for $x = 0{,}37$, instead of being interpreted for what it says about the nature of the stationary point. This is the concept that is missing from the result. Item 7 is equally well done but again the crucial concept is left out. As a result there is an assumption that the value of the radius found (line 74) is the correct one (which happens to be true). Item 8 has one wrong turn in line 83 but this is changed and the correct solution found (line 84). In item 9 the participant makes a choice of trigonometric rearranging which leaves him with the dilemma of finding the integral of a product. He then treats it as if it was the product rule and integrates each term in turn. In item 10 No.16 first attempts to find the derivative of the integrand, abandons that approach and then correctly determines the solution.

Critical indicators: Participant No.16 displays an active mind when working out solutions. Options are tried and abandoned if they don't work. This shows much reflection on what is being attempted. It also shows willingness to risk and certain boldness. It also shows an engagement with the details of the work and not simply a rehash of solutions[24]. These are all positive responses to crises. There is a sense of No.16 grappling with what outcome is expected or appropriate, hence the discarding of certain solutions which others were quite prepared to leave.

Participant No. 19

This participant struggled to find an ease of working. Item 2 was dealt with as if the terms were products. Thus the derivative for the numerator was correct and that for the denominator nearly so, the negative sign being left out (line 7). The deleted line had much the same result except that the answers were written as a quotient. In item 3 the power x is dropped in front of $\sin 2x$ as if it is a logarithm and the presence of the +1 is unclear. This is followed by the identity for $\sin 2x$ (line 9). But then this line of argument is abandoned in favour of a straight use of the power rule for polynomials (line 11). Item 4 sees two approaches: one on the left hand side of the page which is abandoned, and one on the right hand side of the page. The left hand solution has correct derivatives for two of the three terms (line 13). By making y' the subject and substituting the given point, No.19 is left with division by zero. This way is discarded, presumably as a result of that. The equation in line 28 is again manipulated in order to make y' the subject, a different ordering being tried. This is also abandoned because again No.19 is left with division by zero and is not willing to let

[24] it could be a rehash if memory was all that was working and not invention and creativity

that stand. So then a different derivative is attempted, starting from scratch (line 14, right hand side). Again the derivative of $2xy$ is stated equal to zero, but a manipulation with the derivative for x^2 (line 15) means that the division by zero problem is avoided (line 16). This value for y' is then successfully substituted into the equation for the straight line (Lines 19 - 22). Item 5 sees the same successful first derivative and unsuccessful second derivative that has been seen before, but no substitution (lines 33 - 5). Item 6 is taken as an integral and the terms are integrated separately (lines 36 – 7). There is unclear deletion in line 38. Item 7 offers more detail for analysis. There is a useful sketch of a sector in which various options for the values of the radius and arc are tried out: radii of 15 each scratched out and replaced by 2, and arc length 5 or 10. On the left is an attempt to work out the value of the radius *r*. The perimeter is equated with the formula for the area (line 39) and (wrong) substitution for arc length *s* is made (line 43). The value thus found for *r* is substituted into the equation in line 40 (line 48) but this process is then discarded. An alternative attempt follows (line 49) but *r* is again erroneously calculated (lines 51 – 6). S is determined by dividing the equation by 2, but s is not divided by 2 (line 51). Then a substitution is made into line 49 (line 53). The algebraic expression $2r - r$ becomes $r(2-1)$ which is then equal to r^{25}. The substitution mentioned (line 57) is possibly into the equation in line 51. But this would leave a value for s equal to zero. In line 58, however, *s* is given the value 1. Line 59 sees yet another formula, this time for the area, but it is the wrong formula. The equation with θ as subject is then determined (line 60). The procedure followed to determine θ makes logical sense if the normal rules of algebraic manipulation are ignored: participant number 19 multiplies the denominator 2 with values for *r* and *s* and gets 40. This leads to the equation in line 60. This equation leads to the square of the area (line 61) and then the derivative (line

[25] Is this an example of how students abandon their own sense when working within set algorithmic formats? Or is it the case that the participant was “lost” in the mindset of the problem? The answer was not obvious to me.

62). This is equated with zero (line 62), leading to the area being equal to $\frac{1}{2}$, instead of zero. This is the second time that the value zero is ignored. Possibly the participant, despite evidence in the manipulation, refuses to accept a value of zero for the area and arc length since these are shown in sketches to have values other than zero. I interpret this as another example of assimilation as opposed to accommodation. The second derivative is given as 2 (line 63), considered positive (line 62, right hand side), and this ties in with the second sketch, on the right hand side of lines 46 -8. This sketch is a table showing y and its two derivatives and the respective values at the point $\frac{1}{2}$. Notwithstanding the process and the values derived, the table is correct if it stands alone. This probably represents a recall from memory. In item 8 the participant uses a substitution: $u = \frac{3}{x} - 1$ but does not extend this to dx, with the result that the solution is wrong (lines 65 – 7). In item 9 the quotient is separated and the new trigonometric functions are integrated individually. In item 10 a substitution is made without integrating first.

Critical indicators: this participant has stayed close to the surface of the problems, preferring to perform the most elementary procedures, including basic derivatives taken off the formula sheet, even where more were required. Items 4 and 7 saw some real attempts to engage with the questions in an extended way, although item 7 was dealt with somewhat haphazardly. It would appear, from the evidence available, that the participant has only a surface appreciation of the area covered. The participant displayed some procedural fluency, but behaved mostly illogical, even with basic manipulations such as changing the subject of the formula.

Participant No.34

Participant No.34 started well with item 1 and item 2, but left off the numerator in item 2 as indicated by the examiner[26] (line 5). This has implications for the final answer, which is incorrect. However, the logical sequence of steps employed makes for a presentable solution. Item 3 is interpreted as an exponent, with the base being treated as if it was the base as well as the power; the derivative of the base is correctly done (line 12). Item 4 is correctly done, although there is evidence that the participant judged that the derivative of $-2xy$ was done wrongly as it is discarded (lines14 – 5) and replaced by the correct one (line 16). From that point everything works out correctly. Item 5 is also well done as only 30% of the cohort could determine the two derivatives correctly. It is not clear from the text how participant number 34 found the final answer (line 30). Item 6 and 7 test the same concepts, namely the stationary points of a function and their nature. The participant attempted item 6 but stopped short of engaging properly with these concepts. The derivative was wrong (line 33) even though there had been another case where the participant knew to use the product rule (line 28, immediately above). There is an attempt to determine the value of the independent variable when the derivative is zero (line 33), but it is not clear how the solution is arrived at (line 34). There is an attempt to determine the *y*-component (lines 35 – 6) (again not clear how the value is determined) and a point results after that (line 37), presumably a stationary point. The presence of "real, equal, rational" harks back to the school curriculum area, *nature of the roots of a quadratic equation* (line 37). This raises the question whether the participant had made the necessary adaptation to the new conceptual ideas or not. Item 7 is not dealt with apart from a sketch based on the information provided (figure 1). The integration questions (items 8, 9 and 10) are done very well and all done correctly (lines 39 – 48).

Critical indicators: no.34 showed herself able to work reasonably well and to be reasonably organised and methodical in most of her solutions. She appeared procedurally competent. She failed to engage with the concepts dealing with stationary points and maxima and minima, which was a huge gap in her presentation. Her completion of the most recent work, integration, indicates a possible focus on immediate work. This was in contrast to many other participants who did not cope well with the last questions. The indicators are: does not deal with crisis, shows little creativity, is confident about familiar tasks, can be logical and there was not much evidence of reflection.

Participant number 40

This participant stopped after the first stage in item two (line 4). This stage involved many procedures and the use of a formula. The participant got the full score for that stage. Why stop? This could be because the next stage involved complicated calculations and would only score two points. Perhaps the participant calculated that she could afford to lose the points if she saved on time – a reasonable strategy in a time-bound examination.

The next item is treated as an exponent (lines 5 and 6).As shown above, 22% took this approach. So the problem was the recognition of the expression on the right hand side, which has to do with knowing composite functions. However, having decided that it was an exponent, no.40 had no difficulty applying the procedure for differentiating a composite exponent, treating $\sin 2x$ as the power.[27]This would count as an example of a template-driven response, even though the wrong use is made of the "right" template.[28] As was seen

---

[26] Text of the examiner is not analysed in this study.

[27] in other words seeing $\sin 2x$ in place of u in the formula $y = a^u$ where a is a constant

[28] accepting that the "right" template was the wrong one to start with

above, the third item required a combination of procedures and also conceptual understanding of the direction the solution was meant to take. Participant number 40 had ideas about what should be found (a gradient and the equation of a straight line) but not the expected route to follow to get there. Instead participant number 40 did a very interesting thing:

a. she substituted the *x* value into the equation and found two values for *y*, one of which she had already (line 11).
b. Then she substituted the other *y* value and got two *x* values, one of which she had already (line 16).
c. In this way she got three points, satisfying the given equation.
d. Now, however, she constructed a point by taking the *y*-component from one point and the *x*-component from another point. This point, if she checked would not have satisfied the equation.
e. Next she determined, correctly, the slope of the line passing through the given point and her constructed point (line 17 and 18).
f. Then, finally, she determined, correctly, the equation of the straight line passing through those two points in (e) above (line 19).

This method used to determine the equation of a straight line passing through two given points, is a standard one employed in secondary school. The participant showed her familiarity with and competency in the procedure. Using the equation, an unfamiliar one, to determine the two points, was insightful, but then constructing the second point showed a lack of thoroughness: perhaps she should have written the four points down first, after making the substitutions. The style of answering also showed confidence and a certain freedom[29], perhaps indicating the participant had no hesitation in responding to the question. The efficient use of standard procedures is evident in this solution. For example, in line 15 only a summary of using the formula for solving a quadratic equation is given even though the coefficients are fractions.

Items 5 and 6 are done efficiently, supporting the sense of confidence and procedural competence. Item 7 requires the use of the formula for finding the area of a sector. This was not included on the formula sheet the participants were given thus they had to rely on memory to find it. Most used the wrong formula. The issue then became: how did they react to this problem? No.40 used the wrong formula (line 45). Then, in line with her performances elsewhere in the paper, she carried on with confidence, following through, correctly, the conceptual and procedural steps and determined the maximum area. Having found, however, that the area thus determined was a negative value (lines 59 – 60), this was promptly changed to a positive one (line 62). Again this seems to indicate that no.40 was not hesitant; that she had expectations of what is meant to happen in a solution. On the other hand, however, it showed that she did not stop to question her results[30].

Items 8, 9 and 10 were done last. The evidence is that many simply did not consolidate their understanding leading up to the examination. No.40 was one of them. The logic of the solution to item 8 (line 63) seems to be:

a. integrate the power – so she gets power 3
b. integrate $\frac{3}{x}$ so she gets $3\ln x$
c. divide by $\frac{3}{x}$ to cancel the $\frac{3}{x}$ you get when you differentiate $3\ln x$

[29] steps left out, for example leading up to line 17 and also leading up to line 19

This response is almost surely because of a template for finding the solution to $\int(3x-1)^2 dx$.

This template states: use the power rule[31] and divide by the coefficient of *x*.

To solve Item 9 she makes use of the product rule but integrates where one would normally differentiate (line 65): a practical use of a rule, although it is wrong, of course.

It is not clear to me what the use of the "$x+2$" is in item 10; perhaps relating to the initial expression. Creating the perfect square is an inversion of: when integrating a square, first multiply it out.

Critical indicators: participant no.40 is procedurally competent, is able to write logical sequences, takes bold steps, is creative and has a wealth of strategies from former courses in mathematics. She shows a tendency to follow through an initial logic but not to engage in reflection and make changes where deemed necessary. She appears to be confident about what is needed in all questions, even if her responses are sometimes wrong.

Participant No.93

This participant showed herself to be competent procedurally. She scored full points for item 2 (lines 5 – 14), item 4 (lines 23 – 33), item 8 (lines 58 – 63) and item 9 (lines 64 – 68). There were a few indications that she had poor competence in this area. She omitted a further derivative in item 3[32] (line 19) for the full points there. In item 10 she missed a sign in her calculations (line 71), getting the final answer wrong (line 73). In item 5 the initial derivative was not understood although the follow-up derivative and substitution were

---

[30] for example, "why did I get a negative area?"

[31] $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

[32] The derivative of $\sin 2x$ was given as $\cos 2x$ not $2\cos 2x$

correct (lines 36 – 38). There is an indication in item 9 of a search for the correct identity (line 64) with the first one having been rejected.

It is, however, in items 6 and 7 that No.93 reveals some discomfort. As indicated before, both items deal with the concept of stationary points and maxima and minima. No.93 does well in establishing the derivative (line 42), the value of x for which the derivative is zero (line 45) and finding the corresponding y-component (line 47) and hence the stationary point (line 48). She also determines the second derivative correctly (line 51). But then she writes that this derivative equals zero (line 52). The task is to test the sign of the second derivative at the stationary point and draw conclusions about the nature of that point. So the presence of line 52 is a puzzle: clearly No.93 did not know at that point how to make use of the second derivative, but what is the meaning of line 52? In the follow-up item, item 7, which uses the same concepts, No.93 falls short of engaging with the problem (lines 53 – 57). She makes a tentative start, has a formula for the perimeter of the sector (line 55) but soon gives up and erases her attempt (lines 56 – 7). This participant's work is perhaps the best demonstration that test and examination marks do not give the full picture of the mathematical achievement of a student.

Critical indicators: no.93 has good procedural competence. She appears to have some conceptual problems. She has not engaged with a conceptually difficult section but is able to hide behind good work elsewhere. The only items where she could have demonstrated creativity was item 7 and she failed to persist in what she started. This could simply be a case of not wasting time with questions she knows she won't solve. It could therefore be seen as a strategic decision. Her work indicates much confidence: her solutions are set out logically and clearly. She experienced very few crises in her work and avoided the one obvious case (item 7) after a half hearted start.

Participant No.130

This participant showed good procedural competence in dealing with items 2, 3 and 4, all of them intricate, requiring good skill and confidence. He showed, particularly in items 2 and 4 that he is able to successfully maintain procedural and conceptual consistency during a long manipulative process. He did not, however, complete item 5 and ignored the conceptual issues around the stationary point and the maxima-minima problem (item 7).

In item 2 the participant left out the derivative of the argument $1 - x^2$ (line 7). The concept involved here is the chain rule. But he demonstrated his knowledge of the chain rule in item 3 (line 15). In item 5 he did not complete the process of taking natural logs of the initial equation, which probably would have taken him to the correct answer. An alternative would have been to make use of exponential differentiation and the chain rule with argument $-x^2$, the standard algorithmic approach. He did not complete the process he began in item 5 (line 38) partly because he did not differentiate $\ln y$ properly (line 40). The crucial step is the derivative of $\ln y$, which required use of implicit differentiation. Similarly in item 3 he did not determine the derivative of $\ln y$ correctly. He did, however, demonstrate his use of implicit differentiation in item 4. In summary, the participant appears to have problems with the use of the chain rule and the derivative of $\ln y$.

In item 6, the participant again demonstrates his procedural skill in determining the first two derivatives and determining the stationary point. However, he stopped short of determining the nature of the stationary point. Also, he did not attempt item 7 at all, at least in so far as it is evident in the text[33]. This is a possible indication that he felt no confidence to do it and perhaps that he did not understand the underlying concepts. He does, however, attempt to deal with items 8, 9 and 10. In item 8 he tests out a solution and adjusts it (lines 56 -9). By the power rule he changes the power in the solution to one more than the question (2+1) and divides a constant equal to the new power (line 57). Then he also integrates the argument: $\frac{3}{x}-1$ and gets answer: $3\ln x$ and divides by this. The added 3 in the numerator is intended to cancel with this 3. Although this derived solution looks complicated to me, there is no written reflection to indicate that participant no. 130 had any doubts that it was the correct solution. Here the logic seems to be that the differentiation action and integration action are inverted for the argument: the power is meant to be integrated but the argument in the solution is intended to give $3\ln x$, which can only be found by integrating. In item 9 the product derived came from $\frac{\sin x}{\cos^2 x}$ not the one in the question. In item 10 the solution for the integral is correct but No.130 then made an algebraic error (line 66).

Critical indicators: procedurally competent, conceptually incomplete, deferred the crisis (of not knowing certain concepts) and may tend to alter questions to get solutions (items 5 and 9). Showed some creativity (for example in altering certain questions and making a series of adjustments in item 8), but failed to attempt item 7 which required lots of creativity if the standard algorithm was not known.

5.7.2 Follow-up investigation of participants numbers 75, 96 and 100

As discussed in section 5.2, three participants stood out in the analysis of item 10 (multiple choice test): they scored correctly but were drawn from the bottom 25% of the overall performance for that test. I mentioned that I would follow up their performance in the

---

[33] He could have thought about it

examination (tertiary) in the search for possible signs that they had not answered item 10 (multiple choice), by chance. The results which stood out for me are listed in table 48.

Triangulation of individual participants’ response to item 10 (multiple choice)	
Participant number	Stage completed
75	Got item 5 correct (20% did), was conceptually able in item 6 (41,8% were) and had the conceptual idea for item 6 (but did not progress much beyond that)
96	Got the derivatives right for item 5, showed good conceptual understanding of item 6
100	Got item 3 correct (17% did)

Table 48: Triangulation of individual participants’ response to item 10 (multiple choice)

Short of further investigation, I could only surmise from the triangulation that each one of the three participants had some knowledge and facility in the requisite mathematics. Participant number 75 got item 5 (tertiary) correct – this was not by chance. A review of that performance (i.e. looking at the script) showed sound logical steps and correct differentiation, good use of the product rule and chain rule (twice) and good knowledge of functional substitution. Even in item 6 (tertiary), he showed good differentiation technique and understood the conceptual demand of the problem. This was not a participant who did not know what he was doing. So I would surmise that participant number 75 got item 4 (multiple choice) correct not by chance.

Participant number 96, who was the only one of the three to pass the test (52%), nearly answered items 4, 5 and 6 (tertiary) correctly. She was thus on hand to get a pass mark at least for the examination (tertiary). Her correct choice for item 4 (multiple choice) was in all likelihood not by chance either, based on her performance in the tertiary examination. Even when she interpreted the composite function wrongly in item 3 (tertiary) the resultant mathematical steps performed on that function were correct, so chance played no role there.

Participant number 100 got item 3 (tertiary) correct. This was the case for only 17% of the cohort. That item required an initial awareness of the nature of the composite function, as discussed in section 5.4. There I speculated that those who did not understand the composite function and used a wrong procedure correctly may have done equally well had they started off with the right approach, since afterward the problem reduces to a procedural one. Nonetheless, participant number 100 did everything right. My analysis of her script showed that she also had a layout for how she was going to do the problem, which showed order and methodology on her part (see Appendix B). My judgement indicated to me that this was not the work of someone who would choose the solution randomly in the case of item 4 (multiple choice). Elsewhere in her script, she showed clear intention to solve most of the problems but seem to get stuck midway, especially after the derivatives had been determined (items 2, 4 and 5, tertiary). I would thus surmise that she did not get the correct answer for item 4 (multiple choice) by chance.

This investigation of the three participants came about because I surmised that their performance on one multiple choice item, assuming it was not by chance, showed that they had some knowledge and facility in the requisite mathematics, irrespective of their overall performance on that test. The results of the investigation proved to be significant. This has led me to consider my action as an important feature of individual tracking in the assessment of student work.

5.8 Building an integrated profile

When viewed across the different kinds of profiling techniques adopted, the next step was to assess whether those techniques complemented one another. The techniques used were:

analysis of test and examination scores (using a taxonomy); content analysis (with emphasis on procedural and conceptual knowledge); and, textual analysis using semiotics.

In responding to the first question I used a three-prong approach to inquire into the data. I started by presenting the conventional test scores with an overarching analysis in terms of mean and standard deviation. I then dissected the scores and placed them on a scale developed along the lines of a taxonomy of cognitive areas. I further did an analysis of the content of the examinations scripts of the cohort, based on the same cognitive areas, but focussing on the conceptual and procedural knowledge domains. Finally I did a textual analysis of a sample of individual scripts. My summary and discussion in section 5.5 indicated that, apart from the scores for the multiple choice test, the profiles developed provided a cohesive and complementary overview of the achievement of the sample of cohort so selected.

When doing the content analysis I found a myriad of categories. Upon further and deeper analysis I was able to determine broad groups around procedural and conceptual competence. The existence of individual differences encouraged me to seek more trends. I had to make choices about how far I wanted to split the group: my choice was to limit the splits by continuing to use the procedural and conceptual competence as a guide. It was very clear that the existence of so many strategies (many incomplete) for standard constructed response questions should act as a caution against over-generalisation about what groups or individuals can or cannot do.

The mathematical achievement profile of the cohort underpins discussions about systemic issues such as teaching and learning and forms of assessment. I refer back to Neubrand (2005) discussed in the introduction to this study:

> *Different didactical traditions and ways of teaching lead to different "inner structures" of mathematical achievement, made visible by different performance in the types of mathematical activities (p. 82).*

It is these "inner structures" which one attempts to find through "different performance". In the case of this study that performance was almost exclusively limited to "employing only techniques" but had some "problem solving activities using mathematical tools and procedures". That this was the case was not accidental but tied in with Neubrand's comments about systemic issues. The "didactical traditions and ways of teaching" sit at the core of the "performance" which the study attempted to unearth and dissect.

My textual analysis complemented the content analysis to a great extent. This was a good result since the focus of both analyses was the same (the scripts) and the essential core of the analyses was the same: the interrogation of procedural and conceptual competencies. In this regard, the textual analysis was an affirmation of what was revealed through the content analysis done in section 5.4. The textual analysis also extended the results of the content analysis in that close individual scrutiny made possible even greater differences than a group classification could allow. Thus the different strategies as revealed by the content analysis were reinforced and extended in the textual analysis. My analysis reinforced the non-uniform picture of how the participant members of the cohort dealt with the problems (test items) and the varied strategies they used. The standard approaches to dealing with student scripts were revealed as being limited in capturing the full extent of participant responses. These standard responses included procedural and conceptual errors, the use of child methods, misrepresenting and misunderstanding uses of letters and so on. In other words, I'm referring to the range of responses as discussed in section 2.2. The introduction of critical indicators was an attempt to characterise participant responses over and above those methods and labels of section 2.2 as outlined above. The use of these critical indicators was intended to point towards further factors, not easily revealed by standard methods, the consideration of which could enrich an understanding of the achievement of the cohort. To that extent the use of the critical indicators provided an additional form of analysis which enriched the profile of a selected sample of the cohort developed up to that point. Critical indicators such as *creativity, dealing with crisis* and *confidence* extended the profiles developed through the content analysis. These indicators were wrought through the textual analysis and proved to be significant in detailing the various profiles of the participants reviewed. The strategies the cohort adopted could thus more adequately be assessed against the critical indicators, especially in the many cases where their responses were outside the norm or fell short of completion in terms of the norm.

To answer the question of complementarity, I attempted an overview for participant number 130, as an example. Inter- table analysis was also done. For the analysis of test scores the presence of multiple choice individual scores against the established cognitive areas was not entirely satisfactory to me. This was so especially in the light of knowledge that some of the scores in the taxonomy were derived from individual test items. One example was the 0% scored against "perform routine calculations". This made any attempt at being definitive about what participant number 130 could or could not do in certain cognitive areas risky. At the same time, there were some cognitive areas where the range of testing was higher, thus providing more feedback about the achievement of the participant at that level of cognition. This was the case of the cognitive area: problem solving level 2 (with routine problems) (multiple choice column). The tertiary examination column offered more hope when compared with the other two methods of analysis, as described below.

The use of the content analysis for the test items was more satisfactory in that comparisons within the group were possible, as well as individual profiling. Thus it is possible to assess participant number 130's achievement for test item 6 within 10% of the group: all were procedurally competent, but had difficulties with conceptualising the problem. Looking down the profile column it is also possible to get an overarching profile for participant number 130, using the competence scale (table 8, section 4.2.4): conceptually poor, procedurally competent, knowledge competent and poor competence in problem solving.

The textual analysis complements the content analysis. This is not accidental: the process of creating the categories when I did the content analysis informed my analysis of the text. Thus it is possible to read much the same language when I analysed for procedural competence:

> *This participant showed good procedural competence in dealing with item 2, 3 and 4, all of them intricate, requiring good skill and confidence. He showed, particularly in items 2 and 4 that he is able to successfully maintain procedural and conceptual consistency during a long manipulative process (participant no.130, section 5.7).*

What this form of analysis allows for (which the others do not) is an in-depth dissection of individual process, down to forms of text on the page which are other than mathematical forms. As indicated in section 5.4 which deals with methods of providing solutions, these other forms provide further possibilities for inferring affective behaviour. Thus I use the notions of "confidence" and "creativity", which can only be inferred from the text. The advantage of using an individualised analysis as is done here is that there are therefore no limits to how detailed one can be about specific test items, or groups of them or the examination script as a whole. This was done, for example, in the analysis of item 10 (tertiary) where I was able to reflect back to earlier attempted solutions to infer a pattern of working which would not normally be picked up in a typical analysis of content as done is section 5.4:

There have been earlier examples of such adjustments, but there appeared to be logic to them (participant number 130, section 5.7)

At the same time, this method of altering questions to get solutions is seen as creative in this section. What one does with that creativity is open to question, but the point being made is that it is better to "make something" with the problem than nothing at all. If anything, by doing something this allows others some insight into what you may have thought about the problem. It also allowed me to see what the participant is able to do, which is a very important function of assessment in general.

I wanted to compare the relation between the scores in the tertiary examination column in the taxonomy table with the content analysis in terms of complementarity. It was clear that the scores compared very favourably with the content analysis table. For example, there was 0% for the cognitive area "problem solving level 2" for the scores and a "conceptual difficulty" for the content table. The textual analysis was equally clear: the participant made no attempt to answer item 7.

In summary, then, I found that the test scores hid much of what the participant actually achieved. There is thus a danger if this is all that is used in assessment. This was true in particular for the multiple choice test items, but also in the case of the examination results (tertiary). By using the method of triangulation, I found that the use of content analysis told me more about what strategies the group as a whole made use of and how the individuals compared against the group. The textual analysis allowed me to analyse deeper the work of individuals and make inferences based on what was already known from the previous analysis coupled with what was available for interpretation in the text.

6. Chapter 6: Conclusion

6.1 Introduction

Although the study adopts a three- prong approach, the three methods arose out of a single focus: to describe the data which was presented as representative of the mathematical achievement of a cohort of tertiary students at the entry level. The analysis of data scores came first because that is often the springboard from which official statements are made and profiles derived about students. These scores were representative of that level of tertiary student at CPUT. In summary, the statistics reveal positive relationships between the matriculation results and the tertiary results, although those relationships did not appear to be very strong. On account of the number of outliers in the scatterplot the results do not encourage uncritical statements about causality. For example, it would be against the information revealed by the data to speculate that good matriculation (school leaving) results in mathematics lead to good results in mathematics at tertiary entry level.

The classification of the scores along the lines of a taxonomy of cognitive areas expanded the picture of the achievement profile of the cohort. This step enabled me to show the level of achievement within and across test items. From there it was the logical next step to interrogate the content of the assessments, again both within and across items. This time the range of responses to individual test items was quite wide. Simple categories such as initially proposed (section 3.2) proved adequate as a start-up mechanism. It became evident from analysing test item 2 (section 5.4) that the categories hid many more subcategories. It was possible, once a certain number of categories for a section was decided on (based, as always on the data), to further subdivide these sub-divisions, but I decided against it. The reason was simply that I knew that that process would inevitably lead to a form of individual classification. I maintained a number of categories per test item and used the presence or not of conceptual competence as a guide to making the divisions. To assess the mathematical achievement of students at tertiary level, it is a necessary starting point to assess their conceptual and procedural competence. Any further attention to detail, for which methodologies such as semiotic analyses are proving to be conducive, could be the follow up process.[34] The content analysis categories were useful in describing what the participants were able to do in each of the test items. I was also able to draw a profile for any individual member as I indicated (sections 5.5 and 5.5.1).

The process seemed incomplete without an analysis of individuals. This I felt was necessary, firstly to test whether there would be findings which contradicted those from the first two analyses. Secondly, using semiotics as an entrée, I wanted to delve further into the text and seek additional indications about the participant which would enhance the profile I was compiling. Although the method used in the third section differed from those in the first two, there were overlaps. For example I was able to reflect on the procedural and conceptual competencies evident in the individual participant texts. But I was also able, by using a scheme that was sensitive to other, possibly affective indications in the text, to infer the existence of other features of the participant members.

In the end, then, I found that the three methods complemented one another. In fact, I thought that the methods led from one to the other in a logical way and for me to have used the textual analyses, which is presented here as the most modern analytical method, without the build up would not have been as effective and illuminating.

[34] I'm thinking of proponents such as those who inform the philosophy of TIMSS and PISA

6.2 Limitations of the study

This study had methodological limitations. The use of a taxonomy to analyse test scores covered up more about the cohort than it revealed. Although the follow-up content analysis uncovered much, the test scores analysis stood bare in contrast. The use of examination scripts meant that the written mode of expression of the cohort would be analysed; this privileged those who are comfortable in that mode.

The choice of examination scripts and test scores for data was intentional since researchers often only have access to similar kinds of data and I wanted to add to the research results available from that kind of data analysis. The choice of data excluded interpersonal contact, for example, in the form of interviews. Though this choice was intentional, the loss of other methods of determining mathematical achievement is viewed as a limitation of the study.

The lack of response to certain test items by some participant members created a degree of uncertainty for me as I could not assign a consistent interpretation to empty space with any confidence. At best I was left to conjecture what could possibly have led to such behaviour on the part of the participants and seek support for my views elsewhere in the text. To a lesser extent this was also the case where participant members did not respond extensively to the questions.

The lack of non-routine questions in the tertiary entry level course meant that the cohort had to take a multiple choice computer based test with such non-routine questions at the beginning of the course. The analysis would have benefited from a greater interrogation of the non-routine problem solving skills of the cohort, but this was not their experience in the course. This is seen as a limitation of the study.

The research design, which was constructed to address my research questions, includes a cohort of entry-level tertiary mathematics students. These students have diverse backgrounds, almost all of which have been excluded from the study. Although the study was focussed on the cohort as a whole, the differences among them may highlight certain interesting details which could not be revealed in this study. This is a limitation of the study but it is also an opportunity for further exploration and research. There are many questions about language which the study did not consider. These questions have relevance in light of comments about the acquisition of language and mathematical thought. How is the issue of language implicated in the mathematical achievement of a group such as the one in this study?

6.3 Interpretations: responding to research questions

6.3.1 The first question

The first question was: *How did participant members of the cohort respond to constructed response questions?* At the outset it was clear that there was a range of responses to every test item. The responses were not all clearly defined or well articulated. Within items, the spread of responses and the range of half finished strategies were quite wide. For item 4, for example, I ended up with 11 different categories of responses. This fact in itself revealed a distribution of incompleteness in terms of "employing techniques". The distribution across items reveals a similar pattern. A reasonable number responded, for every item, in a pro-forma manner, that is, very close to the solutions provided by me in the thesis. But the group who gave correct answers varied from item to item. Some members did well on some items but poorly on others. This realisation acted as a caution against sweeping generalisations about what a participant member could or could not do. Some participant members who got solutions wrong gave reasonable, meaningful, clear and logical responses, nonetheless. These types of responses showed the importance of following the participant's line of thought even if it was not on the path to the required solution. The participants responded in many ways as predicted by the research literature. For example, the use of child methods was evident; evidence of poor use of reflective abstraction for coordination in the case of using the chain rule for composite functions; accessing the wrong frames; not seeing the underlying structure of an equation and treating letters as objects. In certain cases participants did not respond to test items. There were no obvious indications in the text as to why that would have been the case.

6.3.2 The second question

The second question was: *What is the mathematical achievement profile of the cohort?* Although a significant number of participants were revealed as being procedurally very competent and to have good conceptual competence, the majority were found to have procedural competence and poor conceptual competence. This was the case, despite very important examples where participants revealed good procedural competency even though they were not writing down the correct solution.

The disturbing picture which emerges is one of immediate, quick, shortcut solutions to limited, procedurally heavy questions. Most of these solutions lack depth, reflection and a uniformity of spread of conceptual understanding. This may well have been a direct response to the questions, most of which had similar characteristics.

There were a number of individual cases whose work was also subjected to textual analysis. This analysis made it possible to extend the mathematical achievement profile of those participants by adding notions about their work which was not possible with the other methods. These notions included their response to crises, evidence of confidence in their solutions, signs that they reflected on the steps taken in their solutions and signs of creativity.

6.3.3 The third question

The third question was: *What additional details does the textual analysis of the student produced scripts reveal about participant members of the cohort?* My textual analysis reinforced the non-uniform picture of how the participant members of the cohort dealt with the problems (test items) and the varied strategies they used. However, these were linked to new indicators which I created based on my analysis. The participants, consequently, were revealed as more human, prone to individualistic impulses and less bound by the formalised system within which they were being assessed.

6.3.4 Implications for Research and Pedagogy

Research questions

One of the effects of the study was the realisation that the awareness by a researcher of isolated cases of mathematical knowledge and facility in the requisite mathematics by individuals could act as a catalyst for further exploration of the mathematical achievement of those individuals, who may otherwise remain unobserved in the programme. This was the case of the three participants who scored in the lowest 25% overall for the multiple choice test were part of only 28.57% of the cohort who got the correct answer for item 4 (multiple choice). One question for further research concerns the identification of trends using test items which emphasise specific areas of skill. These items could be used as a stimulus for further exploration of the achievement of individual members of the group, as illustrated in the case of test item 4 (multiple choice).

Radford (2004) expressed the need for "mathematical texts [to be] less committed to the written tradition" and for "predication[35] [to be] capable of integrating into itself the plurality

[35] Predictive judgements such as equations and inequalities

of semiotic systems that teachers use: speech, gestures, graphs, bodily action" (p. 165). My own textual analysis was confined to the written text as a semiotic system as this remains the mainstay of assessment in our South African system to date (2006 -2008). However, other semiotic systems for assessment will have more import as our education system continues to change. This is because the new outcomes based education provisions already in use in schools envisage pedagogical situations which extend interaction and assessment beyond the written formats and other traditions of the past. More exposure is expected to be given to group activities, project work, and oral presentations. This is part of a growing conviction within the educational world that mathematics involves more than content knowledge, that competence in process skills such as investigating, generalising, modelling, making conjectures and proving are all equally necessary (NCTM, 1989; DOE, 2003). A question for research in this regard concerns the kinds of semiotic systems: speech, gestures, graphs and bodily action that would be in use in the new dispensation and how one would conduct assessment in them.

In my analysis of item 3 (tertiary) I introduced the notion of *perceptual* difficulties in relation to members of the cohort who did not recognise the composite function. Those members had demonstrated their procedural competence, thereby showing that this was not what had prevented them from performing the correct procedure. The notion of *perception* in mathematics has to be tied to ideas about concept formation as outlined, for example, by Sfard (2000). The way in which a composite function is described by, for example, Dubinsky (1991), highlights the intricate nature of the concept formation and, I suggest, of the concept perception. One question which could be pursued in this regard is: Is recognition of *form* a pre-requisite for recognition of concept? This is linked to semiotic ideas about signs and symbols and could be explored in that way.

The role of language in the development of mathematical faculty was sketched in brief as part of providing an overarching background to the study in Chapter Two. Again, in South Africa, language issues are at the forefront of educational concerns, including concerns about the learning of mathematics. How do students' mathematical developments parallel the growth in their facility with a language and the complexity of their use of that language?

Pedagogical implications

The use of triangulation to arrive at an achievement profile of participant members of the cohort has implications for pedagogy. The methods employed in the study highlight the need for a range of assessment practices for the same subjects. This is because using a single form of assessment cannot capture all the facets of the student's achievement. Moreover, no single assessment can be complete. As the content analysis demonstrated, a variety of responses to a single test item can lead to further exploration into the mathematical faculty of the subjects. Also, discourse has been shown to reveal thinking patterns of students working in groups which it is not possible to uncover in non-dialogue formats (Radford, 2002, p. 16). Even the use of individual test items from the multiple choice test showed that performance in specific test items could be a useful pointer to overall performance, and could serve as a catalyst for further exploration of individual achievement. The detailed analysis of individual participant members reveals the difficulties inherent in assessing students' abilities from a limited base consisting of tests and written scripts.

The use of students' written texts can be exploited for mathematical learning in a number of ways in pedagogical situations. Students themselves could analyse scripts and suggest insights. In this way, they will be challenged to engage in the mathematics as set out in the scripts and to contrast the concepts and procedures expressed against their own understandings of those concepts and procedures. Such sessions could form the basis for further explorations of the concepts in the texts and related mathematics.

Students' written responses to a test could serve as a basis for a dialogue among peers or between teacher and students. By engaging the participants in dialogue with regard to their written inputs, opportunities are created for clarification of concepts and procedures. Also, the participants are in that way empowered to clarify what they intended if it was not clear to the reader.

Getting students to view different solutions to a test item is another way that they can reflect on alternative ways of thinking about mathematics. This opens up the discipline to variation, flexibility and individuality and moves it away from routine and traditionally accepted formalities.

In the analysis of item 3, assessment examination, section 5.4, I indicated that some of the participants demonstrated their facility with mathematics even though they got the item wrong. I suggested that it was likely that they may have misunderstood the problem because of not understanding functions well enough. The pedagogical implication is that lack of performance in one area of mathematics may cover up a deeper issue: under performance in an area of mathematics which precedes the section under review.

The use of a pedagogical framework for assessing achievement can also act as a tool for understanding what gaps there are in subject matter presented in a course. Since the pedagogical framework shows cells of cognitive processes, such as 'perform routine procedures', these cells can be filled up with those cognitive sections in use in the course. But there may be empty cells, which would reveal an absence of aspects of cognition, for example, 'perform non-routine problem solving'. These should give pause for thought. As Kastberg (2004) puts it, "empty cells in the framework allow for reflection'. The exposure of such gaps could sensitise the facilitator to provide opportunities for exploration in those types of areas. The analysis of the framework can reveal patterns of how the subject matter of the course is being presented.

Many of my findings relate directly to the discourse within which the students worked; the overall aim being for them to acquire the techniques and procedures in order to do the correct algorithmic work. The important implication is that students will develop narrowly if they are narrowly channelled. As has been indicated, the assessment practices within the discourse of traditional schooling are very narrow and limited. This has implications for the change over to a new educational system: other pedagogical methods are being promoted and with that must go a wider range of assessment practices. These findings reveal the existence of students whose comfort with the formalised mathematical system, against which background they were assessed, is not adequate for them to advance deeper into the discipline without structured intervention.

6.4 Summary

This research provided insights into the mathematical achievement of a cohort of tertiary mathematics students. The characteristics of their achievement concern knowledge and understanding of mathematical concepts, procedures and problem solving abilities. This research also developed a mathematical achievement profile for individual members of the cohort. The materials for the inquiry were student produced test scores and examination scripts taken from their entry level mathematics course. The methodology made use of content - and textual analytic methods to profile the cohort.

When viewed across the different kinds of profiling techniques adopted, the findings suggest that these techniques complemented one another: the profiles developed provided a cohesive and complementary overview of the achievement of the cohort. The techniques used were:

analysis of test and examination scores (using a taxonomy); content analysis (with emphasis on procedural and conceptual knowledge); and, textual analysis using semiotics.

The use of these methods revealed interesting patterns when the participant members of the cohort responded to the test and examination questions. The wide range of responses to test items revealed a distribution of incompleteness in terms of employing algorithmic techniques. Some participant members, who got solutions wrong, nonetheless gave reasonable, meaningful, clear and logical responses. The participants responded in many ways as predicted by the research literature. For example, the use of child methods (Booth, 1981) was evident in the text. There was evidence of poor use of reflective abstraction for coordination (Dubinsky, 1991), for example in the case of using the chain rule for composite functions. The theory about accessing the wrong frames (Davis, et al., 1978) was evident. Also observed in the texts were participant members not seeing the underlying structure of equations (Hoch and Dreyfus, 2004) and treating letters as objects (Küchemann, 1983).

The study challenged perceptions that responses to constructed response questions offer little information about the mathematical knowledge of students. Findings suggest that diagnostic uses of paper and pencil tests can be revealing about the achievement of students. The disturbing picture which emerged was one of immediate, quick, shortcut solutions to limited, procedurally dominant questions. Most of these solutions lack depth, reflection and a uniformity of spread of conceptual understanding.

The use of a textual analytic method, which led the creation of critical indicators as a way of sign-posting events, enhanced the achievement profile of the students. The participants, consequently, were revealed through the text as more human, prone to individualistic impulses and less bound by the formalised system within which they were being assessed.

7. References

Adler, J. (1998). Resources as a verb: Recontextualising resources in and for school mathematics. In Olivier, A. and Newstead, K. (Eds.) Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, 1, 1-18.

Anderson, L. W. (2005). Objectives, Evaluation, and the Improvement of Education. *Studies in Educational Evaluation*, 31, 102-113.

Anderson, L.W. (Ed), Krathwohl, D. R. (Ed.), Airasian, P.W., Cruikshank, K.A., Mayer, R.E., Pintrich, P.R., J., and Wittrock, M.C. (2001). A taxonomy for learning, teaching, and assessing: A revision of Bloom's Taxonomy of Educational Objectives. Boston: Allyn and Bacon.

Ansari, D., Donlan, C., Thomas, M. S., Ewing, S. A., Peen, T., and Karmiloff-Smith, A. (2003). What makes counting count? Verbal and visuo-spatial contributions to typical and atypical number development. *Journal of Experimental Child Psychology*, 85(1), 50-62.

Ansari, D., and Karmiloff-Smith, A. (2002). Atypical trajectories of number development: a neuroconstructivist perspective. *Trends in Cognitive Science,* 6(12), 511-516.

Atherton, J. S. (2005). Learning and Teaching: Behaviourism [On-line] UK: Available: http://www.learningandteaching.info/learning/behaviour.htm Accessed: 3 November 2006

Atkinson, P. and Coffey, A. (2004). Analysing documentary realities. In Silverman, David (Ed.) Qualitative Research: Theory, Method and Practice. 2nd Edition. London: Sage Publications.

Baker, C. (2004). Membership categorization and interview accounts. In Silverman, David (Ed.) Qualitative Research: Theory, Method and Practice. 2nd Edition. London: Sage Publications.

Berger, M. (2006). Making mathematical meaning: from preconcepts to pseudoconcepts to concepts. *Pythagoras*, 63, 14-21.

Bloom, B.S. (Ed.), Enlehard, M.D., Furst, E.J., Hill, W.H., and Krathwohl, D.R (1956). Taxonomy of educational objectives: The classification of educational goals, Handbook I: Cognitive domain. New York: David Mckay.

Botha, M., Maree, J. G., and De Witt, M. W. (2005). Developing and piloting the planning for facilitating mathematical processes and strategies for preschool learners. *Early Child Development and Care***,** November, 175(7/8), 697-717. Taken from the online inter- library source: EBSCOhost.

Booth, L. (1981). Child-Methods in Secondary Mathematics. *Educational Studies in Mathematics*, 12, 29-41.

Booth, L. (1984). Misconceptions Leading to Error in Elementary algebra. *Journal of Structural Learning*, 8, 125-138.

Brousseau, G. (1997). Theory of didactical situations in mathematics: Didactique des mathematiques, 1970-1990. Dordrecht, The Netherlands: Kluwer.

Butterworth, B. (1999). The Mathematical Brain. London: Macmillan.

Chamaz, K. (2005). Grounded Theory in the 21st Century: Applications for Advancing Social Justice Studies. In Denzin, N. and Lincoln, Y. (Eds.) Handbook of Qualitative Research. 3rd Edition. London: SAGE Publications.

Chandler, D. (1995). Semiotics for Beginners. Online text retrieved from http://www.aber.ac.uk/media/Documents/S4B/sem09.html on 2 April, 2004.

Cohen, L., Manion, L., and Morrison, K. (2003). Research Methods in Education. 5th Ed. London: RoutledgeFalmer.

Collins Concise Dictionary (2001). General Consultant: Sinclair, J. M. Glasgow: HarperCollins Publishers.

Darling-Hammond, L., and Snyder, J. (1992). Curriculum Studies and the traditions of inquiry: the Scientific tradition. In Jackson, Philip W. (Ed.) Handbook of Research on Curriculum: A project of the American Educational Research Association. New York: Macmillan Publishing Company.

Davie, R. (1972). The longitudinal approach. *Trends in Education,* 28, 8-13.

Davis, R. B., Jockusch, E., and Mcknight, C. (1978). Cognitive Processes in Learning Algebra. *The Journal of Children's Mathematical Behavior*, 2(1), 10-320.

Dehaene, S. (1997). The Number Sense: How the mind creates mathematics. Oxford: Oxford University Press.

Dehaene, S., Dehaene-Lambertz, G., and Cohen, L. (1998). Abstract representations of numbers in the animal and human brain. *Trends in Neuroscience*, 21(8), 355-361.

Dehaene, S., Spelke, E., Pinel, P., Stanescu, R., and Tsivkin, S. (1999). Sources of mathematical thinking: behavioural and brain-imaging evidence science. *Science*, 284(5416), 970-974.

De Jager, C., Fitton, S. And Blake, P. (1992). Just Mathematics. Cape Town:Maskew Miller Longman (Pty) Ltd.

Department of Education (DOE), South Africa (2003). National Curriculum Statement Grades 10-12 (General). Pretoria: Government Printer.

Devlin, K. (2004) The Maths Gene: Why everyone has it, but most people don't use it. Weidenfeld and Nicolson. ISBN: 0297645714.

Dreyfus, T. 7 Hoch, M. (2004) Equations – A Structural Approach. In Proceedings of the Twenty Eighth Conference of the International Group for the Psychology of Mathematics Education (Bergen, Norway).

Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In Tall, D. (Ed.): Advanced Mathematical Thinking. London: Kluiwer Academic Publishers.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D. and Agard, P. (1993). Conceptual Knowledge Falls through the Cracks: Complexities of Learning to Teach Mathematics for Understanding. *Journal for Research in Mathematics Education*, 24(1), pp. 8-40.

Engelbrecht, J., Harding, A., and Potgieter, M. (2005). Undergraduate students' performance and confidence in procedural and conceptual mathematics. *International Journal for Mathematics Education in Science and Technology*, 36(7), 701-712.

Freudenthal, H. (1991) Revisiting mathematics education: China lectures. Dordrecht, The Netherlands: Kluwer.

Gattegno, C. (1973). What we owe children: The subordination of teaching to learning. New York, Avon.

Gonin, A. A.,Archer, I.J.M., Slabber, G.P.L. and De La Rey Nel, G. (1973). Modern Graded Mathematics for Standard 8. Cape Town: Nasou

Gelman, S. (2006). Concept development in preschool children. Taken from the American Association for the Advancement of Science website http://www.project2061.org/publications/earlychild/online/context/gelman.htm on 3 August 2006.

Gelman, R. and Gallistel, C. R. (1978). The Child's understanding of Number. London: Harvard University Press.

Haser, C. and Star, J. R. (2004) Preservice teachers' beliefs about mathematical understanding. Exchange Quarterly, 8(4), 147-151. Taken from the Michigan State University website: http://www.msu.edu/~jonstar/papers/Cigdem.pdf on 4 July 2006.

Hiebert, J., and Lefevre, P. (1986) Conceptual and procedural knowledge in mathematics: an introductory analysis. In J. Hiebert (Ed). Conceptual and procedural knowledge: The case of mathematics (pp. 1- 27). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Hoch, M. and Dreyfus, T. (2004). Structure sense in high school algebra: the effect of brackets. In M.J. Høines and A. B. Fuglestad (Eds.), Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, Vol. 3 , pp 49-56. Bergen, Norway: University of Bergen.

HSRC (1997) Study habits and attitude crucial to achievement in mathematics. Media release. Issued by Corporate Communications, HSRC, Pretoria. Taken from the internet 3 November: http://www.hsrc.ac.za/media/1997/3/19970319.html

Julie, C. (1992) Equations of Inequality: Challenging School Mathematics. *Perspectives in Education,* 13, 3-10.

Kastberg, S.E. (2003). Using Bloom's taxonomy as a framework for classroom assessment. *The Mathematics Teacher,* 96 (6), 402.

Khuzwayo, B. (2005). A history of mathematics education research in South Africa: the apartheid years. In Vithal, R., Adler, J., and Keitel, C. (eds.), Researching Mathematics Education in South Africa: Perspectives, practices and possibilities. Taken from the Human Science Research Council: HSRC Press website (20 September 2006): http://www.hsrcpress.ac.za/full_title_info.asp?id=2034

Kieran, C. (2005). Some results from the PISA 2003 International Assessment of Mathematics Learning: What makes items difficult for students? In Chick, H. L. and Vincent, J. L. (Eds.), Proceedings of the 29th Conference of the International Group for the Pscychology of Mathematics Education, Vol. 1, pp. 83-86. Melbourne: University of Melbourne.

Kilpatrick, J., Swafford, J. and Fidell, B. (2001) Adding it up; Helping children learn mathematics. Washington D. C.:National Acadmies Press. Online text from website of National Academies Press (15-30 July 2006): http://darwin.nap.edu/books/0309069955/html/145.html#p20004b748940145003

Krathwohl, D.R. (2004). Reflection on the taxonomy: Its past, present, and future. In Anderson, L.W. & Sosniak, L.O. (Eds). Bloom's Taxonomy: A forty-year retrospective. Chicago: University of Chicago Press.

Küchemann, D. E. (1981) Cognitive Demand of Secondary School Mathematics Items. *Educational Studies in Mathematics,* 12, 301-316.

Küchemann, D. E. (1983) Quantitative and Formal Methods for Solving Equations. *Mathematics in School*, 12, 5.

Lingefjärd, T. (2000) Assessment and Mathematics Examinations in the CDIO Project. Paper published within the Conception- Design – implementation- Operation Project at Chalmers Technical University.

Lipton, J. S., and Spelke, E. S. (2003) Origins of number sense. Large-number discrimination in human infants. *Psychology of Science,* 14(5), 396-401.

Long, C. (1998). From the intended to the attained curriculum: where are the gaps? *Pythagoras*, 46/47, 13-20.

(The) National Council of Teachers of Mathematics (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, VA: NCTM.

Neubrand, M. (2005). The PISA – Study: Challenge and impetus to research in Mathematics Education. In Chick, H. L. and Vincent, J. L. (Eds.). Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1, pp. 79-82. Melbourne: University of Melbourne.

O'Shaughnessy, M. and Stadler, J. (2002) Media and Society. Melbourne, VIC, Australia: Oxford University Press.

Piaget, J. (1960) The psychology of intelligence. Paterson, NJ: Littlefield, Adams and Co.

Piaget, J. (1973) Comments on mathematical education. In A.G. Howson (Ed.), Developments in mathematical education: Proceedings of the second International Congress on Mathematical Education (pp. 79 – 87). Cambridge: Cambridge University.

PISA (2003) The PISA 2003 Assessment Framework :Mathematics, Reading, Science and Problem Solving Knowledge and Skills OECD. OECD publishing. Available from http://www.oecd.org/document/55/0,2340,en_32252351_32236173_33917303_1_1_1_1,00.html

PISA (2003) Learning for Tomorrow's World :First Results from PISA 2003 OECD. OECD publishing. Available from http://www.oecd.org/document/55/0,2340,en_32252351_32236173_33917303_1_1_1_1,00.html

Prior, L. (2004) Doing things with documents. In Silverman, David (Ed.) Qualitative Research: Theory, Method and Practice. 2nd Edition. London: Sage Publications.

Ramsden, P. (1992) Learning to teach in Higher Education. London: Routledge.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics,* 42, 237-268.

Radford, L. (2002). The Seen, the Spoken and the Written: a Semiotic Approach to the Problem of Objectification of Mathematical knowledge. *For the Learning of Mathematics,* 22, 14-23.

Radford, L. (2004). Syntax and Meaning. In Høines, M.J. and Fuglestad, A. B. (Eds.), Proceedings of the Twenty Eighth Conference of the International Group for the Psychology of Mathematics Education (Bergen, Norway), 1, 162-5.

Reddy, V. (2006). Mathematics and Science Achievement at South African Schools in TIMSS 2003. Pretoria: Human Science Resources Council Press.

Robitaille, D.A (Ed.) (1993) Curriculum Frameworks for Mathematics and Science. TIMSS Monograph No. 1. Vancouver, Canada: Pacific University Press.

Romagnano, L. (2001). The Myth of Objectivity in Mathematics Assessment. The Mathematics Teacher 94 (1), 31-37 ·

Romberg, T. A. (2003) Problematic Features of the School Mathematics Curriculum. In Jackson, Philip W. (Eds.) Handbook of Research on Curriculum: A project of the American Educational Research Association (749-788) New York: Macmillan Publishing Company.

Ross, J.A., McDougall, D., Hogaboam-Gray, A. and LeSage, A. (2003) A survey measuring elementary teacher's implementation of standards-based mathematics teaching. *Journal for Research in Mathematics Education,* 34(4), 344.

Selden, R. (1989) A reader's guide to contemporary literary theory. 2nd Edition. London:Harvester Wheatsheaf.

Scheiber, J., Brown, M., Christelis, M., Lombard, S., Mbatha, L., Randall, I.and Noort, D. (1993). Classroom Mathematics 3. Johannesburg: Lexicon Publishers.

Schoen, H. L. and Hirsh, C. R. (2003). Responding to calls for change in high school mathematics: Implications for collegiate mathematics. *The American Mathematical Monthly,* Washington. February, 110(2), 109-123.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on process and object as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. (2000). Symbolizing Mathematical Reality Into Being – Or How Mathematical Discourse And Mathematical Objects Create Each Other. In Cobb, P., Yackel, E. and McClain, K. (Eds), Symbolizing and Communicating in Mathematics Classrooms: Perspectives on Discourse, Tools and Instructional Design (37-38). Mahwah, NJ: Lawrence Erlbaum Associates.

Sfard, A. and Linchevski, L. (1994). The gains and pitfalls of reification – the case of algebra. *Educational Studies in Mathematics*, 26, 191-228.

Shimizu, Y. (2005). From a profile to the scrutiny of student performance: exploring the research possibilities offered by the international achievement studies. In Chick, H. L. and Vincent, J. L. (Eds.). Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1, pp. 77-78. Melbourne: University of Melbourne.

Shuell,T. (2001). Cognitive Development. Taken on 6 August 2006 from the Graduate School of Education, State University of New York at Buffalo website: http://www.gse.buffalo.edu/fas/shuell/CEP564/Lectures/CogDev.htm

Silverman, D. (2004). Who cares about 'experience'? In Silverman, David (Ed.) Qualitative Research: Theory, Method and Practice. 2nd Edition. London: Sage Publications.

Skemp, R. (1987). The psychology of learning mathematics. Hillsdale: Erlbaum

Spicer, N. (2004). Combining qualitative and quantitative methods. In Seale, C. (Ed.) Researching Society and Culture. 2nd Edition. London: Sage Publications.

Star, J. R. ( ) Re-"Conceptualizing" Procedural Knowledge in Mathematics. Taken from the University of Michigan website on 3 July 2006: http://www.msu.edu/~jonstar/papers/Reconceptualize.pdf

Tall, D. (2004). Thinking through three worlds of Mathematics. In M.J. Høines and A. B. Fuglestad (Eds.), Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, Vol. 4 , pp 281-288. Bergen, Norway: University of Bergen.

Taylor, N. and Vinjevold, P. (1999). Teaching and Learning in South African Schools. In Taylor, N. and Vinjevold, P. (Eds.) Getting Learning Right: Report of the President's Education Initiative Research Project. Johannesburg: The Joint Education Trust.

Taylor, N. (1999). Learnig Outcomes: How Will We Know Them? In Taylor, N. and Vinjevold, P. (Eds.) Getting Learning Right: Report of the President's Education Initiative Research Project. Johannesburg: The Joint Education Trust.

Tomás Ferreira, R. (2004). Improving written tests: impact of research on student teachers' conceptions. Taken from the website http://math.unipa.it/~grim/21_project/CiechFerreira.pdf#search=%22analyzing%20written%20tests%20in%20school%20mathematics%22 14 September 2006.

Tonkiss, F. (2004). Analysing text and speech: content and discourse analysis. In Seale, C. (Ed.) Researching Society and Culture. 2nd Edition. London: Sage Publications.

Venezky, R. L. (1992). Textbooks in School and Society. In Jackson, Philip W. (Eds.) Handbook of Research on Curriculum: A project of the American Educational Research Association (pp. 41-78) New York: Macmillan Publishing Company.

Vile, A. (1998). Development Semiotics: theory and practice. *Pythagoras, 46/47, Aug/Dec, pp.21-27.*

Vygotsky, L. (1986). Thought and language. Cambridge, MA: MIT Press

Webb, N. L. and Coxford, A. F. (1993) (Eds.) Assessment in the mathematics classroom: 1993 Yearbook. Reston, VA: National Council of Teachers of Mathematics.

Wheeler, D. (1982). Mathematization matters. *For the Learning of Mathematics, 3(1), 45-47.*

Wynn, K. (1990). Children's understanding of counting. *Cognition, 36*(2), 155-193.

Xu, F., and Spelke, E. S. (2000). Large number discrimination in 6-month-old infants. *Cognition, 74*, B1-B11.

8. APPENDICES

8.1 APPENDIX A: EXAMINATION QUESTIONS FOR AT CALCULUS

1. Determine $\frac{dy}{dx}$ and simplify where possible:
   a. $y = \ln x^{\pi}$ (1)
   b. $y = \frac{1+x^2}{\sqrt{1-x^2}}$ (6)
   c. $y = (\sin 2x)^x$ (5)
2. Determine the equation of the tangent line to the curve
   $x^2 - 2xy + 3y^2 = 4$ at the point (-2;0). (7)
3. Determine $f''(0)$ if $f(x) = e^{-x^2}$ (5)
4. Determine the coordinates and the nature of the stationary
   points of $y = x \ln x$. (6)
5. Calculate the maximum area that a piece of wire of length
   40cm can enclose if it is bent in the form of a sector of a circle. (7)
6. Determine the following integrals
   a. $\int \left(\frac{3}{x} - 1 \right)^2 dx$ (4)
   b. $\int \frac{\sin^2 x}{\cos x} dx$ (3)
   c. $\int_{-1}^{0} \left(x^2 + 2x + 1 \right) dx$ (4)

Full marks: 48

8.2 APPENDIX B: STUDENT SCRIPTS

EXAMINATION SCRIPTS FOR PARTICIPANTS NUMBER

93, 8, 15, 16, 34, 40, 75, 96 and 100.

No. 93 line

QUESTION 1

(a) $y = \ln x^{\pi} - x^{\pi} + 2^{x\log_2 3}$ 1

$\frac{dy}{dx} \Rightarrow \frac{1}{x}\cdot D(x^{\pi}) - \pi x^{\pi-1} + 2^{x\log_2 3}\ln 2 \cdot D(x\log_2 3)$ 2

$\Rightarrow \frac{1}{x}\cdot \pi x^{\pi-1} - \pi x^{\pi-1} + 2^{x\log_2 3}\ln 2 \cdot \log_2 3$ 3

3

$\therefore \frac{dy}{dx} = \frac{\pi x^{\pi-1}}{x} - \pi x^{\pi-1} + 2^{x\log_2 3}\ln 2 \log_2 3$ 4

(b) $y = \frac{1+x^2}{\sqrt{1-x^2}}$ 5

$\frac{dy}{dx} \Rightarrow \frac{(\sqrt{1-x^2})\cdot D(1+x^2) - (1+x^2)\cdot D(\sqrt{1-x^2})}{(\sqrt{1-x^2})^2}$ 6

$\Rightarrow \frac{(\sqrt{1-x^2})\cdot 2x - (1+x^2)\cdot \frac{-2x}{2\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2}$ 7

$\Rightarrow \frac{2x\sqrt{1-x^2} + \frac{2x(1+x^2)}{2\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2}$ 8

$\Rightarrow \frac{\frac{2x\sqrt{1-x^2}\cdot 2\sqrt{1-x^2} + 2x(1+x^2)}{2\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2}$ 9

6

$\Rightarrow \frac{2x\sqrt{1-x^2}\cdot 2\sqrt{1-x^2} + 2x(1+x^2)}{2\sqrt{1-x^2}\,(\sqrt{1-x^2})^2}$ 10

$\Rightarrow \frac{4x\cdot(1-x^2) + 2x(1+x^2)}{2\cdot(\sqrt{1-x^2})^3}$ 11

$\Rightarrow \frac{4x - 4x^3 + 2x + 2x^3}{2(\sqrt{1-x^2})^3}$ 12

$\Rightarrow \frac{6x - 2x^3}{2(\sqrt{1-x^2})^3}$ 13

$\therefore \frac{dy}{dx} = \frac{3x - x^3}{(\sqrt{1-x^2})^3}$ 14

(e) $y = (\sin 2x)^x$

$\Rightarrow \ln y = \ln(\sin 2x)^x$

$\Rightarrow \ln y = x \ln \sin 2x$

$\Rightarrow \ln y = x \cdot D(\ln \sin 2x) + \ln \sin 2x \cdot D(x)$ ✓ S

$\Rightarrow \ln y = 2x \cdot \frac{\cos 2x}{\sin 2x} + \ln \sin 2x$

$\Rightarrow y \times \cancel{\ln y}\,\frac{1}{y} = 2x \cot 2x + \ln \sin 2x$ ✓

$\Rightarrow \frac{dy}{dx} = (\sin 2x)^x \cdot (x \cot 2x + \ln \sin 2x)$

$\therefore \frac{dy}{dx} = (\sin 2x)^x \cdot (2x \cot 2x + \ln \sin 2x)$ ✓

NO. 93 line

QUESTION ②

⇒ $x^2 - 2xy + 3y^2 = 4$ point $(-2 ; 0)$ 23

$\frac{dy}{dx}$ ⇒ $2x - 2x\frac{dy}{dx} - 2y + 6y\frac{dy}{dx} = 0$ 24

⇒ $2x - 2y = 2x\frac{dy}{dx} - 6y\frac{dy}{dx}$ 25

⇒ $2x - 2y = (2x - 6y)\frac{dy}{dx}$ 26

∴ $\frac{dy}{dx} = \frac{2x-2y}{2x-6y}$ 27

∴ $M = \frac{2(-2)-2(0)}{2(-2)-6(0)}$ ⇒ $\frac{-4-0}{-4-0}$ 28

$= +1$ 29

⇒ $y = +x + c$ 30

∴ c ⇒ $0 = +1(-2) + c$ 31

$c = 2$ 32

∴ equation ⇒ $y = x + 2$ 33

No. 93 line

QUESTION ③

$f''(0)$ if $f(x) = e^{-x^2}$ 34

$f'(x) \Rightarrow -2e^{-x}$ ✗ 35

0

$f''(x) = 2e^{-x}$ ✗ 36

1

$f''(0) = 2e^{-(0)} = 2\cdot(1)$ ✗ 37

$= 2$ ✗ 38

ND 93 line

QUESTION 4

$y = x \ln x$ 39

$\frac{dy}{dx} = x \cdot D(\ln x) + \ln x \cdot D(x)$ 40

$= \frac{x}{x} + \ln x$ 41

$= 1 + \ln x$ 42

$\Rightarrow -1 = \ln x$ 43

$\Rightarrow e^{-1} = e^{\ln x}$ 44

6

$\therefore x = 0.36788$ 45

$\Rightarrow y = (0.36788) \cdot \ln(0.36788)$ 46

4

$= -0.36788$ 47

$\therefore$ point $= (0.4\ ;\ -0.4)$ $\Rightarrow$ 1 signal 48

$\Rightarrow f''(x) = \boxed{1 + \ln x}$ 49

$\Rightarrow D(1) + D(\ln x)$ 50

$= \frac{1}{x}$ 51

$= 0$ 52

NO. 93

line

QUESTION 5

$d = 40cm$ 53

$A_{circle} = \pi r^2$ & $Perimeter_{circle} = \pi d$ 54

$\Rightarrow P_{sector} = \pi 2r\theta - \pi 2r\theta$ 55

$\therefore \cancel{\pi 2r\theta = 40cm}$ 56

$\therefore 2\pi r\theta - 2\pi r = 40cm$ 57

NO. 93

Line

QUESTION 6

(a) $\int \left(\frac{3}{x}-1\right)^2 dx \quad \Rightarrow \int \left(\frac{9}{x^2} - \frac{3}{x} - \frac{3}{x} + 1\right)$ 58

$\Rightarrow \int \frac{9}{x^2} - \frac{6}{x} + 1$ 59

$\Rightarrow 9\int x^{2} - 6\int \frac{1}{x} + \int 1 + c$ 60

$\Rightarrow 9(-x^{-1}) - 6\cdot(\ln x) + x + c$ 61

$\Rightarrow -\frac{9}{x} - 6\ln x + x + c$ 62

$\therefore$ Ans $= x - \frac{9}{x} - 6\ln x + c$ 63

4

(b) $\int \frac{\sin^2 x}{\cos x}\, dx \quad \Rightarrow \int \frac{(1-\cos^2 x)}{\cos x}$ 64

$\Rightarrow \int \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$ 65

$\Rightarrow \int \sec x - \cos x$ 66

$\Rightarrow \ln(\sec x + \tan x) - \sin x + c$ 67

$\therefore$ Ans $= \ln(\sec x + \tan x) - \sin x + c$ 68

3

(c) $\int_{-1}^{0} (x^2 + 2x + 1)\, dx \quad \Rightarrow \int x^2 + \int 2x + \int 1 \Big|_{-1}^{0}$ 69

$\Rightarrow \frac{x^3}{3} + x^2 + x \Big|_{-1}^{0}$ 70

$\Rightarrow \left[\frac{1}{3}\cdot(0)^3 + (0)^2 + (0)\right] - \left[\frac{1}{3}\cdot(-1)^3 + (-1)^2 + (-1)\right] \Big|_{-1}^{0}$ 71

$\Rightarrow 0 - \left(-\frac{1}{3}\right)$ 72

$= -\frac{1}{3}$ 73

4

3

$-\frac{1}{3} + 1 - 1$

MATHEMATICS I

NO. 8 Line

(Done after no. 6) 1
Sorry for the inconvienence 2
3

1) a. $y = \ln x^{\pi} - x^{\pi} + 2^{x \ln 2, 3}$ 1
$\frac{dy}{dx} = 3,14 - 3,14 + 2,197$ 2
$= 2,2$ 3

b. $y = \frac{1+x^2}{\sqrt{1-x^2}}$ 4
$\frac{dy}{dx} = \frac{1\ 2x}{1-(x)^{\frac{1}{2}}}$ 5
$= \frac{2x}{1}$ 6
$= 2x$ 7

c. $y = (\sin 2x)^x$ 8
$y\frac{dy}{dx} = (2\sin x \cos x)^x$ 9
$\frac{dy}{dx} = \cancel{2\sin}(2\cos x)(-\sin x)^x$ 10
$= (2\cos^2 x)^x$ 11
$= 2\cos^3 x$ 12
$= 4\cos x$ 13

2) $x^2 - 2xy + 3y^2 = 4$ 14
$3y^2 = -x^2 - 2xy + 4$ 15
For x - intercept 16
let $y = 0$ 17
$0 = -x^2 + 4$ 18
$x^2 = 4$ 19
$x = \pm 2$ 20
For y - intercept 21
let $x = 0$ 22
$\therefore 3y^2 = -x^2 - 2xy + 4$ 23
$3y^2 = 0 - 0 + 4$ 24
$3y^2 = 4$ 25
$y^2 = \frac{4}{3}$ 26
$y = \pm\sqrt{\frac{4}{3}}$ 27
$y = \pm 1,154$ 28

$\therefore$ tangent line to curve : $3y^2 = -x^2 - 2xy + 4$ 29
$(-2;y)$ & $3y^2 = -(-2)^2 - 2(-2)(y) + 4$
$3y^2 = -4 + 4y + 4$ 31
$3y^2 = 4y$ 32
$3y = 4$ 33
$y = \frac{4}{3}$ 34

NO. 8 Line

3. $f(x) = e^{-x^2}$ 35

$= 2e^{-x}$ 36

$= \frac{2}{e}$ 37

$\therefore f''(0) = \frac{2^2}{2!}$ 38

$= \frac{4}{2!}$ 39

$= 2$ 40

4. $y = x \ln x$ 41

$y = x \cdot \frac{1}{x}$ 42

$= \frac{x}{x}$ 43

$y = 1$ 44

$x \ln x = 1$ 45

$\therefore x = \frac{1}{\ln x}$ 46

$= \frac{1}{x}$ 47

$x = x$ 48

$\therefore$ for $y = 1$ 49

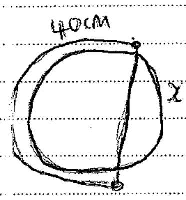
thus $x = 1.8$ 50

$\therefore$ stationery point $= (1.8 ; 1)$ 51

5. Perimeter of circle $= 40 + x$ 52

Perimeter of sector $= 40$cm 53

Perimeter of other sector $= x$ 54



Area of circle $= \pi r^2$ 55

$= (3.14)(40 \pm x)^2$ 56

$= (3.14)(160 \pm 80x + x^2)$ 57

$= 502.4 \pm 251.2x + 3.14x^2$ 58

$= 160 \pm 80x + x^2$ $[\div 3.14]$ 59

$= x^2 \pm 80x + 160$ 60

$= (x \pm 40)$ 61

$x = +40$ cm^2 62

$\therefore$ Area of circle $= 40 + x$ 63

$= 40 + (40)$ 64

$= 80$cm^2 65

No. 8

Line

b) a. $\int \left(\frac{3}{x} - 1\right)^2 dx$ 66

$= \frac{9}{x^2} + 1 - \frac{6}{x}$ 67

$= \frac{9}{x^2} + 1 - \frac{6}{x}$ 68

$= 9x^{-2} - 6x^{-1} + 1$ 69

$\frac{d}{dx} = -18x^{-3} + 6x^{-2}$ 70

b. $\int \frac{\sin^2 x}{\cos x} dx$ 71

$= \frac{2\sin x}{\cos x} dx$ 72

$= 2\tan x$ 73

c. $\int_{-1}^{0} (x^2 + 2x + 1) dx$ 74

$(x+1)(x+1)$ 75

$(x+1)^2$ 76

$x = -1$ 77

$x = j^2$ 78

1a) $y = \ln x^{\pi} - x^{\pi} + 2^{x \log_2 3}$ 79

$= \ln x^{\pi} - x^{\pi} + 2x \log_2 3$ 80

$= \frac{1}{x^{\pi}} - x^{\pi} + 2x \log_2 3$ 81

$= \frac{1 - x^{\pi}}{x^{\pi}} - x^{\pi} + 2\log_2 3$ 82

$= 2\log_2 3$ 83

$3y = 2^2$ 84

$3y = 4$ 85

$y = \frac{4}{3}$ 86

Line | No. 15 | Line

Question 1

1

2 a) $\cancel{y = \ln x^{\pi} - x^{\pi} + 2^{x \ln_2 3}}$

3 $\cancel{\frac{dy}{dx} = \pi \ln x - x^{\pi} + 2^{x \ln_2 3}}$

4

5 $\cancel{\frac{dy}{dx} = \frac{\pi}{x} - \pi x^{\pi - 1} + 2^{x \ln_2 3} \ln 2}$

6

7

a) $y = \ln x^{\pi} - x^{\pi} + 2^{x \log_2 3}$

$\frac{dy}{dx} = \pi . \ln x^{\pi-1} . \frac{1}{x} - \pi x^{\pi-1} + 2^{x \log_2 3} \ln 2$

$= \pi \ln x^{\pi-1} \frac{1}{x} - \pi x^{\pi-1} + 2^{x \frac{\log 3}{\log 2}} \ln 2$

$= \pi \ln x . \frac{1}{x} - \pi x^{\pi-1} + 2^{x \frac{\log 3}{\log 2}} . \ln 2$

$= \frac{\pi}{x} \ln x - \pi x^{\pi-1} + 2^{x \frac{\log 3}{\log 2}} . \ln 2$

8 b) $y = \frac{1+x^2}{\sqrt{1-x^2}}$

9 $= \frac{\sqrt{1-x^2} . 2x - (1+x^2) \frac{-2x}{2\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2}$

10 $= \frac{\sqrt{1-x^2} \left[2x - (1+x^2)\left(-\frac{x}{(\sqrt{1-x^2})^2}\right)\right]}{(\sqrt{1-x^2})^2}$

11 $= \frac{2x - (1+x^2)\left(-\frac{x}{1-x^2}\right)}{\sqrt{1-x^2}}$

12 $= \frac{2x - \left(\frac{-x(1+x^2)}{(1-x^2)}\right)}{\sqrt{1-x^2}} =$

13 $= \frac{2x - \left(\frac{-x+x^3}{1-x^2}\right)}{\sqrt{1-x^2}} = \frac{2x - \left(\frac{-x - x^3}{(1-x)(1+x)}\right)}{\sqrt{1-x^2}}$

14 $= \frac{2x}{\sqrt{1-x^2}}$

3

No. 15 Line

c) $y = (\sin 2x)^x$ 15

$= (2\sin x\cos x)^x$ 16

$\frac{dy}{dx} = (2\sin x\cos x)^x \ln 2\sin x\cos x \cdot \cos x(-\sin x)$ 17

$= (2\sin x\cos x)^x \ln(-2\sin^2 x\cos^2 x)$ 18

reconstructed

$dy/dx = \sin 2x^x \ln \sin 2x \cdot \cos 2x$ 18a

$= \frac{1}{2}(1+\cos 2x)\frac{1}{2}(1-\cos 2x)^x \ln \sin 2x$ 18b

$= (\frac{1}{2} + \frac{1}{2}\cos 2x)(\frac{1}{2} - \frac{1}{2}\cos 2x)^x \ln \sin 2x$ 18c

$= \frac{1}{4} - \frac{1}{4}\cancel{\cos 2x} + \frac{1}{4}\cancel{\cos 2x} - \frac{1}{4}(\cos 2x)^2 \ln \sin 2x$ 18d

$= \frac{1}{4} - \frac{1}{4}(\cos 2x)^{2x} \ln \sin 2x$ 18e

Question 2 19

$x^2 - 2xy + 3y^2 = 4$ 20

$\frac{d}{dy}(x^2 - 2xy + 3y^2) \cdot \frac{dy}{dx} + \frac{d}{dx}(x^2 - 2xy + 3y^2) = 0$ 21

$(-2x + 6y)\frac{dy}{dx} + (2x - 2y) = 0$ 22

$(6y - 2x)\frac{dy}{dx} = -2x + 2y$ 23

$\frac{dy}{dx} = \frac{2y - 2x}{6y - 2x} = \frac{y - x}{3y - x}$ 24

$\frac{0-(-2)}{3(0)-(-2)} = \frac{2}{2} = 1$ 25

Question 3 26

$f(x) = e^{-x^2}$ 27

$f'(x) = -2xe^{-x^2}$ (Product rule) 28

$f''(x) = (-2x)(-2xe^{-x^2})$ 29

$= 4x^2 e^{-x^2}$ 30

$f''(0) = 4(0)^2 e^{-(0)^2}$ 31

$4(0)e^0 = 0 \times 1 = 0$ 32

No. 15

Line

<u>Question 4</u> 33

$y = x \ln x$ 34

$\frac{dy}{dx} = x.\frac{1}{x} + 1 \ln x$ 35

$= \frac{x}{x} + \ln x$ 36

$\frac{dy}{dx} = 1 + \ln x$ ✓ 37

$\ln e = 1$ 38

$\therefore -1 = \ln x$ ✓ 39

$\therefore x = -e$ 40

2

<u>Question 5</u> 41

40cm 42

[illegible] 43

No. 15 line

Question 6. 44

a) $\int \left(\frac{3}{x}-1\right)^2 dx$ 45

$= \int \left[\frac{9}{x^2} - \frac{6}{x} + 1\right] dx$ 46

$= 9\int \frac{1}{x^2} - 6\int \frac{1}{x} + 1\int dx$ 47

$= 9\left(-\frac{1}{x}\right) - 6\ln x + x + C$ 48

$= -\frac{9}{x} - 6\ln x + x + C$ 49

LC

b) $\int \frac{\sin^2 x}{\cos x} dx$ 50

$= \int \frac{1-\cos^2 x}{\cos x} dx$ 51

$= \int \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} dx$ 52

$= \int \sec x - \cos x \, dx$ 53

B

$= \int \sec x \, dx - \int \cos x \, dx$ 54

$= \ln(\sec x + \tan x) - \sin x + C$ 55

c) $\int_{-1}^{0} (x^2 + 2x + 1) dx$ 56

$= \frac{x^{2+1}}{3} + 2\frac{x^2}{2} + x + C \Big|_{-1}^{0}$ 57

LC

$= \frac{1}{3}x^3 + x^2 + x \Big|_{-1}^{0}$ 58

$= \left(\frac{1}{3}(0)^3 - (0)^2 - 0\right) - \left[\frac{1}{3}(-1)^3 + (-1)^2 + (-1)\right)$ 59

$= 0 - \left(-\frac{1}{3} + \cancel{1} \cancel{-1}\right)$ 60

$= 0 - \left(-\frac{1}{3}\right)$ 61

$= \frac{1}{3}$ 62

lines NO. 15 lines

$$y = \frac{1+x^2}{\sqrt{1-x^2}}$$ 63

$$\frac{dy}{dx} = \frac{(2x)\sqrt{1-x^2} - (1+x^2)\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{\left(\sqrt{1-x^2}\right)^2}$$ 64

$$= 2x\sqrt{1-x^2} - \left(\frac{-2x(1+x^2)}{2\sqrt{1-x^2}}\right.$$ 65

$$= \sqrt{1-x^2}\left(2x - \frac{x+x^3}{\left(\sqrt{1-x^2}\right)^2}\right.$$ 66

$$= \frac{2x - \frac{x(1+x^2)}{1-x^2}}{\sqrt{1-x^2}}$$ 67

$$= \frac{\frac{2x-2x^3-x-x^3}{1-x^2}}{\sqrt{1-x^2}}$$ 68

69 $$= \frac{x-3x^3}{1-x^2\sqrt{1-x^2}} = \frac{x(1-3x^2)}{1-x^2\sqrt{1-x^2}}$$ 70

Reconstructed by the author.

No. 16 Line

Vraag 1

1a) $y = \ln x^{\pi} - x^{\pi} + 2^{x \log_2 3}$ 1

$y = \ln x^{\pi} - x^{\pi} + 2^{\log_2 3^x}$ 2

~~$y' = \frac{1}{x^{\pi}} - 0 + 2 \cdot 3^x \cdot 3^x \ln 3 \cdot 1$~~ $y = \ln x^{\pi} - x^{\pi} + 3^x$ 3

$y' = 0 - 0 + 3^x \ln 3$ 4

$y' = 3^x \ln 3$ 5

1

6 b) $y = \frac{1+x^2}{\sqrt{1-x^2}}$ $\begin{matrix} f \\ g \end{matrix}$

7 $y' = \frac{f'g - fg'}{g^2}$

8 $y' = \frac{2x\sqrt{1-x^2} - (1+x^2)\frac{x}{\sqrt{1-x^2}}}{(1-x^2)}$ $g = \sqrt{1-x^2}$

9 $y' = \frac{2x\sqrt{1-x^2} - \left(\frac{x+x^3}{\sqrt{1-x^2}}\right)}{1-x^2}$ $g' = \frac{1}{2\sqrt{1-x^2}} \cdot 2x$

$= \frac{x}{\sqrt{1-x^2}}$

10 $= \frac{2x(1-x^2) - (x+x^3)}{(1-x^2)(\sqrt{1-x^2})}$

11 $= \frac{2x - 2x^3 - x - x^3}{(1-x^2)^1(1-x^2)^{\frac{1}{2}}}$ 4

12 $= \frac{x - 3x^3}{(1-x^2)^{\frac{3}{2}}}$

13 c) $y = (\sin 2x)^x$ $\ln y = \ln(\sin 2x)^x$

14 ~~$y' = \sin 2x^x \ln \sin 2x \cdot \cos 2x \cdot 2x$~~ $\ln y = x \ln(\sin 2x)$

15 $\frac{1}{y} \cdot y' = 1 \cdot \ln(\sin 2x) + x \cdot \frac{1}{\sin 2x} \cdot \cos 2x \times 2$

16 $y' = y\left[\ln(\sin 2x) + \frac{2x \cdot \cos 2x}{\sin 2x}\right]$

17 Vraag 2 $y' = (\sin 2x)^x\left[\ln(\sin 2x) + \frac{2x \cdot \cos 2x}{\sin 2x}\right]$ 4

18 $x^2 - 2xy + 3y^2 = 4$

19 $x^2 + 3y^2 = 4 + 2xy$

20 $f'(x) = 2x + 6y \cdot y' = 2 \cdot y + 2x \cdot y'$

21 $6y \cdot y' - 2x \cdot y' = 2y - 2x$

22 $y'(6y - 2x) = 2y - 2x$

23 $y' = \frac{2y - 2x}{6y - 2x}$

24 $y' = \frac{2(y - x)}{2(3y - x)}$

25 $y' = \frac{y - x}{3y - x}$

26 $(-2; 0)$ $y' = \frac{0+2}{0+2}$

27 $m = 1$

No. 16 Irie

$y = mx + c$ 28

$y = x + c$ 29

$(-2;0)\quad 0 = -2 + c$ 30

$c = 2$ 31

$y = x + 2$ 32

1

Vraag 3 33

$f(x) = e^{-x^2}$ 34

$f'(x) = e^{-x^2} \cdot -2x$ 35

$f''(x) = e^{-x^2} \cdot -2$ 36

$f''(0) = e^{0} \cdot -2$ 37

$= 1 \cdot -2$ 38

$= -2$ 39

1

Vraag 4 40

$y = x \ln x$ 41

~~$y = (\ln x)^2$~~ 42

~~$y' = (\ln x)^2 \ln \ln x \cdot \frac{1}{x}$~~ 43

$y' = 1 \cdot \ln x + x \cdot \frac{1}{x}$ 44

$y' = \ln x + 1$ 45

~~x afsnit~~ waar $y=0$ $\quad y'' = \frac{1}{x}$ 46

~~$0 = x \ln x$~~ $\quad y'' = 0$: 47

$\ln x + 1 = 0$ 48

$\ln x = -1$ 49

$x = 0{,}37$ 50

51

~~Stel x = 0 in~~ 52

Stel $x = 0{,}37$ in y'' 53

54

$y'' = \frac{1}{x}$ 55

$= \frac{1}{0{,}37}$ 56

$= 2{,}70$ 57

$(0{,}37 ; 2{,}70)$ 58

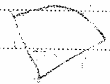
4

No. 16 Line

<u>Vraag 5</u> 59

$A_{sec} = \frac{1}{2}\theta r^2$ 60

$O_{sec} = 2r + s = 40$ $\theta r = s$ 61



$A_{sec} = \frac{1}{2} r s$ 62

$\cancel{O_{sec}} = 2r + s = 40$ 63

~~40 - s~~ 64

$s = 40 - 2r$ 65

$A_{sec} = \frac{1}{2} r s$ 66

$= \frac{1}{2} r (40 - 2r)$ 67

$A'_{sec} = \frac{1}{2} . (40 - 2r) + \frac{1}{2} r . -2$ 68

$= 20 - r$ $-r$ 69

$A'_{sec} = 20 - 2r$ 70

7

$A'_{sec} = 0$ 71

$20 - 2r = 0$ 72

$2r = 20$ 73

$r = 10$ 74

$A_{sec} = \frac{1}{2}(10)(40 - 2(10))$ 75

$= 100 \ cm^2$ 76

<u>Vraag 6</u> 77

a) $\int \left(\frac{3}{x} - 1\right)^2 dx$ 78

$= \int \left(\frac{3 - x}{x}\right)^2 dx$ 79

$= \int \frac{9 - 6x + x^2}{x^2} dx$ 80

4

$= \int \frac{9}{x^2} - \frac{6x}{x^2} + 1 \ dx$ 81

$= \int \frac{9}{x^2} - \frac{6}{x} + 1 \ dx$ 82

= [illegible] 83

$= -9x^{-1} - 6\ln x + x + C$ 84

No. 16 Line

b) $\int \frac{\sin^2 x}{\cos x}\, dx$ 85

$= \int \left(\frac{\sin x}{\cos x} . \sin x\right) dx$ 86

$= \int (\tan x . \sin x)\, dx$ 87

$= \ln \sec x . \sin x + \tan x(-\cos x) + c$ 88

c) $\int_{-1}^{0} (x^2 + 2x + 1)\, dx$ 89

$= \frac{x^3}{3} + \frac{2x^2}{2} + x$ 90

$= \frac{1}{3}x^3 + x^2 + x$ 91

$= \left[\frac{1}{3}(0)^3 + (0)^2 + 0\right] - \left[\frac{1}{3}(-1) + (-1)^2 + (-1)\right]$ 92

$= 0 - \left(-\frac{1}{3} + 1 - 1\right)$ 93

$= \frac{1}{3}$ 94

Line   No. 19   line

Vraag 1: 1

1.) a) $y = \ln x^{\pi} - x^{\pi} + 2^{x \ln_2 3}$ 2

$\therefore y' = \frac{1}{x^{\pi}} - \pi x^{\pi - 1} + x \log_2 3 . 2$ 3

$= \frac{1}{x^{\pi}} - \pi x^{\pi - 1} + 2x \log_2 3$

$= \frac{1}{x^{\pi}} - \pi x^{\pi - 1} + 6x \log_2$ 4

b) $y = \frac{1 + x^2}{\sqrt{1 - x^2}}$ 5

~~$(\frac{dy}{dx} = 2x \ldots)$~~ 6

$\frac{dy}{dx} = \frac{2x}{\sqrt{1-x^2}} . \frac{1}{2\sqrt{1-x^2}} . 2x$ 7

c) $y = (\sin 2x)^x$ 8

~~$(\frac{dy}{} = \frac{x(\sin 2x)^{x+1}}{(\sin 2x + 1)} . 2 \sin x \cos x)$~~ 9

$= x(2 \sin x . \cos x)^{x-1}$ 10

$\therefore \frac{dy}{dx} = x(2 . \cos x . - \sin x)^{x-1}$ 11

Vraag 2: 12

$(\overset{x}{-2}; \overset{y}{0})$

2) $x^2 - 2xy + 3y^2 - 4 = 0$ 13

$\therefore x^2 - 4 = 2xy - 3y^2$ 14

$\therefore \frac{dy}{dx}\ 2x/2 = \cancel{2}0 - 6yy'$ 15

$\therefore y' = \frac{2(-2)}{2 - 6(0)}$ 16

$y' = -\frac{4}{2} = -2$ 17

$\therefore y = -2x + c$ 18

$\therefore 0 = -2(-2) + c$ 19

$\therefore c = -4$ 20

$\therefore y = -2x - 4$ 21

with $y = 0$ 22

23 $x^2 - 2xy' + 3y^2 = 4$

24 $2x - 2\cancel{y'} + \cancel{6}yy' = 0$

25 ~~$\therefore y' =$~~

26 ~~$\therefore y' = \frac{-2x + 2}{6y}$~~

27 ~~$\therefore \frac{dy}{dx} = \frac{-2(-2) + 2}{6(0)}$~~

28 ~~$2x - 2 + 6yy' = 0$~~

29 ~~$\therefore 2x - 2 = -6yy'$~~

30 ~~$\therefore y' = \frac{2x - 2}{-6y}$~~

31 ~~$= \frac{2(-2) - 2}{-6(0)}$~~

32 ~~$= -\frac{6}{0}$~~ error

3

No. 19 line

Vraag 3:

3.) $f(x) = e^{-x^2}$ 33

$\therefore f'(x) = e^{-x^2} \cdot -2x$ 34

$\therefore f''(x) = e^{-x^2} \cdot -2$ 35

Vraag 4:

4.) $y = x \ln x$ 36

$y' = \frac{1}{2}x^2 + x \ln x - x$ 37

~~$y = x + \frac{1}{2}x^2 \quad \frac{1}{x}$~~ 38

No. 19 line:

<u>Vraag 5:</u>

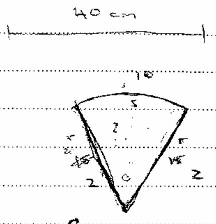


fig 1

5.) Ontrek = $\frac{1}{2}$ r . s 39

∴ 40 = $\frac{1}{2}$ r . s — ① 40

∴ s = $\frac{20}{r}$ 41

subt in ① 42

∴ 40 = $\frac{1}{2}$ r . ($\frac{20}{r}$) 43

∴ $\frac{80}{20}$ = r^2 44

∴ r = 2 45

46 Subt in ①

47

48 ∴ 40 = $\frac{1}{2}$ (2) s

∴

	$\frac{1}{2}$		
y'	0	0	0
y''	Pos		
∴ y	Min →		

fig 2

Omtrek = 2 (r) + s 49

∴ 40 = 2r + s 50

∴ s = 20 − r 51

Subt 52

∴ 40 = 2r + 20 − r 53

∴ 20 = 2r − r 54

20 = r (2 − 1) 55

r = 20 56

Subt 57

∴ s = 1 58

A = θ $\frac{1}{2}$ r s 59

∴ θ = $\frac{40}{A}$ 60

Subt ∴ A^2 = 40 ($\frac{1}{2}$) r . s 61

f' ∴ 2A = 0 − 0 / A = $\frac{1}{2}$ → subt = Pos 62

f'' = 2 63

∴ y = Min 64

No. 19 Line

Vraag 6.1

$u^2 \quad + c$

6. a) $\int \left(\frac{3}{x} - 1 \right)^2 dx$ 65

$= \cancel{\tfrac{1}{2}\left(\tfrac{3}{x} - 1\right) + \tfrac{1}{2} + c} = u^3 + c$ 66

$dx = \frac{1}{3}\left(\frac{3}{x} - 1\right)^3 + c$ 67

b) $\int \frac{\sin^2 x}{\cos x} dx$ 68

$= \int \frac{\sin x}{1} \cdot \frac{\sin x}{\cos x}$ 69

$dx = -\cos x \cdot \ln \sec x + c$ 70

c) $\int_{-1}^{0} (x^2 + 2x + 1)\, dx \qquad dx = g(a) \quad g(b)$ 71

$= (0^2 + 2(0) + 1) - ((-1)^2 + 2(-1) + 1)$ 72

$= 1 - 1 + 2 - 1$ 73

$= 1$ 74

$1 \text{ units}^2 = \text{Area}$ 75

No. 34 line

1 a) $y = \ln x^{\pi} - x^{\pi} + 2^{x \log_2 3}$ 1

$\frac{dy}{dx} = \frac{\pi x^{\pi-1}}{x^{\pi}} - \pi x^{\pi-1} + \cancel{2^{x\log_2 3}} \ln 2$ 2

$= \frac{\pi x^{\pi-1}}{x^{\pi}} - \pi x^{\pi-1} + 3^{x} \ln 3$ 3

4

b) $y = \frac{1+x^2}{\sqrt{1-x^2}}$ 4

$\frac{dy}{dx} = \frac{2x\left(\sqrt{1-x^2}\right) - \frac{-2x\,(1+x^2)}{2\sqrt{1-x^2}}}{1-x^2}$ 5

$= \frac{2x\sqrt{1-x^2} + \frac{x\,(1+x^2)}{\sqrt{1-x^2}}}{1-x^2}$ 6

$= \frac{\frac{2x(1-x^2)+x}{(1-x^2)}}{1-x^2}$ 7

3

$= \frac{2x - 2x^3 + x}{\sqrt{1-x^2}} \times \frac{1}{1-x^2}$ 8

$= \frac{3x - 2x^3}{\sqrt{1-x^2}} \times \frac{1}{1-x^2}$ 9

$= \frac{3x - 2x^3}{(1-x^2)^3}$ 10

c) $y = (\sin 2x)^{x}$ 11

$\frac{dy}{dx} = 2 \cdot \cos 2x + (\sin 2x)^{x} \ln \sin 2x$ 12

No. 34 line

2) $x^2 - 2xy + 3y^2 = 4$ 13

$\cancel{2x - 2 + 1y' + 6y.y' = 0}$ 14

$\cancel{y' + 6yy' = -2x + 2}$ 15

$2x - 2y - 2xy' + 6y.y' = 0$ 16

$6yy' - 2xy' = 2y - 2x$ 17

$(-2;0)$ $-2(-2)y' = -2(-2)$ 18

$4y' = 4$ 19

$y' = 1$ 20

7

$y = mx + c$ 21

$y = x + c$ 22

$0 = -2 + c$ 23

$c = 2$ 24

$\cancel{y = -2x}$ 25

$y = x + 2$ 26

No. 34

line

3) $f(x) = e^{-x^2}$ $\quad$ $f(0) = 1$ 27

$f(x)' = -2xe^{-x^2}$ $\quad$ $f(0)' = 0$ 28

$f(x)'' = -2e^{-x^2} - 2x(-2xe^{-x^2})$ $\quad$ $f(0)'' = 1$ $\quad$ 3 29

$f(0)'' = 1$ 30

4) $y = x \ln x$ 31

$\cancel{= \ln}$ 32

$\frac{dy}{dx} = x\left(\frac{1}{x}\right) = 0$ 33

$= x$ (x = any real number) 34

$y = x \ln x$ 35

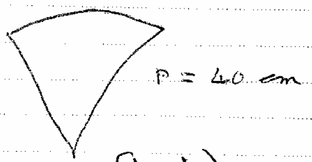
$= 0$ 36

$(x;0)$ Real ; Equal ; Rational 37

No. 34 line

38

5)



(fig 1)

No.34 line

6a) $\int \left(\frac{3}{x} - 1\right)^2 dx = \int \left(\frac{9}{x^2} - \frac{6}{x} + 1\right) dx$ 39

$= -\frac{9}{x} - 6\ln x + x + c$ 4 40

b) $\int \frac{\sin^2 x}{\cos x} dx = \int \frac{1 - \cos^2 x}{\cos x} dx$ 41

$= \int \frac{1}{\cos x} - \frac{\cos^{\cancel{2}} x}{\cancel{\cos x}} dx$ 42

$= \int \sec x - \cos x \, dx$ 43

$= \ln(\sec x + \tan x) - \sin x$ 3 44

c) $\int_{-1}^{0} (x^2 + 2x + 1) dx = \left(\frac{1}{3}x^3 + x^2 + x\right)_{-1}^{0}$ 45

$= \left(\frac{1}{3}(0)^3 + 0^2\right) - \left(\frac{1}{3}(-1)^3 \cancel{+ 1} \cancel{- 1}\right)$ 46

$= 0 - \left(-\frac{1}{3}\right)$ 4 47

$= \frac{1}{3}$ 48

No. 40(a) Line

1.(a) $y = \ln x^{\pi} - x^{\pi} + 2^{x\log_2 3}$ $\ln 2\,[x \cdot \log_2 3]$ 1

$\therefore \frac{dy}{dx} = \left(\frac{1}{x^{\pi}} \cdot \pi x^{\pi-1}\right) - \left(\pi x^{\pi-1}\right) + \left(2^{x\log_2 3} \ln 2\right.$ 2

(b) $y = \frac{1+x^2}{\sqrt{1-x^2}}$ 3

$\therefore \frac{dy}{dx} = \frac{(2x)(\sqrt{1-x^2}) - (1+x^2)\left(\frac{1}{2}[1-x^2]^{-\frac{1}{2}} \cdot -2x\right)}{1-x^2}$ 4 4

(c) $y = (\sin 2x)^x$ 5

$\therefore \frac{dy}{dx} = \left[(\sin 2x)^x \ln(\sin 2x)\right] \cdot [\cos 2x] \cdot [2]$ 6

2. $x^2 - 2xy + 3y^2 = 4$; point $(-2;0)$ 7

subst $x=-2$: $(-2)^2 - 2(-2)y + 3y^2 = 4$ 8

$4 + 4y + 3y^2 = 4$ 9

$\therefore y(3y+4) = 0$ 10

$y = 0$ or $y = \frac{-4}{3}$ 11

subst $y = \frac{-4}{3}$: $x^2 - 2x\left(\frac{-4}{3}\right) + 3\left(\frac{-4}{3}\right)^2 = 4$ 12

$\therefore x^2 + \frac{8}{3}x + \frac{16}{3} - \frac{4}{1} = 0$ 13

$\therefore x^2 + \frac{8}{3}x - \frac{4}{3} = 0$ 14

$\therefore x = \frac{\left(-\frac{8}{3} \pm \frac{4}{3}\right)}{2}$ 15

$\therefore x = -2$ or $x = -\frac{2}{3}$ 16

$\frac{\frac{-4}{3} - 0}{-\frac{2}{3} - (-2)} = m$ 17

$\therefore m = -1$ $\therefore$ gradient is negative 18

$\therefore y = -\frac{2}{3} < -\frac{4}{3}$ 19

No. 40(b) Line

3. $f(x) = e^{-x^2}$ 20

$\therefore f'(x) = e^{-x^2} \cdot -2x$ 21

$f''(x) = (e^{-x^2} \cdot -2x)(-2x) + (e^{-x^2})(-2)$ 22

$\therefore f''(0) = (e^{-(0)^2} \cdot -2(0))(-2(0)) + (e^{-(0)^2})(-2)$ 23

$= (1 \times 0)(0) + (1)(-2)$ 24

$= -2 \rightarrow$ 25

5

4. $y = x \ln x$ 26

$\therefore \frac{dy}{dx} = (\ln x) + (x)(\frac{1}{x})$ 27

$\therefore \frac{dy}{dx} = \ln x + 1$ 28

let $\frac{dy}{dx} = 0$ 29

$\therefore \ln x + 1 = 0$ 30

$\therefore \ln x = -1$ 31

$\therefore x = 0,367879441\ldots$ 32

to get y-int: 33

$\therefore y = x \ln x$ 34

$y = (0,36\ldots) \ln (0,367\ldots)$ 35

$= -0,367879441$ 36

$\therefore$ stationary point co-ordinates $(0,368 ; -0,368) \rightarrow$ 37

$\therefore y'' = \frac{1}{x}$ 38

$\therefore \frac{1}{0,368} > 0$ 39

$\therefore$ nature of stationary point ; 40

that it reaches a maximum $\rightarrow$ 41

6

No. 40(c) Line

5) $\therefore s + 2r = 40cm$ 42

$\therefore s = 40 - 2r$ 43

$= 2(20 - r)$ 44

$\therefore A_{ST} = \frac{1}{2} r^2 s$ 45

$= \frac{1}{2} r^2 (40 - 2r)$ 46

$= r^2 (20 - r)$ 47

2

$= 20r^2 - r^3$ 48

$\therefore \frac{dA}{dr} = 40r - 3r^2$ 49

$\therefore \frac{dA}{dr}$ let it be $= 0$ 50

$\therefore 3r^2 - 40r = 0$ 51

$\therefore r(3r - 40) = 0$ 52

$\therefore r = 0$ or $3r - 40 = 0$ 53

$\therefore r = \frac{40}{3}$ 54

maximum area 55

$\therefore$ we use $r = \frac{40}{3}$ 56

$\therefore A_{ST} = 20\left(\frac{40}{3}\right)^2 - \left(\frac{40}{3}\right)^3$ 57

$= 20\left(\frac{1600}{9}\right) - \left(\frac{64000}{9}\right)$ 58

$= -3\ 555,56 cm^2$ 59

but area is never negative. 60

$\therefore$ maximum Area of sector = 61

$3\ 555\ 56 cm^2$ → 62

No. 40 (d)

Line

6) (a) $\int \left(\frac{3}{x}-1\right)^2 dx = \left[\frac{\left(\frac{3}{x}-1\right)^3}{\left(\frac{3}{x}\right)} \cdot 3 \ln x\right] + c$ → 63

(b) $\int \frac{\sin^2 x}{\cos x} dx$ 64

$\therefore \int \frac{\sin x}{1} \times \tan x \; dx = \left[(-\cos x)(\tan x) + (\sin x)(\ln \sec x)\right] + c$ 65

(c) $\int_{-1}^{0} (x^2 + 2x + 1) dx$ 66

$\therefore \int_{-1}^{0} (x+1)^2 dx = \left[\frac{(x+1)^3}{x+2}\right]_{-1}^{0} + c$ 67

$= \frac{1}{1} + c$ 68

$= 1 + c$ → 69

Participant no 100

1. a) $y = \ln x^{\pi} - x^{\pi} + 2^{x \ln_2 3}$ 1

2 $y = \pi \ln x - x^{\pi} + 2^{x \ln_2 3}$

2

$\ln y = \ln x^{\pi} - \ln x^{\pi} + \ln 2^{x \log_2 3}$

$\frac{1}{y}\frac{dy}{dx} = \pi \ln x - \pi \ln x + x \log_2 3 \ln 2$ 5,6

3 $\frac{dy}{dx} = \pi . \frac{1}{x} - (\pi - 1) . x^{\pi - 1} + 2^{x \ln_2 3} \ln 2$

$\frac{dy}{dx} = y(x \log_2 3 \ln 2)$ 7

4 $= \frac{\pi}{x} - (\pi - 1)(x^{\pi - 1}) + 2^{x \ln_2 3} \ln 2$

$\frac{dy}{dx} = y\left(x . \frac{\log 3}{\log 2} \ln 2\right)$ 8 1

$\frac{dy}{dx} = y(x . 1)$ 9

$\frac{dy}{dx} = xy$ 10

b) $y = \frac{1 + x^2}{\sqrt{1 - x^2}}$ 11

$y = \frac{(1 + x^2)}{(1 - x^2)^{1/2}}$ 12

13

$\ln y = \ln (1 + x^2) - \ln (1 - x^2)^{1/2}$

$\ln y = \ln (1 + x^2) - \frac{1}{2} \ln (1 - x^2)$ 14

$\frac{1}{y}\frac{dy}{dx} = \frac{\cancel{1 \times 2x}}{1 + x^2} - \frac{1}{2} . \frac{1 \times \cancel{2x}}{1 - x^2}$ 15

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1 + x^2} - \frac{1}{2(1 - x^2)}$ 16 2

3

$\frac{1}{y}\frac{dy}{dx} = \left[\frac{2(1 - x^2) - (1 + x^2)}{2(1 + x^2)(1 - x^2)}\right]$ 17

$\frac{1}{y}\frac{dy}{dx} = \left[\frac{2 - 2x^2 - 1 - x^2}{2(1 + x^2)(1 - x^2)}\right]$ 18

$\frac{1}{y}\frac{dy}{dx} = \left[\frac{1 - 3x^2}{2(1 - \cancel{x^2} + \cancel{x^2} - x^4)}\right]$ 19

$\frac{dy}{dx} = y\left(\frac{1 - 3x^2}{2 - 2x^4}\right)$ 20

no. 1150

c) $y = (\sin 2x)^x$ 21

$\ln y = \ln (\sin 2x)^x$ 22

$\ln y = x \ln \sin 2x$ 23

24

$u = x$ $\qquad v = \ln \sin 2x$ 25

$u' = 1$ $\qquad v' = \frac{1}{\sin 2x} \cdot \cos 2x \cdot 2$

$= \frac{\cos 2x}{\sin 2x} \cdot 2$ 26

$= 2 \cot 2x$ 27

5

$uv' + vu'$ 28

$\ln y = x(2\cot 2x) + \ln \sin 2x (1)$ 29

$\frac{dy}{dx} \frac{1}{y} = x . 2\cot 2x + \ln \sin 2x$ 30

$\therefore \frac{dy}{dx} = y(x . 2\cot 2x + \ln \sin 2x)$ 31

$\longrightarrow$

32

$\cancel{\ln y = x . \ln \sin 2x}$ 33

$\cancel{\frac{1}{y} \frac{dy}{dx} = \ln \sin 2x . \frac{1}{\sin 2x} . 2}$

$\cancel{\frac{1}{y} \frac{dy}{dx} = \ln \sin 2x . \sec 2x . 2}$ 34

no. 100

2. $x^2 - 2xy + 3y^2 = 4$ $\quad (\overset{x}{-2};\overset{y}{0})$ 35

36

37

38

43 $2x - 2x\frac{dy}{dx}y + 3\frac{dy}{dx}y^2 = 4$

44 $2x - 2x\frac{dy}{dx} + 3\frac{dy}{dx}2y = 0$

45 $2x - 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$

$-2x\frac{dy}{dx} + 6y\frac{dy}{dx} = -2x$ 46

50 $\frac{dy}{dx}(-2x+6y) = -2x$

51 $\frac{dy}{dx} = \frac{-2x}{(-2x+6y)}$

52 $\frac{dy}{dx} = \frac{-2(-2)}{-2(-2)+6(0)}$

53 $\frac{dy}{dx} = \frac{-4}{-4+0}$

54 $\frac{dy}{dx} = \frac{-4}{-4}$

55 $\frac{dy}{dx} = 1$

$x^2 - 2xy + 3y^2 = 4$ 39

$x^2 - 2x(1) + 3(1)^2 = 4$ 40

$x^2 - 2x + 3 - 4 = 0$ 41

$x^2 - 2x - 1 = 0$ 42

$y = 0 \Rightarrow x^2 = 4$ 47

$x^2 - 4 = 0$ 48

$(x+2)(x-2) = 0$ 49

$\therefore x = 2$ 56

$x = 2 \Rightarrow x^2 - 2xy + 3y^2 = 4$ 57

$(2)^2 - 2(2)y + 3y^2 = 4$ 58

$4 - 4y + 3y^2 = 4$ 59

$-4y + 3y^2 = 4 - 4$ 60

$-4y + 3y^2 = 0$ 61

$y(-4+3y) = 0$ 62

$y = 0$ & $3y = 4$ 63

$y = 4/3$ 64

no. 110

3. $f''(0)$ if $f(x) = e^{-x^2}$ 65

70 $f'(x) = e^{-x^2} . -2x$ ✓ 2 66

71 $= -2x e^{-x^2}$

72 $f''(x) = -2 . e^{-x^2} . -2x$ (prac? $f'(x) = e^{-x^2} -2x$ 67

73 $= 4x e^{-x^2}$ x ? $f''(x) = e^{-x^2} -2x -2$ 68

73 $f''(0) = 4(0) . e^{-(0)^2}$ $f''(0) = e^{-(0)^2} - 2(0) - 2$ 69

74 $= 0 . e^0$ $= 2$

75 $= 0.1$

76 $= 0$

4. $y = x \ln x$ 77

$= x . \frac{1}{x}$ 78

$= \frac{x}{x}$ 79

$y = 1$ $\left(\frac{dy}{dx} = y\right)$ 80

$(0;1)$ → stationary points X 81

5. $l = 40cm$ 82

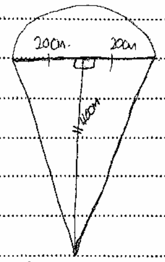


fig. 1

$\cancel{A = \frac{1}{2} l \times b}$ $A = l \times b$ 83

$\cancel{= \frac{1}{2}(40)(20)}$ $= 40 \times 20$ 84

$= 80cm^2$ 85

6. a) $\int \left(\frac{3}{x} - 1\right)^2 . dx$ 87

$= \int \left(\frac{3}{x} - 1\right)\left(\frac{3}{x} - 1\right) . dx$ 88

$= \int \left(\frac{9}{x^2} - \frac{3}{x} - \frac{3}{x} + 1\right) . dx$ 89

$= \int \left(\frac{9}{x^2} - \frac{6}{x} + 1\right) . dx$ 90

$= \int \frac{9}{x^2} . dx - \int \frac{6}{x} . dx + \int 1 . dx$ B 91

$= 9\int \frac{1}{x^2} dx - 6\int \frac{1}{x} . dx + \int 1 . dx$ 92

$= 9 . -\frac{1}{x} - 6 . -\frac{1}{x^2} + x + C$ 93

$= -\frac{9}{x} + \frac{6}{x^2} + x + C$ 94

$= \frac{6}{x^2} - \frac{9}{x} + x + C$ 95

α.

b) $\int \frac{\sin^2 x}{\cos x} . dx$ 96

$= \int \frac{\sin x . \sin x}{\cos x} . dx$ 97

$= \int \sin x . \tan x . dx$ 98

$= -\cos x . \ln \sec x + C$ α. 99

NO. 100

c) $\int_{-1}^{0} (x^2+2x+1).dx$ 100

$= 2x+2 \Big|_{-1}^{0}$ 101

$= [2(0)+2]^{0} - [2(-1)+2]^{1}$ 102

$= [0+2] - [-2+2]$ 103

$= [0] - [0]$ 104

$= 0$ A. 105

NO. 96

1 1.(a) $y = \ln x^{\pi} - x^{\pi} + 2^{x\log_2 3}$ $\qquad \frac{x\ln 3}{\ln 2}$ 15

2 $\cancel{y' = \pi \ln x^{\pi} - x^{\pi} + 2^{x\log_2 3}}$ $\qquad = \frac{(x\ln 3)'\ln 2 - (x\ln 3)(\ln 2)'}{(\ln 2)^2}$ 16

3 $y = \pi \ln x - x^{\pi} + 2^{x\log_2 3}$ $\qquad = (\ln 3 + x/3)\ln 2 - \frac{(x\ln 3)}{2}$ 17

4 $\cancel{y' = \frac{\pi}{x} - \pi x^{\pi-1} + 2^{x\log_2 3}}$

T1

5 $\ln y = e^{x^{\pi}} - \ln x^{\pi} + \ln 2^{x\log_2 3}$

6 $\frac{1}{y} = e^{x^{\pi}}\cdot \pi x^{\pi-1} - \frac{\pi}{x} + \frac{x\log_2 3}{2}$

7. $\frac{1}{y} = e^{x^{\pi}}\cdot \pi x^{\pi-1} - \frac{\pi}{x} + \frac{x\ln 3}{2\ln 2}\left[(\ln 3 + x/3)\ln 2 - (x\frac{\ln 3}{2})\right]$

$\frac{1}{2\sqrt{x}}$ 18

8 (b) $y = \frac{1+x^2}{\sqrt{1-x^2}}$ $\qquad \frac{1}{2\sqrt{(1-x^2}}$ 19

9 $= \frac{1+x^2}{(1-x^2)^{1/2}}$ $\qquad (1-x^2)^{-1/2}$ 20

10 $y' = \frac{(1+x^2)'(1-x^2)^{1/2} - (1+x^2)(1-x^2)^{1/2}}{(\sqrt{1-x^2})^2}$

11 $= \frac{2x(1-x^2)^{1/2} - (1+x^2)(1-x^2)^{-1/2}\cdot 2x}{(1-x^2)}$

12 $= \frac{2x(1-x^2)^{1/2}}{(1-x^2)} - \frac{(1+x^2)(1-x^2)^{-1/2}\cdot -2x}{(1-x^2)}$

4

13 $= \frac{2x}{(1-x^2)^{-1/2}} - \frac{(1+x^2)\cdot -2x}{(1-x^2)^{-3/2}}$

14 $= \frac{2x}{(1-x^2)^{-1/2}} - \frac{\cancel{(1+x^2)}\ -2x - 2x^3}{(1-x^2)^{-3/2}}$

No. 91,

(c) $y = (\sin 2x)^x$ 21

$y' = (\sin 2x)^x \ln \sin 2x \cdot 2\cos 2x \cdot 2$ 22

$= (\sin 2x)^x \ln \sin 2x \cdot 4\cos 2x$ 23

2. $x^2 - 2xy + 3y^2 = 4$ $(-2;0)$ 24

$2x - 2y + 2xy' + 6y \cdot y' = 0$ 25

$\dfrac{y'\cancel{(2x+6y)}}{\cancel{2x+6y}} = \dfrac{-2x+2y}{?-2x+6y}$ 26

$y' = \dfrac{-2(-2)+2(0)}{?-2(-2)+6(0)}$ 27

$y' = \dfrac{4}{-4}$ 28

$m = -1.$ 29

6

$\therefore$ $y = mx + c$ 30

$0 = -1(-2) + c$ 31 5

$0 = 2 + c$ 32

$-2 = c$ 33

$\therefore$ $y = -x - 2.$ 34

3. $f(x) = e^{-x^2}$ 35

$f'(x) = e^{-x^2} \cdot -2x$ 36

$= -2x \cdot e^{-x^2}$ 37

$f''(x) = -2 \cdot e^{-x^2} - 2x e^{-x^2} \cdot -2x$ 38

$= -2e^{-x^2} + 4x^2 \cdot e^{-x^2}$ (4) 39 4

$\therefore f''(0) = -2^{-(0)^2} + 4(0)^2 \cdot \cancel{e^{-(0)^2}}$ 40

$= 1 + 0$ 41

$= 1$ (4) 42

4. $y = x \ln x$. 43

$y' = \ln x + x \cdot \frac{1}{x}$ 44

$\ln x + \frac{x}{x} = 0$ 45

$\ln x + 1 = 0$ 46

$\therefore \ln x = -1$ 47

$x = 1/e$ 48 (6)

$\therefore y = 1/e \ln 1/e$ 49 4

$= 1/e \ln e^{-1}$ 50

$= -1/e$ (1) 51

Turning Point $(1/e ; -1/e)$ 52

$y' = \ln x + 1$ 53

$y'' = \frac{1}{x} = x^{-1}$ 54

$\therefore 1/x \ln 1/x$ 55

NO. 96

6. (a) $\int \left(\frac{3}{x} - 1\right)^2 \, dx$ 57

$= \int \frac{9}{x^2} - 2.\frac{3}{x} + 1 \quad dx$ 58

$= 9\int \frac{1}{x^2} \, dx - 6\int \frac{1}{x} \, dx + \int 1 \, dx$ 59

60

$= 9\int x^{-2} \, dx - 6\int \frac{1}{x} \, dx + \int 1 \, dx$ 60

$= 9\int -x^{-1} - 6\int \ln x + \int x$ 61 3

$= \frac{-9x}{x} - 6 \ln x + x + c$ 62

(b) $\int \frac{\sin^2 x}{\cos x} \, dx$ 63

$= \cancel{\int \frac{1}{2}(\cos 2x - 1)}$ 64

$= \int \frac{\frac{1}{2}(1 - \cos 2x)}{\cos x}$ 65

$= \frac{\frac{1}{2}(1 - \cos 2x) \ln \cos x}{-\sin x} + c$ 66

$= \frac{(\frac{1}{2} - \frac{1}{2}\cos 2x) \ln \cos \cancel{x}}{-\sin x.} + c$ 67

NO. 96

(c) $\int_{-1}^{0} (x^2 + 2x + 1)\, dx.$ 68

$\left. \frac{x^3}{3} + x^2 + x \right|_{-1}^{0}$ C 69

$= [\tfrac{1}{3}(0)^3 + (0)^2 + 0] - [\tfrac{-1}{3}(-1)^3 + (-1)^2 + (-1)]$ 70

$= (0 + 0 + 0) - (-\tfrac{1}{3} + 1 - 1)$ 71

$= 0 + \tfrac{1}{3} - 1 + 1$ 72

$= \tfrac{1}{3}$ 73

Participant no. 75 — line

1. (a) $y = \ln x^{\pi} - x^{\pi} + 2^{x\ln_2 3}$ 1

$y\frac{dy}{dx} = \pi \ln x - \pi \ln x + 2^{x\ln_2 3}$ 2

$\frac{dy}{dx} = \frac{\pi}{x} - \frac{\pi}{x}$ 3

1. (a) $y = \ln x^{\pi} - x^{\pi} + 2^{x\ln_2 3}$ 4

$= \pi \ln x - x^{\pi} + 2^{x\ln_2 3}$ 5

$\frac{dy}{dx} = \frac{\pi}{x} - \pi x^{\pi}\ln x + 2^{x\ln_2 3} \cdot D(x\ln_2 3)$ 6

$= \frac{\pi}{x} - \frac{x^{\pi}}{x} + 2^{x\ln_2 3}$ 7

$= \frac{\pi - x^{\pi} + x2^{x\ln_2 3}}{x}$ 8

2

(b) $y = \frac{1+x^2}{\sqrt{1-x^2}}$ 9

$\frac{dy}{dx} = \frac{2x(1-x^2)^{\frac{1}{2}} - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot 2x(1+x^2)}{(\sqrt{1-x^2})^2}$ 10 11

$\frac{dy}{dx} = \frac{2x(1-x^2)^{\frac{1}{2}} - (1-x^2)(1+x^2) \cdot x}{(1-x^2)}$ 12

$= \frac{(2x-2x^3)^{\frac{1}{2}} - (1-x^4)x}{(1-x^2)}$ 13

$= \frac{(2x-2x^3)^{\frac{1}{2}} - (x-x^5)}{(1-x^2)}$ 14

3

(c) $y = (\sin 2x)^{x}$ 15

$\ln y = x\ln(\sin 2x)$ 16

$\frac{1}{y}\frac{dy}{dx} = \frac{1(\cos 2x) \cdot 2}{\sin 2x}$ 17

$= \frac{2\cos 2x}{\sin 2x}$ 18

$= 2\cot 2x$ 19

1

NO. 75

2. $x^2-2xy+3y^2=4$

$\cancel{f'(x)} = 2x-2+6y=4$

$2x+6y=4+2$

$2x=6-6y$

$x=\frac{6-6y}{2}$

$x=\frac{{}^{3}\cancel{6}(1-y)}{\cancel{2}}$

$x=3-3y$

$2(3-3y)\cancel{-2+6y}=4$

$\cancel{6-6y}$

$(3-3y)^2-2y(3-3y)+3y^2=4$

$9-18y+9y^2-6y+6y^2$

2) $x^2-2xy+3y^2=4$

$2x-2+6y=0$

$6y=2-2x$

$6y=2-2(-2)$

$6y=2+4$

$y=\frac{6}{6}$

$y=1$

P

$2x-2+6y=0$

$2x-2=0$

$2x=2$

$x=1$

$\therefore m=1$

equation is $y=x+1$

No. 75

③. $f''($, $f(x) = e^{-x^2}$

$f'(x) = e^{-x^2} \cdot -2x$

$= -2x \cdot e^{-x^2}$ ✓

$f''(x) = -2x \cdot e^{-x^2} \cdot -2x + (-2e^{-x^2})$

$= 4x^2 e^{-x^2} - 2e^{-x^2}$ ✓

$= e^{-x^2}(4x^2 - 2)$

$= \frac{1}{e^{x^2}}(4x^2 - 2)$ 　 5

$f''(0) = \frac{1}{e^0}(4(0)^2 - 2)$

$= -2$ ✓

④. $y = x \ln x$

$\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x$ ✓

$= 1 + \ln x$

$= \frac{1 + 1}{x}$ 　 $\ln x = -1$ ✓ 　 2

$\cancel{x} = -\frac{1}{\ln}$ 　 $x = -1$

$=$

$= -1 + \ln -1$

$y = -1$ ∴ Stationary points are $(-1, -1)$

⑤. $S = 40\text{cm}$

$A_{st} = \frac{1}{2}\theta r^2$

$S = \frac{r\theta}{2}$

$40 = \frac{r\theta}{2}$

$r\theta = 80$

$\theta = \frac{80}{r}$

$A_{st} = \frac{1}{2} \cdot \frac{80}{r} \cdot r^2$

$A_{st} = 40r$ 　 ✗

$\therefore A'(x) = 40\text{cm}^2$

∴ max Area of wire is $= 40\text{cm}^2$

NO. 75

(a) $\int \left(\frac{3}{x} - 1\right)^2 dx$ 69

$= \int \frac{9}{x^2} - \frac{6}{x} + 1 \; dx$ 70

$= 9\int x^{-2}\,dx \int -6\, x^{-1}\,dx \int +1\,dx + C$ 71

2

$= -18x^{-1} + 6x + x + C$ 72

$= -\frac{18}{x} + x + 6 + C$ 73

(b) $\int \frac{\sin^2 x}{\cos x}\, dx$ 74

$= \int \sin^2 x \, \frac{\ln \cos x}{\sin x} + C$ 75

$= \sin x \ln \cos x + C$ 76

(c) $\int_{-1}^{0} (x^2 + 2x + 1)\, dx$ 77

$\int x^2\,dx \int + 2x\,dx \int + 1\,dx \; \cancel{+C}$ 78

0

$= 2x^3 + 2x^2 + x$ 79

1

$= 2x^3 + 2x^2 + x \Big|_{-1}^{0}$ 80

$=$

$= 2(0)^3 + 2(0)^2 + 0 - [2(-1)^3 + 2(-1)^2 + (-1)]$ 81

$= 2 - 2 + 1$ 82

$= 1$ 83

8.3 APPENDIX C: MULTIPLE CHOICE TEST ITEMS

< Your location: Course Content Home > Calculus > Assessment Manage' > **Maths Comprehension**

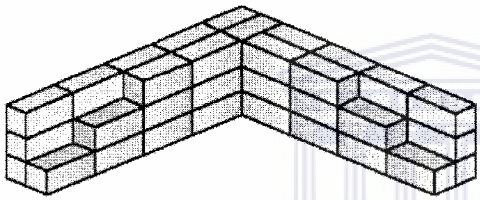
View Attempt 1 of 1

Name:	**ARRIES, MORNE ANDRE**
Title:	Maths Comprehension
Started:	03 February 2006 10:44
Submitted:	03 February 2006 11:11
Time spent:	00:26:44
Total score:	**7/15 = 46.67%** Adjust total score by: 0.0

Update Grade | Reset Attempt | Cancel

1. How many bricks were

How many bricks were used to build the wall in the adjacent sketch?



Student Response	Value	Correct Answer	Feedback
A. 44	0%		
B. 48	100%	✓	
C. 46	0%		
D. 47	0%		

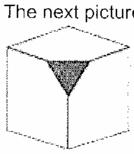
Score: 1/1

Override score: / 1

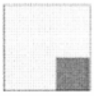
Comments for student:

2. The adjacent picture

The next picture shows a wooden cube with one corner cut off and shaded.



Which one of the following drawings shows how this cube will look when viewed directly from above?

	Student Response	Value	Correct Answer	Feedback
A.		0%		
B.		0%		
C.		100%	☑	
D.		0%		

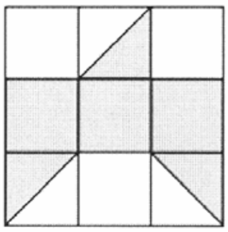
Score: 1/1

Override score: / 1

Comments for student:

3. What fraction of the

What fraction of the following figure is shaded?



	Student Response	Value	Correct Answer	Feedback
A.	$\frac{3}{4}$	0%		
B.	$\frac{2}{3}$	0%		
C.	$\frac{1}{2}$	100%	☑	
D.	$\frac{1}{3}$	0%		

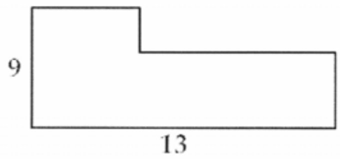
Score: 1/1

Override score: / 1

Comments for student:

4. All sides of this fi

All sides of this figure are either horizontal or vertical.



How far is it around the figure?

Student Response	Value	Correct Answer	Feedback
A. 52	0%		
B. 44	100%	✓	
C. 22	0%		
D. impossible to say	0%		

Score: 0/1

Override score: / 1

Comments for student:

5. The numbers in this

The numbers in this row are listed according to a certain pattern.

7	21	.....	189

Which number is missing?

Student Response	Value	Correct Answer	Feedback
A. 35	0%		
B. 28	0%		
C. 42	0%		
D. 63	100%	✓	

Score: 1/1

Override score: / 1

Comments for student:

6. The mass of a contai

The mass of a container with sugar was 13 kg. After one third of the sugar had been used, the mass of the container with the rest of the sugar was 9 kg. What was the mass of the empty container?

Student Response	Value	Correct Answer	Feedback
A. 1 kg	100%	✓	
B. 4 kg	0%		
C. 10 kg	0%		
D. none of these	0%		

Score: 0/1

Override score: / 1

Comments for student:

7. A piggy-bank holds e

A piggy-bank holds equal numbers of 50-cent coins, 20-cent coins and 10-cent coins. How much money is there in the piggy-bank if it holds eight 50-cent coins?

Student Response	Value	Correct Answer	Feedback
A. R4.00	0%		
B. R6.00	0%		
C. R6.40	100%	✓	
D. 400 cents	0%		

Score: 1/1

Override score: / 1

Comments for student:

8. A pile of 50 sheets

A pile of 50 sheets of paper is 0,5 cm thick.
How thick is one sheet?

Student Response	Value	Correct Answer	Feedback
A. 0,001 cm	0%		
B. 1 mm	0%		

C. 0,025 cm	0%			
D. 0,1 mm	100%	☑		

Score: 0/1

Override score: / 1

Comments for student:

9. Four children play t

Four children play tennis. Each child plays each of the others once. How many matches are played?

	Student Response	Value	Correct Answer	Feedback
	A. 4	0%		
	B. 12	0%		
	C. 5	0%		
☑	D. 6	100%	☑	

Score: 1/1

Override score: / 1

Comments for student:

10. Gina has a 4-digit c

Gina has a 4-digit combination which opens her locker padlock. She remembers that the digits are 3, 5 7 and 9, but has forgotten the correct order. What is the maximum number of guesses she would have to make to try to open her lock?

	Student Response	Value	Correct Answer	Feedback
	A. 256	0%		
☑	B. 16	0%		
	C. 64	0%		
	D. 24	100%	☑	

Score: 0/1

Override score: / 1

Comments for student:

11. A boy and a girl run

A boy and a girl run a 100 m race, which the boy wins by 5 m, so they decide to race again, this time with the boy starting 5 m behind the starting line. If they each run at the same speed they ran before (and their accelerations are ignored):

	Student Response	Value	Correct Answer	Feedback
A.	the boy and the girl will tie	0%		
B.	the boy will win by 50 cm	0%		
C.	the boy will win by 25 cm	100%	☑	
D.	the boy will lose	0%		

Score: 0/1

Override score: / 1

Comments for student:

12. Half of 10-8 is

Half of 10^{-8} is

	Student Response	Value	Correct Answer	Feedback
A.	5^{-8}	0%		
B.	5×10^{-8}	0%		
C.	10^{-4}	0%		
D.	5×10^{-9}	100%	☑	

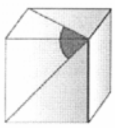
Score: 0/1

Override score: / 1

Comments for student:

13. The diagonals on two

The diagonals on two adjacent faces of a cube meet at one vertex, as shown.



What is the size of the angle between the diagonals?

	Student	Value	Correct	Feedback

	Response		Answer
	A. 120°	0%	
	B. 45°	0%	
☑	C. 90°	0%	
	D. 60°	100%	☑

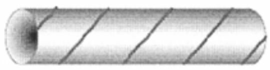
Score: 0/1

Override score: / 1

Comments for student:

14. A string is wound ar

A string is wound around a circular rod exactly four times, creating a helix from one end of the rod to the other.



What is the length of the string if the rod has a **length of 12 cm** and a **circumference of 4 cm**?

	Student Response	Value	Correct Answer	Feedback
	A. 20 cm	100%	☑	
	B. 28 cm	0%		
☑	C. 24 cm	0%		
	D. 16 cm	0%		

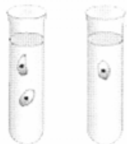
Score: 0/1

Override score: / 1

Comments for student:

15. Two amoebas are plac

Two amoebas are placed in a test tube. They reproduce by splitting themselves in two, a process that takes five minutes. After four hours they have filled the test tube. How long would it take a *single* amoeba in the same quantity of water to do the same?



Student	Value	Correct	Feedback

	Response		Answer
	A. 6 h	0%	
	B. 4 h 5 min	100%	✓
	C. 8 h	0%	
	D. 4 h	0%	

Score: 1/1

Override score: / 1

Comments for student:

Total score: 7/15 = 46.67% Adjust total score by: 0.0

Comments for student:

Audit log comments:

Update Grade | Reset Attempt | Cancel

8.4 APPENDIX D: MULTIPLE CHOICE DESCRIPTIVE STATISTICS

MULTIPLE CHOICE TEST	
Mean	7.87
Standard Error	0.25
Median	8
Mode	8
Standard Deviation	2.87
Sample Variance	8.27
Kurtosis	-0.51
Skewness	-0.17
Range	13
Minimum	1
Maximum	14
Sum	992
Count	126

8.5 APPENDIX E: INDIVIDUAL TEST ITEM ANALYSIS FOR MULITPLE CHOICE TEST

% Answering Correctly				
item no.	Question Title	Whole Group	Upper 25%	Lower 25%
1	How many bricks were	93.94	100	78.57
2	The adjacent picture	83.05	92.86	62.5
3	What fraction of the	83.82	100	58.82
4	All sides of this figure	40.98	62.5	0
5	The numbers in this	91.18	100	85.71
6	The mass of a container	48.75	85.71	12.5
7	A piggy-bank holds	62.3	92.86	18.18
8	A pile of 50 sheets	48.48	87.5	16.67
9	Four children play	43.48	100	4.55
10	Gina has a 4-digit	28.57	52.63	19.05
11	A boy and a girl run	6.56	16.67	0
12	Half of 10-8 is	22.39	47.06	0
13	The diagonals on two	11.67	16.67	0
14	A string is wound around	3.85	0	0
15	Two amoebas are placed	36.07	88.89	13.33

item no.	Question Title	Discrimination	Mean	Standard Deviation
1	How many bricks were	21.43	93.94	24.04
2	The adjacent picture	30.36	83.05	37.84
3	What fraction of the	41.18	83.82	37.1
4	All sides of this figure	62.5	40.98	49.59
5	The numbers in this	14.29	91.18	28.57
6	The mass of a container	73.21	48.75	50.3
7	A piggy-bank holds	74.68	62.3	48.87
8	A pile of 50 sheets	70.83	48.48	50.36
9	Four children play	95.45	43.48	49.94
10	Gina has a 4-digit	33.58	28.57	45.5
11	A boy and a girl run	16.67	6.56	24.96
12	Half of 10-8 is	47.06	22.39	42
13	The diagonals on two	16.67	11.67	32.37
14	A string is wound around	0	3.85	19.42
15	Two amoebas are placed	75.56	36.07	48.42