STOCHASTIC VOLATILITY MODELS FOR CONTINGENT CLAIM PRICING AND HEDGING

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A mini-thesis submitted in partial fulfilment of the requirements for the degree of Magister Scientiae in the Faculty of Natural Sciences, University of the Western Cape.

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STOCHASTIC VOLATILITY MODELS
FOR CONTINGENT CLAIM PRICING
AND HEDGING

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KEYWORDS

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Black-Scholes
Implied Volatility
Stochastic Volatility
Call Option Mixture
Risk-Neutral Pricing
Equity-linked Pension
Brennan-Schwartz
ABSTRACT

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The present mini-thesis seeks to explore and investigate the mathematical theory and concepts that underpins the valuation of derivative securities, particularly European plain-vanilla options. The main argument that we emphasise is that novel models of option pricing, as is suggested by Hull and White (1987) [1] and others, must account for the discrepancy observed on the implied volatility “smile” curve. To achieve this we also propose that market volatility be modeled as random or stochastic as opposed to certain standard option pricing models such as Black-Scholes, in which volatility is assumed to be constant.

We present a generalisation of derivative pricing models and concepts within both complete and incomplete market frameworks. This is supplemented by the investigation of existing models such as Guo (1998) [3], and then innovatively applying such knowledge to price other instruments (e.g. equity-linked pensions) under the context of stochastic volatility. We also conclude that other models can consistently account for the smile effect without directly embedding stochastic volatility.

We then formulate a new follow-up or extended model for the pricing of minimum guarantees that are provided by pension fund managers to minimise the downside risk for pension holders. We establish that the model of Brennan and Schwartz (1976) [2] can be extended so as to capture stochastic volatility and therefore the implied volatility “smile” effect.


DECLARATION

I declare that *Stochastic Volatility Models for Contingent Claim Pricing and Hedging* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

MUZI CHARLES MANZINI

DECEMBER 2008

SIGNED: ...........................
ACKNOWLEDGEMENTS

I devote this mini-thesis to my amazingly supportive and loving mother (Ms. Ntombie P. Manzini). Mom, thank you for your support, “ngiyakubonga kakhulu”. I would also like to express my most sincere gratitude to my Supervisor (Prof. Peter J. Witbooi) for all the support throughout the time. Prof., your supportive effort and dedication is greatly appreciated, “baie dankie”.

UNIVERSITY of the WESTERN CAPE
ABBREVIATIONS

SV - Stochastic Volatility
IV - Implied Volatility
BS - Black-Scholes
Br-S - Brennan-Schwartz
R-N - Risk-neutral
SBM - Standard Brownian Motion
GBM - Geometric Brownian Motion
PDE - Partial Differential Equation
SDE - Stochastic Differential Equation
EMM - Equivalent Martingale Measure
OTC - Over-the-Counter
FRA - Forward Rate Agreement
ROR - Rate of Return
NPV - Net-present Value
DCF - Discounted Cash Flows
IRR - Internal Rate of Return
OTM - Out-of-the-money
ITM - In-the-money
ATM - At-the-money
Log-OU - Log-Ornstein-Uhlenbeck
LIBOR - London Inter-bank Offer Rate
BSOPM - Black-Scholes Option Pricing Model
ELIPAVG - Equity-linked Life Insurance Policy with an Asset-value Guarantee
NASDAQ - National Association of Securities Dealers Automated Quotations
The organisational structure of this mini-thesis is as follows:

Chapter One presents the objectives of the thesis and provides a brief background about the underlying fundamentals that forms the basis of asset pricing.

Chapter Two introduces the financial market system and some of the instruments provided in the market, such as derivatives, bonds, etc. In this chapter we also present a brief literature review of a few of the many applications of option pricing models.

Chapter Three presents some basic mathematical and financial concepts, that allow us to formulate mathematical equations to valuate financial instruments. These include the concepts of martingales, Markov processes, arbitrage, Brownian motion, hedging, etc.

Chapter Four presents the methodology of option pricing under the complete market, such as the Black-Scholes partial differential equation (PDE) method. In this chapter we also introduce in detail the concept of implied volatility, as well as the implied volatility “smile” curve.

Chapter Five introduces the concept of stochastic volatility and the formulation of a stochastic volatility model as is described by Hull and White (1987). Furthermore this chapter reviews the model of Ritchey (1990) and Guo (1998).

Chapter Six is the ultimate chapter. This chapter presents the Brennan and Schwartz (1976) model of pricing equity-linked pensions. We then utilise some of the tools provided in earlier chapters, particularly Guo’s model, to generate an extended Brennan and Schwartz model that captures stochastic volatility.
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Chapter 1

Introduction and Background

1.1 Introduction

The principal theme of the current mini-thesis is based upon the analytical theory underpinning the valuation of financial derivative securities or contingent claims. The thesis gives particular attention to the pricing of European options under the extended framework governed by stochastic volatility. In essence, in the mini-thesis we are concerned with modeling the shift from the conventional Black-Scholes constant volatility (complete market) model to an option pricing model embedding stochastic volatility (incomplete market model).

The fundamental objective of this mini-thesis is thus two-fold, the first is to present an exploratory study and review of existing stochastic volatility pricing models, and the second is to propose an innovative approach or extended model that incorporates stochastic volatility for the pricing of equity-linked life insurance products.

The mathematical pricing methodology underpinning the valuation of contingent claims comprises of primarily two models. The first is the partial differential equation (PDE) method, this method involves the resolution, under suitable boundary conditions, of the following terminal-value problem:

\[
\frac{\partial F_t}{\partial t} + rS_t \frac{\partial F_t}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} = rF_t \tag{1.1}
\]

\[
F_T = f(S_t), \tag{1.2}
\]
see Chapter Four for further discussion. The second method is called the martingale pricing method and is based on the concept of the existence of an equivalent probability measure \( Q \) under which the discounted stock price process is a martingale.

The financial derivatives market thus valuates securities such as stock options using contemporary pricing methodology such as the Black and Scholes (1973) model. These models commonly manifest in the prices being expressed as solutions to stochastic differential equation (SDE) problems or equivalently as conditional expectations of discounted functionals of markov stochastic processes. In Chapter Three we introduce some fundamental concepts that dwell at the center of these pricing models.

1.2 Background

Arguably, the most basic theory of price determination in markets is that of “supply and demand”. This assumes that at each point in time the total amount or stock of each traded asset is fixed. The market price of each asset is then allowed to fluctuate or re-adjust continuously in such a manner that the aggregate demand matches the existing supply of stock on average, see Bailey (2007).

1.2.1 Market Prices and Rates of Return

In Chapter Two we give a brief description of the market system, and we learn that market prices are central in this system. The current market price as observed in the market plays various important roles in the financial market. These include amongst others the following:

(1) Market prices act as signals of information. Current prices are assumed to bear all available information about future prices. Such information is critical to financial managers and investors in relation to policy development and decision making.

(2) In economic analysis, prices also reflect scarcity of the asset relative to other assets. This is also visible in the dynamics of the supply and demand principle. That is, prices of scarcely available assets tend to be higher than those of assets in excess supply.

(3) By convention, current market prices also reflect the opportunity cost. That is, the price
represents the amount that has to be paid or received per unit of the asset.

In comparison, rates of return are also crucial in decision making specifically because they are forward-looking, see Bailey (2007), in that they depend on future prices or expected payoffs. Investors typically hold assets due to the expectation of positive capital gains, that is, they expect to assets to yield a positive rate of return (ROR), \( r_t > 0 \), where

\[
r_t = \frac{S_{t+1} - S_t}{S_t},
\]

such that, \( S_t \) is the current market price and \( S_{t+1} \) is the price one period later.

### 1.2.2 Underlying Asset Price Fundamentals

The price of a financial derivative is dependent on the performance of the underlying assets. Thus an understanding of the theories that govern the valuation of traded assets is central in contingent claim valuation. As is implied by the above discussion, asset pricing models depend largely on the degree to which currently available (price) information can be used to forecast future prices. Some of the most commonly used models of asset prices include the martingale, random walk, and Brownian motion models. The martingale model of asset prices can be expressed in simple terms as

\[
\mathbb{E}[S_t | \mathcal{F}_{t-1}] S_{t-1} = 1,
\]

where \( \mathcal{F}_{t-1} \) (in simpler terms) represents the universal set comprising of all the available relevant past information up to time \( t - 1 \). The martingale model thus asserts that all information leading up to time \( t \) is incorporated in \( S_t \), and at time \( t \) the price \( S_t \) is the best predictor of the asset price at any future date. The martingale model is often associated with a “fair game” since the expected return is zero or

\[
\mathbb{E}[S_t | \mathcal{F}_{t-1}] - S_{t-1} = 0.
\]

However, as mentioned above investors invest in the market because they expect a positive return, hence we often assume that

\[
\mathbb{E}[S_t | \mathcal{F}_{t-1}] S_{t-1} = 1 + \mu,
\]
where $\mu$ is fixed and is assumed to be positive or at least $\mu \geq -1$, see Bailey (2007).

Given a set of price data, empirical studies of asset prices typically investigate the correlation or covariation between a series of observations, at equal distances (lags) apart. Such information can be drawn from the sample autocorrelation function $\rho(k)$, at lag $k$,

$$\rho(k) = \frac{\sum_{t=1}^{N-k} (S_t - \bar{S})(S_{t+k} - \bar{S})}{\sum_{t=1}^{N} (S_t - \bar{S})^2}, \quad (1.7)$$

where $\bar{S}$ is the average stock price, $N$ is the number of cases observed. It turns out that $|\rho(k)| \leq 1$, see Chatfield (1989).

Now, define a random process $W_t$ with fixed mean $\mu$ and variance $\sigma^2$ such that $\rho(0) = 1$ and $\rho(k) = 0 \forall k > 0$, then $W_t = \mu + \sigma \epsilon_t$, where $\epsilon_t \sim \mathcal{N}(0,1)$ are independent identically distributed normal random variables with mean 0 and variance 1. We define a random walk model as:

$$S_t = S_{t-1} + W_t,$$

where $S_0 = 0$. From this we see that $S_t = \sum_{j=0}^{t} W_t$, that is, the random walk model says that the stock price at time $t$ is the accumulative sum of random processes. Moreover, according to the random walk model the price at any time $t$ in the future is given by the most recent price plus some random disturbance. Another process that is often encountered in the modeling of asset prices for derivative pricing is the Brownian motion (BM), in particular, the geometric Brownian motion (GBM), see Chapter Three.

The construction of a derivative pricing model requires the setting of an operational framework under which the model will specifically be defined. As such, basic derivative pricing models make certain assumptions about the market and its instruments. However, as will be shown later in the mini-thesis some of these assumptions tend to be quite stringent in practice and thus restricts the model’s practicality. The aim of any option pricing model is to derive a formula that expresses the option’s price as a function of the underlying market’s and contract’s descriptive variables, see Bailey (2007). The price $p_t$ for instance of a European put option is described by a function $f$ of the form:

$$p_t = f(S, K, r, \sigma, \tau)$$
where $S$ is the underlying asset price, $K$ is the exercise or strike price, $r$ is the risk-free interest rate, $\sigma$ is the volatility parameter and $\tau = T - t$ is the time-to-maturity. To achieve the objective of option pricing most models define a framework by making the following assumptions.

**The underlying asset**

(1) *pays no dividends during the lifetime of the option.* This means that the underlying assets are assumed to yield no stock or cash dividends, at least during the life of the option contract.

**The market:**

(2) *caters for short sales.* Investors can sell assets that they do have in their holdings. This is done by borrowing the assets from the market and then selling them “short” with the aim of purchasing them back when it is favourable to do so. That is, a short usually occurs when an investor expects a future drop in stock prices.

(3) *is information efficient.* That is, all known relevant information is presumed to be already incorporated and reflected in the asset prices. The prices therefore depict the general future sentiments of every investor and are thus unbiased.

(4) *is frictionless.* There are no transaction costs, commissions or taxation. It is worth noting however, that in reality the existence of a completely frictionless market is only hypothetical since in finance there is always a cost associated with any particular trading agreement. Consequently, it is often quoted that “there aint no such a thing as a free lunch”, this is the so called “No Free Lunch” principle, see Mohr and Associates (2002).

(5) *is perfectly competitive.* The market is said to have *perfect competition* when none of the individual market participants can influence market prices, and there is no collusion between the suppliers. The prices are thus only determined by the interaction between the supply and demand forces. The participants have to accept the prices as given by the market. They are therefore called *price takers* and can only decide on what quantities to supply or demand at those prices. Hence all relevant information pertaining to market conditions must be readily available, see also Mohr and Associates (2002) for a more detailed discussion.
1.2.3 The Black-Scholes Argument

In a nutshell the Black-Scholes argument proceeds as follows. Suppose a market investor holds a European call option with maturity premium \( F_T = f(S_T) \). Suppose further that the investor trades in another security \( B_t \), the riskless bond (or bank account). In this model the dynamics of \( S_t \) are given by the geometric Brownian motion model and the bond is assumed to be continuously compounding at the risk-free interest rate \( r \), that is:

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sigma S_t dW_t & S_0 > 0, \\
    dB_t &= r B_t dt & B_0 = 1.
\end{align*}
\]

(1.9) (1.10)

where \( \sigma > 0 \) is the volatility coefficient, \( \mu \in \mathbb{R} \) is the drift coefficient and \( \{W_t\}_{t \geq 0} \) is the standard Brownian motion (SBM) defined on the filtered probability space \( \mathbb{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P}) \) satisfying standard stochastic integrability conditions, and where \( \mathcal{F}_0 \) is the null set completion of the measurable space \( (\Omega, \mathbb{P}) \) (see Chapter Three for a discussion on the mathematical setup underlying derivative pricing).

In addition to the assumptions in the previous section the Black-Scholes model thus makes the following assumptions about the underlying asset price process and market.

(6) **The asset price process varies continuously.** That is, the asset price is assumed to move or change in such a manner that it does not jump from point to point.

(7) **The price process has returns that are independent Gaussian random variables.** The price returns are normal random variables with constant summary statistics or the asset price process \( S_t \) is lognormally distributed. That is, the variance of returns around the mean, \( \mu \), is proportional to time by a constant \( \sigma^2 \) (where \( \sigma \) is called the standard deviation or volatility).

(8) **Market trading occurs continuously over time.**

It follows therefore that if the investor initiates trading with a total initial investment of \( I_0 \) (in the standard unit of account) and trading proceeds in such a manner that the portfolio at time \( T \) mimics or replicates the time \( T \) premium (payoff) \( F_T \) of the claim, then \( I_0 \) is the only “fair-value” for the option.

In this context it can be shown that for \( S_t \) and \( B_t \) there exist a pair \( (\psi_1^t, \psi_2^t) = \psi_t \) of
continuously measurable stochastic processes on $S$ satisfying $I_t = \psi^1_t S_t + \psi^2_t B_t$ and $I_0 = \psi^1_0 S_0 + \psi^2_0$ such that

\[
I_t = I_0 + \int_0^t \psi^1_v dS_v + \int_0^t \psi^2_v dB_v \quad \forall t \in [0, T]
\]

(1.11)

\[
V(I_T) = f(S_T)
\]

(1.12)

where $V(I_T)$ is the value of the portfolio at maturity $T$. In this case the trading strategy $\psi_t$ must satisfy the following standard boundedness conditions,

\[
\mathbb{E}\left[\int_0^T (\psi^1_v)^2 dv\right] \leq \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^T |\psi^2_v| dv\right] \leq \infty.
\]

(1.13)

Thus for the continuous time Black-Scholes framework, the couple $(\psi^1_t, \psi^2_t)$ of measurable processes defines a self-financing portfolio if and only if

\[
d\tilde{I}_t = \psi^1_t d\tilde{S}_t.
\]

(1.14)


Under the auspices of the argument presented above Black and Scholes (1973) showed that in an arbitrage-free market the no-arbitrage price at time $t$ of an attainable European call option is given by the following expression.

\[
c(S, K, r, \sigma, T - t) = \mathbb{E}^Q_B[B_T f(S_T)|\mathcal{F}_t] \quad \forall t \in [0, T]
\]

\[
= \mathbb{E}^Q[e^{-r\tau}(S_T - K)^+|S_t]
\]

(1.15)

where $\tau = T - t$. Therefore, the time zero initial injection into the self-financing portfolio yields the fair price $f_0$ of the option and is given by

\[
f_0 = \mathbb{E}^Q[B_T^{-1}(S_T - K)^+|S_0]
\]

\[
= \mathbb{E}^Q[e^{-rT}(S_T - K)^+].
\]

In this context Black and Scholes (1973) were able to show how to determine the fair price of the option as. For a more comprehensive dicussion on the underlying methodology refer to Chapter Four.
When the special conditions of the Black-Scholes model are not satisfied then the model tends to deviate from the observed market behaviour. One phenomenon that is often quoted to depict this deviation is the so called implied volatility “smile” effect, this is formally introduced in Chapter Four. Further in this mini-thesis we shall consider some of such cases which thus require the extension of the standard option valuation models, see Chapter Five.

1.2.4 Some Empirical Computations

There are several methods of estimating and modeling volatility, these include methods such as the implied volatility, empirical volatility, auto-regressive models (ARIMA, ARCH, GARCH) amongst others. Implied volatility is dealt with in detail starting from Chapter Four. Historical volatility is not directly observed but is filtered or estimated from time-series price data. The estimation of volatility from historical time-series data (i.e. empirical or historical volatility) is carried out as follows:

- first we calculate the periodic relative price returns, \( r_t \), as in equation (1.3).
- next, we calculate the log-relative return, \( L_t \), which is given the natural logarithm of the gross return \( (1 + r_t) \). That is, \( L_t = \ln(1 + r_t) \).
- the periodic (e.g. daily) volatility \( \sigma_d \) is then calculated to equate the standard deviation of the returns \( R_t - R_{t-1} \), where \( R_t = \ln(S_t) \).
- the annualised volatility \( \sigma_a \) is calculated by scaling the daily volatility by the square root of the number of trading days, that is, \( \sigma_a = \sigma_d \sqrt{250} \).

Below we present an example of such a calculation using data from the NASDAQ index. Table (1.1) depicts price returns from monthly index price data sampled from the last 12-months of the period dated from 01-May-71 to 01-January-08, for illustrative purposes we treat the data as representing daily ticks. To illustrate the overall path followed by the index we consider Figure (1.1) below.
Table (1.1): Empirical Volatility Estimation.

Table 1.1: The estimation of empirical volatility from historical time-series data for the NASDAQ Index.

<table>
<thead>
<tr>
<th>Observation</th>
<th>Observed Price</th>
<th>Gross Return</th>
<th>Log-relative</th>
<th>Daily Volatility</th>
<th>Annualised Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$S_t$</td>
<td>$1 + r_t$</td>
<td>$L_t$</td>
<td>$\sigma_d$</td>
<td>$\sigma_a$</td>
</tr>
<tr>
<td>1</td>
<td>2322.57</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2178.88</td>
<td>0.9381</td>
<td>-0.0639</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2172.09</td>
<td>0.9969</td>
<td>-0.0031</td>
<td>0.0430</td>
<td>0.6791</td>
</tr>
<tr>
<td>4</td>
<td>2091.47</td>
<td>0.9629</td>
<td>-0.0378</td>
<td>0.0245</td>
<td>0.3880</td>
</tr>
<tr>
<td>5</td>
<td>2183.75</td>
<td>1.0441</td>
<td>0.0432</td>
<td>0.0573</td>
<td>0.9056</td>
</tr>
<tr>
<td>6</td>
<td>2258.43</td>
<td>1.0342</td>
<td>0.0336</td>
<td>0.0068</td>
<td>0.1068</td>
</tr>
<tr>
<td>7</td>
<td>2366.71</td>
<td>1.0479</td>
<td>0.0468</td>
<td>0.0093</td>
<td>0.1476</td>
</tr>
<tr>
<td>8</td>
<td>2431.77</td>
<td>1.0275</td>
<td>0.0271</td>
<td>0.0139</td>
<td>0.2204</td>
</tr>
<tr>
<td>9</td>
<td>2415.29</td>
<td>0.9932</td>
<td>-0.0068</td>
<td>0.0240</td>
<td>0.3792</td>
</tr>
<tr>
<td>10</td>
<td>2463.93</td>
<td>1.0201</td>
<td>0.0199</td>
<td>0.0189</td>
<td>0.2989</td>
</tr>
<tr>
<td>11</td>
<td>2416.13</td>
<td>0.9806</td>
<td>-0.0196</td>
<td>0.0280</td>
<td>0.4419</td>
</tr>
<tr>
<td>12</td>
<td>2421.64</td>
<td>1.0023</td>
<td>0.0023</td>
<td>0.0155</td>
<td>0.2445</td>
</tr>
</tbody>
</table>
NASDAQ realisation and simulated path of volatility.

Figure 1.1: This is the path realisation followed by the NASDAQ index.

Figure 1.2: This figure displays the path or volatility behaviour corresponding to the underlying period.
Figure (1.2) displays the volatility path of price returns. The figure shows that for the time period considered, NASDAQ volatility seems to fluctuate between periods of high volatility and subsequent periods of low volatility. That is, the volatility appears high for several days and then low for roughly the same number of days. This behaviour of volatility is called clustering or persistence, see for instance Fouque et al. (2001) and Mina and Xiao (2003).

There exists a substantial body of evidence in financial literature (for instance, Das (1998), Das and Sundaram (1999)) that supports the observation that volatility tends to be mean reverting, that is, volatility seems to be regularly pulled back to its long-term mean level. In light of the above characterisation, volatility can thus be said to come in clusters around its mean. In the current study of the NASDAQ index volatility can also be observed to be clustered around the long-term mean level as is depicted by the horizontal line in Figure (1.2). Mean reversion also plays an important to the formulation of a volatility model and, according to Das and Sundaram (1999) when implied volatility is replaced with average expected volatility over the given horizon then the resulting shape of the term structure of implied volatilities is entirely decided by the mean reversion factor. Note that the values on the vertical axis of Figure (1.2) (i.e. volatility values) are given as percentage points such that 0.85 means 85%, this notation is adopted throughout the mini-thesis.
Chapter 2

The Financial Market Economy

2.1 The Market Economy

A market economy is a market in which coordination is achieved through the market mechanism or price system. That is, through the free and spontaneous movement of market prices, as determined by the operation of the forces of supply and demand. The most important components of market mechanism are market prices as these act as signals or indices of various elements such as information, policy, scarcity, etc, see Mohr and Associates (2002). In a market system prices constitute a crucial signaling system that directs and controls economic activity. The two key financial markets encountered are the money market and capital market. Financial transactions involving short-term (maturities of one year or less) debt instruments take place in the money market. Longer-term securities or funds such as bonds and equities are traded in the capital market. The existence of derivative securities offers leverage or opportunity to manage risk (hedging). Financial markets therefore also act as alternative insurance mediums.

2.2 Financial Instruments

Interest Rate, also called required return reflects the cost of money and it performs the role of a regulating instrument that controls the flow of funds between suppliers and demanders. There are various reference services that exist to inform the market of the daily standard
interest rates movements, such as the London Interbank Offer Rate (LIBOR) which is the standard rate that most credit worthy international banks dealing in Eurodollars charge each other for large loans. When comparing interest rates, comparisons are made in fractions of a percent, called basis points. One basis point is equal to one percent of a unit percent or 0.01%.

Equities, since corporations tend to have a large number of owners, shares or stocks represent fractions of ownership. Equity refers to that part of a company that is devoted to the sharing of a company's stock, thus shares and stocks fall under equities. The price of a share can be affected by a number of factors such as for instance, dividends, tax obligations, stock-splits, spin-offs, mergers, etc.

Bonds, a Bond is a fixed term, fixed-interest (or fixed-income) debt instrument that is issued by governments (sovereign bonds) and corporations (corporate bonds). At any time $t$ the bond price $B_t$ depends on the current level of interest rates $r$, term-to-maturity $\tau$ and its yield $y(t)$. The yield or yield-to-maturity is the interest rate that would be effectively earned if the bond is bought now and held till maturity.

### 2.2.1 Financial Derivatives

Derivative securities (Derivatives) are instruments that define a particular contract and whose value depends on the value or dynamics of other less complex underlying assets. Derivatives are also called contingent claims since the claim against any derivative is subject or contingent on another asset such as a stock price of a particular corporation. A brief discussion of some popular derivatives is presented below.

A forward contract is an agreement between two counter-parties to sell or buy a particular asset (the underlying) at a definite future date (delivery date or maturity) for a certain price (delivery price). The delivery price is chosen such that the cost of the contract is set at zero. For an outstanding contract we define the forward price as the effective delivery price which will bring that contract to a cost of zero. Forward contracts are usually traded in the over-the-counter (OTC) market.

Futures contracts are essentially exchange traded forward contracts. Another distinguishing
feature is that for futures contracts the delivery date is not usually specific but rather the
delivery month is specified.

**Swaps** according to Hull (1997), are private agreements between two corporations to ex-
change future cash flows according to a predetermined method. A simple example is an
interest rate swap and this type usually employs the LIBOR as the floating rate.

**Options**, unlike forwards and futures are actively traded on both organised exchanges and
OTC markets. The simplest types of options are call and put options.

A *call* option bestows upon the holder the prerogative (but not the obligation) to purchase
(from the seller or writer of the option) the underlying asset by a certain date in the future
(exercise date) at a certain price (strike or exercise price). A *put* option on the other hand
gives the holder the right to sell the underlying asset. Furthermore, we define a *European
option* as an option which can only be exercised at maturity whereas an *American option*
can be exercised at any time from the time at which it is written up to its maturity date.

In addition to the above-mentioned standard (or plain-vanilla) derivatives there are many
other more complex products. These are often referred to as exotic derivatives or simply *ex-
otics* and they include Barrier options, Forward Rate Agreements (FRA), Flexible forwards,
Caps and Floors, Swaptions, etc. see Hull (1997). The scope to the complexity is literally
interminable.

## 2.3 Literature Review of Applications of Option Pricing Models

Contingent claim valuation theory has in recent times found application in various distinct
areas in the corporate arena, such as for instance corporate development projects, industrial
and manufacturing systems, travel agency and airline industry, oil and gas exploration, life
insurance, moreover this list is by no means exhaustive. Thus basic financial asset pricing
models such as the BSOPM have provided the fundamental theoretical methodology for
corporate and industrial development, expansion, and profit growth option problems. These
have been formalised as *Real Options*, see for instance Agliardi (2006). Option pricing
theory as applied for valuation of real investment opportunities that possess option-related attributes provide useful alternative instruments for managing strategic decision making processes and policy development, specifically during times of uncertainty. The underlying results can thereafter be compared to conventional methods such as Discounted Cash Flows (DCF), Net Present Value (NPV), Internal Rate of Return (IRR), etc. This section provides a brief overview of some of the cases that utilize option pricing methods and these were chosen to show the diversity of the utility of option pricing models.

Dickens and Lorhenz (1996), Option Pricing Methods: Evaluating oil and gas assets. This paper investigates the use of option pricing methods, in particular the BSOPM, in valuating oil and gas assets. The authors achieve this by comparing the solutions from using the option methods against the more conventional investment project valuation methods (such as the NPV and the DCF methods) for an actual gulf of Mexico oil well. According to Dickens and Lorhenz (1996), oil and gas assets exist when there is a view that a particular drilling of an exploration well on searchable property can yield profit making opportunities. The mere existence of such assets opens various option possibilities. For instance, the holder of such an asset has an option to drill the well as soon as possible, postpone or defer the drilling or to sell the searchable asset.

From the calculated values using both the BSOPM and the DCF methods, Dickens and Lorhenz (1996) establishes that the BSOPM yields values greater than the DCF, from which they conclude that option methods capture values that are systematically overlooked by the conventional methods. In addition, the authors note that increased price volatility leads to a high project value because option values reflect the limited downside risk while allowing upside gain. Thus they concluded as a generalisation that, option valuation methods would be more appropriate for oil and gas assets the further downstream the investment is.

Tsai et al. (1996), Option Pricing Methods: Urgency problems between airlines and travel agencies. The basis of the urgency problem arises when the travel agent sells tickets for at least two airlines, as there exists an option for choosing the best airline tickets to sell in order to maximise profits. That is, if \( \alpha \) and \( \beta \) represents the agents profit from airline A
and $B$ respectively then, the payoff function is:

$$\max(\alpha, \beta) = \alpha + \max(\beta - \alpha, 0),$$

$$= \beta + \max(\alpha - \beta, 0).$$

The urgency thus has a European style option at the beginning of the ticket sales period. The authors thus develop an equivalent BSOPM for the assets underlying the problem. The model allows airlines to evaluate the agency costs associated with the ticket price process (GBM) as well as its incentives program distinct from its competitors.

Pak and Ryan (2001), Option Pricing Methods: *Estimation of capacity shortages*. In this investigation the authors research the dynamics of capacity expansion under conditions of uncertain future demand and a positive lead time for adding capacity in the industrial sectors such as, manufacturing, technology, and so on. Under conditions where growth is imminent there exists a risk of capacity shortage during any lead time. Pak and Ryan (2001) studies four option pricing methods (European, Asian, lookback and a weighted sum of European options) to estimate the capacity shortage under a fixed lead time.

A closely related real options problem is that of Agliardi (2006), *Options to expand*. In this volume the author investigates the option to expand a project scale and also studies the interaction between pairs of such options. The model of Agliardi (2006) assumes the usual risk-neutral pricing framework so that the gross project value follows the GBM process and the current option value is just the discounted risk-neutral expectation of future payoffs.

The option to expand is then valued as a call option to acquire part of the project by paying an extra outlay as the strike price.

Other applications of option pricing models can be found in the pricing of certain life insurance and pension products. For instance, in Chapter Six we study the pricing of equity-linked life insurance products with a minimum value guarantee and, we establish that such products have payoffs similar to that of European options, see for instance Brennan and Schwartz (1976), Kurz (1996), Aase and Persson (1994), Miltersen and Persson (1997) or Bacinello (2001).
Chapter 3

Mathematical Framework and Concepts

3.1 Mathematical Model Setup

Let \( X_t \) be a stochastic process. Then \( \{X_t\}_{t \geq 0} \) is a family of random variables indexed by the time parameter \( t \). Under our market economy the stochastic processes and components thereof are described via a sample space \( \Omega \). The subjective views of individual market participants about the status of the market are described by a probability measure \( \mathbb{P} \), which is defined on the sample space \( \Omega \). The configuration of the information hierarchy is described by the filtration \( \mathcal{F} = \{\mathcal{F}_t : t \geq 0\} \) comprising of an expanding sequence \( \mathcal{F}_t \) of \( \sigma \)-algebras on \( \Omega \). Accordingly we define \( \mathcal{F}_0 = \emptyset \) so that \( \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \ldots \mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \ldots \subset \mathcal{F}_T \). In essence, the process is defined via a complete filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \).

A stochastic process \( X_t \) is said to be adapted to a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) of \( \sigma \)-algebras if for every \( t \geq 0 \), \( X_t \) is measurable with regard to \( \mathcal{F}_t \) that is, for every \( x \in \mathbb{R} \) (the set of real numbers) the event \( \{X_t \leq x\} \) is in \( \mathcal{F}_t \).

One of the most important concepts in derivative pricing is the idea of a martingale, which in simple terms describes a fair game.

**Definition 3.1** - A stochastic process \( \{X_t\}_{t \geq 0} \) that is adapted to an increasing family \( \{\mathcal{F}_t\} \) of \( \sigma \)-algebras (defined under the probability measure \( \mathbb{P} \)) is called a *martingale* if \( \mathbb{E}[|X_t|] < \infty \).
and
\[ E[X_{t+1}|\mathcal{F}_t] = X_t. \] (3.1)

**Definition 3.2** - A stochastic process \( X_t : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) is called a Markov process, if for all finite sets of points \( 0 \leq t_0 \leq t_1 \leq \ldots \leq s \leq t \leq \infty \) and \( x \in \mathbb{R} \) we have
\[ P\{X_t \in \mathcal{A}|\mathcal{F}_s\} = P\{X_t \in \mathcal{A}|X_s\} \quad (3.2) \]
\[ = p(s, t; x; \mathcal{A}). \] (3.3)

Equivalently we can write \( E[X_t|\mathcal{F}_s] = E[X_t|X_s] \). The function \( p \) is called the probability transition function of \( X_t \) and reflects the probability that \( X_t \) will be in the subset \( \mathcal{A} \) at time \( t \) given that it was at \( x \) at time \( s \). See Kannan (1979) for definitions (3.1) and (3.2).

Much of the modeling that is dealt with in the pricing of contingent claims involves certain Markov stochastic processes called *diffusions*. Some of these processes include the Brownian motion, Ornstein-Uhlenbeck and the Square-root process. Refer to Prabhu (1966) for the following definition.

**Definition 3.3** - A Markov process \( \{X_t : X_t \in \mathbb{R}\}_{t \in \mathbb{R}_+} \) is called a diffusion process if there exists continuous functions \( \mu(t, x) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \) and \( \sigma(t, x) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+ \) such that for any \( \delta > 0 \) we have the following conditions satisfied.
\[ \lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x|>\delta} p(t, t+h; x, dy) = 0, \]
\[ \lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x|\leq\delta} (y-x)p(t, t+h; x, dy) = \mu(t, x), \]
\[ \lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x|\leq\delta} (y-x)^2p(t, t+h; x, dy) = \sigma(t, x). \]

The function \( \mu(t, x) \) is the aggregate rate of change of \( X_t \) called the *drift* coefficient, and \( \sigma(t, x) \) is the parameter of variation called the diffusion (or *volatility*) coefficient.

### 3.2 Brownian Motion

A *Brownian motion* \( \{z_t\} \) is a regular diffusion process with mean \( \mu(t, x) = 0 \) and variance \( \sigma^2(t, x) = \sigma^2 \) a constant for all \( (t, x) \) such that \( dz_t = \epsilon \sqrt{dt} \) where \( \epsilon \overset{\text{distr}}{=} \mathcal{N}(0, \sigma^2) \) (where
\( X \overset{\text{distr}}{=} N(\mu, \sigma^2) \) means that the random variable \( X \) is normally distributed with mean and variance \( \mu \) and \( \sigma^2 \) respectively. The special case where \( \sigma^2 = 1 \) is called the Standard Brownian motion, see Kannan (1979). Addition of a trend term \( \mu t \) to \( z_t \) produces a Brownian motion with drift \( X_t = \{z_t + \mu t\} \).

If we let \( \{X_t\}_{t \geq 0} \) be a drifted Brownian motion with drift \( \mu \) and variance \( \sigma^2 \) then the process defined by \( S_t = e^{X_t} \) is called the Geometric Brownian motion. The state (sample) space is \((0, \infty)\) and the infinitesimal parameters (mean \( \mu_S(t) = \mathbb{E}[S_t|S_0 = s] \) and variance \( \sigma^2_S(t) = \text{Var}(S_t|S_0 = s) \)) are as follows.

\[
\begin{align*}
\mu(t, s) &= se^{\mu t} \\
\sigma^2(t, s) &= s^2 e^{2\mu t}(e^{\sigma^2 t} - 1)
\end{align*}
\]

Then the dynamics of the GBM are described by the following stochastic differential equation (SDE):

\[
\begin{align*}
dS_t &= \mu S_t dt + \sigma S_t dz_t \\
S_0 &= s > 0, \text{ or } \quad S_t = se^{(\mu - \frac{1}{2}\sigma^2)t + \sigma^2 z_t}
\end{align*}
\]

where \( z_t \) is the SBM. If \( 0 = t_0 < t_1 < t_2 \cdots < t_n < \infty \) are time points then the successive ratios \( \left( \frac{S_1}{S_0} \right), \left( \frac{S_2}{S_1} \right), \cdots, \left( \frac{S_n}{S_{n-1}} \right) \) (where \( S_j = S_{t_j} \)) are independent random variables. Thus roughly speaking, for a GBM the percentage changes over non-overlapping time intervals are independent, refer to Lawler (2000). This is why the GBM is often used to model price behaviour of assets such as stock prices. Stock prices are non-negative and in the long run exhibit exponential growth or decay, two properties borne by the GBM.

### 3.3 Arbitrage

A trading strategy or portfolio is a \( m \)-tuple \( (\psi^1_t, \psi^2_t, \ldots, \psi^m_t) = \psi_t \) of \( \mathcal{F}_t \)-adapted processes for which each entry \( \psi^j_t \) specifies the number of units of asset \( j \in \{1, 2, \ldots, m\} \) held at time \( t \). We thus have an \( \mathbb{R}^m \)-valued adapted process \( S_t = (S^1_t, S^2_t, \ldots, S^m_t) \) where each \( S^j_t \) defines the price of the \( j \)-th asset at time \( t \). In simple terms, an arbitrage portfolio is an investment
strategy which costs nothing (zero cost) to construct, guarantees no loss and yields a positive profit in at least one of the possible states of the market.

Now we assume that there exists an asset \( S_0 = N_t > 0 \) for all values of \( t \) that is freely traded in the market and for any other asset we define \( \tilde{S} = S/N_t \) so that \( \tilde{S} \) is the \textit{price relative}. That is, we assume that all security prices are expressed in terms of a common standard unit of account \( N_t \), called the \textit{numeraire}.

For the purposes of evaluating our contingent claims the numeraire will always be the risk-free asset, bond price \( B_t \). In other words the security prices are expressed as discounted prices. Inevitably as will be shown in the continuous time pricing setting, this leads to the value of our derivative asset (a European call option) to be given in units of \( B_t \).

The total amount of money invested in the portfolio \( \psi_t \) across prices \( S_t \) at time \( t \) is given by the investment process \( I_t = I(t, \psi_t) \), also called the wealth process, see for instance Musiela and Rutkowski (1997). This process is given by the vector inner product of \( \psi_t \) and \( S_t \),

\[
I_t = \psi_t \cdot S_t = \sum_{j=1}^{m} \psi_j t S_j t \quad \forall t \in \{0, 1, \ldots, T\}.
\]

**Definition 3.4** - A trading strategy \( \psi_t \) is called \textit{self-financing} if it satisfies

\[
dI_t = \sum_{j=1}^{m} \psi_j t dS_j t \quad \forall t \in \{0, 1, \ldots, T\}. \tag{3.8}
\]

The self-financing condition ensures that changes in the value of the portfolio over small time intervals results only from changes in the values of the underlying assets, see Musiela and Rutkowski (1997).

**Definition 3.5** - A market economy \( \mathcal{M} = \mathcal{M}(\Psi, S_t) \) is said to admit arbitrage or has an arbitrage opportunity if there exists a self-financing strategy \( \psi_t \in \Psi \) satisfying the following conditions.

\[
I(0, \psi_0) = 0 \quad \text{and} \quad \mathbb{P}\{I(T, \psi_T) \geq 0\} = 1. \tag{3.9}
\]

Accordingly, the strategy \( \psi_t \) satisfying condition (3.9) is called an arbitrage portfolio. The concept of arbitrage possibilities is thus equivalent to the likelihood of making a positive
riskless profit in the marketplace. This is generally assumed to occur as a result of market pricing flaws or mispricing, which open the possibility to simultaneously purchase low and sell high and thereby profiting from the pricing discrepancies in the two markets.

As has been alluded with in Chapter One another key concept in finance is that of market efficiency, in the current context the assumption that the market is efficient essentially renders the market arbitrage-free.

### 3.4 Hedging and Replication

The idea of a riskless hedge is central to contemporary pricing models such as the Black-Scholes, particularly in terms of establishing continuously adjusted replicating portfolios for attainable securities. If a continuously adjusted portfolio, $P_t$, is constructed in such a way that it is invariant to the price movements of its component assets, such that price movements in one asset are offset by counter movements in another component asset, then $P_t$ is a riskless portfolio. In simpler terms, hedging refers to the reduction (or management) of risk.

For our purposes of pricing and hedging of a European call option, a solution to the overall pricing question should provide an answer to the question of how to construct a trading strategy that replicates the option payoff at maturity. Moreover, this portfolio must satisfy the following conditions.

\[
\frac{dP_t}{P_t} = \frac{dB_t}{B_t}, \quad (3.10)
\]

\[
I(T, P) = f(S_T) \quad \text{(hedging condition).} \quad (3.11)
\]

That is, the riskless portfolio must earn the return of the riskless asset, the bond $B_t$ and it must replicate the derivative payoff $f(S_T)$ at maturity $T$. For a European call option with $S_T = x$ we have $f(x) = \max\{(x - K), 0\}$, where $K$ is the exercise price.

**Definition 3.6** - Given a claim $F_t$ with terminal payoff $F_T = f(S_T)$, a trading strategy $\psi_t$ is called a replicating portfolio for $F_t$ if and only if:

\[
I(T, \psi_T) = F_T.
\]
That is $\psi_t = (\psi^1_t, \psi^2_t, \ldots, \psi^m_t)$ is a replicating strategy if $\psi^1_t S^1_T + \psi^2_t S^2_T + \ldots + \psi^m_t S^m_T = F_T$ and we recall that this is the hedging condition. If there exists such a strategy in a market then $F$ is said to be attainable. Thus, $F_t$ is attainable in a market $\mathbb{M}(\Psi, S)$ if $F_t$ allows at least one trading strategy $\psi_t \in \Psi$ such that under any state in the market we have $I(T, \psi_T) = f(S_T)$.

It follows therefore that if $\psi_0$ is commenced with at time zero, then at maturity the value assumed by $\psi_T$ will automatically match $F_T$. This will hold under any state in our arbitrage-free market. Consequently, by this arbitrage-free assertion it follows that the cost $I(0, \psi_t)$ of this portfolio is the fair price, $f_0$, for the claim. That is,

$$f_0 = I(0, \psi_t). \quad (3.12)$$

In other words the fair value of the claim is given by the initial cost of setting the portfolio. If the claim were to trade at any other price $f_1 \neq f_0$, this would lead to the existence of arbitrage opportunities (which by assumption do not exist). The fair price is thus also called the no-arbitrage price. The following definition is extracted from Musiela and Rutkowski (1997).

**Definition 3.7** - A self-financing trading strategy $\psi_t \in \Psi$ is called $Q$-admissible if the discounted investment process $\tilde{I}_t = I_t / B_t$ is a martingale under the probability measure $Q$, for all $t \in \{0, 1, 2, \ldots, T\}$.

It turns out that the measure $Q$ is quite crucial in the pricing of derivative securities. At this stage it suffices to state that $Q$ must be equivalent to the subjective probability measure $P$, that is, $Q$ and $P$ must have the same null sets in $\Omega$. In addition, under the measure $Q$ the following condition must hold.

$$\mathbb{E}_Q[B_t^{-1} S_t | \mathcal{F}_n] = B_n^{-1} S_n \quad \forall n < t,$$

thus under $Q$, the discounted price process must be a martingale.
Chapter 4

Complete-Market Pricing

4.1 Introduction

At this rudimentary stage it suffices to characterise a complete market as a market under which the return matrix \( R = (r_{ij}) \), which specifies the return of each asset in the various possible states of the economy, is a square matrix and there exist a unique equivalent martingale measure (EMM) under which the discounted asset price (stock price) process is a martingale. More importantly, we know from Hand and Jacka (1998) that the existence of an EMM ensures the lack of admission of arbitrage.

In the present section we utilise the Feynman-Kac theorem to derive the Black-Scholes PDE. The Feynman-Kac theorem establishes a connection between PDEs and stochastic processes. The theorem provides a procedure for solving certain SDEs which are usually related to probability transition densities of solutions of other SDEs. This is achieved by replicating the random paths of a stochastic process, a more detailed engagement can be found in Etheridge (2002) and Karatzas and Shrieve (1988).

4.2 Feynman-Kac Representation Theorem

Let \( \{S_t\}_{t \geq 0} \) be a stochastic process defined by

\[
dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t, \quad S_t = s.
\]
Then we define:

\[ F_t = F(t, s) = \mathbb{E}[f(S_T)|S_t = s] \]  

(4.2)

\[ = \int_{\mathbb{R}} f(x)p(t,T;s,x)dx, \]

where \( \mu(t, S_t), \sigma(t, S_t), f(s) \) are known functions and \( p(\cdot) \) is the transition probability function. Then we have

\[ \frac{\partial F}{\partial t} + \mu(t, s)\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2(t, s)\frac{\partial^2 F}{\partial s^2} = 0 \]  

(4.3)

\[ F(t, s) = f(s). \]

This can be restated as: Given the PDE (4.3) such that equation (4.1) is satisfied then the Feynman-Kac formula states that the solution can be written in the form of the expectation (4.2).

**Proposition 4.3.1**: Define a function \( F_t = \mathbb{E}_Q[f_T|S_t] \) under the probability measure \( Q \) such that \( \{S_t\}_{t \geq 0} \) is a Markov process. Then, \( \{F_t\} \) is a martingale.

**Proof**: From the Markov property we know that

\[ F_t = \mathbb{E}_Q[f_T|S_t] = \mathbb{E}_Q[f_T|\mathcal{F}_t]. \]

Thus by the Tower property of expectation on stochastic processes and re-applying the Markov property we can proceed as follows.

\[ \mathbb{E}_Q[F_t|\mathcal{F}_m] = \mathbb{E}_Q[\mathbb{E}_Q[f_T|\mathcal{F}_t]|\mathcal{F}_m], \quad 0 < m < t < T < \infty \]

\[ = \mathbb{E}_Q[f_T|\mathcal{F}_m] \]

\[ = \mathbb{E}_Q[f_T|S_m] \]

\[ = F_m. \]

That is, \( F_t \) is a martingale. \( \square \)
Lemma 4.3.2: Let \( c_t = c(t, s) \) denote the price of a contingent claim at time \( t \) and assume that the function \( F_t = \mathbb{E}_Q[f_T|S_t] \) is given, and \( \{S_t\}_{t \geq 0} \) is a GBM with drift and variance parameters \( rS_t \) and \( \sigma^2S_t^2 \) respectively then, \( F_t \) is a solution to the SDE:

\[
\frac{\partial F}{\partial t} = -rS_t \frac{\partial F}{\partial S_t} - \frac{1}{2} \sigma^2S_t^2 \frac{\partial^2 F}{\partial S_t^2}.
\] (4.4)

Proof: We know from equation (1.15) that

\[
c_t = \mathbb{E}_Q[e^{-r(T-t)} f(S_T)|S_t]
\]

\[
= e^{-r(T-t)} \mathbb{E}_Q[f_T|S_t]
\]

\[
= e^{-r(T-t)} F_t.
\] (4.5)

Now, by the Itô lemma we have

\[
dF_t = \frac{\partial F_t}{\partial t} dt + \frac{\partial F_t}{\partial S_t} dS_t + \frac{\partial^2 F_t}{\partial S_t^2} (dS_t)^2
\]

\[
= \left( \frac{\partial F_t}{\partial t} + rS_t \frac{\partial F_t}{\partial S_t} + \frac{\sigma^2S_t^2}{2} \frac{\partial^2 F_t}{\partial S_t^2} \right) dt + \sigma S_t \frac{\partial F_t}{\partial S_t} dW_t.
\]

We also know from proposition (4.3.1) that \( F_t \) is a martingale. Hence the deterministic part (that is, the drift) of \( dF_t \) must vanish or the coefficient of \( dt \) must be zero so that \( F_t \) takes the form \( F_t = F_0 + \int_0^t \varphi_j dW_j \), for some \( \mathcal{F}_t \)-previsable process \( \varphi_t \). This is only true if:

\[
\frac{\partial F_t}{\partial t} = -rS_t \frac{\partial F_t}{\partial S_t} - \frac{\sigma^2S_t^2}{2} \frac{\partial^2 F_t}{\partial S_t^2}.
\]

This concludes the proof. \( \Box \)

The above PDE turns out to be quite important in options pricing. For instance the Black-Scholes PDE is a consequence of equation (4.4). To see this we rewrite equation (4.4) in terms of the option price \( c_t \). First calculate the partial derivatives in the PDE given by equation (4.4) using the relation of equation (4.5) and then do back-substitution. This gives the following.

\[
\frac{\partial F_t}{\partial t} = \frac{F_t}{c_t} \left( \frac{\partial c_t}{\partial t} - r c_t \right),
\]

\[
\frac{\partial F_t}{\partial S_t} = \frac{F_t}{c_t} \frac{\partial c_t}{\partial S_t},
\]

\[
\frac{\partial^2 F_t}{\partial S_t^2} = \frac{F_t}{c_t} \frac{\partial^2 c_t}{\partial S_t^2}.
\]
Therefore, by substituting back into the Feynman-Kac expression (4.4) we get the Black-Scholes PDE given by equation.

\[
\frac{F_t}{c_t} \left( \frac{\partial c_t}{\partial t} - rc_t \right) = -r S_t \left( \frac{F_t}{c_t} \frac{\partial c_t}{\partial S_t} \right) - \frac{\sigma^2 S_t^2}{2} \left( \frac{F_t}{c_t} \frac{\partial^2 c_t}{\partial S_t^2} \right),
\]

\[
\frac{\partial c_t}{\partial t} + r S_t \frac{\partial c_t}{\partial S_t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 c_t}{\partial S_t^2} = rc_t.
\]

Equation (4.6) is the PDE that must be satisfied by the price of a derivative security. In other words, solving this PDE is one possible way of pricing contingent claims such as stock options, where for a call option we have:

\[
f(S_T) = \begin{cases} 
S_T - K & \text{if } K < S_T \\
0 & \text{otherwise.}
\end{cases}
\]

and \(K\) is the strike price associated with the option contract.

### 4.3 Risk-Neutral Pricing Function

This section presents the pricing function for a European call option on a stock price. It can also be easily shown that this price is the direct solution to the Black-Scholes PDE in equation (4.6), however, the solution is derived here via a different route.

We recall that under the RN pricing framework we are looking for a measure \(Q\) under which \(\tilde{S}_t\) is a martingale. The measure \(Q\) must be equivalent to the subjective measure \(P\) under which the original Brownian motion is defined. From the Itô formula we can compute \(d\tilde{S}_t\).

We know that \(\tilde{S}_t = B_t^{-1}S_t\) and therefore \(dB_t^{-1}S_t = B_t^{-1}dS_t + S_t dB_t^{-1}\), from this we get:

\[
d\tilde{S}_t = (\mu - r)\tilde{S}_tdt + \sigma\tilde{S}_td\tilde{W}_t
\]

That is, the requirement that \(\tilde{S}_t\) must be a martingale is satisfied if and only if the instantaneous rate of return of the stock is equal to that of the riskless asset, that is \(\mu dt = dB_t/B_t\) and \(dS_t = rS_t dt + \sigma S_t d\tilde{W}_t^Q\).

Now, if we define \(\tilde{W}_t^Q = \tilde{W} + \lambda t\) then, equation (4.8) can be written as

\[
d\tilde{S}_t = \sigma\tilde{S}_t d\tilde{W}_t^Q
\]
where $\lambda = \frac{\mu - r}{\sigma}$ is the market price of risk and $(\mu - r)$ is the market risk premium, this quantity is the additional interest charged over the riskfree rate and it reflects the amount of risk that individual investors associate with the risky asset $S_t$, see also Gitman (2003), Ross et al. (2000) and Joshi (2003). Therefore, using Girsanov’s theorem we have found a measure $Q$ under which the process $\tilde{S}_t$ is a martingale and we have shifted from a $P$-Brownian motion $W_t$ to a $Q$-Brownian motion $W_t^Q$. For the following theorem, see also Joshi (2003) or Baxter and Rennie (1996).

**Theorem 4.5.2**  Give a stock price process $\{S_t\}_{t \geq 0}$ which is adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ and a measure $Q$ equivalent to the market measure $P$, such that $W_t^Q$ is a $Q$-Brownian motion, then the R-N price of a European call option contingent on $S_t$, $c_t = c(t, S_t)$, with strike price $K$, maturity $T$ and volatility $\sigma$ is given by:

$$c_t = S_0 \Phi\left( \frac{\ln\left( \frac{S_t}{K} \right) + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \right) - e^{-rT} \Phi\left( \frac{\ln\left( \frac{S_t}{K} \right) + (r - \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \right)$$  \hspace{1cm} (4.10)

where $r$ is the riskfree interest rate, $\tau = (T - t)$ is the time to maturity, $S_{t=0} = S_0$ and $\Phi(\cdot)$ is the cumulative standard normal distribution or $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{z^2}{2}} dz$. Equivalently, the R-N price of a European put option $p_t(t, S_t)$ is given by:

$$p_t = e^{-rT} \Phi\left( \frac{\ln\left( \frac{K}{S_t} \right) - (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \right) - S_0 \Phi\left( \frac{\ln\left( \frac{K}{S_t} \right) - (r - \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \right)$$  \hspace{1cm} (4.11)

**Outline on the Proof of Theorem 4.5.2:** Firstly, we want to find the solution $\tilde{S}_t$ of equation (4.9). We do this intuitively and then verify the result through the Itô lemma. For the GBM we know from equation (3.7) that $S_t = s_0 e^{(r-\frac{1}{2} \sigma^2)t + \sigma W_t^Q}$, where $S_0 = s_0$. We note also that the expression for $d\tilde{S}_t$ is similar to $dS_t$ with $r$ replaced by zero. Thus, for $\tilde{S}_0 = s_0$ we can write

$$\tilde{S}_t = s_0 e^{(\sigma^2 W_t^Q - \frac{1}{2} \sigma^2) t}. \hspace{1cm} (4.12)$$

To validate our supposition we examine $\tilde{S}_t$ by determining $d\tilde{S}_t$ using the Itô lemma. Let $G(Y) = e^{Y}$ and $Y = \sigma W_t^Q - \frac{1}{2} \sigma^2 t$ then, $\frac{\partial G}{\partial Y} = 0$, $\frac{\partial G}{\partial t} = G$ and $\frac{\partial^2 G}{\partial t^2} = G$. Therefore by the Itô lemma we have $d\tilde{S}_t = \sigma \tilde{S}_t dW_t^Q$, subsequently it follows that

$$\frac{\tilde{S}_t}{S_t} = e^{\sigma (W_t^Q - W_0^Q) - \frac{1}{2} \sigma^2 (T-t)}.$$
Now, we know also that $\tilde{S}_T / \tilde{S}_t = S_T / se^{-r(T-t)}$ hence

$$S_T = se^{\sigma(W^Q_T - W^Q_t) + (r - \frac{1}{2}\sigma^2)(T-t)}.$$ 

Then, if we assume that the price of our contingent claim is denoted by $c_t = c(t, S_t)$ then $c_t = \mathbb{E}^Q[e^{-r(T-t)}f(S_T)|S_t = s]$. The process $\{S_t\}_{t \geq 0}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ since $S_t$ is $\mathcal{F}_t$-measurable for every $t$. Also, under the measure $Q$ we have $\mathbb{E}^Q[W^Q_T - W^Q_t|\mathcal{F}_t] = \mathbb{E}^Q[W^Q_T - W^Q_t]$, where $(W^Q_T - W^Q_t) \sim \mathcal{N}(0, \sqrt{T-t})$, and so for a European call option on the stock $S_t$ at strike $K$ we can write

$$c(t, s) = e^{-r\tau}\mathbb{E}^Q[(se^{\sigma\epsilon\sqrt{\tau} + (r - \frac{1}{2}\sigma^2)\tau} - K)1_{\{K < S_T\}}].$$  \hspace{1cm} (4.13)

Where $\epsilon \sim \mathcal{N}(0,1)$, $\tau = T - t$ and

$$1_{\{K < S_T\}} = \begin{cases} 
1 & \text{if } K < S_T \\
0 & \text{otherwise.}
\end{cases}$$

Equivalently, we can write

$$c(t, s) = e^{-r\tau}\int_{\mathbb{R}} f(se^{\sigma\epsilon\sqrt{\tau} + (r - \frac{1}{2}\sigma^2)\tau} - K)1_{\{K < S_T\}}(\frac{1}{\sqrt{2\pi}} e^{-\frac{\epsilon^2}{2}} d\epsilon),$$

which is simplified by first determining the correct limits of integration and then using appropriate change of variables, this gives

$$c(t, s) = s\Phi\left(\frac{\ln\left(\frac{K}{s}\right) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - e^{-r\tau}K\Phi\left(\frac{\ln\left(\frac{K}{s}\right) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right).$$

To prove the BS put price we utilise the put-call-parity relationship, see Hull (1997). Let us first define

$$d1 = \frac{\ln\left(\frac{K}{s}\right) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \hspace{1cm} (4.14)$$

$$d2 = d1 - \sigma\sqrt{\tau} \hspace{1cm} (4.15)$$

The parity states that ceteris paribus the following equation holds:

$$p_t + S_t = c_t + Ke^{-r(T-t)}.$$
From this it follows that

\[ p_t = c_t + Ke^{-r\tau} - S_t \]

\[ = S_t\Phi(d1) - Ke^{-r\tau}\Phi(d2) + Ke^{-r\tau} - S_t \]

\[ = Ke^{-r\tau}(1 - \Phi(d2)) - S_t(1 - \Phi(d1)) \]

\[ = Ke^{-r\tau}\Phi(-d2) - S_t\Phi(-d1). \]

This concludes the proof. \[\square\]

Equations (4.10) and (4.11) are the ubiquitous Black and Scholes (1973) formulae that have since received much attention from financial theorists. In the next chapter we explain the conditions under which the formula fails. The model’s shortcomings are mainly due to the stringent assumptions that the model is based upon. These are listed in the opening chapter. This is one of the reasons that there is continuous research in this field. These investigations usually approach this problem in two ways. The first involves providing complex alternatives of the pricing model that aim to capture the departure of Black-Scholes model from empirical evidence, statistics such as for instance, leptokurtosis, heavy-tailed distribution, etc. The second entails the broadening of the model to accommodate for a wider spectrum of derivative categories.

In the mini-thesis we are mainly based on the former approach and we are particularly focused on models that are geared to capture stochastic volatility and accommodate for the smile curve.

### 4.4 Implied Volatility

The pricing function \( c \) for a European call option has five parameters \( \text{viz. } c = c(S, K, r, \sigma, \tau) \) where \( \tau = T - t \) is the term-to-maturity. The components \( K \) and \( \tau \) are contract specific whereas \( S \) and \( r \) are determined by the market. The only parameter that turns to be quite intimidating to estimate is the volatility \( \sigma \). The author Davis (2004) shows that we necessarily have a mapping \( c : \sigma \mapsto c(\sigma) \).
If we observe the market price \( c_M \) of a particular European call with strike \( K \) and time-to-maturity \( \tau \) then we assume \( c_M = c(\sigma) \). That is, we assume that the market valuates claims according to the Black-Scholes model. This implicitly means that there is a volatility parameter \( I \) associated with any observed market price \( c_M \) and, if we formulate the Black-Scholes pricing model this is the volatility that must go into the model.

**Definition 4.7.1** - The market *Implied Volatility* (IV, denoted by \( I \)) for a European call option is defined by:

\[
\{ I > 0 : c(I) = c_M \}.
\]

In the research papers by Armerin (2004) and Davis (2004) the authors show that the pricing function is an increasing function of \( \sigma \). This essentially means that we can assume that we can find the inverse mapping \( v : c_M \mapsto I \), which maps the volatility image (observed market price \( c_M \)) onto its implied volatility \( I \). But of course the function \( c_M = c_{BS}(\sigma) \) cannot be inverted and expressed as a simple function of \( \sigma \). Hence the implied volatility can only be extracted or recovered through iterative methods.

In Chapter Six we study the pricing of equity-linked life insurance products with an asset value guarantee (ELIPAVGs) and we discover an equilibrium pricing model for such assets derived by Brennan and Schwartz (1976). This model establishes that the cash flows associated with these instruments are similar to those of European options. Brennan and Schwartz (1976) established that the price \( q_t \) of such a contract guarantee is equivalent to the price of a European put option \( p_t \) on the reference portfolio \( X_t \) at strike price equal to the guaranteed amount \( g_T \), which is payable only at maturity \( T \). That is,

\[
q_t = c_{BS} + e^{-rT} g_T - V_0(X_T).
\]

Here \( c_{BS} \) is the related call price and \( V_0(X_T) \) is the market value at time zero of the equity portfolio, see Brennan and Schwartz (1976) or Section (6.3) for the derivation.

**Theorem 4.7.2** Consider a European stock option with strike price \( g_t \) and maturity \( T \), such that the option cost is given as \( q_t \). Let the price \( S_t \) of the underlying asset be distributed as a 2-component normal mixture (see section (5.6.1)) of the form:

\[
\gamma_1 e^{S(\mu_1, \sigma_1)} + \gamma_2 e^{S(\mu_2, \sigma_2)}.
\] (4.16)
then the corresponding implied volatility $I$ (that is the volatility implied by the known price $q_t$ of the option) is obtained by the following iteration:

$$x_{k+1} = x_k - 2\xi(x_k) \int_{-\infty}^{x_k} e^{-\frac{z^2}{2}} dz + \sqrt{2\pi} \xi(x_k) \hat{q}_t,$$

(4.17)

where

$$\xi(x_k) = \frac{1}{\sqrt{\tau}} e^{\frac{x_k^2}{2}} \quad \text{and} \quad \hat{q}_t = 1 + q_t.$$

**Proof of Theorem 4.7.2:** We know that the time $t < T$ price of a European put option at strike $g_t$ and time-to-maturity $\tau = T - t$ is given by a weighted sum of put prices. Now, suppose that at any time $t$ we want to recover the implied volatility $I_t$ associated with the price $q_t$. That is, if given $q(\sigma_t) = q_t$ then, $I_t$ is the solution to the following problem

$$\gamma_1 p_{BS}(X_0, T, g_T, \sigma_1) + \gamma_2 p_{BS}(X_0, T, g_T, \sigma_2) = q(X_0, t, g_t, I_t).$$

(4.18)

In this case we have: $q(\sigma_t) = \sum_{j=1}^{2} \gamma_j p_{BS}(\sigma_j)$ where $\gamma_1, \gamma_2 = 1 - \gamma_1$ are the probability weights and $p_{BS}(\sigma_j)$ is the $j$-th Black-Scholes solution associated to the GBM, $dS_j = \mu_j S_j dt + \sigma_j S_j dW_j$. We can write this as:

$$q(I_t) = \gamma_1 p_{BS}(\sigma_1) + \gamma_2 p_{BS}(\sigma_2).$$

(4.19)

We keep in mind also that with each Black-Scholes price $p_{BS}$ there is an associated volatility parameter $\sigma_j$. For simplicity let us assume that we are given $X_0 = 1 = g_T$ and thus taking the (hypothetical) case where $r = 0$. We know that the left-hand side of equation (4.18) is determined in the Black-Scholes framework. Now recall:

$$q(\sigma_t) = g e^{-rT} \Phi(-d_2) - s_0 \Phi(-d_1),$$

(4.20)

where $d_1 = \frac{\ln\left(\frac{X_0}{g_T}\right) + (r + \frac{1}{2} \sigma^2)\tau}{\sigma \sqrt{\tau}}$, $d_2 = d_1 - \sigma \sqrt{\tau}$ and $\Phi(\cdot)$ is the cumulative standard normal density function or $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{z^2}{2}} dz$.

By simplifying using the given values we obtain the following results, viz. $d_1 = \frac{1}{2} \sigma \sqrt{\tau}$ and $d_2 = -\frac{1}{2} \sigma \sqrt{\tau}$ thus consequently,

$$\Phi\left(\frac{1}{2} \sigma \sqrt{\tau}\right) - \Phi\left(-\frac{1}{2} \sigma \sqrt{\tau}\right) = q_t.$$

(4.21)
From elementary properties of the Normal distribution equation (4.21) simplifies to

\[ 2\Phi\left(\frac{1}{2}\sigma\sqrt{\tau}\right) - 1 = q_t. \]

Now define

\[ F(x) = 2\Phi\left(\frac{1}{2}x\sqrt{\tau}\right) - (1 + q_t). \quad (4.22) \]

Then our intent is to determine the solution(s) of the function \( F \). To this end we utilise the iterative \textit{Newton-Raphson} method. That is, by starting from a suitable estimate \( x_0 \in \mathbb{R} \) we want to find a convergent sequence \( x_0, x_1, x_2, \cdots \) utilising the following iterative procedure.

\[ x_{k+1} = x_k - \frac{F(x_k)}{F'(x_{k+1})}, \quad k = 0, 1, 2, \cdots. \quad (4.23) \]

Let \( d = \frac{\sqrt{\tau}}{2}x \), then

\[
\frac{\partial F}{\partial x} = 2\frac{\partial \Phi}{\partial d} \frac{\partial d}{\partial x} = \sqrt{\frac{\tau}{2\pi}} e^{-\frac{x^2}{2}}. 
\]

Therefore the procedure given by equation (4.23) becomes

\[
x_{k+1} = x_k - \frac{2}{\sqrt{\tau}} e^{\frac{-x^2}{2}} \int_{-\infty}^{\frac{d}{\sqrt{\tau}}} e^{-\frac{1}{2}z^2} dz + \sqrt{\frac{2\pi}{\tau}} e^{\frac{-x^2}{2}} (1 + q_t).
\]

\[ \square \]

Accordingly, \( \mathcal{I} \) must therefore be the volatility parameter that must go into the Black-Scholes model. If we assume that the market supply and demand trade-off actually yields Black-Scholes prices then, for a fixed \( \tau \) we expect \( \mathcal{I} \) to remain fixed across \( K \). Empirical evidence suggests however that \( \mathcal{I} \) varies across strike and term-to-maturity, see Bates (2000), Das and Sundaram (1999), Derman and Kani (1994) and Hull and White (1987). This is reflected through the existence of the volatility curve (smile curve) or the nonflat volatility surface (term-structure).
4.5 The Implied Volatility “Smile” Phenomenon.

Figure 4.1: Simulated implied volatility smile curve, where $r = 27.6\%$, $S_t = 110$, $K = 118$, for a class of options that satisfy the weighted sum model of option pricing such in Guo (1998), see Chapter Five.

Figure (4.1) is an illustrative example of implied volatilities (vertical axis) of identical option contracts but with different strike prices. For instance, we see that the graph is downward sloping for at-the-money (ATM, $K = S_T$) and near-the-money (NTM, $95\% \leq K/S_t \leq 105\%$), it then curves upwards for far out-of-the money (OTM) options. This is a typical “post-crash” smile curve, sometimes called a “smirk”. According to Bates (2000) one consequence of this is that the distribution implicit on option prices since the 1987 crash is substantially negatively skewed, in relation to the relatively symmetric and slightly positively skewed lognormal distribution in the BSOPM.

According to Das (1998), Fouque et al. (2001) and Bates (2000) the volatility smile reflects a wide variety of factors which include amongst others:

- a change in investor’s assessment of the underlying stochastic processes which require adjustments for the distributional assumptions underlying standard option pricing
models;

- added effects of risk management associated to option hedging by traders. These result from a change in investor’s average risk aversion, and mispricing of post-crash options;

- supply and demand effects. The high demand for OTM (put) options tend to increase prices;

- directional expectations of market price movements. The skew or smirk in the volatility smile may be indicative of the market expectations regarding the direction of asset price movements underlying the option price and thus by connotation the implied volatility, see Das (1998).

In addition to the above listed factors a related factor which also plays a role is that of liquidity, in general, the volatility of ATM options is lower and this indicates the higher liquidity associated with such ATM options. The post-crash or “crash-phobic” smile curve tells us that there is a premium charged for OTM put options and in-the-money (ITM) calls ($K < S_T$), over and above the BS price computed with ATM implied volatility, see Fouque et al. (2001).

### 4.5.1 Implied Volatility Term Structure

In general, stock price volatility is higher for options with shorter time to expiration as compared to that of options with longer maturities. Conversely, options that mature later tend to exhibit lower implied volatilities, that those that have early maturities. The relationship between volatility and maturity describes the term structure of volatility.

The term structure of volatility curve exhibits the variation of implied volatilities of representative options against time-to-maturity. Information about the term structure of volatility may also be inferred from the rows implied volatility matrix, see Briys et al. (1998).

According to Briys et al. (1998) implied volatilities can be inferred across different strike prices and different maturities, which means that by using option prices we can construct a matrix of implied volatilities. The matrix is constructed such that the rows are ordered
by the strike price and the columns are ordered by time-to-maturity. In Chapter Five we present a numerical example for such a matrix.

Combining the two aspects, that is, the smile and the term structure of volatility produces what is called the volatility surface. The surface is typically produced with the intention of valuating option portfolios that may contain a range of options with different strikes and maturities, hence we shall not further deal with the surface for the purposes of this mini-thesis.
Chapter 5

Incomplete-Market Pricing: Stochastic Volatility Models

5.1 Introduction

In this section we first briefly discuss some SV models, others will be treated in more detail with illustrative sample computations, later on. The orthodox Black-Scholes option pricing model is premised on a number of model assumptions, as is illustrated in Chapter One. For instance, the model assumes that the stock prices are lognormally distributed with constant mean and volatility parameters.

However, a diverse spectrum of results from empirical studies (such as those by Davis (2004), Cox and Ross (1976), Guo (1998), Ritchey (1990), etc) of actual stock price behavior reflect contradictory evidence to this model. For instance, the volatility corresponding to actual stock price returns (i.e. the volatility implied by market data) varies across the strike price and term-to-maturity. This results in the smile curve observed in Chapter Four.

In addition to this and amongst others, the fact that asset prices jump from time to time rather than moving continuously, has sparked extensive interest in this area. A collection of models aimed at resting some of the intrinsic model assumptions of Black-Scholes have also been proposed by various authors and researchers such as Armerin (2004), Fouque et al. (2000b) amongst others. In particular, the assumption of a constant volatility parameter
has received much deserved attention.

In the endeavour to remedy this restriction in the model and explain the volatility smile, it has been proposed by these authors that volatility be modeled as random or stochastic. In other words, volatility ought to be expressed by a stochastic process. But for a model to be referred to as a stochastic volatility (SV) model it must incorporate or introduce further sources of randomness in addition to that of the Brownian motion driving the underlying asset (stock) prices. This however, will lead to valuation in an incomplete market. That is, there is no unique equivalent probability measure under which the discounted asset price process is a martingale. In fact, we instead have a class or family of equivalent martingale measures (EMMs) \( \mathbb{Q} \) and not all our contingent claims are attainable. For the purposes of this mini-thesis we shall assume the perspective also taken by Fouque et al. (2000b) of a “crash-o-phobic” market, that is, the market selects a unique martingale measure from a class of equivalent measures. It is however important to also note that (as stated by Hand and Jacka (1998)) in an incomplete market some derivative securities can be replicated or hedged and can thus be valued uniquely. The authors Hand and Jacka (1998) also provide a characterisation of such claims.

In an incomplete market, a claim \( F_t \) is hedgeable if \( \mathbb{E}^\mathbb{Q}_{W_t}[F_t | \mathcal{F}_t] \) remains constant as \( \mathbb{Q} \) runs through all EMMs in \( \mathbb{Q} \).

One of the factors which make volatility to be quite a challenging statistical quantity to deal with is that volatility is not directly observable in the market, as it is not a directly traded asset. Thus given the observed market data (such as the market price of a European call option), a question that often arises in finance is that: what does the market reveal to us in terms of volatility \( \sigma \)? It is the direct solution to this question that gives rise to the so called implied volatility, \( I \).

### 5.2 Literature Review of SV Models

As we alluded to above for a pricing model to be called a stochastic volatility model it must introduce new sources of randomness. In modeling the shift from the constant volatility
models to non-constant volatility pricing models some authors have pursued a more gentle route, by first moving to time-dependent volatility $\sigma(t)$ and then moving to state-dependent volatility $\sigma(t, S_t)$, see for instance Fouque et al. (2001) and Armerin (2004). These are called single-factor models and provide some valuable insight but do not introduce any additional sources of randomness. Models that incorporate new sources of randomness in addition to that driving the stock price process are termed multi-factor models and they therefore epitomise true stochastic volatility.

In theory there are various models that are aimed at counteracting some of the Black-Scholes shortcomings and examine the impacts thereof. As examples of such models we have *inter alia*, the Ritchey (1990) discrete normal mixture model in which a mixture call option pricing model is derived to assess the effects of non-Gaussian underlying returns distributions. This model will be further reviewed and presented in this mini-thesis. Also, there is the Guo (1998) finite Markov chain model which aims at modifying the latter model by substituting a finite Markov chain for Ritchey’s binomial probability tree.

Research in this area however is by no means a novelty. For instance, it only took three years after the Black and Scholes (1973) paper for Cox and Ross (1976) to release a research paper studying the valuation of options for alternative stochastic processes such as jump and diffusion processes. This paper identified that asset prices hardly vary continuously but rather jump from time to time. Other models also followed in these footsteps and generalised the Black-Scholes model in order to allow for stochastic volatility. These include Merton (1976), Hull and White (1987), etc.

It is important to note however that notwithstanding the ubiquity of the Black-Scholes model, the story did not begin with Black and Scholes (1973). There is also a diverse spectrum of precursors to the Black-Scholes model. These began as early as in 1900 with Louis Bachelier’s formula, which provided an analytical formula for the valuation of a European call option on a non-dividend paying stock. Others include *inter alia* Sprenkle (1961), Boness (1964), and Samuelson (1965).
5.3 Stochastic Volatility Model Setup

In a stochastic volatility model the volatility parameter is given by a process \( \{ \sigma_t \}_{t \geq 0} \), so that the revised dynamics of the stock price returns process are given by the following GBM.

\[
\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t, \quad S_0 > 0.
\] (5.1)

The volatility process \( \sigma_t = \sigma(Y_t) \) is taken to be a diffusion process and we let \( V_t = (W_t, Z_t) \) be a 2-dimensional Brownian motion where \( W_t \) and \( Z_t \) are uncorrelated Brownian motions.

Now define

\[
\frac{dY_t}{Y_t} = d(t, Y)dt + \nu(t, Y)dV_t, \quad (5.2)
\]

\[
dV_t = \rho dW_t + \sqrt{1 - \rho^2} dZ_t, \quad (5.3)
\]

\[
E[dW_t dV_t] = \rho dt, \quad (5.4)
\]

where \( \rho \in [-1, 1] \) is the instantaneous correlation between the returns \( dS_t/S_t \) and \( dY_t/Y_t \) and \( \sigma(y) > 0 \ \forall \ y \in \mathbb{R} \).

We essentially wish to assess the impact that stochastic volatility has on the Black-Scholes model and consequently we want to retain as much of the structure of this model as possible. We assume herein also that the riskless asset is given by \( dB_t/B_t = rd_t \), \( B_0 = 1 \). where \( r \) is constant and by the concept of change of numeraire (see Jacka (1996), Joshi (2003), or Briys et al. (1998)) we know that there is no loss of generality in choosing \( r \) this way.

5.3.1 Log-Ornstein-Uhlenbeck Stochastic Volatility Model

As was noted in the introduction of this chapter, any model that strictly seeks to embed stochastic volatility in its formulation must introduce new sources of randomness in addition to the Brownian motion governing the stock price returns. In the endeavour to model stochastic volatility researchers in this discipline have suggested various processes, including \textit{inter alia} the the \textit{Square-root} process (Cox-Ingersoll-Ross model), the log-normal \textit{Brownian motion} process (Hull-White model), the mean-reverting \textit{Ornstein-Uhlenbeck} process, and others. Here we briefly introduce the latter process. Under the auspices of this stochastic volatility framework we essentially let the volatility process be a function of the underlying
process, in this case this is the Ornstein-Uhlenbeck process \( \text{viz.} \)

\[
dY_t = \kappa (m - Y_t) dt + \eta dV_t,
\]

(5.5)

where \( \kappa > 0, \eta \) and \( m \) are constants. To put matters in context we write

\[
\sigma_t = \sigma(Y_t)
\]

(5.6)

but as also noted by Fouque et al. (2000b) this opens the likelihood that there exists \( y \in \mathbb{R} \) such that \( \mathbb{P}\{\sigma(y) < 0\} > 0 \), whereas we want \( \mathbb{P}\{\sigma(y) > 0\} = 1 \). To remedy this encumbrance we define \( Y_t = \log(\sigma_t) \) or

\[
\sigma(y) = e^y
\]

(5.7)

and we call \( Y_t \) the log-Ornstein-Uhlenbeck (log-OU) process.

### 5.4 Ritchey (1990): Discrete Normal Mixture Call Option Pricing Model

Mixture distributions commonly arise when attempts are made to model events from which the underlying data appears to emanate from a mixture of multiple populations.

For the purposes of stochastic volatility modeling (and risk management in general), mixture models arise from the universal observation that volatility appears to be moderately low on relatively “quiet days” and abnormally high on very “hectic days” for essentially the same number of days. Each member of the population is allocated a probability weight for instance, we may specify the probability that a given day will be quiet or hectic and then assume that the individual returns are normally distributed conditioned on the above universal observation.

In the endeavour to assess the impact of non-Gaussian returns densities Ritchey (1990) proposes a discrete \( k \)-component independent normal mixture call option pricing model to explain the deviation (“peaked” and “fat-tailed”) of returns relative to the Normal density.
5.4.1 Ritchey’s Model Setup

The normal mixture density \( f(x) \) is formed by a weighted sum of \( k \)-component normal densities \( f_j(x) \). The weights \( \gamma_j \) are assumed to be the relevant probabilities, so that \( \sum_{j=1}^{k} \gamma_j = 1 \). That is,

\[
f(x) = \sum_{j=1}^{k} \gamma_j f_j(x).
\]

In line with BSOPM framework and following specifications of Cox and Ross (1976) we define \( \eta \), the continuously compounded expected rate of return such that

\[
S_T = S_t e^{\eta \tau}, \quad \text{with} \quad \tau = T - t.
\]

The rate of return is thus normally distributed as

\[
\eta \sim \mathcal{N}\left( \frac{1}{2} (2\mu - \sigma^2), \frac{\sigma}{\sqrt{\tau}} \right).
\]

The call option price \( c_t \) is then determined under the Black-Scholes constructs as usual, that is, as the discounted expectation of the expiration value \( c_T = \max\{(S_T - K), 0\} \)

\[
c_t = e^{-\eta \tau} \mathbb{E}[c_T],
\]

By simplifying further for the \( n \)-period case, Ritchey shows that the resulting normal mixture call option valuation model is a weighted sum of the individual Black-Scholes prices,

\[
c_t = e^{-\varphi n} \sum_{j=1}^{m} \omega_j c_{BS}(\mu_j, \sigma_j) \mathbb{E}[c_T^{(j)}] \tag{5.8}
\]

where \( c_{BS}(\mu_j, \sigma_j) = \mathbb{E}[c_T^{(j)}] \) is the expected call option price at time \( T \) according to the Black-Scholes method, \( c_T^{(j)} \) is the call option price on an asset governed by a GBM with drift and volatility parameters \( \mu_j \) and \( \sigma_j \) respectively, \( \varphi \) is the underlying discounting rate, and \( \sum_{j=1}^{m} \omega_j = 1 \).

The attraction which makes the normal mixture model to be quite inviting is its consistency regarding the volatility character of distinct periods of low and high values, and the capacity
to account for the heavy tailed and sharp peaked densities of stock price returns. Moreover, by assuming that the distribution of daily returns is an independent $k$-component normal mixture then, consecutive periodic returns are uncorrelated and are normally distributed with stochastic volatility.

It is particularly this attribute of capturing the stochasticity of returns that is of interest in terms of our pricing demands. This also allows us to utilise the normal mixture distribution to price other instruments within the framework of stochastic volatility.

Ritchey’s model works well for a relatively small number of component distributions, but for multi-period ($n$-period) option pricing the non-combining binomial probability tree expands the number of component normal distributions to $Z^n$, forcing the mixture to converge rapidly to the normal distribution according to the central limit theorem. This compels Ritchey’s option pricing model to hastily converge to the Black-Scholes model. This was the motivation for Guo (1998) to propose a finite Markov chain model to replace Ritchey’s non-combining binomial tree.

5.5 Guo (1998): Finite Markov Chain Stochastic Volatility Model

As is customary of SV models, the model of Guo (1998) is a follow-up to the precursor Ritchey (1990) model. In his paper Chen Guo noted that when the number of component distributions is relatively large the Ritchey (1990) model quickly converges to the Black-Scholes model thereby losing its capacity to capture stochastic volatility. In attempting to remedy this finding Guo (1998) proposes to replace Ritchey’s binomial tree by a finite Markov chain with $k$ discrete volatility states. In this way, Guo (1998) maintains some of the properties of Ritchey (1990) but in Guo’s case the chain produces a fixed number of component distributions.

The model assumes that the underlying asset price follows a process of the form:

$$dS_t = S_t(\mu dt + \sigma(t)dW_t)$$
where the volatility $\{\sigma(t)\}_{t \geq 0}$ is assumed to follow a finite Markov chain specified by a one-step transition probability matrix $(P)$. This matrix is given by

$$
P = \begin{pmatrix}
p_{11}(\Delta t) & p_{12}(\Delta t) & \cdots & p_{1k}(\Delta t) \\
\cdots & \cdots & \cdots & \cdots \\
p_{i1}(\Delta t) & p_{i2}(\Delta t) & \cdots & p_{ik}(\Delta t) \\
\cdots & \cdots & \cdots & \cdots \\
p_{k1}(\Delta t) & p_{k2}(\Delta t) & \cdots & p_{kk}(\Delta t)
\end{pmatrix}
$$

with entries

$$
p_{ij}(\Delta t) = \mathbb{P}\{\sigma(t + \Delta t) = \sigma_j | \sigma(t) = \sigma_i\} \geq 0
$$

representing the probability that the volatility currently at state $\sigma_i$ will transit to state $\sigma_j$ over a fixed time interval $\Delta t$ in such a manner that the $k$ discrete volatility states are ordered from low to high that is, $\sigma_1 < \sigma_2 < \ldots < \sigma_k$. In this discrete time setting the $n$-step transition probability matrix $(P)^n$ becomes the $n$th power of $(P)$ according to the Chapman-Kolmogorov equation (CKE), see Kannan (1979).

Utilising the risk neutral pricing approach Guo (1998) determined the pricing function $F(S_t, K, r, \sigma(t), \tau)$ of a call option on $S_t$ at strike $K$ with time to maturity $\tau = T - t$ as:

$$
F(S_t, K, r, \sigma(t), \tau) = \sum_{j=1}^{k} p_{ij}(\Delta t)c(S_t, K, r, \sigma_j, \tau).
$$

(5.9)

Where $c(S_t, K, r, \sigma_j, \tau)$ is the usual BS price of the $j^{\text{th}}$ call option on the asset $S_t$, which is distinguishable from other options on this asset by its strike price and/or time to maturity. Therefore, according to Guo’s model the option price is a weighted sum of the Black-Scholes prices. The weights are represented by the entries of the probability transition matrix.

### 5.5.1 Numerical Computations

In this section we study numerical examples under the Guo (1998) model with the intention of investigating the volatility smile behaviour and the term structure of volatility. We assume
first that the stochastic volatility is expressed by a three-state Markov chain given by the following one-step probability matrix:

\[
P = \begin{pmatrix}
0.8500 & 0.1000 & 0.0500 \\
0.0000 & 0.7500 & 0.2500 \\
0.0250 & 0.0950 & 0.0250
\end{pmatrix}
\]

Now if a one-step transition equals one day, then according to Kolmogorov’s equation (see Kannan (1979)) for a contract with \( \tau \) days-to-maturity the transition matrix is given by \( P^\tau \). Thus if \( P \) is a one-step equals one day transition matrix and the underlying option contract has 30 days-to-maturity or 0.0822 years, then according to Kolmogorov’s equation the 30 days transition probabilities are given by the matrix \( P^{30} \), and \( P^{90} \) for 90 days and so on.

\[
P^{30} = \begin{pmatrix}
0.0394 & 0.7631 & 0.1975 \\
0.0329 & 0.7684 & 0.1987 \\
0.0330 & 0.7683 & 0.1987
\end{pmatrix}
\]

\[
P^{90} = \begin{pmatrix}
0.0331 & 0.7682 & 0.1987 \\
0.0331 & 0.7682 & 0.1987 \\
0.0331 & 0.7682 & 0.1987
\end{pmatrix}
\]

If for instance we consider the third row of \( P^{30} \) and volatility is currently at state \( \sigma_3 \) then, there is a 19.87% probability that it will maintain state \( \sigma_3 \) after a 30-day period, but there is also a 76.83% or 3.30% probability that the volatility will assume either state \( \sigma_2 \) or state \( \sigma_1 \) after 30-days respectively. The following table gives various Black-Scholes option prices (BS Prices), the option prices (Mix.Price) according to the Guo model using the transition matrix \( P^\tau \), and for the latter case we also calculate the corresponding implied volatilities (IV). In this example we assume that the market gives the following contract variables, that is \( K = 100, r = 10\%, S_0 = 105 \), and the volatility states are \( \sigma_1 = 0.05, \sigma_2 = 0.1, \) and \( \sigma_3 = 0.15 \).

The example in Table (5.1) displays the variation of the implied volatility and the Guo price with time to maturity (\( \tau \)). This also allows us to investigate the term-structure of the implied volatilities for the range of options generated by the model. In his paper Guo (1998) showed that the equilibrium option price depends on the current volatility state \( (\sigma(0), \) the
reference volatility), the one-step transition probabilities, and the time-to-expiration. The current example investigates the effects of varying the latter variable ($\tau$) on the quantities price and implied volatility, from this we also learn about the underlying term-structure of volatility, see also Table (5.2).

In line with our expectations the implied volatility in this model varies across time-to-maturity, and the display also exhibits the existing inverse relation between the volatility and the time to maturity. To further investigate the smile character of implied volatility we consider the model under changes in the strike price, see graphical illustration in Figure (5.5.1).

Table 5.1: Numerical calculations of the weighted sum call option normal mixture price under the model of Guo (1998), with corresponding implied volatilities, assuming that $K = 100$, $r = 10\%$, $S_0 = 105$.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>BS Prices</th>
<th>$\sigma(0) = \sigma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_1 = 5%$</td>
<td>$\sigma_2 = 10%$</td>
</tr>
<tr>
<td>90</td>
<td>7.4367</td>
<td>7.5900</td>
</tr>
<tr>
<td>60</td>
<td>6.6308</td>
<td>6.7242</td>
</tr>
<tr>
<td>30</td>
<td>5.8186</td>
<td>5.8442</td>
</tr>
<tr>
<td>25</td>
<td>5.6826</td>
<td>5.6987</td>
</tr>
<tr>
<td>20</td>
<td>5.5464</td>
<td>5.5547</td>
</tr>
<tr>
<td>15</td>
<td>5.4101</td>
<td>5.4131</td>
</tr>
<tr>
<td>10</td>
<td>5.2736</td>
<td>5.2740</td>
</tr>
<tr>
<td>7</td>
<td>5.1916</td>
<td>5.1916</td>
</tr>
<tr>
<td>5</td>
<td>5.1369</td>
<td>5.1369</td>
</tr>
<tr>
<td>3</td>
<td>5.0822</td>
<td>5.0822</td>
</tr>
</tbody>
</table>
Figure 5.1: Implied volatility “smile” curve generated from Guo mixture prices. For each $K$ we compute the corresponding Mix.Price for a fixed $\tau = 30$ days. This price is then fitted into the BS formula, from which we then iteratively search for a single volatility parameter that equates the price from the BS formula to the relevant Mix.Price.

Table 5.2: Representation of the calculated implied volatility matrix.

<table>
<thead>
<tr>
<th></th>
<th>$\tau = 10$</th>
<th>$\tau = 15$</th>
<th>$\tau = 20$</th>
<th>$\tau = 25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = 80$</td>
<td>0.2397</td>
<td>0.1944</td>
<td>0.1672</td>
<td>0.1485</td>
</tr>
<tr>
<td>$K = 100$</td>
<td>0.3941</td>
<td>0.3291</td>
<td>0.2921</td>
<td>0.2681</td>
</tr>
<tr>
<td>$K = 105$</td>
<td>0.1293</td>
<td>0.1341</td>
<td>0.1381</td>
<td>0.1416</td>
</tr>
<tr>
<td>$K = 110$</td>
<td>0.3624</td>
<td>0.3054</td>
<td>0.2739</td>
<td>0.2542</td>
</tr>
</tbody>
</table>

We recall from Section (4.5) that the matrix of implied volatilities has rows ordered by strike prices ($80, 100, 105$ and $110$) and columns ordered by time-to-maturities ($\tau = 10, 15, 20$, and $25$). For instance, if we consider the option with strike $K = 10.5$ and maturity $\tau = 15$ and $\tau = 20$ days, then the corresponding implied volatilities are $13.4\%$ and $13.8\%$ respectively. Continuing in this order we can thus draw inferences about the term-structure of the implied-
volatilities, as is suggested by Briys et al. (1998).

The exhibit in Figure (5.5.1) displays the variation of implied volatility across the strike price, and also shows how the Guo model produces different values depending on the reference (current) volatility. Consistent with other models in stochastic volatility literature the Guo (1998) model under the current example also bears testimony to the implied volatility smile effect. That is, the implied volatility is not fixed across the range of option prices, but varies with the strike price.

As is customary, relative to out-of-the-money options, at-the-money options are typically observed to trade with lower implied volatility. This effect becomes less pronounced for in-the-money options.

We can thus conclude that other models (such as Ritchey (1990) and Guo (1998) amongst others) have the capacity to account for the implied volatility smile effect, without explicitly utilising stochastic volatility modeling techniques such as those presented in Section (5.3). But, as is presented here, for the purposes of this mini-thesis we consider any model which captures the smile effect to be stochastic.
Chapter 6

Pricing of Equity-Linked Pension Policies

6.1 Introduction

Over the years researchers, actuarial scientists, analysts and financial engineers working for investment firms and other financial institutions have derived various products that allow for the sharing of the associated risk-return trade-off, and that enhance the attractiveness of their financial products. For instance, rate of return (ROR) guarantees are incorporated into various financial assets such as in life insurance agreements or contracts and other guaranteed investment securities. In quite a number of financial markets and all around the world (e.g. Norway, USA, UK, Canada, etc, see Melnikov and Romanyuk (2006) and Brennan and Schwartz (1976)) contemporary life insurance products embed a fixed percentage guarantee on each year’s return.

According to Miltersen and Persson (1997) in theory a minimum guarantee may be associated with any identified ROR such as that of stock prices, unit prices or mutual funds and various indices. We shall only consider here the minimum asset value guarantee (MAVG) connected to equity-linked pensions and in which the ROR guarantee is associated with actively traded assets in the market.
When provisioning for minimum benefit contracts, in for instance a pension fund, one of the parties may want to optimize a certain utility. Such a scenario is considered in the paper by Deelstra et al. (2003), in which the authors consider a pension plan for which the final wealth is subject to a sharing agreement between the pension fund manager and the holding member, together with a minimum guarantee on the holder’s benefit. In this mini-thesis we present a method of calculating the cost of a guarantee on a pension when the reference portfolio is subject to stochastic volatility.

Pension contracts may be financed by a single deposit contribution but in this mini-thesis we concern ourselves with a periodic contract. That is, the pension member is expected to make \( N \) periodic payments or contributions \( k_t \) at time \( t \) (i.e. when there are \( N = T - t \) terms to maturity) during the life of the contract. The fund manager then invests the members contributions in the reference portfolio \( X_t \), and the member is then expected to pay an additional premium \( q_t \) should he or she wish to receive the guaranteed benefit \( g_t \) on this contribution.

Hence Brennan and Schwartz (1976) characterised pure equity-linked life insurance policies as mere financial investment programmes. In return the member is thus guaranteed periodic minimum returns \( g_t \) depending on the market value of the reference portfolio at maturity \( T \) or at death (mortality factor). That is, the member can either receive the market value of the reference portfolio or the minimum guarantee depending on which is greater. The pension fund manager is thus clearly not completely risk-immune as it is both exposed to investment risk and mortality risk moreover, part of the member’s risk exposure is relaxed in this manner.

In their original paper Brennan and Schwartz (1976) derive equilibrium pricing formulae for an equity-linked insurance policy with an asset value guarantee. To this end, they recognised that the payable benefit in this model is equivalent to a fixed amount (the guaranteed amount) plus the risk neutral price of an immediately exercisable call option on the reference portfolio at strike price equal to the guarantee, or equivalently, the benefit is equal to the payoff of a European put option plus the market value of the reference portfolio.

The cost of the guarantee in this model is determined under the assumption that the econ-
omy allows underlying equity to vary according to the GBM with constant interest rate. Other authors such as Aase and Persson (1994) have extended their models to incorporate deterministic interest rates. In the paper by Kurz (1996), the author presents a model for a periodic case in the context of a stochastic interest rate process. This provides the author with necessary framework to derive a quasi-explicit closed form solution.

6.2 Pricing Model Setup

Henceforth we consider the case of pension funds, such that the holder’s entire contribution is invested in the portfolio, the pension holder is then expected to pay an additional premium for the guarantee. Ultimately, the maturity payable benefit $B_T$ will be the accumulated sum of the individual increments that accumulate at each payment period. The following notation are applied:

- $r$ - riskfree interest rate.
- $B_t$ - market price of zero-coupon bond with maturity $T$, $(B_t = e^{r(T-t)})$.
- $X_t$ - the reference portfolio.
- $S_t$ - market value at time $t$ of a unit of stock or fund referencing the portfolio.
- $G_t$ - minimum guarantee benefit at time $t$.
- $B_T$ - ultimate payable benefit at time $T$, (at maturity or mortality).
- $k_t$ - holder’s periodically contributed premium at time $t$.
- $q_t$ - associated accumulated cost of the guarantee.
- $V_t(B_T)$ - market value at time $t \leq T$ of the uncertain benefit.
- $V_t(G_T)$ - market value at time $t$ of the minimum guarantee.
- $V_t(X_T)$ - market value at time $t$ of the reference portfolio.
- $c_t(X,T,G)$ - call option price at time $t$, to buy $X_T$ at time $T$ at the strike price $G_T$.
- $p_t(X,T,G)$ - put option price at time $t$, to sell $X_T$ at time $T$ at the strike price $G_T$.

As noted above, the cashflows of the guarantee are to some extent related to cashflows of European options. To acquire the rights to receive the guarantee the holder of the policy is expected to inject $N$ periodic contributions or payments $k_t$ towards the policy. That is, we have a sequence $k_0, k_1, k_2, ..., k_{N-2}, k_{N-1}$ at premium payment dates $t_i$ such that $0 = t_0, t_1, t_2, ...,
In response the holder is deemed to receive periodic returns $g_0, g_1, g_2, \ldots, g_{N-2}, g_{N-1}$ payable at maturity $T$ such that $\sum_{t=0}^{N-1} g_t = G_T$. The payable benefit $B_T$ is defined by

$$B_T = \begin{cases} X_T & \text{if } G_T < X_T \\ G_T & \text{otherwise.} \end{cases} \quad \text{(6.1)}$$

The objective here is thus two-fold, the first is to determine the amount $q_t$ that the firm charges for the risk borne in providing the guarantee. This price will be determined in the Brennan and Schwartz (1976) framework. The second is to extend this model to embed stochastic volatility to account for the “smile” curve behaviour.

### 6.3 The Brennan and Schwartz Framework

In their treatise Brennan and Schwartz (1976) considered the equilibrium pricing of equity-linked life insurance policies with an asset value guarantee (ELIPAVG). We assume that our market is arbitrage-free, perfectly competitive and frictionless with non-dividend paying assets, and is free from mortality risks during the life of the pension contract. Then the benefit $B_t$ is decomposed into a put option plus the market value of the portfolio, or into a sure amount plus an immediately exercisable call option on the reference portfolio.

Define the time-to-maturity $\tau = T - t$ and let $(x)^+ = \max\{x, 0\}$. Given that $B_T = \max\{X_T, G_T\}$ then we can write, $B_T = G_T + (X_T - G_T)^+$ and similarly, $B_T = X_T + (G_T - X_T)^+$. From these expressions of $B_T$ we identify the terms $(X_T - G_T)^+$ and $(G_T - X_T)^+$ as the maturity values a European call and put option respectively, contingent on the reference portfolio $X_T$ at the strike price equal to the guarantee $G_T$. Upon discounting to the present value of the benefit, that is when the time-to-maturity $\tau = T$ ($t = 0$) we get the following:

$$V_0(B_T) = e^{-r_T}G_T + c(X_0, T, G_T), \quad \text{(6.2)}$$

$$= V_0(X_T) + p(X_0, T, G_T). \quad \text{(6.3)}$$

By first assuming that the assets underlying the investment portfolio follows the usual GBM, and then forming a hedging strategy comprising of the portfolio, the call option and the risk-free asset (bond) in a combination such that the net investment is zero and the portfolios
return is deterministic, Brennan and Schwartz (1976) were able to show how to find the call price satisfying the Black-Scholes PDE.

The quantity $G_T$ is known, and $c(X_0, T, G_T)$ amounts to the Black-Scholes call option price. Moreover, the amount $V_0(X_T)$ reduces to the present value of the pre-determined investments to be made towards the reference portfolio. Thus in essence the unknown $p(X_0, T, G_T)$ is equal to the additional amount ($q_t$) levied by the insurance firm for providing the guarantee. From the equations (6.3) and (6.3) we can thus write

$$p(X_0, T, G_T) = c(X_0, T, G_T) + e^{-rT}G_T - V_0(X_T).$$

(6.4)

Thus $q$ is equivalent to a Black-Scholes put option value $p_{BS}(X_0, T, G_T)$ on the reference portfolio $X_0$ at a strike price equal to the guaranteed amount $G_T$ or $q = p_{BS}(X_0, T, G_T) = c(X_0, T, G_T) + e^{-rT}G_T - V_0(X_T)$. Alternatively,

$$q_t = c_{BS} + e^{-rT}G_T - V_0(X_T).$$

(6.5)

For a contract that is structured in such a way that only part of the members contribution goes into the reference portfolio and the difference is assumed to be the charged premium, then it is important to note as alluded by Kurz (1996) that for such a contract, the invested amount $\delta$ re-adjusts the values of the European options and has to be accounted for in the pricing these pensions. To briefly illustrate this we consider the value of the reference portfolio $X_t$. This quantity depends on the predetermined amount $\delta$, the time $t$ price per unit fund of stocks $S_t$ in the portfolio, and the unit prices at previous contribution payment dates $t_i$. That is,

$$X_t = \sum_{i=0}^{\hat{t}_i-1} \frac{S_t}{S_{t_i}} \delta,$$

where $\hat{t}_i$ is the most immediate contribution payment date after time $t$ or $\hat{t}_i = \min\{i|t_i \in (t, T]\}$. It therefore follows that the values of $c_{BS}$ and $p_{BS}$ are upset since for instance

$$p_{BS}(X, T, G) = (G - X)^+ = (G - \sum_{i=0}^{\hat{t}_i-1} \frac{S_t}{S_{t_i}} \delta)^+. $$

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The author proceeds and shows (using the risk-neutral valuation method) how to determine the option value in these conditions. The option price incorporates the weighted discounted investment \( \delta \) and an adjusted guarantee, this is validated under the assumption that the economy yields a stochastic term structure of interest rates.

### 6.4 Numerical Computations

We consider for example, a 6-period problem. We know that the pension member contributes periodic payments \( k_t \) and expects in return periodic guarantees \( g_t \) at times \( t \in \{0, 1, \cdots, 5\} \), which are payable only at maturity \( T \). We assume also that \( \sum_{t=0}^{5} k_t = \Gamma \) is the total contribution towards the policy. For simplicity and illustrative purposes we also assume that there is no growth factor associated with the \( k_i \)'s and \( g_i \)'s, so that \( k_i = k_j \) and \( g_i = g_j \) for all \( i, j \in \{0, 1, \cdots, 5\} \) such that \( i \neq j \). Thus \( k_t = \Gamma/6 \) and \( g_t = G/6 \).

Thus, in this example we consider the pricing of a simple asset-value guarantee where we assume that the holder makes equal contributions \( k_t \), all of which are invested in the reference portfolio by the fund manager. The fund manager charges additional premiums \( q_t \) against the member’s guarantee \( g_t \). We assume that the initial contribution towards the portfolio is 1000 units, the market interest rate is 3%. Therefore, \( k_t = 1000 \) and \( \Gamma = 6000 \) for a policy with 6 periods (years) to maturity. For argument sake, if \( g_t = 990 \) and market volatility is 5% then the price of the guarantee is 2.8937 units. If we consider the same conditions but a shorter maturity such as three periods, we see that the price jumps to 5.0586 units. This is inline with expectations since short-term contracts tend to be riskier than their longer-maturities counterparts, and hence are priced higher.

The following Table (6.1) displays the effects of varying the guarantee \( g_t \) on the price \( q_t \) against time-to-maturity, and also depicts the consequences on the price resulting from using different volatilities.

Table (6.1) displays the variation of the cost of providing the guarantee with changes in the guaranteed amount and maturity. From the table data we see that as the time-to-maturity increases the guarantee price tends to decrease, as is explained above. In addition
we also note that as the corresponding call option moves further away from the money, the guarantee price increases. This is also anticipated since the underlying cash-flows are associated to put options. Finally, we consider for example the ATM (Guar = 1000) prices 1.1626, 3.5091, 7.1480 and 11.8365 corresponding to 6 years maturity and with volatilities 4%, 5%, 6% and 7% respectively. We see that ceteris paribus the guarantee price varies positively with increasing volatility and vice-versa, thus such a guarantee price also needs to be adjusted to capture the stochastic nature of volatility, this is the object in the next section and is influenced by Guo (1998).

Table 6.1: Computation of premiums charged per asset-value minimum guarantee under the standard Brennan and Schwartz (1976) model, for constant sequences of guarantee increments. The contributions are \( k_t = 1000 \) for \( t = 0, 1, 2, 3, 4, 5 \) and \( r = 3\% \), “Guar.” represents the present value of the guarantee, “Total.guar.” is the minimum guarantee, \( q(j) \) is the guarantee price for a policy maturing in \( j \) years, and “SD.Price” is the guarantee price at time zero and equals the sum of discounted prices \( q(j) \).

<table>
<thead>
<tr>
<th>Volatility</th>
<th>Guar.</th>
<th>Total.Guar.</th>
<th>( q(1) )</th>
<th>( q(2) )</th>
<th>( q(3) )</th>
<th>( q(4) )</th>
<th>( q(5) )</th>
<th>( q(6) )</th>
<th>SD.Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma = 4% )</td>
<td>1000</td>
<td>6000</td>
<td>5.1680</td>
<td>4.0683</td>
<td>3.0204</td>
<td>2.2067</td>
<td>1.6033</td>
<td>1.1626</td>
<td>15.9153</td>
</tr>
<tr>
<td>( \sigma = 5% )</td>
<td>990</td>
<td>5940</td>
<td>5.8797</td>
<td>5.7811</td>
<td>5.0586</td>
<td>4.2579</td>
<td>3.5252</td>
<td>2.8937</td>
<td>25.0077</td>
</tr>
<tr>
<td>( \sigma = 5% )</td>
<td>1000</td>
<td>6000</td>
<td>8.3067</td>
<td>7.5635</td>
<td>6.3939</td>
<td>5.2791</td>
<td>4.3147</td>
<td>3.509</td>
<td>32.3546</td>
</tr>
<tr>
<td>( \sigma = 5% )</td>
<td>1010</td>
<td>6060</td>
<td>11.3884</td>
<td>9.7203</td>
<td>7.9856</td>
<td>6.4829</td>
<td>5.2387</td>
<td>4.2253</td>
<td>41.2923</td>
</tr>
<tr>
<td>( \sigma = 5% )</td>
<td>1050</td>
<td>6300</td>
<td>31.0230</td>
<td>22.8243</td>
<td>17.4130</td>
<td>13.4983</td>
<td>10.5620</td>
<td>8.3170</td>
<td>41.2923</td>
</tr>
<tr>
<td>( \sigma = 5% )</td>
<td>1100</td>
<td>6600</td>
<td>69.8107</td>
<td>50.1861</td>
<td>37.3578</td>
<td>28.3911</td>
<td>21.8666</td>
<td>16.9987</td>
<td>207.3532</td>
</tr>
<tr>
<td>( \sigma = 5% )</td>
<td>1150</td>
<td>6900</td>
<td>116.2740</td>
<td>87.8034</td>
<td>66.6166</td>
<td>51.0380</td>
<td>39.4456</td>
<td>30.7040</td>
<td>361.2746</td>
</tr>
<tr>
<td>( \sigma = 6% )</td>
<td>1000</td>
<td>6000</td>
<td>11.6884</td>
<td>11.6185</td>
<td>10.6059</td>
<td>9.4049</td>
<td>8.2287</td>
<td>7.1480</td>
<td>53.3723</td>
</tr>
<tr>
<td>( \sigma = 7% )</td>
<td>1000</td>
<td>6000</td>
<td>16.0337</td>
<td>15.3902</td>
<td>15.2174</td>
<td>14.2931</td>
<td>13.0649</td>
<td>11.8365</td>
<td>77.7419</td>
</tr>
<tr>
<td>( \sigma = 7% )</td>
<td>1150</td>
<td>6300</td>
<td>117.8641</td>
<td>95.2801</td>
<td>79.1948</td>
<td>66.9022</td>
<td>57.1025</td>
<td>49.0847</td>
<td>425.9751</td>
</tr>
</tbody>
</table>
6.5 Extending the Brennan-Schwartz Model

In order to expand the above scenario and implicitly embed stochastic volatility we apply the Guo (1998) finite Markov chain model on the reference portfolio. To do this we emanate from the premises provided by the following theorem. We assume all the processes and variables declared in the above sections particularly Sections (4.5), (5.7), and (6.2).

**Theorem 6.5.1** Suppose that $X_t$ is an equity portfolio that is referenced by stocks that are distributed as $n$-component mixtures driven by processes of the form:

$$dS_t^{(j)} = \mu^{(j)} S_t^{(j)} dt + \sigma^{(j)}(t) S_t^{(j)} dW_t^{(j)}, \quad (6.6)$$

where each $\sigma^{(j)}(t)$ is assumed to follow a finite Markov chain specified by an $n$-step transition probability matrix $\bar{P}_n$ with probability entries $p_{ij} = \gamma_j$ and $M$ volatility states. Then the stochastic volatility price of providing the minimum guarantee $G_T$ for a periodic ELIPAVG is given by:

$$q_t(\sigma_t, X, T, G) = \sum_{j=1}^{M} \gamma_j (c(\sigma^{(j)}(t), X, T, G) - V_0(X_T^{(j)})) + V_0(G_T). \quad (6.7)$$

**Proof of Theorem (6.5.1):** Suppose that we are given the one-step probability transition matrix $\bar{P}$, we know from the Kolmogorov’s equation that getting $\bar{P}_n$ is straightforward. Now assuming that the volatility is currently at state $\sigma(0) = \sigma_i$ for some $i \in \{1, 2, \ldots, M\}$, we can extract the probability weights $\gamma_j$ from $\bar{P}_n$. Therefore by implication we know from Guo (1998) that:

$$p(\sigma_i) = \sum_{j=1}^{M} \gamma_j p_{BS}(\sigma^{(j)}, X, T, G), \quad (6.8)$$

where $p_{BS}(\sigma^{(i)}, X, T, G) = p_{BS}^{(i)}$ is the Black-Scholes put option price on $X$ and at strike price $G$, that corresponds to the volatility $\sigma^{(i)}(t)$, and $\gamma_M = 1 - \sum_{i=1}^{M-1} \gamma_i$. Now by equation (6.4) we know that $q^{(i)} = p_{BS}^{(i)}$ and therefore from equation (6.5) we can write:

$$q^{(j)} = c_{BS}(\sigma^{(j)}, X, T, G) + e^{-rT} G_T - V_0(X_T^{(j)}).$$

From equation (6.8) we thus have:
\[
q_t(\sigma_t, X, T, G) = \sum_{j=1}^{M} \gamma_j (c_{BS}(\sigma^{(j)}, X, T, G) + e^{-rT}G_T - V_0(X_T^{(j)}))
\]

\[
= \gamma_1 q^{(1)} + \gamma_2 q^{(2)} + \ldots + (1 - \gamma_1 - \gamma_2 - \ldots - \gamma_{M-1})q^{(M)}
\]

\[
= \gamma_1 (c_{BS}^{(1)} - V_0(X_T^{(1)})) + \gamma_2 (c_{BS}^{(2)} - V_0(X_T^{(2)})) + \ldots + \gamma_M (c_{BS}^{(M)} - V_0(X_T^{(M)})) + (\gamma_1 + \gamma_2 + \ldots + \gamma_M)e^{-rT}G_T
\]

\[
= \sum_{j=1}^{M} \gamma_j (c_{BS}^{(j)} - V_0(X_T^{(j)})) + e^{-rT}G_T \sum_{j=1}^{M} \gamma_j.
\]

Now since \(\sum_{j=1}^{M} \gamma_j = 1\) and \(V_0(G_T) = e^{-rT}G_T\) it therefore follows that

\[
q_t(\sigma_t, X, T, G) = \sum_{j=1}^{M} \gamma_j (c_{BS}^{(j)} - V_0(X_T^{(j)})) + V_0(G_T),
\]

which concludes the proof of theorem (6.5.1).

\[\square\]

### 6.6 Incorporating Stochastic Volatility

Let us consider the model of Brennan and Schwartz (1976) such that the dynamics of the investment portfolio are described by a mixture weighted by Guo’s finite Markov transition matrix. Let the one-step transition probability matrix \(P\). We shall fix the matrix \(P\) and write its powers as follows,

\[
P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P^t = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}.
\]

In Chapter Five we presented the model of Guo (1998), which is briefly outlined as follows. Assuming that the length of steps is one year and the transition matrix is as above, then for instance the second row of \(P^t\) indicates that if \(\sigma(0) = \sigma_2\) then, the probability that the volatility will be at \(\sigma_2\) after \(t\) years is \(100d_t\%\) and, there is a \(100c_t\%\) probability that volatility will be at state \(\sigma_1\) after six years. Therefore option prices are calculated over the period \([0, t]\) as follows,

\[
c_t BS(\sigma_1, t) + d_t BS(\sigma_2, t),
\]

where \(BS(\sigma, t)\) is the Black-Scholes price calculated using the value \(\sigma\) for volatility.
In this section we modify this scenario to take into account the contribution to be made at time \(0 < t \leq T\) relative to the term-to-maturity \((T - t)\), such that \(\sigma(t)\) is unknown. That is, there is uncertainty about the state of volatility at time \(t\). In such a case we take the weighted sum of the prices given by the various volatility states. Theorem (6.1) provides the necessary toolkit for such a scenario.

Consider the case with two volatility states \(\sigma_1\) and \(\sigma_2\), and for instance a pension accumulating over \(n\) sub-periods. The following theorem describes how to calculate the cost of an incremental guarantee. In this case we also know that \(b_t = 1 - a_t\), \(d_t = 1 - c_t\). Furthermore, we have \(a_0 = 1\) and \(c_0 = 0\).

**Theorem 6.6.1:** The calculation at time \(t = 0\) of the price of an incremental guarantee, given that \(\sigma(0) = \sigma_1\), is as follows:

\[
\text{SD.Price} = \sum_{t=0}^{n-1} e^{-rt} \{ a_t q_{sv1}(t) + (1 - a_t) q_{sv2}(t) \}
\]

(6.9)

where \(q_{sv1}\) and \(q_{sv2}\) are as follows:

\[
q_{sv1}(t) = a_{n-t} BS(\sigma_1, t) + (1 - a_{n-t}) BS(\sigma_2, t),
\]

(6.10)

\[
q_{sv2}(t) = c_{n-t} BS(\sigma_1, t) + (1 - c_{n-t}) BS(\sigma_2, t).
\]

(6.11)

where \(r\) is the riskfree interest rate, \(BS(\sigma_i, t)\) is the Black-Scholes price calculated at \(\sigma_i\) with \(\tau = T - t\) to maturity. That is

\[
BS(\sigma_i, t) = e^{-r(n-t)} g_t \Phi(-\alpha_2) - k_t \Phi(-\alpha_1),
\]

(6.12)

where \(\alpha_1 = \frac{\ln(k_t g_t) + (r + \frac{1}{2} \sigma_i^2)(n-t)}{\sigma_i \sqrt{t}}\), \(\alpha_2 = \alpha_1 - \sigma_i \sqrt{n-t}\), and \(\Phi(\alpha_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha_i} e^{-z^2/2} dz\).

**Proof of Theorem 6.6.1:**

Suppose that \(\sigma(t) = \sigma_i\). Then the Black-Scholes price of the particular incremental guarantee is \(BS(\sigma_i, t)\), described by equation (6.12). Therefore, the price of the \(i\)-th guarantee is the weighted sum which in the theorem is denoted \(q_{svi}(t)\), calculated at time \(t\) using the entries of the \(i\)-th row of transition matrix \(P^{n-t}\) as weights. At time \(t = 0\), the value of \(\sigma(t)\) is unknown. Thus at time \(t = 0\) we must discount these prices and weight them accordingly.

In the formula (6.9) we reflect first the discounting factor outside the braces and then the weighted sum of the \(q_{svi}\) prices. This concludes the proof. \(\square\)
6.7 Numerical Example

Consider a 6 period case, such that

\[ P = \begin{pmatrix} 0.8500 & 0.1500 \\ 0.1000 & 0.9000 \end{pmatrix} \]

Now if the volatility is currently in state \( \sigma_1 \) or \( \sigma(0) = \sigma_1 \), \( r = 3\% \), \( k_t = 1000 \), \( \sigma_1 = 4\% \), \( \sigma_2 = 6\% \), then for \( g_t = 990 \) units and looking at cashflows 4 years forward then the adjusted guarantee price \( q_{sv}(4) \) is 2.4010 units. The following table expands this scenario and depicts the stochastic volatility adjusted guarantee prices under varying conditions. Table (6.2) also shows the corresponding variation of the time zero premium (SD.Price) charged for the guarantee. In comparison to the standard model of Brennan and Schwartz (1976) in Table (6.1) the extended model shows that the guarantee price is less sensitive to changes in volatility states. This is expected since the model is assumed to capture the stochastic demeanour of volatility.

Table 6.2: The pricing of equity-linked pension policies under the extended model of Brennan and Schwartz (1976) that captures stochastic volatility. The contributions are $k_t = 1000$ for $t = 0, 1, 2, 3, 4, 5, 6, 7$ and $r = 3\%$, $\sigma(0) = \sigma_2$. “Guar.” represents the present value of the guarantee, “Total.guar.” is the minimum guarantee, $\sigma_{it} = \sigma^{(i)}(t)$, $q_{sv}(j)$ is the stochastic volatility adjusted guarantee price for a policy maturing in $T-j$-years, and “SD.Price” is the guarantee price at time zero and equals the sum of discounted prices $q_{sv}(j)$.

<table>
<thead>
<tr>
<th>$\sigma_{1t}$</th>
<th>$\sigma_{2t}$</th>
<th>Guar.</th>
<th>Total.Guar.</th>
<th>$q_{sv}(1)$</th>
<th>$q_{sv}(2)$</th>
<th>$q_{sv}(3)$</th>
<th>$q_{sv}(4)$</th>
<th>$q_{sv}(5)$</th>
<th>$q_{sv}(6)$</th>
<th>SD.Price</th>
</tr>
</thead>
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<tr>
<td>0.04</td>
<td>0.06</td>
<td>990</td>
<td>5940</td>
<td>5.2068</td>
<td>4.4710</td>
<td>3.3770</td>
<td>2.4010</td>
<td>1.5974</td>
<td>0.9427</td>
<td>5.8754</td>
</tr>
<tr>
<td>0.04</td>
<td>0.06</td>
<td>1000</td>
<td>6000</td>
<td>7.2698</td>
<td>5.7670</td>
<td>4.2001</td>
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<td>6.2472</td>
</tr>
<tr>
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<td>1010</td>
<td>6060</td>
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<td>5.2003</td>
<td>3.5315</td>
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<tr>
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<td>0.06</td>
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<td>6900</td>
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Conclusion

In the context of derivative asset pricing we have illustrated how crucial the volatility estimate is in the valuation of option instruments. In particular, in an economy governed by stochastic volatility as is observed through the implied volatility “smile” effect or the stochastic development of the term-structure of implied volatility, we have succinctly demonstrated how to price alternate instruments that exhibit option characteristics (real options) such as equity-linked pensions. Utilising Maple XI programming and simulation methods we generated the “smile” curve and drawn inferences about the term-structure of implied volatility from the computed matrix of implied volatilities, from which we also confirmed that implied volatilities decreases as time-to-maturity increases.

In this mini-thesis we also reviewed the standard works of Guo (1998) option pricing model and Brennan and Schwartz (1976) ELIPAVG pricing model. By modifying the ordinary geometric Brownian motion in Brennan and Schwartz (1976) by Guo’s finite Markov chain that describes the stochastic volatility, we managed to extend the model of Brennan and Schwartz (1976) such that it captures the stochastic character of volatility.

In addition, we showed the deviation in numerical results between the ordinary Brennan-Schwartz prices and the extended stochastic volatility version, through tabulated results. From this we established that the extended model is less sensitive to changes in volatility states, as the model is already conformed for such behaviour.

We thus conclude as in Briys et al. (1998) that other models can consistently account for the implied volatility “smile” effect without directly incorporating stochastic volatility as is directly done in Hull and White (1987) or Section (5.3). Moreover, such models including Guo (1998) can be readily used to adapt the ability of standard constant volatility models to capture the “smile” effect.
Bibliography


