Efficient Variable Mesh Techniques to Solve Interior Layer Problems

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Abstract

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Singularly perturbed problems have been studied extensively over the past few years from different perspectives. The recent research has focussed on the problems whose solutions possess interior layers. These interior layers appear in the interior of the domain, location of which is difficult to determine a-priori and hence making it difficult to investigate these problems analytically. This explains the need for approximation methods to gain some insight into the behaviour of the solution of such problems. Keeping this in mind, in this thesis we would like to explore a special class of numerical methods, namely, fitted finite difference methods to determine reliable solutions. As far as the fitted finite difference methods are concerned, they are grouped into two categories: fitted mesh finite difference methods (FMFDMs) and the fitted operator finite difference methods (FOFDMs). The aim of this thesis is to focus on the former. To this end, we note that FMFDMs have extensively been used for singularly perturbed two-point boundary value problems (TPBVPs) whose solutions possess boundary layers. However, they are not fully explored for problems whose solutions have interior layers. Hence, in this thesis, we intend

firstly to design robust FMFDMs for singularly perturbed TPBVPs whose solutions possess interior layers and to improve accuracy of these approximation methods via methods like Richardson extrapolation. Then we extend these two ideas to solve such singularly perturbed TPBVPs with variable diffusion coefficients. The overall approach is further extended to parabolic singularly perturbed problems having constant as well as variable diffusion coefficients.

KEYWORDS

Efficient variable mesh techniques to solve interior layer problems

Singularly perturbed problems

Two-point boundary problems

Parabolic problems

Turning point problems

Interior layer

Fitted finite difference methods

Piecewise uniform mesh

Graded mesh

Richardson extrapolation

Convergence and stability analysis.

Declaration

I declare that *Efficient variable mesh techniques to solve interior layer problems* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Charles K. Mbayi March 13, 2020

Signed: ....

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Contents

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List of Publications

Part of this thesis has already been published/submitted in the form of the following research papers:

1. C.K. Mbayi, J.B. Munyakazi and K.C. Patidar, A robust fitted numerical method for singularly perturbed turning point problems whose solution exhibits an interior layer, *Quaestiones Mathematicae* (2018), 1-24.

2. C.K. Mbayi, J.B. Munyakazi and K.C. Patidar, A fitted numerical for interior-layer parabolic convection-diffusion problems, submitted for publication.

3. C.K. Mbayi, J.B. Munyakazi and K.C. Patidar, A numerical method for convection-diffusion problems with a power interior layer and variable coefficient diffusion term, ready for submission.

4. C.K. Mbayi, J.B. Munyakazi and K.C. Patidar, Time-dependent for convection-diffusion problems with a power interior layer and variable coefficient diffusion term, ready for submission.

5. C.K. Mbayi, J.B. Munyakazi and K.C. Patidar, A numerical method for interior layer convection-diffusion problems with a variable coefficient diffusion term, in preparation.

6. C.K. Mbayi, J.B. Munyakazi and K.C. Patidar, Time-dependent for interior layer convection-diffusion problems with a variable coefficient diffusion term, in preparation.

Chapter 1

General Introduction

This chapter gives a general overview of the work presented in this thesis. In particular, we provide some background information, review on literature relevant to this work followed by the outline of the rest of the thesis.

1.1 Introduction

There have been several studies on analytical and numerical aspects of singularly perturbed problems. It was noticed that the singular perturbation problems (SPPs) became popular after the Heildeberg conference at the begining of previous century where Prandtl [60] presented his remarkable work. The original ideas in the area on fluid dynamics were subsequently spread over many other areas of science and engineering including geophysical fluid dynamics, ocean and atmospheric circulation, chemical reactions and optimal control [19, 50]. There have been numerous survey articles covering applications of such SPPs in life sciences and engineering.

The SPPs are characterised by a small positive parameter called the singular perturbation parameter, often denoted by ε, known as the diffusion coefficient. This parameter multiplies the highest order derivative term of the differential equation of the underlying problem. When this parameter approaches zero, the solution to the problem displays rapid variation(s) in narrow regions of the domain referred to as boundary or interior layer(s). The smaller the ε, the more difficult the problem becomes, whether one wishes to analyse this qualitatively or quantitatively. The studies over the past three decades have shown that the analytical methods are unable to capture the overall dynamics of the solutions of such problems [25, 26, 28, 69]. This motivated researchers to look for possible numerical approximations for the solutions of these types of problems. To this end, several numerical techniques have been proposed by numerous researchers. The most popular ones are the fitted methods.

As far as the above mentioned fitted methods, in particular, the finite difference analogues, are concerned, there are two categories of these: fitted mesh finite difference methods (FMFDMs) and the fitted operator finite difference methods (FOFDMs). The aim of this thesis is to focus on the former. To this end, we note that FMFDMs have extensively been used for singularly perturbed two-point boundary value problems (TPBVPs) whose solutions possess boundary layers [18, 24, 58]. However, they are not fully explored for problems whose solutions have interior layers. Hence, in this thesis, we intend to do the following: (i) design robust FMFDMs for singularly perturbed TPBVPs whose solutions possess interior layers; (ii) improve accuracy of these approximation methods via methods like Richardson extrapolation; and (iii) extend the ideas presented in (i) and (ii) to TPBVPs with variable diffusion coefficients. The second major contribution is to collectively explore the ideas discussed in (i)-(iii) for parabolic singularly perturbed problems having constant as well as variable diffusion coefficients.

To obtain a better picture of what has transpired over the past few years, we now discuss some approximation methods used for solving singularly perturbed turning point problems. There are various types of numerical methods discussed in the literature which can broadly be classified as Finite Difference Methods, Finite Element Methods, Spline Approximation Methods, and so on [19, 27, 42, 43, 70]. Since we are focusing on a special class of finite difference methods, in what follows, we will only provide details on methods falling under this major category.

1.2 Fitted Numerical methods to solve singularly perturbed problems

The standard finite difference method is not suitable to solve SPPs especially when the perturbation parameter ε is very small unless the mesh is very fine, which unfortunately, increases the round-off error. It is to circumvent this drawback that FMFDMs and FOFDMs were developed. These methods allow one to utilise a reasonable number of mesh points and still achieve reliable accuracy. Moreover, these methods are ε-uniform convergent in the sense of the following definition [42]:

Definition 1.2.1. *Consider a family of mathematical problems parameterized by a singular perturbation parameter ε, where ε, lies in the semi-open interval $0 < \varepsilon \leq 1$. Assume that each problem in the family has a unique solution denoted by U_ε, and that each u_ε is approximated by a sequence of numerical solutions $\{(U_\varepsilon, \tilde{\Omega}^N)\}_{N=1}^{\infty}$, where U_ε is defined on the mesh $\tilde{\Omega}^N$ and N is a discretization parameter. Then, the numerical solutions U_ε is said to converge ε-uniformly to the exact solution u_ε, if there exists a positive integer N_0, and positive numbers C and p, where N_0, C and p are all independent of N and ε, such that, for all $N \geq N_0$,*

$$\sup_{0<\varepsilon\leq 1} ||U_\varepsilon - u_\varepsilon||_{\tilde{\Omega}^N} \leq CN^{-p}.$$

Here p is called the ε-uniform rate of convergence and C is called the ε-uniform error constant.

Below we describe these two categories of methods in more details.

1.2.1 Fitted Operator Finite Difference Methods (FOFDMs)

These methods are divided in two categories, namely: Exponentially Fitted Methods and Non-Standard Fitted Finite Difference Methods. The FOFDMs consist of replacing the standard finite difference operator with a finite difference operator which reflects the singularly perturbed nature of the differential operator [42, 62]. In other words, modifying the difference scheme coefficients in such a way that the scheme becomes more suitable in order to achieve ε-uniform convergent behaviour.

Mickens [41] was the first to introduce the concept denominator function. The principle idea of constructing this FOFDMs is to substitute the denominator functions of the classical derivatives with positive functions derived in order to capture some notable properties of the governing differential equations [8]. Many researchers have been dedicated to construction of FOFDMs for singularly perturbed differential equations (see e.g.,[34, 48, 55, 56]).

1.2.2 Fitted Mesh Finite Difference Methods (FMFDMs)

The FMFDMs involve the use of a mesh that is adapted to the layer regions. They require transforming the continuous problem into a discrete one on a non-uniform mesh which is adapted to the singularly perturbed nature of the problem [42, 62].

Layer-adapted meshes have first been proposed by Bakhvalov in the context of reaction-diffusion problems [3]. In the late 1970s and early 1980s special meshes were investigated by researchers such as [17, 36, 77] in order to achieve uniform convergence. Further investigations and discussions led to the introduction of a special piecewise-uniform meshes by Shishkin [68]. Due to their simple structure, they have attracted attention and are now widely referred to as Shishkin meshes [35]. Parallel to the piecewise uniform meshes, there are also other layer-adapted meshes known as the graded meshes that allow one to achieve uniform convergence We will describe these types of meshes in more detail in Chapter 4.

1.2.3 Richardson extrapolation

This is a post-processing technique which combines two numerical solutions calculated on two embedded meshes to obtain a third and better (in terms of accuracy and rate of convergence) numerical solution [49]. In this work, we will use Richardson extrapolation on fitted mesh finite difference methods constructed on Shishkin meshes as well as on Bakhvalov meshes.

We provide a brief review of works already accomplished on singularly perturbed problems.

1.3 Literature review on Finite difference methods for SPPs

This section presents a survey of the works done on singularly perturbed problems.

Geng *et al.* [19] presented a numerical method for solving singularly perturbed turning point problems exhibiting an interior layer. Through this paper, many methods have been discussed, but the interior layer problem is treated by the method of starching variable and the reproducing kernel method. It is very difficult to extend the application of reproducing kernel method to singularly perturbed differential equations.

Sharma *et al.* [64] surveyed those works until 2011. In that paper, these authors reviewed existing literature on asymptotic and numerical methods techniques for solving singularly perturbed turning point and interior layer problems with the aim of reporting on problems studied, the numerical and asymptotic methods utilised to solve them over the last forty-one years. In this brief survey, we consider some of the works cited in [64] but also some of the works published after 2011.

Existence and asymptotic stability of periodic solutions with an interior layer of reaction-convection-diffusion equation were considered by Nefedov *et al.* [52]. The goal of this paper was to establish the existence of a solution for a problem with an interior layer and to determine the stability of this solution. They constructed sufficiently precise asymptotic lower and upper solutions and applied the results from where they developed an approach to investigate the asymptotic stability of periodic solutions to singularly perturbed reaction-convection-diffusion equations by using the theorem of Krein–Rutman [51].

An asymptotic numerical method to solve singularly perturbed fourth order ordinary differential equations with weak interior layer was presented by Shanti and Ramanujam [63]. The goal for this paper was the construction of a numerical method for a singularly perturbed two-point boundary value problem of convection-diffusion type for fourth ordinary differential equations with the interior layer. The given fourth-order boundary value problem was transformed into a system of two weakly coupled second-order ODEs, one without the parameter and the other with the parameter. In this paper they developed two approaches: firstly an asymptotic numerical method, which dealt with the equation without the parameter, and secondly a Shishkin mesh, which dealt with the parameter.

O'Riordan and Shishkin [54] established numerical methods for a singularly perturbed reaction-diffusion problem with discontinuous source term. In this paper, they used the standard finite difference operator and the piecewise-uniform mesh. They also showed that the mesh is fitted to the boundary and interior layers that occur in the solution of the problem.

Rai and Sharma [66] were concerned with the numerical study of singularly perturbed boundary value problems for delay differential equations with a turning point. They developed the fitted mesh technique to generate a piecewise uniform mesh, condensed in the neighbourhood of the boundary/interior layers. The difference scheme was shown to converge to the continuous solution uniformly with respect to the perturbation parameter.

An initial value method for singularly perturbed system of reaction-diffusion type delay differential equations was considered Subburayan and Ramanujam [75]. The aim was to present a numerical method to solve the singularly perturbed weakly coupled system of reaction-diffusion type second order ordinary differential with negative shift (delay) terms. They developed an asymptotic numerical method named which they referred to as initial value method to solve this problem. In this method, the original problem of solving the second order system of equations was reduced to solving eight first order singularly perturbed differential equations without delay and one system of difference equations. These singularly perturbed problems were solved by the order hybrid finite difference scheme. An error estimate for this method was derived by using the supremum norm and it is almost second order.

Erdogan and Amiraliyey [14] developed a numerical method to solve singularly perturbed delay differential equations. This paper dealt with a singularly perturbed initial value problem for a linear second-order delay differential equation. They presented the completely exponentially fitted scheme on a uniform mesh. The difference scheme was constructed by the method of integral identities with the use of exponentially basis functions and interpolating quadrature rules with weight and remainder terms integral form.

An interior layer in the thermal power-law blown in film model was developed by Bennet and Shepherd [4]. Film blowing is a highly complex industrial process used to manufacture thin sheets of polymer. This paper investigated the structure of typical solutions that arise when the polymer is assumed to be described by a power-law fluid operating under non-isothermal conditions. In this paper, they considered the problem determining the radial bubble of the film as a singular perturbation problem. Asymptotic analysis was used to identify an interior layer in this problem, by applying heuristic techniques along with singular perturbation theory to obtain a closed form approximate expression for the film radius, which was subsequently used to iteratively obtain a numerical solution to the highly nonlinear system determining this radius.

Singularly perturbed parabolic problems with non-smooth data was proposed by O'Riordan and Shishkin [54]. The aim from this paper was to obtain numerical methods for solving a class of singularly perturbed parabolic equations with discontinuous data when the presence of interior layers appears in the solutions. A piecewise-uniform mesh was constructed for a numerical solution of a class of singularly perturbed parabolic differential equations whose solutions exhibit interior layers.

Singular perturbed convection-diffusion problems with boundary and weak interior layers were considered by Farrell *et al.* [15]. They discussed a two point boundary value problem for a singularly perturbed convection-diffusion equation with a singular perturbation parameter. Therefore they constructed a piecewise-uniform mesh for solving this problem. The method was shown to be uniformly convergent with respect to the singular perturbation parameter.

A uniformly convergent method for a singularly perturbed semilinear reaction-diffusion problem with discontinuous data was developed by Boglaev and Pack [7]. The purpose of this paper was to construct a uniform numerical method for solving nonlinear singularly perturbed two point boundary-value problems with discontinuous data of reaction-diffusion type. Long and a piecewise uniform meshes were constructed to solve this problem, which generated uniformly convergent numerical approximations to the solution. They also used a monotone iterative method based on the method of upper and lower solutions for computing the nonlinear difference scheme.

A global maximum norm parameter-uniform numerical method for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient was discussed by Farrell *et al.* [16]. The aim of this paper was to obtain a numerical method for a singularly perturbed convection-diffusion problem, with discontinuous convection-diffusion coefficient and a singular perturbation parameter. Due to the discontinuity an interior layer appears in the solution. They developed a finite difference, on a piecewise uniform mesh, which was fitted to the interior layer, the standard upwind finite difference operator on this mesh.

Some aspects of adaptive grid technology related to boundary and interior layers were discussed by Graham *et al.* [21]. Boundary and interior layers structures in the solution are a familiar feature of certain classes of applications in engineering and science. In this paper, they gave a brief overview of the main adaptive grid strategies in the context of problems with layers. They also explained why numerical approaches must be applied with due care, indicating why some methods fail and others succeed. In the present context, they provided insight into the question of constructing successful adaptive mesh strategies.

A class of nonlinear singular perturbed differential systems with time delays was proposed by Xu and Jin [78]. Singular perturbed differential equations are often used as mathematical models describing processes in biological sciences and physics, such as genetic engineering and the El Nino phenomenon of atmospheric physics. The main purpose of this paper was to deal with the interior layer for a class of nonlinear singularly perturbed differential difference equations and construct its asymptotic expansion formula. They also proved the existence of the smooth interior layer solution and the uniform validity of the asymptotic expansion.

Internal layers of a transient convection-diffusion problem by perturbation methods were considered by Shih and Tung [67]. Understanding pollutant transport mechanisms in water bodies, including surface and subsurface flow, is essential for risk assessment, pollutant clean up, monitoring network design, and various other related activities. The objective of this study was to propose a reasonable approximate solution to a solute transport process using singular perturbation procedures. The accuracy of the approximate solution was examined and its relative performance can be used to compare with other solution techniques. Furthermore the proposed solution allows one to investigate the effect of uncertainties in model parameters, initial and boundary conditions on the pollutant concentration level.

Limitation of adaptive mesh refinement techniques for singularly perturbed problems with interior layer was considered by Shishkin [70]. Numerical analysis of heat and mass transfer with fixed concentrated sources median characterised by small coefficients heat conductivity/diffusion often result in diffraction boundary value problems for singularly perturbed partial differential equations. In this paper they considered an initial value problem on an axis R for a singularly perturbed parabolic reaction-diffusion equation in a composed domain with a moving interface boundary between two sub-domains. In the case of problems with moving transition layers, they developed special numerical methods whose errors depended rather weakly on the parameter and, in particular were independent.

Kadalbajoo-2010 and Patidar [27] introduced a second order numerical method based on cubic splines on a non-uniform mesh to solve such problems.

Liseikin [37] considered the problem: $-(\varepsilon + px)^{\beta}u'' + a(x)u + f(x, \varepsilon) = 0$,
$0 \leq x \leq 1,\ p = 0, 1\ , \beta > 0$. Estimates of the solution and its derivatives were expressed in the form of exponential and polynomial functions both dependent of ε. The same author in [39] considered the equation: $-(\varepsilon + x)^{\beta}u'' - a(x)u + f(x, \varepsilon) = 0,\ 0 \leq x \leq 1\ , \beta > 0$. Bounds on the solution and its derivatives were derived and for $\beta = 1$ a numerical method was presented and its convergence analysed. But in [38], Liseikin treated the boundary value problem: $(\varepsilon + x)u'' + a(x)u' - c(x)u = f(x),\ 0 \leq x \leq 1$. Estimations of the solution and its derivatives were derived and a numerical scheme was developed and its convergence analysed.

Time-dependent parabolic singularly perturbed problems are well studied in the literature. While Riordan *et al.* [72] derived various finite difference schemes using a semi-discrete Petrov-Galerkin finite element method, Clavero *et al.* [11] and Gracia and OŔiordan [20] proposed a upwind finite difference scheme, Kadalbajoo *et al.* [30] designed a B-spline collocation method, Kadalbajoo and Ramesh [29] developed an upwind and midpoint upwind difference methods to discretize the problem in the spatial direction. All these all authors used the implicit Euler method for the time discretization.

The problems considered in the above works are non-turning point problems. Turning point problems are those in which the coefficient of the convection term vanishes inside the spatial-domain by changing signs. This gives rise to the presence of boundary and/or interior layers depending on the number of zeros and the signs of the convection term coefficient. Examples of works where turning points give rise to boundary and/or interior layers include [6, 19, 33, 47, 65].

As far as time-dependent singularly perturbed parabolic problems are concerned, most researchers combine the backward Euler method for the time discretization and an appropriate space discretization that suits the features of the particular problem at hand. For one-dimensional parabolic singularly perturbed reaction-convection-diffusion problems with parameters affecting the diffusion and the convection terms, Clavero et al. [12] constructed a classical upwind finite difference scheme on an piecewise defined mesh of Shishkin type. Dunne and O'Riordan [13] examined singularly perturbed parabolic problems in which the coefficients were discontinuous in the space variable. They designed numerical methods which involved piecewise uniform meshes of Shishkin type which were fitted to both the interior and boundary layers. The methods they proposed differ in the way they discretized derivatives in the differential equation. In [53], O'Riordan and Quinn considered time dependent singularly perturbed convection-diffusion problems in which the convective coefficient contained an interior layer. They constructed a classical upwind finite difference method on a Shishkin mesh.

None of the above works considered the discretization of interior layer problems on Bakhvalov-type and Shishkin-type meshes. These meshes have widely been used in solving singular perturbation problems. Also none of the works has considered problems with a variable coefficient as perturbation parameter whose solution exhibits an interior layer due to the presence of a turning point.

1.4 Outline of the thesis

This thesis deals with the design and analysis of robust fitted finite difference methods to solve various classes of singularly perturbed problems (SPPs) whose solution displays an interior layer due to the presence of a turning point. Moreover, in order to increase the accuracy as well as the order of the convergence of the designed methods, we use the Richardson extrapolation technique on the proposed method. The outline of this thesis is as follows.

In Chapter 2, we construct and analyse a FMFDM on a Shishkin mesh to solve a singularly perturbed turning point problem whose solution has an interior layer with a small positive parameter ε affecting the highest derivative term and we study the performance of Richardson extrapolation on a FMFDM.

A linear singularly perturbed time-dependent convection-diffusion problem is considered in Chapter 3. We use the classical implicit Euler method to discretize the time variable with a constant step-size. Then we construct a FMFDM to solve the resulting system of two-point boundary problems at each time level. This proposed method used an upwind scheme on a piecewise uniform mesh, fine in the (interior) layer and coarse elsewhere. We again consider the effect of Richardson on the fitted mesh finite difference method for this problem.

In Chapter 4, we construct and analyse a FMFDM to solve a singularly perturbed problem with a variable coefficient $(\varepsilon + x^2)$ multiplying the second derivative, whose solution displays an interior layer due the presence of the turning point. This method is applied on both Bakhvalov and Shishkin-type meshes. By post-processing our results using Richardson extrapolation, our overall method is almost second order accurate uniformly with respect to ε. We also present results obtained via Bakhvalov and Shishkin-type meshes. After a thorough comparison, we notice that the results obtained by Shishkin-type meshes before and after extrapolation are little inferior to those obtained by Bakhvalov-type meshes.

In Chapter 5, we consider a class of time-dependent singularly perturbed convection-diffusion problems with a variable coefficient $(\varepsilon + x^2)$ affecting the second derivative. We discretize the time variable with a constant step-size by means of the classical implicit Euler method. This process results in a linear system of equation in space at each time level which we solve using a FMFDM. Richardson extrapolation is applied on a FMFDM.

A family of two-point boundary value singularly perturbed convection-diffusion problems in which the diffusion term is expressed as $(\varepsilon + x)$ is the subject Chapter. We construct and analyse a FMFDM to solve a singularly perturbed turning point problem whose solution has an interior layer. This method is applied on an appropriate piecewise uniform of Shishkin type-mesh. We study the performance of Richardson extrapolation on this method.

Chapter 7 deals with a class of time-dependent for convection-diffusion problems with coefficient $(\varepsilon + x)$ in the highest derivative. Also we study this problem whose solution displays an interior layer due to the presence of a turning point. The proposed numerical scheme comprises the classical Euler method to discretize the time variable. Then we construct and analyse a FMFDM to solve the system of equations obtained from the time dicretization. We apply Richardson extrapolation via FMFDM.

Finally, in Chapter 8, we provide some concluding remarks and direction for further research.

Chapter 2

A fitted numerical method for interior layer turning point problems

In this chapter, we consider singularly perturbed convection-diffusion-reaction problems with a turning point whose solution exhibits an interior layer. After proving bounds on the solution to these problems and their derivatives, we construct a fitted mesh finite difference method (FMFDM) applied on a Shishkin type mesh to solve this problem. In order to improve the accuracy of the proposed FMFDM, we apply Richardson extrapolation.

2.1 Introduction

The research field of singular perturbation problems (SPPs) was born after the Heildeberg conference on Fluid Dynamics where Prandtl [60] presented his remarkable work. The original ideas in the area on fluid dynamics were subsequently spread over many other areas of science and engineering.

Due to the presence of a small parameter ε in the coefficient of the highest derivative of the model equation of singularly perturbed problems, solutions behave abruptly in small parts of the domain called layer regions. These layers may be located at the boundary of the domain or in its interior. In one dimension, a typical SPP consists of determining the solution u to the equation

$$Lu :\equiv \varepsilon u'' + a(x)u' - b(x)u = f(x), x \in \Omega = [-1, 1], \tag{2.1.1}$$

$$u(-1) = \alpha \ \text{ and } \ u(1) = \beta, \tag{2.1.2}$$

where $0 < \varepsilon \leq 1$ and α and β are some constants. If the functions $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth and $a(x)$ does not change sign throughout the domain, then the solution to (2.1.1)-(2.1.2) has a boundary layer near -1 or 1. But, if $a(x)$ happens to change sign, then an interior layer may occur. Interior layers are also present in the solution of the problem above if the coefficient functions are not smooth or if the data function $f(x)$ is discontinuous.

The presence of layers renders classical numerical methods unfit to provide acceptable approximations to the solution of SPPs. Over many decades now, researchers have developed reliable numerical schemes in the case of smooth coefficient functions (see e.g., [25, 26, 28, 45, 46, 56, 57, 69] and the references therein).

All the works listed above are on non-turning point problems. Singularly perturbed turning point problems received systematic attention from late 1960s [64]. These are problems in which the coefficient of the convection term vanishes inside the domain by changing sign. This gives rise to the presence of boundary and/or interior layers depending on the number of zeros and the signs of the coefficient of the convection term.

While the works accomplished for the case of boundary layers are relatively abundant in the literature (see e.g., [18, 24, 58]), very few researchers have studied internal layer problems.

None of the above works considered the discretization of interior layer problems on Shishkin meshes. These meshes have widely been used in solving singular perturbation problems. For more on these meshes, interested readers are referred to [42].

This chapter focusses on studying singularly perturbed problems whose solution exhibits an interior layer due to the presence of a turning point. Thus, we consider problem (2.1.1)-(2.1.2), where α and β are given real constants and $0 < \varepsilon \ll 1$. In addition, the coefficients $a(x)$, $b(x)$ and $f(x)$ of (2.1.1) are assumed to be sufficiently smooth so as to ensure the existence of a unique solution. The point in the domain where $a(x) = 0$ is known as a turning point.

The following assumptions guarantee that the solution to problem (2.1.1)-(2.1.2) exhibits an interior layer at $x = 0$.

$$\left.\begin{array}{ll} a(0) = 0 & a'(0) > 0, \\ b(x) \geq b_0 > 0, & x \in [-1, 1] \\ |a'(x)| \geq |a'(0)|/2, & x \in [-1, 1]. \end{array}\right\} \tag{2.1.3}$$

Further, we assume that $-2\eta \leq a(x) \leq 2\eta$, where η is a positive constant and independent of ε. Note that interior layers may also occur in the case of singularly perturbed convection-diffusion-reaction problems where coefficients are discontinuous or non-smooth (see e.g., [16]). Such problems are discussed elsewhere.

The rest of this chapter is organised as follows. We establish bounds of the solution and its derivatives in the next section. In Section 3 we develop our novel numerical method by first designing a fitted mesh of Shishkin type. This mesh is fine around the turning point (where the interior layer is situated) and coarse away from it. We adopt the upwinding schemes to discretize equation (2.1.1). Section 4 is devoted to error analysis. We prove that the method is almost first order, uniformly convergent with respect to the perturbation parameter ε. In Section 5 we present Richardson extrapolation's method via fitted mesh finite difference method (FMFDM). To see how the proposed method works in practice and to confirm our theoretical results, numerical experiments are presented in Section 6 for two examples. We conclude the chapter in Section 7.

Throughout the chapter, C denotes a generic constant which is independent of the perturbation parameter ε and of the mesh parameter which will be introduced in Section 3.

2.2 A priori estimates of the solution and its derivatives

In this section, we present some qualitative results on the solution to problem (2.1.1)-(2.1.2), including its bounds and that on its derivatives.

Under the assumptions (2.1.3), the operator L admits the following continuous minimum principle.

Lemma 2.2.1. (Minimum principle) Let ξ be a smooth function satisfying $\xi(1) \geq 0$, $\xi(-1) \geq 0$ and $L\xi(x) \leq 0$, $\forall x \in \Omega = (-1, 1)$.Then $\xi(x) \geq 0$, $\forall x \in [-1, 1]$.

Proof. Let $x^* \in [-1, 1]$ such that $\xi(x^*) = \min\limits_{-1 \leq x \leq 1} \xi(x)$ and assume $\xi(x^*) < 0$. Then, obviously $x^* \notin \{-1, 1\}$, $\xi'(x^*) = 0$ and $\xi''(x^*) \geq 0$ and we have

$$L\xi(x^*) = \varepsilon\xi''(x^*) + a(x)\xi'(x^*) - b(x)\xi(x^*) > 0, \forall x \in [-1, 1],$$

which is a contradiction. It follows that $\xi(x^*) \geq 0$ and thus $\xi(x) \geq 0$, $\forall x \in [-1, 1]$.

The minimum principle implies the uniqueness and existence of the solution (as for linear problems, the existence of the solution is implied by its uniqueness). We use this principle to prove the following result which states that the solution depends continuously on the data.

Lemma 2.2.2. [5] If $u(x)$ is the solution of (2.1.1)-(2.1.2), then we have

$$||u(x)|| \leqslant [\max\{||\alpha||_\infty, ||\beta||_\infty\}] + \frac{1}{b_0}||f||_\infty, \forall x \in [-1, 1],$$

Proof. We consider the comparison functions

$$\Gamma^{\pm}(x) = C \pm u(x) \ \forall x \in [-1, 1],$$

where $C = \max\{||\alpha||_\infty, ||\beta||_\infty\} + \frac{1}{b_0}\,||f||_\infty$.
Applying the minimum principle to the comparison functions, we have

$$\begin{aligned}
L\Gamma^{\pm}(x) &= \varepsilon\Gamma^{\pm}(x)]'' + a(x)[\Gamma^{\pm}(x)]' - b(x)[\Gamma^{\pm}(x)] \\
&= \pm\varepsilon u''(x) \pm a(x)u'(x) - b(x)[\pm u(x)] - b(x)C \\
&= \pm f(x) - b(x)\left[\max\{||\alpha||_\infty, ||\beta||_\infty\} + \frac{1}{b_0}\,||f||_\infty\right] \\
&= \pm f(x) - \frac{b(x)}{b_0}\,||f||_\infty - b(x)\max\{||\alpha||_\infty, ||\beta||_\infty\} \\
&= -[||f||_\infty \mp f(x)] - b(x)\max\{||\alpha||_\infty, ||\beta||_\infty\} \leqslant 0,
\end{aligned}$$

implying that

$$\Gamma^{\pm}(x) \geqslant 0\ ,\ \forall x \in [-1, 1],$$

then we have

$$C \pm u(x) \geq 0.$$

It follows immediately that $||u(x)||_\infty \leqslant C$, which completes the proof.

Hereinafter we denote the sub-domains as follows: $\Omega_l = [-1, -\sigma)$, $\Omega_c = [-\sigma, +\sigma] = [-\sigma, 0] \cup (0, +\sigma]$ and $\Omega_r = (\sigma, 1]$, where $0 \leq \sigma \leq 1/2$; the left, central and right part of the domain, respectively. Below we provide the appropriate bounds in the following lemmas.

Lemma 2.2.3. If $u(x)$ is the solution of (2.1.1)-(2.1.2) and a, b and f $\in C^k(\bar{\Omega})$, then there exist positive constants η and C such that

$$|u^{(j)}(x)| \leq C, \forall x \in \Omega_l \text{ or } \Omega_r.$$

Proof. The proof is by induction. Following the ideas of Lemma 8.1 of [5]. A bound on the solution u of the equation (2.1.1)-(2.1.2) is obtained by using the minimum principle as follows. Consider the function:

$$\psi^{\pm}(x) = Cx \pm u(x),$$

where C is a constant chosen sufficiently large such that the following inequality is fulfilled

$$\psi^{\pm}(-1) \geq 0,\ \psi^{\pm}(-\sigma) \geq 0$$

and

$$L\psi^{\pm}(x) = C(x) - Cxb(x) \pm f(x) \geq C\ a(x) - Cxb_0 \pm f(x) \leq 0.$$

Then the minimum principle for L gives $\psi^{\pm} \geq 0$, and so

$$|u^{(j)}(x)| \leq C, \forall x \in \Omega_l \text{ or } \Omega_r.$$

Lemma 2.2.4. Let $u(x)$ is the solution of (2.1.1)-(2.1.2). Then, for $0 \leq j \leq 3$

$$|u^{(j)}(x)| \leq C\left[1 + \varepsilon^{-j} \exp\left(\frac{2\eta x}{\varepsilon}\right)\right], x \in [-1, 0]$$

and

$$|u^{(j)}(x)| \leq C\left[1 + \varepsilon^{-j} \exp\left(\frac{-2\eta x}{\varepsilon}\right)\right], x \in [0, 1].$$

Proof. The proof is by induction by using ideas of (Lemma 8.1 of [5]). To obtain the required estimates of the derivative of u. The first step is to find is the differential equation satisfied by these derivatives by differentiating (2.1.1) j times. This gives

$$Lu^{(j)} = f_j,$$

where

$$f_0 = f \text{ and } 1 \leq j \leq 3,$$

$$f_j = f^{(j)} - \sum_{s=0}^{j-1} \binom{j}{s} a^{(j-s)} u^{(s+1)} + \sum_{s=0}^{j-1} \binom{j}{s} b^{(j-s)} u^{(s)}.$$

Thus, the inhomogeneous term f_j of the equation satisfied by u depends on the j^{th} and lower order derivatives of u and of the coefficient a, and on the j^{th} order derivatives of f. This observation suggests the following argument, which suffices to prove the theorem. We assume that, for $0 \leq k \leq j$, the following estimates hold:

$$|u^{(j)}(x)| \leq C\left[1 + \varepsilon^{-j} \exp\left(\frac{2\eta x}{\varepsilon}\right)\right], \text{for all } x\ [-1, 0],$$

the original equation (2.1.1) $Lu = f(x)$, which gives $Lu^{(j)} = f_j$. From above assumption it is clear that

$$Lu^{(j)} = f_j,$$

where

$$|u^{(j)}(x)| \leq C\left[1 + \varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right)\right]$$

and

$$|f^{(j)}(x)| \leq C\left[1 + \varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right)\right].$$

Let us first determine the following:

$$|u^{(j)}(-1)| \leq C\left[1 + \varepsilon^{-j}\exp\left(\frac{-2\eta}{\varepsilon}\right)\right] \leq C\left[1 + \varepsilon^{-(j-1)}\right]$$

and

$$|u^{(j)}(0)| \leq C\left(1 + \varepsilon^{-j}\right).$$

Note that $\varepsilon^{-1}\exp\left(\frac{-2\eta}{\varepsilon}\right) \leq C$, we obtain

$$|u^{(j)}(-1)| \leq C\varepsilon^{-(j-1)}$$

and

$$|u^{(j)}(0)| \leq C\varepsilon^{-j}.$$

Define the new function

$$\theta_j(x) = \varepsilon^{-1}\int_x^0 f_j \exp\left[\frac{-(A(x) - A(t))}{\varepsilon}\right] dt \tag{2.2.1}$$

where

$$A(x) = \int_{-1}^x a(s)\, ds \tag{2.2.2}$$

and

$$u_p^{(j)}(x) = \int_{-1}^x \theta_j(t)\, dt, \tag{2.2.3}$$

which is the solution of the equation

$$Lu^{(j)} = f_j.$$

Its general solution is written in the form

$$u^{(j)} = u_p^{(j)} + u_h^{(j)},$$

where the homogeneous solution $u_h^{(j)}$ satisfies

$$Lu^{(j)} = 0,\ u_h^{(j)}(-1) = u^{(j)}(-1) - u_p^{(j)}(-1), u_h^{(j)}(0) = u^{(j)}(0).$$

Introduce the function

$$\varphi(x) = \frac{\int_{-1}^{x} \exp\left[\frac{-A(t)}{\varepsilon}\right]\ dt}{\int_{-1}^{0} \exp\left[\frac{-A(t)}{\varepsilon}\right]\ dt}.$$

Firstly we need to obtain the value of $\varphi'(x)$ by using the lower bound on the coefficient $a(x)$. In the present chapter, the value of $\varphi'(x)$ will be obtained in $[-1, 0]$. Therefore, we consider

$$-2\eta \leq a(x). \tag{2.2.4}$$

Integrating both sides of (2.2.4) from -1 to x and using the expression of (2.2.2), we obtain

$$-2\eta(x+1) \leqslant A(x). \tag{2.2.5}$$

Multiplying both sides of (2.2.5) by $-1/\varepsilon$, we obtain

$$\frac{-A(x)}{\varepsilon} \leq \frac{2\eta(x+1)}{\varepsilon}. \tag{2.2.6}$$

The above inequality can also be written as

$$\exp\left[\frac{-A(x)}{\varepsilon}\right] \leq \exp\left[\frac{2\eta(x+1)}{\varepsilon}\right]. \tag{2.2.7}$$

Integrating both sides of (2.2.7) from -1 to 0,

$$\int_{-1}^{0} \exp\left[\frac{-A(t)}{\varepsilon}\right]\ dt \leq \int_{-1}^{0} \exp\left[\frac{2\eta(x+1)}{\varepsilon}\right]\ dt,$$

we obtain

$$\int_{-1}^{0} \exp\left[\frac{-A(t)}{\varepsilon}\right]\ dt \leq \frac{\varepsilon}{2\eta} \exp\left[\frac{4\eta}{\varepsilon}\right].$$

Arranging the above inequality, therefore we obtain

$$-\int_{-1}^{0} \exp\left[\frac{-A(t)}{\varepsilon}\right] dt \geq -\frac{\varepsilon}{2\eta}\exp\left[\frac{4\eta}{\varepsilon}\right].$$

Inverting both sides of the last above inequality, we obtain

$$\frac{1}{-\int_{-1}^{0}\exp\left[\frac{-A(t)}{\varepsilon}\right]\, dt} \leq \frac{1}{\frac{-\varepsilon}{2\eta}\exp\left[\frac{4\eta}{\varepsilon}\right]}.$$

Multiplying both sides of the last above inequalities by $\exp\left[-\frac{A(x)}{\varepsilon}\right]$, then gives

$$\frac{\exp\left[-\frac{A(x)}{\varepsilon}\right]}{-\int_{-1}^{0}\exp\left[\frac{-A(t)}{\varepsilon}\right]\, dt} \leq \frac{\exp\left[-\frac{A(x)}{\varepsilon}\right]}{\frac{-\varepsilon}{2\eta}\exp\left[\frac{4\eta}{\varepsilon}\right]}.$$

Taking into account (2.2.7), the above inequality leads to

$$\frac{-\exp\left[-\frac{A(x)}{\varepsilon}\right]}{\int_{-1}^{0}\exp\left[\frac{-A(t)}{\varepsilon}\right]\, dt} \leq \frac{\varepsilon^{-1}\exp\left[\frac{2\eta(x+1)}{\varepsilon}\right]}{\frac{-1}{2\eta}\exp\left(\frac{4\eta}{\varepsilon}\right)} \leq \frac{\varepsilon^{-1}\exp\left[\frac{2\eta x}{\varepsilon}\right]}{\frac{-1}{2\eta}\exp\left(\frac{2\eta}{\varepsilon}\right)}.$$

Since

$$\varphi'(x) = \frac{-\exp\left[-\frac{A(x)}{\varepsilon}\right]}{\int_{-1}^{0}\exp\left[\frac{-A(t)}{\varepsilon}\right]\, dt},$$

then we obtain

$$\varphi'(x) \leq \frac{\varepsilon^{-1}\exp\left[\frac{2\eta x}{\varepsilon}\right]}{\frac{-1}{2\eta}\exp\left(\frac{2\eta}{\varepsilon}\right)}.$$

Using the upper and lower bounds of $a(x)$, we obtain

$$|\varphi'(x)| \leq C\varepsilon^{-1}\exp\left[\frac{2\eta x}{\varepsilon}\right], \forall x \in [-1, 0]. \tag{2.2.8}$$

It is clear that $L\varphi = 0, \varphi(-1) = 1, \varphi(0)$ and $0 \leq \varphi(x) \leq 1$. Then $u_h^{(j)}$ is given by

$$u_h^{(j)}(x) = (u^{(j)}(-1) - u_p^{(j)}(-1))\varphi(x) + u^{(j)}(0)(0 - \varphi(x)).$$

The above leads to the following expression for $u^{(j+1)}$ as follows:

$$u^{(j+1)}(x) = (u_p^{(j+1)}(-1) + u_h^{(j+1)}(-1)) = \theta_j(-1) + (u^{(j)}(-1) - u_p^{(j)}(-1) - u_\varepsilon^{(j)}(0)\varphi'(x),$$

where

$$\varphi'(x) \leq C\varepsilon^{-1}\exp\left(\frac{2\eta x}{\varepsilon}\right), \forall x \in [-1, 0].$$

The above expressions lead to determine $\theta_j(x)$ from (2.2.1)

$$\theta_j(x) = \varepsilon^{-1} \int_x^0 C \left[1 + \varepsilon^{-j} \exp\left(\frac{2\eta t}{\varepsilon}\right)\right] \exp\left[\frac{-2\eta(t-x)}{\varepsilon}\right] \, dt.$$

Arranging the above integral, we obtain

$$|\theta_j(x)| \leq \varepsilon^{-1} C \exp\left(\frac{2\eta x}{\varepsilon}\right) \int_x^0 \left[\exp\left(\frac{-2\eta t}{\varepsilon}\right) + \varepsilon^{-j}\right] dt,$$

evaluating the integral exactly, and estimating the terms in the resulting expression, we obtain

$$|\theta_j(x)| \leq C \left[1 - \exp\left(\frac{2\eta x}{\varepsilon}\right) - 2\eta x \varepsilon^{-(j+1)} \exp\left(\frac{2\eta x}{\varepsilon}\right)\right].$$

Estimating the terms in the resulting above expression, we obtain

$$|\theta_j(x)| \leq C \left[1 + \varepsilon^{-(j+1)} \exp\left(\frac{2\eta x}{\varepsilon}\right)\right].$$

Since

$$u_p^{(j)}(-1) = -\int_{-1}^0 \theta_j(t) \, dt = -\int_{-1}^0 C \left[1 + \varepsilon^{-(j+1)} \exp\left(\frac{2\eta t}{\varepsilon}\right)\right] \, dt.$$

Evaluating the integral exactly, and estimating the terms in the resulting expression, we obtain $u_p^{(j)}(-1) \leq C\varepsilon^{-j}$. But

$$|u^{(j+1)}(x)| \leq |\theta_j(x)| + (|(u^{(j)}(-1)| + |u_p^{(j)}(-1)| + |u^{(j)}(0)|)\varphi'(x).$$

Substituting and estimating in the resulting expression, we have

$$|u^{(j+1)}(x)| \leq C \left[1 + \varepsilon^{-(j+1)} \exp\left(\frac{2\eta x}{\varepsilon}\right)\right] + \left(C\varepsilon^{-(j-1)} + C\varepsilon^{-j} + C\varepsilon^{-j}\right) C\varepsilon^{-1} \exp\left(\frac{2\eta x}{\varepsilon}\right),$$

$$|u^{(j+1)}(x)| \leq C \left[1 + \varepsilon^{-(j+1)} \exp\left(\frac{2\eta x}{\varepsilon}\right)\right] + C \left(\varepsilon^{-1} + \varepsilon^{-j-1}\right) \exp\left(\frac{2\eta x}{\varepsilon}\right).$$

Estimating the terms in the resulting above expression, we obtain

$$|u^{(j+1)}(x)| \leq C \left[1 + \varepsilon^{-(j+1)} \exp\left(\frac{2\eta x}{\varepsilon}\right)\right],$$

which completes the induction required.

Note that the solution of the SPTPP (2.1.1)-(2.1.2) can be decomposed into two parts namely the smooth component v and the singular component w. Following the ideas of [47], we establish the following lemma which gives bounds on the solution to (2.1.1)-(2.1.2) and its derivatives.

Lemma 2.2.5. We decompose u of the SPTPP (2.1.1)-(2.1.2) into smooth and singular components as $u = v + w$, where, for all j, $0 \le j \le k$, and all $x \in [-1, 1]$, the smooth component v satisfies

$$|v^{(j)}(x)| \le C\left[1 + \varepsilon^{-(j-2)} \exp\left(\frac{2\eta x}{\varepsilon}\right)\right], x \in [-1, 0],$$

$$|v^{(j)}(x)| \le C\left[1 + \varepsilon^{-(j-2)} \exp\left(\frac{-2\eta x}{\varepsilon}\right)\right], x \in [0, 1],$$

and the singular components w satisfies

$$|w^{(j)}(x)| \le C\varepsilon^{-j} \exp\left(\frac{2\eta x}{\varepsilon}\right), x \in [-1, 0],$$

$$|w^{(j)}(x)| \le C\varepsilon^{-j} \exp\left(\frac{-2\eta x}{\varepsilon}\right), x \in [0, 1],$$

for some constants η and C independent of ε.

Proof. The SPTPP (2.1.1)-(2.1.2) can be regarded as concatenation of problems

$$\varepsilon u'' + a(x)u' - b(x)u = f(x), x \in [-1, 0], u(-1) = \alpha, u(0) = A \tag{2.2.9}$$

and

$$\varepsilon u'' + a(x)u' - b(x)u = f(x), x \in [0, 1], u(0) = A, u(1) = \beta \tag{2.2.10}$$

where A is still to be determined.

In the present chapter, we consider the problem (2.2.9), which can be converted to the convection problem by using the same steps of [42]. We can write the problem (2.2.9) as

$$-\varepsilon u'' - a(x)u' = g(x), u(-1) = \alpha, u(0) = A \tag{2.2.11}$$

where

$$g(x) = -f(x) - b(x)u.$$

The solution u of the problem (2.2.11) can be written under the following form:

$$u = v_0 + \varepsilon y_1 + w_0. \tag{2.2.12}$$

Here, v_0 satisfies the reduced problem $-a(x)v_0' = g(x)$, $v_0 = A$ and y_1 satisfies $Ly_1 = v_0''$, $y_1(-1) = -w_0(-1)/\varepsilon$, $y_1(0) = 0$. Moreover, note that w_0 is the solution to the homogeneous problem $Lw_0 = 0$, where $w_0 = w_0(0)\exp\left(\frac{2\eta}{\varepsilon}\right)$, and $w_0(0) = A - v_0$. It is clear that $|w_0(-1)|$, $|w_0(0)|$, $|y_1(-1)|$ and $|v_0''|$ are all bounded by a constant independent of ε. It follows that y_1 is the solution of a problem similar to (2.1.1)-(2.1.2), thus for $j = 1, 2, 3, 4, \cdots, k+1$,

$$|y_1^{(j)}(x)| \leq C\left[1 + \varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right)\right].$$

Following the approach of [47], we are going to obtain the bounds for the singular component w_0 and its derivatives on [-1,0]. First define the comparison functions

$$\Psi^{\pm}(x) = |w_0(0)|\exp\left(\frac{2\eta x}{\varepsilon}\right) \pm w_0(x).$$

Applying the minimum principle to these functions, we see that $\Psi^{\pm}(x) \geqslant 0$ and consequently,

$$|w_0(x)| \leq C\exp\left(\frac{2\eta x}{\varepsilon}\right), \quad x \in [-1, 0].$$

Therefore the singular component $w_0(x)$ of the solution can be written as follows:

$$w_0(x) = w_0(-1)\varphi(x) - w_0(0)\varphi(x),$$

where

$$\varphi(x) = \frac{\int_x^0 \exp\left(\frac{-A(t)}{\varepsilon}\right)\, dt}{\int_{-1}^0 \exp\left(\frac{-A(t)}{\varepsilon}\right)\, dt}$$

and $A(x) = -\int_x^0 a(s)\, ds$. Now

$$w_0'(x) = (w_0(-1) - w_0(0))\varphi'(x),$$

where $\varphi'(x)$ is given in (2.2.8). Therefore, substituting $\varphi'(x)$ into the equation for w_0', we obtain

$$|w_0'| \leq C\varepsilon^{-1}\exp\left(\frac{2\eta x}{\varepsilon}\right).$$

Since $Lw_0 = 0$, the j^{th} derivatives of w_0 can be estimated immediately from the estimates w_0 and w_0', $\forall x \in [-1, 0]$,

$$|w_0^{(j)}(x)| \leq C\varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right),$$

Since $u^{(j)} = v_0^{(j)} + \varepsilon y_1^{(j)} + w_0^{(j)}$, we have

$$|(v_0^{(j)} + \varepsilon y_1^{(j)})| \leq C\left[1 + \varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right)\right]$$

and

$$|w_0^{(j)}(x)| \leq C\varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right),$$

for $0 \leq j \leq k$ and all $x \in [-1, 0]$. In particular, this shows that the smooth component $v_0 + \varepsilon y_1$ and its derivatives are bounded for all values of ε. However, the component y_1 can also be decomposed in the same manner as was u, leading immediately to $y_1 = v_1 + \varepsilon v_2 + w_1$ where for $0 \leq j \leq k$ and all $x \in [-1, 0]$, we have

$$|v_1^{(j)}| \leq C,$$

$$|(v_2^{(j)}| \leq C\left[1 + \varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right)\right],$$

$$|w_1^{(j)}| \leq C\varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right).$$

Combining these two decompositions, we have $u = v + w$, where $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$ and $w = w_0 + \varepsilon w_1$. Since $u^{(j)} = v^{(j)} + w^{(j)}$, and the above estimates hold for $0 \leq j \leq k$ and all $x \in [-1, 0]$, we have

$$|(v^{(j)}(x)| \leq C\left[1 + \varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right)\right] \text{ and } |w^{(j)}(x)| \leq C\varepsilon^{-j}\exp\left(\frac{2\eta x}{\varepsilon}\right).$$

Following the same ideas as above, the solution u of problem (2.2.10) can also be written in the same form as problem (2.2.9), which $u = v + w$ for $0 \leqslant j \leqslant k$ and all $x \in [0, 1]$ where

$$|(v^{(j)}(x)| \leq C\left[1 + \varepsilon^{-(j-2)}\exp\left(\frac{-2\eta x}{\varepsilon}\right)\right] \text{ and } |w^{(j)}(x)| \leqslant C\varepsilon^{-j}\exp\left(\frac{-2\eta x}{\varepsilon}\right).$$

2.3 Construction of the FMFDM

In this section, we develop a difference scheme to solve this problem. We discretize the problem on a special nonuniform mesh. Since the solution has large gradients in a narrow region near $x = 0$, the mesh in this region will be fine and coarse everywhere else. Let

n be a positive integer such that $n = 2^m$ with $m \geq 3$. With this in mind, the transition parameter τ is chosen to be

$$\tau = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\eta}\ln\left(\frac{n}{4}\right)\right\}, \tag{2.3.1}$$

where τ is a positive constant. The sub-intervals $[-1,-\tau]$, $[-\tau,\tau]$ and $[\tau,1]$ of the domain $[-1,1]$ are subdivided uniformly to contain $n/4$, $n/2$ and $n/4$ mesh elements respectively. Note that $x_0 = -1$, $x_{n/2} = 0$, and $x_n = 1$. The mesh spacing $h_j = x_j - x_{j-1}$ is given by

$$h_j = \begin{cases} 4(1-\tau)/n \text{ if } j = 1,2,\cdots,n/4, 3n/4+1,\cdots,n-1,n, \\ 4\tau/n \text{ if } j = n/4+1, n/4+2\cdots 3n/4. \end{cases} \tag{2.3.2}$$

We denote this mesh by Ω_n^τ.

For the rest of the chapter, for any function $S(x)$, we adopt the notation $S(x_j) = S_j$. We discretize the problem (2.1.1)-(2.1.2) on Ω_n^τ in the following manner:

$$L^n U_j := \begin{cases} \varepsilon\tilde{D}U_j + a_j D^- U_j - b_j U_j = f_j & \text{if } a_j < 0, \\ \varepsilon\tilde{D}U_j + a_j D^+ U_j - b_j U_j = f_j & \text{if } a_j \geq 0, \end{cases} \tag{2.3.3}$$

$$U(-1) = \alpha, \ \ U(1) = \beta, \tag{2.3.4}$$

where

$$D^+U_j = \frac{U_{j+1} - U_j}{h_{j+1}}, \ \ D^-U_j = \frac{U_j - U_{j-1}}{h_j} \ \text{ and } \ \tilde{D}U_j = \frac{2}{h_j + h_{j+1}}(D^+U_j - D^-U_j).$$

(2.3.3) can be written in the form:

$$L^n U_j := r^- U_{j-1} + r^c U_j + r^+ U_{j+1} = f_j, \ j = 1,2,3\cdots,n-1, \tag{2.3.5}$$

where, for $j = 1,2,3\cdots,n/2-1$, we have

$$r_j^- = \frac{2\varepsilon}{h_j(h_j+h_{j+1})} - \frac{a_j}{h_j}, \ r_j^c = \frac{a_j}{h_j} - \frac{2\varepsilon}{h_j h_{j+1}} - b_j, \ r_j^+ = \frac{2\varepsilon}{h_{j+1}(h_j+h_{j+1})} \tag{2.3.6}$$

and for $j = n/2, n/2+1,\cdots,n-1$, we have

$$r_j^- = \frac{2\varepsilon}{h_j(h_j+h_{j+1})}, \ r_j^c = -\frac{a_j}{h_{j+1}} - \frac{2\varepsilon}{h_j h_{j+1}} - b_j, \ r_j^+ = \frac{2\varepsilon}{h_{j+1}(h_j+h_{j+1})} + \frac{a_j}{h_{j+1}}. \tag{2.3.7}$$

The discrete operator L^n satisfies the following minimum principle:

Lemma 2.3.1. For any mesh function ξ_j such that $L^n \xi_j \le 0$, $\forall j = 1, 2, \ldots, n-1$, $\xi_0 \ge 0$ and $\xi_n \ge 0$, we have $\xi_j \ge 0$, $\forall j = 0, 1, \cdots, n$.

Proof. Let k be such that $\xi_k = \min_{0 \le j \le n} \xi_j$ and suppose that $\xi_k < 0$. Obviously, $k \ne 0$ and $k \ne n$. Also $\xi_{k+1} - \xi_k \ge 0$, and $\xi_k - \xi_{k-1} \le 0$. We have

$$L^n \xi_k = \begin{cases} \varepsilon \tilde{D} \xi_k + a_k D^- \xi_k - b_k \xi_k > 0, & \text{for } 1 \le k \le \frac{n}{2} - 1, \\ \varepsilon \tilde{D} \xi_k + a_k D^+ \xi_k - b_k \xi_k > 0, & \text{for } \frac{n}{2} \le k \le n-1. \end{cases}$$

Thus $L^n \xi_k > 0, \ 1 \le k \le n-1$, which is a contradiction. It follows that $\xi_k \ge 0$ and therefore $\xi_j \ge 0, \ 0 \le j \le n$.

Lemma 2.3.2. If Z_i is any mesh function such that $Z_0 = Z_n = 0$, then

$$|Z_i| \le \frac{1}{\eta^*} \max_{1 \le j \le n-1} |L^n Z_j| \ \forall 0 \le i \le n,$$

where

$$\eta^* = \begin{cases} -2\eta & \text{if } 0 \le i \le n/2 - 1, \\ 2\eta & \text{if } n/2 \le i \le n. \end{cases}$$

Proof. Let us define

$$|M_i^{\pm}| = \frac{1}{\eta^*} \max_{1 \le j \le n-1} |L^n Z_j|.$$

Introduce the two mesh functions $Y_i^{\pm}$ defined by

$$Y_i^{\pm} = M^{\pm} x_i \pm Z_i.$$

Clearly $Y_0^{\pm} \ge 0$, $Y_n^{\pm} \ge 0$ and $L^n Y^{\pm} = M^{\pm}(a_i - b_i x_i) \pm L^n Z_i \le 0$,
since $\eta^* \ge -2\eta > 0$, $\forall x_i < 0, \ 1 \le i \le n/2,$ and $\eta^* \le -2\eta < 0$,$\forall x_i > 0$, $n/2 + 1 \le i \le n-1$.
The discrete minimum principle 2.2.1 then implies that $Y_i \ge 0$, for $0 \le i \le n$.
With the above continuous and discrete results, we are in a position to provide the ε-uniform convergence result in the following section.

2.4 Convergence analysis

In this section, the convergence of the scheme analyzed on a Shishikin mesh, is proved in the previous section.

Lemma 2.4.1. . Let u be the solution of the continuous problem (2.1.1)-(2.1.2), and U the solution of the corresponding discrete problem (2.3.3) and (2.3.4). Then, for sufficiently large n, we have the following estimate:

$$\sup_{0<\varepsilon\leq 1}\max_{0\leq j\leq n}|u_j - U_j| \leq Cn^{-1}\left[\ln\left(\frac{n}{4}\right)\right]^2. \tag{2.4.1}$$

Proof. We prove the lemma on the interval [-1,0]. The proof on [0,1] follows similar steps.

The solution U of the discrete problem (2.3.3) and (2.3.4) is decomposed in regular part V and singular part W. Thus

$$U = V + W,$$

where V is the solution of the inhomogeneous problem

$$L^n V = f, V(-1) = v(-1), V(-1) = v(-1),$$

and W is the solution of the homogenous problem given by

$$L^n W = 0, W(-1) = w(-1), W(-1) = w(-1).$$

The error can be written in the form:

$$U - u = (V - v) + (W - w), \tag{2.4.2}$$

so the error in the regular and singular components of the solution can be estimated separately. The estimate of the smooth component is obtained using the following stability and consistency argument. From the differential and difference equations, we have

$$L^n(V - v) = f - L^n v.$$

Therefore the local truncation is given by

$$L^n(V-v) = \varepsilon\left(\frac{d^2}{dx^2} - \tilde{D}\right)v + a\left(\frac{d}{dx} - D^-\right)v.$$

Then, by local truncation error estimates (Lemma 4.1 [42]), we obtain

$$|L^n(V_j - v_j)| \leq \frac{\varepsilon}{3}(x_{j+1} - x_{j-1})|v_j'''| + \frac{a_j}{2}(x_j - x_{j-1})|v_j''| \text{ for } 1 \leq j \leq \frac{n}{2} - 1. \tag{2.4.3}$$

Note that $h_j = x_j - x_{j-1} \leq 4n^{-1}$ for any j, therefore using Lemma 2.2.5 in conjunction with Lemma 7 of [48], we have

$$|L^n(V_j - v_j)(x_j)| \leq Cn^{-1}.$$

Now, applying Lemma 2.3.2 to mesh function $(V-v)(x_j)$, we obtain

$$|(V_j - v_j)(x_j)| \leq Cn^{-1} \text{ for } 1 \leq j \leq \frac{n}{2} - 1. \tag{2.4.4}$$

To estimate the local truncation error of the singular component $L^n(W-w)$, the argument depends on whether $\tau = 1/2$ or $\tau = (\varepsilon/\eta)\ln(n/4)$. The mesh is uniform in the first case $1/2 \leq (\varepsilon/\eta)\ln(n/4)$. The local truncation error is bounded in the same manner as done above.

$$|L^n(W_j - w_j)| \leq \frac{\varepsilon}{3}(x_{j+1} - x_{j-1})|w_j'''| + \frac{a_j}{2}(x_j - x_{j-1})|w_j''| \text{ for } 1 \leq j \leq \frac{n}{2} - 1. \tag{2.4.5}$$

Since $h_j = x_j - x_{j-1} \leq 4n^{-1}$ for any j, applying Lemma 2.2.5 in conjunction with Lemma 7 of [48], we obtain

$$|L^n(W_j - w_j)(x_j)| \leq C\varepsilon^{-2}n^{-1}.$$

But the present case, $\varepsilon^{-1} \leq (2/\eta)\ln(n/4)$, we obtain

$$|L^n(W_j - w_j)(x_j)| \leq Cn^{-1}\left[\ln\left(\frac{n}{4}\right)\right]^2.$$

Now, applying Lemma 2.3.2 to the mesh function $(W-w)(x_j)$, we obtain

$$|(W_j - w_j)(x_j)| \leq Cn^{-1}\left[\ln\left(\frac{n}{4}\right)\right]^2 \text{ for } 1 \leq j \leq \frac{n}{2} - 1. \tag{2.4.6}$$

In the second case (namely $\tau = (\varepsilon/\eta)\ln(n/4)$, the mesh is piecewise uniform. A different argument is used to bound $|W-w|$ in each subintervals $[-1, -\tau]$ and $[-\tau, 0]$. There is no interior layer in the subinterval $[-1, -\tau]$, both W and w are small, and because $|W - w| \leq |W| + |w|$, it suffices to bound W and w separately. Note first that w can also be decomposed as $w = w_0 + \varepsilon w_1$. The expression for w_0 can be written in the form

$$w_0(x) = w_0(-1)\varphi(x) + w_0(0)(0 - \varphi(x))$$

and $w_0'(x)$ is given by

$$w_0'(x) = (w_0(-1) - w_0(0))\varphi'(x)$$

where

$$|\varphi'(x)| \leq C\varepsilon^{-1} \exp\left(\frac{-\eta x}{\varepsilon}\right), \forall x \in [-1, 0],$$

and $w_0(-1) = w_0(0) \exp(-\tau/\varepsilon)$. It follows that

$$\frac{w_0'(x)}{w_0(0)} = -\left[1 - \exp\left(\frac{-\eta}{\varepsilon}\right)\right]\varphi'(x) \;\; > 0$$

and

$$\frac{w_0(-1)}{w_0(0)} = \exp\left(\frac{-\eta}{\varepsilon}\right).$$

Thus $w_0(-1)/w_0(0)$ is positive and increasing in the interval $[-1, 0]$. It follows that $\forall x \in [-1, -\tau]$, we have

$$0 \leq \frac{w_0(-1)}{w_0(0)} \leq \frac{w_0(-\tau)}{w_0(0)}$$

and so

$$|w_0(x)| \leq |w_0(-\tau)|.$$

It is also true for $w_1(x)$ and since $w(x) = w_0(x) + \varepsilon w_1(x)$, it follows that $|w(x)| \leq |w(-\tau)|$, $\forall x \in [-1, -\tau]$.

Using the estimate for $w_0(-\tau)$ and $w_1(-\tau)$, we obtain $w(-\tau) \leq C \exp(-\eta/\varepsilon)$. The fact that $\tau = (\varepsilon/\eta)\ln(n/4)$, finally we obtain

$$|w(x_j)| \leq Cn^{-1} \text{ for } 1 \leq j \leq \frac{n}{2} - 1. \tag{2.4.7}$$

To obtain a similar bound on W, an auxiliary mesh function $\tilde{W}$ is defined analogous to W except that the coefficient of $a(x)$ in the difference operator L^n is replaced by $-\eta$. Then, from (Lemma 7.5 of [42]),

$$|W_j| \leq |\tilde{W}|, \text{ for } 0 \leqslant j \leqslant n.$$

Furthermore, using Lemma 2.2.5 leads us immediately to conclude that

$$|w(x_j)| \leq Cn^{-1} \text{ for } 1 \leq j \leq \frac{n}{4} - 1. \tag{2.4.8}$$

By using the estimated obtained by (2.4.7) and (2.4.8), we obtain

$$|(W - w)(x_j)| \leq Cn^{-1} \text{ for } 1 \leq j \leq \frac{n}{4} - 1. \tag{2.4.9}$$

In the subinterval $[-\tau, 0]$, the classical analogue to (2.4.5) leads to

$$|L^n(W_j - w_j)| = C\varepsilon^{-2}|x_{j+1} - x_{j-1}| = 8C\varepsilon^{-2}n^{-1}\tau \text{ for } \frac{n}{4} \leq j \leq \frac{n}{2} - 1.$$

Also, $|(W(-1) - w(-1)| = 0$ and $|(W(n/4) - w(n/4)| \leq |(W(n/4)| + |w(n/4)| \leq Cn^{-1}$ from (2.4.8). We introduce the new function, which is called the barrier function. The barrier function in the subinterval $[-\tau, 0]$ is given by

$$\Phi_j = (x_j + \tau)C_1\varepsilon^{-2}\tau n^{-1} + C_2 n^{-1},$$

it follows that for a suitable choice of C_1 and C_2 the mesh functions

$$\Psi_j^{\pm} = \Phi_j \pm (W_j - w_j)$$

satisfy the inequalities

$$\Psi_{\frac{n}{4}} \geq 0, \quad \Psi_{\frac{n}{2}} = 0$$

and

$$L^n\Psi_j \leq 0, \quad \frac{n}{4} + 1 \leq j \leq \frac{n}{2} - 1.$$

By applying Lemma 2.2.1 on $[-\tau, 0]$ to the function $\Psi_j^{\pm}$, we obtain

$$\Psi_j \geq 0, \quad \frac{n}{4} + 1 \leq j \leq \frac{n}{2} - 1.$$

Therefore, we obtain

$$|W_j - w_j| \leq \Phi_j \leq C_1\varepsilon^{-2}\tau^2 n^{-1} + C_2 n^{-1}.$$

Since $\tau = (\varepsilon/\eta)\ln(n/4)$, we have

$$|W_j - w_j| \leq Cn^{-1}\left[\ln\left(\frac{n}{4}\right)\right]^2. \tag{2.4.10}$$

Combining (2.4.9) and (2.4.10), we obtain the following estimate on the singular component of the error over interval $[-1, 0]$ as follows

$$|W_j - w_j| \leq Cn^{-1}\left[\ln\left(\frac{n}{4}\right)\right]^2, \quad \frac{n}{4} + 1 \leq j \leq \frac{n}{2} - 1. \tag{2.4.11}$$

Given the estimates (2.4.4) and (2.4.11) along with the inequality (2.4.2), we obtain

$$|U_j - u_j| \leq Cn^{-1}\left[\ln\left(\frac{n}{4}\right)\right]^2, \quad 1 \leq j \leq \frac{n}{2} - 1. \tag{2.4.12}$$

Similarly for the subinterval $[0, 1]$, we obtain

$$|U_j - u_j| \leq Cn^{-1}\left[\ln\left(\frac{n}{4}\right)\right]^2, \quad \frac{n}{2} \leq j \leq n. \tag{2.4.13}$$

Combining (2.4.12) and (2.4.13) then gives the required result.

2.5 Richardson extrapolation on the FMFDM

In this section, we use Richardson extrapolation to improve the accuracy of the proposed method. Richardson extrapolation is procedure where a linear combination of two approximations of a some quantity gives a better approximation of the quantity [49].

We focus our attention on the interval $[-1, 0]$ as before. Results on the interval $[0, 1]$ can be obtained in a similar way. Keeping in mind that there is a transition point at $x = -\tau$, we consider the subintervals $[-1; -\tau] \cup (-\tau, 0]$ separately.

With reference to (2.3.2), to simplify the analysis, for $x_j \in \Omega_n^\tau$, $h_j = x_j - x_{j-1}$, we denote $H = h_j$ for $j = 1, 2, \cdots, n/4$ and $h = h_j$ for $j = n/4 + 1, \cdots, n/2$.

We consider the mesh Ω_{2n}^{τ} where τ is given by (2.3.1) where we bisect each mesh subinterval. It is clear that $\Omega_n^{\tau} \subset \Omega_{2n}^{\tau} = \{\tilde{x}_j\}$ and $\tilde{x}_j - \tilde{x}_{j-1} = \tilde{h}_j = h_j/2$. We denote the numerical solution on the mesh Ω_{2n}^{τ} by $\tilde{U}_j$. The estimate (2.4.12) can be written as

$$U_j - u_j = C_1 n^{-1}\left[\ln\left(\frac{n}{4}\right)\right]^2 + R_n(x_j), \;\; \forall x_j \in \Omega_n^{\tau} \tag{2.5.1}$$

and

$$\tilde{U}_j - u_j = C_2(2n)^{-1}\left[\ln\left(\frac{n}{4}\right)\right]^2 + R_{2n}(\tilde{x}_j), \;\; \forall \tilde{x}_j \in \Omega_{2n}^{\tau}, \tag{2.5.2}$$

where C_1 and C_2 are some constants and the remainders term $R_n(x_j)$ and $R_{2n}(\tilde{x}_j)$ are $\mathcal{O}\left[n^{-1}\left(\ln\left(\frac{n}{4}\right)\right)^2\right]$. Note that we have used the same transition parameter τ when computing both U_j and $\tilde{U}_j$. This is seen from the factor $\ln(n/4)$.

A combination of the two equations above gives

$$u_j - (2\tilde{U}_j - U_j) = R_n(x_j) - 2R_{2n}(x_j) = \mathcal{O}\left[n^{-1}\left(\ln\left(\frac{n}{4}\right)\right)^2\right], \;\; \forall x_j \in \Omega_n^{\tau}. \tag{2.5.3}$$

We set

$$U_j^{ext} = 2\tilde{U}_j - U_j, \;\; \forall x_j \in \Omega_n^{\tau}, \tag{2.5.4}$$

as the new approximation of u_j obtained after applying Richardson extrapolation. The error after extrapolation U_j^{ext} can also be decomposed as in (2.4.2),

$$U_j^{ext} - u_j = (V_j^{ext} - v_j) + (W_j^{ext} - w_j), \tag{2.5.5}$$

where V_j^{ext} and W_j^{ext} are the regular and singular components of U_j^{ext}. The local truncation error of the scheme (2.3.3)-(2.3.5) after extrapolation is given by

$$L^n(U_j^{ext} - u_j) = 2L^n(\tilde{U}_j - u_j) - L^n(U_j - u_j), \tag{2.5.6}$$

where

$$L^n(U_j - u_j) = r^- u_{j-1} + r^c u_j + r^+ u_{j+1} - \varepsilon u_j'' - a_j u_j' + b_j u, \tag{2.5.7}$$

and

$$L^n(\tilde{U}_j - u_j) = \tilde{r}^- u_{j-1} + \tilde{r}^c u_j + \tilde{r}^+ u_{j+1} - \varepsilon u_j'' - a_j u_j' + b_j u. \tag{2.5.8}$$

The quantities of r^-, r^c and r^+ are given in (2.3.6), but $\tilde{r}^-$, $\tilde{r}^c$ and $\tilde{r}^+$ are obtained by substituting h_j by $\tilde{h}_j$ and h_{j+1} by $\tilde{h}_{j+1}$ in the expressions of r^-, r^c and r^+ respectively. Taking the Taylor series expansion of u_j around x_j, we obtain the following approximations for u_{j-1} and u_{j+1}:

$$u_{j-1} = u_j - h_j u_j' + \frac{h_j^2}{2} u_j^2 - \frac{h_j^3}{6} u_j^3 + \frac{h_j^4}{24} u^4(\xi_1, j), \tag{2.5.9}$$

$$u_{j+1} = u_j + h_{j+1} u_j' + \frac{h_{j+1}^2}{2} u_j^2 + \frac{h_{j+1}^3}{6} u_j^3 + \frac{h_{j+1}^4}{24} u^4(\xi_2, j), \tag{2.5.10}$$

$$u_{j-1} = u_j - \tilde{h}_j u_j' + \frac{\tilde{h}_j^2}{2} u_j^2 - \frac{\tilde{h}_j^3}{6} u_j^3 + \frac{\tilde{h}_j^4}{24} u^4(\tilde{\xi}_1, j), \tag{2.5.11}$$

$$u_{j+1} = u_j + \tilde{h}_{j+1} u_j' + \frac{\tilde{h}_{j+1}^2}{2} u_j^2 + \frac{\tilde{h}_{j+1}^3}{6} u_j^3 + \frac{\tilde{h}_{j+1}^4}{24} u^4(\tilde{\xi}_2, j), \tag{2.5.12}$$

where

$$\xi_1, j \in (x_{j-1}, x_j), \quad \xi_2, j \in (x_j, x_{j+1}), \quad \tilde{\xi}_1 \in (\frac{x_{j-1} + x_j}{2}, x_j) \text{ and } \tilde{\xi}_2 \in (x_j, \frac{x_j + x_{j+1}}{2}).$$

Substituting (2.5.9) and (2.5.10) into (2.5.7), (2.5.11) and (2.5.12) into (2.5.8), we obtain the following expressions:

$$L^n(U_j - u_j) = k_1 u_j + k_2 u_j' + k_3 u_j^2 + k_4 u_j^3 + k_{5,1} u^4(\xi_1, j) + k_{5,2} u^4(\xi_2, j) \tag{2.5.13}$$

and

$$L^n(\tilde{U}_j - u_j) = \tilde{k}_1 u_j + \tilde{k}_2 u_j' + \tilde{k}_3 u_j^2 + \tilde{k}_4 u_j^3 + \tilde{k}_4 u_j^4 + \tilde{k}_{5,1} u^4(\tilde{\xi}_1, j) + \tilde{k}_{5,2} u^4(\tilde{\xi}_2, j). \tag{2.5.14}$$

The coefficients in (2.5.13) are

$$k_1 = \frac{2\varepsilon}{h_j(h_j + h_{j+1})} - \frac{2\varepsilon}{h_j h_{j+1}} + \frac{2\varepsilon}{h_{j+1}(h_j + h_{j+1})}, \; k_2 = 0, \; k_3 = \frac{\varepsilon h_j}{h_j + h_{j+1}} - \frac{a_j h_j}{2} + \frac{\varepsilon h_{j+1}}{h_j + h_{j+1}} - \varepsilon,$$

$$k_4 = \frac{-\varepsilon h_j^2}{3(h_j+h_{j+1})} + \frac{a_j h_j^2}{6} + \frac{\varepsilon h_{j+1}^2}{3(h_j+h_{j+1})},\ k_{5,1} = \frac{\varepsilon h_j^3}{12(h_j+h_{j+1})} - \frac{a_j h_j^3}{24},\ k_{5,2} = \frac{\varepsilon h_{j+1}^3}{12(h_j+h_{j+1})}.$$

The quantities for $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$, $\tilde{k}_4$, $\tilde{k}_{5,1}$ and $\tilde{k}_{5,2}$ can be obtained by substituting h_j by $\tilde{h}_j$ and h_{j+1} by $\tilde{h}_{j+1}$.

Substituting (2.5.13) and (2.5.14) into (2.5.6), we obtain

$$L^n(U_j^{ext} - u_j) = T_1 u_j + T_2 u_j'' + +T_3 u_j''' + +T_{4,1} u^{(4)}(\xi_1, j) + T_{4,2} u^{(4)}(\xi_2, j), \qquad (2.5.15)$$

where

$$T_1 = \frac{14\varepsilon}{h_j(h_j + h_{j+1})} - \frac{14\varepsilon}{h_j h_{j+1}} + \frac{14\varepsilon}{h_{j+1}(h_j + h_{j+1})},$$

$$T_2 = \frac{\varepsilon h_j}{h_j + h_{j+1}} - \varepsilon + \frac{\varepsilon h_{j+1}}{h_j + h_{j+1}},\ \ T_3 = -\frac{a_j h_j^2}{12},$$

$$T_{4,1} = -\frac{\varepsilon h_j^3}{24(h_j + h_{j+1})} + \frac{a_j h_j^3}{32} \ \text{ and } \ T_{4,2} = -\frac{\varepsilon h_{j+1}^3}{24}.$$

Using the fact that, for $\forall j = 1, \ldots, n/4,\ H = h_j \leqslant 4n^{-1}$ into (2.5.15) in the subinterval $[-1, -\tau]$, we obtain

$$L^n(V_j^{ext} - v_j) = -\frac{a_j H^2}{12} v_j''' + \left[\frac{\varepsilon H^2}{48} + \frac{a_j H^3}{32}\right] v^{(4)}(\xi_1, j) - \frac{\varepsilon H^3}{24} v^{(4)}(\xi_2, j). \qquad (2.5.16)$$

Now applying the triangle inequality, Lemma 2.2.5 in conjunction with Lemma 7 of [48] to (2.5.16), we obtain

$$|L^n(V_j^{ext} - v_j)| \leq Cn^{-2}. \qquad (2.5.17)$$

To estimate $L^n(W_j^{ext}-w_j)$, the argument depends on whether $\tau = 1/2$ or $\tau = (\varepsilon/\eta)\ln(n/4)$.

In the first case the mesh is uniform and $(\varepsilon/\eta)\ln(n/4) \geq 1/2$. The estimate of singular component of the local truncation error is obtained as follows:

$$L^n(W_j^{ext} - w_j) = -\frac{a_j h^2}{12} w_j''' + \left[\frac{\varepsilon h^2}{48} + \frac{a_j h^3}{32}\right] w^{(4)}(\xi_1, j) - \frac{\varepsilon h^3}{24} w^{(4)}(\xi_2, j). \qquad (2.5.18)$$

Now, applying the triangle inequality, Lemma 2.2.5 and using Lemma 7 of [48], we obtain

$$|L^n(W_j^{ext} - w_j)| \leq Cn^{-2}\varepsilon^{-3}\exp\left(2x_j\eta/\varepsilon\right). \qquad (2.5.19)$$

Note that $\varepsilon^{-1}\exp\left(2x_j\eta/\varepsilon\right) \leq C$ and $\varepsilon^{-1} \leqslant (2/\eta)\ln(n/4)$, we obtain

$$|L^n(W_j^{ext} - w_j)| \leq Cn^{-2}\left[\ln(n/4)\right]^2. \qquad (2.5.20)$$

In the second case (viz $\tau = (\varepsilon/\eta)\ln(n/4)$), the mesh is piecewise uniform with the mesh spacing $h = h_j = 4\tau n^{-1}$ for $\forall\ j = n/4+1, \ldots, n/2$ in the subinterval $[-\tau; 0]$. Applying the triangle inequality, Lemma 2.2.5 along with Lemma 7 of [48] to (2.5.18), we obtain

$$|L^n(W_j^{ext} - w_j)| \leq C_1 n^{-2}\tau^2\varepsilon^{-2}. \qquad (2.5.21)$$

Since $\tau = (\varepsilon/\eta)\ln(n/4)$, this gives

$$|L^n(W_j^{ext} - w_j)| \leq Cn^{-2}\left[\ln(n/4)\right]^2. \qquad (2.5.22)$$

On application of Lemma 2.3.2 in (2.5.17), (2.5.20) and (2.5.22) and combining the resulting inequalities, we obtain the following theorem.

Theorem 2.5.1. (Error after extrapolation). Let $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth and $u(x)$ be the solution of (2.1.1). If U^{ext} is the approximation of u obtained using (2.3.1)-(2.3.5) with $u(-1) = U(-1)$, $u(1) = U(1)$, then there is a positive constant C independent of ε and the mesh spacing such that

$$\sup_{0<\varepsilon\leq 1}\max_{0\leq j\leq n}|(U^{ext} - u)_j| \leq Cn^{-2}\left[\ln\left(\frac{n}{4}\right)\right]^2. \qquad (2.5.23)$$

2.6 Numerical examples

In this section, we present two test examples for which numerical results are computed to illustrate the effectiveness of the present method. The maximum errors and order of convergence are calculated by using the exact solution. The solution in the examples has a turning point at $x = 0$ and $x = 0.5$, respectively, which gives rise to an interior layer.

Example 2.6.1. Consider the following singularly perturbed turning point problem:

$$\varepsilon u'' + xu' - u = -(1 + \varepsilon\pi^2)\cos\pi x - \pi x \sin\pi x, \quad x \in [-1, 1],$$

$$u(-1) = -1, \quad u(1) = 1.$$

The exact solution is

$$u(x) = \cos\pi x + x + \frac{x\operatorname{erf}(x/\sqrt{2\varepsilon}) + \sqrt{2\varepsilon/\pi}\exp(-x^2/2\varepsilon)}{\operatorname{erf}(1/\sqrt{2\varepsilon}) + \sqrt{2\varepsilon/\pi}\exp(-1/2\varepsilon)}.$$

This example was solved numerically in [22] using the Exponentially Fitted Weighted-Residual (EFWR) and classical Galerkin (GAL) methods. In this work, the error was not computed inside the turning-point region. Moreover, the author indicates that EFWRs do not yield ε-uniform convergence results for problems with a turning point.

We will calculate the error of the method we propose throughout the entire domain (including in the layer region). Moreover, our numerical results (see tables 2.1 and 2.3) will confirm the theoretical estimates regarding ε-uniform convergence.

Example 2.6.2. Consider the following singular perturbed turning point problem:

$$\varepsilon u'' + 2(x - 0.5)u' - 2u = f(x), \quad x \in [0, 1],$$

where

$$f(x) = 3\varepsilon(x - 0.5)\exp((x - 0.5)^2/\varepsilon) - (\varepsilon\pi^2 + 2)\cos\pi(x - 0.5) - 2(x - 0.5)\pi\sin\pi(x - 0.5)$$

$$u(0) = \frac{\varepsilon}{4}\exp(-1/4\varepsilon), \quad u(1) = -\frac{\varepsilon}{4}\exp(-1/4\varepsilon).$$

The exact solution is

$$u(x) = -\frac{\varepsilon}{2}(x-0.5)\exp\left[-(x-0.5)^2/\varepsilon\right] + \cos\pi(x-0.5).$$

While the maximum errors before extrapolation at all mesh points are evaluated using the formula

$$E_{n,\varepsilon} = \max_{0\le j\le n} |u_j - U_j|,$$

these errors after extrapolation are given by

$$E_{n,\varepsilon}^{ext} = \max_{0\le j\le n} |u_j - U_j^{ext}|.$$

The numerical rate of convergence are found by using the formula

$$r_{\varepsilon,k} = \log_2(\tilde{E}_{n_k}/\tilde{E}_{2n_k}),$$

where $\tilde{E}$ stands for E or E^{ext}.

Table 2.1: Results for Example 2.6.1 Maximum errors before extrapolation

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-5}	2.94E-01	1.09E-01	8.06E-02	4.69E-02	2.48E-02	1.24E-02	5.94E-03
10^{-6}	2.93E-01	1.10E-01	8.13E-02	4.75E-02	2.53E-02	1.30E-02	6.46E-03
10^{-8}	2.93E-01	1.10E-01	8.15E-02	4.77E-02	2.56E-02	1.32E-02	6.69E-03
10^{-10}	2.93E-01	1.10E-01	8.16E-02	4.78E-02	2.56E-02	1.32E-02	6.72E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-15}	2.93E-01	1.10E-01	8.16E-02	4.78E-02	2.56E-02	1.32E-02	6.72E-03

Table 2.2: Results for Example 2.6.1 Maximum errors after extrapolation

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-5}	2.66E-01	7.40E-02	1.89E-02	4.73E-03	2.68E-03	2.54E-03	2.51E-03
10^{-6}	2.67E-01	7.42E-02	1.90E-02	4.78E-03	1.19E-03	8.47E-04	8.10E-04
10^{-8}	2.67E-01	7.43E-02	1.91E-02	4.81E-03	1.20E-03	3.01E-04	9.55E-05
10^{-10}	2.67E-01	7.43E-02	1.91E-02	4.81E-03	1.20E-03	3.01E-04	7.53E-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-15}	2.67E-01	7.43E-02	1.91E-02	4.81E-03	1.20E-03	3.01E-04	7.53E-05

Table 2.3: Results for Example 2.6.1 Rates of convergence before extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-5}	1.42	0.44	0.78	0.92	1.00	1.06
10^{-6}	1.41	0.44	0.78	0.91	0.97	1.00
10^{-8}	1.41	0.44	0.77	0.90	0.95	0.98
10^{10}	1.41	0.44	0.77	0.90	0.95	0.98
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-15}	1.41	0.44	0.77	0.90	0.95	0.98

Table 2.4: Results for Example 2.6.1 Rates of convergence after extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-5}	1.85	1.97	2.00	0.82	0.08	0.02
10^{-6}	1.85	1.96	1.99	2.00	0.50	0.06
10^{-8}	1.84	1.96	1.99	2.00	2.00	1.67
10^{-10}	1.84	1.96	1.99	2.00	2.00	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-15}	1.84	1.96	1.99	2.00	2.00	2.00

Table 2.5: Results for Example 2.6.2 Maximum errors before extrapolation

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-5}	3.83E-01	1.95E-01	9.80E-02	4.90E-02	2.45E-02	1.22E-02	6.10E-03
10^{-6}	3.83E-01	1.95E-01	9.80E-02	4.91E-02	2.45E-02	1.23E-02	6.13E-03
10^{-7}	3.83E-01	1.95E-01	9.80E-02	4.91E-02	2.45E-02	1.23E-02	6.14E-03
10^{-8}	3.83E-01	1.95E-01	9.80E-02	4.91E-02	2.45E-02	1.23E-02	6.14E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-15}	3.83E-01	1.95E-01	9.80E-02	4.91E-02	2.45E-02	1.23E-02	6.14E-03

Table 2.6: Results for Example 2.6.2 Maximum errors after extrapolation

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-5}	3.76E-02	1.07E-02	2.88E-03	7.70E-04	2.19E-04	7.83E-05	4.27E-05
10^{-6}	3.76E-02	1.07E-02	2.86E-03	7.43E-04	1.91E-04	5.06E-05	1.50E-05
10^{-7}	3.76E-02	1.07E-02	2.86E-03	7.40E-04	1.89E-04	4.78E-05	1.22E-05
10^{-8}	3.76E-02	1.07E-02	2.86E-03	7.40E-04	1.88E-04	4.75E-05	1.20E-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-15}	3.76E-02	1.07E-02	2.86E-03	7.40E-04	1.88E-04	4.75E-05	1.19E-05

Table 2.7: Results for Example 2.6.2 Rates of convergence before extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-5}	0.97	1.00	1.00	1.00	1.00	1.00
10^{-6}	0.97	1.00	1.00	1.00	1.00	1.00
10^{-7}	0.97	1.00	1.00	1.00	1.00	1.00
10^{-8}	0.97	1.00	1.00	1.00	1.00	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-15}	0.97	0.99	1.00	1.00	1.00	1.00

Table 2.8: Results for Example 2.6.2 Rates of convergence after extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-5}	1.82	1.89	1.90	1.81	1.48	0.88
10^{-6}	1.82	1.90	1.94	1.96	1.92	1.74
10^{-7}	1.82	1.90	1.95	1.97	1.98	1.97
10^{-8}	1.82	1.90	1.95	1.97	1.99	1.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-15}	1.82	1.90	1.95	1.97	1.99	1.99

2.7 Discussion

In this chapter, we proposed a finite mesh finite difference method (FMFDM) for the class of two-point boundary value singularly perturbed problems whose solution exhibits an interior layer. First, we derived bounds on the solution and its derivatives. Then we constructed a mesh, of Shishkin type, prone to handle the rapid change of the solution near the turning point. On this mesh, a discrete upwind scheme was designed in accordance with the sign of the convection coefficient. Using bounds on the solution and its derivatives, we proved that the developed method was uniformly convergent of order one with respect to the perturbation parameter and the step size.

Further we investigated the effect of Richardson extrapolation on the FMFDM and noticed that it improved the accuracy of the computed solution. In particular, the rate of convergence increased from 1 to 2.

To support the above conclusions based on theoretical analysis, we solved two examples to confirm the findings. Numerical results are displayed in tables 2.1-2.8.

Chapter 3

A numerical method for a stationary interior layer convection-diffusion problems

The aim of this chapter is to construct and analyse a fitted mesh finite difference method (FMFDM) for a class of time-dependent singularly perturbed convection-diffusion-reaction problems with a turning point whose solution exhibits an interior layer. We establish bounds on the solution to these problems and their derivatives. Then we use the classical implicit Euler method to discretize the time variable with a constant step-size. We construct a FMFDM to solve the resulting system of two-point boundary value problems. To improve the accuracy of the proposed method, we apply Richardson extrapolation.

3.1 Introduction

Numerous numerical schemes for singularly problems are available in the literature. These problems are characterised by a small parameter affecting the highest derivative in the differential equations underlying the problem. In this chapter, we examine the linear singularly perturbed time-dependent convection-diffusion problem

$$Lu :\equiv \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)\frac{\partial u(x,t)}{\partial x} - b(x,t)u(x,t) - d(x,t)\frac{\partial u(x,t)}{\partial t} = f(x,t), \quad (3.1.1)$$

$$(x,t) \in Q \equiv \Omega \times (0,T] \equiv (-1,1) \times (0,T], \tag{3.1.2}$$

subject to the initial and boundary conditions

$$u(x,0) = u_0(x), \;\; -1 \leq x \leq 1, \;\; u(-1,t) = \alpha_1(t)\;, \; u(1,t) = \alpha_2(t), \;\; t \in (0,T], \tag{3.1.3}$$

where ε is the diffusion parameter satisfying $0 < \varepsilon \leq 1$. Further, we assume that the functions $a(x,t)$, $b(x,t)$, $d(x,t)$, $f(x,t)$ and $u_0(x)$ are sufficiently smooth with $b(x,t) \geq \beta > 0$ and $d(x,t) \geq \delta > 0$ in $\bar{Q}$ along with the conditions

$$\left.\begin{array}{ll} a(0,t) = 0 & a_x(0,t) > 0, t \in [0;T] \\ b(x,t) \geq b(0,t) > 0, & x \in [-1,1], t \in [0;T] \\ |a_x(x,t)| \geq |a_x(0,t)|/2, & x \in [-1,1], t \in [0;T], \end{array}\right\} \tag{3.1.4}$$

which guarantee that the solution of problem (3.1.1)-(3.1.3) has a unique solution which possesses an interior layer at $x = 0$ [19]. Also, we impose the compatibility conditions

$$u_0(-1) = \alpha_1(0) \;\text{ and }\; u_0(1) = \alpha_2(0),$$

so that the data match at the two corners $(-1,0)$ and $(1,0)$ of the domain $\bar{Q}$. These conditions guarantee that there exists a constant C independent of ε such that [72]

$$|u(x,t) - \alpha_1(t)| \leq C(1+x), \;\; |u(x,t) - \alpha_2(t)| \leq C(1-x), \quad \forall (x,t) \in \bar{Q}$$

and

$$|u(x,t) - u_0(x)| \leq Ct, \;\; \forall (x,t) \in \bar{Q}.$$

In general if the coefficient of the convection term $a(x,t)$ does not change sign throughout the spatial domain, then the solution possesses a boundary layer near -1 or 1. But if $a(x,t)$ does change sign, then an interior layer may occur. Note that the interior layers are also present in the solution of the problem above if the coefficient functions are not smooth or if the data function $f(x,t)$ is discontinuous (see e.g., [13]).

The presence of layers renders classical methods unfit to provide acceptable approximations to the solution for a class of time-dependent singularly perturbed convection-diffusion problems.

Little attention has been given to the study of time-dependent singularly perturbed convection-diffusion problems whose solution exhibits an interior layer.

In this chapter we aim to study the time-dependent problem (3.1.1)-(3.1.3) whose solution exhibits an interior layer due to the presence of a turning point. We propose and analyse a fitted mesh finite difference method (FMFDM). We establish that the method is first order parameter uniform convergent of order one, up to a logarithmic factor.

The rest of this chapter is organised as follows. We establish bounds on the solution $u(x,t)$ and its derivatives in Section 2. In Section 3, We adopt an upwind scheme on the mesh to obtain our FMFDM. In Section 4, we conduct a rigourous error analysis. We prove that the scheme is almost first order uniformly convergent with respect to the perturbation parameter in time and space. To improve the accuracy of the proposed FMFDM, we apply Richardson extrapolation in Section 5 and obtain an almost second order uniform convergence in space. To see how the proposed method works in practice and to confirm our theoretical results, we present numerical experiments in Section 6. In Section 7, we present some concluding remarks.

In the rest of this chapter, C denotes a generic constant which may assume different values in different inequalities but will always be independent of ε, of the space and time discretization parameters.

3.2 A priori estimates of the solution and its derivatives

This section presents the bounds for the solution of problem (3.1.1)-(3.1.3) and its derivatives. We shall denote the subintervals of $[-1, 1]$ as $\Omega_l = [-1, -\tau]$, $\Omega_c = [-\tau, +\tau] = [-\tau, 0) \cup [0, +\tau)$ and $\Omega_r = [\tau, 1]$, where $0 \leq \tau \leq 1/2$.

In the following, we first prove that the operator L as defined in (3.1.1) admits the following continuous minimum principle and then we state stability estimate for the solution of problem (3.1.1)-(3.1.3).

Lemma 3.2.1. (Minimum principle). Let $\xi(x, t)$ be a smooth function satisfying $\xi(\pm 1, t) \geq 0$ and $L\xi(x, t) \leq 0$, $\forall (x, t) \in Q$. Then $\xi(x, t) \geq 0$, $\forall (x, t) \in \bar{Q}$.

Proof. The proof is by contradiction. Let $(x^*, t^*) \in \bar{Q}$ such that $\xi(x^*, t^*) = \min_{\bar{Q}} \xi(x, t)$ and assume that $\xi(x^*, t^*) < 0$. Clearly $(x^*, t^*) \notin Q$, therefore $\xi_x(x^*, t^*) = 0$, $\xi_t(x^*, t^*) = 0$ and $\xi_{xx}(x^*, t^*) \geq 0$ and we have

$$L\xi(x^*, t^*) = \varepsilon \xi_{xx}(x^*, t^*) + a(x^*, t^*)\xi_x(x^*, t^*) - b(x^*, t^*)\xi(x^*, t^*) - d(x^*, t^*)\xi_t(x^*, t^*) > 0,$$

which is a contradiction. It is proved that $\xi(x^*, t^*) \geq 0$ and thus $\xi(x, t) \geq 0$, $\forall (x, t) \in \bar{Q}$.

The minimum principle implies the existence and uniqueness of the solution (as for linear problems, the existence of the solution is implied by its uniqueness). We use this principle to prove the following results which state that the solution depends continuously on the data.

Lemma 3.2.2. (Stability estimate). The solution $u(x, t)$ of problem (3.1.1)-(3.1.3) satisfies:

$$||u(x, t|| \leqslant [\max\{||\alpha_1||_\infty, ||\alpha_2||_\infty\}] + \frac{1}{\beta}||f||_\infty, \forall (x, t) \in \bar{Q}.$$

Proof. We consider the comparison functions

$$\Gamma^{\pm}(x, t) = C \pm u(x, t) \;\; \forall (x, t) \in \bar{Q},$$

where $C = \max\{||\alpha_1||_\infty, ||\alpha_2||_\infty\} + \frac{1}{\beta}\, ||f||_\infty$.

Applying the minimum principle to the comparison functions, we have

$$\begin{aligned} L\Gamma^{\pm}(x,t) &= \varepsilon[\Gamma^{\pm}_{xx}(x,t)] + a(x,t)[\Gamma^{\pm}_{x}(x,t)] - b(x,t)[\Gamma^{\pm}(x,t)] - d(x,t)[\Gamma^{\pm}_{t}(x,t)] \\ &= \pm\varepsilon u_{xx}(x,t) \pm a(x,t)u_x(x,t) - b(x,t)[\pm u(x,t)] - d(x,t)[\pm u_t(x,t)] - b(x,t)C \\ &= \pm f(x,t) - b(x,t)\left[\max\{||\alpha_1||_\infty, ||\alpha_2||_\infty\} + \frac{1}{\beta}\, ||f||_\infty\right] \\ &= \pm f(x,t) - \frac{b(x,t)}{\beta}\, ||f||_\infty - b(x,t)\max\{||\alpha_1||_\infty, ||\alpha_2||_\infty\} \\ &= -[||f||_\infty \mp f(x,t)] - b(x,t)\max\{||\alpha_1||_\infty, ||\alpha_2||_\infty\} \leqslant 0, \end{aligned}$$

implying that

$$\Gamma^{\pm}(x,t) \geqslant 0\,,\ \forall (x,t) \in \bar{Q},$$

then we have

$$C \pm u(x,t) \geq 0.$$

It follows immediately that $||u(x,t)||_\infty \leqslant C$, which completes the proof.

Lemma 3.2.3. The bound on the solution $u(x,t)$ of (3.1.1)-(3.1.3) is given by

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C,\ (x,t) \in \bar{\Omega}.$$

Proof. See [11].

Lemma 3.2.4. The bound on the derivative $u_t(x,t)$ of (3.1.1)-(3.1.3) is given by

$$|u_t(x,t)| \leq C, \quad (x,t) \in \bar{Q}.$$

Proof. For the proof of the lemma, readers are referred to [72].

Lemma 3.2.5. (Continuous results). Let $u(x,t)$ be the solution of (3.1.1)-(3.1.3). There exists *a positive constants* η and C, such that

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq C\left[1 + \varepsilon^{-j}\exp\left(\frac{\eta x}{\varepsilon}\right)\right], x \in [-\tau, 0),\ t \in [0,T]$$

and

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq C\left[1 + \varepsilon^{-j}\exp\left(\frac{-\eta x}{\varepsilon}\right)\right], x \in [0, \tau],\ t \in [0,T],$$

where $j = 0, 1, 2, 3$.

Proof. We prove this Lemma on $[-\tau, 0)$. The proof on $[0, \tau]$ will follow the same lines. (3.1.1) can be written as follows

$$L_x u - d(x,t)\,\frac{\partial u(x,t)}{\partial t} = f(x,t),\ x \in [-\tau, 0),\ t \in [0,T], \tag{3.2.1}$$

where

$$L_x u = \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)\,\frac{\partial u(x,t)}{\partial x} - b(x,t)u(x,t).$$

We obtain from (3.2.1) as follows

$$L_x u = d(x,t)\,\frac{\partial u(x,t)}{\partial t} + f(x,t) = k(x,t). \tag{3.2.2}$$

As we assume $u_0 = u(x,0),\ d(x,t)$ and $f(x,t)$ to be smooth functions and then $k(x,t)$ is continous and ε-uniformly bounded. By using the technique provided in [32] and (3.2.2) one obtains

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \le C\left[1 + \varepsilon^{-j}\exp\left(\frac{\eta x}{\varepsilon}\right)\right],\ x \in [-\tau, 0),\quad j = 0, 1. \tag{3.2.3}$$

Differentiating (3.2.2) with respect to x, we can deduce the similar bound for higher values of j and considering $p(x,t) = \partial u(x,t)/\partial x$, it follows that

$$L_x p - d(x,t)\,\frac{\partial p(x,t)}{\partial t} = s(x,t) = \frac{\partial f(x,t)}{\partial x} - a_x \frac{\partial u(x,t)}{\partial x} + b_x u(x,t) + d_x \frac{\partial u(x,t)}{\partial x},$$

$$p(-1,t) = \frac{\partial u(-1,t)}{\partial x} = \beta_1(t),\ \ p(1,t) = \frac{\partial u(1,t)}{\partial x} = \beta_2(t),\ p(x,0) = \frac{\partial u(x,0)}{\partial x} = p_0(x).$$

Here again, we assume that $s(x,t)$ is smooth function and using the same technique above, we obtain the second bound

$$\left|\frac{\partial p(x,t)}{\partial x}\right| \leqslant C\left[1 + \varepsilon^{-1}\exp\left(\frac{\eta x}{\varepsilon}\right)\right],\ x \in [-\tau, 0),\ t \in [0,T],$$

which complete the prove.

The third condition of (3.1.4) guarantees that $a(x,t) < 0$ for $-1 \leqslant x < 0$ and $a(x,t) > 0$ for $0 < x \leqslant 1$. Therefore, the solution of the problem (3.1.1)-(3.1.3) may be considered as a concatenation of two solutions: One on $-1 \leqslant x < 0$ presenting a layer near $x = 0$

(right end of the interval) and the other one on $0 < x \leqslant 1$ exhibiting a layer near $x = 0$ (left end of the interval) as well. This consideration will be useful firstly in seeking an in-depth understanding of the behaviour of the solution and its derivatives and secondly, in the design of the numerical method in Section 3.3.

In order to prove ε-uniform convergence of the numerical method to be designed in subsequent sections, we need to know the behaviour of the exact solution $u(x,t)$ of (3.1.1)-(3.1.3). This solution can be decomposed in two parts, namely the smooth component $v(x,t)$ and the singular component $w(x,t)$ ([42], pp 47) such that

$$u(x,t) = v(x,t) + w(x,t),$$

where $v(x,t)$ is the solution of the inhomogeneous problem

$$Lv(x,t) = f(x,t), \quad (x,t) \in \Omega_1 = (-1,0) \times (0,T], \tag{3.2.4}$$

$$v(x,0) = u(x,0) = u_0, \quad -1 \leqslant x \leqslant 0, \tag{3.2.5}$$

$$v(-1,t) = u(-1,t), \quad 0 \leqslant t \leqslant T, \tag{3.2.6}$$

and $w(x,t)$ is the solution of the homogeneous problem

$$Lw(x,t) = 0, \quad (x,t) \in \Omega_1, \tag{3.2.7}$$

$$w(x,0) = 0, \quad -1 \leqslant x \leqslant 0, \tag{3.2.8}$$

$$w(-1,t) = 0, \; 0 \leqslant t \leqslant T, \tag{3.2.9}$$

$$w(0,t) = u(0,t) - v(0,t), \; 0 \leqslant t \leqslant T. \tag{3.2.10}$$

We establish the following lemma which gives bounds on the solution to (3.1.1)-(3.1.3) and its derivatives.

Lemma 3.2.6. The smooth and singular components of $u(x,t)$ of problem (3.1.1)-(3.1.3), for $0 \le j \le 3$, and $0 \leqslant t \leqslant T$, satisfy

$$\left|\frac{\partial^j v(x,t)}{\partial x^j}\right| \leqslant C, \quad (x,t) \in \bar{Q}$$

and

$$\left|\frac{\partial^j w(x,t)}{\partial x^j}\right| \le C\varepsilon^{-j} \exp\left(\frac{\eta x}{\varepsilon}\right), \ x \in [-1,0],$$
$$\left|\frac{\partial^j w(x,t)}{\partial x^j}\right| \le C\varepsilon^{-j} \exp\left(\frac{-\eta x}{\varepsilon}\right), \ x \in [0,1].$$

for some constants η and C independent of ε.

Proof. We prove this lemma on $\Omega_1 = [-1,0]$. The proof on $[0,1]$ follows similar steps. We obtain the reduced problem ($\varepsilon = 0$) from (3.1.1) as follows:

$$a(x,t)v_x^0(x,t) - b(x,t)v^0(x,t) - d(x,t)v_t^0(x,t) = f(x,t), \quad (x,t) \in \Omega_1 \tag{3.2.11}$$

$$v^0(x,0) = v_0^0(x), \quad -1 \leqslant x \leqslant 0, \tag{3.2.12}$$

$$v^0(-1,t) = \alpha_1(t), \quad t \in (0,T]. \tag{3.2.13}$$

The smooth component $v(x,t)$ is further split into the sum

$$v(x,t) = v_0(x,t) + \varepsilon v_1(x,t), \quad (x,t) \in \bar{\Omega}, \tag{3.2.14}$$

where v_0 is the solution of the reduced problem in (3.2.11), which is independent of ε, then, for $0 \leqslant j \leqslant 3$, we have

$$\left\|\frac{\partial^j v_0(x,t)}{\partial x^j}\right\|_{\Omega_1} \leqslant C \tag{3.2.15}$$

and v_1 is the solution of (3.1.1), which Lemma 3.2.5 applied. It follows that

$$\left\|\frac{\partial^j v_1(x,t)}{\partial x^j}\right\|_{\Omega_1} \leqslant C, \ 0 \leqslant j \leqslant 3. \tag{3.2.16}$$

Now, applying the triangle inequality and using the estimates (3.2.15) and (3.2.16) into (3.2.14), we have

$$\begin{aligned}
\left\|\frac{\partial^j v(x,t)}{\partial x^j}\right\|_{\Omega_1} &\leqslant \left\|\frac{\partial^j v_0(x,t)}{\partial x^j}\right\|_{\Omega_1} + \varepsilon \left\|\frac{\partial^j v_1(x,t)}{\partial x^j}\right\|_{\Omega_1} \\
&\leqslant C + C\varepsilon \\
&\leqslant C(1+\varepsilon) \\
&\leqslant C.
\end{aligned}$$

Now, let us prove the regular component $w(x,t)$. We define the barrier functions as follows [31].

$$\Psi^{\pm}(x,t) = C\exp(\eta x/\varepsilon)e^{t} \pm w(x,t),\ (x,t) \in \bar{\Omega}_1.$$

Let us find the values of $\Psi^{\pm}(x,t)$ at boundaries:

$$\begin{aligned}
\Psi^{\pm}(-1,t) &= C\exp(-\eta/\varepsilon)e^{t} \pm w(-1,t),\ 0 \leqslant t \leqslant T,\\
&= C\exp(-\eta/\varepsilon)e^{t}, \ \text{using}\ (3.2.9),\\
&\geqslant 0, \quad 0 \leqslant t \leqslant T,\\
\Psi^{\pm}(0,t) &= Ce^{t} \pm w(0,t),\ 0 \leqslant t \leqslant T,\\
&= Ce^{t} \pm (u(0,t) - v(0,t)), \ \text{using}\ (3.2.10),\\
&\geqslant 0, \ \text{for a suitable choice of C}, \ 0 \leqslant t \leqslant T,\\
\Psi^{\pm}(x,0) &= C\exp(\eta x/\varepsilon) \pm w(x,0),\ -1 \leqslant x \leqslant 0,\\
&= C\exp(\eta x/\varepsilon), \ \text{using}\ (3.2.8),\\
&\geqslant 0, \quad -1 \leqslant x \leqslant 0.
\end{aligned}$$

From the above estimates, we notice that $\Psi \geqslant 0, \ (x,t) \in \Omega_2 = \bar{\Omega}_1 \backslash \Omega_1$ therefore, we have

$$\begin{aligned}
L\Psi^{\pm}(x,t) &= \varepsilon\Psi^{\pm}_{xx}(x,t) + a(x,t)\Psi^{\pm}_{x}(x,t) - b(x,t)\Psi^{\pm}(x,t) - d(x,t)\Psi^{\pm}_{t}(x,t)\\
&= C\exp(\eta x/\varepsilon)e^{t}\left[\frac{\eta^2}{\varepsilon} + \frac{\eta a(x,t)}{\varepsilon} - b(x,t) - d(x,t)\right] \pm Lw(x,t)\\
&= C\exp(\eta x/\varepsilon)e^{t}\left[\frac{\eta^2}{\varepsilon} + \frac{\eta a(x,t)}{\varepsilon} - b(x,t) - d(x,t)\right], \ \text{using}\ (3.2.7)\\
&\leqslant 0, \quad (x,t) \in \Omega_1.
\end{aligned}$$

Now, by applying Lemma 3.2.1 to the barrier functions, we obtain $\Psi^{\pm}(x,t) \geqslant 0, \ (x,t) \in \Omega_2$. Then we have

$$C\exp(\eta x/\varepsilon)e^{t} \pm w(x,t) \geqslant 0.$$

It follows that

$$\begin{aligned} w(x,t) &\leqslant C\exp(\eta x/\varepsilon)e^t,\ (x,t)\in\Omega_1 \\ &\leqslant C\exp(\eta x/\varepsilon)e^T \text{ since } e^t\leqslant e^T \\ &\leqslant C\exp(\eta x/\varepsilon)\ \ (x,t)\in\Omega_1. \end{aligned}$$

Since $Lw(x,t)=0$, the j^{th} derivative of $w(x,t)$ can be estimated immediately from the estimate of $w(x,t)$,

$$\left|\frac{\partial^j w(x,t)}{\partial x^j}\right| \leq C\varepsilon^{-j}\exp\left(\frac{\eta x}{\varepsilon}\right),\ 0\leqslant j\leqslant 3.$$

This completes the proof.

We construct a fitted mesh finite difference method to solve time-dependent convection-diffusion problems (3.1.1)-(3.1.3).

3.3 Construction of the FMFDM

Time dicretization

We present the Euler implicit method to discretize problem (3.1.1)-(3.1.3) with uniform step-size $\Delta t = T/K$. The time $[0,T]$ is therefore partitioned as

$$\bar{w}^K = \{t_k = k\Delta t, 0\leqslant k\leqslant K\}. \tag{3.3.1}$$

We discretize problem (3.1.1)-(3.1.3) on $\bar{w}^K$ as follows

$$\varepsilon z_{xx}(x,t_k)+a(x,t_k)z_x(x,t_k)-b(x,t_k)z(x,t_k)-d(x,t_k)\frac{z(x,t_k)-z(x,t_{k-1})}{\Delta t}=f(x,t_k), \tag{3.3.2}$$

subject to

$$z(x,0)=z_0(x),\ \ -1\leq x\leq 1,\ \ z(-1,t_k)=\alpha_1(t),\ \ z(1,t_k)=\alpha_2(t). \tag{3.3.3}$$

Now, (3.3.2) can be written as

$$Lz(x,t_k)=f(x,t_k)-d(x,t)\frac{z(x,t_{k-1})}{\Delta t}. \tag{3.3.4}$$

subject to

$$z(x,0) = z_0(x), \ \ -1 \leq x \leq 1, \ \ z(-1,t_k) = \alpha_1(t), \ \ z(1,t_k) = \alpha_2(t), \tag{3.3.5}$$

where

$$Lz(x,t_k) = \varepsilon z_{xx}(x,t_k) + a(x,t_k)z_x(x,t_k) - \left[b(x,t_k) + \frac{d(x,t_k)}{\Delta t}\right] z(x,t_{k-1}).$$

The local truncation error denoted by e_k at each time level to t_k is given by $e_k = u(x,t_k) - z(x,t_k)$, where $z(x,t_k)$ is the solution of (3.3.4)-(3.3.5).

The local error in the temporal direction is given by [11] as follows

$$\|e_k\|_\infty \leq C(\Delta t)^2, \ \ 1 \leq k \leq K. \tag{3.3.6}$$

The global error is given by [11] as follows

$$\|E_k\|_\infty \leq C\Delta t, \ \ 1 \leq k \leq K. \tag{3.3.7}$$

Space discretization

Let N and K be two positive integers. We consider the following partition in interval $[-1,1]$ which we denote $\bar{\Omega}^N$: $x_0 = -1$, $x_{N/2} = 0$, $x_N = 1$ and let $\bar{Q}^{N,K} = \bar{\Omega}^N \times \bar{w}^K$ be the grid for the (x,t)-variables, and $Q^{N,K} = \bar{Q}^{N,K} \cap Q$. Due to the presence of an interior layer at the point $x_{N/2} = 0$, the transition parameter τ is chosen to be

$$\tau = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\eta}\ln\left(\frac{N}{4}\right)\right\}, \tag{3.3.8}$$

where τ is a positive constant. The space domain is discretized using a piecewise uniform mesh which splits the space domain $[-1,1]$ into the following subintervals $[-1,-\tau]$, $[-\tau,\tau]$ and $[\tau,1]$. These subintervals are subdivided uniformly to contain $N/4$, $N/2$ and $N/4$ mesh elements respectively. Note that the mesh spacing is given by

$$h_j = \begin{cases} 4(1-\tau)/N \text{ if } j = 1,2,\cdots,N/4, 3N/4+1,\cdots,N-1,N, \\ 4\tau/N \text{ if } j = N/4+1, N/4+2\cdots 3N/4. \end{cases} \tag{3.3.9}$$

For the rest of the chapter, we adopt the notation $S(x_j, t_k) := S_j^k$ for any mesh function $S(x_j, t_k)$. We discretize the problem (3.1.1)-(3.1.3) in the following manner:

$$L^{N,K}U_j^k := \begin{cases} \varepsilon \tilde{D}_x U_j^k + \tilde{a}_j^k D_x^- U_j^k - (\tilde{b}_j^k + \frac{d_j^k}{\Delta t})U_j^k = \tilde{f}_j^k - \frac{d_j^k U_j^{k-1}}{\Delta t} & \text{for } j = 0, 1, \cdots, \frac{N}{2} - 1, \\ \varepsilon \tilde{D}_x U_j^k + \tilde{a}_j^k D_x^+ U_j^k - (\tilde{b}_j^k + \frac{d_j^k}{\Delta t})U_j^k = \tilde{f}_j^k - \frac{d_j^k U_j^{k-1}}{\Delta t} & \text{for } j = \frac{N}{2}, \frac{N}{2} + 1, \cdots, N - 1, \end{cases} \tag{3.3.10}$$

with the discrete initial and boundary conditions

$$U_j^0 = u_j^0, \ \ j = 0, 1, \cdots, N, \tag{3.3.11}$$

$$U_0^k = \alpha_1^k \equiv \alpha_1(t_k), \ \ U_N^k = \alpha_2^k \equiv \alpha_2(t_k), \ \ 1 \le k \le K, \tag{3.3.12}$$

where

$$\begin{cases} \tilde{a}_j^k = \frac{a_{j-1}^k + a_j^k}{2} & \text{for } j = 0, 1, \cdots, \frac{N}{2} - 1, \\ \tilde{a}_j^k = \frac{a_j^k + a_{j+1}^k}{2} & \text{for } j = \frac{N}{2}, \frac{N}{2} + 1, \cdots, N - 1, \end{cases}$$

$$\tilde{b}_j^k = \frac{b_{j-1}^k + b_j^k + b_{j+1}^k}{3}, \ \ \tilde{f}_j^k = \frac{f_{j-1}^k + f_j^k + f_{j+1}^k}{3} \text{ for } j = 1, 2, 3, \cdots, N - 1,$$

$$D_x^+ U_j^k = \frac{U_{j+1}^k - U_j^k}{h_{j+1}}, \ \ D_x^- U_j^k = \frac{U_j^k - U_{j-1}^k}{h_j}, \ \tilde{D}_x U_j^k = \frac{2}{h_j + h_{j+1}}(D_x^+ U_j^k - D_x^- U_j^k)$$

and

$$D_t^- U_j^k = \frac{U_j^k - U_j^{k-1}}{\Delta t}.$$

Now (3.3.10) can be written in the form

$$L^{N,K}U_j^k := r^- U_{j-1}^k + r^c U_j^k + r^+ U_{j+1}^k = F_j, \ j = 1, 2, 3 \cdots, N - 1, \tag{3.3.13}$$

where for $j = 1, 2, 3 \cdots, \frac{N}{2} - 1$, we have

$$r_j^- = \frac{2\varepsilon}{h_j(h_j + h_{j+1})} - \frac{\tilde{a}_j^k}{h_j}, \ r_j^c = \frac{\tilde{a}_j^k}{h_j} - \frac{2\varepsilon}{h_j h_{j+1}} - \tilde{b}_j^k - \frac{d_j^k}{\Delta t}, \ r_j^+ = \frac{2\varepsilon}{h_{j+1}(h_j + h_{j+1})}, \tag{3.3.14}$$

for $j = \frac{N}{2}, \frac{N}{2} + 1, \cdots, N - 1$, we have

$$r_j^- = \frac{2\varepsilon}{h_j(h_j + h_{j+1})}, \ r_j^c = -\frac{\tilde{a}_j^k}{h_{j+1}} - \frac{2\varepsilon}{h_j h_{j+1}} - \tilde{b}_j^k - \frac{d_j^k}{\Delta t}, \ r_j^+ = \frac{2\varepsilon}{h_{j+1}(h_j + h_{j+1})} + \frac{\tilde{a}_j^k}{h_{j+1}} \tag{3.3.15}$$

and

$$F_j = \tilde{f}_j^k - \frac{d_j^k U_j^{k-1}}{\Delta t}. \tag{3.3.16}$$

The analysis of the scheme developed above requires the following Lemmas.

Lemma 3.3.1. (Discrete minimum principle) For any mesh function ξ_j^k such that $L^{N,K}\xi_j^k \leq 0$, in $Q^{N,K}$, $\xi_j^0 \geq 0$, $1 \leq j \leq N$, $\xi_0^k \geq 0$, and $\xi_N^k \geq 0$, $1 \leq k \leq K$, we have $\xi_j^k \geq 0$ in $\bar{Q}^{N,K}$.

Proof. Let (p,s) be indices such that $\xi_p^s = \min\limits_{(j,k)} \xi_j^k$, ξ_j^k in $\bar{Q}^{N,K}$ and suppose that $\xi_p^s < 0$. It is clear that $(p,s) \in \{1, 2\cdots, N-1\} \times \{1, 2\cdots, K\}$ otherwise $\xi_p^s \geq 0$. We observe that $\xi_{p+1}^s - \xi_p^s \geq 0$ and $\xi_p^s - \xi_{p-1}^s \leq 0$.
For $p = 1,\ 2,\ \ldots,\ \frac{N}{2} - 1$, we have

$$L^{N,K}\xi_p^s = \varepsilon\tilde{D}_x\xi_p^s + a_p^s D_x^-\xi_p^s - (b_p^s + \frac{d_p^s}{\Delta t})\xi_p^s. \tag{3.3.17}$$

Substituting $D_x^-\xi_p^s$ and $\tilde{D}_x\xi_p^s$ into (3.3.17), we obtain

$$L^{N,K}\xi_p^s = \frac{2\varepsilon}{h_p + h_{p+1}}\left(\frac{\xi_{p+1}^s - \xi_p^s}{h_{p+1}} - \frac{\xi_p^s - \xi_{p-1}^s}{h_p}\right) + a_p^s\left(\frac{\xi_p^s - \xi_{p-1}^s}{h_p}\right) - (b_p^s + \frac{d_p^s}{\Delta t})\xi_p^s > 0.$$

For $p = \frac{N}{2}$, we have

$$L^{N,K}\xi_p^s = -(b_p^s + \frac{d_p^s}{\Delta t})\xi_p^s > 0. \tag{3.3.18}$$

For $p = \frac{N}{2} + 1, \cdots, N+1$, we have

$$L^{N,K}\xi_p^s = \varepsilon\tilde{D}_x\xi_p^s + a_p^s D_x^+\xi_p^s - (b_p^s + \frac{d_p^s}{\Delta t})\xi_p^s. \tag{3.3.19}$$

Substituting $D_x^+\xi_p^s$ and $\tilde{D}_x\xi_p^s$ into (3.3.19), we obtain

$$L^{N,K}\xi_p^s = \frac{2\varepsilon}{h_p + h_{p+1}}\left(\frac{\xi_{p+1}^s - \xi_p^s}{h_{p+1}} - \frac{\xi_p^s - \xi_{p-1}^s}{h_p}\right) + a_p^s\left(\frac{\xi_{p+1}^s - \xi_p^s}{h_{p+1}}\right) - (b_p^s + \frac{d_p^s}{\Delta t})\xi_p^s > 0.$$

Thus $L^{N,K}\xi_p^s > 0,\ 1 \leq p \leq N-1$, which is a contradiction. It follows that $\xi_p^s \geq 0$ and therefore $\xi_j^k \geq 0,\ 0 \leq j \leq N$.

Lemma 3.3.2. (Uniform stability estimate). At time level t_k, if Z_j^k *is any mesh function such* that $Z_0^k = Z_N^k = 0$, then

$$|Z_i^k| \leq \frac{1}{\beta}\max_{1\leq j\leq N-1}|L^{N,K}Z_j^k| \ \ \forall\, 0 \leq i \leq N.$$

Proof. Let us define

$$M = \frac{1}{\beta}\max_{1\leq j\leq N-1}|L^{N,K}Z_j^k|$$

and introduce the two mesh functions $(\Upsilon^{\pm})_i^k$ defined by

$$(\Upsilon^{\pm})_i^k = M \pm Z_i^k.$$

It is clear that $(\Upsilon^{\pm})_0^k \geq 0$ and $(\Upsilon^{\pm})_N^k \geq 0$, for $1 \leq i \leq N-1$. We observe that

$$L^{N,K}(\Upsilon^{\pm})_i^k = -Mb_i^k \pm L^{K,N} Z_i^k \leq 0$$

for $1 \leq i \leq N-1$, because $b_i^k \geq \beta > 0$. By the discrete minimum principle Lemma 3.3.1, we conclude that

$$(\Upsilon^{\pm})_i^k \geq 0, \quad \text{for } \; 0 \leq i \leq N \text{ and } \forall t \in [0,T].$$

With the above continuous and discrete results, we are in a position to provide the ε-uniform convergence result in the next section.

3.4 Convergence analysis

In this section, the convergence of the scheme will be analyzed on a Shishikin mesh.

Theorem 3.4.1. . *Let U_j^k be the numerical solution of problem (3.3.10)-(3.3.12) and denote the solution $z(x_j,t_k)$ of problem (3.3.4)-(3.3.5) at the time level t_k by $z_j^k = z(x_j,t_k)$. Then, we have*

$$\max_{0\leq j\leq N} |U_j^k - z_j^k| \leqslant CN^{-1}\left[\ln\left(\frac{N}{4}\right)\right]^2. \tag{3.4.1}$$

Proof. We prove this Lemma on the interval $[-1,0]$. The proof on $[0,1]$ follows similar steps. The solution U_j^k of the discrete problem (3.3.10)-(3.3.12) can be decomposed into a regular part and a singular part as

$$U_j^k = V_j^k + W_j^k,$$

where V_j^k is the solution of the inhomogeneous problem

$$L^{N,K}V_j^k = f_j^k - \frac{d_j^k \times V_j^{k-1}}{\Delta t}, \; V_j^0 = v_j^0, \; V_0^k = v_0^k$$

and W_j^k is the solution of the homogeneous problem

$$L^{N,K}W_j^k = 0,\ W_j^0 = w_j^0,\ W_0^k = w_0^k,\ W_{N/2}^k = U_{N/2}^k - V_{N/2}^k.$$

The estimate of the smooth component is obtained using the following stability and consistency argument. From the differential equation, we obtain

$$\begin{aligned} L^{N,K}(V_j^k - v_j^k) &= f_j^k - \frac{d_j^k \times V_j^{k-1}}{\Delta t} - L^{N,K}v_j^k \\ &= \varepsilon\left(\frac{d^2}{dx^2} - \tilde{D}_x\right)v_j^k + a_j^k\left(\frac{d}{dx} - D_x^-\right)v_j^k. \end{aligned}$$

Then, by local truncation error estimates (Lemma 4.1 [42]) at each point (x_j, t_k), we obtain

$$|L^{N,K}(V_j^k - v_j^k)| \le \frac{\varepsilon}{3}(x_{j+1} - x_{j-1})\left\|\frac{\partial^3 v_j}{\partial x^3}\right\| + \frac{a_j^k}{2}(x_j - x_{j-1})\left\|\frac{\partial^2 v_j}{\partial x^2}\right\| \text{ for } 1 \le j \leqslant N/2 - 1. \tag{3.4.2}$$

Since $h_j = x_j - x_{j-1} \le 4N^{-1}$ and using the estimates of the derivatives of v_j^k in Lemma 3.2.6 in conjunction with Lemma 7 of [48], we have

$$|L^{N,K}(V_j^k - v_j^k)| \le CN^{-1}.$$

Now, applying Lemma 3.3.2 to the mesh function $(V_j^k - v_j^k)$, we obtain

$$|(V_j^k - v_j^k)| \le CN^{-1} \text{ for } 1 \leqslant j \leqslant N/2 - 1. \tag{3.4.3}$$

Now, let us estimate the local truncation error of the singular component $L^{N,K}(W_j^k - w_j^k)$. In this case the argument depends on whether $\tau = 1/2$ or $\tau = (\varepsilon/\eta)\ln(N/4)$.

The mesh is uniform in the first case $\tau = 1/2 \le (\varepsilon/\eta)\ln(N/4)$. The local truncation error is bounded in the same manner as done above.

$$|L^{N,K}(W_j^k - w_j^k)| \le \frac{\varepsilon}{3}(x_{j+1} - x_{j-1})\left\|\frac{\partial^3 w_j}{\partial x^3}\right\| + \frac{a_j^k}{2}(x_j - x_{j-1})\left\|\frac{\partial^2 w_j}{\partial x^2}\right\| \text{ for } 1 \le j \leqslant N/2 - 1. \tag{3.4.4}$$

Since $h_j = x_j - x_{j-1} \leq 4N^{-1}$ and applying Lemma 3.2.6 in conjunction with Lemma 7 of [48], we obtain

$$|L^{N,K}(W_j^k - w_j^k)| \leq C\varepsilon^{-2}N^{-1}.$$

We know that in the uniform mesh case, $\varepsilon^{-1} \leq (2/\eta)\ln(N/4)$, we obtain

$$|L^{N,K}(W_j^k - w_j^k)| \leq CN^{-1}\left[\ln(N/4)\right]^2.$$

Now, applying Lemma 3.3.2 to the mesh function $(W_j^k - w_j^k)$, we obtain

$$|(W_j^k - w_j^k)| \leq CN^{-1}\left[\ln(N/4)\right]^2 \text{ for } 1 \leq j \leqslant N/2 - 1. \tag{3.4.5}$$

In the second case $\tau = (\varepsilon/\eta)\ln(N/4) \leq 1/2$, the mesh is piecewise uniform. In this case we have two subintervals, namely $[-1, -\tau]$ and $[-\tau, 0]$. Firstly, we compute the error for the singular component in the mesh region $[-1, -\tau]$, i.e., for all $-1 \leq x_j \leq -\tau$. Using the triangle inequality, we get

$$|W_j^k - w_j^k| \leq |W_j^k| + |w_j^k| \tag{3.4.6}$$

With the help of Lemma 3.2.6, we obtain

$$|w_j^k| \leq C\exp(\eta x_j/\varepsilon) \leq C\exp(-\eta\tau/\varepsilon) \leq CN^{-1} \text{ for } 1 \leq j \leqslant N/2 - 1. \tag{3.4.7}$$

To obtain a similar bound on W_j^k an auxiliary mesh function $\tilde{W}_j^k$ is defined analogous to W_j^k except that the coefficient of $a(x,t)$ in the difference operator $L^{N,K}$ is replaced by the lower bound of $a(x,t)$ ([42] pp 72). Then, from (Lemma 7.5 of [42]),

$$|W_j^k| \leq |\tilde{W}_j^k| \text{ for } 0 \leq j \leq N.$$

Furthermore, by using Lemma 3.2.6 which leads us immediately to conclude that

$$|w_j^k| \leq CN^{-1} \text{ for } 1 \leq j \leqslant N/4 - 1. \tag{3.4.8}$$

By using the estimated obtained by (3.4.7) and (3.4.8), we obtain

$$|W_j^k - w_j^k| \leqslant CN^{-1} \text{ for } 1 \leq j \leqslant N/4 - 1. \tag{3.4.9}$$

Now, in the subinterval $[-\tau, 0]$, (3.4.4) leads to

$$|L^{N,K}(W_j^k - w_j^k)| = C\varepsilon^{-2}|x_{j+1} - x_{j-1}| = 4C\varepsilon^{-2}N^{-1}\tau \text{ for } N/4 \leqslant j \leqslant N/2 - 1.$$

Also, $|(W_0^k - w_0^k| = 0$ and $|W_{N/4}^k - w_{N/4}^k| \leqslant |W_{N/4}^k| + |w_{N/4}^k| \leqslant CN^{-1}$ from (3.4.8). Let us introduce the barrier function on $[-\tau, 0]$ as follows

$$\Phi_j^k = (x_j + \tau)C_1\varepsilon^{-2}\tau N^{-1} + C_2N^{-1}.$$

For a suitable choice of C_1 and C_2, the mesh functions

$$(\Psi^{\pm})_j^k = \Phi_j^k \pm (W_j^k - w_j^k)$$

satisfy the inequalities

$$\Psi_{N/4}^k \geq 0, \quad \Psi_{N/2}^k = 0$$

and

$$L^{N,K}\Psi_j^k \leq 0, \quad N/4 + 1 \leqslant j \leqslant N/2 - 1.$$

By applying Lemma 3.3.1 on $[-\tau, 0]$ to the function $(\Psi^{\pm})_j^k$, we obtain

$$\Psi_j^k \geqslant 0, \quad N/4 + 1 \leqslant j \leqslant N/2 - 1.$$

Therefore, we obtain

$$|W_j^k - w_j^k| \leq \Phi_j^k \leq C_1\varepsilon^{-2}\tau^2N^{-1} + C_2N^{-1}.$$

Since $\tau = (\varepsilon/\eta)\ln(N/4)$, we have

$$|W_j^k - w_j^k| \leq CN^{-1}\left[\ln(N/4)\right]^2. \tag{3.4.10}$$

Combining (3.4.9) and (3.4.10), we obtain the following estimate on the singular component of the error over interval $[-1, 0]$

$$|W_j^k - w_j^k| \leq CN^{-1}\left[\ln(N/4)\right]^2, \quad N/4 + 1 \leqslant j \leqslant N/2 - 1. \tag{3.4.11}$$

Noting that

$$U_j^k - z_j^k = (V_j^k - v_j^k) + (W_j^k - w_j^k) \tag{3.4.12}$$

and using the estimates (3.4.3) and (3.4.11), we obtain

$$|U_j^k - z_j^k| \leq CN^{-1}\left[\ln(N/4)\right]^2, \quad 1 \leqslant j \leqslant N/2-1. \tag{3.4.13}$$

A similar analysis on the subinterval $[0,1]$ yields

$$|U_j^k - z_j^k| \leq CN^{-1}\left[\ln(N/4)\right]^2, \quad N/2 \leqslant j \leqslant N. \tag{3.4.14}$$

Combining (3.4.13) and (3.4.14) then gives the required result.

The next theorem provides the main result of this chapter.

Theorem 3.4.2. . *Let u be the exact solution of problem* (3.1.1)-(3.1.2) *and U be its numerical solution obtained via the difference equations* (3.3.10)-(3.3.12). *Then, there exists a constant C independent of the perturbation parameter ε, and of the discretization parameters h_j and Δt such that*

$$\max_{0\leq j\leq N;1\leq k\leq K} \|U_j^k - u_j^k\| \leq C\left[\Delta t + N^{-1}\left[\ln\left(\frac{N}{4}\right)\right]^2\right]. \tag{3.4.15}$$

Proof. The result follows from the triangle inequality

$$\|U_j^k - u_j^k\| \;\; \leq \;\; \|U_j^k - z_j^k\| + \|z_j^k - u_j^k\|,$$

and the combination of (3.3.7) and Theorem 3.4.1.

To improve the accuracy and the rate of convergence of the proposed numerical method, we apply Richardson extrapolation in the next section.

3.5 Richardson extrapolation on the FMFDM

In this section, we use Richardson extrapolation to improve the accuracy of the proposed method. Richardson extrapolation is a procedure where a linear combination of two approximations of a some quantity gives a better approximation of the quantity [49].

We focus our attention on the interval $[-1, 0]$ as before. Results on the interval $[0, 1]$ can be obtained in a similar way. Keeping in mind that there is a transition point at $x = -\tau$, we consider the subintervals $[-1, -\tau] \cup [-\tau, 0]$ separately.

With reference to (3.3.9), to simplify the analysis, for $x_j \in \Omega_N^\tau$, $h_j = x_j - x_{j-1}$ we denote $H = h_j$ for $j = 1, 2, \cdots, N/4$ and $h = h_j$ for $j = N/4 + 1, \cdots, N/2 - 1$.

We consider the mesh Ω_{2N}^τ where τ is given by (3.3.8) where we bisect each mesh subinterval. It is clear that $\Omega_N^\tau \subset \Omega_{2N}^\tau = \{\tilde{x}_j\}$ and $\tilde{x}_j - \tilde{x}_{j-1} = \tilde{h}_j = h_j/2$. We denote the numerical solution on the mesh Ω_{2N}^τ by $\tilde{U}_j^k$. The estimate (3.4.13) can be written as

$$U_j^k - z_j^k = C_1 N^{-1} \ln(N/4)^2 + R_N(x_j), \ \ \forall x_j \in \Omega_N^\tau \tag{3.5.1}$$

and

$$\tilde{U}_j^k - z_j^k = C_2 (2N)^{-1} \ln(N/4)^2 + R_{2N}(\tilde{x}_j), \ \ \forall \tilde{x}_j \in \Omega_{2N}^\tau, \tag{3.5.2}$$

where C_1 and C_2 are some constants and the remainders

$$R_N(x_j) \text{ and } R_{2N}(\tilde{x}_j) \text{ are } \mathcal{O}[N^{-1}(\ln(N/4))^2].$$

Note that we have used the same transition parameter τ when computing both U_j^k and $\tilde{U}_j^k$. This is seen from the factor $\ln(N/4)$.

A combination of the two equations above gives

$$z_j^k - (2\tilde{U}_j^k - U_j^k) = R_N(x_j) - 2R_{2N}(x_j) = \mathcal{O}[N^{-1}(\ln(N/4))^2], \ \ \forall x_j \in \Omega_n^\tau. \tag{3.5.3}$$

We set

$$U_j^{ext,k} = 2\tilde{U}_j^k - U_j^k, \ \ \forall x_j \in \Omega_N^\tau, \tag{3.5.4}$$

as the new approximation of z_j^k obtained after applying Richardson extrapolation. The error after extrapolation $U_j^{ext,k}$ can also be decomposed as in (3.4.12),

$$(U^{ext} - z)_j^k = (V^{ext} - v)_j^k + (W^{ext} - w)_j^k, \tag{3.5.5}$$

where $V_j^{ext,k}$ and $W_j^{ext,k}$ are the regular and singular components of $U_j^{ext,k}$. The local truncation error of the scheme (3.3.10)-(3.3.13) after extrapolation is given by

$$L^{N,K}(U^{ext}-z)_j^k = 2L^{N,K}(\tilde{U}_j^k - z_j^k) - L^{N,K}(U_j^k - z_j^k), \tag{3.5.6}$$

where

$$L^{N,K}(U_j^k - z_j^k) = r^- z_{j-1}^k + r^c z_j^k + r^+ z_{j+1}^k - \varepsilon z_j'' - \tilde{a}_j^k z_j' + \tilde{b}_j^k z_j^k + \frac{d_j^k z_j^k}{\Delta t} \tag{3.5.7}$$

and

$$L^{N,K}(\tilde{U}_j^k - z_j^k) = \tilde{r}^- z_{j-1}^k + \tilde{r}^c z_j^k + \tilde{r}^+ z_{j+1}^k - \varepsilon z_j'' - \tilde{a}_j^k z_j' + \tilde{b}_j^k z_j^k + \frac{d_j^k z_j^k}{\Delta t}. \tag{3.5.8}$$

The quantities of r^-, r^c and r^+ are given in (3.3.14), (3.3.15) and (3.3.16) respectively, but $\tilde{r}^-$, $\tilde{r}^c$ and $\tilde{r}^+$ are obtained by substituting h_j by $\tilde{h}_j$ and h_{j+1} by $\tilde{h}_{j+1}$, in the expressions of r^-, r^c and r^+ respectively. Taking the Taylor series expansion of z_j^k around x_j, we obtain the following approximations for z_{j-1}^k and z_{j+1}^k:

$$z_{j-1}^k = z_j - h_j z_j' + \frac{h_j^2}{2} z_j^2 - \frac{h_j^3}{6} z_j^3 + \frac{h_j^4}{24} z^4(\xi_1, j), \tag{3.5.9}$$

$$z_{j+1}^k = z_j + h_{j+1} z_j' + \frac{h_{j+1}^2}{2} z_j^2 + \frac{h_{j+1}^3}{6} z_j^3 + \frac{h_{j+1}^4}{24} z^4(\xi_2, j), \tag{3.5.10}$$

$$z_{j-1}^k = z_j - \tilde{h}_j z_j' + \frac{\tilde{h}_j^2}{2} z_j^2 - \frac{\tilde{h}_j^3}{6} z_j^3 + \frac{\tilde{h}_j^4}{24} z^4(\tilde{\xi}_1, j), \tag{3.5.11}$$

$$z_{j+1}^k = z_j + \tilde{h}_{j+1} z_j' + \frac{\tilde{h}_{j+1}^2}{2} z_j^2 + \frac{\tilde{h}_{j+1}^3}{6} z_j^3 + \frac{\tilde{h}_{j+1}^4}{24} z^4(\tilde{\xi}_2, j), \tag{3.5.12}$$

where

$$\xi_1, j \in (x_{j-1}, x_j), \;\; \xi_2, j \in (x_j, x_{j+1}), \;\; \tilde{\xi}_1 \in (\frac{x_{j-1}+x_j}{2}, x_j) \text{ and } \tilde{\xi}_2 \in (x_j, \frac{x_j + x_{j+1}}{2}).$$

Substituting (3.5.9) and (3.5.10) into (3.5.7), (3.5.11) and (3.5.12) into (3.5.8), we obtain the following expressions:

$$L^{N,K}(U_j^k - z_j^k) = k_1 z_j + k_2 z_j' + k_3 z_j^2 + k_4 z_j^3 + k_{5,1} z^4(\xi_1, j) + k_{5,2} z^4(\xi_2, j) \tag{3.5.13}$$

and

$$L^{N,K}(\tilde{U}_j^k - z_j^k) = \tilde{k}_1 z_j + \tilde{k}_2 z_j' + \tilde{k}_3 z_j^2 + \tilde{k}_4 z_j^3 + \tilde{k}_4 z_j^4 + \tilde{k}_{5,1} z^4(\tilde{\xi}_1, j) + \tilde{k}_{5,2} z^4(\tilde{\xi}_2, j). \quad (3.5.14)$$

The coefficients (3.5.13) are

$$k_1 = \frac{2\varepsilon}{h_j(h_j+h_{j+1})} - \frac{2\varepsilon}{h_j h_{j+1}} + \frac{2\varepsilon}{h_{j+1}(h_j+h_{j+1})},\ k_2=0,\ k_3 = \frac{\varepsilon h_j}{h_j+h_{j+1}} - \frac{\tilde{a}_j^k h_j}{2} + \frac{\varepsilon h_{j+1}}{h_j+h_{j+1}} - \varepsilon,$$

$$k_4 = \frac{-\varepsilon h_j^2}{3(h_j+h_{j+1})} + \frac{\tilde{a}_j^k h_j^2}{6} + \frac{\varepsilon h_{j+1}^2}{3(h_j+h_{j+1})},\ k_{5,1} = \frac{\varepsilon h_j^3}{12(h_j+h_{j+1})} - \frac{\tilde{a}_j^k h_j^3}{24},\ k_{5,2} = \frac{\varepsilon h_{j+1}^3}{12(h_j+h_{j+1})}.$$

The quantities for $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$, $\tilde{k}_4$, $\tilde{k}_{5,1}$ and $\tilde{k}_{5,2}$ can be obtained by substituting h_j by $\tilde{h}_j$ and h_{j+1} by $\tilde{h}_{j+1}$.

Substituting (3.5.13) and (3.5.14) into (3.5.6), we obtain

$$L^{N,K}(U^{ext} - z)_j^k = T_1 z_j + T_2 z_j'' + T_3 z_j''' + +T_{4,1} z^{(4)}(\xi_1, j) + T_{4,2} z^{(4)}(\xi_2, j), \qquad (3.5.15)$$

where

$$T_1 = \frac{14\varepsilon}{h_j(h_j + h_{j+1})} - \frac{14\varepsilon}{h_j h_{j+1}} + \frac{14\varepsilon}{h_{j+1}(h_j + h_{j+1})},$$

$$T_2 = \frac{\varepsilon h_j}{h_j + h_{j+1}} - \varepsilon + \frac{\varepsilon h_{j+1}}{h_j + h_{j+1}},\quad T_3 = -\frac{\tilde{a}_j^k h_j^2}{12},$$

$$T_{4,1} = -\frac{\varepsilon h_j^3}{24(h_j + h_{j+1})} + \frac{\tilde{a}_j h_j^3}{32} \text{ and } T_{4,2} = -\frac{\varepsilon h_{j+1}^3}{24}.$$

Using the fact that, for all $j = 1, \ldots, N/4$, $H = h_j \leqslant 4N^{-1}$ substituting into (3.5.15) in the subinterval $[-1, -\tau]$, we obtain:

$$L^{N,K}(V^{ext} - v)_j^k = -\frac{\tilde{a}_j^k H^2}{12} v_j''' + \left[\frac{\varepsilon H^2}{48} + \frac{\tilde{a}_j^k H^3}{32}\right] v^{(4)}(\xi_1, j) - \frac{\varepsilon H^3}{24} v^{(4)}(\xi_2, j). \qquad (3.5.16)$$

Now applying the triangle inequality, Lemma 3.2.6 in conjunction with Lemma 7 of [48] to (3.5.16), we obtain:

$$|L^{N,K}(V^{ext} - v)_j^k| \leq CN^{-2}. \tag{3.5.17}$$

To estimate $L^{N,K}(W^{ext} - w)_j^k$, the argument depends on whether $\tau = 1/2$ or $\tau = (\varepsilon/\eta)\ln(N/4)$.

In the first case the mesh is uniform and $(\varepsilon/\eta)\ln(N/4) \geq 1/2$. The estimate of singular component of the local truncation error is obtained as follows:

$$L^{N,K}(W^{ext} - w)_j^k = -\frac{\tilde{a}_j^k h^2}{12} w_j''' + \left[\frac{\varepsilon h^2}{48} + \frac{\tilde{a}_j^k h^3}{32}\right] w^{(4)}(\xi_1, j) - \frac{\varepsilon h^3}{24} w^{(4)}(\xi_2, j). \tag{3.5.18}$$

Now, applying the triangle inequality, Lemma 3.2.6 and using Lemma 7 of [48], we obtain

$$|L^{N,K}(W^{ext} - w)_j^k| \leq CN^{-2}\varepsilon^{-3}\exp\left(2x_j\eta/\varepsilon\right). \tag{3.5.19}$$

Note that $\varepsilon^{-1}\exp\left(2x_j\eta/\varepsilon\right) \leq C$ and $\varepsilon^{-1} \leqslant (2/\eta)\ln(N/4)$, we obtain

$$|L^{N,K}(W^{ext} - w)_j^k| \leqslant CN^{-2}\left[\ln(N/4)\right]^2. \tag{3.5.20}$$

In the second case (viz $\tau = (\varepsilon/\eta)\ln(N/4)$), the mesh is piecewise uniform with the mesh spacing $h = h_j = 4\tau N^{-1}$ for $\forall\, j = N/4+1, \ldots, N/2$ in the subinterval $(-\tau, 0]$. Applying the triangle inequality, Lemma 3.2.6 along with Lemma 7 of [48] to (3.5.18), we obtain

$$|L^{N,K}(W^{ext} - w)_j^k| \leq C_1 N^{-2}\tau^2\varepsilon^{-2}. \tag{3.5.21}$$

Since $\tau = (\varepsilon/\eta)\ln(N/4)$, this gives

$$|L^{N,K}(W^{ext} - w)_j^k| \leqslant CN^{-2}\left[\ln(N/4)\right]^2. \tag{3.5.22}$$

A similar analysis can be performed for $j = N/2+1, \cdots, N-1$.

Using Lemma 3.3.2 in (3.5.17), (3.5.20) and (3.5.22) along with (3.5.5), we obtain the following result:

Theorem 3.5.1. *Let z and U be the solution of (3.3.4)-(3.3.5) and (3.3.10)-(3.3.12), respectively. Then, there exists a constant C, independent of the perturbation parameter ε and the space discretization parameters h_j such that*

$$\max_{0\leq j\leq N} |(U^{ext} - z)_j^k| \leqslant CN^{-2}\left[\ln\left(\frac{N}{4}\right)\right]^2. \tag{3.5.23}$$

Once more, using the triangle inequality and combining (3.3.7) and Theorem 3.5.1, we obtain the error after extrapolation which we state in the following theorem.

Theorem 3.5.2. (Error after extrapolation). *Let u be the exact solution of (3.1.1)-(3.1.2) and U its numerical approximation obtained via the scheme (3.3.10)-(3.3.12). Then, there exists a constant C, independent of the perturbation parameter ε, the time discretization Δt and the space discretization parameters h_j such that*

$$\max_{0\leq j\leq N; 1\leq k\leq K} |(U^{ext} - u)_j^k| \leqslant C\left[\Delta t + N^{-2}\left[\ln\left(\frac{N}{4}\right)\right]^2\right]. \tag{3.5.24}$$

3.6 Numerical examples

This section presents numerical results obtained for test problems. In both examples, we start with $N = 16$ and $\Delta t = 0.1$ and we multiply N by two and divide Δt also by two. The maximum errors and order of convergence are calculated by the exact solution. The solution in the examples has a turning point at $x = 0$ and $x = 0.5$, which gives rise to an interior layer.

Example 3.6.1. We consider problem (3.1.1)-(3.1.2) for

$$a(x,t) = 2x(1+t^2),\ b(x,t) = (3+xt),\ d(x,t) = (1+xt),\ T = 1$$

and the functions $f(x,t)$ and $u_0(x)$ are such that the exact solution is given by

$$u(x,t) = \varepsilon e^{-t/\varepsilon}\left[\operatorname{erf}\left(\frac{x}{\sqrt{\varepsilon}}\right) + 2x\,\frac{e^{x^2/\varepsilon}}{\sqrt{\varepsilon\pi}}\right] - \varepsilon^{3/2}e^{-xt}.$$

Example 3.6.2. Here we consider the following problem (3.1.1) and $0 \leq x \leq 1$ for

$$a(x,t) = (2x-1)(1+t),\ b(x,t) = (1+xt),\ d(x,t) = e^{-xt},\ T = 1$$

and the functions $f(x,t)$ and $u_0(x)$ are such that the exact solution is given by

$$u(x,t) = \varepsilon e^{-t/\varepsilon} \tanh\left(\frac{0.5-x}{\varepsilon}\right) - \varepsilon^{3/2} e^{-(1-2x)t}.$$

Maximum errors at all mesh points are determined by using the formula

$$E^{\varepsilon,N,K} = \max_{0\le j\le N; 0\le k\le K} |u_{j,k}^{\varepsilon,N,K} - U_{j,k}^{\varepsilon,N,K}|, \text{ and we compute } E_{\varepsilon,N,K} = \max_{0\le\varepsilon\le1} E_{\varepsilon,N,K},$$

where $u_{j,k}^{\varepsilon,N,K}$ denotes the exact solution, and $U_{j,k}^{\varepsilon,N,K}$ denotes the numerical solution which is obtained by a constant time step Δt using N mesh intervals in the entire domain $\Omega = [-1,1]$ or $\Omega = [0,1]$. In addition, the numerical rate of uniform is computed as

$$r_l \equiv r_{\varepsilon,l} = \log_2\left(E^{\varepsilon,N_l,K_l}/E^{\varepsilon,2N_l,2K_l}\right).$$

After extrapolation the maximum errors at all mesh points and the numerical rates of convergence are calculated as follows:

$$E_{\varepsilon,N,K}^{ext} = \max_{0\le j\le N; 0\le k\le K} |U_{j,k}^{ext} - u_{j,k}^{\varepsilon,N,K}|, \text{ and } R_{N,K} \equiv R_{\varepsilon,N,K} \equiv \log_2(E_{\varepsilon,N_l,K_l}^{ext}/E_{\varepsilon,2N_l,2K_l}^{ext}).$$

Table 3.1: Results for Example 3.6.1 Maximum errors before extrapolation

ε	$N=16$ $K=10$	$N=32$ $K=20$	$N=64$ $K=40$	$N=128$ $K=80$	$N=256$ $K=160$	$N=512$ $K=320$
10^{-4}	7.55E-02	4.35E-02	2.33E-02	1.20E-02	6.59E-03	3.63E-03
10^{-5}	7.55E-02	4.35E-02	2.33E-02	1.20E-02	6.13E-03	3.11E-03
10^{-6}	7.55E-02	4.35E-02	2.33E-02	1.20E-02	6.13E-03	3.10E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	7.55E-02	4.35E-02	2.33E-02	1.20E-02	6.13E-03	3.10E-03

Table 3.2: Results for Example 3.6.1 Maximum errors after extrapolation

ε	$N=16$ $K=10$	$N=32$ $K=40$	$N=64$ $K=160$	$N=128$ $K=640$	$N=256$ $K=2560$	$N=512$ $K=10240$
10^{-4}	8.61E-02	2.58E-02	6.80E-03	3.60E-03	3.60E-03	3.60E-03
10^{-5}	8.61E-02	2.59E-02	6.81E-03	1.73E-03	7.88E-04	7.88E-04
10^{-6}	8.62E-02	2.59E-02	6.82E-03	1.73E-03	4.33E-04	1.71E-04
10^{-7}	8.62E-02	2.59E-02	6.82E-03	1.73E-03	4.34E-04	1.08E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	8.62E-02	2.59E-02	6.82E-03	1.73E-03	4.34E-04	1.08E-04

Table 3.3: Results for Example 3.6.1 Rates of convergence before extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	0.80	0.90	0.95	0.87	0.86
10^{-5}	0.80	0.90	0.95	0.97	0.98
10^{-6}	0.80	0.90	0.95	0.97	0.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	0.80	0.90	0.95	0.97	0.99

Table 3.4: Results for Example 3.6.1 Rates of convergence after extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-5}	1.74	1.92	1.98	1.13	-0.00
10^{-6}	1.74	1.92	1.98	2.00	1.34
10^{-7}	1.74	1.92	1.98	2.00	2.00
10^{-8}	1.74	1.92	1.98	2.00	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	1.74	1.92	1.98	2.00	2.00

Table 3.5: Results for Example 3.6.2 Maximum errors before extrapolation

ε	$N=16$ $K=10$	$N=32$ $K=20$	$N=64$ $K=40$	$N=128$ $K=80$	$N=256$ $K=160$	$N=512$ $K=320$
10^{-4}	8.85E-02	4.76E-02	2.44E-02	1.23E-02	6.21E-03	3.12E-03
10^{-5}	8.85E-02	4.76E-02	2.44E-02	1.23E-02	6.21E-03	3.12E-03
10^{-6}	8.85E-02	4.76E-02	2.44E-02	1.23E-02	6.21E-03	3.12E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	8.85E-02	4.76E-02	2.44E-02	1.23E-02	6.21E-03	3.12E-03

Table 3.6: Results for Example 3.6.2 Maximum errors after extrapolation

ε	$N=16$ $K=10$	$N=32$ $K=40$	$N=64$ $K=160$	$N=128$ $K=640$	$N=256$ $K=2560$	$N=512$ $K=10240$
10^{-4}	1.07E-01	2.79E-02	8.45E-03	5.76E-03	5.76E-03	5.76E-03
10^{-5}	1.06E-01	2.62E-02	6.78E-03	2.08E-03	1.25E-03	1.25E-03
10^{-6}	1.05E-01	2.58E-02	6.42E-03	1.72E-03	5.08E-04	2.71E-04
10^{-7}	1.05E-01	2.58E-02	6.34E-03	1.64E-03	4.31E-04	1.25E-04
10^{-8}	1.05E-01	2.58E-02	6.32E-03	1.62E-03	4.09E-04	1.03E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	1.05E-01	2.57E-02	6.32E-03	1.62E-03	4.09E-04	1.03E-04

Table 3.7: Results for Example 3.6.2 Rates of convergence before extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	0.88	0.96	0.98	0.99	0.67
10^{-5}	0.89	0.96	0.98	0.99	1.00
10^{-6}	0.89	0.96	0.98	0.99	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	0.89	0.97	0.98	0.99	1.00

Table 3.8: Results for Example 3.6.2 Rates of convergence after extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-5}	2.01	1.95	1.70	0.73	-0.00
10^{-6}	2.03	2.01	1.90	1.75	0.91
10^{-7}	2.03	2.02	1.95	1.93	1.79
10^{-8}	2.03	2.03	1.96	1.97	1.94
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	2.03	2.03	1.97	1.98	1.99

3.7 Discussion

In this chapter, we proposed a fitted mesh finite difference method (FMFDM) for a class of time-dependent singularly perturbed problems whose solution exhibits an interior layer. After establishing bounds on the solution and its derivatives, we applied the classical Euler method to discretize the time variable. This results in a system of interior layer boundary value problems (one at each time level). We constructed a Fitted Mesh Finite Difference Method (FMFDM) to solve the system above. The FMFDM uses an upwind scheme on a piecewise uniform mesh, fine in the (interior) layer and coarse elsewhere. Using bounds on the solution and its derivatives, we proved that the method is uniformly convergent relative to the perturbation parameter ε and the step-size.

In order to support the above conclusions based on a theoretical analysis, we performed numerical investigations on two examples. In each example, we computed the maximum pointwise errors and the corresponding rates of convergence for various values of N and K. The results shown in tables 3.1, 3.3, 3.5 and 3.7,confirmed that the method was uniformly convergent.

Furthermore, we investigated the effect of Richardson extrapolation on the FMFDM in order to improve both its accuracy and order of convergence. Numerical results are displayed in tables 3.2, 3.4, 3.6 and 3.8 for the same values of N and K considered above for comparison purposes.

Chapter 4

Time dependent power interior layer convection-diffusion problems

In this chapter, we consider a class of two-point boundary value singularly perturbed convection-diffusion problems whose solution has an interior layer due to the presence of a turning point. The perturbation parameter is embedded in a quadratic function. We first derive bounds on the solution and its derivatives. Then we design a fitted mesh finite difference method (FMFDM) applied on both Bakhvalov and Shishkin-type meshes for the solution of problem which is $\varepsilon-$uniform convergent of order one. In order to improve the accuracy and the rate of convergence for both methods, we apply Richardson extrapolation.

4.1 Introduction

In one dimension, a typical singularly perturbed problem consists of determining the solution y to the equation

$$\varepsilon y'' + a(x)y' - b(x)y = f(x), \quad x \in [\theta_1, \theta_2], \tag{4.1.1}$$

subject to

$$y(\theta_1) = y_1 \text{ and } y(\theta_2) = y_2, \tag{4.1.2}$$

where ε is a small parameter lying in $(0,1]$, $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions, y_1 and y_2 are given constants, and also that $a(x) \geqslant \eta > 0$, $b(x) \geqslant b_0 > 0$, $x \in [\theta_1, \theta_2]$. Such problems often arise in chemistry, biology, continuum mechanics, aerodynamics, semi-conductor theory, electromagnetic fields, financial mathematics, reaction-diffusion processes [19, 50, 73].

The perturbation parameter $\varepsilon << 1$ in (4.1.1) dictates large gradients of the solution in small parts of the domain called layer regions. These layers may be situated at the boundary of the domain or in its interior depending upon the nature of the coefficient of the convection and the reaction terms. The solution presents a boundary layer at the right end or the left end of the domain if $a(x) < 0$ or $a(x) > 0$, for all $x \in [\theta_1, \theta_2]$ respectively. With such coefficient functions $a(x)$, problems (4.1.1)-(4.1.2) are said to be an non-turning point problems. In recent years, many researchers have successfully developed numerical schemes for such problems [25, 26, 28, 35, 45, 46, 57, 69].

The zeros of $a(x)$, if they exist and with $a(\theta_1)a(\theta_2) \neq 0$, are called the turning points of the problem. Turning points may give rise to boundary and/or interior layers. For more information about the type of layers, interested readers may refer to [5]. Numerous numerical schemes for turning point singularly perturbed problems are available in the literature. Examples of works where turning points give rise to boundary and/or interior layers include [9, 12, 18, 19, 24, 27, 33, 50, 58, 64, 74]. Note that interior layers are also present in the solution to problems (4.1.1)-(4.1.2) if the coefficients are not smooth or if the data function $f(x)$ is discontinuous [16].

The literature presented above proves that problem (4.1.1)-(4.1.2) is well-studied. Parallel to a constant perturbation parameter ε, it is important to study problems when the perturbation parameter is a function of ε and x. Such problems arise in a modelling process with variable viscosity [38].

In this chapter we study the problem in which $l(x, \varepsilon) = \varepsilon + x^2$. More precisely, we seek to determine the solution to the problem whose reduced equation ($\varepsilon = 0$) has the same order

$$Lu :\equiv (\varepsilon + x^2)u'' + a(x)u' - b(x)u = f(x), x \in \Omega = (-1, 1), \tag{4.1.3}$$

$$u(-1) = \alpha,\ u(1) = \beta, \tag{4.1.4}$$

where α and β are given real constants. We assume that the functions $a(x), b(x)$ and $f(x)$ of (4.1.3) are sufficiently smooth with $b(x) \geq b_0 > 0$ in $\bar{\Omega}$ along with the conditions

$$\left.\begin{array}{lll} (i) & a(0) = 0 & a'(0) > 0, \\ (ii) & b(x) \geq b_0 > 0 & x \in [-1, 1], \\ (iii) & |a'(x)| \geq a'(0)/2, & x \in [-1, 1]. \end{array}\right\} \tag{4.1.5}$$

The above hypotheses guarantee that the solution of problem (4.1.3)-(4.1.4) possesses a unique solution exhibiting an interior layer at the point $x = 0$ [19]. In fact, condition (i) guarantees the existence of the turning point, condition (ii) ensures that the problem satisfies a minimum principle and condition (iii) implies that zero is the only turning point in $[-1, 1]$.

In Mbayi *et al.*[40], proposed a numerical method to problem (4.1.1)-(4.1.2) with a turning point whose solution exhibits an interior layer. They used a fitted mesh finite difference method (FMFDM).

In the present chapter, we propose and analyse a fitted mesh finite mesh difference method (FMFDM) to problem (4.1.3)-(4.1.4) as applied on two different meshes, namely Bakhvalov-type and Shishkin-type meshes. The method we propose is an extension of our recent work [40].

The rest of the chapter is organised as follows. In Section 2, we present a set of bounds on the solution $u(x)$ and its derivatives . In Section 3, we design a FMFDM applied on two different meshes, namely a piecewise mesh (Shishkin-type) and graded mesh (Bakhvalov-type). Section 4 is dedicated to the analysis of the scheme. We prove that the scheme is almost first order, uniformly convergent with respect to the perturbation parameter ε for Shishkin-type and Bakhvalov-type meshes. To improve the accuracy of the proposed (FMFDM), we apply Richardson extrapolation in Section 5 to obtain a second order method, uniformly convergent with respect to the perturbation parameter ε, convergence up to a logarithmic factor for a Shishkin-type mesh and also second order for a Bakhvalov-type mesh. To see how the proposed method works in practice and to confirm our theoretical results, we present numerical experiments in Section 6. We conclude this chapter in Section 7.

In the rest of this chapter, C denotes a generic constant which may assume different values in different inequalities but will always be independent of ε and of the mesh parameter.

4.2 Bounds on the solution and its derivatives

Bounds for the solution to problem (4.1.3)-(4.1.4) and its derivatives are given in this section. We shall note the sub-intervals of $[-1, 1]$ as $\Omega_l = [-1, -\tau]$, $\Omega_c = [-\tau, \tau]$ and $\Omega_r = [\tau, 1]$, where $0 < \tau \leq 1/2$.

We first prove that the operator L as defined in (4.1.3) admits the following continuous minimum principle and then we state a stability estimate for the solution of problem (4.1.3)-(4.1.4).

Lemma 4.2.1. (Minimum principle). Assume that ξ is any sufficiently smooth function satisfying $\xi(\pm 1) \geq 0$ and $L\xi(x) \leq 0$, $\forall x \in \Omega$, implies that $\xi(x) \geq 0$, $\forall x \in \bar{\Omega}$.

Proof. The proof is by contradiction. Let x^* be such that $\xi(x^*) = \min_{-1 \leq x \leq 1} \xi(x)$ and assume that $\xi(x^*) < 0$. Clearly, $x^* \notin \Omega$ and therefore $\xi'(x^*) = 0$ and $\xi''(x^*) \geq 0$. Consequently,

$$L\xi(x^*) := (\varepsilon + x^{*2})\xi''(x^*) + a(x^*)\xi'(x^*) - b(x^*)\xi(x^*) > 0,$$

which is a contradiction. It follows that $\xi(x^*) \geq 0$ and thus $\xi(x) \geq 0$, $\forall x \in \bar{\Omega}$.

The minimum principle implies the existence and uniqueness of the solution. We use this principle to prove the next results which state that the solution depends continuously on the data.

Lemma 4.2.2. (Stability estimate). If $u(x)$ is the solution of (4.1.3)-(4.1.4), then we have

$$||u(x)|| \leqslant [\max\{||\alpha||_\infty, ||\beta||_\infty\}] + \frac{1}{b_0}||f||_\infty, \forall x \in \bar{\Omega}.$$

Proof. See Lemma 2.2.2 in Chapter 2.

Lemma 4.2.3. [5]. If $u(x)$ is the solution of (4.1.3)-(4.1.4) and $a(x), b(x)$ and $f(x) \in C^k(\bar{\Omega})$, then there exists a positive constant C such that

$$|u^{(j)}(x)| \leq C, \ \forall x \in \Omega_l \text{ or } \Omega_r, j = 1, 2, \cdots, k,$$

for sufficient small $0 < \tau \leq 1/2$.

Proof. See theorem 2.4 of [5].

Lemma 4.2.4. [39] (Inverse monotonicity.) Let $d(x) = x^2$ and $q(x) = f(x) - a(x)u'(x) + b(x)u(x)$ be continuous in $[-1, 1]$ and $[-1, 1] \times \Re^2$, respectively. Then the operator $T = (L, \Gamma)$ for the functions from $C^2(-1, 1) \cup C[-1, 1]$ is inverse-monotone if one of the following conditions imposed of $q(x)$ is satisfied:

- $q(x, u, u')$ is strictly increasing in u, i.e. $q(x, u_1, z) < q(x, u_2, z)$ if $u_1 < u_2$;
- $q(x, u, u')$ is weakly increasing in u, and there exists a constant $C > 0$ such that

$$|q(x, u, z_1) - q(x, u, z_2)| \leqslant C|z_1 - z_2|.$$

Proof. For the proof of the Lemma, readers may refer to ([39], pp 47).

We adapt the next according to [39]. As we know that the solution to problem (4.1.3)-(4.1.4) exhibits an interior layer at the point $x_{N/2} = 0$. Therefore, the derivatives of $u(x)$ are estimated in the vicinity $x_{N/2} = 0$ by polynomial functions according to the sign of the coefficient convection term at the point $x_0^\star$. Then, we have two cases

$$a = \begin{cases} a(x_0^\star) \leqslant 0, & x_0^\star \in [-\tau, 0] \text{ and} \\ a(x_0^\star) > 0, & x_0^\star \in (0, \tau]. \end{cases} \tag{4.2.1}$$

Lemma 4.2.5. Let $u(x)$ be the solution of (4.1.3)-(4.1.4). Then assuming that $a = a(x_0^\star) > 0,$ for $0 < x \leqslant \tau$ and $j = 1, 2, 3, 4,$ we have the following bounds

$$|u^{(j)}(x)| \leqslant C \begin{cases} 1 + (\varepsilon + x^2)^{1-a-j}, & 0 < a < 1, \\ 1 + (\varepsilon + x^2)^{-j}, & a = 1, \\ 1 + \varepsilon^{a-1}(\varepsilon + x^2)^{1-a-j}, & a > 1, \end{cases} \tag{4.2.2}$$

and $a = a(x_0^\star) \leqslant 0,$ for $-\tau \leqslant x \leqslant 0$. Let p be an integer such that $a + p = 0$ and $a + p - 1 < 0$, then for $j = 1, 2, 3, 4$, we have the following bounds

$$|u^{(j)}(x)| \leqslant C \begin{cases} 1, & a < 0,\ j \leqslant p, \\ 1 + (\varepsilon + x^2)^{1-j-p} \arctan(x/\sqrt{\varepsilon}), & a + p = 0,\ j > p, \\ 1 + (\varepsilon + x^2)^{-a-j}, & a + p > 0,\ j > p. \end{cases} \tag{4.2.3}$$

Proof. We prove this Lemma by following the ideas of ([39], from pp. 107-110). Application of the inverse-monotone pair $T = (L, \Gamma)$ (see pp 49) implies that

$$|u(x)| \leqslant C,\ -1 \leqslant x \leqslant -1. \tag{4.2.4}$$

Combining (4.1.3)-(4.1.4) and (4.2.4), we obtain

$$|u^{(j)}(x)| \leqslant C \begin{cases} 1, & -\tau < x_0 \leqslant x \leqslant 0, \\ \varepsilon^{-j}, & -\tau \leqslant x \leqslant x_0,\ j = 1, 2, 3, 4, \\ 1, & 0 < x_0 \leqslant x \leqslant \tau, \\ \varepsilon^{-j}, & 0 \leqslant x \leqslant x_0,\ j = 1, 2, 3, 4, \end{cases} \tag{4.2.5}$$

for arbitrary $x_0 > 0$, independent of ε and x.

case 1 : $\mathbf{a > 0}$ for $0 < x \leqslant \tau$. In this case, the derivatives of $u(x)$ are estimated according to the value of $a : 0 < a < 1,\ a = 1$ and $a > 1$. Solving (4.1.3) for $u''(x)$, we obtain

$$u''(x) = \frac{f(x) + b(x)u(x)}{(\varepsilon + x^2)} - \frac{a(x)u'(x)}{(\varepsilon + x^2)}. \tag{4.2.6}$$

One can determine $u'(x)$ from (4.2.6) as follows

$$u'(x) = \int_0^x \frac{f(s) + b(s)u(s)}{\varepsilon + s^2}\, ds - \int_0^x \frac{a(s)}{\varepsilon + s^2} u'(s)\, ds. \tag{4.2.7}$$

$u'(x)$ can be expressed as follows

$$u'(x) = u'(0)\left[\frac{\varepsilon}{\varepsilon + x^2}\right]^a \exp[-g_1(x)] + g_2(x), \tag{4.2.8}$$

where

$$g_1(x) = \int_0^x \frac{a(s)}{\varepsilon + s^2}\, ds = \frac{a(x)}{\sqrt{\varepsilon}} \arctan(x/\sqrt{\varepsilon}) - \int_0^x \frac{a'(x)}{\sqrt{\varepsilon}} \arctan(s/\sqrt{\varepsilon})\, ds, \tag{4.2.9}$$

with $a(0) = 0$, and

$$g_2(x) = (\varepsilon + x^2)^{-a} \int_0^x [f(s) + b(s)u(s)](\varepsilon + s^2)^{a-1} \exp[g_1(s) - g_1(x)]\, ds. \tag{4.2.10}$$

We have $|g_1(x)| \leqslant C$ from (4.2.4), therefore we obtain

$$|g_2(x)| \leqslant C(\varepsilon + x^2)^{-a} \int_0^x (\varepsilon + s^2)^{a-1}\, ds \leqslant C.$$

Applying the triangle inequality in (4.2.8) we obtain

$$|u'(x)| \leqslant C\left[1 + |u'(0)|(\varepsilon/(\varepsilon + x^2))^a\right]. \tag{4.2.11}$$

Considering $0 < a < 1$, there is a point x_0 in the interval $(0, \tau)$ such that $|u'(x_0)| \leqslant C$. Thus we have

$$|u'(0)| \left(\frac{\varepsilon}{\varepsilon + x_0^2}\right)^a \leqslant C.$$

This inequality gives

$$|u'(0)| \leqslant C\left(\frac{\varepsilon+x_0^2}{\varepsilon}\right)^a \leqslant C(\varepsilon+x_0^2)^a\varepsilon^{-a} \leqslant C\varepsilon^{-a}.$$

Using the value of $|u'(0)|$ into (4.2.11), we obtain

$$|u'(x)| \leqslant C\left[1+(\varepsilon+x^2)^{-a}\right],\ 0<a<1.$$

Before getting $u''(x)$ let us make first $a(x)u'(x)/(\varepsilon+x^2)$ the subject of the formula from (4.2.6), then we have

$$\frac{a(x)u'(x)}{\varepsilon+x^2} = \frac{f(x)+b(x)u(x)}{(\varepsilon+x^2)} - u''(x). \tag{4.2.12}$$

Differentiating (4.1.3), solving the resulting equation for $u'''(x)$ and taking into account (4.2.12), we obtain

$$u'''(x) = \frac{f'(x)+b'(x)u(x)+b(x)u'(x)-a'(x)u'(x)}{(\varepsilon+x^2)} - \frac{a'(x)+2s}{(\varepsilon+x^2)}u''(x). \tag{4.2.13}$$

From the above equation, we obtain $u''(x)$

$$u''(x) = \int_0^x \frac{f'(s)+b'(s)u(s)+b(s)u'(s)-a'(s)u'(s)}{\varepsilon+s^2}\,ds - \int_0^x \frac{a(s)+2s}{\varepsilon+s^2}u''(s)\,ds. \tag{4.2.14}$$

Below is the second derivative of $u(x)$ proposed by [37]

$$u''(x) = u''(0)\left[\frac{\varepsilon}{\varepsilon+x^2}\right]^{a+1}\exp[-g_3(x)] + g_4(x), \tag{4.2.15}$$

where

$$g_3(x) = \int_0^x \frac{a(s)+2s}{\varepsilon+s^2}\,ds = \frac{a(x)+2s}{\sqrt{\varepsilon}}\arctan(x/\sqrt{\varepsilon}) - \int_0^x \frac{a'(s)+2}{\sqrt{\varepsilon}}\arctan(s/\sqrt{\varepsilon})\,ds, \tag{4.2.16}$$

with $a(0)=0$, and

$$g_4(x) = (\varepsilon+x^2)^{-a-1}\int_0^x [f'(s)+b'(s)u(s)+[b(s)-a'(s)]u'(s)](\varepsilon+s^2)^a\exp[g_3(s)-g_3(x)]\,ds. \tag{4.2.17}$$

Since $|g_3(x)| \leqslant C$ and $|u(x)| \leqslant C$, we find that

$$|g_4(x)| \leqslant C(\varepsilon+x^2)^{-a-1}\int_0^x [1+u'(s)](\varepsilon+s^2)^a\,ds \leqslant C[1+(\varepsilon+x^2)^{-a}]. \tag{4.2.18}$$

In addition, from (4.1.3) we obtain

$$u''(0) \leqslant C\varepsilon^{-1}[1 + u'(0)] \leqslant C\varepsilon^{-a-1}.$$

Using the estimate of $u''(0)$ and $g_4(x)$ into (4.2.15), we obtain

$$u''(x) \leqslant C\varepsilon^{-1-a}\varepsilon^{a+1}(\varepsilon + x^2)^{-a-1} + C\left[1 + (\varepsilon + x^2)^{-a}\right].$$

This inequality gives

$$|u''(x)| \leqslant C\left[1 + (\varepsilon + x^2)^{-a-1}\right].$$

Differentiating equations (4.1.3)-(4.1.4) and taking into account (4.2.4), we obtain the following result

$$|u^{(j)}(x)| \leqslant C\left[1 + (\varepsilon + x^2)^{-a+1-j}\right], \; 0 < a < 1.$$

Let us now consider the case when $a = 1$. On integrating (4.2.8) from 0 to τ, we obtain

$$\begin{aligned} u(\tau) - u(0) = u'(0)\varepsilon^{1/2}\{&\arctan(\tau/\sqrt{\varepsilon})\exp(-g_1(\tau)) \\ &+ \int_0^\tau a(x)(\varepsilon + x^2)^{-1}\arctan(x/\sqrt{\varepsilon})\exp(-g_1(x))\,dx\} + \int_0^\tau g_2(x)\,dx. \end{aligned} \quad (4.2.19)$$

Further

$$|\arctan(\tau/\sqrt{\varepsilon})\exp[-g_1(\tau)] + \int_0^\tau a(x)(\varepsilon + x^2)^{-1}\exp[-g_1(x)]\arctan(x/\sqrt{\varepsilon})\,dx| \leqslant C.$$

Using the triangle inequality in (4.2.19) and taking account the above inequality, we obtain

$$|u'(0)|\,\varepsilon^{1/2} \leqslant C.$$

This inequality yields $|u'(0)| \leqslant C\varepsilon^{-1/2}$. Then from (4.2.11) we obtain

$$|u'(x)| \leqslant C[1 + \varepsilon^{1/2}(\varepsilon + x^2)^{-1}] \leqslant C[1 + (\varepsilon + x^2)^{-1}].$$

Now consider $u''(x)$ for $a = 1$. In this case (4.2.15) gives

$$u''(x) = u''(0)\left[\frac{\varepsilon}{\varepsilon + x^2}\right]^2 \exp[-g_3(x)] + g_4(x). \quad (4.2.20)$$

From (4.2.18), $g_4(x)$ is defined as follows

$$|g_4(x)| \leqslant C(\varepsilon + x)^{-2} \int_0^x [1 + u'(s)](\varepsilon + s^2)\, ds \leqslant C[1 + \varepsilon^{1/2}(\varepsilon + x^2)^{-1}]. \qquad (4.2.21)$$

Further, by (4.1.3) we obtain

$$u''(0) \leqslant C\varepsilon^{-1}[1 + u'(0)] \leqslant C\varepsilon^{-3/2}.$$

Using the estimate of $u''(0)$ and $g_4(x)$ into (4.2.20), we obtain

$$u''(x) \leqslant C[1 + \varepsilon^{1/2}(\varepsilon + x^2)^{-2}] \leqslant C[1 + (\varepsilon + x^2)^{-2}].$$

By differentiating (4.1.3) and with the help of (4.2.5), we come to the following result

$$|u^{(j)}(x)| \leqslant C[1 + (\varepsilon + x^2)^{-j}], \; a = 1.$$

The case when $a > 1$ is easily proved by using $u'(0) \leqslant C\varepsilon^{-1}$ obtained from (4.2.5) for $0 \leqslant x \leqslant x_0$. Substituting into (4.2.11), leads to

$$|u'(x)| \leqslant C[1 + \varepsilon^{a-1}(\varepsilon + x^2)^{-a}], \;\; 0 < x \leqslant \tau.$$

Now consider $u''(x)$ for $a > 1$. Substituting $u''(0) \leqslant C\varepsilon^{-2}$ obtained from (4.2.5) for $0 \leqslant x \leqslant x_0$ and (4.2.18) into (4.2.15), we obtain

$$|u''(x)| \leqslant C[1 + \varepsilon^{a-1}(\varepsilon + x^2)^{-a-1}], \;\; 0 < x \leqslant \tau.$$

Differentiating (4.1.3) and taking into account (4.2.5), we easily obtain

$$|u^{(j)}(x)| \leqslant C[1 + \varepsilon^{a-1}(\varepsilon + x^2)^{1-a-j}].$$

This completes the proof of the estimate (4.2.2) for $0 < x \leqslant \tau$.

case 2 : $\mathbf{a \leqslant 0}$ for $-\tau \leqslant x \leqslant 0$. In this case, $u'(x)$ is expressed by the following formula

$$u'(x) = u'(x_0)\exp[\psi(x_0, x)] + \int_{x_0}^x \frac{f(s) + b(s)u(s)}{\varepsilon + s^2} \exp[\psi(x_0, x)]\, ds, \qquad (4.2.22)$$

where

$$\psi(s,x) = -\int_s^x \frac{a(\kappa)}{\varepsilon+\kappa^2}\,d\kappa.$$

If $a(0) = 0$ then $\psi(s,x) \leqslant C,\ -\tau \leqslant s,\ x \leqslant 0$. Using the triangle inequality in (4.2.22) and choosing a point $x_0 \in [-\tau/2, 0]$ such that $u'(x_0) \leqslant C$, we obtain

$$|u'(x)| \leqslant C[1+\varepsilon^{-1/2}\arctan(x/\sqrt{\varepsilon})] \leqslant C[1+\arctan(x/\sqrt{\varepsilon})],\ \ a(0) = 0,\ j = 1 \text{ since } p = 0.$$

Now determine $u''(x)$ with $p = 0$ for $j = 2$. On differentiating (4.1.3) and solving the resulting equation for $u''(x)$, we obtain

$$u''(x) = u''(x_0)\exp[\psi(x_0,x)] + (\varepsilon+x^2)^{-p-1}\int_{x_0}^x \frac{F(s)}{\varepsilon+s^2}(\varepsilon+s^2)^{p+1}\exp[\psi(s,x)]\,ds, \quad (4.2.23)$$

where

$$\psi(s,x) = -\int_s^x \frac{a(\kappa)}{\varepsilon+\kappa^2}\,d\kappa$$

and

$$F(s) = f(s) + b'(s)u(s) + [b(s) - a'(s)]u'(s).$$

Substituting $\psi(s,x) \leqslant C$ and $u''(x_0) \leqslant C$ into (4.2.23), we obtain

$$|u''(x)| \leqslant C + C(\varepsilon+x^2)^{-p-1}\int_{x_0}^x [1+u'(s)](\varepsilon+s^2)^p\,ds \leqslant C[1+(\varepsilon+x^2)^{-p-1}\arctan(x/\sqrt{\varepsilon})].$$

From (4.1.3)-(4.1.4) with $p = 0$, for $j > 1$, we obtain

$$|u^{(j)}(x)| \leqslant C[1+(\varepsilon+x^2)^{-1-j-p}\arctan(x/\sqrt{\varepsilon})],\ \ a+p = 0,\ j > p.$$

Let $a(0) < 0$. In this case $p > 1$. Then there exists a constant $x_0 > 0$ such that $a(x) < 0$ for $-\tau \leqslant x \leqslant x_0$. Therefore, we have

$$\psi(s,x) \leqslant -x_0\ln[(\varepsilon+s^2)/(\varepsilon+x^2)],\ -\tau \leqslant x \leqslant s \leqslant x_0.$$

Taking exponentials on both sides of the above inequality, we obtain

$$\exp(\psi(s,x)) \leqslant [(\varepsilon+x^2)/(\varepsilon+s^2)]^{-x_0},\ -\tau \leqslant x \leqslant s \leqslant x_0.$$

Substituting this estimate in (4.2.22) with $x = s$ and taking into account (4.2.5), we obtain

$$|u'(x)| \leqslant C, \ -\tau \leqslant x \leqslant x_0, \ a(0) < 0.$$

Differentiating (4.1.3) and taking into account (4.2.5), we obtain

$$|u^{(j)}(x)| \leqslant C, \ -\tau \leqslant x \leqslant 0, \ a < 0, \ k \leqslant p.$$

Consider the case when $j > p$, $a + p > 0$ and $a \leqslant 0$. We define $u'(x)$ from (4.2.8) as follows:

$$u'(x) = u'(0)\left[\frac{\varepsilon}{\varepsilon + x^2}\right]^{a+1} \exp[-g_1(x)] + g_2(x), \ -\tau \leqslant x \leqslant 0. \tag{4.2.24}$$

Analogous to the case $0 < a < 1$, we obtain

$$|u'(x)| \leqslant C[1 + (\varepsilon + x^2)^{-a-1}], \ a \leqslant 0.$$

We estimate $u''(x)$ from (4.2.15), and we obtain

$$u''(x) = u''(0)\left[\frac{\varepsilon}{\varepsilon + x^2}\right]^{a+2} \exp[-g_3(x)] + g_4(x). \tag{4.2.25}$$

Analogous to the case for $0 < a < 1$, we obtain

$$|u''(x)| \leqslant C[1 + (\varepsilon + x^2)^{-a-2}], \ a \leqslant 0.$$

Differentiating (4.1.3) and taking into account (4.2.5), we obtain

$$|u^{(j)}(x)| \leqslant C[1 + (\varepsilon + x^2)^{-a-j}], \ -\tau \leqslant x \leqslant 0, \ a \leqslant 0, \ j > p.$$

This completes the proof of the estimate 4.2.3 for $-\tau < x \leqslant 0$.

By (4.2.2) and (4.2.3), the derivatives of (4.1.3)-(4.1.4) may be estimated by a power function with argument $\varepsilon + x^2$. Therefore (4.1.3)-(4.1.4) are referred to as an equation with power interior layer [38].

The singularly perturbed turning point problem (4.1.3)-(4.1.4) may be regarded as a concatenation of two problems: One on the interval $[-1,0)$ and the other on the interval $(0,1]$. Therefore, the solution of the problem (4.1.3)-(4.1.4) may present a layer near $x=0$ on $[-1,0)$ and a layer near $x=0$ on $(0,1]$. This consideration allows us to understand the behaviour of the solution and its derivatives. The solution can be decomposed into two parts, namely the smooth component $v(x)$ and the singular component $w(x)$ ([42], pp 47) such that

$$u(x) = v(x) + w(x),$$

where $v(x)$ is the solution of the inhomogeneous problem

$$Lv(x) = f(x), \quad x \in \Omega_1 = (0,1], \tag{4.2.26}$$

$$v(0) = 0, \quad v(1) = u(1) = \beta, \tag{4.2.27}$$

and $w(x)$ is the solution of the homogeneous problem

$$Lw(x) = 0, \quad x \in \Omega_1, \tag{4.2.28}$$

$$w(0) = u(0) - v(0), \quad w(1) = 0. \tag{4.2.29}$$

The next lemma gives the bounds on the solution to (4.1.3)-(4.1.4) and its derivatives.

Lemma 4.2.6. The smooth and singular components of $u(x)$ of problem (4.1.3)-(4.1.4), for $0 \le j \leqslant 4$, satisfy

$$|v^{(j)}(x)| \leqslant C \begin{cases} 1 + (\varepsilon + x^2)^{3-j} \arctan(x/\sqrt{\varepsilon}), & x \in [-1,0], \\ 1 + (\varepsilon + x^2)^{3-a-j}, & a < 1, \ x \in [0,1], \\ 1 + (\varepsilon + x^2)^{2-j}, & a = 1, \ x \in [0,1], \\ 1 + \varepsilon^{a-1}(\varepsilon + x^2)^{3-a-j}, & a > 1 \ x \in [0,1], \end{cases} \tag{4.2.30}$$

and

$$|w^{(j)}(x)| \leqslant C \begin{cases} (\varepsilon + x^2)^{1-j} \arctan(x/\sqrt{\varepsilon}), & x \in [-1,0], \\ (\varepsilon + x^2)^{1-a-j}, & a < 1, \ x \in [0,1], \\ (\varepsilon + x^2)^{-j}, & a = 1, \ x \in [0,1], \\ \varepsilon^{a-1}(\varepsilon + x^2)^{1-a-j}, & a > 1, \ x \in [0,1], \end{cases} \tag{4.2.31}$$

where C is constant and independent of ε.

Proof. We prove this lemma on $\Omega_1 = [0, 1]$. The proof on $[-1, 0]$ follows similarly. We obtain the reduced problem ($\varepsilon = 0$) from (4.1.3) as follows:

$$x^2 v_0'' + a(x)v_0' - b(x)v_0 = f(x), \; x \in \Omega_1 \tag{4.2.32}$$

$$v_0(0) = 0, \; v_0(1) = u(1) = \beta. \tag{4.2.33}$$

The smooth component $v(x)$ is further split into the sum ([42], pp 68)

$$v(x) = v_0(x) + (\varepsilon + x^2)v_1(x) + (\varepsilon + x^2)^2 v_2(x), \quad x \in \bar{\Omega}_1, \tag{4.2.34}$$

where v_0 is the solution of the reduced problem in (4.2.32), which is independent of ε, and having smooth coefficients $a(x)$, $b(x)$ and $f(x)$. From these assumptions, for $0 \leqslant j \leqslant 4$, we have

$$|v_0^{(j)}(x)| \leqslant C, \text{ for all } x \in \bar{\Omega}_1. \tag{4.2.35}$$

v_1 and v_2 are the solutions of (4.1.3), where Lemma 4.2.5 is applied.

Now, applying the triangle inequality, using the estimates of v_0 from (4.2.35), v_1 and v_2 from (4.2.2) into (4.2.34), for $0 \leqslant j \leqslant 4$, we obtain the following results

$$|v^{(j)}(x)| \leqslant C \begin{cases} 1 + (\varepsilon + x^2)^{3-a-j}, & a < 1, \\ 1 + (\varepsilon + x^2)^{2-j}, & a = 1, \\ 1 + \varepsilon^{a-1}(\varepsilon + x^2)^{3-a-j}, & a > 1. \end{cases} \tag{4.2.36}$$

Now, let us prove the regular component $w(x)$. We consider the barrier functions as follows [31].

$$\Psi^{\pm}(x) = C \exp(-\eta x/\varepsilon) \pm w(x), \; x \in \bar{\Omega}_1.$$

Let us calculate the values of $\Psi^{\pm}(x)$ at the boundaries:

$$\begin{aligned}
\Psi^{\pm}(0) &= C \pm w(0), \\
&= C \pm [u(0) - v(0)], \ \ \text{using } (4.2.29), \\
&\geqslant 0, \ \ \text{for a suitable choice of C}, \\
\Psi^{\pm}(1) &= C\exp(-\eta/\varepsilon) \pm w(1), \\
&= C\exp(-\eta/\varepsilon), \ \ \text{using } (4.2.29), \\
&\geqslant 0.
\end{aligned}$$

From the above estimates, we notice that $\Psi(x) \geqslant 0, \ \ x \in \Omega_2 = \bar{\Omega}_1 \backslash \Omega_1$. Therefore we have

$$\begin{aligned}
L\Psi^{\pm}(x) &= (\varepsilon + x^2)[\Psi^{\pm}(x)]'' + a(x)[\Psi^{\pm}(x)]' - b(x)\Psi^{\pm}(x), \ x \in \Omega_1 \\
&= C\exp(-\eta x/\varepsilon)\left[\frac{\eta^2(\varepsilon + x^2)}{\varepsilon^2} - \frac{\eta a(x)}{\varepsilon} - b(x)\right] \pm Lw(x) \\
&= C\exp(-\eta x/\varepsilon)\left[\frac{\eta^2(\varepsilon + x^2)}{\varepsilon^2} - \frac{\eta a(x)}{\varepsilon} - b(x)\right], \ \ \text{using } (4.2.28) \\
&\leqslant 0, \ \text{since } (x/\varepsilon)^2 \leqslant b(x), \ \ x \in \Omega_1.
\end{aligned}$$

Now, by applying Lemma 4.2.1 to the barrier functions, we obtain $\Psi^{\pm}(x) \geqslant 0, \ \ x \in \bar{\Omega}_1$. Then we have

$$C\exp(-\eta x/\varepsilon) \pm w(x) \geqslant 0.$$

It follows that

$$w(x) \ \leqslant \ C\exp(-\eta x/\varepsilon), \ x \in \Omega_1.$$

Using the inequality relation, the above inequality can written as follows:

$$|w(x)| \leqslant C\exp(-\eta x/\varepsilon) \leqslant C \begin{cases} (\varepsilon + x^2)^{1-a}, & a < 1, \\ (\varepsilon + x^2)^{-1}, & a = 1, \\ \varepsilon^{a-1}(\varepsilon + x^2)^{1-a}, & a > 1. \end{cases} \tag{4.2.37}$$

Since $Lw(x) = 0$, the j^{th} derivative of $w(x)$ can be estimated immediately from the estimate of $w(x)$. The following estimates hold for $0 \leqslant j \leqslant 4$,

$$|w^{(j)}(x)| \leqslant C \begin{cases} (\varepsilon + x^2)^{1-a-j}, & a < 1, \\ (\varepsilon + x^2)^{-j}, & a = 1, \\ \varepsilon^{a-1}(\varepsilon + x^2)^{1-a-j}, & a > 1. \end{cases} \tag{4.2.38}$$

This completes the proof.

4.3 Construction of the FMFDM

This section develops two fitted fitted diference scheme to solve (4.1.3)-(4.1.4). First, we discretize this problem on Bakhvalov-type and then on Shishkin-type meshes.

Method 1: FMFDM on a Bakhvalov mesh

The idea of a layer-adapted mesh is to construct the mesh generating function. A mesh generating function is a strictly monotone function $\varphi : \to [0,1]$ that maps a uniform mesh in φ onto a layer-adapted mesh in x by $x = \varphi(\xi)$ [35].

Bakhvalov meshes are non-uniform which can be constructed in order to overcome the difficulties encountered by using uniform meshes to solve singularly perturbed problems [76]. Bakhvalov's idea is to use an equidistant ξ near $x = 0$, then to map this grid back onto the $x - axis$ by means of the (scaled) boundary layer function. That is, grid points x_i near $x = 0$ are defined by [35].

$$q\left[1 - \exp\left(\frac{-\eta x_i}{\sigma\varepsilon}\right)\right] = \xi_i = \frac{i}{N} \text{ for } i = 0, 1, 2, 3, \cdots, \tag{4.3.1}$$

where the scaling parameters q lying in $(0, 1]$ and $\sigma > 0$ are user chosen: q is the ratio of mesh points used to resolve the layer, while σ determines the grading of the mesh inside the layer. Away from the layer a uniform mesh x is used with the transition point τ such that the resulting generating function is $C^1[0,1]$, i.e.,

$$x_i = \varphi(\xi) = \begin{cases} \chi(\xi) := -\frac{\sigma\varepsilon}{\eta}\ln\left(1-\frac{\xi_i}{q}\right) & \text{for } \xi_i = i/N,\ i = N/2,\cdots,3N/4, \\ \pi(\xi) := \chi(\tau) + \chi'(\tau)(\xi-\tau) & \text{for } \xi_i = i/N,\ i = 3N/4+1,\cdots,N, \end{cases} \tag{4.3.2}$$

where the point τ verifies

$$\chi(\tau) + \chi'(\tau)(1-\tau) = 1. \tag{4.3.3}$$

Geometrically this means that $(\tau, \chi(\tau))$ is the contact point of the tangent π to $x = \chi(\xi)$ that passes through the point $(1,1)$. The nonlinear differential equation(4.3.3) can be solved by iteration. Therefore the mesh transition points in the ξ and x coordinates are given by

$$\tau_1 = q - \frac{\sigma\varepsilon}{\beta} \quad \text{and} \quad \chi(\tau_1) = \frac{\sigma\varepsilon}{\beta}\ln\frac{\beta q}{\sigma\varepsilon}. \tag{4.3.4}$$

Method 2: FMFDM on a Shishkin type-meshes

We describe this mesh for problem (4.1.3)-(4.1.4) on $[0,1]$. Let $q \in (0,1)$ and $\sigma > 0$. The mesh transition point τ is chosen to be

$$\tau = \min\left\{q, \frac{\sigma\varepsilon}{\eta}\ln N\right\}. \tag{4.3.5}$$

The following sub-intervals $[0,\tau]$ and $[\tau,1]$ are divided into Nq and $(1-q)/N$ equidistant subintervals (assuming that qN is an integer). This mesh may be generated by the mesh generating function

$$x_i = \varphi(\xi) = \begin{cases} \frac{\sigma\varepsilon}{\eta}\,\tilde{\varphi}(\xi) \ \text{ with } \ \tilde{\varphi}(\xi) = \frac{\xi}{q}\ln N & \text{for } \xi_i = i/N,\ i = N/2,\cdots,3N/4, \\ 1-\left(1-\frac{\sigma\varepsilon}{\eta}\ln N\right)\frac{1-\xi}{1-q} & \text{for } \xi_i = i/N,\ i = 3N/4+1,\cdots,N, \end{cases} \tag{4.3.6}$$

if $q \geqslant \tau$. The parameter q is defined as the number of mesh points used to resolve the layer. The mesh transition point τ has been chosen such that the layer term $\exp(-\eta x/\varepsilon)$ is smaller that N^{σ} on $[\tau,1]$. Typically σ will be chosen equal to the formal order of the method or sufficiently large to accommodate the error analysis.

Note that unlike the Bakhvalov mesh (and Vulanovics modification of it) the underlying mesh generating function is only on $C^1[0,1]$ and depends on N, the number of mesh points. For simplicity's sake, it is assumed that $q \geqslant \tau$ as otherwise N is exponentially large compared to $1/\varepsilon$ and a uniform mesh is sufficient to cope with the problem.

Although the structure of Shishkin meshes is simple to manipulate and numerical methods using them are easier to analyse than Bakhvalov's method, they give numerical results that are inferior to those obtained by Bakhvalov-type meshes.

The Shishkin-type mesh is piecewise uniform, finer which finer near the layer(s) and coarser elsewhere [42]. Due to the presence of an interior layer at point $x_{N/2} = 0$, we divide $[0,1]$ into two sub-intervals $[0,\tau]$ and $[\tau,1]$ which are each then divided into $N/4$ equal mesh elements. Assume $N = 2^m, m \geqslant 2$. We choose $0 < \tau \leqslant 1/2$, the transition parameter τ is given by

$$\tau = \min\left\{\frac{1}{2}, \frac{\sigma\varepsilon}{\eta}\ln N\right\}. \tag{4.3.7}$$

The mesh said to be uniform when $\varepsilon \ln N \geqslant 1/2$ for N sufficiently large.

The grid points x_i on the interval $[-1,0]$ are obtained by replacing x_j by $-x_j$ for both cases.

Let the mesh be generated by (4.3.6) with a monotone $\tilde{\varphi}(t)$ satisfying

$$\tilde{\varphi}(0) = 0 \;\text{ and }\; \tilde{\varphi}(1/2) = \ln N.$$

Define the new function $\psi(t)$ by

$$\tilde{\varphi}(t) = -\ln \psi(t). \tag{4.3.8}$$

This function is monotonically decreasing with $\psi(0) = 1$ and $\psi(1/2) = N^{-1}$. One of the examples for the mesh characterizing function

Shishkin-type mesh

$$\psi(t) = \exp(-2t\ln N) \;\text{ with } \max|\psi'| = 2\ln N \;\;, h \leq CN^{-1}, \;\text{ for }\; t \in [0,\tau]. \tag{4.3.9}$$

Two further properties of the mesh generating function that will be assumed later when analysing numerical schemes are

$$\max_{0\le t\le 1/2} \tilde{\varphi}'(t) \leqslant CN \tag{4.3.10}$$

and

$$\int_0^{1/2} \tilde{\varphi}'(t)^2 \, dt \leqslant CN. \tag{4.3.11}$$

The Gauchy-Schwarz inequality yields

$$\sum_{k=1}^{N/2} \left(\frac{\eta h_k}{\varepsilon}\right)^2 = \sigma^2 N^{-1} \int_0^{1/2} \tilde{\varphi}'(t)^2 \, dt \leqslant C. \tag{4.3.12}$$

We adopt the notation $K(x_j) = K_j$. Also let

$$D^+U_j = \frac{U_{j+1} - U_j}{h_{j+1}}, \quad D^-U_j = \frac{U_j - U_{j-1}}{h_j} \text{ and } \tilde{D}U_j = \frac{2}{h_j + h_{j+1}}(D^+U_j - D^-U_j),$$

where D^+U_j, D^-U_j and $\tilde{D}U_j$ are first and second order finite differences respectively. Using the upwind scheme, our problem is discretized in the following way:

$$L^N U_j := \begin{cases} (\varepsilon + x_j^2)\tilde{D}U_j + \tilde{a}_j D^- U_j - \tilde{b}_j U_j = \tilde{f}_j & \text{for } j = 0, 1, \cdots, N/2 - 1, \\ (\varepsilon + x_j^2)\tilde{D}U_j + \tilde{a}_j D^+ U_j - \tilde{b}_(x) U_j = \tilde{f}_j & \text{for } j = N/2, N/2 + 1, \cdots, N - 1, \end{cases} \tag{4.3.13}$$

$$U(-1) = \alpha, \quad U(1) = \beta, \tag{4.3.14}$$

where

$$\begin{cases} \tilde{a}_j = \frac{a_{j-1} + a_j}{2} & \text{for } j = 0, 1, \cdots, N/2 - 1, \\ \tilde{a}_j = \frac{a_j + a_{j+1}}{2} & \text{for } j = N/2, N/2 + 1, \cdots, N - 1, \end{cases}$$

$$\begin{cases} \tilde{b}_j = \frac{b_{j-1} + b_j + b_{j+1}}{3} & \text{for } j = 1, 2, 3, \cdots, N - 1, \\ \tilde{f}_j = \frac{f_{j-1} + f_j + f_{j+1}}{3} & \text{for } j = 1, 2, 3, \cdots, N - 1. \end{cases}$$

Now (4.3.13) can be written in the form:

$$L^N U_j := r^- U_{j-1} + r^c U_j + r^+ U_{j+1} = f_j, \; j = 1, 2, 3 \cdots, N - 1, \tag{4.3.15}$$

where for $j = 1, 2, 3 \cdots, N/2 - 1$, we have

$$r_j^- = \frac{2(\varepsilon + x_j^2)}{h_j(h_j + h_{j+1})} - \frac{\tilde{a}_j}{h_j}, \quad r_j^c = \frac{\tilde{a}_j}{h_j} - \frac{2(\varepsilon + x_j^2)}{h_j h_{j+1}} - \tilde{b}_j \text{ and } r_j^+ = \frac{2(\varepsilon + x_j^2)}{h_{j+1}(h_j + h_{j+1})} \quad (4.3.16)$$

and for $j = N/2, N/2 + 1, \cdots, N - 1$, we have

$$r_j^- = \frac{2(\varepsilon + x_j^2)}{h_j(h_j + h_{j+1})}, \quad r_j^c = -\frac{\tilde{a}_j}{h_{j+1}} - \frac{2(\varepsilon + x_j^2)}{h_j h_{j+1}} - \tilde{b}_j \text{ and } r_j^+ = \frac{2(\varepsilon + x_j^2)}{h_{j+1}(h_j + h_{j+1})} + \frac{\tilde{a}_j}{h_{j+1}}. \quad (4.3.17)$$

We denote $h_i = x_i - x_{i-1}$ for $i = 1, 2, \cdots, N$ the local mesh sizes. For the uniform mesh, we have $h_i \leqslant 4N^{-1}$ and the maximal mesh sizes $h \leqslant CN^{-1}$.

In view of the analysis developed above, we need to prove the following lemma which states that problem (4.3.13)-(4.3.14) satisfies Lemma 4.2.1.

Lemma 4.3.1. For any mesh function ξ_j such that $L^N \xi_j \leq 0$, $\forall j = 1, 2, \ldots, N-1$, $\xi_0 \geq 0$ and $\xi_n \geq 0$, we have $\xi_j \geq 0$, $\forall j = 0, 1, \cdots, N$.

Proof. Let k be such that $\xi_k = \min_{0 \leq j \leq N} \xi_j$ and suppose that $\xi_k < 0$. Obviously, $k \neq 0$ and $k \neq N$. Also $\xi_{k+1} - \xi_k \geq 0$ and $\xi_k - \xi_{k-1} \leq 0$. For $k = 1, 2, \ldots, N/2 - 1$ and $a_k < 0$, we have

$$L^N \xi_k := (\varepsilon + \xi_k^2)\tilde{D}\xi_k + a_k D^- \xi_k - b_k \xi_k, \quad (4.3.18)$$

Substituting $D^- \varepsilon_k$ and $\tilde{D}\varepsilon_k$ into (4.3.18), we obtain

$$L^N \xi_k = \frac{2(\varepsilon + \xi_k^2)}{h_k + h_{k+1}} \left(\frac{\varepsilon_{k+1} - \varepsilon_k}{h_{k+1}} - \frac{\varepsilon_k - \varepsilon_{k-1}}{h_k} \right) + a_k \left(\frac{\varepsilon_k - \varepsilon_{k-1}}{h_k} \right) - b_k \xi_k > 0.$$

For $k = N/2$ and $a_k = 0$, we have

$$L^N \xi_k := -b_k \xi_k > 0. \quad (4.3.19)$$

For $k = N/2 + 1, \ldots, N + 1$ and $a_k > 0$, we have

$$L^N \xi_k := (\varepsilon + \xi_k^2)\tilde{D}\xi_k + a_k D^+ \xi_k - b_k \xi_k. \quad (4.3.20)$$

Substituting $D^+ \varepsilon_k$ and $\tilde{D}\varepsilon_k$ into (4.3.20), we obtain

$$L^N \xi_k = \frac{2(\varepsilon + \xi_k^2)}{h_k + h_{k+1}} \left(\frac{\varepsilon_{k+1} - \varepsilon_k}{h_{k+1}} - \frac{\varepsilon_k - \varepsilon_{k-1}}{h_k} \right) + a_k \left(\frac{\varepsilon_{k+1} - \varepsilon_k}{h_{k+1}} \right) - b_k \xi_k > 0.$$

Thus $L^N\xi_k > 0,\ 1 \le k \le N-1$, which is a contradiction. It follows that $\xi_k \ge 0$ and consequently $\xi_j \ge 0,\ 0 \le j \le N$.

The next lemma will be proved by means of Lemma 4.3.1.

Lemma 4.3.2. If Z_i is any mesh function such that $Z_0 = Z_N = 0$, then

$$|Z_i| \le \frac{1}{b_0} \max_{1\le j\le N-1} |L^N Z_j|, \ \ \forall 0 \le i \le N.$$

Proof. Define

$$|M_i^{\pm}| = \frac{1}{b_0} \max_{1\le j\le N-1} |L^N Z_j|.$$

Introduce the two mesh functions $Y_i^{\pm}$ defined by

$$Y_i^{\pm} = M^{\pm} \pm Z_i.$$

It follows that $Y_0^{\pm} = Y_N^{\pm} = M \ge 0$ and, for $1 \le i \leqslant N-1$. We observe that

$$L^N Y^{\pm} = -M^{\pm} b_i \pm L^N Z_i \le 0$$

for $1 \le i \le N-1$, because $b_i \ge b_0 > 0$. By the discrete minimum principle Lemma 4.3.1, we conclude that $Y_i \ge 0$, for $0 \leqslant i \leqslant N$.

Based on the above results, now we are in a position to provide the ε-uniform convergence result in the next section.

4.4 Convergence analysis

In this section, the convergence of the scheme will be analyzed on both method.

An error estimate for method 1

Theorem 4.4.1. Let Ω_N^{τ} be a B-type mesh with $\sigma \geqslant 2$. Then the error of the upwinding scheme (4.3.13)-(4.3.14) applied to (4.1.3)-(4.1.4) verifies

$$\max_{0\leqslant j\leqslant N} |u_j - U_j| \leqslant CN^{-1}. \tag{4.4.1}$$

Proof. We prove the Lemma on the interval $[0, 1]$. The proof on $[-1, 0]$ follows in a similarly way. Let u be the solution of (4.1.3)-(4.1.4). The solution U of the discrete problems (4.3.13)-(4.3.14) can be decomposed into a regular and a singular parts as

$$U = V + W,$$

where V is the solution of the inhomogeneous problem

$$L^N V = f_j, \text{ for } j = N/2, \cdots, N, \; V(0) = v(0), V(1) = v(1),$$

and W is the solution of the homogeneous problem

$$L^N W_j = 0, \text{ for } j = N/2, \cdots, N, \; W(0) = U(0) - V(0), W(1) = w(1).$$

Now, the error in the regular and singular components of the solution can be computed separately

$$U - u = (V - v) + (W - w). \tag{4.4.2}$$

Combining (4.1.3) and (4.3.13), we obtain the error for the regular component as follows

$$\begin{aligned} L^N(V - v) &= f - L^N v \\ &= (L - L^N)v \\ &= (\varepsilon + x_j^2)\left(\frac{d^2}{dx^2} - \tilde{D}\right)v + \tilde{a}_j\left(\frac{d}{dx} - D^-\right)v. \end{aligned}$$

Applying Lemma 4.1 pp 24 of [42] to the above result, the local truncation error estimates can be written as

$$|L^N(V_j - v_j)| \leq \frac{(\varepsilon + x_j^2)}{3}(x_{j+1} - x_{j-1})|v_j'''| + \frac{\tilde{a}_j}{2}(x_j - x_{j-1})|v_j''| \text{ for } N/2 \leqslant j \leqslant N. \tag{4.4.3}$$

Since $h \leqslant CN^{-1}$ for any j and using Lemma 4.2.6 for different cases of a, we obtain the same result as follows

$$|L^N(V_j - v_j)| \leqslant CN^{-1}.$$

Application of Lemma 4.3.2 for the mesh function $V_j - v_j$, we obtain

$$|V_j - v_j| \leqslant CN^{-1} \text{ for } N/2 \leqslant j \leqslant N. \tag{4.4.4}$$

The estimate of the error for the singular component in the interval $[\tau, 1]$ can be estimated by using the M-matrix property of L^N [35]. Then one can show that

$$|W_i| \leq \tilde{W}_i := \prod_{k=1}^{3N/4} \left(1 + \frac{\eta h_k}{2\varepsilon}\right)^{-1} \text{ for } 3N/4 \leqslant j \leqslant N. \tag{4.4.5}$$

For $t \geqslant 0$, we have $\ln(1+t) \geqslant t - t^2/2$. (4.4.5) can be written as follows

$$\ln \prod_{k=1}^{3N/4} \left(1 + \frac{\eta h_k}{2\varepsilon}\right) \geqslant \sum_{k=1}^{3N/4} \left[\frac{\eta h_k}{2\varepsilon} - \frac{1}{2}\left(\frac{\eta h_k}{2\varepsilon}\right)^2\right] \geqslant \sum_{k=1}^{3N/4} \frac{\eta h_k}{2\varepsilon} - \frac{1}{2}\sum_{k=1}^{3N/4}\left(\frac{\eta h_k}{2\varepsilon}\right)^2. \tag{4.4.6}$$

In this case $h_k = \tau = \tau_1 = (1/2 - 2\varepsilon/\eta)$ from (4.3.4). Then we have

$$\frac{\eta h_k}{2\varepsilon} = \frac{\eta}{2\varepsilon} \times \left(\frac{1}{2} - \frac{2\varepsilon}{\eta}\right) = \frac{\eta}{4\varepsilon} - 1.$$

Substituting this expression into (4.4.6), we obtain

$$\ln \prod_{k=1}^{N/4} \left(1 + \frac{\eta h_k}{2\varepsilon}\right) \geqslant \frac{\eta}{4\varepsilon} - 1 - \frac{1}{2}\sum_{k=1}^{N/4}\left(\frac{\eta h_k}{2\varepsilon}\right)^2.$$

Multiplying by a negative sign on both sides and taking the exponential on both sides in the above inequality, we obtain

$$\prod_{k=1}^{N/4} \left(1 + \frac{\eta h_k}{2\varepsilon}\right)^{-1} \leqslant \exp\left(1 - \frac{\eta}{4\varepsilon}\right) \exp\left[\frac{1}{2}\sum_{k=1}^{N/4}\left(\frac{\eta h_k}{2\varepsilon}\right)^2\right] \leqslant C.$$

Then we have

$$|W_i| \leqslant C, \text{ for } 3N/4 \leqslant j \leqslant N. \tag{4.4.7}$$

Consequently, we obtain

$$|w_i - W_i| \leqslant |w_i| + |W_i| \leqslant C, \text{ for } 3N/4 \leqslant j \leqslant N, \tag{4.4.8}$$

where we have used the bounds of the derivatives of w from Lemma 4.2.6.

The truncation error for the singular component w in the interval $[0, \tau]$ is similar to that of the regular component.

$$|L^N(W_j - w_j)| \leqslant \frac{(\varepsilon + x_j^2)}{3}(x_{j+1} - x_{j-1})|w_j'''| + \frac{\tilde{a}_j}{2}(x_j - x_{j-1})|w_j''| \text{ for } N/2 \leqslant j \leqslant 3N/4 - 1. \tag{4.4.9}$$

Applying the bounds of the derivatives of w_j of Lemma 4.2.6 for each case of a in (4.4.9), we obtain

$$|L^N(W_j - w_j)| \leqslant Ch \begin{cases} (\varepsilon + x^2)^{-a-1}, & a < 1, \\ (\varepsilon + x^2)^{-2}, & a = 1, \\ \varepsilon^{a-1}(\varepsilon + x^2)^{-a-1}, & a > 1. \end{cases} \tag{4.4.10}$$

Using the inequality relation, the above inequalities lead to

$$|L^N(W_j - w_j)| \leqslant Ch\varepsilon^{-2}x_j^2. \tag{4.4.11}$$

Substituting $h \leqslant CN^{-1}$ and $x_j = (-\sigma\varepsilon/\eta)\ln(1 - \tau_1/q)$ into (4.4.11), we obtain

$$\left|L^N(W_j - w_j)\right| \leqslant CN^{-1}. \tag{4.4.12}$$

On application of Lemma 4.3.2 to (4.4.12), we obtain

$$|W_j - w_j| \leqslant CN^{-1} \text{ for } N/2 \leqslant j \leqslant 3N/4. \tag{4.4.13}$$

Combining (4.4.4), (4.4.8) and (4.4.13), we obtain the following result

$$|u_j - U_j| \leqslant CN^{-1} \text{ for } N/2 \leqslant j \leqslant N. \tag{4.4.14}$$

A similar analysis can be obtained in the interval $[-1, 0]$,

$$|u_j - U_j| \leq CN^{-1} \text{ for } 1 \leqslant j \leqslant N/2 - 1. \tag{4.4.15}$$

Collecting (4.4.14) and (4.4.15), we obtain the main result.

An error estimate for method 2

Theorem 4.4.2. Let Ω_N^τ be an S-type mesh with $\sigma \geqslant 2$. Assume that the function $\tilde{\varphi}(t)$ is piecewise differentiable and verifies (4.3.10)-(4.3.11). Then the error of the simple scheme on the given interval satisfies

$$\max_{0\leq j\leq N}|u_j - U_j| \leq CN^{-1}\ln^2 N. \tag{4.4.16}$$

Proof. The proof for S-type meshes will follow the same idea as that for B-type meshes. Recall that

$$U - u = (V - v) + (W - w),$$

by (4.4.2).

The truncation error for the regular part v is defined as given in (4.4.3) and noting that $h_j = x_j - x_{j-1} \leqslant 4N^{-1}$ then we obtain the same result for the different cases of a as follows

$$|V_j - v_j| \leqslant CN^{-1} \text{ for } N/2 \leqslant j \leqslant N. \tag{4.4.17}$$

The truncation error for the singular part W in the interval $[\tau, 1]$ can be obtained from (4.4.5) where $h_k = \tau$. Substituting this expression

$$\frac{\eta h_k}{2\varepsilon} = \frac{\eta}{2\varepsilon} \times \frac{2\varepsilon \ln N}{\eta} = \ln N$$

into (4.4.6), we obtain

$$\left[\ln \prod_{k=1}^{3N/4}\left(1 + \frac{\eta h_k}{2\varepsilon}\right)\right] \geqslant \ln N - \frac{1}{2}\sum_{k=1}^{3N/4}\left(\frac{\eta h_k}{2\varepsilon}\right)^2.$$

Multiplying by a negative sign on both sides and taking the exponential on both sides in the above inequality, we obtain

$$\prod_{k=1}^{3N/4}\left(1 + \frac{\eta h_k}{2\varepsilon}\right)^{-1} \leqslant N^{-1}\exp\left[\frac{1}{2}\sum_{k=1}^{3N/4}\left(\frac{\eta h_k}{2\varepsilon}\right)^2\right]. \tag{4.4.18}$$

Using (4.3.12), then (4.4.18) becomes

$$\prod_{k=1}^{3N/4}\left(1 + \frac{\eta h_k}{2\varepsilon}\right)^{-1} \leqslant CN^{-1}.$$

Then we obtain

$$|W_i| \leqslant CN^{-1} \text{ for } 3N/4 \leqslant j \leqslant N. \tag{4.4.19}$$

Therefore, we obtain

$$|w_i - W_i| \leqslant |w_i| + |W_i| \leqslant CN^{-1} \text{ for } 3N/4 \leqslant j \leqslant N, \tag{4.4.20}$$

where we have used the bounds of the derivatives of w from Lemma 4.2.6.

Substituting $x = (2\varepsilon/\eta)\tilde{\varphi}(t)$ with $\tilde{\varphi}(t = q) = \ln N$ and $h \leqslant CN^{-1}$ into (4.4.11), we obtain the truncation error for the singular part w on $[0, \tau]$:

$$|L^N(W_j - w_j)| \leqslant CN^{-1} \ln^2 N.$$

With application of Lemma 4.3.2 for the mesh function $|L^N(W_j - w_j)|$, we obtain

$$|W_j - w_j| \leqslant CN^{-1} \ln^2 N \text{ for } N/2 \leqslant j \leqslant 3N/4. \tag{4.4.21}$$

Collecting (4.4.17), (4.4.20) and (4.4.21), we obtain the following result in $[0, 1]$

$$|u_j - U_j| \leq CN^{-1} \ln^2 N \text{ for } N/2 \leqslant j \leqslant N. \tag{4.4.22}$$

Similarly, for the sub-interval $[-1, 0]$, we obtain

$$|u_j - U_j| \leq CN^{-1} \ln^2 N \text{ for } 1 \leqslant j \leqslant N/2 - 1. \tag{4.4.23}$$

Combining (4.4.22) and (4.4.23) then gives the required result.

We apply the Richardson extrapolation technique in the next section to improve the accuracy and the rate of convergence of the scheme.

4.5 Richardson extrapolation on the FMFDM

Richardson extrapolation on layer-adapted meshes was first analysed by Natividad and Stynes [49]. We apply this procedure for the proposed scheme. They studied a simple upwind scheme on a Shishkin mesh and proved that Richardson extrapolation improves the accuracy to almost second order by combining discrete solutions calculated on different meshes.

An error estimate for method 1

Theorem 4.5.1. (Error after extrapolation). Let $a(x)$, $b(x)$, and $f(x)$ be sufficiently smooth and $u(x)$ be the solution of (4.1.3)-(4.1.4). If U^{ext} is the approximation of $u(x)$ obtained using (4.3.13)-(4.3.14) with $u(-1) = U(-1),\ u(1) = U(1)$, there exists a positive constant C independent of ε and the mesh spacing such that

$$\max_{0 \leq j \leqslant N} |U_j^{ext} - u_j| \leqslant CN^{-2}. \tag{4.5.1}$$

Proof. We prove this theorem on the interval $[0, 1]$ as before. Results on the interval $[-1, 0]$ can be obtained in a similar way. We consider the mesh Ω_{2N}^{τ} by bisecting each subinterval of Ω_N^{τ}. It is clear that $\Omega_N{}^{\tau} \subset \Omega_{2N}^{\tau}$ and $\tilde{x}_j - \tilde{x}_{j-1} = \tilde{h}_j = h_j/2$. We denote the numerical solution on the mesh Ω_{2N}^{τ} by $\tilde{U}_j$.

From (4.4.14), we have

$$U_j - u_j = C_1 N^{-1} + R_n(x_j) \ \ \forall x_j \in \Omega_N^{\tau} \tag{4.5.2}$$

and

$$\tilde{U}_j - u_j = C_2 (2N)^{-1} + R_{2N}(\tilde{x}_j) \ \ \forall \tilde{x}_j \in \Omega_{2N}^{\tau}, \tag{4.5.3}$$

where C_1 and C_2 are some fixed constants and the remainder terms

$$R_N(x_j) \text{ and } R_{2N}(\tilde{x}_j) \text{ are } \mathcal{O}\left[N^{-1}\right].$$

Note that we have used the same transition parameter τ_1 which is given by (4.3.4) when computing both U_j and $\tilde{U}_j$.

Multiplying (4.5.3) by a factor 2 gives

$$2\left[\tilde{U}_j - u_j\right] = C_1 (N)^{-1} + 2R_{2N}(\tilde{x}_j) \ \ \forall x_j \in \Omega_N{}^{\tau}. \tag{4.5.4}$$

Then the difference of (4.5.4) and (4.5.2) suggests that

$$u_j - (2\tilde{U}_j - U_j) = R_N(x_j) - 2R_{2N}(x_j) = \mathcal{O}\left[N^{-1}\right] \ \ \forall x_j \in \Omega_N^{\tau} \tag{4.5.5}$$

and therefore we shall use

$$U_j^{ext} = 2\tilde{U}_j - U_j \;\; \forall x_j \in \Omega_N^\tau, \tag{4.5.6}$$

as the new approximation of u_j at the point $x_j \in \Omega_N^\tau$ obtaining from Richardson extrapolation.

Recalling the decomposition (4.4.2) of $U - u$, we split the error after extrapolation U_j^{ext} in a similar manner:

$$U_j^{ext} - u_j = (V_j^{ext} - v_j) + (W_j^{ext} - w_j), \tag{4.5.7}$$

where V_j^{ext} and W_j^{ext} are the regular and singular components of U_j^{ext}, respectively. The local truncation error of the scheme (4.3.13)-(4.3.15) after extrapolation is given by

$$\left| L^N U_j^{ext} - (Lv)_j \right| = 2\left(L^N\tilde{U}_j - (Lu)_j\right) - \left(L^N U_j - (Lu)_j\right), \tag{4.5.8}$$

where

$$L^N U_j - (Lu)_j = r^- u_{j-1} + r^c u_j + r^+ u_{j+1} - (\varepsilon + x_j^2)u_j'' - a_j u_j' + b_j u \tag{4.5.9}$$

and

$$L^N\tilde{U}_j - (Lu)_j = \tilde{r}^- u_{j-1} + \tilde{r}^c u_j + \tilde{r}^+ u_{j+1} - (\varepsilon + x_j^2)u_j'' - a_j u_j' + b_j u. \tag{4.5.10}$$

The quantities r^-, r^c and r^+ are given in (4.3.16) while the expressions $\tilde{r}^-$, $\tilde{r}^c$ and $\tilde{r}^+$ are obtained by substituting h_j by $\tilde{h}_j$ and h_{j+1} by $\tilde{h}_{j+1}$ in the expressions r^-, r^c and r^+, respectively.

Taking the Taylor series expansion of u_j about x_j, we obtain the following approximations for u_{j-1} and u_{j+1}.

$$u_{j-1} = u_j - h_j u_j' + \frac{h_j^2}{2}u_j^2 - \frac{h_j^3}{6}u_j^3 + \frac{h_j^4}{24}u^4(\xi_1, j), \tag{4.5.11}$$

$$u_{j+1} = u_j + h_{j+1} u_j' + \frac{h_{j+1}^2}{2}u_j^2 + \frac{h_{j+1}^3}{6}u_j^3 + \frac{h_{j+1}^4}{24}u^4(\xi_2, j), \tag{4.5.12}$$

where

$$(\xi_1, j) \in (x_{j-1}, x_j), \;\; (\xi_2, j) \in (x_j, x_{j+1}).$$

Expansions to be used in (4.5.10):

$$u_{j-1} = u_j - \tilde{h}_j u_j' + \frac{\tilde{h}_j^2}{2} u_j^2 - \frac{\tilde{h}_j^3}{6} u_j^3 + \frac{\tilde{h}_j^4}{24} u^4(\tilde{\xi}_1, j), \tag{4.5.13}$$

$$u_{j+1} = u_j + \tilde{h}_{j+1} u_j' + \frac{\tilde{h}_{j+1}^2}{2} u_j^2 + \frac{\tilde{h}_{j+1}^3}{6} u_j^3 + \frac{\tilde{h}_{j+1}^4}{24} u^4(\tilde{\xi}_2, j), \tag{4.5.14}$$

where

$$\tilde{\xi}_1 \in (\frac{x_{j-1}+x_j}{2}, x_j) \text{ and } \tilde{\xi}_2 \in (x_j, \frac{x_j + x_{j+1}}{2}).$$

Substituting (4.5.11) and (4.5.12) into (4.5.9) and (4.5.13) and (4.5.14) into (4.5.10), (4.5.9) and (4.5.10), respectively become

$$L^N U_j - (Lu)_j = k_1 u_j + k_2 u_j' + k_3 u_j^2 + k_4 u_j^3 + k_{5,1} u^4(\xi_1, j) + k_{5,2} u^4(\xi_2, j) \tag{4.5.15}$$

and

$$L^N \tilde{U}_j - (Lu)_j = \tilde{k}_1 u_j + \tilde{k}_2 u_j' + \tilde{k}_3 u_j^2 + \tilde{k}_4 u_j^3 + \tilde{k}_4 u_j^4 + \tilde{k}_{5,1} u^4(\tilde{\xi}_1, j) + \tilde{k}_{5,2} u^4(\tilde{\xi}_2, j). \tag{4.5.16}$$

The coefficients of (4.5.15) are

$$k_1 = \frac{2(\varepsilon + x_j^2)}{h_j(h_j + h_{j+1})} - \frac{2(\varepsilon + x_j^2)}{h_j h_{j+1}} + \frac{2(\varepsilon + x_j^2)}{h_{j+1}(h_j + h_{j+1})}, \; k_2 = 0,$$

$$k_3 = \frac{(\varepsilon + x_j^2) h_j}{h_j + h_{j+1}} - \frac{a_j h_j}{2} + \frac{(\varepsilon + x_j^2) h_{j+1}}{h_j + h_{j+1}} - (\varepsilon + x_j^2),$$

$$k_4 = \frac{-(\varepsilon + x_j^2) h_j^2}{3(h_j + h_{j+1})} + \frac{a_j h_j^2}{6} + \frac{(\varepsilon + x_j^2) h_{j+1}^2}{3(h_j + h_{j+1})}, \; k_{5,1} = \frac{(\varepsilon + x_j^2) h_j^3}{12(h_j + h_{j+1})} - \frac{a_j h_j^3}{24}, \; k_{5,2} = \frac{(\varepsilon + x_j^2) h_{j+1}^3}{12(h_j + h_{j+1})}.$$

The quantities for $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$, $\tilde{k}_4$, $\tilde{k}_{5,1}$ and $\tilde{k}_{5,2}$ in (4.5.16) can be similarly obtained as in (4.5.15) by substituting h_j with $\tilde{h}_j$ with h_{j+1} by $\tilde{h}_{j+1}$.

Substituting (4.5.15) and (4.5.16) into (4.5.8), then (4.5.8) becomes

$$L^N U_j^{ext} - (Lu)_j = T_1 u_j + T_2 u_j'' + T_3 u_j''' + + T_{4,1} u^{(4)}(\xi_1, j) + T_{4,2} u^{(4)}(\xi_2, j). \tag{4.5.17}$$

In the above,

$$T_1 = \frac{14(\varepsilon + x_j^2)}{h_j(h_j + h_{j+1})} - \frac{14(\varepsilon + x_j^2)}{h_j h_{j+1}} + \frac{14(\varepsilon + x_j^2)}{h_{j+1}(h_j + h_{j+1})},$$

$$T_2 = \frac{(\varepsilon + x_j^2)h_j}{h_j + h_{j+1}} - (\varepsilon + x_j^2) + \frac{(\varepsilon + x_j^2)h_{j+1}}{h_j + h_{j+1}}, \quad T_3 = -\frac{a_j h_j^2}{12},$$

$$T_{4,1} = -\frac{(\varepsilon + x_j^2)h_j^3}{24(h_j + h_{j+1})} + \frac{a_j h_j^3}{32} \text{ and } T_{4,2} = -\frac{(\varepsilon + x_j^2)h_{j+1}^3}{24}.$$

For the sake of simplicity, we use the notation

$$h_j = \begin{cases} H & \text{if } j = 3N/4 + 1, 3N/4 + 2, \cdots, N, \\ h & \text{if } j = N/2, \cdots 3N/4. \end{cases} \tag{4.5.18}$$

Using the fact that, for $\forall j = 3N/4 + 1, \ldots, N$, $H = h_j \leqslant CN^{-1}$ in (4.5.17) on the subinterval $[\tau, 1]$, we obtain

$$L^N V_j^{ext} - (Lv)_j = \left[-\frac{a_j}{12} v_j''' + \frac{(\varepsilon + x_j^2)}{48} v^{(4)}(\xi_1, j) \right] H^2 + \left[\frac{a_j}{32} v^{(4)}(\xi_1, j) - \frac{(\varepsilon + x_j^2)}{24} v^{(4)}(\xi_2, j) \right] H^3. \tag{4.5.19}$$

Applying the triangle inequality and Lemma 4.2.6 for different cases of a in (4.5.19), we obtain the same result

$$L^N (V_j^{ext} - v_j) \leqslant CH^2 \leqslant CN^{-2}. \tag{4.5.20}$$

Applying Lemma 4.3.2 to (4.5.20), we obtain

$$|V_j^{ext} - v_j| \leqslant CN^{-2}. \tag{4.5.21}$$

For the truncation error for the singular part W^{ext} in the interval $[\tau, 1]$, we proceed in the same as we did for V^{ext}. Using (4.5.19) and with the help of Lemma 4.2.6 by considering each case of a, we obtain the same result:

$$L^N W_j^{ext} - (Lw)_j \leqslant C\varepsilon^{-2} x_j^2 H^2. \tag{4.5.22}$$

Using the fact that $H \leqslant CN^{-1}$ and $x_j = 1/2 - (\sigma\varepsilon)/\beta$, (4.5.22) becomes

$$L^N W_j^{ext} - (Lw)_j \leqslant CN^{-2}. \tag{4.5.23}$$

Finally we obtain the following result by application of Lemma 4.3.2

$$|W_j^{ext} - w_j| \leqslant CN^{-2}. \tag{4.5.24}$$

The truncation error for the singular component W^{ext} on $[0,\tau]$ can be obtained in the same way as we did in the previous case. Noting that $h \leqslant CN^{-1}$ and with the help of Lemma 4.2.6, we obtain from (4.5.19):

$$L^N(W_j^{ext} - w_j) \leqslant CN^{-2}\varepsilon^{-2}x_j^2. \tag{4.5.25}$$

Using the value $x_j = (-\sigma\varepsilon/\eta)\ln(1-\tau_1/q)$ and application of Lemma 4.3.2 in (4.5.25) yields

$$|W_j^{ext} - w_j| \leqslant CN^{-2}. \tag{4.5.26}$$

Combining (4.5.21), (4.5.24) and (4.5.26), we obtain the result in the interval $[0,1]$

$$|U_j^{ext} - u_j| \leqslant CN^{-2}. \tag{4.5.27}$$

A similar analysis performed in the interval $[-1,0]$, yields

$$|U_j^{ext} - u_j| \leqslant CN^{-2}. \tag{4.5.28}$$

Putting together (4.5.27) and (4.5.28), then gives the required result.

An error estimate for method 2

Theorem 4.5.2. (Error after extrapolation). Suppose that the piecewise differentiable generating function $\varphi(t)$ satisfies (4.3.9) and let U^{ext} be the approximate solution to (4.1.3)-(4.1.4) obtained by Richardson extrapolation applied to the simple upwind scheme (4.3.13)-(4.3.14). Then

$$\max_{0\leq j\leqslant N} |U_j^{ext} - u_j| \leqslant CN^{-2}\ln^2 N. \tag{4.5.29}$$

Proof. The S-type mesh after extrapolation will follow the same lines as the B-type mesh after extrapolation. Recall that

$$U_j^{ext} - u_j = (V_j^{ext} - v_j) + (W_j^{ext} - w_j),$$

by (4.5.7).

From (4.5.20) and noting that $H = h_j \leqslant CN^{-1}$ and using Lemma 4.3.2, we obtain

$$|V_j^{ext} - v_j| \leqslant CN^{-2}, \text{ for } N/2 \leqslant j \leqslant N. \tag{4.5.30}$$

Using the value $x_j = (2\varepsilon/\eta)\ln N$ and $H = h_j \leqslant CN^{-1}$ in (4.5.22), we obtain the truncation error for the singular component W^{ext} on $[\tau, 1]$

$$L^N(W_j^{ext} - w_j) \leqslant CN^{-2}\ln^2 N. \tag{4.5.31}$$

On application of Lemma 4.3.2 for the above mesh, we obtain

$$|W_j^{ext} - w_j| \leqslant CN^{-2}\ln^2 N. \tag{4.5.32}$$

The truncation error for the singular component W^{ext} on $[0, \tau]$ can be computed by substituting $x = (2\varepsilon/\eta)\tilde{\varphi}(t)$ with $\tilde{\varphi}(t = q) = \ln N$ and $h \leqslant CN^{-1}$ into (4.5.22)

$$L^N W_j^{ext} - (Lw)_j \leqslant CN^{-2}\ln^2 N. \tag{4.5.33}$$

Application of 4.3.2 to the mesh $L^N W_j^{ext} - (Lw)_j$ gives

$$|W_j^{ext} - w_j| \leqslant CN^{-2}\ln^2 N. \tag{4.5.34}$$

Combining (4.5.30), (4.5.32) and (4.5.34), we obtain

$$\max_{N/2 \leq j \leq N} |U_j^{ext} - u_j| \leqslant CN^{-2}\ln^2 N. \tag{4.5.35}$$

A similar analysis performed for $1 \leqslant j \leqslant N/2$, gives

$$\max_{0 \leq j \leq N/2-1} |U_j^{ext} - u_j| \leqslant CN^{-2}\ln^2 N. \tag{4.5.36}$$

Collecting (4.5.35) and (4.5.36), then gives the main result.

4.6 Numerical results

This section presents the numerical results before and after extrapolation obtained in the integration of some problems of type (4.1.3). The maximum errors and order of convergence are estimated by using the exact solution. The solution in both examples has a turning point at $x = 0$, which gives rise to an interior layer.

Example 4.6.1. Consider the following singularly perturbed turning point problem:

$$(\varepsilon + x^2)u'' + xu' - u = 1 + x^2, \ \ x \in [-1, 1],$$

$$u(-1) = 1, \ \ u(1) = 1$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$. The exact solution is

$$u(x) = -\frac{1}{3}\sqrt{x^2+\varepsilon} \times \frac{2\varepsilon - 5}{\sqrt{\varepsilon+1}} + \frac{1}{3}x^2 - 1 + \frac{2\varepsilon}{3}.$$

Example 4.6.2. Consider the following singularly perturbed turning point problem:

$$(\varepsilon + x^2)u'' + 2xu' - 2u = x^2, \ \ x \in [-1, 1],$$

$$u(-1) = 1, \ \ u(1) = 1$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$. The exact solution is

$$u(x) = -\frac{1}{4}\frac{\left[\sqrt{\varepsilon}\ \arctan\left(\frac{x}{\sqrt{\varepsilon}}\right)x + \varepsilon\right](-3+\varepsilon)}{\sqrt{\varepsilon}\ \arctan\left(\frac{1}{\sqrt{\varepsilon}}\right) + \varepsilon} + \frac{x^2}{4} + \frac{\varepsilon}{4}.$$

While the maximum errors before extrapolation at all mesh points are evaluated using the formula

$$E_{n,\varepsilon} = \max_{0 \le j \le n} |u_j - U_j|,$$

these errors after extrapolation are given by

$$E^{ext}_{n,\varepsilon} = \max_{0 \le j \le n} |u_j - U^{ext}_j|.$$

The numerical rates of convergence before and after extrapolation are obtained by using the formula

$$r_{\varepsilon,k} = \log_2(\tilde{E}_{n_k}/\tilde{E}_{2n_k}),$$

where $\tilde{E}$ represents E or E^{ext}.

Table 4.1: Results for Example 4.6.1: Maximum errors before extrapolation: B-mesh

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	7.54E-02	4.22E-02	2.21E-02	1.12E-02	5.65E-03	2.83E-03	1.42E-03
10^{-3}	8.21E-02	4.61E-02	2.47E-02	1.27E-02	6.35E-03	3.17E-03	1.59E-03
10^{-4}	8.28E-02	4.70E-02	2.49E-02	1.29E-02	6.58E-03	3.29E-03	1.64E-03
10^{-13}	8.80E-02	4.80E-02	2.51E-02	1.28E-02	6.50E-03	3.27E-03	1.64E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	8.80E-02	4.80E-02	2.51E-02	1.28E-02	6.50E-03	3.27E-03	1.64E-03

Table 4.2: Results for Example 4.6.1: Maximum errors after extrapolation: B-mesh

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	1.21E-02	2.56E-03	3.17E-04	3.09E-05	7.77E-06	2.39E-06	6.85E-07
10^{-3}	5.15E-03	5.46E-03	2.37E-03	2.76E-04	5.77E-05	1.94E-05	5.47E-06
10^{-4}	7.36E-03	4.76E-03	6.30E-04	1.49E-03	3.90E-04	4.34E-05	1.68E-05
10^{-13}	4.06E-03	1.22E-03	3.37E-04	8.82E-05	2.25E-05	5.67E-06	1.42E-06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	4.06E-03	1.22E-03	3.37E-04	8.82E-05	2.25E-05	5.67E-06	1.42E-06

Table 4.3: Results for Example 4.6.1: Rates of convergence before extrapolation: B-mesh

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-2}	0.84	0.94	0.97	0.99	1.00	1.00
10^{-3}	0.83	0.90	0.96	1.00	1.00	1.00
10^{-4}	0.82	0.92	0.94	0.98	1.00	1.01
10^{-16}	0.87	0.94	0.97	0.98	0.99	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	0.87	0.94	0.97	0.98	0.99	1.00

Table 4.4: Results for Example 4.6.1: Rates of convergence after extrapolation: B-mesh

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-2}	2.24	3.02	3.36	1.99	1.70	1.80
10^{-3}	-0.08	1.21	3.10	2.26	1.58	1.82
10^{-4}	0.63	2.92	-1.25	1.94	3.16	1.37
10^{-16}	1.73	1.86	1.93	1.97	1.99	1.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	1.73	1.86	1.93	1.97	1.99	1.99

Table 4.5: Results for Example 4.6.1: Maximum errors before extrapolation: S-mesh

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	8.37E-02	4.47E-02	2.27E-02	1.14E-02	5.69E-03	2.84E-03	1.42E-03
10^{-3}	9.05E-02	4.88E-02	2.55E-02	1.29E-02	6.40E-03	3.19E-03	1.59E-03
10^{-4}	9.12E-02	4.97E-02	2.56E-02	1.31E-02	6.64E-03	3.31E-03	1.64E-03
10^{-13}	9.81E-02	5.09E-02	2.58E-02	1.30E-02	6.55E-03	3.28E-03	1.64E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	9.81E-02	5.09E-02	2.58E-02	1.30E-02	6.55E-03	3.28E-03	1.64E-03

Table 4.6: Results for Example 4.6.1: Maximum errors after extrapolation: S-mesh

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	1.47E-02	3.12E-03	3.82E-04	3.80E-05	7.68E-06	1.87E-06	4.66E-07
10^{-3}	6.15E-03	5.36E-03	2.54E-03	3.00E-04	5.71E-05	1.91E-05	5.39E-06
10^{-4}	7.22E-03	5.17E-03	6.52E-04	1.51E-03	4.00E-04	4.34E-05	1.68E-05
10^{-13}	5.09E-03	1.38E-03	3.58E-04	9.09E-05	2.28E-05	5.72E-06	1.43E-06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	5.09E-03	1.38E-03	3.58E-04	9.09E-05	2.28E-05	5.72E-06	1.43E-06

Table 4.7: Results for Example 4.6.1: Rates of convergence before extrapolation: S-mesh

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-2}	0.91	0.98	1.00	1.00	1.00	1.00
10^{-3}	0.89	0.94	0.99	1.01	1.01	1.00
10^{-4}	0.88	0.96	0.96	0.99	1.01	1.01
10^{-13}	0.95	0.98	0.99	0.99	1.00	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	0.95	0.98	0.99	0.99	1.00	1.00

Table 4.8: Results for Example 4.6.1: Rates of convergence after extrapolation: S-mesh

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-2}	2.24	3.03	3.33	2.31	2.04	2.01
10^{-3}	0.20	1.07	3.08	2.39	1.58	1.83
10^{-4}	0.48	2.99	-1.21	1.92	3.21	1.37
10^{-13}	1.88	1.94	1.98	1.99	2.00	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	1.88	1.94	1.98	1.99	2.00	2.00

Table 4.9: Results for Example 4.6.2: Maximum errors before extrapolation: B-mesh

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	4.18E-02	2.33E-02	1.23E-02	6.27E-03	3.17E-03	1.59E-03	7.97E-04
10^{-3}	4.23E-02	2.35E-02	1.24E-02	6.39E-03	3.23E-03	1.62E-03	8.14E-04
10^{-8}	4.19E-02	2.32E-02	1.22E-02	6.30E-03	3.19E-03	1.61E-03	8.07E-04
10^{-18}	4.19E-02	2.32E-02	1.22E-02	6.30E-03	3.19E-03	1.61E-03	8.07E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	4.19E-02	2.32E-02	1.22E-02	6.30E-03	3.19E-03	1.61E-03	8.07E-04

Table 4.10: Results for Example 4.6.2: Maximum errors after extrapolation: B-mesh

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	5.14E-03	1.58E-03	2.10E-04	3.45E-05	8.67E-06	2.17E-06	5.58E-0
10^{-3}	2.29E-03	9.59E-04	1.12E-03	2.21E-04	2.31E-05	9.69E-06	3.02E-06
10^{-8}	2.17E-03	6.37E-04	1.73E-04	6.54E-05	5.37E-05	4.65E-05	4.05E-05
10^{-18}	2.17E-03	6.37E-04	1.73E-04	4.49E-05	1.14E-05	2.89E-06	9.40E-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	2.17E-03	6.37E-04	1.73E-04	4.49E-05	1.14E-05	2.89E-06	9.40E-07

Table 4.11: Results for Example 4.6.2: Rates of convergence before extrapolation: B-mesh

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-2}	0.84	0.93	0.97	0.99	0.99	1.00
10^{-3}	0.85	0.91	0.96	0.98	0.99	1.00
10^{-5}	0.85	0.92	0.96	0.98	0.99	0.99
10^{-18}	0.85	0.92	0.96	0.98	0.99	0.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	0.85	0.92	0.96	0.98	0.99	0.99

Table 4.12: Results for Example 4.6.2: Rates of convergence after extrapolation: B-mesh

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-2}	1.70	2.91	2.61	1.99	2.00	1.96
10^{-3}	1.26	-0.22	2.34	3.26	1.25	1.68
10^{-5}	0.90	0.20	0.79	3.15	-0.98	1.66
10^{-18}	1.77	1.88	1.94	1.97	1.99	1.62
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	1.77	1.88	1.94	1.97	1.99	1.62

Table 4.13: Results for Example 4.6.2: Maximum errors before extrapolation: S-mesh

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	4.65E-02	2.47E-02	1.26E-02	6.36E-03	3.19E-03	1.60E-03	7.98E-04
10^{-3}	4.68E-02	2.49E-02	1.28E-02	6.49E-03	3.26E-03	1.63E-03	8.16E-04
10^{-8}	4.65E-02	2.45E-02	1.26E-02	6.39E-03	3.22E-03	1.61E-03	8.09E-04
10^{-18}	4.65E-02	2.45E-02	1.26E-02	6.39E-03	3.22E-03	1.61E-03	8.09E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	4.65E-02	2.45E-02	1.26E-02	6.39E-03	3.22E-03	1.61E-03	8.09E-04

Table 4.14: Results for Example 4.6.2: Maximum errors after extrapolation: S-mesh

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-2}	5.60E-03	1.85E-03	2.51E-04	3.58E-05	8.86E-06	2.21E-06	5.52E-07
10^{-3}	2.94E-03	7.67E-04	1.17E-03	2.35E-04	2.25E-05	9.50E-06	2.95E-06
10^{-8}	2.69E-03	7.19E-04	1.84E-04	6.57E-05	5.38E-05	4.65E-05	4.06E-05
10^{-18}	2.69E-03	7.19E-04	1.84E-04	4.64E-05	1.16E-05	2.91E-06	9.43E-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	2.69E-03	7.19E-04	1.84E-04	4.64E-05	1.16E-05	2.91E-06	9.43E-07

Table 4.15: Results for Example 4.6.2: Rates of convergence before extrapolation: S-mesh

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-2}	0.91	0.97	0.99	1.00	1.00	1.00
10^{-3}	0.91	0.95	0.98	0.99	1.00	1.00
10^{-5}	0.92	0.96	0.98	0.99	1.00	1.00
10^{-18}	0.92	0.96	0.98	0.99	1.00	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	0.92	0.96	0.98	0.99	1.00	1.00

Table 4.16: Results for Example 4.6.2: Rates of convergence after extrapolation: S-mesh

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-2}	1.59	2.89	2.81	2.01	2.00	2.00
10^{-3}	1.94	-0.60	2.31	3.39	1.24	1.69
10^{-5}	1.22	0.17	0.78	3.07	-0.88	1.65
10^{-18}	1.91	1.97	1.99	2.00	2.00	1.63
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-30}	1.91	1.97	1.99	2.00	2.00	1.63

Remark 4.6.1. Tables of numerical results with Richardson extrapolation show that the computed rate of convergence deviates notably from the theoretical rate of convergence with Richardson extrapolation which is two. This is not surprising as performance of Richardson extrapolation may be hindered by the fact that, for non-uniform (such as the Shishkin and Bakhvalov type) meshes, nodes are offset by the fact that the transition point depends on the number of nodes used in computations. This observation corroborates assertions in the literature regarding issues with implementation of Richardson extrapolation (see e.g. [10, 61, 71])

4.7 Discussion

In this chapter, we proposed a FMFDM for singularly perturbed a class of two-point boundary value problems with a variable coefficient multiplying the second derivative, whose solution exhibits an interior layer due the presence of a turning point. After establishing a set of bounds on the derivatives of the solution, we constructed a mesh of Bakhvalov and Shishkin type on which we designed a discrete upwind scheme. We proved that the proposed method is uniformly convergent of order one for both methods. We used Richardson extrapolation to increase the accuracy of the scheme. The theoretical results were supported by numerical investigations that we carried out on two examples. For each example, we computed the maximum point-wise errors and the corresponding rates of convergence for various values of the step-sizes. We observed that the numerical results based on the FMFDM before and after extrapolation on a B-type mesh were found to be a little inferior as compared to those obtained on the S-type mesh as illustrated in tables 4.1, 4.2, 4.5 and 4.6 for example 4.6.1 and tables 4.9, 4.10, 4.13 and 4.14 for example 4.6.2. Furthermore, we investigated the effect of Richardson extrapolation on the FMFDM for both methods and have observed that it improved the accuracy of the computed solution. In particular, the rate of convergence increased from 1 to 2 as shown in tables 4.3, 4.4, 4.7, 4.8, 4.11, 4.12, 4.15 and 4.16.

Chapter 5

Time-dependent convection-diffusion problems with a power interior layer and variable coefficient diffusion term

This chapter deals with singularly perturbed parabolic problems whose solution displays an interior layer due to the presence of a turning point. The diffusion term is embedded in a quadratic function. After providing appropriate bounds on the solution to these problems and their derivatives, we discretize the time variable with a constant step-size using the implicit Euler method. This process results in a linear system of equations at each time level which is solved using a fitted mesh finite difference method (FMFDM). We also discuss the extrapolation technique to improve the order of convergence on the proposed method.

5.1 Introduction

We examine the following problem

$$Lu :\equiv (\varepsilon + x^2)\,\frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)\,\frac{\partial u(x,t)}{\partial x} - b(x,t)u(x,t) - d(x,t)\,\frac{\partial u(x,t)}{\partial t} = f(x,t), \tag{5.1.1}$$

$$(x,t) \in Q \equiv \Omega \times (0,T] \equiv (-1,1) \times (0,T], \tag{5.1.2}$$

subject to the initial and boundary conditions

$$u(x,0) = u_0(x), \;\; -1 \le x \le 1, \;\; u(-1,t) = \alpha_1(t)\;, \; u(1,t) = \alpha_2(t), \;\; t \in (0,T], \tag{5.1.3}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter. We assume that the functions $a(x,t)$, $b(x,t)$, $d(x,t)$,$f(x,t)$ and initial conditions $u_0(x)$ are sufficiently smooth and $d(x,t) \ge \delta > 0$ in $\bar{Q}$. Furthermore, (i) $a(0,t) = 0$ and $a_x(0,t) > 0\ \forall t \in [0,T]$ guarantees the existence of the turning point, (ii) $b(x,t) \ge \beta > 0\ \forall (x,t) \in \bar{Q}$, which ensures that the problem satisfies a minimum principle and (iii) $|a_x(x,t)| \ge |a_x(0,t)|/2\ \forall (x,t) \in \bar{Q}$ implies that the turning point occurs at $(0,t)$, $\forall t \in [0,T]$. Under the assumptions $(i) - (iii)$, the turning point problem (5.1.1)-(5.1.3) possesses a unique solution exhibiting an interior layer at the point $x = 0$ [19]. Also, we impose the compatibility conditions

$$u_0(-1) = \alpha_1(0) \text{ and } u_0(1) = \alpha_2(0),$$

so that the data match at the two corners $(-1,0)$ and $(1,0)$ of the domain $\bar{Q}$.

In [72], it was proved that there exists a constant C independent of ε such that

$$|u(x,t) - \alpha_1(t)| \le C(1+x), \;\; |u(x,t) - \alpha_2(t)| \le C(1-x), \quad \forall (x,t) \in \bar{Q}$$

and

$$|u(x,t) - u_0(x)| \le Ct, \;\; \forall (x,t) \in \bar{Q}.$$

Time-dependent singularly perturbed problems are widely studied in the literature. Such problems arise in several fields of engineering and applied mathematics, including convection-dominated flows in fluid mechanics, heat and mass transfer in chemical and nuclear engineering, electromagnetic theory [11, 23, 30, 44, 59].

A number of authors studied a class of time-dependent singularly perturbed problems with non turning points [11, 30, 72]. Turning point problems are those where the coefficient of the convective term vanishes inside the spatial-domain by changing signs. Examples of works where turning points give rise to boundary and/or interior layers may be found [5, 12, 13, 16, 19, 40, 44, 47, 53, 58, 64].

All the works listed above are characterised by a small parameter ε in front of the highest derivative. Parallel to a constant perturbation parameter ε, it is important to study problems with diffusion terms are functions of x and ε.

However [37] and [39] studied the problem (5.1.1)-(5.1.3) in the space variable only. In [37], Liseikin derived bounds on the solution and its derivatives for the problem $-(\varepsilon + px)^{\beta}u'' + a(x)u + f(x,\varepsilon) = 0,\ 0 \le x \le 1,\ p = 0,1\ ,\beta \ge 1$. While in [39], Liseikin considered the equation $-(\varepsilon + x)^{\beta}u'' - a(x)u + f(x,\varepsilon) = 0,\ 0 \le x \le 1\ ,\beta > 0$. Bounds on the solution and its derivatives were established (see p. 160-111) while on p. 256-262, for $\beta = 1$ a numerical method was presented and its convergence analysed.

In this chapter, we focus on studying the time-dependent problem (5.1.1)-(5.1.3) where the coefficients of the differential equations depend on both space and time, and are smooth. We propose and analyse a fitted mesh finite difference method (FMFDM). It turns out from convergence analysis that the proposed method is uniformly convergent of order one, up to a logarithmic factor with respect to ε.

The rest of this chapter is organised as follows. We establish bounds on the solution

$u(x,t)$ and its derivatives in Section 2. Section 3 is devoted to constructing a FMFDM applied on a Shishkin mesh. We show that the proposed method satisfies a minimum principle. We use this fact to establish a stability result. In Section 4, we conduct a rigourous error analysis. We prove that the proposed numerical method is almost first order uniformly convergent with respect to the perturbation parameter in time and space, up to a logarithmic factor with respect to ε. In order to enhance the accuracy and order of convergence of the proposed FMFDM, we apply Richardson extrapolation in Section 5 to obtain almost second order uniform convergence in space. Numerical experiments are presented in Section 6 for two examples to confirm our theoretical results. Finally, some conclusions are drawn in the last Section.

In the rest of this chapter, we use C as a generic positive constant which may assume different values in different inequalities but will always be independent of ε, the spatial and time discretization parameters.

5.2 A priori estimates of the solution and its derivatives

Bounds on the solution to problem (5.1.1)-(5.1.3) and its derivatives are the subject of this section.
The interval $[-1,1]$ which we denote $\bar{\Omega}$ is divided as follows $\Omega_l = [-1,-\tau], \Omega_c = [-\tau,\tau]$ and $\Omega_r = [\tau, 1]$, where $0 < \tau \leq 1/2$.

The linear operator L as defined in (5.1.1) satisfies the following minimum principle and then we state a stability estimate for the solution of (5.1.1)-(5.1.3).

Lemma 5.2.1. (Minimum principle). Suppose $\xi(x,t)$ is a smooth function satisfying $\xi(\pm 1, t) \geq 0$ and $L\xi(x,t) \leq 0$, $\forall x \in \Omega$. Then $\xi(x,t) \geq 0$, $\forall x \in \bar{\Omega}$.

Proof. Let $(x^*, t^*) \in \bar{Q}$ such that $\xi(x^*,t^*) = \min\limits_{x\in[-1,1]} \xi(x,t)$ and assume that $\xi(x^*,t^*) < 0$.

It follows that $(x^*, t^*) \notin Q$, therefore $\xi_x(x^*, t^*) = 0$, $\xi_t(x^*, t^*) = 0$ and $\xi_{xx}(x^*, t^*) \geq 0$. Then, we obtain

$$L\xi(x^*, t^*) = (\varepsilon + x^{*2})\xi_{xx}(x^*, t^*) + a(x^*, t^*)\xi_x(x^*, t^*) - b(x^*, t^*)\xi(x^*, t^*) - d(x^*, t^*)\xi_t(x^*, t^*) > 0,$$

which is a contradiction. It follows that $\xi(x^*, t^*) \geq 0$ and thus $\xi(x, t) \geq 0$, $\forall x \in \bar{Q}$.

We apply this minimum principle to prove the next results which state that the solution depends continuously on the data.

Lemma 5.2.2. (Stability estimate). If $u(x, t)$ is the solution of (5.1.1)-(5.1.3), then we have

$$||u(x, t|| \leqslant [\max\{||\alpha_1||_\infty, ||\alpha_2||_\infty\}] + \frac{1}{\beta}||f||_\infty, \forall (x, t) \in \bar{Q}.$$

Proof. See Lemma 3.2.2 in Chapter 3.

The next Lemma provide estimates of u and its derivatives in the interval $[-1, -\tau]$ and $[\tau, 1]$.

Lemma 5.2.3. The bound on the solution $u(x, t)$ of (5.1.1) is given by
$|u(x, t)| \leq C$, $(x, t) \in \bar{Q}$.

Proof. See [30].

Lemma 5.2.4. Let $u(x, t)$ be the solution to (5.1.1)-(5.1.3) and $a(x, t), b(x, t)$ and $f(x, t)$ sufficiently smooth function in $\bar{Q}$. Then, there exists a positive constant C independent of ε, such that

$$\left|\frac{\partial^j u(x, t)}{\partial x^j}\right| \leq C, \ \forall x \in \Omega_l \text{ or } \Omega_r \text{ and } (x, t) \in \bar{Q}, 0 \leq j \leq 2.$$

Proof. See [11].

Lemma 5.2.5. Under the assumption of Lemmas 5.2.1 and 5.2.4, the bound on the derivative of u with respect to t is $|u_t(x, t)| \leq C, \quad (x, t) \in \bar{Q}$.

Proof. See [30].

Lemma 5.2.6. $|u_{xt}(x,t)| \leqslant C, \quad (x,t) \in \bar{Q}$.

Proof. See [31].

Based on the ideas of [39], we will be able to establish the next lemma. Note that the solution of (5.1.1)-(5.1.3) has an interior layer at the point $x_{N/2} = 0$. Then, the derivatives of $u(x,t)$ are estimated in the vicinity $x_{N/2} = 0$ by polynomial functions according to the sign of the coefficient of the convection term $a(x,t)$ at the point $x_0^\star$. Therefore, we present two different cases

$$a = \begin{cases} a(x_0^\star, t) \leqslant 0, \quad x_0^\star \in [-\tau, 0], \ t \in [0,T], \\ a(x_0^\star, t) > 0, \quad x_0^\star \in (0, \tau], \ t \in [0,T]. \end{cases} \tag{5.2.1}$$

Lemma 5.2.7. Let $u(x,t)$ be the solution of (5.1.1)-(5.1.3). Then assuming that $a = a(x_0^\star, t) > 0$, for $0 < x \leqslant \tau$, $\forall t \in [0,T]$, we have

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} 1 + (\varepsilon + x^2)^{1-a-j}, & 0 < a < 1, \ j = 1,2,3,4, \\ 1 + (\varepsilon + x^2)^{-j}, & a = 1, \ j = 1,2,3,4, \\ 1 + \varepsilon^{a-1}(\varepsilon + x^2)^{1-a-j}, & a > 1, \ j = 1,2,3,4, \end{cases} \tag{5.2.2}$$

and $a = a(x_0^\star, t) \leqslant 0$, for $-\tau \leqslant x \leqslant 0$, and let p be a whole number such that $a + p = 0$ and $a + p - 1 < 0$, $\forall t \in [0,T]$, then we have the following bounds

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} 1, & a < 0, \ j \leqslant p, \ j = 1,2,3,4, \\ 1 + (\varepsilon + x^2)^{1-j-p} \arctan(x/\sqrt{\varepsilon}), & a + p = 0, \ j > p, \ j = 1,2,3,4, \\ 1 + (\varepsilon + x^2)^{-a-j}, & a + p > 0, \ j > p, \ j = 1,2,3,4. \end{cases} \tag{5.2.3}$$

Proof. This Lemma will be proved by following the ideas of ([39], from pp. 107-110). Application of the inverse-monotone pair $T = (L, \Gamma)$ (see pp 49) implies that

$$|u(x,t)| \leqslant C, \quad (x,t) \in \bar{Q}. \tag{5.2.4}$$

From (5.1.1)-(5.1.3) and (5.2.4) and $\forall t \in [0, T]$, we obtain

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} 1, & -\tau < x_0 \leqslant x \leqslant 0, \\ \varepsilon^{-j}, & -\tau \leqslant x \leqslant x_0,\ j = 1,2,3,4, \\ 1, & 0 < x_0 \leqslant x \leqslant \tau, \\ \varepsilon^{-j}, & 0 \leqslant x \leqslant x_0,\ j = 1,2,3,4, \end{cases} \tag{5.2.5}$$

and arbitrary $x_0 > 0$, independent of ε and x.

case 1 : a > 0 for $0 < x \leqslant 1$, $\forall t \in [0, T]$. In this case, the derivatives of $u(x,t)$ are estimated according to the value of $a : 0 < a < 1$, $a = 1$ and $a > 1$. Solving (5.1.1) for $u_{xx}(x,t)$, we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| = u_{xx}(x,t) = \frac{f(x,t) + b(x,t)u(x,t) + d(x,t)u_t(x,t)}{(\varepsilon + x^2)} - \frac{a(x,t)u_x(x,t)}{(\varepsilon + x^2)}. \tag{5.2.6}$$

Integrating (5.2.6) on both sides from 0 to x, we obtain

$$u_x(x,t) = \int_0^x \frac{f(s,t) + b(s,t)u(s,t) + d(s,t)u_t(s,t)}{\varepsilon + s^2}\, ds - \int_0^x \frac{a(s,t)u_x(s,t)}{\varepsilon + s^2}\, ds. \tag{5.2.7}$$

$u_x(x,t)$ can be expressed as follows

$$u_x(x,t) = u_x(0,t)\left[\frac{\varepsilon}{\varepsilon + x^2}\right]^a \exp[-g_1(x,t)] + g_2(x,t), \tag{5.2.8}$$

where

$$g_1(x,t) = \int_0^x \frac{a(s,t)}{\varepsilon + s^2}\, ds = \frac{a(x,t)}{\sqrt{\varepsilon}} \arctan(x/\sqrt{\varepsilon}) - \int_0^x \frac{a_s(s,t)}{\sqrt{\varepsilon}} \arctan(s/\sqrt{\varepsilon})\, ds, \tag{5.2.9}$$

with $a(0,t) = 0$, and

$$g_2(x,t) = (\varepsilon + x^2)^{-a} \int_0^x [f(s,t) + b(s,t)u(s,t) + d(s,t)u_t(s,t)](\varepsilon + s^2)^{a-1} \exp[g_1(s,t) - g_1(x,t)]\, ds. \tag{5.2.10}$$

Since $|g_1(x,t)| \leqslant C$ from (5.2.4), we find that

$$|g_2(x,t)| \leqslant C(\varepsilon + x^2)^{-a} \int_0^x (\varepsilon + s^2)^{a-1}\, ds \leqslant C.$$

Applying the triangle inequality in (5.2.8) and taking into account the estimates of $g_1(x,t)$ and $g_2(x,t)$, we obtain

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C\left[1 + |u_x(0,t)|(\varepsilon/(\varepsilon+x^2))^a\right]. \tag{5.2.11}$$

Considering $0 < a < 1$, there is a point x_0 in the interval $(0,\tau)$ such that $|u_x(x_0,t)| \leqslant C$, so that

$$|u_x(0,t)|\left(\frac{\varepsilon}{\varepsilon+x_0^2}\right)^a \leqslant C.$$

This inequality yields

$$|u_x(0,t)| \leqslant C\left(\frac{\varepsilon+x_0^2}{\varepsilon}\right)^a \leqslant C(\varepsilon+x_0^2)^a\varepsilon^{-a} \leqslant C\varepsilon^{-a}.$$

Using the estimates obtained for $|u_x(0,t)|$ in (5.2.11), we obtain

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C\left[1 + (\varepsilon+x^2)^{-a}\right],\ 0 < a < 1.$$

Differentiating (5.1.1) with respect to x, solving the resulting equation for $u_{xxx}(x,t)$, we obtain

$$u_{xxx}(x,t) = \frac{b(x,t)-a_x(x,t)}{\varepsilon+x^2}u_x(x,t) - \frac{a_x(x,t)+2s}{(\varepsilon+x^2)}u_{xx}(x,t) + \frac{f_x(x,t)+b_x(x,t)u(x,t)+d_x(x,t)u_t(x,t)-d(x,t)u_{tx}(x,t)}{(\varepsilon+x^2)}. \tag{5.2.12}$$

Below is the expression of $\partial^2 u(x,t)/\partial x^2$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}(x,t) = u_{xx}(0,t)\left[\frac{\varepsilon}{\varepsilon+x^2}\right]^{a+1}\exp[-g_3(x,t)] + g_4(x,t), \tag{5.2.13}$$

where

$$g_3(x,t) = \int_0^x \frac{a(s,t)+2s}{\varepsilon+s^2}\,ds = \frac{a(x,t)+2x}{\sqrt{\varepsilon}}\arctan(x/\sqrt{\varepsilon}) - \int_0^x \frac{a_s(s,t)+2}{\sqrt{\varepsilon}}\arctan(s/\sqrt{\varepsilon}))\,ds, \tag{5.2.14}$$

with $a(0,t) = 0$, and

$$g_4(x,t) = (\varepsilon+x^2)^{-a-1}\int_0^x [f_s(s,t)+b_s(s,t)u(s,t)+d_s(s,t)u_t(s,t)-d(s,t)u_{ts}(s,t)+ (b(s,t)-a_s(s,t)u_s(s,t)](\varepsilon+s^2)^a\exp[g_3(s,t)-g_3(x,t)]\,ds. \tag{5.2.15}$$

Noting that $|g_3(x,t)| \leqslant C$, $|u_t(x,t)| \leqslant C$, $|u_{tx}(x,t)| \leqslant C$, and $|u(x,t)| \leqslant C$, we find that

$$|g_4(x,t)| \leqslant C(\varepsilon + x^2)^{-a-1} \int_0^x [1 + u_s(s,t)](\varepsilon + s^2)^a \, ds \leqslant C[1 + (\varepsilon + x^2)^{-a}]. \qquad (5.2.16)$$

We obtain from (5.1.1)

$$u_{xx}(0,t) \leqslant C\varepsilon^{-1}[1 + u_x(0,t)] \leqslant C\varepsilon^{-a-1}.$$

Substituting the estimate of $u_{xx}(0,t)$ and $g_4(x,t)$ into (5.2.13), we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| = u_{xx}(x,t) \leqslant C\varepsilon^{-1-a}\varepsilon^{a+1}(\varepsilon + x^2)^{-a-1} + C\left[1 + (\varepsilon + x^2)^{-a}\right] \leqslant C\left[1 + (\varepsilon + x^2)^{-a-1}\right].$$

Differentiating equation (5.1.1)-(5.1.3) and taking into account (5.2.4), we obtain the following result

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C\left[1 + (\varepsilon + x^2)^{-a+1-j}\right].$$

Consider the case when $a = 1$. On integrating (5.2.8) from 0 to τ, we obtain

$$u(\tau,t) - u(0,t) = u_x(0,t)\varepsilon^{1/2}\{\arctan(\tau/\sqrt{\varepsilon})\exp[-g_1(\tau,t)]$$
$$+ \int_0^\tau a(x,t)(\varepsilon + x^2)^{-1}\arctan(x/\sqrt{\varepsilon})\exp[-g_1(x,t)]\,dx\} + \int_0^\tau g_2(x,t)\,dx. \qquad (5.2.17)$$

Since

$$|\arctan(\tau/\sqrt{\varepsilon})\exp[-g_1(\tau,t)] + \int_0^\tau a(x,t)(\varepsilon + x^2)^{-1}\exp[-g_1(x,t)]\arctan(x/\sqrt{\varepsilon})\,dx| \leqslant C,$$

using the triangle inequality in (5.2.17) and taking into account the inequality above, we obtain

$$|u_x(0,t)|\,\varepsilon^{1/2} \leqslant C.$$

This inequality yields $|u_x(0,t)| \leqslant C\varepsilon^{-1/2}$. From (5.2.11), we obtain

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C[1 + \varepsilon^{1/2}(\varepsilon + x^2)^{-1}] \leqslant C[1 + (\varepsilon + x^2)^{-1}].$$

Consider $u_{xx}(x,t)$ for $a = 1$. In this case (5.2.13) gives

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}(x,t) = u_{xx}(0,t)\left[\frac{\varepsilon}{\varepsilon + x^2}\right]^2 \exp[-g_3(x,t)] + g_4(x,t). \qquad (5.2.18)$$

From (5.2.16), $g_4(x,t)$ is defined as follows

$$|g_4(x,t)| \leqslant C(\varepsilon+x)^{-2}\int_0^x [1+u'(s)](\varepsilon+s^2)\,ds \leqslant C[1+\varepsilon^{1/2}(\varepsilon+x^2)^{-1}]. \tag{5.2.19}$$

Moreover, by (5.1.1) we have

$$u_{xx}(0,t) \leqslant C\varepsilon^{-1}[1+u_x(0,t)] \leqslant C\varepsilon^{-3/2}.$$

From (5.2.18) we obtain using the estimates of $u_{xx}(0,t)$ and $g_4(x,t)$:

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| = u_{xx}(x,t) \leqslant C[1+\varepsilon^{1/2}(\varepsilon+x^2)^{-2}] \leqslant C[1+(\varepsilon+x^2)^{-2}].$$

By differentiating (5.1.1) and with the help of (5.2.5), we arrive at the following result

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C[1+(\varepsilon+x^2)^{-j}],\ a=1,\ 0\leqslant x\leqslant\tau,\ j=1,2,3,4.$$

This completes the proof of the estimate 5.2.2 for $0 < x \leqslant \tau$.

case 2 : $\mathbf{a \leqslant 0}$ for $-\tau \leqslant x \leqslant 0$. In this case, $u_x(x,t)$ is given as follows [39]

$$u_x(x,t) = u_x(x_0,t)\exp[\psi(x_0,x,t)] + \int_{x_0}^x \frac{f(s,t)+b(s,t)u(s,t)+d(s,t)u_t(s,t)}{\varepsilon+s^2}\exp[\psi(x_0,x,t)]\,ds, \tag{5.2.20}$$

where

$$\psi(s,x,t) = -\int_s^x \frac{a(\kappa,t)}{\varepsilon+\kappa^2}\,d\kappa.$$

If $a(0,t)=0$ then $\psi(s,x,t)\leqslant C,\ -\tau\leqslant s,\ x\leqslant 0$. Using the triangle inequality in (5.2.20) and choosing a point $x_0\in[-\tau/2,0]$ such that $u'(x_0,t)\leqslant C$, we obtain

$$u_x(x,t)| \leqslant C[1+\varepsilon^{-1/2}\arctan(x/\sqrt{\varepsilon})] \leqslant C[1+\arctan(x/\sqrt{\varepsilon})],\ \ a(0)=0,\ j=1 \text{ since } p=0.$$

We wish to determine $u_{xx}(x,t)$ with $p=0$ for $j=2$. On differentiating (5.1.1) and solving the resulting equation for $u_{xx}(x,t)$, we obtain

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}(x_0,t)\exp[\psi(x_0,x,t)] + (\varepsilon+x^2)^{-p-1}\int_{x_0}^x \frac{F(s,t)}{\varepsilon+s^2}(\varepsilon+s^2)^{p+1}\exp[\psi(s,x,t)]\,ds, \tag{5.2.21}$$

where

$$\psi(s,x,t) = -\int_s^x \frac{a(\kappa,t)}{\varepsilon+\kappa^2}\,d\kappa$$

and

$$F(s,t) = f(s,t) + b_s(s,t)u(s,t) + d_s(s,t)u_t(s,t) + [b(s,t) - a_s(s,t)]u_s(s,t).$$

Substituting $\psi(s,x,t) \leqslant C$ and $u_{xx}(x_0,t) \leqslant C$ into (5.2.21), we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leqslant C + C(\varepsilon+x^2)^{-p-1}\int_{x_0}^x [1+u_s(s,t)](\varepsilon+s^2)^p\,ds \leqslant C[1+(\varepsilon+x^2)^{-p-1}\arctan(x/\sqrt{\varepsilon})].$$

From (5.1.1)-(5.1.3) with $p=0,$ for $j>1$, we obtain

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C[1+(\varepsilon+x^2)^{-1-j-p}\arctan(x/\sqrt{\varepsilon})],\;\; a+p=0,\; j>p.$$

Let $a(0,t) < 0$. In this case $p > 1$. Then there exists a constant $x_0 > 0$ such that $a(x,t) < 0$ for $-\tau \leqslant x \leqslant x_0$. Therefore, we have

$$\psi(s,x,t) \leqslant -x_0 \ln[(\varepsilon+s^2)/(\varepsilon+x^2)],\; -\tau \leqslant x \leqslant s \leqslant x_0.$$

Taking exponentials on both sides of the above inequality, we obtain

$$\exp(\psi(s,x,t)) \leqslant [(\varepsilon+x^2)/(\varepsilon+s^2)]^{-x_0},\; -\tau \leqslant x \leqslant s \leqslant x_0.$$

Substituting this estimate into (5.2.20) with $x=s$ and taking into account (5.2.5), we obtain

$$|u_x(x,t)| = \left|\frac{\partial u(x,t)}{\partial x}\right| \leqslant C,\; -\tau \leqslant x \leqslant x_0,\; a(0,t) < 0.$$

Differentiating (5.1.1) and taking into account (5.2.5), we obtain

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C,\; -\tau \leqslant x \leqslant 0,\; a<0,\; k \leqslant p,\; j=1,2,3,4.$$

Consider the case when $j>p,\; a+p>0$ and $a \leqslant 0$. We will estimate $u_x(x,t)$ and $u_{xx}(x,t)$ by following the same steps as we did for $0<a<1$. We define $u_x(x,t)$ from (5.2.8) as follows:

$$u_x(x,t) = u_x(0,t)\left[\frac{\varepsilon}{\varepsilon+x^2}\right]^{a+1}\exp[-g_1(x,t)] + g_2(x,t),\; -\tau \leqslant x \leqslant 0. \tag{5.2.22}$$

Following the same steps as we did for $0 < a < 1$, we obtain

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C[1 + (\varepsilon + x^2)^{-a-1}],\ a \leqslant 0.$$

We estimate $u_{xx}(x,t)$ from (5.2.13) as follows

$$u_{xx}(x,t) = u_{xx}(0,t)\left[\frac{\varepsilon}{\varepsilon + x^2}\right]^{a+2} \exp[-g_3(x,t)] + g_4(x,t),\ -\tau \leqslant x \leqslant 0. \tag{5.2.23}$$

Following the same steps as we did for $0 < a < 1$, we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| = |u_{xx}(x,t)| \leqslant C[1 + (\varepsilon + x^2)^{-a-2}],\ a \leqslant 0.$$

Differentiating (5.1.1) and taking into account (5.2.5), we obtain

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C[1 + (\varepsilon + x^2)^{-a-j}],\ -\tau \leqslant x \leqslant 0,\ a \leqslant 0,\ j > p.$$

This complete the proof of the estimate 5.2.3 for $-\tau < x \leqslant 0$.

According to the sign of the coefficient of the convection term $a(x,t)$, the singularly perturbed turning point problem (5.1.1)-(5.1.3) may be regarded as a concatenation of two problems: One side $a(x,t) < 0$ for $-1 \leqslant x < 0$ and the other side $a(x,t) > 0$ for $0 < x \leqslant 1$. Therefore the solution of the problem (5.1.1)-(5.1.3) may display a layer near $x = 0$ on $[-1, 0)$ and a layer near $x = 0$ on $(0, 1]$. This consideration will be useful firstly in seeking an in-depth understanding of the behaviour of the solution and its derivatives and secondly, in the design of the numerical method in Section 5.4. The solution can be decomposed into two parts, namely the smooth component $v(x,t)$ and the singular component $w(x,t)$ ([42], pp 47) such that

$$u(x,t) = v(x,t) + w(x,t),$$

where $v(x,t)$ is the solution of the inhomogeneous problem

$$Lv(x,t) = f(x,t),\ \ (x,t) \in \Omega_1 = (-1,0) \times (0,T], \tag{5.2.24}$$

$$v(x,0) = u(x,0) = u_0,\ \ -1 \leqslant x \leqslant 0, \tag{5.2.25}$$

$$v(-1,t) = u(-1,t), \;\; 0 \leqslant t \leqslant T, \tag{5.2.26}$$

and $w(x,t)$ is the solution of the homogeneous problem

$$Lw(x,t) = 0, \;\; (x,t) \in \Omega_1, \tag{5.2.27}$$

$$w(x,0) = 0, \;\; -1 \leqslant x \leqslant 0, \tag{5.2.28}$$

$$w(-1,t) = 0, \; 0 \leqslant t \leqslant T, \tag{5.2.29}$$

$$w(0,t) = u(0,t) - v(0,t), \; 0 \leqslant t \leqslant T. \tag{5.2.30}$$

We establish the following lemma which gives bounds on the solution to (5.1.1)-(5.1.3) and its derivatives.

Lemma 5.2.8. The smooth and singular components of $u(x,t)$ of problem (5.1.1)-(5.1.3), for $0 \leq j \leq 4$, and $0 \leqslant t \leqslant T$, satisfy

$$\left|\frac{\partial^j v(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} 1 + (\varepsilon + x^2)^{3-j} \arctan(x/\sqrt{\varepsilon}), & x \in [-1,0], \\ 1 + (\varepsilon + x^2)^{3-a-j}, & a < 1, \; x \in [0,1], \\ 1 + (\varepsilon + x^2)^{2-j}, & a = 1, \; x \in [0,1], \\ 1 + \varepsilon^{a-1}(\varepsilon + x^2)^{3-a-j}, & a > 1, \; x \in [0,1], \end{cases} \tag{5.2.31}$$

and

$$\left|\frac{\partial^j w(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} (\varepsilon + x^2)^{1-j} \arctan(x/\sqrt{\varepsilon}), & x \in [-1,0], \\ (\varepsilon + x^2)^{1-a-j}, & a < 1, \; x \in [0,1], \\ (\varepsilon + x^2)^{-j}, & a = 1, \; x \in [0,1], \\ \varepsilon^{a-1}(\varepsilon + x^2)^{1-a-j}, & a > 1, \; x \in [0,1], \end{cases} \tag{5.2.32}$$

where C is a constant and independent of ε.

Proof. We prove this lemma on $\Omega_1 = [-1,0]$. The proof on $[0,1]$ follows similar steps. We obtain the reduced problem ($\varepsilon = 0$) from (5.1.1) as follows

$$x^2 v^0_{xx} + a(x,t)v^0_x(x,t) - b(x,t)v^0(x,t) - d(x,t)v^0_t(x,t) = f(x,t), \;\; (x,t) \in \Omega_1 \tag{5.2.33}$$

$$v^0(x,0) = v^0_0(x), \;\; -1 \leqslant x \leqslant 0, \tag{5.2.34}$$

$$v^0(-1,t) = \alpha_1(t), \quad t \in (0,T]. \tag{5.2.35}$$

Further, we decompose the smooth component $v(x,t)$ ([42], pp 68) as follows

$$v(x,t) = v_0(x,t) + (\varepsilon + x^2)v_1(x,t) + (\varepsilon + x^2)^2 v_2(x,t), \quad (x,t) \in \bar{\Omega}, \tag{5.2.36}$$

where v_0 is the solution of the reduced problem in (5.2.33), which is independent of ε, and having smooth coefficients $a(x,t)$, $b(x,t)$ and $f(x,t)$. From these assumptions, for $0 \leqslant j \leqslant 4$, we have

$$\left|\frac{\partial^j v_0(x,t)}{\partial x^j}\right| \leqslant C, \text{ for all } x \in \bar{\Omega}_1. \tag{5.2.37}$$

v_1 and v_2 are the solutions of (5.1.1). By using Lemma 5.2.7 for $-1 \leqslant x \leqslant 0$, we have the following bounds

$$\left|\frac{\partial^j v_1(x,t)}{\partial x^j}\right| \leqslant C[1 + (\varepsilon + x^2)^{1-j} \arctan(x/\sqrt{\varepsilon})], \text{ for } \ 0 \leqslant j \leqslant 4 \tag{5.2.38}$$

and

$$\left|\frac{\partial^j v_2(x,t)}{\partial x^j}\right| \leqslant C[1 + (\varepsilon + x^2)^{1-j} \arctan(x/\sqrt{\varepsilon})], \text{ for } \ 0 \leqslant j \leqslant 4. \tag{5.2.39}$$

Now, applying the triangle inequality and substituting these above three estimates into (5.2.36), for $0 \leqslant j \leqslant 4$, we obtain

$$\begin{aligned}
\left|\frac{\partial^j v(x,t)}{\partial x^j}\right| &\leqslant \left|\frac{\partial^j v_0(x,t)}{\partial x^j}\right| + (\varepsilon + x^2)\left|\frac{\partial^j v_1(x,t)}{\partial x^j}\right| + +(\varepsilon + x^2)^2\left|\frac{\partial^j v_2(x,t)}{\partial x^j}\right| \\
&\leqslant C[1 + (\varepsilon + x^2)^{3-j} \arctan(x/\sqrt{\varepsilon})].
\end{aligned}$$

To prove the regular component $w(x,t)$, let us define the barrier functions as follows [31].

$$\Psi^{\pm}(x,t) = C\exp(\eta x/\varepsilon)e^t \pm w(x,t), \ (x,t) \in \bar{\Omega}_1.$$

Compute the values of $\Psi^{\pm}(x,t)$ at the boundaries:

$$\begin{aligned}
\Psi^{\pm}(-1,t) &= C\exp(-\eta/\varepsilon)e^t \pm w(-1,t),\ 0 \leqslant t \leqslant T,\\
&= C\exp(-\eta/\varepsilon)e^t, \ \ \text{using} \ \ (5.2.29),\\
&\geqslant 0, \quad 0 \leqslant t \leqslant T,\\
\Psi^{\pm}(0,t) &= Ce^t \pm w(0,t),\ 0 \leqslant t \leqslant T,\\
&= Ce^t \pm [u(0,t) - v(0,t)], \ \ \text{using} \ \ (5.2.30),\\
&\geqslant 0, \ \ \text{for a suitable choice of C}, \ \ 0 \leqslant t \leqslant T,\\
\Psi^{\pm}(x,0) &= C\exp(\eta x/\varepsilon) \pm w(x,0),\ -1 \leqslant x \leqslant 0,\\
&= C\exp(\eta x/\varepsilon), \ \ \text{using} \ \ (5.2.28),\\
&\geqslant 0, \quad -1 \leqslant x \leqslant 0.
\end{aligned}$$

From the above estimates, we notice that $\Psi(x,t) \geqslant 0, \ (x,t) \in \Omega_2 = \bar{\Omega}_1 \backslash \Omega_1$. Therefore we have

$$\begin{aligned}
L\Psi^{\pm}(x,t) &= (\varepsilon + x^2)\Psi^{\pm}_{xx}(x,t) + a(x,t)\Psi^{\pm}_{x}(x,t) - b(x,t)\Psi^{\pm}(x,t) - d(x,t)\Psi^{\pm}_{t}(x,t)\\
&= C\exp(\eta x/\varepsilon)e^t\left[\frac{\eta^2}{\varepsilon} + \frac{\eta a(x,t)}{\varepsilon} - b(x,t) - d(x,t)\right] \pm Lw(x,t)\\
&= C\exp(\eta x/\varepsilon)e^t\left[\frac{\eta^2(\varepsilon + x^2)}{\varepsilon^2} + \frac{\eta a(x,t)}{\varepsilon} - b(x,t) - d(x,t)\right], \ \ \text{using} \ \ (5.2.27)\\
&\leqslant 0, \ \text{since} \ (x/\varepsilon)^2 \leqslant b(x,t) \ \ (x,t) \in \Omega_1.
\end{aligned}$$

Now applying Lemma 5.2.1 to the barrier functions, we obtain $\Psi^{\pm}(x,t) \geqslant 0, \ \ (x,t) \in \bar{\Omega}_1$. Then we have

$$C\exp(\eta x/\varepsilon)e^t \pm w(x,t) \geqslant 0.$$

It follows that

$$\begin{aligned}
w(x,t) &\leqslant C\exp(\eta x/\varepsilon)e^t, \ (x,t) \in \Omega_1\\
&\leqslant C\exp(\eta x/\varepsilon)e^T \ \text{since} \ \ e^t \leqslant e^T\\
&\leqslant C\exp(\eta x/\varepsilon) \ \ (x,t) \in \Omega_1.
\end{aligned}$$

By using the inequality relation, the above inequality can expressed as follows

$$|w(x,t)| \leqslant C(\varepsilon+x^2)^0 \exp(\eta x/\varepsilon) \leqslant C(\varepsilon+x^2)^1 \arctan(x/\varepsilon) \ \ (x,t) \in \Omega_1.$$

Since $Lw(x,t) = 0$, the j^{th} derivative of $w(x,t)$ can be estimated immediately from the estimate of $w(x,t)$,

$$\left|\frac{\partial^j w(x,t)}{\partial x^j}\right| \leqslant C(\varepsilon+x^2)^{1-j} \arctan(x/\varepsilon), \ 0 \leqslant j \leqslant 4.$$

This completes the proof.

We design a FMFDM to solve time-dependent convection-diffusion problems (5.1.1)-(5.1.3) in the next section.

5.3 Construction of the FMFDM

Time discretization

We use the Euler implicit method to discretize problem (5.1.1)-(5.1.3) with uniform step-size $\Delta t = T/K$. The time $[0,T]$ is therefore partitioned as

$$\bar{w}^K = \{t_k = k\Delta t, 0 \leqslant k \leqslant K\}. \tag{5.3.1}$$

We discretize problem (5.1.1)-(5.1.3) on $\bar{w}^K$ as follows

$$(\varepsilon+x^2)z_{xx}(x,t_k)+a(x,t_k)z_x(x,t_k)-b(x,t_k)z(x,t_k)-d(x,t_k)\frac{z(x,t_k)-z(x,t_{k-1})}{\Delta t} = f(x,t_k), \tag{5.3.2}$$

subject to

$$z(x,0) = z_0(x), \ \ -1 \leq x \leq 1, \ \ z(-1,t_k) = \alpha_1(t), \ \ z(1,t_k) = \alpha_2(t). \tag{5.3.3}$$

Now, (5.3.2) can be written as

$$Lz(x,t_k) = f(x,t_k) - d(x,t)\frac{z(x,t_{k-1})}{\Delta t}. \tag{5.3.4}$$

subject to

$$z(x,0) = z_0(x), \quad -1 \le x \le 1, \quad z(-1,t_k) = \alpha_1(t), \quad z(1,t_k) = \alpha_2(t), \tag{5.3.5}$$

where

$$Lz(x,t_k) = (\varepsilon + x^2)z_{xx}(x,t_k) + a(x,t_k)z_x(x,t_k) - \left[b(x,t_k) + \frac{d(x,t_k)}{\Delta t}\right] z(x,t_{k-1}).$$

The local truncation error e_k at each time level to t_k, is given by

$$e_k = u(x,t_k) - z(x,t_k),$$

where $z(x,t_k)$ is the solution of (5.3.4)-(5.3.5).

The local error estimate of the time discretization

$$\|e_k\|_\infty \le C(\Delta t)^2, \quad 1 \le k \le K. \tag{5.3.6}$$

The global error estimate of the time discretization :

$$\|E_k\|_\infty \le C\Delta t, \quad 1 \le k \le K. \tag{5.3.7}$$

Spatial discretization

We develop a difference scheme to solve the problem (5.1.1)-(5.1.3). We consider the following partition of the interval $[-1,1]$ which we denote $\bar{\Omega}^N$: $x_0 = -1$, $x_{N/2} = 0$, $x_N = 1$ and let $\bar{Q}^{K,N} = \bar{w}^K \times \bar{\Omega}^N$ be the grid for the (x,t)-variables, and $Q^{K,N} = \bar{Q}^{K,N} \cap Q$. Due to the presence of an interior layer at the point $x_{N/2} = 0$, the transition parameter τ is given by

$$\tau = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\eta} \ln\left(\frac{N}{4}\right)\right\}, \tag{5.3.8}$$

where τ is a positive constant. The spatial domain is discretized using a piecewise uniform mesh which splits the space domain $[-1,1]$ into three sub-intervals $[-1,-\tau]$, $[-\tau,\tau]$ and $[\tau,1]$. These sub-intervals are subdivided uniformly to contain $N/4$, $N/2$ and $N/4$ mesh elements respectively. Note that the mesh spacing is given by

$$h_j = \begin{cases} 4(1-\tau)/N \text{ if } j = 1,2,\cdots,N/4, 3N/4+1,\cdots,N-1,N, \\ 4\tau/N \text{ if } j = N/4+1, N/4+2\cdots 3N/4. \end{cases} \tag{5.3.9}$$

For the rest of the chapter, we adopt the notation $B(x_j, t_k) := B_j^k$ for ease of exposition. Also, let

$$D_x^+ U_j^k = \frac{U_{j+1}^k - U_j^k}{h_{j+1}^k}, \quad D_x^- U_j^k = \frac{U_j^k - U_{j-1}^k}{h_j^k}, \quad \tilde{D}_x U_j^k = \frac{2}{h_j^k + h_{j+1}^k}(D_x^+ U_j^k - D_x^- U_j^k)$$

and

$$D_t^- U_j^k = \frac{U_j^k - U_j^{k-1}}{\Delta t},$$

where $D_x^+ U_j^k, D_x^- U_j^k, D_t^- U_j^k$ and $\tilde{D}_x U_j^k$ are first and second order finite differences respectively. Using the upwind scheme both in time and space above, we discretize the problem (5.1.1)-(5.1.3) in the following manner:

$$L^{K,N} U_j^k := \begin{cases} (\varepsilon + x_j^2)\tilde{D}_x U_j^k + \tilde{a}_j^k D_x^- U_j^k - (\tilde{b}_j^k + \frac{d_j^k}{\Delta t})U_j^k = \tilde{f}_j^k - \frac{d_j^k U_j^{k-1}}{\Delta t} & \text{for } j = 0, \cdots, N/2 - 1, \\ (\varepsilon + x_j^2)\tilde{D}_x U_j^k + \tilde{a}_j^k D_x^+ U_j^k - (\tilde{b}_j^k + \frac{d_j^k}{\Delta t})U_j^k = \tilde{f}_j^k - \frac{d_j^k U_j^{k-1}}{\Delta t} & \text{for } j = N/2, \cdots, N - 1, \end{cases} \tag{5.3.10}$$

subject to the discrete initial and boundary conditions

$$U_j^k = u_j^0, \quad j = 0, 1, \cdots, N, \tag{5.3.11}$$

$$U_0^k \equiv \alpha_1^k = \alpha_1(t_k), \quad U_N^k \equiv \alpha_2^k = \alpha_2(t_k), \quad 1 \leqslant k \leq K, \tag{5.3.12}$$

where

$$\begin{cases} \tilde{a}_j^k = \frac{a_{j-1}^k + a_j^k}{2} & \text{for } j = 0, 1, \cdots, N/2 - 1, \\ \tilde{a}_j^k = \frac{a_j^k + a_{j+1}^k}{2} & \text{for } j = N/2, N/2 + 1, \cdots, N - 1, \end{cases}$$

$$\begin{cases} \tilde{b}_j^k = \frac{b_{j-1}^k + b_j^k + b_{j+1}^k}{3} & \text{for } j = 1, 2, 3, \cdots, N - 1, \\ \tilde{f}_j^k = \frac{f_{j-1}^k + f_j^k + f_{j+1}^k}{3} & \text{for } j = 1, 2, 3, \cdots, N - 1. \end{cases}$$

Now, (5.3.10) can be written in the form:

$$L_{x,\varepsilon}^{N,K} U_j^k := r^- U_{j-1}^k + r^c U_j^k + r^+ U_{j+1}^k = F_j, \ j = 1, 2, 3 \cdots, N - 1, \tag{5.3.13}$$

where for $j = 1, 2, 3 \cdots, N/2 - 1$, we have

$$r_j^- = \frac{2(\varepsilon + x_j^2)}{h_j(h_j + h_{j+1})} - \frac{\tilde{a}_j^k}{h_j}, \ r_j^c = \frac{\tilde{a}_j^k}{h_j} - \frac{2(\varepsilon + x_j^2)}{h_j h_{j+1}} - \tilde{b}_j^k - \frac{d_j^k}{\Delta t}, \ r_j^+ = \frac{2(\varepsilon + x_j^2)}{h_{j+1}(h_j + h_{j+1})}, \tag{5.3.14}$$

for $j = N/2, N/2+1, \cdots, N-1$, we have

$$r_j^- = \frac{2(\varepsilon + x_j^2)}{h_j(h_j + h_{j+1})},\ r_j^c = -\frac{\tilde{a}_j^k}{h_{j+1}} - \frac{2(\varepsilon + x_j^2)}{h_j h_{j+1}^k} - \tilde{b}_j^k - \frac{d_j^k}{\Delta t},\ r_j^+ = \frac{2(\varepsilon + x_j^2)}{h_{j+1}(h_j + h_{j+1})} + \frac{\tilde{a}_j^k}{h_{j+1}} \tag{5.3.15}$$

and

$$F_j = \tilde{f}_j^k - \frac{d_j^k U_j^{k-1}}{\Delta t}. \tag{5.3.16}$$

The results of the analysis of the scheme (5.3.10)-(5.3.12) depend on the following minimum principle.

Lemma 5.3.1. (Discrete minimum principle). For any mesh function ξ_j^k such that $L^{N,K}\xi_j^k \leq 0$ in $Q^{N,K}$, $\xi_j^0 \geq 0$, $1 \leq j \leq N$, $\xi_0^k \geq 0$, and $\xi_N^k \geq 0$, $1 \leq k \leq K$, we have $\xi_j^k \geq 0$ in $\bar{Q}^{N,K}$.

Proof. See Lemma 3.3.1 in Chapter 3.

Lemma 5.3.2. (Uniform stability estimate). At any time level t_k, if Z_j^k is any mesh function such that $Z_0^k = Z_N^k = 0$, then

$$|Z_i^k| \leq \frac{1}{\beta} \max_{1 \leq j \leq N-1} |L^{K,N} Z_j^k| \ \ \forall\, 0 \leq i \leq N.$$

Proof. See Lemma 3.3.2 in Chapter 3.

Based on the above continuous (Lemma 5.2.7) and discrete (Lemma 5.3.1) results, we are able to analyse the proposed method for convergence in the next section.

5.4 Convergence analysis

The convergence of the scheme will be analysed on a Shishikin mesh, which was proved in section 5.3.

Theorem 5.4.1. *Let U_j^k be the numerical solution of* (5.3.10)-(5.3.12) *and denote the solution $z(x_j, t_k)$ of problem* (5.3.4)-(5.3.5) *at the level t_k by $z_j^k = z(x_j, t_k)$. Then, we have*

$$\max_{0\leq j\leq N} |U_j^k - z_j^k| \leqslant CN^{-1}\left[\ln\left(\frac{N}{4}\right)\right]^2. \tag{5.4.1}$$

Proof. We prove this above Lemma on the interval $[0,1]$. The proof of $[-1,0]$ follows in a similar way. From (5.3.10)-(5.3.12), the solution U_j^k can be decomposed into a regular part and a singular part as follows

$$U_j^k = V_j^k + W_j^k,$$

where V_j^k is the solution of the inhomogeneous problem

$$L^{N,K}V_j^k = f_j^k - \frac{d_j^k \times V_j^{k-1}}{\Delta t},\ V_j^0 = v_j^0,\ V_0^k = v_0^k,$$

and W_j^k is the solution of the homogeneous problem

$$L^{N,K}W_j^k = 0,\ W_j^0 = w_j^0,\ W_{N/2}^k = U_{N2}^k - V_{N/2}^k.$$

Using (5.1.1) and (5.3.10), we obtain the error of the smooth component

$$\begin{aligned} L^{N,K}(V_j^k - v_j^k) &= f_j^k - \frac{d_j^k \times V_j^{k-1}}{\Delta t} - L^{N,K}v_j^k \\ &= (\varepsilon + x_j^2)\left(\frac{d^2}{dx^2} - \tilde{D}_x\right)v_j^k + a_j^k\left(\frac{d}{dx} - D_x^-\right)v_j^k. \end{aligned}$$

Applying the two estimates in Lemma 4.1 [42] at each point (x_j, t_k) for the above result, we obtain

$$|L^{N,K}(V_j^k - v_j^k)| \leq \frac{(\varepsilon + x_j^2)}{3}(x_{j+1} - x_{j-1})\left\|\frac{\partial^3 v_j}{\partial x^3}\right\| + \frac{a_j^k}{2}(x_j - x_{j-1})\left\|\frac{\partial^2 v_j}{\partial x^2}\right\| \text{ for } 1 \leq j \leqslant N/2-1. \tag{5.4.2}$$

Noting that $h_j = x_j - x_{j-1} \leq 4N^{-1}$ for any j, therefore using the bounds of v_j of Lemma 5.2.8 in inequality (5.4.2), we obtain

$$\begin{aligned} L^{N,K}(V_j^k - v_j^k) &\leqslant Ch[1 + (\varepsilon + x^2) + 2(\varepsilon + x^2)\arctan(x/\varepsilon)] \\ &\leqslant Ch \\ &\leqslant CN^{-1}. \end{aligned}$$

Hence, by Lemma 5.3.2 we obtain

$$|(V_j^k - v_j^k)| \leq CN^{-1} \text{ for } 1 \leqslant j \leqslant N/2 - 1. \tag{5.4.3}$$

The estimate of the error of the singular component depends on whether $\tau = 1/2$ or $\tau = (\varepsilon/\eta)\ln(N/4)$.

• The mesh is uniform: $\tau = 1/2 \leq (\varepsilon/\eta)\ln(N/4)$. The local truncation error $L^{N,K}(W^k - w^k)$ is given by

$$|L^{N,K}(W_j^k - w_j^k)| \leqslant \frac{(\varepsilon + x_j^2)}{3}(x_{j+1} - x_{j-1}) \left\| \frac{\partial^3 w_j}{\partial x^3} \right\| + \frac{a_j^k}{2}(x_j - x_{j-1}) \left\| \frac{\partial^2 w_j}{\partial x^2} \right\| \text{ for } 1 \leq j \leq N/2 - 1. \tag{5.4.4}$$

Using Lemma 5.2.8 and $h_j = x_j - x_{j-1} \leq 4N^{-1}$ in (5.4.4), we obtain

$$|L^{N,K}(W_j^k - w_j^k)| \leqslant Ch(\varepsilon + x^2)^{-1}\arctan(x/\varepsilon) \leqslant ch\varepsilon^{-2}.$$

Since $\varepsilon^{-1} \leq (1/\eta)\ln(N/4)$, we obtain

$$|L^{N,K}(W_j^k - w_j^k)| \leqslant C\,[\ln(N/4)]^2.$$

With the help of Lemma 5.3.2, the above inequality gives

$$|(W_j^k - w_j^k)| \leqslant C[\ln(N/4)]^2 \text{ for } 1 \leqslant j \leqslant N/2 - 1. \tag{5.4.5}$$

• Piecewise uniform: $\tau = (\varepsilon/\eta)\ln(N/4) \leq 1/2$. In this case we have two subintervals, namely $[-1, -\tau]$ and $[-\tau, 0]$. We give separate error estimates in the coarse and fine mesh subintervals. Firstly, we compute the error for the singular component in the mesh region $-1 \leqslant x_j \leqslant -\tau$. Using the triangle inequality, we obtain

$$|(W_j^k - w_j^k)| \leq |W_j^k| + |w_j^k|. \tag{5.4.6}$$

Then by Lemma 5.2.8, we have

$$\begin{aligned} |w_j^k| &\leqslant C(\varepsilon + x^2)\arctan(x/\varepsilon)] \\ &\leqslant C\varepsilon^{-2}x_j^2 \\ &\leqslant C[\ln(N/4)]^2, \text{ since } \ x_j = \tau = (\varepsilon/\eta)\ln(N/4). \end{aligned}$$

To establish a similar bound on W_j^k the interested reader may refer to Lemma 7.3 (p.58) and Lemma 7.5 (p.60) of [42], which leads immediately to

$$|W_j^k| \leqslant C[\ln(N/4)]^2 \text{ for } 1 \leq j \leqslant N/4 - 1. \tag{5.4.7}$$

Combining these above estimates into (5.4.6), we obtain

$$|W_j^k - w_j^k| \leqslant C[\ln(N/4)]^2 \text{ for } 1 \leqslant j \leqslant N/4 - 1. \tag{5.4.8}$$

If $-\tau < x \leqslant 0$, then the estimate of the truncation error is obtained from (5.4.4) as

$$\begin{aligned} |L^{N,K}(W_j^k - w_j^k)| &\leqslant C(x_{j+1} - x_{j-1})(\varepsilon + x^2)^{-1} \arctan(x/\varepsilon) \\ &\leqslant C(x_{j+1} - x_{j-1})\varepsilon^{-2} \\ &= CN^{-1}\varepsilon^{-2}\tau, \text{ since } \ x_{j+1} - x_{j-1} = h = 4\tau/N. \end{aligned}$$

Moreover, $|(W_0^k - w_0^k)| = 0$ and $|(W_{N/4}^k - w_{N/4}^k)| \leqslant |W_{N/4}^k| + |w_{N/4}^k| \leqslant C[\ln(N/4)]^2$ from (5.4.7). Consider the barrier function (p.72) of [42] on $[-\tau, 0]$

$$\Phi_j^k = (x_j - (-\tau))C_1\varepsilon^{-2}\tau N^{-1} + C_2N^{-1} = (x_j + \tau)C_1\varepsilon^{-2}\tau N^{-1} + C_2N^{-1},$$

it follows that for an appropriate choice of C_1 and C_2, the mesh functions

$$(\Psi^{\pm})_j^k = \Phi_j^k \pm (W_j^k - w_j^k)$$

satisfy the inequalities

$$\Psi_{N/4}^k \geq 0, \ \ \Psi_{N/2}^k = 0 \text{ and } L^{N,K}\Psi_j^k \leq 0, \ \ N/4 + 1 \leqslant j \leqslant N/2 - 1.$$

By applying Lemma 5.3.1 on $[-\tau, 0]$ to $(\Psi^{\pm})_j^k$, we obtain

$$\Psi_j^k \geq 0, \ \ N/4 + 1 \leqslant j \leqslant N/2 - 1.$$

Therefore, we obtain

$$|W_j^k - w_j^k| \leq \Phi_j^k \leq C_1\varepsilon^{-2}\tau^2N^{-1} + C_2N^{-1}.$$

Using the value of $\tau = (\varepsilon/\eta)\ln(N/4)$ in the above inequality, we obtain

$$|W_j^k - w_j^k| \leqslant CN^{-1}[\ln(N/4)]^2. \tag{5.4.9}$$

Combining (5.4.8) and (5.4.9), we obtain the following estimate on the singular component of the error over interval $[-1, 0]$

$$|W_j^k - w_j^k| \leq CN^{-1}[\ln(N/4]^2, \ \ N/4 + 1 \leq j \leqslant N/2 - 1. \tag{5.4.10}$$

Note that

$$U_j^k - z_j^k = (V_j^k - v_j^k) + (W_j^k - w_j^k), \tag{5.4.11}$$

using (5.4.3) and (5.4.10), we obtain

$$|U_j^k - z_j^k| \leq CN^{-1}[\ln(N/4)]^2, \;\; 1 \leqslant j \leqslant N/2 - 1. \tag{5.4.12}$$

Results on the subinterval $[0, 1]$ can be obtained in the same way as we did on the interval $[-1, 0]$, then we have

$$|U_j^k - z_j^k| \leq CN^{-1}[\ln\left(\frac{N}{4}\right)]^2, \;\; N/2 \leqslant j \leqslant N. \tag{5.4.13}$$

Combining (5.4.12) and (5.4.13), leads to the required result.

The following theorem provides the main result.

Theorem 5.4.2. . *Let u be the exact solution of (5.1.1)-(5.1.2) and U be its numerical solution obtained via the difference equations (5.3.10)-(5.3.12). Then, there exists constant C independent of the perturbation parameter ε, and of the discretization parameters h_j and Δt such that*

$$\max_{0\leq j\leq N; 1\leq k\leq K} \|U_j^k - u_j^k\| \leq C\left[\Delta t + N^{-1}\left[\ln\left(\frac{N}{4}\right)\right]^2\right]. \tag{5.4.14}$$

Proof. The result follows from the triangle inequality

$$||U_j^k - u_j^k|| = ||U_j^k - z_j^k|| + ||z_j^k - u_j^k||$$

and the combination of the time discretization(5.3.7) and the result of Theorem 5.4.1.

The above theorem shows that the scheme we propose is first order convergent in time and almost first order convergent in space, uniformly with respect to the perturbation parameter ε. In order to improve the accuracy and the rate of convergence of the scheme, we apply Richardson extrapolation in the next section.

5.5 Richardson extrapolation on the FMFDM

Richardson extrapolation is a procedure where a linear combination of two approximations of some quantity gives a third and better approximation of the quantity [49]. We employ this procedure for the proposed scheme.

Let Ω^{τ}_{2N} be the mesh obtained by bisecting each mesh interval in Ω^{τ}_{N}. It is easy to understand that $\Omega^{\tau}_{N} \subset \Omega^{\tau}_{2N} = \{\tilde{x}_j\}$ and $\tilde{x}_j - \tilde{x}_{j-1} = \tilde{h}_j = h_j/2$.

Theorem 5.5.1. *Let $U^{ext,k}_j$ be the numerical solution approximation obtained using (5.3.10)-(5.3.12) and z^k_j be the solution of (5.3.2)-(5.3.3) at the time level t_k. Then, we have*

$$\max_{0\le j\le N} |(U^{ext} - z)^k_j| \le CN^{-2}\left[\ln\left(\frac{N}{4}\right)\right]^2. \tag{5.5.1}$$

Proof. Let U^k_j and $\tilde{U}^k_j$ be the numerical solution of (5.3.10)-(5.3.12) on the mesh Ω^{τ}_{N} and Ω^{τ}_{2N} respectively. The estimate (5.4.12) can be written as

$$U^k_j - z^k_j = C_1 N^{-1}\ln(N/4)^2 + R_N(x_j), \quad \forall x_j \in \Omega^{\tau}_{N} \tag{5.5.2}$$

and

$$\tilde{U}^k_j - z^k_j = C_2(2N)^{-1}\ln(N/4)^2 + R_{2N}(\tilde{x}_j), \quad \forall \tilde{x}_j \in \Omega^{\tau}_{2N}, \tag{5.5.3}$$

where C_1 and C_2 are some constants and the remainder terms

$$R_N(x_j) \text{ and } R_{2N}(\tilde{x}_j) \text{ are } \mathcal{O}[N^{-1}(\ln(N/4))^2].$$

The transition parameter τ remains the same as in (5.3.8) when calculating both U^k_j and $\tilde{U}^k_j$.

Combining (5.5.2) and (5.5.3), we obtain

$$z^k_j - (2\tilde{U}^k_j - U^k_j) = R_N(x_j) - 2R_{2N}(x_j) = \mathcal{O}[N^{-1}(\ln(N/4))^2], \quad \forall x_j \in \Omega^{\tau}_{n}. \tag{5.5.4}$$

Therefore we set

$$U^{ext,k}_j = 2\tilde{U}^k_j - U^k_j, \quad \forall x_j \in \Omega^{\tau}_{N}, \tag{5.5.5}$$

as the new approximation of z_j^k at the point $x_j \in \Omega_N^\tau$ resulting from Richardson extrapolation.

The error after extrapolation $U_j^{ext,k}$ can written as in (5.4.11),

$$(U^{ext} - z)_j^k = (V^{ext} - v)_j^k + (W^{ext} - w)_j^k, \tag{5.5.6}$$

where $V_j^{ext,k}$ and $W_j^{ext,k}$ are the regular and singular components of $U_j^{ext,k}$, respectively. The local truncation error of the scheme (5.3.10)-(5.3.13) after extrapolation is given by

$$L^{N,K}(U^{ext} - z)_j^k = 2L^{N,K}(\tilde{U}_j^k - z_j) - L^{N,K}(U_j^k - z_j), \tag{5.5.7}$$

where

$$L^{N,K}(U_j^k - z_j^k) = r^- z_{j-1} + r^c z_j + r^+ z_{j+1} - (\varepsilon + x_j^2) z_j'' - \tilde{a}_j^k z_j' + \tilde{b}_j^k z_j^k + \frac{d_j^k z_j^k}{\Delta t} \tag{5.5.8}$$

and

$$L^{N,K}(\tilde{U}_j^k - z_j^k) = \tilde{r}^- z_{j-1} + \tilde{r}^c z_j + \tilde{r}^+ z_{j+1} - (\varepsilon + x_j^2) z_j'' - \tilde{a}_j z_j' + \tilde{b}_j^k z_j^k + \frac{d_j^k u_j^k}{\Delta t}. \tag{5.5.9}$$

The expressions for r^-, r^c and r^+ are given in (5.3.14), (5.3.15) and (5.3.16) respectively while the quantities $\tilde{r}^-$, $\tilde{r}^c$ and $\tilde{r}^+$ are obtained by substituting h_j with $\tilde{h}_j$ and h_{j+1} with $\tilde{h}_{j+1}$ in the quantities of r^-, r^c and r^+ respectively. Taking the Taylor series expansion of z_{j-1}^k and z_{j+1}^k about x_j yields

$$z_{j-1}^k = z_j - h_j z_j' + \frac{h_j^2}{2} z_j^2 - \frac{h_j^3}{6} z_j^3 + \frac{h_j^4}{24} z^4(\xi_1, j), \tag{5.5.10}$$

$$z_{j+1}^k = z_j + h_{j+1} z_j' + \frac{h_{j+1}^2}{2} z_j^2 + \frac{h_{j+1}^3}{6} z_j^3 + \frac{h_{j+1}^4}{24} z^4(\xi_2, j), \tag{5.5.11}$$

$$z_{j-1}^k = z_j - \tilde{h}_j z_j' + \frac{\tilde{h}_j^2}{2} z_j^2 - \frac{\tilde{h}_j^3}{6} z_j^3 + \frac{\tilde{h}_j^4}{24} z^4(\tilde{\xi}_1, j), \tag{5.5.12}$$

$$z_{j+1}^k = z_j + \tilde{h}_{j+1} z_j' + \frac{\tilde{h}_{j+1}^2}{2} z_j^2 + \frac{\tilde{h}_{j+1}^3}{6} z_j^3 + \frac{\tilde{h}_{j+1}^4}{24} z^4(\tilde{\xi}_2, j), \tag{5.5.13}$$

where

$$(\xi_1, j) \in (x_{j-1}, x_j), \quad (\xi_2, j) \in (x_j, x_{j+1}), \quad \tilde{\xi}_1 \in (\frac{x_{j-1} + x_j}{2}, x_j) \text{ and } \tilde{\xi}_2 \in (x_j, \frac{x_j + x_{j+1}}{2}).$$

Substituting (5.5.10) and (5.5.11) into (5.5.8), (5.5.12) and (5.5.13) into (5.5.9), we obtain

$$L^{N,K}(U_j^k - z_j^k) = k_1 z_j + k_2 z_j' + k_3 z_j^2 + k_4 z_j^3 + k_{5,1} z^4(\xi_1, j) + k_{5,2} z^4(\xi_2, j) \tag{5.5.14}$$

and

$$L^{N,K}(\tilde{U}_j^k - z_j^k) = \tilde{k}_1 z_j + \tilde{k}_2 z_j' + \tilde{k}_3 z_j^2 + \tilde{k}_4 z_j^3 + \tilde{k}_4 z_j^4 + \tilde{k}_{5,1} z^4(\tilde{\xi}_1, j) + \tilde{k}_{5,2} z^4(\tilde{\xi}_2, j). \tag{5.5.15}$$

The coefficients in (5.5.14) are

$$k_1 = \frac{2(\varepsilon + x_j^2)}{h_j(h_j + h_{j+1})} - \frac{2(\varepsilon + x_j^2)}{h_j h_{j+1}} + \frac{2(\varepsilon + x_j^2)}{h_{j+1}(h_j + h_{j+1})}, \; k_2 = 0,$$

$$k_3 = \frac{(\varepsilon + x_j^2)h_j}{h_j + h_{j+1}} - \frac{\tilde{a}_j^k h_j}{2} + \frac{(\varepsilon + x_j^2)h_{j+1}}{h_j + h_{j+1}} - (\varepsilon + x_j^2),$$

$$k_4 = \frac{-(\varepsilon + x_j^2)h_j^2}{3(h_j + h_{j+1})} + \frac{\tilde{a}_j^k h_j^2}{6} + \frac{(\varepsilon + x_j^2)h_{j+1}^2}{3(h_j + h_{j+1})}, \; k_{5,1} = \frac{(\varepsilon + x_j^2)h_j^3}{12(h_j + h_{j+1})} - \frac{\tilde{a}_j^k h_j^3}{24}, \; k_{5,2} = \frac{(\varepsilon + x_j^2)h_{j+1}^3}{12(h_j + h_{j+1})}.$$

The quantities for $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$, $\tilde{k}_4$, $\tilde{k}_{5,1}$ and $\tilde{k}_{5,2}$ can be obtained by substituting h_j with $\tilde{h}_j$ and h_{j+1} with $\tilde{h}_{j+1}$.

Substituting (5.5.14) and (5.5.15) into (5.5.7), we obtain

$$L^{N,K}(U^{ext} - z)_j^k = T_1 z_j + T_2 z_j'' + +T_3 z_j''' + +T_{4,1} z^{(4)}(\xi_1, j) + T_{4,2} z^{(4)}(\xi_2, j), \tag{5.5.16}$$

where

$$T_1 = \frac{14(\varepsilon + x_j^2)}{h_j(h_j + h_{j+1})} - \frac{14(\varepsilon + x_j^2)}{h_j h_{j+1}} + \frac{14(\varepsilon + x_j^2)}{h_{j+1}(h_j + h_{j+1})},$$

$$T_2 = \frac{(\varepsilon + x_j^2)h_j}{h_j + h_{j+1}} - (\varepsilon + x_j^2) + \frac{(\varepsilon + x_j^2)h_{j+1}}{h_j + h_{j+1}}, \quad T_3 = -\frac{\tilde{a}_j^k h_j^2}{12},$$

$$T_{4,1} = -\frac{(\varepsilon + x_j^2)h_j^3}{24(h_j + h_{j+1})} + \frac{\tilde{a}_j^k h_j^3}{32} \text{ and } T_{4,2} = -\frac{(\varepsilon + x_j^2)h_{j+1}^3}{24}.$$

Given (5.3.9) and for the sake of simplicity, we use the notation

$$h_j = \begin{cases} H & \text{if } j = 1, 2, \cdots, N/4, \\ h & \text{if } j = N/4 + 1, \cdots N/2. \end{cases} \tag{5.5.17}$$

Using the fact that $\forall j = 1, \ldots, N/4,\ H = h_j \leqslant 4N^{-1}$ substituted into (5.5.16) in the subinterval $[-1, -\tau]$, we obtain

$$L^{N,K}(V^{ext} - v)_j^k = -\frac{\tilde{a}_j^k H^2}{12} v_j''' + \left[\frac{(\varepsilon + x_j^2)H^2}{48} + \frac{\tilde{a}_j^k H^3}{32}\right] v^{(4)}(\xi_1, j) - \frac{(\varepsilon + x_j^2)H^3}{24} v^{(4)}(\xi_2, j). \tag{5.5.18}$$

Applying the triangle inequality in the above inequality, we obtain

$$L^{N,K}(V^{ext} - v)_j^k \leqslant -\frac{\tilde{a}_j^k H^2}{12} v_j''' + \left[\frac{(\varepsilon + x_j^2)H^2}{48} + \frac{\tilde{a}_j^k H^3}{32}\right] v^{(4)}(\xi_1, j) - \frac{(\varepsilon + x_j^2)H^3}{24} v^{(4)}(\xi_2, j).$$

By Lemma 5.2.8, the above inequality gives

$$|L^{N,K}(V^{ext} - v)_j^k| \leqslant CN^{-2}. \tag{5.5.19}$$

The estimate on $L^{N,K}(W^{ext} - w)_j^k$ depends on whether $\tau = 1/2$ or $\tau = (\varepsilon/\eta)\ln(N/4)$.

• In the first case the mesh is uniform and $(\varepsilon/\eta)\ln(N/4) \geqslant 1/2$. The estimate of the singular component of the local truncation error is given by

$$L^{N,K}(W^{ext} - w)_j^k \leqslant -\frac{\tilde{a}_j^k h^2}{12} w_j''' + \left[\frac{(\varepsilon + x_j^2)h^2}{48} + \frac{\tilde{a}_j^k h^3}{32}\right] w^{(4)}(\xi_1, j) - \frac{(\varepsilon + x_j^2)h^3}{24} w^{(4)}(\xi_2, j). \tag{5.5.20}$$

With the help of Lemma 5.2.8, we obtain

$$|L^{N,K}(W^{ext} - w)_j^k| \leqslant CN^{-2}(\varepsilon + x_j^2)^{-2}\arctan(x/\varepsilon) \leqslant CN^{-2}\varepsilon^{-2}. \tag{5.5.21}$$

Using this inequality $\varepsilon^{-1} \leqslant (2/\eta)\ln(N/4)$ in (5.5.21), we get

$$|L^{N,K}(W^{ext} - w)_j^k| \leq CN^{-2}[\ln(N/4)]^2. \tag{5.5.22}$$

• In the second case (viz $\tau = (\varepsilon/\eta)\ln(N/4)$, the mesh is piecewise uniform with the mesh spacing $h = h_j = 4\tau N^{-1}$ for $\forall\, j = N/4 + 1, \ldots, N/2$ in the subinterval $[-\tau, 0]$.
On application of Lemma 5.2.8, (5.5.20) gives

$$|L^{N,K}(W^{ext} - w)_j^k| \leq C_1 N^{-2}\tau^2\varepsilon^{-2}. \tag{5.5.23}$$

Using the value of $\tau = (\varepsilon/\eta)\ln(N/4)$ in (5.5.23), then gives

$$|L^{N,K}(W^{ext} - w)_j^k| \leq CN^{-2}[\ln(N/4)]^2. \tag{5.5.24}$$

A similar analysis can also be done for $N/2+1 \leqslant j \leqslant N-1$.

Using Lemma 5.3.2 in (5.5.19), (5.5.22) and (5.5.24) along with the inequality (5.5.6), we obtain the required result.

Once again, using the triangle inequality and combining (5.3.7) and Theorem 5.5.1, we obtain the error after extrapolation which is stated in the following theorem.

Theorem 5.5.2. (Error after extrapolation). *If $U_j^{ext,k}$ is the approximation of u_j^k obtained by using (5.3.10)-(5.3.12) and u_j^k be the exact solution of (5.1.1)-(5.1.2), then, there exists a constant C, independent of the perturbation parameter ε, the time discretization Δt and the space discretization parameters h_j such that*

$$\max_{0\leqslant j\leqslant N; 1\leq k\leqslant K} \|U_j^k - u_j^k\| \leqslant C\left[\Delta t + N^{-2}\left[\ln\left(\frac{N}{4}\right)\right]^2\right]. \tag{5.5.25}$$

We propose two examples to test the proposed method in the following section.

5.6 Numerical examples

In this section we present the numerical results obtained for the difference scheme that has previously been discussed. In both examples, we start with $N = 16$ and $\Delta t = \frac{1}{16}$ and we multiply N by two and divide Δt also by two. The maximum errors and order of convergence are calculated by the exact solution. The solution in both examples has a turning point at point $x = 0$, which yields an interior layer.

Example 5.6.1. We consider the problem (5.1.1)-(5.1.2) for

$$a(x,t) = 2x(1+t^2),\ b(x,t) = 1 + x^2 + \cos\pi xt,\ d(x,t) = 3 + xt,\ T = 1$$

and the functions $f(x,t)$ and $u_0(x)$ are such that the exact solution is given by

$$u(x,t) = \varepsilon\, e^{-t/\varepsilon} \arctan\left(\frac{x}{\sqrt{\varepsilon}}\right) - \varepsilon^{2/3} e^{-xt}.$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$.

Example 5.6.2. Here, we consider problem (5.1.1)-(5.1.2) for

$$a(x,t) = 2x(1+t^2),\ b(x,t) = 1 + x^2 + \cos \pi xt,\ d(x,t) = 3 + xt,\ T = 1$$

and the functions $f(x,t)$ and $u_0(x)$ are such that the exact solution is given by

$$u(x,t) = \varepsilon\ e^{-t/\varepsilon} e^{\arctan(x^2/\sqrt{\varepsilon})}.$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$.

Maximum errors at all mesh points are determined

$$E^{\varepsilon,N,K} = \max_{0\le j\le N; 0\le k\le K} |u_{j,k}^{\varepsilon,N,K} - U_{j,k}^{\varepsilon,N,K}|.$$

where $u_{j,k}^{\varepsilon,N,K}$ denotes the exact solution, and $U_{j,k}^{\varepsilon,N,K}$ represents the numerical solution which is obtained by a constant time step Δt using N mesh intervals in the entire domain $\Omega = [-1,1]$. Furthermore, we calculate the numerical rate of uniform convergence as follows

$$r_l \equiv r_{\varepsilon,l} = \log_2(E^{\varepsilon,N_l,K_l}/E^{\varepsilon,2N_l,2K_l}).$$

After extrapolation the maximum errors at all mesh points and the numerical rates of convergence are computed as follows

$$E_{\varepsilon,N,K}^{ext} = \max_{0\le j\le N; 0\le k\le K} |U_{j,k}^{ext} - u_{j,k}^{\varepsilon,N,K}|, \text{ and } R_{N,K} \equiv R_{\varepsilon,N,K} \equiv \log_2(E_{\varepsilon,N_l,K_l}^{ext}/E_{\varepsilon,2N_l,2K_l}^{ext}).$$

Table 5.1: Results for Example 5.6.1: Maximum errors before extrapolation.

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
	$K=16$	$K=32$	$K=64$	$K=128$	$K=256$	$K=512$
10^{-4}	9.09E-02	4.73E-02	2.40E-02	1.21E-02	6.06E-03	3.55E-03
10^{-5}	9.22E-02	4.78E-02	2.42E-02	1.22E-02	6.10E-03	3.05E-03
10^{-7}	9.28E-02	4.80E-02	2.43E-02	1.22E-02	6.12E-03	3.06E-03
10^{-12}	9.28E-02	4.81E-02	2.43E-02	1.22E-02	6.12E-03	3.06E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	9.28E-02	4.81E-02	2.43E-02	1.22E-02	6.12E-03	3.06E-03

Table 5.2: Results for Example 5.6.1: Maximum errors after extrapolation.

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
	$K=16$	$K=64$	$K=256$	$K=1024$	$K=4096$	$K=16384$
10^{-4}	9.99E-02	2.55E-02	6.23E-03	3.55E-03	3.55E-03	3.55E-03
10^{-5}	1.02E-01	2.63E-02	6.52E-03	1.60E-03	7.82E-04	7.82E-04
10^{-7}	1.03E-01	2.68E-02	6.76E-03	1.69E-03	4.18E-04	1.03E-04
10^{-12}	1.04E-01	2.69E-02	6.79E-03	1.70E-03	4.26E-04	1.07E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.04E-01	2.69E-02	6.79E-03	1.70E-03	4.26E-04	1.07E-04

Table 5.3: Results for Example 5.6.1: Rates of convergence before extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	0.94	0.98	0.99	0.99	0.77
10^{-5}	0.95	0.98	0.99	1.00	1.00
10^{-9}	0.95	0.98	0.99	1.00	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	0.95	0.98	0.99	1.00	1.00

Table 5.4: Results for Example 5.6.1: Rates of convergence after extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	1.97	2.03	0.81	0.00	0.00
10^{-5}	1.96	2.01	2.03	1.03	0.00
10^{-9}	1.94	1.99	2.00	2.00	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.94	1.99	2.00	2.00	2.00

Table 5.5: Results for Example 5.6.2: Maximum errors before extrapolation

ε	$N=16$ $K=16$	$N=32$ $K=32$	$N=64$ $K=64$	$N=128$ $K=128$	$N=256$ $K=256$	$N=512$ $K=512$
10^{-3}	2.50E-01	1.34E-01	6.91E-02	3.52E-02	1.78E-02	8.96E-03
10^{-4}	2.69E-01	1.41E-01	7.23E-02	3.66E-02	1.84E-02	9.24E-03
10^{-5}	2.76E-01	1.45E-01	7.35E-02	3.71E-02	1.86E-02	9.33E-03
10^{-16}	2.85E-01	1.47E-01	7.44E-02	3.74E-02	1.87E-02	9.38E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	2.85E-01	2.32E-01	7.44E-02	3.74E-02	1.87E-02	9.38E-03

Table 5.6: Results for Example 5.6.2: Maximum errors after extrapolation

ε	$N=16$ $K=16$	$N=32$ $K=64$	$N=64$ $K=256$	$N=128$ $K=1024$	$N=256$ $K=4096$	$N=512$ $K=16384$
10^{-3}	2.57E-01	7.03E-02	1.79E-02	4.66E-03	4.77E-03	4.90E-03
10^{-4}	2.85E-01	7.30E-02	1.84E-02	4.64E-03	1.16E-03	5.09E-04
10^{-5}	2.98E-01	7.47E-02	1.86E-02	4.67E-03	1.17E-03	2.93E-04
10^{-16}	3.11E-01	8.07E-02	2.04E-02	5.11E-03	1.28E-03	3.19E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	3.11E-01	8.07E-02	2.04E-02	5.11E-03	1.28E-03	3.19E-04

Table 5.7: Results for Example 5.6.2: Rates of convergence before extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-3}	0.91	0.95	0.97	0.98	0.99
10^{-4}	0.93	0.97	0.98	0.99	0.99
10^{-11}	0.95	0.98	0.99	1.00	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	0.95	0.98	0.99	1.00	1.00

Table 5.8: Results for Example 5.6.2: Rates of convergence after extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-3}	1.87	1.97	1.94	-0.03	-0.04
10^{-4}	1.96	1.98	1.99	2.00	1.19
10^{-11}	1.95	1.99	2.00	2.02	2.02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.95	1.99	2.00	2.00	2.00

5.7 Discussion

In this chapter, we treated a class of time-dependent singularly perturbed problems in which the diffusion term contains the perturbation parameter ε and a quadratic function. We also studied this problem whose solution exhibits an interior layer due to the presence of a turning point. After setting bounds on the solution and its derivative, we used the proposed numerical scheme comprising of the classical Euler method to discretize the time variable. A resulting system of two-point boundary value problems (one at each time level) was solved by using a Fitted Mesh Finite Difference Method (FMFDM). Applying bounds on the solution and its derivative, we showed that the proposed numerical method was uniformly convergent relative to the perturbation parameter ε and the step-size.

In order to validate the above conclusions based on theoretical analysis, we solved two examples to support the findings. In each example, we computed the maximum pointwise errors and the corresponding rates of convergence for different values of N and K. The results displayed in tables 5.1, 5.3, 5.5 and 5.7, confirmed that the proposed method was uniformly convergent. We also investigated the effect of Richardson of extrapolation via FMFDM in order to improve our results. For comparison purposes, we kept the same values of N and K above and numerical results are shown in tables 5.2, 5.4, 5.6 and 5.8.

Chapter 6

A numerical method for interior layer convection-diffusion problems with a variable coefficient diffusion term

In this chapter, we consider singularly perturbed convection-diffusion problems whose solution displays an interior layer due to the presence of a turning point. Moreover, the perturbation parameter ε is embedded in a linear function. After deriving bounds on the solution to these problems and its derivatives, we construct a fitted mesh difference method (FMDM) and analyse its convergence. We investigate the Richardson extrapolation method via FMFDM in order to increase its accuracy and order of convergence.

6.1 Introduction

In recent years, many researchers have studied a class of two-point boundary value singularly perturbed convection-diffusion problems (2.1.1)-(2.1.2).

Due to the presence of a small parameter ε in (2.1.1), the solution to this problem possesses large gradients in narrow regions of the domain referred to as boundary or interior layers depending upon the nature of the coefficient of the convection term $a(x)$. The solution presents a boundary layer at the right end or the left end of the domain if $a(x) < 0$ or $a(x) >, 0$ for $x \in [-1, 1]$ respectively. These problems are called non-turning pint problems. Many authors have developed successful numerical schemes for such problems [25, 26, 28, 45, 46, 56, 57, 69].

But if $a(x) = 0$ with $a(-1)a(1) \neq 0$, these are called turning points of the problem. Turning points may give rise to boundary and/or interior layers.

Several authors have suggested numerous numerical schemes for such problems [18, 19, 24, 27, 40, 58, 64]. It is also to be noted that interior layers may occur in the solution to problems (2.1.1)-(2.1.2) if the coefficients are not smooth or if the data function $f(x)$ is discontinuous [16]. However little attention has been given to the study of problems having a variable coefficient affecting the second derivative. Such problems often arise in fluid and geo-dynamics [37, 39].

Liseikin [37] considered the equation:
$-(\varepsilon + px)^{\beta}u'' + a(x)u + f(x, \varepsilon) = 0,\ x \in [0, 1],\ p = 0, 1\ , \beta > 0$. Estimates of the solution and its derivatives are provided. The same author in [39] considered the problem: $-(\varepsilon + x)^{\beta}u'' - a(x)u + f(x, u) = 0,\ x \in [0, 1],\ \beta > 0$. The solution to this problem can exhibit only a single layer in the vicinity of $x = 0$. It turns out that for $\beta = 1$ the bounds on the derivatives of $u(x)$ in the boundary layer are estimated by three singular power functions according to the value of $a = a(0)$, while for $\beta = 2$ they are estimated by layer-type exponential functions (see [39] pp 106-111). A numerical scheme was developed and its convergence was analysed (see [39] pp 256-262) for $\beta = 1$.

In this chapter, we consider the problem for an equation of the second order with a small ε whose reduced equation ($\varepsilon = 0$) has the same order

$$Lu :\equiv (\varepsilon + x)u'' + a(x)u' - b(x)u = f(x), x \in \Omega = (0, 1), \tag{6.1.1}$$

subject to the boundary conditions

$$u(0) = \alpha_1,\ u(1) = \alpha_2, \tag{6.1.2}$$

where $0 < \varepsilon \leq 1$ and α_1 and α_2 are given real constants.

In the rest of this chapter, we assume the following hypotheses

- $a(0.5) = 0$ and $a'(0.5) > 0$, thus the solution to problem (6.1.1)-(6.1.2) guarantees the existence of the turning point,
- $b(x) \geq b_0 > 0$ for $0 \leq x \leq 1$ which ensures that the solution to (6.1.1)-(6.1.2) satisfies a minimum principle and
- $|a'(x)| \geq |a'(0.5)|/2$ for $0 \leqslant x \leqslant 1$ guarantees the uniqueness of the turning point in the interval $[0, 1]$ [19].

In this chapter, our aim is to propose and analyse a fitted mesh finite difference method (FMFDM) to problem (6.1.1)-(6.1.2) as applied on a Shishkin mesh.

The rest of this chapter is structured as follows. In Section 2, we establish bounds on the solution and its derivatives $u(x)$, which will be used in the analysis of the uniform convergence of the numerical scheme. In Section 3, we develop our numerical method by first designing a piecewise uniform mesh of Shishkin type. We then discretize (6.1.1)-(6.1.2) on the mesh using an upwind scheme. Section 4 is devoted to the convergence analysis of the method. We prove that the proposed numerical method is uniformly convergent almost of order one with respect to the perturbation parameter ε. In order to improve our results of the proposed (FMFDM), we employ the Richardson extrapolation method in Section 5. Section 6 provides detailed numerical results. In Section 7, we present some concluding remarks and scope for future research.

Throughout this chapter, C denotes a positive constant independent of the singular perturbation parameter ε and the discretization parameter N of the discrete problem.

6.2 Bounds on the solution and its derivatives

Here we study the qualitative behaviour of the solution to problem (6.1.1)-(6.1.2) and its derivatives which will be used in the convergence analysis of the numerical method. We denote the sub-intervals of $[0, 1]$ as $\Omega_l = [0, 1/2 - \tau)$,
$\Omega_c = [1/2 - \tau, 1/2 + \tau] = [1/2 - \tau, 1/2] \cup (1/2, 1/2 + \tau]$ and $\Omega_r = (1/2 + \tau, 1]$, where $\tau \in (0, 1/4]$.

Under the requirements mentioned above, the differential operator L as defined in (6.1.1) admits the following continuous minimum principle.

Lemma 6.2.1. (Minimum principle). Suppose that ξ is a smooth function satisfying $\xi(0) \geq 0$, $\xi(1) \geq 0$ and $L\xi(x) \leq 0$, $\forall x \in \Omega$. Then $\xi(x) \geq 0$, $\forall x \in \bar{\Omega}$.

Proof. Let $x^* \in \bar{\Omega}$ such that $\xi(x^*) = \min\limits_{0 \leq x \leq 1} \xi(x)$ and assume $\xi(x^*) < 0$. It follows that $x^* \notin \Omega$, therefore $\xi'(x^*) = 0$ and $\xi''(x^*) \geq 0$, which implies

$$L\xi(x^*) := (\varepsilon + x^*)\xi''(x^*) + a(x^*)\xi'(x^*) - b(x^*)\xi(x^*) > 0,$$

which is a contradiction to our assumption. Hence $\xi(x^*) \geq 0$ and $\xi(x) \geq 0$, for all $x \in \bar{\Omega}$.

We apply Lemma 6.2.1 to prove the next results which state that the solution depends continuously on the data.

Lemma 6.2.2. (Stability estimate). If $u(x)$ is the solution of (6.1.1)-(6.1.2), then we have

$$||u(x)|| \leqslant [\max\{||\alpha||_\infty, ||\beta||_\infty\}] + \frac{1}{b_0}||f||_\infty, \forall x \in \bar{\Omega}.$$

Proof. See Lemma 2.2.2 in Chapter 2.

Lemma 6.2.3. Let $u(x)$ be the solution of (6.1.1)-(6.1.2) and $a(x), b(x)$ and $f(x) \in C^k(\bar{\Omega})$, then there exists a positive constant C such that

$$|u^{(j)}(x)| \leq C, \ \forall x \in \Omega_l \text{ or } \Omega_r \ , j = 1, 2, 3 \cdots , k,$$

for sufficiently small $\tau \in (0, 1/4]$.

Proof. See theorem 2.4 of [5].

Lemma 6.2.4. [39] (Inverse monotonicity.) Let $d(x) = x$ and $g(x) = f(x) - a(x)u'(x) + b(x)u(x)$ be continuous in $[0, 1]$ and $[0, 1] \times \Re^2$, respectively. Then the operator $T = (L, \Gamma)$ for the functions from $C^2(0, 1) \cup C[0, 1]$ is inverse-monotone if one of the following conditions imposed of g is satisfied:

- $g(x, u, u')$ is strictly increasing in u, i.e. $g(x, u_1, z) < g(x, u_2, z)$ if $u_1 < u_2$;
- $g(x, u, u')$ is weakly increasing in u, and there exists a constant $C > 0$ such that

$$|g(x, u, z_1) - g(x, u, z_2)| \leqslant C|z_1 - z_2|.$$

Proof. For the proof of the Lemma, readers may refer to [39], pp 47.

We adapt the following Lemma according to [39]. Note that the solution of problem (6.1.1)-(6.1.2) displays an interior layer at the point $x_{N/2} = 1/2$. Therefore, the derivatives of $u(x)$ are estimated in the layer region by polynomial functions according to the sign of the coefficient of the convection term at the point $x_0^\star$. Then, we have two cases

$$a = \begin{cases} a(x_0^\star) \leqslant 0, & x_0^\star \in [1/2 - \tau, 1/2] \ \text{ and} \\ a(x_0^\star) > 0, & x_0^\star \in (1/2, \tau + 1/2]. \end{cases} \tag{6.2.1}$$

Lemma 6.2.5. Let $u(x)$ be the solution of problem (6.1.1)-(6.1.2). Then assuming that (i) $a = a(x_0^\star) > 0$, for $1/2 < x \leqslant 1/2 + \tau$ and $j = 1, 2, 3, 4$, we have the following bounds

$$|u^{(j)}(x)| \leqslant C \begin{cases} 1 + (\varepsilon + x)^{1-a-j}, & 0 < a < 1, \\ 1 + (\varepsilon + x)^{-j} |\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \\ 1 + \varepsilon^{a-1}(\varepsilon + x)^{1-a-j} & a > 1, \end{cases} \tag{6.2.2}$$

(ii) $a(x_0^\star) = a \leqslant 0,$ for $1/2 - \tau \leqslant x \leqslant 1/2$, and $j = 1, 2, 3, 4$, and let p be an integer such that $a + p = 0$ and $a + p - 1 < 0$, then we have the following bounds

$$|u^{(j)}(x)| \leqslant C \begin{cases} 1, & a < 0,\ j \leqslant p, \\ 1 + (\varepsilon + x)^{1-j-p}|\ln(\varepsilon + x)|, & a + p = 0,\ j > p, \\ 1 + (\varepsilon + x)^{-a-j}, & a + p > 0,\ j > p. \end{cases} \tag{6.2.3}$$

Proof. The proof of this Lemma will follow the same ideas provided by ([39], pp. 107-110). Application of the inverse-monotone pair $T = (L, \Gamma)$ (see pp 49) implies that

$$|u(x)| \leqslant C,\ 0 \leqslant x \leqslant 1. \tag{6.2.4}$$

Combining (6.1.1)-(6.1.2) and (6.2.4), we obtain

$$|u^{(j)}(x)| \leqslant C \begin{cases} 1, & 1/2 - \tau < x_0 \leqslant x \leqslant 1/2, \\ \varepsilon^{-j}, & 1/2 - \tau \leqslant x \leqslant x_0, \\ 1, & 1/2 < x_0 \leqslant x \leqslant \tau + 1/2, \\ \varepsilon^{-j}, & 1/2 \leqslant x \leqslant x_0, \end{cases} \tag{6.2.5}$$

for $j = 1, 2, 3, 4$ and arbitrary $x_0 > 0$.

case 1 : $\mathbf{a > 0}$ for $1/2 < x \leqslant \tau + 1/2$. The derivatives of $u(x)$ are estimated according to the value of a: $0 < a < 1$, $a = 1$ and $a > 1$. Solving (6.1.1) for $u''(x)$, we obtain

$$u''(x) = \frac{f(x) + b(x)u(x)}{(\varepsilon + x)} - \frac{a(x)u'(x)}{(\varepsilon + x)}. \tag{6.2.6}$$

One can determine $u'(x)$ from (6.2.6) as follows

$$u'(x) = \int_{1/2}^{x} \frac{f(s) + b(s)u(s)}{\varepsilon + s}\, ds - \int_{1/2}^{x} \frac{a(s)}{\varepsilon + s} u'(s)\, ds. \tag{6.2.7}$$

By [39], $u'(x)$ is given as follows

$$u'(x) = u'(1/2) \left[\frac{\varepsilon}{\varepsilon + x}\right]^{a} \exp[-g_1(x)] + g_2(x) \tag{6.2.8}$$

where

$$g_1(x) = \int_{1/2}^{x} \frac{a(s)}{\varepsilon + s}\, ds = a(x)\ln(\varepsilon + x) - \int_{1/2}^{x} a'(s)\ln(\varepsilon + s)\, ds \tag{6.2.9}$$

with $a(1/2) = 0$, and

$$g_2(x) = (\varepsilon + x)^{-a} \int_{1/2}^{x} [f(s) + b(s)u(s)](\varepsilon + s)^{a-1} \exp[g_1(s) - g_1(x)]\, ds. \tag{6.2.10}$$

Note that $|g_1(x)| \leqslant C$ from (6.2.4), then $g_2(x)$ becomes

$$|g_2(x)| \leqslant C(\varepsilon + x)^{-a} \int_{1/2}^{x} (\varepsilon + s)^{a-1}\, ds \leqslant C.$$

Applying the triangle inequality in (6.2.8) we obtain

$$|u'(x)| \leqslant C\left[1 + |u'(1/2)|(\varepsilon/(\varepsilon + x))^a\right]. \tag{6.2.11}$$

Considering $0 < a < 1$, there exists a point x_0 in the interval $(1/2, \tau + 1/2)$ such that $|u'(x_0)| \leqslant C$, we have

$$|u'(1/2)|\left(\frac{\varepsilon}{\varepsilon + x_0}\right)^a \leqslant C.$$

This inequality gives

$$|u'(1/2)| \leqslant C\left(\frac{\varepsilon + x_0}{\varepsilon}\right)^a \leqslant C\varepsilon^{-a}.$$

Substituting the estimate obtained for $|u'(1/2)|$ into (6.2.11), we obtain

$$|u'(x)| \leqslant C\left[1 + (\varepsilon + x)^{-a}\right],\ 0 < a < 1.$$

To obtain $u''(x)$, let us make first make $a(x)u'(x)/(\varepsilon + x)$ the subject of the formula in (6.2.6), then we have

$$\frac{a(x)u'(x)}{\varepsilon + x} = \frac{f(x) + b(x)u(x)}{(\varepsilon + x)} - u''(x). \tag{6.2.12}$$

Differentiating (6.1.1), solving the resulting equation for $u'''(x)$ and taking into account (6.2.12), we obtain

$$u'''(x) = \frac{f'(x) + b'(x)u(x) + b(x)u'(x) - a'(x)u'(x)}{(\varepsilon + x)} - \frac{a'(x) + 1}{(\varepsilon + x)}u''(x). \tag{6.2.13}$$

From the above equation, we obtain $u''(x)$ as follows

$$u''(x) = \int_{1/2}^{x} \frac{f'(s) + b'(s)u(s) + b(s)u'(s) - a'(s)u'(s)}{\varepsilon + s}\, ds - \int_{1/2}^{x} \frac{a'(s) + 1}{\varepsilon + s}u''(s)\, ds. \tag{6.2.14}$$

According to [37], $u''(x)$ is given as

$$u''(x) = u''(1/2)\left[\frac{\varepsilon}{\varepsilon + x}\right]^{a+1} \exp[-g_3(x)] + g_4(x) \tag{6.2.15}$$

where

$$g_3(x) = \int_{1/2}^{x} \frac{a'(s)+1}{\varepsilon + s}\, ds = [a(x)+1]\ln(\varepsilon + x) - \ln(\varepsilon + 1/2) - \int_{1/2}^{x} a'(s)\ln(\varepsilon + s)\, ds, \tag{6.2.16}$$

with $a(1/2) = 0$, and

$$g_4(x) = (\varepsilon + x)^{-a-1}\int_{1/2}^{x} [f'(s) + b'(s)u(s) + [b(s) - a'(s)]u'(s)](\varepsilon + s)^{a} \exp[g_3(s) - g_3(x)]\, ds. \tag{6.2.17}$$

Since $|g_3(x)| \leqslant C$ and $|u(x)| \leqslant C$, we obtain

$$|g_4(x)| \leqslant C(\varepsilon + x)^{-a-1}\int_{1/2}^{x} [1 + u'(s)](\varepsilon + s)^{a}\, ds \leqslant C[1 + (\varepsilon + x)^{-a}]. \tag{6.2.18}$$

From (6.1.1), we obtain

$$u''(1/2) \leqslant C(\varepsilon + 1/2)^{-1}[1 + u'(1/2)] \leqslant C[1 + (\varepsilon + 1/2)^{-a-1}].$$

Using the estimate of $u''(1/2)$ and $g_4(x)$ into (6.2.15), we obtain

$$u''(x) \leqslant C\left[1 + (\varepsilon + 1/2)^{-a-1}\right]\left[\frac{\varepsilon}{\varepsilon + x}\right]^{a+1} + C\left[1 + (\varepsilon + x)^{-a}\right]$$

Using $(\varepsilon + 1/2)^{-a-1} \leqslant (\varepsilon)^{-a-1}$ in the above inequality gives

$$|u''(x)| \leqslant C\left[1 + (\varepsilon + x)^{-a-1}\right], \text{ for } 0 < a < 1.$$

Differentiating (6.1.1) and taking into account (6.2.5), we obtain the following result

$$|u^{(j)}(x)| \leqslant C\left[1 + (\varepsilon + x)^{-a+1-j}\right], \ 0 < a < 1.$$

Let us now prove the case when $a = 1$. Equation (6.2.8) can be written in integral form $1/2$ to $1/2 + \tau$ as follows

$$\int_{1/2}^{1/2+\tau} u'(x)\, dx = \int_{1/2}^{1/2+\tau} u'(1/2)\left[\frac{\varepsilon}{\varepsilon + x}\right] \exp[-g_1(x)]\, dx + \int_{1/2}^{1/2+\tau} g_2(x)\, dx. \tag{6.2.19}$$

Evaluating this integral by parts on the right hand we obtain

$$u(1/2+\tau)-u(1/2) = u'(1/2)\varepsilon[\ln(\varepsilon+1/2+\tau)\exp(-g_1(1/2+\tau))-\ln(\varepsilon+1/2)\exp(-g_1(1/2)) + \int_{1/2}^{1/2+\tau} \ln(\varepsilon+x)g_1'(x)\exp(-g_1(x))\,dx] + \int_{1/2}^{1/2+\tau} g_2(x)\,dx, \quad (6.2.20)$$

where $g_1' = a(x)(\varepsilon+x)^{-1}$ and $g_1(1/2) = 0$ are obtained from (6.2.9). Substituting these equations into (6.2.20), we obtain

$$u(1/2+\tau)-u(1/2) = u'(1/2)\varepsilon[\ln(\varepsilon+1/2+\tau)\exp(-g_1(1/2+\tau))-\ln(\varepsilon+1/2)\exp(-g_1(1/2)) - \int_{1/2}^{1/2+\tau} a(x)(\varepsilon+x)^{-1}\ln(\varepsilon+x)\exp(-g_1(x))\,dx] + \int_{1/2}^{1/2+\tau} g_2(x)\,dx, \quad (6.2.21)$$

Using the triangle inequality in (6.2.21) and taking into account the inequality

$$|\ln(\varepsilon+1/2+\tau)\exp(-g_1(1/2+\tau)) - \int_{1/2}^{1/2+\tau} a(x)(\varepsilon+x)^{-1}\exp(-g_1(x))\,dx| \leqslant C,$$

we obtain

$$C \leqslant |u'(1/2)||[\varepsilon - \varepsilon\ln(\varepsilon+1/2)]|.$$

For sufficiently small $\varepsilon \leqslant x_0,\ x_0 > 0$, we have $\varepsilon - \varepsilon\ln(\varepsilon+1/2) \geqslant \varepsilon\ln(\varepsilon+1/2)$. It follows that

$$C \geqslant |u'(1/2)||[\varepsilon - \varepsilon\ln(\varepsilon+1/2)]| \leqslant |u'(1/2)|[\varepsilon\ln(\varepsilon+1/2)^{-1}].$$

Solving this inequality, we obtain

$$|u'(1/2)| \leqslant C\varepsilon^{-1}\ln^{-1}(\varepsilon+1/2)^{-1}.$$

Substituting this estimate into (6.2.11), we obtain

$$|u'(x)| \leqslant C[1+(\varepsilon+x)^{-1}\ln^{-1}(\varepsilon+1/2)^{-1}].$$

To determine $u''(x)$ for $a=1$, (6.2.15) becomes

$$u''(x) = u''(1/2)\left[\frac{\varepsilon}{\varepsilon+x}\right]^2 \exp[-g_3(x)] + g_4(x). \quad (6.2.22)$$

From (6.2.18), $g_4(x)$ is defined as follows

$$|g_4(x)| \leqslant C(\varepsilon+x)^{-2}\int_{1/2}^{x} [1+u'(s)](\varepsilon+s)\,ds \leqslant C[1+(\varepsilon+x)^{-1}\ln^{-1}(\varepsilon+1/2)^{-1}]. \quad (6.2.23)$$

From (6.1.1), we obtain

$$u''(1/2) \leqslant C(\varepsilon+1/2)^{-1}[1+u'(1/2)] \leqslant C(\varepsilon+1/2)^{-2}\ln^{-1}(\varepsilon+1/2)^{-1}.$$

From (6.2.22) using the estimate of $u''(1/2)$ and $g_4(x)$, and taking into account the inequality $(\varepsilon+1/2)^{-2} \leqslant \varepsilon^{-2}$, we obtain

$$u''(x) \leqslant C[1+(\varepsilon+x)^{-2}\ln^{-1}(\varepsilon+1/2)^{-1}].$$

By differentiating (6.1.1) and with the help of (6.2.5), we obtain the following result

$$|u^{(j)}(x)| \leqslant C[1+(\varepsilon+x)^{-j}\ln^{-1}(\varepsilon+1/2)^{-1}], \ a=1.$$

We easily prove the case when $a>1$ by using $u'(1/2) \leqslant C\varepsilon^{-1}$ obtained from (6.2.5) for $1/2 \leqslant x \leqslant x_0$ which substituted into (6.2.11), leads to

$$|u'(x)| \leqslant C[1+\varepsilon^{a-1}(\varepsilon+x)^{-a}].$$

To obtain $u''(x)$ for $a>1$. Using $u''(1/2) \leqslant C\varepsilon^{-2}$ obtained from (6.2.5) for $1/2 \leqslant x \leqslant x_0$ and $|g_4(x)| \leqslant C[1+\varepsilon^{a-1}(\varepsilon+x)^{-a}]$ substituted into (6.2.15), we obtain

$$|u''(x)| \leqslant C[1+\varepsilon^{a-1}(\varepsilon+x)^{-a-1}].$$

Differentiating (6.1.1) and taking into account (6.2.5), we easily obtain

$$|u^{(j)}(x)| \leqslant C[1+\varepsilon^{a-1}(\varepsilon+x)^{1-a-j}].$$

This concludes the proof of the estimate (6.2.2) for $1/2 < x \leqslant 1/2+\tau$.

case 2 : a $\leqslant$ 0 for $1/2-\tau \leqslant x \leqslant 1/2$. In this case, $u'(x)$ is expressed as follows:

$$u'(x) = u'(x_0)\exp[\psi(x_0,x)] + \int_{x_0}^{x} \frac{f(s)+b(s)u(s)}{\varepsilon+s}\exp[\psi(x_0,x)]\,ds, \tag{6.2.24}$$

where

$$\psi(s,x) = -\int_s^x \frac{a(\kappa)}{\varepsilon+\kappa}\,d\kappa.$$

If $a(1/2) = 0$ then $\psi(s,x) \leqslant C,\ 1/2-\tau \leqslant s,\ x \leqslant 1/2$. Using the triangle inequality in (6.2.24) and choosing a point $x_0 \in [(1-\tau)/2, 1/2]$ such that $u'(x_0) \leqslant C$, we obtain

$$|u'(x)| \leqslant C[1 + |\ln(\varepsilon + x)|],\ 1/2-\tau \leqslant x \leqslant 1/2,\ a(1/2) = 0,\ j = 1 \text{ since } p = 0.$$

Let determine $u''(x)$ with $p = 0$ for $j = 2$. On differentiating (6.1.1) and solving the resulting equation for $u''(x)$, we obtain

$$u''(x) = u''(x_0)\exp[\psi(x_0,x)] + (\varepsilon+x)^{-p-1}\int_{x_0}^{x} \frac{F(s)}{\varepsilon+s}(\varepsilon+s)^{p+1}\exp[\psi(s,x)]\,ds \quad (6.2.25)$$

where

$$\psi(s,x) = -\int_s^x \frac{a(\kappa)}{\varepsilon+\kappa}\,d\kappa,$$

and

$$F(s) = f(s) + b'(s)u(s) + [b(s) - a'(s)]u'(s).$$

Substituting $\psi(s,x) \leqslant C$ and $u''(x_0) \leqslant C$ into (6.2.25), we obtain

$$|u''(x)| \leqslant C + C(\varepsilon+x)^{-p-1}\int_{x_0}^{x}[1+u'(s)](\varepsilon+s)^p\,ds \leqslant C[1 + (\varepsilon+x)^{-p-1}\ln(\varepsilon+x)].$$

Differentiating (6.1.1) and taking into account (6.2.5), we obtain

$$|u^{(j)}(x)| \leqslant C[1 + (\varepsilon+x)^{1-j-p}|\ln(\varepsilon+x)|],\ \ a+p=0;\ j>p,\ j=1,2,3,4.$$

Let $a(1/2) < 0$. In this case $p > 1$. Then there exists a constant $x_0 > 0$ such that $a(x) < 0$ for $1/2-\tau \leqslant x \leqslant x_0$. Therefore, we have

$$\psi(s,x) \leqslant -x_0\ln[(\varepsilon+s)/(\varepsilon+x)],\ 1/2-\tau \leqslant x \leqslant s \leqslant x_0.$$

Taking exponentials on both sides of inequality, we obtain

$$\exp(\psi(s,x)) \leqslant [(\varepsilon+x)/(\varepsilon+s)]^{x_0},\ 1/2-\tau \leqslant x \leqslant s \leqslant x_0.$$

Substituting this estimate in (6.2.24) with $x = s$ and taking into account (6.2.5), we obtain

$$|u'(x)| \leqslant C,\ 1/2-\tau \leqslant x \leqslant x_0,\ a(1/2) < 0.$$

Differentiating (6.1.1) and taking into account (6.2.5), we obtain

$$|u^{(j)}(x)| \leqslant C,\ 1/2-\tau \leqslant x \leqslant 1/2,\ a<0,\ k \leqslant p,\ j=1,2,3,4.$$

Consider the case when $j>p,\ a+p>0$ and $a \leqslant 0$. The expression for $u'(x)$ is

$$u'(x) = u'(1/2)\left[\frac{\varepsilon}{\varepsilon+x}\right]^{a+1}\exp[-g_1(x)] + g_2(x),\ 1/2-\tau \leqslant x \leqslant 1/2. \tag{6.2.26}$$

Following the same steps as for $0<a<1$, we obtain

$$|u'(x)| \leqslant C[1+(\varepsilon+x)^{-a-1}],\ a \leqslant 0.$$

$u''(x)$ is given by

$$u''(x) = u''(1/2)\left[\frac{\varepsilon}{\varepsilon+x}\right]^{a+2}\exp[-g_3(x)] + g_4(x),\ 1/2-\tau \leqslant x \leqslant 1/2. \tag{6.2.27}$$

We obtain $u''(x)$ by following the same steps as for $0<a<1$:

$$|u''(x)| \leqslant C[1+(\varepsilon+x)^{-a-2}],\ a \leqslant 0.$$

Differentiating (6.1.1) and taking into account (6.2.5), we obtain

$$|u^{(j)}(x)| \leqslant C[1+(\varepsilon+x)^{-a-j}],\ 1/2-\tau \leqslant x \leqslant 1/2,\ a \leqslant 0,\ j>p,\ j=1,2,3,4.$$

By (6.2.2) and (6.2.3), the derivatives in (6.1.1)-(6.1.2) may be estimated by a power function with argument $\varepsilon+x$, so that (6.1.1)-(6.1.2) is an equation with power interior function [38].

The singularly perturbed turning point problem (6.1.1)-(6.1.2) may be regarded as a concatenation of two problems: One defined on the interval $[0,1/2)$ and the other one on the interval $(1/2,1]$. Therefore, the solution of the problem (6.1.1)-(6.1.2) may display a layer near $x=1/2$ on $[0,1/2)$ and a layer $x=1/2$ on $(1/2,1]$. This consideration allows us to understand the behaviour of the solution and its derivatives. The solution can be decomposed into two parts, namely the smooth component $v(x)$ and the singular component $w(x)$ ([42], pp 47) such that

$$u(x) = v(x) + w(x),$$

where $v(x)$ is the solution of the inhomogeneous problem

$$Lv(x) = f(x), \ \ x \in \Omega_1 = (1/2, 1], \tag{6.2.28}$$

$$v(1/2) = 0, \ \ v(1) = u(1) = \alpha_2, \tag{6.2.29}$$

and $w(x)$ is the solution of the homogeneous problem

$$Lw(x) = 0, \ \ x \in \Omega_1, \tag{6.2.30}$$

$$w(1/2) = u(1/2) - v(1/2), \ \ w(1) = 0. \tag{6.2.31}$$

The following lemma gives the bounds on the solution to (6.1.1)-(6.1.2) and its derivatives.

Lemma 6.2.6. The smooth and singular components of $u(x)$ of problem (6.1.1)-(6.1.2), for $0 \leqslant j \leqslant 4$ satisfies

$$|v^{(j)}(x)| \leqslant C \begin{cases} 1 + (\varepsilon + x)^{2-j}|\ln(\varepsilon + x)|, & x \in [0, 1/2], \\ 1 + (\varepsilon + x)^{3-a-j}, & a < 1, \ x \in [1/2, 1], \\ 1 + (\varepsilon + x)^{2-j}|\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \ x \in [1/2, 1], \\ 1 + \varepsilon^{a-1}(\varepsilon + x)^{3-a-j}, & a > 1, \ x \in [1/2, 1], \end{cases} \tag{6.2.32}$$

and

$$|w^{(j)}(x)| \leqslant C \begin{cases} (\varepsilon + x)^{1-j}|\ln(\varepsilon + x)|, & x \in [0, 1/2], \\ (\varepsilon + x)^{1-a-j}, & a < 1, \ \ x \in [1/2, 1], \\ (\varepsilon + x)^{-j}|\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \ \ x \in [1/2, 1], \\ \varepsilon^{a-1}(\varepsilon + x)^{1-a-j} & a > 1, \ \ x \in [1/2, 1], \end{cases} \tag{6.2.33}$$

where C is constant and independent of ε.

Proof. We prove this lemma on $\Omega_1 = [1/2, 1]$. The proof on $[0, 1/2]$ follows similar steps. The reduced problem ($\varepsilon = 0$), corresponding to problem (6.1.1) has the same order

$$x^2 v_0'' + a(x)v_0' - b(x)v_0 = f(x), \ x \in \Omega_1 \tag{6.2.34}$$

$$v(1/2) = 0, \ v_0(1) = u(1) = \beta. \tag{6.2.35}$$

The smooth component $v(x)$ is further decomposed into the sum ([42], pp 68)

$$v(x) = v_0(x) + (\varepsilon + x)v_1(x) + (\varepsilon + x)^2 v_2(x), \quad x \in \bar{\Omega}_1, \qquad (6.2.36)$$

where v_0 is the solution of the reduced problem in (6.2.34), which is independent of ε, and having smooth coefficients $a(x)$, $b(x)$ and $f(x)$. From these assumptions, for $0 \leqslant j \leqslant 4$, we have

$$|v_0^{(j)}(x)| \leqslant C, \text{ for all } x \in \bar{\Omega}_1 \qquad (6.2.37)$$

and, v_1 and v_2 are the solutions of (6.1.1), where Lemma 6.2.5 used. Now, applying the triangle inequality, using (6.2.37) and the estimates of v_1 and v_2 from (6.2.2) substituted into (6.2.36), which complete the proof.

To prove the regular part $w(x)$, construct the barrier functions as follows [31].

$$\Psi^{\pm}(x) = C\exp(-\eta x/\varepsilon) \pm w(x), \; x \in \bar{\Omega}_1.$$

The values of $\Psi^{\pm}(x)$ at the boundaries are

$$\begin{aligned}
\Psi^{\pm}(1/2) &= C\exp(-\eta/2\varepsilon) \pm w(1/2), \\
&= C\exp(-\eta/2\varepsilon) \pm (u(1/2) - v(1/2)), \quad \text{using (6.2.31)}, \\
&\geqslant 0, \quad \text{C is a constant chosen sufficiently large}, \\
\Psi^{\pm}(1) &= C\exp(-\eta/\varepsilon) \pm w(1), \\
&= C\exp(-\eta/\varepsilon), \quad \text{using (6.2.31)}, \\
&\geqslant 0.
\end{aligned}$$

From the above estimates, it is clear that $\Psi(x) \geqslant 0, \;\; x \in \Omega_2 = \bar{\Omega}_1 \backslash \Omega_1$. Therefore we have

$$\begin{aligned}
L\Psi^{\pm}(x) &= (\varepsilon + x)[\Psi^{\pm}(x)]'' + a(x)[\Psi^{\pm}(x)]' - b(x)\Psi^{\pm}(x), \; x \in \Omega_1 \\
&= C\exp(-\eta x/\varepsilon)\left[\frac{\eta^2(\varepsilon + x)}{\varepsilon^2} - \frac{\eta a(x)}{\varepsilon} - b(x)\right] \pm Lw(x) \\
&= C\exp(-\eta x/\varepsilon)\left[\frac{\eta^2(\varepsilon + x)}{\varepsilon^2} - \frac{\eta a(x)}{\varepsilon} - b(x)\right], \quad \text{using (6.2.30)} \\
&\leqslant 0, \text{ since } (x/\varepsilon^2) \leqslant b(x), \;\; x \in \Omega_1.
\end{aligned}$$

Hence, by Lemma 6.2.1, we obtain $\Psi^{\pm}(x) \geqslant 0, \;\; x \in \bar{\Omega}_1$. Then we have

$$C \exp(-\eta x/\varepsilon) \pm w(x) \geqslant 0.$$

Then, we obtain

$$w(x) \;\; \leqslant \;\; C \exp(-\eta x/\varepsilon), \; x \in \Omega_1.$$

Using the inequality relation, the above inequality can expressed as follows

$$|w(x)| \leqslant C \exp(-\eta x/\varepsilon) \leqslant C \begin{cases} (\varepsilon + x)^{1-a}, & a < 1, \\ (\varepsilon + x)^{-1} |\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \\ \varepsilon^{a-1}(\varepsilon + x)^{1-a}, & a > 1. \end{cases} \tag{6.2.38}$$

Since $Lw(x) = 0$, the j^{th} derivative of $w(x)$ can be estimated immediately from the estimate of $w(x)$. The following estimates hold for $0 \leqslant j \leqslant 4$,

$$|w^{(j)}(x)| \leqslant C \begin{cases} (\varepsilon + x)^{1-a-j}, & a < 1, \\ (\varepsilon + x)^{-j} |\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \\ \varepsilon^{a-1}(\varepsilon + x)^{1-a-j}, & a > 1, \end{cases} \tag{6.2.39}$$

which completes the proof.

6.3 Construction of the FMFDM

We develop a difference scheme to determine the solution of problem (6.1.1)-(6.1.2). It is assumed that there is an interior layer at point $x_{N/2} = 1/2$ with adapted Shishkin mesh Ω_N^{τ}, where N is a multiple of 4. The interval $[0,1]$ is divided into three subintervals

$$[0,1] =: [0, 1/2 - \tau], [1/2 - \tau, 1/2 + \tau] \text{ and } [1/2 + \tau, 1].$$

Each of the intervals $[0, 1/2 - \tau]$ and $[1/2 + \tau, 1]$ is divided uniformly into $N/4$ sub-intervals whist the interval $[1/2 - \tau, 1/2 + \tau]$ is divided into $N/2$ sub-intervals. In this case, we define τ as

$$\tau = \min \left\{ \frac{1}{4}, \frac{\varepsilon}{\eta} \ln (N/4) \right\}. \tag{6.3.1}$$

The mesh size in each of the three sub-intervals is given by

$$x_j - x_{j-1} = \begin{cases} h_j = 4(0.5-\tau)/N & \text{if } j = 1,2,\cdots,N/4 \ , \ 3N/4+1,\cdots,N-1,N, \\ h_j = 4\tau/N & \text{if } j = N/4+1 \ , \ N/4+2\cdots 3N/4. \end{cases} \tag{6.3.2}$$

For the rest of chapter, we use the notation $D(x_j) = D_j$. We discretize (6.1.1)-(6.1.2) using the upwind scheme on Ω_N^τ in the following manner:

$$L^N U_j := \begin{cases} (\varepsilon + x_j)\tilde{D}U_j + \tilde{a}_j D^- U_j - \tilde{b}_j U_j = \tilde{f}_j & \text{for } j = 0,1,\cdots,N/2-1, \\ (\varepsilon + x_j)\tilde{D}U_j + \tilde{a}_j D^+ U_j - \tilde{b}_(x) U_j = \tilde{f}_j & \text{for } j = N/2, N/2+1,\cdots,N-1, \end{cases} \tag{6.3.3}$$

$$U(0) = \alpha_1, \ \ U(1) = \alpha_2, \tag{6.3.4}$$

where

$$\begin{cases} \tilde{a}_j = \frac{a_{j-1}+a_j}{2} & \text{for } j = 0,1,\cdots,N/2-1, \\ \tilde{a}_j = \frac{a_j + a_{j+1}}{2} & \text{for } j = N/2, N/2+1,\cdots,N-1, \end{cases}$$

$$\begin{cases} \tilde{b}_j = \frac{b_{j-1}+b_j+b_{j+1}}{3} & \text{for } j = 1,2,3,\cdots,N-1, \\ \tilde{f}_j = \frac{f_{j-1}+f_j+f_{j+1}}{3} & \text{for } j = 1,2,3,\cdots,N-1. \end{cases}$$

$$D^+U_j = \frac{U_{j+1}-U_j}{h_{j+1}}, \ \ D^-U_j = \frac{U_j - U_{j-1}}{h_j} \ \text{ and } \ \tilde{D}U_j = \frac{2}{h_j + h_{j+1}}(D^+U_j - D^-U_j).$$

(6.3.3) can be written in the form:

$$L^N U_j := r^- U_{j-1} + r^c U_j + r^+ U_{j+1} = f_j, \ j = 1,2,3\cdots,N-1, \tag{6.3.5}$$

where, for $j = 1,2,3\cdots,N/2-1$, we have

$$r_j^- = \frac{2(\varepsilon + x_j)}{h_j(h_j+h_{j+1})} - \frac{\tilde{a}_j}{h_j}, \ \ r_j^c = \frac{\tilde{a}_j}{h_j} - \frac{2(\varepsilon + x_j)}{h_j h_{j+1}} - \tilde{b}_j, \ \ r_j^+ = \frac{2(\varepsilon + x_j)}{h_{j+1}(h_j + h_{j+1})}, \tag{6.3.6}$$

and for $j = N/2, N/2+1,\cdots,N-1$, we have

$$r_j^- = \frac{2(\varepsilon + x_j)}{h_j(h_j+h_{j+1})}, \ \ r_j^c = -\frac{\tilde{a}_j}{h_{j+1}} - \frac{2(\varepsilon + x_j)}{h_j h_{j+1}} - \tilde{b}_j \text{ and } r_j^+ = \frac{2(\varepsilon + x_j)}{h_{j+1}(h_j + h_{j+1})} + \frac{\tilde{a}_j}{h_{j+1}}. \tag{6.3.7}$$

Based on the scheme developed above, we need to validate the next lemma which states that problem (6.3.3)-(6.3.4) satisfies the discrete minimum principle.

Lemma 6.3.1. Let ξ_j be any mesh function such that $L^N\xi_j \leqslant 0$, $1 \leqslant j \leq N-1$, $\xi_0 \geq 0$ and $\xi_N \geq 0$ implies that $\xi_j \geq 0$, $0 \leq j \leqslant N$.

Proof. See Lemma 4.3.1 in Chapter 4.

Lemma 6.3.1 is used to prove the following lemma.

Lemma 6.3.2. Suppose that Z_i is any mesh function such that $Z_0 = Z_N = 0$, then

$$|Z_i| \leqslant \frac{1}{b_0} \max_{1\leqslant j\leq N-1} |L^N Z_j|, \ \ \forall \ \ 0 \leq i \leq N.$$

Proof. See Lemma 4.3.2 in Chapter 4.

With the above results, we are ready to provide the ε-uniform convergence in the next section.

6.4 Convergence analysis

In this section we prove that the proposed method FMFDM is uniformly convergent of order one, up to a logarithmic factor.

Theorem 6.4.1. Let $u(x)$ be the solution of the continuous problem (6.1.1)-(6.1.2) and $U(x)$ is the numerical solution of problem (6.3.3) and (6.3.4). Then, for sufficiently large N, we have the following result

$$\max_{0\leq j\leq N} |u_j - U_j| \leqslant CN^{-1}\left[\ln\left(\frac{N}{4}\right)\right]^2. \tag{6.4.1}$$

Proof. We prove the theorem on $[1/2, 1]$ by considering each case of a. The proof on $[0, 1/2]$ follows in a similar way. We decompose the solution U of the discrete problem (6.3.3) and (6.3.4) into a regular and a singular parts as

$$U = V + W, \tag{6.4.2}$$

where V is the solution of the inhomogeneous problem

$$L^N V = f_j, \text{ for } j = N/2, \cdots, N\ , V(1/2) = v(1/2), V(1) = v(1),$$

and W is the solution of the homogenous problem

$$L^N W_j = 0, \text{ for } j = N/2, \cdots, N, \ W(1/2) = U(1/2) - V(1/2), W(1) = w(1).$$

Equation (6.4.2) can be expressed as follows:

$$U - u = (V - v) + (W - w) \tag{6.4.3}$$

and the components of the solution can be estimated separately. A combination of (6.1.1) and (6.3.3) gives the error for the regular component as follows

$$\begin{aligned} L^N(V - v) &= f - L^N v \\ &= (L - L^N)v \\ &= (\varepsilon + x_j)\left(\frac{d^2}{dx^2} - \tilde{D}\right)v + \tilde{a}_j\left(\frac{d}{dx} - D^-\right)v. \end{aligned}$$

Applying the two estimates in Lemma 4.1 pp 24 of [42], we obtain

$$|L^N(V_j - v_j)| \le \frac{(\varepsilon + x_j)}{3}(x_{j+1} - x_{j-1})|v_j'''| + \frac{\tilde{a}_j}{2}(x_j - x_{j-1})|v_j''| \text{ for } 1 \le j \leqslant N/2 - 1. \tag{6.4.4}$$

Using the estimates of the derivatives of v_j in Lemma 6.2.6 and $h_j = x_j - x_{j-1} \le 4N^{-1}$ in (6.4.4), we obtain

$$|L^N(V_j - v_j)| \leqslant CN^{-1} \begin{cases} 1 + (\varepsilon + x) + (\varepsilon + x)^{1-a}, & a < 1, \\ 1 + (\varepsilon + x) + \ln^{-1}(\varepsilon + 1/2)^{-1}, & a = 1, \\ 1 + (\varepsilon + x) + \varepsilon^{a-1}(\varepsilon + x)^{1-a}, & a > 1. \end{cases} \tag{6.4.5}$$

Using the inequality relation, the inequalities in (6.4.5) lead to the following result

$$|L^N(V_j - v_j)(x_j)| \le CN^{-1}.$$

Hence, by Lemma 6.3.2, we obtain

$$|(V_j - v_j)(x_j)| \leqslant CN^{-1} \text{ for } N/2 \leqslant j \leqslant N. \tag{6.4.6}$$

The estimates on $L^N(W_j - w_j)$ depends on the value of τ, whether $\tau = 1/4$ or $\tau = (\varepsilon/\eta)\ln(N/4)$.

If $\tau = 1/4$, the mesh is uniform, i.e., $\tau = 1/4 \leq (\varepsilon/\eta)\ln(N/4)$. The estimation of the singular component is similar to the equation (6.4.4), then gives

$$|L^N(W_j - w_j)| \leq \frac{(\varepsilon + x_j)}{3}(x_{j+1} - x_{j-1})|w_j'''| + \frac{\tilde{a}_j}{2}(x_j - x_{j-1})|w_j''| \text{ for } 1 \leq j \leqslant N/2-1. \quad (6.4.7)$$

Noting that $x_j - x_{j-1} \leq 4N^{-1}$ and taking into account the bounds on w_j in Lemma 6.2.6, we obtain

$$|L^N(W_j - w_j)| \leqslant CN^{-1}\begin{cases} (\varepsilon + x)^{-a-1}, & a < 1, \\ (\varepsilon + x)^{-2}|\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \\ \varepsilon^{a-1}(\varepsilon + x)^{-a-2}, & a > 1. \end{cases} \quad (6.4.8)$$

The above inequalities lead to

$$|L^N(W_j - w_j)(x_j)| \leqslant CN^{-1}\varepsilon^{-2}. \quad (6.4.9)$$

Since $\varepsilon^{-1} \leqslant (4/\eta)\ln(N/4)$, we obtain the following inequality

$$|L^N(W_j - w_j)(x_j)| \leqslant CN^{-1}\ln^2(N/4).$$

Using Lemma 6.3.2 then we obtain

$$|(W_j - w_j)(x_j)| \leqslant CN^{-1}\ln^2(N/4) \text{ for } N/2 \leqslant j \leqslant N-1. \quad (6.4.10)$$

If $\tau = (\varepsilon/\eta)\ln(N/4)$, the mesh is piecewise uniform. In this case we have two subintervals namely $[1/2, \tau + 1/2]$ and $[\tau + 1/2, 1]$. A different argument is used to bound $W - w$ in each subinterval. Firstly, compute the error for the singular component in the coarse mesh region $[\tau + 1/2, 1]$, i.e., for all $\tau + 1/2 \leqslant x_j \leqslant 1$. Using the triangle inequality, we have

$$|W_j - w_j| \leq |W_j| + |w_j|. \quad (6.4.11)$$

Application of Lemma 6.2.6 in (6.4.11), we obtain

$$|w(x_j)| \leqslant C\varepsilon^{-1}x_j \leqslant C\varepsilon^{-1}\tau.$$

Substituting the value of τ into the above expression, we obtain

$$|w(x_j)| \leqslant C\ln(N/4) \leqslant C \text{ for } 3N/4 \leqslant j \leqslant N. \tag{6.4.12}$$

Now to obtain a similar bound on W, the interested readers can obtain the following inequalities with the help of Lemma 7.3 (p.58) and Lemma 7.5 (p.60) of [42] which lead to

$$|W(x_j)| \leqslant C\ln(N/4) \leqslant C \text{ for } 3N/4 \leqslant j \leqslant N. \tag{6.4.13}$$

Combining the estimated obtained by (6.4.12) and (6.4.13), we obtain

$$|(W-w)(x_j)| \leqslant C\ln(N/4) \leqslant C \text{ for } 3N/4 \leqslant j \leqslant N. \tag{6.4.14}$$

Now the bounds in the interior region, $[1/2, \tau+1/2]$ can be obtained from (6.4.7) by using the bounds of w_j in Lemma 6.2.6 and keeping in mind that $h = 4\tau/N$, we obtain

$$|L^N(W_j - w_j)| \leq Ch\varepsilon^{-2} \leqslant C\tau\varepsilon^{-2}N^{-1}.$$

From (6.4.14), we have

$$|(W(1/2) - w(1/2)| = 0$$

and

$$|(W(3N/4) - w(3N/4)| \leq |(W(3N/4)| + |w(3N/4)| \leqslant C\ln(N/4) \leqslant C.$$

Now, introduce the barrier function Φ_j in $[1/2, \tau+1/2]$ defined by

$$\Phi_j = (x+\tau)C_1\varepsilon^{-2}\tau N^{-1} + C_2N^{-1},$$

where C_1 and C_2 are arbitrary constants.

Now, consider comparison functions Ψ defined by

$$\Psi_j^{\pm} = (x_j+\tau)C_1\varepsilon^{-2}\tau N^{-1} + C_2N^{-1} \pm (W_j - w_j). \tag{6.4.15}$$

For an appropriate choice of C_1 and C_2, (6.4.15) satisfies the following

$$\Psi_{3N/4} \geq 0 \text{ and } \Psi_{N/2} = 0.$$

Note that

$$L^N \Psi_j \leqslant 0, \quad N/2+1 \leqslant j \leqslant 3N/4-1.$$

Application of Lemma 6.3.1 on $[1/2, \tau+1/2]$ for the function $\Psi_j^{\pm}$, we obtain

$$\Psi_j \geqslant 0, \quad N/2+1 \leqslant j \leqslant 3N/4-1.$$

Consequently,

$$|W_j - w_j| \leq \Phi_j \leq C_1 \varepsilon^{-2} \tau^2 N^{-1} + C_2 N^{-1}.$$

Substituting the value of τ into the above inequality, we obtain

$$|W_j - w_j| \leq C N^{-1} \ln^2 (N/4) \,. \tag{6.4.16}$$

Using estimates (6.4.14) and (6.4.16), we obtain the estimate of the singular component of the error on $[1/2, \tau+1/2]$

$$|W_j - w_j| \leq C N^{-1} \ln^2 (N/4) \,, \; N/2+1 \leqslant j \leqslant 3N/4-1. \tag{6.4.17}$$

Combining estimates (6.4.6) and (6.4.17) along with (6.4.3), we obtain

$$|U_j - u_j| \leqslant C N^{-1} \left[\ln \left(\frac{N}{4} \right) \right]^2, \quad N/2-1 \leqslant j \leqslant N. \tag{6.4.18}$$

A similar analysis on the subinterval $[0, 1/2]$ gives

$$|U_j - u_j| \leqslant C N^{-1} \left[\ln \left(\frac{N}{4} \right) \right]^2, \quad 1 \leqslant j \leqslant N/2. \tag{6.4.19}$$

Combining inequalities (6.4.19) and (6.4.18), we obtain the required result.

To increase the accuracy as well as the rate of convergence of the scheme, we use Richardson extrapolation in the next section.

6.5 Richardson extrapolation on the FMFDM

It is a post-processing procedure where a linear combination of two computed solutions approximating a particular quantity results a third and better approximation [49]. We apply this method for the proposed scheme.

We consider a second mesh Ω_{2N}^{τ}, which has the same transition parameter τ given by (6.3.1), and is obtained by bisecting each sub-interval of Ω_N^{τ} defined in (6.3.2). Thus,

$$\Omega_N^{\tau} = x_j \subset \Omega_{2N}^{\tau} = \tilde{x}_j$$

and

$$\tilde{x}_j - \tilde{x}_{j-1} = \tilde{h}_j = h_j/2.$$

Theorem 6.5.1. (Error after extrapolation). Let $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth and $u(x)$ be the solution of (6.1.1). If U^{ext} be the approximation solution of u obtained using (6.3.3)-(6.3.4) with $u(0) = U(0)$, $u(1) = U(1)$, then there exists a positive constant C independent of ε and the mesh spacing such that

$$\max_{0\leq j\leq N} |(U_j^{ext} - u_j)| \leqslant CN^{-2}\left[\ln\left(\frac{N}{4}\right)\right]^2. \tag{6.5.1}$$

Proof. As mentioned in the previous section, Theorem 6.5.1 will be proved only on the interval $[1/2, 1]$. Let U_j, $\tilde{U}_j$ be the numerical solution of the discrete problem (6.3.3)-(6.3.4) on the mesh Ω_N^{τ} and Ω_{2n}^{τ} respectively. Then the inequality (6.4.19) can be written as

$$U_j - u_j = C_1 N^{-1} \ln^2(N/4) + R_N(x_j) \ \ \forall x_j \in {\Omega_N}^{\tau} \tag{6.5.2}$$

and

$$\tilde{U}_j - u_j = C_2(2N)^{-1} \ln^2(N/4) + R_{2N}(\tilde{x}_j) \ \ \forall \tilde{x}_j \in \Omega_{2n}^{\tau}, \tag{6.5.3}$$

where C_1 and C_2 are some fixed constants and where the remainder terms

$$R_N(x_j) \text{ and } R_{2N}(\tilde{x}_j) \text{ are } \mathcal{O}[N^{-1}\ln^2(N/4)].$$

A combination of two equations (6.5.2) and (6.5.3) leads

$$u_j - (2\tilde{U}_j - U_j) = R_N(x_j) - 2R_{2N}(x_j) = \mathcal{O}[N^{-1}\ln^2(N/4)] \ \ \forall x_j \in \Omega_N^{\tau}, \tag{6.5.4}$$

therefore we set

$$U_j^{ext} = 2\tilde{U}_j - U_j \ \ \forall x_j \in \Omega_N^{\tau}, \tag{6.5.5}$$

as the new numerical approximation of u_j at the grid point $x_j \in \Omega_n^\tau$ computed after using the extrapolation process.

The decomposition of the error after extrapolation can be also expressed as in (6.4.3), which gives

$$U_j^{ext} - u_j = (V_j^{ext} - v_j) + (W_j^{ext} - w_j), \tag{6.5.6}$$

where V_j^{ext} and W_j^{ext} are the regular and singular component of U_j^{ext}.

The local truncation error of the scheme (6.3.3)-(6.3.5) after extrapolation is given by

$$L^N(U_j^{ext} - u_j) = 2L^N(\tilde{U}_j - u_j) - L^N(U_j - u_j) \tag{6.5.7}$$

where

$$L^N(U_j - u_j) = r^- u_{j-1} + r^c u_j + r^+ u_{j+1} - (\varepsilon + x_j)u_j'' - a_j u_j' + b_j u, \tag{6.5.8}$$

and

$$L^N(\tilde{U}_j - u_j) = \tilde{r}^- u_{j-1} + \tilde{r}^c u_j + \tilde{r}^+ u_{j+1} - (\varepsilon + x_j)u_j'' - a_j u_j' + b_j u. \tag{6.5.9}$$

The quantities r^-, r^c and r^+ are given in (6.3.6) while the quantities $\tilde{r}^-$, $\tilde{r}^c$ and $\tilde{r}^+$ are obtained by replacing h_j with $\tilde{h}_j$ and h_{j+1} with $\tilde{h}_{j+1}$ in the expressions of r^-, r^c and r^+ respectively. Taking the Taylor series expansion of u_j around x_j, we obtain the following approximations for u_{j-1} and u_{j+1}:

$$u_{j-1} = u_j - h_j u_j' + \frac{h_j^2}{2} u_j^2 - \frac{h_j^3}{6} u_j^3 + \frac{h_j^4}{24} u^4(\xi_1, j), \tag{6.5.10}$$

$$u_{j+1} = u_j + h_{j+1} u_j' + \frac{h_{j+1}^2}{2} u_j^2 + \frac{h_{j+1}^3}{6} u_j^3 + \frac{h_{j+1}^4}{24} u^4(\xi_2, j), \tag{6.5.11}$$

$$u_{j-1} = u_j - \tilde{h}_j u_j' + \frac{\tilde{h}_j^2}{2} u_j^2 - \frac{\tilde{h}_j^3}{6} u_j^3 + \frac{\tilde{h}_j^4}{24} u^4(\tilde{\xi}_1, j), \tag{6.5.12}$$

$$u_{j+1} = u_j + \tilde{h}_{j+1} u_j' + \frac{\tilde{h}_{j+1}^2}{2} u_j^2 + \frac{\tilde{h}_{j+1}^3}{6} u_j^3 + \frac{\tilde{h}_{j+1}^4}{24} u^4(\tilde{\xi}_2, j), \tag{6.5.13}$$

where

$$(\xi_1, j) \in (x_{j-1}, x_j), \quad (\xi_2, j) \in (x_j, x_{j+1}), \quad \tilde{\xi}_1 \in (\frac{x_{j-1} + x_j}{2}, x_j) \text{ and } \tilde{\xi}_2 \in (x_j, \frac{x_j + x_{j+1}}{2}).$$

Substituting (6.5.10) and (6.5.11) into (6.5.8), (6.5.12) and (6.5.13) into (6.5.9), we obtain the following expressions

$$L^N(U_j - u_j) = k_1 u_j + k_2 u_j' + k_3 u_j^2 + k_4 u_j^3 + k_{5,1} u^4(\xi_1, j) + k_{5,2} u^4(\xi_2, j) \tag{6.5.14}$$

and

$$L^N(\tilde{U}_j - u_j) = \tilde{k}_1 u_j + \tilde{k}_2 u_j' + \tilde{k}_3 u_j^2 + \tilde{k}_4 u_j^3 + \tilde{k}_4 u_j^4 + \tilde{k}_{5,1} u^4(\tilde{\xi}_1, j) + \tilde{k}_{5,2} u^4(\tilde{\xi}_2, j). \tag{6.5.15}$$

The coefficients in (6.5.14)

$$k_1 = \frac{2(\varepsilon + x_j)}{h_j(h_j + h_{j+1})} - \frac{2(\varepsilon + x_j)}{h_j h_{j+1}} + \frac{2(\varepsilon + x_j)}{h_{j+1}(h_j + h_{j+1})}, \; k_2 = 0,$$

$$k_3 = \frac{(\varepsilon + x_j)h_j}{h_j + h_{j+1}} - \frac{a_j h_j}{2} + \frac{(\varepsilon + x_j)h_{j+1}}{h_j + h_{j+1}} - (\varepsilon + x_j),$$

$$k_4 = \frac{-(\varepsilon + x_j)h_j^2}{3(h_j + h_{j+1})} + \frac{a_j h_j^2}{6} + \frac{(\varepsilon + x_j)h_{j+1}^2}{3(h_j + h_{j+1})}, \; k_{5,1} = \frac{(\varepsilon + x_j)h_j^3}{12(h_j + h_{j+1})} - \frac{a_j h_j^3}{24}, \; k_{5,2} = \frac{(\varepsilon + x_j)h_{j+1}^3}{12(h_j + h_{j+1})}.$$

The quantities $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$, $\tilde{k}_4$, $\tilde{k}_{5,1}$ and $\tilde{k}_{5,2}$ can be determined by replacing h_j by $\tilde{h}_j$ and h_{j+1} by $\tilde{h}_{j+1}$.

Substituting (6.5.14) and (6.5.15) into (6.5.7), we obtain

$$L^N(U_j^{ext} - u_j) = T_1 u_j + T_2 u_j'' + +T_3 u_j''' + +T_{4,1} u^{(4)}(\xi_1, j) + T_{4,2} u^{(4)}(\xi_2, j), \tag{6.5.16}$$

where

$$T_1 = \frac{14(\varepsilon + x_j)}{h_j(h_j + h_{j+1})} - \frac{14(\varepsilon + x_j)}{h_j h_{j+1}} + \frac{14(\varepsilon + x_j)}{h_{j+1}(h_j + h_{j+1})},$$

$$T_2 = \frac{(\varepsilon + x_j)h_j}{h_j + h_{j+1}} - (\varepsilon + x_j) + \frac{(\varepsilon + x_j)h_{j+1}}{h_j + h_{j+1}}, \quad T_3 = -\frac{a_j h_j^2}{12},$$

$$T_{4,1} = -\frac{(\varepsilon + x_j)h_j^3}{24(h_j + h_{j+1})} + \frac{a_j h_j^3}{32} \text{ and } T_{4,2} = -\frac{(\varepsilon + x_j)h_{j+1}^3}{24}.$$

Given (6.3.2) and for the sake of simplicity, we shall use the following notation

$$h_j = H \text{ if } j = 3N/4, 3N/4 + 1, \cdots, N, \tag{6.5.17}$$

$$h_j = h \text{ if } j = N/2, N/2 + 1, \cdots 3N/4. \tag{6.5.18}$$

Using the fact that, for $\forall j = 3N/4, \ldots, N$, $H = h_j \leqslant 4N^{-1}$ substituted into (6.5.16) on $[\tau + 1/2, 1]$, we obtain:

$$L^N(V_j^{ext} - v_j) = \left[-\frac{a_j}{12} v_j''' + \frac{(\varepsilon + x_j)}{48} v^{(4)}(\xi_1, j) \right] H^2 + \left[\frac{a_j}{32} v^{(4)}(\xi_1, j) - \frac{(\varepsilon + x_j)}{24} v^{(4)}(\xi_2, j) \right] H^3. \tag{6.5.19}$$

Applying the triangle inequality and Lemma 6.2.3 to (6.5.19), we obtain:

$$|L^N(V_j - v_j)| \leqslant CN^{-2} \begin{cases} 1 + (\varepsilon + x)^{-a}, & a < 1, \\ 1 + (\varepsilon + x)^{-1} |\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \\ 1 + \varepsilon^{a-1}(\varepsilon + x)^{-a}, & a > 1. \end{cases} \tag{6.5.20}$$

The above inequalities lead to

$$|L^N(V_j^{ext} - v_j)| \leqslant CN^{-2}.$$

Hence, by Lemma 6.3.2, we obtain

$$|(V_j^{ext} - v_j)| \leqslant CN^{-2}. \tag{6.5.21}$$

The estimation of the nodal error of the singular component depends on whether $\tau = 1/4$ or $\tau = (\varepsilon/\eta) \ln(N/4)$.

In the first case, the mesh is uniform and $(\varepsilon/\eta) \ln(N/4) \geq 1/4$. The estimate of the singular component is given by

$$L^N(W_j^{ext} - w_j) = \left[-\frac{a_j}{12} w_j''' + \frac{(\varepsilon + x_j)}{48} w^{(4)}(\xi_1, j) \right] h^2 + \left[\frac{a_j}{32} w^{(4)}(\xi_1, j) - \frac{(\varepsilon + x_j)}{24} w^{(4)}(\xi_2, j) \right] h^3. \tag{6.5.22}$$

Using the triangle inequality in (6.5.22), we obtain

$$L^N(W_j^{ext} - w_j) \leqslant \left[-\frac{a_j}{12} w_j''' + \frac{(\varepsilon + x_j)}{48} w^{(4)}(\xi_1, j) \right] h^2 + \left[\frac{a_j}{32} w^{(4)}(\xi_1, j) - \frac{(\varepsilon + x_j)}{24} w^{(4)}(\xi_2, j) \right] h^3.$$

By Lemma 6.2.6 and the fact that $h_j = x_j - x_{j-1} \leqslant 4N^{-1}$, the above inequality gives

$$|L^N(W_j - w_j)| \leqslant Ch^{-2} \begin{cases} (\varepsilon + x)^{-a-2}, & a < 1, \\ (\varepsilon + x)^{-3} |\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \\ \varepsilon^{a-1}(\varepsilon + x)^{-a-2}, & a > 1. \end{cases} \tag{6.5.23}$$

The above inequalities lead to

$$|L^N(W_j - w_j)(x_j)| \leqslant Ch^{-2}\varepsilon^{-2} \leqslant CN^{-2}\varepsilon^{-2}. \tag{6.5.24}$$

Using $\varepsilon^{-1} \leqslant (2/\eta)\ln(N/4)$ in (6.5.24), we obtain

$$|L^N(W_j^{ext} - w_j)| \leqslant CN^{-2}\ln^2(N/4).$$

Applying Lemma 6.3.2 to the above inequality, we obtain

$$|(W_j^{ext} - w_j)| \leqslant CN^{-2}\ln^2(N/4). \tag{6.5.25}$$

In the latter case, the mesh is piecewise uniform with the mesh spacing $h = h_j \leqslant 4\tau N^{-1}$ for $\forall j = N/2, \ldots, 3N/4$ on $[1/2, \tau + 1/2]$. We obtain (6.5.24)

$$|L^N(W_j^{ext} - w_j)| \leq C_1 N^{-2}\tau^2\varepsilon^{-2}. \tag{6.5.26}$$

Using the value of $\tau = (\varepsilon/\eta)\ln(N/4)$ and applying Lemma 6.3.2 to the above inequality, we obtain

$$|(W_j^{ext} - w_j)| \leqslant CN^{-2}\ln^2(N/4). \tag{6.5.27}$$

A similar analysis can be performed on $[0, 1/2]$.

Combining of (6.5.21), (6.5.25) and (6.5.27), we obtain the required error after extrapolation.

6.6 Numerical results

Two numerical examples are presented in this section to confirm the theoretical results of problems of the type (6.1.1). The numerical results are displayed in the tables. The maximum errors and order of convergence are estimated by using the exact solution. The solution in both examples has a turning point at point $x = 0$ (or $x = 0.5$), which results an interior layer.

Example 6.6.1. Consider the following singularly perturbed turning point problem:

$$(\varepsilon + x)u'' + xu' - u = f(x),\ x \in [-1, 1], \tag{6.6.1}$$

$$u(-1) = -1,\ \ and\ \ u(1) = 1. \tag{6.6.2}$$

The exact solution is

$$u(x) = \cos \pi x + x + \frac{x\mathrm{erf}(x/\sqrt{2\varepsilon}) + \sqrt{2\varepsilon/\pi}\exp(-x^2/2\varepsilon)}{\mathrm{erf}(1/\sqrt{2\varepsilon}) + \sqrt{2\varepsilon/\pi}\exp(-1/2\varepsilon)}.$$

The expression for $f(x)$ is obtained after substituting $u(x)$ and its derivatives into (6.6.1).

Example 6.6.2. Consider the following singularly perturbed turning point problem:

$$(\varepsilon + x)u'' + 2(x - 0.5)u' - u = f(x),\ x \in [0, 1], \tag{6.6.3}$$

$$u(0) = -(\varepsilon - 1)^2\ \ and\ \ u(1) = -(\varepsilon + 1)^2. \tag{6.6.4}$$

The exact solution is

$$u(x) = (2x - 1 + \varepsilon)^2 \cos[\pi(2x - 1)].$$

The expression for $f(x)$ is obtained after substituting $u(x)$ and its derivatives into (6.6.3).

The maximum errors before extrapolation at all mesh points are evaluated using

$$E_{n,\varepsilon} = \max_{0\le j\le n} |u_j - U_j|,$$

and these errors after extrapolation are given by

$$E^{ext}_{n,\varepsilon} = \max_{0\leqslant j\leqslant n} |u_j - U^{ext}_j|.$$

The numerical rates of convergence before and after extrapolation are obtained by using

$$r_{\varepsilon,k} = \log_2(\tilde{E}_{n_k}/\tilde{E}_{2n_k}),$$

where $\tilde{E}$ represents E or E^{ext}.

Table 6.1: Results for Example 6.6.1: Maximum errors before extrapolation

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-3}	2.84E-01	4.11E-02	1.36E-02	1.13E-02	1.99E-02	1.43E-02	2.29E-02
10^{-4}	2.89E-01	3.54E-02	4.38E-02	3.14E-02	4.89E-03	1.56E-02	7.03E-03
10^{-5}	2.91E-01	2.79E-02	2.53E-02	2.90E-02	2.70E-02	6.67E-03	2.12E-03
10^{-18}	2.93E-01	3.13E-02	8.81E-03	4.62E-03	2.35E-03	1.18E-03	5.95E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	2.93E-01	3.13E-02	8.81E-03	4.62E-03	2.35E-03	1.18E-03	5.95E-04

Table 6.2: Results for Example 6.6.1: Maximum errors after extrapolation

ε	$N=16$	$n=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^{-3}	2.49E-01	5.19E-02	2.00E-02	2.91E-02	2.07E-02	3.76E-02	3.18E-02
10^{-4}	2.49E-01	5.27E-02	1.86E-02	3.93E-02	2.74E-02	9.35E-03	1.02E-02
10^{-5}	2.37E-01	3.22E-02	3.26E-02	2.50E-02	1.37E-02	1.06E-02	2.91E-03
10^{-18}	2.30E-01	1.63E-02	1.93E-03	3.65E-04	8.34E-05	2.03E-05	5.02E-06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	2.30E-01	1.63E-02	1.93E-03	3.65E-04	8.34E-05	2.03E-05	5.02E-06

Table 6.3: Results for Example 6.6.1: Rates of convergence before extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-3}	2.79	1.59	0.27	-0.81	0.48	-0.68
10^{-4}	3.03	-0.31	0.48	2.68	-1.67	1.15
10^{-5}	3.39	0.14	-0.20	0.10	2.02	1.66
10^{-18}	3.22	1.83	0.93	0.98	0.99	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	3.22	1.83	0.93	0.98	0.99	1.00

Table 6.4: Results for Example 6.6.1: Rates of convergence after extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-3}	2.26	1.38	-0.54	0.49	-0.86	0.24
10^{-4}	2.24	1.50	-1.08	0.52	1.55	-0.13
10^{-5}	2.88	-0.02	0.38	0.87	0.38	1.86
10^{-18}	3.82	3.07	2.41	2.13	2.04	2.01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	3.82	3.07	2.41	2.13	2.04	2.01

Table 6.5: Results for Example 6.6.2: Maximum errors before extrapolation

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
10^{-1}	6.46E-03	3.43E-03	1.77E-03	8.97E-04	4.52E-04	2.27E-04
10^{-2}	1.38E-02	7.03E-03	3.57E-03	1.78E-03	8.86E-04	4.38E-04
10^{-4}	1.35E-02	7.08E-03	3.65E-03	1.85E-03	9.35E-04	4.69E-04
10^{-28}	8.87E-03	4.83E-03	2.52E-03	1.29E-03	6.50E-04	3.27E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-35}	8.87E-03	4.83E-03	2.52E-03	1.29E-03	6.50E-04	3.27E-04

Table 6.6: Results for Example 6.6.2: Maximum errors after extrapolation

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
10^{-2}	2.02E-02	1.95E-02	1.96E-02	1.98E-02	2.01E-02	2.05E-02
10^{-3}	3.54E-03	2.23E-03	1.88E-03	1.79E-03	1.77E-03	1.76E-03
10^{-4}	2.04E-03	6.80E-04	3.04E-04	2.07E-04	1.82E-04	1.76E-04
10^{-28}	2.48E-03	6.10E-04	1.52E-04	3.81E-05	9.54E-06	2.39E-06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-35}	2.48E-03	6.10E-04	1.52E-04	3.81E-05	9.54E-06	2.39E-06

Table 6.7: Results for Example 6.6.2: Rates of convergence before extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-2}	0.98	0.98	1.00	1.01	1.02
10^{-3}	0.93	0.96	0.98	0.99	1.00
10^{-4}	0.93	0.96	0.98	0.99	0.99
10^{-29}	0.88	0.94	0.97	0.98	0.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-35}	0.88	0.94	0.97	0.98	0.99

Table 6.8: Results for Example 6.6.2: Rates of convergence after extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-2}	0.05	-0.00	-0.02	-0.02	-0.02
10^{-3}	0.66	0.25	0.07	0.02	0.00
10^{-4}	1.59	1.16	0.56	0.18	0.05
10^{-29}	2.03	2.00	2.00	2.00	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-35}	2.33	2.00	2.00	2.00	2.00

Remark 6.6.1. Tables of numerical results with Richardson extrapolation show that the computed rate of convergence deviate notably from the theoretical rate of convergence with Richardson extrapolation which is two. This is not surprising as performance of Richardson extrapolation may be hindered by the fact that, for non-uniform (such as the Shishkin type) meshes, nodes are offset by the fact that the transition point depends on the number of nodes used in computations. This observation corroborates assertions in the literature regarding issues with implementation of Richardson extrapolation (see e.g. [10, 61, 71])

6.7 Discussion

In this chapter, we constructed a Fitted Mesh Finite Difference Method (FMFDM) for a family of two-point singularly perturbed boundary value problems with diffusion term of the form $l(\varepsilon + x)$. Furthermore, the solution to these problems has an interior layer due to the presence of a turning point. We first provided a set of bounds on the derivatives of the solution. Then we constructed a mesh, of Shishkin type. Depending on the sign of the coefficient of the convection term, a discrete upwind scheme was designed on this mesh. Using bounds on the solution and its derivatives, we showed that the proposed numerical method is uniformly convergent of order one, up to a logarithmic factor. We used Richardson extrapolation via FMFDM in order to improve the accuracy of the scheme.

In order to confirm the above conclusions based on the theoretical analysis, we carried out numerical investigations on two examples. In each example, we calculated the maximum point-wise errors and the corresponding rates of convergence for various values of N. We noticed that the numerical method was uniformly convergent (see tables 6.1, 6.3, 6.5 and 6.7). We provided computed point-wise maximum errors after extrapolation in tables 6.2, 6.4, 6.6 and 6.8, which improved the accuracy.

Chapter 7

Time-dependent for interior layer convection-diffusion problems with a variable coefficient diffusion term

In this chapter, we consider a class of time-dependent singularly perturbed convection-diffusion problems whose solution presents an interior layer due to the presence of a turning point. In addition, the diffusion term is a function of ε and x. After establishing bounds on the solution and their derivatives, we discretize the time variable by means of the classical implicit Euler method with a constant step-size. This process results in a system of two-point boundary problem at each time level which will be solved using a fitted mesh finite difference method (FMFDM). We apply the Richardson extrapolation technique in order to increase the accuracy as well as the order of convergence.

7.1 Introduction

Numerous numerical schemes for a class of time-dependent singular problems are available in the literature. These problems are characterised by a small parameter affecting the highest derivative in the differential equations underlying the problem (see chapter 2).

In this chapter, we seek to determine the solution of the linear singularly perturbed problem

$$Lu :\equiv (\varepsilon + x)\,\frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t)\,\frac{\partial u(x,t)}{\partial x} - b(x,t)u(x,t) - d(x,t)\,\frac{\partial u(x,t)}{\partial t} = f(x,t), \tag{7.1.1}$$

$$(x,t) \in Q \equiv \Omega \times (0,T] \equiv (0,1) \times (0,T], \tag{7.1.2}$$

subject to the initial and boundary conditions

$$u(x,0) = u_0(x),\ \ 0 \le x \le 1,\ \ u(0,t) = \alpha_1(t)\ ,\ u(1,t) = \alpha_2(t),\ \ t \in (0,T]. \tag{7.1.3}$$

In the rest of this chapter, we assume that

- $a(0.5,t) = 0$ and $a_x(0.5,t) > 0$, for $0 \leqslant t \leqslant T$, ensuring that the solution to problem (7.1.1)-(7.1.3) guarantees the existence of a turning point;
- $b(x,t) \ge \beta > 0$, $\forall (x,t) \in \bar{Q}$, which ensures that the problem verifies a minimum principle;
- $|a_x(x,t)| \geqslant |a_x(0.5,t)/2|$, $\forall (x,t) \in \bar{Q}$, implies that the turning point appears at point $(05,t)$, $\forall t \in [0,T]$.

The above conditions guarantee that the solution of problem (7.1.1)-(7.1.3) has a unique solution which possesses an interior layer at $x = 0$ [19].

In order to have compatibility between the boundary and initial conditions, we also assume that

$$u_0(0) = \alpha_1(0) \text{ and } u_0(1) = \alpha_2(0),$$

so that the data match at the two corners $(0,0)$ and $(1,0)$ of the domain $\bar{Q}$. These conditions guarantee that there exists a constant C independent of ε such that [72]

$$|u(x,t) - \alpha_1(t)| \le Cx,\ \ |u(x,t) - \alpha_2(t)| \le C(1-x),\quad \forall (x,t) \in \bar{Q}$$

and

$$|u(x,t) - u_0(x)| \le Ct,\ \ \forall (x,t) \in \bar{Q}.$$

In this chapter, we propose and analyse a fitted mesh finite difference method (FMFDM) to solve a class of time-dependent problem (7.1.1)-(7.1.3) where the coefficients of the differential equations depend on both space and time, are smooth.

The rest of this chapter is structured as follows. We provide bounds of the solution $u(x,t)$ of (7.1.1) and its derivatives in Section 2. In Section 3 we design a FMFDM for solving our problem. We prove that the proposed method satisfies a minimum principle. We use this fact to establish a stability result. In Section 4 we conduct a rigorous error analysis. We prove that the proposed numerical method is uniformly convergent of first order with respect to the perturbation parameter in time and space, up to a logarithmic factor. In order to improve the accuracy as well as the order of convergence of the proposed FMFDM, we apply the Richardson extrapolation method in Section 5. In Section 6 we present one example to see how the proposed method works and confirm our theoretical results. Finally, some conclusions are drawn in the Section 7.

In the rest of this chapter, C denotes a generic constant which may assume different values in different inequalities but will always be independent of ε, as well as the spatial and time discretization parameters.

7.2 A priori estimates of the solution and its derivatives

In this section, we derive appropriate bounds on the solution of problem (7.1.1)-(7.1.3) and its derivatives. The interval $[0,1]$ which we denote by $\bar{\Omega}$ is partitioned as $\Omega_l = [0, 1/2-\tau]$, $\Omega_c = [1/2-\tau, 1/2+\tau]$ and $\Omega_r = [1/2+\tau, 1]$, where $0 < \tau \leq 1/4$.

The linear operator L as defined in (7.1.1) verifies the following minimum principle and then we a state stability estimate for the solution of (7.1.1)-(7.1.3).

Lemma 7.2.1. (Minimum principle.) Assume that $\xi(x,t)$ is a smooth function satisfying $\xi(0,t) \geq 0$, $\xi(1,t) \geq 0$ and $L\xi(x,t) \leq 0$, $\forall x \in \Omega$. Then $\xi(x,t) \geq 0$, $\forall x \in \bar{\Omega}$.

Proof. Suppose that there exists a point $(x^*,t^*) \in \bar{Q}$ such that $\xi(x^*,t^*) = \min_{0\le x\le 1} \xi(x,t)$ and assume that $\xi(x^*,t^*) < 0$. Clearly $(x^*,t^*) \notin Q$. It follows that
$\xi_x(x^*,t^*) = 0,\ \xi_t(x^*,t^*) = 0$ and $\xi_{xx}(x^*,t^*) \geq 0$. This implies

$$L\xi(x^*,t^*) = (\varepsilon+x^*)\xi_{xx}(x^*,t^*)+a(x^*,t^*)\xi_x(x^*,t^*)-b(x^*,t^*)\xi(x^*,t^*)-d(x^*,t^*)\xi_t(x^*,t^*) > 0,$$

which is a contradiction. It follows that $\xi(x^*,t^*) \geq 0$ and thus $\xi(x,t) \geq 0$, $\forall (x,t) \in \bar{Q}$.

We employ the minimum principle above to prove the next result which states that the solution depends continuously on the data.

Lemma 7.2.2. (Stability estimate). The solution $u(x,t)$ of problem (7.1.1)-(7.1.3) satisfies:

$$||u(x,t|| \leqslant [\max\{||\alpha_1||_\infty, ||\alpha_2||_\infty\}] + \frac{1}{\beta}||f||_\infty, \forall (x,t) \in \bar{Q}.$$

Proof. See Lemma 3.2.2 in Chapter 3.

Estimates of u and its derivatives in the interval $[0, 1/2 - \tau]$ and $[1/2 + \tau, 1]$ are given in the next lemmas.

Lemma 7.2.3. The bound on the solution $u(x,t)$ of (7.1.1) is $|u(x,t)| \leq C,\ (x,t) \in \bar{Q}$.

Proof. We refer to [30] for the proof.

Lemma 7.2.4. Let $u(x,t)$ be the solution to (7.1.1)-(7.1.3) and $a(x,t), b(x,t)$ and $f(x,t)$ sufficiently smooth function in $\bar{Q}$. Then, there exists C independent of ε, such that

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq C,\ \forall x \in \Omega_l \text{ or } \Omega_r \text{ and } (x,t) \in \bar{Q}, 0 \leq j \leq 2.$$

Proof. See [11].

Lemma 7.2.5. Under the assumption of Lemmas 7.2.1 and 7.2.4, the bound on the derivative of u with respect to t is $|u_t(x,t)| \leq C, \quad (x,t) \in \bar{Q}$.

Proof. See [30].

Lemma 7.2.6. $|u_{xt}(x,t)| \leqslant C, \ \ (x,t) \in \bar{Q}$.

Proof. See [31].

Based on the ideas of [39], we are in position to establish the following lemma. Due to the presence of an interior layer at point at $x_{N/2} = 1/2$, the solution of the problem (7.1.1)-(7.1.3) may be considered as a concatenation of two solutions: One side on $0 \leqslant x < 1/2$ displaying an interior layer near $x_{N/2} = 1/2$ (right hand of the interval) and the other side on $1/2 < x \leqslant 1$ presenting an interior layer near $x_{N/2} = 1/2$ (left hand of the interval) as well. Consequently, the derivatives of $u(x,t)$ in the interior layer are estimated by three types of singular, power functions according to the sign of the coefficient of the convection term $a(x,t)$ at the point $x_0^\star$. Then, we have two different cases

$$a = \begin{cases} a(x_0^\star, t) \leqslant 0, & x_0^\star \in [\tau - 1/2, 1/2], \ t \in [0,T] \text{ and} \\ a(x_0^\star, t) > 0, & x_0^\star \in (1/2, \tau + 1/2], \ t \in [0,T]. \end{cases} \tag{7.2.1}$$

Lemma 7.2.7. Let $u(x,t)$ be the solution of (7.1.1)-(7.1.3). Then assuming that $a = a(x_0^\star, t) > 0, \text{ for } 1/2 < x \leqslant \tau + 1/2, \ \forall t \ \in [0,T]$, and $j = 1,2,3,4$, we have

$$\left| \frac{\partial^j u(x,t)}{\partial x^j} \right| \leqslant C \begin{cases} 1 + (\varepsilon + x)^{1-a-j}, & 0 < a < 1, \\ 1 + (\varepsilon + x)^{-j} |\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \\ 1 + \varepsilon^{a-1}(\varepsilon + x)^{1-a-j}, & a > 1. \end{cases} \tag{7.2.2}$$

When $a = a(x_0^\star, t) \leqslant 0$, for $1/2 - \tau \leqslant x \leqslant 1/2$, let p be an integer such that $a + p = 0$ and $a + p - 1 < 0, \ \forall t \ \in [0,T]$ and $j = 1,2,3,4$, then we have the following bounds

$$\left| \frac{\partial^j u(x,t)}{\partial x^j} \right| \leqslant C \begin{cases} 1, & a < 0, \ j \leqslant p, \\ 1 + (\varepsilon + x)^{1-j-p} |\ln(\varepsilon + x)|, & a + p = 0, \ j > p, \\ 1 + (\varepsilon + x)^{1-a-j}, & a + p > 0, \ j > p. \end{cases} \tag{7.2.3}$$

Proof. We prove this lemma by following the steps provided by ([39], pp. 107-110). Application of the inverse-monotone pair $T = (L, \Gamma)$ (see pp 49) implies that

$$|u(x,t)| \leqslant C, \quad (x,t) \in \bar{Q}. \tag{7.2.4}$$

From (7.1.1)-(7.1.3) and(7.2.4) and $\forall t \in [0,T]$, we obtain

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} 1, & 1/2-\tau < x_0 \leqslant x \leqslant 1/2, \\ \varepsilon^{-j}, & 1/2-\tau \leqslant x \leqslant x_0,\ j=1,2,3,4, \\ 1, & 1/2 < x_0 \leqslant x \leqslant \tau+1/2, \\ \varepsilon^{-j}, & 1/2 \leqslant x \leqslant x_0,\ j=1,2,3,4, \end{cases} \tag{7.2.5}$$

and arbitrary $x_0 > 0$, independent of ε and x.

case 1 : $\mathbf{a} > \mathbf{0}$ for $0 < x \leqslant 1/2$, $\forall t \in [0,T]$. The derivatives of $u(x,t)$ are estimated according to the value of $a : 0 < a < 1$, $a = 1$ and $a > 1$. Solving (7.1.1) for $u_{xx}(x,t)$, we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| = u_{xx}(x,t) = \frac{f(x,t)+b(x,t)u(x,t)+d(x,t)u_t(x,t)}{(\varepsilon+x)} - \frac{a(x,t)u_x(x,t)}{(\varepsilon+x)}. \tag{7.2.6}$$

Integrating (7.2.6) on both sides from 1/2 to x, we obtain

$$u_x(x,t) = \int_{1/2}^{x} \frac{f(s,t)+b(s,t)u(s,t)+d(s,t)u_t(s,t)}{\varepsilon+s}\, ds - \int_{1/2}^{x} \frac{a(s,t)u_x(s,t)}{\varepsilon+s}\, ds. \tag{7.2.7}$$

By [39], $u_x(x,t)$ is given by

$$u_x(x,t) = u_x(1/2,t)\left[\frac{\varepsilon}{\varepsilon+x}\right]^a \exp[-g_1(x,t)] + g_2(x,t) \tag{7.2.8}$$

where

$$g_1(x,t) = \int_{1/2}^{x} \frac{a(s,t)}{\varepsilon+s}\, ds = a(x,t)\ln(\varepsilon+x) - \int_{1/2}^{x} a_s(s,t)\ln(\varepsilon+s)\, ds \tag{7.2.9}$$

with $a(1/2,t) = 0$, and

$$g_2(x,t) = (\varepsilon+x)^{-a}\int_{1/2}^{x} [f(s,t)+b(s,t)u(s,t)+ \\ d(s,t)u_t(s,t)](\varepsilon+s)^{a-1}\exp[g_1(s,t)-g_1(x,t)]\, ds. \tag{7.2.10}$$

Noting that $|g_1(x,t)| \leqslant C$ from (7.2.4), we find that

$$|g_2(x,t)| \leqslant C(\varepsilon+x)^{-a}\int_0^x (\varepsilon+s)^{a-1}\, ds \leqslant C.$$

Using the triangle inequality in (7.2.8) and taking into account the estimates of $g_1(x,t)$ and $g_2(x,t)$, we obtain

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C\left[1 + |u_x(1/2,t)|(\varepsilon/(\varepsilon+x))^a\right]. \tag{7.2.11}$$

Considering $0 < a < 1$, there exists a point x_0 in the interval $(1/2, \tau + 1/2)$ such that $|u_x(x_0,t)| \leqslant C$. Thus we have

$$|u_x(1/2,t)|\left(\frac{\varepsilon}{\varepsilon + x_0}\right)^a \leqslant C.$$

This inequality yields

$$|u_x(1/2,t)| \leqslant C\left(\frac{\varepsilon + x_0}{\varepsilon}\right)^a \leqslant C(\varepsilon + x_0)^a \varepsilon^{-a} \leqslant C\varepsilon^{-a}.$$

Using the value of $|u_x(1/2,t)|$ in (7.2.11), we obtain

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C\left[1 + (\varepsilon + x)^{-a}\right],\ 0 < a < 1.$$

Differentiating (7.1.1) with respect to x, solving the resulting equation for $u_{xxx}(x,t)$, we obtain

$$\begin{aligned} u_{xxx}(x,t) = \frac{b(x,t) - a_x(x,t)}{\varepsilon + x}u_x(x,t) - \frac{a_x(x,t)+1}{(\varepsilon+x)}u_{xx}(x,t) + \\ \frac{f_x(x,t) + b_x(x,t)u(x,t) + d_x(x,t)u_t(x,t) - d(x,t)u_{tx}(x,t)}{(\varepsilon+x)}. \end{aligned} \tag{7.2.12}$$

$\partial^2 u(x,t)/\partial x^2$ is given by

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}(x,t) = u_{xx}(1/2,t)\left[\frac{\varepsilon}{\varepsilon + x}\right]^{a+1}\exp[-g_3(x,t)] + g_4(x,t) \tag{7.2.13}$$

where

$$g_3(x,t) = \int_{1/2}^{x} \frac{a(s,t)+1}{\varepsilon + s}\, ds = [a(x,t)+1]\ln(\varepsilon+x) - \ln(\varepsilon+1/2) - \int_{1/2}^{x} a_s(s,t)\ln(\varepsilon+x)\, ds, \tag{7.2.14}$$

with $a(1/2,t) = 0$ and

$$\begin{aligned} g_4(x,t) = (\varepsilon+x)^{-a-1}\int_{1/2}^{x} [f_s(s,t) + b_s(s,t)u(s,t) + d_s(s,t)u_t(s,t) - d(s,t)u_{ts}(s,t) + \\ (b(s,t) - a_s(s,t)u_s(s,t)](\varepsilon+s)^a \exp[g_3(s,t) - g_3(x,t)]\, ds. \end{aligned} \tag{7.2.15}$$

Since $|g_3(x,t)| \leqslant C$, $|u_t(x,t)| \leqslant C$, $|u_{tx}(x,t)| \leqslant C$, and $|u(x,t)| \leqslant C$, we find that

$$|g_4(x,t)| \leqslant C(\varepsilon+x)^{-a-1}\int_{1/2}^{x}[1+u_s(s,t)](\varepsilon+s)^a\,ds \leqslant C[1+(\varepsilon+x)^{-a}]. \tag{7.2.16}$$

We obtain from (7.1.1)

$$u_{xx}(1/2,t) \leqslant C(\varepsilon+1/2)^{-1}[1+u_x(1/2,t)] \leqslant C[1+(\varepsilon+1/2)^{-a-1}].$$

Substituting the estimates of $u_{xx}(1/2,t)$ and $g_4(x,t)$ into (7.2.13), we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| = u_{xx}(x,t) \leqslant C\left[1+(\varepsilon+1/2)^{-a-1}\right]\left[\frac{\varepsilon}{\varepsilon+x}\right]^{a+1} + C\left[1+(\varepsilon+x)^{-a}\right].$$

Using $(\varepsilon+1/2)^{-a-1} \leqslant (\varepsilon)^{-a-1}$ in the above inequality, we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| = |u_{xx}(x,t)| \leqslant C\left[1+(\varepsilon+x)^{-a-1}\right], \text{ for } 0<a<1.$$

Differentiating (7.1.1)-(7.1.3) and taking into account (7.2.4), we get the following result

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C\left[1+(\varepsilon+x)^{1-a-j}\right].$$

Consider the case when $a=1$. On integrating (7.2.8) from $1/2$ to $\tau+1/2$, we obtain

$$u(1/2+\tau,t)-u(1/2,t) = u_x(1/2,t)\varepsilon[\ln(\varepsilon+1/2+\tau)\exp(-g_1(1/2+\tau,t)) - \ln(\varepsilon+1/2)\exp[-g_1(1/2,t)] + \int_{1/2}^{1/2+\tau}\ln(\varepsilon+x)\frac{\partial g_1(x,t)}{\partial x}\exp[-g_1(x,t)]\,dx] + \int_{1/2}^{1/2+\tau} g_2(x,t)\,dx, \tag{7.2.17}$$

where

$$\frac{\partial g_1(x,t)}{\partial x} = a(x,t)(\varepsilon+x)^{-1} \text{ and } g_1(1/2,t)=0$$

are obtained from (7.2.9). Substituting these equations into (7.2.17), we obtain

$$u(1/2+\tau,t)-u(1/2,t) = u_x(1/2,t)\varepsilon[\ln(\varepsilon+1/2+\tau)\exp(-g_1(1/2+\tau,t)) - \ln(\varepsilon+1/2)\exp[-g_1(1/2,t)] - \int_{1/2}^{1/2+\tau} a(x,t)(\varepsilon+x)^{-1}\ln(\varepsilon+x)\exp[-g_1(x,t)]\,dx] + \int_{1/2}^{1/2+\tau} g_2(x,t)\,dx. \tag{7.2.18}$$

Using the triangle inequality in (7.2.18) and we obtain

$$|\ln(\varepsilon+1/2+\tau)\exp[-g_1(1/2+\tau,t)] - \int_{1/2}^{1/2+\tau} a(x,t)(\varepsilon+x)^{-1}\exp[-g_1(x,t)]\,dx| \leqslant C,$$

we obtain

$$C \leqslant |u'(1/2,t)[\varepsilon - \varepsilon\ln(\varepsilon+1/2)]|.$$

For sufficiently small $\varepsilon \leqslant x_0,\ x_0 > 0$, we have $\varepsilon - \varepsilon\ln(\varepsilon+1/2) \geqslant \varepsilon\ln(\varepsilon+1/2)$. It follows that

$$C \geqslant |u_x(1/2,t)|[\varepsilon - \varepsilon\ln(\varepsilon+1/2)]| \leqslant |u_x(1/2,t)|[\varepsilon\ln(\varepsilon+1/2)^{-1}].$$

Solving this inequality, we obtain

$$|u_x(1/2,t)| \leqslant C\varepsilon^{-1}\ln^{-1}(\varepsilon+1/2)^{-1}.$$

Substituting this estimate into (7.2.11), we obtain

$$\left|\frac{\partial g_1(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C[1+(\varepsilon+x)^{-1}\ln^{-1}(\varepsilon+1/2)^{-1}].$$

Now determine $u_{xx}(x,t)$ for $a=1$. Now, (7.2.13) becomes

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}(x,t) = u_{xx}(1/2,t)\left[\frac{\varepsilon}{\varepsilon+x}\right]^2 \exp[-g_3(x,t)] + g_4(x,t). \tag{7.2.19}$$

From (7.2.16), $g_4(x,t)$ is defined as follows

$$|g_4(x,t)| \leqslant C(\varepsilon+x)^{-2}\int_{1/2}^{x}[1+u_s(s,t)](\varepsilon+s)\,ds \leqslant C[1+(\varepsilon+x)^{-1}\ln^{-1}(\varepsilon+1/2)^{-1}]. \tag{7.2.20}$$

From (7.1.1), we obtain

$$u_{xx}(1/2,t) \leqslant C(\varepsilon+1/2)^{-1}[1+u_x(1/2,t)] \leqslant C(\varepsilon+1/2)^{-2}\ln^{-1}(\varepsilon+1/2)^{-1}.$$

Using the estimates of $u_{xx}(1/2,t)$ and $g_4(x)$ into (7.1.1) and taking into account $(\varepsilon+1/2)^{-2} \leqslant \varepsilon^{-2}$, we obtain

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}(x,t) \leqslant C[1+(\varepsilon+x)^{-2}\ln^{-1}(\varepsilon+1/2)^{-1}].$$

By differentiating (7.1.1) and with the help of (7.2.5), we arrive at the following result

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C[1+(\varepsilon+x)^{-j}\ln^{-1}(\varepsilon+1/2)^{-1}],\ a=1.$$

We prove the case when $a > 1$ by using $u'(1/2,t) \leqslant C\varepsilon^{-1}$ obtained from(7.2.5) for $1/2 \leqslant x \leqslant x_0$ substituting into (7.2.11), which then leads to

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C[1+\varepsilon^{a-1}(\varepsilon+x)^{-a}].$$

Now determine $u_{xx}(x,t)$ for $a > 1$. Using $u_{xx}(1/2,t) \leqslant C\varepsilon^{-2}$ obtained from(7.2.5) for $1/2 \leqslant x \leqslant x_0$ and $|g_4(x,t)| \leqslant C[1+\varepsilon^{a-1}(\varepsilon+x)^{-a}]$ substituted into (7.2.13), we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leqslant C[1+\varepsilon^{a-1}(\varepsilon+x)^{-a-1}].$$

Differentiating (7.1.1) and taking into account (7.2.5), we easily obtain

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C[1+\varepsilon^{a-1}(\varepsilon+x)^{1-a-j}],\ j=1,2,3,4.$$

This concludes the proof of the estimate (7.2.2) for $1/2 < x \leqslant 1/2+\tau$.

case 2 : a $\leqslant$ 0 for $1/2-\tau \leqslant x \leqslant 1/2$. $u_x(x,t)$ is given

$$u_x(x,t) = u_x(x_0,t)\exp[\psi(x_0,x,t)]+ \int_{x_0}^{x} \frac{f(s,t)+b(s,t)u(s,t)+d(s,t)u_t(s,t)}{\varepsilon+s}\exp[\psi(x_0,x,t)]\,ds \quad (7.2.21)$$

where

$$\psi(s,x,t) = -\int_s^x \frac{a(\kappa,t)}{\varepsilon+\kappa}\,d\kappa.$$

If $a(1/2,t)=0$ then $\psi(s,x,t) \leqslant C,\ 1/2-\tau \leqslant s,\ x \leqslant 1/2$. Using the triangle inequality in (7.2.21) and choosing a point $x_0 \in [-\tau/2, 1/2]$ such that $u_x(x_0,t) \leqslant C$, we obtain

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C[1+|\ln(\varepsilon+x)|],\ 1/2-\tau \leqslant x \leqslant 1/2,\ a(1/2)=0,\ j=1 \text{ since } p=0.$$

Now we determine $u_{xx}(x,t)$ with $p=0$ for $j=2$. On differentiating (7.1.1) and solving the resulting equation for $u_{xx}(x,t)$, we obtain

$$\frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}(x_0,t)\exp[\psi(x_0,x,t)] + (\varepsilon+x)^{-p-1}\int_{x_0}^{x}\frac{F(s,t)}{\varepsilon+s}(\varepsilon+s)^{p+1}\exp[\psi(s,x,t)]\,ds, \quad (7.2.22)$$

where

$$\psi(s,x,t) = -\int_s^x \frac{a(\kappa,t)}{\varepsilon+\kappa}\, d\kappa$$

and

$$F(s,t) = f(s,t) + b_s(s,t)u(s,t) + d_s(s,t)u_t(s,t) + [b(s,t) - a_s(s,t)]u_s(s,t).$$

Substituting $\psi(s,x,t) \leqslant C$ and $u_{xx}(x_0,t) \leqslant C$ into (7.2.22), we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leqslant C + C(\varepsilon+x)^{-p-1}\int_{x_0}^x [1+u_s(s,t)](\varepsilon+s)^p\, ds \leqslant C[1+(\varepsilon+x)^{-p-1}\ln(\varepsilon+x)].$$

From (7.1.1)-(7.1.3) with $p=0,$ for $j>1$, we obtain

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C[1+(\varepsilon+x)^{1-j-p}|\ln(\varepsilon+x)|],\ \ a+p=0,\ j>p.$$

Let $a(1/2,t) < 0$. In this case $p > 1$. Then there exists a constant $x_0 > 0$ such that $a(x,t) < 0$ for $1/2-\tau \leqslant x \leqslant x_0$. Therefore, we have

$$\psi(s,x,t) \leqslant -x_0 \ln[(\varepsilon+s)/(\varepsilon+x)],\ 1/2-\tau \leqslant x \leqslant s \leqslant x_0.$$

Taking exponentials on both sides of the above inequality, we obtain

$$\exp(\psi(s,x,t)) \leqslant [(\varepsilon+x)/(\varepsilon+s)]^{-x_0},\ 1/2-\tau \leqslant x \leqslant s \leqslant x_0.$$

Substituting this estimate into (7.2.21) with $x=s$ and taking into account (7.2.5), we obtain

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C,\ 1/2-\tau \leqslant x \leqslant x_0,\ a(0,t) < 0.$$

Differentiating (7.1.1) and taking into account (7.2.5), we obtain

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C,\ 1/2-\tau \leqslant x \leqslant 1/2,\ a<0,\ k \leqslant p,\ j=1,2,3,4.$$

Consider the case when $j>p,\ a+p>0$ and $a \leqslant 0$. We will estimate $u_x(x,t)$ and $u_{xx}(x,t)$ by following the same steps as for $0<a<1$. We define $u_x(x,t)$ from (7.2.8) as follows

$$u_x(x,t) = u_x(1/2,t)\left[\frac{\varepsilon}{\varepsilon+x}\right]^{a+1}\exp[-g_1(x,t)] + g_2(x,t),\ 1/2-\tau \leqslant x \leqslant 1/2. \quad (7.2.23)$$

Following the same lines as for $0 < a < 1$, we obtain

$$\left|\frac{\partial u(x,t)}{\partial x}\right| = |u_x(x,t)| \leqslant C[1 + (\varepsilon + x)^{-a-1}],\ a \leqslant 0.$$

We estimate $u_{xx}(x,t)$ from (7.2.13) as follows

$$u_{xx}(x,t) = u_{xx}(1/2,t)\left[\frac{\varepsilon}{\varepsilon + x}\right]^{a+2} \exp[-g_3(x,t)] + g_4(x,t),\ 1/2 - \tau \leqslant x \leqslant 1/2. \quad (7.2.24)$$

Following the same lines as for $0 < a < 1$, we obtain

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| = |u_{xx}(x,t)| \leqslant C[1 + (\varepsilon + x)^{-a-2}],\ a \leqslant 0.$$

Differentiating (7.1.1) and taking into account (7.2.5), we obtain

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leqslant C[1 + (\varepsilon + x)^{1-a-j}],\ 1/2 - \tau \leqslant x \leqslant 1/2,\ a \leqslant 0,\ j > p.$$

This complete the proof of the estimate 7.2.3 for $1/2 - \tau < x \leqslant 1/2$.

The above bounds on the derivatives of the solution are estimated in the layer regions, which are not sharp enough for the proof of ε convergence in Section 7.4. Therefore we need to derive stronger bounds, which are obtained on a method originally given by Shishkin. This can be realised by decomposing the solution of problem (7.1.1)-(7.1.3) into two parts, namely the smooth component $v(x)$ and the singular component $w(x)$ ([42], pp 47) such that

$$u(x,t) = v(x,t) + w(x,t),$$

where $v(x,t)$ is the solution of the inhomogeneous problem

$$Lv(x,t) = f(x,t),\ \ (x,t) \in \Omega_1 = (1/2,1) \times (0,T], \quad (7.2.25)$$

$$v(x,0) = u(x,0) = u_0,\ 1/2 \leqslant x \leqslant 1, \quad (7.2.26)$$

$$v(1/2,t) = 0, v(1,t) = u(1,t), \quad (7.2.27)$$

and $w(x,t)$ is the solution of the homogeneous problem

$$Lw(x,t) = 0,\ \ (x,t) \in \Omega_1 = (1/2,1) \times (0,T], \quad (7.2.28)$$

$$w(x,0) = 0,\ 1/2 \leqslant x \leqslant 1, \tag{7.2.29}$$

$$w(1/2,t) = u(1/2,t) - v(1/2,t),\ \ 0 \leqslant t \leqslant T, \tag{7.2.30}$$

$$w(1,t) = 0,\ \ 0 \leqslant t \leqslant T. \tag{7.2.31}$$

The bounds on the solution of (7.1.1)-(7.1.3) are established below.

Lemma 7.2.8. The smooth and singular components of $u(x,t)$ of problem (7.1.1)-(7.1.3), for $0 \leqslant j \leqslant 4$ satisfy

$$\left|\frac{\partial^j v(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} 1 + (\varepsilon + x)^{2-j}|\ln(\varepsilon + x)|, & x \in [0,1/2], \\ 1 + (\varepsilon + x)^{3-a-j}, & a < 1,\ x \in [1/2,1], \\ 1 + (\varepsilon + x)^{2-j}|\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1,\ x \in [1/2,1], \\ 1 + \varepsilon^{a-1}(\varepsilon + x)^{3-a-j}, & a > 1,\ x \in [1/2,1], \end{cases} \tag{7.2.32}$$

and

$$\left|\frac{\partial^j w(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} (\varepsilon + x)^{1-j}|\ln(\varepsilon + x)|, & x \in [0,1/2], \\ (\varepsilon + x)^{1-a-j}, & a < 1,\ \ x \in [1/2,1], \\ (\varepsilon + x)^{-j}|\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1,\ \ x \in [1/2,1], \\ \varepsilon^{a-1}(\varepsilon + x)^{1-a-j}, & a > 1,\ \ x \in [1/2,1] \end{cases} \tag{7.2.33}$$

where C is a constant independent of ε.

Proof. We prove this lemma on $\Omega_1 = [1/2,1]$. The proof on $[0,1/2]$ follows in a similar way. The reduced problem ($\varepsilon = 0$), corresponding to problem (7.1.1) is

$$x^2 v^0_{xx} + a(x,t)v^0_x(x,t) - b(x,t)v^0(x,t) - d(x,t)v^0_t(x,t) = f(x,t),\ \ (x,t) \in \Omega_1, \tag{7.2.34}$$

$$v^0(x,0) = v^0_0(x),\ \ 1/2 \leqslant x \leqslant 1,\ \ v^0(1,t) = u(1,t) = \alpha_1(t),\ \ t \in (0,T]. \tag{7.2.35}$$

Further, we decompose the smooth component $v(x,t)$ ([42], pp 68) as follows

$$v(x,t) = v_0(x,t) + (\varepsilon + x)v_1(x,t) + (\varepsilon + x)^2 v_2(x,t),\ \ (x,t) \in \bar{\Omega}, \tag{7.2.36}$$

where v_0 is the solution of the reduced problem in (7.2.34), which is independent of ε, and having smooth coefficients $a(x,t)$, $b(x,t)$ and $f(x,t)$. From these assumptions, for $0 \leqslant j \leqslant 4$, we have

$$\left|\frac{\partial^j v_0(x,t)}{\partial x^j}\right| \leqslant C,\ \text{for all } x \in \bar{\Omega}_1, \tag{7.2.37}$$

v_1 and v_2 are the solutions of (7.1.1). Applying Lemma 7.2.7 for $1/2 \leqslant x \leqslant 0$ and $j = 1, 2, 3, 4$, results in the following bounds (for $k = 1, 2$)

$$\left|\frac{\partial^j v_k(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} 1 + (\varepsilon + x)^{1-a-j}, & 0 < a < 1, \\ 1 + (\varepsilon + x)^{-j} |\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \\ 1 + \varepsilon^{a-1}(\varepsilon + x)^{1-a-j}, & a > 1. \end{cases} \tag{7.2.38}$$

Now, apply the triangle inequality and substitute tha above three estimates into (7.2.36), for $0 \leqslant j \leqslant 4$, we prove the Lemma (7.2.8) for the smooth part.

For the proof of the regular component $w(x,t)$, define the barrier functions as follows

$$\Psi^{\pm}(x,t) = C\exp(-\eta x/\varepsilon)e^t \pm w(x,t),\ (x,t) \in \bar{\Omega}_1.$$

First, we calculate the values of $\Psi^{\pm}(x,t)$ at the boundaries:

$$\begin{aligned}
\Psi^{\pm}(1/2,t) &= C\exp(-\eta/2\varepsilon)e^t \pm w(1/2,t),\ 0 \leqslant t \leqslant T, \\
&= C\exp(-\eta/2\varepsilon)e^t \pm u(1/2,t) - v(1/2,t), \quad \text{using } (7.2.30), \\
&\geqslant 0, \quad \text{for a suitable choice of C } 0 \leqslant t \leqslant T, \\
\Psi^{\pm}(1,t) &= C\exp(-\eta/\varepsilon)e^t \pm w(1,t),\ 0 \leqslant t \leqslant T, \\
&= C\exp(-\eta/\varepsilon)e^t, \quad \text{using } (7.2.31), \\
&\geqslant 0, \quad 0 \leqslant t \leqslant T, \\
\Psi^{\pm}(x,0) &= C\exp(-\eta x/\varepsilon) \pm w(x,0),\ 1/2 \leqslant x \leqslant 1, \\
&= C\exp(-\eta x/\varepsilon), \quad \text{using } (7.2.29), \\
&\geqslant 0, \quad 1/2 \leqslant x \leqslant 1.
\end{aligned}$$

From the above estimates, we notice that $\Psi(x,t) \geqslant 0, \quad (x,t) \in \Omega_2 = \bar{\Omega}_1 \backslash \Omega_1$. Therefore we have

$$\begin{aligned}
L\Psi^{\pm}(x,t) &= (\varepsilon + x)\Psi^{\pm}_{xx}(x,t) + a(x,t)\Psi^{\pm}_x(x,t) - b(x,t)\Psi^{\pm}(x,t) - d(x,t)\Psi^{\pm}_t(x,t) \\
&= C\exp(-\eta x/\varepsilon)e^t \left[\frac{\eta^2(\varepsilon + x)}{\varepsilon} - \frac{\eta a(x,t)}{\varepsilon} - b(x,t) - d(x,t)\right] \pm Lw(x,t) \\
&= C\exp(-\eta x/\varepsilon)e^t \left[\frac{\eta^2(\varepsilon + x)}{\varepsilon^2} - \frac{\eta a(x,t)}{\varepsilon} - b(x,t) - d(x,t)\right], \quad \text{using } (7.2.28) \\
&\leqslant 0,\ \text{since } (x/\varepsilon) \leqslant [-b(x,t) - d(x,t)], \quad (x,t) \in \Omega_1.
\end{aligned}$$

Now since Lemma 7.2.1 to the barrier functions, we obtain $\Psi^{\pm}(x,t) \geqslant 0, \quad (x,t) \in \bar{\Omega}_1$. Since $C\exp(\eta x/\varepsilon)e^t \pm w(x,t) \geqslant 0$, it follows that

$$\begin{aligned} w(x,t) &\leqslant C\exp(-\eta x/\varepsilon)e^t,\ (x,t) \in \Omega_1 \\ &\leqslant C\exp(-\eta x/\varepsilon)e^T \text{ since } \ e^t \leqslant e^T \\ &\leqslant C\exp(-\eta x/\varepsilon) \ \ (x,t) \in \Omega_1. \end{aligned}$$

By using the inequality relation, the above inequalities can written as

$$|w(x,t)| \leqslant C\exp(-\eta x/\varepsilon) \leqslant C \begin{cases} (\varepsilon+x)^{1-a}, & a<1, \\ (\varepsilon+x)^{-1}|\ln^{-1}(\varepsilon+1/2)^{-1}|, & a=1, \\ \varepsilon^{a-1}(\varepsilon+x)^{1-a}, & a>1. \end{cases} \tag{7.2.39}$$

Since $Lw(x) = 0$, the j^{th} derivative of $w(x)$ can be estimated immediately from the estimate of $w(x)$. The following estimates hold for $0 \leqslant j \leqslant 4$,

$$\left|\frac{\partial^j w(x,t)}{\partial x^j}\right| \leqslant C \begin{cases} (\varepsilon+x)^{1-a-j}, & a<1, \\ (\varepsilon+x)^{-j}|\ln^{-1}(\varepsilon+1/2)^{-1}|, & a=1, \\ \varepsilon^{a-1}(\varepsilon+x)^{1-a-j}, & a>1. \end{cases} \tag{7.2.40}$$

which completes the proof of Lemma (7.2.8)for the regular parts.

In the following section, we design a FMFDM to solve a time-dependent convection-diffusion problem (7.1.1)-(7.1.3).

7.3 Construction of the FMFDM

Time discretization

The Euler implicit method is used to discretize problem (7.1.1)-(7.1.3) with uniform step-size $\Delta t = T/K$. Note the time $[0,T]$ is therefore partitioned as

$$\bar{w}^K = \{t_k = k\Delta t, 0 \leqslant k \leqslant K\}. \tag{7.3.1}$$

We discretize problem (7.1.1)-(7.1.3) on $\bar{w}^K$ as follows

$$(\varepsilon+x)z_{xx}(x,t_k)+a(x,t_k)z_x(x,t_k)-b(x,t_k)z(x,t_k)-d(x,t_k)\frac{z(x,t_k)-z(x,t_{k-1})}{\Delta t}=f(x,t_k) \tag{7.3.2}$$

subject to

$$z(x,0)=z_0(x),\ \ 0\leqslant x\leqslant 1,\ \ z(0,t_k)=\alpha_1(t),\ \ z(1,t_k)=\alpha_2(t). \tag{7.3.3}$$

Now, (7.3.2) can be written as

$$Lz(x,t_k)=f(x,t_k)-d(x,t)\frac{z(x,t_{k-1})}{\Delta t}, \tag{7.3.4}$$

subject to

$$z(x,0)=z_0(x),\ \ 0\leqslant x\leq 1,\ \ z(0,t_k)=\alpha_1(t),\ \ z(1,t_k)=\alpha_2(t), \tag{7.3.5}$$

where

$$Lz(x,t_k)=(\varepsilon+x)z_{xx}(x,t_k)+a(x,t_k)z_x(x,t_k)-\left[b(x,t_k)+\frac{d(x,t_k)}{\Delta t}\right]z(x,t_{k-1}).$$

The local truncation error e_k at each time level to t_k, is given by
$e_k=u(x,t_k)-z(x,t_k)$, where $z(x,t_k)$ is the solution of (7.3.4)-(7.3.5).

The local error estimate is [11]

$$\|e_k\|_\infty\leq C(\Delta t)^2,\ \ 1\leq k\leq K. \tag{7.3.6}$$

The global error estimate is [11]:

$$\|E_k\|_\infty\leq C\Delta t,\ \ 1\leq k\leq K. \tag{7.3.7}$$

Spatial discretization

Let $\bar{\Omega}^N$ denote the following partition in the interval $[0,1]$ such that: $x_0=0$, $x_{N/2}=1/2$, $x_N=1$ and let $\bar{Q}^{K,N}=\bar{w}^K\times\bar{\Omega}^N$ be the grid for the (x,t)-variables, and $Q^{K,N}=\bar{Q}^{K,N}\cap Q$.

Due to the presence of an interior layer at the point $x_{N/2} = 1/2$, the transition parameter τ is to be chosen as

$$\tau = \min\left\{\frac{1}{4}, \frac{\varepsilon}{\eta}\ln\left(\frac{N}{4}\right)\right\}, \tag{7.3.8}$$

where τ is a positive constant. The spatial domain is discretized using a piecewise uniform mesh which divides the space domain $[0,1]$ into the following subintervals $[0, 1/2-\tau]$, $[1/2+\tau, 1/2+\tau]$ and $[1/2+\tau, 1]$. These subintervals are subdivided uniformly to contain $N/4$, $N/2$ and $N/4$ mesh elements respectively. Note that the mesh spacing is given by

$$h_j = \begin{cases} 4(0.5-\tau)/N \text{ if } j = 1, 2, \cdots, N/4, 3N/4+1, \cdots, N-1, N, \\ 4\tau/N \text{ if } j = N/4+1, N/4+2\cdots 3N/4. \end{cases} \tag{7.3.9}$$

We adopt the notation $S(x_j, t_k) := S_j^k$. We construct the following scheme to solve (7.1.1)-(7.1.3) along with appropriate boundary conditions.

$$L^{N,K}U_j^k := \begin{cases} (\varepsilon + x_j)\tilde{D}_x U_j^k + \tilde{a}_j^k D_x^- U_j^k - (\tilde{b}_j^k + \frac{d_j^k}{\Delta t})U_j^k = \tilde{f}_j^k - \frac{d_j^k U_j^{k-1}}{\Delta t}, & j = 0, \cdots, N/2-1, \\ (\varepsilon + x_j)\tilde{D}_x U_j^k + \tilde{a}_j^k D_x^+ U_j^k - (\tilde{b}_j^k + \frac{d_j^k}{\Delta t})U_j^k = \tilde{f}_j^k - \frac{d_j^k U_j^{k-1}}{\Delta t}, & j = N/2, \cdots, N-1, \end{cases} \tag{7.3.10}$$

subject to the discrete initial and boundary conditions

$$U_j^0(x,0) = u_j^0, \;\; j = 0, 1, \cdots, N, \tag{7.3.11}$$

$$U_0^k = \alpha_1^k \equiv \alpha_1(k), \;\; U_N^k = \alpha_2^k \equiv \alpha_2(k), \;\; 1 \le k \le K, \tag{7.3.12}$$

where

$$\begin{cases} \tilde{a}_j^k = \frac{a_{j-1}^k + a_j^k}{2} & \text{for } j = 0, 1, \cdots, N/2-1, \\ \tilde{a}_j^k = \frac{a_j^k + a_{j+1}^k}{2} & \text{for } j = N/2, N/2+1, \cdots, N-1, \end{cases}$$

$$\begin{cases} \tilde{b}_j^k = \frac{b_{j-1}^k + b_j^k + b_{j+1}^k}{3} & \text{for } j = 1, 2, 3, \cdots, N-1, \\ \tilde{f}_j^k = \frac{f_{j-1}^k + f_j^k + f_{j+1}^k}{3} & \text{for } j = 1, 2, 3, \cdots, N-1. \end{cases}$$

$$D_x^+ U_j^k = \frac{U_{j+1}^k - U_j^k}{h_{j+1}^k}, \;\; D_x^- U_j^k = \frac{U_j^k - U_{j-1}^k}{h_j^k}, \;\; \tilde{D}_x U_j^k = \frac{2}{h_j^k + h_{j+1}^k}(D_x^+ U_j^k - D_x^- U_j^k)$$

and

$$D_t^- U_j^k = \frac{U_j^k - U_j^{k-1}}{\Delta t}.$$

Now, (7.3.10) can be expressed as

$$L_{x,\varepsilon}^{N,K}U_j^k := r^- U_{j-1}^k + r^c U_j^k + r^+ U_{j+1}^k = F_j,\ j = 1, 2, 3\cdots, N-1, \tag{7.3.13}$$

where for $j = 1, 2, 3\cdots, N/2 - 1$, we have

$$r_j^- = \frac{2(\varepsilon + x_j)}{h_j(h_j + h_{j+1})} - \frac{\tilde{a}_j^k}{h_j},\ r_j^c = \frac{\tilde{a}_j^k}{h_j} - \frac{2(\varepsilon + x_j)}{h_j h_{j+1}} - \tilde{b}_j^k - \frac{d_j^k}{\Delta t},\ \ r_j^+ = \frac{2(\varepsilon + x_j)}{h_{j+1}(h_j + h_{j+1})}. \tag{7.3.14}$$

For $j = N/2, N/2+1, \cdots, N-1$, we have

$$r_j^- = \frac{2(\varepsilon + x_j)}{h_j(h_j + h_{j+1})};\ r_j^c = -\frac{\tilde{a}_j^k}{h_{j+1}} - \frac{2(\varepsilon + x_j)}{h_j h_{j+1}^k} - \tilde{b}_j^k - \frac{d_j^k}{\Delta t};\ r_j^+ = \frac{2(\varepsilon + x_j)}{h_{j+1}(h_j + h_{j+1})} + \frac{\tilde{a}_j^k}{h_{j+1}} \tag{7.3.15}$$

and

$$F_j = \tilde{f}_j^k - \frac{d_j^k U_j^{k-1}}{\Delta t}. \tag{7.3.16}$$

The scheme (7.3.10)-(7.3.12) is a fitted mesh finite difference method (FMFDM) to solve the problem (7.1.1)-(7.1.3).

The results of the analysis of the scheme (7.3.10)-(7.3.12) depend on the following minimum principle.

Lemma 7.3.1. (Discrete minimum principle). Suppose that $L^{N,K}$ is the discrete given in (7.3.10) and ξ_j^k is any mesh function verifying $L^{N,K}\xi_j^k \leq 0$ in $Q^{N,K}, \xi_j^0 \geqslant 0$, $1 \leqslant j \leqslant N, \xi_0^k \geq 0$ and $\xi_N^k \geqslant 0$, $1 \leq k \leqslant K$, then $\xi_j^k \geq 0$ in $\bar{Q}^{N,K}$.

Proof. See Lemma 3.3.1 in Chapter 3.

Lemma 7.3.2. (Uniform stability estimate). At any time level t_k, if Z_j^k is any mesh function such that $Z_0^k = Z_N^k = 0$, then

$$|Z_i^k| \leqslant \frac{1}{\beta} \max_{1 \leqslant j \leq N-1} |L^{N,K} Z_j^k|\ \ \forall 0 \leqslant i \leqslant N.$$

Proof. See Lemma 3.3.2 in Chapter 3.

Based on the above continuous and discrete results, we are now in a position to provide the ε-uniform convergence

7.4 Convergence analysis

In this section, we prove that the proposed FMFDM is uniformly convergent of order one, up to a logarithmic factor.

Theorem 7.4.1. . *Let U_j^k be the approximation solution of problem (7.3.10)-(7.3.12) and denote the solution $z(x_j, t_k)$ of problem (7.3.4)-(7.3.5) at the time level t_k by $z_j^k = z(x_j, t_k)$. Then, we have*

$$\max_{0\leq j\leq N} |U_j^k - z_j^k| \leqslant CN^{-1}\left[\ln\left(\frac{N}{4}\right)\right]^2. \tag{7.4.1}$$

Proof. We prove this Lemma on the interval $[1/2, 1]$. The proof on $[0, 1/2]$ follows in a similar way. In the case of the discrete problem, the solution U_j^k of(7.3.10)-(7.3.12) can be decomposed into a regular part and a singular part as

$$U_j^k = V_j^k + W_j^k,$$

where V_j^k is the solution of the inhomogeneous problem

$$L^{N,K}V_j^k = f_j^k - \frac{d_j^k \times V_j^{k-1}}{\Delta t},\ V_j^0 = v_j^0, V_0^k = v_0^k,$$

and W_j^k is the solution of the homogeneous problem

$$L^{N,K}W_j^k = 0,\ W_j^0 = w_j^0,\ W_0^k = w_0^k,\ W_{N/2}^k = U_{N/2}^k - V_{N/2}^k.$$

Using (7.1.1) and (7.3.10) the nodal error of the smooth component is computed as

$$\begin{aligned} L^{N,K}(V_j^k - v_j^k) &= f_j^k - \frac{d_j^k \times V_j^{k-1}}{\Delta t} - L^{N,K}v_j^k \\ &= (\varepsilon + x_j)\left(\frac{d^2}{dx^2} - \tilde{D}_x\right)v_j^k + a_j^k\left(\frac{d}{dx} - D_x^-\right)v_j^k. \end{aligned}$$

Then, by local truncation error estimates (Lemma 4.1 [42]) at each point (x_j, t_k), we obtain

$$|L^{N,K}(V_j^k - v_j^k)| \leq \frac{(\varepsilon + x_j)}{3}(x_{j+1} - x_{j-1})\left\|\frac{\partial^3 v_j}{\partial x^3}\right\| + \frac{a_j^k}{2}(x_j - x_{j-1})\left\|\frac{\partial^2 v_j}{\partial x^2}\right\| \text{ for } N/2 \leqslant j \leqslant N-1. \tag{7.4.2}$$

Using the estimates of the derivatives of v_j of Lemma 7.2.8 and $h_j = x_j - x_{j-1} \leq 4N^{-1}$ in (7.4.2), we obtain

$$|L^{N,K}(V_j^k - v_j^k)| \leqslant CN^{-1} \begin{cases} 1 + (\varepsilon + x) + (\varepsilon + x)^{1-a}, & a < 1, \\ 1 + (\varepsilon + x) + \ln^{-1}(\varepsilon + 1/2)^{-1}, & a = 1, \\ 1 + (\varepsilon + x) + \varepsilon^{a-1}(\varepsilon + x)^{1-a}, & a > 1. \end{cases} \tag{7.4.3}$$

The above inequalities lead to:

$$|L^{N,K}(V_j^k - v_j^k)| \leqslant CN^{-1}.$$

Now, applying Lemma 7.3.2 to the mesh function $(V_j^k - v_j^k)$, we obtain

$$|(V_j^k - v_j^k)| \leq CN^{-1} \text{ for } N/2 \leqslant j \leqslant N - 1. \tag{7.4.4}$$

The estimation of the nodal error of the singular component depends on whether $\tau = 1/4$ or

$\tau = (\varepsilon/\eta)\ln(N/4)$. If $\tau = 1/4$, the mesh is uniform, i.e., $\tau = 1/4 \leqslant (\varepsilon/\eta)\ln(N/4)$. The estimate of the singular component similar to equation (7.4.2), then gives

$$|L^{N,K}(W_j^k - w_j^k)| \leqslant \frac{(\varepsilon + x_j}{3}(x_{j+1} - x_{j-1}) \left\| \frac{\partial^3 w_j}{\partial x^3} \right\| + \frac{a_j^k}{2}(x_j - x_{j-1}) \left\| \frac{\partial^2 w_j}{\partial x^2} \right\| \text{ for } 1 \leq j \leqslant N/2 - 1. \tag{7.4.5}$$

Noting that $x_j - x_{j-1} \leq 4N^{-1}$ and taking into account the bounds on w_j of Lemma 7.2.8 for each case of a, we obtain

$$|L^{N,K}(W_j^k - w_j^k)| \leqslant CN^{-1} \begin{cases} (\varepsilon + x)^{-a-1}, & a < 1, \\ (\varepsilon + x)^{-2}|\ln^{-1}(\varepsilon + 1/2)^{-1}|, & a = 1, \\ \varepsilon^{a-1}(\varepsilon + x)^{-a-2}, & a > 1. \end{cases} \tag{7.4.6}$$

The above inequalities lead to

$$|L^{N,K}(W_j^k - w_j^k)| \leqslant CN^{-1}\varepsilon^{-2}. \tag{7.4.7}$$

Since $\varepsilon^{-1} \leqslant (4/\eta)\ln(N/4)$, we obtain the following inequality

$$|L^{N,K}(W_j^k - w_j^k)| \leqslant CN^{-1}\ln^2(N/4).$$

Using Lemma 7.3.2 then we obtain

$$|(W_j^k - w_j^k)| \leqslant CN^{-1} \ln^2(N/4) \text{ for } N/2 \leqslant j \leqslant N-1. \tag{7.4.8}$$

If $\tau = (\varepsilon/\eta) \ln(N/4)$, the mesh is piecewise uniform. In this case we have two sub-intervals namely $[\tau + 1/2, 1]$ and $[1/2, \tau + 1/2]$. A different argument is used to bound $W - w$ in each subinterval. Firstly, we compute the error for the singular component in the coarse mesh region $[\tau + 1/2, 1]$, i.e. for all $\tau + 1/2 \leqslant x_j \leqslant 1$. Using the triangle inequality, we have

$$|W_j^k - w_j^k| \leqslant |W_j^k| + |w_j^k|. \tag{7.4.9}$$

Applying Lemma 7.2.8 to (7.4.9), we obtain

$$|w_j^k| \leqslant C\varepsilon^{-1} x_j \leqslant C\varepsilon^{-1}\tau.$$

Substituting the value of τ in the above expression, we obtain

$$|w_j^k| \leqslant C \ln(N/4) \leqslant C \text{ for } \ 3N/4 \leqslant j \leqslant N. \tag{7.4.10}$$

Now to obtain a similar bound on W_j^k, the interested readers can obtain the following inequalities using of Lemma 7.3 (p.58) and Lemma 7.5 (p.60) of [42] which lead to

$$|W_j^k| \leqslant C \ln(N/4) \leqslant C \text{ for } \ 3N/4 \leqslant j \leqslant N. \tag{7.4.11}$$

Combining the estimates obtained by (7.4.10) and (7.4.11), we obtain

$$|(W_j^k - w_j^k| \leqslant C \ln(N/4) \leqslant C \text{ for } \ 3N/4 \leqslant j \leqslant N. \tag{7.4.12}$$

Now the bounds in the interior region, $[1/2, \tau + 1/2]$ can be obtained from (7.4.5) by using the bounds of w_j in Lemma 7.2.8 keeping in mind that $h = 4\tau/N$, we obtain

$$|L^{N,K}(W_j^k - w_j^k)| \leqslant Ch\varepsilon^{-2} \leqslant C\tau\varepsilon^{-2}N^{-1}.$$

From (7.4.12), we have

$$|(W_{N/2}^k - w_{N/2}^k)| = 0$$

and

$$|W_{3N/4}^k - w_{3N/4}^k| \leqslant |W_{3N/4}^k| + |w_{3N/4}^k)| \leqslant C.$$

Now, introduce the barrier function Φ_j^k in $[1/2, \tau + 1/2]$ defined by

$$\Phi_j^k = (x+\tau)C_1\varepsilon^{-2}\tau N^{-1} + C_2 N^{-1},$$

where C_1 and C_2 are arbitrary constants.

Now consider comparison functions $[\Psi^{\pm}]_j^k$ defined by

$$[\Psi^{\pm}]_j^k = (x_j+\tau)C_1\varepsilon^{-2}\tau N^{-1} + C_2 N^{-1} \pm (W_j^k - w_j^k). \tag{7.4.13}$$

For an appropriate choice of C_1 and C_2, (7.4.13) satisfies the following inequalities

$$\Psi_{3N/4}^k \geqslant 0 \text{ and } \Psi_{N/2}^k = 0.$$

Note that

$$L^{N,K}[\Psi^{\pm}]_j^k \leqslant 0, \quad N/2+1 \leqslant j \leqslant 3N/4-1.$$

By applying Lemma 7.3.1 on $[1/2, \tau+1/2]$ for the function $[\Psi^{\pm}]_j^k$, we obtain

$$[\Psi^{\pm}]_j^k \geqslant 0, \quad N/2+1 \leqslant j \leqslant 3N/4-1.$$

Consequently,

$$|W_j^k - w_j^k| \leqslant \Phi_j \leq C_1\varepsilon^{-2}\tau^2 N^{-1} + C_2 N^{-1}.$$

Substituting the value of τ in the above inequality, we obtain

$$|W_j^k - w_j^k| \leqslant CN^{-1}\ln^2(N/4). \tag{7.4.14}$$

Using estimates (7.4.12) and (7.4.14), we obtain the estimate on the singular component of the error on $[1/2, \tau+1/2]$

$$|W_j^k - w_j^k| \leqslant CN^{-1}\ln^2(N/4), \; N/2+1 \leqslant j \leqslant 3N/4-1. \tag{7.4.15}$$

Noting that

$$U_j^k - z_j^k = (V_j^k - v_j^k) + (W_j^k - w_j^k) \tag{7.4.16}$$

and using (7.4.4) and (7.4.15), we obtain

$$|U_j^k - z_j^k| \leqslant CN^{-1} [\ln(N/4)]^2, \;\; N/2-1 \leqslant j \leqslant N. \tag{7.4.17}$$

A similar analysis on the subinterval $[0, 1/2]$ yields

$$|U_j^k - z_j^k| \leq CN^{-1} [\ln(N/4)]^2, \;\; 1 \leqslant j \leqslant N/2. \tag{7.4.18}$$

Combining the inequalities (7.4.17) and (7.4.18) then gives the required result.

The next theorem provides the main result of this chapter.

Theorem 7.4.2. . *Let u be the exact solution of problem* (7.1.1)-(7.1.3) *and U be its numerical solution obtained via the difference equations* (7.3.10)-(7.3.12). *Then, there exists a constant C independent of the perturbation parameter ε, and of the discretization parameters h_j and Δt such that*

$$\max_{0\leq j\leq N; 1\leq k\leq K} \|U_j^k - u_j^k\| \leq C\left[\Delta t + N^{-1}\left[\ln\left(\frac{N}{4}\right)\right]^2\right]. \tag{7.4.19}$$

Proof. The result follows from the triangle inequality

$$\|U_j^k - u_j^k\| \;\; \leq \;\; \|U_j^k - z_j^k\| + \|z_j^k - u_j^k\|,$$

and the combination of (7.3.7) and Theorem 7.4.1.

To increase the accuracy as well as the rate of convergence of the scheme, we use Richardson extrapolation in the following section.

7.5 Richardson extrapolation on the FMFDM

In order to improve the accuracy of the proposed method, we apply the Richardson extrapolation method. Richardson extrapolation is a procedure where a linear combination of two approximations of some quantity results a third and better approximation of the quantity [49].

We consider the mesh Ω^{τ}_{2N} which is obtained by bisecting each sub-interval of Ω^{τ}_{N}. It is clear that $\Omega^{\tau}_{N} \subset \Omega^{\tau}_{2N} = \{\tilde{x}_j\}$ and $\tilde{x}_j - \tilde{x}_{j-1} = \tilde{h}_j = h_j/2$. Let U^k_j and $\tilde{U}^k_j$ be the numerical solution of (7.3.10)-(7.3.12) on the mesh Ω^{τ}_{N} and Ω^{τ}_{2N} respectively. The estimate (7.4.17) can be written as

$$U^k_j - z^k_j = C_1 N^{-1} \ln(N/4)^2 + R_N(x_j), \ \ \forall x_j \in \Omega^{\tau}_{N} \tag{7.5.1}$$

and

$$\tilde{U}^k_j - z^k_j = C_2 (2N)^{-1} \ln(N/4)^2 + R_{2N}(\tilde{x}_j), \ \ \forall \tilde{x}_j \in \Omega^{\tau}_{2N}, \tag{7.5.2}$$

where C_1 and C_2 are some constants and the remainder terms

$$R_N(x_j) \text{ and } R_{2N}(\tilde{x}_j) \text{ are } \mathcal{O}[N^{-1}(\ln(N/4))^2].$$

Bear in mind that the transition parameter τ remains the same as in (7.3.8) when computing both U^k_j and $\tilde{U}^k_j$. This is seen from the factor $\ln(N/4)$.

Combination of (7.5.1) and (7.5.2) gives

$$z^k_j - (2\tilde{U}^k_j - U^k_j) = R_N(x_j) - 2R_{2N}(x_j) = \mathcal{O}[N^{-1}(\ln(N/4))^2], \ \ \forall x_j \in \Omega^{\tau}_{n}. \tag{7.5.3}$$

Set

$$U^{ext,k}_j = 2\tilde{U}^k_j - U^k_j, \ \ \forall x_j \in \Omega^{\tau}_{N}, \tag{7.5.4}$$

as the new approximation of z^k_j computed after employing Richardson extrapolation. The error after extrapolation $U^{ext,k}_j$ can also be decomposed as in (7.4.16),

$$(U^{ext} - z)^k_j = (V^{ext} - v)^k_j + (W^{ext} - w)^k_j, \tag{7.5.5}$$

where $V^{ext,k}_j$ and $W^{ext,k}_j$ are the regular and singular components of $U^{ext,k}_j$. The local truncation error of the scheme (7.3.10)-(7.3.13) after extrapolation is given by

$$L^{N,K}(U^{ext} - z)^k_j = 2L^{N,K}(\tilde{U}^k_j - z_j) - L^{N,K}(U^k_j - z_j), \tag{7.5.6}$$

where

$$L^{N,K}(U^k_j - z^k_j) = r^- z_{j-1} + r^c z_j + r^+ z_{j+1} - (\varepsilon + x_j^2) z''_j - \tilde{a}^k_j z'_j + \tilde{b}^k_j z^k_j + \frac{d^k_j z^k_j}{\Delta t}, \tag{7.5.7}$$

and

$$L^{N,K}(\tilde{U}_j^k - z_j^k) = \tilde{r}^- z_{j-1} + \tilde{r}^c z_j + \tilde{r}^+ z_{j+1} - (\varepsilon + x_j^2) z_j'' - \tilde{a}_j z_j' + \tilde{b}_j^k z_j^k + \frac{d_j^k u_j^k}{\Delta t}. \quad (7.5.8)$$

The expressions for r^-, r^c and r^+ are given in (7.3.14) (7.3.15) and (7.3.16) respectively while the quantities $\tilde{r}^-$, $\tilde{r}^c$ and $\tilde{r}^+$ are obtained by substituting h_j by $\tilde{h}_j$ and h_{j+1} by $\tilde{h}_{j+1}$ in the expressions r^-, r^c and r^+ respectively. Taking the Taylor series expansion of z_{j-1}^k and z_{j+1}^k about x_j yields

$$z_{j-1}^k = z_j - h_j z_j' + \frac{h_j^2}{2} z_j^2 - \frac{h_j^3}{6} z_j^3 + \frac{h_j^4}{24} z^4(\xi_1, j), \quad (7.5.9)$$

$$z_{j+1}^k = z_j + h_{j+1} z_j' + \frac{h_{j+1}^2}{2} z_j^2 + \frac{h_{j+1}^3}{6} z_j^3 + \frac{h_{j+1}^4}{24} z^4(\xi_2, j), \quad (7.5.10)$$

$$z_{j-1}^k = z_j - \tilde{h}_j z_j' + \frac{\tilde{h}_j^2}{2} z_j^2 - \frac{\tilde{h}_j^3}{6} z_j^3 + \frac{\tilde{h}_j^4}{24} z^4(\tilde{\xi}_1, j), \quad (7.5.11)$$

$$z_{j+1}^k = z_j + \tilde{h}_{j+1} z_j' + \frac{\tilde{h}_{j+1}^2}{2} z_j^2 + \frac{\tilde{h}_{j+1}^3}{6} z_j^3 + \frac{\tilde{h}_{j+1}^4}{24} z^4(\tilde{\xi}_2, j), \quad (7.5.12)$$

where

$$(\xi_1, j) \in (x_{j-1}, x_j), \quad (\xi_2, j) \in (x_j, x_{j+1}), \quad \tilde{\xi}_1 \in (\frac{x_{j-1} + x_j}{2}, x_j) \text{ and } \tilde{\xi}_2 \in (x_j, \frac{x_j + x_{j+1}}{2}).$$

Substituting (7.5.9) and (7.5.10) into (7.5.7), (7.5.11) and (7.5.12) into (7.5.8), we obtain

$$L^{N,K}(U_j^k - z_j^k) = k_1 z_j + k_2 z_j' + k_3 z_j^2 + k_4 z_j^3 + k_{5,1} z^4(\xi_1, j) + k_{5,2} z^4(\xi_2, j) \quad (7.5.13)$$

and

$$L^{N,K}(\tilde{U}_j^k - z_j^k) = \tilde{k}_1 z_j + \tilde{k}_2 z_j' + \tilde{k}_3 z_j^2 + \tilde{k}_4 z_j^3 + \tilde{k}_4 z_j^4 + \tilde{k}_{5,1} z^4(\tilde{\xi}_1, j) + \tilde{k}_{5,2} z^4(\tilde{\xi}_2, j). \quad (7.5.14)$$

The coefficients in (7.5.13) are

$$k_1 = \frac{2(\varepsilon + x_j)}{h_j(h_j + h_{j+1})} - \frac{2(\varepsilon + x_j)}{h_j h_{j+1}} + \frac{2(\varepsilon + x_j)}{h_{j+1}(h_j + h_{j+1})}, \; k_2 = 0,$$

$$k_3 = \frac{(\varepsilon + x_j) h_j}{h_j + h_{j+1}} - \frac{\tilde{a}_j^k h_j}{2} + \frac{(\varepsilon + x_j^2) h_{j+1}}{h_j + h_{j+1}} - (\varepsilon + x_j),$$

$$k_4 = \frac{-(\varepsilon + x_j) h_j^2}{3(h_j + h_{j+1})} + \frac{\tilde{a}_j^k h_j^2}{6} + \frac{(\varepsilon + x_j) h_{j+1}^2}{3(h_j + h_{j+1})}, \; k_{5,1} = \frac{(\varepsilon + x_j) h_j^3}{12(h_j + h_{j+1})} - \frac{\tilde{a}_j^k h_j^3}{24}, \; k_{5,2} = \frac{(\varepsilon + x_j) h_{j+1}^3}{12(h_j + h_{j+1})}.$$

The quantities for $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$, $\tilde{k}_4$, $\tilde{k}_{5,1}$ and $\tilde{k}_{5,2}$ can be obtained by substituting h_j by $\tilde{h}_j$ and h_{j+1} by $\tilde{h}_{j+1}$.

Substituting (7.5.13) and (7.5.14) into (7.5.6), we obtain

$$L^{N,K}(U^{ext} - z)_j^k = T_1 z_j + T_2 z_j'' + +T_3 z_j''' + +T_{4,1} z^{(4)}(\xi_1, j) + T_{4,2} z^{(4)}(\xi_2, j), \quad (7.5.15)$$

where

$$T_1 = \frac{14(\varepsilon + x_j)}{h_j(h_j + h_{j+1})} - \frac{14(\varepsilon + x_j)}{h_j h_{j+1}} + \frac{14(\varepsilon + x_j)}{h_{j+1}(h_j + h_{j+1})},$$
$$T_2 = \frac{(\varepsilon + x_j)h_j}{h_j + h_{j+1}} - (\varepsilon + x_j) + \frac{(\varepsilon + x_j)h_{j+1}}{h_j + h_{j+1}}, \quad T_3 = -\frac{\tilde{a}_j^k h_j^2}{12},$$
$$T_{4,1} = -\frac{(\varepsilon + x_j)h_j^3}{24(h_j + h_{j+1})} + \frac{\tilde{a}_j^k h_j^3}{32} \text{ and } T_{4,2} = -\frac{(\varepsilon + x_j)h_{j+1}^3}{24}.$$

Given (7.3.9) and for the sake of simplicity, we use the notation

$$h_j = \begin{cases} H & \text{if } j = 3N/4, 3N/4 + 1, \cdots, N, \\ h & \text{if } j = N/2, N/2 + 1, \cdots 3N/4. \end{cases} \quad (7.5.16)$$

Using the fact that, for $\forall j = 3N/4, 3N/4 + 1, \cdots, N$, $H = h_j \leqslant 4N^{-1}$ substituted into (7.5.15) in the subinterval $[\tau + 1/2, 1]$, we obtain

$$L^{N,K}(V^{ext} - v)_j^k = -\frac{\tilde{a}_j^k H^2}{12} v_j''' + \left[\frac{\varepsilon H^2}{48} + \frac{\tilde{a}_j^k H^3}{32}\right] v^{(4)}(\xi_1, j) - \frac{\varepsilon H^3}{24} v^{(4)}(\xi_2, j). \quad (7.5.17)$$

Using the triangle inequality and Lemma 7.2.8 substituting into (7.5.17), we obtain:

$$|L^{N,K}(V^{ext} - v)_j^k| \leqslant CN^{-2} \begin{cases} 1 + (\varepsilon + x)^{-a}, & a < 1, \\ 1 + (\varepsilon + x)^{-1} \ln^{-1}(\varepsilon + 1/2)^{-1}, & a = 1, \\ 1 + \varepsilon^{a-1}(\varepsilon + x)^{-a}, & a > 1. \end{cases} \quad (7.5.18)$$

The above inequalities lead to

$$|L^{N,K}(V^{ext} - v)_j^k| \leqslant CN^{-2}.$$

Hence, by Lemma 7.3.2, we obtain

$$|(V^{ext} - v)_j^k| \leqslant CN^{-2}. \quad (7.5.19)$$

The estimate of the nodal error of the singular component depends on whether $\tau = 1/4$ or $\tau = (\varepsilon/\eta)\ln(N/4)$. Firstly, the mesh is uniform and $(\varepsilon/\eta)\ln(N/4) \geq 1/2$. The estimate of the singular part of the local truncation error is obtained as follows

$$L^{N,K}(W^{ext} - w)_j^k = -\frac{\tilde{a}_j^k h^2}{12} w_j''' + \left[\frac{\varepsilon h^2}{48} + \frac{\tilde{a}_j^k h^3}{32}\right] w^{(4)}(\xi_1, j) - \frac{\varepsilon h^3}{24} w^{(4)}(\xi_2, j). \qquad (7.5.20)$$

Now, applying the triangle inequality and Lemma 7.2.8 substituted into (7.5.20), we obtain

$$L^{N,K}(W^{ext} - w)_j^k \leqslant Ch^{-2} \begin{cases} (\varepsilon + x)^{-a-2}, & a < 1, \\ (\varepsilon + x)^{-3} \ln^{-1}(\varepsilon + 1/2)^{-1}, & a = 1, \\ \varepsilon^{a-1}(\varepsilon + x)^{-a-2}, & a > 1. \end{cases} \qquad (7.5.21)$$

The above inequalities lead to

$$L^{N,K}(W^{ext} - w)_j^k \leqslant Ch^{-2}\varepsilon^{-2} \leqslant CN^{-2}\varepsilon^{-2}. \qquad (7.5.22)$$

Since $\varepsilon^{-1} \leqslant (2/\eta)\ln(N/4)$, (7.5.22) becomes

$$|L^{N,K}(W^{ext} - w)_j^k| \leqslant CN^{-2}\ln^2(N/4).$$

Hence, by Lemma 7.2.8 we obtain

$$|(W^{ext} - w)_j^k| \leqslant CN^{-2}\ln^2(N/4). \qquad (7.5.23)$$

Secondly, the mesh is piecewise uniform with the mesh spacing $h = h_j \leqslant 4\tau N^{-1}$ for $\forall j = N/2, \ldots, 3N/4$ on $[1/2, \tau + 1/2]$. We obtain from (7.5.22)

$$|L^{N,K}(W^{ext} - w)_j^k| \leqslant C_1 N^{-2}\tau^2\varepsilon^{-2}. \qquad (7.5.24)$$

Using $\tau = (\varepsilon/\eta)\ln(N/4)$ and Lemma 7.2.8 in (7.5.24) we obtain

$$|(W^{ext} - w)_j^k| \leqslant CN^{-2}\ln^2(N/4). \qquad (7.5.25)$$

A similar analysis can be performed on $[0, 1/2]$.

Combination (7.5.19), (7.5.23) and (7.5.25) along with (7.5.5), gives rise to the following theorem.

Theorem 7.5.1. *Let z and U be the solution of (7.3.4)-(7.3.5) and (7.3.10)-(7.3.12), respectively. Then, there exists a constant C, independent of the perturbation parameter ε and the space discretization parameters h_j such that*

$$\max_{0\leq j\leq N} |(U^{ext} - z)_j^k| \leqslant CN^{-2}\left[\ln\left(\frac{N}{4}\right)\right]^2. \tag{7.5.26}$$

Once more, using the triangle inequality and combining (7.3.7) and Theorem 7.5.1, we obtain the error after extrapolation which we state in the following theorem.

Theorem 7.5.2. (Error after extrapolation). *Let u be the exact solution of (7.1.1)-(7.1.3) and U its numerical approximation obtained via the scheme (7.3.10)-(7.3.12). Then, there exists a constant C, independent of the perturbation parameter ε, the time discretization Δt and the space discretization parameters h_j such that*

$$\max_{0\leq j\leq N; 1\leq k\leq K} |(U^{ext} - u)_j^k| \leqslant C\left[\Delta t + N^{-2}\left[\ln\left(\frac{N}{4}\right)\right]^2\right]. \tag{7.5.27}$$

7.6 Numerical example

In this section, we present the numerical results obtained for the test problem. We begin with $N = 16$ and $\Delta t = 0.1$ and then we multiply N by two and divide Δt also by two. The maximum errors and order of convergence are calculated by the exact solution. The solution in the example has a turning point at $x = 0.5$, which gives rise to an interior layer.

Example 7.6.1. Here we consider the following problem (7.1.1)-(7.1.3) for

$$a(x,t) = (2x-1)(1+t),\ b(x,t) = (1+xt),\ d(x,t) = e^{-xt},\ T = 1$$

and the functions $f(x,t)$ and $u_0(x)$ are such that the exact solution is given by

$$u(x,t) = \varepsilon e^{-t/\varepsilon}(2x-1+\varepsilon)^2 \cos\pi(2x-1).$$

Maximum errors at all mesh points are determined

$$E^{\varepsilon,N,K} = \max_{0\leq j\leq N; 0\leq k\leq K} |u_{j,k}^{\varepsilon,N,K} - U_{j,k}^{\varepsilon,N,K}|, \text{ and we compute } E_{\varepsilon,N,K} = \max_{0\leq \varepsilon\leq 1} E_{\varepsilon,N,K},$$

where $u_{j,k}^{\varepsilon,N,K}$ denotes the exact solution, and $U_{j,k}^{\varepsilon,N,K}$ denotes the numerical solution which is obtained by a constant time step Δt using N mesh intervals in the entire domain $\Omega = [0,1]$. In addition, the numerical rate of uniform convergence is computed as

$$r_l \equiv r_{\varepsilon,l} = \log_2\left(E^{\varepsilon,N_l,K_l}/E^{\varepsilon,2N_l,2K_l}\right).$$

After extrapolation the maximum errors at all mesh points and the numerical rates of convergence are calculated as follows:

$$E_{\varepsilon,N,K}^{ext} = \max_{0\le j\le N;0\le k\le K} |U_{j,k}^{ext} - u_{j,k}^{\varepsilon,N,K}|, \text{ and } R_{N,K} \equiv R_{\varepsilon,N,K} \equiv \log_2(E_{\varepsilon,N_l,K_l}^{ext}/E_{\varepsilon,2N_l,2K_l}^{ext}).$$

Table 7.1: Results for Example 7.6.1 Maximum errors before extrapolation

ε	$N = 16$ $K = 16$	$N = 32$ $K = 32$	$N = 64$ $K = 64$	$N = 128$ $K = 128$	$N = 256$ $K = 1256$	$N = 512$ $K = 512$
10^{-2}	2.23E-02	1.61E-02	1.08E-02	1.02E-02	1.02E-02	1.02E-02
10^{-3}	2.22E-02	1.60E-02	1.10E-02	6.32E-03	3.47E-03	1.82E-03
10^{-4}	2.22E-02	1.61E-02	1.10E-02	6.33E-03	3.48E-03	1.83E-03
10^{-11}	2.21E-02	1.61E-02	1.11E-02	6.34E-03	3.48E-03	1.83E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	2.21E-02	1.61E-02	1.11E-02	6.34E-03	3.48E-03	1.83E-03

Table 7.2: Results for Example 7.6.1 Maximum errors after extrapolation

ε	$N = 16$ $K = 16$	$N = 32$ $K = 432$	$N = 64$ $K = 64$	$N = 128$ $K = 128$	$N = 256$ $K = 256$	$N = 512$ $K = 10240$
10^{-2}	3.34E-02	1.21E-02	1.02E-02	1.02E-02	1.02E-02	1.02E-02
10^{-3}	3.36E-02	1.24E-02	3.63E-03	1.00E-03	1.01E-03	1.01E-03
10^{-4}	3.36E-02	1.24E-02	3.64E-03	9.60E-04	2.44E-04	1.02E-04
10^{-11}	3.36E-02	1.24E-02	3.64E-03	9.74E-04	2.52E-04	6.40E-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	3.36E-02	1.24E-02	3.64E-03	9.74E-04	2.52E-04	6.40E-05

Table 7.3: Results for Example 7.6.1 Rates of convergence before extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	0.47	0.54	0.80	0.68	0.00
10^{-5}	0.47	0.54	0.80	0.87	0.93
10^{-7}	0.46	0.54	0.80	0.86	0.93
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	0.46	0.54	0.80	0.86	0.93

Table 7.4: Results for Example 7.6.1 Rates of convergence after extrapolation

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	1.43	1.77	1.92	1.98	1.26
10^{-5}	1.43	1.77	1.92	1.99	2.00
10^{-7}	1.43	1.77	1.90	1.95	1.98
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-25}	1.43	1.77	1.90	1.95	1.98

7.7 Discussion

In this chapter, we proposed a Fitted Mesh Finite Difference Method (FMFDM) for a class of time-dependent singularly perturbed problems in which the diffusion term is a function ε with x as a linear function. In addition, the solution to this problem displays an interior layer due to the presence of a turning point. After providing appropriate bounds on the solution and its derivatives, we used the classical Euler method to discretize the time variable. This process resulted in a system of interior layer boundary value problems (one at each time level), which we solved by using a FMFDM. The proposed method used an upwind scheme on an appropriate non-uniform mesh of Shishkin type, fine in the (interior) layer and coarse elsewhere. Using bounds on the solution and its derivative, we showed that numerical method was uniformly convergent relative to the perturbation parameter ε and the step-size.

Theoretical results were validated through a numerical experiment. We computed the maximum point-wise errors and the corresponding rates of convergence for different values of N and K. The results displayed in tables 7.1 and 7.3 that the method was uniformly convergent. Furthermore, we investigated the effect of Richardson extrapolation via FMFDM in order to improve our results. For comparison purposes, we kept the same values of N and K considered above with numerical results shown in tables 7.2 and 7.4.

Chapter 8

Concluding remarks and scope for future research

In this thesis, we considered various classes of singularly perturbed two-point boundary value problems and time-dependent parabolic problems whose solution exhibited an interior layer due to the presence of a turning point. Cases of constant and variable diffusion coefficients were investigated. The major objective of this thesis was to construct and analyse fitted mesh finite difference methods (FMFDMs) on Shishkin meshes and to use Richardson extrapolation to increase their accuracy and rate of convergence. Noting that a Shishkin mesh is a piecewise constant mesh, we wanted to investigate how these methods would perform if the mesh was graded. To this end, in Chapter 4, we designed a FMFDM on a Bakhvalov mesh.

For each of the problems, we commenced by establishing bounds on the solutions and their derivatives before embarking on the design of the method. These bounds were instrumental in the convergence analyses of the proposed FMFDMs.

Investigations of the two-point boundary value problems indicated that the proposed methods were ε-uniformly convergent of order one which was improved to order two when applying Richardson extrapolation. Similarly, methods proposed for the parabolic proved to be ε-convergent of order one in the time and space. Richardson extrapolation was applied in space direction to achieve second order accuracy. These theoretical results were supported with extensive simulations. Numerical results were tabulated in relevant chapters.

As far as the **scope for further research** is concerned, we intend to

- Present a mesh-independent analysis to the problems considered.
- Explore the possibility of extending the proposed approach for elliptic singular perturbation problems having a variable diffusion coefficient.
- Construct higher order fitted mesh methods to solve the problems considered.
- Explore the possibility of extending the proposed method to solve fractional order differential equations.
- Explore problems in applied sciences where solutions change in the interior of the domain and are sensitive to the change in the diffusion coefficient.

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