LEFT VERSUS RIGHT CANONICAL WIENER-HOPF FACTORIZATION FOR RATIONAL MATRIX FUNCTIONS: AN ALTERNATIVE VERSION

by

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Chapter 0

INTRODUCTION

It is an established fact that both canonical and non-canonical Wiener-Hopf factorizations of matrix functions play an important role in various aspects of mathematical analysis and its applications. Indeed, for instance, the Fredholm properties of a block Toeplitz operator T, with symbol W from the $m \times m$ matrix Wiener algebra $\mathcal{W}^{m\times m}$ over the unit circle $\mathbb{T}$, may be read off from a *(right) Wiener- Hopf factorization*

$$W(\lambda) = W_-(\lambda)D(\lambda)W_+(\lambda) \quad , \quad \lambda \in \mathbb{T}, \tag{0.1}$$

where W_+ and W_- are in $\mathcal{W}^{m\times m}$, the function W_+ has an analytic extension to the open unit disc $\mathbb{D}$ such that det $W_+(z) \neq 0$ for $z \in \bar{\mathbb{D}}$, the function W_- has an analytic extension to $\mathbb{C} \cup \{\infty\} \setminus \bar{\mathbb{D}}$, such that det $W_-(z) \neq 0$ for $z \in \mathbb{C} \cup \{\infty\} \setminus \mathbb{D}$, and

$$D(\lambda) = \text{diag}\,(\lambda^{\kappa_j})_{j=1}^{m} \ , \tag{0.2}$$

with $\kappa_1, \ldots, \kappa_m$ integers. In particular, T is invertible if and only if the factorization is *canonical*, i.e., the indices $\kappa_1, \ldots, \kappa_m$ are all equal to zero, and in this case the inverse of T may be constructed from the Fourier coefficients of $W_-(.)^{-1}$ and $W_+(.)^{-1}$ (see [GKr]; also[GF]). Analogous results hold for Wiener-Hopf and singular integral operators (see the books [GGK], [GF], [GKr] and [MiPr]).

Also in mathematical systems theory, particularly in the analysis of H_∞-control problems, Wiener-Hopf factorization plays an important role (see, e.g., [BHV],[GGLD], [Fr]). In the latter case the matrix functions are, in general, rational, i.e., their entries are quotients of polynomials.

Up to the late seventies, the standard construction of the Wiener-Hopf factorization (see [GKr], also [GF]) did not yield explicit formulas for the factors W_+ and W_- nor the factorization indices, but only an algorithm which yields the factors and the indices in a finite number of steps. Subsequent to this, a new method, known as the *state space method* (see [BGK4]), was developed to deal with problems involving rational matrix functions. This method largely depends on the notion of realization which originates from mathematical systems theory (see [K]) and allows one to reduce problems in analysis to ones in linear algebra involving matrices.

A realization of a rational matrix function W which is analytic at infinity is a representation of W in the form:

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \tag{0.3}$$

where A is a square matrix of order n say, and B, C and D are matrices of sizes $n \times m$, $m \times n$ and $m \times m$, respectively. Here D is assumed to be invertible. In the papers [BGK1] and [BGK2], canonical and non-canonical Wiener-Hopf factorizations (0.1) of $W(.)$ in the form (0.3) are discussed. Explicit formulas for the right and left factors and the diagonal term $D(\lambda)$ are given in terms of A, B, C, D, the corresponding Riesz projections and various matrices derived from these transformations.

For a rational matrix function W which is not analytic and invertible at infinity, the realization is not of the form (0.3) with D invertible, but may be represented as in [GK1], as

$$W(\lambda) = D + C(\lambda G - A)^{-1}B, \tag{0.4}$$

where A, B and C are as above, and G is of the same size as A. In [GK1] the form (0.4) is used to obtain necessary and sufficient conditions for a canonical Wiener-Hopf factorization and explicit realization formulas for the factors are given in terms of the matrices A, G, B and C and generalized Riesz projections.

This dissertation also concerns another case where the rational matrix function is not analytic and invertible at infinity. The aim is to provide necessary and sufficient conditions for the existence of a right canonical Wiener-Hopf factorization and corresponding explicit formulas for the factors in terms of a given left canonical factorization. We present an alternative version of the construction given in [BR]. Instead of the representation (0.4) we use the form (see [GK1])

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \tag{0.5}$$

where α is a non-zero complex number which is neither a pole nor a zero of W, the matrices A, G, B and C have the same properties as in (0.4) and D is a non-singular $m \times m$ matrix. The construction yields an explicit factorization, with factors of the form (0.5). The factors are described explicitly in terms of the matrices appearing in the realization (0.5) and the corresponding generalized Riesz projections. In the second chapter our main factorization theorem is described in detail. Note that the representation (0.5) may be deduced from classical realization results by applying the Möbius transformation

$$\phi(\lambda) = \alpha\,\frac{2\lambda - 1}{2\lambda + 1} \quad , \quad \phi^{-1}(z) = -\frac{1}{2}\,\frac{z + \alpha}{z - \alpha}\,.$$

Indeed, setting $\widehat{W}(\lambda) = W(\phi(\lambda))$ we have that $\widehat{W}(\lambda)$ is a rational matrix function which is analytic and invertible at infinity and from the discussion earlier, may be represented as

$$\widehat{W}(\lambda) = \widehat{D} + \widehat{C}(\lambda - \widehat{A})^{-1}\widehat{B} \quad ,$$

where $\widehat{D} = \widehat{W}(\infty)$ and $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ are matrices of appropriate sizes. If we define $A = \alpha(\frac{1}{2} - \widehat{A})$,

$G = -\frac{1}{2} - \widehat{A}$, $B = \widehat{B}$, $C = \widehat{C}$ and $D = \widehat{D}$, then it is clear that (0.5) now holds (cf., Theorem 1.9 in [BGK1]).

This dissertation consists of two chapters. Chapter 1 contains preliminaries, the canonical Wiener-Hopf factorization theorem for rational matrix functions represented in the form (0.5), a discussion of a certain operator equation and a derivation of the general inverse formula for rational matrix functions of the form (0.5). The majority of the results in this chapter (see Proposition 2.1, Corollary 2.2, Lemma 2.3, Theorem 2.4 and Corollary 2.5) are generalizations of results in Chapter I of [BGK1], which involve realizations of the form (0.3). Also, Theorem 1.3.1 in the sequel is a natural analogue of Theorem 1.4.1 in [GGK].

In Chapter 2 we provide a statement and proof of our main Wiener- Hopf factorization theorem. This result provides necessary and sufficient conditions for the existence of a right canonical Wiener-Hopf factorization and explicit formulas for the right canonical factors in terms of a given left canonical factorization. We conclude this chapter by considering an application of the aforementioned result to singular integral operators.

Chapter 1

PRELIMINARIES AND CANONICAL FACTORIZATION

In this chapter we discuss preliminaries about spectral properties, canonical factorizations and operator equations of various types.

Throughout this chapter, we shall consider the representation of rational matrix functions of the form

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B. \tag{0.1}$$

The main result is a canonical factorization theorem for rational matrix functions represented in the form (0.1). Many of the results derived below, are analogues of those which involve the realization form

$$W(\lambda) = D + C(\lambda I - A)^{-1}B.$$

1.1 SPECTRAL PRELIMINARIES

This section contains definitions which will appear in the factorization theorems derived later. Firstly, we establish some notation.

A *Cauchy contour* γ is the positively oriented boundary of a bounded Cauchy domain in $\mathbb{C}$. Such a contour consists of a finite number of non-intersecting closed rectifiable Jordan curves.

The set of points inside γ is called the *inner domain* of γ and will be denoted by Δ_+. The *outer domain* of γ is the set $\Delta_- = \mathbb{C}_\infty \setminus \bar{\Delta}_+$. We assume that $0 \in \Delta_+$. By definition $\infty \in \Delta_-$.

Next, we consider operator pencils. Let X be a complex Banach space and let G and A be bounded linear operators on X. For $\lambda \in \mathbb{C}$, the expression $\lambda G - A$ will be known as a *(linear) operator pencil* on X. Given a non-empty subset Δ of the Riemann sphere $\mathbb{C}_\infty$, we say that $\lambda G - A$ is Δ-regular if $\lambda G - A$ (or just G if $\lambda = \infty$) is invertible for each λ in Δ. The *spectrum* of $\lambda G - A$, $\sigma(G, A)$, is the subset of $\mathbb{C}_\infty$ determined by the following properties. $\infty \in \sigma(G, A)$ if and only if G is not invertible, and $\sigma(G, A) \cap \mathbb{C}$ consists of all those $\lambda \in \mathbb{C}$ for which $\lambda G - A$ is not invertible. Its complement (in $\mathbb{C}_\infty$) is the *resolvent set* of $\lambda G - A$, denoted by $\rho(G, A)$.

Next, we look at generalized definitions of concepts associated with the decomposition of $\sigma(G, A)$ (cf.,[GGK], Ch.1). If $\gamma \cap \sigma(G, A) = \emptyset$, i.e., γ splits the spectrum of $\lambda G - A$, then $\sigma(G, A)$ decomposes into two disjoint compact sets σ_1 and σ_2 such that σ_1 is in the inner domain and σ_2 is in the outer domain of γ. Furthermore, if γ splits the spectrum of $\lambda G - A$, then we have *generalized Riesz projections* of X associated with $\lambda G - A$ and γ, namely the projections

$$\begin{aligned} P(G, A, \gamma) &= \frac{1}{2\pi i} \int_\gamma G(\lambda G - A)^{-1} d\lambda, \\ Q(G, A, \gamma) &= \frac{1}{2\pi i} \int_\gamma (\lambda G - A)^{-1} G d\lambda. \end{aligned} \tag{1.1}$$

The subspaces im $P(G, A, \gamma)$ and im $Q(G, A, \gamma)$ are called the *generalized spectral subspaces* for $\lambda G - A$ corresponding to the contour γ. It may be shown that if the spectrum, $\sigma(G, A)$, lies inside

γ, i.e., $\sigma(G, A) \subset \Delta_+$, then the projections P and Q are the identity operators on X. Also, if $\sigma(G, A) \subset \Delta_-$ then P and Q are both zero.

1.2 PRELIMINARIES ABOUT FACTORIZATION

In the main result in this section (see Theorem 2.4) we derive a canonical factorization theorem which is a natural analogue of Theorem 1.5 in [BGK1].

Firstly, we define a number of concepts which will appear in the sequel.

Let $W(\lambda)$ be an $m \times m$ rational matrix function, and let γ be a Cauchy contour in the complex plane $\mathbb{C}$ with inner domain Δ_+ and outer domain Δ_-. Assume that $W(\lambda)$ has no pole or zero on γ and that $0 \in \Delta_+$. Then $W(\lambda)$ admits a *(right) Wiener-Hopf factorization* relative to γ, that is, $W(\lambda)$ factorizes as

$$W(\lambda) = W_-(\lambda)D(\lambda)W_+(\lambda) \quad , \quad \lambda \in \gamma \quad , \tag{2.1}$$

where W_+ and W_- are $m \times m$ rational matrix functions, W_+ has no pole or zero in $\Delta_+ \cup \gamma$ and W_- has no pole or zero in $\Delta_- \cup \gamma$ (which includes the point ∞), and

$$D(\lambda) = \operatorname{diag}(\lambda^{\kappa_j})_{j=1}^m.$$

Here $\kappa_1 \le \kappa_2 \le \cdots \le \kappa_m$ are integers, which are uniquely determined by W (and γ), and are called the *(right) factorization indices* of W relative to γ (see, e.g., [CG]). The factorization is called a *(right) canonical Wiener-Hopf factorization* if and only if the indices $\kappa_1, \cdots, \kappa_m$ are all zero. If W admits such a factorization, then det $W(\lambda) \ne 0$ for each $\lambda \in \gamma$, but in general, this condition is only necessary and not sufficient for the existence of a canonical factorization. We refer to a *(left) Wiener-Hopf factorization* if in (2.1) the order of the factors are interchanged.

We consider a representation of W of the form (see [GK1]) :

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \tag{2.2}$$

where we choose $\alpha \neq 0$ such that α is neither a pole nor a zero of W, the square matrices G and A are both of order n say, and B, C and D are matrices of sizes $n \times m$, $m \times n$ and $m \times m$, respectively. Here we assume D to be invertible. Note that $D = \lim_{\lambda \to \alpha} W(\lambda)$.

Next, we state and prove results which will be useful in the sequel. The first of these results is a natural analogue of Theorem 1.1 in [BGK1] for realizations of the type (2.2).

Proposition 2.1 *Let*

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B$$

be a given realization with invertible external matrix D, let (π_1, π_2) be a pair of projections of a complex unitary space $\mathbb{C}^n$ such that rank π_1 = rank π_2, and let

$$\lambda G - A = \begin{pmatrix} \lambda G_{11} - A_{11} & \lambda G_{12} - A_{12} \\ \lambda G_{21} - A_{21} & \lambda G_{22} - A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

be the matrix representations of $\lambda G - A$, B and C with respect to the decomposition $\mathbb{C}^n = \ker \pi_i \oplus \operatorname{im} \pi_i$, $i = 1, 2$. Assume $D = D_1 D_2$, where D_1 and D_2 are invertible matrices on $\mathbb{C}^m$.
Write

$$W_1(\lambda) = D_1 + (\lambda - \alpha)C_1(\lambda G_{11} - A_{11})^{-1}B_1 D_2^{-1}, \tag{2.3}$$

and

$$W_2(\lambda) = D_2 + (\lambda - \alpha)D_1^{-1}C_2(\lambda G_{22} - A_{22})^{-1}B_2. \tag{2.4}$$

Set $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$. If $(\lambda G - A)[\ker \pi_1] \subset \ker \pi_2$ and $(\lambda G^\times - A^\times)[\operatorname{im} \pi_1] \subset \operatorname{im} \pi_2$, then $W(\lambda) = W_1(\lambda)W_2(\lambda)$, $\lambda \in \rho(G_{11}, A_{11}) \cap \rho(G_{22}, A_{22}) \subset \rho(G, A)$.

Proof. Since $(\lambda G - A)\ker \pi_1 \subset \ker \pi_2$, we have that $\lambda G_{21} - A_{21} = 0$. As

$$\lambda G^\times - A^\times = \begin{pmatrix} \lambda G_{11}^\times - A_{11}^\times & (\lambda G_{12} - A_{12}) + (\lambda - \alpha)B_1 D_2^{-1} D_1^{-1} C_2 \\ (\lambda - \alpha)B_2 D_2^{-1} D_1^{-1} C_1 & \lambda G_{22}^\times - A_{22}^\times \end{pmatrix}$$

maps im π_1 into im π_2, we have $\lambda G_{12} - A_{12} = (\alpha - \lambda) B_1 D_2^{-1} D_1^{-1} C_2$.

Hence, for $\lambda \in \rho(G_{11}, A_{11}) \cap \rho(G_{22}, A_{22}) \subset \rho(G, A)$, we compute that

$$\begin{aligned} W(\lambda) &= D + (\lambda - \alpha) C (\lambda G - A)^{-1} B \\ &= D_1 D_2 + (\lambda - \alpha) \begin{pmatrix} C_1 & C_2 \end{pmatrix} \end{aligned}$$

$$\cdot \begin{pmatrix} (\lambda G_{11} - A_{11})^{-1} & (\lambda - \alpha)(\lambda G_{11} - A_{11})^{-1} B_1 D_2^{-1} D_1^{-1} C_2 (\lambda G_{22} - A_{22})^{-1} \\ 0 & (\lambda G_{22} - A_{22})^{-1} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$\begin{aligned} &= D_1 D_2 + (\lambda - \alpha) C_1 (\lambda G_{11} - A_{11})^{-1} B_1 \\ &\quad + (\lambda - \alpha)^2 C_1 (\lambda G_{11} - A_{11})^{-1} B_1 D_2^{-1} D_1^{-1} C_2 (\lambda G_{22} - A_{22})^{-1} B_2 + (\lambda - \alpha) C_2 (\lambda G_{22} - A_{22})^{-1} B_2. \end{aligned}$$

Also, we have that

$$\begin{aligned} W_1(\lambda) W_2(\lambda) &= [D_1 + (\lambda - \alpha) C_1 (\lambda G_{11} - A_{11})^{-1} B_1 D_2^{-1}] \cdot [D_2 + (\lambda - \alpha) D_1^{-1} C_2 (\lambda G_{22} - A_{22})^{-1} B_2] \\ &= D_1 D_2 + (\lambda - \alpha) C_1 (\lambda G_{11} - A_{11})^{-1} B_1 + (\lambda - \alpha)^2 C_1 (\lambda G_{11} - A_{11})^{-1} B_1 D_2^{-1} D_1^{-1} C_2 \\ &\quad \cdot (\lambda G_{22} - A_{22})^{-1} B_2 + (\lambda - \alpha) C_2 (\lambda G_{22} - A_{22})^{-1} B_2. \end{aligned}$$

So we conclude from the above computations of $W(\lambda)$ and $W_1(\lambda) W_2(\lambda)$ that they are equal. □

It is easy to show that the converse of this proposition also holds. Note that the formulas for the factors are written in terms of the components of the block matrix representations of $\lambda G - A$, B, C and D. Under certain conditions, we may express these formulas in terms of the projections π_1 and π_2. The result is an analogue of the Corollary to Theorem 1.1 in [BGK1].

Corollary 2.2 *Let $W(\lambda) = D + (\lambda - \alpha) C (\lambda G - A)^{-1} B$ be a given realization with invertible external matrix D, and let (π_1, π_2) be a pair of projections of the state space $\mathbb{C}^n$ such that rank π_1 = rank π_2, and*

$$(\lambda G - A)[\ker \pi_1] \subset \ker \pi_2, \quad (\lambda G^\times - A^\times)[\operatorname{im} \pi_1] \subset \operatorname{im} \pi_2.$$

Here we set $G^\times = G + BD^{-1}C$ *and* $A^\times = A + \alpha BD^{-1}C$. *Assume* $D = D_1D_2$, *where* D_1 *and* D_2 *are invertible matrices on* $\mathbb{C}^m$.

Then, for λ *in some open neighbourhood of* α, *we have* $W(\lambda) = W_{\pi_1}(\lambda)W_{\pi_2}(\lambda)$, *where*

$$W_{\pi_1}(\lambda) = D_1 + (\lambda - \alpha)C(\lambda G - A)^{-1}(I - \pi_1)BD_2^{-1},$$

and

$$W_{\pi_2}(\lambda) = D_2 + (\lambda - \alpha)D_1^{-1}C\pi_2(\lambda G - A)^{-1}B.$$

Proof. Let $W_1(\cdot)$ and $W_2(\cdot)$ be defined as in formulas (2.3) and (2.4). Then by Proposition 2.1, $W(\lambda) = W_1(\lambda)W_2(\lambda)$, for λ in some open neighbourhood of α. To complete the proof observe that $W_1(\lambda) = W_{\pi_1}(\lambda)$ and $W_2(\lambda) = W_{\pi_2}(\lambda)$, for λ near α. □

The operator pencil $\lambda G^\times - A^\times$ is often referred to as the *associate operator pencil.*

The next lemma is a natural analogue of Lemma 1.4 in [BGK1]. We assume that X_1 and X_2 are complex Banach spaces.

Lemma 2.3 *Let* $\lambda G - A = \begin{pmatrix} \lambda G_{11} - A_{11} & \lambda G_{12} - A_{12} \\ 0 & \lambda G_{22} - A_{22} \end{pmatrix}$ *be given, and let* π *be a projection of* $\mathbb{C}^n = X_1 \oplus X_2$ *such that* $\ker \pi = X_1$. *Then for the compression* $\lambda\pi G - \pi A\mid_{im\ \pi}$ *and* $\lambda G_{22} - A_{22}$ *there exists an invertible operator* $E : im\ \pi \to X_2$ *such that* $E^{-1}(\lambda G_{22} - A_{22})E = \lambda\pi G - \pi A\mid_{im\ \pi}$.

Furthermore, X_1 *is a spectral subspace for* $\lambda G - A$ *if and only if* $\sigma(G_{11}, A_{11}) \cap \sigma(G_{22}, A_{22}) = \emptyset$, *and in this case* $\sigma(G, A) = \sigma(G_{11}, A_{11}) \cup \sigma(G_{22}, A_{22})$ *and*

$$X_1 = im\ [\frac{1}{2\pi i}\int_\gamma G(\lambda G - A)^{-1}d\lambda], \tag{2.5}$$

where γ *is a Cauchy contour around* $\sigma(G_{11}, A_{11})$ *separating* $\sigma(G_{11}, A_{11})$ *from* $\sigma(G_{22}, A_{22})$.

Proof. Let P be the projection of $\mathbb{C}^n = X_1 \oplus X_2$ onto X_2 along X_1. As ker $P =$ ker π, we have $P = P\pi$ and the map $E = P\mid_{\text{im}\pi}$: im $\pi \to X_2$ is an invertible operator. Denote the compression of $\lambda G - A$ to im π by $\lambda G_0 - A_0$. Take $x = \pi y$. Then

$$\begin{aligned} E(\lambda G_0 - A_0)x &= P\pi(\lambda G - A)\pi y \\ &= P(\lambda G - A)\pi y \\ &= P(\lambda G - A)P\pi y \\ &= (\lambda G_{22} - A_{22})Ex, \end{aligned}$$

and hence $(\lambda G_0 - A_0) = E^{-1}(\lambda G_{22} - A_{22})E$.
Now suppose that $\sigma(G_{11}, A_{11}) \cap \sigma(G_{22}, A_{22}) = \emptyset$. Since $\lambda G - A$ is in upper triangular form, it is easy to verify that $\sigma(G, A) = \sigma(G_{11}, A_{11}) \cup \sigma(G_{22}, A_{22})$. Let γ be a Cauchy contour around $\sigma(G_{11}, A_{11})$ separating $\sigma(G_{11}, A_{11})$ from $\sigma(G_{22}, A_{22})$. Note that γ splits the spectrum of $\lambda G - A$. For the corresponding generalized Riesz projection we have

$$P(G, A, \gamma) = \begin{pmatrix} I & \star \\ 0 & 0 \end{pmatrix},$$

and it is clear that $X_1 = \text{im } P(G, A, \gamma)$.
So X_1 is a spectral subspace for $\lambda G - A$ and (2.5) holds.
Next, assume that $X_1 = \text{im } R$, where R is a generalized Riesz projection associated with $\lambda G - A$ and γ. Put $\pi = I - R$, and let $\lambda G_0 - A_0$ be the restriction of $\lambda G - A$ to im π. Then $\sigma(G_{11}, A_{11}) \cap \sigma(G_0, A_0) = \emptyset$. Since, by the first part of the proof, we have $E^{-1}(\lambda G_{22} - A_{22})E = \lambda G_0 - A_0$, it follows that $\sigma(G_0, A_0) = \sigma(G_{22}, A_{22})$, and hence we have shown that $\sigma(G_{11}, A_{11}) \cap \sigma(G_{22}, A_{22}) = \emptyset$. □

Note that we may obtain a similar result, as the one above, for the generalized spectral subspace of the form,

$$Q(G,A,\gamma) = \text{im}\,[\frac{1}{2\pi i}\int_\gamma (\lambda G - A)^{-1} G d\lambda].$$

In the main result of our present section, we consider the right canonical Wiener-Hopf factorization of a rational matrix function given in realized form (2.2). Necessary and sufficient conditions for the existence of such a factorization and explicit formulas for the factors are given in terms of the data which appear in the realization. In the proof, we will make extensive use of Proposition 2.1, Corollary 2.2 and Lemma 2.3. The theorem may be regarded as a natural analogue of Theorem 1.5 in [BGK1].

Theorem 2.4 *Let $W(\lambda)$ admit a realization of the form $W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B$, where we assume that $D = D_1 D_2$, with D_1 and D_2 invertible matrices on $\mathbb{C}^m$. Set $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$. Let γ be a Cauchy contour that splits the spectra of $\lambda G - A$ and $\lambda G^\times - A^\times$. Assume that*

$$\text{(i)}\ \mathbb{C}^n = im\, P(G,A,\gamma) \oplus ker\, P(G^\times, A^\times, \gamma),$$

$$\text{(ii)}\ \mathbb{C}^n = im\, Q(G,A,\gamma) \oplus ker\, Q(G^\times, A^\times, \gamma),$$

where

$$\begin{aligned} P(G,A,\gamma) &= \frac{1}{2\pi i}\int_\gamma G(\lambda G - A)^{-1} d\lambda, \\ P(G^\times, A^\times, \gamma) &= \frac{1}{2\pi i}\int_\gamma G^\times(\lambda G^\times - A^\times)^{-1} d\lambda \\ Q(G,A,\gamma) &= \frac{1}{2\pi i}\int_\gamma (\lambda G - A)^{-1} G d\lambda, \\ Q(G^\times, A^\times, \gamma) &= \frac{1}{2\pi i}\int_\gamma (\lambda G^\times - A^\times)^{-1} G^\times d\lambda. \end{aligned}$$

Let π_1 be the projection of $\mathbb{C}^n$ onto $ker\, P(G^\times, A^\times, \gamma)$ along $im\, P(G,A,\gamma)$ and π_2 be the projection of $\mathbb{C}^n$ onto $ker\, Q(G^\times, A^\times, \gamma)$ along $im\, Q(G,A,\gamma)$, and let

$$\lambda G - A = \begin{pmatrix} \lambda G_{11} - A_{11} & \lambda G_{12} - A_{12} \\ 0 & \lambda G_{22} - A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

be the matrix representations of $\lambda G - A$, B and C with respect to the decomposition (i) or (ii).

Define

$$W_-(\lambda) = D_1 + (\lambda - \alpha)C_1(\lambda G_{11} - A_{11})^{-1}B_1 D_2^{-1},$$

and

$$W_+(\lambda) = D_2 + (\lambda - \alpha)D_1^{-1}C_2(\lambda G_{22} - A_{22})^{-1}B_2.$$

Then $W(\lambda) = W_-(\lambda)W_+(\lambda)$ for $\lambda \in \rho(G, A)$, and this factorization is a right canonical Wiener-Hopf factorization of W with respect to γ.

Conversely, if $W = W_- W_+$ is a right canonical Wiener-Hopf factorization with respect to γ and $W_-(\infty) = D_1$, where D_1 is an invertible $m \times m$ matrix, then there exists a realization

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B$$

on a neighbourhood of γ and the contour γ splits the spectra of $\lambda G - A$ and $\lambda G^\times - A^\times$. Furthermore, decompositions (i) and (ii) hold. With respect to these decompositions

$$\lambda G - A = \begin{pmatrix} \lambda G_1 - A_1 & (\alpha - \lambda)B_1 D_2^{-1} D_1^{-1} C_2 \\ 0 & \lambda G_2 - A_2 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}, D = D_1 D_2,$$

and if π_1 is a projection of $\mathbb{C}^n$ onto $\ker P(G^\times, A^\times, \gamma)$ along $\operatorname{im} P(G, A, \gamma)$ and π_2 is a projection of $\mathbb{C}^n$ onto $\ker Q(G^\times, A^\times, \gamma)$ along $\operatorname{im} Q(G, A, \gamma)$, the factors $W_-(\lambda)$ and $W_+(\lambda)$ for a right canonical factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$ are given by the formulas

$$W_-(\lambda) = D_1 + (\lambda - \alpha)C_1(\lambda G_1 - A_1)^{-1}B_1 D_2^{-1}, \; \lambda \in \bar{\Delta}_-,$$

and

$$W_+(\lambda) = D_2 + (\lambda - \alpha)D_1^{-1}C_2(\lambda G_2 - A_2)^{-1}B_2, \ \lambda \in \bar{\Delta}_+.$$

Proof. We note that the matching conditions (i) and (ii) are equivalent, (see, e.g., [GK2]). From the upper (respectively, lower) triangular form of $\lambda G - A$ (respectively, $\lambda G^\times - A^\times$) we deduce that $(\lambda G - A)[\ker \pi_1] \subset \ker \pi_2$ and $(\lambda G^\times - A^\times)[\text{im } \pi_1] \subset \text{im } \pi_2$. It follows directly from Proposition 2.1 that $W(\lambda) = W_-(\lambda)W_+(\lambda)$, for each

$$\lambda \in \rho(G_{11}, A_{11}) \cap \rho(G_{22}, A_{22}). \tag{2.6}$$

Since X_1 is a generalized spectral subspace for $\lambda G - A$ we can apply Lemma 2.3 to show that $\sigma(G_{11}, A_{11}) \cap \sigma(G_{22}, A_{22}) = \emptyset$. But then $\rho(G, A) = \rho(G_{11}, A_{11}) \cap \rho(G_{22}, A_{22})$ and it follows that (2.6) holds for each $\lambda \in \rho(G, A)$.

Also, we have from Lemma 2.3 that

$$\sigma(G_{11}, A_{11}) = \sigma(G, A) \cap \Delta_+ \ , \ \sigma(G_{22}, A_{22}) = \sigma(G, A) \cap \Delta_- \ . \tag{2.7}$$

In a similar way, one may show that

$$\sigma(G_{11}, A_{11}) = \sigma(G^\times, A^\times) \cap \Delta_+ \ , \ \sigma(G_{22}, A_{22}) = \sigma(G^\times, A^\times) \cap \Delta_-. \tag{2.8}$$

Since $W_-(\lambda) = D_1 + (\lambda - \alpha)C_1(\lambda G_{11} - A_{11})^{-1}B_1D_2^{-1}$, we know that W_- is defined and analytic on the complement of $\sigma(G_{11}, A_{11})$ and det $W_-(\lambda) \neq 0$ for each $\lambda \notin \sigma(G_{11}^\times, A_{11}^\times)$.
So using the first parts of (2.7) and (2.8), it follows that W_- is an $m \times m$ matrix which is continuous on $\bar{\Delta}_-$; analytic on Δ_- and det $W_-(\lambda) \neq 0$ for each $\lambda \in \bar{\Delta}_-$. In the same way, using the second

parts of (2.7) and (2.8), one proves that W_+ is an $m \times m$ matrix which is continuous on $\bar{\Delta}_+$; analytic on Δ_+ and det $W_+(\lambda) \neq 0$ for each $\lambda \in \bar{\Delta}_+$.

To prove the second part of the theorem, let us assume that $W(\lambda) = W_-(\lambda)W_+(\lambda)$ is a right canonical Wiener-Hopf factorization with respect to γ and $W_-(\infty) = D_1$. As W_- is analytic on a neighbourhood of $\bar{\Delta}_-$ and $W_-(\lambda)$ is invertible for each $\lambda \in \bar{\Delta}_-$, it follows from an analogue of the classical realization theorem (see, [GK2] and [GK3]) that one can find a realization $W_1(\lambda) = D_1 + (\lambda-\alpha)C_1(\lambda G_1 - A_1)^{-1}B_1D_2^{-1}$ for W_- on a neighbourhood of $\bar{\Delta}_-$ such that $\sigma(G_1, A_1)$ and $\sigma(G_1^\times, A_1^\times)$ are subsets of Δ_+. Also, W_+ admits a realization $W_2(\lambda) = D_2 + (\lambda-\alpha)D_1^{-1}C_2(\lambda G_2 - A_2)^{-1}B_2$ such that $\sigma(G_2, A_2)$ and $\sigma(G_2^\times, A_2^\times)$ are subsets of Δ_-.

Put $W_{\mathbb{C}^n}(\lambda) = W_1(\lambda)W_2(\lambda)$, $\lambda \in \rho(G_1, A_1) \cap \rho(G_2, A_2)$. Then $W_{\mathbb{C}^n}(\lambda) = D + (\lambda-\alpha)C(\lambda G - A)^{-1}B$, where $\mathbb{C}^n = X_1 \oplus X_2$ and

$$\lambda G - A = \begin{pmatrix} \lambda G_1 - A_1 & (\alpha-\lambda)B_1D_2^{-1}D_1^{-1}C_2 \\ 0 & \lambda G_2 - A_2 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}, D = D_1D_2.$$

As $\sigma(G_1, A_1) \cap \sigma(G_2, A_2) = \emptyset$, we have $\sigma(G, A) = \sigma(G_1, A_1) \cup \sigma(G_2, A_2)$. But then $\gamma \subset \rho(G, A) = \rho(G_1, A_1) \cap \rho(G_2, A_2)$ and $W_{\mathbb{C}^n}(\lambda) = W_1(\lambda)W_2(\lambda) = W_-(\lambda)W_+(\lambda) = W(\lambda)$, $\lambda \in \rho(G, A)$. So $D + (\lambda-\alpha)C(\lambda G - A)^{-1}B$ is a realization for W on a neighbourhood of γ. Since $\lambda G - A$ is represented in triangular form, we have that γ splits $\sigma(G, A)$. Also, by consideration of Lemma 2.3, it follows that $X_1 = \text{im } P(G, A, \gamma)$. Since

$$\lambda G^\times - A^\times = \begin{pmatrix} \lambda G_1^\times - A_1^\times & 0 \\ (\lambda-\alpha)B_2D_2^{-1}D_1^{-1}C_1 & \lambda G_2^\times - A_2^\times \end{pmatrix}$$

we have that the contour γ splits the spectrum of $\lambda G^\times - A^\times$ too, and $X_2 = \ker P(G^\times, A^\times, \gamma)$. It follows that $\mathbb{C}^n = X_1 \oplus X_2 = \text{im } P(G, A, \gamma) \oplus \ker P(G^\times, A^\times, \gamma)$. In a similar way we may show that the decomposition $\mathbb{C}^n = \text{im } Q(G, A, \gamma) \oplus \ker Q(G^\times, A^\times, \gamma)$ holds. If π_1 is the projection of $\mathbb{C}^n$ onto $X_2 = \ker P(G^\times, A^\times, \gamma)$ along $X_1 = \text{im } P(G, A, \gamma)$ and π_2 is the projection of $\mathbb{C}^n$ onto $\ker Q(G^\times, A^\times, \gamma)$ along $\text{im } Q(G, A, \gamma)$, then $W_-(\lambda) = W_1(\lambda)$ for $\lambda \in \bar{\Delta}_-$ and $W_+(\lambda) = W_2(\lambda)$ for $\lambda \in \bar{\Delta}_+$, and the proof is complete. □

The following Corollary allows us to express the right canonical factors, appearing in Theorem 2.4, in terms of the projections π_1 and π_2.

Corollary 2.5 *Let $W(\cdot)$, D, $G^\times$, $A^\times$ and the Cauchy contour γ be described as in Theorem 2.4. Assume*

(i) $\mathbb{C}^n = im\ P(G, A, \gamma) \oplus ker\ P(G^\times, A^\times, \gamma)$,

(ii) $\mathbb{C}^n = im\ Q(G, A, \gamma) \oplus ker\ Q(G^\times, A^\times, \gamma)$,

where $P(G, A, \gamma)$, $P(G^\times, A^\times, \gamma)$, $Q(G, A, \gamma)$ and $Q(G^\times, A^\times, \gamma)$ are as in Theorem 2.4. Let π_1 be the projection of $\mathbb{C}^n$ onto $ker\ P(G^\times, A^\times, \gamma)$ along $im\ P(G, A, \gamma)$ and π_2 be the projection of $\mathbb{C}^n$ onto $ker\ Q(G^\times, A^\times, \gamma)$ along $im\ Q(G, A, \gamma)$, and define

$$W_-(\lambda) = D_1 + (\lambda - \alpha)C(\lambda G - A)^{-1}(I - \pi_1)BD_2^{-1}, \tag{2.9}$$

and

$$W_+(\lambda) = D_2 + (\lambda - \alpha)D_1^{-1}C\pi_2(\lambda G - A)^{-1}B. \tag{2.10}$$

Then $W(\lambda) = W_-(\lambda)W_+(\lambda)$ for $\lambda \in \rho(G, A)$, and this factorization is a right canonical Wiener-Hopf factorization of W with respect to γ.

Conversely, if $W = W_-W_+$ is a right canonical Wiener-Hopf factorization of W with respect to γ and $W_-(\infty) = D_1$, where D_1 is an invertible matrix, then there exists a realization

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B$$

on a neighbourhood of γ, the contour γ splits the spectra of $\lambda G - A$ and $\lambda G^\times - A^\times$,

$$\mathbb{C}^n = im\ P(G, A, \gamma) \oplus ker\ P(G^\times, A^\times, \gamma),$$

$$\mathbb{C}^n = im\ Q(G, A, \gamma) \oplus ker\ Q(G^\times, A^\times, \gamma),$$

and if π_1 is a projection of $\mathbb{C}^n$ onto ker $P(G^\times, A^\times, \gamma)$ along im $P(G, A, \gamma)$, and π_2 is a projection of $\mathbb{C}^n$ onto ker $Q(G^\times, A^\times, \gamma)$ along im $Q(G, A, \gamma)$, then the factors $W_-(\lambda)$ and $W_+(\lambda)$ for a right canonical factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$ are given by

$$W_-(\lambda) = D_1 + (\lambda - \alpha)C(\lambda G - A)^{-1}(I - \pi_1)BD_2^{-1},\ \lambda \in \bar{\Delta}_-,$$

and

$$W_+(\lambda) = D_2 + (\lambda - \alpha)D_1^{-1}C\pi_2(\lambda G - A)^{-1}B,\ \lambda \in \bar{\Delta}_+.$$

Proof. Let W_- and W_+ be the rational matrix functions defined by (2.9) and (2.10). From the given π_1, we have that $I - \pi_1$ is a projection of $\mathbb{C}^n$ onto im $P(G, A, \gamma)$ along ker $P(G^\times, A^\times, \gamma)$. Thus

$$I - \pi_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \quad \text{im } P(G, A, \gamma) \oplus \text{ker } P(G^\times, A^\times, \gamma) \quad \to \quad \text{im } P(G, A, \gamma) \oplus \text{ker } P(G^\times, A^\times, \gamma).$$

By using the block matrix representations of $\lambda G - A$, B, C, D and $I - \pi_1$ we have that

$$\begin{aligned} W_-(\lambda) &= D_1 + (\lambda - \alpha)(C_1\ C_2) \begin{pmatrix} (\lambda G_{11} - A_{11})^{-1} & (\lambda - \alpha)(\lambda G_{11} - A_{11})^{-1}B_1D^{-1}C_2(\lambda G_{22} - A_{22})^{-1} \\ 0 & (\lambda G_{22} - A_{22})^{-1} \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} D_2^{-1} \\ &= D_1 + (\lambda - \alpha)(C_1\ C_2) \begin{pmatrix} (\lambda G_{11} - A_{11})^{-1}B_1 \\ 0 \end{pmatrix} D_2^{-1} \\ &= D_1 + (\lambda - \alpha)C_1(\lambda G_{11} - A_{11})^{-1}B_1D_2^{-1}. \end{aligned}$$

Similarly, for the computation of W_+, we need the projection

$$\pi_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} : \quad \text{im } Q(G,A,\gamma) \oplus \ker Q(G^\times, A^\times, \gamma) \quad \rightarrow \quad \text{im } Q(G,A,\gamma) \oplus \ker Q(G^\times, A^\times, \gamma),$$

in order to obtain

$$W_+(\lambda) = D_2 + (\lambda - \alpha) D_1^{-1} C_2 (\lambda G_{22} - A_{22})^{-1} B_2.$$

The proof is completed by combining the observations made above and Theorem 2.4. □

1.3 MORE ABOUT REALIZATIONS AND OTHER OPERATOR EQUATIONS

As before, in this section we consider a regular $m \times m$ rational matrix function W which is not necessarily analytic and invertible at infinity. Here we may represent W in the realization form

$$W(\lambda) = D + (\lambda - \alpha) C (\lambda G - A)^{-1} B \ , \ \lambda \in \gamma, \tag{3.1}$$

where γ is a Cauchy contour in $\mathbb{C}$. In the first part of this section we look at the invertibility of (3.1) and under certain given conditions provide an explicit formula for its inverse. Note that a necessary and sufficient condition for the invertibility of W on γ, is that $\lambda G^\times - A^\times$ is γ-regular, where $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$ (cf., [G], Theorem I.2.1).

We now show that we may compute W^{-1} in terms of $G^\times$ and $A^\times$, i.e., we have that

$$W(\lambda)^{-1} = D^{-1} - (\lambda - \alpha) D^{-1} C (\lambda G^\times - A^\times)^{-1} B D^{-1} \ , \ \lambda \in \gamma. \tag{3.2}$$

Indeed, from an earlier note, we may assume for invertible $W(\lambda) = D + (\lambda - \alpha) C (\lambda G - A)^{-1} B \ , \ \lambda \in \gamma$, that $\lambda G^\times - A^\times$ is invertible for each $\lambda \in \gamma$. Set $z = (\lambda G - A)^{-1} Bx$.

Given y we compute x from

$$\begin{cases} \lambda G z &= \quad A z + B x \ , \\ y &= (\lambda - \alpha) C z + D x \ . \end{cases} \tag{3.3}$$

Applying BD^{-1} to the second equation in (3.3) and subtracting the result from the first equation in (3.3) we obtain the following equivalent system

$$\begin{cases} \lambda G^{\times} z &= \quad A^{\times} z + B D^{-1} y \ , \\ y &= (\lambda - \alpha) C z + D x \ . \end{cases} \tag{3.4}$$

Hence $z = (\lambda G^{\times} - A^{\times})^{-1} B D^{-1} y$ and $W(\lambda)^{-1} y = x = D^{-1} y - (\lambda - \alpha) D^{-1} C (\lambda G^{\times} - A^{\times})^{-1} B D^{-1} y$. This proves (3.2). □

From the above, it is easy to see that the formulas (2.9) and (2.10) in the previous section have the inverses

$$W_{-}(\lambda)^{-1} = D_1^{-1} - (\lambda - \alpha) D_1^{-1} C (I - \pi_2)(\lambda G^{\times} - A^{\times})^{-1} B D^{-1},$$

and

$$W_{+}(\lambda)^{-1} = D_2^{-1} - (\lambda - \alpha) D^{-1} C (\lambda G^{\times} - A^{\times})^{-1} \pi_1 B D_2^{-1},$$

respectively.

Next, we consider an operator equation of the form

$$A_1 Z G_2 - G_1 Z A_2 = C. \tag{3.5}$$

Here A_1, G_1, A_2 and G_2 are given operators acting between the Banach space X. In this regard, we will attempt to find $Z \in \mathcal{L}(X)$, for a given $C \in \mathcal{L}(X)$ such that (3.5) holds. The next theorem is the analogue of Theorem I.4.1 in [GGK].

Theorem 3.1 *If the spectra of the pencils $\lambda G_1 - A_1$ and $\lambda G_2 - A_2$ are disjoint, then for any $C \in \mathcal{L}(X)$, equation (3.5) has a unique solution $Z \in \mathcal{L}(X)$. More precisely, we have that*

$$\begin{aligned} Z &= \frac{1}{2\pi i}\int_{\gamma_1}(\lambda G_1 - A_1)^{-1}C(\lambda G_2 - A_2)^{-1}d\lambda \\ &= -\frac{1}{2\pi i}\int_{\gamma_2}(\lambda G_1 - A_1)^{-1}C(\lambda G_2 - A_2)^{-1}d\lambda, \end{aligned} \tag{3.6}$$

where γ_1 and γ_2 are Cauchy contours around $\sigma(G_1, A_1)$ and $\sigma(G_2, A_2)$, respectively, which separate $\sigma(G_1, A_1)$ from $\sigma(G_2, A_2)$.

Proof. Firstly, we validate the choice of the Cauchy contours γ_1 and γ_2. Since $\sigma(G_1, A_1) \cap \sigma(G_2, A_2) = \emptyset$, the point ∞ cannot be in both spectra. So without loss of generality we may assume that $\infty \notin \sigma(G_1, A_1)$. Then $\sigma(G_1, A_1)$ is a compact subset of $\mathbb{C}$ which lies in the open set $V = \mathbb{C} \setminus \sigma(G_2, A_2)$. Choose a bounded Cauchy domain Δ such that $\sigma(G_1, A_1) \subset \Delta \subset \bar{\Delta} \subset V$, and let γ_1 be the oriented boundary of Δ. Then γ_1 is a Cauchy contour, $\sigma(G_1, A_1)$ is in the inner domain of γ_1 and $\sigma(G_2, A_2)$ is in the outer domain of γ_1. In a similar way, one is able to prove the existence of a Cauchy contour γ_2, with $\sigma(G_2, A_2)$ in its inner domain and $\sigma(G_1, A_1)$ in its outer domain.

It suffices to show that (3.6) gives the unique solution of (3.5). As noted earlier, from the location of $\sigma(G_1, A_1)$ and $\sigma(G_2, A_2)$, it follows that

$$\begin{aligned} \frac{1}{2\pi i}\int_{\gamma_1} G_1(\lambda G_1 - A_1)^{-1}d\lambda &= I, \\ \frac{1}{2\pi i}\int_{\gamma_1} (\lambda G_2 - A_2)^{-1}G_2 d\lambda &= 0. \end{aligned} \tag{3.7}$$

Also,

$$\frac{1}{2\pi i}\int_{\gamma_1}(\lambda G_1 - A_1)^{-1}G_1 d\lambda = I,$$

$$\frac{1}{2\pi i}\int_{\gamma_1}G_2(\lambda G_2 - A_2)^{-1} d\lambda = 0, \tag{3.8}$$

where I is the identity operator on X.

Let Z be the first identity in (3.6). Then $Z \in \mathcal{L}(X)$ and because of (3.7)

$$\begin{aligned}
A_1 Z G_2 &= \frac{1}{2\pi i}\int_{\gamma_1} A_1(\lambda G_1 - A_1)^{-1}C(\lambda G_2 - A_2)^{-1}G_2 d\lambda \\
&= \frac{1}{2\pi i}\int_{\gamma_1} -C(\lambda G_2 - A_2)^{-1}G_2 d\lambda \\
&\quad + \frac{1}{2\pi i}\int_{\gamma_1} \lambda G_1(\lambda G_1 - A_1)^{-1}C(\lambda G_2 - A_2)^{-1}G_2 d\lambda \\
&= \frac{1}{2\pi i}\int_{\gamma_1} G_1(\lambda G_1 - A_1)^{-1}C(\lambda G_2 - A_2)^{-1}(\lambda G_2 - A_2 + A_2) d\lambda \\
&= \frac{1}{2\pi i}\int_{\gamma_1} G_1(\lambda G_1 - A_1)^{-1}C d\lambda + G_1 Z A_2 \\
&= C + G_1 Z A_2.
\end{aligned}$$

Hence Z is a solution of (3.5).

Conversely, if Z is a solution of (3.5). Then

$$\begin{aligned}
C &= A_1 Z G_2 - \lambda G_1 Z G_2 + \lambda G_1 Z G_2 - G_1 Z A_2 \\
&= -(\lambda G_1 - A_1) Z G_2 + G_1 Z(\lambda G_2 - A_2).
\end{aligned}$$

Also, it follows from (3.8) that

$$\frac{1}{2\pi i}\int_{\gamma_1}(\lambda G_1 - A_1)^{-1}C(\lambda G_2 - A_2)^{-1}d\lambda$$

$$\begin{aligned} &= -\frac{1}{2\pi i}\int_{\gamma_1} ZG_2(\lambda G_2 - A_2)^{-1}d\lambda + \frac{1}{2\pi i}\int_{\gamma_1}(\lambda G_1 - A_1)^{-1}G_1 Z d\lambda \\ &= Z. \end{aligned}$$

If we replace γ_1 by γ_2 in the above argument, then we obtain the second identity in (3.6). We have now proved that equation (3.5) is uniquely solvable and its solution is given by (3.6). □

Chapter 2

LEFT VERSUS RIGHT CANONICAL WIENER-HOPF FACTORIZATION

2.1 INTRODUCTION AND MAIN THEOREM

Let $W(\lambda)$ be an $m \times m$ rational matrix function, and let γ be a Cauchy contour in the complex plane $\mathbb{C}$ with inner domain Δ_+ and outer domain Δ_-. If we assume that $W(\lambda)$ is analytic and invertible at ∞, we know that W may be represented in the form

$$W(\lambda) = I_m + C(\lambda I_n - A)^{-1} B \ ,$$

where we assumed without loss of generality that the value of W at ∞ is the identity matrix I_m.

Under these conditions, the existence of a right Wiener-Hopf factorization for W may be characterized in terms of a left canonical Wiener-Hopf factorization. Also, formulas for the factors in a right factorization may be given in terms of the formulas for the factors in a given left factorization. These principles are encapsulated in Theorem 2.1 in [BR].

In the main result of this chapter we show that a similar analysis may be done when W is not necessarily analytic and invertible at ∞, that is, where W may be represented in the form

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B. \tag{1.1}$$

In this regard, we shall assume that the factors Y_+ and Y_- of a left canonical Wiener-Hopf factorization $W(\lambda) = Y_+(\lambda)Y_-(\lambda)$ are known. We give a necessary and sufficient condition for the existence of a right canonical Wiener-Hopf factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$; providing explicit formulas for the factors W_- and W_+ in terms of the realizations of Y_+ and Y_-. In order to obtain this result, we make extensive use of Theorem 1.2.4 and Corollary 1.2.5. The result is as follows.

Theorem 1.1 *Suppose that the rational $m \times m$ matrix function $W(\lambda)$ (not analytic and invertible at ∞) has a left canonical Wiener-Hopf factorization with respect to γ, that is, $W(\lambda)$ factorizes as*

$$W(\lambda) = Y_+(\lambda)Y_-(\lambda),$$

where

$$Y_+(\lambda) = D + (\lambda - \alpha)C_1(\lambda G_1 - A_1)^{-1}B_1, \tag{1.2}$$

and

$$Y_-(\lambda) = I_m + (\lambda - \alpha)D^{-1}C_2(\lambda G_2 - A_2)^{-1}B_2, \tag{1.3}$$

for $\alpha \neq 0$ and α neither a pole nor a zero of $W(\lambda)$. Set $G^\times := G + BD^{-1}C$ and $A^\times := A + \alpha BD^{-1}C$. We may assume that $\lambda G_1 - A_1$ and $\lambda G_1^\times - A_1^\times$ are $n_1 \times n_1$ matrices with spectra inside Δ_- and that $\lambda G_2 - A_2$ and $\lambda G_2^\times - A_2^\times$ are $n_2 \times n_2$ matrices with spectra inside Δ_+.

Let U and T be the unique solutions to the Lyapunov equations

$$A_2^\times U G_1^\times - G_2^\times U A_1^\times = -B_2 D^{-1} C_1, \tag{1.4}$$

and

$$A_1TG_2 - G_1TA_2 = B_1D^{-1}C_2. \tag{1.5}$$

Then W has a right canonical Wiener-Hopf factorization if and only if the $n_1 \times n_1$ matrix $I_{n_1} - (\alpha G_1 - A_1)T(\alpha G_2 - A_2)U$ is invertible, or equivalently, if and only if the $n_2 \times n_2$ matrix $I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T$ is invertible, or equivalently, if and only if the $n_1 \times n_1$ matrix $I_{n_1} - T(\alpha G_2 - A_2)U(\alpha G_1 - A_1)$ is invertible, or equivalently, if and only if the $n_2 \times n_2$ matrix $I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)$ is invertible.

In this case, the factors $W_-(\lambda)$ and $W_+(\lambda)$ for a right canonical Wiener-Hopf factorization

$$W(\lambda) = W_-(\lambda)W_+(\lambda)$$

are given by the formulas

$$\begin{aligned} W_-(\lambda) &= D + (\lambda - \alpha)[C_1T(\alpha G_2 - A_2) + C_2](\lambda G_2 - A_2)^{-1} \\ &\cdot[I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T]^{-1}[(A_2 - \alpha G_2)UB_1 + B_2], \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} W_+(\lambda) &= I_m + (\lambda - \alpha)D^{-1}[C_1 + C_2U(\alpha G_1 - A_1)][I_{n_1} - T(\alpha G_2 - A_2)U(\alpha G_1 - A_1)]^{-1} \\ &\cdot(\lambda G_1 - A_1)^{-1}[B_1 + (A_1 - \alpha G_1)TB_2]. \end{aligned} \tag{1.7}$$

Their inverses are given by

$$\begin{aligned} W_-(\lambda)^{-1} &= D^{-1} - (\lambda - \alpha)D^{-1}[C_1T(\alpha G_2 - A_2) + C_2][I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)]^{-1} \\ &\cdot(\lambda G_2^\times - A_2^\times)^{-1}[(A_2 - \alpha G_2)UB_1 + B_2]D^{-1}, \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} W_+(\lambda)^{-1} &= I_m - (\lambda - \alpha)D^{-1}[C_1 + C_2U(\alpha G_1 - A_1)](\lambda G_1^\times - A_1^\times)^{-1} \\ &\cdot[I_{n_1} - (\alpha G_1 - A_1)T(\alpha G_2 - A_2)U]^{-1}[B_1 + (A_1 - \alpha G_1)TB_2] \end{aligned} \tag{1.9}$$

.

Proof. From the realizations (1.2) and (1.3) we compute a realization for their product

$$W(\lambda) = Y_{+}(\lambda) Y_{-}(\lambda)$$

as $W(\lambda) = D + (\lambda - \alpha) C (\lambda G - A)^{-1} B$,

where

$$A = \begin{pmatrix} A_1 & -\alpha B_1 D^{-1} C_2 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

$$G = \begin{pmatrix} G_1 & -B_1 D^{-1} C_2 \\ 0 & G_2 \end{pmatrix}, \qquad C = \begin{pmatrix} C_1 & C_1 \end{pmatrix}.$$

From this we have that

$$A^{\times} = A + \alpha B D^{-1} C = \begin{pmatrix} A_1^{\times} & 0 \\ \alpha B_2 D^{-1} C_1 & A_2^{\times} \end{pmatrix},$$

where $A_1^{\times} := A_1 + \alpha B_1 D^{-1} C_1$ and $A_2^{\times} := A_2 + \alpha B_2 D^{-1} C_2$, and

$$G^{\times} := G + B D^{-1} C = \begin{pmatrix} G_1^{\times} & 0 \\ B_2 D^{-1} C_1 & G_2^{\times} \end{pmatrix},$$

where $G_1^{\times} := G_1 + B_1 D^{-1} C_1$ and $G_2^{\times} := G_2 + B_2 D^{-1} C_2$.

Now, by assumption the spectrum, $\sigma(G_1, A_1)$, of $\lambda G_1 - A_1$ is contained in Δ_- , while that of $\lambda G_2 - A_2$ is contained in Δ_+ . From the triangular form of $\lambda G - A$ we see that $\sigma(G, A) = \sigma(G_1, A_1) \cup \sigma(G_2, A_2)$ and that the spectral subspace for $\lambda G - A$ associated with Δ_- must be im $\begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix}$. The spectral subspaces η and θ for $\lambda G - A$ corresponding to Δ_+ is determined by the fact that they must be complementary to the spectral subspace im $\begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix}$ for Δ_- , and that

$(\lambda G - A)\eta \subset \theta$. These conditions force η to have the form

$$\begin{aligned}
\eta &= \operatorname{im} \frac{1}{2\pi i} \int_{\gamma_2} G(\lambda G - A)^{-1} d\lambda \\
&= \operatorname{im} \frac{1}{2\pi i} \int_{\gamma_2} \begin{pmatrix} G_1(\lambda G_1 - A_1)^{-1} & -(\alpha G_1 - A_1)(\lambda G_1 - A_1)^{-1} B_1 D^{-1} C_2 (\lambda G_2 - A_2)^{-1} \\ 0 & G_2(\lambda G_2 - A_2)^{-1} \end{pmatrix} d\lambda \\
&= \operatorname{im} \begin{pmatrix} (\alpha G_1 - A_1)T \\ I_{n_2} \end{pmatrix},
\end{aligned}$$

for some $n_1 \times n_2$ matrix T, which is the solution of the Lyapunov equation (1.5), and of the form

$$T = -\frac{1}{2\pi i} \int_{\gamma_2} (\lambda G_1 - A_1)^{-1} B_1 D^{-1} C_2 (\lambda G_2 - A_2)^{-1} d\lambda,$$

where γ_2 is a Cauchy contour around $\sigma(G_2, A_2)$ which separates $\sigma(G_2, A_2)$ from $\sigma(G_1, A_1)$. Also, from our assumption that the spectra of $\lambda G_1 - A_1$ and $\lambda G_2 - A_2$ are disjoint, it follows that T is a unique solution of (1.5) (see Theorem 1.3.1). In a similar way, we have that

$$\begin{aligned}
\theta &= \operatorname{im} \frac{1}{2\pi i} \int_{\gamma_2} (\lambda G - A)^{-1} G d\lambda \\
&= \operatorname{im} \begin{pmatrix} T(\alpha G_2 - A_2) \\ I_{n_2} \end{pmatrix}.
\end{aligned}$$

We have thus identified the spectral subspaces η and θ of $\lambda G - A$ for Δ_+ as $\eta = \operatorname{im} \begin{pmatrix} (\alpha G_1 - A_1)T \\ I_{n_2} \end{pmatrix}$ and $\theta = \operatorname{im} \begin{pmatrix} T(\alpha G_2 - A_2) \\ I_{n_2} \end{pmatrix}$, where T is the unique solution of (1.5).

Since, by assumption, $\lambda G_1^\times - A_1^\times$ has its spectrum in Δ_-, while $\lambda G_2^\times - A_2^\times$ has its spectrum in Δ_+ , the same analysis applies to $\lambda G^\times - A^\times$.

We see that the spectral subspaces of $\lambda G^\times - A^\times$ for Δ_- are the spaces

$$\eta^\times = \operatorname{im} \begin{pmatrix} I_{n_1} \\ (\alpha G_2 - A_2)U \end{pmatrix}$$

and

$$\theta^\times = \operatorname{im} \begin{pmatrix} I_{n_1} \\ U(\alpha G_1 - A_1) \end{pmatrix},$$

for the $n_2 \times n_1$ matrix U, which is the unique solution of the Lyapunov equation (1.4), and is of the form

$$U = \frac{1}{2\pi i} \int_{\gamma_1} (\lambda G_2^\times - A_2^\times)^{-1} B_2 D^{-1} C_1 (\lambda G_1^\times - A_1^\times)^{-1} d\lambda \,,$$

where γ_1 is a Cauchy contour around $\sigma(G_1, A_1)$ which separates $\sigma(G_1, A_1)$ from $\sigma(G_2, A_2)$.

Applying Theorem 1.2.4, we have that the matrix function W has a right canonical Wiener-Hopf factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$ if and only if $\mathbb{C}^{n_1+n_2} = \eta \oplus \eta^\times$ or $\mathbb{C}^{n_1+n_2} = \theta \oplus \theta^\times$, that is, if and only if

$$\mathbb{C}^{n_1+n_2} = \operatorname{im} \begin{pmatrix} (\alpha G_1 - A_1)T \\ I_{n_2} \end{pmatrix} \oplus \operatorname{im} \begin{pmatrix} I_{n_1} \\ (\alpha G_2 - A_2)U \end{pmatrix}$$

or

$$\mathbb{C}^{n_1+n_2} = \operatorname{im} \begin{pmatrix} T(\alpha G_2 - A_2) \\ I_{n_2} \end{pmatrix} \oplus \operatorname{im} \begin{pmatrix} I_{n_1} \\ U(\alpha G_1 - A_1) \end{pmatrix},$$

respectively. One easily checks that these direct sum decompositions hold if and only if the square matrices

$$\begin{pmatrix} I_{n_1} & (\alpha G_1 - A_1)T \\ (\alpha G_2 - A_2)U & I_{n_2} \end{pmatrix} \tag{1.10}$$

or

$$\begin{pmatrix} I_{n_1} & T(\alpha G_2 - A_2) \\ U(\alpha G_1 - A_1) & I_{n_2} \end{pmatrix} \tag{1.11}$$

are invertible. We consider the case (1.10). By standard row and column operations this matrix can be diagonalized in either of two ways:

$$\begin{pmatrix} I_{n_1} & (\alpha G_1 - A_1)T \\ (\alpha G_2 - A_2)U & I_{n_2} \end{pmatrix}$$

$$= \begin{pmatrix} I_{n_1} & (\alpha G_1 - A_1)T \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} I_{n_1} - (\alpha G_1 - A_1)T(\alpha G_2 - A_2)U & 0 \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ (\alpha G_2 - A_2)U & I_{n_2} \end{pmatrix}$$

$$= \begin{pmatrix} I_{n_1} & 0 \\ (\alpha G_2 - A_2)U & I_{n_2} \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T \end{pmatrix} \begin{pmatrix} I_{n_1} & (\alpha G_1 - A_1)T \\ 0 & I_{n_2} \end{pmatrix}.$$

Thus we see that the invertibility of the matrix in (1.10) is equivalent to the invertibility of $I_{n_1} - (\alpha G_1 - A_1)T(\alpha G_2 - A_2)U$ and also to the invertibility of $I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T$.

Similarly, we may show that the invertibility of the matrix in (1.11) is equivalent to the invertibility of $I_{n_1} - T(\alpha G_2 - A_2)U(\alpha G_1 - A_1)$ and also to the invertibility of $I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)$.

Now suppose that this necessary and sufficient condition for the existence of a right canonical Wiener-Hopf factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$ holds. Next, we compute explicit formulas for the right factors $W_+(\lambda)$ and $W_-(\lambda)$ and their inverses.

Let ρ be the projection of $\mathbb{C}^{n_1+n_2}$ onto $\eta^\times = \operatorname{im} \begin{pmatrix} I_{n_1} \\ (\alpha G_2 - A_2)U \end{pmatrix}$ along $\eta = \operatorname{im} \begin{pmatrix} (\alpha G_1 - A_1)T \\ I_{n_2} \end{pmatrix}$.

We compute easily that

$$\rho = \begin{pmatrix} I_{n_1} \\ (\alpha G_2 - A_2)U \end{pmatrix} [I_{n_1} - (\alpha G_1 - A_1)T(\alpha G_2 - A_2)U]^{-1}(I_{n_1} \quad (A_1 - \alpha G_1)T),$$

and that

$$I-\rho=\begin{pmatrix}(\alpha G_1-A_1)T\\ I_{n_2}\end{pmatrix}[I_{n_2}-(\alpha G_2-A_2)U(\alpha G_1-A_1)T]^{-1}((A_2-\alpha G_2)U \quad I_{n_2}).$$

Also, if we let τ be the projection of $\mathbb{C}^{n_1+n_2}$ onto $\theta^{\times}=\mathrm{im}\begin{pmatrix}I_{n_1}\\ U(\alpha G_1-A_1)\end{pmatrix}$ along

$\theta=\mathrm{im}\begin{pmatrix}T(\alpha G_2-A_2)\\ I_{n_2}\end{pmatrix}$, we have that

$$\tau=\begin{pmatrix}I_{n_1}\\ U(\alpha G_1-A_1)\end{pmatrix}[I_{n_1}-T(\alpha G_2-A_2)U(\alpha G_1-A_1)]^{-1}(I_{n_1} \quad T(A_2-\alpha G_2)),$$

and that

$$I-\tau=\begin{pmatrix}T(\alpha G_2-A_2)\\ I_{n_2}\end{pmatrix}[I_{n_2}-U(\alpha G_1-A_1)T(\alpha G_2-A_2)]^{-1}(U(A_1-\alpha G_1) \quad I_{n_2}).$$

Assume that $W_-(\infty)=D$. Then, from Corollary 1.2.5, we have that the formulas for the right canonical spectral factors of W are

$$W_-(\lambda)=D+(\lambda-\alpha)C(\lambda G-A)^{-1}(I-\rho)B, \tag{1.12}$$

and

$$W_+(\lambda)=I+(\lambda-\alpha)D^{-1}C\tau(\lambda G-A)^{-1}B. \tag{1.13}$$

From formula (1.12), the matrix representations introduced earlier, and the Lyapunov equation (1.5) we have that

$$W_-(\lambda) \;=\; D+(\lambda-\alpha)(C_1 \;\; C_2)\begin{pmatrix}(\lambda G_1-A_1)^{-1} & (\lambda-\alpha)(\lambda G_1-A_1)^{-1}B_1D^{-1}C_2(\lambda G_2-A_2)^{-1}\\ 0 & (\lambda G_2-A_2)^{-1}\end{pmatrix}$$

$$\cdot \begin{pmatrix} (\alpha G_1 - A_1)T \\ I_{n_2} \end{pmatrix} [I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T]^{-1}((A_2 - \alpha G_2)U \quad I_{n_2}) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

$$\begin{aligned} &= D + (\lambda - \alpha)[C_1(\lambda G_1 - A_1)^{-1}(\alpha G_1 - A_1)T(\lambda G_2 - A_2) + (\lambda - \alpha)C_1(\lambda G_1 - A_1)^{-1}B_1 D^{-1} \\ &\quad \cdot C_2 + C_2](\lambda G_2 - A_2)^{-1}[I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T]^{-1}[(A_2 - \alpha G_2)UB_1 + B_2] \\ &= D + (\lambda - \alpha)[C_1(\lambda G_1 - A_1)^{-1}\{\lambda\alpha G_1 T G_2 + A_1 T A_2 - \lambda G_1 T A_2 - \alpha A_1 T G_2\} + C_2] \\ &\quad \cdot(\lambda G_2 - A_2)^{-1}[I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T]^{-1}[(A_2 - \alpha G_2)UB_1 + B_2] \\ &= D + (\lambda - \alpha)[C_1 T(\alpha G_2 - A_2) + C_2](\lambda G_2 - A_2)^{-1}[I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T]^{-1} \\ &\quad \cdot[(A_2 - \alpha G_2)UB_1 + B_2]. \end{aligned}$$

Similarly, from formula (1.13), the matrix representations introduced earlier and the Lyapunov equation (1.5) we have that

$$\begin{aligned} W_+(\lambda) &= I_m + (\lambda - \alpha)D^{-1}[C_1 + C_2U(\alpha G_1 - A_1)][I_{n_1} - T(\alpha G_2 - A_2)U(\alpha G_1 - A_1)]^{-1} \\ &\quad \cdot[B_1 + (A_1 - \alpha G_1)TB_2] \end{aligned}$$

Next, we calculate the inverses $W_-(\lambda)^{-1}$ and $W_+(\lambda)^{-1}$. As we noted earlier, the inverse formulas for (1.12) and (1.13) are given by

$$W_-(\lambda)^{-1} = D^{-1} - (\lambda - \alpha)D^{-1}C(I - \tau)(\lambda G^\times - A^\times)^{-1}BD^{-1} \tag{1.14}$$

and

$$W_+(\lambda)^{-1} = I_m - (\lambda - \alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}\rho B, \tag{1.15}$$

respectively. From formula (1.14), the matrix representations introduced earlier and the Lyapunov equation (1.4) we have that

$$W_-(\lambda)^{-1} = D^{-1} - (\lambda - \alpha)D^{-1}(C_1 \quad C_2) \begin{pmatrix} T(\alpha G_2 - A_2) \\ I_{n_2} \end{pmatrix}$$

$$
\begin{aligned}
&\cdot[I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)]^{-1}(U(A_1 - \alpha G_1) \quad I_{n_2}) \\
&\cdot\begin{pmatrix} (\lambda G_1^\times - A_1^\times)^{-1} & 0 \\ (\alpha-\lambda)(\lambda G_2^\times - A_2^\times)^{-1}B_2D^{-1}C_1(\lambda G_1^\times - A_1^\times)^{-1} & (\lambda G_2^\times - A_2^\times)^{-1} \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} D^{-1} \\
= \;& D^{-1} - (\lambda-\alpha)D^{-1}[C_1T(\alpha G_2 - A_2) + C_2][I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)]^{-1} \\
&\cdot(\lambda G_2^\times - A_2^\times)^{-1}[(\lambda G_2^\times - A_2^\times)U(A_1^\times - \alpha G_1^\times)(\lambda G_1^\times - A_1^\times)^{-1}B_1 \\
&+(\alpha-\lambda)B_2D^{-1}C_1(\lambda G_1^\times - A_1^\times)^{-1}B_1 + B_2]D^{-1} \\
= \;& D^{-1} - (\lambda-\alpha)D^{-1}[C_1T(\alpha G_2 - A_2) + C_2][I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)]^{-1} \\
&\cdot(\lambda G_2^\times - A_2^\times)^{-1}[\{-\lambda\alpha G_2^\times U G_1^\times - A_2^\times U A_1^\times + \alpha G_2^\times U A_1^\times + \lambda A_2^\times U G_1^\times\} \\
&\cdot(\lambda G_1^\times - A_1^\times)^{-1}B_1 + B_2]D^{-1} \\
= \;& D^{-1} - (\lambda-\alpha)D^{-1}[C_1T(\alpha G_2 - A_2) + C_2][I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)]^{-1} \\
&\cdot(\lambda G_2^\times - A_2^\times)^{-1}[(A_2 - \alpha G_2)UB_1 + B_2]D^{-1}.
\end{aligned}
$$

Similarly, using formula (1.15), we have that

$$
\begin{aligned}
W_+(\lambda)^{-1} \;=\;& I_m - (\lambda-\alpha)D^{-1}[C_1 + C_2U(\alpha G_1 - A_1)](\lambda G_1^\times - A_1^\times)^{-1} \\
&\cdot[I_{n_1} - (\alpha G_1 - A_1)T(\alpha G_2 - A_2)U]^{-1}[B_1 + (A_1 - \alpha G_1)TB_2].
\end{aligned}
$$

This completes the proof. □

The main result above gives a necessary and sufficient condition for a right canonical Wiener-Hopf factorization to exist under the assumption that factors of a left canonical Wiener-Hopf factorization are given in realized form (1.1).

If we suppose that this condition is not met, i.e., $I_{n_1} - (\alpha G_1 - A_1)T(\alpha G_2 - A_2)U$, $I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T$, $I_{n_1} - T(\alpha G_2 - A_2)U(\alpha G_1 - A_1)$ and $I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)$

fail to be invertible, the right factorization is not canonical and an analysis of the right factorization indices of $W(\lambda)$ becomes imperative. However, this case will be the subject of further investigation.

The fact that $W_-(\infty) = D$ allows us to apply the latter part of Theorem 1.2.4 directly. The method of proof of our theorem, differs from the one given in [BR], in that it involves the direct computation of the appropriate Riesz projections and corresponding spectral subspaces. Note that the resulting explicit formulas for $W_-(\lambda)$ and $W_+(\lambda)$ and their inverses are also represented in realized form (1.1).

In the last part of this section, we provide a one-dimensional example to illustrate some of the key concepts in Theorem 1.1.

Let the Cauchy contour be the real axis of the complex plane $\mathbb{C}$, with the inner domain as the upper half-plane and the outer domain as the lower half-plane. We make the following choices for the matrices appearing in the statement of the theorem:

$$C_1 = C_2 = G_1 = G_2 = B_1 = B_2 = I_m = D = 1,\ A_1 = -i \text{ and } A_2 = i.$$

From the above we have that

$$W(\lambda) = 1 + \frac{(\lambda-\alpha)(3\lambda-\alpha)}{(\lambda+i)(\lambda-i)}.$$

Also, from the above we deduce that $G_1^\times = G_2^\times = 2,\ A_1^\times = \alpha - i,\ A_2^\times = \alpha + i.$

Moreover, the solutions of the Lyapunov equations are given by $U = \frac{i}{4}$ and $T = \frac{i}{2}$.

For convenience, we choose $\alpha = 1$. Then, the left canonical factors are given by:

$$Y_+(\lambda) = 1 + (\lambda-1)(\lambda+i)^{-1}, \quad Y_-(\lambda) = 1 + (\lambda-1)(\lambda-i)^{-1}$$

which have no poles or zeroes on the upper and lower half-planes, respectively.

Furthermore, by replacing the appropriate values in the formulas for $W_+(\lambda)$, $W_-(\lambda)$ and their inverses we have that:

$$W_+(\lambda) = 1 + (\lambda - 1)(\lambda + i)^{-1}, \quad W_-(\lambda) = 1 + (\lambda - 1)(\lambda - i)^{-1}$$

and

$$W_+(\lambda)^{-1} = 1 - (\lambda - 1)(2\lambda - 1 + i)^{-1}, \quad W_-(\lambda)^{-1} = 1 - (\lambda - 1)(2\lambda - 1 - i)^{-1}.$$

2.2 APPLICATIONS TO SINGULAR INTEGRAL OPERATORS

In the sequel, we apply the main factorization theorem derived in the previous section in order to determine necessary and sufficient conditions for the invertibility of a singular integral operator with a rational symbol. For p fixed, $1 < p < \infty$, we denote by $L_p^n(\gamma)$ the Banach space of all $\mathbb{C}^n$-valued functions which are p-integrable (w.r.t. Lebesque measure) on the Cauchy contour γ in $\mathbb{C}$. As is usual in the theory of singular integral operators, we assume that the inner domain Δ_+ of γ contains 0, while the outer domain Δ_- of γ contains ∞.

Consider the operator of singular integration,

$$S_\gamma \;: L_p^n(\gamma) \;\rightarrow\; L_p^n(\gamma) \text{ on } \gamma,$$

given by

$$(S_\gamma \phi)(\lambda) = \frac{1}{\pi i} \int_\gamma \frac{\phi(\tau)}{\tau - \lambda} d\tau \qquad , \lambda \in \gamma,$$

where the integral is taken in the sense of the Cauchy principal value and ϕ is a rational function without poles on γ. Note that the operator S_γ has the property that $S_\gamma^2 = I$. Introduce the operators

$$P_\gamma = \frac{1}{2}(I + S_\gamma) \text{ and } Q_\gamma = \frac{1}{2}(I - S_\gamma).$$

It is clear that P_γ and Q_γ are complementary projections on $L_p^n(\gamma)$, i.e. $P_\gamma^2 = P_\gamma$, $Q_\gamma^2 = Q_\gamma$ and $P_\gamma + Q_\gamma = I$.

Next, we consider the singular integral operator

$$S \ : L_p^n(\gamma) \ \ \rightarrow \ \ L_p^n(\gamma) \text{ given by}$$

$$(S\phi)(\lambda) = A(\lambda)(P_\gamma\phi)(\lambda) + B(\lambda)(Q_\gamma\phi)(\lambda), \tag{2.1}$$

where $A(\lambda)$ and $B(\lambda)$ are rational matrix functions without poles or zeroes on γ.

The symbol of S is the function $W(\lambda) = B(\lambda)^{-1}A(\lambda)$ (see, e.g., [CG], Section 1.3). From [CG] we know that S is invertible if and only if $W(\lambda)$ admits a right canonical factorization

$$W(\lambda) = W_-(\lambda)W_+(\lambda), \tag{2.2}$$

in which case

$$(S^{-1}\phi)(\lambda) = W_+^{-1}(\lambda)(P_\gamma W_-^{-1}B^{-1}\phi)(\lambda) + W_-(\lambda)(Q_\gamma W_-^{-1}B^{-1}\phi)(\lambda). \tag{2.3}$$

We may use Theorem 1.1 to investigate the invertibility of S in terms of either one of the following operators:

$$(S_1\phi)(\lambda) = B(\lambda)(P_\gamma\phi)(\lambda) + A(\lambda)(Q_\gamma\phi)(\lambda),$$

$$(S_2\phi)(\lambda) = [B(\lambda)^{-1}]^T(P_\gamma\phi)(\lambda) + [A(\lambda)^{-1}]^T(Q_\gamma\phi)(\lambda).$$

Note that the symbol of S_1 is $W(\lambda)^{-1}$ and that of S_2 is $W(\lambda)^T$. We may formulate the following theorems, which may be proved by considering the remarks above and Theorem 1.1.

Theorem 2.1 *Assume that S_1 is invertible and let the right Wiener-Hopf factorization of the symbol of S_1 be given by*

$$W(\lambda)^{-1} = A(\lambda)^{-1}B(\lambda) = Y_-(\lambda)^{-1}Y_+(\lambda)^{-1},$$

where

$$Y_-(\lambda)^{-1} = I_m - (\lambda - \alpha)D^{-1}C_2(\lambda G_2^\times - A_2^\times)^{-1}B_2,$$

and

$$Y_+(\lambda)^{-1} = D^{-1} - (\lambda - \alpha)D^{-1}C_1(\lambda G_1^\times - A_1^\times)^{-1}B_1D^{-1}.$$

Set $G_i = G_i^\times - B_iD^{-1}C_i, \quad (i = 1, 2)$, and $A_i = A_i^\times - \alpha B_iD^{-1}C_i, \quad (i = 1, 2)$.
Let U and T be the unique solutions of the Lyapunov equations

$$A_2^\times UG_1^\times - G_2^\times UA_1^\times = -B_2D^{-1}C_1, \tag{2.4}$$

and

$$A_1TG_2 - G_1TA_2 = B_1D^{-1}C_2, \tag{2.5}$$

respectively.
Then S is invertible if and only if $I_{n_1} - (\alpha G_1 - A_1)T(\alpha G_2 - A_2)U$ is invertible, or equivalently, if and only if
$I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T$ is invertible, or equivalently, if and only if
$I_{n_1} - T(\alpha G_2 - A_2)U(\alpha G_1 - A_1)$ is invertible, or equivalently, if and only if
$I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)$ is invertible.

Also, we have the following result.

Theorem 2.2 *Assume that S_2 is invertible and let the right canonical Wiener-Hopf factorization of the symbol of S_2 be given by*

$$W(\lambda)^T = A(\lambda)^T B^{-1}(\lambda)^T = Y_-(\lambda)^T Y_+(\lambda)^T,$$

where

$$Y_-(\lambda)^T = D^T + (\lambda - \alpha)B_1^T(\lambda G_1^T - A_1^T)^{-1}C_1^T,$$

and

$$Y_+(\lambda)^T = I_m + (\lambda - \alpha)B_2^T(\lambda G_2^T - A_2^T)^{-1}C_2^T(D^{-1})^T.$$

Set $G_i^\times = G_i + B_iD^{-1}C_i \ \ ,(i = 1,2)$, *and* $A_i^\times = A_i + \alpha B_iD^{-1}C_i \ \ ,(i = 1,2)$.
Let U *and* T *be the unique solutions to the Lyapunov equations* (2.4) *and* (2.5), *respectively. Then* S *is invertible if and only if* $I_{n_1} - (\alpha G_1 - A_1)T(\alpha G_2 - A_2)U$ *is invertible, or equivalently, if and only if* $I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T$ *is invertible, or equivalently, if and only if* $I_{n_1} - T(\alpha G_2 - A_2)U(\alpha G_1 - A_1)$ *is invertible, or equivalently, if and only if* $I_{n_2} - U(\alpha G_1 - A_1)T(\alpha G_2 - A_2)$ *is invertible.*

In the two theorems above, the formulas for the factors W_- and W_+ in the canonical factorization (2.2) of the symbol of S and the formulas for their inverses are given by (1.6) – (1.9) in the previous section. In this case, we have that (2.3) gives an explicit formula for the inverse S^{-1}. Also, we may reformulate Theorems 2.1 and 2.2 entirely in terms of S and its symbol $W(\lambda)$. In this regard, if $W(\lambda)$ admits a left canonical Wiener-Hopf factorization $W(\lambda) = Y_+(\lambda)Y_-(\lambda)$ with factors Y_+ and Y_- as given by (1.2) and (1.3) then the invertibility of S is equivalent to the invertibility of $I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T$ where U and T are the unique solutions of (2.4) and (2.5), respectively. Indeed, from [BGK3] we know that $I_{n_2} - (\alpha G_2 - A_2)U(\alpha G_1 - A_1)T$ is an indicator for the singular integral operator S_1 as well as for the Toeplitz operator with symbol W. Also, we have from Theorem III.2.2 in [BGK3], that an indicator for S is given by the operator

$$\hat{P}^\times |_{\mathrm{im}\ \hat{P}} : \mathrm{im}\ \hat{P} \ \ \rightarrow \ \ \mathrm{im}\ \hat{P}^\times,$$

where $\hat{P}$ (resp $\hat{P}^\times$) is the generalized Riesz projection of $\lambda G - A$ (resp $\lambda G^\times - A^\times$) corresponding to Δ_+, where $\lambda G - A$ and $\lambda G^\times - A^\times$ are derived from the realization of W. Remember, here, we consider generalized Riesz projections of the form

$$P(G,A,\gamma)=\frac{1}{2\pi i}\int_{\gamma}G(\lambda G-A)^{-1}d\lambda \quad ,$$

which, in terms of the notation adopted in Theorem 1.1, means that

$$\hat{P}=\frac{1}{2\pi i}\int_{\gamma_2}G(\lambda G-A)^{-1}d\lambda \quad ,$$

and

$$\hat{P}^{\times}=\frac{1}{2\pi i}\int_{\gamma_2}G^{\times}(\lambda G^{\times}-A^{\times})^{-1}d\lambda,$$

where γ_2 is a Cauchy contour around $\sigma(G_2,A_2)$ (contained in Δ_+), which separates $\sigma(G_2,A_2)$ from $\sigma(G_1,A_1)$.

It is easily seen (from the proof of Theorem 1.1) that

$$\text{im }\hat{P}=\text{im}\begin{pmatrix}(\alpha G_1-A_1)T\\ I_{n_2}\end{pmatrix},$$

and

$$\hat{P}^{\times}=\begin{pmatrix}0 & 0\\ (A_2-\alpha G_2)U & I_{n_2}\end{pmatrix}.$$

It follows that $\hat{P}^{\times}\mid_{\text{im }\hat{P}}$ is given by $I_{n_2}-(\alpha G_2-A_2)U(\alpha G_1-A_1)T$. A similar analysis may be done for the cases where the invertibility of S is equivalent to the invertibility of $I_{n_1}-(\alpha G_1-A_1)T(\alpha G_2-A_2)U$, $I_{n_2}-U(\alpha G_1-A_1)T(\alpha G_2-A_2)$ and $I_{n_1}-T(\alpha G_2-A_2)U(\alpha G_1-A_1)$.

Further applications of our main theorem to spectral and antispectral factorization on the unit circle and symmetrized canonical spectral factorization on the imaginary axis will not be considered here.

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LIST OF SYMBOLS

$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{C}_\infty$	Riemann sphere $\mathbb{C} \cup \{\infty\}$
$\mathbb{T}$	the unit circle
$\mathbb{D}$	the unit disc
γ	Cauchy contour in $\mathbb{C}$
σ	non-empty subset of the complex plane
Δ_+	inner domain of γ
Δ_-	outer domain of γ
$\ker A$	kernel (nullspace) of the operator A
$\operatorname{im} A$	image (range) of the operator A
A^{-1}	inverse of the operator A
$A\mid_X$	restriction of the operator A to the set X
$I_X,\ I_m$	$m \times m$ identity matrix
$\sigma(G, A)$	spectrum of the operator pencil $\lambda G - A$
$\rho(G, A)$	resolvent of the operator pencil $\lambda G - A$
$\operatorname{diag}\ (\lambda_j)_{j=1}^m$	$m \times m$ diagonal matrix with diagonal entries λ_1 up to λ_m
$X \oplus Y$	direct sum of the linear spaces X and Y
η	generalized spectral subspace
$\mathbb{C}^m$	Euclidean space of dimension m over the field $\mathbb{C}$
$\mathcal{L}(X)$	class of bounded linear operators on a space X
$L_p^m(\gamma)$	space of $\mathbb{C}^m$ -valued p-integrable functions on γ
$\mathcal{W}^{m\times m}$	$m \times m$ matrix Wiener algebra

SUMMARY

In this dissertation we have applied the state space method to construct a right canonical Wiener-Hopf factorization of a rational matrix function explicitly from the representation of a matrix function in realization form. A rational matrix function W, which is analytic and invertible at infinity, may be represented in the form

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \tag{1}$$

where A is a $n \times n$ square matrix, say, B and C are $n \times m$ and $m \times n$ matrices, respectively, and D is an invertible $m \times m$ matrix. The process of constructing the factorization and determining explicit formulas for the factors is well known for rational matrix functions in the form (1). However, in our discussion, we have concentrated on the situation where W does not have these properties at infinity and has a realization of the form

$$W(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \tag{2}$$

where A, B, C and D are as above and G is of the same order as A. In the main result in Chapter 2, we have established necessary and sufficient conditions for the existence of a right canonical Wiener-Hopf factorization in terms of a left canonical Wiener-Hopf factorization and the unique solutions of generalized Lyapunov equations. In addition, we have shown that the explicit formulas (in realized form(2)) for the right canonical factors and their inverses may be written in terms of the formulas for the left canonical factors. In the proof of this result, we made extensive use of the Riesz theory associated with the decomposition of the spectrum of $\lambda G - A$ into two disjoint closed subsets. Finally, we apply this result to singular integral operators; while brief mention is also made of Toeplitz operators.