A categorical study of persistent homology for closure space

A thesis submitted by

Bria Aimé Razanaparany

in partial fulfillment of the requirements for the degree of
Master of Science

Department of Mathematics and Applied Mathematics
University of the Western Cape

Supervisor
David Holgate

Co-Supervisor
Ando Razafindrakoto

October 2023

ABSTRACT

We begin with the concept of closure space and show how it relates to other objects like graphs and metric spaces in categorical terms. We also study the notion of simplicial sets and show that there is an adjunction between the category of simplicial sets and the category of closure spaces. We will define the homology and cohomology of simplicial sets and apply that treatment to the construction of various homologies and cohomologies for closure space. Moreover, we also present an investigation of the Dold-Kan correspondence theorem. Finally, we will focus on the categorical foundation of persistent homology and promote a general formalization of a stability theorem; both of these are at the heart of topological data analysis.

ACKNOWLEDGEMENT

I would like to express my special appreciation to my supervisor David Holgate, for his invaluable advice and for giving me the liberty to work in this field. Thanks also for offering me financial support from the UWC Research Chair he holds for this master. My sincere gratitude also goes to my co-supervisor Ando Razafindrakoto, for some insightful discussion in category theory and his support not only academically but for the many things he has done for me. I consider myself very lucky to have met both of you.

I would like to thank all Topology for Tomorrow group members for the regular seminars, some questions and constructive criticism that help me to improve my understanding. In addition, I also want to thank those who made my stay in Cape Town wonderful.

Misaotra an'i Dada sy Neny ary ireo fianakaviana sy namana akaiky izay manohana sy mankahery hatrany.

"MISAOTRA INDRINDRA".

Contents

Chapter 1

Introduction

From the beginning, algebraic topology aims to distinguish topological spaces and continuous maps between them up to homeomorphism. In classical algebraic topology, we associate with a space an algebraic object that is invariant under continuous deformation, which is called *algebraic invariant*, for instance: homology group, betti numbers, and homotopy group. We then use those algebraic invariants to understand the topological properties of a space. Algebraic topology is then an interplay in the construction and use of functors between the category of topological spaces and the category of groups. With numerous advances made by various mathematicians, algebraic topology has been playing an increasingly important role not in theory but also in data analysis. What we call topological data analysis (TDA) is a novel tool and algorithm for the analysis and visualization of data, that uses mostly the techniques of persistent homology, which are, of course, based on homology theory. An algebraic invariant plays an important role in TDA; it has significant meaning about the shape of the data.

Overall, the importance of this work is threefold:

We will begin with the fundamental concepts of closure spaces, including the correspondence between closure spaces, interior spaces, and neighborhood spaces. In addition, we will investigate how far a closure space can be related to other objects like graphs, digraphs, and metric spaces. Most of the definitions and results have been introduced by Čech in [ČFK66] while the categorical formalization has been presented later in [DT95] and [BM22].

In Chapter 3, we will specialize in the theoretical aspects of simplicial sets. A classical method to study a topological space is to approximate or decompose it into a block of simplices: points, segments, triangles, and so on. In 1950, Samuel Eilenberg and Joseph Zilbert introduced the theory of simplicial sets, which can be thought of as an abstraction of the triangulation of a space. Therefore, in Chapter 3, we will focus on these theoretical aspects. We will see that every simplicial set gives rise to a closure space and vice versa, which leads to an adjunction between those corresponding categories. Furthermore, the main core of this chapter will be the construction of various homologies and cohomologies for closure spaces using simplicial set theory. Singular homology and homotopy for closure spaces has been introduced by Davide Carlo and Bogin, Garbaccio Rosanna in [Bog84]. Peter Bubenik and Nickola Milićević redevelop that idea by defining a various singular and cubical homology theory. We then propose an explicit construction of singular (co)homology and Čech (co)homology of closure spaces using the homology of simplicial set, following the approach given in [GM96]. We will close this chapter by investigating the Dold-Kan correspondence theorem, which states that there exists an equivalence between the category of simplicial abelian groups and the category of non-negative chain complex. We will follow mainly the approach given in [GJ09] [Wei94] [BM21] and [GM96] along this chapter.

The last chapter will be devoted mostly to the theoretical foundations of topological data analysis. We will begin with a categorical definition of the basic material in topological data analysis. We will see what kind of mathematical object we can use to encode the shape of the data. After that, we give a general framework of the methods of persistent homology as well as the stability theorem, which are considered the cores of topological data analysis. In addition, it is worthwhile to know that persistent homology can be applied in data analysis after the first version of the stability theorem was proved by Cohen-Steiner, Edelsbrunner and Harer in [CEH07]. We then end this chapter by showing the stability theorem in the generalized case. The stability theorem guarantees the effectiveness of the method of persistence homology and especially the persistence diagram. The framework given along this chapter will be generalized in the category of Čech closure spaces, but we will show clearly some useful particular cases of topological space and simplicial complex. We refer mostly to [Cha+16], [EH10], [BS14] and [BM22] in this chapter.

The last part of this work is an appendix that is devoted to a brief introduction of category theory. The reader who is familiar with category theory can ignore this part, and those who are not familiar should read first before starting Chapter 2. We will recall some basic definitions of category, functor, and natural transformation. After that, we give an overview of the Yoneda lemma, (co)limit, adjunction and the notion of the coreflective category. It is extremely important to note that we only recall in this appendix what we need along with this work, so this is never enough to cover the background of category theory. However, those who might be interested in category theory can read for instance [Mac98], [Rie16] and [AHS90].

Chapter 2

Closure, Interior and Neighborhood Spaces

In this chapter, we shall introduce the most fundamental concepts of a closure space and give some elementary results. We will provide a description of a closure operation in terms of interior and neighborhood operators, and vice versa. Furthermore, we will see the connection between categories of closure spaces and relations that include simple graphs and simple digraphs. We will close this chapter by constructing a functor from the category of metric spaces to the category of closure spaces. Most of the material given here appears in [ČFK66] and [DT95].

2.1 Closure Spaces

Definition 2.1.1. [ČFK66] Let X be a set. A map $c_X : P(X) \to P(X)$ is called a *closure operator* for X if the following conditions are satisfied:

i. (Grounded) $c_X(\emptyset) = \emptyset$,

ii. (Extension) for all $A \subseteq X$, $A \subseteq c_X(A)$,

iii. (Monotone) for all $A, B \subseteq X$ such that $A \subseteq B$, $c_X(A) \subseteq c_X(B)$.

   The pair (X, c_X) is called a *closure space*.

iv. (Additive) for all $A, B \subseteq X$, $c_X(A \cup B) = c_X(A) \cup c_X(B)$.

   If in addition the closure operator c_X satisfies condition iv. we say that c_X is a Čech closure operator on X and the pair (X, c_X) will be called a *Čech closure space*, we will call it again *closure space* for simplicity.

v. (Idempotent) for all $A \subseteq X$, $c_X(c_X(A)) = c_X(A)$.

   Moreover, if the condition v is satisfied, then c_X is called a *Kuratowski topological closure operation* for X and the pair (X, c_X) is a *Kuratowski topological space*.

If we have a set of one element $\{x\}$, we will write simply $c(\{x\})$ as $c(x)$. We also note that for given closure operators $c, d : P(X) \to P(X)$, we say that *c is finer than d* and *d is coarser than c* if and only of $c(A) \subseteq d(A)$ for every $A \subseteq X$.

Definition 2.1.2. [ČFK66] Given a closure space (X, c_X) and $A \subseteq X$. We say that A is *closed* if $c_X(A) = A$, and *open* if $c_X(X \setminus A) = X \setminus A$.

Example 2.1.3. [DT95] [ČFK66] Let X be a set.

- The discrete closure operator for X is the identity map $id_X : P(X) \to P(X)$, which is the finest closure operator on X.
- The indiscrete closure operator for X is the map $ind_X : P(X) \to P(X)$ defined by

$$ind_X(A) = \begin{cases} X & \text{if } A \neq \emptyset \\ \emptyset & \text{if } A = \emptyset. \end{cases}$$

  This is the coarsest closure operator on X.
- Let $(X, \mathcal{T})$ be a topological space. The *Kuratowski closure operator* $k_X : P(X) \to P(X)$ is defined by $k_X(A) = \bar{A}$ where $\bar{A}$ is the smallest closed set (i.e. $X \setminus \bar{A} \in \mathcal{T}$) on X containing A; in other word, $\bar{A}$ is the intersection of all closed sets containing A. The pair (X, k_X) is called the *Kuratowski closure space*.

Definition 2.1.4. [ČFK66] Let (X, c_X) and (Y, c_Y) be closure spaces. A map $f : (X, c_X) \to (Y, c_Y)$ is continuous at $x \in X$ if for all $A \subseteq X$, $x \in c_X(A)$ implies $f(x) \in c_Y(f(A))$. We say that f is *continuous* if it is continuous at each point $x \in X$.

Definition 2.1.5. [DT95]

1. Let (X, c_X) be a closure space and Y be a subset of X. We define the closure operator $c_Y : P(Y) \to P(Y)$ of the subspace (Y, c_Y) to be the finest closure operator on Y that makes $(Y, c_Y) \hookrightarrow (X, c_X)$ continuous; c_Y is given by

$$c_Y(A) = Y \cap c_X(A), \text{ for all } A \subseteq Y.$$

2. Let $f : (X, c_X) \to Y$ be a surjective map from a closure space (X, c_X) to the set Y. We define a quotient closure operator $c_f : P(Y) \to P(Y)$ by

$$c_f(A) = f(c_X(f^{-1}(A)))$$

   for all $A \subseteq Y$. In fact, this is the coarsest closure operator on Y which makes $f : (X, c_X) \to (Y, c_f)$ continuous.

Proposition 2.1.6. *[ČFK66] Let $f : (X, c_X) \to (Y, c_Y)$ and $g : (Y, c_Y) \to (Z, c_Z)$ be continuous maps. Then the composition $gf : (X, c_X) \to (Z, c_Z)$ is continuous. Moreover, the composition is associative and for every continuous map $f : (X, c_X) \to (Y, c_Y)$, $f id_X = f$ and $id_Y f = f$ where id_X and id_Y are identity continuous maps on (X, c_X) and (Y, c_Y) respectively.*

Proof. Let $A \subseteq X$.

$$\begin{aligned} (gf)(c_X(A)) = g(f(c_X(A))) &\subseteq g(c_Y(f(A))) && f \text{ is continuous} \\ &\subseteq c_Z(g(f(A))) = c_Z((gf)(A)) && g \text{ is continuous.} \end{aligned}$$

Therefore, gf is continuous.

It is clear that the composition is associative and for any continuous map $f : (X, c_X) \to (Y, c_Y)$ we have , $f id_X = f$ and $id_Y f = f$. □

Definition 2.1.7. We define the category **CS** (resp. **CL**) to be the category of all (Čech) closure spaces as objects and continuous maps between (Čech) closure spaces as morphisms. The category **CL** is a full subcategory of **CS**.

Proposition 2.1.8. *[ČFK66] Let $(X, c_X), (Y, c_Y) \in$ **CS** and $f : X \to Y$ be a set map. The following conditions are equivalent.*

(i) $f:(X,c_X)\to(Y,c_Y)$ *is continuous.*

(ii) *For all* $A\subseteq X$, $f(c_X(A))\subseteq c_Y(f(A))$.

(iii) *For all* $B\subseteq Y$, $c_X(f^{-1}(B))\subseteq f^{-1}(c_Y(B))$.

Proof. It is easy to check that $f:(X,c_X)\to(Y,c_Y)$ is continuous if and only if for all $A\subseteq X$, $f(c_X(A))\subseteq c_Y(f(A))$.

Assume that f satisfies the condition (ii). Let $B\subseteq Y$. We have $f^{-1}(B)\subseteq X$. By the condition (ii), we get

$$f(c_X(f^{-1}(B)))\subseteq c_Y(f(f^{-1}(B)))\subseteq c_Y(B).$$

The last inclusion comes from the monotonicity of c_Y and the fact that $f(f^{-1}(B))\subseteq B$. It follows by the Galois connection $f(-)\dashv f^{-1}(-)$ that $c_X(f^{-1}(B))\subseteq f^{-1}(c_Y(B))$. This yields the condition (iii).

Suppose that the condition (iii) holds. Let $A\subseteq X$. We have $f(A)\subseteq Y$. We note that $A\subseteq f^{-1}(f(A))$. The monotonicity of c_X implies

$$c_X(A)\subseteq c_X(f^{-1}(f(A))).$$

By the condition (iii), we obtain

$$c_X(f^{-1}(f(A)))\subseteq f^{-1}(c_Y(f(A))).$$

So $c_X(A)\subseteq f^{-1}(c_Y(f(A)))$. Again, by the Galois connection $f(-)\dashv f^{-1}(-)$, we have $f(c_X(A))\subseteq c_Y(f(A))$. Hence we get the condition (ii). □

Remark 2.1.9. As shown in [ČFK66], [Rie21a] we have the following results:

1. Let (X,c) be a Čech closure space. A topological modification of c is a topological closure operator $\tau_c:P(X)\to P(X)$ given by

   $$\tau_c(A)=\bigcap\{F\mid A\subseteq F,\ c(F)=F\}\text{ for all }A\subseteq X.$$

   We note that τ_c is an idempotent closure operator. i.e., $\tau_c\tau_c=\tau_c$.

2. There is an adjunction $\mathbf{Top}\ \underset{\iota}{\overset{\tau}{\underset{\bot}{\leftrightarrows}}}\ \mathbf{CL}$. where τ is a functor mapping a closure space (X,c) to a topological space (X,τ_c), and a continuous map $f:(X,c)\to(Y,d)$ to a continuous function $\tau f:(X,\tau_c)\to(Y,\tau_d)$ in **Top**; and ι is a functor sending a topological space $(X,\mathcal{T}_X)$ to a closure space (X,k_X) and continuous map $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ to a continuous map $f:(X,k_X)\to(Y,k_Y)$ in **CL**.

2.2 Interior Spaces and Neighborhood Spaces

Definition 2.2.1. Let X be a set. An interior operator on X is a map $i_X:P(X)\to P(X)$ satisfying the following conditions:

i. $i_X(X)=X$,

ii. $i_X(A)\subseteq A$ for all $A\subseteq X$,

iii. $i_X(A)\subseteq i_X(B)$ for all $A\subseteq B\subseteq X$.
   The pair (X,i_X) is called *interior space.*

iv. $i_X(A\cap B)=i_X(A)\cap i_X(B)$ for all $A,B\subseteq X$.

v. $i_X(i_X(A))=A$ for all $A\subseteq X$.

Remark 2.2.2. Let $(X,c) \in \mathbf{CS}$. The closure operator c induces an interior operator $i_c : P(X) \to P(X)$ defined by $i_c(A) := X \setminus c(X \setminus A)$, for all $A \subseteq X$. Conversely, given an interior space (X,i), the interior operator i is uniquely determined by the closure operator $c_i : P(X) \to P(X)$ given by $c_i(A) = X \setminus i(X \setminus A)$ for all $A \subseteq X$. In case that $(X,c) \in \mathbf{CL}$, the interior operator i_c satisfies the condition $iv.$ in Definition 2.2.1. Moreover, if the closure operator c is a topological closure, then i_c satisfies the condition $v.$ in Definition 2.2.1.

Definition 2.2.3. Let $(X,c) \in \mathbf{CL}$. A subset U of X is a *neighborhood* of a set $A \subseteq X$ if $A \subseteq i_c(U)$. A *neighborhood system* of a set A is a non empty collection of neighborhoods of A. In particular, if A is a singleton set $\{x\}$ we write simply neighborhood system of x.

Definition 2.2.4. [Rie21a] Let $(X,c) \in \mathbf{CL}$. We say that

1. a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of subsets of X is a *cover* of (X,c) if $X = \bigcup_{\alpha \in A} U_\alpha$, i.e. if for all $x \in X$ there exists $U \in \mathcal{U}$ such that $x \in U$.

2. a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of subsets of X is an *interior cover* of (X,c) if $X = \bigcup_{\alpha \in A} i_c(U_\alpha)$, i.e. if every point $x \in X$ has neighborhood in $\mathcal{U}$.

If in addition every $U \in \mathcal{U}$ is open (closed), then we say $\mathcal{U}$ is an *open (closed) cover*.

Definition 2.2.5. [ČFK66] Let $(X,c) \in \mathbf{CL}$. A *base of the neighborhood system* of $A \subset X$ is a nonempty collection $\mathcal{B}$ of subsets of X such that each set $B \in \mathcal{B}$ is a neighborhood of A and each neighborhood of A contains a set in $\mathcal{B}$. If $A = \{x\}$ is a set with one element, we will use the term *local base* at x.

Proposition 2.2.6. *[ČFK66] Let $\mathcal{U}(x)$ be a local base at x. Then for all $A \subseteq X$,*

$$x \in c(A) \iff \forall U \in \mathcal{U}(X),\ U \cap A \neq \emptyset.$$

Proof. Let $A \subseteq X$. First, assume that $x \in c(A)$. Since $\mathcal{U}(x)$ is a local base at x, then for every $U \in \mathcal{U}(x)$, $x \in X \setminus c(X \setminus U)$. Suppose that $U \cap A = \emptyset$, then $A \subseteq X \setminus U$. Then $c(A) \subseteq c(X \setminus U)$. So $x \in c(X \setminus U)$. This is contradiction because U is a neighborhood of x. Thus $U \cap A \neq \emptyset$.

Conversely, suppose that $x \notin c(A)$, i.e. $x \in X \setminus c(A) = X \setminus c(X \setminus (X \setminus A))$. Then $X \setminus A$ is a neighborhood of x, so $X \setminus A \in \mathcal{U}(x)$, but $(X \setminus A) \cap A = \emptyset$. We must have $x \in c(A)$. □

Proposition 2.2.7. *[ČFK66] Let $x \in X$. If $\mathcal{U}(x)$ is a local base. Then $\mathcal{U}(x)$ satisfies the following conditions:*

(a) $\mathcal{U}(x) \neq \emptyset$,
(b) for all $U \in \mathcal{U}(x)$, $x \in U$,
(c) for each $U, V \in \mathcal{U}(x)$, there exists $W \in \mathcal{U}(x)$ such that $W \subseteq U \cap V$.

Proof. Suppose that $\mathcal{U}(x)$ is a local base at x. It is clear that $\mathcal{U}(x) \neq \emptyset$.

Let $U \in \mathcal{U}(x)$. Then $x \in X \setminus c(X \setminus U)$. Since $X \setminus U \subseteq c(X \setminus U)$, then $X \setminus c(X \setminus U) \subseteq U$. So $x \in U$.

Let $U, V \in \mathcal{U}(x)$. We have $X \setminus U \subseteq X \setminus (U \cap V)$. It follows that

$$x \in X \setminus c(X \setminus U) \subseteq X \setminus c(X \setminus (U \cap V)).$$

So $x \in X \setminus c(X \setminus (U \cap V))$. Therefore $U \cap V$ is a neighborhood of x. It follows that there exists $W \in \mathcal{U}(x)$ such that $W \subseteq U \cap V$. □

Theorem 2.2.8. *[ČFK66] For each $x \in X$, let $\mathcal{U}(x)$ be a collection of sets that satisfies (a), (b) and (c) in Proposition 2.2.7. Then, there exists a unique closure operator c on X such that for each $x \in X$, $\mathcal{U}(x)$ is a local base at x in the closure space (X,c). Precisely, the closure operator c on X is obtained from Proposition 2.2.6 as follows:*

$$c(A) = \{x \in X \mid \forall U \in \mathcal{U}(x),\ U \cap A \neq \emptyset\}, \text{ for all } A \subseteq X.$$

Proof. Let $U \in \mathcal{U}(x)$. It is clear that $c(\emptyset) = \emptyset$.

Let $A \subseteq X$ and $x \in A$. Let $U \in \mathcal{U}(x)$, by the condition (b) we have $x \in U$. Then $x \in U \cap A$. Therefore $x \in c(A)$ and $A \subseteq c(A)$.

Let $A, B \subseteq X$. Let $x \in c(A) \cup c(B)$. For every $U \in \mathcal{U}(x)$ we have $U \cap A \neq \emptyset$ or $U \cap B \neq \emptyset$, so $U \cap (A \cup B) \neq \emptyset$. Thus $x \in c(A \cup B)$ and then $c(A) \cup c(B) \subseteq c(A \cup B)$.
Suppose that $x \in c(A) \cup c(B)$. There exists $U, V \in \mathcal{U}(x)$ such that $U \cap A = V \cap B = \emptyset$. By the condition (c), there exists $W \in \mathcal{U}(x)$ such that $W \subseteq U \cap V$. Since we have

$$W \cap (A \cup B) = (W \cap A) \cup (W \cap B) = \emptyset.$$

Therefore $x \notin c(A \cup B)$. So $c(A \cup B) \subseteq c(A) \cup c(B)$. Hence $c(A \cup B) = c(A) \cup c(B)$. We conclude that c is a closure operator.

Now, we have to show that $\mathcal{U}(x)$ is a local base at x. Let $x \in X$ and $U \in \mathcal{U}(x)$. Suppose that $x \in c(X \setminus U)$. For every $V \in \mathcal{U}(X)$, $V \cap (X \setminus U) \neq \emptyset$, but $U \in \mathcal{U}(x)$ and $U \cap (X \setminus U) = \emptyset$. So, we must have $x \in X \setminus c(X \setminus U)$. Thus U is a neighborhood of x in (X, c).

Let V be a neighborhood of x in (X, c), that is $x \in X \setminus c(X \setminus V)$.
Suppose that for all $U \in \mathcal{U}(x)$, $(X \setminus V) \cap U = \emptyset$. It follows by the definition of the closure operator c that $x \in c(X \setminus V)$. That means V is not a neighborhood of x, but that is a contradiction. Therefore, there exists $U \in \mathcal{U}(x)$ such that $U \subseteq V$. Hence $\mathcal{U}(x)$ is a local base at x. □

Definition 2.2.9. [BM22] Let X be a set. A neighborhood function on X is a map $\mathcal{N} : X \to P(P(X))$ satisfying the following conditions: for all $x \in X$,

i. $\mathcal{N}(x) \neq \emptyset$,

ii. for all $A \in \mathcal{N}(x)$, $x \in A$,

iii. if $A \in \mathcal{N}(x)$ and $A \subseteq B$ then $B \in \mathcal{N}(x)$,

iv. if $A, B \in \mathcal{N}(x)$ then $A \cap B \in \mathcal{N}(x)$.
We note that $\mathcal{N}(x)$ is a filter. For $x \in X$, we call $A \in \mathcal{N}(x)$ a *neighborhood* of x and call $\mathcal{N}(x)$ a *neighborhood filter*. Call $\mathcal{N}$ a *collection of neighborhood filters*. The pair $(X, \mathcal{N})$ is called a *neighborhood space*.

v. if $A \in \mathcal{N}(x)$ then there exists $B \in \mathcal{N}(x)$ with $B \subseteq A$ such that for all $y \in B$, there exists $C \in \mathcal{N}(y)$ such that $C \subseteq B$. If in addition $\mathcal{N}$ satisfies the condition v., then $(X, \mathcal{N})$ is a topological space.

Definition 2.2.10. [BM22] Let $(X, c) \in \mathbf{CL}$ and let i be the corresponding interior operator associated with c. There is a neighborhood function $\mathcal{N} : X \to P(P(X))$ defined by

$$\mathcal{N}(x) = \{A \subseteq X |\ x \notin c(X \setminus A)\} = \{A \subseteq X |\ x \in i(A)\} \text{ for all } x \in X.$$

Definition 2.2.11. [BM22] Given a collection of neighborhood filters $\mathcal{N}$. There exists an interior and closure operator $i, c : P(X) \to P(X)$ defined as follows:

$$i(A) = \{x \in X |\ \exists U \in \mathcal{N}(x), U \subseteq A\} \text{ for all } A \subseteq X,$$
$$c(A) = \{x \in X |\ \forall U \in \mathcal{N}(x), U \cap A \neq \emptyset\} \text{ for all } A \subseteq X.$$

Definition 2.2.12. [ČFK66] Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be a neighborhood spaces. A continuous map $f : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ is a set map $f : X \to Y$ such that for all $x \in X$ and for all $A \in \mathcal{N}_Y(f(x))$, $f^{-1}(A) \in \mathcal{N}_X(x)$. Equivalently, f is continuous if and only if for each $x \in X$ and for each $A \in \mathcal{N}_Y(f(x))$, there is a $B \in \mathcal{N}_X(x)$ such that $f(B) \subseteq A$.

Proposition 2.2.13. *Let $f : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ and $g : (Y, \mathcal{N}_Y) \to (Z, \mathcal{N}_Z)$ be continuous maps. The composition $gf : (X, \mathcal{N}_X) \to (Z, \mathcal{N}_Z)$ is continuous. Moreover, the composition is associative.*

Proof. Let $x \in X$ and $A \in \mathcal{N}_Z(gf(x))$. Set $y = f(x)$, we have $A \in \mathcal{N}_Z(g(y))$. Since g is continuous, then $g^{-1}(A) \in \mathcal{N}_Y(y)$, so $g^{-1}(A) \in \mathcal{N}_Y(f(x))$. Since f is continuous, we have $f^{-1}(g^{-1}(A)) \in \mathcal{N}_X(x)$, that is $(gf)^{-1}(A) \in \mathcal{N}_X(x)$. Thus gf is continuous. It is easy to check that the composition is associative. □

Remark 2.2.14. The identity map id_X on $(X, \mathcal{N}_X)$ is continuous. Moreover, if $f : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ is a continuous map, then $f id_X = f$ and $id_Y f = f$.
Neighborhood spaces together with all continuous maps between neighborhood spaces form a category of neighborhood spaces denoted by **Nb**.

Proposition 2.2.15. *Let $f : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ be a continuous map in* **Nb**. *Then f induces a continuous map $f : (X, c_X) \to (Y, c_Y)$ in* **CL** *where c_X and c_Y are the closure induced by $\mathcal{N}_X$ and $\mathcal{N}_Y$ respectively as in Definition 2.2.11.*

Proof. Let $A \subseteq X$ and $y \in f(c_X(A))$. There exists $x \in c_X(A)$ such that $y = f(x)$. Let $V \in \mathcal{N}_Y(f(x))$. By the continuity of $f : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ in **Nb**, we have $f^{-1}(V) \in \mathcal{N}_X(x)$. It follows from the definition of $c_X(A)$ that $f^{-1}(V) \cap A \neq \emptyset$. Since $\emptyset \neq f(f^{-1}(V) \cap A) = V \cap f(A)$ then $y = f(x) \in c_Y(f(A))$. So $f(c_X(A)) \subseteq c_Y(f(A))$. Hence $f : (X, c_X) \to (Y, c_Y)$ is continuous in **CL**. □

Proposition 2.2.16. *Let $f : (X, c_X) \to (Y, c_Y)$ be a continuous map in* **CL**. *Then f induces a continuous map $f : (X, \mathcal{N}_X) \to (Y, \mathcal{N}_Y)$ in* **Nb** *where $\mathcal{N}_X$ and $\mathcal{N}_Y$ are defined as in Definition 2.2.10.*

Proof. Let $x \in X$ and $A \in \mathcal{N}_Y(f(x))$. We need to prove that $x \notin c_X(X \setminus f^{-1}(A))$. It follows by the definition of $\mathcal{N}_Y$ that $A \in \mathcal{N}_Y(f(x))$ implies $f(x) \in Y \setminus c_Y(Y \setminus A)$.
We note that

$$x \in f^{-1}(f(x)) \subseteq f^{-1}(Y \setminus c_Y(Y \setminus A)) = X \setminus f^{-1}(c_Y(Y \setminus A)).$$

We observe that

$$c_X(X \setminus f^{-1}(A)) = c_X(f^{-1}(Y \setminus A)).$$

Moreover, since $f : (X, c_X) \to (Y, c_Y)$ is continuous

$$c_X(f^{-1}(Y \setminus A)) \subseteq f^{-1}(c_Y(Y \setminus A)).$$

Therefore

$$x \in X \setminus f^{-1}(c_Y(Y \setminus A)) \subseteq X \setminus c_X(f^{-1}(Y \setminus A)).$$

That is $x \notin c_X(f^{-1}(Y \setminus A))$. Then $f^{-1}(A) \in \mathcal{N}_X(x)$. Hence f is continuous in **Nb**. □

Remark 2.2.17. According to Proposition 2.2.15 and 2.2.16, in addition with Proposition A.0.6, there exists a functor from the category N : **CL** → **Nb** and C : **Nb** → **CL** respectively.

Theorem 2.2.18. *[BM22] The category* **Nb** *is isomorphic to* **CL**.

Proof. Let $(X, \mathcal{N}) \in$ **Nb**. We have to prove that $NC(X, \mathcal{N}) = id_{\mathbf{Nb}}(X, \mathcal{N})$.
Set $NC(X, \mathcal{N}) = (X, \tilde{\mathcal{N}})$, where the neighborhood function $\tilde{\mathcal{N}} : X \to P(P(X))$ is given by

$$\begin{aligned}\tilde{\mathcal{N}}(x) &= \{A \subseteq X \mid \exists U \in \mathcal{N}(x),\ U \cap (X \setminus A) = \emptyset\} \\ &= \{A \subseteq X \mid \exists U \in \mathcal{N}(x),\ U \subseteq A\},\end{aligned}$$

for all $x \in X$.
Let $x \in X$ and $A \in \mathcal{N}(x)$. Since $A \subseteq A$ then $A \in \tilde{\mathcal{N}}(x)$. So $\mathcal{N}(x) \subseteq \tilde{\mathcal{N}}(x)$.
On the other hand, let $A \in \tilde{\mathcal{N}}(x)$. There exists $U \in \mathcal{N}(x)$ such that $U \subseteq A$. Since $c(X \setminus A) \subseteq c(X \setminus U)$, then $x \notin c(X \setminus A)$. Therefore $A \in \mathcal{N}(x)$ and $\tilde{\mathcal{N}}(x) \subseteq \mathcal{N}(x)$. Hence $\tilde{\mathcal{N}}(x) = \mathcal{N}(x)$ and it follows that $(X, \tilde{\mathcal{N}}) = (X, \mathcal{N})$.

Let $(X,c) \in \mathbf{CL}$. Now we have to show that $CN(X,c) = id_{\mathbf{CL}}(X,c)$.
Set $CN(X,c) = (X,\tilde{c})$, where the closure operator $\tilde{c} : P(X) \to P(X)$ is defined by

$$\begin{aligned}\tilde{c}(A) =&\{x \in X |\ x \notin c(X \setminus U) \implies U \cap A \neq \emptyset\}\\ =&\{x \in X |\ U \cap A = \emptyset \implies x \in c(X \setminus U)\}\end{aligned}$$

for all $A \subseteq X$.
Let $A \subseteq X$. Let $x \in c(A)$ such that $x \notin c(X \setminus U)$. If $U \cap A = \emptyset$ then $A \subseteq X \setminus U$, so $c(A) \subseteq c(X \setminus U)$, but this contradict the hypothesis, so we must have $U \cap A \neq \emptyset$. Then $x \in \tilde{c}(A)$ and $c(A) \subseteq \tilde{c}(A)$.

Let $x \in \tilde{c}(A)$. Since $(X \setminus A) \cap A = \emptyset$ then $x \in c(X \setminus (X \setminus A)) = c(A)$. Therefore $\tilde{c}(A) \subseteq c(A)$. Thus $\tilde{c}(A) = c(A)$ and that implies $(X,\tilde{c}) = (X,c)$.

Moreover, it is clear that for every morphism f in $\mathbf{Nb}$ and g in $\mathbf{CL}$ we have $CN(g) = id_{\mathbf{Nb}}(g)$ and $NC(f) = id_{\mathbf{CL}}(f)$. We conclude that $CN = id_{\mathbf{Nb}}$ and $NC = id_{\mathbf{CL}}$. □

2.3 (Co)product and (Co)equalizer

This section is devoted to the definition of certain categorical small limits and colimits in **CL**.

Let $\{(X_i,c_i)\}_{i\in I}$ be a collection of closure spaces. Let X be the set given by the cartesian product of X_i, i.e. $X = \prod_{i\in I} X_i$; any element $x \in X$ is then written as $x = (x_i)_{i\in I}$ where $x_i \in X_i$ for every $i \in I$. Consider the natural projection maps $\pi_i : X \to X_i$. We would like to define the coarsest closure operator c on X which makes those projections π_i all continuous.

Proposition 2.3.1. *[ČFK66] For every $x \in X$, a local base $\mathcal{U}(x)$ at x consists of a collection of all sets of the form $\bigcap_{j\in J} \pi_j^{-1}(V_j)$ where J is a finite subset of I and V_j is a neighborhood of $\pi_j(x)$ in (X_j,c_j) for each $j \in J$. Moreover $\mathcal{U}(x)$ satisfies the condition in Proposition 2.2.7. Therefore, by Theorem 2.2.8 there exists a unique closure operator c on X characterized by the local base $\mathcal{U}(x)$ for all $x \in X$.*

Definition 2.3.2. The closure space (X,c) together with the family of projection maps $\{\pi_i : (X,c) \to (X_i,c_i)\}_{i\in I}$ form a categorical product on the category **CL**.
In other words, it is a limit of the diagram F from the discrete category **I** to **CL** that sends each object $i \in \mathbf{I}$ to the closure space $(X_i,c_i) \in \mathbf{CL}$.

In particular, we have the following product of two closure spaces.

Definition 2.3.3. The diagram $(Y,c_Y) \xleftarrow{\pi_Y} (X\times Y, c_{X,Y}) \xrightarrow{\pi_X} (X,c_X)$ form a *product* of (X,c_X) and (Y,c_Y) in the category **CL** where $X \times Y$ is cartesian product and the closure operator $c_{X,Y}$ is obtained from the local base

$$\mathcal{U}(x,y) = \{U \times V \subseteq X \times Y |\ \forall U \in \mathcal{U}(x), \forall V \in \mathcal{U}(y)\},$$

where $\mathcal{U}(x)$ and $\mathcal{U}(y)$ are the neighborhood systems of x and y respectively.
By Theorem 2.2.8, we have the closure operator

$$c_{X,Y}(A) = \{(x,y) \in X \times Y |\ \forall U \times V \in \mathcal{U}(x,y),\ A \cap (U \times V) \neq \emptyset\},$$

for all $A \subseteq X \times Y$.

Let $\{(X_i,c_i)\}_{i\in I}$ be a collection of closure spaces. Let X be the disjoint union $X := \coprod_{i\in I} X_i$. We have to define the finest closure operator c on X which makes all natural inclusion $\iota_i : (X_i,c_i) \hookrightarrow (X,c)$ continuous.

Definition 2.3.4. [ČFK66] Let $A \subseteq X$ such that $A = \coprod_{i\in I} A_i$ where each $A_i \subseteq X_i$ for all $i \in I$. Define the closure of A by

$$c(\coprod_{i\in I} A_i) = \coprod_{i\in I} c_i(A_i).$$

The following is a characterization of the categorical coproduct of closure spaces.

Definition 2.3.5. The closure space (X, c) is characterized by the following universal property:

For every closure space (Z, d) and every family of continuous maps $\{h_i : (X_i, c_i) \to (Z, d)\}_{i \in I}$, there exists a unique continuous map $f : (X, c) \to (Z, d)$ such that $f\iota_i = h_i$ for every $i \in I$.
In other words, the categorical coproduct is then the colimit of the diagram F from the discrete category $\mathbf{I}$ to $\mathbf{CL}$ sending each object $i \in \mathbf{I}$ to the closure space $(X_i, c_i) \in \mathbf{CL}$.

Definition 2.3.6. [BM22] Given continuous maps $(X, c_X) \overset{f}{\underset{g}{\rightrightarrows}} (Y, c_Y)$. Consider a closure space (E, c) where $E := \{x \in X | \; f(x) = g(x)\}$ and c is defined as in Definition 2.1.5. We define the *equalizer* of f and g to be the object $(E, c) \in \mathbf{CL}$ (or the pair $((E, c), \iota)$) together with the continuous inlusion $\iota : (E, c) \hookrightarrow (X, c_X)$.

Definition 2.3.7. [BM22] Given continuous maps $(X, c_X) \overset{f}{\underset{g}{\rightrightarrows}} (Y, c_Y)$. A *coequalizer* of f and g consists of the object $(Y/\sim, c_p) \in \mathbf{CL}$ (or the pair $((Y/\sim, c_p), p)$), where $\sim$ is an equivalence relation generated by $(f(x), g(x))$ for all $x \in X$, $p : (Y, c_Y) \to (Y/\sim, c_p)$ is a continuous map and the closure operator c_p is defined as in Definition 2.1.5.

Theorem 2.3.8. *[BM22] The category* $\mathbf{CL}$ *is complete and cocomplete.*

Proof. Since category $\mathbf{CL}$ has a product and equalizer (resp. coproduct and coequalizer), then it is complete (resp. cocomplete) by Theorem A.0.23. □

2.4 Relations and Closure Spaces

The material in this section were originally introduced by Šlapal in [Šla93]. He developed a construction of closure space from a relational system and vice versa. Moreover, both of these constructions are functorial.
Let $\alpha > 1$ be an ordinal number.

Definition 2.4.1. [Šla93] Let $(X, c) \in \mathbf{CS}$.

i) A closure space (X, c) is called s_α*-closure space* if the following condition is satisfied:

$$\forall A \subseteq X \implies c(A) = \bigcup_{B \subseteq A} \{c(B) | \; |B| < \alpha\}.$$

Here, the notation $|B|$ means the cardinal of the set B.

ii) A closure space (X, c) is *symmetric* if $y \in c(x)$ implies $x \in c(y)$ for all $x, y \in X$.

It is worth to mention that any s_2-closure spaces are called *Alexandroff closure spaces* which agree with the definition in [DT95]. Furthermore, it is clear that any s_2-closure operation is additive. We will denote by $\mathbf{CL}_A$ (resp. $\mathbf{CL}_{sA}$) the category where whose objects are an Alexandroff (resp. symmetric Alexandroff) closure space and whose morphisms are a continuous map between them. Both are full subcategories of $\mathbf{CL}$.

Let X be a set. We denote by X^α the set of all sequences $(x_i)_{0 \le i < \alpha}$ of type α where $x_i \in X$ for all $0 \le i < \alpha$.

Definition 2.4.2. [Šla93]

1. The ordered pair (X, R) where X is a set and R is an α-ary relation on X (i.e. $R \subseteq X^\alpha$) is said to be a *relational system of type* α or α*-ary relational system.*

2. Given an α-ary relational system (X,R) and (Y,S). A map $f:(X,R)\to(Y,S)$ is called a *homomorphism of relational systems* if for all $(x_i)_{i<\alpha}\in R$ implies $(f(x_i))_{i<\alpha}\in S$.

We denote by $\mathbf{Rel}_\alpha$ the category of α-ary relational systems and a homomorphism of α-ary relational systems.

Example 2.4.3. The following is a well-known, particular 2-ary relational system that we will need later. Let X be a set. Set $D := \{(x,x)|\ x\in X\}\subseteq X\times X$.

a) [BM22] A *simple graph* or *graph* is a pair (X,E) where $E\subseteq X\times X$ is a symmetric relation on X such that $E\cap D=\emptyset$.

b) [DT95] A *directed graph* or *digraph* is a pair (X,E) where $E\subseteq X\times X$ is a relation on X.

c) [DT95] A *spatial digraph* is a pair (X,E) where $E\subseteq X\times X$ is a reflexive relation on X.

A *graph (resp. digraph) homomorphism* is a homomorphism between a simple graphs (resp. digraphs) defined as in Definition 2.4.2.
For simplicity, if R is a 2-ary relation we will write xRy by meaning that $(x,y)\in R$.

Furthermore, all simple graphs (resp. digraphs, spatial digraph) together with all graph homomorphisms (resp. digraph homomorphisms. spatial digraph homomorphisms) form a category denoted by **Gph** (resp. **DiGph**, **SDiGph**), which are full subcategories of $\mathbf{Rel}_2$.

Definition 2.4.4. [Šla93] Let $(X,R)\in\mathbf{Rel}_\alpha$. Define a closure operator $c_R: P(X)\to P(X)$ induced by the α-ary relation R as follows: for all $A\subseteq X$,

$$c_R(A)=A\cup\{x\in X|\ \exists(x_i)_{i<\alpha}\in R,\ \exists(i_0<\alpha),\ x=x_{i_0}\text{ and }\ \forall i<i_0,\ x_i\in A\}.$$

Moreover, the pair (X,c_R) is a closure space. As shown in [Šla93], the closure operator c_R satisfied the condition of s_α-closure in Definition 2.4.1.

Theorem 2.4.5. *[Šla93] There is a functor $\Phi_\alpha:\mathbf{Rel}_\alpha\to\mathbf{CS}$ sending an α-ary relational system to a closure space and a homomorphism of an α-relational system to a continuous map between closure spaces.*

Proof. The mapping on object is defined by $\Phi_\alpha(X,R)=(X,c_R)$ for all $(X,R)\in\mathbf{Rel}_\alpha$, and (X,c_R) is the same as in Definition 2.4.4.

Let $f:(X,R)\to(Y,S)$ in $\mathbf{Rel}_\alpha$. We have to prove that f induces a continuous map $f:(X,c_R)\to(Y,c_S)$ in **CS**.
Let $A\subseteq X$ and $y\in f(c_R(A))$. There exists $x\in c_R(A)$ such that $y=f(x)$.
If $x\in A$, then it is clear that $y\in c_S(f(A))$.
Now, suppose that $x\in c_R(A)\setminus A$. There exists $(x_i)_{i<\alpha}\in R$ and $0\le i_0<\alpha$ such that $x=x_{i_0}$ and $x_i\in A$ for all $i<i_0$. Since f is α-ary homomorphism, there exists $f(x_i)_{i<\alpha}\in S$ and $0\le i<\alpha$ such that $y=f(x)=f(x_{i_0})$ and $f(x_i)\in f(A)$ for all $i<i_0$. Then $y\in c_S(f(A))$. Thus $f(c_R(A))\subseteq c_S(f(A))$, that is $f:(X,c_R)\to(Y,c_S)$ continuous.

We set $\Phi_\alpha(f)=f$ for all f in $\mathbf{Rel}_\alpha$. By Proposition A.0.6, Φ_α is a functor. □

Definition 2.4.6. [Šla93] Let $(X,c)\in\mathbf{CS}$. We define an α-ary relation R_c induced by the closure operator c as follows:

$$(x_i)_{i<\alpha}\in R_c \iff x_{i_0}\in c(\{x_i|\ i<i_0\})\text{ for any } i_0 \text{ with } 0<i_0<\alpha.$$

Furthermore, we obtain an α-ary relational system (X,R_c).

Proposition 2.4.7. *[Šla93] Any continuous map $f:(X,c)\to(Y,d)$ in* **CS** *induces a homomorphism of an α-ary relational system $f:(X,R_c)\to(Y,R_d)$ in $\mathbf{Rel}_\alpha$.*

Proof. Let $(x_i)_{i<\alpha} \in R_c$. Then for every i_0 such that $0 < i_0 < \alpha$, $x_{i_0} \in c(\{x_i | i < i_0\})$. Since $f : (X, c) \to (Y, d)$ is continuous, we have

$$f(x_{i_0}) \in f(c(\{x_i|\ i < i_0\})) \subseteq d(f(\{x_i|\ i < i_0\})) \text{ for any } i_0 \text{ with } 0 < i_0 < \alpha.$$

Thus $(f(x_i))_{i<\alpha} \in R_d$. Hence $f : (X, R_c) \to (Y, R_d)$ is an α-ary homomorphism. □

Theorem 2.4.8. *[Šla93] There is a functor $\Psi_\alpha : \mathbf{CS} \to \mathbf{Rel}_\alpha$ that maps a closure space to an α-ary relational system and a continuous map in* **CS** *to an α-ary homomorphism in $\mathbf{Rel}_\alpha$.*

Proof. For all $(X, c) \in \mathbf{CS}$, we define $\Psi_\alpha(X, c) = (X, R_c)$ where $(X, R_c) \in \mathbf{Rel}_\alpha$ is given as in Definition 2.4.6.

By Proposition 2.4.7, for any $f : (X, c) \to (Y, d)$ in **CS** we set $\Psi_\alpha(f) = f$, where the f in the right hand side is a homomorphism from (X, R_c) to (X, R_d) in $\mathbf{Rel}_\alpha$.
It follows from Proposition A.0.6 that Ψ_α is a functor. □

For the rest of this section, we assume that $\alpha = 2$, we will then focus on the investigation of the relation between the categories **CL**, $\mathbf{CL}_A$, $\mathbf{CL}_{sA}$, **SDiGph**, **DiGph** and **Gph**.

2.4.9 Symmetric Closure and Alexandroff Closure

Some of the definitions and results given here have been developed in [ČFK66] and [DT95], while most of the categorical results are recently from [BM22]. We will present how the results may be proved using the coreflective category which is unknown in those papers.

Definition 2.4.10. [BM22] Let $(X, c) \in \mathbf{CL}$. We denote by A_c the *Alexandroff modification* of the closure operator c defined by $A_c(A) = \bigcup_{x\in A} c(x)$ for all $A \subseteq X$. i.e., A_c is s_2-closure. The pair (X, A_c) form an Alexandroff closure space.

Proposition 2.4.11. *The category $\mathbf{CL}_A$ is a coreflective full subcategory of* **CL**.

Proof. Let $(Y, d) \in \mathbf{CL}$. Consider the identity function $id : (Y, A_d) \to (Y, d)$.
Let $A \subseteq Y$. We note that for all $x \in A$, $d(x) \subseteq d(A)$. It follows that

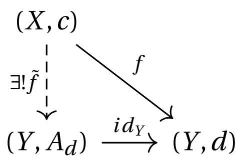
$$A_d(A) = \bigcup_{x\in A} d(x) \subseteq d(A).$$

Then id is continuous.

Now we have to prove that $id : (Y, A_d) \to (Y, d)$ is a coreflection for (Y, d).
Let $(X, c) \in \mathbf{CL}_A$ and let $f : (X, c) \to (Y, d)$ be a morphism in **CL**. Define the map $\tilde{f} : (X, c) \to (Y, A_d)$ given by $\tilde{f}(x) = f(x)$ for all $x \in X$.
Let $A \subseteq X$. Since f is continuous and by the definition of A_d, we have $f(c(x)) \subseteq d(f(x)) = A_d(f(x))$ for all $x \in X$. Then

$$\begin{aligned}\tilde{f}(c(A)) = f(c(A)) &= f(\bigcup_{x\in A} c(x)), \text{ because } c \text{ is Alexandroff closure}\\ &= \bigcup_{x\in A} f(c(x)) \subseteq \bigcup_{x\in A} A_d(f(x)) \subseteq A_d(f(A)) = A_d(\tilde{f}(A)).\end{aligned}$$

So $\tilde{f}(c(A)) \subseteq A_d(\tilde{f}(A))$ and therefore $\tilde{f} : (X, c) \to (Y, A_d)$ is continuous in $\mathbf{CL}_A$.
It is clear that the diagram



commutes, i.e. $id_Y.\tilde{f} = f$ and $\tilde{f}$ is unique.
Therefore, $id_Y : (Y, A_d) \to (Y, d)$ a coreflection for (Y, d). This yields, $\mathbf{CL}_A$ is a coreflective subcategory of $\mathbf{CL}$. □

Corollary 2.4.12. *[BM22] The functor $A : \mathbf{CL} \to \mathbf{CL}_A$ is right adjoint to the inclusion functor $\mathbf{CL}_A \hookrightarrow \mathbf{CL}$. i.e., there is a natural isomorphism*

$$\mathbf{CL}((X, c), (Y, d)) \cong \mathbf{CL}_A((X, c), A(Y, d))$$

for all $(X, c) \in \mathbf{CL}_A$ and $(Y, d) \in \mathbf{CL}$.

Proof. We know by Proposition 2.4.11 that $\mathbf{CL}_A$ is a coreflective full subcategory of $\mathbf{CL}$. By Proposition A.0.28, there exists a unique functor $A : \mathbf{CL} \to \mathbf{CL}_A$ such that $A(X, c) = (X, A_c)$ for each object $(X, c) \in \mathbf{CL}$ and for each morphism $f : (X, c) \to (Y, d)$ in $\mathbf{CL}$, $id_Y.F(f) = f.id_X$. Moreover A is a right adjoint to the inclusion functor by Theorem A.0.29. □

Definition 2.4.13. [BM22] Let $(X, c) \in \mathbf{CL}_A$. Let $x \in X$, set $\rho(x) = \{y \in c(x) | \ x \in c(y)\}$. Define the *symmetrisation* s_c of the closure c to be the symmetric Alexandroff closure given by

$$s_c(A) := \bigcup_{x \in A} \rho(x), \text{ for all } A \subseteq X.$$

Furthermore (X, s_c) is a symmetric Alexandroff closure space.

Proposition 2.4.14. *The category $\mathbf{CL}_{sA}$ is a coreflective full subcategory of $\mathbf{CL}_A$.*

Proof. Let $(Y, d) \in \mathbf{CL}_A$. Consider the identity function $id : (Y, s_d) \to (Y, d)$.
Let $A \subseteq Y$. Note that for every $x \in A$

$$s_d(x) = \rho(x) = \{y \in d(x) | \ x \in d(y)\} \subseteq d(x).$$

It follows that

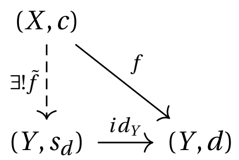
$$s_d(A) = \bigcup_{x \in A} s_d(x) \subseteq \bigcup_{x \in A} d(x) = d(A).$$

Thus id is continuous.

Let $(X, c) \in \mathbf{CL}_{sA}$. Let $f : (X, c) \to (Y, d)$ be a morphism in $\mathbf{CL}_A$.
Consider the map $\tilde{f} : (X, c) \to (Y, s_d)$ defined by $\tilde{f}(x) = f(x)$ for all $x \in X$.
Let $A \subseteq X$ and $y \in \tilde{f}(c(A))$. Since c is Alexandroff closure,

$$\tilde{f}(c(A)) = \bigcup_{x \in A} f(c(x)).$$

Then, there exists $x \in A$ such that $y \in f(c(x))$. It follows by continuity of f that $y \in d(f(x))$. Since $y \in f(c(x))$ then there exists $z \in c(x)$ such that $y = f(z)$. Moreover $x \in c(z)$ because of the symmetricity of the closure c. Then $f(x) \in f(c(z)) \subseteq d(f(z)) = d(y)$. So $\tilde{f}(x) \in d(y)$ and $y \in \tilde{f}(c(x))$.
Therefore $y \in s_d(\tilde{f}(A))$ and $\tilde{f}(c(A)) \subseteq s_d(\tilde{f}(A))$. Hence $\tilde{f} : (X, c) \to (Y, s_d)$ is continuous in $\mathbf{CL}_{sA}$.
The diagram



is commutative, and it is easy to check that $\tilde{f}$ is unique. Then $id_Y : (Y, s_d) \to (Y, d)$ is a coreflection of (Y, d). Hence $\mathbf{CL}_{sA}$ is a coreflective subcategory of $\mathbf{CL}_A$. □

Corollary 2.4.15. *[BM22] The functor $s : \mathbf{CL}_A \to \mathbf{CL}_{sA}$ is right adjoint to the inclusion functor $\mathbf{CL}_{sA} \hookrightarrow \mathbf{CL}_A$.*

Proof. Since $\mathbf{CL}_{sA}$ is a coreflective subcategory of $\mathbf{CL}_A$. By Proposition A.0.28, there exists a unique functor $s : \mathbf{CL}_A \to \mathbf{CL}_{sA}$ that satisfies $s(X,c) = (X,s_c)$ for every object $(X,c) \in \mathbf{CL}_A$, $id_Y.s(f) = f.id_X$ for each morphism $f : (X,c) \to (Y,d)$ in $\mathbf{CL}_A$. By Theorem A.0.29, s is right adjoint to the inclusion functor $\mathbf{CL}_{sA} \hookrightarrow \mathbf{CL}_A$. □

2.4.16 Correspondence between Graphs and Closure Spaces

Following the construction given in Definition 2.4.4 and 2.4.6; we recall that for any $(X,E) \in \mathbf{SDiGph}$, we construct an Alexandroff closure space $(X,c_E) \in \mathbf{CL}_A$ where $c_E(A) = \{y \in X | \exists x \in A,\ xEy\}$ for all $A \subseteq X$. Furthermore, given a closure space $(X,c) \in \mathbf{CL}_A$, we obtain an associated digraph $(X,E_c) \in \mathbf{SDiGph}$ defined by xE_cy if and only if $y \in c(x)$.

Theorem 2.4.17. *[DT95] The functors $\Psi_2 : \mathbf{CL}_A \to \mathbf{SDiGph}$ and $\Phi_2 : \mathbf{SDiGph} \to \mathbf{CL}_A$ define an isomorphism of categories, i.e. $\mathbf{CL}_A \cong \mathbf{SDiGph}$. Furthermore, they restrict to an isomorphism $\mathbf{CL}_{sA} \cong \mathbf{Gph}$.*

Proof. Let $(X,c) \in \mathbf{CL}_A$. We have $\Phi_2\Psi_2(X,c) = \Phi_2(X,E_c) = (X,c_{E_c})$.
Let $A \subseteq X$. We have

$$c_{E_c}(A) = \{y \in X | \exists x \in A,\ xE_cy\} = \{y \in X | \exists x \in A,\ y \in c(x)\} = c(A).$$

So, the closures c and c_{E_c} are equal and $(X,c) = (X,c_{E_c})$. Then $\Phi_2\Psi_2(X,c) = id_{\mathbf{CL}_A}(X,c)$. Furthermore for all $f : (X,c) \to (Y,d) \in \mathbf{CL}_A$, we have $\Phi_2\Psi_2 f = id_{\mathbf{CL}_A} f$. Hence $\Phi_2\Psi_2 = id_{\mathbf{CL}_A}$.

On the other hand, given $(X,E) \in \mathbf{SDiGph}$ we have $\Psi_2\Phi_2(X,E) = \Psi_2(X,c_E) = (X,E_{c_E})$. For all $(x,y) \in E_{c_E}$,

$$xE_{c_E}y \iff y \in c_E(x) \iff xEy$$

Therefore $\Psi_2\Phi_2(X,E) = id_{\mathbf{SDiGph}}(X,E)$. Moreover for all $f : (X,E) \to (Y,F)$ in $\mathbf{SDiGph}$, $\Psi_2\Phi_2 f = id_{\mathbf{SDiGph}} f$. Thus $\Psi_2\Phi_2 = id_{\mathbf{SDiGph}}$. □

2.4.18 Closure Spaces from Metric Spaces

We construct a functor from a category of metric spaces to the category of closure spaces. Denote by **Met** the category of metric spaces where whose objects are metric spaces and whose morphisms are non-expansive maps between them. i.e., every morphism $f : (X,d_X) \to (Y,d_Y)$ in **Met** must satisfies $d_Y(f(x),f(y)) \le d_X(x,y)$ for all $x,y \in X$. We recall that for a given a metric space $(X,d) \in \mathbf{Met}$ and real number $\epsilon \ge 0$,

$$B_\epsilon(x) := \{y \in X | d(x,y) \le \epsilon\}.$$

Definition 2.4.19. Let $(X,d) \in \mathbf{Met}$ and $\epsilon \ge 0$. We have a closure operator defined as follows: for every $A \subseteq X$,

$$c_{\epsilon,d}(A) = \bigcup_{x \in A} B_\epsilon(x)$$

For every $\epsilon \ge 0$, the pair $(X,c_{\epsilon,d})$ is called a *metric closure space*.

Proposition 2.4.20. *Let $\epsilon \ge 0$. Every morphism $f : (X,d_X) \to (Y,d_Y)$ in **Met** induces a continuous map $f : (X,c_{\epsilon,d_X}) \to (Y,c_{\epsilon,d_Y})$ in **CL**.*

Proof. Let $\epsilon \ge 0$ and let $f : (X,d_X) \to (Y,d_Y)$ be a morphism in **Met**.
Let $A \subseteq X$ and $y \in f(c_{\epsilon,d_X}(A))$. There exists $x \in c_{\epsilon,d_X}(A)$ such that $y = f(x)$. Then, there exists $a \in A$ such that $x \in B_\epsilon$, that is $d_X(x,a) \le \epsilon$. Therefore, there exists $b \in f(A)$ such that $b = f(a)$. We note that

$$d_Y(y,b) = d_Y(f(x),f(a)) \le d_X(x,a) \le \epsilon.$$

So $y \in B_\epsilon(f(a))$. That implies

$$y \in \bigcup_{u \in f(A)} B_\epsilon(u) = c_{\epsilon,d_Y}(f(A)).$$

Thus $f(c_{\epsilon,d_X}(A)) \subseteq c_{\epsilon,d_Y}(f(A))$, and $f : (X, c_{\epsilon,d_X}) \to (Y, c_{\epsilon,d_Y})$ is continuous. □

Proposition 2.4.21. *For every $\epsilon \geq 0$ there is a functor M_ϵ :* **Met** → **CL** *sending a metric space (X, d) to a closure space $(X, c_{\epsilon,d})$ and morphism* **Met** *to a continuous map in* **CL**.

Proof. Define $M_\epsilon(X, d) = (X, c_{\epsilon,d})$ for all $(X, d) \in$ **Met** and $M_\epsilon(f) = f$ for every morphism f in **Met**. By Proposition A.0.6, M_ϵ is a functor. □

We note that for every $\epsilon \geq 0$ the closure operator $c_{\epsilon,d}$ is not necessarily idempotent and the functor M_ϵ : **Met** → **CL** is not the composition **Met** → **Top** ↪ **CL**. We can construct a functor from **Met** to **CL** in that manner but in this case we will always have a Kuratowski closure space.

Chapter 3

Simplicial Set Theory and Homology for Closure Space

Our first step is to recapitulate some basic notions of simplicial sets. After that, we will describe the construction of a simplicial set from a closure space and vice versa. We will see that simplicial sets can be useful in the construction of certain homology and cohomology theories. In addition, we will finish this chapter by providing a connection between the category of simplicial abelian groups and the category of non-negative chain complexes.

3.1 Simplicial Set

Definition 3.1.1. [GJ09] We define $\mathbf{\Delta}$ to be the category of finite ordered numbers defined as follows:

- its objects are totally ordered sets of the form $[n] := \{0 < 1 < 2 < \cdots < n\}$ for $n \in \mathbb{N}$,
- its morphisms are non decreasing maps. i.e., for any morphism $\theta : [m] \to [n]$ in $\mathbf{\Delta}$, for all $i, j \in [m]$, if $i < j$ then $\theta(i) \leq \theta(j)$.

Definition 3.1.2. [GJ09] Let $[n] \in \Delta$ and $i \in [n]$.

i) The *i-th coface map* $d^i : [n-1] \rightarrowtail [n]$ is the unique injective monotone map that omits *i-th* entry; it is defined as follows: for all $k \in [n-1]$

$$d^i(k) = \begin{cases} k & if\ k < i, \\ k+1 & if\ k \geq i. \end{cases}$$

$$[0] \overset{d^0}{\underset{d^1}{\rightrightarrows}} [1] \;\rightrightarrows\!\!\!\!\!\!\rightarrow\; [2] \;\rightrightarrows\!\!\!\!\!\!\rightrightarrows\; [3] \cdots$$

ii) The *i-th codegeneracy map* $s^i : [n+1] \twoheadrightarrow [n]$ is the unique surjective map which repeats the *i-th* value. Explicitly, for all $k \in [n+1]$

$$s^i(k) = \begin{cases} k & if\ k \leq i, \\ k-1 & if\ k > i. \end{cases}$$

$$\cdots [3] \;\rightrightarrows\!\!\!\!\!\!\rightarrow\; [2] \overset{s^0}{\underset{s^1}{\rightrightarrows}} [1] \xrightarrow{s^0} [0]$$

Of course, the maps d^i and s^i defined above also depend on n, but we do not indicate that in the notation for simplicity.

Proposition 3.1.3. *[GJ09] We have the following cosimplicial identities:*

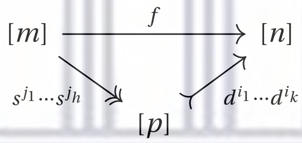
$$\begin{aligned} d^j d^i &= d^i d^{j-1} \qquad if\ i<j,\\ s^j s^i &= s^i s^{j+1} \qquad if\ i \le j,\\ s^j d^i &= \begin{cases} d^i s^{j-1} & if\ i<j,\\ d^{i-1} s^j & if\ i>j+1,\\ id & if\ i=j\ or\ i=j+1.\end{cases}\end{aligned}$$

Proof. We only prove the first identity, the proof of the other identities is analogous.
Let $[n] \in \mathbf{\Delta}$ and $i, j \in [n]$. Notice that $d^j d^i$ and $d^i d^{j-1}$ are mapping from $[n]$ to $[n+2]$ in $\mathbf{\Delta}$. Suppose that $i<j$.
If $k<i$, then $d^j d^i(k) = d^j(k) = k$ and $d^i d^{j-1}(k) = d^i(k) = k$.
If $i \le k < j$, then $d^j d^i(k) = d^j(k+1) = k+1$ and $d^i d^{j-1}(k) = d^i(k+1) = k+1$.
If $j \le k$, then $d^j d^i(k) = d^j(k+1) = k+2$ and $d^i d^{j-1}(k) = d^i(k+1) = k+2$.
Thus, we have $d^j d^i = d^i d^{j-1}$ for all $i<j$. □

Lemma 3.1.4. *[Mac98] Any arrow $f:[m] \to [n]$ in $\mathbf{\Delta}$ has a unique representation $f = d^{i_1} \cdots d^{i_k} s^{j_1} \cdots s^{j_h}$ where*

$$n \ge i_1 > \cdots > i_k \ge 0,\ \ 0 \le j_1 < \cdots < j_h < m$$

and the ordinal numbers h and k satisfy $m-h = n-k = p$.



Proof. Any map f in $\mathbf{\Delta}$ is determined by its image. Let A be a set of elements of $[n]$ that are not in the image of $[m]$. That is

$$A = \{k \in [n] \mid k \notin f([m])\}.$$

Denote by $i_1, \cdots, i_k$ those elements and order it as $0 \le i_k < \cdots < i_1 \le n$. We then have the injection $m = d^{i_1} \cdots d^{i_k} : [p] \to [n]$.
On the other hand, let B be the set of elements of $[m]$ that does not increase, i.e.

$$B = \{k \in [m] \mid f([k]) = f([k+1])\}.$$

Denote by $0 \le j_1 < \cdots < j_h < m$ those elements of B. Note that we set $p = m-h = n-k$. We define the map e to be the surjection $s^{j_1} \cdots s^{j_h} : [m] \to [p]$ and we obtain $f = me$. The unicity of the coface and the degeneracy maps implies the unicity of the factorization. □

Definition 3.1.5. [GJ09]

1. A *simplicial set* is a functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$, also called a presheaf from $\mathbf{\Delta}$. Generally, we say *simplicial object* if the category **Set** is replaced by any category **C**. For every $[n] \in \mathbf{\Delta}$, any element of $X[n]$ is called n-simplice.

2. A morphism $X \to Y$ of simplicial sets, called *simplicial map*, is a natural transformation from the functor X to Y..

We define the category of simplicial sets denoted by **sSet** to be the functor category $\mathbf{Set}^{\mathbf{\Delta}^{op}}$.

Example 3.1.6. Let $n \in \mathbb{N}$. A *standard n-simplex* $\mathbf{\Delta}^n$ is a simplicial set

$$\hom_{\mathbf{\Delta}}(-, [n]) : \mathbf{\Delta}^{op} \to \mathbf{Set}$$

mapping any object $[m] \in \mathbf{\Delta}$ to the set $\hom_{\mathbf{\Delta}}([m],[n])$ and any morphism $f:[p]\to[q]$ in $\mathbf{\Delta}$ to the function

$$\hom(f,[n]) : \hom_{\mathbf{\Delta}}([q],[n]) \to \hom_{\mathbf{\Delta}}([p],[n])$$

given by the composition hf for any $h \in \hom_{\mathbf{\Delta}}([q],[n])$.

Remark 3.1.7. [Mac98]
For each integer $n \geq 0$, we denote by $|\Delta^n| \subset \mathbb{R}^{n+1}$ the *topological standard n-simplex* defined by

$$|\Delta^n| = \left\{ (t_0,\dots,t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\},$$

together with the induced topology from $\mathbb{R}^n$.

We have a functor $^{-} : \mathbf{\Delta} \to \mathbf{Top}$ sending any finite ordered set $[n] \in \mathbf{\Delta}$ to a standard n-simplex $|\Delta^n| \in \mathbf{Top}$ and any order preserving function $\theta : [m] \to [n]$ in $\mathbf{\Delta}$ to a continuous map $\bar{\theta} : |\Delta^m| \to |\Delta^n|$ in $\mathbf{Top}$, where

$$\bar{\theta}(t_0,\dots,t_m) = (s_0,\dots,s_n) \text{ with } s_i = \sum_{\theta(j)=i} t_j.$$

Definition 3.1.8. Let X be a topological space. A *singular simplicial set* is the simplicial set $S_X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ defined by

· $S_X([n]) = \hom_{\mathbf{Top}}(|\Delta^n|, X)$ for all $[n] \in \mathbf{\Delta}$,

· given $\theta : [n] \to [m]$ in $\mathbf{\Delta}$, the map $S_X(\theta) : \hom_{\mathbf{Top}}(|\Delta^m|, X) \to \hom_{\mathbf{Top}}(|\Delta^n|, X)$ is defined by $S_X(\theta)(\sigma) = \sigma\bar{\theta}$ for any $\sigma \in S_X([m])$ and $\bar{\theta}$ is the map induced by θ.

This allows us to define a functor from the category of topological spaces **Top** to the category of simplicial sets **sSet**. Furthermore, it is worth noting that this singular simplicial set is the key of the construction of the singular homology for a given topological space. A well-known reference for more detailed treatement of singular homology is for example [Hat02] and [Mun84].

Proposition 3.1.9. *[GJ09] A simplicial set $X \in$ **sSet** is equivalently specified by a collection of sets $\{X_n\}_{n\in\mathbb{N}}$ together with a map $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i & & if\ i < j, \\ s_i s_j &= s_{j+1} s_i & & if\ i \leq j, \end{aligned}$$
$$d_i s_j = \begin{cases} s_{j-1} d_i & if\ i < j, \\ s_j d_{i-1} & if\ i > j+1, \\ id & if\ i = j \text{ or } i = j+1. \end{cases}$$

Those maps d_i are called face maps while s_i are degeneracy maps.

Proof. Let $X \in$ **sSet**. We define the set X_n by the image of $[n]$ of the simplicial set X, i.e., $X_n := X([n])$ for all $[n] \in \mathbf{\Delta}$. Let $i \in [n]$, define the map $d_i := X(d^i)$ and $s_i := X(s^i)$, where d^i, s^i are the i-th coface and codegeneracy maps in Definition 3.1.2. By functoriality of X and the cosimplicial identities in Proposition 3.1.3 we obtain the identities above.

Conversely, we define the contravariant functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ by assigning any object $[n] \in \mathbf{\Delta}$ to the set $X([n]) := X_n$.
Let $f : [m] \to [n]$ in $\mathbf{\Delta}$. By the lemma 3.1.4, f can be written uniquely as a composition of some s^i and d^i, that is $f = d^{i_1} \cdots d^{i_h} s^{j_1} \cdots s^{j_k}$. We then set $X(f) = s_{j_k} \cdots s_{j_1} d_{i_h} \cdots d_{i_1}$ and it is clear that X contravariant functor. □

Example 3.1.10. We recall that the set of points $\{x_0,\dots,x_n\}\subset\mathbb{R}^m$ is said to be *geometrically independent* if for every real numbers $t_0,...,t_n$ such that $\sum_{i=0}^n t_i=0$ and $\sum_{i=0}^n t_ix_i=0$ implies $t_0=t_1=\cdots=t_n=0$. Let $\{x_0,\dots,x_n\}$ be a geometrically independent set. We define the *n-simplex* σ spanned by $\{x_0,\dots,x_n\}$ to be the smallest convex set containing $\{x_0,\dots,x_n\}$, it is then given by the set

$$\sigma=\left\{x\in\mathbb{R}^m|\ x=\sum_{i=0}^n t_ix_i,\ \sum_{i=0}^n t_i=1,\ t_i\geq 0\right\}.$$

A subset $\tau\subseteq\sigma$ spanned by a subset of $\{x_0,\dots,x_n\}$ is called *face* of σ.
A *(geometric) simplicial complex* K in $\mathbb{R}^m$ is a collection of simplices in $\mathbb{R}^m$ such that:

-For every $\sigma\in K$, if τ is a face of σ then $\tau\in K$,

-The non empty intersection of any two simplexes of K is a face of both of them.

Now, let us associate an orientation for a given simplex σ spanned by $\{x_0,\dots,x_n\}$. For any permutation π of the symmetric group S_{n+1}, we say that

$$\{x_0,\dots,x_n\}\sim\{x_{\pi(0)},\dots,x_{\pi(n)}\}\iff \pi \text{ is even permutation.}$$

The orderings of the vertices of σ is then fall into two equivalence classes. Each of those classes is called *orientation* of σ, and denoted again by $\sigma=[x_0,\dots,x_n]$. We say that σ is an *oriented simplex* if σ is equipped with an orientation. More detailed treatment about simplicial complexes is given in [Mun84].

We are going to define a simplicial set associated to a given simplicial complex.
Let K be a simplicial complex. We define a simplicial set $K:\mathbf{\Delta}^{op}\to\mathbf{Set}$ as follows:
For every $n\in N$,

$$K_n:=\{[x_0,\dots,x_n]|\ [x_0,\dots,x_n] \text{ is an oriented n-simplex of } K\}.$$

For every morphism $\alpha:[m]\to[n]$ in $\mathbf{\Delta}$, $K(\alpha):K_n\to K_m$ is given by

$$K(\alpha)([x_0,\dots,x_n])=[x_{\alpha(0)},\dots,x_{\alpha(n)}].$$

Note that

$$d_i[x_0,\dots,x_n]=[x_0,\dots,x_{i-1},x_{i+1},\dots,x_n] \text{ and } s_i[x_0,\dots,x_n]=[x_0,\dots,x_i,x_i,\dots,x_n].$$

Of course that the maps $d_i:K_n\to K_{n-1}$ and $s_i:K_n\to K_{n+1}$ satisfies the identities in Proposition 3.1.9. One should realize that the map d_i sends n-simplex $[x_0,\dots,x_n]$ to the face $[x_0,\dots,x_{i-1},x_{i+1},\dots,x_n]$ opposite of the vertex x_i.

Example 3.1.11. Let G be a group. A *nerve* of G is a simplicial set BG given by the family $\{BG_n\}_{n\in\mathbb{N}}$ where $BG_0=\{1\}$ and $BG_n=G^n$ for all $n\geq 1$. The face maps $d_i:BG_n\to BG_{n-1}$ and degeneracy maps $s_i:BG_n\to BG_{n+1}$ are defined by

$$d_i(g_1,\dots,g_n)=\begin{cases}(g_2,\dots,g_n) & \text{if } i=1\\ (g_1,\dots,g_ig_{i+1},\dots,g_n) & \text{if } 2\leq i\leq n-1\\ (g_1,\dots,g_{n-1}) & \text{if } i=n\end{cases}$$

and

$$s_i(g_1,\dots,g_n)=(g_1,\dots,g_i,1,g_{i+1},\dots,g_n),\ \text{for each } 1\leq i\leq n.$$

Generally, one can define a nerve of a small category $\mathbf{C}$ to be a functor $N\mathbf{C}:\mathbf{\Delta}^{op}\to\mathbf{Set}$ that sends each $[n] \in \mathbf{\Delta}$ to $N\mathbf{C}_n = \hom_{\mathbf{Cat}}([n], \mathbf{C})$, any element $\sigma_n \in N\mathbf{C}_n$ can be seen as a composition of maps

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n$$

where all those C_i are objects of the category $\mathbf{C}$.
The maps $d_i : N\mathbf{C}_n \to N\mathbf{C}_{n-1}$ sends

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n$$

to

$$\begin{cases} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n & \text{if } i = 0 \\ C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_i f_{i+1}} C_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} C_n & \text{if } 1 \leq i < n \\ C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} & \text{if } i = n. \end{cases}$$

Moreover, the maps $s_i : N\mathbf{C}_n \to N\mathbf{C}_{n+1}$ assigns each

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n$$

to

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} C_i \xrightarrow{id} C_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} C_n.$$

There exists a functor $N : \mathbf{Cat} \to \mathbf{sSet}$ mapping each small category $\mathbf{C}$ to the simplicial sets $N\mathbf{C}$.

Definition 3.1.12. Let $x \in X_{n+1}$. We say that x is degenerate simplice if there exists $y \in X_n$ and $i \in [n]$ such that $s_i(y) = x$.

Lemma 3.1.13. *[GJ09] Given a simplicial set $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ and an object $[n] \in \mathbf{\Delta}$,*

$$\mathbf{sSet}(\mathbf{\Delta}^n, X) \cong X_n.$$

Proof. Since $\mathbf{\Delta}$ is a small category, applying the Yoneda lemma A.0.16 we get the result. This lemma tells us that any n-simplice $x \in X_n$ can be identified as a simplicial map $\mathbf{\Delta}^n \to X$. □

3.2 Nerves and Geometric Realization

The excellent reference for the material given in this section is [GJ09], where we can find the theory of nerves and geometric realization in the case of the category of topological spaces. There exists an analogous generalization of this theory in [BM21] for the category of Čech closure spaces **CL**. Here, we will give an explicit detailed construction of what Bubenik and Milićević outlined in [BM21].

Definition 3.2.1. [GJ09] Given a simplicial set $X \in \mathbf{sSet}$. A *simplicial category* $\mathbf{\Delta} \downarrow X$ is a category where:

- whose objects are morphisms of the form $\mathbf{\Delta}^n \to X$,
- whose morphisms are a commutative diagram of a simplicial map

$$\begin{array}{ccccc} & & X & & \\ & \overset{y}{\nearrow} & & \overset{x}{\nwarrow} & \\ \mathbf{\Delta}^m & & \xrightarrow{\alpha} & & \mathbf{\Delta}^n \end{array}$$

Lemma 3.2.2. *Let X be a simplicial set. Then*

$$X \cong \varinjlim_{\mathbf{\Delta}^n \to X \in \mathbf{\Delta}\downarrow X} \mathbf{\Delta}^n,$$

Proof. We use the lemma A.0.20. Since $\mathbf{\Delta}$ is a small category, any functor $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ is a colimit of representable functors $\mathbf{\Delta}^n = \hom_{\mathbf{\Delta}}(-,[n])$ for some objects $\mathbf{\Delta}^n \to X$ in $\mathbf{\Delta}\downarrow X$. □

Remark 3.2.3. Given a simplicial map $f : X \to Y$ in **sSet**. The simplicial map f induces a functor $f^* : \mathbf{\Delta}\downarrow X \to \mathbf{\Delta}\downarrow Y$ given by the composition of f. Furthermore, for every simplicial set X there is a forgetful functor $\mathcal{U}_X : \mathbf{\Delta}\downarrow X \to \mathbf{\Delta}$ that sends each object $\mathbf{\Delta}^n \to X$ to the object $[n]$.

Definition 3.2.4. [BM21] Given a simplicial set $X \in$ **sSet**. The *geometric realization* $|X|$ of X is defined by the colimit

$$|X| := \varinjlim_{\mathbf{\Delta}^n \to X \in \mathbf{\Delta}\downarrow X} |\Delta^n|,$$

in the category of closure space **CL**.

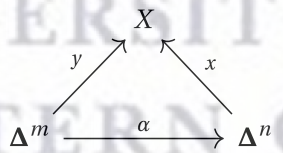
Consider the diagram functor $F_X : \mathbf{\Delta}\downarrow X \to \mathbf{CL}$ mapping each $\mathbf{\Delta}^n \to X$ to the closure space $(|\Delta^n|, k_n)$ where k_n is the Kuratowski closure operator from the topology of $|\Delta^n|$ as a subspace of $\mathbb{R}^{n+1}$. Precisely the functor F_X is given by following composition

$$\mathbf{\Delta}\downarrow X \xrightarrow{\mathcal{U}_X} \mathbf{\Delta} \xrightarrow{\ \overline{\ \ }\ } \mathbf{Top} \xhookrightarrow{\iota} \mathbf{CL}.$$

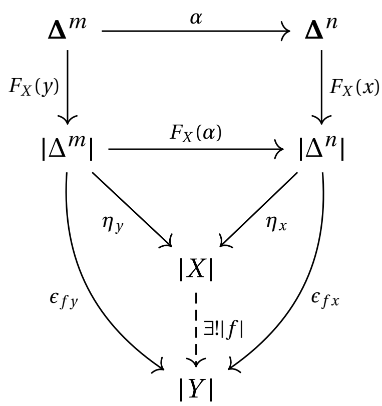
Since $\mathbf{\Delta}\downarrow X$ is a small category and **CL** is cocomplete by Theorem 2.3.8, then there exists a unique colimit of the diagram F_X in the category **CL**. Thus $|X|$ is well defined.

Proposition 3.2.5. *Any simplicial map $f : X \to Y$ in* **sSet** *induces a continuous map $|f| : |X| \to |Y|$ in* **CL**.

Proof. Let $f : X \to Y$ be a simplicial map in **sSet**. Let



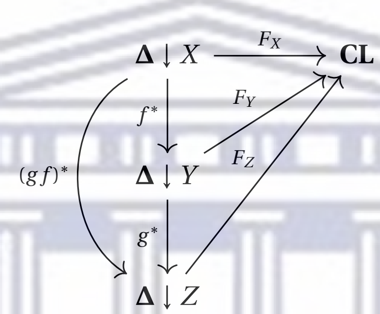
be a morphism in $\mathbf{\Delta}\downarrow X$.
Consider the functor $F_X : \mathbf{\Delta}\downarrow X \to \mathbf{CL}$ mapping $\mathbf{\Delta}^n \to X$ to $(|\Delta^n|, k_n)$. By the definition of the colimit $|X|$, there exists a cocone $(|X|,\eta)$ where $\eta : F_X \Longrightarrow |X|$ such that $\eta_x.F_X(\alpha) = \eta_y$. By the definition of the colimit $|Y|$, there exists a cocone $(|Y|,\epsilon)$ where $\epsilon : F_Y \Longrightarrow |Y|$. Furthermore, we define the natural transformation $\epsilon f^* : F_Y f^* \Longrightarrow |Y|$, where the components are given by $(\epsilon f^*)_x = \epsilon_{fx}$ for all $x \in \mathbf{\Delta}\downarrow X$.



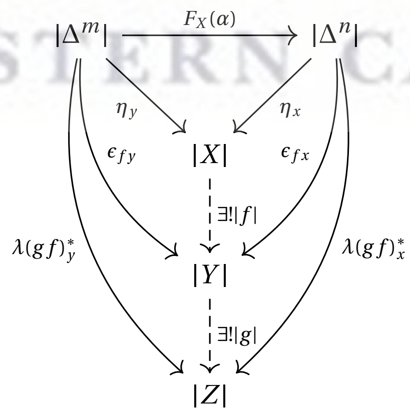
Applying the functor f^* to the morphism α in $\mathbf{\Delta} \downarrow X$, we have a morphism $f^*\alpha$ in $\mathbf{\Delta} \downarrow Y$, which satisfies $\epsilon_{fx} F_Y(f^*\alpha) = \epsilon_{fy}$ because $(|Y|, \epsilon)$ is a cocone.
Since $F_Y f^* = F_X$ then $(\epsilon f^*)_x F_X(\alpha) = (\epsilon f^*)_y$. Therefore $(|Y|, \epsilon f^* : F_X \Longrightarrow |Y|)$ is also a cocone. Moreover, it follows from the definition of the colimit $|X|$ that there exists a unique continuous maps $|f| : |X| \to |Y|$ in the **CL** such that $|f|\eta_y = \epsilon_{fy}$ and $|f|\eta_x = \epsilon_{fx}$. □

Proposition 3.2.6. *[BM21] There is a functor* $|-| : \mathbf{sSet} \to \mathbf{CL}$ *mapping a simplicial set* $X \in \mathbf{sSet}$ *to a closure space* $|X| \in \mathbf{CL}$*, and simplicial map to a continuous map in* **CL**. *This functor is called realization functor.*

Proof. The mapping on objects is defined as in Definition 3.2.4, while on morphisms it is defined as in Proposition 3.2.5.
Let $f : X \to Y$ and $g : Y \to Z$ in **sSet**. We proceed the same analogy of the proof of Proposition 3.2.5.
By the definition of the colimit $|Z|$, there exists a cocone $(|Z|, \lambda)$ where $\lambda : F_Z \Longrightarrow |Z|$.
We note that the following diagram is commutative.



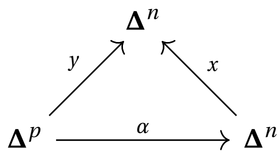
Applying the functor $g^* : \mathbf{\Delta} \downarrow Y \to \mathbf{\Delta} \downarrow Z$ to the morphism $f^*\alpha$ in $\mathbf{\Delta} \downarrow Y$, we obtain the morphism $g^*(f\alpha)$ in $\mathbf{\Delta} \downarrow Z$.
Since $(|Z|, \lambda)$ is a cocone then $\lambda_{(gf)x} F_Z(g^*(f^*\alpha)) = \lambda_{(gf)y}$. Moreover, we have $F_Z g^* = F_Y$ and $\lambda_g = \lambda g^*$ then $\lambda g^*_{fx} F_Y(f(\alpha)) = \lambda g^*_{fy}$. Therefore $(|Z|, \lambda g^* : F_Y \Longrightarrow |Z|)$ is also a cocone.
By the definition of the colimit $|Y|$, there exists a unique continuous map $|g| : |Y| \to |Z|$ such that $|g|\epsilon_{fx} = \lambda(gf)^*_x$ and $|g|\epsilon_{fy} = \lambda(gf)^*_y$.
However, we also have a cocone $(|Z|, \lambda(gf)^* : F_X \Longrightarrow |Z|)$.



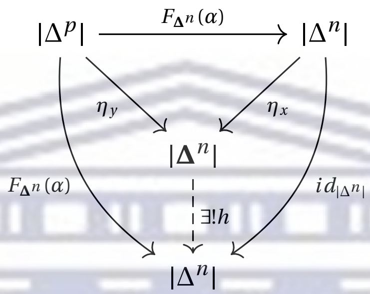
Furthermore, the definition of the colimit $|X|$ implies that there exists a unique continuous map from $|X|$ to $|Z|$ such that $\eta_x|gf| = \lambda(gf)^*_x$ and $\eta_y|gf| = \lambda(gf)^*_y$. Then, we have $|gf| = |g||f| : |X| \to |Z|$.
Finally, the identity simplicial map id_X in **sSet** induces a continuous identity map $id_{|X|}$ in **CL**. Thus $|-|$ is a functor. □

Proposition 3.2.7. *The geometric realization of the standard n-simplex set* $\mathbf{\Delta}^n \in \mathbf{sSet}$ *is homeomorphic to* $(|\Delta^n|, k_n)$.

Proof. Consider the diagram functor $F_{\mathbf{\Delta}^n} : \mathbf{\Delta} \downarrow \mathbf{\Delta}^n \to \mathbf{CL}$ sending each object $\mathbf{\Delta}^p \to \mathbf{\Delta}^n$ to the closure space $(|\Delta^p|, k_p)$.
Consider the morphism



in $\mathbf{\Delta} \downarrow \mathbf{\Delta}^n$.
By the definition of the colimit $|\mathbf{\Delta}^n|$, there exists a natural transformation $\eta : F_{\mathbf{\Delta}^n} \Longrightarrow |\Delta^n|$ such that $(|\Delta^n|, \eta)$ is a cocone. We then have $\eta_x F_{\mathbf{\Delta}^n} \alpha = \eta_y$. Therefore, there exists a continuous map $h : |\mathbf{\Delta}^n| \to |\Delta^n|$ such that $h\eta_x = id_{|\Delta^n|}$ and $h\eta_y = F_{\mathbf{\Delta}^n}(\alpha)$.



So we have

$$h\eta_y = F_{\mathbf{\Delta}^n}(\alpha) \implies (\eta_x h)\eta_y = \eta_x(h\eta_y) = \eta_x F_{\mathbf{\Delta}^n}(\alpha) = \eta_y.$$

Then $\eta_x h = id_{|\mathbf{\Delta}^n|}$.
Thus $h : |\mathbf{\Delta}^n| \to |\Delta^n|$ is homeomorphism with inverse $\eta_x : |\Delta^n| \to |\mathbf{\Delta}^n|$. □

Definition 3.2.8. [BM21] Let (X, c) be a closure space. A *nerve* of X is a simplicial set $\mathcal{J}(X) : \mathbf{\Delta}^{op} \to$ **Set** defined as the following:

- for all $n \geq 0$, $\mathcal{J}(X)_n = \mathbf{CL}((|\Delta^n|, k_n), (X, c))$ and

- for each morphism $f : [m] \to [n]$ in $\mathbf{\Delta}$,

$$\mathcal{J}(X)(f) : \mathbf{CL}((|\Delta^n|, k_n), (X, c)) \longrightarrow \mathbf{CL}((|\Delta^m|, k_m), (X, c))$$
$$\sigma \longmapsto \sigma \bar{f}$$

where $\bar{f} : (|\Delta^m|, k_m) \to (|\Delta^n|, k_n)$ is continuous map in **CL** induced by f.

Proposition 3.2.9. *[BM21] There is a functor $\mathcal{J} : \mathbf{CL} \to \mathbf{sSet}$ mapping each closure space (X, c) to a simplicial set $\mathcal{J}(X)$ and continuous map $f : (X, c) \to (Y, d)$ to a simplicial map $\mathcal{J}(f) : \mathcal{J}(X) \to \mathcal{J}(Y)$.*

Proof. The mapping on objects is defined as in Definition 3.2.8.
Let $f : (X, c) \to (Y, d)$ be a morphism in **CL**.
Let $[n] \in \mathbf{\Delta}$. We define the components of the natural transformation $\mathcal{J}(f)$ by

$$\mathcal{J}(f)_n : \mathbf{CL}(|\Delta^n|, X) \longrightarrow \mathbf{CL}(|\Delta^n|, Y).$$
$$\sigma \longmapsto f\sigma$$

Let $\theta : [m] \to [n] \in \mathbf{\Delta}$.

$$\begin{array}{ccc} \mathbf{CL}((|\Delta^n|, k_n), (X, c)) & \xrightarrow{\mathcal{J}(f)_n} & \mathbf{CL}((|\Delta^n|, k_n), (Y, d)) \\ {\scriptstyle -\cdot\theta}\downarrow & & \downarrow{\scriptstyle -\cdot\theta} \\ \mathbf{CL}((|\Delta^m|, k_m), (X, c)) & \xrightarrow[\mathcal{J}(f)_m]{} & \mathbf{CL}((|\Delta^m|, k_m), (Y, d)) \end{array}$$

Let $\alpha : (|\Delta^n|, k_n) \to (X, c)$.

$$(-\cdot\theta)\cdot\mathcal{J}(f)_n(\alpha) = (-\cdot\theta)\cdot f\alpha = (f\alpha)\cdot\theta = f\alpha\theta$$
$$\mathcal{J}(f)_m\cdot(-\cdot\theta)(\alpha) = \mathcal{J}(f)_m\cdot(\alpha\cdot\theta) = f\alpha\theta$$

Then the diagram above is commutative and $\mathcal{J}(f)$ is a natural transformation.
On the other hand, a routine computation proves that $\mathcal{J}(id_X) = id_{\mathcal{J}(X)}$ and $\mathcal{J}(fg) = \mathcal{J}(f)\mathcal{J}(g)$. □

Theorem 3.2.10. *[BM21] Given a simplicial set X and a closure space (Y, d). We have*

$$\mathbf{CL}(|X|, Y) \cong \mathbf{sSet}(X, \mathcal{J}(Y)),$$

which is natural in both variables. That is the realization functor $|-|$ is left adjoint of the functor $\mathcal{J}$.

Proof. Let $X \in \mathbf{sSet}$ and $(Y, d) \in \mathbf{CL}$. We have

$$\begin{aligned} \mathbf{CL}(|X|, Y) &= \mathbf{CL}(\varinjlim_{\Delta^n \to X \in \mathbf{\Delta}\downarrow X} |\Delta^n|, Y) && \text{Definition 3.2.4} \\ &\cong \varprojlim_{\Delta^n \to X \in \mathbf{\Delta}\downarrow X} \mathbf{CL}(|\Delta^n|, Y) && \text{Lemma A.0.21} \\ &= \lim_{\Delta^n \to X \in \mathbf{\Delta}\downarrow X} \mathcal{J}(Y)_n. && \text{Definition 3.2.8} \end{aligned}$$

$$\begin{aligned} \mathbf{sSet}(X, \mathcal{J}(Y)) &= \mathbf{sSet}(\varinjlim_{\Delta^n \to X \in \mathbf{\Delta}\downarrow X} \mathbf{\Delta}^n, \mathcal{J}(Y)) && \text{Lemma 3.2.2} \\ &\cong \varprojlim_{\Delta^n \to X \in \mathbf{\Delta}\downarrow X} \mathbf{sSet}(\mathbf{\Delta}^n, \mathcal{J}(Y)) && \text{Lemma A.0.21} \\ &\cong \varprojlim_{\Delta^n \to X \in \mathbf{\Delta}\downarrow X} \mathcal{J}(Y)_n. && \text{Yoneda lemma A.0.16} \end{aligned}$$

Thus $\mathbf{CL}(|X|, Y) \cong \mathbf{sSet}(X, \mathcal{J}(Y))$. □

It is important to note that the geometric realization allows us to solve one of the fundamental inverse problem of homotopy: Given any group G. Is there a topological space X such that $\pi_i(X, *) = G$? Eilenberg and Mac Lane discovered that the *classification space* $|BG|$ which is the geometric realization of the nerve of G as in Example 3.1.11 is an *Eilenberg Mac Lane space of type* $K(G, 1)$. i.e.,

$$\pi_i(|BG|, *) = \begin{cases} G & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

More advanced investigation into that problem is presented in [May92].

3.3 Homology and Cohomology of Simplicial Set

Our goal in this section is to define the homology and cohomology of simplicial sets, in particular simplicial abelian groups. We recall that a *chain complex* $C_\bullet$ of abelian group is a family $\{(C_n, \partial_n)\}_{n\in\mathbb{Z}}$

of abelian groups, together with group homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_{n-1}\partial_n = 0$ for all $n \in \mathbb{Z}$. A *chain map* $f : C_\bullet \to D_\bullet$ is a family of group homomorphisms $\{f_n : C_n \to D_n\}_{n\in\mathbb{Z}}$ such that $f_{n-1}\partial_n = \partial_{n-1} f_n$ for all $n \in \mathbb{Z}$. We then denote by **Ch**(**Ab**) the category of chain complexes of abelian group, whose objects are chain complexes and whose morphisms are chain maps; we will write only **Ch** to simplify the notation. We restrict our study to the category of non negative chain complexes $\mathbf{Ch}_+$ where the indices of the family of abelian groups vary over $\mathbb{N}$ instead of $\mathbb{Z}$, i.e. $\{(C_n, \partial_n)\}_{n\in\mathbb{N}}$. Those who are interested to the subject of homological algebra can read for example [CE56] and [Wei94].

Definition 3.3.1. A *simplicial abelian group* is a contravariant functor $X : \mathbf{\Delta}^{op} \to \mathbf{Ab}$. We denote by $\mathbf{sAb} := \mathbf{Ab}^{\mathbf{\Delta}^{op}}$ the category of simplicial abelian groups together with simplicial maps between them.

Remark 3.3.2. Since we have the well-known free functor **Set** → **Ab**. By Lemma A.0.14, there is a functor $\mathbb{Z} : \mathbf{sSet} \to \mathbf{sAb}$, which is defined explicitly as the following:

· Any simplicial set X to the simplicial abelian group $\mathbb{Z}X$ given by $\mathbf{\Delta}^{op} \xrightarrow{X} \mathbf{Set} \xrightarrow{\mathbb{Z}} \mathbf{Ab}$. Explicitly, for each $[n] \in \mathbf{\Delta}$ define $\mathbb{Z}X_n = \mathbb{Z}X([n]) := \langle X_n \rangle$, i.e. the free abelian group generated by X_n and for each $\alpha : [m] \to [n] \in \mathbf{\Delta}$,

$$\begin{aligned} \mathbb{Z}X(\alpha) : \mathbb{Z}X_n &\longrightarrow \mathbb{Z}X_m \\ \sum n_x x &\longmapsto \sum n_x (X\alpha)(x), \end{aligned}$$

· Any simplicial map $f : X \to Y$ to the natural transformation $\mathbb{Z}f : \mathbb{Z}X \to \mathbb{Z}Y$, where the components are given for each $[n] \in \mathbf{\Delta}$ by

$$\begin{aligned} \mathbb{Z}f_n : \mathbb{Z}X_n &\longrightarrow \mathbb{Z}Y_n \\ \sum_x n_x x &\longmapsto \sum_x n_x f_n(x). \end{aligned}$$

Moreover, there is a forgetful functor $\mathcal{U} : \mathbf{sAb} \to \mathbf{sSet}$.

Proposition 3.3.3. *The functor $\mathbb{Z}$ is left adjoint of the functor $\mathcal{U}$.*

Proof. Let $X \in \mathbf{sSet}$ and $Y \in \mathbf{sAb}$. Consider the maps

- $\Psi_{X,Y} : \mathbf{sAb}(\mathbb{Z}X, Y) \to \mathbf{sSet}(X, \mathcal{U}Y)$ defined as follows: let $\eta : \mathbb{Z}X \to Y$ with components $\eta_n : \langle X_n \rangle \to Y_n$ for each $[n] \in \mathbf{\Delta}$. The simplicial map η induces a simplicial map $\eta_{|X} : X \to \mathcal{U}Y$ defined by $(\eta_{|X})_n(x) = \eta_n(x)$ for all $x \in X_n$. We then define $\Psi_{X,Y}(\eta) = \eta_{|X}$.
- $\Phi_{X,Y} : \mathbf{sSet}(X, \mathcal{U}Y) \to \mathbf{sAb}(\mathbb{Z}X, Y)$ defined as follows: let $\mu : X \to \mathcal{U}Y$ with components $\mu_n : X_n \to \mathcal{U}Y_n$ for all $[n] \in \mathbf{\Delta}$, the simplicial map μ induces a simplicial map $\bar{\mu} : \mathbb{Z}X \to Y$ defined for each $[n] \in \mathbf{\Delta}$ by $\bar{\mu}_n(\sum n_x x) = \sum n_x \mu_n(x)$ where $n_x \in \mathbb{Z}$. We define $\Phi_{X,Y}(\mu) = \bar{\mu}$.

Let $\mu \in \mathbf{sSet}(X, \mathcal{U}Y)$. We denote by μ_n the components of μ for all $[n] \in \mathbf{\Delta}$. We have

$$\Psi_{X,Y}\Phi_{X,Y}(\mu) = \Psi_{X,Y}(\bar{\mu}) = \bar{\mu}_{|X},$$

where the components of $\bar{\mu}_{|X} : X \Longrightarrow \mathcal{U}Y$ is defined by $(\bar{\mu}_{|X})_n(x) = \mu_n(x)$ for all $[n] \in \mathbf{\Delta}$. Then $(\bar{\mu}_{|X})_n = \mu_n$ for all $[n] \in \mathbf{\Delta}$. It follows that $\Psi_{X,Y}\Phi_{X,Y}(\mu) = \mu$. Thus $\Psi_{X,Y}\Phi_{X,Y} = id_{\mathbf{sSet}(X,\mathcal{U}Y)}$.

On the other hand, let $\eta \in \mathbf{sAb}(\mathbb{Z}X, Y)$. We have

$$\Phi_{X,Y}\Psi_{X,Y}(\eta) = \Phi_{X,Y}(\eta_{|X}) = \overline{\eta_{|X}},$$

where the component of $\overline{\eta_{|X}} : \mathbb{Z}X \Longrightarrow Y$ is defined by $(\overline{\eta_{|X}})_n(\sum n_x x) = \sum n_x \eta(x)$ for all $[n] \in \mathbf{\Delta}$. We then obtain $(\overline{\eta_{|X}})_n = \eta_n$ for all $[n] \in \mathbf{\Delta}$. Therefore $\Phi_{X,Y}\Psi_{X,Y}(\eta) = \eta$. Then $\Phi_{X,Y}\Psi_{X,Y} = id_{\mathbf{sAb}(\mathbb{Z}X,Y)}$.

Hence $\Psi_{X,Y}$ is bijective with inverse $\Phi_{X,Y}$.

It remains us to check the naturality on **sAb** and **sSet**.
Let $Y \in$ **sAb** and $f : B \to A$ be a morphism in **sSet**. We have to prove that the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{sAb}(\mathbb{Z}A, Y) & \xrightarrow{\Psi_{A,Y}} & \mathbf{sSet}(A, \mathcal{U}Y) \\ {\scriptstyle -\cdot\mathbb{Z}f}\downarrow & & \downarrow{\scriptstyle -\cdot f} \\ \mathbf{sAb}(\mathbb{Z}B, Y) & \xrightarrow[\Psi_{B,Y}]{} & \mathbf{sSet}(B, \mathcal{U}Y) \end{array}$$

Let $\eta : \mathbb{Z}A \to Y$ be a morphism in **sAb**. We have

$$(- \cdot f) \cdot \Psi_{A,Y}(\eta) = (- \cdot f) \cdot \eta_{|A} = \eta_{|A} \cdot f$$

and

$$\Psi_{B,Y} \cdot (- \cdot \mathbb{Z}f)(\eta) = \Psi_{B,Y} \cdot (\eta \cdot \mathbb{Z}f) = (\eta \cdot \mathbb{Z}f)_{|B}.$$

Let $[n] \in \mathbf{\Delta}$ and $x \in B_n$.

$$(\eta_A \cdot f)_n(x) = (\eta_A \cdot f_n)(x) = (\eta|A)_n \cdot f_n(x) = \eta_n \cdot f_n(x) = (\eta_n \cdot \mathbb{Z}f_{|B})_n(x).$$

Let $X \in$ **sSet** and $f : A \to B$ be a morphism in **sAb**. We have

$$\begin{array}{ccc} \mathbf{sAb}(\mathbb{Z}X, A) & \xrightarrow{\Phi_{X,A}} & \mathbf{sSet}(X, \mathcal{U}A) \\ {\scriptstyle f\cdot -}\downarrow & & \downarrow{\scriptstyle \mathcal{U}f\cdot -} \\ \mathbf{sAb}(\mathbb{Z}X, B) & \xrightarrow[\Phi_{X,B}]{} & \mathbf{sSet}(X, \mathcal{U}B) \end{array}$$

Let $\eta : \mathbb{Z}X \to A$ be a morphism in **sAb**.

$$(\mathcal{U}f \cdot -) \cdot \Psi_{X,A}(\eta) = (\mathcal{U}f \cdot -) \cdot \eta_{|X} = \mathcal{U}f \cdot \eta_{|X}$$

and

$$\Psi_{X,B} \cdot (f \cdot -)(\eta) = \Psi_{X,B}(f \cdot \eta) = (f \cdot \eta)_{|X} = f \cdot \eta_{|X}.$$

Therefore, for every $[n] \in \mathbf{\Delta}$ and $x \in X_n$, we have

$$(\mathcal{U}f \cdot \eta_{|X})_n(x) = \mathcal{U}f_n \cdot (\eta_{|X})_n(x) = \mathcal{U}f_n \cdot \eta_n(x) = f_n(\eta_n(x)) = (f \cdot \eta_{|X})_n(x).$$

Hence, $\mathbb{Z}$ is left adjoint to $\mathcal{U}$. □

Proposition 3.3.4. *[GJ09] Let $A \in$ **sAb**. There is a non-negative chain complex of abelian groups $A_\bullet$ given by the family of abelian groups $\{A_n\}_{n\in\mathbb{N}}$ together with the boundary map*

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1} \text{ for every } n \in \mathbb{N}.$$

This complex is also called Moore complex or unnormalized complex.

$$\cdots \xrightarrow{\partial_4} A_3 \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

Proof. Let $n \geq 1$.

$$\begin{aligned}
\partial_{n-1}\partial_n &= \sum_{i=0}^{n-1}\left(\sum_{j=0}^{n}(-1)^{i+j}d_i d_j\right)\\
&= \sum_{0\leq j<i\leq n-1}(-1)^{i+j}d_i d_j + \sum_{0\leq i<j\leq n}(-1)^{i+j}d_i d_j\\
&= \sum_{0\leq j<i\leq n-1}(-1)^{i+j}d_i d_j + \sum_{0\leq i<j\leq n}(-1)^{i+j}d_{j-1} d_i \quad \text{(Proposition 3.1.9)}\\
&= \sum_{0\leq j<i\leq n-1}(-1)^{i+j}d_i d_j + \sum_{0\leq i<k\leq n-1}(-1)^{i+k+1}d_k d_i \quad \text{(Set } k=j-1\text{)}\\
&=0.
\end{aligned}$$

Then $\partial_{n-1}\partial_n = 0$ for all $n \geq 1$. □

Remark 3.3.5. Any simplicial map $f : A \to B$ in **sAb** induces a chain map $\bar{f} : A_\bullet \to B_\bullet$ in $\mathbf{Ch}_+$. Let $\{f_n : A_n \to B_n\}_n$ be the components of the simplicial map $f : A \to B$. Then the chain map $\bar{f}$ is given by the family of group homomorphisms $\{\bar{f}_n : A_n \to B_n\}_n$ where $\bar{f}_n(x) = f_n(x)$. The fact that $f : A \to B$ is a natural transformation implies $\bar{f}_{n+1}\partial_{n+1} = \partial'_{n+1}\bar{f}_n$ for all $n \geq 0$. Moreover, if A_n and B_n are free abelian groups then $\bar{f}_n(\sum_{x\in A_n} n_x x) = \sum_{x\in A_n} n_x f_n(x)$.

Remark 3.3.6. An obvious generalization of this construction is possible. Given an abelian group G. The chain complex $A_\bullet(G)$ is defined by the tensor product

$$A_n(G) = \begin{cases} A_n \otimes G & \text{if } n \geq 0. \\ 0 & \text{if } n < 0 \end{cases}$$

Dually, a *cochain complex* $A^\bullet(G)$ with a coefficient in G can be defined as follows:

$$A^n(G) = \begin{cases} \hom_{\mathbf{Ab}}(A_n, G) & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

and the *coboundary map* $\delta^n : A^n(G) \to A^{n+1}(G)$ is given by

$$\delta^n(f) = f\partial_{n+1} = \sum_{i=0}^{n+1}(-1)^i f d_i.$$

We have applied the hom-set contravariant functor $\hom_{\mathbf{Ab}}(-,G) : \mathbf{Ab} \to \mathbf{Ab}$ to the chain complex $A_\bullet(G)$ in order to obtain a cochain complex $A^\bullet(G)$.
In case where $G = \mathbb{Z}$ we write simply $A^\bullet$ for the cochain complex with a coefficient in $\mathbb{Z}$. The coboundary map satisfies $\delta^{n+1}\delta^n = 0$ for all $n \geq 0$.

$$A^0 \xrightarrow{\delta^0} A^1 \xrightarrow{\delta^1} A^2 \xrightarrow{\delta^2} A^3 \xrightarrow{\delta^3} \cdots$$

Definition 3.3.7. Let $A \in$ **sAb**. For every $n \in \mathbb{N}$. We define a *n-(co)cycle group* of A denoted by $Z_n(A)$ (resp. $Z^n(A)$) to be the kernel of the group homomorphism ∂_n (resp. δ^n). i.e.,

$$Z_n(A) := \{x \in A_n |\ \partial_n(x) = 0\} \text{ and } Z^n(A) := \{x \in A_n |\ \delta^n(x) = 0\}.$$

The *n-(co)boundary group* of A denoted by $B_n(A)$ (resp. $B^n(A)$) is the image of ∂_{n+1} (resp. δ^{n-1}). i.e.,

$$B_n(A) := \{y \in A_n |\ \exists x \in A_{n+1},\ y = \partial_{n+1}(x)\}$$

and

$$B^n(A) := \{y \in A_n |\ \exists x \in A_{n-1},\ y = \delta^{n-1}(x)\}.$$

It is easy to show that B_n and Z_n are both subgroup of A_n. Furthermore, for every $n \in \mathbb{N}$ the identity $\partial_n \partial_{n+1} = 0$ is equivalent to saying that $B_n(A)$ is a subgroup of $Z_n(A)$. The same results for the coboundary group and cocycle group. Therefore we can define the following homology and cohomology group.

Definition 3.3.8. [GM96] Given a simplicial abelian group A. We define the *n-th homology of A with a coefficient in* $\mathbb{Z}$ to be the homology of the chain complex $A_\bullet$. That is

$$H_n(A;\mathbb{Z}) = H_n(A_\bullet) := \frac{Z_n(A)}{B_n(A)}, \text{ for every } n \geq 0.$$

Moreover, we define $H_n(A;G) := H_n(A_\bullet(G))$ for each $n \geq 0$ the homology with a coefficient in an abelian group G.
Similarly, the *n-th cohomology of A with a coefficient in G* is defined by

$$H^n(A;G) = H^n(A^\bullet(G)) := \frac{Z^n(A)}{B^n(A)}, \text{ for every } n \geq 0.$$

In fact, the homology (resp. cohomology) measures the failure of the sequence $A_\bullet(G)$ (resp. $A^\bullet(B)$) to be exact.

Remark 3.3.9. Let A be a simplicial abelian group. Denote $A_\bullet$ its corresponding chain complex. The elements of A_n are called *n-chains* of A, while the elements of $Z_n(A)$ (resp. $B_n(A)$) are *n-cycles* (resp. *n-boundaries*) of A. Elements of the group $H_n(A,G)$ are called *n-homology classes*; those are exactly all n-cycles which are not a $(n+1)$-boundaries.

Generally, for every simplicial set $X \in$ **sSet** we define the (co)homology of X to be the (co)homology of the simplicial abelian group $\mathbb{Z}X$.

Example 3.3.10. Let K be a finite simplicial complex. As in Example 3.1.10, we obtain a simplicial set denoted by $K : \mathbf{\Delta}^{op} \to$ **Set**. Its corresponding simplicial abelian group $\mathbb{Z}K : \mathbf{\Delta}^{op} \to$ **Ab** is then characterized by the family of free abelian groups

$$\mathbb{Z}K_n := \langle K_n \rangle = \left\{ \sum_{\sigma \in K_n} n_\sigma \sigma |\ n_\sigma \in \mathbb{Z} \right\}, \text{ for each } n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, if $\sigma = [x_0, \dots, x_n] \in K_n$, we define $\partial_n : K_n \to \mathbb{Z}K_{n-1}$ by

$$\partial_n[x_0, \dots, x_n] = \sum_{i=0}^{n} (-1)^i d_i[x_0, \dots, x_n].$$

Extending it linearly, we have a group homomorphism $\partial_n : \mathbb{Z}K_n \to \mathbb{Z}K_{n-1}$ given by

$$\partial_n(\sum_{\sigma \in K_n} n_\sigma \sigma) = \sum_{\sigma \in K_n} n_\sigma \partial_n(\sigma).$$

It is easy to see that $\partial_{n-1}\partial_n = 0$ for all $n \geq 1$.
A *simplicial (co)homology* of K is the (co)homology group of the simplicial abelian group $\mathbb{Z}K$ as in Definition 3.3.8.

3.3.11 Homology and Cohomology with a Coefficient System

Now, we want to construct a chain complex and a cochain complex with coefficients in a more general way than an abelian group. We can assign that the coefficients at different simplices are taken from a different abelian groups.

Definition 3.3.12. [GM96] Let $X \in \mathbf{sSet}$. A homological coefficient system $\mathcal{A}$ on X consists of

- a family of abelian groups $\mathcal{A}_x$, one for each simplex $x \in X_n$

- a family of group homomorphisms $\mathcal{A}(f,x) : \mathcal{A}_x \to \mathcal{A}_{X(f)x}$, one for each pair $(x, f : [m] \to [n])$ where $x \in X_n$, such that the following conditions are satisfied:

$$\begin{aligned} \mathcal{A}(id,x) &= id; \\ \mathcal{A}(fg,x) &= \mathcal{A}(g,X(f)x)\mathcal{A}(f,x). \end{aligned}$$

Definition 3.3.13. [GM96] Let $X \in \mathbf{sSet}$. A cohomological coefficient system $\mathcal{B}$ on X is

- a family of abelian groups $\mathcal{B}_x$, one for each simplex $x \in X_n$,

- a family of homomorphisms $\mathcal{B}(f,x) : \mathcal{B}_{X(f)x} \to \mathcal{B}_x$, one for each pair $(x, f : [m] \to [n])$ where $x \in X_n$, satisfying

$$\begin{aligned} \mathcal{B}(id,x) &= id \\ \mathcal{B}(fg,x) &= \mathcal{B}(f,x)\mathcal{B}(g,X(f)x). \end{aligned}$$

The homological and cohomological coefficient system on a simplicial set X can be thought of as an object of the functor categories $\mathbf{Ab}^{\Delta\downarrow X^{op}}$ and $\mathbf{Ab}^{\Delta\downarrow X}$ respectively. Not surprisingly, by Yoneda lemma.

Definition 3.3.14. [GM96] Let $\mathcal{A}$ be a homological coefficient system on a simplicial set X. *A chain complex $C_\bullet(X,\mathcal{A})$ of X with a coefficient in $\mathcal{A}$* is a chain complex given by the family of a free abelian groups $\{C_n(X,\mathcal{A})\}_{n\in\mathbb{N}}$, where

$$C_n(X,\mathcal{A}) := \sum_{x\in X_n} \mathcal{A}_x x.$$

The boundary map is the group homomorphism

$$\begin{aligned} \partial_n : C_n(X,\mathcal{A}) &\longrightarrow C_{n-1}(X,\mathcal{A}) \\ \sum_{x\in X_n} a_x x &\longmapsto \partial_n c = \sum_{x\in X_n}\sum_{i=0}^{n} \mathcal{A}(d^i,x)(a_x)\,(-1)^i d_i(x) \end{aligned}$$

for every $n \in \mathbb{N}$. As usual, $\partial_n\partial_{n+1} = 0$ for all $n \geq 0$.

Definition 3.3.15. [GM96] Let $\mathcal{B}$ be a cohomological coefficient system on a simplicial set X. *A cochain complex $C^\bullet(X,\mathcal{B})$ of X with a coefficient in $\mathcal{B}$* is given by the family of abelian groups $\{C^n(X,\mathcal{B})\}_{n\in\mathbb{N}}$ where

$$C^n(X,\mathcal{B}) := \hom_{\mathbf{Set}}(X_n, \bigcup_{x\in X_n} \mathcal{B}_x).$$

For every $n \in \mathbb{N}$, the coboundary map $\delta^n : C^n(X,\mathcal{B}) \longrightarrow C^{n+1}(X,\mathcal{B})$ is defined by

$$\delta^n(f)(x) = \sum_{i=0}^{n+1} (-1)^i \mathcal{B}(d^i,x)(f d_i), \text{ for all } x \in X_{n+1}.$$

Moreover, the coboundary map satisfies $\delta^{n+1}\delta^n = 0$ for all $n \geq 0$.

Definition 3.3.16. [GM96] Let $X \in \mathbf{sSet}$ and $\mathcal{A}$ be a homological system of coefficient on X. *A homology of X with a coefficient in $\mathcal{A}$* is defined by

$$H_n(X,\mathcal{A}) := H_n(C_\bullet(X,\mathcal{A})) \quad \text{for all } n \geq 0.$$

The cohomology of the simplicial set X with a coefficient in a cohomological system $\mathcal{B}$ is defined dually in a similar way.

$$H^n(X,\mathcal{B}) := H^n(C^\bullet(X,\mathcal{B})) \quad \text{for all } n \geq 0.$$

We note that if $\mathcal{A}_x = \mathbb{Z}$ and $\mathcal{A}(f,x) = id$ for all $x \in X_n$, then the homology group $H_\bullet(X,\mathcal{A})$ coincides with the homology defined in Definition 3.3.8.

Now, it is convenient to verify the functoriality of those homology and cohomology groups. Precisely, for every integer $n \geq 0$, $H_n : \mathbf{sAb} \to \mathbf{Ab}$ and $H^n : \mathbf{sAb} \to \mathbf{Ab}$ define a functor, known as *(co)homology functor*.

Proposition 3.3.17. *Let $f : A \to B$ be a morphism in* **sAb**. *The chain map $\bar{f} : A_\bullet \to B_\bullet$ induces a group homomorphism*

$$\begin{aligned} f_* : H_n(A_\bullet) &\longrightarrow H_n(B_\bullet) \\ x + B_n(A) &\longmapsto \bar{f}(x) + B_n(B). \end{aligned}$$

Proof. Recall the properties of a chain map in Remark 3.3.5.
Let $x \in Z_n(A)$. We have

$$\partial'_n \bar{f}_n(x) = \bar{f}_{n-1}\partial_n(x) = \bar{f}_{n-1}(0) = 0.$$

Then $\bar{f}_n(x) \in Z_n(B)$ and therefore $\bar{f}(Z_n(A)) \subseteq Z_n(B)$.
Let $y \in \bar{f}_n(B_n(A))$. There exists $x \in B_n(A)$ such that $y = \bar{f}_n(x)$. There exists $a \in A_{n+1}$ such that $x = \partial_{n+1}(a)$. We have $y = \bar{f}_n(x) = \bar{f}_n(\partial_{n+1}(a)) = \partial'_{n+1}(\bar{f}_{n+1}(a))$. So $y \in B_n(B)$. Thus $\bar{f}(B_n(A)) \subseteq B_n(B)$. Consequently, the mapping f_* given by $f_*(x + B_n(A)) = \bar{f}(x) + B_n(B)$ is well defined. Moreover, since $\bar{f}$ is a group homomorphism, we have

$$f_*(x + y + B_n(A)) = \bar{f}_n(x+y) + B_n(B) = \bar{f}_n(x) + \bar{f}_n(y) + B_n(B) = f_*(x) + f_*(y).$$

It is clear that the image of the identity element of $H_n(A_\bullet)$ is the identity element of $H_n(B_\bullet)$. Therefore, f_* is a homomorphism. □

Proposition 3.3.18. *There exists a functor H_n* : **sAb** → **Ab**.

Proof. Define the mapping on objects by $H_n(A) = H_n(A;\mathbb{Z})$ for every simplicial abelian group A. For any simplicial map $f : A \to B$ in **sAb**, we define $H_n(f) = f_*$.

Let $f : A \to B$ and $g : B \to C$ be simplicial maps in **sAb**. Let us prove first that the chain map induced by the composition gf is the composition of chain maps $\bar{g}\bar{f}$.
Let $x \in A_n$. We have

$$(\bar{g}_n\bar{f}_n)(x) = \bar{g}_n\bar{f}_n(x) = \bar{g}_n(f_n(x)) = g_n(f_n(x)) = (g_nf_n)(x) = (\overline{g_nf_n})(x).$$

Let $x + B_n(A) \in H_n(A)$. We have

$$\begin{aligned} g_*f_*(x + B_n(A)) = g_*(\bar{f}_n(x) + B_n(B)) &= \bar{g}_n(\bar{f}_n(x)) + B_n(C) \\ &= (\bar{g}_n\bar{f}_n)(x) + B_n(C) \\ &= (\overline{g_nf_n})(x) + B_n(C) \\ &= (gf)_*(x + B_n(A)). \end{aligned}$$

Therefore $H_n(gf) = H_n(g)H_n(f)$.
Let $A \in$ **sAb**. We have

$$(id_A)_*(x + B_n(A)) = \bar{id}(x) + B_n(A) = x + B_n(A).$$

So $H_n(id_A) = id_{H_n(A)}$. Consequently, H_n : **sAb** → **Ab** is a functor. □

Dually, we can define by analogy the contravariant functor H^n : **sAb** → **Ab** that maps each simplicial abelian group A to the cohomology group $H^n(A;\mathbb{Z})$.

3.4 Homology and Cohomology for Closure Spaces

Let us begin with the construction of the singular homology for a closure space. We will use the theory of simplicial sets we have defined above. Given a closure space $(X,c) \in \mathbf{CL}$. By the definition 3.2.8, we have the nerve functor $\mathcal{J}(X) : \mathbf{\Delta}^{op} \to \mathbf{Set}$. Applying the functor in Remark 3.3.2 we have the simplicial abelian group $\mathbb{Z}\mathcal{J}(X) : \mathbf{\Delta}^{op} \to \mathbf{Ab}$.

Definition 3.4.1. Let $(X,c) \in \mathbf{CL}$. We define the chain complex $C_\bullet(X)$ associated to the closure space (X,c) to be the chain complex $\mathbb{Z}\mathcal{J}(X)_\bullet$ as in Proposition 3.3.4.

Explicitly, for each $n \geq 0$, we have the free abelian group $C_n(X;\mathbb{Z}) = \mathbb{Z}\mathcal{J}(X)_n$, and the boundary operator ∂_n is defined by

$$\begin{aligned} \partial_n : \mathcal{J}(X)_n &\longrightarrow C_{n-1}(X) \\ \sigma &\longmapsto \partial_n\sigma := \sum_{i=0}^{n}(-1)^i d_i\sigma. \end{aligned}$$

We extend this linearly to a group homomorphism $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ given by $\partial_n(\sum_\sigma n_\sigma \sigma) = \sum_\sigma n_\sigma \partial_n \sigma$, with $n_\sigma \in \mathbb{Z}$.

We then build a cochain complex $C^\bullet(X)$ given by the family of abelian groups $\{C^n(X;\mathbb{Z})\}_{n\in\mathbb{Z}}$ where $C^n(X;\mathbb{Z}) = \hom_{\mathbf{Ab}}(C_n(X;\mathbb{Z}),\mathbb{Z})$ for each $n \in \mathbb{N}$. The coboundary operator $\delta^n : C^n(X;\mathbb{Z}) \to C^{n+1}(X;\mathbb{Z})$ is defined by $\delta^n f = f \cdot \partial_{n+1}$ for each $f \in C_n(X;\mathbb{Z})$.

Definition 3.4.2. Let (X,c) be a closure space. The *singular homology group* of the closure space (X,c) with a coefficient in $\mathbb{Z}$ is defined by

$$H_n(X;\mathbb{Z}) := H_n(\mathbb{Z}\mathcal{J}(X);\mathbb{Z}) = H_n(\mathbf{C}_\bullet(X)), \text{ for every } n \in \mathbb{N}.$$

Dually, the *n-th singular cohomology group* of the closure space (X,c) with a coefficient in $\mathbb{Z}$ is given by

$$H^n(X;\mathbb{Z}) := H^n(C^\bullet(X)), \text{ for every } n \in \mathbb{N}.$$

Whenever we need to emphasize on the closure operator, will write $H_n((X,c);\mathbb{Z})$ or $H_n(X,c)$ to mean the nth homology of the closure space (X,c) with coefficient in $\mathbb{Z}$. Otherwise, we keep writing $H_n(X;\mathbb{Z})$ for simplicity.

We should notice that there exists a singular homology functor $H_n : \mathbf{CL} \to \mathbf{Ab}$ given by the following composition of the functors $\mathcal{J}$, $\mathbb{Z}$ and H_n

$$\mathbf{CL} \xrightarrow{\mathcal{J}} \mathbf{sSet} \xrightarrow{\mathbb{Z}} \mathbf{sAb} \xrightarrow{H_n} \mathbf{Ab}.$$

Remark 3.4.3. Now, one might wonder about some comparison between homology of the simplicial set and singular homology:

The first question is: If $X \in \mathbf{sSet}$, is the homology of the simplicial set X isomorphic to the singular homology of its geometric realization $|X|$? The answer is positive. In fact, extending the approach given by Milnor in case of the category **Top** [Mil57], the simplicial map $X \to \mathcal{J}(|X|)$ is a weak equivalence or quasi-isomorphism, in the sense that it induces an isomorphism $H_n(X;G) \to H_n(\mathcal{J}(|X|);G)$ for all $n \in \mathbb{N}$. Therefore, since $H_n(|X|;G) := H_n(\mathbb{Z}\mathcal{J}(|X|);G)$ we obtain the desired isomorphism $H_n(X;G) \cong H_n(|X|;G)$ for all $n \in \mathbb{N}$.

Another question is formalized as follow: If $(X,c) \in \mathbf{CL}$, is the singular homology of X isomorphic to the singular homology of $|\mathcal{J}(X)|$? Again, Milnor proved that the continuous $|\mathcal{J}(X)| \to X$ is a weak equivalence, i.e. it induces an isomorphism $H_n(|\mathcal{J}(X)|;G) \to H_n(X;G)$ for all $n \in \mathbb{N}$.

We should be careful that this fact does not imply that the space X is homeomorphic to $|\mathcal{J}(X)|$.

Definition 3.4.4. Let (X,c) be a closure space. The *nth Betti number* $\beta_n(X)$ of X is the rank of the homology group $H_n(X;\mathbb{Z})$.

$$\beta_n(X) := \operatorname{rank} H_n(X;\mathbb{Z}).$$

One can notice that Betti numbers are a *topological invariant*, they are preserved under homeomorphisms but the converse is not true. It is also convenient to note that one interprets the nth Betti number as the number of n-dimensional holes in the space. Therefore, $\beta_0(X)$ counts the number of connected components in (X, c), $\beta_1(X)$ is the number of 1-dimensional holes (X, c) which can be seen as a loops or a circle, and $\beta_2(X)$ corresponds to the number of voids in (X, c), which can be seen as the empty space inside a ball. In general, for higher dimension those kind of holes can not be visualized.

Proposition 3.4.5. *If the closure space (X, c) is one point space, then $H_0(X;\mathbb{Z}) = \mathbb{Z}$ and $H_n(X;\mathbb{Z}) = 0$ for all $n > 0$.*

Proof. Denote by x the unique element of X. For each $n \in \mathbb{N}$, there is exactly one n-simplex $\sigma_n : |\Delta^n| \to X$ that sends everything to x. Then $\mathcal{J}_n(X) = \hom_{\mathbf{CL}}(|\Delta^n|, \{x\}) = \{\sigma_n\}$. Therefore, for every integer $n \geq 0$ we have $C_n(X;\mathbb{Z}) = \mathbb{Z}\mathcal{J}_n(X) = \langle\{\sigma_n\}\rangle$.
For $n \geq 1$, we have

$$\partial_n(\sigma_n) = \sum_{i=0}^{n}(-1)^i d_i\sigma_n = \sum_{i=0}^{n}(-1)^i \sigma_{n-1} = \begin{cases} \sigma_{n-1} & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}$$

Where σ_{n-1} is the only simplex in $\mathcal{J}_{n-1}(X)$.
We then obtain the following chain complex:

$$\cdots \longrightarrow \mathbb{Z}\sigma_3 \xrightarrow{0} \mathbb{Z}\sigma_2 \longrightarrow \mathbb{Z}\sigma_1 \xrightarrow{0} \mathbb{Z}\sigma_0 \xrightarrow{0} 0$$

. If $n = 0$, we have $Z_0(X) = \mathbb{Z}\sigma_0$ and $B_0(X) = 0$ then $H_0(X) = \mathbb{Z}$.
. If n is even $Z_n(X) = 0$ and $B_n(X) = 0$ then $H_n(X) = 0$.
. If n is odd $Z_n(X) = \mathbb{Z}\sigma_n \cong \mathbb{Z}$ and $B_n(X) = \mathbb{Z}\sigma_n \cong \mathbb{Z}$ then $H_n(X) = 0$. □

We are going to define the notion of path connected as in topological spaces.

Definition 3.4.6. Let $(X, c) \in \mathbf{CL}$ and $a, b \in X$. We associate the interval $[0,1]$ with the Kuratowski closure operator k induced by the usual topology on $[0,1]$. A *path* in X with origin a and extremity b is a continuous map $f : ([0,1], k) \to (X, c)$ such that $f(0) = a$ and $f(1) = b$. If such path exists, we say that a and b are path connected. A subspace Y of X is called *path connected component* if every pair of points in Y is path connected.

Theorem 3.4.7. *Let (X, c) be a closure space. Suppose that there exists a family of path connected component $\{(X_i, c_i)\}_{i\in I}$ such that $X = \coprod_{i\in I} X_i$. Then $H_n(X;\mathbb{Z}) = \bigoplus_{i\in I} H_n(X_i;\mathbb{Z})$ for all $n \in \mathbb{N}$.*

Proof. Let $\sigma \in \mathcal{J}_n(X)$. We note the image of a path connected component by continuous map is a path connected component. Then $\sigma(|\Delta^n|)$ is path connected and $\sigma(|\Delta^n|) \subseteq X_i$ for some $i \in I$. Therefore $C_n(X;\mathbb{Z}) = \bigoplus_{i\in I} C_n(X_i;\mathbb{Z})$. Since $\partial_n(C_n(X_i;\mathbb{Z})) \subseteq C_{n-1}(X_i;\mathbb{Z})$ then $Z_n(X) = \bigoplus_{i\in I} Z_n(X_i)$ and $B_n(X) = \bigoplus_{i\in I} B_n(X_i)$.
Consider the canonical projection

$$p : \bigoplus_{i\in I} Z_n(X_i) \longrightarrow \bigoplus_{i\in I} Z_n(X_i)/B_n(X_i).$$
$$(\sigma_i)_{i\in I} \longmapsto (\sigma_i + B_n(X_i))_{i\in I}$$

It is clear that the map p is well defined and surjective.
If $(\sigma_i)_{i\in I} \in \bigoplus_{i\in I} Z_n(X_i)$ such that

$$p(\sigma_i)_{i\in I} = (\sigma_i + B_n(X_i))_{i\in I} = (0 + B_n(X_i))_{i\in I},$$

then $(\sigma_i)_{i\in I} \in \bigoplus_{i\in I} B_n(X_i)$. Therefore $\ker p = \bigoplus_{i\in I} B_n(X_i)$. By the First Isomorphism Theorem, we have

$$\bigoplus_{i\in I} H_n(X_i;\mathbb{Z}) \cong Z_n(X)/\ker p.$$

This yields $H_n(X;\mathbb{Z}) \cong \bigoplus_{i\in I} H_n(X_i;\mathbb{Z})$. □

The construction above is not only for the case of $\mathbb{Z}$. Using the remark 3.3.6, we obtain a generalization of singular homology and cohomology for closure space with a coefficient in abelian group G. More generally, we can define a singular homology with coefficient in homological coefficient system $\mathcal{A}$.

It is also important to note that the construction of the singular homology group above satisfies the Eilenberg-Steenrod axiom, as we can see in [Bog84] and [BM21].

Another form of (co)homology theory has been described for a topological space without using any theory of simplicial sets in [Wal70], namely Čech (co)homology.

In the rest of this section, we would like to define a Čech (co)homology for a closure space using the theory of simplicial sets. We have defined in Definition 2.2.4 that an interior cover $\mathcal{U}$ of a closure space (X,c) is a family $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ such that $X = \bigcup_{\alpha\in I} i_c(U_\alpha)$.

Definition 3.4.8. Let (X,c) be a closure space. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ and $\mathcal{V} = \{V_\beta\}_{\beta\in J}$ be interior covers of the closure space (X,c). We say that $\mathcal{U}$ is *a refinement* of $\mathcal{V}$ if there exists a map $\varphi : I \to J$ such that $U_\alpha \subseteq V_{\varphi(\alpha)}$ for all $\alpha \in I$, where i_c is the interior operator corresponding to c. If $\mathcal{U}$ is a refinement of $\mathcal{V}$, we write this relation $\mathcal{U} \prec \mathcal{V}$.

Let (X,c) be a Čech closure space. We denote by $Cov(X)$ the set of all interior covers of (X,c).

Proposition 3.4.9. *Let (X,c) be a closure space in* **CL**. *Then, $(Cov(X),\prec)$ is a directed preordered set.*

Proof. It is clear that $(Cov(X),\prec)$ is a preordered set.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ and $\mathcal{U} = \{V_\beta\}_{\beta\in J}$ be interior covers of (X,c). Consider the set

$$\mathcal{U}\wedge\mathcal{V} := \{U\cap V \mid U\in\mathcal{U},\ V\in\mathcal{V}\}.$$

Let $x\in X$. Since $\mathcal{U}$ and $\mathcal{V}$ are interior covers of (X,c), there exists $U\in\mathcal{U}$ and $V\in\mathcal{V}$ such that both of U and V are neighborhoods of x, i.e $x\in i_c(U)$ and $x\in i_c(V)$. We have $i_c(U)\cap i_c(V) = i_c(U\cap V)$, then $x\in i_c(U\cap V)$. It means again that $U\cap V$ is a neighborhood of x. Since $U\cap V\in\mathcal{U}\wedge\mathcal{V}$, then $X\subseteq\bigcup_{W\in\mathcal{U}\wedge\mathcal{V}} i_c(W)$.
Therefore $X = \bigcup_{W\in\mathcal{U}\wedge\mathcal{V}} i_c(W)$. Thus $\mathcal{U}\wedge\mathcal{V}$ is also an interior cover of (X,c).

For all $U\cap V\in\mathcal{U}\wedge\mathcal{V}$ such that $U\in\mathcal{U}$ and $V\in\mathcal{V}$, we have $U\cap V\subseteq U$ and $U\cap V\subseteq V$. Then $\mathcal{U}\wedge\mathcal{V}$ is a refinement of $\mathcal{U}$ and $\mathcal{V}$.
We then conclude that $(Cov(X),\prec)$ is a directed preordered set. □

We note that if (X,c) is just a closure spaces in **CS**, that is we drop the additivity, then the relation $\prec$ fail to be directed preorder relation on the covers of (X,c), in that case $(Cov(X),\prec)$ is just a preordered set.

Let (X,c) be a closure space in **CL**. We define $\mathbf{Cov}(X)$ to be the category given by the directed preordered set $(Cov(X),\prec)$.

Definition 3.4.10. Let $(X,c)\in$ **CL**. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be an interior cover of (X,c). A *nerve* of the cover $\mathcal{U}$ is a simplicial set $\tilde{X} : \mathbf{\Delta}^{op} \to \mathbf{Set}$ defined as follows:

- for each $[n]\in\mathbf{\Delta}$, $\tilde{X}_n = \{(\alpha_0,\dots,\alpha_n)\in I^{n+1} \mid U_{\alpha_0}\cap U_{\alpha_1}\cap\cdots\cap U_{\alpha_n}\neq\emptyset\}$,
- for each $f:[m]\to[n]$ in $\mathbf{\Delta}$, we have

$$\tilde{X}(f) : \tilde{X}_n \longrightarrow \tilde{X}_m.$$
$$(\alpha_0,\dots,\alpha_n) \longmapsto (\alpha_{f(0)},\dots,\alpha_{f(n)})$$

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ and $\mathcal{V} = \{V_\beta\}_{\beta\in J}$ be covers of the closure space (X, c) such that $\mathcal{U} \prec \mathcal{V}$. Denote by $\tilde{X}$ and $\tilde{Y}$ the nerves of covers $\mathcal{U}$ and $\mathcal{V}$ respectively. The maps $\varphi : I \to J$ induce naturally a simplicial map $\Phi : \tilde{X} \to \tilde{Y}$ given for each $n \in \mathbb{N}$

$$\Phi_n : \tilde{X}_n \longrightarrow \tilde{Y}_n.$$
$$(\alpha_0, \dots, \alpha_n) \longmapsto (\varphi(\alpha_0), \dots, \varphi(\alpha_n))$$

The maps Φ make sense because $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \emptyset$ implies $U_{\varphi(\alpha_0)} \cap \cdots \cap U_{\varphi(\alpha_n)} \neq \emptyset$.

Definition 3.4.11. Let $(X, c) \in \mathbf{CL}$. A (co)homology of (X, c) with respect to $\mathcal{U}$ is defined respectively by the (co)homology of the simplicial abelian group $\mathbb{Z}\tilde{X}$. That is

$$\check{H}_n(\mathcal{U}; \mathbb{Z}) = H_n(\mathbb{Z}\tilde{X}; \mathbb{Z}) \quad \text{and} \quad \check{H}^n(\mathcal{U}; \mathbb{Z}) = H^n(\mathbb{Z}\tilde{X}; \mathbb{Z}), \text{ for every } n \in \mathbb{N}.$$

We then define a *Čech (co)homology of* (X, c) *with a coefficient in* $\mathbb{Z}$ as follows: for every $n \in \mathbb{N}$,

$$\check{H}_n(X; \mathbb{Z}) = \varprojlim_{\mathcal{U}\in \mathbf{Cov}(X)} \check{H}_n(\mathcal{U}; \mathbb{Z}) \text{ and } \check{H}^n(X; \mathbb{Z}) = \varinjlim_{\mathcal{U}\in \mathbf{Cov}(X)} \check{H}^n(\mathcal{U}; \mathbb{Z}).$$

Let $n \in \mathbb{N}$. Consider the functor $H^n : \mathbf{Cov}(X) \to \mathbf{Ab}$ sending each cover $\mathcal{U}$ of (X, c) to the homology group $\check{H}^n(\mathcal{U}; \mathbb{Z})$. Since $\mathbf{Cov}(X)$ is a small category and $\mathbf{Ab}$ is a cocomplete category, then the colimit of the diagram H^n exists and is unique in $\mathbf{Ab}$ by Theorem A.0.23. Therefore, the nth Čech cohomology group $\check{H}^n$ is well defined.

We might be wondering about the relation between the singular (co)homology and the Čech (co)homology of closure space. When do those two (co)homologies coincide? We leave that question for those who are interested to investigate.

It is also important to mention the notable work of Antonio Rieser in [Rie21b], he developed a construction of a *sheaf cohomology* for Čech closure space.

Finally, one should be aware about the theory of *homotopy group* for closure space, it has been defined in [Bog84]. Recently, some other results about *fundamental groups* for closure space have been developed in [Rie21a].

3.5 Dold-Kan Correspondence

Following [GJ09], we will provide the Dold-Kan correspondence theorem in this section, which states that there is an equivalence between the categories of simplicial abelian groups and non-negative chain complexes. In order to do that, we first need to construct a functor from **sAb** to $\mathbf{Ch}_+$ and vice versa. Along this section, we then provide in detail the treatment that Goerss and Jardine outlined in [GJ09].

3.5.1 Functor from sAb to Ch+

Proposition 3.5.2. *Let $A \in$ **sAb**. There exists a chain complex $NA_\bullet = \{(NA_n, \partial_n)\}_{n\geq 0}$ called normalized chain complex given by*

$$NA_n = \bigcap_{i=0}^{n-1} \ker d_i \subseteq A_n, \text{ for every } n \in \mathbb{N}$$

and the boundary map is defined by

$$\partial_n : NA_n \to NA_{n-1}, \text{ where } \partial_n = (-1)^n d_n, \text{ for every } n \in \mathbb{N}.$$

Proof. On the one hand, let $n \geq 1$ be an integer and $x \in NA_n$. Then $x \in \ker d_i$ for all $0 \leq i \leq n-1$.

Moreover, by Proposition 3.1.9 we have $d_i d_n = d_{n-1} d_i$ for $i < n$, so

$$d_i \partial_n(x) = (-1)^n d_i d_n(x) = (-1)^n d_{n-1} d_i(x) = 0$$

Therefore $\partial_n(x) \in \ker d_i$ for all $0 \le i < n-1$ that is $\partial_n(x) \in NA_{n-1}$. Thus $\partial_n(NA_n) \subseteq NA_{n-1}$ and ∂_n is well defined.

On the other hand, we have to prove that $\{(NA_n, \partial_n)\}_{n \ge 0}$ form a chain complex. i.e., for all $n \ge 0$, $\partial_n \partial_{n+1} = 0$.
Let $x \in NA_{n+1}$ and $n \ge 1$. By the identities in Proposition 3.1.9 and since $x \in \ker d_n$, we have

$$\partial_n \partial_{n+1}(x) = (-1)^{2n+1} d_n d_{n+1}(x) = (-1)^{2n+1} d_n d_n(x) = 0.$$

Hence $\partial_n \partial_{n+1}(x) = 0$ and $NA_\bullet$ is a chain complex. □

Proposition 3.5.3. *[GJ09] There exists a functor $N : \mathbf{sAb} \to \mathbf{Ch}_+$ sending a simplicial abelian group to a normalized chain complex and a simplicial map to a chain map.*

Proof. By Proposition 3.5.2, we define $N(A) = NA_\bullet$ for every $A \in \mathbf{sAb}$.

Let $f : A \to B$ be a simplicial map in **sAb** defined by the component $f_n : A_n \to B_n$ for any $[n] \in \mathbf{\Delta}$. We define $Nf : NA_\bullet \to NB_\bullet$ to be a chain map where the components $Nf_n : NA_n \to NB_n$ given by the restriction of f_n in NA_n, for every $n \in \mathbb{N}$. i.e., $Nf_n(x) = f_n(x)$ for all $x \in NA_n$.

First, for every $n \in \mathbb{N}$ let us prove that the map $Nf_n : NA_n \to NB_n$ is well defined.
Let $x \in NA_n$ and $i < n$. Since f is a natural transformation and $x \in \ker d_i$ for every $i < n$, we have $d_i f_n(x) = f_{n-1} d_i(x) = 0$. Then $f_n(x) \in \ker d_i$ for all $i < n$ and $x \in NA_n$, and that implies $f_n(x) \in NB_n$. Thus $Nf_n(NA_n) \subseteq NB_n$.

Second, we need to prove that Nf is a chain map, which means the following diagram commutes

$$\begin{array}{ccc} NA_n & \xrightarrow{(-1)^n d_n} & NA_{n-1} \\ {\scriptstyle Nf_n}\downarrow & & \downarrow{\scriptstyle Nf_{n-1}} \\ NB_n & \xrightarrow[(-1)^n d_n]{} & NB_{n-1} \end{array}$$

Since f is a natural transformation, we have

$$\partial_n N f_n = (-1)^n d_n f_n = (-1)^n f_{n-1} d_n = f_{n-1} (-1)^n d_n = N f_{n-1} \partial_n.$$

Therefore Nf is a chain map.

Finally, a routine computation shows that $Nid_A = id_{NA_\bullet}$ for all $A \in \mathbf{sAb}$ and $N(gf) = Ng.Nf$, for all $f : A \to B$ and $g : B \to C$ in **sAb**. Thus, $N : \mathbf{sAb} \to \mathbf{Ch}_+$ is a functor. □

It was shown, for example, in [Wei94] or [GJ09] that for a given simplicial abelian group, the Moore complex and the normalized complex have the same homology groups. i.e., for every $A \in \mathbf{sAb}$,

$$H_n(A_\bullet) = H_n(NA_\bullet) \text{ for all } n \in \mathbb{N}.$$

3.5.4 Functor from Ch+ to sAb

Now, we want to construct a functor from the category of non-negative chain complexes $\mathbf{Ch}_+$ to the category of simplicial abelian groups **sAb**. In order to do that, we need the following facts.
We denote by $\mathbf{\Delta}_{mono}$ a subcategory of $\mathbf{\Delta}$, whose objects are finite ordinal numbers, and we take all monomorphisms in $\mathbf{\Delta}$ to be the morphisms of $\mathbf{\Delta}_{mono}$.
Given a chain complex $C_\bullet = \{(C_n, \partial_n)\}_{n \in \mathbb{N}}$. We have the contravariant functor

$$\Phi_C : \mathbf{\Delta}_{mono}^{op} \to \mathbf{Ab}$$

defined as follows:

- for each $[n] \in \mathbf{\Delta}$, we assign $\Phi_C([n]) = C_n$ and
- for each $f : [m] \rightarrowtail [n]$, we define

$$\Phi_C(f) = \begin{cases} id_{C_n} : C_n \to C_n & \text{if } n = m, \\ (-1)^n \partial_n : C_n \to C_{n-1} & \text{if } f = d^n : [n-1] \rightarrowtail [n], \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for each integer $n \geq 0$, we define the abelian group

$$\bar{C}_n := \bigoplus_{[n] \twoheadrightarrow [k]} C_k$$

where the sum goes through all surjective maps with domain $[n]$. Any element $x \in \bar{C}_n$ can be written as $x = (x_1, \cdots, x_p)$ where p is the number of all surjective maps with domain $[n]$. We should notice that the summand C_k can appear many times in the expression $\bar{C}_n$.

Proposition 3.5.5. *Given a chain complex $C_\bullet \in \mathbf{Ch}_+$. Then $C_\bullet$ induces a simplicial abelian group $C : \mathbf{\Delta}^{op} \to \mathbf{Ab}$ defined as follows:*

- *for each $[n] \in \Delta$, $C([n]) = \bar{C}_n$,*
- *for each $\alpha : [m] \to [n]$, we define $C(\alpha)$ to be the map*

$$\bar{\alpha} : \bigoplus_{[n] \twoheadrightarrow [k]} C_k \to \bigoplus_{[m] \twoheadrightarrow [l]} C_l.$$

Proof. Let $\alpha : [m] \to [n]$ be a morphism in $\mathbf{\Delta}$. We need to prove first that the map $\bar{\alpha}$ is well defined.
Let $\sigma : [n] \twoheadrightarrow [k]$ be a surjection in $\mathbf{\Delta}$ that corresponds to a summand C_k in $\bar{C}_n$.
By Lemma 3.1.4, there exists $t : [m] \twoheadrightarrow [s]$ and $d : [s] \rightarrowtail [k]$ such that the composition map $[m] \xrightarrow{\alpha} [n] \overset{\sigma}{\twoheadrightarrow} [k]$ can be factorized as $\sigma\alpha = dt$.

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \sigma} \\ [s] & \underset{d}{\rightarrowtail} & [k] \end{array}$$

We then obtain a summand C_s in $\bar{C}_m$ associated to the surjection $t : [m] \twoheadrightarrow [s]$. The mapping is defined on the summand by the map $\Phi_C(d) : C_k \to C_s$. Therefore for any x_k in the summand C_k of $\bar{C}_n$, we have $\Phi_C(d)(x_k)$ in the summand C_s of $\bar{C}_m$. Thus $\bar{\alpha}$ is well defined.

Let $id_{[n]} : [n] \to [n]$ be an identity map in $\mathbf{\Delta}$. Let $\sigma : [n] \twoheadrightarrow [k]$ be a surjective map in $\mathbf{\Delta}$ corresponding to the summand C_k in $\bar{C}_n$.

$$\begin{array}{ccc} [n] & \xrightarrow{id_{[n]}} & [n] \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ [k] & \underset{id_{[k]}}{\rightarrowtail} & [k] \end{array}$$

The mapping of the summand C_k is given by the map $\Phi_C(id_{[k]}) : C_k \to C_k$.
We then have $C(id_{[n]}) = id_{\bar{C}_n}$ for any $[n] \in \mathbf{\Delta}$.

Finally, let $\alpha : [n] \to [m]$ and $\beta : [m] \to [h]$. Let $\theta : [h] \twoheadrightarrow [i]$ be a surjection in $\mathbf{\Delta}$ corresponding to the summand C_i in $\bar{C}_h$.

We have the factorization

$$\begin{array}{ccc} [m] & \xrightarrow{\beta} & [h] \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle \theta} \\ [s] & \underset{s_1}{\rightarrowtail} & [i] \end{array}$$

Then the surjection $t_1 : [m] \twoheadrightarrow [s]$ corresponds to the summand C_s in $\bar{C}_m$ and the mapping is given by $\Phi_C(s_1) : C_i \to C_s$. Therefore $C(\alpha) : \bar{C}_m \to \bar{C}_n$ is defined on the summand by the map Φ_C.
Similarly, we have

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle t_1} \\ [p] & \underset{s_2}{\rightarrowtail} & [i] \end{array}$$

So the surjection $t_2 : [n] \to [p]$ corresponds to the summand C_p in $\bar{C}_n$ and the mapping is then given by $\Phi_C(s_2) : C_s \to C_p$. Thus, the map $C(\alpha)C(\beta) : \bar{C}_h \to \bar{C}_n$ is given as follows: for any $x \in C_i$ summand of $\bar{C}_h$ we define

$$(C(\alpha)C(\beta))(x) := (\Phi_C(s_2)\Phi_C(s_1))(x).$$

Combining these two diagrams above, we obtain the following commutative diagram

$$\begin{array}{ccccc} [n] & \xrightarrow{\alpha} & [m] & \xrightarrow{\beta} & [h] \\ {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle \theta} \\ [p] & \underset{s_2}{\rightarrowtail} & [s] & \underset{s_1}{\rightarrowtail} & [i] \end{array}$$

Then

$$\theta\beta\alpha = s_1 t_1 \alpha = s_1 s_2 t_2.$$

We then have the surjection $\theta : [h] \twoheadrightarrow [i]$ corresponding to the summand C_i in $\bar{C}_h$ while $t_2 : [n] \twoheadrightarrow [p]$ corresponds to C_p in $\bar{C}_n$. Therefore, we have the map $\Phi_C : C_i \to C_p$ which defines on the summand the map $C(\beta\alpha) : \bar{C}_h \to \bar{C}_n$. Then $C(\beta\alpha)(x) = \Phi_C(s_1 s_2)(x)$ for each $x \in C_i$ and for all summands C_i in $\bar{C}_h$. Moreover, since Φ_C is a contravariant functor then $\Phi_C(s_1 s_2) = \Phi_C(s_2)\Phi_C(s_1)$, then

$$C(\beta\alpha)(x) = C(\alpha)C(\beta)(x)$$

for each $x \in C_i$ and for all summands C_i in $\bar{C}_h$. Hence $C : \mathbf{\Delta}^{op} \to \mathbf{Ab}$ is a contravariant functor, so it is a simplicial abelian group. □

Proposition 3.5.6. *Let $f : C_\bullet \to D_\bullet$ be a morphism in* $\mathbf{Ch}_+$ *defined by the family of group homomorphisms $\{f_n : C_n \to D_n\}_{n\in\mathbb{N}}$. Then, there is a simplicial map $\dot{f} : C \to D$ in* **sAb** *with components $\dot{f}_n : \bar{C}_n \to \bar{D}_n$ for any object $[n] \in \mathbf{\Delta}$.*

Proof. Let $f : C_\bullet \to D_\bullet$ be a morphism in $\mathbf{Ch}_+$ defined by the family of group homomorphisms $\{f_n : C_n \to D_n\}_{n\in\mathbb{N}}$ where $C_\bullet = \{(C_n, \partial_n)\}_{n\geq 0}$ and $D_\bullet = \{(D_n, \partial'_n)\}_{n\geq 0}$.
Note that we have the following commutative diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\partial_k} & C_{k-1} & \xrightarrow{\partial_{k-1}} & C_{k-2} & \xrightarrow{\partial_{k-2}} & \cdots \\ & & {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle f_{k-1}} & & \downarrow{\scriptstyle f_{k-2}} & & \\ \cdots & \underset{\partial'_{k+1}}{\longrightarrow} & D_k & \underset{\partial'_k}{\longrightarrow} & D_{k-1} & \underset{\partial'_{k-1}}{\longrightarrow} & D_{k-2} & \underset{\partial'_{k-2}}{\longrightarrow} & \cdots \end{array}$$

We have to show that $\dot{f} : C \to D$ is a natural transformation with components $\dot{f}_n : \bar{C}_n \to \bar{D}_n$ for all $n \geq 0$.

Let $\alpha : [n] \to [m]$ be a morphism in $\mathbf{\Delta}$.

$$\begin{array}{ccc} \bar{C}_m & \xrightarrow{\dot{f}_m} & \bar{D}_m \\ {\scriptstyle C(\alpha)}\downarrow & & \downarrow{\scriptstyle D(\alpha)} \\ \bar{C}_n & \xrightarrow[\dot{f}_n]{} & \bar{D}_n \end{array} \tag{3.1}$$

In order to prove that the diagram (3.1) is commutative, we need to check that it is commutative on each summand.

Let $\theta : [m] \twoheadrightarrow [k]$ be a surjection in $\mathbf{\Delta}$ corresponding to the summands C_k and D_k in $\bar{C}_m$ and $\bar{D}_m$ respectively. We have the factorization

$$\begin{array}{ccc} [n] & \xrightarrow{\alpha} & [m] \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \theta} \\ [s] & \underset{s}{\rightarrowtail} & [k] \end{array}$$

Then, the surjection $t : [n] \twoheadrightarrow [s]$ corresponds to the summand C_s and D_s in $\bar{C}_n$ and $\bar{D}_n$ respectively. Therefore, there exists a maps $\Phi_C(s) : C_k \to C_s$ and $\Phi_D(s) : D_k \to D_s$.
It is enough to prove that the following diagram is commutative

$$\begin{array}{ccc} C_k & \xrightarrow{f_k} & D_k \\ {\scriptstyle \Phi_C(s)}\downarrow & & \downarrow{\scriptstyle \Phi_D(s)} \\ C_s & \xrightarrow[f_s]{} & D_s \end{array} \tag{3.2}$$

1^{st} case: If the map s is of the form $d^k : [k-1] \to [k]$ that is $s = k-1$, then $\Phi_D(s) = (-1)^k \partial'_k : D_k \to D_{k-1}$ and $\Phi_C(s) = (-1)^k \partial_k : C_k \to C_{k-1}$. Since f is a chain map, the following equality holds

$$\Phi_D(s) f_k = (-1)^k \partial_k f_k = (-1)^k f_{k-1} \partial'_k = f_{k-1} (-1)^k \partial'_k = f_s \Phi_C(s).$$

2^{nd} case: If the map s is identity $s = id_{[k]} : [k] \to [k]$ and $s = k$, then $\Phi_D(s) = \Phi_D(id_{[s]}) = id_{D_s}$ and $\Phi_C(s) = \Phi_C(id_s) = id_{C_s}$.

$$\Phi_D(s) f_k = id_{D_s} f_k = f_k = f_s = f_s . id_{C_s} = f_s \Phi_C(s)$$

3^{rd} case: If s is neither the identity nor of the form d^k then $\Phi_D(s) = \Phi_C(s) = 0$.
Therefore, the diagram (3.2) is commutative and the commutativity of the diagram (3.1) follows. Consequently $\dot{f} : C \to D$ is a natural transformation, so it is a simplicial map in **sAb**. □

Proposition 3.5.7. *[GJ09] There exists a functor* $\Gamma : \mathbf{Ch}_+ \to \mathbf{sAb}$ *sending a chain complex* $C_\bullet$ *to a simplicial abelian group* C *and a chain map* f *to a simplicial map* $\dot{f}$.

Proof. For every $C_\bullet \in \mathbf{Ch}_+$, set $\Gamma(C_\bullet) = C$ as defined in Proposition 3.5.5. For any chain map $f : C_\bullet \to D_\bullet$ in $\mathbf{Ch}_+$, we define $\Gamma(f) = \dot{f}$, where $\dot{f}$ is a simplicial map as in Proposition 3.5.6.

Let $C_\bullet \in \mathbf{Ch}_+$. We have

$$\Gamma(id_{C_\bullet}) = i\dot{d}_{C_\bullet} = id_{\Gamma(C_\bullet)}.$$

Let $f : C_\bullet \to D_\bullet$ and $g : D_\bullet \to E_\bullet$ be morphisms in $\mathbf{Ch}_+$. We note that we have the simplicial maps $\Gamma(f) : C \to D$ and $\Gamma(g) : D \to E$.

On the one hand, let us prove that the composition of those simplicial maps is again a simplicial map $\Gamma(g)\Gamma(f) : C \to E$ with components

$$(\Gamma(g)\Gamma(f))_n = \Gamma_n(g)\Gamma_n(f) \text{ for all } [n] \in \mathbf{\Delta}.$$

We then need to show that the following diagram is commutative for any $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$.

$$\begin{array}{ccccc} \bar{C}_m & \xrightarrow{\Gamma_m(f)} & \bar{D}_m & \xrightarrow{\Gamma_m(g)} & \bar{E}_m \\ {\scriptstyle C(\alpha)}\downarrow & & \downarrow{\scriptstyle D(\alpha)} & & \downarrow{\scriptstyle E(\alpha)} \\ \bar{C}_n & \xrightarrow[\Gamma_n(f)]{} & \bar{D}_n & \xrightarrow[\Gamma_n(g)]{} & \bar{E}_n \end{array} \tag{3.3}$$

Let $\theta : [m] \twoheadrightarrow [k]$ be a surjective map corresponding to the summands C_k, D_k and E_k in $\bar{C}_m, \bar{D}_m$ and $\bar{E}_m$ respectively. By Lemma 3.1.4 there exist maps $t : [n] \twoheadrightarrow [s]$ and $s : [s] \rightarrowtail [k]$ such that $\theta\alpha = st : [n] \to [k]$. Therefore, we have the surjective map t corresponding to the summands C_s, D_s and E_s in $\bar{C}_n, \bar{D}_n$ and $\bar{E}_n$ respectively.

$$\begin{array}{ccccc} C_k & \xrightarrow{f_k} & D_k & \xrightarrow{g_k} & E_k \\ {\scriptstyle \Phi_C(s)}\downarrow & & \downarrow{\scriptstyle \Phi_D(s)} & & \downarrow{\scriptstyle \Phi_E(s)} \\ C_s & \xrightarrow[f_s]{} & D_s & \xrightarrow[g_s]{} & E_s \end{array} \tag{3.4}$$

Since the squares in the diagram (3.4) commute separately, we have

$$(\Phi_E(s)g_k)f_k = (g_s\Phi_D(s))f_k = g_s(\Phi_D(s)f_k) = g_s(f_s\Phi_C(s))$$

Therefore, the diagram (3.4) is commutative, so is the diagram (3.3). Thus $\Gamma(g)\Gamma(f) : C \to E$ is a natural transformation and so a simplicial map given by $(\Gamma(g)\Gamma(f))_n = \Gamma_n(g)\Gamma_n(f)$ for all $[n] \in \mathbf{\Delta}$.

On the other hand, we have the composition of chain maps $gf : C_\bullet \to E_\bullet$. We want to define the simplicial map $\Gamma(gf) : C \to E$. Let $\alpha : [n] \to [m]$, we have to prove that the following diagram is commutative.

$$\begin{array}{ccc} \bar{C}_m & \xrightarrow{\Gamma_m(gf)} & \bar{E}_m \\ {\scriptstyle C(\alpha)}\downarrow & & \downarrow{\scriptstyle E(\alpha)} \\ \bar{C}_n & \xrightarrow[\Gamma_n(gf)]{} & \bar{E}_n \end{array} \tag{3.5}$$

In a similar way as above, we have the following commutative diagram

$$\begin{array}{ccc} C_k & \xrightarrow{(gf)_k} & E_k \\ {\scriptstyle \Phi_C(s)}\downarrow & & \downarrow{\scriptstyle \Phi_E(s)} \\ C_s & \xrightarrow[(gf)_s]{} & E_s \end{array} \tag{3.6}$$

Using the fact that the component $(gf)_k : C_k \to E_k$ is the composition of $g_k : C_k \to D_k$ and $f_k : D_k \to E_k$, i.e. $(gf)_k = g_k f_k$ for all $k \in \mathbb{N}$, we have the component $(\Gamma(gf))_n = \Gamma_n(g)\Gamma_n(f)$. Therefore $(\Gamma(gf))_n = (\Gamma(g)\Gamma(f))_n$ for all $n \in \mathbb{N}$. Hence $\Gamma(gf) = \Gamma(g)\Gamma(f)$ and $\Gamma : \mathbf{Ch}_+ \to \mathbf{sAb}$ is a functor. □

3.5.8 Dold-Kan Correspondence

Definition 3.5.9. [GJ09] Let $A \in \mathbf{sAb}$. We define DA_n as the subgroup of A_n that is generated by the degenerate simplices.

We note that the boundary map $\partial_n : A_n \to A_{n-1}$ of the Moore complex associated to the simplicial abelian group A induces a homomorphism

$$\begin{aligned} \partial_n : A_n/DA_n &\longrightarrow A_{n-1}/DA_{n-1}. \\ x + DA_n &\longmapsto \partial_n(x) + DA_{n-1} \end{aligned}$$

The sequence $\{(A_n/DA_n, \partial_n)\}_{n\in\mathbb{N}}$ forms a chain complex denoted by $A/DA_\bullet$.

Moreover, we have chain maps

$$NA_\bullet \xhookrightarrow{\iota} A_\bullet \xrightarrow{p} A/DA_\bullet.$$

For every $n \geq 0$ and $j \leq n$, define

$$D_jA_n := \langle \bigcup_{i\leq j} s_i(A_{n-1})\rangle = \langle \{x \in A_n |\ \exists y \in A_{n-1},\ \exists\, 0 \leq i \leq j,\ x = s_i y\}\rangle$$

and

$$N_jA_n := \bigcap_{i\leq j} \ker d_i \subseteq A_n.$$

It is clear that for each $n \in \mathbb{N}$

$$N_jA_n \subseteq N_{j-1}A_n \subseteq \cdots \subseteq N_1A_n \subseteq N_0A_n$$

and

$$D_jA_n \subseteq D_{j+1}A_n \subseteq \cdots \subseteq D_nA_n.$$

Proposition 3.5.10. *The following diagram is commutative.*

$$\begin{array}{ccc} N_{j-1}A_{n-1} & \xrightarrow{\phi} & A_{n-1}/D_{j-1}A_{n-1} \\ {\scriptstyle \overline{s}_j}\downarrow & & \downarrow{\scriptstyle \underline{s}_j} \\ N_{j-1}A_n & \xrightarrow[\phi]{} & A_n/D_{j-1}A_n \end{array} \tag{3.7}$$

where $\overline{s}_j(x) = s_jx$ *for all* $x \in N_{j-1}A_{n-1}$ *and* $\underline{s}_j([x]) = [s_jx]$ *for all* $[x] \in A_{n-1}/D_{j-1}A_{n-1}$.

Proof. The map ϕ is defined trivialy by $\phi(x) = [x]$ for $x \in N_{j-1}A_{n-1}$ or $N_{j-1}A_n$.

Let $x \in N_{j-1}A_{n-1}$. That is $x \in \ker d_i$ for all $i \leq j-1$, which also implies that

$$d_is_jx = s_{j-1}d_ix = s_{j-1}0 = 0.$$

Then $s_jx \in \ker d_i$ for all $i \leq j-1$, that is $s_jx \in N_{j-1}A_n$. Thus $\overline{s}_j$ is well defined.

Let $[x], [y] \in A_{n-1}/D_{j-1}A_{n-1}$ such that $[x] = [y]$. We have

$$x + D_{j-1}A_{n-1} = y + D_{j-1}A_{n-1} \implies x - y \in D_{j-1}A_{n-1}.$$

Then, there exists $u \in A_{n-2}$ such that $x - y = s_ku$ for some $k \leq j-1$. So

$$s_j(x-y) = s_js_ku \implies s_jx - s_jy = s_js_ku \implies [s_jx] = [s_jy] \implies \underline{s}_j[x] = \underline{s}_j[y].$$

Hence $\underline{s}_j$ is well defined.

Let $x \in N_{j-1}A_{n-1}$. We have to prove that $\underline{s}_j\phi x = \phi\overline{s}_jx$. We have

$$\phi\overline{s}_jx = \phi(\overline{s}_jx) = \phi(s_jx) = [s_jx] \text{ and } \underline{s}_j\phi x = \underline{s}_j[x] = [s_jx].$$

Thus, the diagram 3.7 is commutative. □

Proposition 3.5.11. *Let n be a positive integer and $j < n$. The sequence*

$$0 \xrightarrow{0} A_{n-1}/D_{j-1}A_{n-1} \xrightarrow{\underline{s}_j} A_n/D_{j-1}A_n \xrightarrow{i} A_n/D_jA_n \xrightarrow{0} 0 \tag{3.8}$$

is exact.

Proof. Let us show first that the map

$$i : A_n/D_{j-1}A_n \to A_n/D_jA_n$$
$$x + D_{j-1}A_n \mapsto x + D_jA_n$$

is surjective.
Let $0 \neq [x] := x + D_jA_n \in A_n/D_jA_n$. Then $x \notin D_jA_n$, i.e. $x \notin s_i(A_{n-1})$ for all $i \leq j$, so $[x] \notin A_n/D_{j-1}A_n$. Since $s_id_ix \in s_i(A_{n-1})$, there exists $[x] = x + s_id_ix \in A_n/D_{j-1}A_n$ such that $i(x + s_id_ix) = x + D_jA_n$. Therefore i is surjective. Thus $\operatorname{im} i = \ker 0$.

Next, we have to prove that the map $\underline{s}_j$ is injective. Let $[x] := x+D_{j-1}A_{n-1} \in A_{n-1}/D_{j-1}A_{n-1}$ such that $\underline{s}_j([x]) = 0$. Then $s_jx + D_{j-1}A_n = 0$ that is $s_jx \in D_{j-1}A_n$. There exists $i \leq j-1$ and $y \in A_{n-1}$ such that $s_jx = s_iy$. Moreover, we have

$$x = d_js_jx = d_js_iy = s_i(d_{j-1}y).$$

Then $x \in s_i(A_{n-2})$ for some $i \leq j-1$. That means $x \in D_{j-1}A_{n-1}$. So $[x] = 0$ and $\ker \underline{s}_j = 0$. Therefore $\underline{s}_j$ is injective and $\ker \underline{s}_j = \operatorname{im} 0$.

Now let us prove that $\operatorname{im} \underline{s}_j = \ker i$. Let $y \in \operatorname{im} \underline{s}_j$. There exists $[x] \in A_{n-1}/D_{j-1}A_{n-1}$ such that $y = \underline{s}_j([x]) = s_jx + D_{j-1}A_n$. Since $s_jx \in D_jA_n$, we have

$$i(y) = i(s_jx + D_{j-1}A_n) = s_jx + D_jA_n = 0$$

Thus $y \in \ker i$ and $\operatorname{im} \underline{s}_j \subseteq \ker i$.

Let $[x] \in \ker i$. We have

$$i[x] = 0 \iff x + D_{j-1}A_n = 0 \iff x \in D_{j-1}A_n.$$

Then, there exists $i \leq j-1$ and $y \in A_{n-1}$ such that $x = s_iy$. It follows that $[x] = [s_iy] = \overline{s}_j[y]$. Therefore $[x] \in \operatorname{im} \underline{s}_j$ and $\ker i \subseteq \operatorname{im} \underline{s}_j$. Hence $\ker i = \operatorname{im} \underline{s}_j$. We conclude that the sequence (3.8) is exact. □

Theorem 3.5.12. *[GJ09] The composite $p\iota : NA_\bullet \to A/DA_\bullet$ is an isomorphism of chain complexes. i.e., $N_jA_n \longrightarrow A_n/D_jA_n$ is an isomorphism for all n and $j < n$.*

Proof. We want to prove it by induction on j.

- For $j = 0$, let us prove that the group homomorphism

$$\phi : N_0A_n \longrightarrow A_n/D_0A_n$$
$$x \longmapsto [x] = x + D_0A_n$$

is an isomorphism.
Let $x \in N_0A_n$ such that $\phi(x) = 0$. We have

$$x + D_0A_n = 0 \iff x \in D_0A_n \iff \exists y \in A_{n-1},\ x = s_0y.$$

Since $x \in \ker d_0$, using the identity in Proposition 3.1.3 we have

$$0 = d_0x = d_0s_0y = id\, y = y.$$

Then $y = 0$ and so $x = 0$. Hence $\ker \phi = \{0\}$ and ϕ is injective.

Let $[x] \in A_n/D_0A_n$. Every class $[x]$ can be represented by $x - s_0d_0x$. We have

$$d_0(x - s_0d_0x) = d_0x - d_0s_0d_0x = d_0x - id(d_0x) = d_0x - d_0x = 0.$$

Then $x - s_0d_0x \in N_0A_n$ and $\phi(x - s_0d_0x) = [x]$. So ϕ is surjective. Consequently ϕ is an isomorphism.

- Assume that the group homomorphism

$$\begin{aligned} \phi : N_k A_n &\longrightarrow A_n / D_k A_n \\ x &\longmapsto [x] := x + D_k A_n \end{aligned}$$

is an isomorphism for all $k < j$.

Consider the following diagram

$$\begin{array}{ccc} N_{j-1}A_n & \xrightarrow{\phi} & A_n/D_{j-1}A_n \\ \iota \big\uparrow & & \big\downarrow i \\ N_j A_n & \xrightarrow[\varphi]{} & A_n/D_j A_n \end{array} \tag{3.9}$$

where the map ι is an inclusion map and the map i is as in 3.5.11.
Let $x \in N_j A_n$. We have

$$i\phi\iota(x) = i\phi(x) = i(x + D_{j-1}A_n) = x + D_j A_n = \varphi(x).$$

Thus, the diagram (3.9) is commutative.

We now want to prove that φ is an isomorphism. We have proved that i is surjective in Proposition 3.5.11. Moreover, by hypothesis, ϕ is bijective particularly surjective, it follows that the composition $i\phi : N_{j-1}A_n \to A_n/D_j A_n$ is surjective. Then, for every $[x] \in A_n/D_j A_n$, there exists $x \in N_{j-1}A_n$ (i.e., $d_i x = 0$ for all $i < j$) such that $i\phi(x) = [x]$.
Since

$$\begin{aligned} d_i(x - s_j d_j x) =& d_i x - d_i s_j d_j x \\ =& \begin{cases} -d_i s_j d_j x = -s_{j-1} d_i d_j x = -s_{j-1} d_{j-1} d_i x = -s_{j-1} d_{j-1} 0 = 0 & \text{if } i < j \\ d_j x - id\, d_j x = d_j x - d_j x = 0 & \text{if } i = j \end{cases} \end{aligned}$$

then $x - s_j d_j x \in \ker d_i$ for all $i \leq j$ and so $x - s_j d_j x \in N_j A_n$.
Therefore, for every $[x] \in A_n/D_j A_n$, there exists $x - s_j d_j x \in N_j A_n$ such that

$$\varphi(x - s_j d_j x) = i\phi\iota(x - s_j d_j x) = [x]$$

by commutativity of the diagram (3.9). Hence φ is surjective.

Next, we have to prove that φ is injective. Let $x \in N_j A_n$ such that $\varphi(x) = 0$. By commutativity of the diagram (3.9), we have

$$\varphi(x) = 0 \iff i\phi\iota(x) = 0 \iff i(\phi(x)) = 0 \iff \phi(x) \in \ker i.$$

Since, the sequence (3.8) in Proposition 3.5.11 is exact, then $\phi(x) \in \operatorname{im} \underline{s}_j$. There exists $z \in A_{n-1}/D_{j-1}A_{n-1}$ such that $\phi(x) = \underline{s}_j(z)$. By the induction hypothesis ϕ is bijective particularly surjective for every $k < j$, then there exists $y \in N_{j-1}A_{n-1}$ such that $z = \phi(y)$. Furthermore, by commutativity of the diagram (3.7) in Proposition 3.5.10

$$\phi(x) = \underline{s}_j(z) = \underline{s}_j(\phi(y)) = \phi(\overline{s}_j(y)).$$

Since ϕ is injective then we obtain $x = \overline{s}_j(y)$. We know that $x \in N_j A_n$, then we get

$$0 = d_j x = d_j s_j y = id\, y = y.$$

So $y = 0$ and it follows that $x = 0$. Hence φ is injective. We conclude that φ is isomorphism. □

It is important to know that the theorem 3.5.12 together with a *cubical set theory* have been used to construct certain homology theory in a different model of simplices, for example a *cubical homology* as we can see in [BM21]; the authors defined three different cubical homologies for given closure spaces.

Theorem 3.5.13 (Dold-Kan correspondence)**.** *[GJ09] The functors N and Γ form an equivalence of categories* $\mathbf{Ch}_+$ *and* $\mathbf{sAb}$.

Proof. We only give the sketch of the proof here. First of all, we need prove that $\Gamma N \cong id_{\mathbf{sAb}}$. Let $A \in \mathbf{sAb}$. We have to show that $\eta_A : \Gamma N(A) \to A$ is a bijection. We note that $\Gamma N(A)$ and A are simplicial abelian group. As seen in [GJ09], for each $[n] \in \mathbf{\Delta}$ the map

$$\eta_{A,n} : \bigoplus_{[n]\twoheadrightarrow[k]} NA_k \to A_n$$

is a natural isomorphism. Therefore $\eta : \Gamma N \Longrightarrow 1_{\mathbf{sAb}}$ is natural isomorphism.

Let $C_\bullet \in \mathbf{Ch}_+$. We have $N\Gamma(C_\bullet) = N(C) = NC_\bullet$. The normalized chain complex $NC_\bullet$ is given by

$$NC_n := \bigcap_{i\leq n-1} \ker d_i, \text{ where } d_i : \bigoplus_{[n]\twoheadrightarrow[k]} C_k \to \bigoplus_{[n-1]\twoheadrightarrow[l]} C_l$$

and $\partial_n : NC_n \to NC_{n-1}$ is difined by $\partial_n := (-1)^n d_n$.
Using Theorem 3.5.12, one can prove that the normalized chain complex $NC_\bullet$ is equal to the chain complex $C_\bullet$. This fact implies the existence of the natural isomorphism $id_{\mathbf{Ch}_+} \Longrightarrow N\Gamma$. □

We have mostly referred to [GJ09] for the description of the Dold-Kan correspondence theorem. Our contribution in that section was to provide in detail the proof of the assertions stated by Goerss and Jardine in [GJ09], that is needed for understanding the Dold-Kan correspondence theorem.

It is relevant to note that several interesting results in homotopy theory, homological algebra, and indeed algebraic topology involve this theorem. Since we are not going through those topics, we would recommend those who are interested to continue [GJ09],[Kan58] and [May92].

Chapter 4

Construction of the Persistent Homology for Closure Spaces

Persistent homology gives a way of tracking the shape of *point cloud data* (usually a finite metric space, but not necessarily). The fundamental concept is that we begin to generate a sequence of simplicial complexes from a point cloud, as we can see in Example 4.3.10. We call this sequence of simplicial complexes a *filtration* and it can be seen diagrammatically as

$$K_0 \hookrightarrow K_1 \hookrightarrow K_2 \hookrightarrow K_3 \hookrightarrow \cdots \hookrightarrow K_p.$$

After that, one can compute the simplicial homology of these sequences to obtain a new sequence of homology groups connected by the group homomorphisms induced by the inclusion from the filtration:

$$H_n(K_0) \to H_n(K_1) \to H_n(K_2) \to H_n(K_3) \to \cdots \to H_n(K_p)$$

for each n.
The technique of persistent homology is then applied to this algebraic sequence to capture the evolution of the homology classes along this sequence itself. Any n-homology class represents what we called *n-dimensional holes* or *features*. By analyzing this sequence of homology groups, one can understand which holes persist longer and then deduce how the shape of the data behaves. This technique is developing quickly; several algorithms for computing persistent homology can be found for example in [EL02], [ZC04] and [EH10]. Those filtration above can be thought of as a particular functor $[n] \to$ **Top** or $[n] \to$ **Simp** whereas the sequence of group homomorphisms can be seen as a functor $[n] \to$ **Ab**.

This chapter is devoted to the categorical generalization of these theories for closure spaces. We then start to construct a simplicial complex from a closure space and give different common methods that can be applied to generate a simplicial complex from a point cloud. An advantage of the use of the closure operator is that it allows us to build a simplicial complex from a sample of points that are not equipped with or suitable for a metric. After that, we will build a filtration and later, the functionality of homology will be applied to this filtration to obtain a persistent homology. Along this chapter, we will refer mostly to [EL02], [ZC04], [BS14], [BM22] and [Cha+16].

4.1 Relation between Simplicial Complexes and Closure Spaces

Frequently, it is convenient to work with combinatorics objects such as graphs and simplicial complexes. Those are the most important mathematical objects and the most practicable in terms of computation. A simple starting point is then to construct a simplicial complex from a given closure space.

Definition 4.1.1. A *(abstract) simplicial complex* is a pair (X, E) where X is a set and E is a collection of non-empty finite subsets of X satisfying the following conditions:

- if $\tau \in E$ and $\emptyset \neq \sigma \subseteq \tau$, then $\sigma \in E$,
- if $x \in X$, then $\{x\} \in E$.

Definition 4.1.2. A *homomorphism of simplicial complexes* $f : (X, E) \to (Y, F)$ is a set map $f : X \to Y$ such that for every $e \in E$, $f(e) \in F$.

All simplicial complexes together with homomorphisms between them form a category of simplicial complexes denoted by **Simp**.

Definition 4.1.3. Let $(X, E) \in$ **Simp**. We define a graph $(X, \mathrm{T}(E))$ where

$$\mathrm{T}(E) = \{\sigma \in E | \ |\sigma| = 2\}.$$

Proposition 4.1.4. *There is a functor T : **Simp** $\to$ **Gph** sending each simplicial complexe (X, E) to the graph $(X, \mathrm{T}(E))$ and each morphism $f : (X, E) \to (Y, F)$ to the graph homomorphism $f : (X, \mathrm{T}(E)) \to (Y, \mathrm{T}(F))$.*

Proof. The mapping on objects is given by the definition 4.1.3. i.e., $\mathrm{T}(X, E) = (X, \mathrm{T}(E))$ for all $(X, E) \in$ **Simp**.

In order to define the mapping on morphisms, we have to prove that every simplicial map $f : (X, E) \to (Y, F)$ induces a graph homomorphism $f : (X, \mathrm{T}(E)) \to (Y, \mathrm{T}(F))$. Let $(x, y) \in \mathrm{T}(E)$. i.e., $x\mathrm{T}(E)y$. We have $\{x, y\} \in E$ and since f is a simplicial map then $f(\{x, y\}) = \{f(x), f(y)\} \in F$. Therefore, $(f(x), f(y)) \in \mathrm{T}(F)$. Thus $f : (X, \mathrm{T}(E)) \to (Y, \mathrm{T}(F))$ is a graph homomorphism.

Define $\mathrm{T}(f) = f$. By the proposition A.0.6, T : **Simp** $\to$ **Gph** is a functor. □

Definition 4.1.5. Let $(X, E) \in$ **Gph**. We define a simplicial complex $(X, \mathrm{K}(E))$ where

$$\mathrm{K}(E) = \{\tau \subseteq X | \ \tau \neq \emptyset, \ |\tau| < \infty, \ \forall x \neq y \in \tau \implies (x, y) \in E\}.$$

Proposition 4.1.6. *There is a functor* K : **Gph** $\to$ **Simp** *sending each graph (X, E) to a simplicial complex $(X, \mathrm{K}(E))$ and a graph homomorphism $f : (X, E) \to (Y, F)$ to a simplicial map $f : (X, \mathrm{K}(E)) \to (Y, \mathrm{K}(F))$.*

Proof. We define $\mathrm{K}(X, E) = (X, \mathrm{K}(E))$ for every $(X, E) \in$ **Gph**.

Let $f : (X, E) \to (Y, F)$ be a morphism in **Gph**. We have to prove that the graph homomorphism f induces a simplicial map $f : (X, \mathrm{K}(E)) \to (Y, \mathrm{K}(F))$. Let $\sigma \in \mathrm{K}(E)$. It is clear that $f(\sigma) \neq \emptyset$ and $|f(\sigma)| < \infty$. Let $y_1, y_2 \in f(\sigma)$ such that $y_1 \neq y_2$. There exists $x_1 \neq x_2 \in \sigma$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since f is a graph homomorphism, then $(x_1, x_2) \in E$ implies $(y_1, y_2) \in F$. Thus $f(\sigma) \in \mathrm{K}(F)$ and $f : (X, \mathrm{K}(E)) \to (Y, \mathrm{K}(F))$ is a simplicial map.

We then define $\mathrm{K}(f) = f$. So K : **Gph** $\to$ **Simp** is a functor using Proposition A.0.6. □

Proposition 4.1.7. *[BM22] Let $(X, E) \in$ **Simp** and $(Y, F) \in$ **Gph**. Given a set map $f : X \to Y$. Then $f : (X, \mathrm{T}(E)) \to (Y, F)$ is a graph homomorphism if and only if $f : (X, E) \to (Y, \mathrm{K}(F))$ is a simplicial map. Thus, the functor* T *is a left adjoint to the functor* K. *For all $(X, E) \in$ **Simp** and $(Y, F) \in$ **Gph**,*

$$\mathbf{Gph}((X, \mathrm{T}(E)), (Y, F)) \cong \mathbf{Simp}((X, E), (Y, \mathrm{K}(F)))$$

which is natural in both variables.

Proof. Let $(X, E) \in$ **Simp** and $(X, F) \in$ **Gph**.

Assume that $f : (X, \mathrm{T}(E)) \to (Y, F)$ is a graph homomorphism. Let $\sigma \in E$. Let $y_1, y_2 \in f(\sigma)$ such that $y_1 \neq y_2$. There exists $x_1, x_2 \in \sigma$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. It is clear that $\{x_1, x_2\} \in \mathrm{T}(E)$. Moreover, since $f : (X, \mathrm{T}(E)) \to (Y, F)$ is a graph homomorphism then $\{y_1, y_2\} = f(\{x_1, x_2\}) \in F$, so $(y_1, y_2) \in F$. Thus $f(\sigma) \in \mathrm{K}(F)$ and $f : (X, E) \to (Y, \mathrm{K}(F))$ is a simplicial map.

Conversely, suppose that $f : (X, E) \to (Y, \mathrm{K}(F))$ is a simplicial map. Let $\sigma = \{x, y\} \in \mathrm{T}(E)$, i.e. $x\mathrm{T}(E)y$. It is clear by the definition of T that $\sigma \in E$. Since f is a simplicial map, then $f(\sigma) = \{f(x), f(y)\} \in \mathrm{K}(F)$. It follows that $(f(x), f(y)) \in F$, i.e. $f(x)\mathrm{T}(E)f(y)$. Thus $f : (X, \mathrm{T}(E)) \to (Y, F)$ is a graph homomorphism. □

By using the functor we have defined in Section 2.4.16 and 2.4.9. We can define a functor from **CL** to **Simp** and vice versa.

Proposition 4.1.8. *[BM22] The following compositions define a functor from* **CL** *to* **Simp** *and vice versa:*

$$\mathbf{CL} \underset{A}{\overset{\iota}{\underset{\perp}{\leftrightarrows}}} \mathbf{CL_A} \underset{s}{\overset{\iota}{\underset{\perp}{\leftrightarrows}}} \mathbf{CL_{sA}} \underset{\Phi_2}{\overset{\Psi_2}{\underset{\cong}{\leftrightarrows}}} \mathbf{Gph} \underset{\mathrm{K}}{\overset{\mathrm{T}}{\underset{\perp}{\leftrightarrows}}} \mathbf{Simp}.$$

4.2 Vietoris-Rips Complexes and Čech Complexes

We have constructed a functor from the category **CL** to the category **Simp**. Furthermore, we will provide another methods which allows us to build a simplicial complex from a closure space. The construction we will present is essential in a topological data analysis because usually, via Vietoris-Rips and Čech complexes one can obtain a simplicial complex built from a point cloud data.

Definition 4.2.1. Let (X, c) be a closure space. We define $VR(c)$ to be the collection of non-empty subsets $\sigma \subseteq X$ such that for all $x \in \sigma$, $\sigma \subseteq c(x)$.

$$VR(c) = \{\sigma \subseteq X \mid 0 < |\sigma| < \infty,\ \forall x \in \sigma,\ \sigma \subseteq c(x)\}.$$

Proposition 4.2.2. *[BM22] There is a functor VR :* **CL** → **Simp** *sending each closure space (X, c) to the simplicial complex $(X, VR(c))$ and each continuous map $f : (X, c) \to (Y, d)$ to the simplicial map $f : (X, VR(c)) \to (Y, VR(d))$.*

Proof. Let $(X, c) \in$ **CL**. We have to prove first that $(X, VR(c))$ is a simplial complex.

Let $x \in X$. We have $\{x\} \in c(x)$, then $\{x\} \in VR(c)$. Let $\tau \in VR(c)$ and $\emptyset \neq \sigma \subseteq \tau$. For every $x \in \tau$, we have $0 < |\tau| < \infty$ and $\tau \subseteq c(x)$. It follows that for every $x \in \sigma$, $0 < |\sigma| < \infty$ and $\sigma \subseteq c(x)$. Then $\sigma \in VR(c)$. Hence $(X, VR(c))$ is a simplicial complex.

Let $f : (X, c) \to (Y, d)$ be a morphism in **CL**. We want to prove that f induces a simplicial map $f : (X, VR(c)) \to (Y, VR(d))$.
Let $\sigma \in VR(c)$. We have $|f(\sigma)| < \infty$ because $|\sigma| < \infty$. Let $y \in f(\sigma)$. There exists $x \in \sigma$ such that $y = f(x)$. Since $\sigma \in VR(c)$, then $\sigma \in c(x)$ for every $x \in \sigma$. It follows by the continuity of f that $f(\sigma) \subseteq f(c(x)) \subseteq d(f(x)) = d(y)$. This yields $f(\sigma) \in VR(d)$. Hence $f : (X, VR(c)) \to (Y, VR(d))$ is a simplicial map.

We define $VR(X, c) = (X, VR(c))$ for all $(X, c) \in$ **CL** and $VR(f) = f$ for every morphism f in **CL**. Then using the proposition A.0.6, $VR :$ **CL** → **Simp** is a functor. □

Definition 4.2.3. Let (X, c) be a closure space. We define

$$\check{C}(c) = \{\sigma \subseteq X \mid 0 < |\sigma| < \infty,\ \exists x \in X,\ \sigma \subseteq c(x)\}.$$

Proposition 4.2.4. *[BM22] There is a functor $\check{C}$:* **CL** → **Simp** *mapping each closure space (X, c) to the simplicial complex $(X, \check{C}(c))$ and each continuous map $f : (X, c) \to (Y, d)$ to a simplicial map $f : (X, \check{C}(c)) \to (Y, \check{C}(d))$.*

Proof. We prove that $(X, \check{C}(c))$ is a simplial complex.

For every $x \in X$ we have $\{x\} \in c(x)$ then $\{x\} \in \check{C}(c)$. Let $\tau \in \check{C}(c)$ and $\emptyset \neq \sigma \subseteq \tau$. There exists $x \in X$ such that $\tau \subseteq c(x)$, moreover $0 < |\tau| < \infty$. It follows that $\sigma \subseteq c(x)$ and $0 < |\sigma| < \infty$. So $\sigma \in \check{C}(c)$. Hence $(X, \check{C}(c))$ is a simplicial complex.

Let $f : (X, c) \to (Y, d)$ be a continuous map in **CL**. Let $\sigma \in \check{C}(c)$. There exists $x \in X$ such that $\sigma \subseteq c(x)$ and $|\sigma| < \infty$. Therefore, there exists $y = f(x) \in Y$ such that $f(\sigma) \subseteq f(c(x))$ and $|f(\sigma)| < \infty$. Since f is continuous, then $f(\sigma) \subseteq f(c(x)) \subseteq d(f(x)) = d(y)$. So $f : (X, \check{C}(c)) \to (Y, \check{C}(d))$ is a simplicial map.

Define $\check{C}(X, c) = (X, \check{C}(c))$ for every $(X, c) \in$ **CL** and $\check{C}(f) = f$ for every continuous map f in **CL**. One can see easily by Proposition A.0.6 that $\check{C}$: **CL** → **Simp** is a functor. □

Remark 4.2.5. From the section 2.4.21, we have defined a functor M_ϵ : **Met** → **CL**. Now, one can construct directly a functor **Met** → **Simp** by composition of M_ϵ with one of $\check{C}$ or VR. The functor **Met** → **Simp** is crucial in topological data analysis because it allows us to give a simple representation of the data.

The following examples are some illustrations of the generation of a simplicial complex from finite data. Our first example is for the case where the data is not associated to a metric distance, while the last would be a data that can be represented as finite metric spaces.

Example 4.2.6. Consider the set $X = \{h_1, h_2, h_3, h_4, c_1, c_2, c_3, \}$. Set $H = \{h_1, h_2, h_3, h_4\}$ and $C = \{c_1, c_2, c_3\}$, we can think of those two sets as a classes or legends of the elements of X. Define a closure operator $c : P(X) \to P(X)$ by

$$c(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ H & \text{if } A \subseteq H \\ C & \text{if } A \subseteq C \\ X & \text{otherwise.} \end{cases}$$

Applying the definition 4.2.3 and 4.2.1 to the closure space (X, c), we obtain a simplicial complex $(X, \check{C}(c))$ where $\check{C}(c) = (P(H) \setminus \emptyset) \cup (P(C) \setminus \emptyset)$. In addition, we also have $VR(c) = \check{V}(c)$.

Figure 4.1: Point cloud X(left). Simplicial complex(right).

Here, one can observe that we obtained two separate simplicial complexes, one (in green) for the class H and the other (in red) for the class C. Any point from the class H will never connect to any point in the class C. However, the disadvantage of this method is that we always obtain the largest simplicial complex for each class. It would be interesting if we can construct a closure operator which allows us to create enough simplicial complex that fits the data, not necessarily this large simplicial complex.

Example 4.2.7. Let (X, d) be a metric space and $\epsilon \geq 0$. For every $\epsilon \geq 0$, we have the closure space $(X, c_{\epsilon,d})$ corresponding to the metric space (X, d) as in Section 2.4.18. We then obtain the usual *Vietoris-Rips complex* and *Čech complex* as follows:

$$\begin{aligned} \check{C}(c_{\epsilon,d}) &= \{\sigma \subseteq X \mid 0 < |\sigma| < \infty,\ \exists x \in X,\ \sigma \subseteq c_{\epsilon,d}(x)\} \\ &= \{\{x_0, \ldots, x_n\} \subseteq X \mid \exists x \in X,\ \forall 0 \leq i \leq n,\ d(x, x_i) \leq \epsilon\} \\ &= \{\{x_0, \ldots, x_n\} \subseteq X \mid \bigcap_{0 \leq i \leq n} B(x_i, \epsilon) \neq \emptyset\} \\ VR(c_{\epsilon,d}) &= \{\sigma \subseteq X \mid 0 < |\sigma| < \infty,\ \forall x \in \sigma,\ \sigma \subseteq c_{\epsilon,d}(x)\} \\ &= \{\{x_0, \ldots, x_n\} \subseteq X \mid \forall 0 \leq i, j \leq n,\ d(x_i, x_j) \leq \epsilon\}. \end{aligned}$$

One can check that for every $\epsilon \geq 0$, we have $\check{C}(c_{\epsilon,d}) \subseteq VR(c_{\epsilon,d}) \subseteq \check{C}(c_{2\epsilon,d})$.

Let us illustrate these two methods by the following figures. Suppose that we have point a cloud (X,d) where

$$X = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

and d is an Euclidean metric. We fix a radius $\epsilon \in [0,+\infty)$.

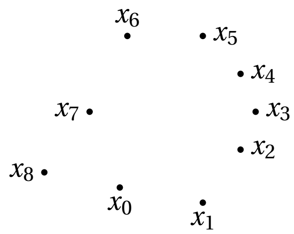


Figure 4.2: Point cloud (X,d).

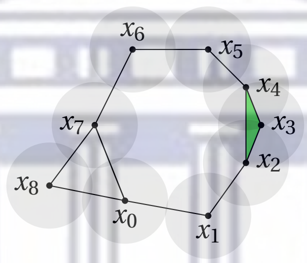


Figure 4.3: Čech complex $\check{C}(c_{\epsilon,d})$. Since $B(x_2,\epsilon) \cap B(x_3,\epsilon) \cap B(x_4,\epsilon) \neq \emptyset$, we have a 2-simplex $\{x_2, x_3, x_4\}$.

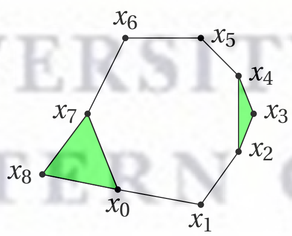


Figure 4.4: Vietoris-Rips complex $VR(c_{2\epsilon,d})$. We have the two 2-simplexes $\{x_2, x_3, x_4\}$ ($\{x_0, x_7, x_8\}$) because $d(x_i, x_j) \leq 2r$ for all $i, j \in \{2,3,4\}$ ($i, j \in \{0,7,8\}$) respectively.

If we look at the point cloud X in Figure 4.2, the only topological information we can deduce from it is that it is a set of points. When we look at the Čech complex built from that point cloud in Figure 4.3 and 4.4, we can deduce that there exists one 1-dimensional hole, and we have only one connected component. However, that topological information may vary whenever we change the value of the parameter ϵ.

In the following section we will construct a sequence of spaces built from different values of ϵ.

4.3 Filtration

We present here various filtrations; they can be a sequence of closure spaces, a sequence of graphs, or a sequence of simplicial complexes connected by inclusion maps. In addition, we will build a filtration of simplicial complexes from metric spaces, which is crucial in topological data analysis. Along this section, we fix **P** to be the category given by the partially ordered set $(P, \leq)$.

Definition 4.3.1. Let **C** be a category. A *filtration* F is a functor from **P** to the category **C** mapping each $p \in \mathbf{P}$ to the object $F(p)$ of **C** and each morphism $p \leq q$ to the inclusion morphism $F(p) \hookrightarrow F(q)$ in **C**.

Definition 4.3.2. Let $F, G : \mathbf{P} \to \mathbf{C}$ be filtrations. A *morphism* between the filtration F and G is a natural transformation $F \Longrightarrow G$.

Usually, such a filtration is represented diagrammatically as

$$\cdots \hookrightarrow F(p_1) \hookrightarrow F(p_2) \hookrightarrow F(p_3) \hookrightarrow \cdots$$

whenever the composition of morphisms $\cdots \leq p_1 \leq p_2 \leq p_3 \leq \cdots$ make sense in **P**.

The category of filtrations in **C** together with all morphisms between them will be denoted by $\mathbf{F_P C}$. It is a subcategory of a $\mathbf{C^P}$.

The following are some explicit examples of filtration, which we will use later.

Definition 4.3.3. [BM22] A **P**-*filtered closure space* $(X_\bullet, c_\bullet)$ is a functor $\mathbf{P} \to \mathbf{CL}$ mapping every object $p \in \mathbf{P}$ to the morphisms $(X_p, c_p) \in \mathbf{CL}$ and each morphism $p \leq q$ in **P** to the continuous inclusion $(X_p, c_p) \hookrightarrow (X_q, c_q)$, i.e. $X_p \subseteq X_q$ and for all $A \subseteq X_p$, $c_p(A) \subseteq c_q(A)$.

A morphism $f : (X_\bullet, c_\bullet) \to (Y_\bullet, d_\bullet)$ between two **P**-filtered closure spaces is then a natural transformation $(X_\bullet, c_\bullet) \Longrightarrow (Y_\bullet, d_\bullet)$. It is characterized by the family of continuous maps $\{f_p : (X_p, c_p) \to (Y_p, d_p)\}_{p \in P}$ such that the following diagram commutes

$$\begin{array}{ccc} (X_p, c_p) & \hookrightarrow & (X_q, c_q) \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle f_q} \\ (Y_p, d_p) & \hookrightarrow & (Y_q, d_q) \end{array}$$

for every $p \leq q$ in **P**.

We denote by $\mathbf{F_P CL}$ the category of **P**-filtered closure spaces together with morphisms between them. Of course it is a subcategory of $\mathbf{CL^P}$.

Definition 4.3.4. [BM22] A **P**-*filtered simplicial complex* $(X_\bullet, E_\bullet)$ is a functor $\mathbf{P} \to \mathbf{Simp}$ that maps each $p \in \mathbf{P}$ to the simplicial complex (X_p, E_p) and each morphism $p \leq q$ to the inclusion simplicial map $(X_p, E_p) \hookrightarrow (X_q, E_q)$. i.e. $X_p \subseteq X_q$ and $E_p \subseteq E_q$ for every $p \leq q$.

A morphism $f : (X_\bullet, E_\bullet) \to (Y_\bullet, F_\bullet)$ between **P**-filtered simplicial complexes is then given by the collection of simplicial maps $\{f_p : (X_p, E_p) \to (Y_p, F_p)\}_{p \in P}$ such that $f_p = f_{q|X_p}$ for every $p \leq q$ in **P**.

All **P**-filtered simplicial complexes together with all morphisms between them form a category $\mathbf{F_P Simp}$ which is a subcategory of $\mathbf{Simp^P}$.

The most useful filtration in topological data analysis is the case when we restrict to the category of topological spaces; very often, we use the objects of $\mathbf{F_{[n]}Top}$, $\mathbf{F_{(\mathbb{R},\leq)}Top}$ and $\mathbf{F_{(\mathbb{R},\leq)}Simp}$.

Example 4.3.5. Let (X, c) be a closure space and $f : X \to P$ be a set map. Set

$$D_p = \{q \in P |\ q \leq p\} \text{ and } X_p = f^{-1}(D_p) \text{ for all } p \in P$$

and define a closure operator $c_p : P(X_p) \to P(X_p)$ by $c_p(A) = c(A) \cap X_p$ for all $A \subseteq X_p$. We have a **P**-filtered closure space

$$\begin{aligned} \mathrm{Sub}(f) : \mathbf{P} &\longrightarrow \mathbf{CL} \\ p &\longmapsto (X_p, c_p) \\ p \leq q &\longmapsto (X_p, c_p) \hookrightarrow (X_q, c_q) \end{aligned}$$

Example 4.3.6. Particularly, let $(X,\mathcal{T})$ be a topological space and assume that $\mathbf{P} = (\mathbb{R},\leq)$. Let $f : X \to \mathbb{R}$ be a real valued function on X. For every $a \in \mathbb{R}$ we define the sublevel set

$$X_a = f^{-1}((-\infty,a]) := \{x \in X|\ f(x) \leq a\}.$$

Additionally, we can define the induced topological space $(X_a,\mathcal{T}_{X_a})$ with

$$\mathcal{T}_{X_a} := \{U \cap X_a|\ U \in \mathcal{T}\}.$$

Then, the *sublevel set filtration* of the pair (X, f) is given as follows:

$$\begin{aligned} \mathrm{Sub}(f) : (\mathbb{R},\leq) &\longrightarrow \mathbf{Top} \\ a &\longmapsto (X_a,\mathcal{T}_{X_a}) \\ a \leq b &\longmapsto (X_a,\mathcal{T}_{X_a}) \hookrightarrow (X_b,\mathcal{T}_{X_b}) \end{aligned}$$

We note that for every $(X,c) \in \mathbf{CL}$ and partially ordered set $\mathbf{P}$ there is a function $\mathrm{Sub} : \hom_{\mathbf{Set}}(X,P) \to Ob(\mathbf{CL}^{\mathbf{P}})$ mapping a set map $f : X \to P$ to a $\mathbf{P}$-filtered closure space $\mathrm{Sub}(f)$.

Now, we are going to build a $[0,\infty)$-filtered closure space and $[0,\infty)$-filtered simplicial complex from a metric space.

Definition 4.3.7. Let (X,d) be a metric space. We obtain a $[0,\infty)$-filtered closure spaces $(X,c_{\bullet,d})$, which is a functor $([0,\infty),\leq) \to \mathbf{CL}$ characterized by the family of closure spaces $\{(X,c_{\epsilon,d})\}_{\epsilon\in[0,\infty)}$ together with all continuous inclusion maps $(X,c_{\epsilon,d}) \hookrightarrow (X,c_{\epsilon',d})$ whenever $\epsilon \leq \epsilon'$.

Remark 4.3.8. There is a functor $\mathbf{Met} \to \mathbf{F}_{[0,\infty)}\mathbf{CL}$ which assigns to each metric space (X,d) the $[0,\infty)$-filtered closure space $(X,c_{\bullet,d})$, and each morphism $f : (X,d_X) \to (Y,d_Y)$ in $\mathbf{Met}$ to the morphism of $[0,\infty)$-filtered closure space $f : (X,c_{\bullet,d_X}) \to (Y,c_{\bullet,d_Y})$.

Proposition 4.3.9. *There is a functor* $\mathbf{Met} \to \mathbf{F}_{[0,\infty)}\mathbf{Simp}$.

Proof. Consider the functor $\check{C} : \mathbf{CL} \to \mathbf{Simp}$. Using the lemma A.0.14, there exists a function $\bar{C} : \mathbf{F}_{[0,\infty)}\mathbf{CL} \to \mathbf{F}_{[0,\infty)}\mathbf{Simp}$. By the remark 4.3.8, we have the functor $\mathbf{Met} \to \mathbf{F}_{[0,\infty)}\mathbf{CL}$. Therefore, the functor $\mathbf{Met} \to \mathbf{F}_{[0,\infty)}\mathbf{Simp}$ is defined by the composition of those functor; the mapping on objects and morphisms is then defined as follows:

- for every $(X,d_X) \in \mathbf{Met}$

$$(X,d_X) \longmapsto (X,c_{\bullet,d_X}) \longmapsto (X,\check{C}(c_{\bullet,d_X})),$$

- for every morphism $f : (X,d_X) \to (Y,d_Y)$ in $\mathbf{Met}$

$$\begin{array}{ccc} (X,d_X) \to (Y,d_Y) & \longmapsto & (X,c_{\bullet,d_X}) \to (Y,c_{\bullet,d_Y}) \\ & & \downarrow \\ & & (X,\check{C}(c_{\bullet,d_X})) \to (Y,\check{C}(c_{\bullet,d_Y})). \end{array}$$

□

Example 4.3.10. Here is an illustration of a filtration build from a point cloud with a samples of points. We take five different values of ϵ and apply the Čech complex construction for each value.

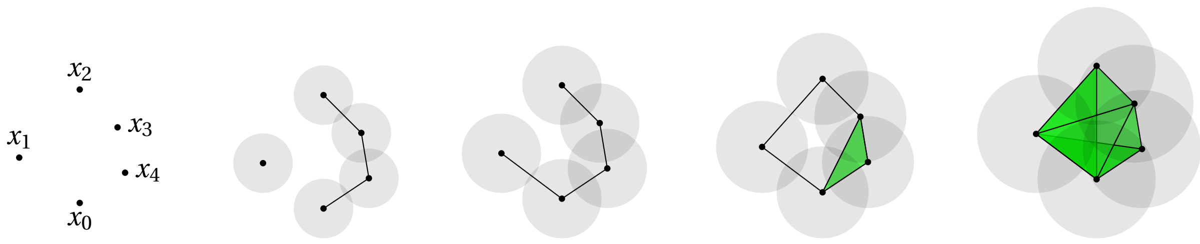


Figure 4.5: Filtration obtained by using a Čech complex $\check{C}(c_{\epsilon,d})$ at a different scales of ϵ.

4.4 Persistence Modules

We have formalized the notions of a filtration. In this section, we will give a fundamental concepts of how we can study the algebraic structure of such a filtration. We will see for example, how the homology group may vary across a filtration.
Let R be a commutative ring with multiplicative identity element. We recall that R**Mod** is the category of all small left modules over the ring R and all module homomorphisms.

Definition 4.4.1. [Cha+16] Given a partially ordered set **P**. A *persistence module* V over **P** is a functor from **P** to R**Mod** that sends each object $p \in \mathbf{P}$ to the left R module $V(p) \in R\mathbf{Mod}$ and each morphism $p \leq q$ in **P** to the module homomorphism $V(p) \to V(q)$.
A persistence module over **P** is then an object of the functor category $R\mathbf{Mod}^{\mathbf{P}}$. If there is no confusion we will write simply V_p instead of $V(p)$ for every $p \in \mathbf{P}$.

Definition 4.4.2. [Cha+16] Let $U, V \in R\mathbf{Mod}^{\mathbf{P}}$. A homomorphism $\phi : U \to V$ between two persistence modules is a natural transformation $U \Longrightarrow V$.

One could notice that if R is the ring of integers $\mathbb{Z}$ then the category of persistence modules is the functor category $\mathbf{Ab}^{\mathbf{P}}$. The ring R also can be replaced by any field $\mathbb{K}$, in this case persistence modules are objects of the functor category $\mathbf{Vec}^{\mathbf{P}}$, and we will say persistence vector spaces over **P**.

Additionally, in topological data analysis situations, we often use the persistence module to be the objects of one the following categories $\mathbf{Vec}^{(\mathbb{R},\leq)}$, $\mathbf{Vec}^{(\mathbb{N},\leq)}$ and $\mathbf{Vec}^{[n]}$.

A persistence module $V : (\mathbb{N}, \leq) \to R\mathbf{Mod}$ can be depicted as the following diagram:

$$V_0 \xrightarrow{V(0\leq 1)} V_1 \xrightarrow{V(1\leq 2)} V_2 \xrightarrow{V(2\leq 3)} \cdots$$

Example 4.4.3. We can apply any corresponding homology functor to the filtration in Example 4.3.5 and 4.3.6 to obtain a persistence module. For instance, given a closure space $(X, c) \in \mathbf{CL}$, we obtain the **P**-filtered closure space $\{(X_p, c_p)\}_{p\in P}$. After that, we apply the singular homology functor $H_n(-;\mathbb{Z}) : \mathbf{CL} \to \mathbf{Ab}$ in Section 3.4 to obtain the family of $\mathbb{Z}$-modules $\{H_n(X_p, c_p)\}_{p\in P}$, together with all group homomorphisms of the form $H_n(X_p, c_p) \to H_n(X_q, c_q)$ induced by $(X_p, c_p) \hookrightarrow (X_q, c_q)$ for all $p \leq q$. This last describes a persistence module $\mathbf{P} \to \mathbf{Ab}$.

Proposition 4.4.4. *There exists a functor* $\mathbf{Met} \to \mathbf{Ab}^{([0,\infty),\leq)}$.

Proof. Let $\epsilon \geq 0$ and $n \in \mathbb{N}$. Consider the homology functor $H_n(-;\mathbb{Z}) : \mathbf{CL} \to \mathbf{Ab}$. From the remark 4.3.8, we have a functor $\mathbf{Met} \to \mathbf{F}_{[0,\infty)}\mathbf{CL}$. By the lemma A.0.14, there exists a functor $\mathbf{F}_{[0,\infty)}\mathbf{CL} \to \mathbf{F}_{[0,\infty)}\mathbf{Ab}$. The functor $\mathbf{Met} \to \mathbf{F}_{[0,\infty)}\mathbf{Ab}$ is then defined by the composition of those functors, i.e.

- for every metric space (X, d_X),

$$(X, d_X) \longmapsto (X, c_{\bullet,d_X}) \longmapsto H_n(X, c_{\bullet,d_X})$$

- for every morphism $f : (X, d_X) \to (Y, d_Y)$,

$$\begin{array}{ccc} (X,d_X) \to (Y,d_Y) & \longmapsto & (X, c_{\bullet,d_X}) \to (Y, c_{\bullet,d_Y}) \\ & & \downarrow \\ & & H_n(X, c_{\bullet,d_X}) \to H_n(Y, c_{\bullet,d_Y}). \end{array}$$

□

We note that this functor is not unique. For instance, by using the simplicial homology functor $H_n(-,\mathbb{Z}) : \mathbf{Simp} \to \mathbf{Ab}$ we can obtain another functor $\mathbf{Met} \to \mathbf{Ab}^{[0,\infty)}$.

4.4.5 Persistent Homology

Given a filtration $F \in \mathbf{F}_{(\mathbb{R},\leq)}\mathbf{CL}$. Let $n \in \mathbb{N}$. Consider the homology functor $H_n(-,\mathbb{Z}) : \mathbf{CL} \to \mathbf{Ab}$. We obtain a persistence module $H_nF : (\mathbb{R}, \leq) \to \mathbf{Ab}$ by composing those two functors. It can be represented as the following diagram:

$$\cdots \to H_nF(a_1) \to H_nF(a_2) \to H_nF(a_3) \to H_nF(a_4) \to \cdots$$

whenever the sequence $\cdots \leq a_1 \leq a_2 \leq a_3 \leq \cdots$ exist.

It is worthwhile to mention that the diagram above is connected by the group homomorphism induced by a sequence of inclusion maps from the filtration. Therefore, some n-homology classes that appear at $H_nF(a_i)$ are still present along $H_nF(a_j)$ but some of them will disappear. The persistent homology helps us to recognize that appearance and disappearance.

Definition 4.4.6. Given a filtration $F \in \mathbf{F}_{(\mathbb{R},\leq)}\mathbf{CL}$. A *(b-a)-persistent nth homology group* of $F(a)$ denoted by $H_n^{a,b}F$ is the image of the morphism $H_nF(a \leq b)$. The *nth persistent Betti number* $\beta_n^{a,b}$ is given by the rank of $H_n^{a,b}F$.

For every $a, p \in \mathbb{R}$ and $n \in \mathbb{N}$, the cycle group and boundary group of the closure space $F(a)$ will be denoted by $Z_n^a := Z_n^a(F(a))$ and $B_n^a := B_n^a(F(a))$ respectively. We also denote by $\eta_n^{a,b}$ the group homomorphism $H_nF(a \leq b) : H_nF(a) \to H_nF(b)$.

The following theorem gives another view and more significant meaning to the definition of persistent homology.

Theorem 4.4.7. *[EH10] Given a filtration $F \in \mathbf{F}_{(\mathbb{R},\leq)}\mathbf{CL}$. For each $a \in \mathbb{R}$ and $n \in \mathbb{N}$. There is an isomorphism*

$$H_n^{a,b}F \cong Z_n^a/(Z_n^a \cap B_n^b).$$

Proof. Consider the map

$$\begin{aligned} \eta_n^{a,b} : Z_n^a/B_n^a &\longrightarrow Z_n^b/B_n^b \\ x + B_n^a &\longmapsto x + B_n^b. \end{aligned}$$

Let $x + B_n^a \in Z_n^a/B_n^a$ such that $\eta_n^{a,b}(x + B_n^a) = 0$. That implies $x \in B_n^b$. Therefore $\ker \eta_n^{a,b} = Z_n^a \cap B_n^b$. We also note that $B_n^a \subseteq Z_n^a \cap B_n^b$. It follows by the First Isomorphism Theorem that we have

$$\operatorname{im} \eta_n^{a,b} \cong (Z_n^a/B_n^a)/(Z_n^a \cap B_n^b) = Z_n^a/(Z_n^a \cap B_n^b).$$

□

Remark 4.4.8. It is extremely important to know that the (b-a)-persistent nth homology group contains the n-homology class of $F(a)$ that still exists at $F(b)$. It allows us to capture the evolution of the n-homology classes along the filtration. In addition, one may interpret the Betti number $\beta_n^{a,b}$ as the number of n-dimensional classes born at, or before $F(a)$ and dies after $F(b)$. Particularly, we have $\beta_n^{0,b} = 0$ for every $b \geq 0$.

Definition 4.4.9. [EH10] Let $F \in \mathbf{F}_{(\mathbb{R},\leq)}\mathbf{CL}$ be a filtration. Given a morphism $a \leq b$ in $(\mathbb{R},\leq)$ and $n \in \mathbb{N}$.

i. We say that a class $x \in H_nF(a)$ is *born* at $F(a)$ if $x \notin H_n^{a-1,a}F$. In this case, we will say that x is *born at time (or scale) a*

ii. If x is born at $F(a)$, we say that x *dies* entering $F(b)$ if $\eta^{a,b-1}(x) \notin H_n^{a-1,b-1}F$ but $\eta^{a,b}(x) \in H_n^{a-1,b}F$. We also say that x *dies at time b*.

iii. If a class x is born at $F(a)$ and dies entering $F(b)$, we define by $\text{pers}(x) = b - a$ *the persistence of x*. If x is born at $F(a)$ but never dies, then we say that x has persistence to infinity. We will write those pairing of birth and death of x as a pair (a,b) and (a,∞).

iv. The number of n-homology classes that are born at $F(a)$ and die entering $F(b)$ is given by

$$\mu_n^{a,b} := (\beta_n^{a,b-1} - \beta_n^{a,b}) - (\beta_n^{a-1,b-1} - \beta_n^{a-1,b}).$$

4.4.10 Visualization

We will briefly review two fundamental visualizations of persistent homology. Recall that a *multiset* is a pair $M = (A,\mu)$ where a A is a set and $\mu : A \to \mathbb{N}$ is a multiplicity function that indicates how many times each element of A occurs in M. The cardinality of M is then given by $|M| = \sum_{a\in A}\mu(a)$. We denote by $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ [Cha+16].

Definition 4.4.11. Let $F : (\mathbb{R},\leq) \to \mathbf{CL}$ be a $(\mathbb{R},\leq)$-filtered closure space. Consider $HF : (\mathbb{R},\leq) \to \mathbf{Ab}$ its corresponding persistence module. A *persistence diagram* $\text{dgm}HF$ is a multiset consisting of all pairs $(a,b) \in \bar{\mathbb{R}}^2$ such that there exists a n-homology classe $x \in H_nF(a)$ that is born at time a and dies at time b across the filtration with multiplicity $\mu_n^{a,b}$.

If we denote by $\text{dgm}_n HF$ the set of all pairs (a,b) consisting of birth and death of all n-dimensional holes along the filtration F, then $\text{dgm}HF$ is the union of all $\text{dgm}_n HF$.

Taking the filtration given in Example 4.3.10, we can see its persistence diagram as follows.

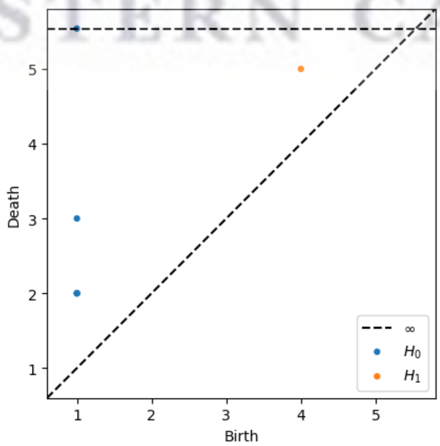


Figure 4.6: Persistence diagram of the filtration in 4.3.5; $\text{dgm}_0 HF = \{(1,2), (1,2), (1,2), (1,3), (1,\infty)\}$ and $\text{dgm}_1 HF = \{(4,5)\}$. This diagram is obtained by using the python libraries `dyonisus` and `persim`.

Here in Figure 4.6, every 0-dimensional hole and 1-dimensional hole are represented by the blue and orange dots respectively.

There exist three 0-dimensional holes generated by x_2, x_3 and x_4 which are born at time 1 and die early at time 2 because of merging with x_0 to form a single component generated by x_0; $\mu_0^{1,2} = 3$. There is also one 0-dimensional hole generated by x_1 that is born at time 0 but die at time 3, since it is combined with the component generated by x_0 at time 3 to form a single component x_0. So $\mu_0^{1,3} = 1$. In addition, there exists exactly one 1-dimensional hole which is born at time 4 and die at time 5, i.e. $\mu_1^{4,5} = 1$.

The other representation of persistence is called *barcode diagram* $\mathrm{bar}HF$, it is exactly the same multiset $\mathrm{dgm}HF$. In the barcode diagram we represent the pair of appearance (a, b) as a line segment $[a, b[$, which is often called a bar. The pair of the form (a, ∞) is represented as an infinite bar $[a, \infty[$. In the representation of the persistence barcode diagram we will emphasize the multiplicity.

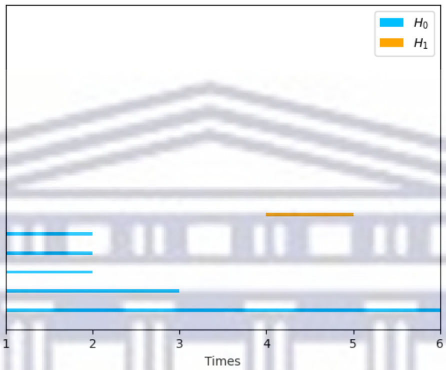


Figure 4.7: Persistence barcode diagram of the filtration in Example 4.3.10; $\mathrm{bar}_0 HF = \{[1,2[, [1,2[, [1,2[, [1,3[, [1,\infty[\}$ and $\mathrm{bar}_1 HF = \{[4,5[\}$.

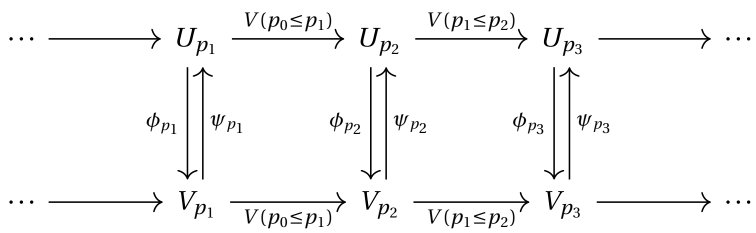
It is worth mentioning that in practice, we will deal mostly with finite point cloud data. In this case, we obtain a finite simplicial complex, we then need to work with simplicial homology as in Example 3.3.10. The advantage of simplicial homology is that one can use some tools from linear algebra like echelon form and smith normal form to compute it. Further treatement of these approaches can be seen in [Mun84].

Furthermore, several variants of the algorithm and efficient implementation has been developed to compute a persistent homology, that can be practically useful for a visualization and exploration of the data. We refer the reader to [EH10] and [ZC04].

Lastly, one should notice that the modification of a filtration may change the persistence module and the persistence diagram. That causes an instability of the features and we don't have a guarantee that our topological information is relevant. Therefore, we should be wondering about when is that machinery stable. The following theory will provide an investigation of the stability of the persistent homology.

4.5 Interleaving and Stability

Given two persistence modules $U, V \in R\mathbf{Mod}^{\mathbf{P}}$. We say that U and V are isomorphic if there exists a morphism $\phi : U \to V$ and $\psi : V \to U$ such that $\psi\phi = id_U$ and $\phi\psi = id_V$.



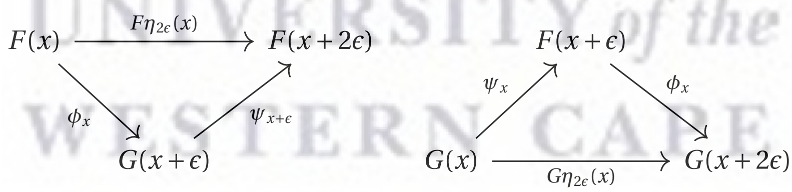
In topological data analysis this condition is too strong because the persistence modules are constructed from uncertain and noisy data, so it can be never reach isomorphism. We then define a weaker condition by introducing a theory of interleaving. The theory of interleaving aims to measure how far two persistence modules are isomorphic. The theory of interleaving was introduced first by Chazal et al in [Cha+08]. Since then, numerous advances have been made for generalization of this theory. In this section, we will present in detail some of the results given in [BS14] and [Cha+16].

Let $\epsilon \ge 0$. Define a functor $T_\epsilon : (\mathbb{R}, \le) \to (\mathbb{R}, \le)$ by $T_\epsilon(x) = x + \epsilon$ for all $x \in \mathbb{R}$ and a natural transformation $\eta_\epsilon : id_{(\mathbb{R},\le)} \Longrightarrow T_\epsilon$ given by $\eta_\epsilon : x \to x+\epsilon$ for all $x \in \mathbb{R}$. We note that $T_b T_c = T_{b+c}$ and $\eta_b \eta_c = \eta_{b+c}$ for all $b, c \in \mathbb{R}$.

We would like to mention that we will use very often Lemma A.0.13 along this section.
Let $\mathbf{D}$ be any category and let $\epsilon \ge 0$.

Definition 4.5.1. [BS14] Let $F, G \in \mathbf{D}^{(\mathbb{R},\le)}$. An ϵ-*interleaving* of F and G consists of natural transformations $\phi : F \Longrightarrow GT_\epsilon$ and $\psi : G \Longrightarrow FT_\epsilon$ such that

$$(\psi T_\epsilon)\phi = F\eta_{2\epsilon} \text{ and } (\phi T_\epsilon)\psi = G\eta_{2\epsilon}.$$



If such ϵ-interleaving exist, we say that F and G are ϵ-*interleaved*.

We note that if $\epsilon = 0$ we have the original definition of isomorphism between F and G.

Definition 4.5.2. [Cha+16] The *interleaving distance* between F and G is defined by

$$d_I(F,G) = \inf_\epsilon \{\epsilon \ge 0 |\ F \text{ and } G \text{ are } \epsilon\text{-interleaved}\}.$$

If there is no ϵ-interleaved between F and G for any value $\epsilon \ge 0$, then we set $d_I(F,G) = \infty$.

Lemma 4.5.3. *[BS14] Let $F, G \in \mathbf{D}^{(\mathbb{R},\le)}$. If F and G are ϵ-interleaved then they are also ϵ'-interleaved for any $\epsilon' \ge \epsilon$.*

Proof. Suppose that F and G are ϵ-interleaved. There exists a natural transformation $\phi : F \Longrightarrow GT_\epsilon$ and $\psi : G \Longrightarrow FT_\epsilon$ such that $(\psi T_\epsilon)\phi = F\eta_{2\eta}$ and $(\phi T_\epsilon)\psi = G\eta_{2\epsilon}$.
Let $\epsilon' \ge \epsilon$ and set $\bar{\epsilon} = \epsilon' - \epsilon$. We have $\eta_{\bar{\epsilon}} : id_{(\mathbb{R},\ge)} \Longrightarrow T_{\bar{\epsilon}}$. Moreover, we know that $\eta_{\bar{\epsilon}} T_\epsilon : T_\epsilon \Longrightarrow T_{\epsilon'}$ is a natural transformation.

Let $x \le y$ be a morphism in $(\mathbb{R}, \le)$. Since $\eta_{\bar{\epsilon}} T_\epsilon : T_\epsilon \Longrightarrow T_{\epsilon'}$ is a natural transformation, there is exactly one morphism $x+\epsilon \le y+\epsilon'$. Moreover, there is a unique morphism $G(x+\epsilon) \to G(y+\epsilon')$. Thus $G\eta_{\bar{\epsilon}} T_\epsilon : GT_\epsilon \Longrightarrow GT_{\epsilon'}$ is a natural transformation.

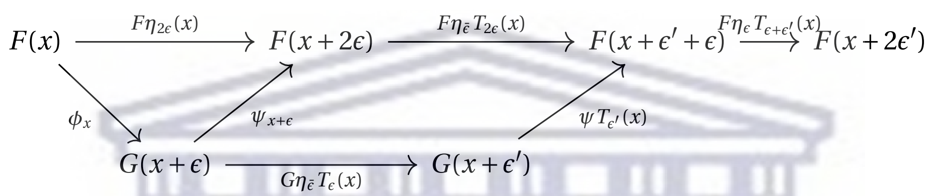
We define the natural transformation $\hat{\phi} = (G\eta_{\bar{\epsilon}} T_\epsilon)\phi : F \Longrightarrow GT_{\epsilon'}$ by the component $\hat{\phi}_x = (G\eta_{\bar{\epsilon}} T_\epsilon)_x \phi_x$ for all $x \in \mathbb{R}$.

$$\hat{\phi}_x : F(x) \xrightarrow{\ \phi_x\ } G(x+\epsilon) \xrightarrow{(G\eta_{\bar{\epsilon}} T_\epsilon)_x} G(x+\epsilon')$$

Similarly, we define $\hat{\psi} = (F\eta_{\bar{\epsilon}} T_\epsilon)\psi : G \Longrightarrow FT_{\epsilon'}$.

$$\hat{\psi}_x : G(x) \xrightarrow{\ \psi_x\ } F(x+\epsilon) \xrightarrow{(F\eta_{\bar{\epsilon}} T_\epsilon)_x} F(x+\epsilon')$$

It remains for us to prove that $(\hat{\psi} T_{\epsilon'})\hat{\phi} = F\eta_{2\epsilon'}$ and $(\hat{\phi} T_{\epsilon'})\hat{\psi} = G\eta_{2\epsilon'}$.
Let $x \in \mathbb{R}$. We consider the following diagram:



(4.1)

By the hypothesis, F and G are ϵ-interleaved so the triangle diagram above is commutative. Moreover, since ψ is a natural transformation then the square diagram in (4.1) is commutative . It is clear that $F\eta_\epsilon T_{\epsilon+\epsilon'} = F\eta_{\bar{\epsilon}} T_\epsilon$. A routine calculation implies that the composition of the horizontal diagram on the top in (4.1) is equal to $F\eta_{2\epsilon'}(x) : F(x) \to F(x+2\epsilon')$. That is also equal to $(\hat{\psi} T_{\epsilon'})\hat{\phi}(x)$ because of the commutativity of the diagram (4.1). Thus $(\hat{\psi} T_{\epsilon'})\hat{\phi} = F\eta_{2\epsilon'}$.

Analogously for the proof of $(\hat{\phi} T_{\epsilon'})\hat{\psi} = G\eta_{2\epsilon'}$. □

Theorem 4.5.4. *[BS14] The map $d_I : Ob(\mathbf{D}^{(\mathbb{R},\le)}) \times Ob(\mathbf{D}^{(\mathbb{R},\le)}) \longrightarrow [0,\infty)$ given by $(F,G) \longmapsto d_I(F,G)$ is a pseudo-metric.*

Proof. Let $F, G \in \mathbf{D}^{(\mathbb{R},\le)}$ such that F and G are isomorphic. So F and G are 0-interleaved. Then $d_I(F,G) = 0$.

Let $F, G \in \mathbf{D}^{(\mathbb{R},\le)}$. By symmetry of the definition of ϵ-interleaving, $d_I(F,G) = d_I(G,F)$.

Let $F, G, H \in \mathbf{D}^{(\mathbb{R},\le)}$ such that F and G are ϵ'-interleaved and G and H are ϵ''-interleaved. Let $a = d_I(F,G)$ and $b = d_I(G,H)$. Given $\epsilon > 0$. By the lemma 4.5.3, F and G are $(a+\epsilon)$-interleaved, G and H are $(b+\epsilon)$-interleaved. There exist natural transformations $\phi' : F \Longrightarrow GT_{a+\epsilon}$ and $\psi' : G \Longrightarrow FT_{a+\epsilon}$ such that

$$(\psi' T_{a+\epsilon})\phi' = F\eta_{2(a+\epsilon)} \text{ and } (\phi' T_{a+\epsilon})\psi' = G\eta_{2(a+\epsilon)}. \tag{4.2}$$

And other natural transformations $\phi'' : G \Longrightarrow HT_{b+\epsilon}$ and $\psi'' : H \Longrightarrow GT_{b+\epsilon}$ such that

$$(\psi'' T_{b+\epsilon})\phi'' = G\eta_{2(b+\epsilon)} \text{ and } (\phi'' T_{b+\epsilon})\psi'' = H\eta_{2(b+\epsilon)}. \tag{4.3}$$

Note that

$$HT_{b+\epsilon} T_{a+\epsilon} = HT_{a+b+2\epsilon} \text{ and } FT_{b+\epsilon} T_{a+\epsilon} = FT_{a+b+2\epsilon}.$$

Consider the natural transformations

$$\phi = (\phi'' T_{a+\epsilon})\phi' : F \Longrightarrow GT_{a+\epsilon} \Longrightarrow HT_{a+b+2\epsilon} \text{ and } \psi = (\psi' T_{b+\epsilon})\psi'' : H \Longrightarrow GT_{b+\epsilon} \Longrightarrow FT_{a+b+2\epsilon}$$

defined as follows: for any object $x \in (\mathbb{R}, \le)$,

$$\phi_x : F(x) \xrightarrow{\ \phi'_x\ } G(x+a+\epsilon) \xrightarrow{\phi''_{x+a+\epsilon}} H(x+a+b+2\epsilon)$$

and

$$\psi_x : H(x) \xrightarrow{\psi''_x} G(x+b+\epsilon) \xrightarrow{\psi'_{x+b+\epsilon}} F(x+a+b+2\epsilon).$$

Using the identities (4.2) and (4.3) and the fact that ϕ and ψ are natural transformations, a routine computation yields that $(\psi T_{a+b+2\epsilon})\phi = F\eta_{2(a+b+2\epsilon)}$ and $(\phi T_{a+b+2\epsilon})\psi = H\eta_{2(a+b+2\epsilon)}$.
Therefore F and H are $(a+b+2\epsilon)$-interleaved for all $\epsilon > 0$. It follows by definition of interleaving distance that $d_I(F,H) \le a+b+2\epsilon$. Since $\epsilon > 0$ is arbitrary then $d_I(F,H) \le a+b = d_I(F,G)+d_I(G,H)$. □

We note that the identity $d_I(F,G) = 0$ only implies that F and G are ϵ-interleaved for every $\epsilon > 0$. One can notice that $\mathbf{D}^{(\mathbb{R},\le)}$ is an enriched category over the monoidal category $((\mathbb{R},\le),+)$, following the definition introduced by Lawvere in [Law73].

Remark 4.5.5. It is important to know that there exists another distance called *Bottleneck distance* which is defined in terms of persistence diagrams, this distance measures how far two persistence diagrams differ from each other. Let dgmHF and dgmHG be persistence diagrams. Consider the set of all bijections $\mathcal{B} = \{\pi : \text{dgm}HF \to \text{dgm}HG | \ \pi \text{ is a bijection}\}$. The bottleneck distance is given by

$$d_B(\text{dgm}HF, \text{dgm}HG) := \inf_{\pi\in\mathcal{B}} \sup_{(a,b)\in\text{dgm}HF} ||(a,b)-\pi(a,b)||_\infty$$

where $||(x,y)||_\infty := \max\{|x|,|y|\}$ for all $(x,y) \in \mathbb{R}^2$.

It was shown in [Cha+16] that if HF and HG are persistence vector spaces over $(\mathbb{R},\le)$ and the dimension of $HF(a)$, $HG(a)$ are finite for all $a \in \mathbb{R}$, then $d_B(\text{dgm}HF, \text{dgm}HG) = d_I(HF,HG)$. This is called *isometry theorem*, and it establishes an equivalence between the interleaving distance and the bottleneck distance. The proof of the isometry theorem requires another theory, we then refer the reader to [Cha+16] and [Les15] for further treatment of this subject.

We recall that for any real valued functions f,g on a set X, we define the sup norm by $||f-g||_\infty = \sup_{x\in X}|f(x)-g(x)|$. Note that if f and g are continuous and X is a compact set, then the supremum is attained and the function $f-g$ is bounded, i.e. there exists a real number $M > 0$ such that $||f-g||_\infty \le M$.

Basically, we can interpret the stability theorem as follows: whenever we make a small change of the filtration we will have a small change of persistence modules and of course the persistence diagram. Fortunately, thanks to a valuable works of Cohen-Steiner, Edelsbrunner and Harer in [CEH07], they manage to prove the first stability theorem which can be formalized as follows: Let X be triangulable space and $f,g : X \to \mathbb{R}$ continuous maps which are "tame" (see [EH10]). Assume H is a singular homology functor on a topological space with coefficient in a field. Then, the bottleneck distance between two persistence diagrams dgm$H$Sub(f) and dgm$H$Sub(g) is bounded by $||f-g||_\infty$, i.e.

$$d_B(\text{dgm}H\text{Sub}(f), \text{dgm}H\text{Sub}(g)) \le ||f-g||_\infty.$$

Several successful works appear to attempt a generalization of this theorem by removing some conditions on the hypothesis. One of the general formalizations is given by Bubenik and Scott in [BS14] as follows: Let $f,g : X \to \mathbb{R}$ be set maps on the space $(X,\mathcal{T})$. If H is a functor from the category of topological spaces then $d_I(H\text{Sub}(f), H\text{Sub}(g)) \le ||f-g||_\infty$. In other words $H\text{Sub} : \hom_{\mathbf{Set}}(X,\mathbb{R}) \to Ob(\mathbf{Top}^{(\mathbb{R},\le)}) \to Ob(\mathbf{Ab}^{(\mathbb{R},\le)})$ is non-expansive map.

The following theorem will be useful to prove the stability of persistence modules obtained from filtered closures spaces.

Theorem 4.5.6. *[BS14] Let $F,G \in \mathbf{D}^{(\mathbb{R},\le)}$ and $H : \mathbf{D} \to \mathbf{E}$. If F and G are ϵ-interleaved, then so are HF and HG. Thus,*

$$d_I(HF,HG) \le d_I(F,G).$$

Proof. Suppose that F and G are ϵ-interleaved. There exist natural transformations $\phi : F \Longrightarrow GT_\epsilon$ and $\psi : G \Longrightarrow FT_\epsilon$ such that $(\psi T_\epsilon)\phi = F\eta_{2\epsilon}$ and $(\phi T_\epsilon)\psi = G\eta_{2\epsilon}$. We consider the natural transformations $H\phi : HF \Longrightarrow HGT_\epsilon$ and $H\psi : HG \Longrightarrow HFT_\epsilon$ given by the following components: for every object $x \in (\mathbb{R}, \leq)$

$$(H\phi)_x : HF(x) \to HG(x+\epsilon) \text{ with } (H\phi)_x(a) = H(\phi_x)(a)$$

and

$$(H\psi)_x : HG(x) \to HF(x+\epsilon) \text{ with } (H\psi)_x(a) = H(\psi_x)(a).$$

We then define the natural transformations $H\psi T_\epsilon : HGT_\epsilon \Longrightarrow HFT_{2\epsilon}$ and $H\phi T_\epsilon : HFT_\epsilon \Longrightarrow HGT_{2\epsilon}$ by the components

$$(H\psi T_\epsilon)_x = (H\psi)_{T_\epsilon(x)} = H(\psi_{T_\epsilon(x)}) : HG(x+\epsilon) \to HF(x+2\epsilon)$$

and

$$(H\phi T_\epsilon)_x = (H\phi)_{T_\epsilon(x)} = H(\phi_{T_\epsilon(x)}) : HF(x+\epsilon) \to HG(x+2\epsilon).$$

We have to prove that $(H\psi T_\epsilon)H\phi = (HF)\eta_{2\epsilon}$ and $(H\phi T_\epsilon)H\psi = (HG)\eta_{2\epsilon}$.
Let x be an object of $(\mathbb{R}, \leq)$. Using the functoriality of H and the fact that F and G are ϵ-interleaved, we have

$$(H\psi T_\epsilon)_x(H\phi)_x = H(\psi T_\epsilon(x))H(\phi_x) = H((\psi T_\epsilon \phi)_x) = H(F\eta_{2\epsilon}(x))$$

and

$$(H\phi T_\epsilon)_x(H\psi)_x = H(\phi T_\epsilon(x))H(\psi_x) = H((\phi T_\epsilon \psi)_x) = H(G\eta_{2\epsilon}(x))$$

We thus have $(H\psi T_\epsilon)(H\phi) = (HF)\eta_{2\epsilon}$ and $(H\phi T_\epsilon)H\psi = (HG)\eta_{2\epsilon}$. Therefore HF and HG are ϵ-interleaved. □

Finally, we are going to prove the stability of persistence modules obtained from filtered closure spaces, which is an extension of what Bubenik and Scott achieved in [BS14], later on formalized again by Bubenik with Milićević in [BM22].

Theorem 4.5.7. *[BM22] Let $f, g : X \to \mathbb{R}$ be set maps on the closure space (X, c). If $\epsilon = ||f - g||_\infty$, then* $\text{Sub}(f), \text{Sub}(g)$ *are ϵ-interleaved. Moreover, if H is a functor from* **CL** *(particularly homology functor from* **CL***) then $d_I(H\text{Sub}(f), H\text{Sub}(g)) \leq ||f - g||_\infty$.*

Proof. Suppose that $\epsilon = ||f - g||_\infty$. Let $a \in (\mathbb{R}, \leq)$.

On the one hand, let $x \in \text{Sub}(f)_a$. Then $-\infty < f(x) \leq a$. By hypothesis, we have $-\epsilon \leq f(x) - g(x) \leq \epsilon$ for all $x \in X$. We then obtain $-\infty \leq g(x) - \epsilon \leq f(x) \leq a$. So $-\infty \leq g(x) \leq a + \epsilon$. That means $x \in \text{Sub}(g)_{a+\epsilon}$. Thus $\text{Sub}(f)_a \subseteq \text{Sub}(g)_{a+\epsilon}$.

One the other hand, let $x \in \text{Sub}(g)_a$. Similarly, we have $x \in \text{Sub}(f)_{a+x}$. Then $\text{Sub}(g)_a \subseteq \text{Sub}(f)_{a+\epsilon}$. Therefore, there exists a natural transformations

$$\phi : \text{Sub}(f) \Longrightarrow \text{Sub}(g)T_\epsilon \text{ and } \psi : \text{Sub}(g) \Longrightarrow \text{Sub}(f)T_\epsilon$$

where the components are given by the inclusion map

$$\phi_a : \text{Sub}(f)_a \hookrightarrow \text{Sub}(g)_{a+\epsilon} \text{ and } \psi_a : \text{Sub}(g)_a \hookrightarrow \text{Sub}(f)_{a+\epsilon}$$

respectively.

Now, we need to show that $(\psi T_\epsilon)\phi = F\eta_{2\epsilon}$ and $(\phi T_\epsilon)\psi = G\eta_{2\epsilon}$.
Let $x \in \text{Sub}(g)_{a+\epsilon}$. We have $-\infty \leq g(x) \leq a + \epsilon$. Then $-\infty \leq f(x) - \epsilon \leq g(x) \leq a + \epsilon$ by hypothesis. So $f(x) \leq a + 2\epsilon$ and $x \in \text{Sub}(f)_{a+2\epsilon}$. Thus

$$\text{Sub}(f)_a \subseteq \text{Sub}(f)_{a+\epsilon} \subseteq \text{Sub}(f)_{a+2\epsilon}.$$

It follows that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Sub}(f)_a & \xrightarrow{F\eta_{2\epsilon}(a)} & \mathrm{Sub}(f)T_{2\epsilon} \\ & \searrow^{\phi_a} \quad \nearrow^{\psi_{x+\epsilon}} & \\ & \mathrm{Sub}(g)T_\epsilon & \end{array}$$

Therefore $(\psi T_\epsilon)_a \phi_a = F\eta_{2\epsilon}(a)$ for all $a \in \mathbb{R}$. This yields $(\psi T_\epsilon)\phi = F\eta_{2\epsilon}$. Analogously for the proof of $(\phi T_\epsilon)\psi = G\eta_{2\epsilon}$. Hence $\mathrm{Sub}(f)$ and $\mathrm{Sub}(g)$ are ϵ-interleaved.
Furthermore, if $H : \mathbf{CL} \to \mathbf{C}$ is a functor from the category **CL** to any category **C** then by Theorem 4.5.6, we have

$$d_I(H\mathrm{Sub}(f), H\mathrm{Sub}(g)) \leq d_I(\mathrm{Sub}(f), \mathrm{Sub}(g)) \leq ||f - g||_\infty.$$

In other words $H\mathrm{Sub} : \hom_{\mathbf{Set}}(X, P) \to Ob(\mathbf{C}^{(\mathbb{R},\leq)})$ is a non- expansive map. □

Appendix A

Review of Category Theory

We recall some elementary background of category theory. We only state what we need, so this chapter is never longer enough for those who want to learn category theory. We are not giving any proof here. However, one can read more about category theory in [Mac98], [Rie16] and [AHS90].

Definition A.0.1. [AHS90] A *category* **C** consists of

1. a class of objects, those members are called *objects* of **C**,
2. a class of morphisms, those members are called *morphisms*; if $f : X \to Y$ is a morphism in **C** we call X and Y *domain* and *codomain* of f respectively,
3. for each object X, a morphism $id_X : X \to X$, called the identity on X,
4. a composition law associating with each morphism $f : X \to Y$ and $g : Y \to Z$ a morphism $gf : X \to Z$, called *composite* of f and g,

   subject to the following conditions:

5. a composition is associative; i.e., for morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, the equation $h(gf) = (hg)f$,
6. for every $f : X \to Y$, we have $id_Y f = f$ and $f id_X = f$.

The class of objects of **C** is usually denoted by $Ob(\mathbf{C})$, while the class of morphisms is denoted by $Mor(\mathbf{C})$. For simplicity, we will write $X \in \mathbf{C}$ to say that X is an object of **C** and $f : X \to Y$ in **C** to say that $f : X \to Y$ is a morphism in **C**. For any pair of objects X, Y in **C**, we write

$$\mathbf{C}(X,Y) = \hom_{\mathbf{C}}(X,Y) := \{f \mid f : X \to Y \text{ is a morphism in } \mathbf{C}\}.$$

Definition A.0.2. [Mac98] A category **C** is said *small* if $Ob(\mathbf{C})$ is a set. We say that the category **C** is a *locally small* or *with small hom-set* if for each pair of objects X and Y, $\hom_{\mathbf{C}}(X,Y)$ is a set. In addition, a category **C** is said *discrete* if every morphism is an identity.

Example A.0.3. Here are some examples of the usual categories:
- **Set**: Whose objects are all small sets and morphisms are all functions between them.
- **Grp**: Whose objects are all small groups and morphisms are all group homomorphisms.
- **Rng**: Whose objects are all small rings and morphisms are all ring homomorphisms.
- R**Mod**: Whose objects are all small left R-modules over a unital ring R and R-module homomorphisms.
- **Mod**R: Whose objects are all small right R-modules over a unital ring R and R-module homomorphisms.
- $\mathbf{Vec}_{\mathbb{K}}$: Whose objects are all $\mathbb{K}$-vector spaces and morphisms are all linear maps between them.

- **Top**: Whose objects are all topological spaces and morphisms are all continuous maps between them.

-**P** = $(P, \leq)$: Category defined by the partialy ordered set $(P, \leq)$. An object is an element of P and an unique morphism $x \to y$ if and only if the relation $x \leq y$ exists.

Definition A.0.4. [AHS90] Let **C** be a category. We say that **S** is a *subcategory* of **B** if the following conditions are satisfied:

1. $Ob(\mathbf{S}) \subseteq Ob(\mathbf{C})$,
2. for each objects $X, Y \in Ob(\mathbf{S})$, $\hom_{\mathbf{S}}(X, Y) \subseteq \hom_{\mathbf{C}}(X, Y)$,
3. for each object $X \in \mathbf{S}$, the identity $id_X : X \to X$ in **C** is the identity $id_X : X \to X$ in **S**,
4. the composition law in **S** is the restriction of the composition law in **C** to the morphisms of **S**.

Definition A.0.5. [Mac98] Let **C** and **D** be categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ is a morphism from the category **C** to **D**. It consists of two suitably related functions: The *object function* F, which assigns to each object $X \in \mathbf{C}$ an object $FX \in \mathbf{D}$ and the *morphism function* F which assigns to each $f : X \to Y$ in **C** a morphism $F(f) : FX \to FY$ in **D**, in such a way that

$$F(id_X) = id_{FX}, \text{ and } F(gf) = F(g)F(f),$$

whenever the composite gf is defined in **C**. It is also called a *covariant functor.*

We also recall that *a forgetful (or underlying) functor* is a functor defined by forgetting or dropping some structure. For instance $\mathcal{U} : \mathbf{Top} \to \mathbf{Set}$ is a functor which sends every topological space to its underlying set and every continuous function to its underlying set maps.

Proposition A.0.6. *Let* **C** *and* **D** *be categories. Let* $\mathcal{U} : \mathbf{C} \to \mathbf{Set}$ *and* $\mathcal{U}' : \mathbf{D} \to \mathbf{Set}$ *be forgetful functors. If the following conditions are satisfied:*

- for all $X \in \mathbf{C}$*, there exists* $FX \in \mathbf{D}$ *such that* $\mathcal{U}(X) = \mathcal{U}'(FX)$*,*

- for all $f : X \to Y$ *in* **C***, there exists a morphism* $F(f) : FX \to FY$ *in* **D** *such that* $\mathcal{U}(f) = \mathcal{U}'(F(f))$*.*

Then, there exists a functor $F : \mathbf{C} \to \mathbf{D}$ *mapping each object* X *of* **C** *to the object* FX *of* **D***, and every morphism* $f : X \to Y$ *in* **C** *to the morphism* $F(f) : FX \to FY$ *in* **D***.*

Remark A.0.7. a) For each category **C**, we associate the *opposite category* $\mathbf{C}^{op}$. The objects of $\mathbf{C}^{op}$ are the objects of C and the morphisms are morphisms of **C** in reversed direction, i.e. if $f : X \to Y$ is a morphism in **C**, then $f^{op} : Y \to X$ is a morphism in $\mathbf{C}^{op}$. The composite $f^{op}g^{op} = (gf)^{op}$ is defined in $\mathbf{C}^{op}$ whenever the composite gf is defined in **C**.

b) A *contravariant functor* $F : \mathbf{C} \to \mathbf{D}$ is a functor that assigns to each object $X \in \mathbf{C}$ an object $FX \in \mathbf{D}$ and to each morphism $f : X \to Y$ a morphism $F(f) : FY \to FX$ (in the opposite direction), all in such a way that

$$F(id_X) = id_{FX}, \text{ and } F(fg) = F(g)F(f),$$

whenever the composite fg is defined in **C**.

Example A.0.8. 1. For each $X \in \mathbf{C}$, we have the *covariant hom-functor*

$$\mathbf{C}(X, -) = \hom(X, -) : \mathbf{C} \to \mathbf{Set}$$

sending each object Y to the set $\hom(X, Y)$, and each morphism $k : A \to B$ to the function $\hom(X, k) : \hom(X, A) \longrightarrow \hom(X, B)$ defined by $\hom(X, k)(f) = kf$ for all $f \in \hom(X, A)$.

2. For each $Y \in \mathbf{C}$, we have the *contravariant hom-functor*

$$\mathbf{C}(-,Y) = \hom(-,Y) : \mathbf{C}^{op} \to \mathbf{Set}$$

sending each object $X \in \mathbf{C}$ to the set $\hom(X,Y)$, and each morphism $h : A \to B$ to the function $\hom(h,Y) : \hom(B,Y) \longrightarrow \hom(A,Y)$ defined by fh for all $f \in \hom(B,Y)$.

Definition A.0.9. [Mac98] Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We say that $\mathbf{C}$ is isomorphic to $\mathbf{D}$ written $\mathbf{C} \cong \mathbf{D}$ if and only if there exists a functor $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{D}$ such that $GF = id_{\mathbf{C}}$ and $FG = id_{\mathbf{D}}$.

Definition A.0.10. [Mac98] Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Given a pair of objects $X, Y \in \mathbf{C}$, we consider the function

$$F_{X,Y} : \hom_{\mathbf{C}}(X,Y) \to \hom_{\mathbf{D}}(FX,FY).$$
$$f \mapsto Ff$$

We say that:

1. F is *full* if $F_{X,Y}$ is surjective for every pair of objects $X, Y \in \mathbf{C}$.

2. F is *faithful* if $F_{X,Y}$ is injective for every pair of objects $X, Y \in \mathbf{C}$.

3. F is *fully faithful* if it is full and faithful.

Definition A.0.11. [Mac98] We say that $\mathbf{S}$ is a *full subcategory* of $\mathbf{C}$ when the inclusion functor $\mathbf{S} \hookrightarrow \mathbf{C}$ is full.

Definition A.0.12. [Mac98] Given two functors $F, G : \mathbf{C} \to \mathbf{D}$. A *natural transformation* $\tau : F \Longrightarrow G$ is a function which assigns to each object X of $\mathbf{C}$ a morphism $\tau_X : FX \to GX$ of $\mathbf{D}$ in such a way that for every morphism $f : X \to Y$ in $\mathbf{C}$ we have $\tau_Y F(f) = G(f)\tau_X$. When this last identity holds, we say that τ_X is *natural* in $\mathbf{C}$. We denote by Nat(F,G) the set of natural transformations from F to G.

We say that τ is a *natural equivalence* or *natural isomorphism* if every component τ_X is invertible for all $X \in \mathbf{C}$. We then write $F \cong G$.

An *equivalence between categories* $\mathbf{C}$ and $\mathbf{D}$ is defined to be a pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ together with natural isomorphisms $id_{\mathbf{C}} \cong GF$ and $id_{\mathbf{D}} \cong FG$.

Let $\mathbf{C}$ and $\mathbf{D}$ be categories. There is a category denoted by $\mathbf{D}^{\mathbf{C}}$ whose objects are a functors from $\mathbf{C}$ to $\mathbf{D}$ and whose morphisms are natural transformations between them.

The following lemma appeared in [Rie16] in the form of exercise.

Lemma A.0.13. *[Rie16] Given a natural transformation $\tau : H \to K$ and functors F and L as*

$$\mathbf{C} \xrightarrow{F} \mathbf{D} \underset{K}{\overset{H}{\rightrightarrows}}{\scriptstyle \Downarrow\tau}\ \mathbf{E} \xrightarrow{L} \mathbf{F}$$

There exists a natural transformation $L\tau F : LHF \to LKF$ defined by $(L\tau F)_X = L\tau_{F(C)}$ for all $X \in \mathbf{C}$. This is called the whiskered composite of τ with L and F.

Using Lemma A.0.13, we obtain the following lemma.

Lemma A.0.14. *Let $\mathbf{P}$ be a category. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. There exists a functor $\bar{F} : \mathbf{C}^{\mathbf{P}} \to \mathbf{D}^{\mathbf{P}}$. The object function maps each $G \in \mathbf{C}^{\mathbf{P}}$ to the object $\bar{F}(G) = FG \in \mathbf{D}^{\mathbf{P}}$ where FG is a composition of F and G. The morphism function assigns to each $\eta : G \to H$ in $\mathbf{C}^{\mathbf{P}}$ to the morphism $F\eta : FG \to FH$ in $\mathbf{D}^{\mathbf{P}}$ where $F\eta$ is defined as in A.0.13.*

Definition A.0.15. [Mac98] Let $\mathbf{C}$ be a locally small category. A *representation* of a functor $F : \mathbf{C} \to \mathbf{Set}$ is a pair (X, ψ), with $X \in \mathbf{C}$ and $\psi : \mathbf{C}(X,-) \cong F$ is a natural isomorphism. If such a representation exists then F is said *representable*.

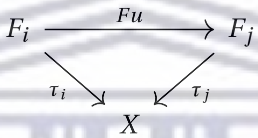
Lemma A.0.16. *[Mac98][Yoneda lemma] Let* **C** *be a category with small hom-sets. If* $F : \mathbf{C} \to \mathbf{Set}$ *is a functor from* **C** *and* C *is an object in* **C***, there is a bijection*

$$y : \mathrm{Nat}(\mathbf{C}(C,-), F) \cong F(C),$$

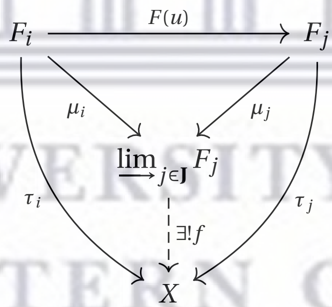
which sends each natural transformation $\alpha : \mathbf{C}(C,-) \Longrightarrow F$ *to* $\alpha_C(id_C)$*, the image of the identity* $id_C : C \to C$.

Let **C** and **J** be categories; (usually **J** is small and often finite). The *diagonal functor* $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ sends each object X to the constant functor $\Delta X : \mathbf{J} \to \mathbf{C}$ which has the value X for all object of **J** and value id_X for all morphism of **J**.

Definition A.0.17. [Mac98] Let $F : \mathbf{J} \to \mathbf{C}$ be a functor. A *cocone* (X, τ) from the base F to the vertex X is a pair of natural transformation $\tau : F \Longrightarrow \Delta X$, often written as $\tau : F \Longrightarrow X$ together with the object $X \in \mathbf{C}$. i.e., for each morphism $u : i \to j$ in **J**, the following diagram is commutative



Definition A.0.18. [Mac98] Let $F : \mathbf{J} \to \mathbf{C}$. A *colimit* (or *direct limit* or *inductive limit*) of F consists of an object $\varinjlim_{j \in \mathbf{J}} F_j \in \mathbf{C}$ and a cocone $\mu : F \Longrightarrow \varinjlim_{j\in\mathbf{J}} F_j$ from the base F to the vertex $\varinjlim_{j\in\mathbf{J}} F_j$ which is universal:
For any cocone $\tau : F \Longrightarrow X$ there exists a unique morphism $f : \varinjlim_{j\in\mathbf{J}} F_j \to X$ in **C** with $\tau_i = f\mu_i$ for every $i \in J$.



Example A.0.19. 1. Let **J** be a discrete category and **C** any category. Consider the diagram functor $F : \mathbf{J} \to \mathbf{C}$ that sends each object $j \in \mathbf{J}$ to the object $X_j \in \mathbf{C}$. A *coproduct* $\coprod_{i\in J} X_i$ of a family of objects of **C** is given by the colimit of the functor F.

2. Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be a pair of parallel morphism in **C**. Consider the category with two elements and two non identity morphism denoted by $\downarrow\downarrow = \{\ \bullet \rightrightarrows \bullet\ \}$. Let $F : \downarrow\downarrow \to \mathbf{C}$ be a diagram functor given by $X \overset{f}{\underset{g}{\rightrightarrows}} Y$. A *coequalizer* of f and g is the colimit of the functor $F : \downarrow\downarrow \to \mathbf{C}$.

The analog definition of *cone* and *limit* (or *inverse limit* or *projective limit*) can be defined dually, as well as *product* and *equalizer*.

Theorem A.0.20. *[Mac98] Any functor* $F : \mathbf{C} \to \mathbf{Set}$ *from small category* **C** *to the category of sets can be represented as a colimit of a diagram of representable functors* $\hom_{\mathbf{C}}(C,-)$ *for some objects* $C \in \mathbf{C}$.

$$F \cong \varinjlim_{C\in\mathbf{C}} \hom_{\mathbf{C}}(C,-)$$

Lemma A.0.21. *[Mac98] Let $F : \mathbf{J} \to \mathbf{C}$ be a diagram functor. Then, there exists a natural isomorphism*

$$\mathbf{C}(\varinjlim_{j \in \mathbf{J}} F, C) \cong \varprojlim_{j \in \mathbf{J}} \mathbf{C}(F-, C).$$

Definition A.0.22. A category $\mathbf{C}$ is *complete* if for each small diagram in $\mathbf{C}$ there exists a limit.

Theorem A.0.23. *[AHS90] For each category $\mathbf{C}$ the following conditions are equivalent:*

(i) $\mathbf{C}$ *is complete,*

(ii) $\mathbf{C}$ *has products and equalizers,*

(iii) $\mathbf{C}$ *has products and finite intersections.*

The dual of the above definition and theorem are also valid for *cocomplete*.

Theorem A.0.24. *[AHS90] A small category is complete if and only if it is cocomplete.*

Definition A.0.25. [Mac98] Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be functors. We say that F is *left adjoint* for G (or G is *right adjoint* for F) if for every pair of objects $X \in \mathbf{C}$ and $A \in \mathbf{D}$ there an isomorphism

$$\phi = \phi_{X,A} : \mathbf{D}(FX, A) \cong \mathbf{C}(X, GA)$$

which is natural in X and A. We write $F \dashv G$ or $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$.

One can see another characterization of adjunction via unit and counit, see for instance [Mac98].

Proposition A.0.26. *[Mac98] [Galois connection] Let $F : (P, \leq) \to (Q, \leq)$ and $G : (Q, \leq) \to (P, \leq)$ be order preserving maps between preorders $(P, \leq)$ and $(Q, \leq)$. Then F is left adjoint to G, $(F \dashv G)$ if and only if for all $p \in P$ and $q \in Q$,*

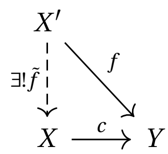
$$p \leq G(q) \iff F(p) \leq q.$$

Furthermore, $p \leq GF(p)$ and $FG(q) \leq q$ holds for all $p \in P$ and $q \in Q$.

Definition A.0.27. [AHS90] Let $\mathbf{C}$ be a subcategory of $\mathbf{D}$ and let Y be an object of $\mathbf{D}$.

(1) A *coreflection* ($\mathbf{C}$-coreflection) for Y is a morphism $c : X \to Y$ from an object $X \in \mathbf{C}$ to Y with the following universal property:

for any morphism $f : X' \to Y$ in $\mathbf{D}$ from some object $X' \in \mathbf{C}$ to Y, there exists a unique morphism $\tilde{f} : X' \to X$ in $\mathbf{C}$ such that the diagram



commutes.

(2) The subcategory $\mathbf{C}$ is called a *coreflective subcategory* of $\mathbf{D}$ provided that each object of $\mathbf{D}$ has a coreflection.

Proposition A.0.28. *[AHS90] If $\mathbf{C}$ is a coreflective subcategory of $\mathbf{D}$ and for each object $Y \in \mathbf{D}$, $c_Y : X \to Y$ is a coreflection for Y, then there exists a unique functor $F : \mathbf{D} \to \mathbf{C}$ (often called a coreflector for $\mathbf{C}$) such that* *(i)* $F(Y) = X$ *for each object* $Y \in \mathbf{D}$,

(ii) *for each morphism* $f : Y \to Y'$ *in* $\mathbf{D}$, *the diagram*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_Y} & Y \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle f} \\ F(Y') & \xrightarrow[c_{Y'}]{} & Y' \end{array}$$

is commutative.

Theorem A.0.29. *[AHS90] Let* $\mathbf{C}$ *be a coreflective subcategory of* $\mathbf{D}$. *Then the functor* $F : \mathbf{D} \to \mathbf{C}$ *is right adjoint to the inclusion functor* $\mathbf{C} \hookrightarrow \mathbf{D}$ *if and only if* F *is a coreflector for* $\mathbf{C}$.

Analogously, the definition and theorem in case of a *reflector* can be defined dually.

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