GENERALIZATION OF A THEOREM OF FITTING ON THE PRODUCT OF TWO NORMAL NILPOTENT SUBGROUPS OF A GROUP

by

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To Daddy and Mummy

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S U M M A R Y

H. Fitting proved that the product of two normal nilpotent subgroups H and K of a group, is itself nilpotent.

Several authors have proved statements of the following type:

(A) If H and K are normal subgroups of a group G and if $H \in P$, $K \in P$ then $HK \in P$, where P is a group theoretical property.

We have considered the question of to what extent the requirement that H and K be normal can be relaxed in (A). This is done by replacing normal by subnormal or serial.

CONTENTS

CHAPTER 1

FITTING'S THEOREM FOR NILPOTENT SUBGROUPS

§1.1 INTRODUCTION

H. Fitting proved that if H and K are normal nilpotent subgroups of G, then so is HK ([1]` Hilfssatz 10, p. 100). The question arises if this result could be generalized to other group theoretical properties.

If a group G has normal E-groups (groups with property E) H and K and if HK is also an E-group then E is called a *multiproperty*. (1.1)

Theorems of this type have been proved by a number of authors. We have the well-known Hirsch-Plotkin Theorem (See [10] and [13]) that local nilpotence is a multiproperty. P. Hall ([6]) proved hypercentrality is a multiproperty. FC - nilpotency and FC - hypercentrality turn out to be multiproperties. This was shown by K.K. Hickin and J.A. Wenzel ([9]). H. Heineken and I.J. Mohamed ([8]) proved that both the normalizer condition and the subnormality condition are not multiproperties.

The question we are to consider is whether the requirement that H and K be normal in (1.1) can be relaxed. This will be done by replacing normality by subnormality or serial in some of the results mentioned above.

§1.2 NOTATION

Let H and K be subgroups of a group G.

If there exists a series

$$H = H_0 \lhd H_1 \lhd H_2 \lhd \dots \lhd H_n = G$$

we say that H is *n-step subnormal* in G and follow the well-known notation due to P. Hall ([7]) by writing $H \lhd^n G$.

If there exists an ascending series of subgroups H_α linking H to G such that

$$H_\alpha \lhd H_{\alpha+1}$$

and

$$H_\alpha = \bigcup_{\gamma<\alpha} H_\gamma$$ for all limit ordinals α, we say that H is *serial* in G and following Gruenberg ([2]) write $H \infty\lhd G$.

For $x_1, x_2 \in G$ the *commutator* $x_1^{-1}x_2^{-1}x_1x_2$ would be denoted by $[x_1, x_2]$ and more generally for k>1

$$[x_1, \dots, x_{k+1}] = [[x_1, \dots, x_k], x_{k+1}].$$

The convention is adopted that for k=0 , $[x_1, \dots, x_{k+1}] = x_1$.

The following well-known standard commutator identities ([4]) will often be referred to:

$$[xy,z] = [x,z]^y [y,z] \qquad (1.2)$$

$$[x,yz] = [x,z][x,y]^z \qquad (1.3)$$

$$[x^{-1},y] = [y,x]^{x^{-1}} \qquad (1.4)$$

$$[x,y^{-1}] = [y,x]^{y^{-1}} \qquad (1.5)$$

The *commutator group* $[H,K,K,\ldots,K]$ with n terms K, is written $[H,_nK]$ with the convention that $[H,_oK] = H$.

The notation $\gamma_m(H)$ denotes $[H,_{m-1}H]$, $m\geq 1$, the *terms of the lower central series* of H.

Thus H is *nilpotent* of class n if $\gamma_{n+1}(H) = 1 \neq \gamma_n(H)$.

As usual the *terms of the upper central series* of H shall be written $1 = Z_o(H)$, $Z_1(H),\ldots,$ $Z_i(H)$ or simply Z_i if H is understood, where

$$Z_1 = \text{the centre of } H.$$

$$\frac{Z_{i+1}}{Z_i} = \text{the centre of } \frac{H}{Z_i}$$

$$Z_\gamma = \bigcup_{\alpha<\gamma} Z_\alpha \text{ if } \gamma \text{ is a limit ordinal.}$$

A group G is a *ZA-group* if and only if its upper central chain, possibly continued transfinitely, leads to the group G.

The *normal closure* of H in G is the smallest normal subgroup of G which contains H and is denoted by H^G. Clearly $H^G = H[H,G]$.

A group G is *locally - nilpotent* if every finitely-generated subgroup of G is nilpotent.

Let G be a group:

$F_o(G) = 1$, the unit subgroup.

$F_1(G)$ is the set of elements of G which posses a finite number of conjugates.

$F_{\alpha+1}(G)$ is defined inductively to be the complete inverse image of $F_1\left(\frac{G}{F_\alpha(G)}\right)$, for all ordinals α.

$F_\alpha(G) = \cup\{F_\beta(G) : \beta<\alpha\}$, if α is a limit ordinal.

For all ordinals α, $F_\alpha(G)$ is a characteristic subgroup of G.

A group G is called *FC-nilpotent* of class n if there exists an integer n such that $F_{n-1}(G) \neq G$ and $F_n(G)=G$.

G is called *FC-hypercentral* of class α if there exists an ordinal α such that $F_\beta(G) \neq G$ for $\beta<\alpha$ and $F_\alpha(G) = G$.

§1.3 FITTING'S THEOREM

Fitting's Theorem that the product MN of normal nilpotent subgroups M and N of a group G is nilpotent, is well-known and easy proofs can be found in textbooks (see for example [4]).

The question, however, arises if it is possible to describe the lower central series (upper central series) of MN in terms of the lower central series (upper central series) of M and the lower central series (upper central series) of N. We give an inclusion relation for the lower central series in Theorem 1.4 below. To facilitate the proof of this, we give a set of generators for $\gamma_k(\langle M,N\rangle)$ for subgroups M and N of a group G in Lemma 1.1 and its corollaries.

Lemma 1.1

If M and N are subgroups of the group G, then $\gamma_k(\langle M,N\rangle) = \langle [x_1,\ldots,x_k]^y : \forall y\in\langle M,N\rangle, \forall x_i \ni \text{ either } x_i\in M \text{ or } x_i\in N\rangle$.

Proof:

The proof is by induction on k. Clearly for k=1, the lemma is trivially true by definition of commutators. Assume the result is true for $1\leq r<k$. Then by the commutator identities in §1.2 $\gamma_k(\langle M,N\rangle)$ is generated by $[[x_1,\ldots,x_{k-1}],y]$ and all their conjugates in $\langle M,N\rangle$ for all x_i such that either $x_i\in M$ or $x_i\in N$ and $y\in\langle M,N\rangle$. By the commutator identities

$[x_1,\ldots,x_{k-1},y]$ is a product of commutators $[x_1,x_2,\ldots,x_{k-1},x_k]$ and their conjugates in $\langle M,N\rangle$, where either $x_k \in M$ or $x_k \in N$. This proves the lemma.□

The following two corollaries are but special cases of the lemma.

Corollary 1.2

If M and N are subgroups of G and if $N \triangleleft G$ then

$$\gamma_k(MN) = \langle [x_1,\ldots,x_k]^y : \forall y \in N, \text{ either } x_i \in M \text{ or } x_i \in N\rangle.$$

Corollary 1.3

If $M \triangleleft G$, $N \triangleleft G$ then

$$\gamma_k(MN) = \langle [x_1,\ldots,x_k] : \forall\ x_i \ni \text{ either } x_i \in M \text{ or } x_i \in N\rangle.$$

These corollaries follow since conjugation is a homomorphism.□

Theorem 1.4

If M and N are normal, nilpotent subgroups of G of nilpotency class m and n respectively, then

$$\gamma_k(MN) \leq \begin{cases} \gamma_k(M)\gamma_k(N) \prod_{s=1}^{k-1} \gamma_s(M) \cap \gamma_{k-s}(N) & \text{for } k>1 \\ \gamma_k(M)\gamma_k(N) & \text{for } k=1 \end{cases}$$

and MN is nilpotent of class at most m+n.

Proof:

The proof is by induction on k. The result is trivially true for k=1. Suppose true for k-1 (k>1).

By Corollary 1.3 of Lemma 1.1, $\gamma_k(MN)$ is generated by the commutators $[x_1,x_2,\ldots x_k]$ for all x_i such that either $x_i \in M$ or $x_i \in N$.

Consider the generator $[x_1,\ldots,x_k]$. If none of the x_i is an element of M, then $[x_1,\ldots,x_k] \in \gamma_k(N)$. On the other hand if none of the x_i is an element of N, then $[x_1,\ldots,x_k] \in \gamma_k(M)$.

Suppose now that s, $(s<k)$, be the number of x_i which are elements of M. Then k-s of the x_i are elements of N and so clearly since $M \triangleleft G$, $N \triangleleft G$, $[x_1,\ldots,x_k] \in \gamma_s(M) \cap \gamma_{k-s}(N)$. Thus

$$\gamma_k(NM) \leq \gamma_k(M)\gamma_k(N) \prod_{s=1}^{k-1} \gamma_s(M) \cap \gamma_{k-s}(N).$$

If we put $k = m+n+1$ then

$$\gamma_{m+n+1}(MN) \leq \prod_{s=1}^{m+n} \gamma_s(M) \cap \gamma_{m+n+1-s}(N) = 1.$$

For if $s \geq m+1$ then $\gamma_s(M) \cap \gamma_{m+n+1-s}(N) = 1$ since M is nilpotent of class m, while if $s<m+1$ then $m+n+1-s \geq n+1$ and so again $\gamma_s(M) \cap \gamma_{m+n+1-s}(N) = 1$, since N is nilpotent of class n. Thus MN is nilpotent of class $\leq m+n$. □

It appears unlikely that the equality holds in the inclusion relations in Theorem 1.4 for $1<k<m+n+1$ and this question will not be considered any further. However, a few simple consequences of the theorem must be noted. These give some conditions under which the bound m+n for the nilpotency class of MN is not attained.

Corollary 1.5

If $\gamma_s(M) \cap \gamma_{k-s}(N) = 1$ for $1 \leqslant s \leqslant k-1$ then MN is nilpotent of class at most max (m,n).

This result is immediately clear if one notes that if $k = \max(m+1, n+1)$ then

$$\gamma_k(MN) \leqslant \prod_{s=1}^{k-1} \gamma_s(M) \cap \gamma_{k-s}(N). \quad \square$$

Corollary 1.6

If $\gamma_m(M) \cap \gamma_n(N) = 1$, then MN is nilpotent of class $< m+n$.

If we choose $k = m+n$ then

$$\gamma_{m+n}(MN) \leqslant \gamma_m(M) \cap \gamma_n(N). \quad \square$$

Corollary 1.7

If $M \cap \gamma_n(N) = 1$ and M is abelian or $\gamma_m(M) \cap N = 1$ and N is abelian then MN is nilpotent of class at most n or m.

In the first case chosing $k=n+1$

$$\gamma_{n+1}(MN) \leqslant M \cap \gamma_n(N)$$

while in the second case one chooses $k = m+1$ and

$$\gamma_{m+1}(MN) \leqslant \gamma_m(M) \cap N. \quad \square$$

The bound obtained in Theorem 1.4 is a least upper bound. As no example of this could be found in the literature, such an example will be given here. To do this the following result which is due to P. Hall ([5]), is needed.

Lemma 1.8 (P. Hall. [5]).

If V is a vector space over the prime field of p elements with basis (v_n), n=0, ± 1, ± 2,... and ξ and η are linear transformations of V defined by

$$v_n\xi = v_{n+1} \quad \text{for all } n$$

and

$$v_o\eta = v_o + v_1 ; \; v_n\eta = v_n \text{ if } n \neq 0$$

then the group $\widetilde{G} = \langle \eta_1, \eta_2, \ldots, \eta_{m+n} \rangle$ of linear transformations of V, where $\eta_i = \xi^{-i}\eta\xi^i$ and η, ξ are defined above, is nilpotent of class at least m+n.

Proof:

The first step is to show that

$$v_i\eta_i = v_i + v_{i+1}$$

and

$$v_j\eta_i = v_j \quad \text{if } j \neq i.$$

Now we have

$$\begin{aligned} v_i\eta_i &= v_i(\xi^{-i}\eta\xi^i) \\ &= v_{i-i}(\eta\xi^i) \\ &= v_o(\eta\xi^i) \\ &= (v_o + v_1)\xi^i \\ &= v_i + v_{i+1} \end{aligned}$$

and
$$v_j\eta_i = v_{j-i}(\eta\xi^i)$$
$$= v_{j-i}(\xi^i)$$
$$= v_{j-i+i}$$
$$= v_j$$

Next one has to show that $\eta_i^p = 1$, for each i.

Now
$$v_i\eta_i^p = (v_i\eta_i)\eta_i^{p-1}$$
$$= (v_i+v_{i+1})\eta_i^{p-1}$$
$$= v_i+pv_{i+1}$$
$$= v_i$$

Therefore $\eta_i^p = 1$, for each i.

An easy induction shows that
$$v_1[\eta_1,\ldots,\eta_{m+n}] = v_1+v_{m+n+1}.$$

It follows in the same way as above that
$$v_1[\eta_1,\ldots,\eta_m] = v_1+v_{m+1}$$
and
$$v_{m+1}\ [\eta_{m+1},\ldots,\ \eta_{m+n}] = v_{m+1}+v_{m+n+1}.$$

It can be shown that each η_j commutes with all its conjugates in $\tilde{G}$.

Let Y_i be the subgroup generated by the conjugates of η_i in $\tilde{G}$. Then $Y_i \triangleleft \tilde{G}$ for each i, and $\tilde{G} = Y_1Y_2\ldots Y_{m+n}$. Since each η_i commutes with all its conjugates in $\tilde{G}$, the

Y_i are all abelian. By Fitting's Theorem $\tilde{G}$ is nilpotent of class at most m+n. But $[\eta_1,\ldots,\eta_{m+n}]$ maps v_1 onto v_1+v_{m+n+1} and so $\tilde{G}$ is of class at least m+n.

Let $A = Y_1Y_2\ldots Y_m$ and $B = Y_{m+1}\ldots Y_{m+n}$. By Fitting's Theorem, A is nilpotent of class at most m and B is nilpotent of class at most n. □

Theorem 1.9

There exists a group G with normal, nilpotent subgroups M and N of classes m and n respectively such that MN is nilpotent of class precisely m+n.

Proof:

Let G be the group generated by the elements $x_1,x_2,\ldots,x_{m+n}$ subject to the defining relations

$$x_i^p = 1,\ i = 1,2,\ldots,m+n,\ p \text{ a prime} \qquad (1.6)$$

and x_i commutes with all its conjugates in G for each $i = 1,2,\ldots,m+n$. (1.7)

Such a group G exists because x_i commutes with all its conjugates in G if and only if $[g,\ {}_2x_i] = 1\ \forall\ g\in G$ and so G is the group with defining relations

$$x_i^p = 1, \quad [g,{}_2x_i] = 1 \quad \forall \quad g\in G$$

and has factor group the elementary abelian group

$$\bar{G} = \langle x_i : x_i^p = 1 = [x_i,x_j]\rangle.$$

Let X_i be the subgroup generated by the conjugates of x_i

in G.

Then $X_i \lhd G$ for each i, and $G = X_1X_2\ldots X_{m+n}$.

By (1.7) the X_i are all abelian. Hence G is nilpotent of class at most m+n, by Fitting's Theorem.

Let $M = X_1X_2\ldots X_m$ and $N = X_{m+1} \ldots X_{m+n}$. By Fitting's Theorem M is nilpotent of class at most m and N is nilpotent of class at most n.

Let $\tilde{G}$, A and B be the groups defined in Lemma 1.8.

The mapping $\phi: x_i \to \eta_i$ $i=1,2,\ldots,m+n$ defines a homomorphism of G onto $\tilde{G}$. Consequently the nilpotency class of G cannot be less than m+n.

The mapping $\phi_1: x_i \to \eta_i$, $i=1,2,\ldots,m$ defines a homomorphism of M onto A. But $[\eta_1,\ldots,\eta_m]$ maps v_1 onto v_1+v_{m+1} and so A is of class at least m and consequently the class of M cannot be less than m.

Similarly the mapping $\phi_2 : x_i \to \eta_i$, $i=m+1,\ldots,m+n$ defines a homomorphism of N onto B. But $[\eta_{m+1},\ldots,\eta_{m+n}]$ maps v_{m+1} onto $v_{m+1} + v_{m+n+1}$ and so B is of class at least n.

Consequently the class of N cannot be less than n.

Hence we have proved that the nilpotency classes of MN, M and N are precisely m+n, m and n respectively. This proves the theorem. □

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CHAPTER 2

GENERALIZATION OF FITTING'S THEOREM FOR NILPOTENT GROUPS

Fitting's Theorem cannot be generalized by replacing $M \triangleleft G$ (or $N \triangleleft G$) by an arbitrary nilpotent subgroup M of G (or N of G). The symmetric group on three symbols shows this clearly since it can be generated by two cyclic subgroups, one of which is a normal subgroup.

In view of this example it seems natural to enquire if the conclusion of Fitting's Theorem remains true by replacing N and M normal subgroups of G by generalizations of normal subgroups. Thus we would like to consider replacing N and M normal by N and M subnormal or even serial. Robinson ([14]) proved that if M is subnormal in r steps in G and N normal in G then the conclusion of Fitting's Theorem still holds. An alternative proof of this result is given here.

Theorem 2.1

If $N \triangleleft G$, $M \triangleleft^r G$, $\gamma_{n+1}(N) = 1 = \gamma_{m+1}(M)$ then MN is nilpotent of class at most rn+m.

Proof:

The case r=1 is Fitting's Theorem and thus provides a basis for induction on r. Assume the result is true for all groups in which M is subnormal in fewer than r steps.

Since for any two subgroups H and K of a group G, $[H,K] \triangleleft \langle H,K\rangle$, we have that

$$[N,_r M] \triangleleft [N,_{r-1}M] \triangleleft \ldots \triangleleft [N,M] \triangleleft \langle N,M\rangle$$

and therefore

$$M = M[N,_r M] \triangleleft M[N,_{r-1}M] \triangleleft \ldots \triangleleft M[N,M] \triangleleft \langle N,M\rangle$$

Thus M is subnormal in at most r-1 steps in M[N,M], while $[N,M] \triangleleft M[N,M]$. But M and [N,M] are nilpotent of classes m and n at most and so by the induction hypothesis the product M[N,M] is nilpotent of class (r-1)n+m at most. N and M[N,M] are normal nilpotent subgroups of $\langle N,M\rangle$ and so by Fitting's Theorem their product MN is nilpotent of class (r-1)n+m+n=rn+m at most. □

Theorem 2.1 suggests that the least upper bound of the nilpotency class of G = MN with $N \triangleleft G$, $M \triangleleft^r G$ is an increasing function of r (as well as of n and m). Thus it appears unlikely that the condition $M \triangleleft\triangleleft G$ can be relaxed to $M \infty\triangleleft G$. The next example shows that the condition $M \triangleleft\triangleleft G$ cannot be relaxed to $M \infty\triangleleft G$.

Theorem 2.2

There exists a non-nilpotent group G with abelian subgroups H and K, $H \triangleleft G$, $K \infty\triangleleft G$ and G = HK.

Proof:

Let H be the free abelian group on an infinite set of generators a_0, a_1, a_2, ...

The map b which maps

$$a_j \to a_j a_{j-1},\ a_o \to a_o \quad j=1,2,\ldots$$

can be extended to a homomorphism of H. b maps the generators onto a set of generators.

Let b^{-1} denote the inverse of b then

$$b^{-1} : a_j \to a_j \prod_{i=1}^{j} a_{j-i}^{(-1)^i}, \quad j \geq 1$$

$$a_o \to a_o.$$

Hence b defines an automorphism of H. Denote the subgroup of Aut(H) generated by b by K. Let G be the holomorph of H with respect to K and identify H and K with their images in G. Then G = HK and satisfies the relations

$$[a_i,a_j] = 1,\ [a_i,b] = a_{i-1},\ [a_o,b] = 1$$

Since

$$K\langle a_o,a_1,\ldots,a_n\rangle \triangleleft K\langle a_o,a_1,\ldots, a_{n+1}\rangle,$$
$$K^{\infty} \triangleleft G.$$

Thus G is a product of the normal abelian group H and the serial abelian subgroup K but is not nilpotent since $\gamma_n(G) = H$ for n>1. □

Robinson's result proved in Theorem 2.1 can be stated in a more general form, namely:

Thoerem 2.3

If P is a multiproperty of groups and is also inherited by subgroups then if $N \triangleleft G$, $M \triangleleft\triangleleft G$ and $N \in P$, $M \in P$ then $MN \in P$.

Proof:

Suppose M is subnormal in r steps in G. For r=1 the theorem is true since P is a multiproperty. Assume the result is true for all groups in which M is subnormal in fewer than r steps.

Since for any two subgroups H and K of a group G, $[H,K] \triangleleft \langle H,K\rangle$, we have that

$$[N,_r M] \triangleleft [N,_{r-1}M] \triangleleft \ldots \triangleleft [N,M] \triangleleft \langle N,M\rangle$$

and therefore

$$M = M[N,_r M] \triangleleft M[N,_{r-1}M] \triangleleft \ldots \triangleleft M[N,M] \triangleleft \langle N,M\rangle$$

Thus M is subnormal in at most r-1 steps in M[N,M], while $[N,M] \triangleleft M[N,M]$. But $M \in P$ and $[N,M] \in P$ and so by the induction hypothesis the product $M[N,M] \in P$.

M[N,M] and N are normal subgroups of $\langle M,N\rangle$. Hence $MN = M[N,M]N \in P$ since P is a multiproperty. □

The conclusion of Fitting's Theorem, however, does not hold if one insists that both N and M are subnormal of indices of subnormality greater than one.

D.S. Robinson ([14] section 5; page 155) defines C to be the class of all groups in which each pair of subnormal subgroups generates a subnormal subgroup. He then constructs an example of a group which is not in the class C.

Robinson attributes this kind of construction to P. Hall. This example is to be used to establish the following result.

Theorem 2.4

There exists a non-nilpotent group G with abelian subgroups P and Q such that $P \triangleleft^2 G$ and $Q \triangleleft^2 G$ and $G = \langle P,Q \rangle$.

Proof:

Let Z denote the set of all integers and let S be the set of all subsets X of Z such that there exists integers $\ell = \ell(X)$ and $L = L(X)$, $\ell \leq L$, with the property that X contains all integers $\leq \ell$ and no integer $> L$. Roughly speaking, X contains all large negative integers but no large positive integers.

Let A and B be two elementary abelian 2-groups with sets of basis elements respectively

$$(a_X)_{X\in S} \text{ and } (b_X)_{X\in S}.$$

For each $n \in Z$ two maps of $M = A \times B$, u_n and v_n, are defined by the rules

$$[A,u_n] = 1 = [B,v_n] \qquad (2.1)$$

$$[b_X,u_n] = a_{X+n} \text{ and } [a_X,v_n] = b_{X+n} \qquad (2.2)$$

for each $X \in S$. Our notation here is as follows:

If $n_1, n_2, \ldots, n_r$ are integers, (r being finite), and

$X \in S$, $a_{X+n_1+n_2+...+n_r}$ is to mean a_Y where $Y = X \cup (n_1) \cup (n_2) \ldots \cup (n_r)$ if the n_i's are all different and none of them belong to X; otherwise $a_{X+n_1+n_2+...+n_r} = 1$.

Similar remarks apply to $b_{X+n_1+n_2+...+n_r}$. Also $[b_X, u_n]$ is used to denote $b_X^{-1} b_X^{u_n}$.

The maps u_n and v_n can be extended to homomorphisms of M and they map the generators on to a set of generators.

The inverses of u_n and v_n exist

and

$$b_X^{u_n^{-1}} = b_X a_{X+n}$$

$$a_X^{v_n^{-1}} = a_X b_{X+n}$$

Thus the mappings u_n and v_n are automorphisms of M. Denote the subgroup of Aut(M) generated by the u_n, by H and the subgroup of Aut(M) generated by the v_n, by K. Let G be the split extension of M by the group of automorphisms $J = \langle H, K \rangle$.

H centralises the factors of the series

$$I \lhd A \lhd M = A \times B$$

and so H is abelian.

K centralises the factors of the series

$$I \lhd B \lhd M = A \times B$$

and so K is abelian.

It is immediately clear that

$$u_n^2 = 1 = v_m^2.$$

Let $z_{mn} = [u_m, v_n]$. It will now be shown that

$$[z_{mn}, a_X] = a_{X+m+n} \text{ and } [z_{mn}, b_X] = b_{X+m+n} \qquad (2.3)$$

$$[z_{mn}, a_X] = [u_m^{-1} v_n^{-1} u_m v_n, a_X]$$

$$= [u_m^{-1}, a_X]^{v_n^{-1} u_m v_n} [v_n^{-1}, a_X]^{u_m v_n} [u_m, a_X]^{v_n} [v_n, a_X]$$

$$= [a_X, u_m]^{z_{mn}} [a_X, v_n]^{v_n^{-1} u_m v_n} [u_m, a_X]^{v_n} [v_n, a_X]$$

$$= 1 \, . \, b_{X+n}^{v_n^{-1} u_m v_n} \, . \, 1 \, . \, b_{X+n}$$

$$= (b_{X+n} \, a_{X+m+n})^{v_n} \, . \, b_{X+n}$$

$$= b_{X+n}^2 \, a_{X+m+n}$$

$$= a_{X+m+n}.$$

Similarly $[z_{mn}, b_X] = b_{X+m+n}$.

Furthermore $z_{mn}^2 = 1$ since:

$$a_X^{z_{mn}} = a_X a_{X+m+n}$$

$$a_X^{z_{mn}^2} = (a_X^{z_{mn}})^{z_{mn}}$$

$$= (a_X a_{X+m+n})^{z_{mn}}$$

$$= a_X a_{X+m+n}^2$$

$$= a_X.$$

and

$$b_X^{z_{mn}^2} = (b_X b_{X+m+n})^{z_{mn}}$$

$$= b_X b^2_{X+m+n}$$

$$= b_X.$$

Therefore $z^2_{mn} = 1$.

The next step is to show that

$$[z_{mn}, u_\ell] = 1 = [z_{mn}, v_\ell].$$

It is immediately clear that since z_{mn} maps $A \to A$ and u_ℓ acts as an identity on A, $[z_{mn}, u_\ell]$ acts like an identity on A.

So we need only consider $b_X^{[z_{mn}, u_\ell]}$.

Now

$$b_X^{z^{-1}_{mn} u^{-1}_\ell z_{mn} u_\ell}$$

$$= (b_X b_{X+m+n})^{u^{-1}_\ell z_{mn} u_\ell}$$

$$= (b_X a_{X+\ell} b_{X+m+n} a_{X+m+n+\ell})^{z_{mn} u_\ell}$$

$$= (b_X b_{X+m+n} a_{X+\ell} a_{X+m+n+\ell} b_{X+m+n} a_{X+m+n+\ell})^{u_\ell}$$

$$= (b_X a_{X+\ell})^{u_\ell}$$

$$= b_X a^2_{X+\ell}$$

$$= b_X.$$

Thus $[z_{mn}, u_\ell] = 1$.

Similarly for $[z_{mn}, v_\ell]$.

Let $P = \langle a_X, u_n : X \in S, n \in Z \rangle$

and

$$Q = \langle b_X, v_m : X \in S,\ m \in Z \rangle.$$

By the rules (2.1) and (2.2) P and Q are abelian.

It will be shown that $P \lhd^2 G$ and $Q \lhd^2 G$.

The normal closure of P in G is $P_1 = P[P,G]$.

$[P,G]$ is generated by $[a_X, v_n]$, $[b_X, u_n]$, $[u_m, v_n]$ and all their conjugates in G.

So P_1 is generated by a_X, u_n, $[a_X, v_n]^g$, $[b_X, u_n]^g$, z^g_{mn} where $g \in G$.

Thus P_1 is generated by a_X, u_n, z^g_{mn}, b^g_{X+n}, a^g_{X+n}

Define $P_2 = P^{P_1} = P[P,P_1]$.

Since $M \lhd G$ it follows that b^g_{X+n}, $a^g_{X+n} \in M$ and hence $[P,P_1]$ is generated by

$$[a_X, z^g_{mn}] \ , \ [u_n, z^g_{mn}] \ , \ [u_n, b^g_{X+m}] \ , \ [u_n, a^g_{X+m}]$$

and all their conjugates in P_1.

So P_2 is generated by

$$a_X,\ u_n,\ [a_X,\ z^g_{mn}]^{g_1} \ , \ [u_n,\ z^g_{mn}]^{g_1} \ , \ [u_n,\ b^g_{X+m}]^{g_1} \ , \ [u_n, a^g_{X+m}]^{g_1}$$

$\forall\ g \in G$, $\forall\ g_1 \in P_1$.

Now let $g \in G$ then $g=xy$ where $x \in M$ and $y \in J$.

x is a word in the $(a_X)_{X \in S}$ and $(b_Y)_{Y \in S}$

and

$$y = u_{q_1}^{\sigma_1} v_{\omega_1}^{\varepsilon_1} \ldots u_{q_r}^{\sigma_r} v_{\omega_r}^{\varepsilon_r}$$

where $\sigma_i = 0$ or 1, $\varepsilon_i = 0$ or 1.

Also
$$z_{mn}^g = z_{mn}\,[z_{mn},g]$$
$$= z_{mn}\,[z_{mn}.xy]$$
$$= z_{mn}\,[z_{mn},y]\,[z_{mn},x]^y$$

It was proved that z_{mn} commutes with all u_ℓ and v_ℓ and since y is a word in u_ℓ and v_ℓ, $[z_{mn},y] = 1$.
Also by (2.3), and since x is a word in $(a_X)_{X\in S}$ and $(b_Y)_{Y\in S}$, $[z_{mn},x]\in M$ and since $M\triangleleft G$, $[z_{mn},x]^y \in M$.

From what has just been proved it follows that $[a_X,z_{mn}^g]$, $[u_n,z_{mn}^g]$, $[u_n,b_{X+m}^g]$, $[u_n,a_{X+n}^g]$ all lie in $A\leqslant P$.

It is thus sufficient to show that
$$x_1^{g_1} \in P \quad \forall \quad x_1\in A$$
where x_1 is a word in the a_X $(X\in S)$, $\forall$ $g_1\in P_1$.

Let $g_1 = x_{i_1}x_{i_2}\ldots x_{i_s}$ where x_{i_j}, $j=1,2,\ldots,s$ is any one of the above generators of P_1 and these x_{i_r} are their own inverses.

Now
$$a_X^{x_{i_r}} = a_X\in A \quad \text{if} \quad x_{i_r} = a_Y \ (Y\in S)$$
$$= a_X\in A \quad \text{if} \quad x_{i_r} = u_m$$
$$= a_X \quad \text{if} \quad x_{i_r} = b_{X+m}^g \text{ or } a_{X+m}^g$$

$$= a_X a_{X+m+n} \quad \text{if} \quad x_{i_r} = z_{mn}\,[z_{mn}, g]$$

since $[z_{mn}, g] \in M$ and M is abelian.

So one can conclude that $P_2 \leq P$ and thus $P \triangleleft^2 G$.
By applying the same argument as above to Q, it can be shown that $Q \triangleleft^2 G$.
To see that G is not nilpotent one need only note that for any integer $n>0$

$$1 \neq [a_X, x_{s_2}, x_{s_3}, \ldots, x_{s_{n+1}}] \in \gamma_n(G)$$

where $s_i \in Z$, $i=2,3, \ldots ,n+1$ are all different and none of them belong to the set X and furthermore

$$x_{s_i} = v_{s_i} \quad \text{if } i \text{ is even}$$

and

$$x_{s_i} = u_{s_i} \quad \text{if } i \text{ is odd.} \square$$

Let G be a group generated by subnormal subgroups H and K. If a and b are non-negative integers then Roseblade ([15]) proved that there is an integer c such that

$$G^{(c)} \leq H^{(a)} K^{(b)}$$

where $G^{(c)}$ is the c-th term of the derived series of G.

No such relation exists between the terms of the lower central series of G, H and K. This is shown by theorem 2.4, since $Q \triangleleft^2 G$, $P \triangleleft^2 G$,

$$\gamma_2(Q) = 1 = \gamma_2(P) \text{ but } \gamma_3(G) = \gamma_\omega(G) = M \neq 1$$

and $G = \langle P,Q \rangle$.

However, there are circumstances under which such a relation exists. This is shown by the next theorem which is due to S.E. Stonehewer.

Theorem 2.5 (S.E. Stonehewer [16])

Suppose that the subgroups H, K are subnormal in their join G and that G = HK. Then given any positive integers c_1, c_2, there exists an integer d such that

$$\gamma_d(G) \leq \gamma_{c_1}(H)\, \gamma_{c_2}(K)$$

Proof:

Let $H \triangleleft^m G$ and proceed by induction on m. Thus suppose m=1, so that $H \triangleleft G$. Then $\gamma_{c_1}(H) \triangleleft G$ and hence without loss of generality, we may assume that $\gamma_{c_1}(H) = 1$.

Let

$$G = K_0 \geq K_1 \geq \ldots \geq K_n = K$$

be the normal closure series of K in G that is, $K_{i+1} = K^{K_i}$ for $0 \leq i \leq n-1$.

Suppose that for some i, $1 \leq i \leq n-1$ there is an integer d_{i+1} such that

$$\gamma_{d_{i+1}}(K_{i+1}) \leq \gamma_{c_2}(K).$$

For example this is the case if i=n-1.

Let $Y = \gamma_{d_{i+1}}(K_{i+1})$. Then $Y \triangleleft K_i$.

Also since $G = HK_{i+1}$, we have

$$K_i = (H \cap K_i)\, K_{i+1}$$

with both factors normal in K_i. Moreover $H \cap K_i$, as a subgroup of H, is nilpotent; and $\frac{K_{i+1}}{Y}$ is nilpotent. Thus by Fitting's Theorem $\frac{K_i}{Y}$ is nilpotent. Therefore there is an integer d_i such that

$$\gamma_{d_i}(K_i) \leq Y \leq \gamma_{c_2}(K).$$

It follows, by induction on i decreasing, that there is an integer $d(=d_0)$ such that

$$\gamma_d(G) \leq \gamma_{c_2}(K) \quad \text{as required.}$$

Now suppose that $m \geq 2$ and that the theorem is true for smaller values of m.

Let $H_1 = H^G$ so that $H \triangleleft^{m-1} H_1$ and $H_1 = H(H_1 \cap K)$.

Then by induction on m, there is an integer c_3 such that

$$\gamma_{c_3}(H_1) \leq \gamma_{c_1}(H)\, \gamma_{c_2}(H_1 \cap K).$$

But $G = H_1K$ and hence by the case m=1, with H_1 replacing H, there is an integer d such that

$$\gamma_d(G) \leq \gamma_{c_3}(H_1)\, \gamma_{c_2}(K) \leq \gamma_{c_1}(H)\, \gamma_{c_2}(K). \quad \square$$

In conclusion it can be mentioned that D.S. Robinson ([14]) proved that if H and K are two subnormal subgroups of a group G and if J = <H,K> can be finitely generated then J is nilpotent. This result has also been proved by P. Hall ([5]).

It shall be shown in chapter 3 that this result is in fact an easy consequence of the Hirsch-Plotkin Theorem.

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CHAPTER 3

FITTING'S THEOREM FOR LOCALLY-NILPOTENT SUBGROUPS AND ZA-SUBGROUPS

§3.1 THE HIRSCH-PLOTKIN THEOREM

The Hirsch-Plotkin theorem states that the product MN of normal locally-nilpotent subgroups M and N of a group G is itself locally-nilpotent. The theorem was proved independently by K.A. Hirsch ([10]) and B. Plotkin ([13]) and is well-known. In this section the proof of K.A. Hirsch will be given. It is then shown that the theorem can be generalized by replacing normal by sub-normal and even serial.

Theorem 3.1 (K.A. Hirsch [10]).

The group generated by two locally-nilpotent normal subgroups A and B of an arbitrary group G,is itself locally-nilpotent.

Proof:

Let

$$a_1b^{(1)},\ a_2b^{(2)}\ ,\ldots,\ a_nb^{(n)}$$

be any arbitrary finite system of elements in $\langle A,B\rangle$.

The group

$$\bar{G} = \langle a_1b^{(1)},\ a_2b^{(2)}\ ,\ldots,\ a_nb^{(n)}\rangle$$

will be nilpotent if one can embed it in a nilpotent subgroup of $<A,B>$.

Let

$$A_o = <a_1,\dots a_n>$$

and

$$B^* = <b^{(1)},\dots,b^{(n)}>.$$

Since B^* is a finitely generated subgroup of B, it is nilpotent and therefore satisfies the maximal condition for subgroups.

Therefore B^* has a principal series

$$1 = B_o < B_1 < B_2 < \dots < B_k = B^* \qquad (3.1)$$

where the groups B_i $(i = 1,2,\dots k)$ are all normal subgroups of B^* and the factor groups $\frac{B_{i+1}}{B_i}$ are cyclic (of finite or infinite order).

Let b_j be a generating element of $\frac{B_j}{B_{j-1}}$, $j=1,2,\dots,k$, so that in particular

$$B_j = <b_1,b_2,\dots,b_j>.$$

For each j $(j=1,2\dots,k)$ construct a group A_j which satisfies the following conditions:

(1) A_j is a finitely-generated subgroup of A which contains A_o

(2) In the ascending chain

$$A_j \lhd <A_j,B_1> \lhd <A_j,B_2> \lhd \dots \lhd <A_j,B_j> \qquad (3.2)$$

all members are nilpotent.

Begin by putting j=1.

Form repeated commutators of b_1 with all the generating elements of A_o.

We get

$$a_1,\ a_1^{(1)},\ a_1^{(2)}\ ,\ldots;$$
$$a_2,\ a_2^{(1)},\ a_2^{(2)}\ ,\ldots;$$
$$\vdots$$
$$a_n,\ a_n^{(1)},\ a_n^{(2)}\ ,\ldots,$$

where

$$a_j^{(i)} = [a_j^{(i-1)},\ b_1] \quad j=1,2,\ldots,n$$
$$a_j^{o} = a_j \tag{3.3}$$

There are only finitely many elements

$$a_i,\ a_i^{(1)},\ a_i^{(2)}\ ,\ldots,\ a_i^{(k)},\ i=1,2,\ldots,n$$

since $a_i^{(N)} = 1$ for some $N = N(i)$.

This is so since $B \triangleleft G$ and B is locally-nilpotent.

Let A_1 be the group generated by

$$a_i,\ a_i^{(1)},\ a_i^{(2)}\ ,\ldots,\ a_i^{(k)}\ ,\ldots \qquad i=1,2,\ldots,n$$

Furthermore $A_1 \triangleleft \langle A_1,B_1\rangle$ since for each element a_m, $m=1,2,\ldots,n$ we have

$$b_1^{-1}\ a_m^{(j)}\ b_1 = a_m^{(j)}\ a_m^{(j+1)} \in A_1 \tag{3.4}$$

One now has to show that $\langle A_1, B_1 \rangle$ is nilpotent. Since A_1 is nilpotent, it has a non-trivial centre, $Z(A_1)$. If $1 \neq z \in Z(A_1)$ then as above, form repeated commutators of b_1 with z, giving z, $z^{(1)}$, $z^{(2)}$, ... and after a finite number of steps one obtains

$$z^{(n)} = [z^{(n-1)}, b_1] = 1$$

Thus $z^{(n-1)} \in Z(\langle A_1, B_1 \rangle)$

Assume that

$$z^{(n-i-1)} \in Z_{i+1} (\langle A_1, B_1 \rangle)$$

then

$$[z^{(n-i-1)}, b_1] = z^{(n-i)} \in Z_i(\langle A_1, B_1 \rangle)$$

$$z^{(o)} = z.$$

Therefore

$$z \in Z_n (\langle A_1, B_1 \rangle)$$

and hence

$Z(A_1) \leq Z_n(\langle A_1, B_1 \rangle)$, since $Z(A_1)$ is finitely generated.

Let $$Q = \langle A_1, B_1 \rangle = A_1 B_1$$

and assume that

$$Z_i(A_1) \leq Z_{m_1}(Q).$$

Letting bars denote cosets modulo $Z_i(A_1)$ (which is normal in Q), we have by the argument above that $Z(\bar{A}_1) \leq Z_{n_2}(\bar{Q})$ for some integer n_2. Then by the induction hypothesis

$$Z_{n_2}(\bar{Q}) \subseteq Z_{n_2}\left(\frac{Q}{Z_{m_1}(Q)}\right)$$

so $Z_{i+1}(A_1) \subseteq Z_{n_2+m_1}(Q) = Z_{m_2}(Q)$ say.

Since A_1 is nilpotent, it follows by induction that

$$A_1 \leq Z_{m_r}(Q).$$

Therefore

$$\frac{Z_{m_r+1}(Q)}{A_1} = Z\left(\frac{Q}{A_1}\right).$$

But $\frac{Q}{A_1}$ is cyclic and so $Z\left(\frac{Q}{A_1}\right) = \frac{Q}{A_1}$ and $A_1 \leq Z_{m_r}(Q)$.

It follows that

$$Z_{m_r+1}(Q) = Q.$$

Hence Q is nilpotent.

In the general case A_i is taken to be the group generated by the a_m $(m = 1,2,\ldots,n)$ and all commutators of the form

$$a = [a_m, b_{\alpha_1}, b_{\alpha_2}, \ldots, b_{\alpha_s}] \qquad (3.5)$$

where $i \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_s \geq 1$.

There are in fact finitely many different commutators of this type so that condition (1) is satisfied for A_i.

In exactly the same way as above it can be shown that A_i is normal in $<A_i, B_1>$ and that $<A_i, B_1>$ is nilpotent. Assume that condition (2) in the chain (3.2) is satisfied up to $<A_i, B_{j-1}>$. One has to prove that

$$<A_i, B_{j-1}> \triangleleft <A_i, B_j> = <A_i, B_{j-1}, b_j>.$$

Since $B_{j-1} \triangleleft B_j$, it will be sufficient to prove that for each commutator (3.5)

$$b_j^{-1} \, ab_j \in <A_i, B_{j-1}>. \qquad (3.6)$$

Choose r such that $\alpha_r \geq j > \alpha_{r+1}$.

Put

$$[a_m, b_{\alpha_1}, \ldots, b_{\alpha_r}] = \bar{a}$$

where $\bar{a}$ is a generating element of A_i.

Thus

$$b_j^{-1} ab_j = b_j^{-1} [\bar{a}, b_{\alpha_{r+1}}, \ldots, b_{\alpha_s}] \, b_j$$

$$= [b_j^{-1} \bar{a} b_j, \, b_j^{-1} \, b_{\alpha_{r+1}} \, bj, \ldots, b_j^{-1} b_{\alpha_s} \, b_j]$$

Here $b_j^{-1} \, \bar{a} b_j = \bar{a} \, [\bar{a}, b_j]$ is a product of two generators of A_i and all other elements, that is, $b_j^{-1} \, b_{\alpha_i} \, b_j$ $(i=r+1, \ldots, s)$ are in B_{j-1} since $\alpha_{r+1} \leq j-1$ and $B_{j-1} \triangleleft B_j$ and this proves (3.6).

In a similar way it follows that $<A_i, B_j>$ is nilpotent.