Mathematical Models of Credit Management and Credit Derivatives

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A thesis submitted in fulfillment of the requirements for the degree of Master of Science (Computational Finance) in the Department of Mathematics and Applied Mathematics, University of the Western Cape.

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Declaration

I declare that *Mathematical Models of Credit Management and Credit Derivatives* is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Thembalethu Khatywa

December, 2009

Signed: ....................
Acknowledgement

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Abstract

The first two chapters give the background, history and overview of the dissertation, together with the necessary mathematical preliminaries. Thereafter, the next four chapters deal with credit risk and credit derivatives. The final part of the dissertation is devoted to the Basel II bank regulatory framework and the mathematical modeling of asset allocation in bank management, pertaining to credit risk.

Credit risk models can be categorized into two groups known as structural models and reduced form models. These models are used in pricing and hedging credit risk. In this thesis we review a variety of credit risk instruments described by models of the said types. One of the strategies utilized by companies to mitigate credit risk is by using credit derivatives. In this thesis, five main types of risk derivatives have been considered: credit swaps, credit linked notes, credit spreads, total return swaps and collateralized debt obligations. Valuation models for the first three derivatives that are mentioned above, are also presented in this dissertation.
The material presented include some of the most recent developments in the literature. Our methods range from single-period modeling to application of stochastic optimal control theory. We expand on the material presented from the literature by way of simplifying or clarifying proofs, and by adding illustrative examples in the form of calculations, tables and simulations. Also, the entire Chapter 6 is a new original contribution to the existing literature on mathematical modeling of credit risk.

*Key words:* credit risk; default risk; structural approach; reduced form approach; incomplete information approach; investment strategy; Basel II regulatory framework
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Chapter 1

Introduction, background and scope of the thesis

1.1 Introduction

Modern credit risk models can be partitioned into two groups known as structural models and reduced form models. Structural Models were pioneered by Merton [30, 1974]. The basic idea, common to all structural-type models, is that a company defaults on its debt if the value of the assets of the company falls below a certain threshold value, the default point. In these models it has been demonstrated that default can be modeled as an option and as a result, researchers were able to apply the same principles used for option pricing to the valuation of risky corporate securities.

The application of option pricing theory avoids the use of risk premium and tries to use other marketable securities to price the option. The use
of the option pricing theory set forth by Merton provides a significant improvement over traditional methods for valuing default risky bonds. It also offers not only much more accurate prices but provides information about how to hedge out the default risk which was not obtainable from traditional methods. Subsequent to the work of Merton [30], there have been many extensions and a few of these extensions will be described in this thesis.

The second group of credit risk models are known as reduced form models and are more recent. These models, most notably the Jarrow-Turnbull [25, 1995] and Duffie-Singleton [14, 1997] models, do not look inside the firm. Instead, they model directly the likelihood of default or downgrade. Not only is the current probability of default modeled, but some researchers attempt to model a forward curve of default probabilities that can be used to price instruments of varying maturities.

1.2 Background

Credit risk is the possibility of financial losses due to unexpected changes in the credit quality of a counterparty in a financial agreement.

1.2.1 Types of credit risk

There are three main types of credit risk, namely Default risk, Downgrade risk, and Credit spread risk. These are described in Anson [1, 2004] as follows.
**Default risk** - is the risk that the issuer of a bond or a debtor on a loan will not repay the outstanding debt in full. Default risk can be complete in that no amount of the bond or loan will be repaid, or it can be partial in that some portion of the debt will be recovered.

**Downgrade risk** - is the risk that a nationally recognized statistical organisations such as Standard & Poor’s, Moody’s Investors, or Fitch Ratings reduces its outstanding credit rating for an issuer based on an evaluation of that issuer’s current earning power versus its capacity to pay its debt obligations as they become due.

**Credit spread risk** - is the risk that the spread over a reference rate will increase for an outstanding debt obligation. Credit spread risk and downgrade risk differ in that the latter pertains to a specific, formal credit review by an independent rating agency, while the former is the financial markets’ reaction.

### 1.2.2 Components of credit risk

Schonbucher [35, 2003] define components of credit risk as follows,

**Arrival risk** - is a term for the uncertainty whether a default will occur or not. To enable comparison it is specified with respect to a given time horizon, usually one year. The measure of arrival risk is the *probability of default*. The probability of default describes the distribution of the indicator variable
default before the time horizon.

Timing risk - refers to the uncertainty about the precise time of default. Knowledge about the time of default includes knowledge about the arrival risk for all possible time horizons. Thus timing risk is more detailed than arrival risk. The underlying unknown quantity of timing risk is the time of default, and its risk is described by the probability distribution function of the time of default. If a default never happens, we set the time of default to infinity.

Recovery risk - describes the uncertainty about the severity of the losses if a default has happened. In recovery risk, the uncertainty quantity is the actual payoff that a creditor receives after a default. Recovery rate of a bond or loan can be expressed as the fraction of the notional value of the claim that is actually paid to the creditor.

1.3 Scope of the thesis

In chapter 2 we cover the basic financial mathematics. In chapter 3 we discussed structural models. Firstly we review the model of Merton [30, 1974] and then we consider two extensions of the Merton model which were made by Geske [18, 1974] and Black and Cox [6, 1976]. In chapter 4 we discuss reduced-form credit risk models. We review the work by Jarrow and Turnbull [25, 1995] and Duffie and Singleton [14, 1997]. Chapter 5 focuses on credit derivatives. In this chapter we define the most popularly used credit
derivatives and explore their valuation methods. In chapter 6 we discuss two related methods of pricing credit risk premium, the spread method and the options method. In Chapter 7 we discuss the some of the main points of Basel II accord, and this chapter does not contain any mathematics. Finally in chapter 8 we discuss the mathematics of portfolio management. In particular we make simulations of two results on portfolio management.
Definitions and Mathematical Tools

In this chapter we record some of the mathematical tools and technical terminology that will be used in the thesis. A general reference for the basics of stochastic calculus used in this dissertation is the book [33] of Oksendal. There are several good introductory textbooks to mathematical finance. The author has consulted the books by Etheridge [16], Wilmott et al [37] and the book [4] of Baz and Chazko.

2.1 Stochastic Process

A stochastic process \( \{ X_t \}_{t \in J} \) is a family of random variables indexed by a parameter \( t \), which runs over an index set \( J \). Two common classifications of stochastic processes are based upon distinctions about the state space, which is the range of the random variables, and the index set \( J \). We say
that a process is a discrete time process if its index set is discrete (usually a
subset of the natural numbers), and a continuous time process if the index
set is an interval.

2.2 Stochastic Process Properties

The following conditions, or some of them, are sometimes imposed on a
stochastic process.
(a) If for any \( t_1 < t_2 < t_3 \) the random variables \( X_2 - X_1 \) and \( X_3 - X_2 \) are
independent, then \( \{X_t\} \) is said to be a process with *independent increments*.
(b) If for any \( t_i \) and \( h > 0 \) the distributions of \( X_{t_i + h} - X_{t_i} \) depend only
on \( h \), then \( \{X_t\} \) is said to have *stationary increments*.
(c) The process \( \{X_t\} \) is said to have *Markov property* if given \( X_t \) and \( s > t \),
then \( X_s \) is independent of \( X_u \) for all \( u < t \).

2.3 Brownian Motion

A Brownian motion \( W_t \) is a continuous-time stochastic process which has
the following characteristics:
(a) \( W_t = 0 \)
(b) \( W_t \) is continuous
(c) \( W_t \) has independent increments with distribution
\( W_t - W_s \sim N(0,t-s) \) for \( 0 < s < t \) and \( N(\mu, \sigma^2) \) denotes the normal
distribution with expected value \( \mu \) and variance \( \sigma^2 \). In particular, if \( \mu = 0 \)
and \( \sigma^2 = 0 \), then \( W_t \) is said to be a standard Brownian motion or Wiener
process.
A Geometric Brownian Motion (GBM) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion. It is used particularly in the field of option pricing because a quantity that follows a GBM may take only positive values, and only the fractional changes of the random variate are significant. This is a reasonable approximation of stock price dynamics. A stochastic process $S_t$ is said to follow a GBM if it satisfies the following stochastic differential equation:

$$dS_t = S_t[\mu dt + \sigma dW_t]$$

where $W_t$ is a Wiener process or Brownian motion $\mu$ (the percentage drift) and $\sigma$ (the percentage volatility) are constants. For an arbitrary initial value
\( S_0 \), the equation has an analytic solution

\[
S_t = S_0 \exp \left[ (\mu - \frac{1}{2} \sigma^2) t + \sigma W_t \right].
\]

### 2.5 Itô Process

A stochastic process \( X = \{X_t, t > 0\} \) that solves the following equation:

\[
X_t = X_0 + \int_0^t \mu(X_s, t) ds + \int_0^t \sigma(X_t, t) dW_s
\]

is called an Itô process. The corresponding stochastic differential equation is given by

\[
dX_t = \mu(X_t, t) dt + \sigma(X_t, t) dW_t
\]

where \( \mu \) is a drift and \( \sigma \) is a standard deviation and \( W_t \) is a Wiener process with \( dW_t \) its differential.

### 2.6 Itô Lemma

Let \( F(S, t) \) be a twice differentiable function of \( t \) and of the random process \( S_t \), and suppose that \( S_t \) follows the Itô process

\[
dS_t = S_t[\mu_t dt + \sigma_t dW_t], t \geq 0 \tag{2.1}
\]

where \( \mu \) is a drift and \( \sigma \) is a variance and \( W_t \) is a Wiener process. Then, by Itô’s lemma we have

\[
dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma^2 dt. \tag{2.2}
\]

If we substitute equation (2.1) into (2.2) for \( dS_t \), then

\[
dF_t = \left( \frac{\partial F}{\partial S_t} \mu_t S_t + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma^2 \right) dt + \frac{\partial F}{\partial S_t} \sigma_t S_t dW_t \tag{2.3}
\]
which follows from Itô’s lemma. Thus, $F_t$ follows an Itô process with the drift rate

$$\frac{\partial F}{\partial S_t} dS_t \mu_t + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2 t$$

and variance rate

$$\left(\sigma_t S_t \frac{\partial F}{\partial S_t}\right)^2.$$

### 2.7 European Options

A European call option is a contract that gives the option holder the right to buy the underlying asset ($S$) for the strike price ($K$). The premium is paid at the beginning and if at maturity ($T$), the asset value is less than the strike price, then the option is worthless and the holder would decide not to exercise the option. If it is greater, then the holder would exercise the option and buy at the strike price. At maturity the value of the European call option on an asset is

$$C(S_T, T) = \max(S_T - K, 0).$$

A European put option gives the option holder the right to sell the underlying asset for the strike price. The holder pays the premium at the beginning. If at maturity the market value of the asset is greater than the strike price, then the holder would not exercise the put option. If the market value of the asset is less than the strike price, the holder would exercise the option and the asset would be sold at the strike price. At maturity the value of the European put option is

$$P(S_T, T) = \max(K - S_T, 0).$$
2.7.1 Black-Scholes Model for European Options Prices

Black-Scholes [7, 1973] gives the price of an European call with exercise price $K$ on a stock trading at price $S$ is given by the formula
\[ C(S_t, t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2) \] (2.4)

where
\[
d_1 = \frac{(\ln S - \ln K) + (r + \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}},
\]
\[
d_2 = d_1 - \sigma \sqrt{T - t}.
\]

\( r \) is risk-free interest rate and is constant, \( \sigma \) is the standard deviation and \( N(.) \) is the standard normal distribution function.

**Proposition 2.1 (Put-Call Parity)** Let \( S_t \) be a stock that pays no dividends. The following relation holds between the prices of European call and put options, both with strike price \( K \) and maturity time \( T \)
\[ C(S_t, t) = P(S_t, t) - S_t + K e^{-r(T-t)}. \] (2.5)

**The put-call parity in Proposition 2.1 allows us to compute the price of put option as**
\[ P(S_t, t) = e^{-r(T-t)} K N(-d_2) - S N(-d_1). \] (2.6)

**2.8 The finite time horizon stochastic control problem**

We follow the approach of Fleming and Soner [17]. This section is rather more for the sake of notation, and not meant to be a proper introductory
section. Detail can be found in [17, Chapter III]. We consider a time horizon 
\([t, T]\) for fixed \(t\) and \(T < \infty\). The notion of Markov Process is important
in this regard. A process \(x(t)\) is said to be a Markov Process if for any
sequence of time ticks \(l_1 < l_2 < \ldots < l_{n-1} < l_n\) with \(l < l_1\) and \(l_n < T\), and
an \(l_n < t\) and any \(B \subseteq \mathbb{R}\), we have
\[
P[x(t) \in B|x(l_1), x(l_2), \ldots, x(l_n)] = P[x(t) \in B|x(l_n)].
\]

We consider an \(n\) dimensional stochastic process of the form:
\[
dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t), \tag{2.7}
\]
where \(u(t)\) is a control and \(W(t)\) is a Brownian motion of dimension \(n\).
The values of \(u(t)\) are restricted to a given set \(Z\), and the functions \(u\) are
restricted to a set \(A\). Let \(\Upsilon\) be a function of two real variables. Now let us
define the quantity \(J\) as a function of three variables:
\[
J(l, x, u) = \mathbb{E}_{lx}\left[\int_t^T L(t, x, u(t))dt + \Upsilon(T, x(T))\right]. \tag{2.8}
\]
Here \(\mathbb{E}_{lx}\) means expectation conditional on the event \(x(l) = x\). Our problem
is to find the maximum value \(V\) of \(J\) with respect to different choices of \(u(t)\):
\[
V = \max_{u(.)} J(l, x, u). \tag{2.9}
\]
Now let us define \(H\) as follows:
\[
H = \sup(-f(t, x, u)(\text{grad}V) - \frac{1}{2} \sum_{i,j} \sigma_{ij}(\partial x)^2 \partial x_i \partial x_j V + L(t, x, u)). \tag{2.10}
\]
In the latter sum, the summation is taken over all the pairs \((i, j)\), a total
of \(n^2\), and the numbers \(\sigma_{ij}\) are the entries of the \(n \times n\) matrix \(\sigma\). More
precisely, \(\sigma_{ij}\) is the coefficient of \(dW_j(t)\) in the expression for \(dx_i(t)\) in the
equation (2.7). The solution of the problem (2.9) can be shown (see [17]) to satisfy the so called Hamilton-Jacobi-Bellman (HJB) equation:

$$-\frac{\partial V}{\partial t} + H = 0.$$  \hspace{1cm} (2.11)
Chapter 3

Structural Credit Risk

Models

In chapter 1, we have mentioned that there are two groups of credit risk models, namely structural and reduced form models. The idea of a structural model is that a company defaults on its debt if the value of the assets of the company falls below a certain default value.

The original structural model dates back to the early 70’s and the papers of Black and Scholes [7] and Merton [30]. Their work seeks to relate credit events to economic fundamentals by modelling the dynamics of the assets of a firm with default occurring if the value of the firm drops below some threshold level. In this chapter, we begin by outlining Merton’s original approach. There have been many extensions to the original work, one of the extensions was proposed by Geske [18]. Geske included multiple debts. Recently a variety of barrier models have appeared and they are quite useful.
for analysing the risky debt problem. Black and Cox [6] were among the first people to develop a barrier model. In this chapter we will also review their paper on barrier models. In conclusion of this section, we illustrate Merton’s equation for credit spread by way of graphs.

### 3.1 Merton’s Model

Note, in this section, subscripts will denote partial derivatives, except when the subscript refers to time.

The landmark work of Merton in 1974 on modeling of credit risk is still quite useful and deserves some attention. In this section we show how the equity value $F$ of a firm subject to debt can be viewed in terms of a European option on the firm. Merton [30] considered a firm with the following characteristics:

(a) Two funding sources - equity and a single homogeneous class of debt
(b) The debt is considered as a zero coupon bond, par value $K$, maturity $T$
(c) In the event of non-payment of the debt at time $T$, the bondholders take control of the firm and equityholders receive nothing
(d) The firm cannot issue any senior claims, and pay cash dividends prior to the maturity of the debt.
(e) The are no transactions cost or taxes
(f) Short selling is permitted
(g) There are no problems with the divisibility of assets
(h) Interest rates are assumed to be constant.
If $S_t$ is defined to be the value of the firm at time $t$, if $D_t$ is the value of the debt and $E_t$ is the value of the equity, then

$$S_t = E_t + D_t$$

and both equity and debt can be viewed as contingent claims on the firm’s assets. If $S_T > K$ then the bondholders are repaid and the balance of the firm goes to the equity holders. If $S_T < K$, default occurs, the bondholders take over the firm and the equity holders receive nothing. In other words

$$E_T = \max(S_T - K, 0). \tag{3.1}$$

This payoff is identical to that of a European call option on the value of the firm’s assets struck at the face value of the debt. The holders of the risky corporate debt get paid either the face value under no default or take over the firm under default. Table 3.1 summarises the three different claims under the two alternatives states at maturity.

<table>
<thead>
<tr>
<th>State</th>
<th>Assets</th>
<th>Equity</th>
<th>Bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>Default</td>
<td>$S_T &lt; K$</td>
<td>0</td>
<td>$S_T$</td>
</tr>
<tr>
<td>No Default</td>
<td>$S_T &gt; K$</td>
<td>$S_T - K$</td>
<td>$K$</td>
</tr>
</tbody>
</table>

Hence the value $D_T = \min(S_T, K)$ of the debt on the maturity date is given by any of the following forms (which are of course identical in value).

$$D_T = S_T - \max(S_T - K, 0), \tag{3.2}$$

$$D_T = K - \max(K - S_T, 0). \tag{3.3}$$
The two equations provide two interpretations. Equation (3.2) decomposes the risky debt into the asset and a short call. This interpretation was first given by Black and Scholes [7] and means that equity owners essentially own a call option of the company. If the company performs well, then the equity owners should call the company, otherwise the equity owners let the debt owners own the company.

Equation (3.3) decomposes the risky debt into a risk-free debt and a short put. This interpretation explains the default risk of the corporate debt. The issuer (equity owners) can put the company back to the debt owner when the performance is bad. When the value of equity and debt are added together they equal the assets of the firm at all times

\[ S_t = E_t + D_t. \]

At maturity we have,

\[ E_T = \max(S_T - K, 0) + \min(S_T, K) \]

\[ E_T = S_T \]

The value of the firm is modelled as a geometric Brownian motion on a probability space \((\omega, \Psi, \mathbb{P})\)

\[ dS_t = S_t[rdt + \sigma dW_t] \] (3.4)

where \(r\) is the instantaneous risk-free interest rate which is assumed constant, \(\sigma\) is the percentage volatility, and \(W_t\) is the Wiener process under the risk neutral measure. If \(Y_1 = F_1(S, t)\) and \(Y_2 = F_2(S, t)\) are two functions
of the value of the firm and time, where $S = S_t$ in all that follows, then for $i \in \{1, 2\}$, using Ito’s lemma we obtain

$$dS_t = \frac{\partial F_i}{\partial S} dS + \frac{\partial F_i}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F_i}{\partial S^2} dt$$  \quad (3.5)$$

since $(dS)^2 = \sigma^2 S^2 dt$. If we consider a portfolio $P$ consisting of the entity $Y_1$ hedged with $-\Delta$ lots of the entity $Y_2$,

$$P = Y_1 - \Delta Y_2$$

then from equations (3.4) and (3.5) we obtain the following. To this end we abbreviate $F_1$ to $F$ and $F_2$ we write as $G$, and we use subscripts to denote partial derivatives.

Therefore,

$$dP = dY_1 - \Delta dY_2$$

$$dP = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \sigma S \frac{\partial F}{\partial t} \right) dt - \Delta \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + \sigma S \frac{\partial G}{\partial t} \right) dt$$

$$+ (\sigma SF - \Delta \sigma SG) dW .$$

Taking $\Delta = \frac{F}{G_S}$ ensures that the portfolio is risk-free. Since in the absence of arbitrage the portfolio must grow at the risk-free rate, we can deduce that in fact:

$$dP = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t} \right) dt - \Delta \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + \frac{\partial G}{\partial t} \right) dt$$

$$+ (\sigma SF - \Delta \sigma SG) dW .$$

Thus

$$dP = r_f P dt = r(F - \Delta G) dt .$$

Thus

$$\frac{1}{F_S} \left( \frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF \right) = \frac{1}{G_S} \left( \frac{\partial G}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} - rG \right) .$$  \quad (3.6)$$
This holds for any two functions $F(S, t)$ and $G(S, t)$ of $S$ and $t$, and therefore each side of the equation must be equal to some function $a(S, t)$. We have the following pde:

$$\frac{1}{2} \sigma^2 S^2 F_{SS} - aF_S - rF + \frac{\partial F}{\partial t} = 0.$$  

Writing $a(S, t) = (\lambda - \sigma \lambda)S$, then $\lambda = \lambda(S, t)$ is the market price of risk and the equation can be written as

$$\frac{1}{2} \sigma^2 S^2 F_{SS} - (\lambda - \sigma \lambda)SF_S - rF + \frac{\partial F}{\partial t} = 0. \quad (3.7)$$

Merton [30] assumes that $S$ is a tradeable asset, in which case $F = S$ is a solution to this equation, and

$$(\lambda - \sigma \lambda)S - rS = 0.$$ 

Hence $\lambda = \frac{\alpha - r}{\sigma}$ is the market price of risk and equation (3.7) reduces to the Black-Scholes equation.

**Proposition 3.1.** Under the given assumptions, the value of the firm satisfies the Black-Scholes pde:

$$\frac{1}{2} \sigma^2 S^2 F_{SS} - rSF_S - rF + F_t = 0. \quad (3.8)$$

(Here $F_t$ is used to denote $\frac{\partial F}{\partial t}$).

Equity $E$ is a function of $S$ and $t$ and therefore it satisfies equation (3.8) with appropriate boundary conditions. Since it is equivalent to a European call option on the value of the firm, strike price $K$, from standard theory we have

$$E_t = S_t N(d_1) - e^{-r(T-t)} KN(d_2) \quad (3.9)$$
where
\[ d_1 = \frac{\ln S - \ln K + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \]
and \[ d_2 = d_1 - \sigma\sqrt{T - t}. \]

The current value of the debt is a covered call value (we assume debt \( D \) to have fixed time of maturity \( T \))
\[ D_t = S_t - E_t \tag{3.10} \]
\[ D_t = S_t - [S_tN(d_1) - e^{-r(T-t)}KN(d_2)] \tag{3.11} \]
or equivalently
\[ D(T, t) = S_t[1 - N(d_1)] + e^{-r(T-t)}KN(d_2). \tag{3.12} \]

The first term of equation (3.12) represents the recovery value. The second term in equation (3.12) is the present value of probability-weighted face value of the debt. It means that if default does not occur (with probability \( N(d_2) \)), the debt owner receives the face value \( K \). Since the probability is risk neutral, the probability-weighted value is discounted by the risk-free rate. The two values together make up the value the value of debt. The yield of the debt is calculated by solving
\[ D_t = Ke^{yt(T-t)} \]
for \( y \) to give us
\[ y = \frac{\ln k - \ln D_t}{T - t}. \tag{3.13} \]

Therefore the spread is
\[ \text{spr}(t, T) = -\frac{1}{T - t} \ln \left[ \frac{1}{d} N(-d_1) + N(d_2) \right] \tag{3.14} \]
where \( d = \frac{K}{S} e^{-r(T-t)} \) is a measure of leverage. Equation (3.14) defines the term structure of credit risk, which depends on the time to maturity of the debt, firm’s asset volatility and leverage \( d \).

### 3.2 Geske’s Compound Model

Geske [18] was the first to relax the capital structure assumption made in Merton (1974). He allowed the firm to be financed with several coupon-bearing bonds, of different priorities. In the Geske paper equity is also priced as a compound option.

Geske [18] made the following assumptions:

(a) the firm has only the common stock and coupon bonds outstanding.

(b) the coupon bond has \( n \) interest payments of \( X \) dollars each, of which \( n - 1 \) are due at equal intervals before maturity \( T \). The common stock receives no dividend payments.

(c) The firm refines each coupon payment with equity and bankruptcy occurs when the firm fails to make an interest payment because it is unable to issue new equity.

Geske’s [18] solution for valuing coupon bonds in discrete time follows the theory of Rubenstein [19] for discounting uncertain income streams and Geske’s approach for valuing compound options. If \( \{Z\} \) is a set of random variables, this can be seen as a set of contingent claims, and \( X[m(t)] \) is the dividend received in state \( m \) at time \( t \).
By defining $Z'(t) = \frac{Z[m(t)]}{\pi[m(t)]}$, where $\pi[m(t)]$ is the relevant probability measure, the current price of the security is given as

$$P_0 = \sum_{t} \mathbb{E}[X(t)Z'(t)].$$

Any uncertain income stream can be valued using the adjusted expectation. The random variables $Z[m(t)]$ can be identified as a transformation of the return on the market portfolio. The final result depends on the assumed stochastic relationship between the uncertain cash flows $X(t)$ and the uncertain risk adjusted discount factor.

Consider the stock at time $t_{n-1}$, just after the final coupon payment. At the bond’s maturity $T$, the stock price will be $\max[V(T) - D, 0]$, where $V(T)$ is the value of the firm at time $T$, and $D$ is the face value of the debt at maturity plus the interest over the period $(T - t_{n-1})$. The stock will have zero value when the firm cannot repay the principal debt ($V(T) < D$), and the stock’s value will equal the difference between the value of the firm and the face value of debt when ($V(T) > D$). Thus

$$S(t_{n-1}) = \mathbb{E}[(V(T) - D)Z'(T)V(T) > M]$$

The boundary condition for the coupon bond at time $T$, is $B_T = \min(V_T, D)$, where the bond holders get the assets of the firm, $V_T$, if the stockholders can not repay the principal ($V(T) < D$). Since the firm has no payments between $t_{n-1}$ and $T$, its value at $t_{n-1}$ is

$$V(t_{n-1}) = \mathbb{E}[V(T)Z'(T)].$$
The value of the firm follows a geometric Brownian motion, the interest is constant, $\bar{V}(T)$ and $Z'(T)$ are jointly lognormal, and variance is agreed upon by investors. The solution to this expectation for $S(t_{n-1})$ is the following Black-Scholes option pricing formula:

$$S(t_{n-1}) = V(t_{n-1})N_1(k_n) - Dr^{-(T-t_{n-1})}N_1(h_n) \quad (3.15)$$

where

$$h_n = \frac{\ln V(t_{n-1}) - \ln D + (\ln r - \frac{\sigma^2}{2})(T-t_{n-1})}{\sigma \sqrt{T-t_{n-1}}}$$

$$k_n = h_n + \sigma \sqrt{T-t_{n-1}}$$

$r$ is the risk-free rate of interest, $\sigma^2$ is the variance, and $N_1(.)$ is the univariate cumulative normal distribution.

The value of the coupon bond at time $t_{n-1}$ is found by subtracting the value of the stock from the value of the firm at $t_{n-1}$. Thus

$$B(t_{n-1}) = V(t_{n-1}) - S(t_{n-1}) \quad (3.16)$$

$$B(t_{n-1}) = V(t_{n-1})[1 - N_1(h_n + \sigma \sqrt{T-t_{n-1}})] + Dr^{-(T-t_{n-1})}N_1(h_n)$$

This is the expression for the value of a pure discount bond.

Let $X_t$ be the coupon payment at time $t$. The stock is a compound option. At the final coupon payment, $t_{n-1}$, the value of the stock is the max[$S(t_{n-1}), 0$], where $S(t_{n-1})$, is as given in equation (3.15). $S(t_{n-1})$ will be the stock’s value when the firm’s value is of sufficient size to assume new equity ($V(t_{n-1}) > \bar{V}(t_{n-1})$) and when the coupon is paid.
The stock will be written when the firm’s value is insufficient to issue new equity \( (V(t_{n-1}) > \bar{V}(t_{n-1})) \), and the firm is then bankrupt. \( \bar{V}(t_{n-1}) \) solves the equation

\[
S(t_{n-1}, V) - X(t_{n-1}) = 0.
\]

At time \( t_{n-1} \), the value of the coupon bond is equal to

\[
\min[V(t_{n-1}), X(t_{n-1}) + B(t_{n-1})],
\]

where the bond holders receive the firm’s coupon payment plus the future value of the bond if the firm can pay the coupon \( (V(t_{n-1}) > \bar{V}(t_{n-1})) \).

The boundary conditions for the stock and the coupon bond take the same form at all earlier coupon dates, \( t_{n-2}, t_{n-3}, \ldots, t_2 \) and \( t_1 \). Thus an expression for the value of a risky bond can be derived by recursively solving for the value at each boundary encountered in terms of the immediate solution to the previous boundary. Using this approach, the following result for pricing a risky coupon bond in discrete time is obtained. The proof in Geske [18] is in detail and we omit it.

**Theorem 3.1.** Suppose that changes in the value of the firm follow geometric Brownian motion, that the value \( \bar{V}(t) \) of the firm and the random variable \( \bar{Z}(t) \) are jointly lognormal for each \( t \), that the firm pays no dividend, and that investors agree on the variance \( \sigma \). Then

\[
B = V[1 - N_n(h_i + \sigma \sqrt{i}; \{\rho_{ij}\})] + G + H \quad (3.17)
\]
where
\[
G = \sum_{i=1}^{n-1} X_i r - T(\frac{1}{n}) N_i(h_i; \{\rho_{ij}\})
\]
\[
H = Dr^T N_n(h_i; \{\rho_{ij}\})
\]
and also \(\tilde{V}(i)\) is the value of \(V\) which solves the equation \(S(i, V) - X(i) = 0\)

\[
h(i) = \frac{(\ln V - \ln \tilde{V}(i)) + (\ln r - \frac{\sigma^2}{2}) T(\frac{1}{n})}{\sigma \sqrt{T(\frac{1}{n})}}
\]
\[
h(n) = \frac{(\ln V - \ln D) + (\ln r - \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}}
\]
\[
\rho_{ij} = \sqrt{\frac{i}{j}}
\]
for all \(i, j\) pairs, \(i < j\) and

\[
N_n(h(i); \{\rho_{ij}\}) = \int_{-\infty}^{h(i)} N_1(h_1') N_{n-2}(h'(i); \{\rho_{ij}\}) f(w_2) dw_2
\]
where
\[
N_n(h(i); \{\rho_{ij}\}) = \int_{-\infty}^{h(i)} N_1(h_1') N_{n-2}(h'(i); \{\rho_{ij}\}) f(w_2) dw_2
\]

\[
\rho_{ij,2} = \frac{(\rho_{ij} - \rho_{i2}\rho_{j2})}{\sqrt{1 - \rho_{i2}^2} \sqrt{1 - \rho_{j2}^2}}
\]

and

\[
N_{n-2}(h'(i); \{\rho_{ij,2}\}) = \int_{-\infty}^{h'(i)} N_1(h_3') N_{n-4}(h''(i); \{\rho_{ij,2,4}\}) f(w_4) dw_4
\]
where
\[
h''(i) = \frac{h'(i) - \rho_{i4,2} w_4}{\sqrt{1 - \rho_{i4,2}^2} \sqrt{1 - \rho_{i2,4}^2}}
\]
\[
\rho_{ij,2,4} = \frac{\rho_{ij,2} - \rho_{i4,2}\rho_{ij,2}}{\sqrt{1 - \rho_{i4,2}^2} \sqrt{1 - \rho_{i2,4}^2}}
\]
and \(B\) is the current value of the risky coupon bond, \(V\) is the current of the firm, \(N_n\) is an \(n\)-dimensional multivariate normal distribution function.
3.3 Black and Cox Model

In addition to Geske’s (compound option) model, Black and Cox [6] have made an attempt to relax some of Merton’s assumptions. Their approach make account for such specific features of debt contract as safety covenants and debtholders. Since they assume that bondholders receive a continuous dividend payment proportional to the current value of the firm, the risk-neutral dynamics of the firm value are

\[ dS_t = S_t[(r - k)dt + \sigma dW_t] \]

where the constant \( k > 0 \) and \( \sigma > 0 \) represent the payout ratio and the volatility co-efficient respectively.

3.3.1 Bonds with safety covenants

Safety covenants provide the firm’s bondholders with the right to force the firm to bankruptcy or reorganisation if the firm is doing poorly according to a set standard. The standard for a poor performance is set in Black and Cox [6] in terms of a time-depended deterministic barrier \( \bar{v} = K e^{-\gamma(T-t)}, \quad t \in [0, T] \). Black and Cox [6], state that as soon as the value of the firm assets crosses this lower threshold, the bondholders are entitled to force the firm to bankruptcy and obtain ownership of the firm. Since the interest rate \( r \) is assumed to be constant, the pricing formula of a defaultable bonds solve the following PDE

\[ D_t - rD + (r - k)V D_v + \frac{1}{2}\sigma^2 V^2 D_{vv} = 0 \quad (3.18) \]
with the boundary conditions

\[ D(V, T) = \min(V, P) \]

and

\[ D(K e^{-\gamma(T-t)}, t) = K e^{-\gamma(T-t)}. \]

Similarly, the value of the stock, \( S \), must satisfy

\[ S_t + aV - rS + (r - a)VS_v + \frac{1}{2}\sigma^2 V^2 S_{vv} = 0 \quad (3.19) \]

with boundary conditions

\[ S(V, T) = \max(V - P, 0) \quad \text{and} \quad S(K e^{-\gamma(T-t)}, t) = 0. \]

**Proposition 3.5** The price of a bond is given as

\[
D(V, t) = P e^{-r(T-t)} \left[ N(z_1) - y^2 N(z_2) \right] + V e^{-a(T-t)} \left[ N(z_3) + y \zeta e^{\sigma^2(T-t)} N(z_5) \right] \\
+ y^2 N(z_4) + y \theta + y e^{\sigma^2(T-t)} N(z_6) - y \theta N(z_7) - y \eta N(z_8) \quad (3.20)
\]

where

\[
y = \frac{K e^{\gamma(T-t)}}{V} \]

\[
\theta = \frac{r - a - \gamma + \frac{1}{2}\sigma^2}{\sigma^2} \]

\[
\delta = (r - a - \gamma - \frac{1}{2}\sigma^2)^2 + 2\sigma^2(r - \gamma) \]

\[
\zeta = \frac{\sqrt{\delta}}{\sigma^2} \]

\[
\eta = \frac{\sigma\delta - 2\sigma^2a}{\sigma^2} \]

\[
z_1 = \frac{\ln V - \ln P + (r - a) - \frac{1}{2}\sigma^2(T - t)}{\sqrt{\sigma^2(T - t)}} \]

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3.3.2 Subordinated Bonds

Another form of indenture agreement involves subordination of the claims of one class of debt holders, the junior bonds, to those of second class, of the senior bonds. A senior bond is a debt security that has a prior or superior claim on the issuer’s assets than the other bonds issued by the same entity. A junior bond is a debt security that is paid after all the other debt obligations have been made. At the maturity date of the bonds, payments can be made to the junior debt holders only if the full promised payment to the senior debt holders has been made.

Suppose that both classes of bonds are discount bonds, and let the promised payments to senior and junior debt be, \( P \) and \( Q \). Then at the maturity date the value of each of the firm’s securities will be shown in Table 3.2.
Table 3.2: Values of Claims at Maturity

<table>
<thead>
<tr>
<th>Claim</th>
<th>V &lt; P</th>
<th>P = V&lt;P+Q</th>
<th>V &gt; P+Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior Bond</td>
<td>V</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>Junior Bond</td>
<td>0</td>
<td>V-P</td>
<td>Q</td>
</tr>
<tr>
<td>Stock</td>
<td>0</td>
<td>0</td>
<td>V-P-Q</td>
</tr>
</tbody>
</table>

Let \( B(V, t; P, \rho P e^{-r(T-t)}) \) denote the formula given in equation (3.20) for single bond issue with promised payment \( P \) and a safety covenant boundary given by \( \rho P e^{-r(T-t)} \). Then the value of the junior debt, \( J \) can be written as

\[
J(V, t) = B(V, t; P + Q, \rho P e^{-r(T-t)}) - B(V, t; P, \rho P e^{-r(T-t)})
\]

\[
= B(V, t; P + Q, \rho P e^{-r(T-t)}) - P e^{-r(T-t)}
\]

\[
= \frac{Q e^{-r(T-t)}}{P} > \frac{P + Q}{P_{thr}}
\]

(3.21)

3.4 Default Spread for Corporate Debt

In this section we illustrate Merton’s equation for credit spread (equation (3.14)) to calculate credit spread for different various levels of volatility and different strike values. We take the following parameters, \( S_0 = 100, T = 1 \) and \( r = 0.1 \)
Figure 3.1: Default Spread for Corporate Debt
Chapter 4

Reduced Form Credit

Models

The name reduced form was first given by Darrel Duffie to differentiate from the structural form models of the Merton type. Reduced form models are mainly represented by the Jarrow-Turnbull [25] and Duffie-Singleton [14] models. Both types of models are arbitrage free and employ the risk-neutral measure to price securities. The principal difference is that default is endogeneous in the Merton model while it is exogeneous in the Jarrow-Turnbull and Duffie-Singleton models.

In the literature on pricing models of loan commitments it is usually the structural models that are being used. In the paper [11] of Chava and Jarrow, the authors propose one of the rare occasions of reduced form models of pricing loan commitments. This approach leads to pricing methods that are more flexible and yet sufficiently analytically tractable. In this chapter
we review both the Jarrow-Turnbull and Duffie-Singleton models and we make a contribution by using the Jarrow-Turnbull model to price a risky coupon bond.

4.1 The Jarrow-Turnbull Model

Jarrow and Turnbull [25] provide a methodology for pricing and hedging derivative securities that have credit risk. They apply the foreign currency analogy of Jarrow and Turnbull [24, 1991] to decompose the dollar payoff from a risky security into a certain payoff and a spot interest rate. In this section we will consider a two-period discrete economy.

4.1.1 Notation

Let \( P_0(t, T) \) be the time \( t \) dollar value of the default-free zero-coupon paying a certain dollar at time \( T \). From this term structure a money market accent can be constructed. Let \( B(t) \) be the time \( t \) value of this money market account initialized with a dollar at time 0. With respect to the risky zero coupon bond, let us define \( V_1(t, T) \) as the time \( t \) value of the risky bond promising one dollar at time \( T \). Define

\[
e_1(t) = V_1(t, t).
\]

\( e_1(t) \) represent the time \( t \) dollar value of one promised X dollars delivered immediately (at time \( t \)) and is analogous to a spot exchange rate. A hypothetical, XY paying zero-coupon bond is introduced, with its value having the properties

\[
P_1(T, T) = 1,
\]
\[ P_1(t, T) = \frac{V_1(t, T)}{e_1(t)}. \] (4.1)

This quantity is the time \( t \) value in units of \( X \), one of the \( X \)'s is delivered at time \( T \). Re-arranging equation (4.1) we get

\[ V_1(t, T) = P_1(t, T)e_1(t). \] (4.2)

### 4.1.2 XY Zero-Coupon Bonds

In Jarrow and Turnbull [25] expected payoff ratios are calculated as shown in equations 4.3 - 4.5.

\[
E_1(e_1(2)) = \begin{cases} 
\delta & \text{if defaulted at time 1} \\
\lambda \mu_1 \delta + (1 - \lambda \mu_1) & \text{if not defaulted at time 1}
\end{cases} \tag{4.3}
\]

\[
E_0(e_1(2)) = \lambda \mu_1 \delta + (1 - \lambda \mu_1) \left[ \lambda \mu_1 \delta + (1 - \lambda \mu_1) \right] \tag{4.4}
\]

\[
E_0(e_1(1)) = \lambda \mu_0 \delta + (1 - \lambda \mu_0) \tag{4.5}
\]

where \( E(\cdot) \) is the time \( t \) conditional expected value.

Equation (4.3) states that at time 1, the payoff ratio at time 2 is either \( \delta \) if the bond issuer has defaulted at time 1. If the firm has not defaulted then the payoff is \( \lambda \mu_1 \delta + (1 - \lambda \mu_1) \). The payoff ratio at time 2 viewed from time 0 is a weighted average of the payments under default \( \delta \). Equation (4.4) shows face value under no default at time 1 and 2. Equation (4.5) shows that the expected payoff is ratio at time 1 as seen at time 0, is the weighted average of the payments under default and no default but only for time 1.
The price of risky zero coupon bond, is given as

\[ V_1(t, T) = P_0(t, T)E_t(e_1(T)). \] (4.6)

### 4.1.3 XY Coupon Bonds

Jarrow and Turnbull [25] also gives a price of a coupon-bond. A coupon bond is a bond which. Consider an XY coupon-bearing bond with promised dollar coupons of \( m_1 \) at time 1 and \( m_2 \) at time 2, where \( m_2 \) includes the principal repayment. Let \( D(t) \) represents the time \( t \) dollar value of this XY coupon-bond price equals its discounted expected payoff, i.e,

\[ D(0) = \mathbb{E} \left( \frac{m_1e_1(1)}{B(1)} + \frac{m_2e_1(2)}{B(2)} \right) \] (4.7)

\[ D(0) = m_1V_1(0, 1) + m_2V_1(0, 2). \] (4.8)

### 4.2 Duffie and Singleton Model

Duffie and Singleton [14] developed a reduced form model for the valuations of contingent claims subject to default risk. They focused on applications to the term structure of interests rates for corporate or sovereign bond.

#### 4.2.1 Notation

Let \( L_t \) denote the expected fractional loss in market value if default were to occur at time \( t \), conditional on the information available up to time \( t \) and \( h_t \) is the hazard rate for default at time \( t \). Let \( \Delta \) be a process which is zero
before default and 1 after. This process can be written as

\[ d\Delta_t = (1 - \Delta)h_t dt + dM_t \quad (4.9) \]

where \( M \) is a martingale under measure \( Q \) and \( h_t \) is a jump arrival at time \( t \) (under \( Q \)) of a Poisson process whose first jump occurs at default.

4.2.2 Exogenous expected loss rate

Let the market value of the defaultable claim to \( X \) at time \( t \) be

\[ V_t = E^Q_t \left[ \exp \left( - \int_t^T R_s ds \right) \right] \quad (4.10) \]

where

\[ R_t = r_t + h_t L_t \quad (4.11) \]

\( R_t \) is the default-adjusted short rate process, \( hL \) is the neutral mean loss rate of the instruments to default and \( r \) is the short term interest rate. In order to confirm equation (4.3), they use the fact that the gain process, after discounting at the short-rate process \( r \), must be a martingale under \( Q \). This discounted gain is defined by

\[ G_t = \exp \left( - \int_0^t r_s ds \right) V_t (1 - \Delta_t) + \int_0^t \exp \left( - \int_u^t r_u du \right) (1 - L_s) dV_s - d\Delta_s \quad (4.12) \]

The first term is the discounted price of the claim; the second term is the discounted payout of the claim upon default. The property that \( G \) is a \( Q \)-martingale and the fact that \( V_T = X \) together provide complete characterization of arbitrage-free pricing of the defaulted claim.

\[ V_t = \int_0^t R_s dV_s + m_t \quad (4.13) \]

for some \( Q \)-martingale \( m \).
4.2.3 Price-dependent expected loss rate

If the risk-neutral expected loss rate $h_t$ is price dependent, then the valuation model is non-linear in the promised cash flows. This can be accommodated in a model in which default at time $t$ implies a fractional loss $L_t = L(Y_t, U_t)$ of market value and hazard rate $H_t = H(Y_t, U_t)$ that may dependent on the current price $U_t$ of the defaultable claim. In Duffie and Singleton [14], they consider a Markov setting and they write

$$R_t = \rho(Y_t, U_t)$$

where $\rho(y, u) = H(y, u)L(y, u) + \hat{r}(y)$ and $\{\hat{r}(y) : t \geq 0\}$ is the state-dependent default-free short rate process. The price $U_t$ of the defaultable claim at any time before default is given by the Feynmann-Kac functional

$$J(Y_t, t) = \mathbb{E}^Q\left[\exp\left(-\int_t^T \rho(Y_s, J(Y_s, s))ds\right) g(Y_T) \mid Y_t\right]$$ (4.14)

and $J$ solves the following linear equation

$$DJ(y, t) + \rho(y, J(y, t))J(y, t) = 0, \quad y \in \mathbb{R}$$ (4.15)

where

$$DJ(y, t) = D^{x,\sigma}J(y, t) + \lambda(y) \int_{\mathbb{R}} [J(y + z, t) - J(y, t)] d\nu_y(z)$$ (4.16)

with $\lambda : \mathbb{R} \to [0, \infty)$ being a given function determining the arrival density $\lambda(y_t)$ of jump in $Y$ at time $t$, under $Q$ and for each $y$, $\nu_y$ is a probability distribution for size($z$) with the boundary condition of $J(y, T) = g(y)$.

4.2.4 Valuation of defaultable bonds

In [14] they used equation (4.7) in valuation of defaultable bonds. They assume that the risk-neutral expected recovery at time $s$ in the event of
default at time \( s + 1 \) to be a fraction of the risk-neutral expected survival-contingent market value at time \( s + 1 \). Under this assumption the price of an \( n \)-year bond with semi-annual coupon payments of \( c \), \( V_t \) at any time \( t \) will be given as

\[
V_t = cE_t^Q \left[ \sum_{j=1}^{2n} \exp \left( -\int_t^{i+0.5j} R_s ds \right) \right] + E_t^Q \left[ \exp \left( \int_t^{i+n} R_s ds \right) \right]
\]

(4.17)

where \( R_t = r_t + h_t \bar{L} \).

### 4.3 Other Models

In addition to the models given above, there are a number of modeling approaches being used in industry to model risk. In this section we look at hazard models.

#### 4.3.1 Hazard Models

Shumway [36] extends Merton model by proposing a hazard model which incorporates both theoretical and empirical factors. Define a survival function as

\[
S(t, x_i; \theta) = 1 - \sum_{j < t} f(j, x; \theta)
\]

where \( x \) is a collection of explanatory (empirical) variables and \( \theta \) is a collection of parameters to be estimated. Hence, the hazard function is

\[
h(t, x; \theta) = \frac{f(t, x; \theta)}{S(t, x; \theta)}
\]

and then the maximum likelihood function (log) can be written as

\[
L = \prod_{i=1}^{n} h(t_i, x_i; \theta)^y_i S(t_i, x_i; \theta)
\]

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where \( y_i \) is a dummy variable equal to 1 if default occurs at time \( t_i \) and 0 otherwise. The functional form of \( f \) is determined empirically by maximising the likelihood function.

### 4.4 Pricing a Risky Coupon Bond

In this section we use the Jarrow-Turnbull model to price a zero-coupon bond. In [25] the price of a risky coupon bond is calculated as a product of a default-free zero coupon bond \((P_0(t, T))\) and a conditional expected ratio \((E_t(e_t(T)))\). In mathematical terms this is expressed as

\[
V_1(t, T) = P_0(t, T)E_t(e_t(T)).
\]

Let \( t = 1, T = 2, \lambda \mu_0 = 0.01, \lambda \mu_1 = 0.02, \delta = 0.32 \).

#### 4.4.1 When there is default at time 1

From equation (4.6) we have

\[
V_1(1, 2) = P_0(1, 2)E_1(e_1(2)).
\]

Since there is default at time 1 then the expected payoff is given as \( E_t(e_1(2)) = \delta = 0.32 \), then we have

\[
V_1(1, 2) = 0.6 \times 0.32
\]

\[
V_1(1, 2) = 0.18.
\]

#### 4.4.2 When there is no default at time 1

From equation (4.6) we have

\[
V_1(1, 2) = P_0(1, 2)E_1(e_1(2)).
\]
Given that there is no default at time 1 then the expected payoff is given as
\[ \lambda \mu_1 \delta + (1 - \lambda \mu_1), \]
with \( \lambda \mu_1 = 0.02, \delta = 0.32, \) then we have

\[ V_1(1, 2) = 0.6 \times \left[ (0.03 \times 0.32) + (1 - 0.03) \right] \]

\[ V_1(1, 2) = 0.58776. \]
Chapter 5

Credit Risk Derivatives

A credit derivative is a contract the payoff of which depends on the creditworthiness of one or more commercial or sovereign entities. In this chapter we explain how the most popular credit derivatives work and discuss how they can be valued. The chapter starts by providing a detailed discussion on credit default swaps. These are the most popular credit derivatives. They provide a market in which default insurance can be bought and sold. The chapter then moves on to discuss a number of other types of credit derivatives: total return swap, credit spread options, and collateralized debt obligations. Evaluation methods for credit default swaps together with a sample calculation appears towards the end of the chapter. We wish to note that single period models of credit risk management remains to be very important, see for instance [9].

Our example is single period, and we include a calculation of risk-neutral default probabilities in the 2-period case. The latter demonstrates the general method for calculating risk-neutral default probabilities when determining
hedging strategies for instance.

5.1 Credit Default Swaps (CDS)

A credit default swap (CDS) is a contract that provides insurance against the risk of a default by a particular company. The company is known as the reference entity and a default by the company is known as a credit event. The buyer of the insurance obtains the right to sell a particular bond issued by the company for its par value when a credit event occurs. The bond is known as the reference obligation and the total par value of the bond that can be sold is known as the swap’s notional principal.

In CDSs, the protection buyer pays a fee, the swap premium, to the protection seller in return of the reference obligation. The payments made by protection buyer are called the premium leg, the contingent payments that might have to be made by the protection seller are called the protection leg. Should a credit event occur, the protection seller must make a payment. This is shown in Figure 5.1.

In a later section we shall discuss the valuation methods for credit default swaps and include computations.

5.2 Credit Spread Options

A credit spread option is an option contract in which the decision to exercise is based on the credit spread of the reference credit relative to some strike spread. This spread may be the yield of a bond quoted relative to
a Treasury or may be a LIBOR spread. In the latter case, exercising the credit spread option can involve the physical delivery of an asset swap, a floating-rate note, or a default swap.

This reference asset may be either a floating rate note or a fixed rate bond via an asset swap. As with standard options, one must specify whether the option is a call or put, the expiry date of the option, the strike price or strike spread, and whether the option exercise is European (single exercise date), American (continuous exercise period), or Bermudan style (multiple exercise dates). The option premium is usually paid up front, but can be converted into a schedule of regular payments.

A call on the spread (put on the bond price), expressing a negative view on the credit, will usually be exercisable in the event of a default. In this case, it would be expected to be at least as expensive as the corresponding default swap premium. For a put on the spread (call on the bond price),
expressing a positive view on the credit, the option to exercise on default is worthless and hence irrelevant.

Two pricing models of credit spread are discussed here, firstly the Black model and then Longstaff-Schwartz model.

5.2.1 Black Model

We derive an expression for the pay-off of a credit spread as presented in Giacommetti et al. [20, 2004] Let $\tau$ be the time to maturity of the option and let $E^Q[.]$ be the expectation operator under the risk neutral measure $Q$.

**Proposition 5.1** The payoff of a credit spread can be expressed as follows:

$$c_t(\tau, s(t, T), K) = e^{-r(\tau-1)} \max(0; s(t, T) - K) \quad (5.1)$$

**Proof.** Let $s(t, T)$ be the credit spread observed at time $t$ and referred to maturity $T$. The credit spread $s(t, T)$ can be seen as:

$$s(t, T) = \lambda_T (1 - \phi_T) \quad (5.2)$$

where $\lambda_T$ is the default probability referred at time $T$ and $\phi_T$ is the *recovery rate* with $0 \leq \phi_T \leq T$. Hence it is reasonable to assume that the spread is always positive and lognormally distributed at the maturity of the option.

In a risk-neutral world, assuming a constant risk-free rate $r$, the expected value of the spread coincides with the forward spread $s_f(t, T)$ and the pay-off of a credit spread call can be expressed as

$$c_t(\tau, s(t, T), K) = e^{-r(\tau-1)} E^Q[\max(0; s(t, T) - K)]$$

50
\[ c_t(\tau, s(t, T), K) = e^{-r(\tau-1)} \max(0; s_t(t, T) - K). \]  

(5.3)

5.2.2 Longstaff and Schwartz Model

Longstaff and Schwartz [27] assume the following dynamics for the logarithm of the spread (5.4) and the interest rates (5.5).

\[ d \ln s(t, T) = (a - b \ln s(t, T)) dt + \omega dZ_1 \]  

(5.4)

\[ dr(t, T) = (\alpha - \beta(t, T)) dt + \sigma dZ_2 \]  

(5.5)

where \( a, b, \omega, \alpha, \beta \) and \( \sigma \) are time-constant parameters, and \( Z_1 \) and \( Z_2 \) are correlated Wiener processes, with correlation \( \rho \).

For simplicity we denote the log spread as \( X = \ln s(t, T) \). Let \( C(X, r, T) \) be the price of a European claim on a credit spread with payoff function \( H(X) \). At expiration time \( T \), \( C(X, r, T) \) must solve the following partial differential equation:

\[
\frac{\omega^2}{2} \frac{\partial^2 C}{\partial X^2} + \rho \sigma \omega \frac{\partial C}{\partial X} \frac{\partial C}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 C}{\partial r^2} + \frac{\partial C}{\partial X} (a - bX) + \frac{\partial C}{\partial r} (\alpha - \beta r) - \frac{\partial C}{\partial t} - rC = 0
\]

(5.6)

subject to the initial condition \( C(X, r, T) = H(X) \).

The solution to this partial differential equation can be obtained in a closed-form using the certainty-equivalent representation

\[ C(X, r, T) = D(r, T) \mathbb{E}[H(X)] \]  

(5.7)
where $D(r, T)$ is the price of a riskless discount bond with maturity $T$ and the expectation is taken with respect to the risk-adjusted process $X$,

$$dX = \left[ a - bX - \frac{\rho \sigma \omega}{\beta} \times (1 - \exp(-\beta(T-t))) \right] dt + \omega dZ_1. \quad (5.8)$$

The solution implies that $X_T$ is conditionally normally distributed with mean $\mu$ and variance $\eta^2$, where

$$\mu = \exp(-bT)X + \frac{1}{b}(a - \frac{\rho \sigma \omega}{\beta})(1 - \exp(-bT))$$

$$+ \frac{\rho \sigma \omega}{\beta(b + \beta)}(1 - \exp(-(b + \beta)(\tau - t)T)), \quad (5.9)$$

$$\eta^2 = \frac{\omega^2[1 - \exp(-2bT)]}{2b}. \quad (5.10)$$

**Proposition 5.2** The value of a European call option on a credit spread, with strike price $K$ is

$$C(X, r, T) = D(r, T) \left[ \exp(\mu + \frac{\eta^2}{2})N(d_1) - KN(d_2) \right] \quad (5.11)$$

where $N(.)$ is the cumulative standard normal distribution function and

$$d_1 = \frac{-\ln K + \mu + \eta^2}{\eta},$$

$$d_2 = d_1 - \eta,$$

and with $\mu$ and $\eta^2$ as in equation 5.9 and 5.10 respectively.
**Sketch of proof.** Since $X$ denotes the log of the spread, the payoff function for this option is simply

$$H(X) = \max(0, e^x - K)$$

Applying the certainty-equivalent valuation operator in equation (5.7) results in the closed-form solution for the value of the call option as in equation (5.11).

\[\square\]

## 5.3 Credit-Linked Notes (CLN)

For investors who wish to take exposure to the credit derivatives market and who require a cash instrument, one possibility is to buy it in a funded credit linked note form. A credit-linked note is a security issued by a corporate entity (bank or otherwise) agreed upon by the investor and the financial institution.

### 5.3.1 Valuation of Credit-Linked Notes (CLN)

**Proposition 5.3** The Value of the Credit-Linked Notes is given as

$$CLN = \sum_{i=1}^{n} \Delta(t_{i-1}, t_{i})cB_{0}^{i}(1-p(t_{i}))+B_{0}^{T}(1-p(T)) + R \sum_{j=1}^{m} [p(t_{j})-p(t_{j-1})]B_{0}^{j}$$

**Proof.** Suppose the credit-linked notes pays coupons at dates $t_{1} < t_{2} < \ldots = T$, where $T$ is the maturity and the annual coupon rate is $c$. The notional is 1. The value of the notes is

$$CLN = CLN_{n} + CLN_{d}$$
the sum of the present values of no-default payments and default payments.

We have

$$CLN_n = E\left[ \sum_{i=1}^{n} \Delta(t_{1-i}, t_i)cB_{0}^{I_i}1_{\{\tau>T\}} + \bar{B}_{0}^{T}1_{\{\tau>T\}} \right]$$

$$= \sum_{i=1}^{n} \Delta(t_{1-i}, t_i)cB_{0}^{I_i}(1 - p(t_i)) + \bar{B}_{0}^{T}(1 - p(T))$$

where $p(t) = \tilde{P}(\tau \leq t)$ is the risk-neutral default probability. Suppose the recovery rate is a constant $R \in [0, 1]$. Then

$$CLN_d = E[\bar{B}_{0}^{T}R1_{\{\tau\leq T\}}]$$

$$= R \int_{0}^{T} B_{0}^{u} \tilde{P}[\tau \in du]$$

$$\approx R \sum_{j=1}^{m} \tilde{P}[t_{j-1} \leq \tau < t_{j}]B_{0}^{I_j}$$

$$= R \sum_{j=1}^{m} [p(t_{j}) - p(t_{j-1})]B_{0}^{I_j}.$$ 

Letting $m \to \infty$, the approximation is exact. □

### 5.4 Other credit derivatives

In this section define other credit derivatives, that are traded in the credit derivatives market. These are Total Return Swaps and Collaterized Debt Obligations

#### 5.4.1 Total Return Swaps

In Bielecki and Rutkowski [5, 2002] a total return swap is defined as a derivative contract that simulates the purchase of an instrument (note, share, etc) with 100% financing. The contract may be marked to market
at each reset date, with the total return receiver (TR Receiver) receiving (or paying) any increase in the value of the underlying instruments, and the total return payer (TR Payer) receiving (or paying) any decrease in the value of the underlying instrument. This is illustrated in figure 5.

![Diagram of Total Return Swap](image)

Figure 5.2: Total Return Swap

5.4.2 Collateralized Debt Obligations (CDO)

A collateralized debt obligation (CDO) is a structure of fixed income securities whose cash flows are linked to the incidence of default in a pool of debt instruments. These debts may include loans, revolving lines of credit, other asset-backed securities, emerging market corporate and sovereign debt, and subordinate debt from structured transactions. When the collateral is mainly made up of loans, the structure is called a collateralised loan obligation (CLO), and when it is mainly bonds, the structure is called a collateralised bond obligation (CBO).
5.5 Applications of Credit Derivatives

Some of the applications of credit derivatives have already been mentioned when they were specific to the credit derivative discussed. General fields of application common to most credit derivatives are:

(1.) Applications in the management of credit exposures: These include the reduction of credit concentration, easier diversification of credit risk and the direct hedging of default risk.

(2.) In trading, credit derivatives can be used for the arbitrage of mispricing in defaultable bonds.

(3.) The largest group of credit derivative users are banks who use credit derivatives to free up or manage credit lines, manage loan exposure without needing the consent of the debtor, manage (or arbitrage) regulatory capital or exploit comparative advantages in costs of funding. Another important application here is the securitisation of loan portfolios in form of CLOs.

(4.) The specification of the credit derivatives can be adjusted to the needs of the counterparties: Denomination, currency, form of coupon, maturity or even the general payoff need not match the reference asset. This is especially useful for the management of counterparty exposures from derivatives transactions.
5.6 Valuation of Credit Default Swaps

The pricing technique discussed below was derived by Hull and White [23, 2000]. We present the theory of [23, 2000], and present a sample calculation.

The following assumptions were made in their pricing model.

Assumptions
(a) default events, interest rates and recovery rates are mutual exclusive
(b) the claim in the in the event of default is the face value plus accrued interest.

Notation—
Suppose that default can occur only at time $t_1, t_2, \ldots, t_n$. Define

$T$: Life of credit default swap

$q(t)$: Risk-neutral default probability density at time $t$

$\hat{R}$: Expected recovery rate on the reference obligation in a risk-neutral world.

$u(t)$: Present value of payments at the rate of 1 unit of cash per year on payment dates between time zero and time $t$

$e(t)$: Present value of an accrual payment at time $t$ where $t$ is the payment date immediately preceding time $t$

$v(t)$: Present value of 1 unit of cash received at time $t$

$w$: Total payments per year made by credit default swap buyer

$s$: Value of $w$ that causes the credit default swap to have a value of zero

$\pi$: The risk-neutral probability of no credit event during the life of the swap
\( A(t) \): Accrued interest on the reference obligation at time \( t \) as a percent of face value

**Proposition 5.4** - The expected present value of payments, takes the form:

\[
\mathbb{E}[P] = w \int_0^T q(t)[u(t) + e(t)]dt + w\pi u(T).
\]

**Proof.** The value of \( \pi \) is one minus the probability that the credit event will occur. It can be calculated from \( q(t) \):

\[
\pi = 1 - \int_0^T q(t)dt.
\]

The payments last until a credit event or until time \( T \), whichever is sooner. If a default occurs at time \( t \) \((t < T)\), the present value of the payments is \( w[u(t) + e(t)] \). If there is no default prior to time \( T \), the present value of the payments is \( wuT \). The expected present value of the payments is therefore:

\[
\mathbb{E}[P] = w \int_0^T q(t)[u(t) + e(t)]dt + w\pi u(T).
\]

□

**Proposition 5.5** - The CDS spread, \( s \), is given by the following formula:

\[
s = \frac{\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)v(t)dt}{\int_0^T q(t)[u(t) + e(t)]d(t) + \pi wu(T)}.
\] (5.12)

**Proof.** Given our assumption about the claim amount, equation (1) shows that the risk-neutral expected payoff from the CDS is

\[
1 - [1 + A(t)\hat{R}] = 1 - \hat{R} - A(t)\hat{R}
\]

The present value of the expected payoff from the CDS is
\[
\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)v(t)dt
\]

and the value of the credit default swap to the buyer is the present value of
the expected payoff minus the present value of the payments made by the
buyer or

\[
\int_0^T [1 - \hat{R} - A(t)\hat{R}]q(t)v(t)dt - w\int_0^T q(t)[u(t) + e(t)]d(t) - \pi w u(T)
\]

The CDS spread \( s \) is the value of \( w \) that makes this expression zero.
Therefore the formula as claimed in the proposition can be seen to hold.

The variable \( s \) is referred to as the default swap spread or CDS spread. It is
the total of the payments per year, as a percentage of the notional principal,
for a newly issued credit default swap. We include a sample computation.

**Example 5.6.** (a) We make a sample computation of \( s \) for the case of a
1-period problem with the following parameters.
We assume that the riskfree rate is \( r = 0.02 \) and a basic rate \( \rho = 0.05 \) is
levied on the loan, and that the recovery rate is \( \hat{R} = 0.25 \).
Then \( v(t) = u(1) = \exp(-r) \) and \( A(1) = \exp(\rho) \).
We further assume \( e(1) = \exp(\rho - r) \), and then \( e(t) \) determines also the
value of \( u(t) \).
Now we can calculate \( q \), the risk-neutral probability of default.
Using the elementary 1-period arguments, as for instance in the book [16]
of Etheridge, the following equation is satisfied by \( q \):
\[(1 - q)e^\rho + qRe^\rho = e^r.\]

This yields the formula:

\[q = \frac{1 - \exp(r - \rho)}{1 - \hat{R}}.\]

We can now calculate the credit default spread, obtaining the value

\[s = 0.018762.\]

(b) We show how to calculate default probabilities beyond the single period case. Thus, let us consider the following 2-period problem.

A debt of principal value 2 is to be amortized by an instalment of size \(\exp(\rho)\) at time \(t = 1\), and another instalment of size \(\exp(2\rho)\) at time \(t = 2\).

The riskfree interest rate is \(r\). Let us calculate the risk-neutral default probabilities \(q(1)\) and \(q(2)\) of default.

Again using the basic single-period arguments, the following formula is obtained for \(q(1)\):

\[q(1) = \frac{2e^r + e^\rho - e(2r - \rho)}{e^r + e^\rho - 2Re^r}.\]

Then, with \(q\) as in (a) above, we have

\[q(2) = (1 - q(1))q.\]

With \(r, \rho\) and \(\hat{R}\) as in (a), we get \(q(1) = 0.692611; \ q(2) = 0.012113\) (and \(q = 0.039406\)).
Chapter 6

An options approach to pricing credit risk premium

As shown in Merton [1973], insuring a single, homogeneous-term debt issue against default is equivalent to acquiring a European put option on the value of the company, before insurance. In this Isomorphic relationship, the maturity of the put option is the same as that of the debt issue, and the strike price is equal to the maturity value of the debt. In this chapter we propose a method of minimising the value of this put option on the value of the company. We also propose a method for determining a credit spread. This credit spread is the interest rate over and above the opportunity costs.

6.1 Spread Method

Options theory applied to amortization of a loan suggests the following method for pricing a credit spread. We assume that a rate $\mu$ is charged
on the loan as the opportunity cost of the loan. In addition to this, a credit
spread will be charged at a rate $\gamma$ along with the basic opportunity cost.
We propose a method for determining $\gamma$. Now $\mu$ together with the volatility
of the lender’s wealth process, combines to create a virtual commodity of
which the value $S$ evolves according to the geometric Brownian motion, with
$\mu$ and $\sigma$ assumed constant:

$$dS(t) = S(t)[\mu dt + \sigma dW(t)].$$

This imaginary commodity serves as the underlying asset in the Black-
Scholes model, as we explain below. Along with the amortization instalment
$a_t$ which is paid at time $t$, the lender will pay an amount $e^{\mu t} p_t$, where $p_t$
is the time-$(t=0)$ price of a European put option on the commodity $S$ with
maturity $t$, starting with $S(0) = e^{-\mu t} a_t$ and with strike price equal to $a_t$.
This implies that

$$p_t = e^{-r t} a_t N(-k_t) - e^{-\mu t} a_t N(-h_t), \quad (6.1)$$

where

$$h_t = \frac{\ln e^{-\mu t} a_t - \ln a_t + (r + \frac{\sigma^2}{2}) t}{\sigma \sqrt{t}}$$

$$k_t = \frac{\ln e^{-\mu t} a_t - \ln a_t + (r - \frac{\sigma^2}{2}) t}{\sigma \sqrt{t}} = h_t - \sigma \sqrt{t}$$

$$N(m) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} e^{-\frac{1}{2}z^2} dz.$$ 

For uniformity we assume that the term $[0, T]$ has been decided, with $N$
instalments (i.e., over $N$ periods of equal duration), and that instalments
will be of the form $a_t = be^{\mu t}$, with $b = \frac{P}{N}$. 
The total cost of the options is

$$C = \Sigma_{t=1}^{N} p_t.$$ 

Now we calculate as follows, noting that the unknown $\gamma$ must satisfy the identity:

$$C = P e^{\gamma T}.$$ 

This yields

$$\gamma = \frac{1}{T} \ln \frac{C}{P}.$$ 

Consequently we obtain the following formula for $\gamma$

$$\gamma = \frac{1}{T} \ln \left( \frac{1}{P} \Sigma_{t=1}^{N} p_t \right).$$

In particular we note that ultimately the particular value we propose here for credit spread, makes it dependent on only the time horizon $[0, T]$ and $\sigma$, as the following proposition shows.

**Proposition 6.1.** *The value of $\gamma$ is independent of $P$.***

**Proof.** Consider two loans, one of principal value $P$, and another one with principal value equal to $Q = 1$ and with “$p_t$-values” denoted by $p_t$ and $q_t$ respectively. Let us denote the spread values by $\gamma_0$ and $\gamma_1$ respectively. Then we have

$$\gamma_0 = \frac{1}{T} \ln \left( \frac{1}{P} \Sigma_{t=1}^{N} p_t \right),$$

$$\gamma_1 = \frac{1}{T} \ln \left( \frac{1}{Q} \Sigma_{t=1}^{N} q_t \right).$$
In the formula for $p_t$, the value of $a_t$ is

$$a_t = be^{\mu t},$$

where

$$b = \frac{P}{N}.$$

Then

$$a_t = \frac{P}{N}e^{\mu t}.$$

In the corresponding formula for $q_t$ the value of the amortization $\alpha_t$ is

$$\alpha_t = \frac{Q}{N}e^{\mu t}.$$

It is now a routine check to confirm that in fact for each $t$, we have

$$\frac{1}{P}p_t = \frac{1}{Q}q_t.$$

Therefore $\gamma_1 = \gamma_0$. □

6.2 The more general options method

Let $D$ be the time $t_0$ value of the amount to be amortized (discounted w.r.t the risk-free interest rate $r$) and $D$ does not cover credit risk.

We make the following assumptions:

(a) the periodical amortizations will be amount of the form $a_1 + c_1$, $a_2 + c_2$, $\ldots$, $a_T + c_T$ such that that $a_t$ is the amortization and $c_t$ is the time $t_0$ value of a risk payment associated with the amount $a_t$. (b) Since an interest
rate $\mu$ ($\mu > r$) will be charged as opportunity cost, the firm’s asset $S_t$ follows a geometric Brownian Motion.

$$dS_t = S_t[\mu dt + \sigma dB_t].$$ \hfill (6.2)

Let us write $D$ in the form

$$D = \sum_{t=1}^{T} d_t.$$ \hfill (6.2.1)

We shall regard $c_t$ as the price of an European put option on $S_t$ with $S_0 = d_t$ and strike price $K = a_t$, maturing at time $t$, then

$$c_t = e^{-r\tau} a_t N(-h_t) - d_t N(-k_t),$$ \hfill (6.3)

where

$$h_t = \frac{(\ln d_t - \ln a_t) + (r + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}},$$

$$k_t = \frac{(\ln d_t - \ln a_t) + (r - \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}} = h_t - \sigma \sqrt{\tau},$$

$$N(m) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} e^{-\frac{z^2}{2}} dz,$$

$$\tau = T - t.$$ \hfill (6.3.1)

Also, the sum of the put options will written as $U$, i.e.,

$$U = \sum_{t=1}^{T} c_t.$$ \hfill (6.3.2)

6.2.1 The optimization problem

Consider a fixed debt $D$ to be amortized over a time horizon $[0, T]$. For a given sequence of company amortization payments $a_1, a_2, \ldots, a_T$, is there a corresponding sequence $d_1, d_2, d_3, \ldots, d_T$ with $D = \sum_{t=1}^{T} d_t$ for which
the sum of the time zero present value of the insurance premiums \( c_1, c_2, c_3, \ldots, c_T \) is at its minimum?

For smaller values of the integer \( T \) we shall illustrate by way of graphs that there is a positive answer to the question above. Thereafter we shall present an analytic solution.

### 6.2.2 2-Period Loan

Let us consider a two period loan contract, i.e. \( T = 2 \) with \( D = 2, r = 0, \mu = 0.04 \) and \( \sigma = 0.05 \). We sketch the graph of \( U \) as a function of \( d_1 \) as an independent variable. Note that having \( d_1 \) as independent, then the variable \( d_2 \) is not independent, being determined by the constraint: \( d_1 + d_2 = D \).

Figure 6.1 indicates that there is a minimum value for \( U \). The minimum value, say \( U_{\text{min}} \), of \( U \) and the associated value of \( d_1 \) can be read off the graph.

### 6.2.3 3-Period Loan

Now, let us consider a 3-period loan contract, i.e \( T = 3 \), with \( D = 3, \sigma = 0.05, r = 0 \) and \( \mu = 0.03 \). We sketch the graph of \( U \) as a function of \( d_1 \) and \( d_2 \). The variable \( d_3 \) is not independent, being determined by the constraint: \( d_1 + d_2 + d_3 = D \).

Figure 6.2 indicates that there is a minimum value for \( U \). The minimum value, of \( U \) and associated values of \( d_1 \) and \( d_2 \) can be read off the graph.
6.2.4 N Period Loan

If the number of periods exceed 3, graphical methods will not possible. This requires us to utilize Lagrange multipliers to solve for N-periods loan contracts. The lagrangian is as follows

\[
L = U - \lambda \left[ \sum_{t=1}^{T} d_t - D \right] \tag{6.4}
\]

This gives the optimization problem as:

Minimise \( U = \sum_{t=1}^{T} c_t \), subject to \( \sum_{t=1}^{T} d_t = D \).

The theorem that follows allows us to solve the problem of finding the optimal sequence of incremental values of the \( d_t \)'s.
Theorem 6.1. The following conditions are necessary to solve the problem

(a) The constraint equation

$$\sum_{t=1}^{T} d_t = D$$

(b) For each $t$,

$$\lambda e^{rt} = \frac{a_t e^{-rt} - \frac{1}{2} h_t^2}{\sqrt{2\pi}d_t\sigma\sqrt{\tau}} - N(-h_t) - \frac{e^{-\frac{1}{2} h_t^2}}{\sqrt{2\pi}\sigma\sqrt{\tau}}$$

$$\tau = T - t$$

$$h_t = \frac{\ln a_t - \ln d_t + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$
\[ k_t = h_t - \sigma \sqrt{\tau} \]

\[ N(.) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\alpha} e^{-\frac{z^2}{2}} dz. \]

**Proof.** Condition (a) holds, following from the condition that

\[ \frac{\partial L}{\partial \lambda} = 0. \]

To prove (b) we compute the partial derivatives

\[ \frac{\partial L}{\partial d_t} = 0, \]

\[ \frac{\partial U}{\partial d_t} - \lambda = 0. \]

The latter condition can be written as

\[ \frac{\partial e^{-rt} c_t}{\partial d_t} = \lambda e^{-rt}. \]

Also, recall that the guarantee premium given by

\[ e^{-rt} c_t = e^{-rt} a_t N(-k_t) - d_t N(-h_t), \quad (6.5) \]

Therefore,

\[ \frac{\partial e^{-rt} c_t}{\partial d_t} = \frac{\partial a_t e^{-rt} N(-k_t)}{\partial d_t} - \frac{\partial d_t N(-h_t)}{\partial d_t}. \]

\[ \frac{\partial e^{-rt} c_t}{\partial d_t} = a_t e^{-rt} \frac{\partial N(-k_t)}{\partial d_t} - N(-h_t) - \frac{d_t \partial N(-h_t)}{\partial d_t} \]

We know that,

\[ N(-\alpha) = \int_{-\infty}^{\alpha} (\sqrt{2\pi})^{-1} e^{-\frac{z^2}{2}} dz, \]

thus

\[ \frac{\partial N(-\alpha)}{\partial \alpha} = -(2\pi)^{-\frac{1}{2}} e^{-\frac{\alpha^2}{2}}. \]

Also,

\[ \frac{\partial h_t}{\partial d_t} = -\frac{1}{d_t \sigma \sqrt{\tau}}. \]
Since \( k_t = h_t - \sigma \sqrt{\tau} \), we have
\[
\frac{\partial k_t}{\partial d_t} = \frac{\partial h_t}{\partial d_t} = -\frac{1}{d_t \sigma \sqrt{\tau}}.
\]

Using the chain rule of differentiation we obtain
\[
\frac{\partial N(-k_t)}{\partial d_t} = \frac{\partial N(-k_t)}{\partial k_t} \frac{\partial k_t}{\partial d_t},
\]
\[
\frac{\partial N(-k_t)}{\partial d_t} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} k_t^2} \left(-\frac{1}{d_t \sigma \sqrt{\tau}}\right),
\]
\[
\frac{\partial N(-k_t)}{\partial d_t} = \frac{e^{-\frac{1}{2} k_t^2}}{\sqrt{2\pi} d_t \sigma \sqrt{\tau}}.
\]
(6.6)

Also,
\[
\frac{\partial N(-h_t)}{\partial d_t} = \frac{\partial N(-h_t)}{\partial h_t} \frac{\partial h_t}{\partial d_t},
\]
\[
\frac{\partial N(-h_t)}{\partial d_t} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} h_t^2} \left(-\frac{1}{d_t \sigma \sqrt{\tau}}\right),
\]
\[
\frac{\partial N(-h_t)}{\partial d_t} = \frac{e^{-\frac{1}{2} h_t^2}}{\sqrt{2\pi} d_t \sigma \sqrt{\tau}}.
\]
(6.7)

Now substituting equation (6.4) and (6.5) into (6.3) yields
\[
\frac{\partial e^{-rt} c_t}{\partial d_t} = a_t e^{-rt} \frac{e^{-\frac{1}{2} k_t^2}}{\sqrt{2\pi} d_t \sigma \sqrt{\tau}} - N(-h_t) - d_t \frac{e^{-\frac{1}{2} h_t^2}}{\sqrt{2\pi} d_t \sigma \sqrt{\tau}}
\]
\[
= a_t e^{-rt-\frac{1}{2} k_t^2} \frac{1}{\sqrt{2\pi} d_t \sigma \sqrt{\tau}} - N(-h_t) - \frac{e^{-\frac{1}{2} h_t^2}}{\sqrt{2\pi} d_t \sigma \sqrt{\tau}}.
\]

Also,
\[
\frac{\partial e^{-rt} c_t}{\partial d_t} = \lambda e^{rt},
\]

Thus,
\[
\lambda e^{rt} = a_t e^{-rt-\frac{1}{2} k_t^2} \frac{1}{\sqrt{2\pi} d_t \sigma \sqrt{\tau}} - N(-h_t) - \frac{e^{-\frac{1}{2} h_t^2}}{\sqrt{2\pi} \sigma \sqrt{\tau}}.
\]
Chapter 7

Credit Risk Management

Under Basel II

The changes that have happened in the banking and financial markets has increased the exposure of banks to risks. To protect these risks, the Bank for International Settlements created the Basel Committee on Banking Supervision (BCBS), which established the Basel accords, (Basel I and Basel II). This committee spearheaded the the framework to minimize credit risk by introducing capital adequacy standards for large active banks. This chapter presents a historical overview of the Basel accords, and we provide a brief description of Basel II. The material in this chapter has been taken from Basel Committee on Banking Supervision (2004) [2] and Basel Committee on Banking Supervision (2005) [3]. We also review the paper by Decamps et al [12].
7.1 Basel I

The current banking regulations for internationally operating banks are the results of the Basel II Capital Accord, published in 2004, which is an amendment of Basel I Capital Accord which was published in 1988. The custodians of these regulations is the so-called Basel Committee on Banking Supervision (BCBS). The main objectives of Basel I were to promote the soundness and stability of the banking system and adopt a standard approach across banks in different countries.

Although it was initially intended to be only used by banks in the G-10 countries, it was finally adopted by over 120 countries and recognized as a global standard. However, shortcomings of the Basel I became increasingly obvious over time.

7.2 Overview of Basel II

The BCBS agreed in 2004 on a revised capital adequacy framework (Basel II) (see [2]). The main objective is to further strengthen the soundness and stability of the international banking system by adopting better risk management. Basel II consists of a broad set of supervisory standards to improve risk management practices, which are structured along three mutually reinforcing elements or pillars. These Pillars are discussed in detail in the Basel II framework [2]. There have been many research papers on Basel II in the recent history. One of these papers is by Decamps et al [12]. The authors of this paper developed a continuous-time model of commercial banks be-
behavior where interaction between the three pillars can be analyzed. In the following section we provide a brief description of each Pillar.

7.2.1 Pillar 1 - Minimum Regulatory Capital Requirements

In the first pillar of the Basel II Accord, capital requirements are proposed for three categories of risk:

(a) Credit Risk
Credit risk is the possibility of a loss as a result of clients not fulfilling their obligation. Three methods are used to determine credit risk. These are the Standardized Approach, the Foundation Internal Ratings Based (IRB) Approach and the Advanced Ratings Based (IRB) Approach. The standardized approach provides improved risk sensitivity compared to Basel I. The two IRB approaches rely on banks own internal risk ratings.

(b) Operational Risk
Operational risk is defined as the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events. In Basel II three methods of measuring operational risk are given as the Basic Indicator Approach, the Standardized Approach and the Advanced Measurement Approach.

(c) Market Risk
Market risk is the risk that the value of an investment will decrease due to moves in market factors. Two methods of measuring this risk discussed, the Standardized Approach and the Internal Models Approach.
7.2.2 Pillar 2 - Supervisory Review of Capital Adequacy

The second pillar of Basel II is a supervisory review of capital adequacy. National supervisors must ensure that banks develop an internal capital assessment process and set capital targets consistent with the risk profiles. National supervisors are responsible for evaluating how well banks are assessing their capital adequacy needs relative to their risks.

7.2.3 Pillar 3 - Market Discipline and Disclosure

The third pillar of Basel II, is about market discipline and disclosure. The main impact of this pillar is to promote the development of financial reporting about risks.

7.3 Basel II Credit Risk Approaches

In contrast to Basel I that applies one approach to all banks, Basel II offers a menu of options under Pillar 1 for calculating the credit capital requirements of banking book exposures. In particular, two main methodologies can be used for most exposures: the Standard Approach and the Internal Ratings Based (IRB) approach; securitization exposures are subject to a separate (but similar) capital treatment. Each approach has different characteristics and requirements that are briefly described below.

7.3.1 The Standardised Approach

This approach measures credit risk similar to Basel I, but has greater risk sensitivity because it uses the credit ratings of external credit assessment
institutions to define the weights used when calculating risk-weighted assets.

Table 7.1 illustrates the risk weights proposed in Basel II.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Banks</th>
<th>Corporates</th>
<th>Sovereign</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA to AA-</td>
<td>20%</td>
<td>20%</td>
<td>0%</td>
</tr>
<tr>
<td>A+ to A-</td>
<td>50%</td>
<td>50%</td>
<td>20%</td>
</tr>
<tr>
<td>BBB+ to BBB-</td>
<td>50%</td>
<td>100%</td>
<td>50%</td>
</tr>
<tr>
<td>BB+ to BB-</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>B+ to B-</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>Below B</td>
<td>150%</td>
<td>150%</td>
<td>150%</td>
</tr>
<tr>
<td>Unrated</td>
<td>50%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

The following equation is used to calculate the minimum regulatory capital for credit risk and is the same as in the Basel I Accord.

\[
\text{Regulatory Capital} = \text{Risk-Weight} \times \text{Exposure} \times 8\%.
\]

### 7.3.2 IRB Approach

The IRB approach relies on the bank’s own internal estimates of certain risk parameters to determine credit capital requirements. However, the capital figure itself is still derived from a supervisory formula provided by the Basel Committee. This formula has been calibrated to reflect the risk of specific asset types. Under IRB approach, all banking book exposures must be categorized into broad asset classes using specific definitions and criteria provided by the Basel Committee. For example, the RWA’s for corporate,
sovereign and bank exposures, are calculated as follows:

\[ R = 0.12 \times \frac{1 - e^{-50 \times PD}}{1 - e^{-50}} + 0.24 \times \left[ 1 - \frac{1 - e^{-50 \times PD}}{1 - e^{-50}} \right] \]

\[ b = (0.08451 - 0.05898 \times \ln(PD))^2 \]

\[ K = LGD \times N \left[ \frac{N^{-1}(PD)}{\sqrt{1 - R}} + \sqrt{\frac{R}{1 - R}} N^{-1}(0.999) \right] \times \frac{1 + (M - 2.5) \times b(PD)}{1 - 1.5 \times b(PD)} \]

\[ RWA = K \times 12.5 \times EAD \]

\[ RCP = RWA \times 8\% \]

where \( R \) is the correlation, \( b \) is the maturity adjustment, \( K \) is the capital requirement, \( RWA \) is the risk-weighted asset and \( RCP \) is the regulatory capital charge. There are other formulas for different exposures. These are presented in [2] and [3].

### 7.4 Credit Derivatives in Basel II

Basel II permits the reduction of credit risk by means that include the use of collateral, credit derivatives, guarantees, or netting agreements. Banks can take account of such credit protection in calculating capital requirements provided that the supervisors are satisfied that banks fulfil certain minimum operational conditions relating to risk management processes. Only credit default swaps and total return swaps that provide credit protection equivalent to guarantees will be eligible for recognition.

The protected portion is assigned the risk weight of the protection provider. The uncovered portion of the exposure is assigned the risk weight of the underlying counterparty.
Chapter 8

Portfolio Management with Respect to Bank Lending

Portfolio management is a major activity in the finance industry, whether banking, insurance or pension. In this section we present two cases of investment portfolio management. Firstly, we consider the paper Devolder et al. [13] and their method of optimal portfolio selection, as an example of a rather classical methodology. Secondly, we present the more modern results of Mukuddem-Petersen and Petersen [31, 2006] pertaining specifically to the banking industry. In particular we make a new contribution by including simulated graphs to illustrate the relevant investment processes.

8.1 Optimal portfolio management

In investment projects one of the key problems facing the manager is that of selecting the appropriate portfolio. That is, among the various possible
investment choices with varying risk levels and varying expected returns, the most appropriate combination must be found. This process may be a dynamic one, i.e., the combination may change over time. The paper of Merton [29] was a milestone in this regard. The stochastic control method of Merton remains to be applied after almost half a century. In what follows we present the portfolio selection problem from the paper of Devolder et al [13]. We present the problem briefly, without discussing or even describing all the different variables. The aim is to show the main features of a very fundamental method in financial modeling, and to run a simulation.

**Example 8.1** (Devolder et al [13]). We consider the case of exponential utility. Let $F(t)$ be the total value of the investment, and let $u(t)$ be the fraction of $F$ which is invested in the risky asset. The process $F$ is a solution of the stochastic differential equation

$$dF(t) = F(t)[u(t)\alpha + (1 - u(t))r]dt + F(t)u(t)\sigma dw(t)$$

(8.1)

with

$$F(0) = P(0 \geq t \leq N).$$

The particular expected utility we consider is, for some $c > 0$:

$$J(u(.), F) = \mathbb{E}\left[-\frac{1}{c} \exp\{-cF(T)\}|F(0)\right].$$

The problem is to find the $u = u^*$ which maximizes $J$, i.e., so that

$$J(u^*, F) = \max_{\{u\}} \mathbb{E}\left[-\frac{1}{c} \exp\{-cF(T)\}|F(0)\right].$$
Towards solving this risk-sensitive optimal control problem we introduce a more general function, the value function

$$W(t, F(.)) = \max_{\{u\}} \mathbb{E} \left[ -\frac{1}{c} \exp\{-cF(T)\} | F(t) \right].$$

The solution of this problem amounts to a maximization problem of an expression in the associated HJB equation:

$$\max_{\{u\}} \left[ \frac{\partial W}{\partial t} + \left( u(t)(\alpha - r) + r \right) F \frac{\partial W}{\partial F} + \frac{1}{2} u^2(t) \sigma^2 F^2 \frac{\partial^2 W}{\partial F^2} \right] = 0 \quad (8.2)$$

with \( \frac{\partial^2 F}{\partial F^2} < 0 \)

Maximizing the given quantity with respect to \( u \), requires a derivative with respect to \( u \) and then finding its critical value. The problem therefore becomes that of solving for \( u \) in the following equation:

$$(\alpha - r) F \frac{\partial W}{\partial F} + u(t) \sigma^2 F^2 \frac{\partial^2 W}{\partial F^2} = 0, \quad (8.3)$$

which eventually yields:

$$u^*(t) = -\frac{\partial W/\partial F}{F(\partial^2 W/\partial F^2)} \frac{-\alpha - r}{\sigma^2}.$$

At this stage we test a particular form for \( W \). A test function is substituted into the HJB-equation. We skip the detail of this process as it is sufficiently clearly shown in [13, pp231-232]. Eventually we find the optimal portfolio proportion \( u \) :

$$u^*(t) = e^{r(t-N)} \frac{\alpha - r}{\sigma^2 c}.$$ 

(8.5)
The total amount of the portfolio sitting in the risky asset is then

\[ u^*(t)F = e^{\left(t-N\right)} \frac{\alpha - r}{\sigma^2 c}. \]  

(8.6)

We show a simulation of this, based on equation (8.5)... Simulation 8.2. We take the parameters: \( N = 120, \ r = 0.03, \ \alpha = 0.05, \ \sigma = 0.12, \) and \( c = 1. \) Figure 8.1 shows the results.

![Figure 8.1: Optimal value of \( u \)](image-url)
8.2 Optimal capital allocation in banking

One of key issues in Basel II, is the management of risk the banking sector at large. Researchers have propose different solution in managing risk in banks. Mukuddem-Petersen and Petersen [31, 2006] examine a problem of optimal risk management of banks in a continuous-time stochastic dynamics. The authors minimise market and capital adequacy risk that involves the safety of assets held. In this section we present the main results of Mukuddem-Petersen and Petersen [31], and we perform numerical simulations of these results.

In [31] they denote the value of the bank securities by $S$. The bank is allowed to invest in a financial market with $n + 1$ financial securities. These securities include riskless and risky securities. The stochastic dynamics of the optimal value, $S^*$, of bank securities are shown in proposition 8.1.

**Proposition 8.1** The optimal value, $S^*$, of bank securities may be represented by

$$dS^*(t) = \left\{ \left( r - \xi^T \hat{\xi} - \frac{\delta_{ss}}{\omega}\right) S^*(t) + \left( -r + \xi^T \hat{\xi} + \frac{\delta_{ss}}{\omega} + r_1\right) l_r(t) \right\} dt$$

$$+ \left\{ -\hat{\xi}^T S^*(t) + (\hat{\xi}^T + \sigma q^T) l_r(t) \right\} dW(t)$$

(8.7)

where $\delta_{ss}$ is the unique positive solution of the equation.

$$\delta_{ss}^2 + \omega (r_i - 2r + \xi^T \hat{\xi}) \delta_{ss} - \omega (1 - \omega) = 0$$

**Proof.** For a complete proof see [31].
Mukuddem-Petersen and Petersen [31] proposes an optimal bank security allocation strategy, $\tilde{\pi}^\ast$. This strategy shows the weight of risky assets in the total portfolio. This strategy is shown in proposition 8.2.

**Proposition 8.2** The optimal risky securities allocation may be given as

$$\tilde{\pi}(t)^\ast = [\chi^{-1}\sigma\tilde{\xi} + \sigma_I\sigma^{-T}\tilde{\eta}] (l_u(t) + S(t)) - \chi^{-1}\sigma\xi S(t)$$  \hspace{1cm} (8.8)

where $l_u$ is the unfunded loans.

**Proof.** For a complete proof see [31].

8.2.1 Numerical Simulations

In this section we perform a numerical simulation of Proposition 8.1, with the following values. $n = 1, r = 0.1, r_I = 0.1, \omega = 0.2, \tilde{q} = 0.4, l_r = 0.23, \tilde{\xi} = 0.2$ and $\sigma = 0.05$. The results of this simulation are shown figure 8.2.

![Figure 8.2: Optimal Value of Bank Securities](image-url)
Chapter 9

Conclusion

Credit risk models are divided into two groups, structural and reduced form models. We have discussed three models under the structural approach, these are Merton [30], Geske [18] and Black and Cox [6] models. The latter two models being the extensions of Merton model. Under reduced form models we reviewed models by Jarrow and Turnbull [25] and Duffie and Singleton [14].

Credit derivatives have the capacity to transfer the credit risk associated with lending agreements, enabling the risk to be managed or traded independently of ownership of the underlying asset. However, the responsibility for managing the asset has been stripped of its credit risk still remains with the lender.

In Chapter 6 we proposed a new method for determining a credit spread and we also introduced a method for minimising the guarantee on a debt contract.
Basel II consists of a broad set of supervisory standards to improve risk management practices, which are structured along three mutually reinforcing elements or pillars. These pillars are Minimum Regulatory Capital Requirements (Pillar 1), Supervisory Review of Capital Adequacy (Pillar 2) and Market Discipline and Disclosure (Pillar 3). Under Pillar 1 different methods for calculating the credit capital capital requirements of banking book exposures are given. In particular, two main methodologies can be used for most exposures: the *Standard Approach* and the *Internal Ratings Based* approach.

In Chapter 8, we have presented two cases of portfolio management. Firstly we presented work by Devolder et al [13] and we also presented work by Mukuddem-Petersen and Petersen [31].
Bibliography


