SOME COGNITIVE DIFFICULTIES THAT STANDARD SIX AND SEVEN PUPILS AT A TYPICAL CAPE FLATS SCHOOL EXPERIENCE WITH THE GENERALIZED NUMBER PATTERN APPROACH TO ALGEBRA

by

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ABSTRACT

Mathematics teaching in South African schools is currently in a state of ferment. The recent political transformation in the country has had a ripple effect on the curriculum in the form a more "pupil centered" approach. One of the more noteworthy curricular changes is the proposed generalized number pattern approach to algebra (e.g., in the Draft Syllabus for Mathematics of the Western Cape Education Department for the Junior Secondary Phase, 1996). This thesis sets out to explore, against a constructivist background, some of the cognitive difficulties that standard six and seven pupils at a typical Cape Flats School experience with the new approach.

The research was conducted at the Bellville South Senior Secondary School. By virtue of its location and feeding areas, this school is in my opinion a typical Cape Flats school and its pupils are representative of the broader Cape Flats population.

This developmental research effort went through three phases. During the first phase all the pupils in the mathematics department were given a project on generalized number patterns in the form of a take-home worksheet. This was followed up by conducting three interviews with different standard six pupils in which they were questioned about their responses to the questions in the worksheet. For the second phase three standard six and three standard seven pupils were interviewed on a matchstick pattern that was generated by physically building it with matchsticks. The pupils were given the option to participate in the building process. During the third phase three standard six and three standard seven pupils were interviewed as they attempted to generalize from a functional table.

During the interviews and in the analyses a conscious attempt was made to reveal some of the cognitive difficulties that the pupils might have experienced in the course of their attempts to generalize from the patterns. The interviews were structured around certain core questions that were designed to gradually guide the pupils from the perception of the pattern to the writing of a symbolic rule in terms of the algebraic code.

A number of cognitive difficulties that occurred across the sample of subjects were identified, for example: a fixation with a recurrence rule; a tendency to "over-generalize"; difficulties with the articulation of the rule in a natural language, These were discussed within the context of particular examples from the interviews. A few teaching recommendations based upon the research findings are suggested, for example: teachers should familiarize themselves with the cognitive difficulties that their pupils are likely to experience with generalizing from number patterns in order to equip themselves to address these; exposing pupils to a variety of patterns over a period of time; encouraging pupils to see that the same pattern can be expressed in more than one way; etc.

In conclusion, the research findings seems to suggest that pupils on the Cape Flats are not yet ready for the generalized number pattern approach to algebra.

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CHAPTER 1

1. INTRODUCTION AND MOTIVATION

1.1 Background to the Study

Current trends in South African mathematics education include the use of number patterns as a suggested introduction to algebra. In the latest Draft Syllabus for Mathematics in the Junior Secondary Course of the Western Cape Education Department (for implementation: 1996-1997) the study of number patterns, generalizations, and relationships between variables are suggested as precursors for the introduction of algebra in the Junior Secondary Phase of schooling. The examples of number pattern generators given in the document include functional tables, flow diagrams and match stick patterns. These topics are to be introduced in standard five through a number of activities aimed at:

1. The investigation of number patterns.
2. Generalization and description of patterns:
   (a) in words;
   (b) in a flow diagram; and
   (c) with the aid of a formula: (i) in words and (ii) letter symbols.
3. Solving problems.
4. Generating number sequences and tables.

These patterning activities are intended to lay the foundations on which the introductory concepts of algebra (e.g., the concept of variable and relationships between variables; transformations of algebraic expressions into equivalent expressions; etc.) can be built in standard six. In standard seven the focus of the algebra syllabus shifts to symbolic manipulations. A tacit assumption underlying the approach seems to be that the pupils, having been exposed to the above mentioned activities in standards five and six, would have attained the necessary conceptual frameworks to enable them to deal with the purely syntactic-manipulative side of algebra in standard seven and thereafter.

Researchers into mathematics education (e.g., Cortes, Vergnaud and Kavafian, 1990; Herscovics and Linchevski, 1994) have identified a cognitive gap between the arithmetic framework and the algebraic framework that is sufficiently wide for pupils to experience difficulties in trying to make a transition from the one to the other. Some of these difficulties have been documented by them and other researchers (e.g., Herscovics, 1989; Herscovics and Linchevski, 1994; Kieran, 1989; MacGregor and Stacey, 1993; Orton and Orton, 1994; Reggiani, 1994). I hope to use their work as a theoretical background for my own explorations into the cognitive difficulties that standard six and standard seven pupils at a typical Cape Flats school experience when they attempt to generalize from number patterns.

The study reported on here was conducted as a small scale developmental research project within the mathematics department of the Bellville South Senior Secondary School. Pupils who attend this school are drawn not only from Bellville South itself, but also from other residential areas such as: Belhar, Blackheath, Delft, Eerste River, Ravensmead and Khayalitsha. These areas form part of what is known as the Cape Flats, a name used to describe the regions in and around Cape Town where the non-white people were forced to live during the Apartheid era; and where many of them still live. The sample of pupils who were interviewed for this study were found to be more or less evenly distributed amongst the residential areas mentioned.

This was not by design, but happened spontaneously because of the wide feeding area of the school itself.

At this stage I need to explain why I see Bellville South Senior Secondary as a typical Cape Flats school. The school shows a number of features that are common amongst schools on the Cape Flats. It is a dual medium school, catering for both Afrikaans and English speakers. A growing number of Xhosa speakers also attend the school; receiving instruction in their second language, which is English. Many of the pupils come from sub-economic areas where poverty and other related social problems are rife. The school has a lack of adequate teaching and learning resources that can be ascribed to an ever diminishing state budget and other financial constraints. These are just some of the features that, in my own opinion, contribute to the school's typical Cape Flats character.

Motivated by the above mentioned curricular innovations in introductory algebra, I embarked on this study hoping to illuminate some of the cognitive difficulties that standard six and seven pupils on the Cape Flats experience with the generalized number pattern approach to algebra.

1.2 The Constructivist Perspective of Learning and Teaching

Constructivism can have different meanings for different people (Simon, 1995; Wheatley, 1991). Take, for example, the distinction between "radical constructivism" and "social constructivism". Radical constructivists, in keeping with the psychological or cognitive perspective, focus on the individual learner's cognitive constructs. From this perspective social interaction and consensus is of secondary importance; the primary concern is the reconstruction of the individual's cognition. Social constructivists, on the other hand, perceive cognitive processes as socially determined. The focus of attention from this perspective is the role of the sociocultural environment in the reconstruction of the individual's cognition. What is to count as the ultimate truth is determined by sociocultural consensus. In this thesis, I follow Simon's (1995) coordination of the two perspectives in order to make sense of how learning takes place in everyday classrooms. I agree with Wheatley's (1991) interpretation that the theory of constructivism is supported by two main pillars:

1. The principle that knowledge is not passively received, but is actively reconstructed by the cognizing receiver, forms the first pillar. That is, that the learner does not passively absorb knowledge in an intact form; instead the learner uses the received knowledge to construct his or her own meaning. In our attempts to convey our own meanings we are more likely to evoke meanings in others. Sometimes the evoked meanings may differ radically from what we intended.
2. The principle that there exists no such thing as an independent, objective reality, forms the second pillar. That is, we construct our own reality based on our experiences. Our knowledge of the world is constructed from our own perceptions and experiences, which in turn are mediated through our previous knowledge (Simon, 1995). Since we know the world only through our sensory experiences of the past and the present, no ultimate truth is attainable; at best we can only hope to construct viable explanations for our experiences (Simon, 1995; Wheatley, 1991). Without an ultimate truth to strive for, we must settle for what is viable. As Simon (1995) explained:

> A concept works or is viable to the extent that it does what we need it to do: to make sense of our perceptions or data, to make an accurate prediction, to solve a problem, or to accomplish a personal goal (p.115).

When our concepts are not viable by our own standards, our adaptive processes are triggered, and we are ready to learn. By reflecting on successful adaptive operations, we put ourselves in a position to modify our existing concepts; or to build new ones.

From a constructivist perspective, knowledge evolves as the result of a learner's activity performed on mental constructs variously referred to as "objects" (Wheatley, 1991), "schemas" (Dubinsky, 1991) or "frames" (Davis, 1986). That is, knowledge is always intimately related to the actions and experiences of the learner -- always within the particular schematic context of the learner's experiential knowledge. The learner's activity is transformed into a mental "object" when he or she is able to think it through, come up with a result, and take the result as a given (Wheatley, 1991). Once the actions had been transformed into objects, the learner is able to reflect on them; and, through the process of "reflective abstraction" (Dubinsky, 1991; Simon, 1995; Wheatley, 1991), learning can take place. The process of "reflective abstraction", having its source in the actions of the learner and operating completely internally, allows for the isolation and coordination of properties and relationships amongst the learner's cognitive structures. Two other kinds of abstraction should also be mentioned (Dubinsky, 1991; Vergnaud, 1990): "Empirical abstraction", having its source in the properties of external objects; and "pseudoempirical" abstraction, which teases out the properties that the actions of the subject have introduced into the objects. Of all these kinds of abstraction, according to Dubinsky (1991), "reflective abstraction" is the most important for the development of mathematical thought -- providing a description of the mechanism for the development of intellectual thought. For Dreyfus (1991), the ability of a student to consciously make abstractions from non-obvious mathematical situations signals an advanced level of mathematical thinking. He (ibid.) made the claim that this ability to abstract might well be "the single most important goal of advanced mathematical thinking" (p.34).

Constructivists perceive learning as the adaptations that learners make in their existing schemes to neutralize perturbations that arise through their interactions with the world -- constructing new schemes and elaborating on old ones based on their new experiences. Wheatley (1991) wrote:

> At one time we believe we have something figured out. But if we are reflective and inquiring, it is likely that we will encounter events which call into question our conceptualizations and we will be forced to reorganize our ideas. This reorganization may require throwing out much of what we have constructed and reconstructing our schemes of knowledge (p.12).

Constructivism provides us with a theoretical framework for discussing the adaptation of existing cognitive constructs and the creation of new ones in the minds of our students. Simon (1995) sums it all up by making the following point which is often overlooked by educators and policy makers alike:

> Constructivism, as an epistemological theory, does not define a particular way of teaching. It describes knowledge development, whether or not there is a teacher present or teaching going on (p.117).

Parker (1995) reports the following changes in the discourse about the mathematics curriculum in South Africa:

1. The child is no longer seen as an "empty vessel", but as an active, mathematical thinker who enters school with powerful informal mathematical methods.
2. A shift from the transmission of esoteric knowledge to the exploration of pupils' everyday knowledge for generalizable mathematical knowledge that can be recontextualized into school mathematics.
3. The teacher is no longer seen as an external regulator and knowledge disseminator, but as a consultant and facilitator.

These changes in the discourse about mathematics education are indicative of the current trend towards a more constructivist based curriculum discourse in South Africa.

1.3 "Algebra for Some" or "Algebra for All"?

Algebra has traditionally been regarded to be the domain of the gifted -- only they can "do algebra" and dare to further their studies in the subject. The ability to "do algebra" is often regarded as a sure sign of intelligence; especially by those who do not use it. Educators, discouraged by the high failure rate in algebra, often discard the idea of "algebra for all" either as an irrelevant slogan or too difficult a task to accomplish. Usiskin (1992) made the following points related to the question of "algebra for all":

- The whole of society needs algebra. Not only, the engineers, scientists, statisticians, etc.; but also, the carpenters, plumbers, builders, etc.
- The economic well-being of a country depends upon having jobs for its people; and the creation of new jobs in the 21st century depends upon achievements in sectors such as biotechnology, telecommunications, computers and software, micro-electronics, robotics and machine tools. Advancements in these areas demands considerable amounts of mathematical knowledge; of which algebra is an indispensable part.
- There now exists technology that makes the graphing of functions and data, and even curve fitting and data analysis, accessible to all. One no longer need to know huge amounts of mathematics to do all these things. Algebra has become more accessible; and so also has elementary analysis.
- The available technology does not yet cover all of algebra, so there is still a need to know some algebra.

These are just some of the reasons why the notion of "algebra for some" should not be acceptable for the serious mathematics educator.

Usiskin (1992) also argued that the learning of algebra is like the learning of a language and that anyone who can learn to read, write and comprehend his or her native language should be able to learn how to read, write and comprehend algebraic symbolism. What one must bear in mind, however, is that a native language is learned in a particular context that gives it immediate meaning. It is my contention that, if algebra was to be taught in a context that gives it immediate meaning, it would be learned like a language and would indeed become accessible to all.

1.4 Summary

Chapter one sketched a brief background to the study against the South African curriculum innovations. The study is located within the constructivist perspective of teaching and learning. The notion of a "typical Cape Flats school" was explained as it is used in this thesis. Arguments are forwarded that algebra is indeed accessible to more people than what is generally believed.

In chapter two will be a brief review of the literature on algebra teaching pertaining to number patterns in algebra. Some difficulties that arise when people learn algebra will also be discussed.

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CHAPTER 2

2. LITERATURE REVIEW

2.1 What is this Thing Called "Algebra"?

There is no simple definition for algebra. Usiskin (1988), for example, points out that what is taught as algebra in school is different from what is taught as algebra to mathematics majors at university. At school algebra deals with the rules for manipulating "letters" standing for numbers; at university "letters" are still used, but they may not stand for numbers any more, for example, in abstract algebra they may stand for structures such as vector spaces, groups or rings. These "letters" are what is known to mathematicians as "variables"; the *raison d'être* for algebra's existence as a separate discipline of mathematics. Usiskin (ibid.) also points out that the variable is a multifaceted concept in itself; that is, to claim that "algebra is the study of variables" is simply not enough. To substantiate this point, Usiskin (ibid.) has highlighted four conceptions of algebra that can be related to different uses of the variable:

1. *Algebra as generalized arithmetic*, i.e., thinking of variables as *pattern generalizers*. For example, the pattern, 2 + 1.3 ; 2 + 2.3 ; 2 + 3.3 ; 2 + 4.3 ; ... The student is expected to translate and generalize the pattern to $2 + 3n$.
2. *Algebra as the study of procedures for solving certain kinds of problems*, i.e., thinking of variables as *unknowns* or *constants*. For example, when the student is given the equation $2 + 3x = 17$ with the instruction to simplify and solve for x.

3. *Algebra as the study of relationships among quantities*, i.e., the variable is either an *argument* (domain value of a function) or a *parameter* (number on which other numbers depend). For example, when the student is given the function $f(x) = 2 + 3x$ with the instruction to find

(i) $f(x)$ for $x = a$ (x is an *argument*); and

(ii) x so that $f(x) = a$ (x is a *parameter*).

4. *Algebra as the study of structures*, i.e., thinking of the variable as an *arbitrary object* in a structure related by certain properties. The instruction is to treat the variable as an *arbitrary mark on paper* without any numerical referents. For example, the groups, rings, and vector spaces of advanced mathematics are thought of and treated in this way.

I have asked a number of former students about their recollections of school algebra and most of them remember it as a branch of mathematics that deals with the rules for manipulating symbols with yet to be discovered meanings. When this typical response about the nature of algebra is considered within the context of how algebra has traditionally been taught in schools, it comes as no surprise. In the past teachers and textbook authors alike have focused on the manipulation of abstract symbols. Sometimes these symbol systems taught have only other symbols as referents (Meira, 1990), which is in line with Usiskin's (1988) description of the conception of "algebra as a study of structures". This traditional "syntactic-manipulative approach" (Furinghetti and Paola, 1994) of introducing algebra by means of a formal definition of the variable, followed by a few examples, is unlikely to be of much assistance to pupils (Briggs, Demana and Osborne, 1986).

According to Pegg and Redden (1990), the assumption that algebraic language forms a natural part of a pupil's vocabulary is not necessarily true. If it was, then the traditional approach would have more success in producing significant numbers of pupils capable of constructing meaning for this new kind of symbolism. In fact, as Herscovics and Linchevski (1994) have pointed out, the traditional approach limits pupils to the performance of meaningless operations on symbols that they do not understand at all. That explains why many ex-algebra students will describe their experience of the subject as a meaningless manipulation of symbols.

For those initiated into its mysteries, algebra becomes a powerful tool; able to serve at least three purposes:

1. It permits the concise representation of quantities, general relationships and procedures.
2. The concise system of representation enables the initiated to solve a wide range of problems related to such relationships and procedures.
3. It also allows for the derivation of new relationships and procedures by the appropriate manipulation of the known ones.

Once the initiate attains a workable knowledge of these three purposes of algebra, then, for him or her, in the words of McQualter (1983): "Algebra provides the medium of mathematical discourse..." (p.3). Or as Polya (1962) so aptly described it:

> Algebra is a language which does not consist of words but of symbols. If we are familiar with it we can translate into it appropriate sentences of everyday language (p.24).

2.2 The History of Algebra and the Evolution of Symbolic Awareness

Classical algebra was introduced to the world by an Arabian mathematician known as Al-Khowarizmi (*c.* 830 A.D.). The title of his book, "Hisab Al-jabr w'al Mugabalah", from which the name Algebra was derived, means reorganising or regrouping terms or parts, while changing their state (McQualter, 1983; Streefland, 1994). It was presented as a list of rules and procedures that could be used to solve specific linear and quadratic equations.

In its evolution symbolic algebra has gone through three distinct phases (Kieran, 1990, 1994; Kramer, 1981; McQualter, 1983) in the following order: rhetorical algebra, syncopated algebra and functional algebra.

2.2.1 Rhetorical Algebra

This first phase spans the period from the Babylonians to the time of Diophantus of Alexandria (*c.* 250 A.D.). It was distinctly characterized by its rhetorical (verbal) approach, i.e., results were obtained by means of verbal argument, without the use of abbreviations or symbols of any kind (Kramer 1981). The system was not perfect. McQualter (1983) draws our attention to the fact that the lack of adequate symbolism precluded any attempts at generalization.

2.2.2 Syncopated Algebra

The second phase was introduced by Diophantus through his work "*Arithmetica*" in which there is the first evidence of the "syncopation" (abbreviation) of algebra (Kramer, 1981; McQualter, 1983). Syncopated algebra, as it became known, is a form of shorthand rather than a system of complete abstract symbolism -- that is, a sort of substitute for lengthy verbal expressions. Kramer (1981) points out that Diophantus had a particular affinity for the initial-letter type of shorthand. The concern of the algebraïsts of this period was exclusively that of discovering the identity of the letter or letters, as opposed to an attempt to express the general (Kieran, 1990).

2.2.3 Functional Algebra

The third phase was initiated by the introduction of proper algebraic symbolism towards the end of the sixteenth century by a French mathematician called François Viète (1540-1603). Viète reportedly was the first person to use letters to represent unknowns (Kramer, 1981; Streefland, 1994; Struik, 1969). He was also, as Struik (1969) points out, the first man to introduce, in a systematic way, general letters instead of numbers into the theory of equations. In doing so, Viète had shown that a general solution, in symbolic form, was possible for many classes of equations (McQualter, 1983). Algebra had thus become a tool, not only for expressing general solutions, but also for proving rules governing numerical relations (Kieran, 1990).

2.2.4 Algebra and Symbolism

According to Sfard and Linchevski (1994) the history of algebra is not necessarily the history of symbols. In fact, symbolic representation is not a necessary characteristic of algebra. Both the history and the learning of algebra provide evidence thereof that algebraic thinking appears long before algebraic notation is introduced. Both Rhetorical algebra and Syncopated algebra showed evidence of algebraic thinking, but without the symbolism that later became the hallmark of Functional algebra. After the introduction of algebraic symbolism, the history of algebra and symbols became so intertwined, that it has become almost impossible to tell them apart (Sfard and Linchevski, 1994). It has already been pointed out that an adequate system of symbolic notation is a necessary prerequisite for generalization. For that reason, and because this thesis deals with generalization from patterns, I prefer to treat the history of algebra and the history of symbols as the same thing.

The history of algebra teaches us that algebraic knowledge was not attained quickly; it developed slowly over many centuries (Herscovics, 1989). Likewise the transition from arithmetic to algebra will take time and should not be rushed (Streefland, 1994). There is evidence (e.g., Kieran, 1990; Mason, Pimm, Graham and Gowar, 1985) that algebra students pass through the same historical evolutionary stages in their acquisition of algebraic symbolism. The point is that it would be unrealistic to expect of our students to grasp the basic tenets of algebra in the course of a few formal classroom presentations. In the words of McQualter (1983):

> The algebraic knowledge a child must obtain from school mathematics should be permitted to grow slowly and inductively and not be presented as a formal set of rules and operations often resulting in confusion of thinking and obscurity of purpose (p.3).

This comment becomes especially relevant if one seriously considers the above mentioned account (Kieran, 1990; Mason, et.al., 1985) that students progress through similar evolutionary stages as the historical ones as they move along the axis from using words to using symbols to record generality.

The use of symbols and conventions in algebra to represent quantities and operations with those quantities (McQualter, 1983) provides the mathematician with a medium of mathematical discourse capable not only of expressing formulae succinctly, but also of manipulating those formulae in order to derive new conclusions.

2.3 Some Explanations for Pupils' Difficulties with School Algebra

According to Herscovics (1989) two types of arguments are often used to explain the high failure rate in school algebra:

1. inadequate teaching practices -- relating the problem to the teacher's instructional methods; and
2. that mathematics, especially algebra, is "too difficult" to be learned by all of the general population and should only be taught to the elite few capable of coping with its traditional presentation -- relating the problem to the pupils themselves.

The first of these arguments is still acceptable; offering a hint of optimism in the form of room for possible improvement. That is, if we as educators strive to improve our instructional methods, then there will be hope for more of our students. The second argument, however, is pessimistic, defeatest and far from being acceptable. Given the needs of our modern society for more and more people with the necessary algebraic skills to translate real life problems into solvable mathematical models (Usiskin, 1992), we cannot allow room for such pessimism.

The difficulties that pupils experience with school algebra have been well documented in the literature (e.g., Booth, 1986; 1988; Chalouh and Herscovics, 1988; Glencross, 1995; Herscovics and Linchevski, 1994; Kieran, 1989; Küchemann, 1981; Linchevski and Herscovics, 1994). This part of my treatise will be limited to a cursory overview of selected examples from the literature that have a bearing on my own research work.

2.3.1 Difficulties arising from the shift to a set of conventions different from those used in arithmetic

Several researchers (e.g., Booth, 1986; Herscovics, 1989; Olivier, 1984) have found that pupils' prior arithmetic experiences can be a source of difficulties when the transition to algebra has to be made. This is ironical, because arithmetic should provide the cognitive basis from the pupil's background (Chalouh and Herscovics, 1988) on which to build the notions of algebra. Since school algebra principally has to do with the formulation and manipulation of generalized statements about numbers, it is to be expected that pupils' prior experiences with arithmetic should have an important effect on their ability to make sense of it (Kieran, 1989).

In order to describe the difficulties of overcoming existing cognitive frameworks in order to construct new ones, and/or the unresolved conflict that sometimes exist between the new frameworks and the old ones, Herscovics (1989) introduced the notion of a "cognitive obstacle". Several examples of cognitive obstacles related to the arithmetic framework are discussed in the literature (e.g., Booth 1986; 1988). Here are a few to illustrate:

- The obsession with finding of a numerical answer; even where no such answer is desirable. Booth (1986; 1988) gave the example of a fourteen year old girl who calculated the perimeter of a figure with n sides, each of length 2, by assigning a value of 14 to n to obtain the answer of 28. The value of n as 14 was obtained by counting along the alphabet up to the letter n; and assigning an integer value to each letter as she went along.

- The persistent need for a single term answer sometimes leads to conjoining; that is, when an algebraic expression like $2a + 5b$ is simplified to the single term $7ab$. Booth (1988) explained that conjoining is acceptable in arithmetic where it implies addition, e.g., $2\frac{1}{2} = 2 + \frac{1}{2}$, and 43 = 4tens + 3units. Pupils, drawing on their experiences from arithmetic, tend to conjoin algebraic terms.
- Booth (1988) also pointed out that pupils often have a cognitive difficulty in accepting a "lack of closure" because they have certain expectations, derived from arithmetic, concerning what "well-formed answers" are supposed to look like. This prevents them from appreciating that unclosed algebraic expressions are not only legitimate answers; but may also represent the procedure or relationship by which the answer was obtained, as well as the answer itself.
- Olivier (1984) contended that, with the emphasis on computational algorithms in the primary standards, the important groundwork that could facilitate generalization are likely to be neglected. For example, in the addition algorithm, pupils are expected to add columns mechanically, i.e.,

$$\begin{array}{r} 2\ 2\ 2 \\ +\ \underline{3\ 3\ 3} \\ 5\ 5\ 5 \end{array}$$

instead of; 2 ones + 3 ones = 5 ones

2 tens + 3 tens = 5 tens

2 hundreds + 3 hundreds = 5 hundreds.

The background provided by the latter approach would have paved the way for a more meaningful generalization, viz., $2x + 3x = 5x$.

The tendency for pupils to fall back into the arithmetic frame of reference illustrates just how difficult it is to overcome existing conceptual frameworks in order to construct new ones and how old and new frames of reference sometimes conflict with each other (Herscovics, 1989).

2.3.2 Difficulties related to the recognition and use of structure

In arithmetic the emphasis is on "finding the answer" and many pupils are able to survive with informal and intuitive procedures. In algebra, by contrast, pupils are required to recognize and use the structures that they have been able to avoid in arithmetic (Kieran, 1989). Kieran (ibid.) made the distinction between *surface structure* and *systemic structure* when referring to the structure of an algebraic or arithmetic expression. The surface structure of an expression refers to the given form or arrangement of the terms and operations, subject to the constraints of the order of operations. When referring to an equation, surface structure comprises the given terms and operations on both sides of the equal sign, as well as the condition of equality. The systemic structure (relating to the system from which the expression inherits its properties) refers to the properties of the operations, e.g., commutativity and associativity, and to relationships between the operations, e.g., distributivity. Both the surface structure and the systemic structure are important considerations when an existing algebraic expression is manipulated to create a new one.

MacGregor and Stacey (1994) use the concept "metalinguistic awareness" (from linguistics) to describe an awareness of the structure of an algebraic expression. They (ibid.) identified two metalinguistic components that have a strong bearing on algebra learning:

1. *Symbol awareness*, i.e., knowing that numerals, letters and other mathematical signs can be manipulated in order to rearrange or simplify an algebraic expression, regardless of their original referents.
2. *Syntax awareness*, i.e., the ability to recognize when an algebraic expression is well formed, and when it is not. For example, $2x = 10 \Rightarrow x = 5$ is well formed, whereas $2x = 10 = 5$ is not.

2.3.3 Difficulties related to the meaning of the letters

Kieran (1989) pointed out that pupils' past experiences with placeholders in open sentences and letters used in formulas such as the area of a rectangle cannot easily be related to the many uses of the variable to which they are exposed in high school algebra. The uses of the variable include, according to Usiskin (1988), that of pattern generalizers; unknowns; arguments or parameters; and arbitrary objects (see section 2.1). Booth(1986) argued that,

> ... until a student does appreciate the use of letters as variables, or at least as 'generalised' number, then algebra can have little real meaning (p.3).

2.4 The Generalized Number Pattern Approach

2.4.1 The growing interest in the generalized number pattern approach to the teaching of algebra

Currently the generalized number pattern approach to the teaching of algebra is fast gaining momentum nationally as well as internationally. In the past decade there has been a steady growth in the number of mathematics educators and researchers who are arguing for an alternative approach to algebra via generalized number patterns generated from concrete situations, e.g., Abbot (1992); Andrew (1992); Andrews (1990); Booth (1989); Eagle (1986); Hale (1981); Mason (1988); Mason, et al. (1985); Morgan (1994); Pagni (1992); Pegg and Redden (1990); Redden (1994) and Richardson (1984). The current interest in the number pattern approach should be seen as part of a growing worldwide quest for more meaningful alternatives to replace the ineffective, traditional introduction into algebra via exercises in substitution, simplification and other manipulations of abstract, and otherwise meaningless symbols. One such suggested alternative is an approach via the micro-world of computers (see section 2.4.3). According to Booth (1989), algebraic notation was invented to represent general statements and she suggests that pupils should be introduced to the use of letters in algebra via the same ("more natural") route of generalization if they are to appreciate the purposes of their studies in algebra. In an earlier paper Booth (1986) has pointed out how the traditional method of introducing algebra "via exercises in substitution, simplification, or elementary equations may tend to encourage students' view of letters as standing for specific unknown values"(p.4). This leads to a restricted perception of the variable as a "letter used as a specific unknown" (Kücheman,

1981) which may be an obstacle when a different notion of the variable is required.

The very notion of a general rule immediately presupposes a pattern that the rule describes. The obvious starting point in the process of generalization would thus be a pattern from which a general rule can be derived. Some researchers arguing in favor of an approach to algebra via pattern finding and generalization (e.g., Arnold, 1992; Booth, 1989; Hale, 1981; Mason, et al., 1985; Pegg and Redden, 1990) claim that pupils will find the introduction to the process of pattern recognition and recording easier if the situations embodying the pattern are concrete and more "obvious" in the sense of having an easy visual representation. This claim emphasizes the importance of the exploration of concrete shape patterns (e.g., match stick patterns) that can be easily extended to form number patterns and "guess my rule" patterns in the number pattern approach. In their studies Orton and Orton (1994) provided match sticks to pupils in the hope that the experience of using the match sticks to build the next shape in a match stick pattern would help pupils to focus on the structure of the shapes. Contrary to their expectations, pupils ignored the match sticks once the numbers had been made explicit. Their interpretation of the phenomenon was that the task of coping with large numbers seemed to steer the pupils' thinking away from the matches and towards a search for a quick method of obtaining an answer.

2.4.2 Towards a proposed teaching methodology

Amongst the many studies in the literature exploring the link between generalized arithmetic and early algebra, only two papers (viz., Booth, 1989; Pegg and Redden, 1990) were found to explicitly suggest a teaching methodology. Although still in its infant stage of development, and in need of serious research reflection, the proposed methodology is useful because it offers to both teacher and researcher a point of departure.

The following methodological stages were proposed by Pegg and Redden (1990) and Booth (1989):

1. *Experiencing activities with number patterns.* This involves finding or "seeing" a pattern in a concrete situation -- which is not always as obvious as it may seem (Booth, 1989). The teaching objective, however, should not be the teaching of number patterns per se, but rather to use the number patterns as an alternative route towards the meaningful introduction of algebra.
2. *Expressing the rules which govern particular number patterns in full sentences.* During this stage students should be involved in clarifying and expressing accurately in their own words the rule(s) which determine or explain a given number pattern. At this point Booth (1989) reminds us that there are more than one way of seeing the same pattern, and subsequently more than one way of describing it. Orton (1993) also reported individual differences in children's perception of a pattern.
3. *Rewriting the rule(s) which govern a number pattern in an abbreviated form.* The emphasis here is on replacing the rule previously written out in full sentences with a more succinct symbolic alternative. This is also the stage where symbols first emerge and where algebra, as it is traditionally perceived, commences. Da Rocha Falcão (1995) contends that, "algebraic procedure ... begins by a formal transposition from empirical domain or natural language to an specific representational system" (p.71). This is a crucial stage --

the formulation of useful expressions and equations from a problem situation -- if students are to derive any real power from the algebra they learn in school (Stacey and MacGregor, 1995). Morgan (1994) wrote: "Symbolisation is not merely a process of translation from one language into another but is the starting point for developing new ways of looking at a problem and for enabling manipulations that may lead to new discoveries and further generalizations" (p.298). In section 2.2.4 McQualter(1983) is quoted for expressing the same sentiments about symbolism in algebra.

Pegg and Redden (1990) suffice with the first three stages, but Booth (1989) actually took the methodological approach one step further by suggesting a fourth stage:

4. *Using the pattern rule to solve problems more efficiently.* This step is to give the activity of pattern finding and description "a purpose which the students can readily appreciate" (Booth, 1989, p.12) because, students often see algebra as a language for expressing mathematical relationships without realizing that it is also useful for problem solving (Stacey and MacGregor, 1995).

2.4.3 The micro-world counterpart of the number pattern approach

The generalized number pattern approach is not the only way of introducing algebra to pupils. Another approach that also enjoys extensive coverage in the literature is via the micro-world of computers (e.g., Arnold, 1992; Ernest, 1989; Yerushalmy and Shterenberg, 1994). Given the South African context where most schools do not have adequate computer facilities, I thought that it would make more sense to investigate the viability of the number pattern approach to algebra.

2.4.4 Number patterns in the curriculum: International trends

Morgan's (1994) description of current trends in mathematics teaching in the United Kingdom towards the assessment by means of investigative work resembles similar trends in South Africa towards assessment by continuous evaluation. That is that a part of the assessment mark should include teacher assessments of aspects of mathematical achievement that are not amenable to timed written papers. This involves engaging the pupils in mathematical investigations through projects and model building, which in turn have other benefits. Through such activities students can gain first hand experience of the heuristics for problem solving. They can be encouraged to actively take part in mathematical thinking rather than passively receiving mathematical thoughts. The area of number patterns offers some excellent opportunities for investigative work, but more importantly, through the process of generalization it forms a natural route towards the use of algebraic notation.

In some countries the generalized number pattern approach already forms part of their mathematics curriculum. According to reports by Orton and Orton (1994) from England, mathematical activities that involve recognizing, exploring, continuing, and patterning has already become a significant element of the mathematics curriculum there. Reports from Australia (e.g., Arnold, 1992; Pegg and Redden, 1990) are that number patterns are also forming an increasingly more important part of the mathematics curriculum there.

2.4.5 Number patterns in the curriculum: South African trends

According to the Draft Syllabus for Mathematics in the Junior Secondary Course of the Western Cape Education Department (for implementation: 1996-1997), algebra in standards 5 to 7 is based on two different, but related views:

(a) Algebra as a study of relationships between variables

This is based on situations involving two variables where the one variable is dependent on the other variable. Algebra provides the models with which to describe and analyze such situations and it also provides the analytical tools with which to obtain additional, unknown information about the situation. The mathematical model may be presented in a number of different ways: in words, as a table of values, as a graph or as a computational procedure (i.e., as a formula or expression).

(b) Algebra as generalized arithmetic

This approach is based on the view that, historically, algebra grew out of arithmetic and ought to be introduced to pupils via that same historical path. Algebra is perceived as the generator of "new" mathematical knowledge. This it does through the mathematical processes of induction, generalization and proof. This is where generalized number patterns have an important role to play.

> Most pupils will easily recognise(abstract) the pattern ... and <u>generalise</u> by <u>conjecturing</u>... However, pupils should have adequate experience of the pitfals of induction to realise that it is necessary to <u>prove</u> the validity of the conjecture. This requires the introduction of a "generalised number" to cater for <u>any</u> natural number... (Draft Syllabus for Mathematics in the Junior Secondary Course of the Western Cape Education Department, 1996, p.7).

2.5 Some difficulties with the "seeing" of a pattern and the writing of a rule

2.5.1 "Seeing" a pattern

According to Mason, et al.(1985), 'Seeing' refers to the act of "grasping mentally a pattern or relationship, and is often accompanied by a sense of elation or insight" (p.8). This 'seeing' of the pattern is not the same for everyone. Different people often 'see' the same pattern differently (MacGregor and Stacey, 1993; Mason, et al., 1985; Orton, 1993; Orton and Orton, 1994).

2.5.2 Writing a rule

MacGregor and Stacey (1993) have identified four critical steps in moving from a function table to an algebraic rule:

- looking beyond recurrence patterns and finding a relationship linking the two variables;
- being able to formally articulate the relationship used for calculating numerical values (e.g., being able to say "Add 3" rather than "you count three places");
- knowing what can and cannot be said in elementary algebra (e.g. "Every time x goes up by 1, y goes up by 3" cannot be easily translated to an equation); and
- knowing the syntax of algebra (e.g., "$x = 3y$" does not mean "Start with x, and add 3 to get y").

The formulation of a general rule for a pattern is not an easy task. MacGregor and Stacey (1993) have found that more pupils could detect and use a relationship for calculations than could describe it verbally or algebraically. Similarly, Herscovics (1989) has reported how most pupils tested in a national assessment in the USA could recognize a simple pattern in a table of ordered pairs connected by a simple relationship (e.g., add 7), but that the majority were unable to generate the corresponding algebraic rule (e.g., $y = x + 7$). Others, like Pegg and Redden (1990), have also shown that it is difficult for pupils to generate algebraic rules from patterns and tables. This inability to formulate a general rule from a perceived pattern is a major obstacle, because it is the ability to perceive a relationship and then formulate it algebraically that is fundamental to being able to use algebra (MacGregor and Stacey, 1993).

According to MacGregor and Stacey (1993) pupils are likely to start of by searching for a recurrence rule that would enable them to predict a number from the value of its predecessor rather than by searching for a functional relationship linking pairs of numbers. That is, they are likely to focus on the difference between successive values of each variable. Similar observations have been made by others, for example, Pegg and Redden (1990); Orton and Orton (1994; 1996); etc. In fact, Orton and Orton (1994) made the point that "differencing" on its own is inadequate for finding the general rule, and that it often sets the pupils of on the wrong track. They (ibid.) identify it as an obstacle in the sense that pupils are unable to express the universal rule until they are prepared to discard the recursive pattern.

Most pupils can perceive patterns in tables easily, but few of them can perceive the functional relationship. Many only 'see' the pattern in one variable at a time. Even amongst those who may 'see' the functional relationship sufficiently clearly to calculate with it, many cannot express it in their natural language. Those who cannot express the functional relationship in their natural language also cannot write the relationship in the symbolic code of elementary algebra (MacGregor and Stacey, 1993). On the other hand, a verbal description is often available to pupils when an algebraic statement is not (Orton and Orton, 1996). Only some of the patterns that pupils so readily observe in tables are useful in other parts of the questions.

MacGregor and Stacey (1993) reported that one of the most striking findings of their study of number patterns was the variety of patterns perceived and the large proportion of generalizations expressed verbally that cannot be expressed in the elementary algebra that pupils are learning. Many of these patterns, although valid, are not helpful and do not lead to an idea that can easily be expressed with the algebra that the pupils are learning (MacGregor and Stacey, 1993). That is why it is important that pupils need to discuss why some patterns and relationships are more useful than others. Teachers must also be aware of the variety of patterns that their pupils perceive.

According to MacGregor and Stacey (1993) a high success rate for extending tables indicate that most pupils easily perceive the recurrence relations, either as a link between the two variables ("Top row" and "bottom row") or as two separate sequences ("Top row" separate from "bottom row").

Even the pupils' sometimes immature ways of expressing simple addition also enjoyed some coverage in the literature. For example, MacGregor and Stacey (1993) who noted that pupils will sometimes express addition as: "In between x and y there is four" or "There is three numbers missing", etc.

Some pupils use one rule for simple calculations and another rule for larger values of variables (MacGregor and Stacey, 1993; Orton and Orton, 1994). MacGregor and Stacey(1993) wrote that it is likely that many pupils do not clearly understand that they are not applying the same rule to larger numbers, and think that by using direct proportion they are merely taking a shortcut to the answer.

MacGregor and Stacey (1993) argue that the fact that many pupils are able to calculate values, but are unable to write explanations or algebraic equations is a clear indication thereof that recognizing and articulating the structure of a relationship is a major stumbling block. One essential prerequisite for using algebra is that pupils must be able to put their informal arithmetic knowledge into a formal structure -- to know, for example, that doubling ("You plus it by the same number") is the same as multiplication by two (ibid.). MacGregor and Stacey (1993) also argue that pupils who are able to give a correct verbal description are more likely to succeed in writing a correct algebraic rule. That is not to say that the ability to formulate a correct verbal description will guarantee the formulation of a correct algebraic rule. In some of the reported cases MacGregor and Stacey (ibid.) found that the pupils' verbal descriptions were not helpful in providing a basis for clear thinking about the function and its algebraic representation.

Orton and Orton (1994) also reported a reluctance of pupils to check their patterns or rules for possible errors. This points at a lack of cognitive awareness on the part of the pupils. Even in instances where their "rules" obviously do not fit the first few cases in a sequence, the pupils appear to completely ignore it. When left undetected, errors like these can become serious obstacles in the search for a generalized rule (Orton and Orton, 1996).

This completes the literature review.

2.6 Summary

Chapter two reviews selected aspects from the literature on algebra teaching that are of relevance to the generalized number pattern approach to algebra. The chapter starts of with a brief overview of some of the different meanings commonly associated with the concept "Algebra" as defined by the context in which it is used. It goes on to claim that the particular use of the variable also has an influence on the conceptualization of algebra.

A brief overview of the history of Algebra and the parallel evolution of symbolic awareness is included. It is suggested that the individual algebra learner, *en route* to symbolic awareness, goes through the same evolutionary stages that algebra has gone historically, albeit within a much shorter time span.

Some common difficulties with school Algebra that might be related to pupils' difficulties with generalizing algebraically from number patterns were discussed.

The growing local and international interest in the generalized number pattern approach to Algebra were highlighted; as well as a proposed teaching methodology for introducing Algebra to pupils via generalized number patterns.

Chapter two concludes with an overview of some of the issues from the literature concerning pupils' problems with the "seeing" of a pattern and the writing of a rule.

In chapter three the research methodology used to collect the data concerning the pupils' cognitive processing during their attempts to generalize from the given number patterns will be discussed. The focus will be on the three different phases of the research and the improvements in the clinical interviews as the research progressed.

CHAPTER 3

3. RESEARCH METHODOLOGY

3.1 Adopting the Clinical Interview as a Methodological Tool: Reflections on My Own Approach and Experiences

The main methodological tool that was used to collect the data for this research is the clinical interview. This method of collecting data in the form of verbal reports (Ericsson and Simon, 1980) is currently widely used as by, amongst others, professionals like doctors, psychologists, educators, and also researchers into cognition. All of these people have one thing in common; an interest in qualitative rather than quantitative data. The clinical interview is especially well suited for the collection of qualitative data.

One of the major advantages of the clinical interview as a cognitive assessment device, as Hunting and Doig (1995) point out, is that it allows the data source (the pupil) and the data analyzer and interpreter (the interviewer) to engage directly in an interactive mode of communication. As the interviewer conducts an interview, he or she inadvertently also engages in an active analysis of the interviewee's responses. It is my belief that this direct, interactive mode of communication provides the interviewer with a unique opportunity to verify, through skillful probing, the interviewee's thoughts as they are being processed. This also allows the interviewer to trace the cognitive path of a thought through its sporadic surfacings in the verbalizations of the interviewee. More than that, the interviewer can make use of strategic probes to force a thought to surface more often, thereby affording himself or herself with an opportunity to look at it from different angles and hence to learn more about the cognitive processes behind it. This level of clinical proficiency does not come easy to the novice interviewer. Piaget (1960) wrote that the clinical method can only be learned by long practice. Ericsson and Simon (1980) were able to identify three forms of probing:

1. *The instruction to think aloud or talk aloud.* With this type of instruction the heeded information may be verbalized either through a direct articulation or by a verbal encoding of information that was originally stored in a nonverbal code. In this way a direct trace can be obtained of the heeded information, and hence, an indirect one of the internal stages of the cognitive process.
2. *Concurrent probing.* The subjects are probed, concurrently with their performance of a task, for specific information. This information is usually of the kind that they presumably need to guide their succeeding behavior. Examples include requests to report the hypotheses they are using while attempting to solve a problem.
3. *Retrospective verbalization probing.* These are probes used to collect information from the subject after the completion of the task-induced processes. For example, asking the subject for a verbal report immediately after the process has been completed.

In this research I have made use of all three kinds of probing in an attempt to collect as much information about the subjects' cognitive processing as possible.

The nature of this research effort -- to illuminate some cognitive difficulties that standard six and seven pupils at a typical Cape Flats school experience with the generalized number pattern approach to algebra -- required the interviews to focus on the cognitive processes of the pupils from the time of their first glance at a pattern to their eventual transition or non-transition to an algebraic description of the pattern. This demanded a semi-structured approach to the interviews. On the one hand it had to be structured in order to cover the four main questions evident in most exercises on number patterns (Booth, 1989), viz.:

(a) finding or "seeing" the pattern;

(b) describing the pattern in words;

(c) recording the pattern rule using appropriate mathematical symbolism; and

(d) using the pattern to solve problems more efficiently.

These four questions, in that order, are suggestive of four developmental stages that the pupil has to pass through *en route* to an algebraic conceptualization of a pattern. Booth (1989) subsequently suggested that the learner should be exposed to all four questions in that order; while Pegg and Redden (1990) sufficed with only the first three (see the proposed teaching methodology discussed in section 2.4.2). On the other hand, the interviews should not be so rigidly structured that it resembles a verbal questionnaire; it should be fixed with respect to the core questions to be asked, but remain flexible enough to allow for creativity on the part of the interviewee and in depth probing on the part of the interviewer where necessary. Civil (1995) wrote:

> It is through cognitive conflict and by looking into students' "errors" that we are most likely to learn about their thinking. Interviewing in mathematics should enable teachers to prod students' ideas and walk into murky areas. Shying away from these areas may lead to lost

opportunities for learning for both teacher and students (p.157).

According to Goldin, De Bellis, De Windt, King, Passantino and Zang (1993) questioning can be broken up into the following four stages:

(i) Posing the question (free problem solving);

(ii) Heuristic suggestions (if not spontaneously evident), e.g., "Can you show me by using some of these materials?";

(iii) Guided heuristic suggestions, e.g., "Do you see a pattern in the numbers?"; and

(vi) Exploratory (metacognitive) questions, e.g., "Do you think you could explain how you thought about the problem?".

In order generate as much qualitative data as possible with each of the core questions I decided that, where necessary, I would break the core questions up into some of the stages as suggested by Golden, et al. (1993). Where further explication was necessary, I would supplement my questioning with deeper probing to elicit the finer details of the subject's cognitive processing.

A useful strategy for the interviewer is to wait for a complete, coherent, verbal explanation; and, if applicable, a complete, external representation before proceeding with the next question (Golden, et al., 1993). This exercise in patience proved to be very difficult for me as a novice interviewer -- there was often the temptation to interrupt the interviewee's explanation to clarify something that, at that stage, seemed unclear or ambiguous. Although clarification is important and prompts to effect clarification is necessary, I soon realized that it should rather be left until after the interviewee has finished his or her explanation. An untimely interruption may alter the original line of thought; that is, if it does not result in a complete loss of the original thought. Ericsson and Simon (1980) wrote, "Inaccurate reports ... are shown to result from requesting information that was never directly heeded, thus forcing subjects to infer rather than remember their mental processes" (p.215).

Sometimes it took great effort to restrain myself from switching to the tutorial mode of interviewing. This can of course be ascribed to my background as an educator. Piaget (1960) commented on the same difficulty when he wrote:

> It is hard not to talk too much when questioning a child, especially for a pedagogue. It is so hard not to be suggestive (p.20).

However, I did my best not to talk too much and to stick to the guidelines as discussed above.

The subjects that were chosen to be interviewed had to give their consent first and no-one was interviewed against his or her will. A small audio tape was used to record the interviews and the subjects were asked for their consent before they were taped. They were guaranteed of the confidentiality of the interviews. The times of the interviews were scheduled to suit the subjects so as to minimize any interruptions their personal schedules. An quiet, empty classroom was used as the venue of the interviews, to provide the subjects with the comfort of familiar territory and to minimize any tensions that may arise from unfamiliar territory. During the interviews the interviewer was seated at a right angle to the subject, across the corner of a table, in order have a clear view of all the subject's writings, calculator work, manipulations of the match sticks in the pattern, and other activities. All non-verbal activities were recorded manually in a set of field notes.

3.2 The Research Plan

This thesis is the result of a developmental research effort that went through three phases:

- The *first phase* was regarded as a pilot study aimed at getting a cursory overview of the pupils' cognitive reactions to the main questions on number patterns, as described by Booth (1989), when presented with these in the form of a worksheet. The focus of attention was the reactions of the standard six group, because this is the standard where symbolic algebra is first introduced to pupils in South African schools.
- The *second phase* widened up to include both the standard six and seven groups and involved an in depth study of these pupils' cognitive processing when they deal with questions about a number pattern generated from match sticks.
- The *third phase* also included both the standard six and seven groups and focused on the cognitive processing involved in coming to grips with a number pattern arising from a function table.

Through this study I hoped to gain some understanding of the cognitive difficulties that the standard six and seven pupils experience with the generalized number pattern approach to algebra.

3.3. Phase One: The Worksheet

In March 1996 worksheets based on problems leading to the recognition of number patterns and the derivation of generalized rules were prepared and distributed to all of the mathematics students at the Bellville South Senior Secondary School, a school situated in the city of Bellville. The difficulty level of the problems varied across standards so that higher standards received more difficult problems than lower standards. No distinction was made between higher grade and standard grade students within the same standard group -- all of them received the same worksheet. For examples of the worksheets given to the pupils, see appendix I(a)-(e). The worksheets were distributed by the mathematics teachers who were also given the responsibility of collecting them again after one week. Teachers were asked to inform the students that they could use whatever resources they had at their disposal, including their peers, but that they should not consult their teachers or other experts for help. The teachers agreed not to assist the students in any way with the solutions. Teachers were also asked to inform the students that some of them might be interviewed by myself afterwards.

Not all the worksheets were handed out on the same day as some of the teachers do not have contact with their classes every day of the week and others forgot to hand out the worksheets and had to be reminded. Neither had all the worksheets been returned by the time I went by to collect them and I had to go back several times in an effort to retrieve as many of the worksheets as possible. Many students asked for an extension of time either because they have not completed, wanted to review their methods, or have not had time to look at worksheet at all. Extension was granted, because the exercise was extra-curricular and I did not want it to interfere too much with the normal curricular activities of the school and, besides, I did not want to antagonize the students whom I still needed for further research. Extending the time would also enable to collect more of the worksheets that would otherwise be lost. Nevertheless, many of the worksheets were never returned and the students made excuses like: "I've lost mine" or "I didn't know what to do because we were never taught these sums". However, the unrecovered worksheets was no major obstacle as this was only a pilot study to get a general overview of how pupils reacted to a number pattern problem. The actual study would focus on the qualitative data from a few interviews rather than on the quantitative data from a large number of worksheets.

Symbolic algebra in the tradition of x and y is formally introduced to South African pupils in standard six. That is why, with the worksheets, the attempts of the standard six group were given more attention than the others. Apart from making an analysis of the percentage of correct responses to each of the questions in the worksheet, three standard six pupils were randomly selected to be interviewed. The selection was made from amongst those who handed in their worksheets. In the selection of the interviewees I had to ensure that their participation would be completely voluntarily. The whole group was therefore asked for their participation. The purpose of and procedure for the interviews was explained. Then volunteers were requested. The time and place of the interviews were arranged with the interviewees at their own convenience -- well in advance of the actual interviews. During the interviews of the first phase the pupils were supplied with a pencil, a sheet of paper and a calculator (which was placed within reach); but no match sticks were supplied. The interviews were intended to elicit some of the cognitive processes and difficulties involved in the pupils' attempts to answer the questions posed in the worksheet. As it turned out, valuable inferences could be drawn from their responses.

3.4 Phase Two: The Match Stick Pattern

This follow-up study was also conducted amongst the standard six and seven pupils of the Bellville South Senior Secondary School in July 1996. Three standard six and three standard seven pupils were selected at random from amongst a number of volunteers to be interviewed. I hoped that a critical reflection on my experiences from the first phase would enable me to improve my research methodology for this second phase. This time, instead of giving the pupils a worksheet to complete before they were interviewed, they were interviewed as they attempted to solve a problem on a novel match stick pattern (Appendix III). The core questions contained in the worksheet of the first phase were also modified slightly in accordance with my experiences from the first set of interviews.

3.4.1 The Stimulus Item One

The stimulus item used as a number pattern generator during this phase of the investigation (Stimulus Item One) was a match stick pattern adapted from a GEC Examination item discussed by Schäfer (1996). Adopting the terminology of the original item, I called the individual match stick constructs in the row "patterns" instead of "figures". Reflecting on my use of terminology afterwards, I felt that it might have led to an ambiguity in the understanding of the concept of "pattern". One interpretation might be that "pattern" refers to an isolated match stick figure from the whole sequence of match stick figures. This is the sense in which it was intended to be interpreted for the purpose of this study. An alternative interpretation might be that "pattern" refers to the whole of the sequence itself. By the time I discovered the ambiguity the interviews had already been conducted and there was nothing that I could do to change the terminology. In order to distinguish between the different uses of the notion inverted commas will henceforth be used to indicate the difference. That is, "pattern" will be used to indicate an individual construct from the sequence, and pattern (without the inverted commas) the whole of sequence itself.

Stimulus Item One was presented to the pupils as a physical model of a match stick pattern, consisting out of the first three individual match stick figures making up the pattern. These "patterns" were arranged with the first one right on top, the second one underneath it, the third one underneath the second one, and so on. In addition the pupils were given a pictorial version of the pattern on a sheet containing the drawings of the first three "patterns", each numbered according to their position in the sequence, e.g., "pattern 1", "pattern 2", "pattern 3", etc. See appendix III for a sample. Spaces were left for "pattern 4" and "pattern 5" with the intention of asking the pupils to draw them in after they had build them physically. I hoped that this double exposure to the pattern would increase their chances of "seeing" the pattern.

3.4.2 The Prepared Interview Questions

As before, a set of core questions for the interview had been prepared. These questions were intended to:

(a) serve as a set of heuristic guidelines to help the pupils discover a general rule for determining the number of matches in any one of the "patterns" in the row;

(b) stimulate the formulation of a more succinct expression of the general rule by using the notation of symbolic algebra; and

(c) elucidate some aspects of the pupils' cognitive processing as they moved along the axis from the "seeing" of the pattern to the writing of a rule.

The interviewees were not given a copy of the questions -- these were kept on a separate sheet of paper for the attention of the interviewer only. These prepared questions would form the core around which other questions and prompts would be structured to elucidate the finer details of the pupils' cognitive processing as they dealt with it. The core questions were:

1. Can you build the fourth "pattern"?
2. How many match sticks do you think you will need for the fifth "pattern"?
3. How many match sticks would you need for the 80th "pattern"?
4. Describe, in your own words, a rule that you can use to find the number of match sticks in any one of the "patterns".
5. Try to write your rule by using the symbolic notation of mathematics.
6. Which of the following symbolic rules can be used to find the number of match sticks in any of the "patterns"?

   (i) $5 + 4(n - 1)$

   (ii) $1 + 4n$

   (iii) $6n - 1$

   (iv) $1 + 3n + n$

3.4.3 The Interview Questions Revisited

The core questions for the interviews of the second phase were structured differently from the questions in the worksheet of the first phase in a number of ways. A reflection on the responses to the questions posed in the worksheet and during the interviews about the worksheet revealed a number of weaknesses. What follows is a brief discussion of what I perceived to be weaknesses in the questioning and how I tried to improve on them:

- The table completion exercise was left out because it appeared to cue pupils towards what Booth (1989) calls a "building on from the term before approach" (p.13). This approach often prevents pupils from discovering a functional relationship which connects the position of the 'pattern' in the sequence with the number of matches out of which it consists.
- The first question that was to be asked, "Can you build the 4th 'pattern'?", was a deliberate suggestion to physically build the "pattern" with match sticks. With the take-home worksheet it was practically impossible to supply all the pupils with match sticks. Now it was possible to have match sticks available during the interviews and a box of match sticks was always available and well within sight and reach of the interviewee; together with the other aids like pencils, paper and a calculator. According to Reynolds and Wheatley (1994) pupils use these materials not only to symbolize and explain the patterns and relationships they had constructed, but also reflexively to elaborate on their schemes. Booth (1989) claimed that some students may find the process of pattern-finding and recording easier if the situations embodying the pattern are more concrete and obvious. For this reason physical constructions of the first three match stick patterns were build and shown to the interviewees who were then encouraged to carry on by building the 4th and 5th "patterns".
- In the second question, the 5th "pattern" that was asked for follows directly after the 4th "pattern"; which the pupil

presumably would have built by the time the question is asked. This is unlike question 2(a) of the worksheet where the 17^{th} figure do not follow directly after the 6^{th} figure, which is the last one in the table. The intention was to make it easier for the student, in the absence of a table, to notice that four match sticks have to be added every time to make up the next "pattern" in the sequence.

- The third question corresponded with question 2(b) in the worksheet in the sense that it will also lead to a long and tedious search to exhaustion in the absence a workable rule. This type of question is intended to encourage the formulation of a generalized rule for computing the number of match sticks in any one "pattern" in the row. At first the question was about the number of matches in the 80^{th} "pattern", but this was subsequently changed to the 83^{rd} "pattern". The reason for the change was that some pupils who were interviewed, once they had found the number of matches in a factor of 80, would tend to over-generalize. That is, they would simply assume that by adding up the number of matches in the correct number of factors they could find the number of matches in the 80^{th} "pattern". For example, if there are 41 matches in the 10^{th} "pattern" (10 being a factor of 80), then the pupil would add up 41 eight times and conclude that the 80^{th} "pattern" consists of 328 matches. This over-generalization sets the pupils off on the wrong track and often precludes the discovery of a valid generalized rule. In an attempt to avoid this over-generalization, I changed from asking for the number of matches in the 80^{th} "pattern" to asking for the number of matches in the 83^{rd} one -- 83 being a prime number with only two factors: one and itself.

- The fourth question was intended to elicit a verbal description of a generalized rule for finding the number of match sticks in any one of the "patterns" -- presuming that such a rule had already been discovered. One of the more serious shortcomings of the worksheet of the first phase is that it neglects to ask, explicitly, for a verbal description of a generalized rule. By explicitly asking for such a verbal description I hoped to:
  (a) find out more about the pupil's own perception of the pattern; and
  (b) assist the pupil in the formulation of a symbolic version of the rule which I hoped would come easier after a verbal description had been formulated.
- The fifth question asked the pupil to make use of the symbolic notation of mathematics to formulate a symbolic version of the generalized rule. Evidence of the different interpretations of the meaning of the *n*-th figure in the worksheet (see the discussion of question 3) prompted a reformulation of this question. This time any reference to the *n*-th figure was left out of the question. I also felt that asking the pupil to "apply this formula to the *n*-th figure" would restrict him or her to use *n* as a symbol for the variable -- leaving them no freedom of choice as to what they would like to use as a symbol to represent the variable.
- The question prompting the application of the symbolic rule to obtain the position of the figure in the row when the number of matches out of which it consists is given (question 4 in the worksheet) was left out. There are various reasons for omitting this question. Firstly, it would require of the interviewee to formulate an algebraic equation, make a

substitution, and solve the equation for the unknown. The first requirement is what the fifth question asked for -- formulating an algebraic equation -- so there was no need to ask for it again. The second requirement involves the substitution of a numerical value into an algebraic equation where the dependent variable had become the independent variable and vice versa. That is, whereas the number of match sticks was always the dependent variable and the position of the "pattern" in the sequence the independent variable, in this question they switch roles. It might confuse the pupil if he or she is expected to cope with the role switching at a too early stage of concept formation. Besides, this issue was never intended to be part of this investigation, but should be a worthwhile research question to explore. A separate question was designed to investigate the third requirement -- the ability to algebraically manipulate an expression. This involved posing the pupil with the task of finding the correct symbolic representation from amongst a number of alternatives.

- A question that focused on the ability to identify different ways of representing the pattern symbolically was added to the list. This question would be presented to the interviewees as a number of alternative symbolic representations for the generalized rule on a sheet of paper. They would then be asked to verify which of the representations are valid and which are not. One of the symbolic alternatives given was deliberately designed to work only for the first "pattern" in the sequence. The idea behind this was to find out whether the pupils actually verified their algebraic representations by checking against terms other than the first one in the sequence.

3.5 Phase Three: The Function Table

The third phase followed soon after the second one. In August 1996 three standard six and three standard seven pupils, also of the Bellville South Senior Secondary School, were interviewed. As before, the interviewees were selected from a group of volunteers; a set of core questions were prepared in advance; appropriate times and venues for the interviews were arranged, etc. The major difference between the third phase and the second one was the interview problem.

3.5.1 The Stimulus Item Two

For the third phase of the investigation the number pattern generator was a function table (Stimulus Item Two). See appendix V. The pupils were told that it (the function table) works like a computer that uses the numbers in the "top row" to generate the numbers in the "bottom row". Note that "top row" and "bottom row" were used instead of x and y to indicate the independent and the dependent variable respectively. The reason for this use of terminology was that I did not want to impose my own choice of symbols on the pupils. This is in line with Reynolds and Wheatley's (1994) suggestion that pupils should be encouraged to express mathematical relationships in their own meaningful ways rather than having symbolizations imposed on them. By inventing their own symbols to represent the quantities the pupils:

1. would see that there is no pre-set conventions guiding the choice of "letters" to represent the variable, and
2. that their symbolic version of the rule would have that personal touch, which would give it that extra significance.

This should contribute towards the meaning-making process in Algebra.

3.5.2 The Prepared Interview Questions

The structure of the core questions for the interviews of this third phase of the investigation was in essence the same as that used for the second phase. The core questions that would be asked were:

1. Can you fill in the "bottom row" values for 5 and 6 in the "top row"?
2. What do you think will be the value in the "bottom row" if you have 80 in the "top row"?
3. Describe, in your own words, a rule that can be used to find the "bottom number" if the "top number" is known.
4. Try to write down your rule by using the symbolic notation of mathematics.
5. Which of the following symbolic rules could be used to find the "bottom number" from the "top number"?

   (i) $n + (2n - 1)$

   (ii) $3n - 1$

   (iii) $2 + 3(n - 1)$

   (iv) $4n - 2$

These core questions were supplemented with additional questions of clarification and probes as discussed above.

As before the interviews were audio-taped with the consent of the interviewees and supplementary field notes were taken to record non-verbal actions.

3.6 Summary

In this chapter an account is given of how the clinical interview was used as the dominant research methodology to collect data about the cognitive difficulties that pupils experience when they have to generalize from number patterns. The types of questions that were used during the interviews were discussed in detail; as well as how the questioning had improved as a result of the continuous reflection on results they produced. Three distinct phases in the research were identified and discussed. The first phase involved take-home worksheets for all the standards. These all the pupils in the entire Mathematics Department were required to complete. Interviews were afterwards conducted with three pupils from the standard six group. For second and third phases the stimulus items were a physically constructed match stick pattern and a function table respectively. Interviews were conducted with three standard six and three standard seven pupils as they attempted to solve problems based on the stimulus items in each of the second and the third phases. Full lists of the core questions for the interviews of the second and third phases have been supplied

In chapter four the data collected during the interviews will be discussed. Special emphasis will be placed on the cognitive difficulties that standard six and seven pupils experience when they have to generalize algebraically from a number pattern.

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CHAPTER 4

4. RESULTS AND DISCUSSION

4.1 Phase One: The Worksheet

4.1.1 The problem of retrieving the worksheets

Despite of a number of extensions of the deadline in order to allow as many pupils as possible to hand in their worksheets, not all the worksheets were returned. Table 1.1 gives an indication of the percentage of worksheets returned per standard group.

***Table 1.1* : Percentage of worksheets returned per standard group**

Standard	Number of pupils	Number of worksheets returned	Percentage of worksheets returned
6	58	48	82,8
7	155	89	57,4
8	59	36	61,0
9	41	20	48,8
10	28	22	78,6

As table 1.1 shows, there was a significant variation in the percentage return of the worksheets across the standard groups. The lowest percentage return (less than half of the worksheets) was in standard nine. A deeper investigation into the underlying reasons for the variation in percentage return across the standard groups might render valuable insights into, for example, the pupils' changing attitude towards mathematics assignments across the different standards. Such an investigation, however, falls outside of the scope of this particular research effort.

4.1.2 Some details of the standard six group

Some of the details of the group of standard six pupils are provided in table 1.2 as a gauge of how representative they might be of the typical standard six group at any Cape Flats school.

***__Table 1.2__* : Details of the standard six group (n = 51)**

	Passed maths in standard 5	First year in standard 6	Repeating standard 6	Want to do maths till standard 10
Boys	28	22	7	14
Girls	22	18	4	16
Total	50	40	11	30

By the time the data for table 1.2 was collected, in September 1996, seven pupils in standard six had already dropped out of school (cf. table 1.1). A high drop-out rate, especially in standard six and seven, is not uncommon in schools on the Cape Flats.

4.1.3 Discussion of the questions in the worksheet

For a copy of the standard six worksheet, see appendix I(a). Table 1.3 provides a summary of how the standard six pupils performed in each of the questions of the worksheet. Only the worksheets that were retrieved were used in the calculation of the percentages. In this case 48 worksheets were retrieved out of the 58 that were handed out (see table 1.1).

<u>Table 1.3</u> : Summary of the standard six group's performance on the questions of the worksheet (n = 48)

	Question 1	Question 2(a)	Question 2(b)	Question 3	Question 4
Correct	97,9%	66,7%	33,3%	20,8%	52,1%
Wrong	2,1%	31,2%	64,6%	72,9%	43,7%
Not answered	0%	2,1%	2,1%	6,3%	4,2%

<u>Question 1</u> seems to have posed no particular problems. All of the respondents filled in the table of which 97,9% were able to provide the correct answers. This question could easily be solved by spotting a recurrence rule, i.e., that by adding on three matches to number of matches in the last figure one would get the number of matches in the following one.

<u>Question 2(a)</u> was slightly more difficult; it was solved by only 66,7% of the respondents. This question could be solved by continuing the recurrence rule of adding on three every time till the seventeenth figure. However, doing so would require one to keep careful track of one's progress through the sequence of figures up to the seventeenth one. This already strongly suggests a functional relationship between the position of the figure in the sequence and the number of matches out of which it consists.

<u>Question 2(b)</u> can also be solved by continuing the recurrence rule of adding on three every time till the fifty-seventh figure, but the process is much more exhaustive and even more suggestive of the need for a generalized method. When compared to question 4, for which more than half of the pupils had found the correct solution, it comes as a surprise that only a third of the pupils were able to find the solution of question 2(b). And 2(b) is supposed to be an easier question! Perhaps when one is using the "adding on" strategy, the task of finding the position of the figure made up of 98 matches(32nd figure) is easier than finding the number of matches in the 57th figure.

Question 3 proved to be the most difficult one, with only 20,8% of the respondents being able to solve it. This question is in fact where the link between number patterns and symbolic algebra must be made. If number patterns is to be considered as a meaningful alternative for the traditional mode of introducing algebra via exercises in the manipulation of meaningless symbols, then the problems surrounding this question needs to be thoroughly investigated and addressed.

Question 4 was solved by 52,1% of the respondents, but almost all of them responded by writing only the answer, without any calculation or explanation to substantiate it. This made it very difficult, if not impossible, to make any inferences about the methods that they used to find the answer. Perhaps this is just another manifestation of the pupils' fixation with earlier arithmetic frameworks where a single-term answer is the thing to go for. I hoped that the interviews would provide me with more information about their reactions to this question.

4.1.4 Discussion of the interviews concerning the worksheet

The worksheet was subsequently followed up by conducting three interviews with three pupils that were selected at random from amongst the group of interview volunteers in standard six. The purpose of the interviews was to gather more data about the pupils strategies and the difficulties that they have experienced with the questions in the worksheet. The following volunteers were interviewed:

(a) Gerald: 13 years old, male, and a first language speaker;

(b) Tracy: 13 years old, female, and a first language speaker; and

(c) Sello: 13 years old, male, and a second language speaker.

Note that these are not the pupils' true names. The names had been changed to conserve the confidentiality agreed upon prior to the interviews.

First language speakers, as they are referred to here, are pupils who are taught and were interviewed in their mother tongue, which is English. Second language speakers do not have English as their mother tongue, for example, Sello who speaks Xhosa at home. At school he is taught in English and he was also interviewed in English.

In the transcripts the following abbreviations and conventions will be used:

I for interviewer;

G for Gerald (the first letter of the pupil's name); and

... to indicate a pause of more than two seconds.

For a full transcript of the interview with Sello, see appendix II. This transcript was selected as an appendix because the interviewee is a second language speaker and one is thus tempted to anticipate communication problems. I thought the reader would find it interesting to see how a second language speaker copes with an interview of this kind. The transcript also reveals an interesting interpretation of the meaning of the letter n as it is used in the n-th figure.

In this discussion of the interview responses I will focus on the questions one-by-one in the order in which they appear in the worksheet.

Question 1 required of the pupil to fill in the given table by writing in the numbers of matches for the 4^{th}, 5^{th} and 6^{th} figure in the sequence. The high percentage of correct answers(97,9%) indicates that this question did not pose much difficulties. What the high percentage of correct answers does not show, however, is whether the intended development of a functional relationship between the position of the figure in the sequence and the number of matches out of which it consists is indeed achieved. The interviews that were subsequently conducted seem to provide evidence to the contrary. That is, instead of developing the intended functional relationship, pupils are likely to develop their own strategies based on the way they perceive the pattern.

Amongst the sample of three pupils that were interviewed, one of them described how he filled in the table by "adding three every time" to find the next number. This pupil, Gerald (13 years old; std.6), described his method as follows:

1) I: What did you have in mind with number one? I see you have drawn four figures.

2) G: Every time <u>I add three</u> sir... <u>It is only the bottom three that actually counts</u>... so... so I just counted them all up sir... I figured it out... <u>I was drawing everything so I said this is too difficult</u>... so I'd rather use the calculator to plus every time... that's how I got 173 and 53.

3) I: You used a calculator for that...?

4) G: Yes sir.

5) I: Where did you start first of all?

6) G: By drawing it sir... by adding three every time.

...

11) I: Why didn't you go further than the fourth one?

12) G: I don't know sir I just looked here... then I saw four and I didn't know that I must do five and six sir. I just done until four there sir.

13) I: And then what happened after that?

14) G: <u>So every time I only added three... whole time sir... and then try to get to your answer</u>.

Gerald, apparently after having drawn the first four figures, discovered that by adding three matches to the bottom of a figure the next one could be made up. He says, "It's only the bottom three that actually counts..." (line 2). After having made this discovery, and realizing that drawing all the figures can be an exhaustive exercise, he abandoned the drawing of the figures altogether, "I was drawing everything so I said this is to difficult..." (line 2). Instead he opted for the strategy of adding on the difference of three between successive terms, "So every time I only added three... whole time sir... and then try to get to your answer" (line 14). At a later stage in the interview it was discovered that it took him more than four drawings to make his discovery [See the earlier discussion of question 2(a)]. Instead of discovering a functional relationship Gerald had discovered a recurrence rule that he was able to use successfully.

In a number of the retrieved worksheets, however, all six of the figures were drawn in full. One of the interviewees who did this, Sello (13 years old; std.6), explained why he drew them all:

1) I: I see you drew a number of figures for number one (*i.e., question 1*). What was your plans with that?

2) S: I did that to get the answer quickly.

3) I: Okay, and did it help you?

4) S: Yes it did help me... to get the answer.

5) I: How did you get the answer from the picture?

6) S: It says to me I must just do it... then I just do it... it tells me the answer to the... I forgot now... What is the answer?

7) I: You can have a look at your answer if it will help you.

(*Hands the pupil's worksheet back to him*)

8) S: Sir can you... I must...

(*Pupil reads out question 2(a) aloud*)

9) S: Sir again you must find way to work out how many matches you are going to need to make up the picture...

10) I: Okay... so you actually drew some of the figures I see... and then? From there... what did you do then?

11) S: I just drew the pictures... and I get the answers.

12) I: How did you get the answers from that?

13) S: I just drew the pictures and after I drew it... <u>it says I must draw the pictures to six than I do it</u>... from five to six than I do it... from five to six... then I get the answers.

14) I: <u>Did you count each of the matches</u>?

15) S: <u>Yes, yes</u>...

From the response: "... it says I must draw the pictures to six than I do it..." (line 13), it seems as if the pupil sees the drawings given in the worksheet as a suggestion that the other figures in the table should also be drawn. Although Sello did not spontaneously admit that he counted the matches in the drawn figures, he confirmed this after being directly asked about it (lines 14-15). His almost impatient response (line 15) shows that he must have considered the counting of the matches in the drawn figures so obvious, almost trivial, that it was not even worth mentioning. This phenomenon of drawing and counting is evident in the work of a number of other pupils who also drew all the figures in the row -- some till the seventeenth one. None were found to have drawn only the figure in question -- always including the whole row of preceding figures. I neglected to prompt them to find out why, e.g., "Why didn't you only draw the seventeenth figure?". Reflecting upon that now, I think that it could be explained in terms of the pupils' perception that one would have to know the number of matches in the sixteenth figure so that one can add three matches to it to form the seventeenth figure. But, likewise, the number of matches in the sixteenth figure depends upon the number of matches in the fifteenth figure, and so on. Once again the recurrence rule, linking the number of matches in a figure to its predecessor, was used; albeit disguised in the pupil's drawings.

Another interviewee, Tracy (13 years old; Std.6), on the other hand, completely ignored the diagrams and focused on the table only:

1) I: Perhaps you'd like to tell me what... how you did the first one.

2) T: The one over here(*points at question 1*)?

3) I: Uhm...

4) T: Just add three every time.

5) I: How did you know to do that?

6) T: Five... three... You must add three over there to get eight... and eight plus three to get eleven... eleven plus three to get fourteen... fourteen plus three to get seventeen.

7) I: Right... Uhm... When you did that, did you look at the pictures or did you look at the table?

8) T: At the table.

9) I: So you never had a look at the pictures actually?

10) T: No.

Both Tracy and Gerald used the same strategy -- a recurrence rule -- of adding on the difference between successive terms. Note that Gerald developed his strategy from an analysis of the drawings of the figures; whereas Tracy developed hers from an analysis of the numbers in the table.

Differencing might be a viable strategy for finding the number of matches in the first few figures in the sequence, but it becomes a tedious and exhaustive process for figures further down the sequence. And the more exhaustive the process becomes, the more prone to mistakes it becomes. Once the pupils had discovered a recurrence rule that works, they do not search for any alternative rule that might be easier to apply and that are less likely to result in errors. In this way the discovery of a functional relationship is precluded even before the search has started.

Question 2(a) required of the pupils to find the number of matches in the seventeenth figure in the sequence. As before, the pupils invented their own strategies. In the three interviews at least two different strategies were evident:

(1) Draw all the figures from the first one right up till the seventeenth one and then count the number of matches in the seventeenth one.

The sequence of figures up till the seventeenth one, although tedious to draw and count, was still within the tolerance level of most pupils. Gerald (13 years old; Std.6) explained his use of the strategy as follows:

15) I: And how did you get the seventeenth one... using a calculator?

(*The interviewer's mentioning of a calculator should not be interpreted as a suggestion; rather as a verification of the pupil's earlier reference(line 3) to a calculator. Besides, this is a retrospective probe -- the task had been completed*)

16) G: No... I drawed it up.

17) I: You drew a figure?

18) G: I drew it... every time I added three...

(G. *scratches in school bag and takes out drawings on a piece of scratch paper.*)

19) I: Wow! What is this?

20) G: <u>There I got my seventeenth figure... I drawed it up sir. I started to count them... and that's how I figured out that... every time I must add three</u>... and that is why I started using a calculator.

21) I: So you only went up to the seventeenth one and then you didn't go further than that?

22) G: Yes... and then I started using a calculator.

From this explanation it seems that the pupil had to draw all the figures and count each one's number of matches, till the seventeenth figure, before he realized that the number of matches in a figure can be found by simply adding three to the number of matches in the preceding one (line 20). This comes as a surprise, because, judging from the drawings of the four figures in the worksheet and his earlier responses (see the earlier discussion of question 1, lines: 2, 12 and 14), it appeared at first that he discovered his rule of differencing after having only drawn the first four figures in the row. As it turned out now, not all his written efforts were handed in with the worksheet. Some of it he kept aside, not even mentioning it until he needed it to substantiate an explanation.

As it turned out, Sello (13 years old; std.6) did the same; also keeping his drawings of the figures up till the seventeenth one aside and not handing it in with his worksheet. This is how I found out about it:

16) I: For number two they want the number of matches in the seventeenth figure... Now I don't see that you drew the seventeenth figure... How did you do it then?

17) S: I drew it in another book... in another book... I didn't think I must do it to this page.

18) I: Now can you tell me what you did in that book... because I can't see it.

19) S: I just did the pictures... then I get the answer... The answer was fifty-three.

The tendency of pupils to exclude some of their "workings" from presentation is an interesting phenomenon not exclusive to this particular worksheet experience. It is not uncommon to see pupils doing the bulk of their calculations in the classroom on scrap paper before copying a "watered-down" version of the solutions into their books -- sometimes excluding the gist of a solution itself. This phenomenon could be ascribed to a fixation in classrooms with elegant, textbook-like solutions.

(2) Counting on from the last figure in the table -- adding three every time -- up until the seventeenth figure.

Tracy (13 years old; std.6) explained her use of this strategy as follows:

11) I: Okay, fine... and number two?

12) T: Actually I tried it out by like just adding... three every time.

13) I: Can you explain more... what did you do?

14) T: I added three...

15) I: Three to what now?

16) T: Every time I just added three to every answer I get.

17) I: Where did you start?

18) T: At twenty, because I got my sixth one.

19) I: Okay...

20) T: And then I moved on until I got my seventeenth one.

This is an example of the use of "adding on" which, according to Orton and Orton (1994) is on its own inadequate for finding the universal rule and often sets the problem solvers off on the wrong track. By "adding on" every time the problem solver often misses the opportunity to discover a functional relationship between the number of matches in a figure and its position in the sequence.

From the responses in the worksheet another strategy was apparent, viz., the use of the arithmetic calculation 17 x 3 + 2 to find the answer. However, none of the interviewee volunteers had used this method, and an opportunity was lost to further investigate the cognitive processing beyond the response.

Question 2(b) required the pupil to find the number of matches in the fifty-seventh figure in the row. The strategy of "adding on" will work here, but requires an even longer and more tedious calculation than for the seventeenth figure. One third of the pupils managed to find the correct answer, presumably using the "adding on" strategy.

Tracy (13 years old; std.6) confirmed this presumption:

25) I: Okay that I understand (*referring to her explanation for 2(a)*)... and then 2(b)?

26) T: I practically did the same... it's just that!

<u>Question 3</u> required the statement of a general rule to work out the number of matches in any figure in terms of *n*. The 20,8% of the solutions that were correct were all from one particular class -- arising suspicion that not all of the pupils might have genuinely come up with the correct solution. The possibility that this group of pupils might have received help from a tutor or some other expert could not be ruled out. On the other hand, it is also quite possible that for some of them, if not for all, the solution might be their own. One pupil might have arrived at the answer and shared it with the others. The method of assessment by take-home assignment makes it difficult to tell.

There were four different types of responses to this question in the worksheet:

(1) A number of pupils expressed the "formula" as an arithmetic operation. For example, 26 + 3 = 29. At first I could not explain why they did this, but it soon became clear when I came across two written responses stating that: "The ninth figure in the row is equal to 29 matches". Then it dawned to me that the pupils thought that the *n*-th figure was the *ninth* figure. They obviously thought of the *n* as a first letter abbreviation for the word *nine*. None of them, however, used other values for *n* like nineteen; ninety; ninety-nine; nine-hundred; etc.

Another example of an arithmetic operation was:3 + 3 + 5 = 11. This appeared to be a crude attempt to explain, by means of the example of the third figure given in pictorial form in the worksheet, how the number of matches in a figure can be found by grouping the matches and then adding the groups.

None of the interviewee volunteers had responded in any one of these ways and again an opportunity was lost to further investigate the cognitive processing beyond the arithmetic operational responses.

(2) Some pupils expressed the "formula" as "Add three every time". This crude rule, which does not even specify to what the three should be added every time, is probably their attempt to describe recurrence rule used for the long and tedious process of adding three every time till, after the correct number of additions, the number of matches in a particular figure is found. Although the process of "adding three every time" was mentioned during the interviews, none of the interviewees actually wrote it down as a response to the question in the worksheet.

(3) Only 35,4% of the pupils attempted a symbolic "formula". Of the pupils who attempted written symbolic rule, 58,2% wrote "$n \times 3 + 2$"; the rest wrote "$n + 3$". These were the only written attempts of formulating a symbolic rule.

By writing "$n + 3$" the pupil reveals that he or she does not yet realize the meaning of the variable n and how it is related to the number of the figure. The variable n is interpreted as the number of matches in any one figure; in which case $n + 3$ must be the number of matches in the consecutive figure. This seems to be a direct translation of the rule to "add three every time" into symbolic form. What it reveals, however, is that the pupil does not yet perceive connection between the variable n and the position of the figure in the sequence. Instead, they have a rather episodic perception of the pattern, focusing on two figures at a time. That is, on obtaining the number of matches in any particular figure by adding three to the number of matches in the one immediately preceding it in the sequence.

The pupils who wrote *"n x 3 + 2"* appear to have succeeded in connecting the variable *n* to the position of the figure in the sequence, and have subsequently discovered a viable functional relationship between the number of matches in a figure and its position in the sequence. None of the interviewee volunteers were able to come up with one of these symbolic versions of a general rule. None of the pupils interviewed used this method.

(4) Some pupils viewed the *"n"* in the *n*-th figure as an initial letter abbreviation. These included Tracy (13 years old; std.6) who explained herself as follows:

31) I: Let's go to number three... Now there they say you must work out something that will give you the n-th figure... Do you know what the n-th figure is?

32) T: <u>The ninth figure</u>.

33) I: The ninth figure?

34) T: Yes.

35) I: Why you say ninth figure?

36) T: I think so!

37) I: But why?

38) T: Ain't it the ninth figure!... Sir for what does "n" stand?

39) I: Oh so you thought that "n" stands for nine?

40) T: Yes that's what I thought.

Tracy's written response to question 3 was 26 + 3, which she explained as follows:

46) T: I thought the sixth one was already twenty...

47) I: Right...

48) T: And then I started counting...

49) I: So the seventh one will be...?

50) T: Twenty-three... and then twenty-six.

So her written response makes perfect sense when it is explained in terms of her interpretation of the letter *n* as a first-letter abbreviation for the word *nine*.

Sello(13 years old; std.6), on the other hand, did not even attempt a written solution. He explained why:

21) I: And number three (*referring to question 3 in the worksheet*)...?

22) S: I didn't do it.

23) I: Do you understand the question?

24) S: Yes I do understand the question.

25) I: What does it say?

26) S: It says I must do the pictures to... to <u>northwards</u>... but I didn't do it.

27) I: To what?

28) S: To <u>northwards</u>.

29) I: What do you think is the meaning of this little part here that says "n-th"?

30) S: I didn't know.

31) I: What did you think it was?

32) S: I think it is to <u>north</u>.

...

39) I: Do you think that's possible?

40) S: No.

41) I: Why not?

42) S: <u>Because in maths you don't use north</u>.

Sello interpreted the letter *n* as a first-letter abbreviation for the word "*northwards*" (lines 26 and 28). He explained that he did not answer the question "Because in maths you don't use north" (line 42). (*There are probably as many interpretations of the meaning of the letter n as there are words starting with the letter n.*)

Question 4 required the pupil to work out the position of an unknown figure in the sequence; given the condition that it consists out of 98 matches. Most of the pupils who found the correct answer just wrote it down as 32, without any calculation or explanation to substantiate. Presumably they had extended the table until they reached 98 matches in the bottom row and then found that it corresponded with the 32^{nd} figure in the top row. Again this could not be verified in the interviews as none of the interviewee volunteers had attempted to answer the question.

Two pupils did the following calculations: $98 - 2 = 96$, followed by $96 \div 3 = 32$. They apparently understood that their initial arithmetic operations on the number of the figure, to obtain the number of matches out of which it consists, had to be reversed to obtain the number of the figure again. In fact, this indicates that they can actually "see" the pattern, but are unable to represent and manipulate it algebraically.

Two more pupils wrote: $n \times 3 + 2 = 98$, but did not attempt to solve the equation for *n*. These pupils were successful in identifying the relationship of equality and in setting up an algebraic equation; but they did not possess the algebraic manipulative skills to follow the calculation through and solve the equation for n.

Of the pupils who responded to the third question by writing the "formula" as $5 + 3 + 3 = 11$, most responded to the fourth question by writing: $98 - 5 \div 3 = 31$. This revealed a serious problem with the structure of the expression, i.e., that they do not know that certain arithmetical operations enjoy precedence over others and that the operations in an expression are not necessarily carried out from left to right. In their expression, for example, division should enjoy precedence over subtraction, and hence the order of the operations are not from left to right as they seem to suggest. The use of brackets would have been a valuable aid in this case.

4.2 Phase Two: The Match Stick Pattern

4.2.1 Some Details of the Interviewees

In the selection of interviewees for this phase, I tried to choose pupils from the higher, middle and lower mathematical ability range (judged by their performance in the June 1996 examination) in each standard group. This was done in an attempt to make the sample of interviewees as representative as possible; and, knowing very well that pupils' performance in tests and examinations might be influenced by factors other than their cognitive abilities; e.g., social problems at home, not having studied the work, sickness, etc. Also knowing that this brings into disrepute any assessment of cognitive abilities based upon a pupil's performance in a single examination and a few class tests. Nevertheless, this is how assessments have traditionally been made in schools; and, to be consistent with school standards, I have adopted it into my own research.

Three pupils from the standard six group and three pupils from the standard seven group were approached individually and asked to be interviewed. All of them agreed, after an explanation of the purpose of the interviews and methodology that would be followed; and, having been reassured about its confidentiality. Proper arrangements were then made for the interviews to be conducted at each interviewee's convenience. In full acknowledgment thereof that my categorization of the pupils into higher, middle, and lower ability groups is based on superficial criteria (as I have pointed out earlier), as well as for ethical reasons, ability grouping will not form part of the description of the interviewees.

The interviewees were:

(a) Andrea: 13 years old, female, first language speaker, std.6

(b) Edwin: 13 years old, male, second language speaker, std.6

(c) Nigel: 15 years old, male, second language speaker, std.6

(d) Bonnie: 14 years old, female, first language speaker, std.7

(e) Claud: 14 years old, male, first language speaker, std.7

(f) Ester: 15 years old, female, second language speaker, std.7

In order to honour my promise of confidentiality, the real names of the interviewees have been replaced with pseudonyms. Those indicated as first language speakers speak English at home and are taught in English at school (their first language). Second language speakers, as they are indicated here, speak Xhosa at home but are taught in English (which is their second language) at school.

4.2.2 Discussion of the Cognitive Difficulties with the Match Stick Pattern

Unlike with the analysis of the interview responses in the previous section (4.2.1), where the questions in the worksheet were used as a structural framework for the subsequent discussion, here I structured my analysis around certain cognitive difficulties observed during the interviews. The same interview conventions as explained in 4.2.1 will be used in the excerpts from the interviews.

(a) The pupils tend to calculate the number of matches in a given "pattern" from the value of its predecessor rather than to try to find a functional relationship between the number of matches in the "pattern" and its position in the sequence.

Note, for example, how Andrea (13 years old; std.6) responded to the question of how many matches she would need to build the fifth 'pattern':

3) I: Now how many matches from this bunch do you think you will need to build the fifth 'pattern'?

4) A: (*After a short pause during which she counted the number of matches in the fourth 'pattern'*) Twenty-one.

5) I: How did you get that?

6) A: I added the fourth 'pattern'... to the four that I had to add to it.

7) I: What four? Could you show me?

8) A: I need <u>four matches to make up another one</u> (*points at the four matches that she added to the third 'pattern' to make up the fourth one*).

Andrea's response that she needs "four matches to make up another one" (line 8) merely demonstrates the natural tendency of pupils to opt for "adding on" -- a strategy of using the number of matches in the preceding 'pattern' and adding the difference between two consecutive 'patterns' to it in order to calculate the number of matches in the 'pattern' in question.

Ester (14 years old; std.7) also used "adding on". When she was asked about the number of matches in the sixth 'pattern', after having just drawn the fifth 'pattern', she explained as follows:

25) I: How many match sticks do you think you need for the sixth 'pattern'?

26) E: (*Speaking very softly -- whispering almost*) Twenty-five.

27) I: How many?

28) E: Twenty-five(*more affirmative this time*).

29) I: Can you explain how you got to twenty-five?

30) P: Because I counted here (*indicates how she counted the match sticks in the fifth pattern*)... and then I need four... if I want to make up a 'pattern' of six... I need four... then I get twenty-five.

In both these examples the correct calculation of the number of matches needed for a given 'pattern' by "adding on" depended upon knowledge of the number of matches in the preceding 'pattern'. The students had to go back to count the number of matches in the preceding 'pattern' and then added the difference of four to it. In no instance was it found that the pupils spontaneously counted the number of matches in each one of the preceding 'patterns' in an attempt to establish a number pattern. Counting only started after they were asked for the number of matches in a particular 'pattern' and then it was restricted to the counting of the number of matches in the preceding 'pattern' only. The pupils appear to have an episodic perception of the pattern, focusing only on two 'patterns' at a time -- the one in question and its predecessor. This episodic perception prevents them form finding a functional relationship between the number of matches in a 'pattern' and its position in the sequence.

(b) Pupils tend to "over-generalize" from the known to the unknown.

Nigel (15 years old; std.6) explained his method of finding the number of matches in the eightieth 'pattern' as follows:

171) N: Okay, for instance if I try like this... This is now ten, hey sir (*referring to the tenth figure*)... All right, all right, all right... <u>For ten I use forty-one</u>, right sir...

172) I: Uhm...

173) N: Okay I want to do it <u>my way</u> sir...

174) I: You just do it your way.

175) N: All right, I will do it <u>my way</u> now sir (*Writes an addition sum consisting out of ten forty-one's underneath each other*)... I'm going to do it <u>the long way</u> now. <u>I want not to confuse myself</u>.

176) I: How many of those groups must you have? Remember we want the eightieth 'pattern'...

177) N: Okay, actually sir... I try it this way now. I'm using the tenth 'pattern' now sir. Okay, I'll try to do it <u>my way</u> to get the eightieth 'pattern'.

178) I: So you're counting ten forty-one's now... Why then?

179) N: <u>It have to be eight. Sorry sir. I mean ten times eight gives you eighty</u>, right sir...?

180) I: So how many do you get?

181) N: (*Adds up*)... Eighty-eight.

182) I: Show me again how you did that.

183) N: (*Adds again*)... Actually I want to be dead sure of my answer sir... (*Adds again*)... My answer is 328.

Nigel used his knowledge that the tenth 'pattern' consists out of forty-one matches (line 172) in the generalization that eight similar 'patterns' would have the same number of matches as the eightieth 'pattern' (line 179). This led him to add together ten forty-ones yielding 328. He explained that he preferred to use "the long way" (repeated addition) instead of multiplication because he felt that it was less likely to confuse him (line 175). He probably felt more comfortable with addition and repeatedly spoke about it as "my way" (lines 173, 175 and 177). The use of repeated addition as an alternative for multiplication was so common that I chose to make it a separate point.

(c) Pupils prefer repeated addition as an alternative for multiplication in their calculations.

Ester (15 years old; std.7) when asked about the number of matches in the twenty-seventh 'pattern', explained her method as follows:

41) I: How many match sticks do you need for the twenty-seventh one?

(*She reaches for a calculator and starts to add.*)

42) I: What are you doing?

43) E: I'm <u>telling</u> (*counting?*) them sir... I'm busy calculating this... No sir I got it... In the one match stick... you have five né ... and if you need another one... another two pattern... <u>I can't explain this</u> (*softly*)... You take another four match sticks né... and then you put it there... Okay... so you have to say <u>five plus four</u>... <u>plus four</u>... <u>plus four</u> ... How much did you say sir?

44) I: Twenty-seven.

(*She uses a calculator to do the successive additions.*)

45) E: One-oh-nine... One-oh-nine match sticks...

It should be remembered that Ester is a second language speaker who often finds difficulty in expressing herself in English. Note her use of the word "telling" (line 43) an adaptation of the Afrikaans "om te tel" instead of the English word "counting". After trouble with the first attempt to explain herself she says, "I can't explain this..." (line 43), but then her use of successive additions of four becomes apparent when she says, "... five plus four... plus four... plus four..." (line 43). Using the method of successive additions, without the aid of a calculator, could be a long and tedious process, prone to mistakes. Even more so if the 'pattern' in question is further down the row, e.g., the eighty-third one. Ester used the calculator to good effect, obtaining the correct solution of 109. By using multiplication, instead of successive additions, the problem could be simplified to the solving of the arithmetic expression: $5 + 4 \times 26$, which is also a forerunner of the symbolic version of the generalized rule, viz., $5 + 4n$. I wish to contend that the formulation of a symbolic version of a generalized rule are prohibited by the sustained use of successive additions instead of multiplication. That is because the operation of multiplication is easier to represent algebraically than the operation of successive additions.

(d) Some pupils' strategy is to search for and operate on the first two discernible numerals by either multiplying them or dividing the one into the other.

When Andrea (13 years old; std.6) was asked about the number of matches in the eightieth 'pattern' she explained her method as follows:

14) I: Now for the eightieth pattern; how many matches do you need?

15) A: I divide the ... I divide... I multiply...

16) I: Wow, you're going too fast here.

17) A: I multiply eighty and five... four!. Eighty and four.

18) I: Why four?

19) A: Because... (*Mumbles something inaudible*).

20) I: Now I'm not sure if I know what you are saying; perhaps you should do that to show me.

21) A: Uhm... Like say there is eighty of this... the eightieth pattern. Uhm... then there will be eighty...

22) I: Eighty of what?

23) A: Eighty little squares.

24) I: Eighty little squares, yes.

25) A: Now, uhm... If you... If you multiply eighty with these four that you need to add on all the time... then... (*She reaches for the calculator*)

26) I: Now what are you doing now?

27) A: I'm dividing... (*Mumbles something inaudible as she operates with the calculator*)

28) I: Excuse me...

29) A: I'm dividing forty into eighty.

30) I: Forty?

31) A: I mean... Oh yes! Okay... Okay... No... I must multiply... I must multiply...

32) I: Now you have me mixed up.

33) A: I must multiply this, because... I must multiply eighty... four by eighty...

34) I: Four by eighty.

35) A: Uhm, because you need four every time to build... to build on to make up the eightieth block. So... so... so that is it.

(*Fiddles with the calculator.*)

36) A: Four goes into eighty (*Does the calculation on the calculator*) Eighty times four is three-hundred-and-twenty.

Note how Andrea searched for a suitable arithmetic operation first; limiting herself to a choice between division and multiplication (line 15). She started of by considering division, pondered it for a while, and then decided that multiplication was a more plausible choice. As it turned out, she later reverted back to division (lines 27-29), indicating that there was still a certain degree of uncertainty about her choice of operation. But eventually multiplication got the upper hand (lines 31-33). In line 33 she was on the verge of explaining why, but then she abandoned the explanation to focus on the numerals on which to operate. I realize now that I should have probed her for the explanation. Her first choice of eighty is an obvious one - after all "the eightieth pattern" (line 21) is what the question is all about. There are "eighty little squares" (line 23). As a second choice she first considered five; then discarded it in favor of four (line 17). She explained her second choice as follows: "... you multiply eighty with these four that you need to add on all the time..."(line 25). This strategy is probably the result of: (i) earlier learned frames from arithmetic where a good strategy normally involves the selection of the correct one of the four arithmetic operations (i.e., + , - , x or ÷), together with two numbers on which to apply it; and (ii) a recognition that the pattern is expanded by adding four matches every time -- but not taking into account that the first "block" consists out of five matches.

(e) Pupils find it very difficult to express, in a natural language, the generalized rule. To compensate they would use a specific example to show how the rule should be applied.

Note in the following example how Ester (15 years old; std.7), when asked to write down a general rule, resorts to the example of the twenty-seventh 'pattern' discussed just before that to explain herself.

58) I: Okay... Can you write down a general rule?

59) E: A general rule?

60) I: Yes... Write down the method so that you can use it for any pattern.

She then wrote: "If I want to get up to 27 patter (*sic*) I say 5 match stick (*sic*) plus four plus anather (*sic*) four up antill (*sic*) I get to 27 anthen (*sic*) I get my answer". Note how she too used repeated addition in preference to multiplication. One would have expected Ester to come up with a general rule description like: Take five and keep on adding fours to it until you have added one less fours than the number of the 'pattern'. Yet she chose to explain her rule by means of the example of the twenty-seventh 'pattern'.

When Bonnie (14 years old; std.7) was asked for a general rule, this is how she responded:

56) I: Okay... Now can you describe, perhaps, the general rule that you used?

57) B: I used the multiplication rule...

58) I: Uhm...

59) I: But what is the method that you used to find the number of matches?

60) B: Eighty-three times five...

61) I: Yes... And if I said... for instance... the twenty-seventh 'pattern'... How many matches would you need then?

62) B: I'll multiply by five sir... because in the first 'pattern'... in the first 'pattern' there were five...

63) I: Write down the rule... write down the rule...

64) B: The rule sir?

65) I: The general rule that you used.

66) B: The multiplication rule... I'd have to multiply by five... if I had to multiply by eighty-three...

She then wrote: "if I had to find the answer of this question how many mathes (*sic*) stick will you use to find the 83rd parten (*sic*)? I would multiply 83 by 5 because 5 was the 1st Parten(sic) given". Note how Bonnie, like Andrea (see the example in (*d*) above), also opts for multiplication of the first two discernible numbers, viz., 83, the position of the 'pattern' in the sequence, and 5, which is the number of matches in the first 'pattern'. One would have expected Bonnie to come up with a less specific general rule, for example: Multiply the number of the 'pattern' by five. Yet she uses the example of the eighty-third 'pattern' as a description of her rule, which she calls the "multiplication rule" (lines 57 and 66).

It is common practice for the teacher to explain a difficult new mathematical rule in the classroom by means of carefully chosen numerical examples. The teacher, in many ways, acts as a role-model for the pupils who are likely to copy his or her behavior. One possible reason why pupils are more likely to try and explain their generalized rules by means of numerical examples of specific instances to show how the rule is applied, is that they are copying their teacher's way of explaining mathematical rules in class. Another reason might be that the pupils do not possess the necessary verbal skills to express their generalized rule in the absence of a specific numerical example to which they can refer.

4.3 Phase Three: The Function Table

4.3.1 Some Details of the Interviewees.

Three pupils from the standard six group and three pupils from the standard seven group were interviewed. The interviewees were selected from amongst the group of volunteers; ignoring their earlier, superficial categorization into higher, middle and lower ability groups as a criterion during the selection process. Volunteers who had not yet been interviewed during the previous phases of the investigation received preference during the selection for this phase. Second-time interviewees were only considered once I had run out of first-time interviewees. This happened on three occasions; twice in the standard six group and once in the standard seven group. The reason for giving preference to first-time interviewees was that I did not want to exclude any of the pupils from the small group who so generously and unselfishly availed themselves for the interviews.

The interviewees were:

(a) Andrea, 13 years old, female, first language speaker, std.6

(b) George, 13 years old, male, first language speaker, std.6

(c) Tracy, 13 years old, female, first language speaker, std.6

(d) Anna, 14 years old, female, first language speaker, std.7

(e) Claud, 14 years old, male, first language speaker, std.7

(f) Claudette, 14 years old, female, first language speaker, std.7

As before these are not the pupils' real names. Where a pupil was interviewed for the second time, I have retained his or her pseudonym for the sake of consistency. All the pupils that were interviewed are first language speakers; that is, they speak English at home and are taught in English at school as well. English is their first language.

4.3.2 Discussion of the cognitive difficulties with the function table.

The ensuing discussions will center around the particular cognitive difficulties that were detected in the course of the interviews, viz.:

(a) Pupils perceive a variety of patterns -- not all of them feasible for translation into the algebraic code.

Andrea (13 years old; std.6) perceived a complicated pattern in the numbers of the functional table:

11) A: Count like one... than you see its one... two... one plus one is two... so you add one... Here two times two is four plus one is five... Here by number three is... is three times three is nine minus one is eight... and if it stands here by four... four times three is twelve minus one... is eleven... Now I don't think that counts...

12) I: Why?

13) A: Because it's two additions and two subtractions... I think here it will be... uhm... nineteen...

14) I: At number five?

15) A: No twenty-one! ... twenty-one...

16) I: How did you get twenty-one?

17) A: Uhm... I see that this is two additions and two subtractions... and I go again two additions... uhm... add one...

18) I: Now would you perhaps do number five?

19) A: Five times four is twenty... plus one is twenty-one...

20) I: Okay... Number six in the top row...?

21) A: Uhm... twenty-four... plus one is twenty-five.

22) I: Explain again...

23) A: Uhm... six times four... is... twenty-four... plus one is twenty-five.

24) I: So if the top row number is seven, what will the bottom row number be?

25) A: I see that you bring up the...the numbers also that you're supposed to multiply... because here you can't multiply this by four... it's one this... this is three... this is four...

26) I: I don't think I understand what you are trying to say... can you explain a little more?

27) A: Uhm... this is two... The first two numbers one and two... and at the bottom there's three and five... you have to multiply by two... one you have to multiply by two... and two you have to multiply by two... And here by three and four...eight and eleven... you multiply by three... and here you multiply by four... and the others will probably go on...five; six; seven; ...

Andrea's rule only works for numbers 2, 3 and 4 in the "top row". It does not work for number 1 in the "top row", but she either did not notice that, or deliberately ignored that. She also did not notice that her rule does not yield the correct "bottom row" numbers beyond the fifth on in the "top row", but this is to be expected, because she seems to have no other reference other than her own rule.

Tracy (13 years old, std.6) perceived the pattern differently:

2) T: Here he pushes out one... always he kicks out odd numbers... then he pushes out one...

3) I: When you put in one... then what comes out?

4) T: Two.. but, he... one... he adds one.... Add two and he pushes out five... he takes three... Add three and he pushes out eight... he takes five... Add four and he pushes out eleven... he takes seven... Add five and... I don't know what type of number, but he pushes out nine... and he goes on like that...

5) I: Okay, now what would be the number that he kicks out when you feed in five?

(Silence)

6) I: You wrote down fourteen... seventeen... Where do you get those numbers?

7) T: Because when he pushes out five... ne'... I'm sorry... When you put in five... he pushes out fourteen... and he skips nine... When you put in six... he pushes out seventeen... and he skips eleven. It takes on odd numbers... then he pushes out another number... and he add the number... the top row's number and.. to the odd numbers and the you get this... the bottom row.

...

13) I: Explain how you get the bottom number every time.

14) T: Okay... one plus one is two... two plus three is five... three plus five is eight... every time its pushes out another number it takes a odd number... in that certain... how you call it? How do you say? ... In a certain way... like when you count one, two, three... it is in the same order, but odd numbers... then he adds it to the top row's numbers, then he gets the bottom row.

15) I: Now how do you know what odd number to use?

16) T: Because it's in the descending order... the odd number, one... and then three... five... and so on. And then you just take the numbers like they come... for that's the same as one, two, three, four, five, six... and then you take the odd numbers ... and then you add it... to the top numbers and then you get the bottom numbers.

Note how Tracy used the pronoun "he" to refer to the functional table. That was after it had been explained to her that the table works like a computer that turns the numbers in the "top row" into the numbers in the "bottom row" by using a to-be-discovered rule. She probably confused the term "descending order" (line 16) with ascending order which means the opposite. Her explanation, however, makes it is clear that what she meant was ascending order.

Tracy's perception of the pattern links the "top row" numbers to the "bottom row" numbers as follows:

$$1 + 1 = 2$$
$$2 + 3 = 5$$
$$3 + 5 = 8$$
$$4 + 7 = 11, \text{ etc.}$$

This generalized rule for generating the numbers in the "bottom row" can be expressed algebraically as: $n + (2n - 1)$, where n is the number in the "top row". I wish to contend, however, that this expression requires a level of algebraic sophistication that might just be too much for the ordinary pupil at the introductory level of algebra. As it turned out, Tracy was unable to translate her rule into an algebraic form.

Claud (14 years old; std.7) explained his perception of the pattern as follows:

5) C: With the 'bottom row'...uhm... there's two numbers gone like in between the... the numbers...

6) I: Explain that again.

7) C: The 'bottom row' is like... is two numbers... uhm... two numbers gone in between the numbers here like... 2, 5, 8, 11... between 2 and 5 there must be 3 and 4...

....

11) I: What do you think will be the 'bottom row' numbers for 5 and 6 in the top row? That is fill in the blank spaces.

12) C: 14 and 17.

13) I: Write it in.

(C. writes 14 and 17 in blank spaces in the table)

14) I: Can you explain how you got to 14 and 17?

15) C: Here in between 2 and 5, 3 and 4 is gone sir... like here between 5 and 8, 6 and 7 is gone... and the same here by 8 and 11 sir... so it will be the same here by this two... 14 and 17.

16) I: How did you get to 14?

17) C: I left out 12 and 13 sir.

18) I: And 17?

19) C: 15 and 16 I left out.

Claud made it clear that he focused only on the numbers in the "bottom row" (lines 5 and 7). This fixation with the numbers in the "bottom row" would certainly not aid him in perceiving the functional relationship between the "top row" and the "bottom row". His rule is the same as adding on three every time, but from his way of expressing it, it seems that it still had to mature into being seen as such. It would certainly be difficult to translate it into the algebraic code from this crude description -- he never could.

(b) Pupils often set of in pursuit of a recurrence rule as a first strategy and often revert back to this rule if all else fails.

For example, George (13 years old; std.6), who responded to the function table by immediately filling in the missing "bottom row" numbers corresponding to 5 and 6 in the "top row". When prompted, he explained his reasoning as follows:

1) I: You say five turns into fourteen and six turns into seventeen. How did you do that?

2) G: You start with the two... start with adding three all the time; three to make five; three to make eight and three to make eleven. That's why I added three to make fourteen and three to make seventeen.

The recurrence rule that he had used, was to "start with adding three all the time" (line 2) and he obviously took only the "bottom row" numbers into account. He then went on to explore short cuts for implementing this rule, e.g., applying direct proportions (e.g., the explanation in line 26). Note how he reverted back to "adding three" (line 30) when his short cut proved to be inadequate:

26) G: And if you go... If you use hundred... numbers like hundred, the you just times with ten. With ten if you... Sir said hundred then you just use ten times ten... and then you times the bottom number also with ten. And then you get the answer for hundred, two-hundred, three-hundred ...

27) I: And what if I have eighty-nine... in the 'top row'?

28) G: Eighty-nine...? Eighty-nine...?

29) I: Now what are you going to do now?

30) G: I'll just start adding three again.

31) I: Are you gonna go all the way up to eighty-nine?

32) G: Yes sir.

The eighty-nine (line 27), a prime number, was deliberately chosen because the strategy of using direct proportions as explained (e.g., in line 26) would not work on it. George, when he noticed this breakdown in his strategy, reverted back to the original strategy of "adding three" (line 30).

(c) Pupils tend to over-generalize, e.g., they show a strong tendency to use direct proportions as a short cut to the solution.

George (13 years old; std.6) gave a clear explanation of how he used direct proportions to speed up his solution:

3) I: Okay... Now if I had eighty in the 'top row', what do you think will be the value that comes out in the 'bottom row'?

4) G: Twenty-three sir... Two-hundred-and-thirty.

5) I: Can you write down how you got to your answer?

(G. opts for a verbal explanation rather than a written one)

6) G: Eight times ten is eighty. Twenty-three times ten is two-hundred-and-thirty.

7) I: And if you had one-hundred in the 'top row'; what will come out in the 'bottom row'?

8) G: Two-hundred-and-ninety.

9) I: How did you get two-hundred-and-ninety?

10) G: Just adding three all the time...

11) I: Now I'm not sure if I understand what you are saying?

12) G: Five was fourteen, six was seventeen, seven was twenty, eight was twenty-three, nine was twenty-six, ten was twenty-nine... just adding threes all the time.

13) I: Now how did you get to one-hundred so fast... to the number that comes out of one-hundred?

14) G: Sir said for one-hundred sir... so I just used twenty-nine and ten... because when you times ten with ten you get a hundred... When you times twenty-nine with ten... you get two-ninety.

15) I: Did you do the same for eighty?

16) G : Yes sir.

His method does not yield the correct answers, because no direct proportion between the "top row" and the "bottom row" exists in the stimulus item. George simply assumed the existence of such a direct proportion, without even having checked for it between different sets of top and bottom row numbers.

When Andrea (13 years old; std.6) discovered that her rule (described in lines 11- 27 in (a) above) was less than ideal for calculating with the larger numbers, she also opted for the method of using direct proportions:

32) I: Now what do you think will happen if I give you a large number... let's say in the 'top row' you have eighty... what do you think will be the 'bottom number'?

(*Pause as A ponders the question*)

33) I: What do you think now?

34) A: And the 'bottom number' will be?

(*Pause again*)

35) I: What are you thinking?

36) A: I think three-hundred-and-ninety.

37) I: Can you explain?

38) A: I think because this... uhm... you <u>add a naught to the eight</u> and a naught to the 'bottom number' too...

When she spoke about adding a naught (line 38), what she probably implied was multiplication by ten, e.g., 8 x 10 = 80. Note how the multiplication by ten converts the 8 into 80, giving rise to the description of the process as "add(ing) a naught to the eight" (line 38). She also simply assumed a direct proportion between the numbers in the "top row" and the numbers and the "bottom row", without actually having verified it first.

(d) Pupils fail to check spontaneously for the validity of their assumed rules.

This was a common phenomenon found amongst all of the interviewees. Examples of how some interviewees simply assumed a direct proportion between the numbers in the "top row" and the numbers in the "bottom row", without having verified it first, are discussed in (c) above.

(e) Immature ways of expressing arithmetic operations, e.g., expressing addition as the skipping of numbers.

For example, Claud (14 years old; std.7), whose description of the "adding on" strategy (lines 5-7 and 11-19) is discussed in (a) above.

(f) Pupils are often not ready to start creating their own symbolic expressions by the time they are expected to, but will nevertheless attempt to do so when required of them. That is, for them a spontaneous cognitive need for a more succinct representational system and its advantages has not yet developed.

Anna (14 years old; std.7) had found the rule for generating the "bottom number" from the "top number". She wrote: "Multiply the top row number by 3 and subtract one". She was then asked to write her rule by using mathematical symbols.

13) I: Now can you try to write the rule by using mathematical symbols.

14) A: Can sir say it in English (giggles)? ... Is sir serious? ... What mathematical symbols?

15) I: That is a good question. What is mathematical symbols?

16) A: Like so (*writes "80 x 3 - 1"*) ... or is it... uhm... like with figures; like with *x* and *y* and that?

17) I: You can use that as well, yes... I want you to use what you think is appropriate... as mathematical symbols to write your rule with... That's like asking you to invent your own mathematical system of writing.

18) A: Give an example(*irritated*)...

19) I: What are you looking for in an example?

20) A: I don't know what sir would expect.

21) A: Can I say like the top row, for instance is the numbers on top, whatever the number is... is 'a' and the bottom row's number is... say 'b'?

22) I: I suppose you can do that...

23) A: Okay...

(*A. writes," Top Row - a" and "Bottom Row - b". And then also: " b = 1 - (a x 3)"*)

24) A: There is something I don't understand...

25) I: Explain it to me...

26) A: One minus ...

27) I: When I say explain then explain where a person would start... How I would read this...

28) A: I did it... start with the brackets... and then the brackets say 'a' times three ... so it's the top row number times three...

29) I: Uhm...

30) A: And then... and then I get my answer... and I subtract one from it... and I get 'b', the bottom row's answer.

As Anna has pointed out correctly (line 16), "mathematical symbols" includes both arithmetic symbols (e.g., + and -) and algebraic symbols (e.g., x and y). Note how she asked for an example to see what is expected of her (lines 18-20). Mason (1989) explained that pupils know that the teacher (interviewer) is looking for a particular behavior as a manifestation of their understanding of the concepts or the topic. Tension arises when the teacher (interviewer) is not explicit about that behavior, as was the case in this instance. If I had to give an example at this stage I would forsake an opportunity to learn more about the interviewee's own symbolizing strategies.

Note how she wrote $b = 1 - (a \times 3)$ instead of the correct $b = (a \times 3) - 1$. When asked how her equation should be read, she does so correctly (lines 28 and 30). In line 28 she explained that one should "start with the brackets" which can probably be traced back to an earlier learned arithmetic frame that has to do with the order of operations. That also explains why she does not read the expression from left to right.

Then Anna was asked to apply her symbolic rule:

37) I: Now how would you use that symbolic rule of yours to find the bottom row number for 46 in the top row?

She replied by writing: "46 x 3 = 138 - 1 = 137(b)". This shows how she carried the operations out in the order that she read the symbolic equation; that is, doing the part in the brackets first, getting its answer, and subtracting one from it. Her numerical version of the rule reveals two difficulties. One is that of viewing the equal sign as the introduction of a result, that is, the number immediately on the right of the equal sign is the result of the arithmetic operation on the left of it. The other is that she is obviously not aware that the use of the brackets in mathematics differ from its use in a natural language. In a natural language it would have been in order to put the "b" in brackets next to the 137 to indicate that it is the value of "b"; whereas in mathematics it is not.

Anna was then asked to calculate the number in the "top row" for number 38 in the "bottom row".

38) I: Now for another question... Let's say, for instance, you had 38 in your bottom row; what will be the number in the top row?

39) A: Oh! ... What is the number?

40) I: You have 38 in your bottom row... what will be the number in the top row?

41) A: So... Uhm... must we write this out first?

42) I: You can do that if you want to.

43) A: Uhm... 48...

44) I: 38...

45) A: 38... must I do the same

(A. writes, " a = 1 + (b ÷ 3)", and underneath it "39 = 13")

46) I: Wait... but this is not the same rule... is it?

47) A: No, but you must switch it around because you want the bottom number now. If you wanted the top row's answer... That (*the first symbolic equation*) only applies for if you want the top row's answer... You must switch it around if you want the bottom row's answer...

48) I: So how did you how know to switch it... to come up with that?

49) A: Because they give you... uhm... Here I... Here by this first rule, they ... uhm... give you... the top row's number and they ask the bottom row's number... and here (*the second symbolic equation*) they give you the bottom row's number and they ask the top row's number... so you must just switch your stuff around...

50) I: But how did you switch from the one symbolic form to the other symbolic form?

51) A: I just put... uhm... switched '*a*' and '*b*' around; and put '*a*' in front of the equal signs... and then; instead of subtracting one from... Let's start with the brackets... And instead of multiplying... uhm... the number, I divided it because they give you the number already and you can't multiply that... you must divide it... and then I added one onto that, because its like you're switching the whole thing around...

52) I: Now let's see what comes up...

53) A: I think I must first add this... add the one (*writes "38 = 13" then scratches out the 38 and writes, "39 = 13"*) ... then it's 39...

54) I: Explain quickly what you did.

55) A: Uhm.. I took the... the... bottom row's number and then I added one onto it before I... before I divided by three... into three... so the answer wasn't 38; it was 39... and then I divided 39 by three... it's 13.

Anna realized that the subject of the formula had to be changed from "*b*" to "*a*". She did this by switching the places of "*a*" and "*b*" and replacing the arithmetic operations with their opposites, writing: "*a = 1 + (b ÷ 3)*". This is a serious error when interpreted from an algebraic viewpoint. However, Anna was able to bypass the difficulty of an erroneously converted algebraic equation by reverting back to the arithmetic frame (see her explanation in line 55) and obtaining the correct answer. Note how the equivalence relation normally indicated by an equal sign was completely ignored when she wrote "*39 = 13*". The equal sign was most probably used to introduce the answer of 13 rather than to indicate an equivalence relation.

Andrea (13 years old; std.6) explained her general rule as follows (for a full transcript see appendix VI):

40) A: Uhm... It's the top row... okay... nothing added or nothing to the top row... uhm... it's just 1; 2; 3; 4; 6; 7; 8;... common numbers... The bottom row... uhm... you must... the first two numbers, one and two, you must multiply by two... and... uhm... one times two is two... and then you must... here by number two... two times two is four, plus one... you must add one... plus one gives you five... here by number three and number four... you must multiply by three again... uhm... three times three is nine, then you must subtract one... then it gives you eight... and here the same (*for four in the top row*), its twelve, you must subtract one and it gives you eleven... and here you start again by addition... and so you go on...

....

47) A: It goes like this... three and four is subtract one... and five and six is add one... and seven and eight is subtract one... it's like that... So five and six will be... five

times four equals twenty, plus one, equals twenty-one... that will be your bottom number... same with six (*A. continues to write down the example with five in the top row as she explains*)... Five is your top number (*referring the example with five in the top row*) and it will always be first... uhm... the first thing that you write down... it will be five (*emphasizing the word*) times four... equals twenty... and twenty plus one is twenty-one... that's your bottom number... that is your last number that you write down... and the same with six... six is your first number, that's your top number, times four equals twenty-four... plus one equals twenty-five and that is your bottom number... that is the last number...

I chose to include her verbal explanation of the rule to show how difficult it would be to translate it into symbolic form. Note her proposed convention of a hierarchical order of writing to distinguish between the independent variable("top number") and the dependent variable("bottom number"). The "top number" is what you start of with and should always be written first; the "bottom number" is what you end up with and is therefore written last. In an algebraic equation this is just the other way round with the dependent variable written on the left of the equal sign (and therefore written first) and the independent variable on the right thereof. She then explained further by means of written examples (see appendix VI): e.g., "*3 x 3 = 9 -1 = 8*" and "*4 x 3 =12 -1 = 11*". Note that she too (like Anna above) used the equal sign to introduce the answer to the operation on the left of it rather than to indicate an equality. She was then asked to translate her rule into symbolic form.

48) I: Okay... Can you perhaps write it in mathematical form... using symbols?

49) A: Must I give my own symbols and so on?

50) I: Yes, if you wish you can give your own symbols.

51) A: Okay... Take three and four again... The top number... is three, so... the three...

52) I: Can you write the symbol down?

53) A: You make an 'x'... so... no I can't see... three times 'x' ... because three and four will be multiplied by three... both of them will be multiplied by three... so it's 'x'... Three is 'x'... three... the number that you multiply... the top number is three... like four times three... and three times three...

(*A. writes: 3 x X = 9 - Y = 8*)

54) I: But didn't you say you write the top number first?

55) A: Yes... so this is three...

56) I: Now show with a little arrow which is the top number again.

57) A: This is the top number (*indicates it with a the letter T*).

58) I: What is '*x*'?

59) A: Times '*x*'... <u>'*x*' is the number that you multiply the three (*top number*) by</u>.

60) I: Okay... go on...

61) A: This is equal to... three time three is equal to nine (*writes nine*) and you subtract one... subtract '*y*'... '*y*' will be one (*indicates it with the number 1*), because it will be used in the top number in number three and number four... equals eight... and eight is your bottom number (*indicates it with the letter B*)...

62) I: Now what did you say the '*y*' was again?

63) A: *y* is a minus one... is one... '*y*' will be one... and '*x*' is (*writes down "number you multiply by"*)... '<u>*x*' is the number that you multiply by and '*y*' is the number one</u>... so it reads like this: <u>The top number is three, so three will always be first</u>... on top... and the bottom number is eight... because... in <u>the bottom it will always be last because it's at the bottom</u>... Uhm... so it reads this way: Three times '*x*', and '<u>*x*' is the number that you multiply by</u>, and in this case '*x*' is three... so three times three, that is '*x*', equals nine... and minus '*y*', and '*y*' will be one... and that will equal eight... and that is the bottom number...

The problem with Andrea's rule is that it involves too many variables: the "top numbers"; the "number that you multiply by"; the one that is sometimes subtracted and sometimes added; and the "bottom numbers". She seemed to understand, intuitively, that this should be narrowed down to only two variables. Hence she introduced "*x*" and "*y*" to represent the "number that you multiply by" and the "number one" respectively. I think she deliberately overlooked the "top numbers" and "bottom numbers" as candidates for variables in the tradition of "x" and "y" because the "top numbers" are treated as given (and hence known) and the "bottom numbers" are treated as the sought after answers (and hence unknown). In order to learn more about the way she used these symbols she was then prompted to give another example:

64) I: Can you make another example? Just one more example... Use another number... say thirteen.

65) A: Thirteen and fourteen (*A draws a little table and writes 13 an 14 in the top row*).

66) I: Why do you always use two numbers?

67) A: It's because... Uhm... I always use two numbers because... here its like in groups... it's in groups of two... its the amount you multiply by... the one and two... okay, hold on... take five and six is multiplied by four... three and four is multiplied by three... it's like... this is 2; 3; 4 (*indicates what numbers to multiply by above the groups in the table*)... in groups... groups of two (*extends the table -- writing the numbers in a group together*)... 6; 7; 8 (*writing this above the groups in the extension to indicate what to multiply by*)...

68) I: I don't understand...times eight.

69) A: This will be... uhm... say you go on here (*referring to the extended table*)... 7 and 8... 9 an 10 will be together... 11 and 12... 13 and 14... So this will be... this is 1 and 2 multiplied by two; 3 and 4 multiplied by three; 5 and 6 multiplied by four; 7 and 8 multiplied by five... and this is six (*9 and 10*); this is seven (*11 and 12*); and this is eight (*13 and 14*)...

70) I: Okay...

71) A: So you multiply... uhm... (*A does the calculation 13 x 8 = 104 in writing*)... So in this case, number thirteen, the bottom number will be hundred-and-four.

72) I: Can you explain again... through all the steps.

73) A: Okay... thirteen you must multiply by eight... thirteen and fourteen is grouped together in two... thirteen and

fourteen are together, so they will be multiplied by one number, and that is eight... now I multiplied thirteen by eight and that gives me hundred-and-four... so the bottom number of number thirteen will be hundred-and-four...

74) I: And what about the *y*-number?

75) A: That you must... It's also grouped in two's... The first... Okay... The even numbers... is always you must add one...

76) I: What even numbers?

78) A: If you must... uhm... <u>multiply by an even number, then it will be plus one</u>... so <u>if you must multiply by an uneven number, then it will be subtract one</u>...

79) I: Okay, so what will you do with thirteen?

80) A: Here you must multiply by eight, so you must add one... plus one.

81) I: So what is your bottom number now?

82) A: It's... hundred-and-five... So it's thirteen times eight... it's hundred-and-four... plus one... it's hundred-and-five... so I'm gonna write it down...

(*A writes: 13 x X = 104 + Y = 105*)

83) A: Thirteen times '*x*'... and '*x*' is the number you multiply by... will give you hundred-and-four... so it will be plus '*y*', and '*y*' is number one... that will give you hundred-and-five... so hundred-and-five is your bottom number... and thirteen will be your top number!

Note how Andrea paired the numbers of in order to keep track of which pair to multiply with what number; as well as from which pair to subtract one or add on after multiplication. She even formulated a rule for when to subtract one and when to add one (line 78). Her use of "*y*" to represent the "number one" is akin to Küchemann's (1981) notion of the "letter evaluated" where a letter is assigned a numerical value from the outset. It is clear from her given examples (see appendix VI) that she uses "*x*" in the same way, with a numerical value assigned to it from the outset. I do not think that Andrea has yet acquired the concept of variable which is necessary for setting up a symbolic rule for generating the "bottom number" from the "top number".

(g) Pupils sometimes make arithmetic errors which, when left undetected, prohibit the discovery of a generalized rule.

Claudette (14 years old; std.7) filled in the following numbers in the table: 14, 17 and 21. It is easy to see that the addition of three every time should give 20 instead of 21. This error went undetected throughout the interview and led to other mistakes and wrong conclusions, e.g.:

16) C: The 8^{th} one will be 24... the 9^{th} one will be 27... the 10^{th} one will be 30... So by every tenth one there will be a round number... with a naught... It won't be... How can I say now? Over there a whole number...

17) I: Well, all of those are whole numbers.

18) C: But I mean, it will be a... How can I say? ... It will end ten sir... It won't be having another digit next to it... It will only have a naught...

She then continued with the successive addition of three every time, building on her mistake, to find that for number 20 in the "top row" the "bottom row" number will be 60. This she used to make the generalization that for every multiple of ten in the "top row", the "bottom row" will contain a multiple of thirty.

46) I: What will be the 80^{th} number... the bottom number for the 80^{th} top number?

47) C: The 10^{th} one is 30 sir(*starts to write down the top row numbers and next to it the bottom row numbers: 10*30; 20*60; 30*90;...; 90*270*) ... so if the 20^{th} one is 60 sir... That is, <u>30 plus 30 gives you 60</u>; so if you want to come to the 80^{th} one, then it's gonna be... 60 is the 20^{th} one... and... 30 is the 10^{th} one... then the 30^{th} one will be... 90... the 40^{th} one will be... 120... no, 110 (*C. mumbles the numbers as the writes them down*)... 140; 170; 210; 240... 80^{th} will be 240...

With the 40^{th} one she made another mistake. Assuming that she generated the "bottom row" values by adding 30 every time, because she started of by adding 30 and 30 to get the 60 (line 47), one would expect the 40^{th} one to be 120. However, she changed her initial "bottom row" value of 120 to 110. I think a plausible explanation for this is that at the 40^{th} one she started to look at the table for an easier way out and that she changed her strategy from adding on 30 every time to the strategy of using direct proportions. This was confirmed later by her in a further explanation:

55) C: So then it will be.... sir here I can show sir something sir.. here by the 8^{th} one sir... if you add a naught there sir... that's gonna be the 80^{th} one sir... if you add a naught there than it will be 240... same here (*7^{th} one*).

56) I: Does it work in all cases?

57) C: On the 70^{th} one it will be 210 sir... there is 21 sir (*Referring to the 7^{th} one*).

58) I: Now explain that again.

59) C: If you add a naught to the number... even if it's gonna be 10 and then it's gonna be 20 sir (*Converting the first*

column with 1 in the top row and 2 in the bottom row)... because look there (*refers back to the tenth column with 10 in the top row and 30 in the bottom row*)... No! ... It's not in all cases like that (*softly*) ... The 20^{th} one is not 50 (*softly*)... It's just some of them sir...

60) I: Do you think you have a rule already?

61) C: You add three sir... and then odd numbers...

62) I: Can you describe the rule... or just describe your thoughts so far... those that you think work...

63) C: Only number 8; number 7; and number 6; and number 5 will work ... works.

64) I: Works...?

65) C: If you add the naught sir...

66) I: Explain more... If you add the naught...

67) C: If you add the naught to the top one... and to the bottom one ... that will be your answer... So if it's gonna be the 7^{th} one then it's 21... then you just add a naught to both bottom and top...

68) I: And you get the 70^{th} one...?

69) C: Yes sir.

70) I: And the bottom number will be...?

71) C: 210 sir

72) I: But you say it only works for... for what numbers?

73) C: Only from... number 4 onwards, sir, it works... only from 4 up it works sir.

74) I: Can you describe the rule... write it down perhaps?

(*C. writes: "From the fourth number in the top row add a 0 then you can determine the bottom number. In the seventh* *number on the top you add a 0 then it will be 70. Do the same to the bottom. Then you will arrive at your answer".)*

This shows how an undetected arithmetic error can lead to a failure to find the generalized rule in a number pattern. Claudette had innovatively come up with some rule, but had to admit that it is far from being general; working "only from ... number 4 onwards..." (line 73). In her written statement of the rule she took care to include this condition.

4.4 Summary

Chapter four discussed the data that was collected during the research. It highlighted a number of strategies that were used to deal with the questions in the worksheet of the first phase; which in turn revealed some of the cognitive difficulties that prevented pupils from finding the correct solutions. These difficulties were subsequently discussed. The interviews on the match stick pattern of the second phase and the function table of the third phase also revealed a number of cognitive difficulties that were discussed.

In chapter five an attempt will be made to use the difficulties identified in chapter four to make some general conclusions about the cognitive difficulties that standard six and standard seven pupils are anticipated to experience with the generalized number pattern approach to algebra. The implications for teaching will be discussed; as well as some suggestions for further research.

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CHAPTER 5

5. CONCLUSIONS AND RESEARCH RECOMMENDATIONS

5.1 Conclusions

The data collected during this research provide evidence of a number of cognitive difficulties that standard six and seven pupils on the Cape Flats experience with the generalized number pattern approach to algebra. Many of these cognitive difficulties are not unique to pupils from the Cape Flats as some have also been documented by researchers in other parts of the world, for example: MacGregor and Stacey (1993) in Australia and Orton and Orton (1994; 1996) in England. I wish to agree with MacGregor and Stacey (1993) when they say: "The route from perceiving a pattern to writing an algebraic rule is complex" (p.187). To be able to travel that route, the pupil needs to be equipped with certain cognitive skills which are necessary to overcome the difficulties. These necessary cognitive skills include:

1. **The ability to transcend the fixation with recurrent patterns in pursuit of a functional relationship between the dependent and the independent variable.** For some pupils the interview never progressed beyond the finding of the next few numbers by using a recurrence pattern. Any attempts to prompt them beyond that resulted in over-generalizations.
2. **The ability to formally articulate, in a natural language, the generalized rule.** Some pupils are able to calculate with a generalized rule, but they cannot give a verbal explanation of how it works. Instead they are likely to work out a numerical example to show how the rule is applied. If there is to be any hopes of expressing the generalized rule in terms of algebraic

symbolism, the pupils must be able to express it verbally first. This will afford them with the opportunity to reflect critically on their construction, paraphrase it, reorganize it, and reinterpret it before any attempt is made to express it symbolically. Doing this will develop an awareness of what can and cannot be easily translated into algebra. An acceptable verbal description will reflect a conscious awareness of the structures and relationships involved in the solution process. Besides, the formulation of a verbal description is where one should start if the need for a more concise representational system is to be developed.

3. **A metacognitive awareness that will enable pupils to check the validity of their assertions against the available data.** A number of the pupils invented "general rules" that worked in one or two cases. These were then "over-generalized" and presented as the generalized rule. This happened despite obvious evidence to the contrary in the immediate data available. With the necessary metacognitive awareness this would have been detected and the search for a generalized would have been continued. The likelihood of undetected arithmetic errors precluding the finding of a generalized rule would also be reduced.

4. **Some facility in the proper use of mathematical syntax,** e.g., knowing that the equal sign denotes equality and does not only signal the introduction of a solution as in: "*3 x 3 = 9 - 1 = 8*". This includes having some appreciation for the structure of an algebraic expression, e.g., knowing that "*1 - (a x 3)*" does not translate into "*46 x 3 = 138 -1 = 137*" for *a = 46*. Without a basic knowledge of the mathematical syntax and

conventions any attempts at constructing an algebraic representation of a generalized rule would be fruitless.

5. **A sound understanding of the variable as a pattern generalizer.** This should allow the pupil to set of in pursuit of a variable that can be used to generalize from the pattern. In fact, I perceive this as a prerequisite for the eventual generalization of the pattern into an algebraically represented rule. The pupils' interpretations of the letter *n* in the *n-th* figure of the worksheet (see section 4.1.4, Question 3) shows that, in some instances, the understanding of the variable as a pattern generalizer had not yet been attained.

5.2 Implications for Teaching

In view of the proposed introduction of generalized number patterns into the mathematics curriculum of the Western Cape (Draft Syllabus for Mathematics in the Junior Secondary Course of the Western Cape Education Department, 1996) and the results of the research documented in this thesis, I wish to make the following teaching suggestions:

1. **Teachers should familiarize themselves with the cognitive difficulties that their pupils are most likely to experience with generalizing from number patterns. This would enable them to design instructional activities aimed specifically at addressing these difficulties.** For example, analyzing a particular number pattern prior to its presentation to the pupils with the aim of identifying and resolving some of the cognitive difficulties that might prevent its successful generalization and translation into algebraic notation.

2. **It is advisable not to introduce number patterns as abruptly as algebra has traditionally been introduced to pupils.** Instead, pupils should be exposed to a variety of patterns over a period of time to give them practice in the "seeing" of patterns first. During this time they might be encouraged, if there is no evidence of a spontaneous need to do so, to express their perceived patterns in a generalized form; using whatever representational means they might have at their disposal. This should be done without pressing the pupils; allowing ample time for the need for a more concise and elegant representational system to mature.
3. **Pupils should be encouraged to see that the same pattern can often be expressed in more than one way.** During group work sessions different perceptions of the same pattern is almost certain to arise among the pupils. When this happens, it should be used as an opportunity to encourage pupils to investigate and evaluate each others' perceptions of the pattern. During the ensuing debates pupils will be forced into a metacognitive evaluation of their own representations in order to be able defend it. Without this kind of challenge, few pupils are likely to engage into a metacognitive evaluation of their own work.
4. **Pupils should be allowed to choose their own symbols for representing the variables.** According to Booth (1989) pupils are able to construct meaning for the symbols more readily when they themselves have decided what the symbols are to represent. I wish to take this further by adding that using their own symbols will enhance the pupils' feeling of ownership of their own constructs.

5. **To counter the spontaneous, and sometimes persistent, search for a recurrent rule of finding the successive term from the one before by "differencing", I suggest alternating patterns in which this strategy might work with others in which it does not.** For example, patterns like 5; 8; 11; 14 ... can be alternated with patterns like 1; 8; 27; 64 ..., etc.

6. **When making the jump from the first few consecutive terms in the pattern to term further down the pattern (e.g., the 83rd term) try to avoid multiples of the terms that had already been dealt with**, e.g., do not ask for the 80th term if the 8th term had been dealt with as this will encourage "over-generalization" [see sections 4.2.2(b) and 4.3.2(c)]. Pupils are likely to respond like this: $T(80) = T(8) \times 10$, where $T(n)$ denotes the n-th term in the pattern. To be on the safe side it is advisable to use larger prime numbers further down the pattern, e.g., T(83) instead of T(80).

7. **Always start of with concrete patterns that the pupils can physically build for themselves.** Number patterns can be generated from various other patterns, e.g., geometric patterns, functional tables, match stick patterns, etc. While it is advisable to introduce pupils to all of these, one would do well to start in the early stages of the introduction with concrete and manipulable patterns to give pupils some form of concrete referent that might aid them in the conceptualization of the pattern.

8. **Group work is strongly suggested as the way to deal with the suggested number pattern work.** This way pupils will be exposed to perceptions other than their own; addressing the

problems of the lack of metacognitive awareness, the fixation with only one correct answer, as well as enhancing their verbal communicative skills.

9. **Extra attention has to be paid to the preference for repeated addition instead of multiplication; as this makes the process of generalization from number patterns more difficult.** See, for example, the discussion in section 4.2.2(c). With the problem-centered approach in mathematics pupils are being encouraged to work with their own processes rather than to use algorithms (e.g., the multiplication algorithm). In some instances, however, their processes may be less than ideal. When this happens, it might be better to encourage them to look for alternatives, rather than to impose our own processes on them.

These teaching suggestions would hopefully help to address some of the cognitive difficulties that pupils experience when they have to generalize from a number pattern.

5.3 Suggestions for Further Research

The research reported on in this thesis is based on clinical interviews that were conducted with individual pupils as they attempted to generalize from the given number patterns within the time constraint of an interview (which is about 20 minutes). An investigation stretched out over a longer period of time and utilizing more interviews might yield interesting, if not different, results. Alternatively, for example, the subjects could be left alone with a number pattern for some time before the actual interview to give them ample time for the "seeing" of the pattern and to structure their thoughts before they are interviewed. By attempting to squeeze a number of questions into the limited time of the interview one might not allow enough time for the subjects to fully explore each question and they might develop a sense of just getting a question over and done with, in as little time as possible, in order to start on the next one. Another methodological alternative is to interview a subject on one question per interview session, using different patterns and exploring each to the full. In this way some of the pressure of having a set number of questions to squeeze into the limited time of a single interview can be relieved.

Generalized number patterns is only to be introduced into the formal mathematics curriculum of the Western Cape in standards six and seven as from January 1997. Therefore exposure to number patterns during the interviews was a novel experience to the subjects. Further research is needed to establish how pupils who have had some prior experiences with number patterns would react when they have to deal with the same tasks.

A limited number of number patterns were used during this research: one in the worksheet, one functional table, and one match stick pattern. All of these lend themselves to a recurrent rule in the sense of a constant difference between the terms. Further research is necessary to see how pupils would react if they have to deal with patterns in which there are no constant differences between the patterns.

Group work has long been established as a valuable teaching strategy in a variety of disciplines. In section 5.2 I suggest a number of reasons why group work should be a viable teaching strategy pertaining to number patterns. However, the specific effects of group work on pupils' perception of number patterns and its implications for generalization and algebraic rule writing still needs research verification.

The difficulties that second language speakers might find with the verbal expression of their generalizations and the influence that this might have on their ability to translate their generalizations into the algebraic code is another area for wider, more detailed research.

Finally, table 1.1 in section 4.1.1 shows a significant variation in the percentage return of the worksheets across the standard groups. Pupils' changing attitudes towards mathematics assignments across the different standards was suggested as a possible explanation. This phenomenon opens up further opportunities for research into this and other explanations for it.

5.4 Final Conclusion

Reflecting on the observed cognitive difficulties, I wish to conclude that the standard six and seven pupils in my study were not yet ready for the generalized number pattern approach to algebra. I contend that they will be ready, once their cognitive difficulties had been addressed via an informed mediational intervention; and, with sufficient exposure to number patterns over a period of time to allow the cognitive need for a more succinct symbolic representation system to mature.

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APPENDIX I(a)

Std.6... **Bellville South Senior Secondary** **1996**
Mathematics Project

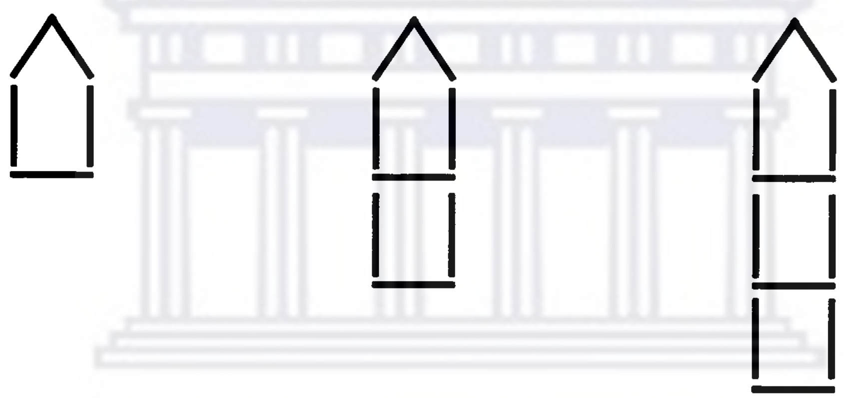
Name:..Age:....

<u>Instructions</u>:

- Do all your work on a loose sheet of paper and staple it to the back of this worksheet.
- Your solutions should be as complete as possible, showing the finest detail of all the processes and calculations involved. No calculation should be seen as unimportant if it helps you towards the finding of a solution.
- Hand in your attempt even if you think that you will never find the solution, as you may be only a short step away from the actual solution.

<u>Problem</u>

The following figures are made up with matches.



The first figure is made up of 5 matches, the second figure of 8 matches, the third figure of 11 matches, and so on.

<u>What you must do</u>.

1. Fill in the following table for the first six figures in the sequence.

Position of figure in sequence.	1	2	3	4	5	6
Number of matches	5	8	11			

2 (a) Find a way to work out how many matches you need to make up the 17th figure in the sequence.

(b) How many matches do you need to make up the 57th figure in the sequence?

3. If you have not done so yet, work out a general formula that you can use to determine the number of matches in any one of the figures in the sequence. Apply this formula to the n-th figure.

4. In what position in the sequence would you find the figure made up of exactly 98 matches?

APPENDIX I(b)

Std.7... **Bellville South Senior Secondary** **1996**
Mathematics Project

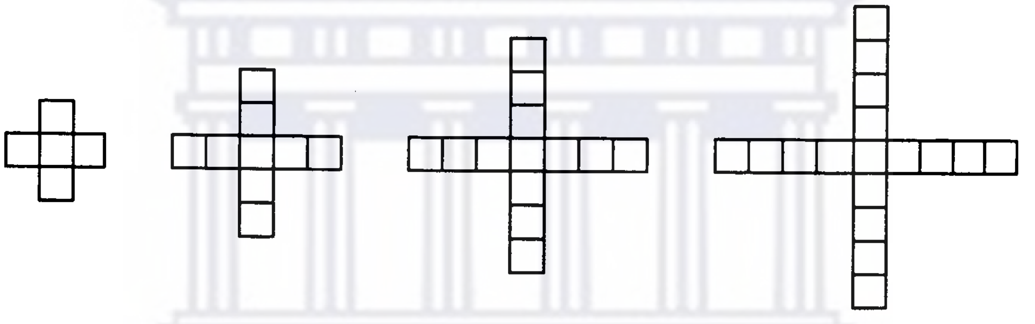
Name:..Age:.......

Instructions:

- Do all your work on a loose sheet of paper and staple it to the back of this worksheet.
- Your solutions should be as complete as possible, showing the finest detail of all the processes and calculations involved. No calculation should be seen as unimportant if it helps you towards the finding of a solution.
- Hand in your attempt even if you think that you will never find the solution, as you may be only a short step away from the actual solution.

Problem

A sequence of crosses of increasing size is made up of squares as shown.



The first cross consists of 5 squares, the second cross of 9 squares, the third cross of 13 squares, etc.

What you must do.

1. Fill in the following table for the first six crosses in the sequence.

Position of cross in sequence	1	2	3	4	5	6
Number of matches	5	9	13			

2(a) Find a way to work out how many squares you need for the 50-th cross in the sequence.
(b) How many squares do you need for the 504-th cross in the sequence?

3. If you have not done so yet, work out a general formula that can be used to determine the number of squares in any one of the crosses in the sequence. Apply this formula to the n-th cross in the sequence.

4. A cross consists out of 481 squares. In what position in the sequence would you find it?

APPENDIX I(c)

Std.8... **Bellville South Senior Secondary** **1996**

Mathematics Project

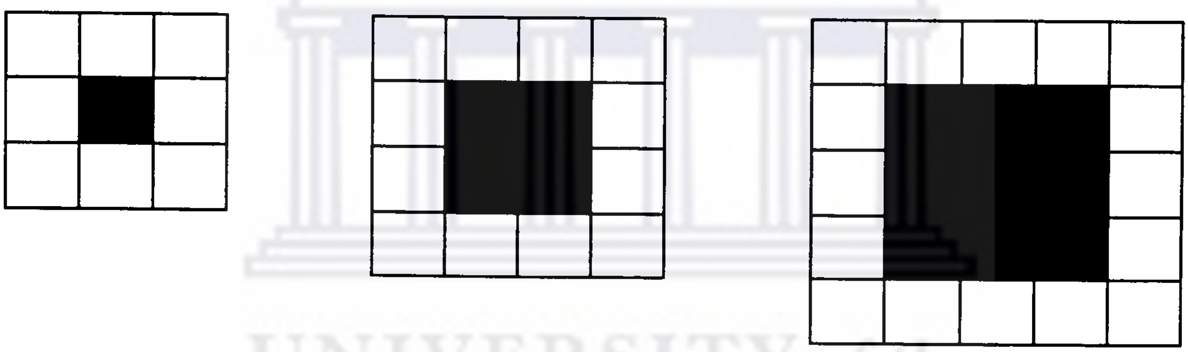
Name:...Age:.......

Instructions:

- Do all your work on a loose sheet of paper and staple it to the back of this worksheet.
- Your solutions should be as complete as possible, showing the finest detail of all the processes and calculations involved. No calculation should be seen as unimportant if it helps you towards the finding of a solution.
- Hand in your attempt even if you think that you will never find the solution, as you may be only a short step away from the actual solution.

Problem

The standard nines of Bellville South Secondary are planning to decorate their school. They want to build square flower beds of different sizes bordered with square tiles like in the sketches. The dark areas represent the flower beds.



What you must do.

1(a) Calculate how many tiles you need to form a border around a 10 by 10 flower bed.

(b) How many tiles do you need to form a border around a 100 by 100 flower bed?

2. 500 tiles are available for a border around a single flower bed. What is the largest flower bed that can be enclosed by it?

3. The tiles for a new flower bed had already been delivered when the decision is taken to increase the size of the bed by one tile-length per side. How many additional tiles must be ordered?

4. If you have not done so yet, construct a formula that can be used to calculate the number of tiles needed for a border around a ***n by n*** flower bed.

APPENDIX I(d)

Std.9... **Bellville South Senior Secondary** 1996

Mathematics Project

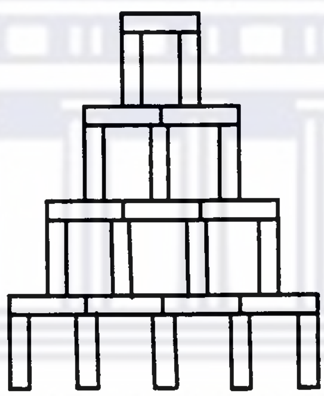
Name:..Age:.......

Instructions:

- Do all your work on a loose sheet of paper and staple it to the back of this worksheet.
- Your solutions should be as complete as possible, showing the finest detail of all the processes and calculations involved. No calculation should be seen as unimportant if it helps you towards the finding of a solution.
- Hand in your attempt even if you think that you will never find the solution, as you may be only a short step away from the actual solution.

Problem

A tower that has been constructed with dominoes is shown in the sketch. The tower is four stories high and 24 dominoes were used in the construction.



What you must do.

1. The following table gives the number of dominoes needed to complete a certain number of stories in the tower. Complete the table by filling in the missing values.

Height of tower in storeys	1	2	3	4	5	6	7
Number of dominoes	3	8	15	...	...	...	...

2. How many dominoes are needed to build the tower
(a) 10 storeys high;
(b) 70 storeys high?

3. Develop a general formula that can be used to calculate the number of dominoes needed to build the tower up to the n-th storey(the n-th storey can be any storey in the tower).

4. A complete set of dominoes consists out of 28 pieces. How high(in storeys) can the tower be built with 6 complete sets of dominoes?

APPENDIX I(e)

Std.10... **Bellville South Senior Secondary** **1996**
Mathematics Project

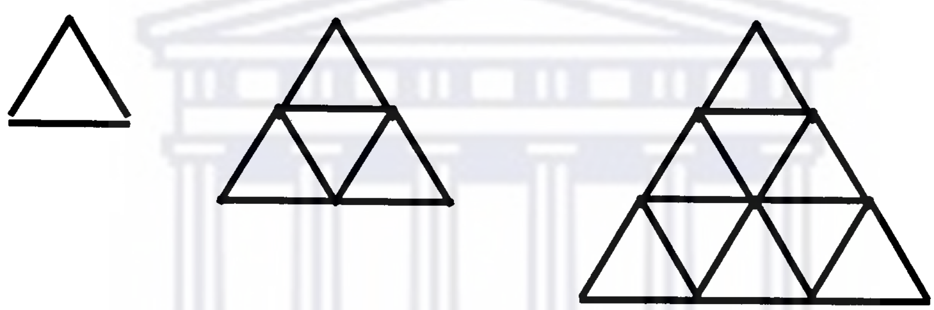
Name:..Age:.......

Instructions:

- Do all your work on a loose sheet of paper and staple it to the back of this worksheet.
- Your solutions should be as complete as possible, showing the finest detail of all the processes and calculations involved. No calculation should be seen as unimportant if it helps you towards the finding of a solution.
- Hand in your attempt even if you think that you will never find the solution, as you may be only a short step away from the actual solution.

Problem

The following figures are made with matches.



The first figure consists out of 3 matches, the second figure out of 9 matches, the third figure out of 18 matches, etc.

What you must do.

1. Complete the following table for the first six figures in the sequence.

Position of the figure in the sequence	1	2	3	4	5	6
Number of matches	3	9	18	....	....	....

2(a) Find a way to work out how many matches you would need to construct the 10th figure of the sequence.
(b) How many matches would there be in the 100th figure of the sequence?

3. If you have not done so yet, work out a general formula that can be used to calculate the number of matches in the n-th figure in the sequence. The n-th figure can be any one of the figures in the sequence.

4. You are supplied with 360 matches and instructed to construct the largest possible figure that would fit into the sequence. At what position would it fit into the sequence?

APPENDIX II

Fully Transcribed Interview on the Worksheet

I: Interviewer

S: Sello; 13 years old; Std.6

1) I: I see you drew a number of figures for number one. What was your plans with that?
2) S: I did that to get the answer quickly.
3) I: Okay, and did it help you.
4) S: Yes it did help me... to get the answer.
5) I: How did you get the answer from the picture?
6) S: It says to me I must just do it... then I just do it... it tells me the answer to the... I forgot it now... what is the answer.
7) I: You can have a look at your answer if it will help you. (Hands the answer sheet to the pupil)
8) S: Sir can you... I must(Pupil reads out question 2(a) out aloud)...
9) S: Sir again you must find way to work out how many matches you are going to need to make up the picture of the seventeenth.
10) I: Okay... so you actually drew some of the figures I see... and then? ... From there... what did you do then?
11) S: I just drew the pictures... and I get the answers.
12) I: How did you get the answers from that?
13) S: I just drew the pictures and after I drew it... it says I must draw the pictures to six than I do it... from five to six then I get the answers.
14) I: Did you count each of the matches?
15) S: Yes... yes...
16) I: For number two they want the number of matches in the seventeenth figure... Now I don't see that you drew the seventeenth figure... How did you do it then?
17) S: I drew it in the other book... in the other book... I didn't think I must do it to this page.
18) I: Now you have to tell me what you did in that book... because I can't see it.
19) S: I just did the pictures... then I get the answer... The answer was fifty-three.
20) I: Now wait...

21) I: And number three...?

(Pupil reads the question silently)

22) S: I didn't do it.

23) I: Do you understand the question?

24) S: Yes I do understand the question.

25) I: What does it say?

26) S: It says I must do the pictures to... to northwards... but I didn't do it.

27) I: To what?

28) S: To northwards.

29) I: What do you think is the meaning of that little part there that says "n-th" ?

30) S: I didn't know.

31) I: What did you think it was?

32) S: I think it is to north.

33) I: To north?

34) S: Yeah...

35) I: What is north?

36) S: To north... north!

37) I: You thought it was north... like in south, east, north?

38) S: Yeah...

39) I: Do you think that's possible?

40) S: No.

41) I: Why not?

42) S: Because in maths you don't use north.

43) I: And number four?

44) S: I didn't do it.

45) I: And why not?

46) S: Because... I didn't do it because... I can't count ninety-eight matches... I can't write down that... because I don't know what figure must I do... what number of matches must I use to make that figure.

47) I: Explain that again quickly because I don't really understand?

48) S: I didn't know how to do it... really... but I just used my matches... I didn't know what to do!

49) I: How did you use the matches?

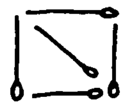
50) S: I did not...

...END OF THE INTERVIEW

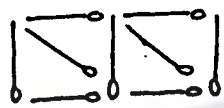
The Matchstick Pattern

Item one:

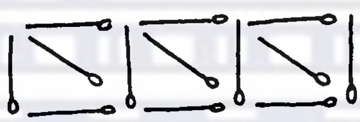
Pattern 1



Pattern 2



Pattern 3



Pattern 4

. . . .

Pattern 5

. . . .

APPENDIX IV

Fully Transcribed Interview on the Match Stick Pattern

I: Interviewer

B: Bonnie; 14 years old; Std.7

1) I: Bonnie tell me what you see.
2) B: Here sir...?
3) I: Yeah...
4) B: In the 'pattern' one there are five... match sticks, sir... in 'pattern' two there are nine... so the difference is... five and nine... is four... and you add it up to two... the difference is still four...
5) I: Okay...
6) B: And in 'pattern' four... there should be seventeen sir.
7) I: And how did you get that seventeen?
8) B: I add... more four sir... and more four stick mathces.
9) I: Okay... and in 'pattern' five?
10) B: It's twenty-one.
11) I: Okay... Do you know what the next 'pattern' looks like... 'pattern' four?
12) B: Yes sir.
13) I: Can you draw it?
14) B: Yes sir.
15) I: Would you like to draw it?
16) B: Yes sir!
17) I: Okay, draw it for me.
(*Pupil draws a sketch of 'pattern' four.*)
18) I: Bonnie ... now you just told me how many matches you had in each 'pattern'... Would you like to go throught that again? Perhaps write it in next to the figure.
19) B: Okay sir... Here there are...
20) I: 'Pattern' one...
21) B: 'Pattern' one there are five sir... in 'pattern' two there are nine... 'pattern' three there are thirteen... and 'pattern' four seventeen *(writing the numbers next to the corresponding 'patterns')*...
22) I: How many would you have 'pattern' five ?
23) B: Twenty-one.
24) I: Write it down as well.

(B. writes 21 on the dotted line.)

25) I: Now how many matchsticks would you need for the eighty-third 'pattern'?
26) B: Eighty-three sir?
27) I: Uhm...
28) B: Can I count it down sir?
29) I: Okay, but you must tell me what you will do.
30) B: Okay sir...
31) I: What will you do?
32) B: I'll... I'll multiply... of what multiply?... Yes.... Five... five.. eighty three... No!
33) I: Wow... I don't understand that... Explain again.
34) B: I'm going to multiply five by eighty-three sir... No(*softly*)...
35) I: Why do you say no?
36) B: It will be more... more than enough sir.
37) I: More than enough?
38) B: Yes sir.
39) I: But I need the exact amount of matches that you will need ... for your eighty-third 'pattern'.
40) B: Can I count it down sir?
41) I: Yeah...
42) B: (*Calculates on sheet 83 x 5 = 95.*) Ninety-five sir...
43) I: Why do you say eighty-three times... What did you do just now?
44) B: Eighty-three times five...
45) I: Wait... wait, I don't understand...
46) B: Eighty-three times five sir... sorry(*sees the mistake*)...
47) I: Scrath it out and do it over.
(*Pupil does the calculation again.*)
48) I: What did you do?
49) B: I multiplied sir... eighty-three by five...
50) I: Now why did you multiply by five?
51) B: Because in the first 'pattern' there are five sir.
52) I: Can you explain more?
53) B: Here sir?
54) I: Uhm...
55) B: In the first 'pattern' there are five... so I have to multiply by five sir...
56) I: Okay... Now can you describe, perhaps, the general rule that you used?

57) B: I used the multiplication rule.

58) I: Uhm...

59) I: But what is the method that you used to find the number of matches?

60) B: Eighty-three times five...

61) I: Yes... And if I said...for instance...the twenty-seventh 'pattern'... How many matches would you need then?

62) B: I'll multiply by five sir... because in the first 'pattern'... in the first 'pattern' there were five...

63) I: Write down the rule... write the rule...

64) B: The rule sir?

65) I: The general rule... that you used.

66) B: The multiplication rule... I'd have to multiply by five... if I had to multiply by eighty-three...

(*Pupil writes the rule down pronouncing the words as she progresses.*)

67) I: Very well... Now can you write that in mathematical terms?

(*No response.*)

68) I: All that you wrote down there is very well... but how do you write that in mathematics... using mathematics symbols?

69) B: Okay sir... (*Writes 83 x 5*)

70) I: Do you think that rule will work for any 'pattern'?

71) B: This rule sir?

72) I: Uhm...

73) B: For any 'pattern' given?

74) I: Yeah...

75) B: Yes sir.

76) I: Does it work for the second 'pattern'?

77) B: No sir... no sir! It doesn't work.

78) I: How can that be?

(*Pupil looks around, apparently for paper on which to write*)

79) I: You can write on this here.... (gives a sheet of paper).

(*Pupil appears to ponder something -- does not write immediately*)

80) I: What are you thinking?

81) B: Okay sir... I'll try... Here in the first 'pattern'... there are five matchsticks... so here... I don't feel I

have to add another five sir... because here is the one... who stands on this corner... so I have to add more... four only sir.

82) I: Uhm...

83) B: So that's why it becomes nine(*refers to the second pattern*)... that's why it doesn't become ten.

84) I: So what are you thinking?

85) B: (*Sighs*) Must I write this down sir?

86) I: I want to know what you're thinking.

87) B: Okay sir...

(*Pupil thinks quietly for a while.*)

88) B: Sir!

89) I: Yes...

90) B: Here... I've added four more... and here... up to the eighty-third sir...

91) I: Do I understand you?

92) B: Don't understand sir? Sir said so sir... Here I don't have to add the other five... and here again...(*mumbles something*).

93) I: So what are you saying?

(*No response. Pupil seems to be thinking very deeply.*)

94) I: What are you thinking?

(*Still no response. Pupil slowly shakes her head form side to side.*)

95) I: Why are you shaking your head?

(*Still no response.*)

96) I: Do you think that your rule of multiplying by five works?

97) B: Here? No sir.

98) I: In general?

99) B: No sir... It doesn't work in all the sums that is here.

100) I: Uhm... So what can you do?

101) B: What should I do sir?

102) B: If I'm given this sum only sir... then I'm asked to find this sum... what should I do?

103) I: I don't understand what you're asking.

104) B: Maybe if I'm given 'pattern' one only... then I'm asked to find the second 'pattern'... then what should I do?

105) I: What do you think?

(*No response.*)

106) I: What are your thoughts?

(*Still no response.*)

107) I: Come now... be honest.

108) B: (*Sighs*) Sir...

109) I: Uhm...

110) B: Here I have to add...

111) I: Yes... How many?

112) B: Four...

113) I: Okay...

114) B: Because I'm given five so I can't add another five...

115) I: Now I'm with you... Now how many mathcsticks would you need for the eighty-third 'pattern'?

116) B: Can I count it down sir?

117) I: How would you count it down... can you show me?

118) B: Okay I will show you sir...

(*Pupil thinks for a while, sighs and does not say or write anything.*)

119) I: What are you thinking?

120) B: I can't think now sir.

121) I: Do you have any idea of how to get to the number of matches in the eighty-third one?

122) B: No sir.

123) I: Why do you think you can't find the answer? You said:" Must I count it down sir?"... So I'm waiting for you to show me, but you do nothing. What do you mean with "count down"?

124) B: Sir I'm going to count it down...

(*Starts a written calculation*)

125) I: What are you doing now Bonnie.?

126) B: I'm counting sir...

(*Goes on and finishes calculation.*)

127) B: It's three-thirty-three sir.

128) I: Can you explain what you just did?

129) B: Sir... I've said eighty-three equals the eighty-third... multiplied by four... because I add four here(*points at second pattern*)... then my answer was three-thirty-two...

130) I: Uhm...

131) B: Then I add one... because... in the first 'pattern' there wasn't only one... there weren't only four matchsticks... but five...

132) I: Yeah...

133) B: So I added that one... Because here(*Points at the 83 x 4 part of the calculation*) I didn't add that matchstick... because I said its eighty-three divided by four...

134) I: Divided by four?

135) B: Sorry... multiplied by four.

136) I: Uhm...

137) B: So then I added this one... for the first piece over here.

138) I: Now quickly explain it right through... from the start.

139) B: From here sir(*points at the start of the calculation*)?

140) I: Uhm...

141) B: Okay... I said eighty-three times four... then I got three-thirty-two... but... the first 'pattern' had five matchsticks... so I had to add that fifth one... then the answer was three-thirty-three sir.

142) I: Okay... Can you write that rule down in words?

143) B: Okay sir...

144) I: Do you think there is a general rule? ... Does that apply to all the figures?

145) B: Sir? ... No sir... Yes sir... It does.

146) I: Okay, write it down... in general.

(*Writes down the rule in words*)

147) I: Can you write that down as in mathematics?

148) B: In mathematical order?

149) I: Uhm... using mathematical symbols...

(*Pupil writes 83 x 4 + 1 = ...*)

150) I: Do you think it's possible that you can use symbols?

(*No response.*)

151) I: Are you happy with that?

152) B: Yes sir.

153) I: Uhm...but that is only the answer to one particular sum... to the eighty-third one... Can you write down something more general... something that will apply to all sums... all 'patterns'?

154) B: It's this one sir...(*points at the one in the written expression and underlines it.*)

155) I: What about the one?

156) B: I'll have to add it... its all you always add... in all of these patterns... because here sir... in the third 'pattern'... maybe if you ask me to find the third 'pattern'... then I'll say four times three... then I'll get twelve... plus one... I'll have to add that one... because only in the first 'pattern' there are five matchsticks...

157) I: Now let me ask you something... Do you think you could use an '*x*' somewhere in the problem?

158) B: An 'x'(*emphasizing the x*) sir!

159) I: Why you say an 'x'(*emphasizing the x*)? Don't you work with x's in class.

160) B: Yes sir.

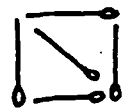
162) I: Do you want to write something down?

163) B: No sir.

END OF THE INTERVIEW

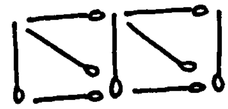
Item one:

Pattern 1



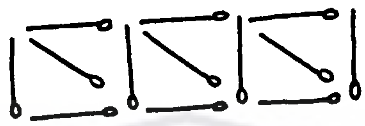
5

Pattern 2



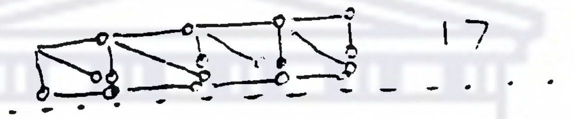
9

Pattern 3



13

Pattern 4



17

Pattern 5

21

83
×5

If I had to find the answer of this question how will many matches stick will yo use to find 83 the 83rd parten?

83
5
415

I would multiply 83 by 5 because 5 was the 1st parten given

$$\begin{array}{r} 83 \\ \times\ 4 \\ \hline 332 \\ +\ 1 \\ \hline 333 \end{array}$$

first I had to multipthy 83 by 4 and add 1 for the first patten because the first patten had 5 matthes stick

$$\begin{array}{r} 83 \\ \times\ 1 \end{array}$$

$83 \times 4 + 1 =$

APPENDIX V

The Function Table

Item two:

Top row	1	2	3	4	5	6	...
Bottom row	2	5	8	11			

19) A: Five times four is twenty... plus one is twenty-one...

20) I: Okay... Number six in the top row...?

21) A: Uhm... twenty-four... plus one is twenty-five.

22) I: Explain again...

23) A: Uhm... six times four... is... twenty-four... plus one is twenty-five.

24) I: So if the top row number is seven, what will the bottom row number be?

25) A: I see that you bring up the...the numbers also that you're supposed to multiply... because here you can't multiply this by four... it's one this... this is three... this is four...

26) I: I don't think I understand what you are trying to say... can you explain a little more?

27) A: Uhm... this is two... The first two numbers one and two... and at the bottom there's three and five... you have to multiply by two... one you have to multiply by two... and two you have to multiply by two... And here by three and four...eight and eleven... you multiply by three... and here you multiply by four... and the others will probably go on...five; six; seven; ...

28) I: Now could you do seven and eight... just to show me?

29) A: Okay(*softly*)... Uhm... Seven times five is... is thirty-five...minus one is...thirty-four...

30) I: And number eight?

31) A: Is... Eight times five is forty... minus one is thirty-nine...

32) I: Now what do you think will happen if I give you a large number... let's say in the top row you have eighty...what do you think will be the bottom number?

(*Pause as she ponders the question*)

33) I: What do you think now?

34) A: And the bottom number will be?

(*Pause again*)

35) I: What are you thinking?

36) A: I think three-hundred-and-ninety.

37) I: Can you explain?

38) A: I think because this... uhm... you add a naught to the eight and a naught to the bottom number too...

39) I: Okay... The rule that you used up to number eight... can you describe that?

40) A: Uhm... It's the top row... okay... nothing added or nothing to the top row... uhm... it's just 1; 2; 3; 4; 6; 7; 8;... common numbers... The bottom row... uhm... you must... the first two numbers, one and two, you must multiply by two... and... uhm... one times two is two... and then you must... here by number two... two times two is four, plus one... you must add one... plus one gives you five... here by number three and number four... you must multiply by three again... uhm... three times three is nine, then you must subtract one... then it gives you eight... and here the same(*four in the top row*), its twelve, you must subtract one and it gives you eleven... and here you start again by addition... and so you go on...

41) I: Do you think you can write it down?
(*No response*)

42) I: Write down how to get the bottom number from the top row number.

43) A: Okay(*writes the examples as she explains*)... Take three and four... Uhm... Here by number three and number four... three times three will give you nine, minus one equals eight and that's your bottom number... And here by four...

44) I: Perhaps you should indicate which is the top number and which is the bottom number.
(*A indicates by circling the numbers and writing the words "top number" or "bottom number"*)

45) A: It's always the first number is the top number that you write down and the last number is the bottom number.
(*A writes the example of the fourth number in the top row*)

46) A: The same with the five and the six...
(*A starts to write down the example with five in the top row*)

47) A: It goes like this... three and four is subtract one... and five and six is add one... and seven and eight is subtract one... it's like that... So five and six will be... five times four equals twenty, plus one, equals twenty-one... that will be your bottom number... same with six(*A continues to write down the example with five in the top*

row as she explains)... Five is your top number (*referring the example with five in the top row)* and it will always be first... uhm... the first thing that you write down... it will be five(*emphasizing the word*) times four... equals twenty... and twenty plus one is twenty-one... that's your bottom number... that is your last number that you write down... and the same with six... six is your first number, that's your top number, times four equals twenty-four... plus one equals twenty-five and that is your bottom number... that is the last number...

48) I: Okay... Can you perhaps write it in mathematical form... using symbols?

49) A: Must I give my own symbols and so on?

50) I: Yes, if you wish you can give your own symbols.

51) A: Okay... Take three and four again... The top number... is three, so... the three...

52) I: Can you write the symbol down?

53) A: You make an *x*... so... no I can't see... three times *x* ... because three and four will be multiplied by three... both of them will be multiplied by three... so it's *x*... Three is x... three... the number that you multiply... the top number is three... like four times three... and three times three...

(*A. writes:* $3 \times x = 9 - y = 8$)

54) I: But didn't you say you write the top number first?

55) A: Yes... so this is three...

56) I: Now show with a little arrow which is the top number again.

57) A: This is the top number(*indicates it with a the letter T*).

58) I: What is *x*?

59) A: Times *x*... *x* is the number that you multiply the three(*top number*) by.

60) I: Okay... go on...

61) A: This is equal to... three time three is equal to nine(*writes nine*) and you subtract one... subtract *y*... *y* will be one(*indicates it with the number 1*), because it will be used in the top number in number three and number

four... equals eight... and eight is your bottom number(*indicates it with the letter B*)...

62) I: Now what did you say the *y* was again?

63) A: *y* is a minus one... is one... *y* will be one... and *x* is(*writes down "number you multiply by"*)... *x* is the number that you multiply by and *y* is the number one... so it reads like this: The top number is three, so three will always be first... on top... and the bottom number is eight... because... in the bottom it will always be last because it's at the bottom... Uhm... so it reads this way: Three times *x*, and *x* is the number that you multiply by, and in this case *x* is three... so three times three, that is *x*, equals nine... and minus *y*, and *y* will be one... and that will equal eight... and that is the bottom number...

64) I: Can you make another example? Just one more example... Use another number... say thirteen.

65) A: Thirteen and fourteen(*draws a little table and writes 13 an 14 in the top row*).

66) I: Why do you always use two numbers?

67) A: It's because... Uhm... I always use two numbers because... here its like in groups... it's in groups of two... its the amount you multiply by... the one and two... okay, hold on... take five and six is multiplied by four... three and four is multiplied by three... it's like... this is 2; 3; 4(*indicates what numbers to multiply by above the groups in the table*)... in groups... groups of two(*extends the table -- writing the numbers in a group together*)... 6; 7; 8(*writing this above the groups in the extension to indicate what to multiply by)*...

68) I: I don't understand...times eight.

69) A: This will be... uhm... say you go on here(*referring to the extended table*)... 7 and 8... 9 an 10 will be together... 11 and 12... 13 and 14... So this will be... this is 1 and 2 multiplied by two; 3 and 4 multiplied by three; 5 and 6 multiplied by four; 7 and 8 multiplied by five... and this is six(*9 and 10*); this is seven(*11 and 12*); and this is eight(*13 and 14*)...

70) I: Okay...

71) A: So you multiply... uhm.... (*A does the calculation 13 x 8 = 104 in writing*)... So in this case, number thirteen, the bottom number will be hundred-and-four.

72) I: Can you explain again... through all the steps.

73) A: Okay... thirteen you must multiply by eight... thirteen and fourteen is grouped together in two... thirteen and fourteen are together, so they will be multiplied by one number, and that is eight... now I multiplied thirteen by eight and that gives me hundred-and-four... so the bottom number of number thirteen will be hundred-and-four...

74) I: And what about the *y*-number?

75) A: That you must... It's also grouped in two's... The first... Okay... The even numbers... is always you must add one...

76) I: What even numbers?

77) A: If you must... uhm... multiply by an even number, then it will be plus one... so if you must multiply by an uneven number, then it will be subtract one...

78) I: Okay, so what will you do with thirteen?

79) A: Here you must multiply by eight, so you must add one... plus one.

80) I: So what is your bottom number now?

81) A: It's... hundred-and-five... So it's thirteen times eight... it's hundred-and-four... plus one... it's hundred-and-five... so I'm gonna write it down...

(*A writes: 13 x X = 104 + Y = 105*)

82) A: Thirteen times *x*... and *x* is the number you multiply by... will give you hundred-and-four... so it will be plus *y*, and *y* is number one... that will give you hundred-and-five... so hundred-and-five is your bottom number... and thirteen will be your top number!

83) I: Thank you very much!

END OF THE INTERVIEW

-oOox2 x3 ~~x3~~x4 x5

Top row	1	2	3	4	5	6	7.	8
Bottom row	2	5	8	11	21	25	34	39

+1 -1 -1

+1 +1 -1 -1 +1

x6 x7 x8

9 | 10   11 | 12   13 | 14

80

~~39~~

390

3 | 4

3	4
8	11

5	6
21	25

TOP NUMBER 3 x 3 = 9 - 1 = 8 BOTTOM NUMBER

TOP 4 x 3 = 12 - 1 = 11 BOTTOM

T 5 x 4 = 20 + 1 = 21 B

T 6 x 4 = 24 + 1 = 25 B

3	4
8	11

$$\begin{array}{r} {}^{2}13 \\ \times\ 8 \\ \hline 104 \end{array}$$

x8

x8

13	14
105	

NUMBER YOU MULTIPLY BY

1

T 3 x x = 9 - y = 8 B

+1

~~T 13 x 5 =~~

T 13 x x = 104 + y = 105 B