

COMPUTATION OF THE CHARACTER TABLES OF CERTAIN GROUP EXTENSIONS

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Chapter 1

THE CONJUGACY CLASSES OF GROUP EXTENSIONS

In this chapter some basic theory on group extensions is first given in section 1.1 and then a method for finding the conjugacy classes of group extensions is described in section 1.2. In section 1.3 we look at an example due to Whitley[19] to illustrate how the theory developed in section 1.2 is used to calculate the conjugacy classes of the group $2^3 : GL_3(2)$. For section 1.1 , the books by Rotman[17] and Gorenstein[8] were used as references while for section 1.2 we used the works of Whitley[19], Moori[15], Moori and Mpono[16] and Salleh[18].

1.1 Definitions and Basic Results

Definition 1.1.1 *If N and G are groups, an extension of N by G is a group \bar{G} that satisfies the following properties*

1. $N \triangleleft \bar{G}$

$$2. \overline{G}/N \cong G.$$

We say that \overline{G} is a *split extension* of N by G if \overline{G} contains subgroups N and G_1 with $G_1 \cong G$ such that

$$1. N \triangleleft \overline{G}$$

$$2. NG_1 = \overline{G}$$

$$3. N \cap G_1 = 1_{\overline{G}}$$

In this case \overline{G} is also called a *semi-direct* product of N by G , and we identify G_1 with G .

Note 1 If \overline{G} is a semi-direct product of N by G , then every $\overline{g} \in \overline{G}$ can be uniquely expressed in the form $\overline{g} = ng$, where $n \in N$ and $g \in G$. Multiplication in \overline{G} satisfies $(n_1g_1)(n_2g_2) = n_1n_2^{g_1}g_1g_2$, where n^g denotes gng^{-1} .

Definition 1.1.2 *The automorphism group of a group G , denoted by $\text{Aut}(G)$, is the set of all automorphisms of G under the binary operation of composition.*

If \overline{G} is a *split extension* of N by G , then there is a homomorphism $\theta : G \rightarrow \text{Aut}(N)$ given by $\theta_g(n) = gng^{-1} = n^g (n \in N, g \in G)$, where we denote $\theta(g)$ by θ_g . Thus G acts on N , and we say that the extension \overline{G} realizes θ .

Conversely, given any groups N and G , and $\theta : G \rightarrow \text{Aut}(N)$, we can define a semi-direct product of N by G that realizes θ as follows. Let \overline{G} be the set of ordered pairs $(n, g) (n \in N, g \in G)$ with

multiplication $(n_1, g_1)(n_2, g_2) = (n_1\theta_{g_1}(n_2), g_1g_2)$. Then \overline{G} is a semi-direct product of N by G .

Hence a split extension of N by G is completely described by the map $\theta : G \rightarrow \text{Aut}(N)$, that is, it is described by the way G acts on N .

We use the ATLAS [3] notation and let $N.G$ denote an arbitrary extension of N by G . A split extension is denoted by $N : G$ or $N : {}^\theta G$, where $\theta : G \rightarrow \text{Aut}(N)$ determines the extension. A non-split extension is denoted by $N \cdot G$.

If \overline{G} is a split extension of N by G , then $\overline{G} = NG = \bigcup_{g \in G} Ng$, so G may be regarded as a right transversal for N in \overline{G} (that is, a complete set of right coset representatives of N in \overline{G}). Now suppose \overline{G} is any extension of N by G , not necessarily split. Since $\overline{G}/N \cong G$, there is an epimorphism $\lambda : \overline{G} \rightarrow G$ with kernel N . For $g \in G$, define a lifting of g to be an element $\overline{g} \in \overline{G}$ such that $\lambda(\overline{g}) = g$. Then choosing a lifting of each element of G , we get the set $\{\overline{g} : g \in G\}$ which is a transversal for N in \overline{G} .

We now show that even for a non-split extension \overline{G} of N by G , where N is abelian, G acts on N . This result can be obtained from Rotman[17].

Lemma 1.1.3 *Let \overline{G} be an extension of an abelian group N by G , then there is a homomorphism $\theta : G \rightarrow \text{Aut}(N)$ such that $\theta_g(n) = \overline{g}n\overline{g}^{-1}$ ($n \in N$), and θ is independent of the choice of liftings $\{\overline{g} : g \in G\}$.*

Proof: For $a \in \overline{G}$, denote conjugation by a by γ_a . Since N is normal in \overline{G} , $\gamma_a|_N$ is an automorphism of N and the function $\mu : \overline{G} \rightarrow \text{Aut}(N)$ defined by $\mu(a) = \gamma_a|_N$ is a homomorphism.

If $a \in N$, then $\mu(a) = 1_N$, since N is abelian. Therefore there is a homomorphism $\mu^* : \overline{G}/N \rightarrow \text{Aut}(N)$ defined by $\mu^*(Na) = \mu(a)$.

Now $G \cong \overline{G}/N$ and for any lifting $\{\overline{g} : g \in G\}$, the map $\phi : G \rightarrow \overline{G}/N$ defined by $\phi(g) = N\overline{g}$ is an isomorphism. If $\{\overline{h} : h \in G\}$ is another choice of liftings, then $\overline{g}\overline{h}^{-1} \in N$ so that $N\overline{g} = N\overline{h}$. Therefore the isomorphism ϕ is independent of the choice of liftings. Now let $\theta : G \rightarrow \text{Aut}(N)$ be the composite $\mu^* \circ \phi$. If $g \in G$ and \overline{g} is a lifting, then $\theta(g) = \mu^*(\phi(g)) = \mu^*(N\overline{g}) = \mu(\overline{g}) \in \text{Aut}(N)$, so for $n \in N$, $\theta_g(n) = \mu(\overline{g})(n) = \overline{g}n\overline{g}^{-1} = n^{\overline{g}}$, as required. \square

Note 2 Let \overline{G} be an extension of an abelian group N by G . For each $g \in G$ we choose a lifting $\overline{g} \in \overline{G}$, and for convenience we take $\overline{1} = 1$. We identify G with \overline{G}/N under the isomorphism $g \rightarrow N\overline{g}$. Now $\{\overline{g} : g \in G\}$ is a right transversal for N in \overline{G} , so every element $h \in \overline{G}$ has a unique expression of the form $h = n\overline{g}$ ($n \in N, g \in G$), and we have the following relations.

1. $\overline{g}n = n^{\overline{g}}\overline{g}$, where $n \in N$ and $g \in G$.
2. $\overline{g}\overline{h} = f(g, h)\overline{g}h$ for some $f(g, h) \in N$, where $g, h \in G$.

1.2 The Conjugacy Classes of Group Extensions

Let $\overline{G} = N.G$, where N is abelian. Then for each conjugacy class $[g]$ in G with representative $g \in G$, we analyse the coset $N\overline{g}$, where \overline{g} is a lifting of g in \overline{G} and $\overline{G} = \bigcup_{g \in G} N\overline{g}$. To each class representative $g \in G$ with lifting $\overline{g} \in \overline{G}$, we define

$$C_g = \{x \in \overline{G} : x(N\overline{g}) = (N\overline{g})x\}.$$

Then C_g being the set stabilizer of $N\overline{g}$ in \overline{G} under the action by conjugation of \overline{G} on $N\overline{g}$, is a subgroup of \overline{G} . The following lemmas and their proofs due to Whitley[19] and Moori and Mpono[16] will be required in the next section .

Lemma 1.2.1 $N \triangleleft C_g$.

Proof: For any $n \in N$

$$n(N\bar{g})n^{-1} = N\bar{g}n^{-1} = N\bar{g}n^{-1}\bar{g}^{-1}\bar{g} = N\bar{g},$$

the last step following from the fact that $(n^{-1})\bar{g} \in N$ since $N \triangleleft \bar{G}$.

Hence $N \subseteq C_g$. From $N \leq C_g \leq \bar{G}$ and $N \triangleleft \bar{G}$, we obtain $N \triangleleft C_g$. \square

Lemma 1.2.2 $C_g/N = C_{\bar{G}/N}(N\bar{g})$.

Proof: Consider $Nk \in \bar{G}/N$. Then

$$\begin{aligned} Nk \in C_{\bar{G}/N}(N\bar{g}) &\iff Nk(N\bar{g})(Nk)^{-1} = N\bar{g} \\ &\iff NkN\bar{g}Nk^{-1} = N\bar{g} \\ &\iff NkN\bar{g}k^{-1} = N\bar{g} \\ &\iff NkNn\bar{g}k^{-1} = N\bar{g} \quad \forall n \in N \\ &\iff Nkn\bar{g}k^{-1} = N\bar{g} \quad \forall n \in N \\ &\iff kn\bar{g}k^{-1} \in N\bar{g} \quad \forall n \in N \\ &\iff k \in C_g. \end{aligned}$$

Thus we obtain that $C_g/N = C_{\bar{G}/N}(N\bar{g})$. \square

From the two preceding lemmas, we have that $C_g = N.C_{\bar{G}/N}(N\bar{g})$. For a lifting $\bar{g} \in \bar{G}$ of $g \in G$, we can identify $C_{\bar{G}/N}(N\bar{g})$ with $C_G(g)$ and write $C_g = N.C_G(g)$ in general. If $\bar{G} = N : G$ then we can identify C_g with $C_g = \{x \in \bar{G} : x(Ng) = (Ng)x\}$ and in this case we obtain the following corollary.

Corollary 1.2.3 *Let $\bar{G} = N : G$. Then $C_g = N : C_G(g)$.*

Proof: We have already shown in the Lemma 1.2.1 that $N \triangleleft C_g$. Now we show that $C_G(g) \leq C_g$ and that $N \cap C_G(g) = \{1_G\}$. Let $x \in C_G(g)$. Then we obtain $(Ng)^x = x(Ng)x^{-1} = xNgx^{-1} = Nxgx^{-1} = Ng$. Thus $x \in C_g$ and hence $C_G(g) \leq C_g$. Since $N \cap C_G(g) \leq N \cap G = \{1_G\}$, then

we have that $N \cap C_G(g) = \{1_G\}$. This completes the proof. \square

The conjugacy classes of \bar{G} will be determined by the action by conjugation of C_g , for each conjugacy class $[g]_G$ of G , on the elements of $N\bar{g}$ or in the case of a split extension on the elements of Ng . Since $C_g = N : C_G(g)$, we act first N and then act $\{\bar{h} : h \in C_G(g)\}$ on the elements of $N\bar{g}$. The outline of this action is given in two steps by Moori and Mpono [16,page 5] as follows:

STEP 1: *The action of N on $N\bar{g}$:*

Let $C_N(\bar{g})$ be the stabilizer of \bar{g} in N . Then for any $n \in N$ we have

$$\begin{aligned}
 x \in C_N(n\bar{g}) &\Leftrightarrow x(n\bar{g})x^{-1} = n\bar{g} \\
 &\Leftrightarrow xnx^{-1}x\bar{g}x^{-1} = n\bar{g} \\
 &\Leftrightarrow n(x\bar{g}x^{-1}) = n\bar{g}, \quad \text{since } N \text{ is abelian} \\
 &\Leftrightarrow x\bar{g}x^{-1} = \bar{g} \\
 &\Leftrightarrow x \in C_N(\bar{g}).
 \end{aligned}$$

Thus $C_N(\bar{g})$ fixes every element of $N\bar{g}$. Now let $|C_N(\bar{g})| = k$. Then under the action of N , $N\bar{g}$ splits into k orbits Q_1, Q_2, \dots, Q_k , where

$$\begin{aligned}
 |Q_i| &= [N : C_N(\bar{g})] \\
 &= \frac{|N|}{k}, \quad \text{for } i \in \{1, \dots, k\}.
 \end{aligned}$$

STEP 2: *The action of $\{\bar{h} : h \in C_G(g)\}$ on $N\bar{g}$*

Since the elements of $N\bar{g}$ are now in the orbits Q_1, \dots, Q_k from step 1 above, we need only to act $\{\bar{h} : h \in C_G(g)\}$ on the k orbits. Suppose that under this action f_j of the orbits Q_1, \dots, Q_k fuse together to form one orbit Δ_i , then the f_j 's obtained this way must satisfy

$$\sum_j f_j = k$$

and we have

$$|\Delta_i| = f_j \times \frac{|N|}{k}$$

Thus for $x = d_i \bar{g} \in \Delta_i$, we obtain that

$$\begin{aligned} |[x]_{\bar{G}}| &= |\Delta_i| \times |[g]_G| \\ &= f_j \times \frac{|N|}{k} \times \frac{|G|}{|C_G(g)|} \\ &= f_j \times \frac{|\bar{G}|}{k|C_G(g)|} \end{aligned}$$

and thus we obtain that

$$\begin{aligned} |C_{\bar{G}}(x)| &= \frac{|\bar{G}|}{|[x]_{\bar{G}}|} \\ &= |\bar{G}| \times \frac{k|C_G(g)|}{f_j|\bar{G}|} \\ &= \frac{k|C_G(g)|}{f_j}. \end{aligned}$$

Thus to calculate the conjugacy classes of $\bar{G} = N.G$, we need to find the values of k and the f_j 's for each class representative $g \in G$. We note that the values of k can be determined from the action of G on N (given in lemma 1.1.3). If $\bar{G} = N : G$ (a split extension) however, we analyse the coset Ng instead of $N(\bar{g})$ since in the split case $G \leq \bar{G}$. Under the action of N on Ng , we always assume that $g \in Q_1$. Since $C_G(g)$ fixes g , Q_1 does not fuse with any other Q_i . Hence we will always have that $f_1 = 1$. Hence

$$\begin{aligned}
k &= \sum_j f_j \\
&= 1 + \sum_m f_m,
\end{aligned}$$

where the sum is taken over all m such that $g \notin Q_m$.

We now apply the method described in the Step 1 and Step 2 in the next section.

1.3 The Conjugacy Classes of a Group of the Form

$$2^3 : GL_3(2)$$

In this section we give the conjugacy classes of the group $\overline{G} = N : G$ where N is an elementary abelian group of order 8 and $G \cong GL_3(2)$, as calculated by Whitley[19], where G acts naturally on N .

We regard N as the vector space $V_3(2)$ of dimension three over a field of two elements. Let N be generated by $\{e_1, e_2, e_3\}$ with $e_i^2 = 1$ for $1 \leq i \leq 3$, so

$$N = \{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}$$

To determine the conjugacy classes of \overline{G} we analyse the cosets Ng where g is a representative of a class of G . (Note that the extension is split, so $\overline{G} = \bigcup_{g \in G} Ng$). Now

$$|C_{\overline{G}}(x)| = \frac{k \cdot |C_G(g)|}{f_j},$$

where f_j of the k blocks of the coset Ng have fused to give a class of \overline{G} containing x . We need the conjugacy classes of G , so we exhibit it here (obtained from ATLAS [3]).

class	(1A)	(2A)	(3A)	(4A)	(7A)	(7B)
centralizer	168	8	3	4	7	7

Table 1.3.1: The conjugacy table of $GL_3(2)$.

The representatives thus must come from the classes mentioned in the table above:

- $g = 1_G$:

For g the identity of G , g fixes all elements of N , so $k = 8$. Since G is transitive on $N - \{1\}$ under the action of $C_G(g) = G$, we have two orbits with $f_1 = 1$ and $f_2 = 7$, so this coset gives two classes of \overline{G} :

$$x = 1, \text{ class}(1), \quad |C_{\overline{G}}(x)| = 8 \times 168 = 1344$$

$$x = e_1, \text{ class}(2_1), \quad |C_{\overline{G}}(x)| = \frac{8 \times 168}{7} = 192$$

- $g \in (2A)$:

We take

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with $|C_G(g)| = 8$. The action of g on N is represented by the cycle structure

$$(1)(e_1)(e_1e_2e_3)(e_2e_3)(e_2e_3)(e_1e_2e_1e_3), \text{ so } k = 4.$$

The four orbits of N on Ng are $\{g, e_2e_3g\}$, $\{e_1g, e_1e_2e_3g\}$, $\{e_2g, e_3g\}$ and $\{e_1e_2g, e_1e_3g\}$.

Now we act

$$C_G(g) = \left\langle \left(\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right) \right\rangle$$

on these orbits.

For $eg \in Ng, h \in C_G(g), (eg)^h = e^h g^h = e^h g$ so we obtain the following orbits:

$$\begin{aligned} \{g, e_2e_3g\}^{C_G(g)} &= \{g, e_2e_3g\}, \{e_1g, e_1e_2e_3g\}^{C_G(g)} = \{e_1g, e_1e_2e_3g\}, \{e_2g, e_3g\}^{C_G(g)} \\ &= \{e_2g, e_3g, e_1e_2g, e_1e_3g\} \end{aligned}$$

Therefore we get three classes of \overline{G} :

$$f_1 = 1, x = g, \text{class}(2_2), \quad |C_{\overline{G}}(x)| = 4 \times 8 = 32;$$

$$f_2 = 1, x = e_1g, \text{class}(2_3), \quad |C_{\overline{G}}(x)| = 32;$$

$$f_3 = 2, x = e_2g, \text{class}(4_1), \quad |C_{\overline{G}}(x)| = \frac{4 \times 8}{2} = 16.$$

- $g \in (3A)$:

We take

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with $|C_G(g)| = 3$. The action of g on N is represented by $(1)(e_1e_2e_3)(e_1 e_2 e_3)(e_1e_2 e_1e_3 e_2e_3)$, so $k = 2$ which means we must have two blocks. These cannot fuse together under $C_G(g)$, since $g^{C_G(g)} = \{g\}$. Therefore we have two classes of \overline{G} , with $f_1 = 1$ and $f_2 = 1$:

$$x = g, \text{ class}(3_1), \quad |C_{\overline{G}}(x)| = 2 \times 3 = 6;$$

$$x = e_1g, \text{ class}(6_1), \quad |C_{\overline{G}}(x)| = 6.$$

- $g \in (4A)$:

We get two classes of \overline{G} once more:

$$x = g, \text{ class}(4_2), \quad |C_{\overline{G}}(x)| = 8;$$

$$x = e_1g, \text{ class}(4_3), \quad |C_{\overline{G}}(x)| = 8.$$

- $g \in (7A)$:

For the class $(7A)$, we have $k = 1$, so each coset has just one class in \overline{G} . We thus get the class (7_1) of \overline{G} , with centralizer of order 7.

- $g \in (7B)$:

This case works the same as for the previous class and we obtain class (7_2) of \overline{G} , with centralizer of order 7.

class of G	(1A)		(2A)			(3A)		(4A)		(7A)	(7B)
class of \bar{G}	(1)	(2 ₁)	(2 ₂)	(2 ₃)	(4 ₁)	(3 ₁)	(6 ₁)	(4 ₂)	(4 ₃)	(7 ₁)	(7 ₂)
centralizer	1344	192	32	32	16	6	6	8	8	7	7

Table 1.3.2: The conjugacy table of $2^3 : GL_3(2)$.

Chapter 2

REPRESENTATIONS AND CHARACTERS

Two ways of approaching representation and character theory are through the use of modules on the one hand (for instance, the approach used by James and Liebeck [10]), and through the classical approach used by Feit[5] for example, on the other hand. Our discussion is along the classical approach and for this purpose we follow the class notes of Moori[15].

We give some basic results on the representations and characters of finite groups in this chapter as well as some examples of how these results are used to determine the character tables of some finite groups. In the first section, theorems and lemmas will almost always be stated without proofs. Section 2.2 deals with the relationship between characters of groups and the characters of their subgroups, while in section 2.3 we shall look at the role of normal subgroups in the calculation of characters of a group. In the last two sections mentioned, only the proofs of the main results (that is those results dealing more directly with the techniques of finding the characters of a group) are given. These proofs are mainly taken from Moori's notes [15]. In the last three sections we calculate the character tables of three group extensions, which are all split extensions.

2.1 Basic Concepts

Definition 2.1.1 *Let G be a group. Let $f : G \rightarrow GL_n(F)$ be a homomorphism. Then we say that f is a matrix representation of G of degree n (or dimension n), over the field F .*

If $\text{Ker}(f) = \{1_G\}$, then we say that f is a *faithful* representation of G . In this situation $G \cong \text{Image}(f)$, so that G is isomorphic to a subgroup of $GL_n(F)$.

Definition 2.1.2 *Let $f : G \rightarrow GL_n(F)$ be a representation of G over the field F . The function $\chi : G \rightarrow F$ defined by $\chi(g) = \text{trace}(f(g))$ is called the character of f .*

Definition 2.1.3 *If $\phi : G \rightarrow F$ is a function from a group G to a field F which is constant on conjugacy classes of G , that is $\phi(g) = \phi(xgx^{-1}), \forall x \in G$, then ϕ is a class function.*

Lemma 2.1.4 *A character is a class function.*

Proof: See [15, Lemma i.4]

Definition 2.1.5 *Two representations $\rho, \phi : G \rightarrow GL_n(F)$ are said to be equivalent if there exists an $n \times n$ matrix P over F such that*

$$P^{-1}\rho(g)P = \phi(g), \quad \forall g \in G.$$

Theorem 2.1.6 *Equivalent representations have the same character.*

Proof: See [15, Theorem i.5]

Before defining the concepts of reducibility and irreducibility of representations and characters, we need to say what is meant by a reducible and an irreducible set of matrices. If S is a set of matrices, then S is *reducible* if $\exists m, k \in \mathbb{N}$, and $\exists P \in GL_n(F)$ such that $\forall A \in S$ we have

$$P^{-1}AP = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B is an $m \times m$ matrix, D is a $k \times k$ matrix, C is a $k \times m$ matrix and 0 is the zero matrix. If no such P exists, we say that S is *irreducible*. Furthermore if $C = 0 \forall A \in S$, we say that S is fully reducible and if $\exists P \in GL_n(F)$ such that

$$P^{-1}AP = \begin{pmatrix} B_1 & 0 & \dots & \dots & 0 \\ 0 & B_2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & B_k \end{pmatrix}, \quad \forall A \in S,$$

where each B_i is irreducible, we say S is completely reducible.

Definition 2.1.7 Let $f : G \rightarrow GL_n(F)$ be a representation of G over F and let $S = \{f(g) : g \in G\}$. We say that f is *reducible*, *fully reducible*, or *completely reducible* if S is *reducible*, *fully reducible*, or *completely reducible*, respectively.

Definition 2.1.8 If χ_ρ is a character afforded by a representation ρ of G , then we say that χ_ρ is an *irreducible character* of G if ρ is an irreducible representation.

Definition 2.1.9 Let $\rho : G \rightarrow GL_n(F)$ and $\phi : G \rightarrow GL_m(F)$ be two representations of G over F . Define $\rho + \phi : G \rightarrow GL_{n+m}(F)$ by

$$(\rho + \phi)(g) = \begin{pmatrix} \rho(g)_{n \times n} & 0_{n \times m} \\ 0_{m \times n} & \phi(g)_{m \times m} \end{pmatrix} = \rho(g) \oplus \phi(g), \quad \forall g \in G.$$

Then $\rho + \phi$ is a representation of G over F , of degree $n + m$.

If χ_1 and χ_2 are the characters of ρ and ϕ respectively and χ is the character of $\rho + \phi$, then for all $g \in G$ we have $\chi(g) = \chi_1(g) + \chi_2(g)$.

Theorem 2.1.10 (*Maschke's theorem*) Let G be a finite group. Let f be a representation of G over a field F whose characteristic is either equal to zero or is a prime that does not divide $|G|$. If f is reducible, then f is fully reducible.

Proof: See [15, Theorem i.6]

Theorem 2.1.11 (*The general form of Maschke's theorem*)

Let G be a finite group and F be a field whose characteristic is either equal to zero or is a prime that does not divide $|G|$. Then every representation of G over F is completely reducible.

Proof: See [5, (1.1)]

Theorem 2.1.12 (*Schur's lemma*) Let $\rho : G \rightarrow GL_n(F)$ and $\phi : G \rightarrow GL_m(F)$ be two representations of a group G over a field F . Assume there exists an $m \times n$ matrix P such that $P\rho(g) = \phi(g)P$ for all $g \in G$. Then either $P = 0_{m \times n}$ or P is non-singular so that $\rho(g) = P^{-1}\phi(g)P$ (that is, ρ and ϕ are equivalent representations).

Proof: See [5,(1.2)]

Definition 2.1.13 Let G be a finite group and assume that the characteristic of the field F does not divide $|G|$. If ρ and ϕ are two functions from G into F , we define an innerproduct \langle, \rangle by the following rule:

$$\langle \rho, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \rho(g) \phi(g^{-1}) ,$$

where $\frac{1}{|G|}$ stands for $|G|^{-1}$ in F .

Theorem 2.1.14 The innerproduct \langle, \rangle is bilinear:

$$(i) \langle \rho_1 + \rho_2, \phi \rangle = \langle \rho_1, \phi \rangle + \langle \rho_2, \phi \rangle$$

$$(ii) \langle \rho, \phi_1 + \phi_2 \rangle = \langle \rho, \phi_1 \rangle + \langle \rho, \phi_2 \rangle$$

$$(iii) \langle a\rho, \phi \rangle = a\langle \rho, \phi \rangle = \langle \rho, a\phi \rangle, \quad \forall a \in F$$

and symmetric:

$$\langle \rho, \phi \rangle = \langle \phi, \rho \rangle$$

Proof:

(i)

$$\begin{aligned}\langle \rho_1 + \rho_2, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} (\rho_1 + \rho_2)(g) \phi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} (\rho_1(g) + \rho_2(g)) \phi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} (\rho_1(g) \phi(g^{-1}) + \rho_2(g) \phi(g^{-1})), \text{ } \mathbb{F} \text{ being an additive abelian group} \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_1(g) \phi(g^{-1}) + \frac{1}{|G|} \sum_{g \in G} \rho_2(g) \phi(g^{-1}), \\ &= \langle \rho_1, \phi \rangle + \langle \rho_2, \phi \rangle\end{aligned}$$

(ii) Similar to (i).

(iii)

$$\begin{aligned}\langle a\rho, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} (a\rho)(g) \phi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} a(\rho(g)) \phi(g^{-1}) \\ &= a \frac{1}{|G|} \sum_{g \in G} \rho(g) \phi(g^{-1}) \\ &= a \langle \rho, \phi \rangle\end{aligned}$$

and

$$\begin{aligned}\langle a\rho, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} (a\rho)(g) \phi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} a\rho(g) \phi(g^{-1})\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|G|} \sum_{g \in G} \rho(g) a \phi(g^{-1}), \text{ F being a multiplicative abelian group} \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(g) (a \phi)(g^{-1}) \\
&= \langle \rho, a \phi \rangle
\end{aligned}$$

To complete the proof, see [15, Theorem i.11]. \square

Note 1 If $\rho : G \rightarrow GL_n(\mathbb{C})$ is a representation of a group G , then we denote the (i, j) entry of $\rho(g)$ by $\rho_{ij}(g)$. Hence $\rho_{ij}(g)$ is a map from G into \mathbb{C} .

For the rest of this chapter we shall mean finite groups when mentioning groups, unless explicit exceptions are made and all representations will be over the field \mathbb{C} of complex numbers.

Theorem 2.1.15 *Let G be a finite group and let ρ and ϕ be two irreducible representations of G .*

(i) *If ρ and ϕ are inequivalent, then*

$$\langle \rho_{rs}, \phi_{ij} \rangle = 0, \quad \forall i, j, r, \text{ and } s.$$

$$(ii) \langle \rho_{rs}, \phi_{ij} \rangle = \frac{\delta_{is} \cdot \delta_{jr}}{\deg(\rho)}.$$

Proof: See [15, Theorem ii.1]

Theorem 2.1.16 *Let G be a finite group and let ρ and ϕ be two irreducible representations of G , with characters χ_ρ and χ_ϕ .*

(i) If ρ and ϕ are equivalent, then

$$\langle \chi_\rho, \chi_\phi \rangle = 1$$

(ii) If ρ and ϕ are not equivalent, then

$$\langle \chi_\rho, \chi_\phi \rangle = 0$$

(iii) $\langle \chi_\rho, \chi_\rho \rangle = 1$

Proof: See [15, Theorem ii.2]

Theorem 2.1.17 *Two representations of a group G are equivalent if and only if they have the same characters.*

Proof: See [15, Corollary ii.4]

Lemma 2.1.18 (i) If

$$\chi = \sum_{i=1}^k \lambda_i \chi_i$$

where χ_i are distinct irreducible characters of a group G and λ_i are nonnegative integers, then

$$\langle \chi, \chi \rangle = \sum_{i=1}^k \lambda_i^2.$$

(ii) If χ is a character of G , then χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Proof:

(i)

$$\begin{aligned}\langle \chi, \chi \rangle &= \left\langle \sum_{i=1}^k \lambda_i \chi_i, \sum_{j=1}^k \lambda_j \chi_j \right\rangle \\ &= \sum_{i=1}^k \lambda_i \sum_{j=1}^k \lambda_j \langle \chi_i, \chi_j \rangle \\ &= \sum_{i=1}^k \lambda_i^2 \langle \chi_i, \chi_i \rangle \\ &= \sum_{i=1}^k \lambda_i^2\end{aligned}$$

(ii) By theorem 2.1.7, we have that if χ is irreducible, then $\langle \chi, \chi \rangle = 1$.

For the converse, assume that $\langle \chi, \chi \rangle = 1$. Let

$$\chi = \sum_{i=1}^k \lambda_i \chi_i$$

where χ_i are distinct irreducible characters of G and λ_i are nonnegative integers, then by (i), we have

$$\sum_{i=1}^k \lambda_i^2 = \langle \chi, \chi \rangle = 1$$

$$\begin{aligned}\Rightarrow \lambda_j^2 &= 1, \text{ for some } j = 1, 2, \dots, k \\ \text{and } \lambda_i^2 &= 0 \quad \forall i \neq j.\end{aligned}$$

Hence $\lambda_j = 1$. Thus $\chi = \chi_j$ is irreducible. \square

Note 2 If C_i is a conjugacy class of G , then

$$C_{i'} = \{ g \in G : g^{-1} \in C_i \}$$

is also a conjugacy class of G and $C_i = C_{i'}$ if and only if $g \sim g^{-1}$ for all $g \in C_i$.

Theorem 2.1.19 *Let $\text{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_k\}$. Then*

$$(i) \quad \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_j(g) = \delta_{ij}, \quad (\text{row orthogonality})$$

$$(ii) \quad \sum_{s=1}^k \chi_s(g_i) \chi_s(g_j) = \delta_{ij'} |C_G(g_j)|, \quad (\text{column orthogonality})$$

Proof: See [15, Theorem ii.17]

Theorem 2.1.20 *The number of irreducible characters of a group G equals the number of conjugacy classes of G .*

Proof: See [15, Theorem ii.18]

Proposition 2.1.21 *Let $G = \langle x \rangle$ be a cyclic group of order n . Let $e^{\frac{2k\pi}{n}i}$ be the n -th roots of unity in \mathbb{C} , $k = 0, 1, 2, \dots, n-1$. Define $\rho_k : G \rightarrow \mathbb{C}^*$ by*

$$\rho_k(x^m) = [e^{\frac{2k\pi}{n}i}]^m.$$

For $k = 0, 1, 2, \dots, n-1$, ρ_k defines the n distinct irreducible representations of G .

Proof: We first show that ρ_k is well defined:

Let $x^m = x^{m'}$, where $m = sn + t$, $m' = s'n + t'$, $s, s' \in \mathbb{Z}$ and $t, t' = 0, 1, 2, \dots, n-1$.
 From which we get $x^t = x^{t'} \Rightarrow t = t'$.

If for contradiction, $[e^{\frac{2k\pi}{n}i}]^m \neq [e^{\frac{2k\pi}{n}i}]^{m'}$, then we have

$$\begin{aligned}
 [e^{\frac{2k\pi}{n}i}]^{m-m'} \neq 1 &\Rightarrow [e^{\frac{2k\pi}{n}i}]^{(s-s')n + (t-t')} \neq 1 \\
 &\Rightarrow [e^{\frac{2k\pi}{n}i}]^{(s-s')n} \neq 1 \\
 &\Rightarrow \rho_k(x^{(s-s')n}) \neq 1 \\
 &\Rightarrow \rho_k(x^0) \neq 1 \\
 &\Rightarrow [e^{\frac{2k\pi}{n}i}]^0 \neq 1,
 \end{aligned}$$

giving us the contradiction. Hence ρ_k is well defined.

Next we show that ρ_k is a homomorphism:

$$\begin{aligned}
 \rho_k(x^m)\rho_k(x^{m'}) &= \rho_k(x^t)\rho_k(x^{t'}) \\
 &= [e^{\frac{2k\pi}{n}i}]^t [e^{\frac{2k\pi}{n}i}]^{t'} \\
 &= [e^{\frac{2k\pi}{n}i}]^{t+t'} \\
 &= \rho_k(x^{t+t'}) \\
 &= \rho_k(x^t \cdot x^{t'}) \\
 &= \rho_k(x^m \cdot x^{m'})
 \end{aligned}$$

So ρ_k is a homomorphism and hence a representation.

ρ_k is unique:

Let $\rho_k = \rho_{k'}$ with $k, k' \leq n$. Now $\forall g \in \langle x \rangle$, $g = x^r$ where $r = 0, 1, 2, \dots, n-1$. So we have

$$\begin{aligned}
\rho_k(x^r) = \rho_{k'}(x^r) &\Rightarrow [e^{\frac{2k\pi}{n}i}]^r = [e^{\frac{2k'\pi}{n}i}]^r \\
&\Rightarrow e^{(\frac{2k\pi}{n}r - \frac{2k'\pi}{n}r)i} = 1 \\
&\Rightarrow e^{\frac{2\pi r}{n}(k-k')i} = 1 \\
&\Rightarrow \rho_{(k-k')}(x^r) = 1, \quad \forall r = 0, 1, 2, \dots, n-1. \\
&\Rightarrow k - k' = 0, \text{ so that } k = k'.
\end{aligned}$$

Lastly we must show that ρ_k is irreducible:

We use lemma 2.1.2.

$$\begin{aligned}
\langle \rho_k, \rho_k \rangle &= \frac{1}{|\langle x \rangle|} \sum_{g \in \langle x \rangle} \rho_k(g) \rho_k(g^{-1}) \\
&= \frac{1}{n} \sum_{g \in \langle x \rangle} \rho_k(gg^{-1}) \\
&= \frac{1}{n} \sum_{g \in \langle x \rangle} \rho_k(1_{\langle x \rangle}) \\
&= \frac{1}{n} \sum_{g \in \langle x \rangle} 1_{\mathbb{C}} \\
&= \frac{1}{n} n \\
&= 1.
\end{aligned}$$

Hence ρ_k is irreducible.

This completes the proof of the proposition. \square

Definition 2.1.22 Let $P = (p_{ij})_{m \times m}$ and $Q = (q_{ij})_{n \times n}$ be two matrices. Then the $mn \times mn$ matrix $P \otimes Q$ is defined by

$$P \otimes Q := (p_{ij}Q) = \begin{pmatrix} p_{11}Q & p_{12}Q & \dots & \dots & \dots & p_{1m}Q \\ p_{21}Q & p_{22}Q & \dots & \dots & \dots & p_{2m}Q \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{m1}Q & p_{m2}Q & \dots & \dots & \dots & p_{mm}Q \end{pmatrix}$$

From this definition, we can show that

$$(P \otimes Q)(P' \otimes Q') = (PP') \otimes (QQ') \quad (*):$$

$$\begin{aligned} (P \otimes Q)(P' \otimes Q') &= \left(\sum_{k=1}^m p_{ik}Q p'_{ki}Q' \right)_{mn \times mn} \\ &= \left(\sum_{k=1}^m p_{ik}p'_{ki}QQ' \right)_{mn \times mn} \\ &= (PP') \otimes (QQ'). \end{aligned}$$

Definition 2.1.23 Let T and U be representations of a group G , then the tensor product $T \otimes U$ is defined by:

$$(T \otimes U)(g) := T(g) \otimes U(g)$$

Theorem 2.1.24 Let T and U be representations of a group G , then

(i) $T \otimes U$ is a representation of G .

(ii) if $\chi_{(T \otimes U)}$ is the character afforded by $T \otimes U$ then

$$\chi_{(T \otimes U)} = \chi_T \chi_U$$

Proof: See [15, Theorem iii.1]

Definition 2.1.25 Let $G = H \times K$ be the direct product of two groups H and K and let $T : H \rightarrow GL_m(\mathbb{C})$ and $U : K \rightarrow GL_n(\mathbb{C})$ be representations of H and K respectively. Since for every element g in G , $g = hk$ uniquely, for some $h \in H$ and some $k \in K$, the direct product $T \times U$ can be defined by

$$(T \times U)(g) : = T(h) \otimes U(k)$$

From the uniqueness of $g = hk$ and because of the property of representations T and U of being well defined, it can be shown that $T \times U$ is well defined. Also for $g = hk$ and $g' = h'k'$ with $h, h' \in H$ and $k, k' \in K$, we have

$$\begin{aligned} (T \times U)(g)(T \times U)(g') &= (T(h) \otimes U(k))(T(h') \otimes U(k')) \\ &= T(h)T(h') \otimes U(k)U(k'), \text{ by } (*) \\ &= T(hh') \otimes U(kk') \\ &= (T \times U)(gg'), \end{aligned}$$

which means $T \times U$ is a homomorphism and therefore a representation.

From definition 2.1.22, we can deduce that for two matrices P and Q , that

$$\text{Trace}(P \otimes Q) = \text{Trace}(P).\text{Trace}(Q).$$

So we show the following

$$\begin{aligned}
\chi_{(T \times U)}(g) &= \text{Trace}((T \times U)(g)) \\
&= \text{Trace}(T(h) \otimes U(k)) \\
&= \text{Trace}(T(h)).\text{Trace}(U(k))
\end{aligned}$$

and the next theorem tells us that all the characters of a direct product are constructed in this way.

Theorem 2.1.26 *Let $G = H \times K$ be the direct product of two groups H and K . Then the direct product of any irreducible character of H and any irreducible character of K is an irreducible character of G . Moreover, every irreducible character of G can be constructed in this way.*

Proof: See [15, Theorem iii.2]

Definition 2.1.27 *Let χ be a character of a group G . For $n \in (\mathbb{N} \cup \{0\})$, we define χ^n by*

$$\chi^n(g) := (\chi(g))^n, \quad \forall g \in G.$$

If G is a group and H is a subgroup of G , then we can use the irreducible characters of G to find at least some of the characters of H and vice versa. We deal with the methods of doing this in the following section and use the notes of Moori[15] again.

2.2 Restriction and Induction of Characters

Definition 2.2.1 *Let G be a group and H be a subgroup of G . If $\rho : G \rightarrow GL_n(\mathbb{C})$ is a representation of G , then $(\rho \downarrow H) : H \rightarrow GL_n(\mathbb{C})$ given by*

$$(\rho \downarrow H)(h) = \rho(h), \quad \forall h \in H,$$

is a representation of H . We say that $\rho \downarrow H$ is the restriction of ρ to H . If χ_ρ is the character of ρ , then $\chi_\rho \downarrow H$ is the character of $\rho \downarrow H$. We refer to $\chi_\rho \downarrow H$ as the restriction of χ_ρ to H .

Theorem 2.2.2 Let G be a group and $H \leq G$. If ψ is a character of H , then there is an irreducible character χ of G such

$$\langle \chi \downarrow H, \psi \rangle_H \neq 0.$$

Proof: See [15, Theorem iv.1.1].

Theorem 2.2.3 Let G be a group and $H \leq G$. If

$$\chi \in \text{Irr}(G) \quad \text{and} \quad \text{Irr}(H) = \{\psi_1, \psi_2, \dots, \psi_r\},$$

then

$$\begin{aligned} \chi \downarrow H &= \sum_{i=1}^r \delta_i \psi_i, \text{ where } \delta_i \in (\mathbb{N} \cup \{0\}) \text{ and} \\ \sum_{i=1}^r \delta_i^2 &\leq [G : H] \quad (**) \end{aligned}$$

Moreover, we have equality in (**) if and only if $\chi(g) = 0, \quad \forall g \in (G \setminus H)$.

Proof: Since $\chi \downarrow H$ is a character of H , $\exists \delta_i \in (\mathbb{N} \cup \{0\})$ such that

$$\chi \downarrow H = \sum_{i=1}^r \delta_i \psi_i.$$

Now

$$\begin{aligned}
\langle \chi \downarrow H, \chi \downarrow H \rangle_H &= \left\langle \sum_{i=1}^r \delta_i \psi_i, \sum_{i=1}^r \delta_i \psi_i \right\rangle_H \\
&= \sum_{i=1}^r \delta_i^2 \langle \psi_i, \psi_i \rangle_H \\
&= \sum_{i=1}^r \delta_i^2
\end{aligned}$$

and

$$\langle \chi \downarrow H, \chi \downarrow H \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h) \cdot \overline{\chi(h)}.$$

Hence we get

$$\begin{aligned}
\sum_{i=1}^r \delta_i^2 &= \frac{1}{|H|} \sum_{h \in H} \chi(h) \cdot \overline{\chi(h)} \quad \text{so that} \\
|H| \sum_{i=1}^r \delta_i^2 &= \sum_{h \in H} \chi(h) \cdot \overline{\chi(h)} \quad (***)
\end{aligned}$$

From

$$\begin{aligned}
1 &= \langle \chi, \chi \rangle_G \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot \overline{\chi(g)} \\
&= \frac{1}{|G|} \sum_{g \in H} \chi(g) \cdot \overline{\chi(g)} + \frac{1}{|G|} \sum_{g \in (G \setminus H)} \chi(g) \cdot \overline{\chi(g)} \\
&= \frac{|H|}{|G|} \sum_{i=1}^r \delta_i^2 + \frac{1}{|G|} \sum_{g \in (G \setminus H)} \chi(g) \cdot \overline{\chi(g)} \quad \text{by (***)} \\
&= \frac{|H|}{|G|} \sum_{i=1}^r \delta_i^2 + \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2
\end{aligned}$$

we obtain that

$$\frac{|H|}{|G|} \sum_{i=1}^r \delta_i^2 = 1 - \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2 \leq 1$$

and therefore

$$\sum_{i=1}^r \delta_i^2 \leq \frac{|G|}{|H|} = [G : H]$$

Also

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2 &= 0 \quad \text{if and only if} \\ |\chi(g)|^2 &= 0 \quad \forall g \in (G \setminus H). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2 &= 0 \quad \text{if and only if} \\ \chi(g) &= 0 \quad \forall g \in (G \setminus H) \end{aligned}$$

and we have the equality in (**). \square

We have seen how the irreducible characters G can be used to find characters of a subgroup H and can now look at a technique of finding the characters of G from the irreducible characters of any subgroup. We start with the following definition.

Definition 2.2.4 *Let H be a subgroup of G . The right transversal of H in G is a set of representatives for the right cosets of H in G .*

The following theorem tells us how a representation of H can be extended to a representation of G .

Theorem 2.2.5 *Let H be a subgroup of G and T be a representation of H of degree n . Extend T to G by $T^0(g) = T(g)$ if $g \in H$ and $T^0(g) = 0_{n \times n}$ if $g \notin H$. Let $\{x_1, x_2, \dots, x_r\}$ be a right transversal of H in G . Define $T \uparrow G$ by*

$$(T \uparrow G)(g) := \begin{pmatrix} T^0(x_1 g x_1^{-1}) & T^0(x_1 g x_2^{-1}) & \dots & \dots & \dots & T^0(x_1 g x_r^{-1}) \\ T^0(x_2 g x_1^{-1}) & T^0(x_2 g x_2^{-1}) & \dots & \dots & \dots & T^0(x_2 g x_r^{-1}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ T^0(x_n g x_1^{-1}) & T^0(x_n g x_2^{-1}) & \dots & \dots & \dots & T^0(x_n g x_r^{-1}) \end{pmatrix}$$

$$= \left(T^0(x_i g x_j^{-1}) \right)_{i,j=1,2,\dots,r}, \quad \forall g \in G.$$

Then $T \uparrow G$ is a representation of G of degree nr .

Proof: See [15, theorem iv.2.1].

Definition 2.2.6 The representation $T \uparrow G$ defined in the previous theorem said to be induced from the representation T of H . Let ϕ be the character afforded by T . Then the character afforded by $T \uparrow G$ is called the induced character from ϕ and is denoted by ϕ^G . If we extend ϕ to G by $\phi^0(g) = \phi(g)$ if $g \in H$ and $\phi^0(g) = 0$ if $g \notin H$, then

$$\begin{aligned} \phi^G(g) &= \text{Trace}\left((T \uparrow G)(g)\right) \\ &= \sum_{i=1}^r \text{Trace}\left((T^0(x_i g x_i^{-1}))\right) \\ &= \sum_{i=1}^r \phi^0(x_i g x_i^{-1}) \end{aligned}$$

In order to construct a formula to find the induced character, the next two propositions are needed.

Proposition 2.2.7 *If $H \leq G$ and ϕ is a character of H , then ϕ^G is independent of the choice of transversal.*

Proof: See [15, Proposition iv. 2.2].

Proposition 2.2.8 *The values of the induced character are given by*

$$\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \phi^0(xgx^{-1}), \quad g \in G$$

Proof: See [15, Proposition iv.2.3].

The following proposition provides us with a formula to calculate the induced character and the proof is provided by Moori [15, Proposition iv.2.4].

Proposition 2.2.9 *Let $H \leq G$, ϕ be a character of H and $g \in G$. Let $[g]$ denote the conjugacy class containing g .*

(i) *If $H \cap [g] = \emptyset$, then $\phi^G(g) = 0$,*

(ii) *If $H \cap [g] \neq \emptyset$, then*

$$\phi^G(g) = |C_G(g)| \sum_{i=1}^m \frac{\phi(x_i)}{|C_H(x_i)|}$$

where x_1, x_2, \dots, x_m are representatives of classes of H that fuse to $[g]$. (That is $H \cap [g]$ breaks up into m conjugacy classes of H with representations x_1, x_2, \dots, x_m .)

Proof: By Proposition 2.2.8, we have

$$\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \phi^0(xgx^{-1}).$$

If $H \cap [g] = \emptyset$, then $xgx^{-1} \notin H$ for all $x \in G$, so $\phi^0(xgx^{-1}) = 0 \quad \forall x \in G$ and $\phi^G(g) = 0$.

If $H \cap [g] \neq \emptyset$, then as x runs over G , xgx^{-1} covers $[g]$ exactly $|C_G(g)|$ times, so

$$\begin{aligned} \phi^G(g) &= \frac{1}{|H|} \times |C_G(g)| \sum_{y \in [g]} \phi^0(y) \\ &= \frac{1}{|H|} \times |C_G(g)| \sum_{y \in ([g] \cap H)} \phi(y) \\ &= \frac{|C_G(g)|}{|H|} \times \sum_{i=1}^m [H : C_H(x_i)] \cdot \phi(x_i) \\ &= |C_G(g)| \sum_{i=1}^m \frac{\phi(x_i)}{|C_H(x_i)|} \quad \square \end{aligned}$$

The restriction and induction of characters are related and can be expressed by means of a matrix which we call the Frobenius Reciprocity table. To obtain this relationship, we shall take the route through class functions. We shall use the proof given by Moori [15] for the main result(the Frobenius Reciprocity theorem) in establishing the relationship.

Definition 2.2.10 *Let H be a subgroup of G and ϕ be a class function on H then the induced class function ϕ^G on G is defined by*

$$\phi^G(g) = \frac{1}{|H|} \sum_{x \in G} \phi^0(xgx^{-1}), \quad g \in G$$