COMPUTATION OF THE CHARACTER TABLES OF CERTAIN GROUP EXTENSIONS

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Contents

Chapter 1

THE CONJUGACY CLASSES OF GROUP EXTENSIONS

In this chapter some basic theory on group extensions is first given in section 1.1 and then a method for finding the conjugacy classes of group extensions is described in section 1.2. In section 1.3 we look at an example due to Whitley[19] to illustrate how the theory developed in section 1.2 is used to calculate the conjugacy classes of the group $2^3 : GL_3(2)$. For section 1.1 , the books by Rotman[17] and Gorenstein[8] were used as references while for section 1.2 we used the works of Whitley[19], Moori[15], Moori and Mpono[16] and Salleh[18].

1.1 Definitions and Basic Results

Definition 1.1.1 *If N and G are groups, an extension of N by G is a group $\overline{G}$ that satisfies the following properties*

1. $N \triangleleft \overline{G}$

2. $\overline{G}/N \cong G$.

We say that $\overline{G}$ is a *split extension* of N by G if $\overline{G}$ contains subgroups N and G_1 with $G_1 \cong G$ such that

1. $N \lhd \overline{G}$
2. $NG_1 = \overline{G}$
3. $N \cap G_1 = 1_{\overline{G}}$

In this case $\overline{G}$ is also called a *semi-direct* product of N by G, and we identify G_1 with G.

Note 1 If $\overline{G}$ is a semi-direct product of N by G, then every $\overline{g} \in \overline{G}$ can be uniquely expressed in the form $\overline{g} = ng$, where $n \in N$ and $g \in G$. Multiplication in $\overline{G}$ satisfies $(n_1g_1)(n_2g_2) = n_1n_2^{g_1}g_1g_2$, where n^g denotes gng^{-1}.

Definition 1.1.2 *The automorphism group of a group G, denoted by $Aut(G)$, is the set of all automorphisms of G under the binary operation of composition.*

If $\overline{G}$ is a *split extension* of N by G , then there is a homomorphism $\theta : G \to \mathrm{Aut}(N)$ given by $\theta_g(n) = gng^{-1} = n^g (n \in N, g \in G)$, where we denote $\theta(g)$ by θ_g. Thus G acts on N, and we say that the extension $\overline{G}$ realizes θ.

Conversely, given any groups N and G, and $\theta : G \to \mathrm{Aut}(N)$, we can define a semi-direct product of N by G that realizes θ as follows. Let $\overline{G}$ be the set of ordered pairs $(n, g)(n \in N, g \in G)$ with multiplication $(n_1, g_1)(n_2, g_2) = (n_1\theta_{g_1}(n_2), g_1g_2)$. Then $\overline{G}$ is a semi-direct product of N by G.

Hence a split extension of N by G is completely described by the map $\theta : G \to \mathrm{Aut}(N)$, that is, it is described by the way G acts on N.

We use the ATLAS [3] notation and let $N.G$ denote an arbitrary extension of N by G. A split extension is denoted by $N : G$ or $N :^{\theta} G$, where $\theta : G \to \mathrm{Aut}(N)$ determines the extension. A non-split extension is denoted by $N \cdot G$.

If $\overline{G}$ is a split extension of N by G, then $\overline{G} = NG = \bigcup_{g\in G} Ng$, so G may be regarded as a right transversal for N in $\overline{G}$ (that is, a complete set of right coset representatives of N in $\overline{G}$). Now suppose $\overline{G}$ is any extension of N by G, not necessarily split. Since $\overline{G}/N \cong G$, there is an epimorphism $\lambda : \overline{G} \to G$ with kernel N. For $g \in G$, define a lifting of g to be an element $\overline{g} \in \overline{G}$ such that $\lambda(\overline{g}) = g$. Then choosing a lifting of each element of G, we get the set $\{\overline{g} : g \in G\}$ which is a transversal for N in $\overline{G}$.

We now show that even for a non-split extension $\overline{G}$ of N by G, where N is abelian, G acts on N. This result can be obtained from Rotman[17].

Lemma 1.1.3 *Let $\overline{G}$ be an extension of an abelian group N by G, then there is a homomorphism $\theta : G \to Aut(N)$ such that $\theta_g(n) = \overline{g}n\overline{g}^{-1} (n \in N)$, and θ is independent of the choice of liftings $\{\overline{g} : g \in G\}$.*

Proof: For $a \in \overline{G}$, denote conjugation by a by γ_a. Since N is normal in $\overline{G}, \gamma_a|_N$ is an automorphism of N and the function $\mu : \overline{G} \to \mathrm{Aut}(N)$ defined by $\mu(a) = \gamma_a|_N$ is a homomorphism.

If $a \in N$, then $\mu(a) = 1_N$, since N is abelian. Therefore there is a homomorphism $\mu^* : \overline{G}/N \to \mathrm{Aut}(N)$ defined by $\mu^*(Na) = \mu(a)$.

Now $G \cong \overline{G}/N$ and for any lifting $\{\overline{g} : g \in G\}$, the map $\phi : G \to \overline{G}/N$ defined by $\phi(g) = N\overline{g}$ is an isomorphism. If $\{\overline{h} : h \in G\}$ is another choice of liftings , then $\overline{g}\overline{h}^{-1} \in N$ so that $N\overline{g} = N\overline{h}$. Therefore the isomorphism ϕ is independent of the choice of liftings. Now let $\theta : G \to \mathrm{Aut}(N)$ be the composite $\mu^* \circ \phi$. If $g \in G$ and $\overline{g}$ is a lifting , then $\theta(g) = \mu^*(\phi(g)) = \mu^*(N\overline{g}) = \mu(\overline{g}) \in \mathrm{Aut}(N)$, so for $n \in N$, $\theta_g(n) = \mu(\overline{g})(n) = \overline{g}n\overline{g}^{-1} = n^{\overline{g}}$, as required. $\square$

Note 2 Let $\overline{G}$ be an extension of an abelian group N by G. For each $g \in G$ we choose a lifting $\overline{g} \in \overline{G}$, and for convenience we take $\overline{1} = 1$. We identify G with $\overline{G}/N$ under the isomorphism $g \to N\overline{g}$. Now $\{\overline{g} : g \in G\}$ is a right transversal for N in $\overline{G}$, so every element $h \in \overline{G}$ has a unique expression of the form $h = n\overline{g}$ $(n \in N, g \in G)$, and we have the following relations.

1. $\overline{g}n = n^{\overline{g}}\overline{g}$, where $n \in N$ and $g \in G$.

2. $\overline{g}\overline{h} = f(g,h)\overline{gh}$ for some $f(g,h) \in N$, where $g,h \in G$.

1.2 The Conjugacy Classes of Group Extensions

Let $\overline{G} = N.G$, where N is abelian. Then for each conjugacy class $[g]$ in G with representative $g \in G$, we analyse the coset $N\overline{g}$, where $\overline{g}$ is a lifting of g in $\overline{G}$ and $\overline{G} = \bigcup_{g \in G} N\overline{g}$. To each class representative $g \in G$ with lifting $\overline{g} \in \overline{G}$, we define

$$C_g = \{x \in \overline{G} : x(N\overline{g}) = (N\overline{g})x\}.$$

Then C_g being the set stabilizer of $N\overline{g}$ in $\overline{G}$ under the action by conjugation of $\overline{G}$ on $N\overline{g}$, is a subgroup of $\overline{G}$. The following lemmas and their proofs due to Whitley[19] and Moori and Mpono[16] will be required in the next section .

Lemma 1.2.1 $N \lhd C_g$.

Proof: For any $n \in N$

$$n(N\overline{g})n^{-1} = N\overline{g}n^{-1} = N\overline{g}n^{-1}\overline{g}^{-1}\overline{g} = N\overline{g},$$

the last step following from the fact that $(n^{-1})^{\overline{g}} \in N$ since $N \lhd \overline{G}$.

Hence $N \subseteq C_g$. From $N \leq C_g \leq \overline{G}$ and $N \lhd \overline{G}$, we obtain $N \lhd C_g$. $\square$

Lemma 1.2.2 $C_g/N = C_{\overline{G}/N}(N\overline{g})$.

Proof: Consider $Nk \in \overline{G}/N$. Then

$$\begin{aligned}
Nk \in C_{\overline{G}/N}(N\overline{g}) &\iff Nk(N\overline{g})(Nk)^{-1} = N\overline{g} \\
&\iff NkN\overline{g}Nk^{-1} = N\overline{g} \\
&\iff NkN\overline{g}k^{-1} = N\overline{g} \\
&\iff NkNn\overline{g}k^{-1} = N\overline{g} \quad \forall\, n \in N \\
&\iff Nkn\overline{g}k^{-1} = N\overline{g} \quad \forall\, n \in N \\
&\iff kn\overline{g}k^{-1} \in N\overline{g} \quad \forall\, n \in N \\
&\iff k \in C_g.
\end{aligned}$$

Thus we obtain that $C_g/N = C_{\overline{G}/N}(N\overline{g})$. $\square$

From the two preceding lemmas, we have that $C_g = N.C_{\overline{G}/N}(N\overline{g})$. For a lifting $\overline{g} \in \overline{G}$ of $g \in G$, we can identify $C_{\overline{G}/N}(N\overline{g})$ with $C_G(g)$ and write $C_g = N.C_G(g)$ in general. If $\overline{G} = N : G$ then we can identify C_g with $C_g = \{x \in \overline{G} : \ x(Ng) = (Ng)x\}$ and in this case we obtain the following corollary.

Corollary 1.2.3 *Let $\overline{G} = N : G$. Then $C_g = N : C_G(g)$.*

Proof: We have already shown in the Lemma 1.2.1 that $N \lhd C_g$. Now we show that $C_G(g) \leq C_g$ and that $N \cap C_G(g) = \{1_G\}$. Let $x \in C_G(g)$. Then we obtain $(Ng)^x = x(Ng)x^{-1} = xNgx^{-1} = Nxgx^{-1} = Ng$. Thus $x \in C_g$ and hence $C_G(g) \leq C_g$. Since $N \cap C_G(g) \leq N \cap G = \{1_G\}$, then we have that $N \cap C_G(g) = \{1_G\}$. This completes the proof. $\square$

The conjugacy classes of $\overline{G}$ will be determined by the action by conjugation of C_g, for each conjugacy class $[g]_G$ of G, on the elements of $N\overline{g}$ or in the case of a split extension on the elements of Ng. Since $C_g = N : C_G(g)$, we act first N and then act $\{\overline{h} : h \in C_G(g)\}$ on the elements of $N\overline{g}$. The outline of this action is given in two steps by Moori and Mpono [16,page 5] as follows:

STEP 1: *The action of N on $N\overline{g}$:*

Let $C_N(\overline{g})$ be the stabilizer of $\overline{g}$ in N. Then for any $n \in N$ we have

$$\begin{aligned} x \in C_N(n\overline{g}) &\Leftrightarrow x(n\overline{g})x^{-1} = n\overline{g} \\ &\Leftrightarrow xnx^{-1}x\overline{g}x^{-1} = n\overline{g} \\ &\Leftrightarrow n(x\overline{g}x^{-1}) = n\overline{g}, \quad \textit{since } N \textit{ is abelian} \\ &\Leftrightarrow x\overline{g}x^{-1} = \overline{g} \\ &\Leftrightarrow x \in C_N(\overline{g}). \end{aligned}$$

Thus $C_N(\overline{g})$ fixes every element of $N\overline{g}$. Now let $|C_N(\overline{g})| = k$. Then under the action of N, $N\overline{g}$ splits into k orbits $Q_1, Q_2, ..., Q_k$, where

$$\begin{aligned} |Q_i| &= [N : C_N(\overline{g})] \\ &= \frac{|N|}{k}, \quad \text{for } i \in \{1, ..., k\}. \end{aligned}$$

STEP 2: *The action of $\{\overline{h} : h \in C_G(g)\}$ on $N\overline{g}$*

Since the elements of $N\overline{g}$ are now in the orbits $Q_1, ..., Q_k$ from step 1 above, we need only to act $\{\overline{h} : h \in C_G(g)\}$ on the k orbits. Suppose that under this action f_j of the orbits $Q_1, ..., Q_k$ fuse together to form one orbit $\triangle_l$, then the $f_j's$ obtained this way must satisfy

$\sum_j f_j = k$

and we have

$| \triangle_l| = f_j \times \frac{|N|}{k}$

Thus for $x = d_l\overline{g} \in \triangle_l$, we obtain that

$$\begin{aligned}
|[x]_{\overline{G}}| &= |\triangle_l| \times |[g]_G| \\
&= f_j \times \frac{|N|}{k} \times \frac{|G|}{|C_G(g)|} \\
&= f_j \times \frac{|\overline{G}|}{k|C_G(g)|}
\end{aligned}$$

and thus we obtain that

$$\begin{aligned}
|C_{\overline{G}}(x)| &= \frac{|\overline{G}|}{|[x]_G|} \\
&= |\overline{G}| \times \frac{k|C_G(g)|}{f_j|\overline{G}|} \\
&= \frac{k|C_G(g)|}{f_j}.
\end{aligned}$$

Thus to calculate the conjugacy classes of $\overline{G} = N.G$, we need to find the values of k and the $f_j's$ for each class representative $g \in G$. We note that the values of k can be determined from the action of G on N(given in lemma 1.1.3). If $\overline{G} = N : G$ (a split extension) however, we analyse the coset Ng instead of $N(\overline{g})$ since in the split case $G \leq \overline{G}$. Under the action of N on Ng, we always assume that $g \in Q_1$. Since $C_G(g)$ fixes g, Q_1 does not fuse with any other Q_i. Hence we will always have that $f_1 = 1$. Hence

$$
\begin{aligned}
k &= \sum_j f_j \\
&= 1 \quad + \quad \sum_m f_m,
\end{aligned}
$$

where the sum is taken over all m such that $g \notin Q_m$.

We now apply the method described in the Step 1 and Step 2 in the next section.

1.3 The Conjugacy Classes of a Group of the Form $2^3 : GL_3(2)$

In this section we give the conjugacy classes of the group $\overline{G} = N : G$ where N is an elementary abelian group of order 8 and $G \cong GL_3(2)$, as calculated by Whitley[19], where G acts naturally on N.

We regard N as the vector space $V_3(2)$ of dimension three over a field of two elements. Let N be generated by $\{e_1, e_2, e_3\}$ with $e_i^2 = 1$ for $1 \leq i \leq 3$, so

$$N = \{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}$$

To determine the conjugacy classes of $\overline{G}$ we analyse the cosets Ng where g is a representative of a class of G. (Note that the extension is split, so $\overline{G} = \bigcup_{g \in G} Ng$). Now

$$|C_{\overline{G}}(x)| = \tfrac{k.|C_G(g)|}{f_j},$$

where f_j of the k blocks of the coset Ng have fused to give a class of $\overline{G}$ containing x.We need the conjugacy classes of G, so we exhibit it here (obtained from ATLAS [3]).

class	(1A)	(2A)	(3A)	(4A)	(7A)	(7B)
centralizer	168	8	3	4	7	7

Table 1.3.1: The conjugacy table of $GL_3(2)$.

The representatives thus must come from the classes mentioned in the table above:

- $\underline{g = 1_G :}$

For g the identity of G, g fixes all elements of N, so k = 8. Since G is transitive on $N - \{1\}$ under the action of $C_G(g) = G$, we have two orbits with $f_1 = 1$ and $f_2 = 7$, so this coset gives two classes of $\overline{G}$:

$x = 1$, class(1), $\quad |C_{\overline{G}}(x)| = 8 \times 168 = 1344$

$x = e_1$, class(2_1), $\quad |C_{\overline{G}}(x)| = \frac{8 \times 168}{7} = 192$

- $\underline{g \in (2A) :}$

We take

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with $|C_G(g)| = 8$. The action of g on N is represented by the cycle structure

$$(1)(e_1)(e_1e_2e_3)(e_2e_3)(e_2\ e_3)(e_1e_2\ e_1e_3), \text{ so } k = 4.$$

The four orbits of N on Ng are $\{g,\ e_2e_3g\}, \{e_1g,\ e_1e_2e_3g\}, \{e_2g,\ e_3g\}$ and $\{e_1e_2g,\ e_1e_3g\}$.

Now we act

$$C_G(g) = \left\langle \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle$$

on these orbits.

For $eg \in Ng, h \in C_G(g), (eg)^h = e^h g^h = e^h g$ so we obtain the following orbits:
$\{g, e_2e_3g\}^{C_G(g)} = \{g, e_2e_3g\}, \{e_1g, e_1e_2e_3g\}^{C_G(g)} = \{e_1g, e_1e_2e_3g\}, \{e_2g, e_3g\}^{C_G(g)}$
$= \{e_2g, e_3g, e_1e_2g, e_1e_3g\}$

Therefore we get three classes of $\overline{G}$:

$f_1 = 1,\ x = g$, class(2_2), $\quad |C_{\overline{G}}(x)| = 4 \times 8 = 32;$

$f_2 = 1,\ x = e_1g$, class(2_3), $\quad |C_{\overline{G}}(x)| = 32;$

$f_3 = 2,\ x = e_2g$, class(4_1), $\quad |C_{\overline{G}}(x)| = \frac{4\times 8}{2} = 16.$

- $\underline{g \in (3A):}$

We take

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with $|C_G(g)| = 3$. The action of g on N is represented by $(1)(e_1e_2e_3)(e_1\ e_2\ e_3)(e_1e_2\ e_1e_3\ e_2e_3)$, so $k = 2$ which means we must have two blocks. These cannot fuse together under $C_G(g)$, since $g^{C_G(g)} = \{g\}$. Therefore we have two classes of $\overline{G}$, with $f_1 = 1$ and $f_2 = 1$:

$x = g$, class(3_1), $\quad |C_{\overline{G}}(x)| = 2 \times 3 = 6$;

$x = e_1g$, class(6_1), $\quad |C_{\overline{G}}(x)| = 6$.

- $\underline{g \in (4A):}$

We get two classes of $\overline{G}$ once more:

$x = g$, class(4_2), $\quad |C_{\overline{G}}(x)| = 8$;

$x = e_1g$, class(4_3), $\quad |C_{\overline{G}}(x)| = 8$.

- $\underline{g \in (7A):}$

For the class $(7A)$, we have $k = 1$, so each coset has just one class in $\overline{G}$. We thus get the class (7_1) of $\overline{G}$, with centralizer of order 7.

- $\underline{g \in (7B):}$

This case works the same as for the previous class and we obtain class (7_2) of $\overline{G}$, with centralizer of order 7.

class of G	(1A)		(2A)			(3A)		(4A)		(7A)	(7B)
class of $\overline{G}$	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(3_1)	(6_1)	(4_2)	(4_3)	(7_1)	(7_2)
centralizer	1344	192	32	32	16	6	6	8	8	7	7

Table 1.3.2: The conjugacy table of $2^3 : GL_3(2)$.

Chapter 2

REPRESENTATIONS AND CHARACTERS

Two ways of approaching representation and character theory are through the use of modules on the one hand (for instance, the approach used by James and Liebeck [10]), and through the classical approach used by Feit[5] for example, on the other hand. Our discussion is along the classical approach and for this purpose we follow the class notes of Moori[15].

We give some basic results on the representations and characters of finite groups in this chapter as well as some examples of how these results are used to determine the character tables of some finite groups. In the first section, theorems and lemmas will almost always be stated without proofs. Section 2.2 deals with the relationship between characters of groups and the characters of their subgroups, while in section 2.3 we shall look at the role of normal subgroups in the calculation of characters of a group. In the last two sections mentioned, only the proofs of the main results (that is those results dealing more directly with the techniques of finding the characters of a group) are given. These proofs are mainly taken from Moori's notes [15]. In the last three sections we calculate the character tables of three group extensions, which are all split extensions.

2.1 Basic Concepts

Definition 2.1.1 *Let G be a group. Let $f : G \to GL_n(F)$ be a homomorphism. Then we say that f is a matrix representation of G of degree n (or dimension n), over the field F.*

If $Ker(f) = \{1_G\}$, then we say that f is a $faithful$ representation of G. In this situation $G \cong Image(f)$, so that G is isomorphic to a subgroup of $GL_n(F)$.

Definition 2.1.2 *Let $f : G \to GL_n(F)$ be a representation of G over the field F. The function $\chi : G \to F$ defined by $\chi(g) = trace(f(g))$ is called the character of f.*

Definition 2.1.3 *If $\phi : G \to F$ is a function from a group G to a field F which is constant on conjugacy classes of G, that is $\phi(g) = \phi(xgx^{-1}), \forall x \in G$, then ϕ is a class function.*

Lemma 2.1.4 *A character is a class function.*

Proof: See [15, Lemma i.4]

Definition 2.1.5 *Two representations $\rho, \phi : G \to GL_n(F)$ are said to be equivalent if there exists an $n \times n$ matrix P over F such that*

$$P^{-1}\rho(g)P = \phi(g), \quad \forall g \in G.$$

Theorem 2.1.6 *Equivalent representations have the same character.*

Proof: See [15,Theorem i.5]

Before defining the concepts of reducibility and irreducibility of representations and characters, we need to say what is meant by a reducible and an irreducible set of matrices. If S is a set of matrices, then S is *reducible* if $\exists\ m,\ k\ \in\ \mathbb{N}$, and $\exists\ P \in\ GL_n(F)$ such that $\forall\ A \in S$ we have

$$P^{-1}AP = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B is an $m \times m$ matrix, D is a $k \times k$ matrix, C is a $k \times m$ matrix and 0 is the zero matrix. If no such P exists, we say that S is *irreducible.* Furthermore if $C = 0\ \forall\ A \in S$, we say that S is fully reducible and if $\exists\ P \in\ GL_n(F)$ such that

$$P^{-1}AP = \begin{pmatrix} B_1 & 0 & \dots & \dots & 0 \\ 0 & B_2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & B_k \end{pmatrix}, \quad \forall\ A \in S,$$

where each B_i is irreducible, we say S is completely reducible.

Definition 2.1.7 *Let $f : G \to GL_n(F)$ be a representation of G over F and let $S = \{f(g) : g \in G\}$. We say that f is reducible, fully reducible, or completely reducible if S is reducible, fully reducible, or completely reducible, respectively.*

Definition 2.1.8 *If χ_ρ is a character afforded by a representation ρ of G, then we say that χ_ρ is an irreducible character of G if ρ is an irreducible representation.*

Definition 2.1.9 *Let $\rho : G \to GL_n(F)$ and $\phi : G \to GL_m(F)$ be two representations of G over F. Define $\rho + \phi : G \to GL_{n+m}(F)$ by*

$$(\rho + \phi)(g) = \begin{pmatrix} \rho(g)_{n\times n} & 0_{n\times m} \\ 0_{m\times n} & \phi(g)_{m\times m} \end{pmatrix} = \rho(g) \oplus \phi(g), \quad \forall g \in G.$$

Then $\rho + \phi$ is a representation of G over F, of degree $n + m$.

If χ_1 and χ_2 are the characters of ρ and ϕ respectively and χ is the character of $\rho + \phi$, then for all $g \in G$ we have $\chi(g) = \chi_1(g) + \chi_2(g)$.

Theorem 2.1.10 *(Maschke's theorem) Let G be a finite group. Let f be a representation of G over a field F whose characteristic is either equal to zero or is a prime that does not divide $|G|$. If f is reducible, then f is fully reducible.*

Proof: See [15,Theorem i.6]

Theorem 2.1.11 *(The general form of Maschke's theorem)*
Let G be a finite group and F be a field whose characteristic is either equal to zero or is a prime that does not divide $|G|$. Then every representation of G over F is completely reducible.

Proof: See [5,(1.1)]

Theorem 2.1.12 *(Schur's lemma) Let $\rho : G \to GL_n(F)$ and $\phi : G \to GL_m(F)$ be two representations of a group G over a field F. Assume there exists an $m \times n$ matrix P such that $P\rho(g) = \phi(g)P$ for all $g \in G$. Then either $P = 0_{m\times n}$ or P is non-singular so that $\rho(g) = P^{-1}\phi(g)P$ (that is, ρ and ϕ are equivalent representations).*

Proof: See [5,(1.2)]

Definition 2.1.13 *Let G be a finite group and assume that the characteristic of the field F does not divide $|G|$. If ρ and ϕ are two functions from G into F, we define an innerproduct $\langle , \rangle$ by the following rule:*

$$\langle \rho, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \rho(g)\phi(g^{-1}) ,$$

where $\frac{1}{|G|}$ stands for $|G|^{-1}$ in F.

Theorem 2.1.14 *The innerproduct $\langle , \rangle$ is bilinear:*

(i) $\langle \rho_1 + \rho_2, \phi \rangle = \langle \rho_1, \phi \rangle + \langle \rho_2, \phi \rangle$

(ii) $\langle \rho, \phi_1 + \phi_2 \rangle = \langle \rho, \phi_1 \rangle + \langle \rho, \phi_2 \rangle$

(iii) $\langle \ a\rho, \phi \ \rangle = a\langle \rho, \phi \rangle = \langle \rho, a\phi \rangle, \quad \forall \ a \in F$

and symmetric:

$$\langle \rho, \phi \rangle = \langle \phi, \rho \rangle$$

Proof:

(i)

$$
\begin{aligned}
\langle \rho_1 + \rho_2, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} \Big(\rho_1 + \rho_2\Big)(g)\phi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \Big(\rho_1(g) + \rho_2(g)\Big)\phi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \Big(\rho_1(g)\phi(g^{-1}) + \rho_1(g)\phi(g^{-1})\Big), \text{F being an additive abelian group} \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_1(g)\phi(g^{-1}) + \frac{1}{|G|} \sum_{g \in G} \rho_2(g)\phi(g^{-1}), \\
&= \langle \rho_1, \phi \rangle + \langle \rho_2, \phi \rangle
\end{aligned}
$$

(ii) Similar to (i).

(iii)

$$
\begin{aligned}
\langle a\rho, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} \Big(a\rho\Big)(g)\phi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} a\Big(\rho(g)\Big)\phi(g^{-1}) \\
&= a\frac{1}{|G|} \sum_{g \in G} \rho(g)\phi(g^{-1}) \\
&= a\langle \rho, \phi \rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\langle a\rho, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} \Big(a\rho\Big)(g)\phi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} a\rho(g)\phi(g^{-1})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{|G|}\sum_{g\in G}\rho(g)a\phi(g^{-1}), \text{ F being a multiplicative abelian group} \\
&= \frac{1}{|G|}\sum_{g\in G}\rho(g)\Big(a\phi\Big)(g^{-1}) \\
&= \langle \rho, a\phi \rangle
\end{aligned}
$$

To complete the proof, see [15,Theorem i.11]. □

Note 1 If $\rho : G \to GL_n(\mathbb{C})$ is a representation of a group G, then we denote the (i,j) entry of $\rho(g)$ by $\rho_{ij}(g)$. Hence $\rho_{ij}(g)$ is a map from G into $\mathbb{C}$.

For the rest of this chapter we shall mean finite groups when mentioning groups, unless explicit exceptions are made and all representations will be over the field $\mathbb{C}$ of complex numbers.

Theorem 2.1.15 *Let G be a finite group and let ρ and ϕ be two irreducible representations of G.*

(i) If ρ and ϕ are inequivalent, then

$$\langle \rho_{rs}, \phi_{ij} \rangle = 0, \quad \forall i,j,r, \text{ and } s.$$

(ii) $\langle \rho_{rs}, \phi_{ij} \rangle = \frac{\delta_{is}.\delta_{jr}}{deg(\rho)}$.

Proof: See [15, Theorem ii.1]

Theorem 2.1.16 *Let G be a finite group and let ρ and ϕ be two irreducible representations of G, with characters χ_ρ and χ_ϕ.*

(i) If ρ and ϕ are equivalent, then

$$\langle \chi_\rho, \chi_\phi \rangle = 1$$

(ii) If ρ and ϕ are not equivalent, then

$$\langle \chi_\rho, \chi_\phi \rangle = 0$$

(iii) $\langle \chi_\rho, \chi_\rho \rangle = 1$

Proof: See [15,Theorem ii.2]

Theorem 2.1.17 *Two representations of a group G are equivalent if and only if they have the same characters.*

Proof: See [15,Corollary ii.4]

Lemma 2.1.18 *(i) If*

$$\chi = \sum_{i=1}^{k} \lambda_i \chi_i$$

where χ_i are distinct irreducible characters of a group G and λ_i are nonnegative integers, then

$$\langle \chi, \chi \rangle = \sum_{i=1}^{k} \lambda_i^2.$$

(ii) If χ is a character of G, then χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Proof:

(i)

$$\begin{aligned}
\langle \chi, \chi \rangle &= \langle \sum_{i=1}^{k} \lambda_i \chi_i\ ,\ \sum_{j=1}^{k} \lambda_j \chi_j \rangle \\
&= \sum_{i=1}^{k} \lambda_i \sum_{j=1}^{k} \lambda_j \langle \chi_i\ ,\ \chi_j \rangle \\
&= \sum_{i=1}^{k} \lambda_i^2 \langle \chi_i\ ,\ \chi_i \rangle \\
&= \sum_{i=1}^{k} \lambda_i^2
\end{aligned}$$

(ii)By theorem 2.1.7, we have that if χ is irreducible, then $\langle \chi, \chi \rangle = 1$.

For the converse, assume that $\langle \chi\ ,\ \chi \rangle = 1$. Let

$$\chi = \sum_{i=1}^{k} \lambda_i \chi_i$$

where χ_i are distinct irreducible characters of G and λ_i are nonnegative integers, then by (i), we have

$$\sum_{i=1}^{k} \lambda_i^2 = \langle \chi\ ,\ \chi \rangle = 1$$

$$\begin{aligned}
\Rightarrow \quad \lambda_j^2 &= 1, \text{ for some } \quad j = 1, 2, ..., k \\
\text{and} \quad \lambda_i^2 &= 0 \quad \forall\, i \neq j.
\end{aligned}$$

Hence $\lambda_j = 1$. Thus $\chi = \chi_j$ is irreducible. $\square$

Note 2 If C_i is a conjugacy class of G, then

$$C_{i'} = \{\ g \in G : g^{-1} \in C_i\}$$

is also a conjugacy class of G and $C_i = C_{i'}$ if and only if $g \sim g^{-1}$ for all $g \in C_i$.

Theorem 2.1.19 *Let $Irr(G) = \{\chi_1, \chi_2, ..., \chi_k\}$. Then*

(i) $\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_j(g) = \delta_{ij}$, *(row orthogonality)*

(ii) $\sum_{s=1}^{k} \chi_s(g_i)\chi_s(g_j) = \delta_{ij'}\ |C_G(g_j)|$, *(column orthogonality)*

Proof: See [15, Theorem ii.17]

Theorem 2.1.20 *The number of irreducible characters of a group G equals the number of conjugacy classes of G.*

Proof: See [15, Theorem ii.18]

Proposition 2.1.21 *Let $G = \langle\ x\ \rangle$ be a cyclic group of order n. Let $e^{\frac{2k\pi}{n}i}$ be the n-th roots of unity in $\mathbb{C}$, $k = 0, 1, 2, ..., n-1$. Define $\rho_k : G \to \mathbb{C}^*$ by*

$$\rho_k(x^m) = [e^{\frac{2k\pi}{n}i}\ ]^m.$$

For $k = 0, 1, 2, ..., n-1$, ρ_k defines the n distinct irreducible representations of G.

Proof: We first show that ρ_k is well defined:

Let $x^m = x^{m'}$, where $m = sn + t$, $m' = s'n + t'$, $s, s' \in \mathbb{Z}$ and $t,\ t' = 0, 1, 2, \ldots, n-1$. From which we get $x^t = x^{t'} \Rightarrow t = t'$.

If for contradiction, $[e^{\frac{2k\pi}{n}\, i}]^m \neq [e^{\frac{2k\pi}{n}\, i}]^{m'}$, then we have

$$
\begin{aligned}
[e^{\frac{2k\pi}{n}\, i}]^{m-m'} \neq 1 \quad & \Rightarrow \quad [e^{\frac{2k\pi}{n}\, i}]^{(s-s')n + (t-t')} \neq 1 \\
& \Rightarrow \quad [e^{\frac{2k\pi}{n}\, i}]^{(s-s')n} \neq 1 \\
& \Rightarrow \quad \rho_k(x^{(s-s')n}) \neq 1 \\
& \Rightarrow \quad \rho_k(x^0) \neq 1 \\
& \Rightarrow \quad [e^{\frac{2k\pi}{n}\, i}]^0 \neq 1,
\end{aligned}
$$

giving us the contradiction. Hence ρ_k is well defined.

Next we show that ρ_k is a homomorphism:

$$
\begin{aligned}
\rho_k(x^m)\rho_k(x^{m'}) &= \rho_k(x^t)\rho_k(x^{t'}) \\
&= [e^{\frac{2k\pi}{n} i}]^t [e^{\frac{2k\pi}{n} i}]^{t'} \\
&= [e^{\frac{2k\pi}{n}\, i}]^{t+t'} \\
&= \rho_k(x^{t+t'}) \\
&= \rho_k(x^t.x^{t'}) \\
&= \rho_k(x^m.x^{m'})
\end{aligned}
$$

So ρ_k is a homomorphism and hence a representation.

ρ_k is unique:

Let $\rho_k = \rho_{k'}$ with $k, k' \leq n$. Now $\forall g \in \langle x \rangle, \quad g = x^r$ where $r = 0, 1, 2, ..., n-1$. So we have

$$\begin{aligned}
\rho_k(x^r) = \rho_{k'}(x^r) \quad &\Rightarrow \quad [e^{\frac{2k\pi}{n} i}]^r = [e^{\frac{2k'\pi}{n} i}]^r \\
&\Rightarrow \quad e^{(\frac{2k\pi.r}{n} i \ - \ \frac{2k'\pi.r}{n} i)} = 1 \\
&\Rightarrow \quad e^{\frac{2\pi.r}{n} \ (k-k')i} = 1 \\
&\Rightarrow \quad \rho_{(k-k')}(x^r) = 1, \quad \forall\, r = 0, 1, 2, ..., n-1. \\
&\Rightarrow \quad k - k' = 0, \text{ so that } k = k'.
\end{aligned}$$

Lastly we must show that ρ_k is irreducible:

We use lemma 2.1.2.

$$\begin{aligned}
\langle \rho_k, \rho_k \rangle &= \frac{1}{|\langle x \rangle|} \sum_{g \in \langle x \rangle} \rho_k(g) \rho_k(g^{-1}) \\
&= \frac{1}{n} \sum_{g \in \langle x \rangle} \rho_k(gg^{-1}) \\
&= \frac{1}{n} \sum_{g \in \langle x \rangle} \rho_k(1_{\langle x \rangle}) \\
&= \frac{1}{n} \sum_{g \in \langle x \rangle} 1_{\mathbb{C}^*} \\
&= \frac{1}{n} n \\
&= 1.
\end{aligned}$$

Hence ρ_k is irreducible.

This completes the proof of the proposition. $\square$

Definition 2.1.22 *Let $P = (p_{ij})_{m \times m}$ and $Q = (q_{ij})_{n \times n}$ be two matrices. Then the $mn \times mn$ matrix $P \otimes Q$ is defined by*

$$P \otimes Q := (p_{ij}Q) = \begin{pmatrix} p_{11}Q & p_{12}Q & \cdots & \cdots & \cdots & p_{1m}Q \\ p_{21}Q & p_{22}Q & \cdots & \cdots & \cdots & p_{2m}Q \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m1}Q & p_{m2}Q & \cdots & \cdots & \cdots & p_{mm}Q \end{pmatrix}$$

From this definition, we can show that

$(P \otimes Q)(P' \otimes Q') = (PP') \otimes (QQ') \quad (*)$:

$$\begin{aligned} (P \otimes Q)(P' \otimes Q') &= \left(\sum_{k=1}^{m} p_{il} Q p'_{li} Q' \right)_{mn \times mn} \\ &= \left(\sum_{k=1}^{m} p_{il} p'_{li} Q Q' \right)_{mn \times mn} \\ &= (PP') \otimes (QQ'). \end{aligned}$$

Definition 2.1.23 *Let T and U be representations of a group G, then the tensor product $T \otimes U$ is defined by:*

$$(T \otimes U)(g) := T(g) \otimes U(g)$$

Theorem 2.1.24 *Let T and U be representations of a group G, then*

(i) $T \otimes U$ is a representation of G.

(ii) if $\chi_{(T \otimes U)}$ is the character afforded by $T \otimes U$ then

$$\chi_{(T \otimes U)} = \chi_T \chi_U$$

Proof: See [15,Theorem iii.1]

Definition 2.1.25 *Let $G = H \times K$ be the direct product of two groups H and K and let $T : H \longrightarrow GL_m(\mathbb{C})$ and $U : K \longrightarrow GL_n(\mathbb{C})$ be representations of H and K respectively. Since for every element g in $G, g = hk$ uniquely, for some $h \in H$ and some $k \in K$, the direct product $T \times U$ can be defined by*

$$(T \times U)(g) : = T(h) \otimes U(k)$$

From the uniqueness of $g = hk$ and because of the property of representations T and U of being well defined, it can be shown that $T \times U$ is well defined. Also for $g = hk$ and $g' = h'k'$ with $h, h' \in H$ and $k, k' \in K$, we have

$$\begin{aligned}
(T \times U)(g)(T \times U)(g') &= (T(h) \otimes U(k))(T(h') \otimes U(k')) \\
&= T(h)T(h') \otimes U(k)U(k'), \text{ by } (*) \\
&= T(hh') \otimes U(kk') \\
&= (T \times U)(gg'),
\end{aligned}$$

which means $T \times U$ is a homomorphism and therefore a representation.

From definition 2.1.22, we can deduce that for two matrices P and Q, that

$$Trace(P \otimes Q) = Trace(P).Trace(Q).$$

So we show the following

$$
\begin{aligned}
\chi_{(T \times U)}(g) &= Trace(\,(T \times U)(g)\,) \\
&= Trace(\,T(h) \otimes U(k)\,) \\
&= Trace(T(h)).Trace(U(k))
\end{aligned}
$$

and the next theorem tells us that all the characters of a direct product are constructed in this way.

Theorem 2.1.26 *Let $G = H \times K$ be the direct product of two groups H and K. Then the direct product of any irreducible character of H and any irreducible character of K is an irreducible character of G. Moreover, every irreducible character of G can be constructed in this way.*

Proof: See [15,Theorem iii.2]

Definition 2.1.27 *Let χ be a character of a group G. For $n \in (\mathbb{N} \cup \{0\})$, we define χ^n by*

$$\chi^n(g) := (\chi(g))^n, \quad \forall g \in G.$$

If G is a group and H is a subgroup of G, then we can use the irreducible characters of G to find at least some of the characters of H and vice versa. We deal with the methods of doing this in the following section and use the notes of Moori[15] again.

2.2 Restriction and Induction of Characters

Definition 2.2.1 *Let G be a group and H be a subgroup of G. If $\rho : G \longrightarrow GL_n(\mathbb{C})$ is a representation of G, then $(\rho \downarrow H) : H \longrightarrow GL_n(\mathbb{C})$ given by*

$$(\rho \downarrow H)(h) \quad = \quad \rho(h), \quad \forall\, h \in H,$$

is a representation of H. We say that $\rho \downarrow H$ is the restriction of ρ to H. If χ_ρ is the character of ρ, then $\chi_\rho \downarrow H$ is the character of $\rho \downarrow H$. We refer to $\chi_\rho \downarrow H$ as the restriction of χ_ρ to H.

Theorem 2.2.2 *Let G be a group and $H \leq G$. If ψ is a character of H, then there is an irreducible character χ of G such*

$$\langle \chi \downarrow H, \psi \rangle_H \quad \neq \quad 0.$$

Proof: See [15,Theorem iv.1.1].

Theorem 2.2.3 *Let G be a group and $H \leq G$. If*

$$\chi \in Irr(G) \quad \text{and} \quad Irr(H) = \{\psi_1, \psi_2, \ldots, \psi_r\},$$

then

$$\chi \downarrow H \quad = \quad \sum_{i=1}^{r} \delta_i \psi_i, \text{ where } \quad \delta_i \in (\mathbb{N} \cup \{0\}) \text{ and}$$
$$\sum_{i=1}^{r} \delta_i^2 \quad \leq \quad [G : H] \ (**)$$

Moreover, we have equality in $(**)$ if and only if $\chi(g) = 0, \quad \forall g \in (G \setminus H)$.

Proof: Since $\chi \downarrow H$ is a character of H, $\exists \delta_i \in (\mathbb{N} \cup \{0\})$ such that

$$\chi \downarrow H \quad = \quad \sum_{i=1}^{r} \delta_i \psi_i.$$

Now

$$\begin{aligned}
\langle \chi \downarrow H, \chi \downarrow H \rangle_H &= \left\langle \sum_{i=1}^{r} \delta_i \psi_i, \sum_{i=1}^{r} \delta_i \psi_i \right\rangle_H \\
&= \sum_{i=1}^{r} \delta_i^2 \, \langle \psi_i, \psi_i \rangle_H \\
&= \sum_{i=1}^{r} \delta_i^2
\end{aligned}$$

and

$$\langle \chi \downarrow H, \chi \downarrow H \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h).\overline{\chi(h)}.$$

Hence we get

$$\begin{aligned}
\sum_{i=1}^{r} \delta_i^2 &= \frac{1}{|H|} \sum_{h \in H} \chi(h).\overline{\chi(h)} \quad \textit{so that} \\
|H| \sum_{i=1}^{r} \delta_i^2 &= \sum_{h \in H} \chi(h).\overline{\chi(h)} \quad (***)
\end{aligned}$$

From

$$\begin{aligned}
1 &= \langle \chi, \chi \rangle_G \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g).\overline{\chi(g)} \\
&= \frac{1}{|G|} \sum_{g \in H} \chi(g).\overline{\chi(g)} + \frac{1}{|G|} \sum_{g \in (G \setminus H)} \chi(g).\overline{\chi(g)} \\
&= \frac{|H|}{|G|} \sum_{i=1}^{r} \delta_i^2 + \frac{1}{|G|} \sum_{g \in (G \setminus H)} \chi(g).\overline{\chi(g)} \quad \textit{by } (***) \\
&= \frac{|H|}{|G|} \sum_{i=1}^{r} \delta_i^2 + \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2
\end{aligned}$$

we obtain that

$$\frac{|H|}{|G|} \sum_{i=1}^{r} \delta_i^2 = 1 - \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2 \leq 1$$

and therefore

$$\sum_{i=1}^{r} \delta_i^2 \quad \leq \quad \frac{|G|}{|H|} = [G:H]$$

Also

$$\begin{aligned} \frac{1}{|G|}\sum_{g\in(G\setminus H)} |\chi(g)|^2 &= 0 \quad \textit{if and only if} \\ |\chi(g)|^2 &= 0 \quad \forall\, g \in (G \setminus H). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{|G|}\sum_{g\in(G\setminus H)} |\chi(g)|^2 &= 0 \quad \textit{if and only if} \\ \chi(g) &= 0 \quad \forall\, g \in (G \setminus H) \end{aligned}$$

and we have the equality in (**). $\square$

We have seen how the irreducible characters G can be used to find characters of a subgroup H and can now look at a technique of finding the characters of G from the irreducible characters of any subgroup. We start with the following definition.

Definition 2.2.4 *Let H be a subgroup of G. The right transversal of H in G is a set of representatives for the right cosets of H in G.*

The following theorem tells us how a representation of H can be extended to a representation of G.

Theorem 2.2.5 *Let H be a subgroup of G and T be a representation of H of degree n. Extend T to G by $T^0(g) = T(g)$ if $g \in H$ and $T^0(g) = 0_{n\times n}$ if $g \notin H$. Let $\{x_1, x_2, ..., x_r\}$ be a right transversal of H in G. Define $T \uparrow G$ by*

$$(T \uparrow G)(g) \ := \begin{pmatrix} T^0(x_1 g x_1^{-1}) & T^0(x_1 g x_2^{-1}) & \dots & \dots & \dots & T^0(x_1 g x_r^{-1}) \\ T^0(x_2 g x_1^{-1}) & T^0(x_2 g x_2^{-1}) & \dots & \dots & \dots & T^0(x_2 g x_r^{-1}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ T^0(x_n g x_1^{-1}) & T^0(x_n g x_2^{-1}) & \dots & \dots & \dots & T^0(x_n g x_r^{-1}) \end{pmatrix}$$

$$= \quad \Big(T^0(x_i g x_j^{-1})\Big)_{i,j=1,2,\dots,r}, \quad \forall\, g \in G.$$

Then $T \uparrow G$ is a representation of G of degree nr.

Proof: See [15, theorem iv.2.1].

Definition 2.2.6 *The representation $T \uparrow G$ defined in the previous theorem said to be induced from the representation T of H. Let ϕ be the character afforded by T. Then the character afforded by $T \uparrow G$ is called the induced character from ϕ and is denoted by ϕ^G. If we extend ϕ to G by $\phi^0(g) = \phi(g)$ if $g \in H$ and $\phi^0(g) = 0$ if $g \notin H$, then*

$$\begin{aligned} \phi^G(g) &= Trace\Big((T \uparrow G)(g)\Big) \\ &= \sum_{i=1}^{r} Trace\Big((T^0(x_i g x_i^{-1})\Big) \\ &= \sum_{i=1}^{r} \phi^0(x_i g x_i^{-1}) \end{aligned}$$

In order to construct a formula to find the induced character, the next two propositions are needed.

Proposition 2.2.7 *If $H \leq G$ and ϕ is a character of H, then ϕ^G is independent of the choice of transversal.*

Proof: See [15, Proposition iv. 2.2].

Proposition 2.2.8 *The values of the induced character are given by*

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1}), \quad g \in G$$

Proof: See [15, Proposition iv.2.3].

The following proposition provides us with a formula to calculate the induced character and the proof is provided by Moori [15, Proposition iv.2.4].

Proposition 2.2.9 *Let $H \leq G$, ϕ be a character of H and $g \in G$. Let $[g]$ denote the conjugacy class containing g.*

(i) If $H \cap [g] = \emptyset$, then $\phi^G(g) = 0$,

(ii) If $H \cap [g] \neq \emptyset$, then

$$\phi^G(g) = |C_G(g)|\sum_{i=1}^{m}\frac{\phi(x_i)}{|C_H(x_i)|}$$

where $x_1, x_2, ..., x_m$ are representatives of classes of H that fuse to $[g]$. (That is $H \cap [g]$ breaks up into m conjugacy classes of H with representations $x_1, x_2, ..., x_m$.)

Proof: By Proposition 2.2.8, we have

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1}).$$

If $H \cap [g] = \emptyset$, then $xgx^{-1} \notin H$ for all $x \in G$, so $\phi^0(xgx^{-1}) = 0 \quad \forall\, x \in G$ and $\phi^G(g) = 0$.

If $H \cap [g] \neq \emptyset$, then as x runs over G, xgx^{-1} covers $[g]$ exactly $|C_G(g)|$ times, so

$$\begin{aligned}\phi^G(g) &= \frac{1}{|H|} \times |C_G(g)|\sum_{y\in[g]}\phi^0(y)\\ &= \frac{1}{|H|} \times |C_G(g)| \sum_{y\in([g]\cap H)}\phi(y)\\ &= \frac{|C_G(g)|}{|H|} \times \sum_{i=1}^{m}[H : C_H(x_i)].\phi(x_i)\\ &= |C_G(g)|\sum_{i=1}^{m}\frac{\phi(x_i)}{|C_H(x_i)|} \quad \square\end{aligned}$$

The restriction and induction of characters are related and can be expressed by means of a matrix which we call the Frobenius Reciprocity table. To obtain this relationship, we shall take the route through class functions. We shall use the proof given by Moori [15] for the main result(the Frobenius Reciprocity theorem) in establishing the relationship.

Definition 2.2.10 *Let H be a subgroup of G and ϕ be a class function on H then the induced class function ϕ^G on G is defined by*

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1}), \quad g \in G$$

where ϕ^0 coincides with ϕ on H and is zero otherwise. Notice that

$$\begin{aligned}
\phi^G(ygy^{-1}) &= \frac{1}{|H|}\sum_{x\in G}\phi^0(xygy^{-1}x^{-1}) \\
&= \frac{1}{|H|}\sum_{x\in G}\phi^0((xy)g(xy)^{-1}) \\
&= \frac{1}{|H|}\sum_{z\in G}\phi^0(zgz^{-1}) \\
&= \phi^G(g)
\end{aligned}$$

Thus ϕ^G is also a class function on G.

Note 3 If $H \leq G$ and ϕ is a class function on G, then $\phi \downarrow H$ is a class function on H.

Theorem 2.2.11 *(Frobenius Reciprocity)*
Let $H \leq G$, ϕ be a class function on H and ψ a class fuction on G. Then

$$\langle\, \phi\,,\ \psi \downarrow H\rangle_H \;=\; \langle\, \phi^G,\ \psi\rangle_G$$

Proof:

$$\begin{aligned}
\langle\, \phi^G,\ \psi\rangle_G &= \frac{1}{|G|}\sum_{g\in G}\phi^G(g).\overline{\psi(g)} \\
&= \frac{1}{|G|}\sum_{g\in G}\Big(\frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1})\Big).\overline{\psi(g)} \\
&= \frac{1}{|G|.|H|}\sum_{g\in G}\sum_{x\in G}\phi^0(xgx^{-1}).\overline{\psi(g)} \qquad (****)
\end{aligned}$$

Let $y = xgx^{-1}$. Then as g runs over G, xgx^{-1} runs through G. Also since ψ is a class function on G, $\psi(y) = \psi(xgx^{-1}) = \psi(g)$. Thus by $(****)$ we have

$$\begin{aligned}
\langle \phi^G, \psi \rangle_G &= \frac{1}{|G|.|H|} \sum_{y\in G} \sum_{x\in G} \phi^0(y).\overline{\psi(y)} \\
&= \frac{1}{|G|.|H|} \sum_{x\in G} \left(\sum_{y\in G} \phi^0(y).\overline{\psi(y)} \right) \\
&= \frac{1}{|G|.|H|} |G| \sum_{y\in G} \phi^0(y).\overline{\psi(y)} \\
&= \frac{1}{|H|} \sum_{y\in H} \phi(y).\overline{\psi(y)} \\
&= \langle \phi , \psi \downarrow H \rangle_H \quad \square
\end{aligned}$$

Corollary 2.2.12 *Let $H \leq G$. Assume that $Irr(G) = \{\chi_1, \chi_2, ..., \chi_r\}$ and $Irr(H) = \{\psi_1, \psi_2, ..., \psi_s\}$. Suppose that*

$$\begin{aligned}
\chi_j \downarrow H &= \sum_{i=1}^{s} b_{ij}\psi_i \text{ and} \\
\psi_i^G &= \sum_{j=1}^{r} a_{ij}\chi_j, \text{ then} \\
a_{ij} &= b_{ij}, \quad \forall i, j.
\end{aligned}$$

Proof: See [15,Corollary iv.3.2].

Remark 1 (Frobenius Reciprocity table)

Let $H \leq G$. Assume that $Irr(G) = \{\chi_1, \chi_2, ..., \chi_r\}$and $Irr(H) = \{\psi_1, \psi_2, ..., \psi_s\}$, then by the previous corollary we have

$$\chi_j \downarrow H = \sum_{i=1}^{s} a_{ij}\psi_i \quad \text{and}$$
$$\psi_i^G = \sum_{j=1}^{r} a_{ij}\chi_j, \quad \text{then}$$

the matrix $A = (a_{ij})_{sr}$ is called the Frobenius Reciprocity table for G and H.

2.3 Normal Subgroups

In this section we shall look mainly at how the irreducible characters of a quotient group of a group G can be used to find some of the characters of G itself .

In order to justify a definition for the concept $ker(\chi)$, where χ is a character of G , we state lemma 2.3.1 and lemma 2.3.2 and prove the lemma 2.3.2 using the thesis of Whitley [19].

Lemma 2.3.1 *Let χ be a character of a group G afforded by the representation T. Then for $g \in G$, $T(g)$ is similar to a diagonal matrix $diag(e_1, e_2, ..., e_n)$ where each e_i is a complex root of unity. Then $\chi(g) = e_1 + e_2 + ... + e_n$ and $\chi(g^{-1}) = \overline{\chi(g)}$, where $\overline{x}$ denotes the complex conjugate of x.*

Proof: See [19,Lemma 2.2.1].

Lemma 2.3.2 *Let χ be a character of a group G afforded by the representation T. Then $g \in ker(T)$ if and only if $\chi(g) = \chi(1)$.*

Proof:

Let $n = \chi(1)$, so n is the degree of T. If $g \in ker(T)$ then $T(g) = I_n = T(1)$, where I_n is the $n \times n$ identity matrix, so $\chi(g) = n = \chi(1)$. Conversely, assume $\chi(g) = \chi(1) = n$. By lemma 2.3.1, $\chi(g) = e_1 + e_2 + ... + e_n$, where each e_i is a complex root of unity. Therefore, $e_1 + e_2 + ... + e_n = n$. But $|e_i| = 1$ for all i, so we must have $e_i = 1 \quad \forall i$. Hence $T(g)$ is similar to $diag(e_1, e_2, ..., e_n) = I_n$, so $g \in ker(T)$. □

Definition 2.3.3 *Let χ be a character of a group G. We define*

$$ker(\chi) = \{g \in G : \chi(g) = \chi(1)\}.$$

We note from lemma 2.3.2 $ker(\chi)$ is a normal subgroup of G. The next two theorems taken from the Moori-notes[15, pages 78 and 79] will tell us how the normal subgroups of G can be determined from its character table and how we can tell whether G is simple or not.

Theorem 2.3.4 *Let N be a normal subgroup of G. Then there exists irreducible characters $\chi_1, \chi_2, ..., \chi_s$ of G such that*

$$N = \bigcap_{i=1}^{s} ker(\chi_i).$$

Proof: See [15, Theorem v.3].

Theorem 2.3.5 *A group G is simple if and only if $\chi(g) \neq \chi(1)$ for all nontrivial irreducible characters of G and for all non-identity elements g of G.*

Proof: See [15, Theorem v.4].

The following results form the basis for another tool in finding the characters of a group.

Theorem 2.3.6 *Let N be a normal subgroup of G.*

(a) Let $\hat{\chi}$ be a character of G/N and $\chi : G \to \mathbb{C}$ be defined by

$$\chi(g) = \hat{\chi}(gN) \quad for\ g \in G,$$

Then χ is a character of G and χ has the same degree as $\hat{\chi}$.

(b) Let χ be a character of G, $N \leq ker(\chi)$ and $\hat{\chi} : G/N \to \mathbb{C}$ be defined by

$$\hat{\chi}(gN) = \chi(g) \quad for\ g \in G,$$

Then $\hat{\chi}$ is a character of G/N.

(c) In both of the statements above, $\hat{\chi}$ is an irreducible character of G/N if and only if χ is an irreducible character of G.

Proof:

(a) Let $\hat{T}$ be the representation of degree n that affords $\hat{\chi}$ and define $T : G \rightarrow GL_n(\mathbb{C})$ by $T(g) = \hat{T}(gN)$. Then for $g_1, g_2 \in G$,

$$\begin{aligned} g_1 = g_2 &\implies g_1 N = g_2 N \\ &\implies \hat{T}(g_1 N) = \hat{T}(g_2 N) \\ &\implies T(g_1) = T(g_2). \end{aligned}$$

So T is well-defined. Also

$$\begin{aligned} T(g_1g_2) &= \hat{T}(g_1g_2N) \\ &= \hat{T}(g_1Ng_2N) \\ &= \hat{T}(g_1N)\hat{T}(g_2N) \\ &= T(g_1)T(g_2) \end{aligned}$$

Hence T is a homomorphism and therefore a representation.

Now $Trace(T(g)) = Trace(\hat{T}(gN)) = \hat{\chi}(gN) = \chi(g)$ for all $g \in G$, so T affords χ. Moreover

$$I_m = T(1) = \hat{T}(N) = I_n$$

and so the degree of χ is the same as that of $\hat{\chi}$.

(b) Let T be the representation that affords χ and define $\hat{T} : G/N \to GL_n(\mathbb{C})$ by $\hat{T}(gN) = T(g)$. Then for $g_1, g_2 \in G$,

$$\begin{aligned} g_1N = g_2N &\implies g_1^{-1}g_2 \in N \leq ker(\chi) = ker(T) \\ &\implies T(g_1^{-1}g_2) = I, \text{the identity matrix} \\ &\implies T(g_1^{-1})T(g_2) = I \\ &\implies T(g_1) = T(g_2) \\ &\implies \hat{T}(g_1N) = \hat{T}(g_2N) \end{aligned}$$

thus $\hat{T}$ is well-defined and

$$\begin{aligned} \hat{T}(g_1Ng_2N) &= \hat{T}(g_1g_2N) \\ &= T(g_1g_2) \\ &= T(g_1)T(g_2) \\ &= \hat{T}(g_1N)\hat{T}(g_2N) \end{aligned}$$

Hence T a representation.

$Trace(\hat{T}(gN)) = Trace(T(g)) = \chi(g) = \hat{\chi}(gN)$ for all $g \in G$, so $\hat{T}$ affords $\hat{\chi}$.

(c) For this part, we use the proof by Whitley [19]:

$$\begin{aligned}
\langle \chi, \chi \rangle_G &= |G|^{-1} \sum_{g \in G} |\chi(g)|^2 \\
&= |G|^{-1} \sum_{g \in G} |\hat{\chi}(gN)|^2 \\
&= |G|^{-1}|N| \sum_{gN \in G/N} |\hat{\chi}(gN)|^2 \\
&= |G/N|^{-1} \sum_{gN \in G/N} |\hat{\chi}(gN)|^2 \\
&= \langle \hat{\chi}, \hat{\chi} \rangle_{G/N}.
\end{aligned}$$

By lemma 2.1.2,

$$\begin{aligned}
\chi \in Irr(G) &\iff \langle \chi, \chi \rangle_G = 1 \\
&\iff \langle \hat{\chi}, \hat{\chi} \rangle_{G/N} = 1 \\
&\iff \hat{\chi} \in Irr(G/N) \quad \square
\end{aligned}$$

We end this section with a definition from James and Liebeck [10, Definition 17.2].

Definition 2.3.7 *Let N be a normal subgroup of G and let $\hat{\chi}$ be a character of G/N, then the character χ which is given by*

$$\chi(g) = \hat{\chi}(gN) \quad \text{for } g \in G$$

is called the lift of $\hat{\chi}$ to G. The process of obtaining characters of a group from the characters of any of its quotient groups using theorem 2.3.5 is called the lifting process.

In each of the remaining sections we shall try to illustrate in a group extension how some of the concepts discussed in this chapter are used to calculate the character table of the specific group in discussion.

2.4 The Character Table of a Group of the form $2^3 : 7$

Let $\overline{G}$ be a split extension of N, an elementary abelian two-group of order 8, by G, a cyclic subgroup of $GL(3,2)$ of order 7. As with the example in chapter 1 (section 1.3), we use the method described in section 1.2 of chapter 1 to calculate the conjugacy classes of $\overline{G}$.

G can be generated by the following element of order 7 in $GL_3(2)$

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and $N \cong V_3(2)$, the vector space of dimension three over a field of two elements. G, being cyclic, has 7 conjugacy classes of which each class consists of a power of x. In this example, we thus work with seven cosets, namely Nx^j where $j = 0, 1, 2, ..., 6$. For each j we must consider the action of $N \cong \langle e_1, e_2, e_3 \rangle$ and $C_G(x^i)$ on Nx^j.

Action of N and $C_G(1_G)$ on $N1_G$:
1_G fixes all elements N so that $C_N(1_G) = N$. Thus $k = 8$. That is we have eight orbits, Q_s with $s = 1, 2, ..., 8$, each consisting of one element. Now $C_G(1_G) = G$ so we only need to look at the action x on N. This action is represented by the cycle structure ($e_1 \quad e_1e_2e_3 \quad e_3 \quad e_2e_3 \quad e_1e_2 \quad e_2 \quad e_1e_3$). So

$$\triangle_1 = \{1\} = Q_1 \quad and \quad \triangle_2 = \bigcup_{s=2}^{8} Q_s.$$

Hence $f = 1$ and $f = 7$. We obtain the following:

$$\begin{aligned} |C_{\overline{G}}(1_G)| &= \frac{8 \times 7}{f_1} = 56; \\ |C_{\overline{G}}(e_1)| &= \frac{8 \times 7}{f_2} = 8; \end{aligned}$$

Action of N and $C_G(x)$ on Nx :

$C_N(x) = \{1_G\}$. So $k = 1$ and therefore $f = 1$. Also $C_G(x) = G$ so we have $|C_{\overline{G}}(x)| = 7$. In fact $|C_{\overline{G}}(x^j)| = 7$ for all $j = 1, 2, ..., 6$ because the action of x^j is represented by a $7 - cycle$ and hence x^j $(j \neq 0)$ fixes only 1_N. We thus have $C_N(x^j) = \{1\}$, $j \neq 0$ and so $k = 1$ and again $f = 1$. With $C_G(x^j) = G$, $j \neq 0$ we have $|C_{\overline{G}}(x)| = 7$, $\forall\, i = 1, 2, ..., 6$. With that, the conjugacy table of $\overline{G}$ is completed:

class	(1)	(e_1)	(x)	(x^2)	(x^3)	(x^4)	(x^5)	(x^6)
no. of elements	1	7	8	8	8	8	8	8
order	1	2	7	7	7	7	7	7
centralizer	56	8	7	7	7	7	7	7

Table 2.4.1: The conjugacy table of $2^3 : 7$.

To calculate the character table of $\overline{G}$ we use the method of inducing characters of subgroups of $\overline{G}$ (discussed in section 2.2). In this case we shall use the irreducible characters of N and G.

The character table of N is easily calculated from the character table of $\mathbb{Z}_2 = \langle a : a^2 = 1 \rangle$ by using the product of these characters (theorem 2.1.13). We give the character tables of $\mathbb{Z}_2$ and N.

class	(1)	(a)
centralizer	2	2
ψ_1	1	1
ψ_2	1	-1

Table 2.4.2: The character table of $\mathbb{Z}_2$.

class	(1)	(e_1)	(e_2)	(e_3)	(e_1e_2)	(e_1e_3)	(e_2e_3)	$(e_1e_2e_3)$
order	1	2	2	2	2	2	2	2
centralizer	8	8	8	8	8	8	8	8
τ_1	1	1	1	1	1	1	1	1
τ_2	1	1	-1	1	-1	1	-1	-1
τ_3	1	-1	1	1	-1	-1	1	-1
τ_4	1	-1	-1	1	1	-1	-1	1
τ_5	1	1	1	-1	1	-1	-1	-1
τ_6	1	1	-1	-1	-1	-1	1	1
τ_7	1	-1	1	-1	-1	1	-1	1
τ_8	1	-1	-1	-1	1	1	1	-1

Table 2.4.3: The character table of the group 2^3.

We have seen in proposition 2.1.11 that if $H = \langle x : x^n = 1 \rangle$, then $\rho_k : H \longrightarrow \mathbb{C}^*$ defined by

$$\rho_k(x^m) = [e^{\frac{2k\pi i}{n}}]^m$$

defines n irreducible representations of H. So the character table of $G = \langle x : x^7 = 1 \rangle$ is completely determined by its representatives of this type. The character table of G is as follows:

and we obtain the following characters of $\overline{G}$:

class	(1)	(e_1)	(x)	(x^2)	(x^3)	(x^4)	(x^5)	(x^6)
no. of elements	1	7	8	8	8	8	8	8
order	1	2	7	7	7	7	7	7
centralizer	56	8	7	7	7	7	7	7
$\tau_1^{\overline{G}}$	7	7	0	0	0	0	0	0
$\tau_2^{\overline{G}}$	7	-1	0	0	0	0	0	0

Table 2.4.5

If $\rho \in Irr(G)$, then

$$\begin{aligned}
\rho^{\overline{G}}(1) &= 56(\frac{1.\rho(1)}{7}) = 8.\rho(1) \\
\rho^{\overline{G}}(e_1) &= 0 \\
\rho^{\overline{G}}(x^i) &= 7(\frac{\rho(x^i)}{7}) = \rho(x^i), \qquad for\ each\ i = 1, 2, ..., 6.
\end{aligned}$$

The characters of $\overline{G}$ induced from G are :

class	(1)	(e_1)	(x)	(x^2)	(x^3)	(x^4)	(x^5)	(x^6)
no. of elements	1	7	8	8	8	8	8	8
order	1	2	7	7	7	7	7	7
centralizer	56	8	7	7	7	7	7	7
$\rho_0^{\overline{G}}$	8	0	1	1	1	1	1	1
$\rho_1^{\overline{G}}$	8	0	b_1	b_2	b_3	b_4	b_5	b_6
$\rho_2^{\overline{G}}$	8	0	b_2	b_4	b_6	b_1	b_3	b_5
$\rho_3^{\overline{G}}$	8	0	b_3	b_6	b_2	b_5	b_1	b_4
$\rho_4^{\overline{G}}$	8	0	b_4	b_1	b_5	b_2	b_6	b_3
$\rho_5^{\overline{G}}$	8	0	b_5	b_3	b_1	b_6	b_4	b_2
$\rho_6^{\overline{G}}$	8	0	b_6	b_5	b_4	b_3	b_2	b_1

Table 2.4.6.

where for each $k = 1, 2, ..., 6, \quad b_k = e^{\frac{2k\pi i}{7}}$.

Besides the trivial character χ_0, we have another irreducible character of $\overline{G}$ in $\tau_2^{\overline{G}}$, because

$$\langle \tau_2^{\overline{G}}, \tau_2^{\overline{G}} \rangle = 1.$$

For each $i = 1, 2, ..., 6$,

$$\langle \rho_i^{\overline{G}}, \rho_i^{\overline{G}} \rangle = 2.$$

Hence none of these characters are irreducible, but for each i,

$$\langle \rho_i^{\overline{G}}, \tau_2^{\overline{G}} \rangle = 1.$$

This means that for each $i = 1, 2, ..., 6$, $\rho_i^{\overline{G}}$ is the sum of two irreducible characters of $\overline{G}$ of which one is $\tau_2^{\overline{G}}$. Hence for each i, $\rho_i^{\overline{G}} - \tau_2^{\overline{G}}$ is an irreducible character of $\overline{G}$. With this, we now have all the irreducible characters of $\overline{G}$.

class	(1)	(e_1)	(x)	(x^2)	(x^3)	(x^4)	(x^5)	(x^6)
no. of elements	1	7	8	8	8	8	8	8
order	1	2	7	7	7	7	7	7
centralizer	56	8	7	7	7	7	7	7
χ_1	1	1	1	1	1	1	1	1
$\chi_2 = \rho_1^{\overline{G}} - \tau_2^{\overline{G}}$	1	1	b_1	b_2	b_3	b_4	b_5	b_6
$\chi_3 = \rho_2^{\overline{G}} - \tau_2^{\overline{G}}$	1	1	b_2	b_4	b_6	b_1	b_3	b_5
$\chi_4 = \rho_3^{\overline{G}} - \tau_2^{\overline{G}}$	1	1	b_3	b_6	b_2	b_5	b_1	b_4
$\chi_5 = \rho_4^{\overline{G}} - \tau_2^{\overline{G}}$	1	1	b_4	b_1	b_5	b_2	b_6	b_3
$\chi_6 = \rho_5^{\overline{G}} - \tau_2^{\overline{G}}$	1	1	b_5	b_3	b_1	b_6	b_4	b_2
$\chi_7 = \rho_6^{\overline{G}} - \tau_2^{\overline{G}}$	1	1	b_6	b_5	b_4	b_3	b_2	b_1
$\chi_8 = \tau_2^{\overline{G}}$	7	-1	0	0	0	0	0	0

Table2.4.7: The character table of $2^3 : 7$.

where for each $k = 1, 2, ..., 6, \quad b_k = e^{\frac{2k\pi i}{7}}$.

2.5 The Character Table of a Group of the form $2^3 : GL_3(2)$

Once we knew what the irreducible characters of N and G in the example in section 2.4 was, we solely applied the method of induction to calculate the character table of $\overline{G}$. To calculate the character table of $\overline{G} = 2^3 : GL_3(2)$ where $GL_3(2)$ acts naturally on 2^3, we shall in addition to the method of induction, also use the methods of restriction (discussed in section 2.2) and of lifting of characters (discussed in section 2.3). The character table of this group has also been calculated by Whitley [19] but through the use of Fisher matrices.

The conjugacy classes of $\overline{G}$ has been discussed in chapter 1 (section 1.3), so we start immediately with the business of finding the irreducible characters of G. As in section 1.3 we let N be the group 2^3 and G be the group $GL_3(2)$. Now $G \cong \overline{G}/N$, which implies that some of the irreducible characters of $\overline{G}$ can be found by lifting the irreducible characters of G to $\overline{G}$. The character table of G is obtained from ATLAS[3], so our first six irreducible characters of $\overline{G}$ are the lifts χ_i, $i = 1, 2, ..., 6$ of $\hat{\chi}_i \in Irr(G)$:

class	$(1A)$	$(2A)$	$(3A)$	$(4A)$	$(7A)$	$(7B)$
centralizer	168	8	3	4	7	7
$\hat{\chi}_1$	1	1	1	1	1	1
$\hat{\chi}_2$	3	-1	0	1	a	$\overline{a}$
$\hat{\chi}_3$	3	-1	0	1	$\overline{a}$	a
$\hat{\chi}_4$	6	2	0	0	-1	-1
$\hat{\chi}_5$	7	-1	1	-1	0	0
$\hat{\chi}_6$	8	0	-1	0	1	1

Table 2.5.1:The character table of $G = GL_3(2)$

where $a = \frac{1}{2}(-1 + \sqrt{7}i)$

class	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(3_1)	(6_1)	(4_2)	(4_3)	(7_1)	(7_2)
no. of elements	1	7	42	42	84	224	224	168	168	192	192
centralizer	1344	192	32	32	16	6	6	8	8	7	7
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	-1	-1	-1	0	0	1	1	a	$\overline{a}$
χ_3	3	3	-1	-1	-1	0	0	1	1	$\overline{a}$	a
χ_4	6	6	2	2	2	0	0	0	0	-1	-1
χ_5	7	7	-1	-1	-1	1	1	-1	-1	0	0
χ_6	8	8	0	0	0	-1	-1	0	0	1	1

Table 2.5.2.

where $a = \frac{1}{2}(-1 + \sqrt{7}i)$.

The induction of the characters of G to $\overline{G}$ will put us in a position to find more irreducible characters of $\overline{G}$:

If $\hat{\chi} \in Irr(G)$, then by using the formula for induced characters, we find that

$$\begin{aligned}
(\hat{\chi})^{\overline{G}}(1) &= 8\hat{\chi}(1) \\
(\hat{\chi})^{\overline{G}}(g) &= 4\hat{\chi}(g) \quad for\ g \in (2_2); \\
(\hat{\chi})^{\overline{G}}(g) &= 2\hat{\chi}(g) \quad for\ g \in (3_1) \cup (4_2); \\
(\hat{\chi})^{\overline{G}}(g) &= \hat{\chi}(g) \quad for\ g \in (7_1) \cup (2_2); \\
and\ (\hat{\chi})^{\overline{G}}(g) &= 0 \text{ for } g \notin G.
\end{aligned}$$

Inducing $\hat{\chi}_1$, $\hat{\chi}_2$, $\hat{\chi}_4$, $\hat{\chi}_5$, we obtain $(\hat{\chi}_1)^{\overline{G}}$, $(\hat{\chi}_2)^{\overline{G}}$, $(\hat{\chi}_4)^{\overline{G}}$ *and* $(\hat{\chi}_5)^{\overline{G}}$:

class	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(3_1)	(6_1)	(4_2)	(4_3)	(7_1)	(7_2)
no. of elements	1	7	42	42	84	224	224	168	168	192	192
centralizer	1344	192	32	32	16	6	6	8	8	7	7
$(\hat{\chi}_1)^{\overline{G}}$	8	0	4	0	0	2	0	2	0	1	1
$(\hat{\chi}_2)^{\overline{G}}$	24	0	-4	0	0	0	0	2	0	a	$\overline{a}$
$(\hat{\chi}_4)^{\overline{G}}$	48	0	8	0	0	0	0	0	0	-1	-1
$(\hat{\chi}_5)^{\overline{G}}$	56	0	-4	0	0	2	0	-2	0	0	0

Table 2.5.3.

Now

$$\begin{aligned} \langle\, (\hat{\chi}_1)^{\overline{G}},\ (\hat{\chi}_1)^{\overline{G}}\,\rangle &= 2 \ \ \textit{and} \\ \langle\, \chi_1\ ,\ (\hat{\chi}_1)^{\overline{G}}\,\rangle &= 1\ ,\ \ \textit{so that} \\ \chi_7 &= \Big((\hat{\chi}_1)^{\overline{G}} - \chi_1 \Big) \in Irr(\overline{G}) \end{aligned}$$

Similarly,

$$\begin{aligned} \langle\, (\hat{\chi}_2)^{\overline{G}},\ (\hat{\chi}_2)^{\overline{G}}\,\rangle &= 2 \ \ \textit{and} \\ \langle\, \chi_2\ ,\ (\hat{\chi}_2)^{\overline{G}}\,\rangle &= 1\ ,\ \ \textit{so that} \\ \chi_8 &= \Big((\hat{\chi}_2)^{\overline{G}} - \chi_2 \Big) \in Irr(\overline{G}) \end{aligned}$$

$\overline{G}$ is a maximal subgroup of the group A_8. Thus by restricting the characters of A_8 to $\overline{G}$ we may find more irreducible characters of $\overline{G}$. We shall use the following character, say τ of A_8 obtained from its character table (in ATLAS, page [22]):

class	$1A$	$2A$	$2B$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$	$7A$	$7B$	$15A$	$15B$
centralizer	20160	192	96	180	18	16	8	15	12	6	7	7	15	15
τ	21	-3	1	6	0	1	-1	1	-2	0	0	0	1	1

Table 2.5.4.

Using the fusion map of $\overline{G}$ into A_8 and restricting τ to $\overline{G}$, we obtain $\tau \downarrow \overline{G}$:

$\overline{G}$	A_8
(1)	$(1A)$
(2_1)	$(2A)$
(2_2)	$(2B)$
(2_3)	$(2A)$
(4_1)	$(4A)$
(3_1)	$(3B)$
(6_1)	$(6B)$
(4_2)	$(4B)$
(4_3)	$(4A)$
(7_1)	$(7A)$
(7_2)	$(7B)$

Table 2.5.5.

class	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(3_1)	(6_1)	(4_2)	(4_3)	(7_1)	(7_2)
no. of elements	1	7	42	42	84	224	224	168	168	192	192
centralizer	1344	192	32	32	16	6	6	8	8	7	7
$\tau \downarrow \overline{G}$	21	-3	1	-3	1	0	0	-1	1	0	0

Table 2.5.6.

Because

$$\begin{aligned} \langle\, \tau\downarrow\overline{G}\,,\ \tau\downarrow\overline{G}\rangle &= 1 \quad \textit{we have} \\ \chi_9 &= \tau\downarrow\overline{G} \in Irr(\overline{G}) \end{aligned}$$

Furthermore

$$\begin{aligned} \langle\, (\hat{\chi}_4)^{\overline{G}},\ (\hat{\chi}_4)^{\overline{G}}\,\rangle &= 4\,; \\ \langle\, (\hat{\chi}_4)^{\overline{G}},\ \chi_4\,\rangle &= 1\,; \\ \langle\, (\hat{\chi}_4)^{\overline{G}},\ \chi_7\,\rangle &= 1 \quad \textit{and} \\ \langle\, (\hat{\chi}_4)^{\overline{G}},\ \chi_9\,\rangle &= 1 \quad \text{so that} \\ \chi_{10} &= \Big(\, (\hat{\chi}_4)^{\overline{G}} - (\chi_4 + \chi_7 + \chi_9)\Big) \in Irr(\overline{G}) \end{aligned}$$

and also

$$\begin{aligned} \langle\, (\hat{\chi}_5)^{\overline{G}},\ (\hat{\chi}_5)^{\overline{G}}\,\rangle &= 4\,; \\ \langle\, (\hat{\chi}_5)^{\overline{G}},\ \chi_5\,\rangle &= 1\,; \\ \langle\, (\hat{\chi}_5)^{\overline{G}},\ \chi_8\,\rangle &= 1 \quad \textit{and} \\ \langle\, (\hat{\chi}_5)^{\overline{G}},\ \chi_9\,\rangle &= 1. \quad \textit{Thus} \\ \chi_{11} &= \Big(\, (\hat{\chi}_4)^{\overline{G}} - (\chi_4 + \chi_7 + \chi_9)\Big) \in Irr(\overline{G}). \end{aligned}$$

And so the character table of $\overline{G}$ is completed.

class	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(3_1)	(6_1)	(4_2)	(4_3)	(7_1)	(7_2)
no. of elements	1	7	42	42	84	224	224	168	168	192	192
centralizer	1344	192	32	32	16	6	6	8	8	7	7
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	-1	-1	-1	0	0	1	1	a	$\overline{a}$
χ_3	3	3	-1	-1	-1	0	0	1	1	$\overline{a}$	a
χ_4	6	6	2	2	2	0	0	0	0	-1	-1
χ_5	7	7	-1	-1	-1	1	1	-1	-1	0	0
χ_6	8	8	0	0	0	-1	-1	0	0	1	1
χ_7	7	-1	3	-1	-1	1	-1	1	-1	0	0
χ_8	21	-3	-3	1	1	0	0	1	-1	0	0
χ_9	21	-3	1	-3	1	0	0	-1	1	0	0
χ_{10}	14	-2	2	2	-2	-1	1	0	0	0	0
χ_{11}	7	-1	-1	3	-1	1	-1	-1	1	0	0

Table 2.5.7: The character table of $2^3 : GL_3(2)$.

2.6 The Character Table of a Group of the form $(A_5 \times 3) : 2$

Let N be the direct product of the groups A_5 and the cyclic group $\mathbb{Z}_3$ and let G be a cyclic group of order two. For the calculation of the character table of $\overline{G} = N : G$, a maximal subgroup of A_8, we shall use the methods of restriction and induction of characters. For this purpose we shall make use of the character tables of the groups $H = S_5 \times S_3$ and N.

Since N is non-abelian we cannot use the method discussed in section 1.2 of chapter 1 to calculate the conjugacy table of $\overline{G}$. By regarding A_5 as the alternating on the set $\{1, 2, 3, 4, 5\}$, $\mathbb{Z}_3$ as $\langle(6\ 7\ 8)\rangle$ and G as the group $\langle(1\ 2)(6\ 7)\rangle$, we can determine the conjugacy classes of $\overline{G}$ by acting $(1\ 2)(6\ 7)$ on N. We first show the conjugacy classes of N:

class	(1)	$(3A)$	$(3B)$	$(3C)$	$(3D)$	$(3E)$
class representative	1	(6 7 8)	(6 8 7)	(1 2 3)	(1 2 3) (6 7 8)	(1 2 3) (6 8 7)
no. of elements	1	1	1	20	20	20

Table 2.6.1 : The conjugacy table of $A_5 \times 3$.

class	$(2A)$	$(6A)$	$(6B)$
class representative	(1 2)(3 4)	(1 2)(3 4) (6 7 8)	(1 2)(3 4)(6 8 7)
no. of elements	15	15	15

Table 2.6.1 : The conjugacy table of $A_5 \times 3$(continued).

class	$(5A)$	$(15A)$	$(15B)$
class representative	(1 2 3 4 5)	(1 2 3 4 5)(6 7 8)	(12345)(687)
no. of elements	12	12	12

Table 2.6.1 : The conjugacy table of $A_5 \times 3$(continued).

class	$(5B)$	$(15C)$	$(15D)$
class representative	(1 3 4 5 2)	(1 3 4 5 2)(6 7 8)	(1 3 4 5 2)(6 8 7)
no. of elements	12	12	12

Table 2.6.1 : The conjugacy table of $A_5 \times 3$(continued).

By the action of (1 2)(6 7) on N we obtain the following fusion table

N	$(A_5 \times 3):2$
(1)	(1)
$(3A)$	(3_1)
$(3B)$	(3_1)
$(3C)$	(3_2)
$(3D)$	(3_3)
$(3E)$	(3_3)
$(2A)$	(2_1)
$(6A)$	(6_1)
$(6B)$	(6_1)
$(5A)$	(5_1)
$(15A)$	(15_1)
$(15B)$	(15_2)
$(5B)$	(5_1)
$(15C)$	(15_2)
$(15D)$	(15_1)

Table 2.6.2.

and hence complete the conjugacy table of $\overline{G}$.

class	(1)	(3_1)	(3_2)	(3_3)	(2_1)
class representative	1	(6 7 8)	(1 2 3)	(1 2 3)(6 7 8)	(1 2)(3 4)
no. of elements	1	2	20	40	15
centralizer	360	180	18	9	24

Table 2.6.3 : The conjugacy table of $(A_5 \times 3)) : 2$.

class	(6_1)	(5_1)	(15_1)
class representative	(1 2)(3 4)(6 7 8)	(1 2 3 4 5)	(1 2 3 4 5) (6 7 8)
no. of elements	30	24	24
centralizer	12	15	15

Table 2.6.3 : The conjugacy table of $(A_5 \times 3)) : 2$ (continued).

class	(15_2)	(2_2)	(6_2)	(4_1)
class representative	(1 2 3 4 5)(6 8 7)	(1 2) (6 7)	(1 2) (3 4 5) (6 7)	(1 2 3 4)(6 7)
no. of elements	24	30	60	90
centralizer	15	12	6	4

Table 2.6.3 : The conjugacy table of $(A_5 \times 3) : 2$ (continued).

We start the calculation of the character table of $\overline{G}$ by restricting the characters of H to $\overline{G}$. We show the character table of H on the next two pages.

class	$1\overline{A}$	$2\overline{A}$	$3\overline{A}$	$2\overline{B}$	$2\overline{C}$	$6\overline{A}$	$3\overline{B}$	$6\overline{B}$	$3\overline{C}$	$4\overline{A}$	$4\overline{B}$
no. of elements	1	3	2	10	30	20	20	60	40	30	90
centralizer	720	240	360	72	24	36	36	12	18	24	8
ϕ_1	1	1	1	1	1	1	1	1	1	1	1
ϕ_2	2	0	-1	2	0	-1	2	0	-1	2	0
ϕ_3	1	-1	1	1	-1	1	1	-1	1	1	-1
ϕ_4	1	1	1	-1	-1	-1	1	1	1	-1	-1
ϕ_5	2	0	-1	-2	0	1	2	0	-1	-2	0
ϕ_6	1	-1	1	-1	1	-1	1	-1	1	-1	1
ϕ_7	5	5	5	1	1	1	-1	-1	-1	-1	-1
ϕ_8	10	0	-5	2	0	-1	-2	0	1	-2	0
ϕ_9	5	-5	5	1	-1	1	-1	1	-1	-1	1
ϕ_{10}	6	6	6	0	0	0	0	0	0	0	0
ϕ_{11}	12	0	-6	0	0	0	0	0	0	0	0
ϕ_{12}	6	-6	6	0	0	0	0	0	0	0	0
ϕ_{13}	5	5	5	-1	-1	-1	-1	-1	-1	1	1
ϕ_{14}	10	0	-5	-2	0	1	-2	0	1	2	0
ϕ_{15}	5	-5	5	-1	1	-1	-1	1	-1	1	-1
ϕ_{16}	4	4	4	2	2	2	1	1	1	0	0
ϕ_{17}	8	0	-4	4	0	-2	2	0	-1	0	0
ϕ_{18}	4	-4	4	-2	-2	-2	1	1	1	0	0
ϕ_{19}	4	4	4	-2	-2	-2	1	1	1	0	0
ϕ_{20}	8	0	-4	-4	0	2	2	0	-1	0	0
ϕ_{21}	4	-4	4	-2	2	-2	1	-1	1	0	0

Table 2.6.4 : The character table of $S_5 \times S_3$.

class	$12\overline{A}$	$2\overline{D}$	$2\overline{E}$	$6\overline{C}$	$6\overline{D}$	$6\overline{E}$	$6\overline{F}$	$5\overline{A}$	$10\overline{A}$	$15\overline{A}$
no. of elements	60	15	45	30	20	60	40	24	72	48
centralizer	12	48	16	24	36	12	18	30	10	15
ϕ_1	1	1	1	1	1	1	1	1	1	1
ϕ_2	-1	2	0	-1	2	0	-1	2	0	-1
ϕ_3	1	1	-1	1	1	-1	1	1	-1	1
ϕ_4	-1	1	1	1	-1	-1	-1	1	1	1
ϕ_5	-1	2	0	-1	-2	0	1	2	0	-1
ϕ_6	-1	1	-1	1	-1	1	-1	1	-1	1
ϕ_7	-1	1	1	1	1	1	1	0	0	0
ϕ_8	1	2	0	-1	2	0	-1	0	0	0
ϕ_9	0	-2	-2	-2	0	0	0	1	1	1
ϕ_{10}	0	-2	-2	-2	0	0	0	1	1	1
ϕ_{11}	0	-4	0	2	0	0	0	2	0	-1
ϕ_{12}	0	-2	2	-2	0	0	0	1	-1	1
ϕ_{13}	1	1	1	1	-1	-1	-1	0	0	0
ϕ_{14}	-1	2	0	-1	-2	0	1	0	0	0
ϕ_{15}	1	1	-1	1	-1	1	-1	0	0	0
ϕ_{16}	0	0	0	0	-1	-1	-1	-1	-1	-1
ϕ_{17}	0	0	0	0	-2	0	-1	-2	0	1
ϕ_{18}	0	0	0	0	-1	1	-1	-1	1	-1
ϕ_{19}	0	0	0	0	1	1	1	-1	-1	-1
ϕ_{20}	0	0	0	0	2	0	-1	2	0	-1
ϕ_{21}	0	0	0	0	1	-1	1	-1	1	-1

Table 2.6.4 : The character table of $S_5 \times S_3$(continued).

In the process of restricting the characters of H to $\overline{G}$ we first have to see how the conjugacy classes of $\overline{G}$ fuse to the classes of H:

$\overline{G}$	H
(1)	$1\overline{A}$
(3_1)	$3\overline{A}$
(3_2)	$3\overline{B}$
(3_3)	$3\overline{C}$
(2_1)	$2\overline{D}$
(6_1)	$6\overline{C}$
(5_1)	$5\overline{A}$
(15_1)	$15\overline{A}$
(15_2)	$15\overline{A}$
(2_2)	$2\overline{C}$
(6_2)	$6\overline{E}$
(4_1)	$4\overline{B}$

Table 2.6.5.

By restricting ϕ_1, ϕ_2, ϕ_3, ϕ_7, ϕ_8, ϕ_9, ϕ_{10}, ϕ_{16}, ϕ_{17} and ϕ_{18} of $Irr(H)$, we obtain ten irreducible characters of $\overline{G}$.

We now look at the character table of N for the induction of some of its irreducible characters to $\overline{G}$.

class	(1)	$(3A)$	$(3B)$	$(3C)$	$(3D)$	$(3E)$	$(2A)$	$(6A)$
no. of elements	1	1	1	20	20	20	15	15
centralizer	180	180	180	9	9	9	12	12
ψ_1	1	1	1	1	1	1	1	1
ψ_2	1	c	$\overline{c}$	1	c	$\overline{c}$	1	c
ψ_3	1	$\overline{c}$	c	1	$\overline{c}$	c	1	$\overline{c}$
ψ_4	3	3	3	0	0	0	-1	-1
ψ_5	3	$3c$	$3\overline{c}$	0	0	0	-1	$-c$
ψ_6	3	$3\overline{c}$	$3c$	0	0	0	-1	$-\overline{c}$
ψ_7	3	3	3	0	0	0	-1	-1
ψ_8	3	$3c$	$3\overline{c}$	0	0	0	-1	$-c$
ψ_9	3	$3\overline{c}$	$3c$	0	0	0	-1	$-\overline{c}$
ψ_{10}	4	4	4	1	1	1	0	0
ψ_{11}	4	$4c$	$4\overline{c}$	1	c	$\overline{c}$	0	0
ψ_{12}	4	$4\overline{c}$	$4c$	1	$\overline{c}$	c	-1	$-\overline{c}$
ψ_{13}	5	5	5	-1	-1	-1	1	1
ψ_{14}	5	$5c$	$5\overline{c}$	-1	$-c$	$-\overline{c}$	1	c
ψ_{15}	5	$5\overline{c}$	$5c$	-1	$-\overline{c}$	$-c$	1	$\overline{c}$

Table 2.6.6 : The character table of $A_5 \times 3$.

class	$(6B)$	$(5A)$	$(15A)$	$(15B)$	$(5B)$	$(15C)$	$(15D)$
no. of elements	15	12	12	12	12	12	12
centralizer	12	15	15	15	15	15	15
ψ_1	1	1	1	1	1	1	1
ψ_2	$\overline{c}$	1	c	$\overline{c}$	1	c	$\overline{c}$
ψ_3	c	1	$\overline{c}$	c	1	$\overline{c}$	c
ψ_4	-1	a	a	a	b	b	b
ψ_5	$-\overline{c}$	a	ac	$a\overline{c}$	b	bc	$b\overline{c}$
ψ_6	$-c$	a	$a\overline{c}$	ac	b	$b\overline{c}$	bc
ψ_7	-1	b	b	b	a	a	a
ψ_8	$-\overline{c}$	b	bc	$b\overline{c}$	a	ac	$a\overline{c}$
ψ_9	$-c$	b	$b\overline{c}$	bc	a	$a\overline{c}$	ac
ψ_{10}	0	-1	-1	-1	-1	-1	-1
ψ_{11}	0	-1	$-c$	$-\overline{c}$	-1	$-c$	$-\overline{c}$
ψ_{12}	0	-1	$-\overline{c}$	$-c$	-1	$-\overline{c}$	$-c$
ψ_{13}	1	0	0	0	0	0	0
ψ_{14}	$\overline{c}$	0	0	0	0	0	0
ψ_{15}	c	0	0	0	0	0	0

Table 2.6.6 : The character table of $A_5 \times 3$ (continued).

where

$$\begin{aligned} a &= \frac{1+\sqrt{5}}{2}; \\ b &= \frac{1-\sqrt{5}}{2} \quad \textit{and} \\ c &= -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i \end{aligned}$$

If $\psi \in Irr(N)$, then by using the formula for induced characters, we have

$$
\begin{aligned}
\psi^{\overline{G}}(1_{\overline{G}}) &= 2.\psi(1_N) \\
\psi^{\overline{G}}(g) &= \psi(z_1) + \psi(z_2); \quad g \in (3_1); \quad z_1 \in (3A) \; and \quad z_2 \in (3B) \\
\psi^{\overline{G}}(g) &= 2.\psi(z_3) \quad ; \quad g \in (3_2) \; and \quad z_3 \in (3C) \\
\psi^{\overline{G}}(g) &= \psi(z_4) + \psi(z_5); \quad g \in (3_3); \quad z_4 \in (3D) \; and \quad z_5 \in (3E) \\
\psi^{\overline{G}}(g) &= 2.\psi(z_6) \quad ; \quad g \in (2_1) \; and \quad z_6 \in (2A) \\
\psi^{\overline{G}}(g) &= \psi(z_7) + \psi(z_8); \quad g \in (6_1); \quad z_7 \in (6A) \; and \quad z_8 \in (6B) \\
\psi^{\overline{G}}(g) &= \psi(z_9) + \psi(z_{10}); \quad g \in (5_1); \quad z_9 \in (5A) \; and \quad z_{10} \in (5B) \\
\psi^{\overline{G}}(g) &= \psi(z_{11}) + \psi(z_{12}); \quad g \in (15_1); \quad z_{11} \in (15A) \; and \quad z_{12} \in (15D) \\
\psi^{\overline{G}}(g) &= \psi(z_{13}) + \psi(z_{14}); \quad g \in (15_2); \quad z_{13} \in (15B) \; and \quad z_{12} \in (15C) \\
\psi^{\overline{G}}(g) &= 0. \quad if \quad g \in (2_2) \cup (6_2) \cup (4_1)
\end{aligned}
$$

From the character table of N we induce the characters ψ_5 and ψ_6 to $\overline{G}$ to obtain the irreducible characters χ_{11} and χ_{12} of $\overline{G}$ and so complete the character table of $\overline{G}$:

class	1	(3_1)	(3_2)	(3_3)	(2_1)	(6_1)	(5_1)
no. of elements	1	2	20	40	15	30	24
centralizer	360	180	18	9	24	12	15
χ_1	1	1	1	1	1	1	1
χ_2	2	-1	2	-1	2	-1	2
χ_3	1	1	1	1	1	1	1
χ_4	5	5	-1	-1	1	1	0
χ_5	10	-5	-2	1	2	-1	0
χ_6	5	5	-1	-1	1	1	0
χ_7	6	6	0	0	-2	-2	1
χ_8	4	4	1	1	0	0	-1
χ_9	8	-4	2	-1	0	0	-2
χ_{10}	4	4	1	1	0	0	-1
χ_{11}	6	-3	0	0	-2	1	1
χ_{12}	6	-3	0	0	-2	1	1

Table 2.6.7 : The character table of $A_5 \times 3:2$.

class	(15_1)	(15_2)	(2_2)	(6_2)	(4_1)
no. of elements	24	24	30	60	90
centralizer	15	15	12	6	4
χ_1	1	1	1	1	1
χ_2	-1	-1	0	0	0
χ_3	1	1	-1	-1	-1
χ_4	0	0	1	1	-1
χ_5	0	0	0	0	0
χ_6	0	0	-1	-1	1
χ_7	1	1	0	0	0
χ_8	-1	-1	2	-1	0
χ_9	1	1	0	0	0
χ_{10}	-1	-1	-2	1	0
χ_{11}	$ac + b.\bar{c}$	$a.\bar{c} + bc$	0	0	0
χ_{12}	$a.\bar{c} + bc$	$ac + b.\bar{c}$	0	0	0

Table 2.6.7 : The character table of $A_5 \times 3 : 2$(continued).

where

$$\begin{aligned} a &= \frac{1+\sqrt{5}}{2} \; ; \\ b &= \frac{1-\sqrt{5}}{2} \quad and \\ c &= -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i \end{aligned}$$

To conclude this chapter we give as examples for our discussion on how the methods of restriction and induction of characters are related (section 2.3) , the following two Frobenius reciprocity tables:

$N \setminus \overline{G}$	$\downarrow$ χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}
ψ_1 $\uparrow$	0	1	0	0	0	0	0	0	0	0	0	0
ψ_2	0	1	0	0	0	0	0	0	0	0	0	0
ψ_3	0	1	0	0	0	0	0	0	0	0	0	0
ψ_4	0	0	0	0	0	0	1	0	0	0	0	0
ψ_5	0	0	0	0	0	0	0	0	0	0	1	0
ψ_6	0	0	0	0	0	0	0	0	0	0	1	0
ψ_7	0	0	0	0	0	0	1	0	0	0	0	0
ψ_8	0	0	0	0	0	0	0	0	0	0	0	1
ψ_9	0	0	0	0	0	0	0	0	0	0	0	1
ψ_{10}	0	0	0	0	0	0	0	1	0	1	0	0
ψ_{11}	0	0	0	0	0	0	0	0	1	0	0	0
ψ_{12}	0	0	0	0	0	0	0	0	1	0	0	0
ψ_{13}	0	0	0	1	0	1	0	0	0	0	0	0
ψ_{14}	0	0	0	0	1	0	0	0	0	0	0	0
ψ_{15}	0	0	0	0	1	0	0	0	0	0	0	0

Table 2.6.8.

From the table above we can easily express $\psi^{\overline{G}}$ as a sum of irreducible characters χ_i of $\overline{G}$ for every $\psi \in N$ and likewise express $\chi \downarrow N$ as a sum of irreducible characters ψ_j of N for every $\chi \in \overline{G}$.

$\overline{G} \setminus H$	$\downarrow$ ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}
χ_1 $\uparrow$	1	0	0	0	0	1	0	0	0	0	0
χ_2	0	1	0	0	1	0	0	0	0	0	0
χ_3	0	0	1	1	0	0	0	0	0	0	0
χ_4	0	0	0	0	0	0	1	0	0	0	0
χ_5	0	0	0	0	0	0	0	1	0	0	0
χ_6	0	0	0	0	0	0	0	0	1	0	0
χ_7	0	0	0	0	0	0	0	0	0	1	0
χ_8	0	0	0	0	0	0	0	0	0	0	0
χ_9	0	0	0	0	0	0	0	0	0	0	0
χ_{10}	0	0	0	0	0	0	0	0	0	0	0
χ_{11}	0	0	0	0	0	0	0	0	0	0	1
χ_{12}	0	0	0	0	0	0	0	0	0	0	1

Table 2.6.9.

$\overline{G} \setminus H$	$\downarrow$ ϕ_{12}	ϕ_{13}	ϕ_{14}	ϕ_{15}	ϕ_{16}	ϕ_{17}	ϕ_{18}	ϕ_{19}	ϕ_{20}	ϕ_{21}
χ_1 $\uparrow$	0	0	0	0	0	1	0	0	0	0
χ_2	0	0	0	0	0	0	0	0	0	0
χ_3	0	0	0	0	0	0	0	0	0	0
χ_4	0	0	0	1	0	0	0	0	0	0
χ_5	0	0	1	0	0	0	0	0	0	0
χ_6	0	1	0	0	0	0	0	0	0	0
χ_7	1	0	0	0	0	0	0	0	0	0
χ_8	0	0	0	0	1	0	0	0	0	1
χ_9	0	0	0	0	0	1	0	0	1	0
χ_{10}	0	0	0	0	0	0	1	1	0	0
χ_{11}	0	0	0	0	0	0	0	0	0	0
χ_{12}	0	0	0	0	0	0	0	0	0	0

Table 2.6.9(continued)

The Frobenius table (above) in this case tells us how to express χ^H as a sum of irreducible characters ϕ_i of H for every $\chi \in \overline{G}$ and how to express $\phi \downarrow \overline{G}$ as a sum of irreducible characters χ_j of $\overline{G}$ for every $\phi \in H$.

Chapter 3

FISCHER MATRICES

In this chapter we discuss the theory of Fischer matrices and show how it is applied in finding the irreducible characters of three finite groups which are all split extensions. We shall first look at results which are necessary for our discussion of Fischer matrices. This theory, called Clifford theory, is discussed in section 3.1. Section 3.2 deals with the properties of Fischer matrices and in the rest of the chapter we calculate the character tables of the three groups as mentioned. For the first two sections we make use of the thesis of Whitley [19].

3.1 Clifford Theory

We consider the characters of $\overline{G}$, an extension of N by G, with N not necessarily abelian.

Let $\theta \in Irr(N)$, where $N \lhd \overline{G}$ and for $g \in \overline{G}$, $n \in N$ we let θ^g be defined by $\theta^g(n) = \theta(gng^{-1})$. Then θ^g is a character of N and is said to be conjugate to θ in $\overline{G}$. $\overline{G}$ permutes $Irr(N)$ by $g : \theta \to \theta^g$. Since N acts trivially on $Irr(N)$, $Irr(N)$ is permuted by $\overline{G}/N$, by $gN : \theta \to \theta^g$.

The next result, due to Clifford [2], is fundamental to the work that follows in this and the next section. The proof is from Isaacs[9].

Theorem 3.1.1 *(Clifford's theorem) Let* $N \triangleleft \overline{G}$ *and* $\chi \in Irr(\overline{G})$*. Let* θ *be an irreducible constituent of* $\chi|_N$ *and suppose that* $\theta = \theta_1, \theta_2, ..., \theta_t$ *are the distinct conjugates of* θ *in* $\overline{G}$*.*
Then $\chi|_N = e\sum_{i=1}^{t}\theta_i$ *where* $e = \langle \chi|_N, \theta\rangle$*.*

Proof: We compute $\theta^{\overline{G}}|_N$. Define θ^0 on $\overline{G}$ by

$$\theta^0(x) = \begin{cases} \theta(x) & , \quad \text{if } x \in N \\ 0 & , \quad x \notin N \end{cases}$$

For $n \in N$, we have

$$\theta^{\overline{G}}(n) \quad = \quad |N|^{-1}\sum_{x\in\overline{G}}\theta^0(xnx^{-1}).$$

Since $xnx^{-1} \in N \; \forall \; x \in \overline{G}$ we have

$$\theta^{\overline{G}}(n) \quad = \quad |N|^{-1}\sum_{x\in\overline{G}}\theta^x(n). \text{ Therefore}$$
$$|N|.\theta^{\overline{G}}|_N \quad = \quad \sum_{x\in\overline{G}}\theta^x,$$

and if $\phi \in Irr(N)$ and $\phi \notin \{\theta_i : 1 \leq i \leq t\}$ then

$$0 \;=\; \langle\sum_{x\in\overline{G}}\theta^x, \phi\rangle\,, \quad \text{so } \langle\theta^{\overline{G}}|_N, \phi\rangle \quad = \quad 0.$$

Since χ is an irreducible constituent of $\theta^{\overline{G}}$ by Frobenius reciprocity, it follows that $\langle \chi|_N, \phi \rangle = 0$. Thus all the irreducible constituents of $\chi|_N$ are among the θ_i, so

$$\chi|_N = \sum_{i=1}^{t} \langle \chi|_N, \theta_i \rangle \theta_i.$$

But $\langle \chi|_N, \theta_i \rangle = \langle \chi|_N, \theta \rangle$ since θ_i and θ are conjugate and so the proof is complete. $\square$

Definition 3.1.2 *Let $N \triangleleft \overline{G}$ and $\theta \in Irr(N)$. Then $I_{\overline{G}}(\theta) = \{ g \in \overline{G} : \theta^g = \theta \}$ is the inertia group of θ in $\overline{G}$.*

Since $I_{\overline{G}}(\theta)$ is the stabilizer of θ in the action of $\overline{G}$ on $Irr(N)$, we have that $I_{\overline{G}}(\theta)$ is a subgroup of $\overline{G}$ and $N \subseteq I_{\overline{G}}(\theta)$. Also $[\overline{G} : I_{\overline{G}}(\theta)]$ is the size of the orbit containing θ, so in the formula $\chi|_N = e \sum_{i=1}^{t} \theta_i$, we have $t = [\overline{G} : I_{\overline{G}}(\theta)]$.

As a consequence of Clifford's theorem, we have the following theorem.

Theorem 3.1.3 *Let $N \triangleleft \overline{G}$, $\theta \in Irr(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$. Then induction to $\overline{G}$ maps the irreducible characters of $\overline{H}$ that contain θ in their restriction to N faithfully onto the irreducible characters of $\overline{G}$ which contains θ in their restiction to N.*

Proof: See [19,Theorem 3.3.2]

Theorem 3.1.3 shows that to find the irreducible characters of $\overline{G}$ that contain θ in their restriction to N, it suffices to find the irreducible characters $\overline{H} = I_{\overline{G}}(\theta)$ that contain θ in their restriction. If θ can be extended to an irreducible character ψ of $\overline{H}$ (that is $\psi \in Irr(\overline{H})$ with $\psi|_N = \theta$), then the relevant characters of $\overline{H}$ can be obtained by using the following theorem.

Theorem 3.1.4 *(Gallaghar [6]) With N, $\overline{G}$, θ and $\overline{H}$ as above, if θ extends to a character $\psi \in Irr(\overline{H})$ then as β ranges over all irreducible characters of $\overline{H}$ that contain N in their kernel,*

$\beta\psi$ ranges over all irreducible characters of $\overline{H}$ that contain θ in their restriction.

Proof: By definition of $\overline{H}$, θ is the only $\overline{H}$-conjugate of θ, so by Clifford's theorem $\theta^{\overline{H}}|_N = f\theta$ for some integer f. Comparing degrees, $\theta^{\overline{H}}|_N = [\overline{H} : N]\theta$, so

$$\begin{aligned}\langle \theta^{\overline{H}}, \theta^{\overline{H}} \rangle &= \langle \theta, \theta^{\overline{H}}|_N \rangle \\ &= [\overline{H} : N].\end{aligned}$$

Now we claim that $\theta^{\overline{H}} = \sum_\beta \beta(1)\beta\psi$, where β runs over all irreducible characters of $\overline{H}$ that contain N in their kernel, or, equivalently, over all irreducible characters of $\overline{H}/N$. Both $\theta^{\overline{H}}$ and $\sum_\beta \beta(1)\beta\psi$ are zero off N because for $g \notin N$, $\theta^{\overline{H}}(g) = 0$ since $xgx^{-1} \notin N \ \forall\ x \in \overline{G}$, and by the column orthogonality for the character table of $\overline{H}/N$ since g does not belong to N, we have

$$\sum_\beta \beta(1)(\beta\psi)(g) = \sum_\beta (\beta(1)\beta(g))\psi(g) = 0.$$

Also

$$\theta^{\overline{H}}|_N = [\overline{H} : N]\theta = (\sum_\beta \beta(1)\beta\psi)|_N$$

because for $g \in N$,

$$\begin{aligned}\sum_\beta \beta(1)\beta(g)\psi(g) &= \sum_\beta (\beta(1))^2.\psi(g) \\ &= [\overline{H} : N]\psi(g) \\ &= [\overline{H} : N]\theta(g).\end{aligned}$$

Therefore $\theta^{\overline{H}} = \sum_\beta \beta(1)\beta\psi$ as claimed. Now

$$\begin{aligned}[\overline{H}:N] &= \langle \theta^{\overline{H}}, \theta^{\overline{H}} \rangle \\ &= \langle \sum_\beta \beta(1)\beta\psi, \sum_\gamma \gamma(1)\gamma\psi \rangle \\ &= \sum_{\beta,\gamma} \beta(1)\gamma(1)\langle \beta\psi, \gamma\psi \rangle.\end{aligned}$$

The diagonal terms contribute at least $\sum \beta(1)^2 = [\overline{H}:N]$ so the $\beta\psi$ are irreducible and distinct. These $\beta\psi$ are all the irreducible constituents of $\theta^{\overline{H}}$, so are all the irreducible characters of $\overline{H}$ that contain θ in their restriction, since for $\phi \in Irr(\overline{H}), \langle \phi|_N, \theta \rangle = \langle \phi, \phi^{\overline{H}} \rangle$. $\square$

Note 1 Now suppose $\overline{G}$ is an extension of N by G. If every irreducible character of N can be extended to its inertia group in $\overline{G}$, then by application of theorems 3.1.3 and 3.1.4 the characters of $\overline{G}$ can be obtained as follows:

Let $\theta_1, \theta_2, ..., \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$. For each i, let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and let $\psi_i \in Irr(\overline{H}_i)$ with $\psi_i|_N = \theta_i$. Now each irreducible character of $\overline{G}$ contains some θ_i in its restriction N by Clifford's theorem, so by theorems 3.1.3 and 3.1.4 we have

$$Irr(\overline{G}) = \dot{\bigcup_{i=1}^{t}} \left\{ (\beta\psi_i)^{\overline{G}} : \beta \in Irr(\overline{H}_i),\ N \subset ker(\beta) \right\}$$

Hence the characters of $\overline{G}$ fall into blocks, with each block corresponding to an inertia group.

We now quote some results which give sufficient conditions for the irreducible characters of N to be extendible to their respective inertia groups, so that the above method can be used to calculate the characters of $\overline{G}$.

The following result and proof was obtained from Curtis and Reiner ([4, page 353]).

Theorem 3.1.5 *(Mackey's theorem) Suppose that N is a normal subgroup of $\overline{H}$ such that N is abelian and $\overline{H}$ is a semi-direct product of N and H for some $H \leq \overline{H}$. If $\theta \in Irr(N)$ is invariant in $\overline{H}$ (that is, $\theta^h = \theta, \forall\, h \in \overline{H}$) then θ can be extended to a linear character of $\overline{H}$.*

Proof: Since $\overline{H}$ is a semi-direct product, any $h \in \overline{H}$ can be written uniquely as $h = nk, n \in N, k \in H$. Define χ on $\overline{H}$ by $\chi(nk) = \theta(n)$. Since N is abelian, θ has degree 1, hence is linear, and the fact that $\theta = \theta^h$ for all $h \in \overline{H}$ implies that $\theta(n) = \theta(hnh^{-1})$ for all $h \in \overline{H}$. Then if $h_1 = n_1k_1,\ h_2 = n_2k_2$, we have

$$\begin{aligned}
\chi(h_1h_2) &= \chi(n_1k_1n_2k_2) \\
&= \chi(n_1n_2^{k_1}k_1k_2) \\
&= \theta(n_1n_2^{k_1}) \\
&= \theta(n_1)\theta(n_2^{k_1}) \\
&= \theta(n_1)\theta(n_2) \\
&= \theta(n_1n_2) = \chi(h_1)\chi(h_2).
\end{aligned}$$

Therefore χ is a linear character of $\overline{H}$, and $\chi|_N = \theta$. □

Since in all our examples that we will consider, N is abelian and the extension is split, Mackey's theorem will apply. Mackey's theorem is a corollary of a more general result by Karpilovsky [11] which we state without proof.

Theorem 3.1.6 *Let the group $\overline{H}$ contain a subgroup H of order n such that $\overline{H} = NH$ for N normal in $\overline{H}$ and let $\chi \in Irr(N)$ be invariant in $\overline{H}$. Then χ extends to an irreducible character of $\overline{H}$ if the following conditions hold:*

1. $(m,n) = 1$ *where* $m = \chi(1)$,

2. $N \cap H \leq N'$ *where* N' *is the derived subgroup of* N.

Another extension theorem which can be found in [7] is the following:

Theorem 3.1.7 *If* N *is a normal subgroup of* $\overline{H}$ *and* θ *is an irreducible character of* N *that is invariant in* $\overline{H}$, *then* θ *is extendable to an irreducible character of* $\overline{H}$ *if*

$$([\overline{H} : N], \tfrac{|N|}{\theta(1)}) = 1.$$

3.2 Properties of Fischer Matrices

In this section we give some properties of the Fischer matrices which will enable us to compute the character tables of three finite group extensions in the last three sections. We however need to look at some background material first.

Let $\overline{G}$ be an extension of N by G, with the property that every irreducible character of N can be extended to its inertia group. With the notation of the previous chapter we have that $[Irr(\overline{G}) = \bigcup_{i=1}^{t} \{(\beta\psi_i)^{\overline{G}} : \beta \in Irr(\overline{H_i})$ with $N \subset ker(\beta)\}]$ Now we show how the character table $\overline{G}$ can be constructed using this result. We construct a matrix for each conjugacy class of G (the Fischer matices). Then the character table of $\overline{G}$ can be constructed using these matrices and the character tables of factor groups of the inertia groups. These constructions of Fischer matrices have been discussed by Salleh [18], List [13] and List and Mahmoud [14].

As previously, let $\theta_1, ..., \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$, and let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and $H_i = \overline{H}_i/N$. Let ψ_i be an extension of θ_i to $\overline{H}_i$. We take $\theta_1 = 1_N$, so $\overline{H}_1 = \overline{G}$ and $H_1 = G$. We consider a conjugacy class $[g]$ of G with representative g. Let $X(g) = \{x_1, ...x_{c(g)}\}$ be representatives of $\overline{G}$-conjugacy classes of elements of the coset $N\overline{g}$. Take $x_1 = \overline{g}$. Let $R(g)$ be a set of pairs (i, y)

where $i \in \{1, ..., t\}$ such that H_i contains an element of $[g]$, and y ranges over representatives of the conjugacy classes of H_i that fuse to $[g]$. Corresponding to this $y \in H_i$, let $\{y_{l_k}\}$ be representatives of conjugacy classes of $\overline{H_i}$ that contain liftings of y.

If $\beta \in Irr(\overline{H_i})$ with $N \subset \ker(\beta)$, then β has been lifted from some $\hat{\beta} \in Irr(H_i)$, with $\hat{\beta}(y) = \beta(y_{l_k})$ for any lifting y_{l_k} of y. For convenience we write $\beta(y)$ for $\hat{\beta}(y)$.

Now, using the formula for induced characters given in Proposition 2.2.9., we have

$$\begin{aligned}(\psi_i\beta)^{\overline{G}}(x_j) &= \sum_{y:(i,y)\in R(g)} \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}(\psi_i\beta)(y_{l_k}) \\ &= \sum_{y:(i,y)\in R(g)} \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k})\hat{\beta}(y) \\ &= \sum_{y:(i,y)\in R(g)} \left(\sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k})\right)\beta(y)\end{aligned}$$

By $\Sigma_k{}'$ we mean that we sum over those k for which y_{l_k} is conjugate to x_j in $\overline{G}$. Now we define the Fischer matrix $M(g) = (a^j_{(i,y)})$ with columns indexed by $X(g)$ and rows indexed by $R(g)$ by

$$a^j_{(i,y)} = \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k})$$

Then

$$(\psi_i\beta)^{\overline{G}}(x_j) = \sum_{y:(i,y)\in R(g)} a^j_{(i,y)}\beta(y).$$

The rows of $M(g)$ can be divided into blocks, each block corresponding to an inertia group. Denote the submatrix corresponding to H_i by $M_i(g)$, and let $C_i(g)$ be the fragment of the character table of H_i consisting of the columns corresponding to classes that fuse to $[g]$. Then, by the above relation , the characters of $\overline{G}$ at the classes represented by $X(g)$ obtained from inducing characters of $\overline{H_i}$ are given by the matrix product $C_i(g).M_i(g)$.

We now state a result of Brauer and prove a lemma which will be needed later.

Lemma 3.2.1 *(Brauer) Let A be a group of automorphisms of a group K. Then A also acts on Irr(K) and the number of orbits of A on Irr(K) is the same as that on the conjugacy classes of K.*

Proof: See [8, 4.5.2]

Lemma 3.2.2 *Let A be a group of automorphisms of a group K , so A acts on Irr(K) and on the conjugacy classes of K with the same number of orbits on each by the previous lemma. Suppose we have the following matrix describing these actions:*

$$\begin{array}{cc} & \begin{array}{cccccc} 1=l_1 & l_2 & \dots & l_j & \dots & l_t \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_i \\ \vdots \\ s_t \end{array} & \left(\begin{array}{cccccc} 1 & 1 & \dots & 1 & \dots & 1 \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2t} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{it} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{t1} & a_{t2} & \dots & a_{tj} & \dots & a_{tt} \end{array}\right) \end{array}$$

where $a_{ij} = 1$ for $j = 1, ...t$, l_j 's are lengths of orbits A on the conjugacy classes of K,
s_i 's are lengths of orbits A on $Irr(K)$,
a_{ij} is the sum of s_i irreducible characters of K on the element x_j, where x_j be an element of the orbit of length l_j.
Then the following relation holds for $i, i' \in \{1, ...t\}$:

$$\sum_{j=1}^{t} a_{ij}\overline{a_{i'j}}l_j = |K|s_i\delta_{ii'}$$

.

Proof: Let $\underline{s_i}$ denote the sum of s_i irreducible characters of K, so $\underline{s_i}(x_j) = a_{ij}$. Then

$$< \underline{s_i}, \underline{s_{i'}} >= |K|^{-1}\sum_{j=1}^{t} l_j\underline{s_i}(x_j)\overline{\underline{s_{i'}}(x_j)} = |K|^{-1}\sum_{j=1}^{t} l_j a_{ij}\overline{a_{i'j}}$$

But by orthogonality of irreducible characters, $< \underline{s_i}, \underline{s_{i'}} >= \delta_{ii'} s_i$, so

$$\sum_{j=1}^{t} l_j a_{ij} \overline{a_{i'j}} = |K| s_i \delta_{ii'}. \quad \square$$

Now let $M(g) = (a^j_{(i,y)})$ be the Fischer matrix for $\overline{G} = N.G$ at $g \in G$. We present $M(g)$ with corresponding "weights" for columns and rows as follows:

$$
\begin{array}{cc}
 & \begin{array}{cccc} |C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & \ldots & |C_{\overline{G}}x_{c(g)}| \end{array} \\
\begin{array}{c} |C_{H_1}(g)| \\ |C_{H_2}(y)| \\ |C_{H_2}(y')| \\ \vdots \\ |C_{H_i}(y)| \\ \vdots \\ |C_{H_t}(y)| \\ \vdots \end{array} &
\left(\begin{array}{cccc}
1 & 1 & \ldots & 1 \\ \hline
a^1_{(2,y)} & a^2_{(2,y)} & \ldots & \\
a^1_{(2,y')} & a^2_{(2,y')} & \ldots & \\
\vdots & & & \\ \hline
a^1_{(i,y)} & a^2_{(i,y)} & \ldots & \\
\vdots & & & \\ \hline
a^1_{(t,y)} & a^2_{(t,y)} & \ldots & \\
\vdots & & &
\end{array}\right)
\end{array}
$$

The matrix $M(g)$ is divided into blocks (separated by horizontal lines), each corresponding to an inertia group. Note that $a^j_{(1,g)} = 1$ for all $j \in \{1, ..., c(g)\}$. Fischer has shown that $M(g)$ is square and nonsingular(see[14]). In the following propositions and note we give further properties of Fischer matrices.

Proposition 3.2.3 *(column orthogonality)*

$$\sum_{(i,y)\in R(g)} |C_{H_i}(y)| a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|$$

.

Proof: The partial character table of $\overline{G}$ at classes $x_1, ..., x_{c(g)}$ is

$$\begin{bmatrix} C_1(g)M_1(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$$

where $C_i(g), M_i(g)$ are as defined earlier in this section.

By column orthogonality of the character table of $\overline{G}$, we have

$$\begin{aligned}
|C_{\overline{G}}(x_j)|\delta_{jj'} &= \sum_{i=1}^{t} \sum_{\beta_i \in Irr(H_i)} \left(\sum_{y:(i,y)\in R(g)} a^j_{(i,y)} \beta_i(y)) (\overline{\sum_{y':(i,y')\in R(g)} a^{j'}_{(i,y')} \beta_i(y')} \right) \\
&= \sum_{i=1}^{t} \sum_{\beta_i \in Irr(H_i)} \left(\sum_y a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} \beta_i(y) \overline{\beta_i(y)} + \sum_y \sum_{y'\neq y} a^j_{(i,y)} \overline{a^{j'}_{(i,y')}} \beta_i(y) \overline{\beta_i(y')} \right) \\
&= \sum_{i=1}^{t} \left(\sum_y a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} \sum_{\beta_i \in Irr(H_i)} \beta_i(y) \overline{\beta_i(y)} + \sum_y \sum_{y'\neq y} a^j_{(i,y)} \overline{a^{j'}_{(i,y')}} \sum_{\beta_i \in Irr(H_i)} \beta_i(y) \overline{\beta_i(y')} \right) \\
&= \sum_{i=1}^{t} \left(\sum_y a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} |C_{H_i}(y)| + 0 \right) \\
&= \sum_{(i,y)\in R(g)} a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} |C_{H_i}(y)|. \quad \square
\end{aligned}$$

Proposition 3.2.4 *(List [13]) At the identity of G, the matrix M(1) is the matrix with rows equal to orbit sums of the action of $\overline{G}$ on Irr(N) with duplicate columns discarded.*

For this matrix we have $a^j_{(i,1)} = [G : H_i]$*, and an orthogonality relation for rows:*

$$\sum_{j=1}^{t} a^j_{(i,1)} a^j_{(i',1)} |C_{\overline{G}}(x_j)|^{-1} = \delta_{ii'} |C_{H_i}(1)|^{-1} = \delta_{ii'} |H_i|^{-1}$$

Proof:: The $(i,1), j^{th}$ entry of $M(1)$ is

$$a^j_{(i,1)} = \sum_k \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|} \psi_i(y_{l_k})$$

where we sum over representatives of conjugacy classes of $\overline{H}_i$ that fuse to $[x_j]$ in $\overline{G}$. Therefore $a^j_{(i,1)} = \psi_i^{\overline{G}}(x_j)$. By theorem 3.1.3 $\psi_i^{\overline{G}}$ is an irreducible character of $\overline{G}$, and $< \psi_i^{\overline{G}}|_N, \theta_i > = < \psi_i|_N, \theta_i > = 1$. Therefore, by Clifford's Theorem (Theorem 3.1.1), $\psi_i^{\overline{G}}|_N = \sum_\alpha \chi_\alpha$, where we sum over all $\chi_\alpha \in Irr(N)$ in the orbit containing θ_i. Now $x_j \in N$, and $a^j_{(i,1)} = \sum_\alpha \chi_\alpha(x_j)$. The orthogonality relation follows by Lemma 3.2.2. □

Note 1 If N is an elementary abelian group (which is the case for our calculations), then List[13] has also shown the following for $M(g)$, where $g \neq 1$:

If $\overline{G}$ is a split extension of N by G, then $M(g)$ is the matrix of orbit sums of C_g (as defined in section 1.2) acting on the rows of the character table for a certain factor group of N with duplicate columns discarded.

If the extension is not split, $M(g)$ is the matrix of orbit sums of C_g acting on the rows of the character table with duplicate columns discarded and with each row multiplied by a $p-th$ root of unity where $|N| = p^n$ for some n. It may be that the root of unity for each row is 1.
For these matrices (N elementary abelian, any extension) $a^1_{(i,y)} = \frac{|C_G(g)|}{|C_{H_i}(y)|}$, and we have an orthogonality relation for rows (as a consequence of Lemma 3.2.2.):

$$\sum_{j=1}^{c(g)} m_j a^j_{(i,y)} \overline{a^j_{(i',y')}} = \delta_{(i,y)(i',y')} |C_G(g)||C_{H_i}(y)|^{-1}|N| = \delta_{(i,y)(i',y')} a^1_{(i,y)} |N|$$

where $m_j = [C_g : C_{\overline{G}}(x_j)]$.

(In the notation of section 1.2, m_j is the length of the orbit Δ_l of C_g, so $m_j = \frac{f|N|}{k}$)

The relations given in the above propositions and note will be used later in our calculations of Fischer matrices, so for convenience we list them in a theorem.

Theorem 3.2.5 *For a Fischer matrix $M(g) = (a^j_{(i,y)})$ of $\overline{G} = N.G$ we have the following relations.*

1. $a^j_{(1,g)} = 1$ *for all* $j \in \{1, ..., c(g)\}$.

2. $\displaystyle\sum_{(i,y)\in R(g)} |C_{H_i}(y)| a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|$.

3. *If N is elementary abelian, then* $a^1_{(i,y)} = \frac{|C_G(g)|}{|C_{H_i}(y)|}$, *and*

4. $\displaystyle\sum_{j=1}^{c(g)} m_j a^j_{(i,y)} \overline{a^j_{(i',y')}} = \delta_{(i,y)(i',y')} a^1_{(i,y)} |N|$.

3.3 The Character Table of a Group of the form $2^4 : S_3 \times S_3$

Let $\overline{G} = N : G$ where N is an elementary abelian 2-group of order 16 and $G = S_3 \times S_3$. We start with the conjugacy classes of $\overline{G}$ and use the facts that $S_3 \cong GL_2(2)$ and that N is isomorphic to $V_4(2)$, the vector space of dimension four over a field of two elements. Now

$$GL_2(2) = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle,$$

so we consider the following 4×4 matrices over $GF(2)$:

$$1_G = (\,(1),(1)\,) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x_1 = (\,(1\ 2),(1)\,) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$y_1 = (\,(1\ 2\ 3),(1)\,) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_2 = (\,(1),(1\ 2)\,) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$x_3 = (\,(1\ 2), (1\ 2)\,) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$w_1 = (\,(1\ 2\ 3), (1\ 2)\,) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$y_2 = (\,(1), (1\ 2\ 3)\,) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$w_2 = (\,(1\ 2), (1\ 2\ 3)\,) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$y_3 = (\,(1\ 2\ 3), (1\ 2\ 3)\,) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

We let $G = < x_1, y_1, x_2, y_2 >$. Then $\{1_G, x_1, x_2, x_3, y_1, y_2, y_3, w_1, w_2\}$ is a complete set of the class representatives for G. N is generated by $\{e_1, e_2, e_3, e_4\}$ i.e.

$$N = \langle\ (1\ 0\ 0\ 0),\ (0\ 1\ 0\ 0),\ (0\ 0\ 1\ 0),\ (0\ 0\ 0\ 1)\ \rangle$$

Let G act naturally on N. Using the method discussed in chapter 1, section 1.2, we act N and $C_G(g)$ on the cosets Ng where $g \in \{1_G, x_1, x_2, x_3, y_1, y_2, y_3, w_1, w_2\}$.

- $\underline{g = 1_G:}$

If g is the identity of G, then g fixes all elements of N, so $k = 16$. Under the action of $C_G(1_G) = G$ these orbits are fused as follows:

$\Delta_1 = 1^G = 1 \implies f_1 = 1,$
$\Delta_2 = \{e_1\}^G = \{e_1,\ e_2, e_1e_2\} \implies f_2 = 3,$
$\Delta_3 = \{e_3\}^G = \{e_3,\ e_4, e_3e_4\} \implies f_3 = 3$ and
$\Delta_4 = \{e_1, e_3\}^G = N \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3) \implies f_4 = 9,$

so this coset gives four classes of $\overline{G}$:

$|C_{\overline{G}}(1)| = 16 \times 36 = 576$
$|C_{\overline{G}}(e_1)| = 16 \times 36 \div 3 = 192$
$|C_{\overline{G}}(e_3)| = 16 \times 36 \div 3 = 192$
$|C_{\overline{G}}(e_1e_3)| = 16 \times 36 \div 9 = 64$

- $\underline{g = x_1:}$

x_1 fixes the elements of $\langle\ e_3,\ e_4,\ e_1e_2\ \rangle$ so $k = 8$. The orbits are

$Q_1 = \{x_1,\ e_1e_2x_1\}$, $Q_2 = \{e_1x_1,\ e_2x_1\}$, $Q_3 = \{e_3x_1,\ e_1e_2e_3x_1\}$, $Q_4 = \{e_4x_1,\ e_1e_2e_4x_1\}$,
$Q_5 = \{e_1e_3x_1,\ e_2e_3x_1\}$, $Q_6 = \{e_1e_4x_1,\ e_2e_4x_1\}$, $Q_7 = \{e_3e_4x_1,\ e_1e_2e_3e_4x_1\}$,

$Q_8 = \{e_1e_3e_4x_1,\ e_2e_3e_4x_1\}$.

Under the action of $C_G(x_1) = \langle x_1, x_2, y_2\rangle$, $Q_1 = \Delta_1$ and $Q_2 = \Delta_2$ are fixed while $Q_3 \cup Q_4 \cup Q_7$ becomes Δ_3 and $Q_5 \cup Q_6 \cup Q_8$ becomes Δ_4 and we obtain $f_1 = 1,\ f_2 = 1,\ f_3 = 3$ and $f_4 = 3$, so this coset gives us four classes of $\overline{G}$:

$|C_{\overline{G}}(x_1)| = 8 \times 12 = 96$
$|C_{\overline{G}}(e_1x_1)| = 8 \times 12 = 96$
$|C_{\overline{G}}(e_3x_1)| = 8 \times 12 \div 3 = 32$
$|C_{\overline{G}}(e_1e_3x_1)| = 8 \times 12 \div 3 = 32$

- $\underline{g = y_1:}$

$C_N(y_1) = \langle\ e_3,\ e_4\ \rangle$, so $k = 4$. Under the action of $C_G(y_1) = \langle y_1, x_2, y_2\rangle$ three of the orbits are fused into one and we obtain $f_1 = 1$, and $f_2 = 3$, so this coset gives us two more classes of $\overline{G}$:

$|C_{\overline{G}}(y_1)| = 4 \times 18 = 72$
$|C_{\overline{G}}(e_3y_1)| = 4 \times 18 \div 3 = 24$

- $\underline{g = x_2:}$

Here we have $C_N(x_2) = \langle\ e_1,\ e_2,\ e_3e_4\ \rangle$, so $k = 8$. Under the action of $C_G(x_2) = \langle x_1, x_2, y_1\rangle$ we obtain $f_1 = 1,\ f_2 = 3\ f_3 = 1$ and $f_4 = 3$, so we obtain four more classes of $\overline{G}$:

$|C_{\overline{G}}(x_2)| = 8 \times 12 = 96$
$|C_{\overline{G}}(e_1x_2)| = 8 \times 12 \div 3 = 32$
$|C_{\overline{G}}(e_3x_2)| = 8 \times 12 = 96$

$|C_{\overline{G}}(e_1e_3x_2)| = 8 \times 12 \div 3 = 32$

- $\underline{g = y_2:}$

$C_N(g) = \langle e_1, e_2 \rangle$, so $k = 4$. Under the action of $C_G(g) = \langle y_1, x_2, x_1 \rangle$ we obtain $f_1 = 1$ and $f_2 = 3$ and we obtain another two classes of $\overline{G}$:

$|C_{\overline{G}}(y_2)| = 4 \times 18 = 72$
$|C_{\overline{G}}(e_1y_2)| = 4 \times 18 \div 3 = 24$

- $\underline{g = x_3:}$

Here we have $C_N(g) = \langle e_1e_2, e_3e_4 \rangle$, hence $k = 4$. Under the action of $C_G(g) = \langle x_1, x_2, \rangle$ we obtain $f_1 = 1$, $f_2 = 1$ $f_3 = 1$ and $f_4 = 1$ and so there are four more classes of $\overline{G}$:

$|C_{\overline{G}}(x_3)| = 4 \times 4 = 16$
$|C_{\overline{G}}(e_1x_3)| = 16$
$|C_{\overline{G}}(e_3x_3)| = 16$
$|C_{\overline{G}}(e_1e_3x_3)| = 16$

- $\underline{g = y_3:}$

We have $C_N(g) = \{1_N\}$, therefore $k = 1$, hence $f_1 = 1$. We thus gained one class:
$|C_{\overline{G}}(y_3)| = |C_G(y_3)| = 9$

- $\underline{g = w_1:}$

$C_N(g) = \langle e_3e_4 \rangle$, so $k = 2$. Under the action of $C_G(g) = \langle y_1, x_2 \rangle$ we obtain $f_1 = 1$ and $f_2 = 1$. We have obtained another two classes of $\overline{G}$:

$|C_{\overline{G}}(w_1)| = 2 \times 6 \ = \ 12$

$|C_{\overline{G}}(e_3w_1)| = 2 \times 6 \ = \ 12$

- $\underline{g = w_2:}$

$C_N(g) \ = \ \langle \ e_1e_3 \ \rangle$ and so $k = 2$. Under the action of $C_G(g) \ = \ \langle x_1, y_2 \rangle$ we get $f_1 = 1$ and $f_2 = 1$ and so obtain the last two classes of $\overline{G}$:

$|C_{\overline{G}}(w_1)| = 2 \times 6 \ = \ 12$

$|C_{\overline{G}}(e_3w_1)| = 2 \times 6 \ = \ 12$

The conjugacy classes of $\overline{G}$ are given below and h_i denotes the number of elements in a conjugacy class.

class	1	e_1	e_3	e_1e_3	x_1	e_1x_1	e_3x_1	$e_1e_3x_1$	y_1	e_3y_1	x_2	e_1x_2	e_3x_2
h_i	1	3	3	9	6	6	18	18	8	24	6	18	6
$C_{\overline{G}}(x)$	576	192	192	64	96	96	32	32	72	24	96	32	96

Table 3.3.1 : The conjugacy table of $2^4 : S_3 \ \times \ S_3$.

class	$e_1e_3x_2$	y_2	e_1y_2	x_3	e_1x_3	e_3x_3	$e_1e_3x_3$	y_3	w_1	e_3w_1	w_2	e_1w_2
h_i	18	8	24	36	36	36	36	64	48	48	48	48
$C_{\overline{G}}(x)$	32	72	24	16	16	16	16	9	12	12	12	12

Table 3.3.1 : The conjugacy table of $2^4 : S_3 \ \times \ S_3$(continued).

We proceed to calculate the Fischer matrices. From the action of G on $Irr(N)$ we obtain the same number of orbits as when G acts on N. From each of the four orbits, we determine the inertia groups $\overline{H}_i$ where $i \ = \ 1, 2, 3, 4$. Then we let $H_i \ = \ \overline{H}_i/N$ and we obtain the following inertia factors

$H_1 = G$; $H_2 = \langle x_1,\ x_2,\ y_2 \rangle$; $H_3 = \langle x_1,\ x_2,\ y_1 \rangle$ and $H_4 = \langle x_1,\ x_2,\ \rangle$.

The character tables of these inertia factors are:

class	1	x_1	x_2	x_3	y_1	y_2	y_3	w_1	w_2
h_i	1	3	3	9	2	2	4	6	6
$C_{\overline{G}}(x)$	36	12	12	4	18	18	9	6	6
ψ_1	1	1	1	1	1	1	1	1	1
ψ_2	2	2	0	0	2	-1	-1	0	-1
ψ_3	1	1	-1	-1	1	1	1	-1	1
ψ_4	2	0	2	0	-1	2	-1	0	-1
ψ_5	4	0	0	0	-2	-2	1	0	0
ψ_6	2	0	-2	0	-1	2	-1	1	0
ψ_7	1	-1	1	-1	1	1	1	1	-1
ψ_8	2	-2	0	0	2	-1	-1	0	1
ψ_9	1	-1	-1	1	1	1	1	-1	-1

Table 3.3.2 : The character table of $H_1 = S_3 \times S_3$.

class	1	x_1	x_2	x_3	y_2	w_2
h_i	1	1	3	3	2	2
$C_{\overline{G}}(x)$	12	12	4	4	6	6
ϕ_1	1	1	1	1	1	1
ϕ_2	2	2	0	0	-1	-1
ϕ_3	1	1	-1	-1	1	1
ϕ_4	1	-1	1	-1	1	-1
ϕ_5	2	-2	0	0	-1	1
ϕ_6	1	-1	-1	1	1	-1

Table 3.3.3 : The character table of H_2.

class	1	x_1	x_2	x_3	y_1	w_1
h_i	1	3	1	3	2	2
$C_{\overline{G}}(x)$	12	4	12	4	6	6
τ_1	1	1	1	1	1	1
τ_2	2	0	2	0	-1	-1
τ_3	1	-1	1	-1	1	1
τ_4	1	1	-1	-1	1	-1
τ_5	2	0	-2	0	-1	1
τ_6	1	-1	-1	1	1	-1

Table 3.3.4 : The character table of H_3.

class	1	x_1	x_2	x_3
h_i	1	1	1	1
$C_{\overline{G}}(x)$	4	4	4	4
Φ_1	1	1	1	1
Φ_3	1	1	-1	-1
Φ_4	1	-1	1	-1
Φ_6	1	-1	-1	1

Table 3.3.5 : The character table of H_4.

and their fusion maps into G are:

H_2	G
1	1
x_1	x_1
x_2	x_2
x_3	x_3
y_2	y_2
w_2	w_2

Table 3.3.6.

H_3	G
1	1
x_1	x_1
x_2	x_2
x_3	x_3
y_1	y_1
w_1	w_1

Table 3.3.7.

H_4	G
1	1
x_1	x_1
x_2	x_2
x_3	x_3

Table 3.3.8.

To calculate the Fischer matrices we use the relations of Theorem 3.2.5. For every g in Ng, we have the Fischer matrix $M(g)$. For each matrix $M(g)$, we index the columns by the orders of the centralizers of the class representatives of $\overline{G}$ which comes from Ng and the rows by the orders of the centralizers of the class representatives of the inertia factors which fuse to $[g]$ in G. Corresponding to 1_G, we let

$$M(1_G) = \begin{array}{c} \\ 36 \\ 12 \\ 12 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 576 & 192 & 192 & 64 \end{array} \\ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \end{array}$$

because we obtained four orbits from the action of G on $N1$ and so we have a 4×4 matrix. Now by relation 3.2.5 (1), $a_i = 1$ for each $i = 1, 2, 3, 4$ and by relation 3.2.5 (3) we have $b_1 = 3;\ c_1 = 3;\ d_1 = 9$. By column orthoganality given by relation 3.2.5 (2), we have for example

$$\begin{aligned} 36 + 12|b_2|^2 + 12|c_2|^2 + 4|d_2|^2 &= 192; \\ 36 + 12.3.b_2 + 12.3.c_2 + 4.9d_2 &= 0; \\ 36 + 12.b_1.b_2 + 12.c_1.c_2 + 4.d_1.d_2 &= 0; \\ 36 + 12|b_3|^2 + 12|c_3|^2 + 4|d_3|^2 &= 192; \text{ and so on.} \end{aligned}$$

Manipulating these equations we obtain the matrix:

$$M(1_G) = \begin{array}{c} \\ 36 \\ 12 \\ 12 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 576 & 192 & 192 & 64 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & 3 & -1 \\ 3 & 3 & -1 & -1 \\ 9 & -3 & -3 & 1 \end{pmatrix} \end{array}$$

Similarly, we determine the other Fischer matrices. They appear below.

- $\underline{g = x_1:}$

$$M(x_1) = \begin{array}{c} \\ 12 \\ 12 \\ 4 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 96 & 96 & 32 & 96 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 3 & 3 & -1 & -1 \\ 3 & -3 & 1 & -1 \end{pmatrix} \end{array}$$

- $\underline{g = y_1:}$

$$M(y_1) = \begin{array}{c} \\ 18 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} 72 & 24 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \end{array}$$

- $\underline{g = x_2:}$

$$M(x_2) = \begin{array}{c} \\ 12 \\ 4 \\ 12 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 96 & 32 & 96 & 32 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & 3 & -1 \\ 1 & 1 & -1 & -1 \\ 3 & -1 & -3 & 1 \end{pmatrix} \end{array}$$

- $\underline{g = y_2:}$

$$M(y_2) = \begin{array}{cc} & \begin{array}{cc} 72 & 24 \end{array} \\ \begin{array}{c} 18 \\ 6 \end{array} & \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \end{array}$$

- $\underline{g = x_3:}$

$$M(x_3) = \begin{array}{cc} & \begin{array}{cccc} 16 & 16 & 16 & 16 \end{array} \\ \begin{array}{c} 4 \\ 4 \\ 4 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \end{array}$$

- $\underline{g = y_3:}$

$$M(y_3) = \begin{array}{cc} & 9 \\ 9 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

- $\underline{g = w_1:}$

$$M(w_1) = \begin{array}{c} \\ 6 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} 12 & 12 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$$

- $\underline{g = w_2:}$

$$M(w_2) = \begin{array}{c} \\ 6 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} 12 & 12 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$$

We are now ready to determine the character table $\overline{G}$. There are four inertia factors, so the characters of $\overline{G}$ fall into four blocks. The characters are calculated from the Fischer matrices and the character tables of the inertia factors. This is achieved by multiplying rows of the matrix $M(g)$ with sections of the character tables of the inertia factors fusing to $[g]$.

For $g = 1_G$ we have

$$M(1) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & 3 & -1 \\ 3 & 3 & -1 & -1 \\ 9 & -3 & -3 & 1 \end{pmatrix}$$

By multiplying each row of $M(1)$ by the columns in the character tables of the inertia factors which correspond with the classes fusing to $1_{\overline{G}}$ respectively, we obtain the values of the characters of $\overline{G}$ on the $\overline{G}$-classes with representatives 1, e_1, e_3 and e_1e_3 :

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 4 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 3 & -1 \\ 6 & -2 & 6 & -2 \\ 3 & -1 & 3 & -1 \\ 3 & -1 & 3 & -1 \\ 6 & -2 & 6 & -2 \\ 3 & -1 & 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 & -1 \\ 6 & 6 & -2 & -2 \\ 3 & 3 & -1 & -1 \\ 3 & 3 & -1 & -1 \\ 6 & 6 & -2 & -2 \\ 3 & 3 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 9 & -3 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -3 & -3 & 1 \\ 9 & -3 & -3 & 1 \\ 9 & -3 & -3 & 1 \\ 9 & -3 & -3 & 1 \end{pmatrix}$$

We determine the values the characters of $\overline{G}$ corresponding to the class of G with representative x_1 in a similar fashion:

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ -2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ -2 & -2 & -2 & -2 \\ -1 & -1 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ -2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & -2 & -2 & 2 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 1 & 1 \\ 3 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 & -1 \\ 3 & -3 & 1 & -1 \\ -3 & 3 & -1 & 1 \\ -3 & 3 & -1 & 1 \end{pmatrix}$$

With this we now also know the values of the characters of $\overline{G}$ on the $\overline{G}$-classes with representatives x_1, e_1x_1, e_3x_1 and $e_1e_3x_1$.

Continuing this process with the other classes of G, we complete the character table of $\overline{G}$.

class	1	e_1	e_3	e_1e_3	x_1	e_1x_1	e_3x_1	$e_1e_3x_1$	y_1	e_3y_1	x_2	e_1x_2	e_3x_2	$e_1e_3x_2$
h_i	1	3	3	9	6	6	18	18	8	24	6	18	6	18
$C_{\overline{G}}(x)$	576	192	192	64	96	96	32	32	72	24	96	32	96	32
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	2	2	2	2	2	2	2	2	2	2	0	0	0	0
χ_3	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_4	2	2	2	2	0	0	0	0	-1	-1	2	2	2	2
χ_5	4	4	4	4	0	0	0	0	-2	-2	0	0	0	0
χ_6	2	2	2	2	0	0	0	0	-1	-1	-2	-2	-2	-2
χ_7	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1
χ_8	2	2	2	2	-2	-2	-2	-2	2	2	0	0	0	0
χ_9	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1
χ_{10}	3	-1	3	-1	1	-1	-1	1	0	0	3	-1	3	1
χ_{11}	6	-2	6	-2	2	-2	-2	2	0	0	0	0	0	0
χ_{12}	3	-1	3	-1	1	-1	-1	1	0	0	-3	1	-3	-1
χ_{13}	3	-1	3	-1	-1	1	1	-1	0	0	3	-1	3	1
χ_{14}	6	-2	6	-2	-2	2	2	-2	0	0	0	0	0	0
χ_{15}	3	-1	3	-1	-1	1	1	-1	0	0	-3	1	-3	-1
χ_{16}	3	3	-1	-1	3	3	-1	-1	3	-1	1	1	-1	-1
χ_{17}	6	6	-2	-2	0	0	0	0	-3	1	2	2	-2	-2
χ_{18}	3	3	-1	-1	-3	-3	1	1	3	-1	1	1	-1	-1
χ_{19}	3	3	-1	-1	3	3	-1	-1	3	-1	-1	-1	1	1
χ_{20}	6	6	-2	-2	0	0	0	0	-3	1	-2	-2	2	2
χ_{21}	3	3	-1	-1	-3	-3	1	1	3	-1	-1	-1	1	1
χ_{22}	9	-3	-3	1	3	-3	1	-1	0	0	3	-1	-3	1
χ_{23}	9	-3	-3	1	3	-3	1	-1	0	0	-3	1	3	-1
χ_{24}	9	-3	-3	1	-3	3	-1	1	0	0	3	-1	-3	1
χ_{25}	9	-3	-3	1	-3	3	-1	1	0	0	-3	1	3	-1

Table 3.3.9 : The character table of $2^4 : S_3 \times S_3$.

class	y_2	e_1y_2	x_3	e_1x_3	e_3x_3	$e_1e_3x_3$	y_3	w_1	e_3w_1	w_2	e_1w_2
h_i	8	24	36	36	36	36	64	48	48	48	48
$C_{\overline{G}}(x)$	72	24	16	16	16	16	9	12	12	12	12
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	-1	-1	0	0	0	0	-1	0	0	-1	-1
χ_3	1	1	-1	-1	-1	-1	1	-1	-1	1	1
χ_4	2	2	0	0	0	0	-1	-1	-1	0	0
χ_5	-2	-2	0	0	0	0	-1	0	0	0	0
χ_6	2	2	0	0	0	0	-1	1	1	0	0
χ_7	1	1	-1	-1	-1	-1	1	1	1	-1	-1
χ_8	-1	-1	0	0	0	0	-1	0	0	1	1
χ_9	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_{10}	3	-1	1	1	-1	-1	0	0	0	1	-1
χ_{11}	-3	1	0	0	0	0	0	0	0	-1	1
χ_{12}	3	-1	-1	-1	1	1	0	0	0	1	-1
χ_{13}	3	-1	-1	-1	1	1	0	0	0	-1	1
χ_{14}	-3	1	0	0	0	0	0	0	0	1	-1
χ_{15}	3	-1	1	1	-1	-1	0	0	0	-1	1
χ_{16}	0	0	1	-1	1	-1	0	1	-1	0	0
χ_{17}	0	0	0	0	0	0	0	-1	1	0	0
χ_{18}	0	0	-1	1	-1	1	0	1	-1	0	0
χ_{19}	0	0	-1	1	-1	1	0	-1	1	0	0
χ_{20}	0	0	0	0	0	0	0	1	-1	0	0
χ_{21}	0	0	1	-1	1	-1	0	-1	1	0	0
χ_{22}	0	0	1	-1	-1	1	0	0	0	0	0
χ_{23}	0	0	-1	1	1	-1	0	0	0	0	0
χ_{24}	0	0	-1	1	1	-1	0	0	0	0	0
χ_{25}	0	0	1	-1	-1	1	0	0	0	0	0

Table 3.3.9 : The Character Table of $2^4 : S_3 \times S_3$(continued).

3.4 The character table of a Group of the form $2^4 : S_4$

Again we let $\overline{G} = N : G$ where N is an elementary abelian 2-group of order 16 and $G = S_4$. The symmetric group S_4 is generated by (1 2) *and* (1 2 3 4). By identifying (1 2) and (1 2 3 4) with

$$g_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

respectively, we can regard S_4 as the subgroup $\langle g_1,\ g_2 \rangle$ of $GL_4(2) \cong S_6$. Then we act the group $\langle g_1,\ g_2 \rangle$ naturally on $V_4(2) \cong N$.

To determine the conjugacy classes of $\overline{G}$ we need the conjugacy table of S_4 for the cosets of $\overline{G}/N$ and for this purpose, we use the character table of $G = S_4$. We may again use the method discussed in chapter 1, section 1.2. We act N and $C_G(g)$ on the cosets Ng as follows:

- $\underline{g = 1:}$

  The identity of G fixes all elements of N, so $k = 16$. Under the action of $C_G(1_G) = G$ on $N1$, we obtain

$$f_1 = 1,\ f_2 = 4,\ f_3 = 6,\ f_4 = 4 \textit{ and } f_5 = 1$$

  and so the following classes of $\overline{G}$ from the coset N:

  $|C_{\overline{G}}(1)| = 16 \times 24 = 384$

  $|C_{\overline{G}}(e_1)| = 16 \times 24 \div 4 = 96$

  $|C_{\overline{G}}(e_1e_2)| = 16 \times 24 \div 6 = 64$

  $|C_{\overline{G}}(e_1e_2e_3)| = 16 \times 24 \div 4 = 96$

  $|C_{\overline{G}}(e_1e_2e_3e_4)| = 16 \times 24 = 384$

- $\underline{g \in (2A):}$

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With the action of g on N we get $k = 8$ and the action of $C_G(x_1)$ gives us

$$f_1 = 1, \ f_2 = 1, \ f_3 = 2, \ f_4 = 1 \ f_5 = 1 \ and \ f_6 = 2$$

Also $|C_G(g)| = 4$ and we obtain

$|C_{\overline{G}}(g)| = 32$
$|C_{\overline{G}}(e_1g)| = 32$
$|C_{\overline{G}}(e_3g)| = 16$
$|C_{\overline{G}}(e_1e_3g)| = 16$
$|C_{\overline{G}}(e_3e_4g)| = 32$
$|C_{\overline{G}}(e_1e_3e_4g)| = 32$

- $\underline{g \in (3A):}$

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This case gives us $k = 4$ and $f_i = 1$, for each $i = 1, \ 2, \ 3, \ 4$. $|C_G(g)| = 3$ and we obtain

$|C_{\overline{G}}(g)| = 12$

$|C_{\overline{G}}(e_1g)| = 12$

$|C_{\overline{G}}(e_4g)| = 12$

$|C_{\overline{G}}(e_1e_4g)| = 12$

- $g \in (2B)$:

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We have $k = 4$ and

$$f_1 = 1, \ f_2 = 2, \quad \text{and } f_3 = 1$$

$|C_G(g)| = 8$ and we obtain

$|C_{\overline{G}}(g)| = 32$

$|C_{\overline{G}}(e_1g)| = 16$

$|C_{\overline{G}}(e_1e_3g)| = 32$

- $g \in (4A)$:

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This case gives us $k = 2$ and $f_i = 1$, for each $i = 1, 2$. We have $|C_G(g)| = 4$ and so we obtain

$|C_{\overline{G}}(g)| = 8$

$|C_{\overline{G}}(e_1g)| = 8$

which gives us the conjugacy table of $\overline{G}$.

class	1	(2_1)	(2_2)	(2_3)	(2_4)	(2_5)	(2_6)	(2_7)	(2_8)	(2_9)	(2_{10})
no. of elements	1	4	6	4	1	12	12	24	12	12	24
centralizer	384	96	64	96	384	32	32	16	32	32	16

Table 3.4.1 : The conjugacy table of $2^4 : S_4$.

class	(3_1)	(3_2)	(3_3)	(3_4)	(2_{11})	(2_{12})	(2_{13})	(4_1)	(4_2)
no. of elements	32	32	32	32	12	24	12	48	48
centralizer	12	12	12	12	32	16	32	8	8

Table 3.4.2 : The conjugacy table of $2^4 : S_4$(continued).

We can now calculate the Fischer matrices. From the action of G on $Irr(N)$ we obtain five orbits. From each of these orbits, we determine the inertia groups $\overline{H}_i$ where $i = 1,2,3,4,5$. Then we obtain the following inertia factors

$$H_1 = H_5 = G; \quad H_2 = H_3 = S_3 \quad \text{and} \quad H_4 = \langle(12),\ (34)\rangle.$$

The character tables of these inertia factors are:

class	1	$(2A)$	$(3A)$	$(2B)$	$(4A)$
no. of elements	1	6	3	3	6
centralizer	24	4	3	8	4
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	-1	2	0
ψ_4	3	1	0	-1	-1
ψ_5	3	-1	0	-1	1

Table 3.4.3 : The character table of H_1.

class	1	$(2\overline{A})$	$(3\overline{A})$
no. of elements	1	3	2
centralizer	6	2	3
ϕ_1	1	1	1
ϕ_2	1	-1	1
ϕ_3	2	0	-1

Table 3.4.4 : The character table of H_2

class	1	$(2\overline{A})$	$(2\overline{B})$	$(2\overline{C})$
no. of elements	1	1	1	1
centralizer	4	4	4	4
Φ_1	1	1	1	1
Φ_2	1	1	-1	-1
Φ_3	1	-1	1	-1
Φ_4	1	-1	-1	1

Table 3.4.5 : The character table of H_4.

and their fusion maps into G are:

S_3	G
1	1
$2\overline{A}$	$2A$
$3\overline{A}$	$3A$

Table 3.4.6.

H_4	G
1	1
$2\overline{A}$	$2A$
$2\overline{B}$	$2A$
$2\overline{C}$	$2B$

Table 3.4.7

Next we use the relations of Theorem 3.2.5. again to calculate the Fischer matrices which are:

- $\underline{g = 1_G:}$

$$M(1) = \begin{array}{c} \\ 24 \\ 6 \\ 4 \\ 6 \\ 24 \end{array} \begin{array}{c} \begin{array}{ccccc} 384 & 96 & 64 & 96 & 384 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix} \end{array}$$

- $\underline{g \in (2A):}$

$$M(g) = \begin{array}{cc} & \begin{array}{cccccc} 32 & 32 & 16 & 32 & 32 & 16 \end{array} \\ \begin{array}{c} 4 \\ 2 \\ 2 \\ 4 \\ 4 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & -2 & 0 & 2 & -2 & 0 \\ 2 & -2 & 0 & -2 & 2 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix} \end{array}$$

- $\underline{g \in (3A):}$

$$M(g) = \begin{array}{cc} & \begin{array}{cccc} 12 & 12 & 12 & 12 \end{array} \\ \begin{array}{c} 3 \\ 3 \\ 3 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \end{array}$$

- $\underline{g \in (4A):}$

$$M(g) = \begin{array}{cc} & \begin{array}{ccc} 32 & 16 & 32 \end{array} \\ \begin{array}{c} 8 \\ 4 \\ 8 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \end{array}$$

- $\underline{g \in (2B):}$

$$M(g) = \begin{array}{c c} & \begin{array}{cc} 8 & 8 \end{array} \\ \begin{array}{c} 4 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$$

We can now determine the character table $\overline{G}$. As with the example in section 3.3, we just need to multiply the rows of the matrix $M(g)$ with sections of the character tables of the inertia factors corresponding to g.

There are five inertia factors, so the characters of $\overline{G}$ fall into five blocks.

The character table of $\overline{G}$ is as follows:

class	1	(2_1)	(2_2)	(2_3)	(2_4)	(2_5)	(2_6)	(2_7)	(2_8)	(2_9)	(2_{10})
no. of elements	1	4	6	4	1	12	12	24	12	12	24
centralizer	384	96	64	96	384	32	32	16	32	32	16
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1	-1	-1	-1	-1
χ_3	2	2	2	2	2	0	0	0	0	0	0
χ_4	3	3	3	3	3	1	1	1	1	1	1
χ_5	3	3	3	3	3	1	1	1	1	1	1
χ_6	4	2	0	-2	-4	2	-2	0	2	-2	0
χ_7	4	2	0	-2	-4	-2	2	0	-2	2	0
χ_8	8	4	0	-4	-8	0	0	0	0	0	0
χ_9	6	0	-2	0	6	2	2	-2	0	0	0
χ_{10}	6	0	-2	0	6	0	0	0	2	2	-2
χ_{11}	6	0	-2	0	6	0	0	0	-2	-2	2
χ_{12}	6	0	-2	0	6	-2	-2	2	0	0	0
χ_{13}	4	-2	0	2	-4	2	-2	0	-2	2	0
χ_{14}	4	-2	0	2	-4	-2	2	0	2	-2	0
χ_{15}	8	-4	0	4	-8	0	0	0	0	0	0
χ_{16}	1	-1	1	-1	1	1	1	1	-1	-1	-1
χ_{17}	1	-1	1	-1	1	-1	-1	-1	1	1	1
χ_{18}	2	-2	2	-2	2	0	0	0	0	0	0
χ_{19}	3	-3	3	-3	3	1	1	1	-1	-1	-1
χ_{20}	3	-3	3	-3	3	-1	-1	-1	1	1	1

Table 3.4.8 : The character table of $2^4 : S_4$.

class	(3_1)	(3_2)	(3_3)	(3_4)	(2_{11})	(2_{12})	(2_{13})	(4_1)	(4_2)
no. of elements	32	32	32	32	12	24	12	48	48
centralizer	12	12	12	12	32	16	32	8	8
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	-1	-1
χ_3	-1	-1	-1	-1	-2	-2	-2	0	0
χ_4	0	0	0	0	-1	-1	-1	-1	-1
χ_5	0	0	0	0	-1	-1	-1	1	1
χ_6	1	1	-1	-1	0	0	0	0	0
χ_7	1	1	-1	-1	0	0	0	0	0
χ_8	-1	-1	1	1	0	0	0	0	0
χ_9	0	0	0	0	2	0	-2	0	0
χ_{10}	0	0	0	0	-2	0	2	0	0
χ_{11}	0	0	0	0	-2	0	2	0	0
χ_{12}	0	0	0	0	2	0	-2	0	0
χ_{13}	1	-1	1	-1	0	0	0	0	0
χ_{14}	1	-1	1	-1	0	0	0	0	0
χ_{15}	-1	1	-1	1	0	0	0	0	0
χ_{16}	1	-1	-1	1	1	-1	1	1	-1
χ_{17}	1	-1	-1	1	1	-1	1	-1	1
χ_{18}	-1	1	1	-1	-2	2	-2	0	0
χ_{19}	0	0	0	0	-1	1	-1	-1	1
χ_{20}	0	0	0	0	-1	1	-1	1	-1

Table 3.4.8 : The character table of $2^4 : S_4$(continued).

3.5 The Character Table of a Group the form $2^4 : S_3 \times S_3$

Let $\overline{G} = N : G$ where N is as defined in the previous two examples and $G = S_3 \times S_3$. The action of G on N, given by CAYLEY [1], is different from the action in section 3.3, so $\overline{G}$ is a different extension of N by G.

We start to determine the conjugacy classes of $\overline{G}$ by giving the character table of $S_3 \times S_3$ again.

class	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$3C$	$6A$	$6B$
no. of elements	1	3	3	9	2	2	4	6	6
centralizer	36	12	12	4	18	18	9	6	6
ψ_1	1	1	1	1	1	1	1	1	1
ψ_2	2	2	0	0	2	-1	-1	0	-1
ψ_3	1	1	-1	-1	1	1	1	-1	1
ψ_4	2	0	2	0	-1	2	-1	0	-1
ψ_5	4	0	0	0	-2	-2	-1	0	0
ψ_6	2	0	-2	0	-1	2	-1	1	0
ψ_7	1	-1	1	-1	1	1	1	1	-1
ψ_8	2	-2	0	0	2	-1	-1	0	1
ψ_9	1	-1	-1	1	1	1	1	-1	-1

Table 3.5.1 : The character table of $S_3 \times S_3$.

Using the same method as that in the previous sections we determine the the conjugacy classes of $\overline{G}$ by acting N and $C_G(g)$ on the cosets Ng as follows:

- $\underline{g = 1_G :}$

  All elements of $N = \{ (0,0,0,0), (1,1,1,1), (1,0,1,0), (1,0,0,1), (0,1,1,0),$ $(0,1,0,1), (0,0,1,0), (1,1,1,0), (1,0,0,0), (1,0,1,1), (0,0,0,1), (1,1,0,0),$ $(0,1,1,1), (1,1,0,1), (0,0,1,1), (0,1,0,0) \}$, are fixed, so $k = 16$. Under the action of $C_G(1_G) =$

G on $N1_G$, we obtain the following blocks:

$\{(0,0,0,0)\}$,

$\{(1,1,1,1),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(0,0,1,0),(1,1,1,0),$

$(1,0,0,0),(1,0,1,1),\}$,

$\{\ (0,0,0,1),(1,1,0,0),(0,1,1,1),(1,1,0,1),(0,0,1,1),(0,1,0,0)\}$. So we have

$$f_1 = 1,\ \ f_2 = 9, \text{ and } f_3 = 6$$

and so the following classes of $\overline{G}$ from the coset N:

$|C_{\overline{G}}(1)| = 16 \times 36 \ = \ 576$

$|C_{\overline{G}}(\ (1,1,1,1)\ )| = 16 \times 36 \ \div \ 9 \ = \ 64$

$|C_{\overline{G}}(\ (0,0,0,1)\ )| = 16 \times 36 \ \div \ 6 \ = \ 96$

- $\underline{g \in (2A):}$

$$g \ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

With the action of g on N we get the orbits

$\{\ (0,0,0,0),(1,0,1,0),(0,0,1,0),(1,0,0,0)\ \}$

$\{\ (0,1,0,1),(1,1,1,1),(1,1,0,1),(0,1,1,1)\ \}$

$\{\ (0,0,1,1),(1,0,0,1),(1,0,1,1),(0,0,0,1)\ \}$

$\{\ (1,1,0,0),(0,1,1,0),(0,1,0,0),(1,1,1,0)\ \}$

so that $k = 4$

and the by the action of $C_G(g)$

$\{\ (0,0,0,0),(1,0,1,0),(0,0,1,0),(1,0,0,0)\ \}$ is fixed while the other orbits are fused into one, giving us $f_1 = 1$, and $f_2 = 3$. Also $|C_G(g)| = 12$ and we obtain

$|C_{\overline{G}}(g)| = 48$

$|C_{\overline{G}}(\ (1,1,1,1)\ g)| = 16$

- $\underline{g \in (2B):}$

$$g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

This case gives us $k = 4$, $f_1 = 1$ and $f_2 = 3$. $|C_G(g)| = 12$ and we obtain

$|C_{\overline{G}}(g)| = 48$

$|C_{\overline{G}}(\ (1,1,1,1)\ g)| = 16$

- $\underline{g \in (2C):}$

$$g = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

We have $k = 4$, $f_1 = 1$, $f_2 = 1$, and $f_3 = 2$. $|C_G(g)| = 4$ and we obtain

$|C_{\overline{G}}(g)| = 16$

$|C_{\overline{G}}(\ (1,1,1,1)g\ )| = 16$

$|C_{\overline{G}}(\ (1,0,1,0)g\ )| = 8$

- $\underline{g \in (3A):}$

$$g = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

We have $k = 1$, and $f_1 = 1$. $|C_G(g)| = 18$ and we obtain $|C_{\overline{G}}(g)| = 18$

- $\underline{g \in (3B):}$

$$g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

We have $k = 1$, and $f_1 = 1$. $|C_G(g)| = 18$ and we obtain $|C_{\overline{G}}(g)| = 18$

- $\underline{g \in (3C):}$

$$g = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

We have $k = 4$, $f_1 = 1$, and $f_2 = 3$. $|C_G(g)| = 9$ and we obtain

$|C_{\overline{G}}(g)| = 36$

$|C_{\overline{G}}(\ (1,1,1,1)g\ )| = 12$

- $\underline{g \in (6A):}$

$$g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We have $k = 1$, and $f_1 = 1$. $|C_G(g)| = 6$ and we obtain $|C_{\overline{G}}(g)| = 6$

- $\underline{g \in (6B):}$

$$g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We have $k = 1$, and $f_1 = 1$. $|C_G(g)| = 6$ and we obtain $|C_{\overline{G}}(g)| = 6$

We have completed the conjugacy table of $\overline{G}$. We show it below.

class	1	(2_1)	(2_2)	(2_3)	(2_4)	(2_5)	(2_6)	(2_7)	(2_8)	(2_9)
$C_{\overline{G}}(x)$	576	64	96	48	16	48	16	16	16	8

Table 3.5.2 : The conjugacy table of $2^4 : S_3 \times S_3$.

class	(3_1)	(3_2)	(3_3)	(3_4)	(6_1)	(6_2)
$C_{\overline{G}}(x)$	18	18	36	12	6	6

Table 3.5.2 : The conjugacy table of $2^4 : S_3 \times S_3$(continued).

From the action of G on $Irr(N)$ we obtain three orbits. From each of these orbits, we determine the inertia groups $\overline{H}_i$ where $i = 1, 2, 3, 4, 5$ and hence the following inertia factors

$H_1 = G$, H_2 a non-cyclic subgroup of G of order four which is generated by

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and H_3 a non-abelian subgroup of G of order six which is generated by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The character tables of these inertia factors are that of G and:

class	1	$(2\overline{A})$	$(3\overline{A})$	$(2\overline{B})$
no. of elements	1	1	1	1
centralizer	4	4	4	4
ψ_1	1	1	1	1
ψ_2	1	-1	1	-1
ψ_3	1	1	-1	-1
ψ_4	1	-1	-1	1

Table 3.5.3 : The character table of H_2.

class	1	$(2\overline{A})$	$(3\overline{A})$
no. of elements	1	3	2
centralizer	6	2	3
ϕ_1	1	1	1
ϕ_2	1	-1	1
ϕ_3	2	0	-1

Table 3.5.4 : The character table of H_3.

and their fusion maps into G are:

H_3	G
1	1
$2\overline{A}$	$2C$
$3\overline{A}$	$3C$

Table 3.5.5

H_2	G
1	1
$2\overline{A}$	$2C$
$2\overline{B}$	$2B$
$2\overline{C}$	$2A$

Table 3.5.6.

We use the relations of Theorem 3.2.5. to calculate the Fischer matrices which are:

- $\underline{g = 1_G:}$

$$M(1) = \begin{array}{c} \\ 36 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 576 & 64 & 96 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 9 & a & b \\ 6 & c & d \end{pmatrix} \end{array}$$

From the equations

$$\begin{aligned} 36 + 4|a|^2 + 6|c|^2 &= 64 \quad and \\ 36 + 36a + 36c &= 0; \end{aligned}$$

we obtain $a = 1$ and $c = -2$. Then from

$$\begin{aligned} 36 + 4b - 12d &= 0 \quad \text{and} \\ 36 + 36b + 36d &= 0; \end{aligned}$$

we get $b = -3$ and $c = 2$.

By using the appropriate relations, the other Fischer matrices are determined:

- $\underline{g \in (2A):}$

$$M(g) = \begin{array}{c} \\ 12 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} 48 & 16 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \end{array}$$

- $\underline{g \in (2B):}$

$$M(g) = \begin{array}{c} \\ 12 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} 48 & 16 \end{array} \\ \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \end{array}$$

- $g \in (2C)$:

$$\begin{array}{cc} & \begin{array}{ccc} 16 & 16 & 8 \end{array} \\ M(g) = \begin{array}{c} 4 \\ 4 \\ 2 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix} \end{array}$$

- $g \in (3A)$:

$$\begin{array}{cc} & 18 \\ M(g) = 18 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

- $g \in (3B)$:

$$\begin{array}{cc} & 18 \\ M(g) = 18 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

- $g \in (2C)$:

$$\begin{array}{cc} & \begin{array}{cc} 36 & 12 \end{array} \\ M(g) = \begin{array}{c} 9 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \end{array}$$

- $\underline{g \in (6A):}$

$$M(g) = \begin{array}{cc} & 6 \\ 6 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

- $\underline{g \in (6B):}$

$$M(g) = \begin{array}{cc} & 6 \\ 6 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

To determine the character table of $\overline{G}$, we just need to multiply the rows of the matrix $M(g)$ with sections of the character tables of the inertia factors corresponding to g. The characters of $\overline{G}$ fall into three blocks and are shown in the following table.

class	1	(2_1)	(2_2)	(2_3)	(2_4)	(2_5)	(2_6)	(2_7)	(2_8)	(2_9)
$C_{\overline{G}}(x)$	576	64	96	48	16	48	16	16	16	8
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1	-1	-1	-1
χ_3	1	1	1	-1	-1	-1	-1	1	1	1
χ_4	1	1	1	-1	-1	1	1	-1	-1	-1
χ_5	2	2	2	2	2	0	0	0	0	0
χ_6	2	2	2	0	0	2	2	0	0	0
χ_7	2	2	2	0	0	-2	-2	0	0	0
χ_8	2	2	2	-2	-2	0	0	0	0	0
χ_9	4	4	4	0	0	0	0	0	0	0
χ_{10}	9	1	-3	3	-1	3	-1	1	1	-1
χ_{11}	9	1	-3	-3	1	3	-1	-1	-1	1
χ_{12}	9	1	-3	3	-1	-3	1	-1	-1	1
χ_{13}	9	1	-3	-3	1	-3	1	1	1	-1
χ_{14}	6	-2	2	0	0	0	0	2	-2	0
χ_{15}	6	-2	2	0	0	0	0	-2	2	0
χ_{16}	12	-4	4	0	0	0	0	0	0	0

Table 3.5.7 : The character table of $2^4 : S_3 \times S_3$.

class	(3_1)	(3_2)	(3_3)	(3_4)	(6_1)	(6_2)
$C_{\overline{G}}(x)$	18	18	36	12	6	6
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1
χ_3	1	1	1	1	-1	-1
χ_4	1	1	1	1	-1	1
χ_5	2	-1	-1	-1	-1	0
χ_6	-1	2	-1	-1	0	-1
χ_7	-1	2	-1	-1	0	1
χ_8	2	-1	-1	-1	1	0
χ_9	-2	-2	1	1	0	0
χ_{10}	0	0	0	0	0	0
χ_{11}	0	0	0	0	0	0
χ_{12}	0	0	0	0	0	0
χ_{13}	0	0	0	0	0	0
χ_{14}	0	0	3	-1	0	0
χ_{15}	0	0	3	-1	0	0
χ_{16}	0	0	-3	1	0	0

Table 3.5.7 : The character table of $2^4 : S_3 \times S_3$(continued).

For the completion of the character table of $\overline{G}$ most of the calculations were done by CAYLEY[1].

Summary

The work done in this mini thesis deals mainly with different methods of calculating character tables of split extensions of finite groups. Three of the six character tables that are calculated are done with the use of Fischer matrices. In this work the method of Fischer is applied on groups of the form $N.G$ where N is an elementary abelian group. In fact, only one of the six groups of which the character tables are calculated, is not of this form and so Fischer matrices could easily have been used to calculate five of the character tables. The aim of the work done here however is to exhibit a variety of methods to calculate the character tables of split extensions.

In Chapter one a review of basic definitions and results on group extensions and a description of a method for finding the conjugacy tables of group extensions is given. An example on the application of this method is also given. Chapter two deals with basic concepts and results on representa- tion and character theory as well as the application of some of these results in calculating the character tables of some group extensions. In Chapter three we discuss Fischer matrices and how it is used to calculate the character tables of group extensions of the form $N.G$ where N is an elementary abelian group.

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