GROUPOIDS OF HOMOGENEOUS FACTORISATIONS OF GRAPHS

By

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Abstract

This thesis is a study on the confluence of algebraic structures and graph theory. Its aim is to consider groupoids from factorisations of complete graphs. We are especially interested in the cases where the factors are isomorphic. We analyse the loops obtained from homogeneous factorisations and ask if homogeneity is reflected in the kind of loops that are obtained. In particular, we are interested to see if we obtain either groups or quasi-associative Cayley sets from these loops.

November 2008.

Declaration

I declare that Groupoids of Homogeneous Factorisations of Graphs is my own work, that it has not been submitted before for any degree or examination at any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Okitowamba Onyumbe November 2008

Signed:..

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Chapter 1

Introduction

1.1 Introduction and background

There are many generalisations of Cayley graphs, prototypes of transitive graphs. The most general is that of groupoid graphs introduced by Mwambene [19]. However, before the general case was introduced, Gauyacq had looked at quasi-Cayley graphs. Dörfler [12] defined quasi-group graphs by a different approach that is not inherited from Cayley graph constructions.

Recently, Mwambene [21] constructed graphs on algebraic structures in which the set of vertices are the elements of the algebraic structure and the adjacency is defined by a multiplication of a well chosen subset of the vertices (Cayley sets) with the whole structure. Based on them, he has shown that every graph can be represented as a groupoid graph and has distinguished various classes of graphs for certain algebraic properties of underlying groupoids. Furthermore, he has proved that every regular graph is a loop graph, and identified groupoids that represent vertex-transitive graphs [21].

For us, our interest is in the following. We consider homogeneous factorisations, that is, factorisations that admit isomorphic factors. We are interested in the following question: what kind of algebraic structures can we obtain from a given class of homogeneous factorisations? For a given class of factorisations, the question is: what kind of loops do we obtain? In other words, are these loop graphs the classical Cayley graphs, quasi-Cayley graphs or merely loop graphs? To answer this question, we use the classical result of group theory in which loops are groups whenever they are associative. If this fails, we ask if the Cayley sets that describe adjacency are weakly associative. Gauyacq has shown that if it is the case, then the graphs obtained are quasi-Cayley, a class of graphs which are intimately connected to Cayley graphs.

In what shall follow, we use the same method for the various factorisations. We consider Harary, Hamiltonian, 1-rotational and regular factorisations and their variants. The chapters of our study will mainly be according to the various types of factorisations that are at issue.

1.2 Overview of the thesis

In Chapter 2 we introduce basic concepts of algebraic structures and graph theory that we will use in our discussion. Chapter 3 briefly describes the characterisation of regular graphs as loop graphs. Indeed, we will describe how regular graphs represent loop graphs and vice-versa. In Chapter 4, we consider Harary's isomorphic factorisations; highlighting necessary and sufficient conditions for the existence of such factorisations. We investigate by some examples the kind of groupoids which are generated. In Chapter 5, we discuss groupoids from Hamiltonian factorisations. We focus on two kinds of Hamiltonian factorisations; namely Walecki's factorisations and 1-rotational Hamiltonian cycle systems. In Chapter 6, we present the general 1-rotational factorisations and their corresponding groupoids. In Chapter 7, we describe regular factorisations and discuss by some examples the groupoids that are generated from them. We will also comment on a non-isomorphic case.

Chapter 2

Basic concepts

In this chapter, we present some basic algebraic structures and some concepts in graph theory that are relevant to this study. We closely follow [23] for the former and [3] for the latter. In addition, we introduce some general classical fundamental results that are useful to our discussion. As far as possible, we adhere to the commonly used notation and terminology.

2.1 Elementary concepts of algebraic structures

One of the key concepts in this study is groupoids. Although we do not delve into groupoid theory in this study, we need to recall briefly some of the basic notions of algebraic structures which will be used throughout.

2.1 Definition Let A be a set. A function $* : A \times A \longrightarrow A$ is called a *binary operation* on A. A *groupoid* $(A, *)$ is a set A together with the binary operation $*$. When there is no possibility of ambiguity, it is convenient to write a groupoid as A in place of $(A, *)$. We may also dispense with the notation $x * y$ and write xy.

An element $u \in A$ is a *unit* of the groupoid A if

$$xu = x, \; ux = x \text{ for all } x \in A.$$

2.2 Definition A *quasi-group* $(A, *)$ is a groupoid such that (i) for every ordered pair $(a, b) \in A \times A$, there exists a unique element $x \in A$ such that $a * x = b$; (ii) for every ordered pair $(a, b) \in A \times A$ there exists a unique element $y \in A$ such that $y * a = b$.

If only (i) holds, then A is called a *left quasi-group*.

A *loop* is a quasi-group that has a unit element. There is a possibility of terminological ambiguity since both a groupoid and an edge $[x, x]$ are referred to as loops. To avoid confusion, we will emphasize the distinction by calling an edge a *loop-edge*. In any case, our graphs do not have loop-edges.

A left quasi-group with a unit is called a *left loop*.

A *group* is a loop in which the binary operation is associative. Aczel provides the necessary and sufficient conditions for a loop to be associative [1]. The conditions are as follows. In the multiplication table of the loop, choose any four places forming the vertices of a rectangle. Suppose that the entries in these places are

$$\begin{array}{cc} q & r \\ p & s. \end{array}$$

If the loop is a group, then all other rectangles having p, q, r as entries at successive points, with p and q sharing a column, will have s as the entry at the fourth point. The converse is also true.

Let us elaborate these kind of rectangles. It is clear from a Cayley table of a loop (Table 2.1) that we have the following equalities.

$\times$	y_1	y_2	y_3	y_4
x_1	.	q	.	r
x_2	q	.	r	.
x_3	.	p	.	s
x_4	p	.	s	

Table 2.1: Associativity of loops

$$x_1y_2 = x_2y_1 (= q), x_1y_4 = x_2y_3 (= r), x_3y_2 = x_4y_1 (= p), x_3y_4 = x_4y_3 (= s)$$

The rule states that if $x_1y_2 = x_2y_1 (= q)$, $x_1y_4 = x_2y_3 (= r)$ and $x_3y_2 = x_4y_1 (= p)$, then $x_3y_4 = x_4y_3 (= s)$.

2.2 Elementary concepts of graph theory

For our purposes, the definition of a graph is made to reflect the pertinent characterisation of our discussion. To facilitate the discussion of various classes of groupoid graphs, our graphs are defined as follows.

2.3 Definition Let X be a finite set and R a relation on X. The relation R on X defines a *digraph* $D = (X, R)$ if R is irreflexive, i.e, for all $x \in X$, $(x, x) \notin R$. X defines the *vertex-set* of the digraph $D = (X, R)$ and $x \in X$ is called a *vertex* of the digraph. An element of R is called an *arc* and R is the *arc-set.*

Now, a graph is a digraph in which the relation R is symmetric. In this sense we have the following definition.

2.4 Definition Let $\Gamma = (V, E)$ be a digraph. We say that Γ is a *graph* if the relation E is symmetric. In this case, if for any $x, y \in V$, (x, y) is an arc, (y, x) is also an arc. The two arcs together are identified into an *edge* $[x, y]$. Vertices x and y are *adjacent* to each other in Γ if there is an edge of Γ with x and y as its ends. The *out-degree* and *in-degree* of $x \in V$, respectively denoted $d^+(x)$ and $d^-(x)$, are defined by the order of the sets $\{y \in V : (x, y) \in E\}$ and $\{y \in V : (y, x) \in E\}$.

We adopt the usual notation $V(\Gamma)$ for the set of vertices of Γ, $E(\Gamma)$ for the edge set and $\vec{E}(\Gamma)$ the arc set when Γ is a digraph. A graph is called *finite* if both $V(\Gamma)$ and $E(\Gamma)$ are finite. All graphs considered in this thesis are finite. A graph Γ is said to be *complete* if every pair of distinct vertices of Γ are adjacent in Γ. A complete graph on $V = \{1, 2, \cdots, n\}$ is denoted by K_n.

As a preliminary for other concepts, let us now define the neighbours of a vertex $x \in V$ and the degree of a vertex in a graph.

2.5 Definition (a) Let x be a vertex of a graph Γ. The *neighbours* of x are the vertices $y \in V$ that are adjacent to x, i.e. $[x, y] \in E(\Gamma)$. The set of all neighbours of x is denoted by $N(x)$ and the number of neighbours of x is called the degree of x denoted by $d_\Gamma(x)$. Obviously, $d_\Gamma(x) = |N(x)|$. If every vertex of Γ has degree k, Γ is called *k-regular.*
(b) For an edge $[x, y]$, x, y are *incident* with it.
(c) A graph Γ' is a *subgraph* of Γ if $V(\Gamma') \subset V(\Gamma)$ and $E(\Gamma') \subset E(\Gamma)$. If $V(\Gamma') = V(\Gamma)$, Γ' is a *spanning* subgraph of Γ.

A sequence of vertices $x_0, x_1, ..., x_{k-1}$ form a *cycle* $C = (x_0, x_1, ..., x_{k-1})$ if $[x_i, x_{i+1}] \in E(\Gamma)$ for $i = 0, 1, ..., k-1$, where the subscripts are taken modulo k.

The core of our work hinges on the factorisations of graphs. In what follows, we therefore present what is meant by a decomposition or a k-factorisation of a graph.

2.6 Definition Let Γ be a graph. Then a family of subgraphs $\mathcal{H} = \{H_i, i \in I\}$ is a *decomposition* of Γ if the following conditions are satisfied:
(i)$E(\Gamma) = \cup_{i \in I} E(H_i)$;
(ii)$E(H_i) \cap E(H_j) = \emptyset$ for any $i \neq j$.

$\mathcal{H}$ is therefore a partition of the edges of Γ. If each H_i is a cycle, then we refer to the decomposition as a *cycle decomposition*.

2.7 Remark Consider a decomposition of Γ, i.e. a partition of the edge set $E(\Gamma)$ of Γ into subgraphs $H_1, H_2, \cdots, H_t$. We employ the notation

$$\Gamma = H_1 \oplus H_2 \oplus \cdots \oplus H_t \tag{2.1}$$

to denote such a decomposition.

2.8 Definition (a) Let Γ be a graph and k a non-negative integer. Then H is a *k-factor* of Γ if H is k-regular and is a spanning subgraph of Γ.
(b) A graph Γ is *k-factorisable* if there exists a set of factors $\mathcal{F} = \{F_1, F_2, \cdots, F_t\}$ such that F_i is k-factor for each $i = 1, 2, \cdots, t$ and $\mathcal{F}$ is a decomposition of Γ. In this case, $\mathcal{F}$ is called a *k-factorisation* of Γ.
(c) A *k-cycle system* of a graph $\Gamma = (V, E)$ is a set $\mathcal{B}$ of cycles whose vertices belong to V with the additional condition that the number of vertices in each cycle is k and any $[x, y] \in E$ is an edge of exactly one cycle of $\mathcal{B}$. A k-cycle system is *Hamiltonian* if $k = |V|$.

2.2.1 Homomorphisms of graphs

2.9 Definition Let Γ and Γ' be two graphs. A map $f : V(\Gamma) \longrightarrow V(\Gamma')$ is a *homomorphism* from Γ to Γ' if it preserves edges. That is, if $e = [x, y]$ is an edge of Γ, then $[f(x), f(y)] \in E(\Gamma')$. When f is a homomorphism, we write $f : \Gamma \longrightarrow \Gamma'$ for short.

If a homomorphism is one to one, onto and its inverse preserves edges, it is called an *isomorphism.*

Recaptured in another sense, we have the following.

2.10 Definition Let Γ and Γ' be two graphs and let f be a map $f : V(\Gamma) \longrightarrow V(\Gamma')$. We say that f is an *isomorphim* if the following conditions are satisfied:
(i) f is a bijection;
(ii) f preserves edges, that is, if $[x, y] \in E(\Gamma)$, then $[f(x), f(y)] \in E(\Gamma')$;
(iii) f^{-1} is a homomorphism.

If there is an isomorphism f from a graph Γ to a graph Γ', we say Γ and Γ' are isomorphic and write $\Gamma \cong \Gamma'$.

The concept of automorphisms has extensively been used to distinguish and explore degrees of symmetry in graphs and is a key element is defining vertex-transitive graphs. An *automorphism* is an isomorphism from a graph Γ to itself. The set of automorphisms of Γ forms a group under composition, and is denoted by Aut Γ.

An automorphism can also be thought of as a permutation of vertices of a graph that can be characterised in the following way.

2.11 Lemma *A permutation σ of $V(\Gamma)$ is an automorphism if and only if $\sigma N(x) = N(\sigma x)$ for any $x \in V(\Gamma)$.*

Proof
Let $\sigma \in$ Aut Γ, $y \in \sigma N(x)$. Then there exists $z \in V(\Gamma)$ such that $[x, z] \in E(\Gamma)$ and $y = \sigma z$. Applying σ to the edge, we have $[\sigma x, \sigma z] = [\sigma x, y] \in E(\Gamma)$. Hence $y \in N(\sigma x)$. On the other hand, suppose $y \in N(\sigma x)$. Then there exists an edge $[\sigma x, y]$. Applying σ^{-1} to the edge, we obtain $[x, \sigma^{-1}y] \in E(\Gamma)$. Hence $\sigma^{-1}y \in N(x)$ and therefore $y \in \sigma N(x)$. Conversely, let $[x, y] \in E(\Gamma)$. We have $\sigma y \in N(\sigma x)$. Hence we have $[\sigma x, \sigma y] \in E(\Gamma)$. This means that σ preserves edges. Hence it is an automorphism. ∎

We now introduce vertex-transitivity.

2.2.2 Vertex transitive graphs

Let Γ be a graph. Define a relation on $V(\Gamma)$ by

$$x \sim y \text{ if there exists } \sigma \in \text{Aut } \Gamma \text{ such that } \sigma x = y.$$

Clearly $x \sim x$, for all $x \in V(\Gamma)$. If there is an automorphism σ mapping x to y, the inverse automorphism σ^{-1} will map y to x. Hence $\sim$ is symmetric. Also, if there are automorphisms σ, τ such that $\sigma(x) = y$, $\tau(y) = z$ then $\tau \circ \sigma(x) = z$. $\sim$ is therefore transitive. So, we have an equivalence relation on $V(\Gamma)$. The equivalence classes are called *orbits*. Given $x \in V(\Gamma)$, *x-orbit* is denoted by Ω_x. The set of orbits of a graph Γ will be denoted by $\mathcal{P} := \Gamma/\sim$.

2.12 Definition A graph $\Gamma = (V, E)$ is *vertex-transitive* if $\Omega_x = V(\Gamma)$ for any $x \in V(\Gamma)$.

If A is a subgroup of Aut Γ, the *stabilizer* of x in A, is defined by

$$A_x := \{\sigma \in A : \sigma(x) = x\}\,. \tag{2.2}$$

An important and extensively studied class of vertex-transitive graphs are the so-called *Cayley graphs*. In this case, elements of a group form the vertex-set of the graph and a Cayley set has to be chosen to define the arc-set. We now give the definition of Cayley set.

2.13 Definition Let A be a group. A subset S of A is a *Cayley set* if it satisfies the following properties:
(i) the identity element 1_A is not in S;
(ii) if $s \in S$ then so is s^{-1}.

The Cayley graphs are defined as follows.

2.14 Definition Let A be a group and S a Cayley set of A. The *Cayley graph* $\text{Cay}(A, S)$ has the elements of A as vertices and the arcs are given by

$$(x, y) \text{ is an arc, if there is } s \in S \text{ such that } y = xs, \text{ for any } x, y \in A.$$

2.15 Remark We note that the two properties defining a Cayley set are necessary to satisfy irreflexivity and symmetry of the graph $\text{Cay}(A, S)$. As 1_A is not in the Cayley set S, we see that there are no loop-edges in a Cayley graph. In addition, as the inverse of an element $s \in S$ is in S, we have that if (x, y) is an arc of $\text{Cay}(A, S)$, then so is (y, x). In fact we have that $y = xs$ is equivalent to $x = ys^{-1}$. Therefore we have an edge-set and $\text{Cay}(A, S)$ is a graph as defined above.

2.16 Lemma *Let A be a group and S a Cayley set of A and let* $\Gamma = \text{Cay}(A, S)$ *be the corresponding Cayley graph. Then*
(i) *The graph Γ is k-regular, where $k = |S|$;*
(ii) *The graph Γ is vertex-transitive.*

Proof
(i) By definition for any $x \in A, N(x) = \{xs, s \in S\} = xS$. Hence $|N(x)| = k,$ for any $x \in V(\Gamma)$.
(ii) It is easy to see that for a fixed $a \in A$, the map $\lambda_a : A \longrightarrow A$ given by $\lambda_a(x) = ax$, for all $x \in A$, is an automorphism of $\text{Cay}(A, S)$. Define $\Lambda_a :=$ $\{\lambda_a : a \in A\}$. It is easy to see that Λ_a acts transitively on $V(\text{Cay}(A, S))$. ∎

In fact, Cayley graphs exhibit a much stronger symmetry in terms of automorphisms; namely, regularity which is defined below.

2.17 Definition Let A be a group acting on a set X. The action is said to be *regular* if the following conditions are satisfied.
(i) A is transitive on X;
(ii) given $x, y \in X$, there exists a unique element $\sigma \in A$ such that $\sigma(x) = y$.

Regularity provides a characterisation of Cayley graphs proved by Sabidussi [24].

2.18 Theorem *A graph Γ is a Cayley graph if and only if there is a subgroup B of* Aut Γ *acting regularly on $V(\Gamma)$.*

Our interest is in isomorphic factorisations. They are defined in the following way.

2.19 Definition (a) A factorisation $\mathcal{F}$ is said to be *isomorphic* if for any factors $F_i, F_j \in \mathcal{F} : F_i \cong F_j$.

The definition of isomorphic factorisations is intimately related to homogeneous factorisations of Li and Praeger [18]. Their definition reads:

(b) A *homogeneous factorisation* of a complete graph K_n is a partition of the edge set that is invariant under a subgroup G of S_n such that G is transitive on the parts of the partition and induces a vertex-transitive automorphism group on the graph corresponding to each part. It is from these factorisations, we will define a suitable multiplication for our construction.
(c) A graph is *k-homogeneous* if there exists a k-factorisation $\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$ such that F_i is isomorphic to F_j for any $i, j \in \{1, 2, 3, \cdots k\}$.

In the next chapter, we are going now to characterize in general graphs and regular graphs as respectively groupoid graphs and loop graphs.

Chapter 3

Characterisation of regular graphs as loop graphs

3.1 Introduction

In this chapter, we present loop graphs defined on loops and described by factorisations on complete graphs. This is based on the work of Mwambene [20]. The starting point in these constructions are groupoid graphs, a generalisation of Cayley graphs in groups.

Our focus, however, is loop graphs. The definition of a Cayley set in a group is given using the unit element of the group and inverses. It is not the same in a general groupoid in which we must use only products. Thus, the main thrust of the generalisation lies in defining a Cayley set on a general groupoid that enables us to define a relevant relation on a given algebraic structure.

It is required that the generalised Cayley set describes a relation that is both irreflexive and symmetric. The generalised Cayley set is defined as follows.

3.1 Definition Let $(A, *)$ be a groupoid. A subset S of A is called a *Cayley set* if
(i) for any $a \in A, a \notin aS$;
(ii) for any $a \in A$ and $s \in S, a \in (as)S$.

We note that Cayley sets can be combined in the following.

3.2 Proposition *Let $(A, *)$ be a groupoid and I an arbitrary indexing set. If $\{S_i\}_{i \in I}$ is a family of Cayley set, then $\cup_{i \in I} S_i$ is again a Cayley set.*

Proof
(i) Assume $a \in A$. As S_i is a Cayley set for all $i \in I$, we have $a \notin aS_i$; Hence $a \notin \cup_{i \in I}(aS_i) = a(\cup_{i \in I} S_i)$. Therefore, $a \notin a(\cup_{i \in I} S_i)$.
(ii) Let $a \in A$ and $s \in \cup_{i \in I} S_i$. Therefore $s \in S_i$, for some $i \in I$. Since S_i is a Cayley set, there exists $s^{'} \in S_i$ such that $(as)s^{'} = a$. Therefore $a \in (as)(\cup_{i \in I} S_i)$. ∎

As for intersections, the situation is different.

3.3 Remark In general, the intersection of Cayley sets is not necessarily a Cayley set.

For instance, let Q be a group. Consider the groupoid $A = (Q \times \{0, 1\})$ with the binary operation defined by

$$(a, i) * (b, j) = (ab, i)$$

with ab the product in Q. Now suppose that Q contains an element x of order ≥ 3, and let S be a Cayley set in Q such that $x \in S$. Then,

$$S_0 = (S \times \{0\}) \cup \{(x, 1)\}, S_1 = S \times \{1\}$$

are Cayley sets in A (condition (ii) is satisfied for S_0 because $(x^{-1}, 0) \in S_0$ and $(a, i)(x, 1)(x^{-1}, 0) = (a, i)$ for any $(a, i) \in A$). It is clear that $S_0 \cap S_1 = \{(x, 1)\}$ is not Cayley because $x^{-1} \neq x$.

In what follows, we only consider loops. In this instance, the intersection of Caylet sets is again Cayley, as is proved below.

3.4 Lemma *If $S_1,\ S_2$ are Cayley sets in a loop A, then so is $S_1 \cap S_2$.*

Proof
Let S_1 and S_2 be two Cayley sets in the loop A. Then
(i) for any $a \in A$, we have $a \notin aS_1$ and $a \notin aS_2$ because S_1 and S_2 are Cayley. Therefore $a \notin aS_1 \cap aS_2$, which may be written as $a \notin a(S_1 \cap S_2)$ because A is a loop.
(ii) Let $a \in A$ and $s \in S_1 \cap S_2$. Then $a \in (as)S_1$ because S_1 is Cayley. Similarly, $a \in (as)S_2$. Therefore $a \in (as)S_1 \cap (as)S_2 = (as)(S_1 \cap S_2)$. Hence $S_1 \cap S_2$ is a Cayley set. ∎

A Cayley set S is *proper* if S $\neq \emptyset$, and $S \neq A \setminus \{u\}$ where u is a unit. Cayley sets are used to define adjacency between the elements of groupoids.

3.5 Definition Let $(A, *)$ be a groupoid and $S \subset A$ be a Cayley set. Define a binary relation $\vec{E}$ on A by

$$(x, y) \in \vec{E} :\Leftrightarrow y = xs \text{ for some } s \in S,$$

an adjacency relation defined by S on A.

3.6 Lemma [19, p. 3] *Let A be a groupoid, and $S \subset A$. The adjacency relation $\vec{E}$ defined above is irreflexive and symmetric if and only if S is a Cayley set.*

Proof
1. Irreflexivity: If $(x, x) \in \vec{E}$ then $x = xs$ for some $s \in S$, i.e $x \in xS$, a contradiction to condition (i) of Definition 3.1.
Conversely, suppose there exists $s \in S$ such that $x = xs$. Then $(x, x) \in \vec{E}$ contradicting the irreflexivity of $\vec{E}$.
2. Symmetry: $(x, y) \in \vec{E}$ implies that $y = xs_1, s_1 \in S$. If the condition (ii) of Definition 3.1 holds, then we have $x = (xs_1)s_2 = ys_2, s_2 \in S$ so that $(y, x) \in \vec{E}$. Thus $\vec{E}$ is symmetric.
Conversely, suppose $\vec{E}$ is symmetric. Let $x \in A$ and $s \in S$. Then $(x, xs) \in \vec{E}$. By symmetry, $(xs, x) \in \vec{E}$. This implies that $x = (xs)s'$, where $s' \in S$, so that $x \in (xs)S$. The Conditon (ii) of Definition 3.1 holds. ∎

Having defined the adjacency relation S described by Cayley sets in general groupoids, we now define groupoid graphs.

3.7 Definition Let A be a groupoid and S be a Cayley set. The *groupoid graph* denoted by $\Gamma =$ GG(A, S) is defined by
(i) $V(\Gamma) := A$;
(ii)$E(\Gamma) := \{[x, xs] : x \in A, s \in S\}$.

A is called the *underlying groupoid*. When the underlying groupoid A is a left quasi-group (respectively loop), the groupoid graph obtained GG(A, S) is a *quasi-group graph* (respectively *loop graph*).

3.2 Characterisation of graphs as groupoid graphs

Groupoids are algebraic structures of very great generality. We state the result of Mwambene [19] which says that every graph Γ is a groupoid graph. We give the alluded to characterisation theorem of groupoid graphs without proof. The representation reads:

3.8 Theorem (Representation theorem of graphs)[19, p. 4] *Let Γ be a graph with vertex-set A, and $S \neq A$ a subset of A such that Γ has a proper edge-colouring with $|S|$ colours. Then A can be endowed with the structure of a groupoid such that S is a Cayley set, and* $\mathrm{GG}(A, S) = \Gamma$.

To illustrate the representation we present the Petersen graph, denoted P_{10}, as a groupoid graph.

The Petersen graph as a groupoid graph

Consider the vertex set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ of the Petersen graph with its corresponding adjacency as below. It is a classical result that the Petersen graph is 4-edge colourable [22]. We colour the edges with colours $S = \{2, 6, 5, 3\}$.

Denote by S_a the set of colours of the edges incident with the vertex a.

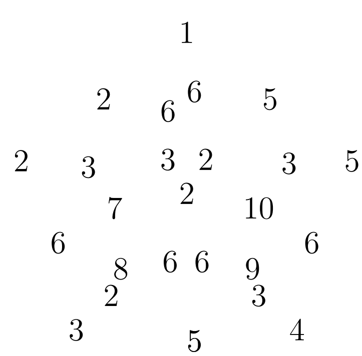

Figure 3.1: The Petersen Graph

We define multiplication on A as follows.

Step 1: If $[x, y] \in E(\Gamma)$ and has colour s then

$$x * s = y.$$

For instance, $e = [2, 7] \in E(P_{10})$ and has colour 3. Therefore $2*3 = 7,\ 7*3 = 2$.

Step 2: Let s be a colour that has not been used in edges incident with the vertex x. Then

$$x * s = y,$$

where $y \in N(x)$.

Step 3: Let $y \notin S$. Then $x * y$ is arbitrarily defined. Table 3.1 below is the groupoid defined on the Petersen graph with the corresponding labels given in Figure 3.1.

$*_1$	1	2	3	4	5	6	7	8	9	10
1	1	2	6	4	5	6	7	8	9	10
2	2	1	7	5	3	3	4	10	8	9
3	6	8	8	9	4	2	3	5	7	1
4	7	5	9	1	3	5	4	2	10	6
5	5	10	10	2	1	4	9	3	6	7
6	6	9	8	3	8	1	5	4	7	10
7	1	10	2	4	9	9	1	5	3	8
8	5	3	6	2	6	10	4	1	9	2
9	10	6	4	8	4	7	3	9	1	5
10	9	7	5	10	7	8	2	6	4	1

Table 3.1: A groupoid of the Petersen graph

3.3 Characterisation of regular graphs as loop graphs

We now present how loop graphs are obtained from regular graphs.

3.3.1 G-compatible factorisation

3.9 Definition Let Γ be a regular graph of degree k. Replace each edge $[x, y]$ by the corresponding arcs (x, y), (y, x). Hence Γ becomes a bi-directed regular digraph such that $d_\Gamma^+(x) = d_\Gamma^-(x) = k$ for all $x \in V(\Gamma)$.

Now, it is possible to factorise the bi-directed Γ into k 1-regular directed spanning subgraphs [4]. Let us denote that factorisation by $\mathcal{F}_\Gamma$. Note that the factorisation satisfies the following conditions:
(i) for any $F \in \mathcal{F}_\Gamma$, $V(F) = V(\Gamma)$ because F is spanning subgraph of Γ;
(ii) for any two distinct factors $F, F' \in \mathcal{F}_\Gamma$
$\vec{E}(F) \cap \vec{E}(F') = \emptyset$;
(iii) $\vec{E}(\Gamma) = \cup_{F \in \mathcal{F}_\Gamma} \vec{E}(F)$;
(iv) for any $x \in V(\Gamma)$, $F \in \mathcal{F}_\Gamma$, $d_F^+(x) = d_F^-(x) = 1$.

Consider $\overline{\Gamma}$, the complement of Γ. In a similar way, we obtain a factorisation $\mathcal{F}_{\overline{\Gamma}}$.

Combining $\mathcal{F}_\Gamma$ and $\mathcal{F}_{\overline{\Gamma}}$, we obtain a factorisation $\mathcal{F}$ of the complete graph $K_{V(\Gamma)}$. We call such a factorisation of $K_{V(\Gamma)}$ Γ-*compatible* (it is, of course, also $\overline{\Gamma}$-*compatible*).

Now, let us define the multiplication between a vertex and an index of a factor. It is, of course, a vertex also by our construction. In addition, let us choose an arbitrary vertex $u \in \Gamma$ as the base point.

3.10 Definition Let Γ be a regular graph with vertex-set A, and $\mathcal{F}$ a Γ-compatible factorisation of $K_{V(\Gamma)}$. Choose an arbitrary vertex u of Γ (the base point) and define a multiplication $*_u$ on A as follows. Let $x, y \in A$,
(i) if $y \neq u$, let $F_y \in \mathcal{F}$ be the (unique) factor such that $(u, y) \in F_y$, then

$$x *_u y := z \text{ where } (x, z) \in F_y; \tag{3.1}$$

(ii) if $y = u$ define

$$x *_u u := x \text{ for all } x \in A. \tag{3.2}$$

We call $*_u$ the $\mathcal{F}$-multiplication on A based on u.

In the following theorem, we present the characterisation of regular graphs as loop graphs.

3.3.2 A regular graph identifying a loop graph

3.11 Theorem (Representation Theorem of Regular Graphs)[19, p. 12] *A graph is regular if and only if it is a loop graph.*

To simplify matters, the proof is presented as a proposition and two lemmas below.

3.12 Proposition *Let Γ be regular graph with vertex-set A, $\mathcal{F}$ a Γ-compatible factorisation of the complete graph K_A and $u \in A$. Then $(A, *_u)$ as defined in Definition 3.10 is a loop with unit u.*

Proof
For $x, y, y' \in A$, let us show that $x *_u y = x *_u y^{'}$ implies $y = y^{'}$, i.e., x is a left-cancellable element.
Case 1. If $y = u$, then $x *_u y = x$. Therefore $x *_u y' = x$. By Definition 3.10, $y' = u$. Hence $y = y'$.
Case 2. Suppose that $y \neq u$. Then by the definition of the multiplication, $x *_u y = z$, i.e $(x, z) \in F_y$ and similarly $(x, z) \in F_{y'}$, then $y = y'$. Hence we have left cancellability.
Now suppose $x *_u y = x^{'} *_u y$, $x \neq x^{'}$. Then $(x, x *_u y) \in F_y$ and $(x^{'}, x^{'} *_u y) = (x^{'}, x *_u y) \in F_y$. This contradicts the fact that $d^{-}_{F_y}(x) = 1$. Hence $x = x^{'}$, so we have right cancellability.
By part (2) of the definition of $*_u$, u is a right unit. Given $x \in A$, $x \neq u$, we have $(u, x) \in F_x$, hence $u *_u x = x$, i.e. u is also a left unit. ∎

3.13 Lemma *Let the notation be as in Proposition 3.12. Then the neighbourhood $N_\Gamma(u)$ of u is a Cayley set in the loop $(A, *_u)$.*

Proof
(i) Let $x \in A$, $y \in N_\Gamma(u)$ and consider $x *_u y$. Since $y \neq u$ we have by the definition of $*_u$ that $(x, x *_u y) \in F_y$, whence $x *_u y \neq x$ because Γ (and hence F_y) has no loop-edges. Thus $x \notin x *_u N_\Gamma(u)$.
(ii) Let $x \in A$ and $z = x *_u y$, where $y \in N_\Gamma(u)$. Then $(x, z) \in F_y \subset \vec{E}(\Gamma)$. Hence $(z, x) \in \vec{E}(\Gamma)$ by the symmetry of the adjacency relation. This means that $(z, x) \in F_{y'}$ for some $y^{'} \in N_\Gamma(u)$. Therefore

$$x = z *_u y^{'} = (x *_u y) *_u y^{'} \in (x *_u y) *_u N_\Gamma(u).$$

∎

3.14 Lemma *With the hypotheses and notation as above, consider the loop* $L = (A, *_u)$. *Then the loop graph* $\mathrm{GG}(L, N_\Gamma(u))$ *is* Γ.

Proof
Let $\mathrm{GG}(L, N_\Gamma(u)) = H$. By the definition of a groupoid graph, $V(H) = A = V(\Gamma)$. For the edges we have that, given $x, y \in A$,

$$\begin{aligned} [x,y] \in E(H) &\iff y = x *_u z, \text{ where } z \in N_\Gamma(u) \\ &\iff (x,y) \in F_z, \text{where } F_z \in \mathcal{F} \text{ is the factor containing} \\ &\qquad \text{the arc } (u,z) \\ &\implies (x,y) \in \vec{E}(\Gamma) \\ &\iff [x,y] \in E(\Gamma). \end{aligned}$$

Conversely, any arc (x, y) of Γ belongs to a unique factor $F \in \mathcal{F}$. Let z be the out-neighbor of u in F. Then $y = x *_u z$, and hence $[x, y] \in E(\Gamma)$. Thus $E(H) = E(\Gamma)$. ∎

Proposition 3.12 together with Lemma 3.14 prove the sufficiency of Theorem 3.11.

For necessity we have

3.15 Lemma *Suppose that* A *is a loop and* S *a Cayley set of* A. *Then* $GG(A, S)$ *is regular.*

Proof
Let $x \in A$. Then $N(x) = S$, and $|N(x)| = |S|$ since we have cancellability. ∎

Various constructions of Proposition 3.12 will be discussed in Chapters 5, 6 and 7.

3.3.3 Representation of vertex transitive graphs

As alluded to in Definition 2.12, Cayley graphs are exactly the groupoid graphs whose underlying groupoid is a group. Now, if the underlying groupoid is not a group, we explore groupoids that represent the class of vertex-transitive graphs. A weak form of associativity is required to represent vertex-transitive graphs on groupoids. We now give a definition of such an associativity.

3.16 Definition Let $(A, *)$ be a loop and S a Cayley set in A. S is said to be *quasi-associative* if the following property holds:

$$(ab)s = a(bs^{'}) \text{ for all } a, b \in A \text{ and } s \in S \text{ and some } s^{'} \in S. \tag{3.3}$$

It is easy to see that $\emptyset$ and $A \setminus \{u\}$ are quasi-associative Cayley sets in a loop A where u is a unit. Now, we consider the necessary condition of the theorem of representation of vertex-transitive graphs, as proved by Mwambene [21].

3.17 Theorem (Representation of vertex-transitive graphs) *If a graph Γ is vertex-transitive then there is a left quasi-group A with a right unit, and a quasi-associative Cayley set $S \subset A$ such that $\Gamma \cong GG(A, S)$.*

Proof
Let Γ be a vertex transitive graph and a vertex $u \in V(\Gamma)$, arbitrarily chosen. Let B be a subgroup of Aut Γ which acts transitively on $V(\Gamma)$, and consider the stabilizer of u in B :

$$B_u = \{\alpha \in B : \alpha(u) = u\}.$$

Let T be a transversal of the left cosets of B_u in B (we shall simply say that T is a *left transversal* for B_u). For all $\sigma, \tau \in T$,

$$\sigma = \tau \Longleftrightarrow \sigma(u) = \tau(u). \tag{3.4}$$

Define a binary operation $*$ on T as follows: Given $\sigma, \tau \in T$, let $\sigma * \tau \in T$ be the representative of the coset $\sigma\tau B_u$. Thus,

$$(\sigma * \tau)B_u = \sigma\tau B_u,$$

Hence,

$$(\sigma * \tau)(u) = \sigma\tau(u). \tag{3.5}$$

Denote by ϵ_T the representative of B_u in T. We show now that
(i) $A_T := (T, *)$ *is a left quasi-group with right unit* ϵ_T. Indeed, suppose $\sigma, \tau, \tau^{'} \in T$ satisfy $\sigma * \tau = \sigma * \tau^{'}$. Then $(\sigma\tau)(u) = (\sigma * \tau)(u) = (\sigma * \tau^{'})(u) = \sigma\tau^{'}(u)$, so that $(\sigma\tau)(u) = (\sigma\tau^{'})(u)$. Hence by (3.4), $\tau = \tau^{'}$. Also, for all $\tau \in T$, $(\tau * \epsilon_T)(u) = \tau\epsilon_T(u) = \tau(u)$. Therefore, $\tau * \epsilon_T = \tau$, again by (3.4). Since A_T is left cancellative, the right unit ϵ_T is unique. Again, because of left cancellativity, every $\tau \in T$ has a unique right inverse which we denote

by τ^*. Thus $\tau * \tau^* = \epsilon_T$. Now consider $S := \{\alpha \in B : [u, \alpha(u)] \in E(G)\}$, and put $S_T := S \cap T$.
(ii) *S_T is quasi-associative in A_T.* Let $\sigma, \tau \in T, \alpha \in S_T$. Trivially, $[u, \alpha(u)] \in E(G)$ implies

$$\sigma\tau\alpha(u) \in N((\sigma\tau)(u)). \tag{3.6}$$

By (3.5), $((\sigma * \tau)^{-1}(\sigma\tau))(u) = u$, hence applying $(\sigma * \tau)^{-1}$ to 3.6 we get $((\sigma * \tau)^{-1}(\sigma\tau\alpha))(u) \in N(u)$. Therefore $((\sigma * \tau)^{-1}(\sigma\tau\alpha))(u) = \alpha^{'}(u)$ for some (unique) $\alpha^{'} \in S_T$. Thus
$(\sigma\tau\alpha)(u) = (\sigma * \tau)(\alpha^{'}(u)) = ((\sigma * \tau) * \alpha^{'})(u)$ (because $\sigma * \tau, \alpha' \in T$).
On other hand, using (3.5)twice,
$(\sigma * (\tau * \alpha))(u) = (\sigma(\tau * \alpha))(u) = \sigma((\tau * \alpha)(u)) = \sigma((\tau\alpha)(u)) = (\sigma\tau\alpha)(u) = ((\sigma * \tau) * \alpha^{'})(u)$.
Therefore, $\sigma * (\tau * \alpha) = (\sigma * \tau) * \alpha^{'}$. Hence S_T is quasi-associative.
(iii) *S_T is a Cayley set in A_T.* Indeed, the right unit ϵ_T is not in S_T. Also, if $\tau \in T$ and $\alpha \in S_T$, then $\tau = \tau * \epsilon_T = \tau * (\alpha * \alpha^{'}) = (\tau * \alpha) * \alpha^{''}$ for some $\alpha^{''} \in S_T$. Hence $\tau \in (\tau * \alpha) * S_T$.
(iv) *$GG(T, S_T) \cong \Gamma$.* Consider the map $f : T \longrightarrow V(G)$ defined by $f(\tau) = \tau(u), \tau \in T$, f is an isomorphism $GG(A_T, S_T) \longrightarrow \Gamma$. We have that, f is obviously bijective because T is a left transversal for A_u. Also, f preserves adjacency because the edge $[\tau, \tau * \alpha]$ (where $\alpha \in S_T$) is mapped to $[\tau(u), (\tau * \alpha)(u)] = [\tau(u), (\tau\alpha)(u)] = \tau[u, \alpha(u)] \in E(\Gamma)$. In addition, f^{-1} preserves adjacency: let $[x, y] \in E(\Gamma)$. There is a unique $\tau \in T$ such that $\tau(u) = x$. Let $\alpha \in T$ be such that $\alpha(u) = \tau^{-1}(y) \in N(u)$. We have that $\alpha \in S_T$, and $[x, y] = [\tau(u), (\tau * \alpha)(u)] = f[\tau, (\tau * \alpha)]$. ∎

If the transversal T is so chosen that $\epsilon_T = 1_\Gamma$ (the identity permutation of $V(\Gamma)$), then ϵ_T is a two-sided unit of A_T; so, any vertex-transitive graph can be represented as the groupoid graph of a left loop (a left quasi-group with two-sided unit).

3.18 Definition A vertex-transitive graph $\Gamma = GG(A, S)$ is quasi-Caley if A admits right cancellability as well.

In the next chapters, we discuss the main theme of this sequel. That is, we address the kind of algebraic structures that are obtained for a given method of factorisation.

Chapter 4

Groupoids of Harary's isomorphic factorisations

In this chapter, we present Harary's isomorphic factorisations of complete graphs and its conditions of existence [16]. With some examples we will discuss the kind of groupoids which are generated by these isomorphic factorisations.

4.1 Isomorphic factorisations

4.1 Definition Let $\Gamma||t$ denote the set of graphs which occur as factors in isomorphic factorisations of Γ into exactly t factors. If F is one such a factor, it is written $\Gamma||F$ and it is said that F *divides* Γ. Γ is said to be *divisible* by t, written $t||\Gamma$, if $\Gamma||t$ is not empty.

4.2 Remark If Γ has m edges, then $\Gamma||t$ will be empty unless t divides m. In general, this necessary condition is not sufficient. For example, the subdivision graph T of the star $K_{1,3}$ has 6 edges and yet $T||2$ is empty.

4.2 Necessary and sufficient conditions for the existence of an isomorphic factorisation

The necessary and sufficient conditions for the existence of isomorphic factorisations is the content of the following theorem of divisibility.

4.3 Theorem [16, p. 244] *The complete graph K_n is divisible by t if and only if t divides $\frac{n(n-1)}{2}$.*

For our purpose, we are interested only in the case where n is odd. We need every factor to be a cycle so that consequently we can define a suitable multiplication to form groupoids. That is, we are interested in 2-factorisations (the decomposition in which each vertex of a factor has a degree two). In that case, every factor must have n edges. Therefore the number of the factor must be $t = \frac{n-1}{2}$. Hence, $\frac{n-1}{2}$ must be a integer and $n-1$ must be an even number.

The sufficient condition of Theorem 4.3 was proved by Guidotti [14] as a special case under certain restrictive number theoretical conditions and later by Harary [15] in full generality.

Now our discussion depends on whether t is odd or even. The following lemma is used in the proof of the sufficiency result.

4.4 Lemma *Let K_n be a complete graph of order n and $\phi \in S_n$ such that the length of every cycle of ϕ is a multiple of a positive integer t. Consider the induced action ϕ' of ϕ on $E(K_n)$ defined by $\phi'[x,y] = [\phi x, \phi y]$ for any $x, y \in K_n$. Then the length of every cycle of ϕ' is also a multiple of t.*

Proof
There are two possibilities for each edge in K_n. For an edge $[x,y] \in E(K_n)$ we have:
Case 1. If x is a vertex of $\phi_i = (xx_2 \cdots x_{st})$ and y a vertex of $\phi_j = (yy_2 \cdots y_{s't})$ with $i \neq j$. Then $[x,y]$ is permuted in cycles of ϕ' of length equal to the least common multiple of st and $s't$. Hence t divides the length.
Case 2. Consider x, y vertices of $\phi_i = (x_1 \cdots x_{st})$. If st is odd, then clearly the orbit of $[x,y]$ has size st. If st is even, with x and y at distance $\frac{st}{2}$, then $[x,y]$ is in an orbit of length $\frac{st}{2}$. Since t is odd, t divides $\frac{st}{2}$. ∎

4.5 Theorem [16, p. 244] *Let t and n be positive integers. If t divides $\frac{n(n-1)}{2}$ and $(t, n) = 1$ or $(t, n-1) = 1$, then K_n is divisible by t.*

Proof

Let $V(K_n) = \{1, 2, ..., n\}$.

Case 1. $(t, n-1)$ =1 and t is odd. Since $(t, n-1) = 1$, it follows that $t|n$. Let ϕ be such a permutation as in Lemma 4.4. Consider the corresponding permutation ϕ'. Every cycle length of ϕ' is also a multiple of t. Write $\phi' = \gamma_1, \gamma_2, ..., \gamma_r$. From each cycle γ_i choose an edge $e_i \in \gamma_i$. Set F to be the graph induced by

$$E = \left\{(\phi')^{lt}(e_i) \text{ such that } l \geq 0, 1 \leq i \leq r\right\}. \tag{4.1}$$

In multiplying successively by the induced permutation we get a partition of $E(K_n)$ and we set

$$\mathcal{F} = \left\{E, \phi' E, ..., (\phi')^{t-1} E\right\}. \tag{4.2}$$

The subgraph F induced by E is isomorphic to the subgraphs of K_n induced by each of $\phi' E, ..., (\phi')^{t-1} E$. The isomorphisms between F and these subgraphs are provided by the corresponding powers of the permutation ϕ. Hence, $\mathcal{F}$ constitutes an isomorphic factorisation of the complete graph K_n. It is clear that F is a factor in the factorisation of K_n among t factors.

Case 2. $(t, n-1) = 1$ and t is even. In this case since $(t, n-1) = 1$ and t is even, then we have that $n-1$ is odd. So $t|\frac{n(n-1)}{2}$ implies that $2t$ divides n. Take a permutation ϕ of V for which the length of every cycle is a multiple of $2t$. The induced permutation ϕ' has the property that the length of every cycle is divisible by t. We now apply the same construction as in Case 1 to obtain a graph in $K_n||t$.

Case 3 $(t, n) = 1$. In this case $n \equiv 1(\text{mod } t)$ if t is odd and $n \equiv 1(\text{mod } 2t)$ if t is even. Take a permutation ϕ acting on V with just one fixed point. If t is odd, let all other cycles of ϕ have lengths which are multiples of t; if t is even let these lengths be multiples of $2t$. The induced permutation ϕ' has all cycle lengths divisible by t. The same construction works again to provide a graph in $K_n||t$ and we have all the required isomorphic factorisations. ∎

4.3 Groupoids generated by Harary's factorisations

We now consider groupoids generated by Harary's factorisations.

(a) Harary's factorisation of K_7 and its groupoid
For K_7, n is prime and t is odd.

It is convenient to write the 21 edges of K_7 in the form ij for all $i, j \in \{1, 2, 3, 4, 5, 6, 7\}$. The edges are

12	13	14	15	16	17
	23	24	25	26	27
		34	35	36	37
			45	46	47
				56	57
					67,

presented in lexicographic order.

For $t = 3$, consider the permutation $\phi = (123)(456)$, and ϕ' its corresponding induced permutation acting on edges of K_7. The choice of t is such that each factor must be 2-regular.

Thus, let us choose the edge 12. We have $\phi(12) = 23, \phi(23) = 13$. This gives the first cycle (12 23 13) of ϕ'. We now consider the edge 14; it follows that the second cycle of ϕ' is (14 25 36) . When we choose the edge 15, we get (15 26 34). In the same way, when we choose the edge 16, we get (16 24 35). For the choice of 17, we get (17 27 37). For the choice 45, we obtain (45 56 46). Finally for the last edge 47, we have (47 57 67) as the last cycle of ϕ'.

Therefore we now write $\phi' =$ (12 23 13)(14 25 36)(15 26 34)(16 24 35)(45 56 46)(17 27 37)(47 57 67).

Next, let us choose one edge from each of the seven cycles. We select 13, 25, 34, 56, 17, 67 and 24. We set

$$E = \{13, 25, 34, 56, 17, 67, 24\}\,.$$

The application of ϕ' on E gives

$$\phi' E = \{12, 36, 15, 46, 27, 47, 35\}\,.$$

Finally, if we apply the permutation ϕ'^2 we get

$$\phi'^2 E = \{23, 14, 26, 45, 37, 57, 16\}.$$

For our construction of groupoids, we need to decompose each of these factors in 2 directed factors. So, from E we obtain $F_3 = \{(1,3), (3,4), (4,2), (2,5), (5,6), (6,7), (7,1)\}$ and $F_7 = \{(1,7), (7,6), (6,5), (5,2), (2,4), (4,3), (3,1)\}$.

$\phi' E$ gives $F_2 = \{(1,2), (2,7), (7,4), (4,6), (6,3), (3,5), (5,1)\}$ and $F_5 = \{(1,5), (5,3), (3,6), (6,4), (4,7), (7,2), (2,1)\}$.

$\phi'^2 E$ gives $F_4 = \{(1,4), (4,5), (5,7), (7,3), (3,2), (2,6), (6,1)\}$ and $F_6 = \{(1,6), (6,2), (2,3), (3,7), (7,5), (5,4), (4,1)\}$.

These directed factors are presented in Figures 4.1, 4.2 and 4.3.

Figure 4.1: Factors F_3 and F_7

Now, the multiplication defined in 3.10 gives us the Cayley Table 4.1, representing a loop, which we denote by Q_1.

We now check if the loop defined in the table is a group. So, let us check all the rectangles that correspond to the conditions required in 2.1 [1].

In the fourth column, fifth column, fourth row and fifth row, we have the rectangle

$$\begin{matrix} 4 & 2 \\ 2 & 5 \end{matrix}$$

having 4, 2, 2 as entries at successive vertices, with 4 and 2 sharing the fourth column, we have 5 as the entry at the fourth vertex. We see the same in the

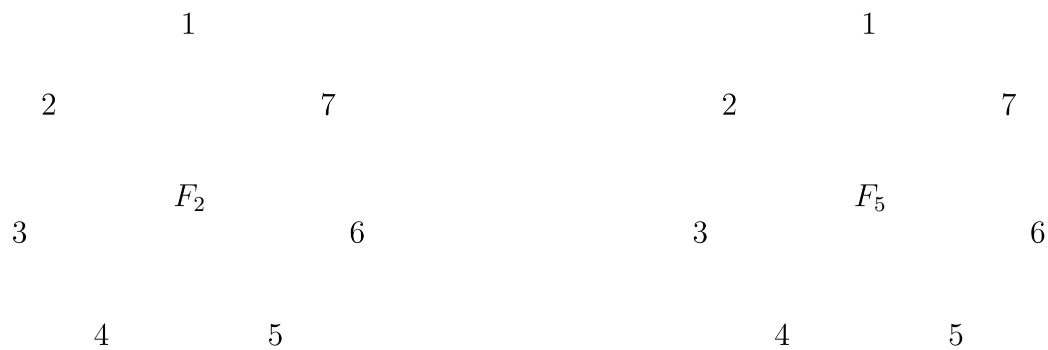

Figure 4.2: Factors F_2 and F_5

Figure 4.3: Factors F_4 and F_6

$*_1$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	7	5	6	1	3	4
3	3	5	4	2	6	7	1
4	4	6	2	5	7	1	3
5	5	1	6	7	3	4	2
6	6	3	7	1	4	2	5
7	7	4	1	3	2	5	6

Table 4.1: Groupoid of isomorphism factorisation of K_5

fifth and the sixth column where 4,2 share these columns. We have also 5 as the entry at the fourth vertex. In the fourth column, fifth column, sixth row and seventh row, we have the rectangle

$$\begin{matrix} 6 & 7 \\ 7 & 1 \end{matrix}$$

having 6, 7, 7 as entries at successive vertices, with 6, 7 sharing the fourth column. We have everywhere 1 as the entry at the fourth vertex.
We see the same in the sixth column where 6 and 7 sharing this column, 1 also is the entry at the fourth vertex. Finally, in the fourth column, fifth column, seventh row and eighth row, we have the rectangle

$$\begin{matrix} 7 & 1 \\ 1 & 3 \end{matrix}$$

having 7, 1, 1 as entries at successive vertices, where 7 and 1 share the fourth column and we have 3 as the entry at the fourth vertex.

We get the same in the seventh column where 7 and 1 share this column, and also 3 is the entry at the fourth vertex.

Therefore, Q_1 is a group, because in the multiplication table of this loop, any four places forming the vertices of a rectangle such that the entries in these places are

$$\begin{matrix} q & r \\ p & s, \end{matrix}$$

all the rectangles having q, r, p as entries at successive vertices, with q and p sharing a column and having s as the entry at the fourth vertex.

(b) Harary's factorisation of K_9 and its groupoid
In this case n is odd and t is even.

As before we have the edges

12	13	14	15	16	17	18	19
	23	24	25	26	27	28	29
		34	35	36	37	38	39
			45	46	47	48	49
				56	57	58	59
					67	68	69
						78	79
							89,

presented in lexicographic order.

Let $\phi = (2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$ be a permutation that fixes 1 and for which the length of every cycle is a multiple of $2t = 8$. The induced permutation $\phi^{'}$ which acts on the edges of K_9 is given by

$$\begin{aligned}\phi^{'} &= (12\ 13\ 14\ 15\ 16\ 17\ 18\ 19)(23\ 34\ 45\ 56\ 67\ 78\ 89\ 29)\\ &\quad (24\ 35\ 46\ 57\ 68\ 79\ 28\ 39)(25\ 36\ 47\ 58\ 69\ 27\ 38\ 49)\\ &\quad (26\ 37\ 48\ 59).\end{aligned}$$

The next step is to choose one edge from each orbit and to form a graph factor in the decomposition. We select 12, 23, 35, 49, 58, 79, 67, 16 and 48 . Then we have

$$E = \{12, 23, 35, 49, 58, 79, 67, 16, 48\}\,.$$

If we let ϕ' act successively, we get the factors:

$$\phi' E = \{13, 34, 46, 25, 69, 28, 78, 17, 59\}\,;$$

$$\phi'^2 E = \{14, 45, 57, 36, 27, 39, 89, 18, 26\}\,;$$

$$\phi'^3 E = \{15, 56, 68, 47, 38, 24, 29, 19, 37\}\,.$$

In the same manner as previously done,
$F_2 = \{(1,2),(2,3),(3,5),(5,8),(8,4),(4,9),(9,7),(7,6),(6,1)\}$;
$F_6 = \{(1,6),(6,7),(7,9),(9,4),(4,8),(8,5),(5,3),(3,2),(2,1)\}$;
$F_3 = \{(1,3),(3,4),(4,6),(6,9),(9,5),(5,2),(2,8),(8,7),(7,1)\}$;
$F_7 = \{(1,7),(7,8),(8,2),(2,5),(5,9),(9,6),(6,4),(4,3),(3,1)\}$;
$F_4 = \{(1,4),(4,5),(5,7),(7,2),(2,6),(6,3),(3,9),(9,8),(8,1)\}$;
$F_8 = \{(1,8),(8,9),(9,3),(3,6),(6,2),(2,7),(7,5),(5,4),(4,1)\}$;
$F_5 = \{(1,5),(5,6),(6,8),(8,3),(3,7),(7,4),(4,2),(2,9),(9,1)\}$;
$F_9 = \{(1,9),(9,2),(2,4),(4,7),(7,3),(3,8),(8,6),(6,5),(5,1)\}$.

The multiplication defined in 3.10 gives the Cayley table 4.2.

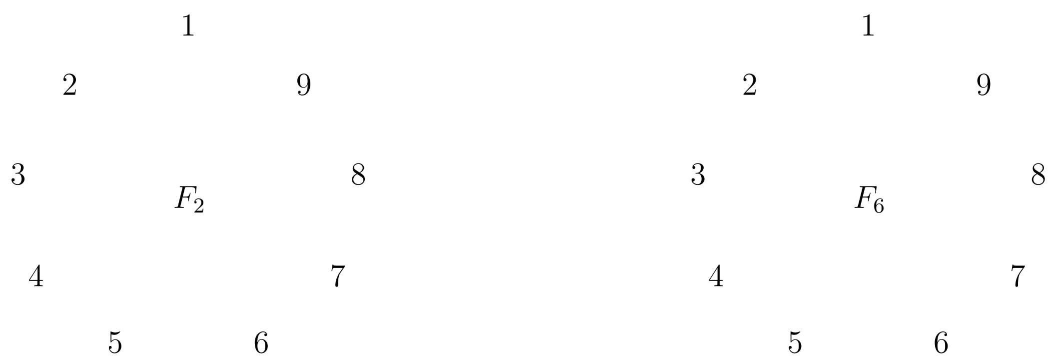

Figure 4.4: Factors F_2 and F_6.

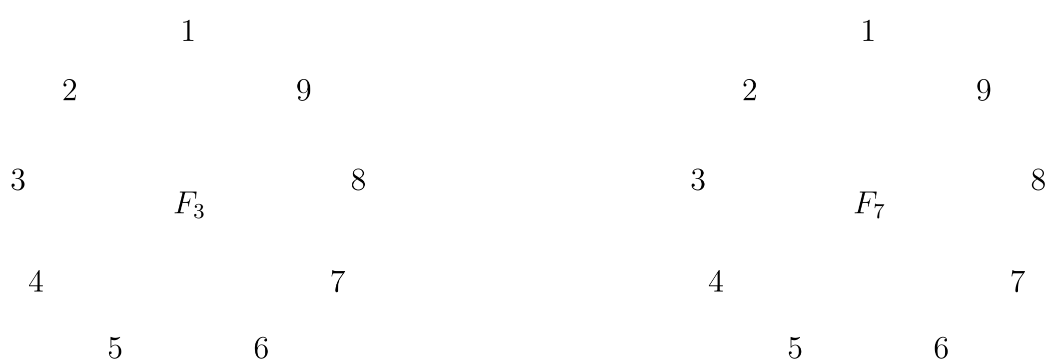

Figure 4.5: Factors F_3 and F_7.

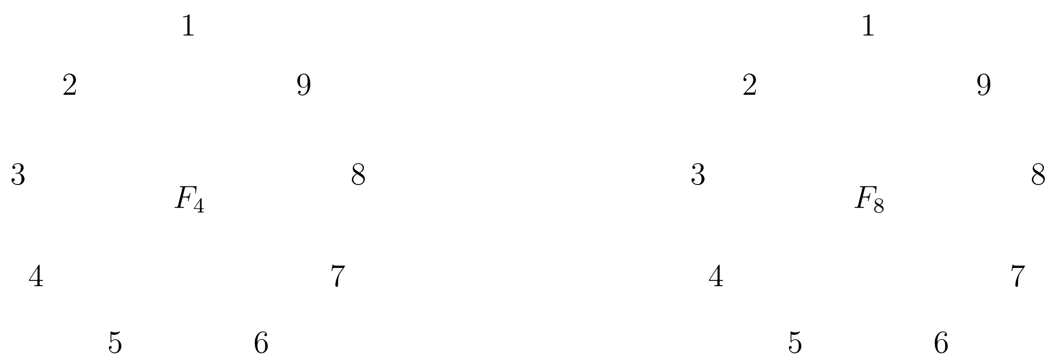

Figure 4.6: Factors F_4 and F_8.

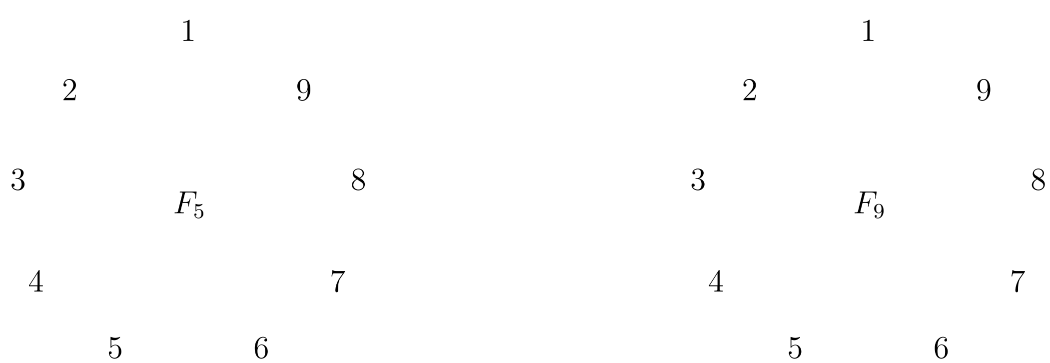

Figure 4.7: Factors F_5 and F_9.

$*_1$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3	8	6	9	1	5	7	4
3	3	5	4	9	7	2	1	6	8
4	4	9	6	5	2	8	3	1	7
5	5	8	2	7	6	3	9	4	1
6	6	1	9	3	8	7	4	2	5
7	7	6	1	2	4	9	8	5	3
8	8	4	7	1	3	5	2	9	6
9	9	7	5	8	1	4	6	3	2

Table 4.2: Groupoid of isomorphism factorisation of K_9

The loop that is represented by this Cayley table, which we denote by Q_2, is not associative. We see that for 9, 4, 7 it holds that $(9 *_1 4) *_1 7 \neq 9 *_1 (4 *_1 7)$;

$(9 *_1 4) *_1 7 = 8 *_1 7 = 2 \qquad 9 *_1 (4 *_1 7) = 9 *_1 3 = 5$ and $2 \neq 5$.

Now, since Q_2 is not a group, let us explore the possibility of quasi-associativity of its Cayley sets, i.e. let us explore if it is possible to get quasi-Cayley graphs. Thus we must present all its Cayley-sets and explore their quasi-associativity.

1. **Caylet sets of Q_2**

   By Definition 3.1, and from the previous Cayley table we have 16 Cayley sets, namely:

   $\emptyset,\ S_1 = \{2,6\},\ S_2 = \{3,7\},\ S_3 = \{4,8\},\ S_4 = \{5,9\},$

   $S_1 \cup S_2 = \{2,6,3,7\},\ S_1 \cup S_3 = \{2,6,4,8\},\ S_1 \cup S_4 = \{2,6,5,9\},$

   $S_2 \cup S_3 = \{3,7,4,8\},\ S_2 \cup S_4 = \{3,7,5,9\},\ S_3 \cup S_4 = \{4,8,5,9\},$

   $S_1 \cup S_2 \cup S_3 = \{2,6,3,7,4,8\},\ S_1 \cup S_2 \cup S_4 = \{2,6,3,7,5,9\},$

   $S_1 \cup S_3 \cup S_4 = \{2,6,4,8,5,9\},\ S_2 \cup S_3 \cup S_4 = \{3,7,4,8,5,9\},$

   $S_1 \cup S_2 \cup S_3 \cup S_4 = \{2,6,3,7,4,8,5,9\} = Q_2 - \{1\}.$

2. **Quasi-associativity**

   4.6 Proposition *All proper Cayley sets of Q_2 (the loop defined in Table 4.2) are not quasi-associative.*

Proof
It is enough to identify $a, b \in Q_2$ and $s \in S$ where S is a Cayley set such that $(ab)s = a(bk)$ and $k \notin S$. The following tabulates such a, b, s and k. (see Table 4.3) ∎

Cayley set	a	b	s	k
S_1	3	4	2	4
S_2	2	6	3	8
S_3	3	5	4	5
S_4	3	2	5	1
$S_1 \cup S_2$	4	9	2	8
$S_1 \cup S_3$	3	9	6	9
$S_1 \cup S_4$	4	8	9	7
$S_2 \cup S_3$	5	9	7	6
$S_2 \cup S_4$	4	8	5	2
$S_3 \cup S_4$	3	7	8	6
$S_1 \cup S_2 \cup S_3$	2	6	7	5
$S_1 \cup S_2 \cup S_4$	2	6	3	8
$S_1 \cup S_3 \cup S_4$	2	6	4	3
$S_2 \cup S_3 \cup S_4$	7	3	4	2

Table 4.3: The non quasi-associativity table of Cayley sets of Q_2

In the next chapter, we consider Hamiltonian factorisations and their groupoids.

Chapter 5

Groupoids from Hamiltonian factorisations

Hamiltonian factorisations have been considered by many people. In this chapter we consider Hamiltonian factorisations that are obtained from the solution to "problème de ronde" of Walecki as given by Alpspach [2] and the 1-rotational Hamiltonian systems, as done by Buratti and Fra [8].

Having obtained the Hamiltonian factorisations, we discuss groupoids that are constructed from them with examples. It had been anticipated that such groupoids would exhibit the symmetry of the Hamiltonian factorisations. We show that in general, utility and truth (as adage has it) are not the same concepts in these instances. We neither obtain groups nor quasi-Cayley graphs.

By Theorem 4.3, trivial counting shows that the number of cycles of a Hamiltonian cycle system of a complete graph K_n is $\frac{n-1}{2}$. It is clear that a necessary condition for its existence is that n must be odd. The condition is also sufficient by the next lemma. It was provided by Buratti and Fra [8] as a method of obtaining the 1-rotational Hamiltonian cycle system, which is a special case of cycle decompositions, and is the content of Lemma 5.3.

5.1 The "problème de ronde" and its Hamiltonian factorisations

5.1.1 The "problème de ronde"

One of the earliest Hamiltonian factorisations was obtained as the result of a solution to the "problème de ronde" posed by Lucas. In its original terms, the problem is the following. Given $2n+1$ people, is it possible to arrange them around a single table on n successive nights so that nobody is seated next to the same person on either side more than once?

There is a natural graph theoretic formulation of the problem. If we let the $2n+1$ people correspond to the vertices of K_{2n+1}, the complete graph of order $2n+1$, then an arrangement of them around a single table corresponds to a Hamilton cycle in K_{2n+1}. Because each vertex of K_{2n+1} has exactly $2n$ neighbours, the "problème de ronde" is asking whether the complete graph K_{2n+1} has a Hamilton decomposition for all $n > 1$.

5.1.2 Walecki's construction

In this section, we present a Hamiltonian decomposition given by Lucas and attributed to Walecki. Our work is wholly that of Alspach [2].

Let us do the construction for all values of $2n+1$. Label the vertices of K_{2n+1} as $x_0, x_1, x_2, \cdots, x_{2n}$ and let ϕ be the permutation whose disjoint cycle decomposition representation is

$$\phi = (x_1 x_2 \cdots x_{2n}).$$

Suppose a Hamilton cycle C_1 is given by

$$C_1 = x_0 x_1 x_2 x_{2n} x_3 x_{2n-1} \cdots x_n x_{n+2} x_{n+1} x_0.$$

Then denote

$$C_i = \phi^{i-1}(C_1), i = 1, 2, \cdots, n. \tag{5.1}$$

5.1 Proposition [2, p. 9] *Let C_i be defined as in (5.1). Then K_{2n+1} admits a decomposition $\mathcal{C} = \{\phi^{i-1}(C_1) : i = 1, 2, \cdots, n\}$, i.e.*

$$K_{2n+1} = C_1 \oplus C_2 \oplus \cdots \oplus C_n. \tag{5.2}$$

Proof
We consider a directed Hamiltonian cycle $\vec{C}_i$. Define the length of an arc $[x_i, x_j]$ by $j - i \pmod{2n}$ where $i, j \neq 0$ and $i, j \in \{1, 2, \cdots, 2n\}$. The arcs of odd length r are $[x_{\frac{(4n-r+1)}{2}}, x_{\frac{(r+1)}{2}}]$ and $[x_{\frac{(2n-r+1)}{2}}, x_{\frac{(2n+r+1)}{2}}]$, $1 \leq r < n$. Every arc of odd length r appears once in $\vec{C}_1, \vec{C}_2, \cdots, \vec{C}_n$, since ϕ^{i-1} preserves length. The arcs $[x_{\frac{(4n-r+2)}{2}}, x_{\frac{(r+2)}{2}}]$ and $[x_{\frac{(2n-r+2)}{2}}, x_{\frac{(2n+r+2)}{2}}]$, $1 < 2 < n$, have even length r, so that every arc of even length r appears once as well. The arcs of length n are $[x_i, x_{i+n}]$, $1 \leq n$, and there is one of them in each Hamilton cycle. Finaly, the edges incident with x_0 in C_i are $[x_0, x_i]$ and $[x_0, x_{n+i}]$ so that all edges incident with x_0 are used. ∎

5.1.3 Groupoids of Walecki's construction

We present a factorisation of K_5 and K_9 and produce their corresponding groupoids.

(a) Walecki's factorisation of K_5 and its groupoid

For K_5, $n = 2$. Let $\phi = (1\ 2\ 3\ 4)$. The factors are $C_1 = 012430$, $C_2 = \phi(C_1) = \phi(012430) = 023140$.

Now, for our construction, as in Chapter 4 we consider the two directions of every factor cycle. We obtain four factor digraphs denoted by $F_1,\ F_3$ from C_1 and $F_2,\ F_4$ from C_2. The factors are
$F_1 = \{(0,1), (1,2), (2,4), (4,3), (3,0)\}$;
$F_3 = \{(0,3), (3,4), (4,2), (2,1), (1,0)\}$;
$F_2 = \{(0,2), (2,3), (3,1), (1,4), (4,0)\}$;
$F_4 = \{(0,4), (4,1), (1,3), (3,2), (2,0)\}$.
(See Figures 5.1, 5.2.)

In addition, the Cayley table representing the multiplication of this loop, which we denote by Q_3, is as below.

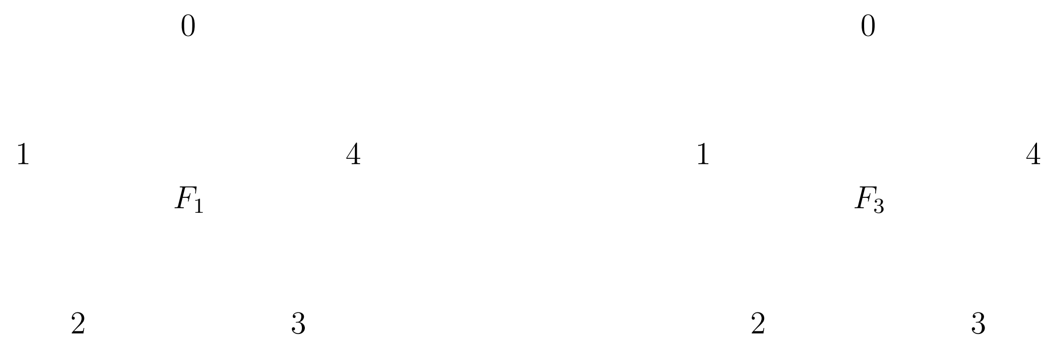

Figure 5.1: Factors F_1 and F_4

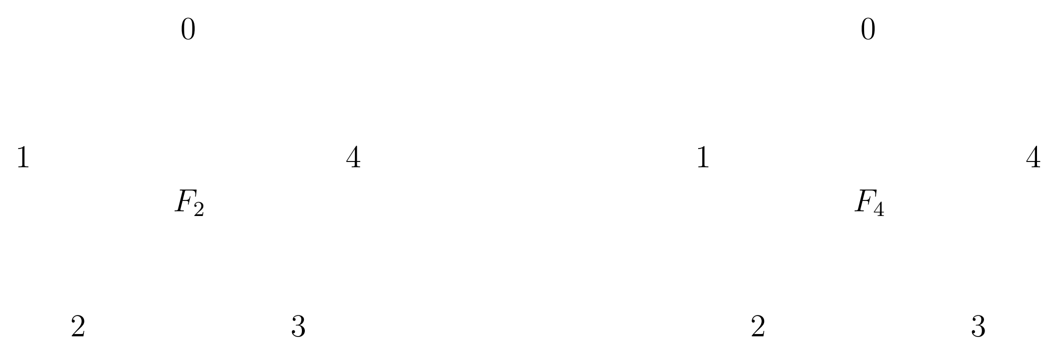

Figure 5.2: Factors F_2 and F_4

$*_0$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	4	0	3
2	2	4	3	1	0
3	3	0	1	4	2
4	4	3	0	2	1

Table 5.1: Groupoid of Walecki's factorisation of K_5

Note that the table is symmetric. Hence Q_3 is commutative.

Calculating from Table 5.1 gives

$(1 *_0 1) *_0 1 = 2 *_0 1 = 4,$ $\quad 1 *_0 (1 *_0 1) = 1 *_0 2 = 4;$

$(2 *_0 2) *_0 2 = 3 *_0 2 = 1,$ $\quad 2 *_0 (2 *_0 2) = 2 *_0 3 = 1;$

$(3 *_0 3) *_0 3 = 4 *_0 3 = 2,$ $\quad 3 *_0 (3 *_0 3) = 3 *_0 4 = 2;$

$(4 *_0 4) *_0 4 = 1 *_0 4 = 3,$ $\quad 4 *_0 (4 *_0 4) = 4 *_0 1 = 3;$

$(1 *_0 2) *_0 3 = 4 *_0 3 = 2,$ $\quad 1 *_0 (2 *_0 3) = 1 *_0 1 = 2;$

$(1 *_0 2) *_0 4 = 4 *_0 4 = 1,$ $\quad 1 *_0 (2 *_0 4) = 1 *_0 0 = 1;$

$(1 *_0 3) *_0 4 = 0 *_0 4 = 4,$ $\quad 1 *_0 (3 *_0 4) = 1 *_0 2 = 4;$

$(2 *_0 3) *_0 4 = 1 *_0 4 = 3,$ $\quad 2 *_0 (3 *_0 4) = 2 *_0 2 = 3.$

Since Q_3 is commutative, the cases considered above are enough to satisfy associativity of Q_3. Therefore, Q_3 is indeed a group.

(b) Walecki's factorisation of K_9 and its groupoid

For $K_9, n = 4$, $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$ and

$$C_1 = x_0x_1x_2x_8x_3x_7x_4x_6x_5x_0.$$

The factors are $C_1 = 0128374650,\ C_2 = \sigma(C_1) = \sigma(012837465) = 0231485760,$ $C_3 = 0342516870,\ C_4 = 0453627180.$

Again, considering the directions of every cycle of our constructions, we obtain the directed factors F_1, F_2 from C_1; F_2, F_6 from C_2; F_3, F_7 from C_3; and F_4, F_8 from C_4. We have
$F_1 = \{(0,1),(1,2),(2,8),(8,3),(3,7),(7,4),(4,6),(6,5),(5,0)\};$
$F_5 = \{(0,5),(5,6),(6,4),(4,7),(7,3),(3,8),(8,2),(2,1),(1,0)\};$

$F_2 = \{(0,2),(2,3),(3,1),(1,4),(4,8),(8,5),(5,7),(7,6),(6,0)\}$;
$F_6 = \{(0,6),(6,7),(7,5),(5,8),(8,4),(4,1),(1,3),(3,2),(2,0)\}$;
$F_3 = \{(0,3),(3,4),(4,2),(2,5),(5,1),(1,6),(6,8),(8,7),(7,0)\}$;
$F_7 = \{(0,7),(7,8),(8,6),(6,1),(1,5),(5,2),(2,4),(4,3),(3,0)\}$;
$F_4 = \{(0,4),(4,5),(5,3),(3,6),(6,2),(2,7),(7,1),(1,8),(8,0)\}$;
$F_8 = \{(0,8),(8,1),(1,7),(7,2),(2,6),(6,3),(3,5),(5,4),(4,0)\}$.
(See Figures , 5.3, 5.4, 5.5 and 5.6.)

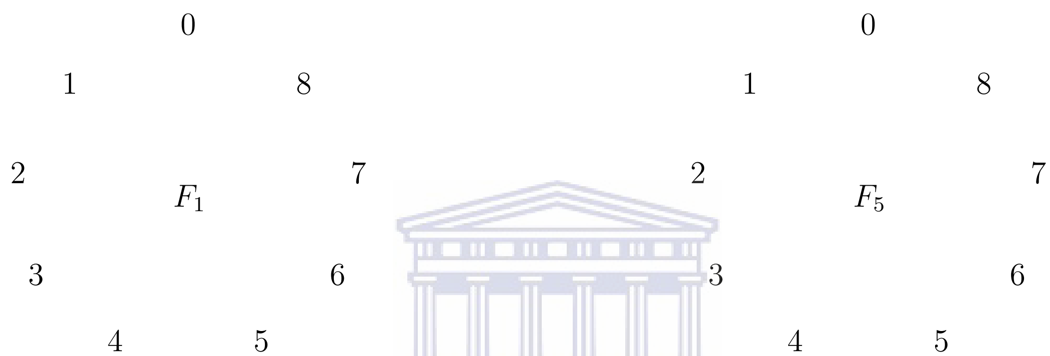

Figure 5.3: Factors F_1 and F_5.

Figure 5.4: Factors F_2 and F_6.

The Cayley table representing the multiplication of the loop, which we denote by Q_4, is as below.

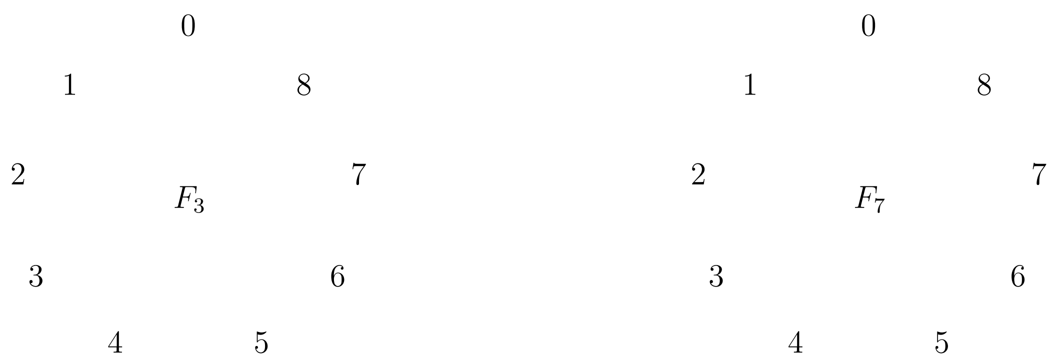

Figure 5.5: Factors F_3 and F_7.

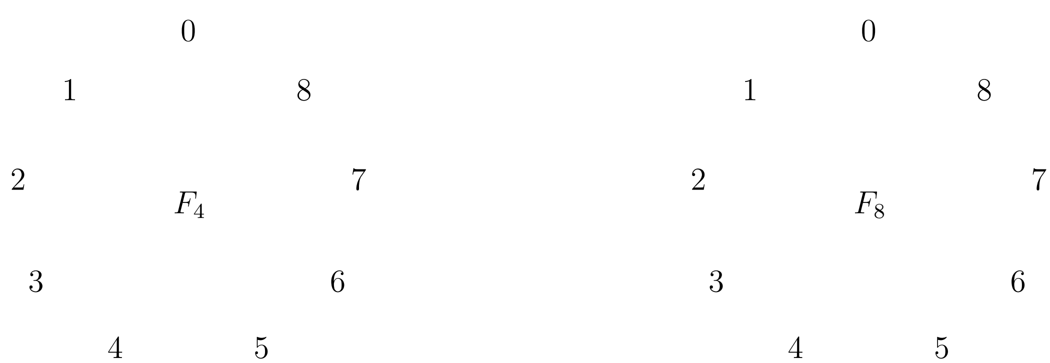

Figure 5.6: Factors F_4 and F_8.

$*_0$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	4	6	8	0	3	5	7
2	2	8	3	5	7	1	0	4	6
3	3	7	1	4	6	8	2	0	5
4	4	6	8	2	5	7	1	3	0
5	5	0	7	1	3	6	8	2	4
6	6	5	0	8	2	4	7	1	3
7	7	4	6	0	1	3	5	8	2
8	8	3	5	7	0	2	4	6	1

Table 5.2: Groupoid of Walecki's factorisation of K_9

Note that we have $1, 2, 3 \in Q_4$ such that

$(1 *_0 2) *_0 3 = 4 *_0 3 = 2,$ $\quad$ $1 *_0 (2 *_0 3) = 1 *_0 5 = 0.$

Q_4 is therefore not a group.

The question of quasi-associativity of Cayley sets therefore comes into play.

5.1.4 Quasi-associativity in Q_4

We will check if there are instances when we obtain quasi-Cayley graphs being represented by Q_4. We first present all its Cayley-sets and check their quasi- associativity.

1. **Caylet-sets of Q_4**

   By Definition 3.1 of a Caylet set in a loop and from the Cayley table, we have 16 Cayley sets, namely:

   $\emptyset,\ S_1 = \{1, 5\},\ S_2 = \{2, 6\},\ S_3 = \{3, 7\}, S_4 = \{4, 8\},$

   $S_1 \cup S_2 = \{1, 5, 2, 6\},\ S_1 \cup S_3 = \{1, 5, 3, 7\},\ S_1 \cup S_4 = \{1, 5, 4, 8\},$

   $S_2 \cup S_3 = \{2, 6, 3, 7\},\ S_2 \cup S_4 = \{2, 6, 4, 8\},\ S_3 \cup S_4 = \{3, 7, 4, 8, \},$

   $S_1 \cup S_2 \cup S_3 = \{1, 5, 2, 6, 3, 7\},\ S_1 \cup S_2 \cup S_4 = \{1, 5, 2, 6, 4, 8\},$

   $S_1 \cup S_3 \cup S_4 = \{1, 5, 3, 7, 4, 8\},\ S_2 \cup S_3 \cup S_4 = \{2, 6, 3, 7, 4, 8\},$

   $S_1 \cup S_2 \cup S_3 \cup S_4 = \{1, 5, 2, 6, 3, 7, 4, 8, \} = Q_4 - \{0\}.$

2. **Quasi-associativity**

   5.2 Proposition *All proper Cayley sets of Q_4, the loop defined in the Table 5.2, are not quasi-associative.*

   Proof
   As in Proposition 4.6, in this case we can identify values for all the variables a, b, s, and k in Table 5.3. ∎

Cayley-set	a	b	s	k
S_1	2	3	1	4
S_2	1	3	2	8
S_3	4	5	3	6
S_4	2	3	4	6
$S_1 \cup S_2$	3	7	1	8
$S_1 \cup S_3$	4	6	1	8
$S_1 \cup S_4$	3	7	5	7
$S_2 \cup S_3$	5	8	6	1
$S_2 \cup S_4$	5	1	3	2
$S_3 \cup S_4$	5	6	7	1
$S_1 \cup S_2 \cup S_3$	1	2	7	8
$S_1 \cup S_2 \cup S_4$	2	6	8	7
$S_1 \cup S_3 \cup S_4$	1	5	7	6
$S_2 \cup S_3 \cup S_4$	4	3	6	5

Table 5.3: The non-quasi-associativity table of Cayley sets of Q_4

In the next section, we consider another kind of Hamiltonian factorisation, that is, 1-rotational Hamiltonian cycle systems.

5.2 1-rotational Hamiltonian cycle systems and their groupoids

We now present 1-rotational Hamiltonian cycle systems as given by Buratti and Fra [8]. From there, we discuss the kind of groupoids that are generated by this method of factorisation, i.e. we investigate whether we obtain Cayley graphs or quasi-Cayley graphs, as in the other cases.

As alluded to, the necessary condition of the existence of a Hamiltonian cycle system is the lemma below.

5.3 Lemma *Let Γ be a complete graph of odd order n , defined on $\mathbb{Z}_{n-1} \cup \{\infty\}$. Then Γ admits a Hamiltonian factorisation .*

Consider $\mathcal{C}$ given by Buratti and Fra. $\mathcal{C}$ is defined by

$$\{(\infty, i, i+1, i-1, i+2, i-2, ..., i+(k-1), i-(k-1), i+k) | 0 \leq i < k\} \tag{5.3}$$

where $n = 2k+1$. $\mathcal{C}$ is a 1-rotational Hamiltonian cycle system.

Note first that the 1-rotational Hamiltonian cycle system of K_5 gives the same result of a Cayley graph as given by Walecki's factorisation. Now, we illustrate the properties (or lack thereof) of groupoids generated by a 1-rotational Hamiltonian cycle system with cases $n = 7$ and $n = 9$, with of course, respectively $k = 3$ and $k = 4$ in equation (5.3). So, we consider the groups $\mathbb{Z}_6$ and $\mathbb{Z}_8$.

5.2.1 Goupoid from 1-rotational Hamiltonian cycle systems of $K_{\mathbb{Z}_6 \cup \{\infty\}}$

The 1-rotational Hamiltonian cycle system of the complete graph $K_{\mathbb{Z}_6 \cup \{\infty\}}$ is given by

$\mathcal{C} = \{(\infty, i, i+1, i-1, i+2, i-2, i+3)\}$ where $0 \leq i < 3$. Hence $\mathcal{C}$ is $\{(\infty, 0, 1, 5, 2, 4, 3),\ (\infty, 1, 2, 0, 3, 5, 4),\ (\infty, 2, 3, 1, 4, 0, 5)\}$

The factors are $C_1 = (\infty, 0, 1, 5, 2, 4, 3)$, $C_2 = (\infty, 1, 2, 0, 3, 5, 4)$ and $C_3 = (\infty, 2, 3, 1, 4, 0, 5)$.

The directed factors are
$F_1 = \{(0, 1), (1, 5), (5, 2), (2, 4), (4, 3), (3, \infty), (\infty, 0)\}$;
$F_\infty = \{(0, \infty), (\infty, 3), (3, 4), (4, 2), (2, 5), (5, 1), (1, 0)\}$;
$F_2 = \{(0, 2), (2, 1), (1, \infty), (\infty, 4), (4, 5), (5, 3), (3, 0)\}$;
$F_3 = \{(0, 3), (3, 5), (5, 4), (4, \infty), (\infty, 1), (1, 2), (2, 0)\}$;
$F_4 = \{(0, 4), (4, 1), (1, 3), (3, 2), (2, \infty), (\infty, 5), (5, 0)\}$;
$F_5 = \{(0, 5), (5, \infty), (\infty, 2), (2, 3), (3, 1), (1, 4), (4, 0)\}$.
(See Figures 5.7, 5.8, 5.9.)

The Cayley table of multiplication representing the loop, which we denote by Q_5, is given by the Cayley Table 5.4.

Figure 5.7: Factors F_1 and F_∞

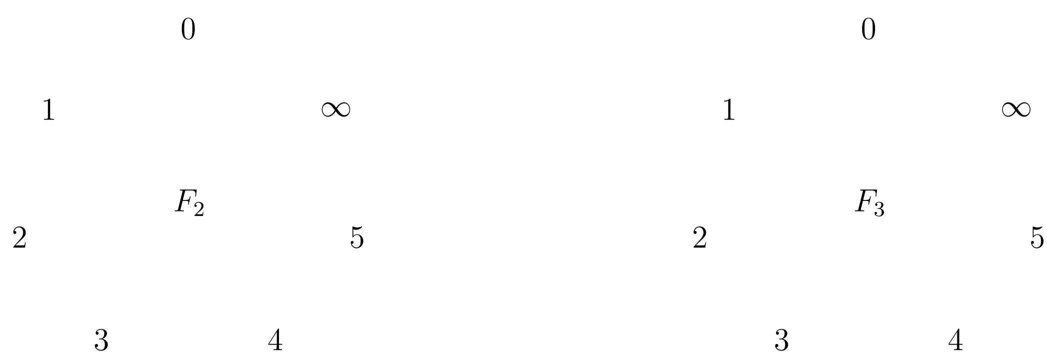

Figure 5.8: Factors F_2 and F_3

$*_0$	0	1	2	3	4	5	∞
0	0	1	2	3	4	5	∞
1	1	5	∞	2	3	4	0
2	2	4	1	0	∞	3	5
3	3	∞	0	5	2	1	4
4	4	3	5	∞	1	0	2
5	5	2	3	4	0	∞	1
∞	∞	0	4	1	5	2	3

Table 5.4: Groupoid of 1-rotational Hamiltonian cycle systems of $K_{\mathbb{Z}_6\cup\{\infty\}}$

Note that we have $3, 5, \infty \in \mathbb{Z}_6 \cup \{\infty\}$ such that

$(3 *_0 5) *_0 \infty = 1 *_0 \infty = 0, \qquad 3 *_0 (5 *_0 \infty) = 3 *_0 1 = \infty.$

It is therefore clear that Q_5 is not a group.

5.2.2 Quasi-associativity in Q_5

Since Q_5 is not a group, let us explore the possibility of quasi-associativity of its Cayley sets.

1. **Caylet-sets of Q_5**

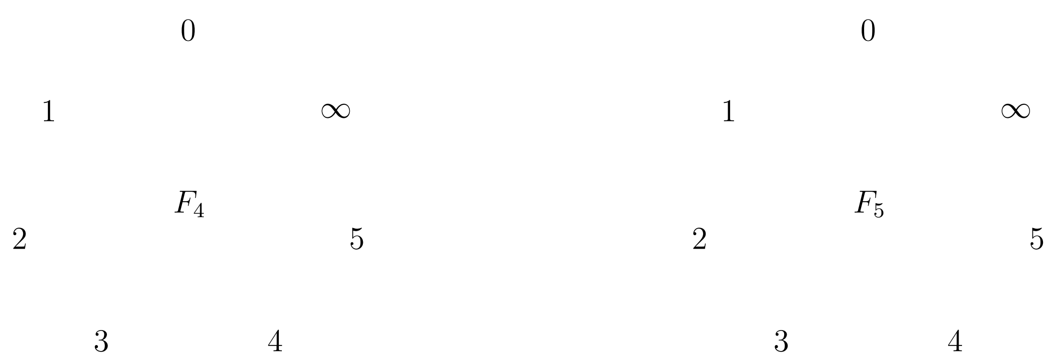

Figure 5.9: Factors F_4 and F_5

By the same argument as in Section 5.1.4, we have the following Cayley sets.

$\emptyset,\ S_1 = \{1, \infty\}\,,\ S_2 = \{2, 3\}\,,\ S_3 = \{4, 5\}\,,\ S_1 \cup S_2 = \{1, \infty, 2, 3\}\,,$

$S_1 \cup S_3 = \{1, \infty, 4, 5\}\,,\ S_2 \cup S_3 = \{2, 3, 4, 5\}\,,$

$S_1 \cup S_2 \cup S_3 = \{1, 2, 3, 4, 5, \infty\} = Q_6 - \{0\}\,.$

2. **Quasi-associativity**

5.4 Proposition *All proper Cayley sets of Q_5, the loop defined in Table 5.4, are not quasi-associative.*

Proof
Again, as in Proposition 4.6, we have the following non-quasi-associativity table (Table 5.5). ∎

Cayley-set	a	b	s	k
S_1	2	3	1	4
S_2	4	5	2	5
S_3	2	3	5	1
$S_1 \cup S_2$	1	∞	∞	5
$S_1 \cup S_3$	1	4	∞	2
$S_2 \cup S_3$	1	∞	2	∞

Table 5.5: The non quasi-associativity table of Cayley sets of Q_5

5.2.3 Groupoid from 1-rotational Hamiltonian cycle system of $K_{\mathbb{Z}_8 \cup \{\infty\}}$

For $K_{\mathbb{Z}_8 \cup \{\infty\}}$, we have

$$\begin{aligned}
\mathcal{C} &= \{(\infty, i, i+1, i-1, i+2, i-2, i+3, i-3, i+4) | 0 \le i < 4\}; \\
&= \{(\infty, 0, 1, 7, 2, 6, 3, 5, 4), (\infty, 1, 2, 0, 3, 7, 4, 6, 5), \\
&\quad (\infty, 2, 3, 1, 4, 0, 5, 7, 6), (\infty, 3, 4, 2, 5, 1, 6, 0, 7)\}.
\end{aligned}$$

We obtain factors $C_1 = (\infty, 0, 1, 7, 2, 6, 3, 5, 4)$, $C_2 = (\infty, 1, 2, 0, 3, 7, 4, 6, 5)$, $C_3 = (\infty, 2, 3, 1, 4, 0, 5, 7, 6)$ and $C_4 = (\infty, 3, 4, 2, 5, 1, 6, 0, 7)$.

The directed factors are
$F_1 = \{(0,1), (1,7), (7,2), (2,6), (6,3), (3,5), (5,4), (4,\infty), (\infty,0)\}$;
$F_\infty = \{(0,\infty), (\infty,4), (4,5), (5,3), (3,6), (6,2), (2,7), (7,1), (1,0)\}$;
$F_2 = \{(0,2), (2,1), (1,\infty), (\infty,5), (5,6), (6,4), (4,7), (7,3), (3,0)\}$;
$F_3 = \{(0,3), (3,7), (7,4), (4,6), (6,5), (5,\infty), (\infty,1), (1,2), (2,0)\}$;
$F_4 = \{(0,4), (4,1), (1,3), (3,2), (2,\infty), (\infty,6), (6,7), (7,5), (5,0)\}$;
$F_5 = \{(0,5), (5,7), (7,6), (6,\infty), (\infty,2), (2,3), (3,1), (1,4), (4,0)\}$;
$F_6 = \{(0,6), (6,1), (1,5), (5,2), (2,4), (4,3), (3,\infty), (\infty,7), (7,0)\}$;
$F_7 = \{(0,7), (7,\infty), (\infty,3), (3,4), (4,2), (2,5), (5,1), (1,6), (6,0)\}$.
(See Figures 5.10, 5.11, 5.12 and 5.13.)

In addition, the Cayley table representing the loop, which we denote by Q_6, is given in Table 5.6.

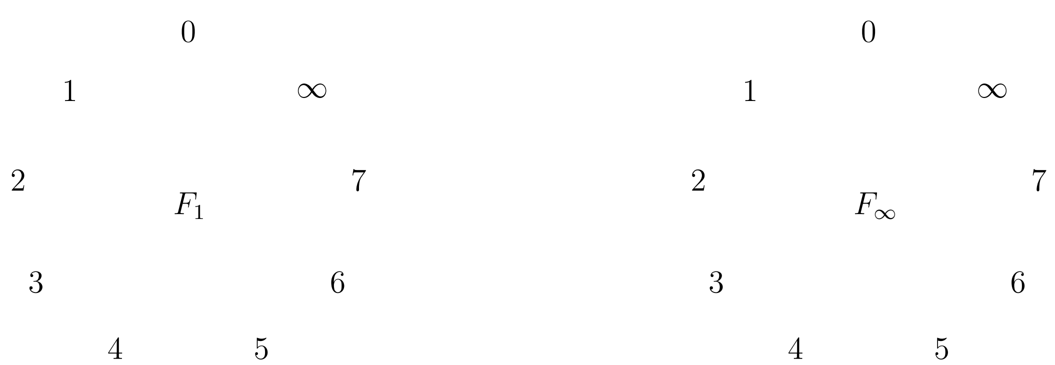

Figure 5.10: Factors F_1 and F_∞.

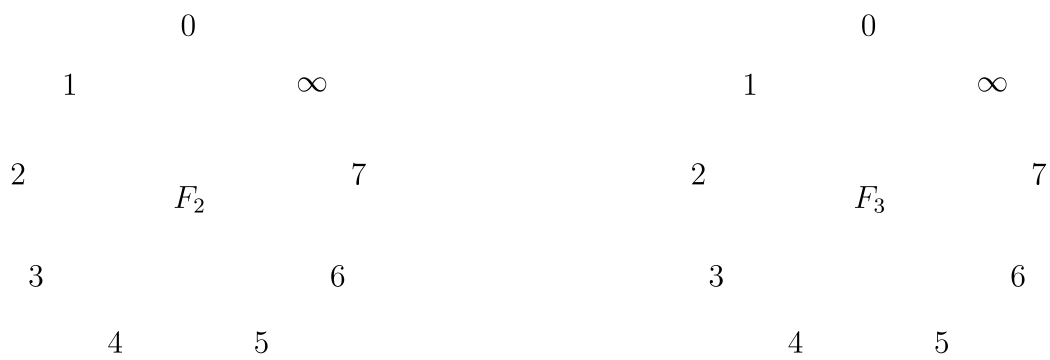

Figure 5.11: Factors F_2 and F_3.

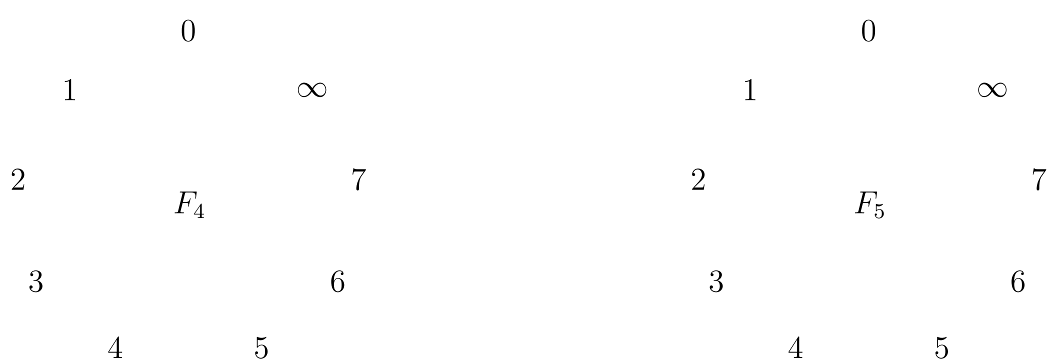

Figure 5.12: Factors F_4 and F_5.

$*_0$	0	1	2	3	4	5	6	7	∞
0	0	1	2	3	4	5	6	7	∞
1	1	7	6	5	∞	4	3	2	0
2	2	∞	1	0	7	6	4	3	5
3	3	2	0	7	6	∞	5	4	1
4	4	3	∞	2	1	0	7	5	6
5	5	4	3	1	0	7	∞	6	2
6	6	5	4	∞	3	2	1	0	7
7	7	6	5	4	2	1	0	∞	3
∞	∞	0	7	6	5	3	2	1	4

Table 5.6: Groupoid of 1-rotational Hamiltonian factorisation of $K_{\mathbb{Z}_8 \cup \{\infty\}}$

In this Cayley table, we have $1, 3, 6 \in \mathbb{Z}_8 \cup \{\infty\}$ such that

$(1 *_0 3) *_0 6 = 5 *_0 6 = \infty,$ $\qquad$ $1 *_0 (3 *_0 6) = 1 *_0 5 = 4.$

Again, Q_6 is not a group.

5.2.4 Quasi-associativity in Q_6

It has been shown that Q_6 is not a group. We check whether it is possible to get a quasi-Cayley graph. Thus we must present all its Cayley sets and

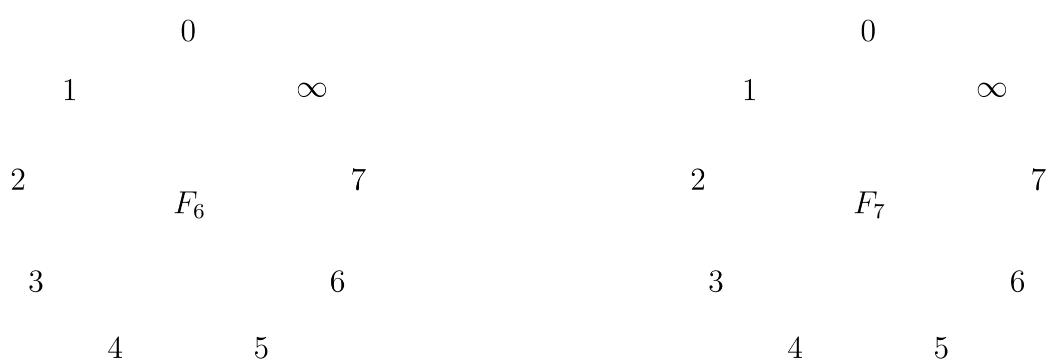

Figure 5.13: Factors F_6 and F_7.

check their quasi- associativity.

1. **Caylet-sets of Q_6**

   Here we have the following Cayley sets.

   $\emptyset,\ S_1 = \{1, \infty\}\,,\ S_2 = \{2, 3\}\,,\ S_3 = \{4, 5\}\,,\ S_4 = \{6, 7\}\,,$

   $S_1 \cup S_2 = \{1, \infty, 2, 3\}\,,\ S_1 \cup S_3 = \{1, \infty, 4, 5\}\,,\ S_1 \cup S_4 = \{1, \infty, 6, 7\}\,,$

   $S_2 \cup S_3 = \{2, 3, 4, 5\}\,,\ S_2 \cup S_4 = \{2, 3, 6, 7\}\,,\ S_3 \cup S_4 = \{4, 5, 6, 7\}\,,$

   $S_1 \cup S_2 \cup S_3 = \{1, \infty, 2, 3, 4, 5\}\,,\ S_1 \cup S_2 \cup S_4 = \{1, \infty, 2, 3, 6, 7\}\,,$

   $S_1 \cup S_3 \cup S_4 = \{1, \infty, 4, 5, 6, 7\}\,,\ S_2 \cup S_3 \cup S_4 = \{2, 3, 4, 5, 6, 7\}\,,$

   $S_1 \cup S_2 \cup S_3 \cup S_4 = \{1, \infty, 2, 3, 4, 5, 6, 7\} = Q_6 - \{0\}\,.$

2. **Quasi-associativity**

   5.5 Proposition *All proper Cayley sets of Q_6, the loop defined in Table 5.6, are not quasi-associative.*

   Proof
   The table, similar to the one in the proof of Proposition 5.4, is now as below. (See Table 5.7.) ∎

Cayley-set	a	b	s	k
S_1	6	5	1	2
S_2	6	5	2	7
S_3	6	3	4	∞
S_4	7	4	6	1
$S_1 \cup S_2$	1	5	1	7
$S_1 \cup S_3$	4	6	1	3
$S_1 \cup S_4$	∞	6	6	3
$S_2 \cup S_3$	2	1	5	1
$S_2 \cup S_4$	1	3	2	4
$S_3 \cup S_4$	1	5	6	3
$S_1 \cup S_2 \cup S_3$	6	2	2	7
$S_1 \cup S_2 \cup S_4$	4	2	2	5
$S_1 \cup S_3 \cup S_4$	1	5	6	3
$S_2 \cup S_3 \cup S_4$	∞	5	6	1

Table 5.7: The non quasi-associativity table of Cayley sets of Q_6

The next chapter is the generalisation of 1-rotational factorisations.

Chapter 6

General 1-rotational 2-factorisations and the corresponding groupoids

6.1 Introduction

In this chapter we consider the case in which a group G of permutations fixes one vertex of a graph and acts regularly on the others, as it acts on a given factorisation which arises from G. This is called by Buratti and Rinaldi [7] the *1-rotational k-factorisation* under the action of G, where k is the degree of each factor. We will focus on the case of 1-rotational 2- factorisations and investigate the kind of groupoids that are generated therefrom. In the first section we introduce the concept of *k-starters* in a group G whose order is divisible by k, as defined in [7]. This concept enables us to describe algebraically k-factorisations which are 1-rotational under G.

6.2 k-starters and 1-rotational k-factorisations

Let us set $\overline{G} = G \cup \{\infty\}$ with $\infty \notin G$ and $g \cdot \infty = \infty$ for $g \in G$. We deal with k-factorisations of complete graphs which are 1-rotational under the action of a group G. It is then clear that the complete graph can be identified with $K_{\overline{G}}$, the action of G on its vertices being defined by $a \rightarrow ga$, for any $(g, a) \in G \times \overline{G}$. That is, $(g, a) \longrightarrow ga$ is an action of G on $\overline{G}$ which fixes ∞ and acts regularly on $\overline{G} \setminus \{\infty\}$.

If $g \in G$ is an arbitrary group element and $[x, y]$ is any edge of $E(K_{\overline{G}})$ then $g[x, y] = [gx, gy]$. By an induced action, G extends to cycles and k-factors of $K_{\overline{G}}$. The order of G must be divisible by k, as is seen in Definition 6.3. It is even in the special case $k = 2$, which is our focus of attention.

For the sake of brevity, instead of speaking of a "factorisation of the complete graph that is 1-rotational under the action of G", we will often speak of a "1-rotational factorisation of $K_{\overline{G}}$."

6.1 Theorem [7, p. 4] *Let $\mathcal{F}$ be a 1-rotational k-factorisation of $K_{\overline{G}}$. Then, any factor $F \in \mathcal{F}$ has the stabilizer G_F of order k so that the induced action on $\mathcal{F}$ is transitive.*

Proof
Let us first show that G acts transitively on $\mathcal{F}$. Consider two factors F, F', and let v, v' be vertices connected to ∞ by an edge in F, F' respectively. That is, $[v, \infty] \in E(F)$ and $[v^{'}, \infty] \in E(F^{'})$. Since G acts transitively on $K_{\overline{G}\setminus\{\infty\}}$, there is a $g \in G$ such that $g(v) = v'$. Hence, $g(F)$ shares an edge with $F^{'}$, that is $g(F) = F^{'}$. This shows G acts transitively on $\mathcal{F}$. Now, since $k|\mathcal{F}| = |G|$ and as $|\mathcal{F}| = |G|/|G_F|$, it is clear that the stabilizer subgroup of the action of G on F must have k elements and the result is complete. ∎

6.2.1 Example Consider the complete graph $K_{\mathbb{Z}_6 \cup \{\infty\}} = K_{\overline{\mathbb{Z}_6}}$ and let $\mathcal{F} = \{F_0, F_1, F_2\}$ be a factorisation such that $F_0 = \{(\infty, 0, 3), (1, 5, 4, 2)\}$, $F_1 = F_0 + 1 = \{(\infty, 1, 4), (2, 0, 5, 3)\}$, $F_2 = F_0 + 2 = \{(\infty, 2, 5), (3, 1, 0, 4)\}$.

The stabilizer of F_0, F_1 and F_2 denoted respectively by G_{F_0}, G_{F_1} and G_{F_1} are
(i) $G_{F_0} = \{g \in G : g + F_0 = F_0\}$, $g + F_0 = F_0 \Rightarrow g + \{(\infty, 0, 3), (1, 5, 4, 2)\} = \{(\infty, 0, 3), (1, 5, 4, 2)\}$. It is clear that $g = 0$ or 3. Therefore, $G_{F_0} = \{0, 3\}$.
(ii) $G_{F_1} = \{g \in G : g + F_1 = F_1\}$. So, $G_{F_1} = \{0, 3\}$.
(iii) $G_{F_2} = \{g \in G : g + F_2 = F_2\}$. Hence, again $G_{F_2} = \{0, 3\}$.

The G-orbit is $\mathcal{F}$. Indeed, the G -orbit of F_0 is $\{g + F_0, g \in G\} = \mathcal{F}$.

6.2 Definition Given a group G and a simple graph Γ with vertices in $\overline{G}$, the *list of differences* of Γ is the multiset $\Delta\Gamma$ of all differences xy^{-1} where

(x, y) is an arc of Γ not passing through ∞. That is,

$$\Delta\Gamma = \left\{xy^{-1}, yx^{-1} | [x, y] \in E(\Gamma); x \neq \infty \neq y\right\}.$$

For illustration, let us find a list of differences of the simple graph F_0, a factor in the above decomposition $\mathcal{F}$ of the complete graph $K_{\mathbb{Z}_6 \cup \{\infty\}}$. We have that $\Delta F_0 = \{1, 2, 3, 4\}$.

The following concept describes algebraically any 1-rotational k-factorisation of $K_{\overline{G}}$.

6.3 Definition Let G be a group of order divisible by k (respectively a group of odd order when $k = 1$) and let F be a factor of $K_{\overline{G}}$. We say that F is a *k-starter* in G if the following conditions are satisfied:
(i) The G-stabilizer of F has order k;
(ii) ΔF covers all elements of $\overline{G} - \{1\}$, where 1 is the identity.

6.4 Theorem [7, p. 5] *The existence of a 1-rotational k-factorisation of a complete graph under the action of a group G with identity* 1 *is equivalent to the existence of a k-starter in G.*

Proof
Suppose that $\mathcal{F}$ is a 1-rotational k-factorisation of $K_{\overline{G}}$. Let G be a group of order divisible by k. Assume G has odd order when $k = 1$. Take an arbitrary F in $\mathcal{F}$. By Theorem 6.1 the stabilizer G_F has order k and its G-orbit is just $\mathcal{F}$. For any element x of $G - \{1\}$, let F' be the factor of $\mathcal{F}$ where 1 and x are adjacent. Of course $F = gF'$ for a suitable $g \in G$ so that $[g, gx]$ is an edge of F. It is then obvious that $x = (xg)g^{-1}$ appears in ΔF. So, every element of $G - \{1\}$ is covered by ΔF. Then by Definition 6.3 (i) and (ii), F is a k-starter in G.
Conversely, assume that F is k-starter in G and let $\mathcal{F}$ be its orbit under G. We must show that $\mathcal{F}$ is a 1- rotational k-factorisation of $K_{\overline{G}}$. For every $g \in G$, it is obvious that $[\infty, g]$ is a edge of $(gx^{-1})F$ where x is an arbitrary neighbour of ∞ in F. Now, given $[a, b] \in E(K_{\overline{G}})$ with $a \neq \infty \neq b$, there is an edge $[x, y]$ of F such that $b^{-1}a = y^{-1}x$ by definition of a k-starter in G. This implies that $[(ax^{-1})x, (ax^{-1})y] = [a, b]$ and hence we have that $[a, b]$ is an edge if $(x^{-1}a)F \in \mathcal{F}$. So, every edge of $K_{\overline{G}}$ appears at least once as an edge of some k-factor of $\mathcal{F}$. On the other hand, considering that the G-stabilizer of F has order k, we have $|\mathcal{F}| = |G|/k$ and hence, by the pigeon-hole principle,

we can replace "at least once" with "exactly once". This means that $\mathcal{F}$ is a k-factorisation of $K_{\overline{G}}$. Obviously, it is 1-rotational under the action of G. ∎

From the proof of the above theorem we have that any factor of a 1-rotational k-factorisation $\mathcal{F}$ of $K_{\overline{G}}$ is a k-starter in G. The factor of $\mathcal{F}$ in which 1 and ∞ are adjacent is called a *normalized k-starter.*

The following section will describe the case of a 1-rotational 2-factorisation.

6.3 1-rotational 2-factorisations of complete graphs

6.5 Definition Let G be a group of even order and let F be a 2-factor of $K_{\overline{G}}$. Then F is a 2-starter in G if the following conditions are satisfied:
(i) The only non-identity element of G fixing F is an involution;
(ii) ΔF covers each element of $G - \{1\}$ exactly twice.

It is clear that the 2-factors of a 1-rotational 2-factorisation of a complete graph are pairwise isomorphic . The next proposition is about the structure of a 2-factor of $K_{\overline{G}}$ that is fixed by an involution of G. So, in particular, any 2-starter of G has such a structure.

6.6 Proposition [7, p. 7] *Let G be a group of even order and let F be a 2-factor of $K_{\overline{G}}$ that is fixed by an involution j of G. Then the set of cycles of F is the disjoint union of the sets $\{A\}, B, \mathcal{C}, \mathcal{D}$ where*

(a) *A is the cycle of F through ∞. It has odd length and j acts on it as a reflection. It has the form*

$$A = (a_1, a_2, \cdots, a_k, \infty, a_kj, \cdots, a_2j, a_1j)$$

for a suitable $k \geq 1$ and suitable elements $a_1, \cdots, a_k$ of G belonging to pairwise distinct left cosets of $\{1, j\}$ in G.

(b) *$\mathcal{B}$ is the set of cycles of F on which j acts as a rotation so that each $B \in \mathcal{B}$ has the form*

$$B = (b_1, b_2, \cdots, b_l, b_1j, b_2j, \cdots, b_lj)$$

for a suitable $l \geq 2$ and suitable elements $b_1, \cdots, b_l$ of G belonging to pairwise distinct left cosets of $\{1, j\}$ in G.

(c) *$\mathcal{C}$ is the set of cycles of $F - \{A\}$ on which j acts as a reflection so that each $c \in \mathcal{C}$ has the form*

$$C = (c_1, c_2, \cdots, c_m, c_m j, \cdots, c_2 j, c_1 j)$$

for a suitable $m \geq 2$ and suitable elements $c_1, c_2, \cdots, c_m$ of G belonging to pairwise distinct left cosets of $\{1, j\}$ in G.

(d) *$\mathcal{D}$ is the set of cycles of F having the trivial G-stabilizer so that $D \in \mathcal{D}$ implies that D and Dj are distinct cycles of $\mathcal{D}$.*

In the next section, we present the case that is non-Hamiltonian. So, we will specially restrict our attention to the construction of non-Hamiltonian 1-rotational 2-factorisations as examples using the concept of a 2-starter.

6.4 1-rotational 2-factorisations under the dihedral group

Consider the dihedral group of order $2n$ with defining relations

$$D_{2n} = < x, y | x^n = y^2 = 1; yx = x^{-1} y > .$$

To obtain factors, Buratti and Rinaldi [7] provide the following theorem that is a development of 1-rotational 2-factorisations under the action of D_{2n} .

6.7 Theorem ([7], p. 12) *There exists a 1-rotational 2-factorisation of $K_{\overline{D}_{m-1}}$ if and only if $m = 4t + 3$ for some positive integer t. It is generated by a 2-factor of the form*

$$F_0 = \{(\infty, 1, y)\} \cup \{(x^{a_i}, x^{b_i}, x^{b_i} y, x^{a_i} y) | i = 1, \cdots, t\}$$

where $\{[a_i, b_i] | i = 1, \cdots, t\}$ is a starter in $\overline{\mathbb{Z}_{2t-1}}$. Conversely, every 2-factor of the above form generates a 1-rotational 2-factorisation of $K_{\overline{D}_{m-1}}$.

To apply Theorem 6.7, using as a starter of $\mathbb{Z}_{2t+1}$ the patterned starter $\{[i,-i]|i=1,\cdots,t\}$, the 2-factorisation $\mathcal{F}$ admits as an automorphism the involution β which acts on $D_{4t+2}\cup\{\infty\}$ fixing ∞, y and 1 and interchanging x^i with x^{-i} and $x^i y$ with $x^{-i}y$, for $i=1,\cdots,t$. So, if $T=\{\tau_g|g\in D_{4t+2}\}$ is the automorphism group of $\mathcal{F}$ consisting of all right translations under D_{4t+2} (so τ_g is defined by $\tau_g(a)=ga$ for every $a\in D_{4t+2}\cup\{\infty\}$). The group $<T,\beta>$ generated by T and β is an automorphim group of $\mathcal{F}$. This group is isomorphic to D_{8t+4} and its cyclic subgroup of order $4t+2$ is generated by the bijection $\alpha=\tau_{xy}\circ\beta$. Also, $<\alpha>$ acts regularly on D_{4t+2}. Indeed, up to isomorphisms, $\mathcal{F}$ can be presented as a 2-factorisation of K_{4t+3}, that is, 1-rotational under $\mathbb{Z}_{4t+2}$. It is obtainable by the 2-starter

$$\{(\infty,0,2t+1)\}\cup\{(i,-i,i+2t+1,-i+2t+1)|i=1,...,t\}.$$

In the next section we shall use this method in some examples and with a suitable multiplication identify the groupoids that can be generated.

6.5 Groupoids of 1-rotational 2-factorisations

We will show that groupoids from 1-rotational 2-factorisation are not necessarily groups. Here, we show that there exists a 2-factorisation of K_n obtainable using the patterned 2-starter of $\mathbb{Z}_{n-1}$ that generates a non associative loop. In general, consider the complete graph K_{4t+3} defined on the set $\mathbb{Z}_{4t+2}\cup\{\infty\}$.

First, if $t=1$, we have $\mathbb{Z}_6\cup\{\infty\}$.

6.5.1 Groupoid of 1-rotational 2-factorisation of $K_{\mathbb{Z}_6\cup\{\infty\}}$

The decomposition is indeed obtainable by developing the 2-starter in the following way.

$$\mathcal{B}=\{(\infty,0,2t+1)\}\cup\{(i,-i,i+2t+1,-i+2t+1)|i=1,...,t\}.$$

Since $t=1$, the factors are

$$\mathcal{B}=\{(\infty,0,2(1)+1)\}\cup\{(i,-i,i+2(1)+1,-i+2(1)+1)|i=1\}.$$

$\mathcal{B} = \{(\infty, 0, 3)\} \cup \{(1, -1, 1+5, -1+5)\} = \{(\infty, 0, 3)\} \cup \{(1, 5, 4, 2)\}$.

We obtain three isomorphic factors, namely:
$B_0 = \{(\infty, 0, 3), (1, 5, 4, 2)\}$, $B_1 = B_0 + 1 = \{(\infty, 1, 4), (2, 0, 5, 3)\}$, $B_2 = B_0 + 2 = \{(\infty, 2, 5), (3, 1, 0, 4)\}$.

These factors give us the following directed factors.
$F_3 = \{(0,3), (3,\infty), (\infty,0), (1,2), (2,4), (4,5), (5,1)\}$;
$F_\infty = \{(0,\infty), (\infty,3), (3,0), (1,5), (5,4), (4,2), (2,1)\}$;
$F_2 = \{(0,2), (2,3), (3,5), (5,0), (1,4), (4,\infty), (\infty,1)\}$;
$F_5 = \{(0,5), (5,3), (3,2), (2,0), (1,\infty), (\infty,4), (4,1)\}$;
$F_1 = \{(0,1), (1,3), (3,4), (4,0), (2,5), (5,\infty), (\infty,2)\}$;
$F_4 = \{(0,4), (4,3), (3,1), (1,0), (2,\infty), (\infty,5), (5,2)\}$. (See Figures 6.1, 6.2 and 6.3.)

Also, the Cayley table of this loop, which we denote by Q_8, is given in Table 6.1.

0 0

1 ∞ 1 ∞

F_3 F_∞.

2 5 2 5

3 4 3 4

Figure 6.1: Factors F_3 and F_∞.

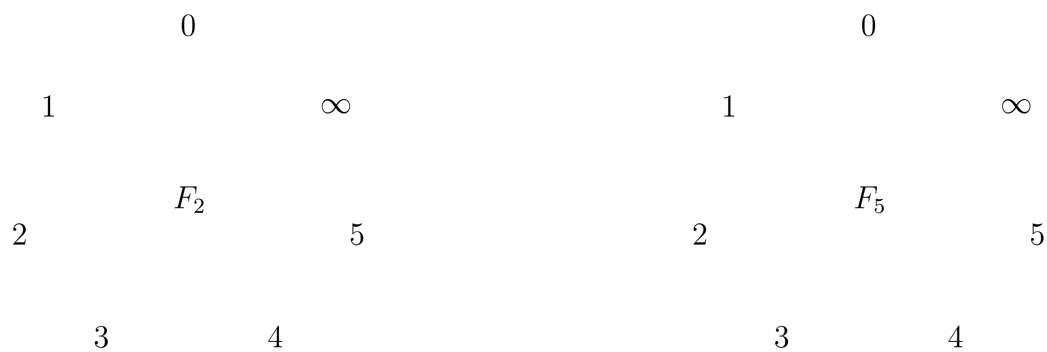

Figure 6.2: Factors F_2 and F_5.

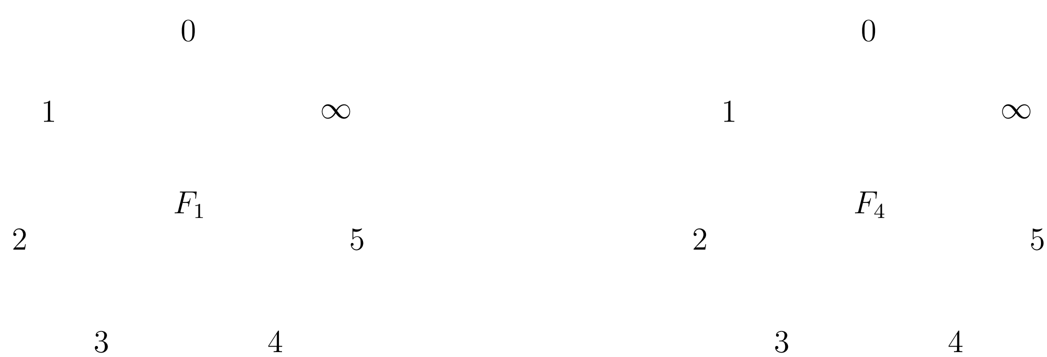

Figure 6.3: Factors F_1 and F_4.

$*_0$	0	1	2	3	4	5	∞
0	0	1	2	3	4	5	∞
1	1	3	4	2	0	∞	5
2	2	5	3	4	∞	0	1
3	3	4	5	∞	1	2	0
4	4	0	∞	5	3	1	2
5	5	∞	0	1	2	3	4
∞	∞	2	1	0	5	4	3

Table 6.1: Groupoid of 1-rotational factorisation of $K_{\mathbb{Z}_6\cup\{\infty\}}$

From the Cayley Table 6.1, we have that in Q_8,

$(1 *_0 2) *_0 3 = 4 *_0 3 = 5,$ $\quad 1 *_0 (2 *_0 3) = 1 *_0 4 = 0$ and $0 \neq 5$.

Therefore, the loop Q_8 is not a group.

As Q_8 is not a group, let us now explore the possibility of quasi-asssociativity of its Cayley sets.

6.5.2 Quasi-associativity in Q_8

Consider Q_8. Let us first present all its Cayley sets and check their quasi-associativity.

1. **Caylet-sets of Q_8**

   By Definition 3.1 of a Cayley set in a loop and from its Cayley table, we have 8 Cayley sets

   $\emptyset,\ S_1 = \{1,4\},\ S_2 = \{2,5\},\ S_3 = \{3,\infty\},\ S_1 \cup S_2 = \{1,4,2,5\},$

   $S_1 \cup S_3 = \{1,4,3,\infty\},\ S_2 \cup S_3 = \{2,5,3,\infty\},$

   $S_1 \cup S_2 \cup S_3 = \{1,4,2,5,3,\infty\} = V(K_{\mathbb{Z}_6\cup\{\infty\}}) - \{0\}.$

2. **Quasi-associativity**

   6.8 Proposition *All proper Cayley sets of Q_8, the loop defined in Table 6.1, are not quasi-associative.*

Proof
It is enough to identify $a, b \in Q_8$ and $s \in S$, where S is a Cayley set such that $(ab)s = a(bk)$ where $k \notin S$. The following tabulates a, b, s and k. (See Table 6.2.) ∎

Cayley-set	a	b	s	k
S_1	2	3	1	2
S_2	1	3	2	4
S_3	4	5	3	1
$S_1 \cup S_2$	1	2	1	3
$S_1 \cup S_3$	1	4	3	5
$S_2 \cup S_3$	2	5	3	4

Table 6.2: The non-quasi-associativity table of Cayley sets of Q_{10}

6.5.3 Groupoid of 1-rotational 2-factorisation of $K_{\mathbb{Z}_{10}\cup\{\infty\}}$

The decomposition is obtainable by developing a 2-starter in the following way. $\mathcal{B} = \{(\infty, 0, 2t+1)\} \cup \{(i, -i, i+2t+1, -i+2t+1)|i = 1, ..., t\}$.

In our case $t = 2$ we consider $\mathbb{Z}_{10} \cup \{\infty\}$. Then factors are

$\mathcal{B} = \{(\infty, 0, 2(2)+1)\} \cup \{(i, -i, i+2(2)+1, -i+2(2)+1)|i = 1, 2\}$, i.e.,

$\mathcal{B} = \{(\infty, 0, 5)\} \cup \{(i, -i, i+5, -i+5)|i = 1, 2\}$.

When i varies from 1 to 2, we get

$\mathcal{B} = \{(\infty, 0, 5)\} \cup \{(1, -1, 6, 4) \cup (2, -2, 7, 3)\}$.

So, we finally have

$\mathcal{B} = \{(\infty, 0, 5)\} \cup \{(1, 9, 6, 4) \cup (2, 8, 7, 3)\}$.

Therefore, we have the factors

$B_0 = \{(\infty, 0, 5), (1, 9, 6, 4), (2, 8, 7, 3)\}$

By use of the cosets we obtain the other factors as follows:

$$
\begin{aligned}
B_1 &= B_0 + 1 = \{(\infty, 0+1, 5+1),\ (1+1, 9+1, 6+1, 4+1),\\
&\quad (2+1, 8+1, 7+1, 3+1)\};\\
B_2 &= B_0 + 2 = \{(\infty, 0+2, 5+2), (1+2, 9+2, 6+2, 4+2),\\
&\quad (2+2, 8+2, 7+2, 3+2)\};\\
B_3 &= B_0 + 3 = \{(\infty, 0+3, 5+3), (1+3, 9+3, 6+3, 4+3)\\
&\quad (2+3, 8+3, 7+3, 3+3)\};\\
B_4 &= B_0 + 4 = \{(\infty, 0+4, 5+4), (1+4, 9+4, 6+4, 4+4),\\
&\quad (2+4, 8+4, 7+4, 3+4)\}.
\end{aligned}
$$

So, the obtained five isomorphic factors are

$B_0 = \{(\infty, 0, 5), (1, 9, 6, 4), (2, 8, 7, 3)\}$;

$B_1 = B_0 + 1 = \{(\infty, 1, 6), (2, 0, 7, 5), (3, 9, 8, 4)\}$;

$B_2 = B_0 + 2 = \{(\infty, 2, 7), (3, 1, 8, 6), (4, 0, 9, 5)\}$;

$B_3 = B_0 + 3 = \{(\infty, 3, 8), (4, 2, 9, 7), (5, 1, 0, 6)\}$;

$B_4 = B_0 + 4 = \{(\infty, 4, 9), (5, 3, 0, 8), (6, 2, 1, 7)\}$.

Likewise the corresponding directed factors are

$$
\begin{aligned}
F_5 &= \{(0,5), (5,\infty), (\infty,0), (1,4), (4,6), (6,9), (9,1), (2,3), (3,7),\\
&\quad (7,8), (8,2)\};\\
F_\infty &= \{(0,\infty), (\infty,5), (5,0), (1,9), (9,6), (6,4), (4,1), (2,8), (8,7),\\
&\quad (7,3), (3,2)\};\\
F_2 &= \{(0,2), (2,5), (5,7), (7,0), (1,6), (6,\infty), (\infty,1), (3,4), (4,8),\\
&\quad (8,9), (9,3)\};\\
F_7 &= \{(0,7), (7,5), (5,2), (2,0), (1,\infty), (\infty,6), (6,1), (3,9), (9,8),\\
&\quad (8,4), (4,3)\};\\
F_4 &= \{(0,4), (4,5), (5,9), (9,0), (1,3), (3,6), (6,8), (8,1), (2,7),\\
&\quad (7,\infty), (\infty,2)\}\\
F_9 &= \{(0,9), (9,5), (5,4), (4,0), (1,8), (8,6), (6,3), (3,1), (2,\infty),\\
&\quad (\infty,7), (7,2)\};\\
F_1 &= \{(0,1), (1,5), (5,6), (6,0), (2,4), (4,7), (7,9), (9,2), (3,8),\\
&\quad (8,\infty), (\infty,3)\}\\
F_6 &= \{(0,6), (6,5), (5,1), (1,0), (2,9), (9,7), (7,4), (4,2), (3,\infty),\\
&\quad (\infty,8), (8,3)\};\\
F_3 &= \{(0,3), (3,5), (5,8), (8,0), (1,2), (2,6), (6,7), (7,1), (4,9),\\
&\quad (9,\infty), (\infty,4)\};\\
F_8 &= \{(0,8), (8,5), (5,3), (3,0), (1,7), (7,6), (6,2), (2,1), (4,\infty),\\
&\quad (\infty,9), (9,4)\}.\ \text{(See Figures 6.4, 6.5, 6.6, 6.7, 6.8.)}
\end{aligned}
$$

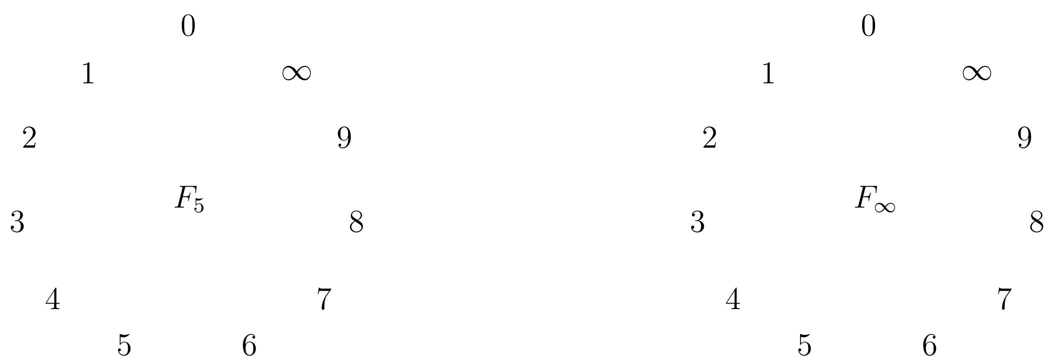

Figure 6.4: Factors F_5 and F_∞.

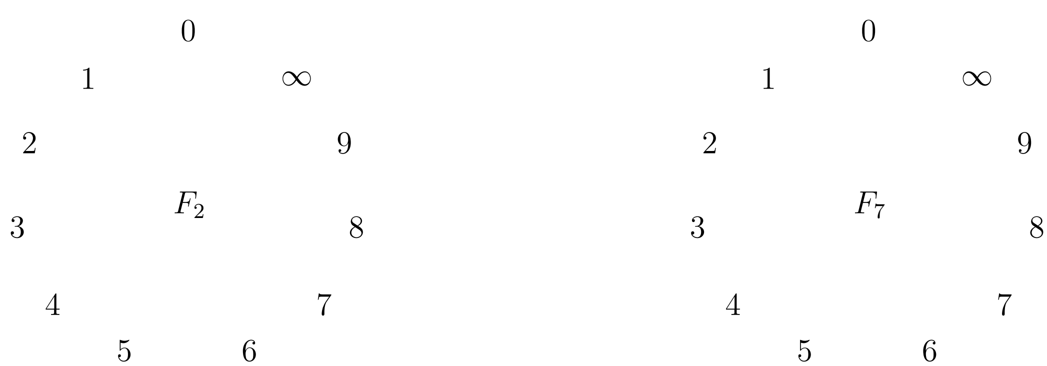

Figure 6.5: Factors F_2 and F_7.

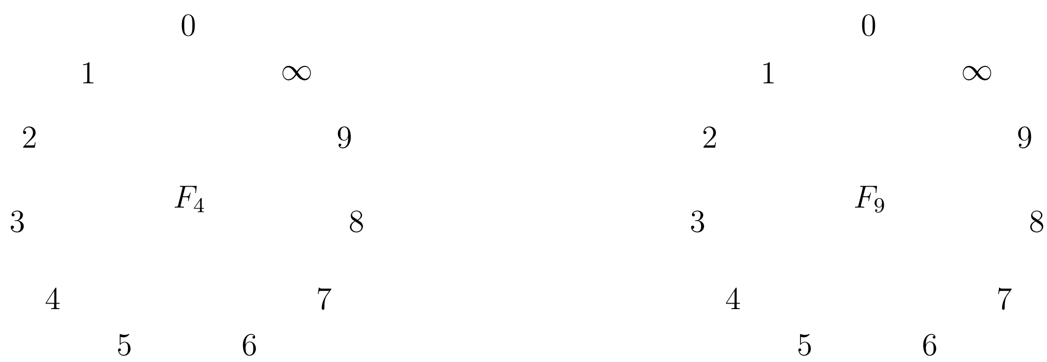

Figure 6.6: Factors F_4 and F_9.

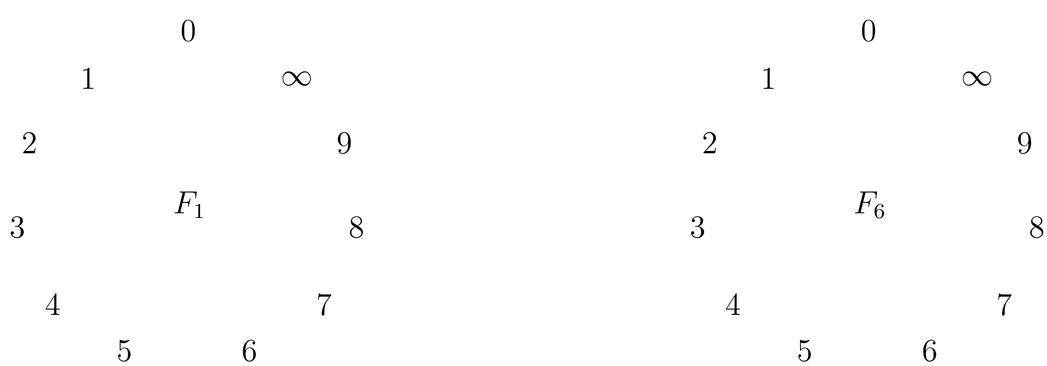

Figure 6.7: Factors F_1 and F_6.

The Cayley graph of this loop, which is denoted by Q_9, is presented in Table 6.3.

$*_0$	0	1	2	3	4	5	6	7	8	9	∞
0	0	1	2	3	4	5	6	7	8	9	∞
1	1	5	6	2	3	4	0	∞	7	8	9
2	2	4	5	6	7	3	9	0	1	∞	8
3	3	8	4	5	6	7	∞	9	0	1	2
4	4	7	8	9	5	6	2	3	∞	0	1
5	5	6	7	8	9	∞	1	2	3	4	0
6	6	0	∞	7	8	9	5	1	2	3	4
7	7	9	0	1	∞	8	4	5	6	2	3
8	8	∞	9	0	1	2	3	4	5	6	7
9	9	2	3	∞	0	1	7	8	4	5	6
∞	∞	3	1	4	2	0	8	6	9	7	5

Table 6.3: Groupoid of 1-rotational factorisation of $K_{\mathbb{Z}_{10}\cup\{\infty\}}$

From this Cayley table, we see there exist 1, 5, 9 such that

$(1 *_0 5) *_0 9 = 4 *_0 9 = 0, \qquad 1 *_0 (5 *_0 9) = 1 *_0 4 = 3.$

So, Q_9 is not a group.

We are now interested in checking the quasi-associativity of its Cayley sets.

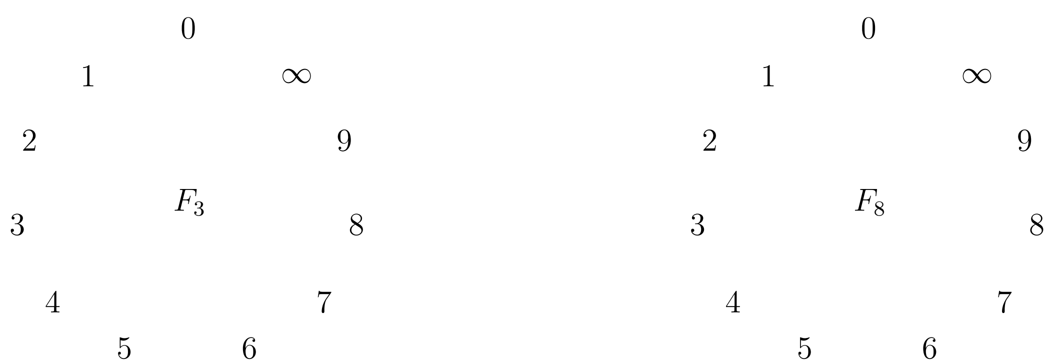

Figure 6.8: Factors F_3 and F_8.

6.5.4 Quasi-associativity in Q_9

We explore the possibility of quasi-Cayley graphs defined on Q_9, the loop obtained from 1-rotational factors of $K_{\mathbb{Z}_{10}\cup\{\infty\}}$. Thus we must first present all its Cayley sets and show their quasi- associativity.

1. **Caylet sets of Q_9**

   By Definition 3.1 of a Cayley set in a loop and from the Cayley table 6.3, we have 32 Cayley sets:

   $\emptyset,\ S_1 = \{1, 6\},\ S_2 = \{2, 7\},\ S_3 = \{3, 8\},\ S_4 = \{4, 9\},\ S_5 = \{5, \infty\},$
   $S_1 \cup S_2 = \{1, 6, 2, 7\},\ S_1 \cup S_3 = \{1, 6, 3, 8\},\ S_1 \cup S_4 = \{1, 6, 4, 9\},$
   $S_1 \cup S_5 = \{1, 6, 5, \infty\},\ S_2 \cup S_3 = \{2, 7, 3, 8\},\ S_2 \cup S_4 = \{2, 7, 4, 9\},$
   $S_2 \cup S_5 = \{2, 7, 5, \infty\},\ S_3 \cup S_4 = \{3, 8, 4, 9\},\ S_3 \cup S_5 = \{3, 8, 5, \infty\},$
   $S_4 \cup S_5 = \{4, 9, 5, \infty\},\ S_1 \cup S_2 \cup S_3 = \{1, 6, 2, 7, 3, 8\},$
   $S_1 \cup S_2 \cup S_4 = \{1, 6, 2, 7, 4, 9\},\ S_1 \cup S_2 \cup S_5 = \{1, 6, 2, 7, 5, \infty\},$
   $S_1 \cup S_3 \cup S_4 = \{1, 6, 3, 8, 4, 9\},\ S_1 \cup S_3 \cup S_5 = \{1, 6, 3, 8, 5, \infty\},$
   $S_1 \cup S_4 \cup S_5 = \{1, 6, 4, 9, 5, \infty\},\ S_2 \cup S_3 \cup S_4 = \{2, 7, 3, 8, 4, 9\},$
   $S_2 \cup S_3 \cup S_5 = \{2, 7, 3, 8, 5, \infty\},\ S_2 \cup S_4 \cup S_5 = \{2, 7, 4, 9, 5, \infty\},$
   $S_3 \cup S_4 \cup S_5 = \{3, 8, 4, 9, 5, \infty\},\ S_1 \cup S_2 \cup S_3 \cup S_4 = \{1, 6, 2, 7, 3, 8, 4, 9\},$
   $S_1 \cup S_2 \cup S_3 \cup S_5 = \{1, 6, 2, 7, 3, 8, 5, \infty\},$
   $S_1 \cup S_2 \cup S_4 \cup S_5 = \{1, 6, 2, 7, 4, 9, 5, \infty\},$
   $S_1 \cup S_3 \cup S_4 \cup S_5 = \{1, 6, 3, 8, 4, 9, 5, \infty\},$
   $S_2 \cup S_3 \cup S_4 \cup S_5 = \{2, 7, 3, 8, 4, 9, 5, \infty\},$
   $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 = \{1, 6, 2, 7, 3, 8, 4, 9, 5, \infty\} = Q_8 - \{0\}.$

2. **Quasi-associativity**

   6.9 Proposition *All proper Cayley sets of Q_9, the loop defined in Table 6.3, are not quasi-associative.*

   Proof
   It is enough to identify $a, b \in Q_9$ and $s \in S$, where S is a Cayley set such that $(ab)s = a(bk)$ where $k \notin S$. Again the following tabulates a, b, s and k. (See Figures 6.4.) ∎

Cayley-set	a	b	s	k
S_1	2	7	1	5
S_2	3	8	2	1
S_3	4	9	3	6
S_4	5	∞	4	8
S_5	1	6	5	7
$S_1 \cup S_2$	9	8	1	9
$S_1 \cup S_3$	4	5	1	4
$S_1 \cup S_4$	3	5	1	2
$S_1 \cup S_5$	3	4	1	2
$S_2 \cup S_3$	4	5	7	5
$S_2 \cup S_4$	3	5	2	3
$S_2 \cup S_5$	6	4	2	4
$S_3 \cup S_4$	6	1	3	∞
$S_3 \cup S_5$	9	1	3	7
$S_4 \cup S_5$	3	6	4	2
$S_1 \cup S_2 \cup S_3$	2	6	3	5
$S_1 \cup S_2 \cup S_4$	2	6	4	3
$S_1 \cup S_2 \cup S_5$	1	2	5	9
$S_1 \cup S_3 \cup S_4$	1	3	9	5
$S_1 \cup S_3 \cup S_5$	1	4	∞	7
$S_1 \cup S_4 \cup S_5$	1	4	∞	7
$S_2 \cup S_3 \cup S_4$	3	1	9	5
$S_2 \cup S_3 \cup S_5$	3	2	5	1
$S_2 \cup S_4 \cup S_5$	3	2	5	1
$S_3 \cup S_4 \cup S_5$	3	2	5	1
$S_1 \cup S_2 \cup S_3 \cup S_4$	2	1	1	5
$S_1 \cup S_2 \cup S_3 \cup S_5$	2	1	5	4
$S_1 \cup S_2 \cup S_4 \cup S_5$	2	1	4	3
$S_1 \cup S_3 \cup S_4 \cup S_5$	1	3	∞	7
$S_2 \cup S_3 \cup S_4 \cup S_5$	2	3	7	1

Table 6.4: The non-quasi-associativity table of Cayley sets of Q_9

Chapter 7

Regular factorisations and their groupoids

7.1 Introduction

We now consider groupoids that are generated by regular factorisations. To present regular 2-factorisations, we need to recall the method of partial differences defined in Chapter 6. By using this, it is possible to construct regular 2-factorisations of a complete graph, as done by Buratti and Rinaldi [6], [9]. It is proved that, for a given group G, a G-regular 2-factorisation of K_G exists if and only if G has a suitable 2-starter in G. From this, we obtain the G-regular 2-factorisation and discuss their groupoids.

In this chapter, all groups are additive abelian of odd order.

7.2 Regular factorisations

Denote by K_G the complete graph with vertex-set $V(K_G) = G$. Consider the regular action of G on $V(K_G)$ defined by $x \to x + g$, for any $(x, g) \in V(K_n) \times G$.

7.1 Definition A cycle decomposition $\mathcal{D}$ of K_G is *regular under the action of* G if we have $C + g \in \mathcal{D}$, for any $C \in \mathcal{D}$ and for any $g \in G$.

In the same manner, if $\mathcal{F}$ is a 2-factorisation of K_G, we say that $\mathcal{F}$ is *regular* under the action of G, or G-*regular*, if we have $F + g \in \mathcal{F}$, for any $F \in \mathcal{F}$ and any $g \in G$.

7.3 Regular 2-factorisations and 2-starters in groups of odd order

7.2 Definition (a) Consider H a subgroup of G. A system of distinct representatives for the left (respectively right) cosets of H in G is called a *left transversal* (respectively a *right transversal*) for H in G.
(b) Given a k-cycle $C = (x_0, x_1, ..., x_{k-1})$ with vertices in G, the *orbit* of C is the set defined by

$$\Omega_C = \{C + g,\ g \in G\}.$$

(c) Consider the group G acting on Ω_G transitively. The *stabilizer* of C under the action of G is the subgroup G_C of G defined by

$$G_C = \{g \in G | C + g = C\}.$$

The factorisations we discuss depend on the following proposition.

7.3 Proposition ([6], p. 247) *Let $C = (x_0, x_1, \cdots, x_{lk-1})$ be an lk-cycle with vertices in G and let k be the order of G_C. Then there is an element $g \in G$ of order t such that the following condition holds:*

$$x_{i+l} - x_i = g,\ \ for\ all\ i \tag{7.1}$$

or, more explicitly,

$C = (x_0, x_1, \cdots, x_{l-1}, x_0 + g, x_1 + g, \cdots, x_{l-1} + g, \cdots, x_0 + (l-1)g, x_1 + (l-1)g, \cdots, x_{l-1} + (k-1)g)$.

Conversely, if g is an element of G of order k, a sequence $C = (x_0, x_1, \cdots, x_{lk-1})$ of lk vertices of G satisfying (7.1) is an lk-cycle with $|G_C| = k$, if the following extra conditions are satisfied:
(i) l is the least divisor of lk such that $x_{i+l} - x_i$ does not depend on i;
(ii) $x_0, x_1, \cdots, x_{l-1}$ lie in pairwise distinct left cosets of $< g >$ in G.

Let C be a cycle as in Proposition 7.3. We define the *list of partial differences* of C to be the multiset

$$\partial C = \{\pm(x_{i+1} - x_i) | 0 \leq i \leq l\}$$

and we set

$$\phi(C) = \{x_0, x_1, \cdots, x_{l-1}\}.$$

If the stabilizer of C is trivial, then ∂C coincides with ΔC, the list of differences of C in the usual sense. In this case $\phi(C) = V(C)$, the set of vertices of C. More generally, if $\mathcal{C} = \{C_1, \cdots, C_q\}$ is a collection of cycles (in particular, a 2-regular graph) with vertices in G, then we set $\partial\mathcal{C} = \partial C_1 \cup \partial C_2 \cup \cdots \cup \partial C_q$ and $\phi(\mathcal{C}) = \phi(C_1) \cup \phi(C_2) \cup \cdots \cup \phi(C_q)$ (where in the union the elements have to be counted with their multiplicity).
The G-*orbit* of a cycle C is the set $\Omega_G(C)$ of all *distinct* cycles in the collection $\{C + g | g \in G\}$. Its size (or *length*) is $|G : G_C|$ the index of the stabilizer of C under G and $\Omega_G(C) = \{C + k | k \in T\}$ where T is the right transversal for G_C in G.

The key feature of regular factorisation are partial differences, which is given in Definition 6.2. Partial differences are used in the following way.

7.4 Proposition ([6], p. 247) *Let $\mathcal{C} = \{C_1, \cdots, C_q\}$ be a collection of cycles with vertices in G. Then*

$$\mathcal{D} = \bigcup_{i=1}^{q} \Omega_G(C_i)$$

is a cycle decomposition of K_G if and only if $\partial\mathcal{C} = G \setminus \{0\}$, where $\Omega_G(C_i)$ is the G-orbit of cycle C_i.

Proof
Denote $n = |G|$. For $i = 1, 2, \cdots, q$, let l_i be the length of C_i and let d_i be the order of the G-stabilizer of C_i. Assume $\mathcal{D}$ is a cycle decomposition of K_G. The size of $\Omega_G(C_i)$ is n/d_i, so that the number $|E(K_G)| = n(n-1)/2$ of edge covered by $\mathcal{D}$ may be also expressed as $n \sum_{i=1}^{q}(l_i/d_i)$. It follows that $2\sum_{i=1}^{q}(l_i/d_i) = n - 1$. Now note that these two sides of the last equality are the sizes of $\partial\mathcal{C}$ and $G - \{0\}$ respectively. Therefore, it is enough to show that any $a \in G \setminus \{0\}$ appears at least once in $\partial\mathcal{C}$. Given any non-zero element $a \in G$, we may claim by assumption that $[0, a]$ is a edge of $C_i + k$ for a suitable pair $(i, k) \in \{1, 2, \cdots, q\} \times G$. It follows that $[0, a] = [x, y] + k =$

$[x+k, y+k]$. where $C_i = (x, y, \cdots)$. This implies that $k = -x$ and hence $a = y + k = y - x \in \partial C_i \subset \partial \mathcal{C}$.
Conversely, assume that $\partial \mathcal{C} = G - \{0\}$. So we have that
$|\partial \mathcal{C}| = 2\sum_{i=1}^{q}(l_i/d_i) = n - 1,$

$$\Rightarrow \quad n\sum_{i=1}^{q}(l_i/d_i) = n(n-1)/2$$
$$\Rightarrow \quad n\sum_{i=1}^{q}(l_i/d_i) = |E(K_G)|.$$

Note that by the action of G on $\mathcal{D}$, the cycles are disjoint.
The left hand side of this equality gives the number of edges covered by the cycles of $\mathcal{D}$. So, to prove that each edge of K_G is covered by the cycles of $\mathcal{D}$ exactly once, it is sufficient to prove that this happens at least once.
Let $[a, b] \in E(K_G)$. By using partial differences, there is a suitable i such that $C_i = (x, y, \cdots)$ with $x - y = a - b$. This implies $a = x - y + b$. Then we have

$$[a, b] = [x - y + b, b] = [x - y + b, y - y + b] = [x, y] + (-y + b).$$

We therefore claim that $[a, b]$ is an edge of $C_i + (-y + b) \in \Omega_G(C_i) \subset \mathcal{D}$. ∎

Proposition 7.4 is similar to Theorem 6.4. However in Proposition 7.4 we are primarily considering 2-starters and the action is regular.

In what follows we introduce a concept which allows us to describe algebraically any G-regular 2-factorisation of a complete graph K_G.

7.5 Definition A 2-starter in G is a collection $\sum = \{S_1, \cdots, S_q\}$ of 2-regular graphs with vertices in G satisfying the following conditions:
(i) $\partial S_1 \cup \cdots \cup \partial S_q = G - \{0\}$;
(ii) $\phi(S_i)$ is a left transversal for some subgroup H_i of G containing the stabilizers of all cycles of $S_i, i = 1, \cdots, q$.

We now present, without proof, the characterisation of a 2-regular factorisation by a 2-starter as described by Buratti and Fra.

7.6 Theorem ([6], p. 248) *The existence of a G-regular 2-factorisation of K_G is equivalent to the existence of a 2-starter in G.*

The heart of our decomposition is the theorem below.

7.7 Theorem ([6], p. 250) *For any group G of odd order, a G-regular 2-factorisation of K_G exists.*

Proof
Consider $G - \{0\} = X \cup -X$. Of course, X and $-X$ are disjoint because G has odd order. For any $x \in X$ denote by k_x the order of x in G and by S_x the cycle $(0, x, \cdots, (k_x - 1)x)$. Observe that $\partial S_x = \pm x$, and that $\phi(S_x) = 0$ is a left transversal for G in G. In addition, theses cycles have to be edge-disjoint. Therefore, the set $\sum = \{S_x | x \in X\}$ is a 2-starter in G. ∎

Let us now investigate the kind of groupoids that are generated by regular 2-factorisations.

7.4 Groupoids from regular 2-factorisations

We will check the groupoids obtained from regular 2-factorisations of K_7 as the case for which factors are isomorphic. We will also consider K_{15} as the case for which factors are non-isomorphic. With this as background, we will characterise their results by a theorem.

7.4.1 Groupoids of regular isomorphic factorisations

Regular 2-factorisation of K_7 and its groupoid

Let us present the regular 2-factorisation of K_7.

By Theorem 7.7, we write

$$\begin{aligned} G - \{0\} &= \{1,2,3,4,5,6\} \\ &= \{1,2,3\} \cup \{6,5,4\} \\ &= X \cup -X \text{ where } X = \{1,2,3\} \text{ and } -X = \{4,5,6\}\,. \end{aligned}$$

For each $x \in X$ denote by k_x the order of x in $\mathbb{Z}_7$. The cycle

$S_x = (0, x, \cdots, (k_x - 1)x)$ defining the factors gives
$S_1 = (0, 1 \cdot 1, 2 \cdot 1, 3 \cdot 1, 4 \cdot 1, 5 \cdot 1, 6 \cdot 1) = (0, 1, 2, 3, 4, 5, 6)$,
$S_2 = (0, 2 \cdot 1, 2 \cdot 2, 2 \cdot 3, 2 \cdot 4, 2 \cdot 5, 2 \cdot 6) = (0, 2, 4, 6, 1, 3, 5)$,
$S_3 = (0, 3 \cdot 1, 3 \cdot 2, 3 \cdot 3, 3 \cdot 4, 3 \cdot 5, 3 \cdot 6) = (0, 3, 6, 2, 5, 1, 4)$.
The partial differences are respectively $\partial S_1 = \{1, 6\}$, $\partial S_2 = \{2, 5\}$, $\partial S_3 = \{3, 4\}$.

It is clear that we have $\sum_i \partial S_i = \{1,2,3,4,5,6\} = G - \{0\}$.
The directed factors are

$F_1 = \{(0,1),(1,2),(2,3),(3,4),(4,5),(5,6),(6,0)\}$;
$F_6 = \{(0,6),(6,5),(5,4),(4,3),(3,2),(2,1),(1,0)\}$;
$F_2 = \{(0,2),(2,4),(4,6),(6,1),(1,3),(3,5),(5,0)\}$;
$F_5 = \{(0,5),(5,3),(3,1),(1,6),(6,4),(4,2),(2,0)\}$;
$F_3 = \{(0,3),(3,6),(6,2),(2,5),(5,1),(1,4),(4,0)\}$;
$F_4 = \{(0,4),(4,1),(1,5),(5,2),(2,6),(6,3),(3,0)\}$.
(See Figures 7.1, 7.2 and 7.3.)

Figure 7.1: Factors F_1 and F_6

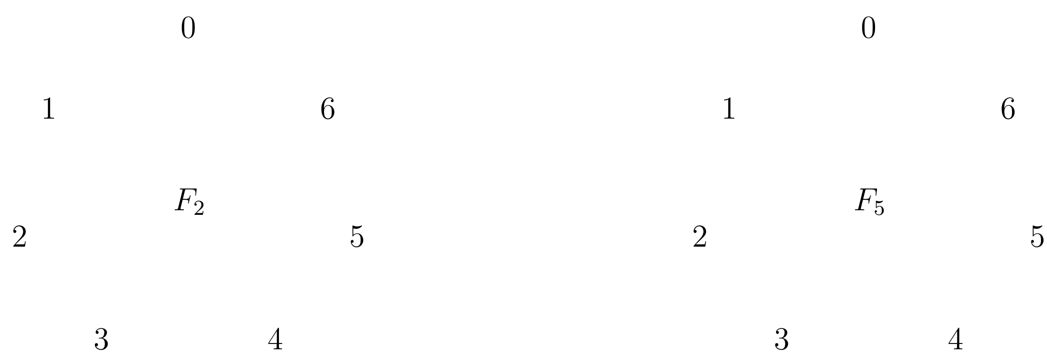

Figure 7.2: Factors F_2 and F_5

Figure 7.3: Factors F_3 and F_4

The Cayley table that represents the generated loop, which we denote by Q_{10}, is given by the Cayley Table 7.1 of Q_{10} and coincides with that of $\mathbb{Z}_7$.

$*_0$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Table 7.1: Groupoid of regular 2-factorisation of K_7

It is clear that $Q_{10} = \mathbb{Z}_7$.

We now illustrate that this result does not change even when the factors are non-isomorphic. Let us take the complete graph $K_{\mathbb{Z}_{15}}$.

7.4.2 Groupoids of regular non isomorphic factorisations: the case of $K_{\mathbb{Z}_{15}}$

By Theorem 7.7, we obtain

$$\begin{aligned} G - \{0\} &= \{1,2,3,4,5,6,7,8,9,10,11,12,13,14\} \\ &= \{1,2,3,4,5,6,7\} \cup \{14,13,12,11,10,9,8\} \\ &= X \cup -X \text{ where } X = \{1,2,3,4,5,6,7\} \end{aligned}$$

The factors are

$$
\begin{aligned}
B_1 &= (0,1,2\cdot 1,3\cdot 1,4\cdot 1,5\cdot 1,6\cdot 1,7\cdot 1,8\cdot 1,9\cdot 1,\\
&\quad 10\cdot 1,11\cdot 1,12\cdot 1,13\cdot 1,14\cdot 1)\\
&= (0,1,2,3,4,5,6,7,8,9,10,11,12,13,14)
\end{aligned}
$$

$B_2 = (0,2,4,6,8,10,12,14,1,3,5,7,9,11,13)$,
$B_3 = (0,3,6,9,12)(1,4,7,10,13)(2,5,8,11,14)$,
$B_4 = (0,4,8,12,1,5,9,13,2,6,10,14,3,7,11)$,
$B_5 = (0,5,10)(1,6,11)(2,7,12)(3,8,13)(4,9,14)$,
$B_6 = (0,6,12,3,9)(1,7,13,4,10)(2,8,14,5,11)$,
$B_7 = (0,7,14,6,13,5,12,4,11,3,10,2,9,1,8)$.

The partial differences are respectively $\partial B_1 = \{1,14\}$, $\partial B_2 = \{2,13\}$, $\partial B_3 = \{3,12\}$, $\partial B_4 = \{4,11\}$, $\partial B_5 = \{5,10\}$, $\partial B_6 = \{6,9\}$, $\partial B_7 = \{7,8\}$. The directed factors are
$F_1 = \{(0,1),(1,2),(2,3),(3,4),(4,5),(5,6),(6,7),(7,8),(8,9),(9,10),(10,11),(11,12),(12,13),(13,14),(14,0)\}$,

$F_{14} = \{(0,14),(14,13),(13,12),(12,11),(11,10),(10,9),(9,8),(8,7),(7,6),(6,5),(5,4),(4,3),(3,2),(2,1),(1,0)\}$,

$F_2 = \{(0,2),(2,4),(4,6),(6,8),(8,10),(10,12),(12,14),(14,1),(1,3),(3,5),(5,7),(7,9),(9,11),(11,13),(13,0)\}$,

$F_{13} = \{(0,13),(13,11),(11,9),(9,7),(7,5),(5,3),(3,1),(1,14),(14,12),(12,10),(10,8),(8,6),(6,4),(4,2),(2,0)\}$,

$F_3 = \{(0,3),(3,6),(6,9),(9,12),(12,0),(1,4),(4,7),(7,10),(10,13),(13,1),(2,5),(5,8),(8,11),(11,14),(14,2)\}$,

$F_{12} = \{(0,12),(12,9),(9,6),(6,3),(3,0),(1,13),(13,10),(10,7),(7,4),(4,1),(2,14),(14,11),(11,8),(8,5),(5,2)\}$,

$F_4 = \{(0,4),(4,8),(8,12),(12,1),(1,5),(5,9),(9,13),(13,2),(2,6),(6,10),(10,14),(14,3),(3,7),(7,11),(11,0)\}$,

$F_{11} = \{(0,11),(11,7),(7,3),(3,14),(14,10),(10,6),(6,2),(2,13),(13,9),(9,5),(5,1),(1,12),(12,8),(8,4),(4,0)\}$,

$F_5 = \{(0,5),(5,10),(10,0),(1,6),(6,11),(11,1),(2,7),(7,12),(12,2),(3,8),(8,13),(13,3),(4,9),(9,14),(14,4)\}$,

$F_{10} = \{(0,10),(10,5),(5,0),(1,11),(11,6),(6,1),(2,12),(12,7),(7,2),(3,13),(13,8),(8,3),(4,14),(14,9),(9,4)\}$,

$F_6 = \{(0,6),(6,12),(12,3),(3,9),(9,0),(1,7),(7,13),(13,4),(4,10),(10,1),$
$(2,8),(8,14),(14,5),(5,11),(11,2)\}$,

$F_9 = \{(0,9),(9,3),(3,12),(12,6),(6,0),(1,10),(10,4),(4,13),(13,7),(7,1),$
$(2,11),(11,5),(5,14),(14,8),(8,2)\}$,

$F_7 = \{(0,7),(7,14),(14,6),(6,13),(13,5),(5,12),(12,4),(4,11),(11,3),$
$(3,10),(10,2),(2,9),(9,1),(1,8),(8,0)\}$,

$F_8 = \{(0,8),(8,1),(1,9),(9,2),(2,10),(10,3),(3,11),(11,4),(4,12),(12,5),$
$(5,13),(13,6),(6,14),(14,7),(7,0)\}$.

The representation of the directed factors are in Figures 7.4, 7.5, 7.6, 7.7, 7.8, 7.9 and 7.10.

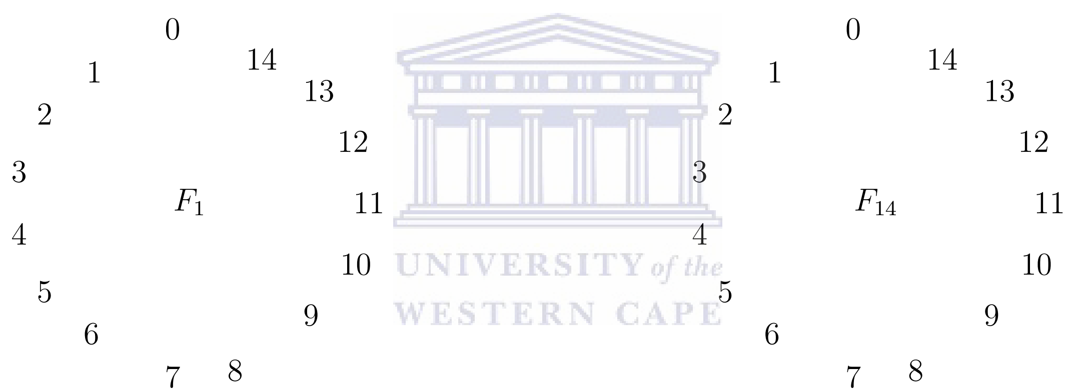

Figure 7.4: Factors F_1 and F_{14}

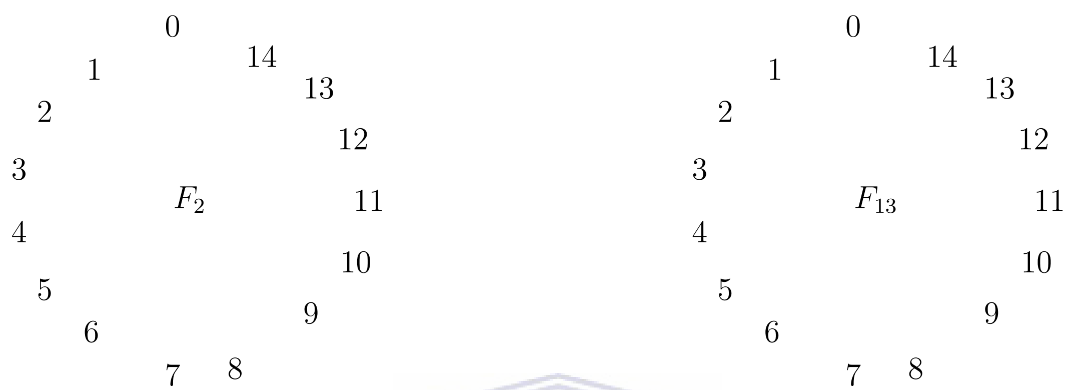

Figure 7.5: Factors F_2 and F_{13}

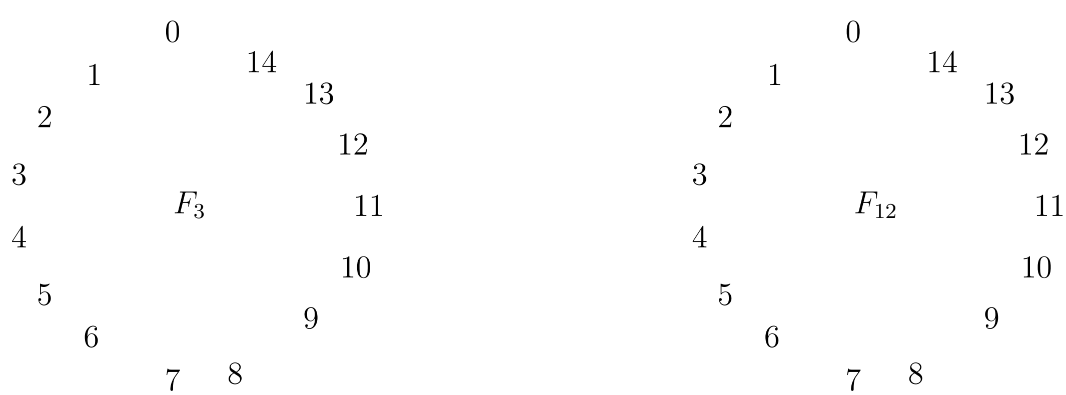

Figure 7.6: Factors F_3 and F_{12}

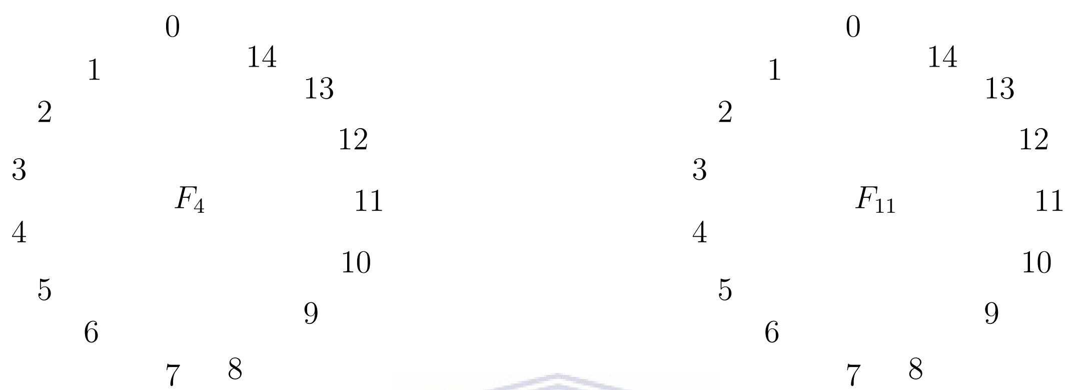

Figure 7.7: Factors F_4 and F_{11}

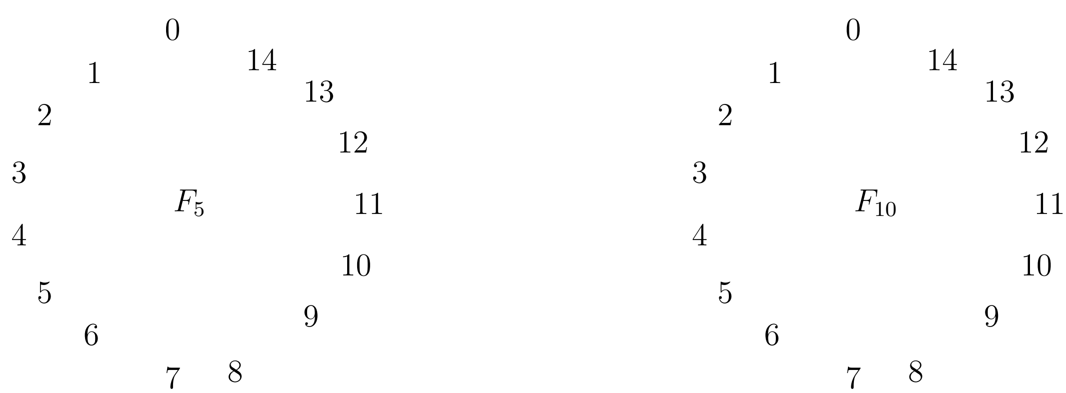

Figure 7.8: Factors F_5 and F_{10}

The Cayley table representing the loop, denoted Q_{11}, is given in Table 7.2.

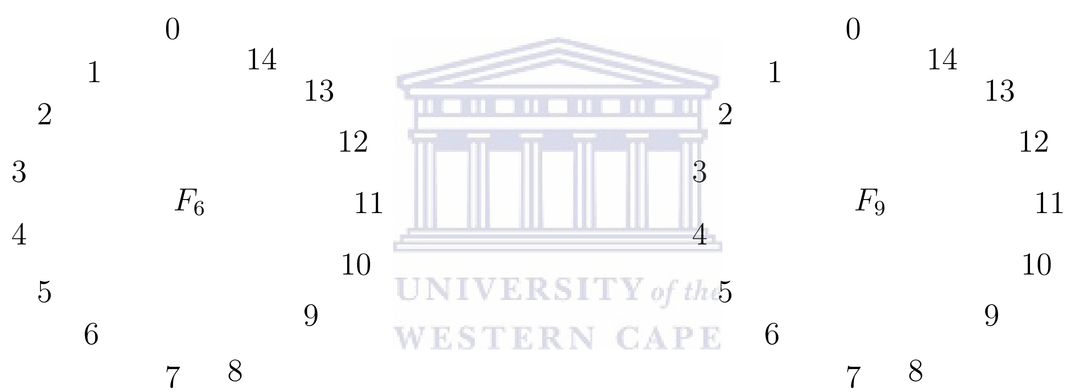

Figure 7.9: Factors F_6 and F_9

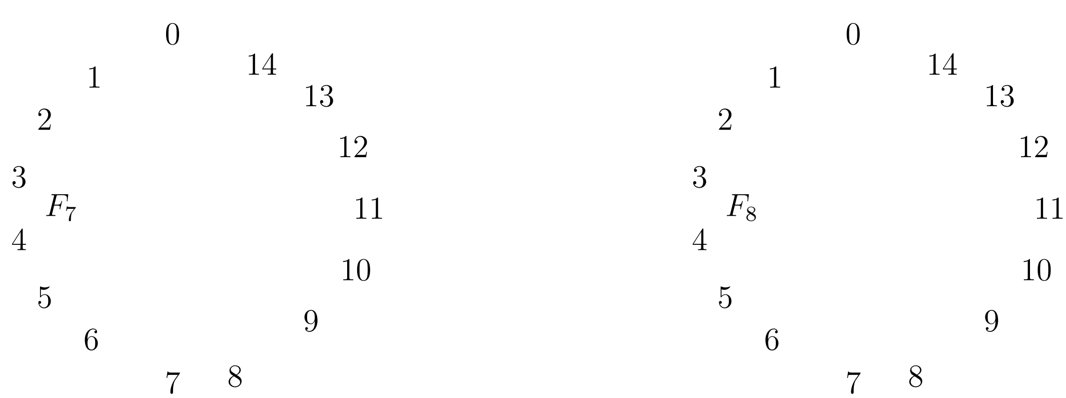

Figure 7.10: Factors F_7 and F_8

$*_0$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
2	2	3	4	5	6	7	8	9	10	11	12	13	14	0	1
3	3	4	5	6	7	8	9	10	11	12	13	14	0	1	2
4	4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
5	5	6	7	8	9	10	11	12	13	14	0	1	2	3	4
6	6	7	8	9	10	11	12	13	14	0	1	2	3	4	5
7	7	8	9	10	11	12	13	14	0	1	2	3	4	5	6
8	8	9	10	11	12	13	14	0	1	2	3	4	5	6	7
9	9	10	11	12	13	14	0	1	2	3	4	5	6	7	8
10	10	11	12	13	14	0	1	2	3	4	5	6	7	8	9
11	11	12	13	14	0	1	2	3	4	5	6	7	8	9	10
12	12	13	14	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	0	1	2	3	4	5	6	7	8	9	10	11	12
14	14	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Table 7.2: Groupoid of regular 2-factorisation of K_{15}

This loop Q_{11} is the group $\mathbb{Z}_{15}$ because Cayley Table 7.2 of Q_{11} coincides with that of $\mathbb{Z}_{15}$.

The next result is a generalisation of groupoids from regular 2-factorisations.

7.8 Theorem *Groupoids from regular 2-factorisations are groups.*

Proof

In view of Theorem 3.11 we only need to show associativity. Let a, b, c, d, x, y, z be the elements in the vertex set A. Take an element u fixed in A. We must show that

$$(x *_u y) *_u z = x *_u (y *_u z). \tag{7.2}$$

If one of the three elements is u it clearly holds.

If all of the three elements are different from u, we have on the left hand side of the equation (7.2)

$$
\begin{aligned}
(x *_u y) *_u z \quad &= a *_u z && \text{with } (x,a) \in F_y \\
& && \Longrightarrow (y^{-1} *_u x, y^{-1} *_u a) \in F_u \\
& && \Longrightarrow y^{-1} *_u x + 1 = y^{-1} *_u a \\
& && \Longrightarrow a = x + y \\
& && \text{since, } a *_u z = b \text{ with } (a,b) \in F_z \\
& && \Longrightarrow (x + y, b) \in F_z \\
& && \Longrightarrow (z^{-1} *_u (x+y), z^{-1} *_u b) \in F_1 \\
& && \Longrightarrow z^{-1} *_u (x+y) + 1 = z^{-1} *_u b) \\
& && \Longrightarrow b = x + y + z.
\end{aligned}
$$

Similarly, on the right hand of Equation 7.2 we have

$$
\begin{aligned}
x *_u (y *_u z) \quad &= x *_u c && \text{with } (y,c) \in F_z \\
& && \Longrightarrow (z^{-1} *_u y, z^{-1} *_u c) \in F_u \\
& && \Longrightarrow z^{-1} *_u y + 1 = z^{-1} *_u c \\
& && \Longrightarrow c = y + z \\
& && \text{since, } x *_u c = d \text{with } (x,d) \in F_c \\
& && \Longrightarrow x *_u (y+z) = d \\
& && \text{if } y + z = 0 \text{ it is fine by the definition.} \\
& && \text{if } y + z \neq 0 \text{ we get,} (x,d) \in F_{y+z} \\
& && \Longrightarrow (y+z)^{-1}x, (y+z)^{-1}d) \in F_u \\
& && \Longrightarrow (y+z)^{-1} *_u d = (x+z)^{-1} *_u x + 1 \\
& && \Longrightarrow d = x + y + z.
\end{aligned}
$$

∎

Note that the proof does not use the fact that the factorisations are isomorphic.

Concluding remarks

This thesis is developed from the question: What kind of groupoids are obtained from a given kind of factorisation of a complete graph? The isomorphic Harary's factorisations, Hamiltonian factorisations, general 1-rotational factorisations and Buratti's regular factorisations are considered. We have shown that the isomorphic factorisations of Harary, in general, do not give groups. Like-wise, among the non-Cayley graphs, the two quasi-Cayley graphs obtained are the trivial ones, a not so surprising result. Again, in the case of Hamiltonian factorisations, groupoids generated are not groups, except the group obtained from the 1-rotational Hamiltonian cycle systems of the complete graph having five vertices. The general 1-rotational factorisations give similar results.

On the other hand, groupoids from regular 2-factorisations are all groups, even when the factors are not isomorphic.

As can be seen, this thesis does not provide a complete study of groupoids from homogeneous factorisations of graphs. It is hoped that this undertaking continues.

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