Modelling the Basel III capital adequacy and net stable funding ratios for a commercial bank following an optimal investment strategy

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Contents

Declaration

I declare that

Modelling the Basel III capital adequacy and net stable funding ratios for a commercial bank following an optimal investment strategy

is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Richard Stephen Hercules 28 February, 2024

Signed: ....

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Abstract

Commercial banks have a significant impact on a country's economy as they raise capital, create liquidity by converting their customers' deposits into loans, and deliver essential services such as loans, certificates of deposits and savings accounts to their clients. As a result, commercial banks are heavily regulated in most countries. The Basel Committee on Banking Supervision (BCBS) introduced an international set of capital standards, known as the Basel Accords, in an attempt to improve the regulation of internationally active banks. These accords resulted from a series of international banking regulatory meetings that established capital and risk management measurements for internationally active banks. Under the accords, banks are required to maintain a minimum level of capital as a buffer to protect their depositors, and the financial market, in the event of severe losses caused by financial risk. The latest of these accords, that is, the Basel III Accord, consists of three key pillars. These are firstly, minimum capital requirements, secondly, supervisory review, and lastly, market discipline. In this regard, the BCBS introduced, respectively, the Capital Adequacy Ratio (CAR) and the Net Stable Funding Ratio (NSFR). The purpose of the CAR is to determine whether or not an absolute amount of a bank's capital is adequate when compared to its absolute risk. The purpose of the NSFR, on the other hand, is to determine whether the bank has enough stable funding to cover its long term assets. Furthermore, government regulators aim to maintain the confidence and trust of the general public through the use of a deposit insurance scheme (DIS). In the event of a bank failure, deposit insurance (DI) has the effect of reducing the probability of mass deposit withdrawals. An insuring agent is tasked with estimating a fairly priced premium for DI coverage.

Bank capital is the difference between the total assets and total liabilities of a bank. In this thesis we model a commercial bank that invests its capital in a constant interest rate financial market where its asset portfolio is a combination of riskless and risky assets, while its liabilities consist of borrowings and deposits. For the aforementioned bank, we study a range of related problems that can be summarized as follows. The first problem involves modelling the CAR and NSFR of the commercial bank described above. In particular, we model the aforementioned ratios by applying well-known techniques from stochastic calculus. In the second problem we use the method of stochastic optimal control to derive an optimal investment strategy in the bank's assets so as to maximize an expected utility of the bank's capital at a future date $T > 0$. Lastly, we study a DI pricing problem involving the underlying commercial bank. It entails using a Monte Carlo simulation method to estimate the premium the bank should be charged for DI coverage for a period of T years. This approach enables us to estimate the price for DI coverage for the bank while it follows the optimal investment strategy on the interval $[0, T]$. We consider varying levels of volatility for the asset portfolio and observe how increasing the volatility in the asset portfolio affects the DI premium. We present various numerical simulations throughout the thesis. These include illustrating graphically how the optimal investment strategy, the CAR and the NSFR evolve over time.

Keywords: Basel Regulatory Framework; Capital Adequacy Ratio; Deposit Insurance; Hamilton-Jacobi-Bellman Equation; Monte Carlo simulation; Net Stable Funding Ratio; Optimal investment strategy; Stochastic differential equation; Stochastic optimal control theory; Partial differential equation

List of Acronyms

Available Stable Factor (ASF)

Basel Committee on Banking Supervision (BCBS)

Capital Adequacy Ratio (CAR)

Capital Conservation Buffer (CCB)

Countercyclical Buffer Capital (CBC)

Deposit Insurance (DI)

Deposit Insurance Scheme (DIS)

Federal Deposit Insurance Corporation (FDIC)

Inverse Net Stable Funding Ratio (INSFR)

Net Stable Funding Ratio (NSFR)

Required Stable Factor (RSF)

Shareholder Cash Flow Rights (SCFR)

Stock of High Quality Liquid Assets (SHQLAs)

Total-Risk Weighted Assets (TRWAs)

List of Notations

A_1 - The price of the treasury security/ treasury

A_2 - The price of the marketable security

A_3 - The value of the loan

A - The value of the asset portfolio

B - The value of the borrowings

D - The value of the cash deposits

L - The value of the total liabilities

C - The value of the total capital

O - The value of the off-balance sheet activities

Y - The value of the total risk-weighted assets

X_C - The value of the Total Capital Ratio/ Capital Adequacy Ratio

F_A - The value of the Available Amount of Stable Funding

F_R - The value of the Required Amount of Stable Funding

X_N - The value of the Net Stable Funding Ratio

ϕ_1 - The optimal amount of capital invested in the treasury security

ϕ_2 - The optimal amount of capital invested in the marketable security

ϕ_3 - The optimal amount of capital invested in the loan

f_1 - The optimal proportion of capital invested in the treasury security

f_2 - The optimal proportion of capital invested in the marketable security

f_3 - The optimal proportion of capital invested in the loan

$\hat{D}$ - The value of the total insured deposits

$\hat{\omega}$ -The value of the fairly priced deposit insurance premium

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Chapter 1

Introduction and scope

Commercial banks are financial institutions that accept deposits, grant loans, and offer basic financial products such as, for example, certificates of deposits and savings accounts to businesses and individuals. They primarily make a profit by offering different types of loans to customers on which they charge interest. Examples of bank customers can be the general public, businesses and companies. Commercial banks ensure economic stability and the sustainable growth of a country's economy [15]. These banks generally grant loans to creditors who are more likely to pay high interest rates and are less likely to default on their loans. In addition, commercial banks try to buy securities with low risk and high returns. They manage their assets by attempting to lower risk by diversifying their asset portfolios [55]. Bank capital is a fundamental building block of the banking industry and is essential to the growth and survival of a bank [20].

The Basel Committee on Banking Supervision (BCBS) was established in 1974 by central bankers from the G10 countries who were working towards building new international structures to replace the collapsed Bretton Woods system [40]. The latter system dissolved between 1968 and 1973. In 1971, the president of the Untied States (U.S.) announced the suspension of the dollar's convertibility to gold. On the other hand, throughout the 1960s, the dollar had struggled within the parity established at Bretton Woods. This caused a crisis that marked the breakdown of the Bretton Woods system [35]. The BCBS currently administers banking regulations and supervises the international banking system in order to improve the stability of the said system. In this regard, the Basel Accords were introduced by the BCBS in order to provide recommendations on international banking regulations pertaining to capital risk, market risk and operational risk. The goal of these accords is to ensure that banks hold enough capital to meet obligations and to absorb unexpected losses [13].

The BCBS issued the Basel Capital Accord, also known as the Basel I Accord, in 1988. The BCBS, under the Basel I regime, aimed to assess banks' capital in relation to their credit risk, or the risk of a loss that occurs should a party not be able to satisfy its obligations. As a result, the Basel I Accord led towards an increase in research on risk modelling by creating a bank asset classification system whereby assets are placed in groups or categories. Under Basel I, banks were required to maintain their total capital, that is, the sum of their Tier 1 and Tier 2 capital, equal to at least 8% of their *total risk-weights assets* (TRWAs) [13]. Here Tier 1 capital consists of equity and reserves, while Tier 2 capital consists of general loan loss, hybrid capital instruments and subordinated term debt, and undisclosed reserves [56]. TRWAs are generally utilized to determine the minimum amount of capital that a bank has to have on hand in relation to the risk profile of its lending activities and other assets [62]. However, since the Basel I Accord was based on simplified calculations and classifications, the BCBS in 2004 introduced the Basel II Capital Accord and further agreements as a sign of the continuous refinement of risk and capital [66].

Capital adequacy management involves the decision of how much capital a bank should hold, and how the bank should access it [20, 55]. The BCBS, with its Basel II Accord, laid down regulations which led them to seek incentives to provide more awareness of differences in risk through more risk-sensitive minimum capital requirements based on numerical formulas [5]. The *Total Capital Ratio* or *Capital Adequacy Ratio* (CAR) is a measure of the amount of banks' capital relative to the amount of their credit exposures. An international standard was created that requires banks to maintain their CAR at a minimum prescribed level. If banks adhere to these minimum requirements, then according to the BCBS, they are guaranteed the ability to adsorb reasonable levels of losses before becoming insolvent. Hence, the CAR ensures that the banking system is stable and safe [50]. The CAR is defined as:

$$\text{Capital Adequacy Ratio} = \frac{\text{Indicator of Absolute Amount of Bank Capital}}{\text{Indicator of Absolute Level of Bank Risk}}$$

Under the Basel II regime, banks were required to maintain a minimum CAR of at least 8%. The CAR can be used by a bank, or the regulator, to determine whether or not an absolute amount of a bank's capital is adequate when compared to a measure of its absolute risk [25]. The CAR is thus a comparison between a bank's total capital and its TRWAs. TRWAs are constituted by the capital charged for credit risk, market risk and operational risk. While credit risk is the risk of loans not being repaid, market risk is the risk of losses on balance sheet and off-balance sheet activities positions resulting from fluctuations in market prices. Operational risk, on the other hand, is the risk of losses arising from inadequate or failed internal processes, people and systems or from external events [25]. Under Basel II, Tier 3 capital was defined as tertiary capital. Banks held this form of capital to support their market risk, commodities risk and foreign currency risk derived from trading activities [13].

The BCBS introduced a comprehensive set of reform measures in response to the subprime mortgage crisis of 2007-2008. The aforementioned crisis involved an extreme decrease of liquidity in the international financial markets that started in the U.S. as a result of the collapse of the U.S. housing market, and that threatened to break down the international financial system [22]. In an attempt to remedy this situation the BCBS in 2010 released the Basel III Accord. Internationally it dictates the most recent regulatory standard on bank capital adequacy, stress testing and market liquidity risk. The Basel III Accord builds on the Basel I and Basel II Accords and aims to improve the banking sector's ability to deal with economic and financial stress, improve risk management and strengthen banks' transparency. Basel III is more rigorous than both the Basel I and II Accords and has two purposes. Firstly, it aims to strengthen international regulation of capital and liquidity with the objective of promoting a more resilient banking sector. Secondly, it aims to improve the banking sector's ability to absorb shocks resulting from economic and financial stress [6, 7, 8, 36, 60]. Basel III includes many different measures that aim to improve the quality and quantity of capital. Its objective is to ultimately improve the loss of absorption capacity in regards to both going concerns and liquidation situations. The Basel III Accord suggests that the minimum prescribed value of the banks' CAR remains unchanged at 8%. However, new concepts of *Capital Conservation Buffer* (CCB) and *Countercyclial Buffer Capital* (CBC) are introduced under Basel III [6]. The CCB guarantees that banks are able to absorb losses without breaching the minimum capital requirement, and ensures that they carry on business even in a downturn without delveraging. The CCB, on the other hand, does not form part of the regulatory prescribed minimum, but is stipulated at 2.5% of the TRWAs. The CBC is a pre-emptive measure that requires bank capital to gradually build up as imbalances in the credit market develop. The CBC is required to be between 0% and 2.5 % of the TRWAs, which could be imposed on banks during periods of excess credit growth [36].

Basel III further aims to strengthen the counter-party credit risk framework in market risk instruments, which includes the use of stressed input parameters to determine the capital requirement for counter-party credit default risk. New capital requirements are also introduced under Basel III. These are known as the credit valuation adjustment risk charges for the over-the-counter derivatives. The purpose of this is to protect banks against the risk of a decrease in the credit quality of the counter-party [6, 36, 60]. Tier 3 capital is being completely abolished under Basel III, as this form of capital is a short-term subordinated debt and was used to support market risk from trading activities under Basel II [38].

Liquidity management involves the decisions banks have to make about how to maintain sufficiently liquid assets to meet their obligations to depositors [48]. Basel III also puts forth a liquidity framework with the purpose of further strengthening the two minimum standards for quantifying funding liquidity. In this regard, Basel III introduces the *Liquidity Coverage Ratio* (LCR) and the *Net Stable Funding Ratio* (NSFR). The purpose of the LCR is to ensure that a bank has an adequate *Stock of High Quality Liquid Assets* (SHQLAs) [8]. SHQLAs consists of assets that can be converted into cash at a slight or no loss of value in private markets to meet its liquidity requirements in a 30 calendar day liquidity stress scenario. The 30 calendar day stress period is the minimum period considered for corrective action to be taken by the banks' management or supervisors [7, 8]. The NSFR needs a minimum amount of stable source funding relative to the liquidity profiles of assets, with the potential for contingent liquidity needs that occurs from off-balance sheet commitments over a one year horizon [8]. Consequently, the assumptions of this would be in relation to the type of current short term markets that are available for banks to provide liquidity, the type of long term markets required and the impact on the profitability of banks. Thus, the NSFR tends to reduce the exposure of funding liquidity risk. The development of the NSFR was reviewed by the BCBS after they had come to an agreement to do so after an observation period. The focus of the review was based on addressing any unintended consequences for the financial market functioning and the economy, and on improving its design with regards to a few important issues. Those key issues being firstly, the impact on retail business activities, secondly, the treatment of short-term matched funding of both assets and liabilities and, thirdly, analysis of sub-one year buckets for both assets and liabilities [8]. In business and finance, the term "bucket" is used to describe the grouping of related assets into several different categories [41]. The NSFR is defined as

$$\text{Net Stable Funding Ratio} = \frac{\text{Available Amount of Stable Funding}}{\text{Required Amount of Stable Funding}}.$$

This ratio is required to be equal to at least 100% under Basel III. The *Required Amount of Stable Funding* (RASF) relies mostly on the characteristics of the instruments' liquidity, which in turns determines the *Available Stable Factor* (ASF) and the *Required Stable Factor* (RSF). The ASF factors define the amount of assets that would be expected to remain with the bank over a long period in a specific stressed event, and is based on a number of characteristics of the relative stability of a financial institution's funding sources [8]. The RSF factors approximate the amount of a particular asset that would have to be funded, either because the asset will be rolled over, or during a liquidity event up to a year that can not be monetized [8]. The NSFR became a minimum standard applicable to all internationally active banks that are active on a consolidated basis on 1 January 2018 [7]. However, national supervisors may also apply it to any subset of entities of large internationally active banks or to all other banks. The NSFR requirement must be met by banks on an ongoing basis and reported on a quarterly basis.

Because of the impact of the NSFR on maturity transformation, and since its implementation may have unintended consequences, the NSFR was subjected to an observation period which began in 2011. While developing the NSFR, one of the aims of the BCBS was to support financial stability by ensuring that funding shocks do not substantially increase the probability of distress for individual banks, a potential source of systemic risk [4].

The stochastic optimal control method is commonly used to solve optimization problems in finance. The aforementioned method involves solving the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE) which results from the principle of dynamic programming under the real-world probability measure [65, 50]. It was first applied in the seminal paper of Merton [45] in which an investor wishes to allocate capital between a risk-free bond and a risky stock in order to maximize the expected utility of his/her terminal wealth. The author explicitly solved a HJB PDE with a fixed volatility of the risky stock. Another optimization method, known as the Martingale method, was developed by Cox and Huang [16] in a setting of complete markets. This method depends on the theory of Lagrange multipliers and involves a risk-neutral measure and the solving of a PDE. Witbooi et al. [65] applied the Martingale method to study an asset portfolio optimization problem in banking. Examples of authors who applied stochastic optimal control theory to solve optimization problems in banking can be Mukuddem-Petersen and Petersen [54], Muller and Witbooi [50], Chakroun and Abid [12], van Schalkwyk and Witbooi [63] and Muller [52]. We will be following the stochastic optimal control approach in this thesis in order to determine the optimal allocation of capital among a commercial bank's assets.

Deposit insurance (DI) is a measure implemented in many countries to protect bank depositors from losses arising from a bank's inability to pay debts when due. DI systems are a component of a financial system's safety net that promotes financial stability [1]. In banking, safety nets are meant to provide assistance and promote prudent risk taking. Safety nets also avoid disintermediation from failing banks and the banking system. They are used to maintain confidence in and the soundness of the financial sector. A deposit insurance scheme (DIS) is a mechanism that reimburse depositors when a bank defaults and, in addition, serve main functions in bank resolution proceedings [33].

According to Abubakar et al. [1], implicit DIS (IDIS) is a system of DI that is not clearly provided for by law or regulation. This is a system of a government guarantee to avoid complete failure of other banks when a bank experiences a bank failure due to insolvency or a bank run. DI is implicit when its implication builds public confidence to stop a bank run on banks that become economically insolvent. Hence, there is no formal communication by the government to the public or bankers on the DI coverage or even the amount of coverage. Thus, the government is not legally bound to deliver deposit guarantees to depositors. Explicit DIS (EDIS) is well-defined by the government laws and other regulations which specify the existence of a DIS and the amount covered. In this scheme, the government openly outlined its assurance by explicitly stating the amount guaranteed. Both IDIS and EDIS can co-exist mainly in a serious financial crisis to reduce the social costs involved.

With his paper [47], Merton suggested an analogy between DI and a put option to value DI contacts. More specifically, he derived a formula to evaluate the cost of DI coverage by suggesting that the option's strike price is equivalent to the value of the insured deposits, while the stock price in the option is equivalent to the value of the bank's assets. The expiration date of the DI contract is equivalent to the duration of time until the next bank audit occurs. If the value of the bank's assets is below the value of the insured deposits at the time of the audit, then the bank has the right to sell the assets at the value of the insured deposits, or else the option is not exercised. Following the publication of the paper [47], other authors also modelled DI as a one-period European put option. Allen and Saunders [2], however, departed from the tradition of modelling DI as a one-period European put option and other researchers soon followed suit. In the paper [2] DI is modelled as a callable American put option in the sense that it is a perpetual put option that can be terminated prematurely. The latter model considered both the regulatory and closure policy, as well as self-closure policy. The model of [2] was extended by Hwang et al. [31] who introduced bankruptcy costs as an additional factor. Duan and Yu [21] proposed a multiperiod framework for modelling and pricing DI. The defaulting banks in [21] are assumed to have their assets reset to the level of outstanding deposits plus accrued interests in the event that insolvency resolution occurs. Muller [52] developed a DI pricing model based on the framework of Duan and Yu [21]. More specifically, the model of [52] incorporates the explicit solution of an optimal capital control problem in conjunction with an asset value reset rule comparable to the usual practice of insolvency resolution by insuring agencies.

In this thesis we model a commercial bank that is assumed to invest its capital in a constant interest rate financial market consisting of a treasury security, a marketable security and a loan. Its liabilities consists of borrowings and deposits. Generally speaking, bank capital is the difference between the values of a bank's asset portfolio and total liabilities. We study three different commercial banking problems involving the bank described above, of which we now give a breakdown. The first problem is in a continuous time setting and involves modelling the CAR and NSFR of the bank. We derive stochastic differential equations (SDEs) for the aforementioned ratios by using Itô's Lemma and Itô's Product rule. Since the CAR is computed from the total capital and TRWAs, we also derive SDEs for those quantities. The NSFR, on the other hand, is computed from the Available Amount of Stable Funding (AASF) and the RASF, hence we will also need to derive SDEs for the latter quantities. In the second problem, which is also in continuous time, we study an optimal control problem involving the bank's capital. In particular, we use the stochastic optimal control approach to derive an optimal investment strategy in the bank's assets that will maximize an expected exponential utility of the bank's capital at a future date $T > 0$. We present a numerical simulation study to characterize the behaviour of the optimal investment strategy by illustrating graphically the optimal proportions of the capital invested in the assets. We also illustrate, by way of graphs, the behaviours of the asset portfolio, the bank capital, the CAR and the NSFR under the optimal investment strategy. In the last problem, which is in discrete time, we study a DI pricing problem for the underlying commercial bank. It involves using a Monte Carlo simulation method to estimate the premium the underlying bank should be charged for entering into a DI contract while following the optimal investment strategy. The DI contract is assumed to be written on the bank's asset portfolio, while the strike price is assumed to be the insured deposits plus accrued interest. In fact, in this problem we consider different levels of volatility for the bank's asset portfolio and estimate the DI premium for each. This allows us to determine the effect of an increase in the asset portfolio's volatility on the DI premium. A similar study was done by Muller [52]. Given the similarity between our DI pricing problem and that of [52], we employ the pricing algorithm of [52] in order to estimate the said premiums for DI coverage for our bank.

The thesis is organized as follows. The current chapter provides discussions on the regulation of the international banking industry with regard to capital adequacy and liquidity management. It also introduces some background on the CAR, the NSFR, optimization problems studied in finance, the techniques used to solve them, and the concept of DI pricing. In Chapter 2 we summarize some of the works of authors who studied problems related to the CAR, the NSFR, optimization problems in finance, and DI pricing. Chapter 3 provides concepts and ideas from probability and measure theory, as well as from finance that are required to formulate our banking model and the banking problems studied in this thesis. To get a feel for the stochastic optimal control technique, we present an optimal portfolio selection problem from the book Øksendal [58] in Chapter 3. Our contributions to the latter problem include numerical simulations of the assets that make up the fund, the wealth, the optimal proportion of wealth invested in the assets, and the optimal amounts of wealth invested in the assets. Simulations of the latter quantities are not shown in [58]. We also include a discussion on option pricing theory in Chapter 3, as this forms part of the background of the DI pricing problem studied in Chapter 6. In Chapter 4 we present some theory pertaining to commercial banking in general. In particular, we describe the general commercial bank's assets, liabilities and capital, which comprise its stylized balance sheet. We also give a brief description of the off-balance sheet activities here. Furthermore, we specify models, and at the same time, present numerical simulations for the assets, liabilities and off-balance sheet activities of our underlying banking model in Chapter 4. This is also the chapter in which we derive SDEs for the bank's asset portfolio and total liabilities, which allows us to derive an SDE for the capital of the bank in question. Towards the end of Chapter 4 the SDEs for the CAR and NSFR are derived. Chapter 5 presents the optimal control problem and the derivation of its solution. In Chapter

5 we simulate the evolution of the optimal investment strategy, the asset portfolio and bank capital under the optimal investment strategy. In addition, we observe numerically the levels of the CAR and NSFR under the optimal investment strategy. Chapter 6 is devoted to the DI pricing problem involving the underlying bank. This is where we employ the multiperiod DI pricing algorithm of Muller [52] to estimate the premiums the underlying bank should be charged for the DI coverage under different levels of volatility in its asset portfolio. We also present a numerical simulation in which we characterize the evolution of the insured deposits and the bank's asset portfolio under the asset value reset rule in this chapter. We conclude the thesis with Chapter 7.

Chapter 2

Literature review

This chapter summarizes some of the contributions made by authors who studied problems related to the CAR, the NSFR, stochastic optimization in finance, and DI pricing. In regards to the CAR, we summarize the contributions of Estrella et al. [24], Fouche et al. [25], Witbooi et al. [65], Danjuma et al. [17], Muller and Witbooi [50], Chakroun and Abid [12], and Mili et al. [49]. As for the discussion on the NSFR, we summarize the works of Gideon et al. [28], Gobat et al. [29], Arvantis and Drakos [3], Wei et al. [64], Ly et al. [44], Le et al. [43], and Papadamou et al. [59]. We then turn our attention to some of the authors who studied problems related to optimization in finance. This includes Merton [46], Devolder et al. [19], Danjuma [18], Keganneg and Basimanebotlhe [39], Mukuddem-Petersen and Petersen [54], Gideon et al. [27], van Schalkwyk and Witbooi [63] and Muller [52]. Finally, we discuss the works of authors who contributed to the development of DI pricing models. Here we discuss the papers Merton [47], Ronn and Verma [61], Allen and Suanders [2], Hwang et al. [31], Duan and Yu [21], Chiang and Tsai [14], Hariati et al. [30], Camara et al. [11] and Muller [53].

2.1 Analyses of the Capital Adequacy and Net Stable Funding ratios

As stated above, we first discuss the work of authors who studied problems related to the CAR. Estrella et al. [24] compared the effectiveness of various types of capital ratios in predicting bank failure. The aforementioned authors found that simple ratios, in particular the leverage ratio and the ratio of capital to gross revenue, as well as the more complex risk-weighted ratio, can be used to predict bank failure. This suggests that bank regulators may discover a useful role for the simple ratios in the design of regulatory capital frameworks, in particular indicators of the need for prompt supervisory action. On the other hand, the risk-weighted ratios tend to perform better over longer time horizons. Their study is not to argue against utilizing additional sophisticated measures of capital adequacy in regulation, but it suggests that simple capital ratios may not be well suited for the determination of optimum levels of bank capital. However, these simple capital ratios contain useful information and are virtually costless to calculate. Moreover, it may be possible to derive substantial benefits from the use of simple ratios as, for example, supplementary or backstop requirements, even when more sophisticated measures are available to use in formulating the primary requirements.

Fouche et al. [25] studied risk-based and non-risk-based CARs. The authors of [25] constructed continuous-time stochastic models for not only the dynamics of the Equity, Leverage and Tier 1 ratios, but for the Basel II CAR as well. They further studied an optimal control problem in which an optimal asset allocation strategy is derived for the Leverage Ratio which is specified on a time interval. More specifically, they derived the optimal expected terminal utility of the Leverage Ratio and determined the asset allocation strategy that makes it possible to maximize the expected terminal utility of the Leverage Ratio on the specified time interval. Their results conform to the qualitative and quantitative standards prescribed by the Basel II Capital Accord.

Witbooi et al. [65] employed stochastic optimization theory to study an asset and capital adequacy management problem in banking. More specifically, Witbooi et al. [65] addressed the problem of obtaining an optimal equity allocation strategy that would optimize the terminal utility of a bank's asset portfolio consisting of a treasury security, a marketable security and a loan under the Cox-Huang [16] methodology. They also constructed a stochastic continuous time model of the Basel II CAR from the banks' capital and TRWAs, and presented a numerical simulation of the optimal equity investment strategy. Witbooi et al. [65] found that the optimal proportion invested in the treasury increases over time. However, the optimal proportion invested in the loans slowly decreases, while the proportion invested in marketable security remains constant. They also found that the CAR resembles a mean-reverting process subject to the optimal allocation strategy.

Danjuma et al. [17] considered a financial market consisting of various assets and where the interest rate is stochastic. They applied the dynamic programming principle for the case of a constant relative risk aversion (CRRA) utility function so as to derive an optimal investment strategy for the bank modelled in their paper. The authors further derived an SDE for the CAR under Basel II Accord and Central Bank of Nigeria standards. They presented a numerical simulation study based on the optimal investment strategy in which they found that the optimal investment strategy is to diversify the asset portfolio away from the risky assets and towards a riskless asset. They also found that the higher the percentage of the CAR, the more capital is required to maintain the prescribed CAR.

Muller and Witbooi [50] studied an investment problem that involves the maximization of an expected logarithmic utility of a commercial bank's asset portfolio at a future date. The aforementioned authors considered a bank that trades in a stochastic interest rate financial market consisting of a treasury security, a marketable security and a loan. They derived formulas for the optimal amount of bank capital invested in each of its assets, as well as a formula for the Basel III CAR. Furthermore, the authors considered the optimal investment strategy subject to a constant CAR at the minimum prescribed level set forth by the BCBS, and derived an expression for the bank's asset portfolio that will fix the CAR at the minimum Basel III prescribed level. Furthermore, they presented numerical simulations based on different situations with their results showing that the asset portfolio at a constant (minimum) CAR value grows considerably slower than the original asset portfolio of the investment problem.

Chakroun and Abid [12] investigated issues of bank capital adequacy and risk management in a stochastic setting. In addition, an explicit risk aggregation and capital expression is provided in regards to the portfolio choice and capital requirement. This framework results in a nonlinear stochastic optimal control problem whose solution may be derived by means of the dynamic programming algorithm. Their analysis depends mainly on stochastic modelling of balance items such as securities, loans and regulatory capital with stochastic interest rates. In their analysis, the special Kalmam filter approach is used for the purpose of estimating the model parameters. Their findings show that the Tunisian bank on which their study is based, typically surpasses the minimum requirements and is adequately capitalized to maintain the relevant amount of capital when compared with the aggregate risk.

Mili et al. [49] investigated the factors that influenced the CAR of foreign banks. The authors of [49] tested whether the CAR subsidiaries and branches in developed and developing countries rely on the same factors. They used data from 310 subsidiaries and 265 branches to examine the effect of the parent bank's fundamentals on subsidiaries and branches' capital ratios. They also studied the economic condition and regulatory environment in a bank's home country to determine foreign banks' CARs. The authors provided strong evidence that the CAR of subsidiaries and branches operating in developed and developing counties does not rely on the same set of explanatory factors. Their study revealed that the regulatory framework of a parent bank's home country affects the capitalization of its foreign subsidiaries in the host counties. Lastly, they illustrated that specific variables of the parent bank has a stronger effect for foreign banks in relation to the interbank market.

We now proceed to discuss the work of some of the authors who studied problems related to the NSFR. Gideon et al. [28] quantitatively validated the Basel III liquidity standards as encapsulated by the NSFR. The aforementioned authors considered the Inverse Net Stable Funding

Ratio (INSFR) as a measure quantifying the banks' prospects for stable funding over a one year period. In particular, they derived a stochastic model for the dynamics of the INSFR that relied solely on AASF and RASF, and also the liquidity provisioning rate. Gideon et al. [28] further studied an optimal control problem involving the INSFR, considering a quadratic objective function. Moreover, they made optimal choices for the INSFR targets in order to formulate its cost. This was obtained by finding an analytical solution for the value function. Their study includes a simulation for the trajectory of the INSFR, from which they found that the bank experienced a few problems to secure some stable funding over the first three months and also over month 7 to month 9. On the other hand, there there was a higher liquidity ratio between month 4 to month 7 due to growth in the RSF being more than that of the ASF.

Gobat et al. [29] complemented earlier quantitative impact studies by discussing the potential impact of the NSFR's introduction on the empirical analysis of the financial data at the end of 2012 for over 2000 banks across 128 countries. Their calculations revealed that a larger percentage of the banks in many countries would meet the minimum NSFR prescribed requirement at the end of 2012, and further, that larger banks tend to be more susceptible to the introduction of the NSFR. In addition, they compared the NSFR to other structural funding mismatch indicators, and found that the NSFR is a consistent regulatory measure for encapsulating banks' funding risk.

Arvantis and Drakos [3] calculated the NSFR metric for U.S. Bank Holding Companies throughout the period from 2001 to 2013 to assess retrospectively whether banks satisfied the recommended requirements. They found that for the most of cases, the NSFR was compatible with the Basel III threshold. Moreover, they documented a significant decline of about 10% of the NSFR during the post-financial crisis period under analysis. In addition, they found that the NSFR exhibits significant heterogeneity across size segments, with its mean level lowering at a decay rate.

Wei et al. [64] developed a theoretical framework in which the bank manager chooses the asset composition and debt maturity structure. The authors of [64] modelled the incongruence of goals between the bank management and the bank stakeholders by letting the bank manager receive only a share of the bank's profit. The aforementioned authors showed that the bank manager's choices result in socially inefficient outcomes, which leaves room for welfare improvement in government regulation. They discuss, within the theoretical framework, the impacts of the NSFR requirement on the bank manager's choices of asset composition structure, with consequences for the banks' profitability and on social welfare. They showed that if short-term debt is given a sufficiently low weight in available stable funding, the NSFR can decrease the use of short term debt and as a result, reduce the banks' exposure to roll-over risk. Under this set of conditions social welfare can be enhanced, but may be reduced when short term debt is given a sufficiently high weight. Under the same set of conditions, they also discovered that the NSFR can increase the banks' probability of survival and unconditional expected profits, due to the constraint on the debt maturity structure alleviating the goal in congruence problem between the bank owner and the manager. The NSFR, however, will reduce the probability of bank failures and actual returns of profit of surviving banks. The results of their study are robust for the case where asset structure with the conditional variance on the interim information is constant. In addition, their theoretical framework can be shown to help study issues on the regulation of liquidity risks.

Ly et al. [44] employed a partial adjustment model and annual data sample of U.S. bank holding companies from 1991 to 2012 to examine the effect of the NSFR adjustment speeds on systemic risk. They noticed that banks with the immediate trading equilibrium tend to adjust the NSFR quickly in response to the Basel III liquidity requirement, which reduces systemic risk. With the same level of the NSFR, their findings showed that only the adjustment speed exerts a negative impact on systemic risk. Ly et al. [44] further found that small banks strengthen the effects of the negative impact of the NSFR adjustment speed on systemic risk. Their study sheds light on a real-time indicator of the NSFR for Basel III review before its implementation in 2018.

Le et al. [43] investigated the empirical relation between liquidity on bank profit efficiency for commercial banks in the U.S. from 2001 to 2015 by using data from two sources: that is, the Bankscope and Federal Financial Institutions Examination Council Call reports. Efficiency scores that were estimated by using the Bankscope dataset are lower than corresponding to using the call reports. The authors of [43] delve more deeply into the non-linear relationship between bank efficiency and the NSFR under the Basel III framework. The authors' empirical results demonstrate that there exists a non-linear relationship between the NSFR and bank efficiency. More specifically, their results suggest that modest intensification in liquidity aids to improve bank profit efficiency. However, profit efficiency could be ruined due to excess liquidity enlargement. The results are robust in both data sets.

Papadamou et al. [59] empirically investigated the effect that the implementation of the NSFR has on an economy. By using data from the European Union banking sector, the aforementioned authors conducted a retrospective analysis by simulating and examining the NSFR index historically as well as its role in the implementation of a common monetary policy. The authors intervened on the traditional bank lending channel of Bernanke and Blinder by using the interaction term between liquidity and interest rates. Their analysis was conducted both at an aggregated loan supply level and by loan category, since it incorporates, as well to the interaction term, traditional asset pricing approaches with the adoption of self-financing trading strategies identifying nonlinearities in the relationship between liquidity provisions and bank lending channel. Their analysis shows that there is evidence of a heterogeneous response of financial intermediaries' loan supply (due to interest rates) across various NSFR levels. High NSFR banks respond positively to an interest rate increase by reorganisation of their loan portfolios to obtain higher risk-adjuated returns, conditional on the presence of an effective asset allocation. On the other hand, low NSFR banks decrease loan supply as a response to higher interest rates.

2.2 Optimization problems in finance

We now turn our attention to the work of authors who studied problems related to optimization in finance. In particular, we first discuss the works of those authors who studied optimization problems related to pension funds, then those who studied similar problems in banking.

Merton's [46] seminal work set in motion the dynamic programming and stochastic control method for continuous-time portfolio optimization. He used the HJB equation of the dynamic programming method to explicitly solve the question of optimal portfolio allocation in a market with a risky asset and a riskless bond as an investment alternative. The stock price process in [46] is assumed to be driven by a geometric Brownian motion. Here it is also assumed that an investor wishes to maximize their terminal wealth under a power utility function. Numerous authors have employed the method of stochastic optimal control since then.

Devolder et al. [19] illustrated how stochastic optimal control theory can be applied to find an optimal investment policy for a defined contribution pension plan before and after retirement. The benefits of this pension plan are paid in the form of annuities that are guaranteed during a certain fixed period of time. The aforementioned authors considered a financial market consisting of two assets. Throughout the activity period of the contract, the contributions of the participant is invested in either a risky asset or a riskless one. They studied the problem of finding the optimal investment strategy for the assets backing the pension liabilities during the whole life of the participant in the plan. At the retirement age of the members, the reserve obtained is the accumulated amount given to the insurer without any special guarantee. The insurer uses the guarantee to purchase a paid up annuity at retirement. He/she is responsible for paying the annuity and is faced with the decision of how much of the mathematical reserve should be invested in the financial market in question. The authors split the problem into two periods because of the presence of liabilities only at retirement. For the first period (i.e., the period before retirement without liability), they optimized the expected utility function of the final wealth at retirement. For the second period (i.e., the period after retirement without liability), they optimized the expected utility function of the final surplus. They considered both the power law utility and exponential utility function for either period.

Danjuma and Ibidoja [18] studied a portfolio optimization problem based on a stochastic interest rate financial market that consists of a treasury, a security and a loan. They applied the stochastic optimal control method for the case of a CRRA function, to derive an optimal investment strategy for the three assets. They present numerical examples based on the optimal investment strategy, where they study the effect of time, risk aversion parameter and market price of risk parameter on the optimal investment strategy. The authors found that the optimal investment strategy is to diversify the asset portfolio of the financial institution away from the risky assets and towards the riskless treasury. They also found that the investor invests more in the risky assets as the investor adopts a less risk averse investment strategy. As the reward of the risks associated with the risky assets increases, the more the investor invests more in the risky assets.

Keganneg and Basimanebotlhe [39] studied an optimal control problem involving asset allocation for a defined contribution plan. The aforementioned paper considered a financial market consisting of three assets, namely, riskless asset, a risky asset and an inflation-linked bond. They constructed the dynamics of the wealth that takes into account a certain proportion of the client's salary paid as the contribution towards the pension fund. They employed the HJB equation to determine the explicit solutions for the constant absolute risk aversion (CARA) and CRRA utility functions, which enable them to compute investment strategies associated with the three assets. In addition, they also presented a numerical simulation of the the behaviour of the model. They found that over time the pension fund manager is to diversify its portfolio away from the risky asset and towards the riskless asset and inflation-linked bond under the CARA function. On the other hand, under the CRRA utility function the pension fund manager is to diversify the portfolio away from the risky asset and riskless asset and towards the inflation-linked bond. They also found that as the degree of risk preference increases, so does the optimal amounts invested in the inflation-linked bond and and riskless asset, but the optimal amounts invested in the risky asset decreases under the CARA utility function. They further observed that the higher the degree of risk preference the lower the optimal amounts invested in the inflation-linked bond and risky asset, and that when the degree of risk aversion increases, then so does the optimal amount invested in the riskless asset under the CRRA utility function.

We now discuss the work of some of the authors who studied optimization problems related to banking specifically. Mukuddem-Petersen and Petersen [54] investigated, in a stochastic dynamic setting, a banking problem related to the optimal risk management of banks. In particular, the aforementioned authors minimized market and capital adequacy risk, which, respectively, involves the stability of sources of funds and the safety of the securities held. In this regard they suggest an optimal portfolio choice and rate of bank capital inflow that will maintain the loan level as close as possible to an actuarially determined reference process. This leads to a non-linear stochastic optimal control problem whose solution can be determined by using the dynamic programming algorithm. Their analysis depends on the construction of continuous-time stochastic models for bank behaviour upon which a spread technique for loan capitalization is imposed. The main novelty of the paper [54] is the solution of an optimal control problem that minimizes bank market and capital adequacy risks by making decisions about the security allocation and capital requirements. The former is measured by the deviations of the bank's securities from the loan issuing process, as it is an indicator of the bank's safety. This gives information regarding the size of the deviation of bank capital requirements from the bank capital reference process and is in relation to the financial stability of the bank.

Gideon et al. [27] studied the stochastic dynamics of bank liquidity parameters which includes liquidity assets and nett cash flow with regards to the global financial crisis. In [54] the aforementioned parameters were used to find the LCR which is one of the metrics utilized in the ratio analysis to measure bank liquidity. The authors presented numerical results that show that the behaviour of bank in relation to liquidity was mostly fluctuating throughout the financial crisis. They also considered a theoretical quantitative approach to bank liquidity. In this regard, they provide an explicit formula for the aggregate liquidity risk when a locally risk-minimizing strategy is used.

van Schalkwyk and Witbooi [63] studied an optimization problem that involves bank liquidity management within a jump diffusion framework. The aforementioned authors studied the interaction between a commercial bank and a central bank, and how this interaction affects (i) the supply of money between those two financial institutions and (ii), the LCR. The main reason behind investigating the dynamics of the LCR is to show how banks are able to control their liquidity through following an appropriate strategy. This will ensure that the LCR level does not drop below an acceptable level.

Muller [52] used the method of stochastic optimal control to determine an optimal investment strategy for maximizing an expected exponential utility of a commercial bank's capital at a future date. He considered a bank that trades in a constant interest rate financial market that consists of three assets, namely a treasury, a marketable security and a loan. Muller [52] provided numerical simulations based on the optimal proportions of capital invested in the treasury, the marketable security and the loan. The results from the simulation study show that the optimal investment strategy is to diversify the asset portfolio of the bank away from the risky assets and towards the riskless treasury. In addition, he also derived a multiperiod DI pricing model that incorporates the aforementioned optimal investment strategy. In the latter problem, the author found that for a fixed initial leverage the DI premium will rise when either the risk in the asset portfolio of the bank or the DI coverage horizon is raised. Furthermore, by increasing the initial leverage levels, the DI premium will increase as the risk in the asset portfolio is increased, but the DI premium decreases as the coverage horizon is raised.

2.3 Pricing deposit insurance as put options

We now discuss some papers on the topic of DI pricing, the last type of banking problem we study in this thesis. Merton [47] suggested an analogy between a put option and DI to value DI contracts. More specifically, he suggested that DI can be modelled as a put option with the strike price of the option being equivalent to the value of the banks' deposits, and the bank assets being equivalent the underlying asset of the option. He assumed that the asset portfolio follows a geometric Brownian motion with the expiration date of the DI contract being the duration of time until the bank audit. In addition, he revealed that by employing the Black-Scholes formula [10] it is possible to find the value of the option, which in this case, is observed to be the DI premium. If the bank is found to be insolvent during this audit, the option can be exercised. A few important variables to this model are the value of the banks' assets, the impact of the stochastic interest rate on the total bank assets, and the volatility of the return on the assets.

Ronn and Verma [61] discovered that Federal Deposit Insurance Corporation (FDIC) insurance is under-priced. The authors of [61] took the same assumption as that of [47], which is that the time until expiration of the debt is equal to the time until the next bank audit. In addition, they assumed that the strike price of the put option is equal to the total debt of the bank, instead of only the total deposit. Their model depends on two variables: the bank's asset value and the equity volatility. The bank's asset value can be observed, but the equity volatility must be estimated. The sample standard deviation of the equity returns, must therefore be taken to be the equity volatility.

Allen and Suanders [2] modelled DI a callable perpetual American put option concerning both self-closure policy and regulatory closure policy. The aforementioned authors argued that DI can be described by as a callable put option, since DI is a perpetual put option with the insuring agent holding the right to terminate the put option prematurely. Their assumption is that the FDIC's closure rule is strictly observed and that there is no extra forbearance, except in the case of the largest banks. The DI is not a standard put option when it comes to the right to exercise. If the option expires in the money, then the bank's shareholders may decide to not exercise since it implies voluntary bank closure. The closure decision is employed to regulate the timing to exercise.

Hwang et al. [31] extended the callable perpetual American put option model of [2] by explicitly incorporating bankruptcy costs and additional realistic rules, so that possible forbearance can be accounted for. In the model of [2] the bankruptcy cost plays a critical role and is set as a function of asset return volatility. By using the isomorphic relationship between DI and a put option, they obtain a closed form solution for the pricing model with bankruptcy costs and closure policies. Subsequently, they modified the barrier option approach to price the DI. Hwang et al. [31] assume that at the time of bank solvency, deposit holders are authorized to a prorated fraction of the asset value with all debt holders. Thus, their model assumes that all debts are of the same liquidation. The authors reveal that the big problem in fair pricing of DI is how to create the premium correctly so as to reflect the risk of the insured bank.

Duan and Yu [21] developed a multiperiod DI pricing model that incorporates an asset value reset rule comparable with the usual practice of insolvency resolution by insuring agencies. The fairly-priced premium rate of their model can differ considerably from that of Merton [47]. They investigated the effect of varying the constant premium rate coverage horizon and found that it affects the fairly-priced DI premium rate. Their model also ensures that it is possible to formally investigate forbearance and the accompanying risk-taking behaviour. Their findings reveal that the fairly priced premium rate is not neutral to that of the capital forbearance policy. The risk-taking intensity of banks determines how the fairly priced rate responds to the forbearance policy. Their model also formalized the process of how excessive risk-taking under capital forbearance results in instability in the DI system.

Chiang and Tsai [14] presents a model for evaluating a DI premium based on a specific official default probability. Their pricing formula can be used to flexibly compute the DI premium that reflects changes in economic circumstances. The authors of [14] provide a new estimation method to compute the implied asset risk based on the efficient frontier between asset value and asset risk. The latter avoids the issue for estimating a bank's assets and asset risk using market equity data. They suggest that the DI premium should be lower for banks that fully meet the financial supervisory regulations, which should incentivize these banks to lower their likelihood of default by strictly using financial regulations. Chiang and Tsai [14] also suggest a new dynamic technique to assist in determining reasonable DI premiums and that maintains the target level of DI fund reserves.

In managing DI in Indonesia, a DI agency uses the flat rate premium system, which is equivalent to that of the premium imposition system for each bank without taking into account the various risk levels of each bank. The implementation of the flat rate can cause moral hazard which could lead to a monetary crisis. In this regard, Hariati et al. [30] studied analytical solutions to determine the risk adjusted DI premiums on the Heston Model by applying the Fourier transform. They present a simulation study that shows that an increase in volatility caused the value of the DI premiums to increase, and the decrease in the value of interest rates increases the value of the DI premiums. They also found that an increase in the value of DI premiums is also caused by the value of debt obligations and dividends.

Camara et al. [11] modelled DI as a European put option on the value of a bank's assets follows a lognormal diffusion process. They obtained closed-form solutions for the value of the bank equity holders, depositors, and the deposit insurer under three different DISs, that are representative of DI across the world. This enables them to compute actuarially fair premiums that are risk-adjusted, include market information, and explicitly account for the diverging effects of riskless assets when compared against risky assets on the bank risk. Camara et al. [11] demonstrates the use of the model on a sample of U.S. bank holding companies and discuss practical considerations for employing their model. Implications for their model as a market-based, and early indicator of bank risk are considered. They found that computed values of DI are not sensitive to model calibration, but differs across those DISs.

Muller [53] considered a bank that invests its capital in a constant interest rate financial market consisting of a treasury security, a marketable security and a loan. The author derived formulas for the rate of capital influx, the asset portfolio and capital with which the bank will maintain its CAR at the prescribed minimum level of 8%. In addition, he derived a multiperiod DI pricing model that incorporates an asset value reset rule to price the DI for the bank while it maintains the fixed (minimum) CAR level. His simulation study suggests that in order for the CAR to be fixed at under its prescribed minimum level, there should be a upward trend in the capital influx. The same applies for the asset portfolio and capital. His simulation study further reveals that the DI price becomes more expensive as the coverage horizon is increased. He also found that the DI price becomes less expensive when the volatility of the asset portfolio is increased, while the opposite holds for initial leverage levels of 0.8 and 0.9 when volatility increases.

Chapter 3

Mathematical preliminaries

This chapter serves to introduce and present concepts from probability and measure theory, stochastic optimal control and finance that are relevant to our study. More specifically, we present the concepts that are required to formulate and solve all of the banking problems studied in this thesis. Our main references are the books Øksendal [58], Nielsen [57], Etheridge [23], Baz and Chacko [9], and Hull [34].

3.1 Concepts from probability and measure theory

We begin by introducing the concepts from probability and measure theory.

Definition 3.1.1. (σ-algebra) *(see [57, p.317]) Let Ω be any non-empty set. A σ-algebra or σ-field on Ω is a class $\mathcal{F}$ of subsets of Ω with the following three properties:*

1. $\Omega \in \mathcal{F}$;

2. *If $\{A(t)\}$ is a finite or infinite sequence of sets in $\mathcal{F}$, then $\bigcup A(t) \in \mathcal{F}$;*

3. *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.*

Definition 3.1.2. (Filtration) *(see [57, p.14]) A filtration is a family $\{\mathcal{F}(t)\}_{t\in J}$ of σ-algebras $\mathcal{F}(t) \subset \mathcal{F}$ which is increasing in the sense that whenever $s, t \in J$ and $s \leq t$, then $\mathcal{F}(s) \subset \mathcal{F}(t)$. Here J is a time interval such that $J = [0, \infty)$ or $J = [0, T]$ for some $T > 0$.*

Definition 3.1.3. (Probability triple) *(see [23, p.29]) A probability triple* $(\Omega, \mathcal{F}, \mathbb{P})$*, consists of a set* Ω *(sample space), a collection of subsets* $\mathcal{F}$ *of* Ω *(events) and a probability measure* $\mathbb{P}$*, which specifies the probability of each event* $A \in \mathcal{F}$*. The collection* $\mathcal{F}$ *is assumed closed under the operations of countable unions and taking complements (*σ*-field). The probability measure* $\mathbb{P}$ *must satisfy the following axioms:*

1. $0 \leq \mathbb{P}[A] \leq 1$ *for all* $A \in \mathcal{F}$*;*

2. $\mathbb{P}[\Omega] = 1$*;*

3. $\mathbb{P}[A \bigcup B] = \mathbb{P}[A] + \mathbb{P}[B]$ *for any disjoint* A *and* B *in* $\mathcal{F}$*;*

4. *If* $A(n) \in \mathcal{F}$*, for all* $n \in N$*, and* $A(1) \subseteq A(2) \subseteq \ldots$ *then* $\mathbb{P}[A(n)] \uparrow \mathbb{P}[\bigcup_n A(n)]$ *as* $n \uparrow \infty$*.*

Definition 3.1.4. (Stochastic process) *(see [57, p.2]) Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A k-dimensional stochastic process is a mapping* $X : \Omega \times J \to \mathbb{R}^k$ *such that for each fixed* $t \in J$*, the mapping*

$$X(t) : \omega \to X(\omega, t) = X(t)(\omega) : \Omega \to \mathbb{R}^k$$

is measurable. A stochastic process is said to be adapted to the filtration $\{\mathcal{F}(t)\}_{t \in J}$ *if for each* $t \in J$*, the random variable or vector resulting from the latter mapping is measurable with respect to* $\mathcal{F}(t)$*. This means that the value* $X(t)$ *of the process* X *at time* t *depends only on information available at time* t*.*

Definition 3.1.5. (Simple random walk) *(see [23, p.34]) A stochastic process* $\{S(n)\}_{t \geq 0}$ *is a simple random walk under the probability measure* $\mathbb{P}$ *if* $S(n) = \sum_{i=1}^{n} \xi(i)$*, where* $\xi(i)$ *can only take the values* $\{+1, -1\}$*, are independent and identically distributed under* $\mathbb{P}$*.*

Definition 3.1.6. (Standard Brownian motion) *(see [57, p.5]) A k-dimensional standard Brownian motion is a k-dimensional process* $\{W(t)\}_{t \geq 0}$ *such that:*

1. $W(0) = 0$ *with probability one;*

2. W *is continuous;*

3. if $0 \leq t(0) \leq \cdots \leq t(n)$, then the increments $W(t(1)) - W(t(0)), W(t(2)) - W(t(1)), \ldots,$ $W(t(n)) - W(t(n-1))$ are independent;

4. if $0 \leq s < t$, then the increments $W(t) - W(s)$ is normally distributed with mean zero and covariance matrix $(t-s)I$, where I is the $k \times k$ identity matrix.

If W is a one-dimensional standard Brownian motion, and if $0 \leq s < t$, then the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$. A one-dimensional process is called a geometric Brownian motion if it has the form e^Z, where Z is a one-dimensional generalized Brownian motion with constant initial value $Z(0)$.

Definition 3.1.7. (Martingale) *(see [57, p.16]) Let $\{\mathcal{F}(t)\}_{t\geq 0}$ be a filtration. A process X is a martingale if it is integrable and adapted and whenever $s, t \in J$ and $0 \leq s \leq t$,*

$$\mathbb{E}[X(t)|\ \mathcal{F}(s)] = X(s).$$

Definition 3.1.8. (One-dimensional Itô process) *(see [58, p.44]) Let $W(t)$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. A (one-dimensional) Itô process (or stochastic integral) is a stochastic process $X(t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form*

$$X(t) = X(0) + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dW(s), \tag{3.1}$$

where $v \in \mathcal{W}_{\mathcal{H}}$, so that

$$\mathbb{P}\left[\int_0^t v(s,\omega)^2 ds < \infty \ \ \forall\, t \geq 0\right] = 1.$$

We also assume that u is $\mathcal{H}(t)$-adapted, where $\mathcal{H}(t)$ is an increasing family of σ-algebras, $\{\mathcal{H}(t)\}_{t\geq 0}$, such that $W(t)$ is a martingale with respect to $\mathcal{H}(t)$, and

$$\mathbb{P}\left[\int_0^t |u(s,\omega)| ds < \infty \ \ \forall\, t \geq 0\right] = 1.$$

If $X(t)$ is an Itô process of the form (3.1), then Eq.(3.1) is sometimes written in the shorter differential form

$$dX(t) = udt + vdW(t). \tag{3.2}$$

In Eq.(3.2), dt denotes increment in time while $dW(t)$ denotes increment in Brownian motion.

Remark 3.1.9. (Itô's formula) *(see [58, p.44]) Let $X(t)$ be an Itô process given by*

$$dX(t) = udt + vdW(t),$$

with $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then $Y(t) = g(t, X(t))$ is again an Itô process, and

$$dY(t) = \frac{\partial g}{\partial t}\Big(t, X(t)\Big)dt + \frac{\partial g}{\partial x}\Big(t, X(t)\Big)dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\Big(t, X(t)\Big)[d(X(t)]^2.$$

Here $[d(X(t)]^2 = dX(t)dX(t)$ is computed according to the rules

$$dtdt = dtdW(t) = dW(t)dt = 0 \text{ and } dW(t)dW(t) = dt.$$

Note:

(i) $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$ means that $g(t,x)$ is twice continuously differentiable on $[0,\infty) \times \mathbb{R}$.

(ii) We will use the alternative form of Itô's formula, given by

$$dY(t) = \dot{f}(Y(t))dt + f'(Y(t))dY(t) + \frac{1}{2}f''(Y(t))[dY(t)]^2,$$

in Chapter 4 where we derive SDEs for the CAR and NSFR.

Remark 3.1.10. (Itô's Product Rule) *(see [58, p.55]) For Itô processes $X(t)$ and $Y(t)$ in $\mathbb{R}$, Itô's Product Rule gives*

$$d[X(t)Y(t)] = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t).$$

3.2 Stochastic optimal control and optimal asset allocation

We now introduce the HJB PDE for stochastic optimal control and illustrate how it is applied via the optimal portfolio problem from the book [58] by Øksendal.

Theorem 3.2.1. *(see [9, p.247]) The Hamilton-Jacobi-Bellman equation (hereafter the HJB equation) of optimal control for an Itô process X for the optimization problem*

$$J(0,X) = \max_y \mathbb{E}\left[\int_0^T f(t,X,y)dt + \bar{B}(T,X(T))\middle|\mathcal{F}(0)\right],$$

subject to the constraints

$$dX = \mu(t,X,y)dt + \sigma(t,X,y)dW$$

and with $X(0)$ fixed, is given by:

$$-\frac{\partial J(t,X)}{\partial t} = \max_y \left\{ f(t,X,y) + \frac{\partial J(t,X)}{\partial X}\mu(t,X,y) + \frac{1}{2}\frac{\partial^2 J(t,X)}{\partial X^2}\sigma^2(t,X,y)\right\}.$$

This is a partial differential equation with boundary condition

$$J(T,X(T)) = \bar{B}(T,X(T)).$$

The variable y is called the control or decision variable and the variable X is the variable of state.

Note: In what follows, we will use the alternative form of the HJB PDE:

$$0 = J_t + \max_y \left\{ f(t,X,y) + J_X\mu(t,X,y) + \frac{1}{2}J_{XX}\sigma^2(t,X,y)\right\}$$

Example 3.2.2. *(see [58, p.236])* We consider an investor who trades in a complete and frictionless financial market that is continuously open over a fixed time interval $[0,T]$. If $t \in [0,T]$, where T is the final date of the investor's wealth, then the problem involves optimizing an expected power utility of the final wealth. We assume here that we are working with a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the real world probability measure. The Brownian motion B appearing in this problem is assumed to be defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$. The filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is generated by the aforementioned Brownian motion and satisfies the usual conditions. The financial market is assumed to be described by two different assets.

The first of these is a risky asset whose price at time $t \geq 0$ is denoted by $p_1(t)$. We assume its dynamics to evolve according to the SDE

$$\frac{dp_1(t)}{p_1(t)} = adt + \alpha dB(t), \tag{3.3}$$

where $a, \alpha > 0$ and B is a one-dimensional standard Brownian motion as stated.

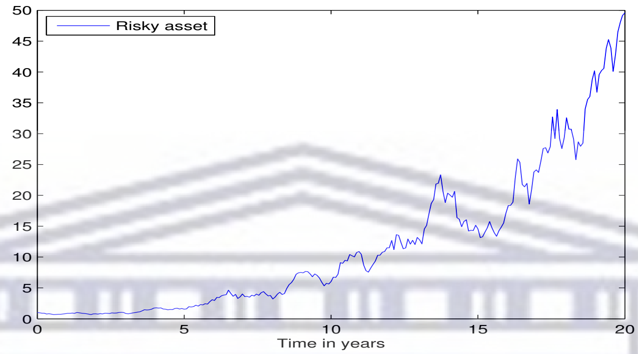

Figure 3.1: A simulation of the evolution of the price of the risky asset, $p_1(t)$, with $a = 0.2$, $\alpha = 0.3$, $p_1(0) = 1$ and $T = 20$ years.

The second asset is a riskless asset whose price at time $t \geq 0$ is denoted by $p_2(t)$. We assume that its dynamics evolve according to the ordinary differential equation (ODE)

$$\frac{dp_2(t)}{p_2(t)} = bdt. \tag{3.4}$$

Here $b > 0$ is the continuously compounded riskless rate.

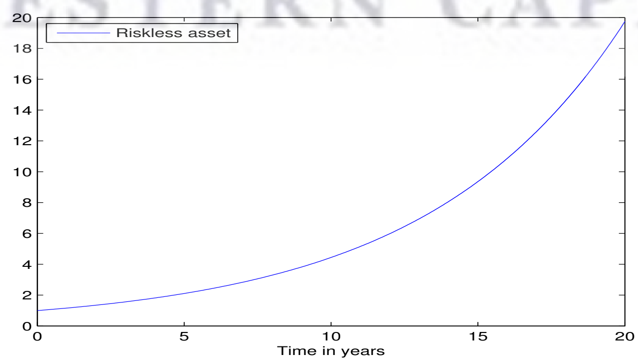

Figure 3.2: A simulation of the evolution of the price of the riskless asset, $p_2(t)$, with $b = 0.15$, $p_2(0) = 1$ and $T = 20$ years.

If we let $X(t)$ denote the investor's wealth at time $t \geq 0$, then the wealth dynamics can be described by the SDE

$$\begin{aligned} dX(t) &= X(t)\left\{u(t)\frac{dp_1(t)}{p_1(t)} + \left[1-u(t)\right]\frac{dp_2(t)}{p_2(t)}\right\} \\ &= X(t)\{u(t)[adt + \alpha dB(t)] + [1-u(t)]bdt\} \\ &= u(t)X(t)adt + u(t)X(t)\alpha dB(t) + b(1-u(t))X(t)dt \\ &= X(t)[au(t) + b(1-u(t))]dt + \alpha u(t)X(t)dB(t). \end{aligned} \tag{3.5}$$

The quantities $u(t)$ and $1-u(t)$ denote the proportions of wealth invested in the risky asset and riskless asset, respectively.

The objective of the problem is to maximize the expected utility of the wealth, X, at a future date $T > 0$. That is,

$$\max_u \mathbb{E}[N(X(T))],$$

with the dynamics of the wealth, $X(t)$, given by the SDE in Eq.(3.5). Here $0 \leq t \leq T$ and $X(0)$ denotes the initial value of the wealth, assumed to be a positive constant.

The value function of this problem can be considered as a kind of utility function. It is given by

$$W(t,X) = \max_u \mathbb{E}[N(X(T)|X(t) = X)],$$

with $0 < t < T$. The marginal utility of the above value function is a constant, and the marginal utility of the original utility function $N(\cdot)$ decreases to zero as $X \to \infty$ (see Kramkov and Schachermayer [42] for instance). The value function inherits the convexity of the utility function and is strictly convex for $t < T$ even if $N(\cdot)$ is not (see Jonsson and Sircar [37] for instance).

The HJB equation of this problem (see Theorem 3.2.1) can be written as

$$0 = W_t + \max_u \left\{X[u(a-b)+b]W_X + \frac{1}{2}u^2\alpha^2X^2W_{XX}\right\}, \tag{3.6}$$

where the time variable t has been suppressed. The variables W_t, W_X and W_{XX} represent the first and second order partial derivatives with respect to time and wealth, respectively.

Now taking the derivative of Eq.(3.6) with respect to u yields

$$0 = (a-b)XW_X + uX^2\alpha^2 W_{XX},$$

leading to a first explicit form of the optimal investment proportion u in the risky asset given by

$$u = -\frac{a-b}{\alpha^2}\frac{W_X}{XW_{XX}}. \tag{3.7}$$

By substituting Eq.(3.7) into Eq.(3.6) the value function transforms into the PDE

$$W_t + X\left[-\frac{a-b}{\alpha^2}\frac{W_X}{XW_{XX}}(a-b) + b\right]W_X + \frac{1}{2}\left[-\frac{a-b}{\alpha^2}\frac{W_X}{XW_{XX}}\right]^2 \alpha^2 X^2 W_{XX} = 0,$$

from which we obtain

$$W_t + X\left[-\frac{(a-b)^2}{\alpha^2}\frac{W_X}{XW_{XX}} + b\right]W_X + \frac{1}{2}\frac{(a-b)^2}{\alpha^4}\frac{W_X^2}{X^2W_{XX}^2}\alpha^2 X^2 W_{XX} = 0.$$

The latter equation simplifies to

$$W_t - \frac{(a-b)^2}{\alpha^2}\frac{W_X^2}{W_{XX}} + bXW_X + \frac{1}{2}\frac{(a-b)^2}{\alpha^2}\frac{W_X^2}{W_{XX}} = 0,$$

or

$$W_t + bXW_X - \frac{1}{2}\frac{(a-b)^2 W_X^2}{\alpha^2 W_{XX}} = 0. \tag{3.8}$$

We must now solve Eq.(3.8) for the function W and put this into Eq.(3.7) to obtain the optimal investment proportion u. The PDE given in Eq.(3.8) admits an explicit solution for the utility function of the form

$$N(X) = X^r,$$

where $0 < r < 1$ (see Øksendal [58]). We try to find an explicit solution for the PDE in Eq.(3.8) with the structure

$$W(t,X) = f(t)X^r, \tag{3.9}$$

for which $f(T) = 1$. By computing the partial derivatives of Eq.(3.9), we obtain:

$$\begin{aligned} W_t &= f'(t)X^r; \\ W_X &= f(t)rX^{r-1}; \\ W_{XX} &= f(t)r(r-1)X^{r-2}. \end{aligned}$$

By substituting the above derivatives into Eq.(3.8), we obtain for $f(t)$ the differential equation

$$f'(t)X^r + brX[f(t)X^{r-1}] - \frac{1}{2}\left[\frac{(a-b)^2}{\alpha^2}\frac{(f(t)rX^{r-1})^2}{f(t)r(r-1)X^{r-2}}\right] = 0.$$

This leads to

$$f'(t)X^r + brX[f(t)X^{r-1}] - \frac{1}{2}\frac{(a-b)^2}{\alpha^2}\frac{f^2(t)r^2X^{2r-2}}{f(t)r(r-1)X^{r-2}} = 0$$

or

$$f'(t)X^r + brf(t)X^r + f(t)X^r\frac{(a-b)^2r}{2\alpha^2(1-r)} = 0. \tag{3.10}$$

Dividing through by X^r, yields

$$f'(t) + f(t)br + f(t)\frac{(a-b)^2r}{2\alpha^2(1-r)} = 0$$

or

$$f'(t) + f(t)\left[br + \frac{(a-b)^2r}{2\alpha^2(1-r)}\right] = 0. \tag{3.11}$$

If we let

$$\lambda = br + \frac{(a-b)^2r}{2\alpha^2(1-r)},$$

then Eq.(3.11) can be written as the ODE

$$f'(t) + f(t)\lambda = 0,$$

or

$$f'(t) = -f(t)\lambda. \tag{3.12}$$

By integrating Eq.(3.12) with respect to the time variable t, the solution is

$$f(t) = e^{-\lambda t + \kappa},$$

or

$$f(t) = \tau e^{-\lambda t}. \tag{3.13}$$

By imposing the condition that $f(T) = 1$, we solve for τ in Eq.(3.13) as follows

$$\begin{aligned} f(T) = 1 \Rightarrow \tau e^{-\lambda T} &= 1 \\ \Rightarrow \tau &= e^{\lambda T}. \end{aligned} \tag{3.14}$$

Substitution of Eq.(3.14) into Eq.(3.13), we see that f has the form

$$\begin{aligned} f(t) &= e^{\lambda T} e^{-\lambda t} \\ &= e^{\lambda(T-t)}. \end{aligned} \tag{3.15}$$

By substituting Eq.(3.15) into Eq.(3.9) the value function becomes

$$W(t, X) = e^{\lambda(T-t)} X^r.$$

The second-order condition is also satisfied since

$$\begin{aligned} \alpha^2 X^2 W_{XX} &= \alpha^2 X^2 (e^{\lambda(T-t)}) r(r-1) X^{r-2} \\ &= r(r-1)\alpha^2 e^{\lambda(T-t)} X^r < 0. \end{aligned}$$

From Eq.(3.7) the optimal proportion of wealth invested in the risky asset can be written explicitly as

$$\begin{aligned} u &= -\frac{f(t)X^{r-1}}{Xf(t)(r-1)X^{r-2}} \frac{a-b}{\alpha^2} \\ &= \frac{a-b}{\alpha^2} \frac{1}{1-r}. \end{aligned} \tag{3.16}$$

Thus the optimal amount of the person's wealth to be invested in the risky asset is

$$Xu = X \frac{a-b}{\alpha^2} \frac{1}{1-r}. \tag{3.17}$$

Let us now present a numerical simulation based on the optimal solution. In particular, we simulate the optimized wealth, the optimal amounts of wealth invested in the risky asset and the riskless asset, as well as the optimal proportions of wealth invested in the risky asset and the riskless asset. We consider the parameter values

$$a = 0.2, b = 0.15, \alpha = 0.3, r = 0.13, T = 20,$$

and the initial condition $X(0) = 1$.

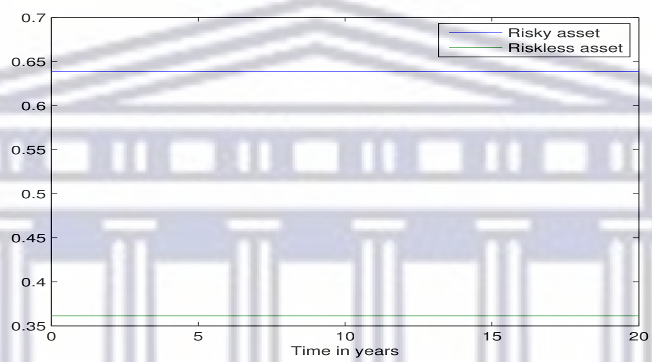

Figure 3.3: A simulation of the evolution of the optimal proportions of wealth invested in the risky asset, given by $u(t)$, and riskless asset, given by $1 - u(t)$.

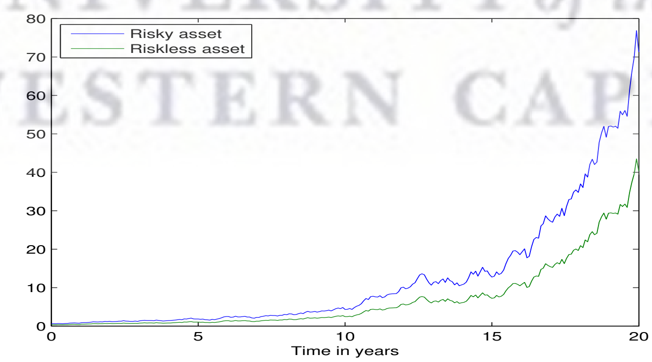

Figure 3.4: A simulation of the evolution of the optimal amounts of wealth invested in respectively, the risky asset given by, $X(t)u(t)$, and riskless asset given by, $X(t)(1 - u(t))$.

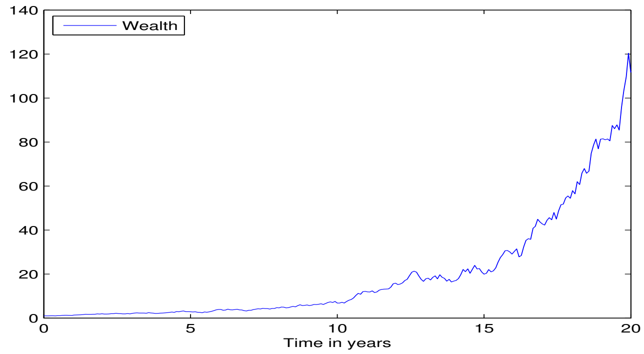

Figure 3.5: A simulation of the evolution of the wealth, $X(t)$, under the optimal investment strategy.

In Figure 3.3 we present a numerical simulation of the optimal investment strategy by illustrating the optimal proportions of wealth invested in the risky asset and riskless asset. We note that there is always a bigger proportion of wealth invested in the risky asset than in the riskless asset. Furthermore, these quantities remain constant over time. In Figure 3.4, we present a numerical simulation of the optimal amounts of wealth invested in the risky asset and the riskless asset. We observe that, for both the risky asset and the riskless asset the optimal amounts invested exhibit upward trends. In Figure 3.5 we present a numerical simulation of the investor's wealth under the optimal investment strategy. We observe that the wealth exhibits an upward trend as well.

3.3 Concepts from finance

The concepts we require from finance include the definitions below.

Definition 3.3.1. *(see [34, p.8])* **(Option)** *An option is a contract which gives the holder the right, but not obligation, to buy (call option) or sell (put option) an underlying asset by a certain date (expiration date) for a certain price (strike price).*

Definition 3.3.2. *(see [34, p.9])* **(European option)** *A European option is an option that can be exercised only on its expiration date.*

Definition 3.3.3. *(see [34, p.216])* **(Payoff of a European option)** *The payoffs (both long and short position) of European options with strike price K, expiration date T, and final price of the underlying asset $S(T)$, are as follows:*

1. *The payoff from a long position in a European call option is* $\max(S(T)-K,0)$, *which can be represented graphically as in Figure 3.6.*

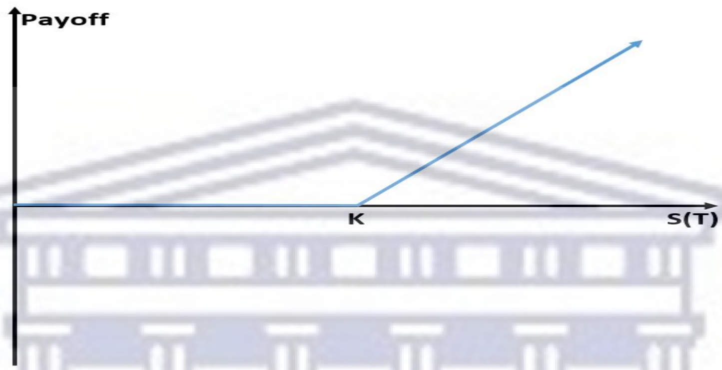

Figure 3.6: Payoff from a long position on a European call option.

2. *The payoff from a short position in a European call option is* $-\max(S(T)-K,0) = \min(K-S(T),0)$. *Graphically, this can be illustrated as in Figure 3.7.*

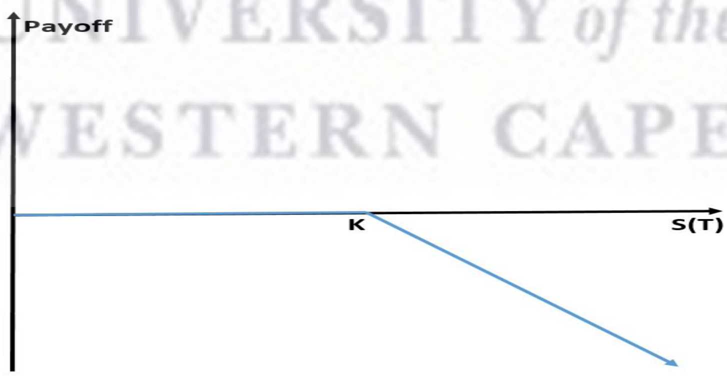

Figure 3.7: Payoff from a short position on a European call option.

3. *The payoff from a long position in a European put option is* $\max(K-S(T),0)$. *Its graph is given in Figure 3.8.*

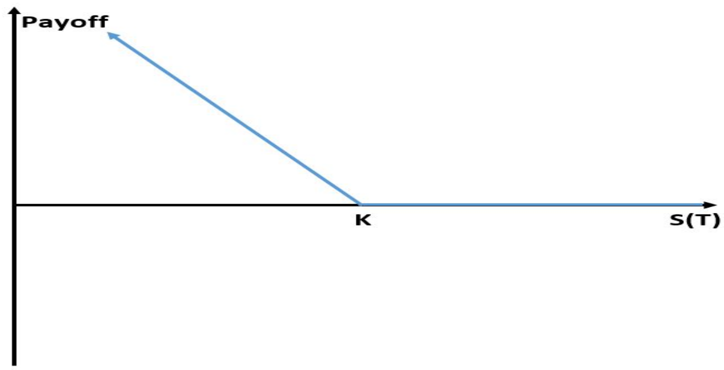

Figure 3.8: Payoff from a long position on a European put option.

4. The payoff from a short position in a European put option is $-\max(K - S(T), 0) = \min(S(T) - K, 0)$. *The payoff can be illustrated graphically as in Figure 3.9.*

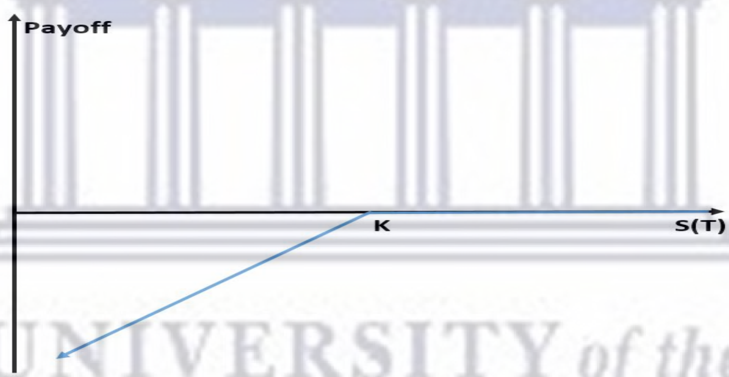

Figure 3.9: Payoff from a short position on a European put option.

Definition 3.3.4. *(see [34, p.278])* **(Risk-Neutral World)** *A world where investors are assumed to require no extra return on average for bearing risks.*

Definition 3.3.5. *(see [34, p.278])* **(Risk-Neutral Valuation)** *The valuation of an option or other derivative under the assumption that the world is risk-neutral. Risk-neutral valuation gives the correct price for a derivative in all worlds, not just in a risk-neutral world.*

Definition 3.3.6. *(see [34, p.310])* **(Monte Carlo Simulation)** *A Monte Carlo simulation of a stochastic process is a procedure for sampling random outcomes for the process. For each* *outcome, the payoff is computed and discounted at a risk-free rate. The average of the discounted payoffs is the estimate value of the process.*

Definition 3.3.7. *(see [34, p.241])* **(Put-Call Parity)** *Suppose that the price of a risky asset evolves according to the geometric Brownian motion*

$$\frac{dS(t)}{S(t)} = rdt + \sigma dB(t), \quad S(0) > 0.$$

Here r is the risk-free rate per annum and σ is the volatility of the asset price. If K denotes the strike price of the option at expiration date T, then

$$c + Ke^{-rT} = p + S(0)$$

is known as the put-call parity. It shows the value of a European call option, c, with a certain strike price and expiration date can be deduced from the value of a European put option, p, with the same strike price and expiration date.

Remark 3.3.8. *(see [34, p.335]) Consider a risky asset whose price evolves according to the geometric Brownian motion*

$$\frac{dS(t)}{S(t)} = rdt + \alpha dB(t), \quad S(0) > 0.$$

Here r is the continuously compounded risk-free rate and σ is the volatility in the asset price. If K denotes the strike price of the option at expiration date T, then by the Black-Scholes formula the price, $\hat{C}$, of a European call option written on the asset is

$$\hat{C} = S(0)N(d_1) - Ke^{-rT}N(d_2).$$

The price of a European put option, $\hat{P}$, on this asset is

$$\hat{P} = Ke^{-rT}N(-d_2) - S(0)N(-d_1),$$

where

$$d_1 = \frac{\ln(\frac{S(0)}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and $d_2 = d_1 - \sigma\sqrt{T}$. The function N is the cumulative probability distribution function for the standardized normal distribution.

We apply the above option pricing formulas in the Example 3.3.9 in order to determine the fair prices of European call and put options based on certain parameters values.

Example 3.3.9. *(see [34, p.338])* The stock price 6 months from expiration of an option is \$42, the strike price of the option \$40, the risk-free interest rate 10% per annum, and the volatility σ, is 0.2. Hence, $S(0) = 42, K = 40, r = 0.1, \sigma = 0.2$, and $T = 0.5$. If the option is a European call option, its value $\hat{C}$ is

$$\begin{aligned}
\hat{C} &= S(0)N(d_1) - Ke^{-rT}N(d_2) \\
&= 42N\left(\frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}}\right) - 40e^{-0.1\times 0.5}N\left(\frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}}\right) \\
&= 42 \times N(0.7693) - 38.049 \times N(0.6278) \\
&= 42 \times 0.7791 - 38.049 \times 0.7349 \\
&= \$4.76.
\end{aligned}$$

If the option is a European put option, its value $\hat{P}$ is given by

$$\begin{aligned}
\hat{P} &= Ke^{-rT}N(-d_2) - S(0)N(-d_1) \\
&= 40e^{-0.1\times 0.5}N\left(-\frac{\ln(42/40) + (0.1 - 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}}\right) - 42N\left(-\frac{\ln(42/40) + (0.1 + 0.2^2/2) \times 0.5}{0.2\sqrt{0.5}}\right) \\
&= 38.049 \times N(-0.6278) - 42 \times N(-0.7693) \\
&= 38.049 \times 0.2209 - 42 \times 0.2651 \\
&= \$0.81.
\end{aligned}$$

Chapter 4

The commercial banking model

In this chapter we introduce the general commercial banking model by breaking down and explaining each of the stylized balance sheet variables of commercial banks. We also provide a brief explanation for the off-balance sheet activities, as this information is needed to derive the NSFR. For the underlying bank of our study, we specify models, by means of differential equations, for its assets, liabilities and off-balance sheet activities. At the same time, we present numerical simulations in order to characterize the behaviour of the aforementioned items. In addition, we derive SDEs for our bank's asset portfolio and total liabilities which enables us to derive an SDE for the bank's capital. The final items we will derive are the SDEs for bank's CAR and NSFR.

The commercial bank underlying our study is assumed to trade in a complete and frictionless financial market that is continuously open over a fixed time interval $[0, T]$. It is assumed throughout that we are working with a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the real world probability measure. The Brownian motions W_B, W_D, W_O and W_i for $i = 1, 2$, appearing in the dynamics of the bank items from the underlying model, that we will introduce later in this chapter, are assumed to be defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$. The filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is generated by the aforementioned Brownian motions. It, of course, satisfies the usual conditions.

4.1 The stylized balance sheet of commercial banks

Generally speaking, to understand the management and operation of a commercial bank, we need to study its stylized balance sheet as well as its off-balance sheet activities. The stylized balance sheet records the assets (uses of funds) and liabilities (sources of funds) of the bank. Assets show how the bank used the funds it has attracted, while liabilities are non-owner claims on the bank's assets [32].

The stylized balance sheet, at time $t \geq 0$, can be described by the equation

$$R(t) + S(t) + L(t) = D(t) + B(t) + C(t), \tag{4.1}$$

where the variables R, S, L, D, B and C are the values of reserves, securities, loans, deposits, borrowings and capital, respectively. The aforementioned quantities are regarded as stochastic processes [54, 65, 50]. The off-balance sheet activities, which will be denoted by O, can also be regarded as a stochastic process [25]. Each bank item is discussed in more detail below.

Reserves refers to the portion of acquired funds that are held as deposits in an account at the central bank in the form of settlement balances. Reserves are those settlement balances, and also the currency that is physically held by banks (known as vault cash since it is stored in bank vaults overnight). Commercial banks are, however, not required to keep reserves in some proportion to their deposits. There is a requirement of zero settlement balances with the central bank at the end of each banking day. Commercial banks also hold a form of reserves known as desired reserves that enables them to meet potentially large and unpredictable withdrawals made by their clients [48].

Securities are any type of eligible debt instruments that are owned by banks. They can be of any maturity and are valued at market value (for securities that are available for sale) or at the price that the bank paid for them, plus or minus an amortized adjustment toward the maturity of the principle (for securities kept to maturity) [32]. Securities can be categorized into treasury securities (treasuries) and marketable securities. *Treasuries* are bonds issued by national treasuries in many countries as a way of borrowing money to meet government expenditures not covered by tax revenues. *Marketable securities*, on the other hand, are bonds and stocks that can be easily and quickly sold in the secondary market when a bank requires more cash. It is usually also referred to as secondary reserves [50, 52].

Loans are the banks' primary earning assets. Here funds are lent to a customer and the bank in return receives a promissory note from the customer who promises to pay interest, either at a variable or fixed rate and to pay back the principle balance of the loan. Loans are typically classified by the type of user and by use of the funds. The three main categories for most banks are commercial loans, consumer loans and real estate loans. A *commercial loan* is a short or intermediate term loan to a business usually for seasonal buildup of inventory, accounts receivable, or for permanent working capital or fixed assets. A *consumer loan* is used to finance personal expenditures and includes automobile loans, credit card loans, home improvement loans, other consumer durable loans, and other installment and single payment loans. A *real estate loan* is used to finance single and multifamily residence, construction and commercial real estate such as factories, office buildings and retail outlets [32].

Deposits are the funds that banks' customers place into a bank account. Deposits can be regarded as the main liabilities of a bank. There are two general types of bank deposits, namely, demand deposits and time deposits. A *demand deposit* is a noninterest-bearing transaction deposit that has no predetermined maturity date and which should be paid by banks when a negotiable instrument, usually in the form of a check or an electronic impulse, is presented. An example of a demand deposit is a checking account that is offered by a bank. A *time deposit* is an interest-bearing bank deposit account that has a predetermined maturity date. Time deposits that are withdrawn before a set date are usually subject to interest penalties. An example of a time account is a certificate of deposit or saving account that is offered by the bank [32].

Borrowings refer to funds borrowed from the central bank, other banks (also referred to as the interbank market), and corporations. Borrowings from the central bank are known as overdraft loans (also referred to as advances). Banks also borrow reserves overnight in the overnight market from other banks and financial institutions. They engage in this type of activity to ensure that they to have adequate settlement balances at the central bank to facilitate the clearing of cheques and other transfers. Other sources of borrowed funds include loan arrangements with corporations (such as repurchase agreements) and borrowings of reserves (deposits denominated in reserves residing in foreign banks or foreign branches of local banks) [48].

As stated earlier, bank capital is the difference between the total assets and total liabilities, which is known as the equity capital or net worth of banks' shareholders. Bank capital is funds raised by either selling new equity (stock) in the bank or that from retained earnings (profits). Bank capital is a cushion against a decrease in the value of its assets, which could force the bank into insolvency [48]. The dynamics of bank capital is stochastic in nature since it depends in part on the uncertainty with regards to debt and shareholder contributions. The banks, in theory, can decide on the rates at which they raise debt and equity [55, 50, 52]. According to [6], the regulatory bank capital is split into different tiers based on subordination and the ability to absorb losses with the clear difference of capital instruments when a bank is still solvent versus after it goes bankrupt. The more capital the bank has on hand, the better it can absorb losses on its assets before it becomes insolvent. Under Basel III, bank capital C has the form

$$C(t) = C_{T1}(t) + C_{T2}(t),$$

where $C_{T1}(t)$ and $C_{T2}(t)$ are Tier 1 and Tier 2, capital respectively [6, 60].

Tier 1 capital consists of retained earnings and shareholders' equity. It is the banks' primary source of funds, and is used to measure banks' financial health. Banks use Tier 1 capital to absorb losses without ceasing business operations. *Tier 2 capital* includes general loan loss reserves, hybrid capital instruments and subordinated term debt, and undisclosed reserves. Tier 2 capital is considered supplementary capital since it is less reliable than Tier 1 capital. Tier 2 capital is more difficult to accurately measure due to its composition of assets that are difficult to liquidate. These funds will often be categorized into either upper or lower pools depending on the characteristics of the individual assets [56].

Off-balance sheet activities are those assets and liabilities that do not appear on banks' balance sheets. There are two broad classes of off-balance sheet activities. The first involves the activities that create expenses or income without holding or creation of an underlying asset or liability. An example of this would be cases where banks acted as a broker (i.e., taking a fee for the agreement of funds to be provided to borrowers without making loans or raising deposits) instead as a dealer (i.e., creating and holding loans and the funding source). The other class of off-balance sheet activities consists of the banks' contingent claims and commitments. A *contingent claim* is an obligation by a bank to take action (i.e., to buy securities or lend funds) if the contingency is realized. A *commitment* means that the bank commits to some future action and gets a fee for doing so [32].

4.2 The models pertaining to the underlying commercial bank

Having explained the balance sheet items for the general commercial banking model, we can now introduce differential equations that describe the evolution of the dynamics of the balance sheet items and the off-balance sheet activities for the bank underlying our study. We simulate the evolution of the aforementioned items by representing one sample path of each. The simulation parameter values are similar to those of Muller [52]. We take the assumption that it is possible for the bank to continuously raise small amounts of capital at a rate $dK(t)/dt$ as was done by Muller and Witbooi [50] and Muller [52, 53]. Our bank invests this capital in a constant interest rate financial market consisting of three different assets.

The first of these is a treasury whose price is denoted by $A_1(t)$. We assume that its dynamics evolve according to the ODE

$$\frac{dA_1(t)}{A_1(t)} = rdt, \quad A_1(0) > 0, \tag{4.2}$$

where $r > 0$ is the continuously compounded interest rate.

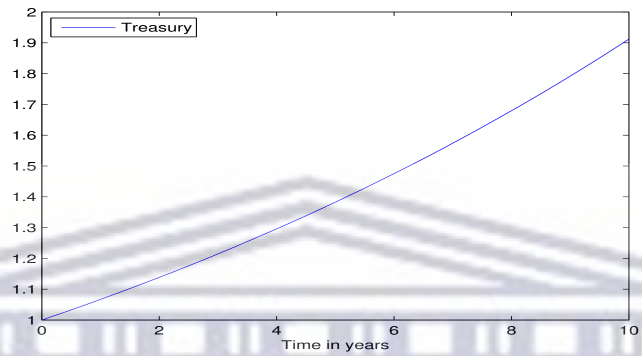

Figure 4.1: A simulation of the evolution of the price of the treasury security, $A_1(t)$, over a 10-year period with $r = 0.065$ and $A_1(0) = 1$.

The second asset is a marketable security whose price is denoted by $A_2(t)$. We assume that its dynamics evolve according to the SDE

$$\frac{dA_2(t)}{A_2(t)} = (r + m_1)dt + \sigma_1 dW_1(t), \quad A_2(0) > 0. \tag{4.3}$$

In the equation above, $m_1, \sigma_1 > 0$ and W_1 is a one-dimensional standard Brownian motion.

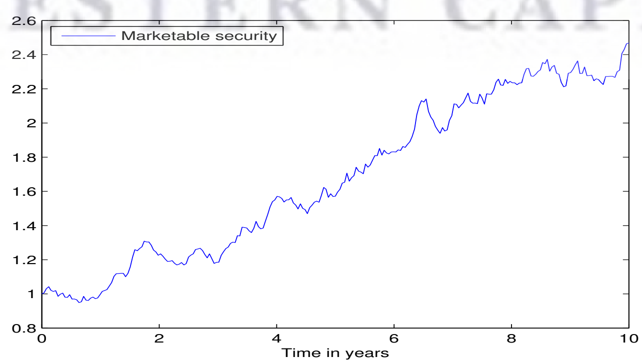

Figure 4.2: A simulation of the evolution of the price of the marketable security, $A_2(t)$, over a 10-year period with $r = 0.065, m_1 = 0.035, \sigma_1 = 0.08$ and $A_2(0) = 1$.

The third asset is a loan which is to be amortized over the interval of $[0, T]$. Its value at time t is denoted by $A_3(t)$ and we assume that its dynamics can be modelled by the SDE

$$\frac{dA_3(t)}{A_3(t)} = (r + m_2)dt + \sigma_2 dW_2(t), \quad A_3(0) > 0. \tag{4.4}$$

Here $m_2, \sigma_2 > 0$, but $m_2 > m_1$ and $\sigma_2 > \sigma_1$. The quantity W_2 is also one-dimensional standard Brownian motion.

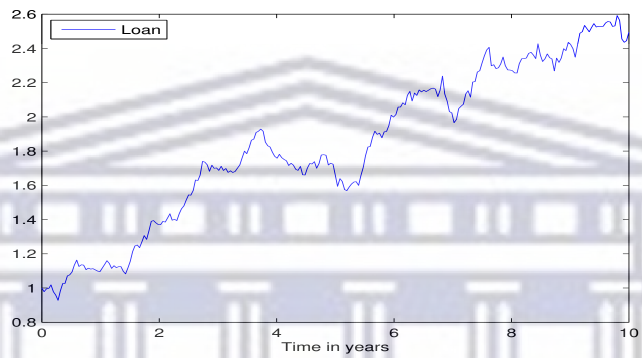

Figure 4.3: A simulation of the evolution of the value of the loan, $A_3(t)$, over a 10-year period with $r = 0.065, m_2 = 0.045, \sigma_2 = 0.095$ and $A_3(0) = 1$.

We assume that the rate of capital influx, $dK(t)/dt$, is $dK(t)/dt = c$, so that $dK(t) = cdt$, for $c > 0$.

To model the total asset value or asset portfolio of the bank, we follow the approach of Gao [26], Muller and Witbooi [50] and Muller [52]. That is, we let $A(t)$ denote the value of the bank's asset portfolio and describe its dynamics by

$$\begin{aligned}
dA(t) &= \sum_{i=1}^{3} \phi_i(t) \frac{dA_i(t)}{A_i(t)} + dK(t) \\
&= \phi_1(t)\frac{dA_1(t)}{A_1(t)} + \phi_2(t)\frac{dA_2(t)}{A_2(t)} + \phi_3(t)\frac{dA_3(t)}{A_3(t)} + dK(t) \\
&= \phi_1(t)rdt + \phi_2(t)[(r + m_1)dt + \sigma_1 dW_1(t)] + \phi_3(t)[(r + m_2)dt \ + \sigma_2 dW_2(t)] + cdt \\
&= \phi_1(t)rdt + \phi_2(t)(r + m_1)dt + \phi_2(t)\sigma_1 dW_1(t) + \phi_3(t)(r + m_2)dt + \phi_3(t)\sigma_2 dW_2(t) + cdt \\
&= [\phi_1(t)r + \phi_2(t)(r + m_1) + \phi_3(t)(r + m_2) + c]dt + \phi_2(t)\sigma_1 dW_1(t)
\end{aligned}$$

$$+ \quad \phi_3(t)\sigma_2 dW_2(t). \tag{4.5}$$

The quantities $\phi_1(t), \phi_2(t)$ and $\phi_3(t)$ denote the amounts of capital invested in the treasury, marketable security and loan, respectively. The optimal form of these quantities will be determined in Chapter 5 via a optimal capital allocation problem.

Next we introduce the liabilities of the underlying bank. We denote the value of the borrowings by $B(t)$, and assume that its dynamics evolve according to the SDE

$$dB(t) = \mu_B dt + \sigma_B dW_B(t), \quad B(0) > 0. \tag{4.6}$$

Here $\mu_B, \sigma_B > 0$ and W_B is also one-dimensional standard Brownian motion.

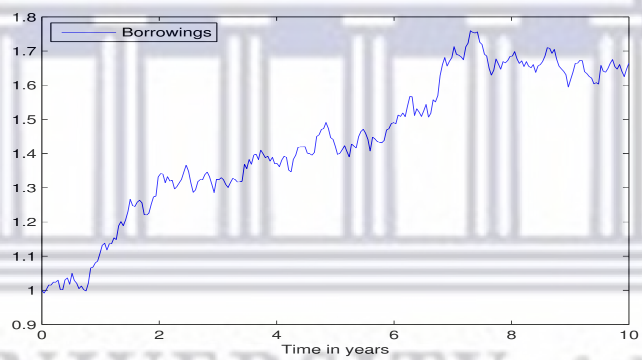

Figure 4.4: A simulation of the evolution of the value of the bank's borrowings, $B(t)$, over a 10-year period with $\mu_B = 0.07$, $\sigma_B = 0.14$ and $B(0) = 1$.

Similarly, we denote the value of the bank's deposits by $D(t)$ and assume that the value of deposits evolves according to the equation

$$dD(t) = \mu_D dt + \sigma_D dW_D(t), \quad D(0) > 0. \tag{4.7}$$

In the equation above, $\mu_D, \sigma_D > 0$ and W_D is also one-dimensional standard Brownian motion.

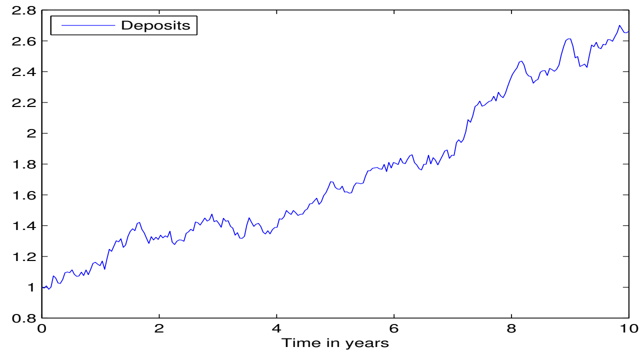

Figure 4.5: A simulation of the evolution of the value of the bank's deposits, $D(t)$, over a 10-year period with $\mu_D = 0.12$, $\sigma_D = 0.15$ and $D(0) = 1$.

Recall that by definition the bank's capital is

$$C(t) = A(t) - L(t).$$

Here $A(t)$ is the total assets and $L(t)$ the total liabilities. Thus, according to Muller [52], the SDE governing $C(t)$ can be obtained via

$$\begin{aligned} dC(t) &= d[A(t) - L(t)] \\ &= dA(t) - dL(t) \\ &= dA(t) - d[B(t) + D(t)] \\ &= dA(t) - dB(t) - dD(t), \end{aligned} \tag{4.8}$$

which is equivalent to:

$$\begin{aligned} dC(t) &= [\phi_1(t)r + \phi_2(t)(r + m_1) + \phi_3(t)(r + m_2) + c]dt + \phi_2(t)\sigma_1 dW_1(t) \\ &+ \phi_3(t)\sigma_2 dW_2(t) - [\mu_B dt + \sigma_B dW_B(t)] - [\mu_D dt + \sigma_D dW_D(t)] \\ &= [\phi_1(t)r + \phi_2(t)(r + m_1) + \phi_3(t)(r + m_2) + c]dt + \phi_2(t)\sigma_1 dW_1(t) \\ &+ \phi_3(t)\sigma_2 dW_2(t) - \mu_B dt - \sigma_B dW_B(t) - \mu_D dt - \sigma_D dW_D(t) \\ &= [\phi_1(t)r + \phi_2(t)(r + m_1) + \phi_3(t)(r + m_2) + c - \mu_B - \mu_D]dt + \phi_2(t)\sigma_1 dW_1(t) \\ &+ \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t). \end{aligned} \tag{4.9}$$

We denote the value of the bank's off-balance sheet activities by $O(t)$, and assume that it can be modelled by the SDE

$$dO(t) = \mu_O dt + \sigma_O dW_O(t), \quad O(0) > 0. \tag{4.10}$$

In Eq.(4.10), $\mu_O, \sigma_O > 0$ and W_O is a one-dimensional standard Brownian motion.

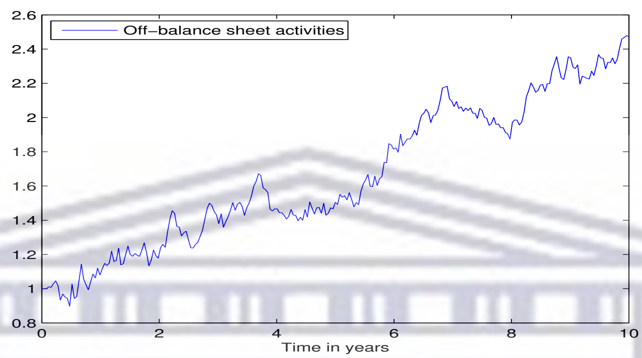

Figure 4.6: A simulation of the evolution of the value of the off-balance sheet activities, $O(t)$, over a 10-year period with $\mu_O = 0.11$, $\sigma_O = 0.2$ and $O(0) = 1$.

Having specified models for the stylized balance sheet items of the underlying bank, we can finally derive the SDEs for the CAR and NSFR.

4.3 Deriving the Capital Adequacy and Net Stable Funding ratios

This section is devoted to the derivation for SDEs of the underlying bank's CAR and NSFR. Before we can derive the dynamics of the CAR, we must first derive the dynamics of the bank's TRWAs. Similarly, in order to derive the dynamics of the NSFR, we need to first derive SDEs for the dynamics of the AASF and RASF. These derivations are presented in the remarks and propositions that follow.

In deriving the dynamics of the CAR, we refer to the Basel document [6] and follow the approach of Muller and Witbooi [50].

Remark 4.3.1. *We assume that the dynamics of the Total Risk-Weighted Assets of the bank evolve according to the SDE*

$$\begin{aligned}
dY(t) &= 0\times\phi_1(t)\frac{dA_1(t)}{A_1(t)} + 0.2\times\phi_2(t)\frac{dA_2(t)}{A_2(t)} + 0.5\times\phi_3(t)\frac{dA_3(t)}{A_3(t)} + dK(t)\\
&= 0.2\phi_2(t)[(r+m_1)dt + \sigma_1 dW_1(t)] + 0.5\phi_3(t)[(r+m_2)dt + \sigma_2 dW_2(t)] + cdt\\
&= [0.2\phi_2(t)(r+m_1) + 0.5\phi_3(t)(r+m_2) + c]dt + 0.2\phi_2(t)\sigma_1 dW_1(t)\\
&+ 0.5\phi_3(t)\sigma_2 dW_2(t), \qquad (4.11)
\end{aligned}$$

where $0, 0.2$ *and* 0.5 *are the risk-weights associated with the treasury, marketable security and loan respectively under the Basel III regime [6, 60, 50].*

Proposition 4.3.2. *With the dynamics of the total bank capital,* $C(t)$*, given by the SDE in Eq.(4.9) and with the dynamics of the Total Risk-Weighted Assets,* $Y(t)$*, given by the SDE in Eq.(4.11), we can write the dynamics of the Capital Adequacy Ratio,* $X_C(t)$*, as:*

$$\begin{aligned}
dX_C(t) &= \Bigg\{\Bigg[\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - \mu_D\Bigg]dt\\
&+ \phi_2(t)\sigma_1 dW_1(t) + \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t)\Bigg\}\frac{1}{Y(t)}\\
&+ C(t)\Bigg\{-\frac{1}{Y^2(t)}\Bigg\{\Bigg[0.2\phi_2(t)(r+m_1) + 0.5\phi_3(t)(r+m_2) + c\Bigg]dt\\
&+ 0.2\phi_2(t)\sigma_1 dW_1(t) + 0.5\phi_3(t)\sigma_2 dW_2(t)\Bigg\} + \frac{1}{Y^3(t)}\Bigg\{\Bigg[[0.2\phi_2(t)\sigma_1]^2\\
&+ [0.5\phi_3(t)\sigma_2]^2\Bigg]dt\Bigg\}\Bigg\} - \frac{1}{Y^2(t)}\Bigg\{\Bigg[0.2[\phi_2(t)\sigma_1]^2 + 0.5[\phi_3(t)\sigma_2]^2\Bigg]dt\Bigg\}. \qquad (4.12)
\end{aligned}$$

Proof. We derive Eq.(4.12) by mainly using Itô's Formula. We let $f(Y(t)) = 1/Y(t)$. Applying Itô's Formula to f yields

$$\begin{aligned}
df(Y(t)) &= \dot{f}(Y(t))dt + f'(Y(t))[dY(t)] + \frac{1}{2}f''(Y(t))[dY(t)]^2\\
&= 0dt - \frac{dY(t)}{Y^2(t)} + \frac{[dY(t)]^2}{Y^3(t)}\\
&= -\frac{1}{Y^2(t)}\Bigg\{\Bigg[0.2\phi_2(t)(r+m_1) + 0.5\phi_3(t)(r+m_2) + c\Bigg]dt + 0.2\phi_2(t)\sigma_1 dW_1(t)\\
&+ 0.5\phi_3(t)\sigma_2 dW_2(t)\Bigg\} + \frac{1}{Y^3(t)}\Bigg\{\Bigg[\Bigg[0.2\phi_2(t)(r+m_1) + 0.5\phi_3(t)(r+m_2) + c\Bigg]dt
\end{aligned}$$

$$+ \quad 0.2\phi_2(t)\sigma_1 dW_1(t) + 0.5\phi_3(t)\sigma_2 dW_2(t)\Bigg]^2\Bigg\}.$$

Since $dtdt = dtdW(t) = dW(t)dt = 0$ and $dW(t)dW(t) = dt$, where W is a standard Brownian motion that has independent increments, we obtain

$$\begin{aligned} df(Y(t)) &= -\frac{1}{Y^2(t)}\Bigg\{\Bigg[0.2\phi_2(t)(r+m_1) + 0.5\phi_3(t)(r+m_2) + c\Bigg]dt + 0.2\phi_2(t)\sigma_1 dW_1(t) \\ &+ 0.5\phi_3(t)\sigma_2 dW_2(t)\Bigg\} + \frac{1}{Y^3(t)}\Bigg\{\Bigg[[0.2\phi_2(t)\sigma_1]^2 + [0.5\phi_3(t)\sigma_2]^2\Bigg]dt\Bigg\}. \end{aligned}$$

If we let $X_C(t)$ denote the CAR at time t, then by definition,

$$X_C(t) = \frac{C(t)}{Y(t)} = C(t)f(Y(t)).$$

If we apply Itô's Product Rule to $X_C(t) = C(t)f(Y(t))$, we get:

$$\begin{aligned} dX_C(t) &= dC(t)f(Y(t)) + C(t)df(Y(t)) + dC(t)df(Y(t)) \\ &= \Bigg\{\Bigg[\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - \mu_D\Bigg]dt + \phi_2(t)\sigma_1 dW_1(t) \\ &+ \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t)\Bigg\}\frac{1}{Y(t)} \\ &+ C(t)\Bigg\{-\frac{1}{Y^2(t)}\Bigg\{\Bigg[0.2\phi_2(t)(r+m_1) + 0.5\phi_3(t)(r+m_2) + c\Bigg]dt + 0.2\phi_2(t)\sigma_1 dW_1(t) \\ &+ 0.5\phi_3(t)\sigma_2 dW_2(t)\Bigg\} + \frac{1}{Y^3(t)}\Bigg[[0.2\phi_2(t)\sigma_1]^2 + [0.5\phi_3(t)\sigma_2]^2\Bigg]dt\Bigg\} \\ &+ \Bigg\{\Bigg[\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - \mu_D\Bigg]dt + \phi_2(t)\sigma_1 dW_1(t) \\ &+ \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t)\Bigg\}\Bigg\{-\frac{1}{Y^2(t)}\Bigg\{\Bigg[0.2\phi_2(t)(r+m_1) \\ &+ 0.5\phi_3(t)(r+m_2) + c\Bigg]dt + 0.2\phi_2(t)\sigma_1 dW_1(t) + 0.5\phi_3(t)\sigma_2 dW_2(t)\Bigg\} \\ &+ \frac{1}{Y^3(t)}\Bigg\{\Bigg[[0.2\phi_2(t)\sigma_1]^2 + [0.5\phi_3(t)\sigma_2]^2\Bigg]dt\Bigg\}\Bigg\}. \end{aligned}$$

This simplifies to

$$\begin{aligned} dX_C(t) &= \Bigg\{\Bigg[\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - \mu_D\Bigg]dt \\ &+ \phi_2(t)\sigma_1 dW_1(t) + \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t)\Bigg\}\frac{1}{Y(t)} \end{aligned}$$

$$
\begin{aligned}
&+ \quad C(t)\Bigg\{-\frac{1}{Y^2(t)}\Bigg\{\Bigg[0.2\phi_2(t)(r+m_1)+0.5\phi_3(t)(r+m_2)+c\Bigg]dt \\
&+ \quad 0.2\phi_2(t)\sigma_1 dW_1(t)+0.5\phi_3(t)\sigma_2 dW_2(t)\Bigg\}+\frac{1}{Y^3(t)}\Bigg\{\Bigg[[0.2\phi_2(t)\sigma_1]^2 \\
&+ \quad [0.5\phi_3(t)\sigma_2]^2\Bigg]dt\Bigg\}\Bigg\}-\frac{1}{Y^2(t)}\Bigg\{\Bigg[0.2[\phi_2(t)\sigma_1]^2+0.5[\phi_3(t)\sigma_2]^2\Bigg]dt\Bigg\},
\end{aligned}
$$

which concludes the proof. □

The derivation of the NSFR is similar to that of the CAR. Before we can derive it, we first have to derive the dynamics of the AASF and RASF for underlying bank. In deriving the dynamics of the NSFR, we refer to the Basel document [8], and follow the approach of Muller [51].

Remark 4.3.3. *We take the assumption that the dynamics of the Available Amount of Stable Funding evolve according to the SDE*

$$
\begin{aligned}
dF_A(t) &= 1.0 \times dC(t) + 0.95 \times dD(t) \\
&= [\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - \mu_D]dt + \phi_2(t)\sigma_1 dW_1(t) \\
&+ \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t) + 0.95[\mu_D dt + \sigma_D dW_D(t)] \\
&= [\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - \mu_D]dt + \phi_2(t)\sigma_1 dW_1(t) \\
&+ \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t) + 0.95\mu_D dt + 0.95\sigma_D dW_D(t) \\
&= [\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - 0.05\mu_D]dt + \phi_2(t)\sigma_1 dW_1(t) \\
&+ \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - 0.05\sigma_D dW_D(t), \qquad (4.13)
\end{aligned}
$$

where 1.0 *and* 0.95 *are the Available Stable Funding factors associated with the total capital and stable deposits in the Basel III Accord [8].*

Remark 4.3.4. *We assume the dynamics of the Required Amount of Stable Funding evolve according to the SDE*

$$
\begin{aligned}
dF_R(t) &= 0.05 \times \phi_1(t)\frac{dA_1(t)}{A_1(t)} + 0.15 \times \phi_2(t)\frac{dA_2(t)}{A_2(t)} + 0.85 \times \phi_3(t)\frac{dA_3(t)}{A_3(t)} + 0.05 \times dO(t) \\
&+ dK(t) \\
&= 0.05\phi_1(t)rdt + 0.15\phi_2(t)[(r+m_1)dt + \sigma_1 dW_1(t)] + 0.85\phi_3(t)[(r+m_2)dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \sigma_2 dW_2(t)] + 0.05[\mu_O dt + \sigma_O dW_O(t)] + cdt \\
&= \quad [0.05\phi_1(t)r + 0.15\phi_2(t)(r+m_1) + 0.85\phi_3(t)(r+m_2) + 0.05\mu_O + c]dt \\
&+ \quad 0.15\phi_2(t)\sigma_1 dW_1(t) + 0.85\phi_3(t)\sigma_2 dW_2(t) + 0.05\sigma_O dW_O(t), \qquad (4.14)
\end{aligned}
$$

where the weights $0.05, 0.15,$ *and* 0.85 *are the Required Stable Funding factors associated with, respectively, the treasury, off-balance sheet activities, marketable security and loan under Basel III [8].*

Proposition 4.3.5. *With the dynamics of the Available Amount of Stable Funding,* $F_A(t)$*, given by the SDE in Eq.(4.13), and with the dynamics of the Required Amount of Stable Funding,* $F_R(t)$*, given by the SDE in Eq.(4.14), the dynamics of the Net Stable Funding Ratio,* $X_N(t)$*, can be written as:*

$$
\begin{aligned}
dX_N(t) &= \frac{1}{F_R(t)}\Bigg\{\Bigg[\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - 0.05\mu_D\Bigg]dt \\
&+ \phi_2(t)\sigma_1 dW_1(t) + \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - 0.05\sigma_D dW_D(t)\Bigg\} \\
&+ F_A(t)\Bigg\{-\frac{1}{F_R^2(t)}\Bigg\{\Bigg[0.05\phi_1(t)r + 0.15\phi_2(t)(r+m_1) + 0.85\phi_3(t)(r+m_2) \\
&+ 0.05\mu_O + c\Bigg]dt + 0.15\phi_2(t)\sigma_1 dW_1(t) + 0.85\phi_3(t)\sigma_2 dW_2(t) + 0.05\sigma_O dW_O(t)\Bigg\} \\
&+ \frac{1}{F_R^3(t)}\Bigg\{\Bigg[[0.15\phi_2(t)\sigma_1]^2 + [0.85\phi_3(t)\sigma_2]^2 + [0.05\sigma_O]^2\Bigg]dt\Bigg\}\Bigg\} \\
&- \frac{1}{F_R^2(t)}\Bigg\{\Bigg[0.15[\phi_2(t)\sigma_1]^2 + 0.85[\phi_3(t)\sigma_2]^2\Bigg]dt\Bigg\}. \qquad (4.15)
\end{aligned}
$$

Proof. In deriving Eq.(4.15) we employ Itô's formula and Product Rule. We let $f(F_R(t)) = 1/F_R(t)$. Then, by Itô's Lemma, we have

$$
\begin{aligned}
df(F_R(t)) &= \dot{f}(F_R(t))dt + f'(F_R(t))[dF_R(t)] + \frac{1}{2}f''(F_R(t))[dF_R(t))]^2 \\
&= 0dt - \frac{dF_R(t)}{F_R^2(t)} + \frac{[dF_R(t)]^2}{F_R^3(t)} \\
&= -\frac{1}{F_R^2(t)}\Bigg\{\Bigg[0.05\phi_1(t)r + 0.15\phi_2(t)(r+m_1) + 0.85\phi_3(t)(r+m_2) + 0.05\mu_O \\
&+ c\Bigg]dt + 0.15\phi_2(t)\sigma_1 dW_1(t) + 0.85\phi_3(t)\sigma_2 dW_2(t) + 0.05\sigma_O d_O W(t)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{F_R^3(t)}\Bigg\{\Bigg[\Bigg[0.05\phi_1(t)r + 0.15\phi_2(t)(r+m_1) + 0.85\phi_3(t)(r+m_2) + 0.05\mu_O \\
&+ \quad c\Bigg]dt + 0.15\phi_2(t)\sigma_1 dW_1(t) + 0.85\phi_3(t)\sigma_2 dW_2(t) + 0.05\sigma_O dW_O(t)\Bigg]^2\Bigg\},
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
df(F_R(t)) \quad &= \quad -\frac{1}{F_R^2(t)}\Bigg\{\Bigg[0.05\phi_1(t)r + 0.15\phi_2(t)(r+m_1) + 0.85\phi_3(t)(r+m_2) + 0.05\mu_O \\
&+ \quad c\Bigg]dt + 0.15\phi_2(t)\sigma_1 dW_1(t) + 0.85\phi_3(t)\sigma_2 dW_2(t) + 0.05\sigma_O dW_O(t)\Bigg\} \\
&+ \quad \frac{1}{F_R^3(t)}\Bigg\{\Bigg[[0.15\phi_2(t)\sigma_1]^2 + [0.85\phi_3(t)\sigma_2]^2 + [0.05\sigma_O]^2\Bigg]dt\Bigg\}.
\end{aligned}
$$

Suppose now that we let $X_N(t)$ denote the NSFR at time $t \geq 0$. Then according to the definition of the NSFR, we can write

$$
X_N(t) = \frac{F_A(t)}{F_R(t)} = F_A(t)f(F_R(t)).
$$

Applying Itô's Product Rule to the above expression yields:

$$
\begin{aligned}
dX_N(t) \quad &= \quad f(F_R(t))dF_A(t) + F_A(t)df(F_R(t)) + dF_A(t)df(F_R(t)) \\
&= \quad \frac{1}{F_R(t)}\Bigg\{\Bigg[\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - 0.05\mu_D\Bigg]dt \\
&+ \quad \phi_2(t)\sigma_1 dW_1(t) + \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - 0.05\sigma_D dW_D(t)\Bigg\} \\
&+ \quad F_A(t)\Bigg\{-\frac{1}{F_R^2(t)}\Bigg\{\Bigg[0.05\phi_1(t)r + 0.15\phi_2(t)(r+m_1) + 0.85\phi_3(t)(r+m_2) \\
&+ \quad 0.05\mu_O + c\Bigg]dt + 0.15\phi_2(t)\sigma_1 dW_1(t) + 0.85\phi_3(t)\sigma_2 dW_2(t) + 0.05\sigma_O dW_O(t)\Bigg\} \\
&+ \quad \frac{1}{F_R^3(t)}\Bigg\{\Bigg[[0.15\phi_2(t)\sigma_1]^2 + [0.85\phi_3(t)\sigma_2]^2 + [0.05\sigma_O]^2\Bigg]dt\Bigg\}\Bigg\} \\
&+ \quad \Bigg\{\Bigg[\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - 0.05\mu_D\Bigg]dt + \phi_2(t)\sigma_1 dW_1(t) \\
&+ \quad \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - 0.05\sigma_D dW_D(t)\Bigg\}\Bigg\{-\frac{1}{F_R^2(t)}\Bigg\{\Bigg[0.05\phi_1(t)r \\
&+ \quad 0.15\phi_2(t)(r+m_1) + 0.85\phi_3(t)(r+m_2) + 0.05\mu_O + c\Bigg]dt + 0.15\phi_2(t)\sigma_1 dW_1(t) \\
&+ \quad 0.85\phi_3(t)\sigma_2 dW_2(t) + 0.05\sigma_O dW_O(t)\Bigg\}
\end{aligned}
$$

$$+ \quad \frac{1}{F_R^3(t)}\left\{\left[[0.15\phi_2(t)\sigma_1]^2 + [0.85\phi_3(t)\sigma_2]^2 + [0.05\sigma_O]^2\right]dt\right\}\Bigg\},$$

or

$$\begin{aligned}
dX_N(t) &= \frac{1}{F_R(t)}\left\{\left[\phi_1(t)r + \phi_2(t)(r+m_1) + \phi_3(t)(r+m_2) + c - \mu_B - 0.05\mu_D\right]dt\right. \\
&+ \left.\phi_2(t)\sigma_1 dW_1(t) + \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - 0.05\sigma_D dW_D(t)\right\} \\
&+ F_A(t)\Bigg\{ - \frac{1}{F_R^2(t)}\Bigg\{\Bigg[0.05\phi_1(t)r + 0.15\phi_2(t)(r+m_1) + 0.85\phi_3(t)(r+m_2) \\
&+ 0.05\mu_O + c\Bigg]dt + 0.15\phi_2(t)\sigma_1 dW_1(t) + 0.85\phi_3(t)\sigma_2 dW_2(t) + 0.05\sigma_O dW_O(t)\Bigg\} \\
&+ \frac{1}{F_R^3(t)}\left\{\left[[0.15\phi_2(t)\sigma_1]^2 + [0.85\phi_3(t)\sigma_2]^2 + [0.05\sigma_O]^2\right]dt\right\}\Bigg\} \\
&- \frac{1}{F_R^2(t)}\left\{\left[0.15[\phi_2(t)\sigma_1]^2 + 0.85[\phi_3(t)\sigma_2]^2\right]dt\right\},
\end{aligned}$$

and the proof is complete. □

Chapter 5

The optimal control problem

We now present the optimal control problem and derive its solution. That is, we determine the investment strategy that maximizes an expected exponential utility of the bank's capital at a future time $T > 0$. We also present a numerical simulation study to characterize the behaviour of the optimal investment strategy by illustrating the optimal proportions of the capital invested in the assets graphically. In addition, we simulate the behaviours of the asset portfolio and optimized bank capital, and observe the levels of the CAR and NSFR under the optimal investment strategy.

In solving the optimal control problem we refer to the references Devolder et al. [19] and Muller [52]. The optimal control problem is presented below.

Problem 5.1. *The objective is to maximize the expected utility of the bank capital, C, at a future date $T > 0$. That is,*

$$\max_{\phi_2,\phi_3} \mathbb{E}[U(C(T))],$$

with the dynamics of bank's capital, $C(t)$, given by the SDE

$$\begin{aligned} dC(t) &= [\phi_1(t)r + \phi_2(t)(r + m_1) + \phi_3(t)(r + m_2) + c - \mu_B - \mu_D]dt + \phi_2(t)\sigma_1 dW_1(t) \\ &+ \phi_3(t)\sigma_2 dW_2(t) - \sigma_B dW_B(t) - \sigma_D dW_D(t). \end{aligned}$$

Here $0 \leq t \leq T$ *and* $C(0)$ *denotes the initial value of the capital, assumed to be a positive constant.*

The value function of the control problem is

$$H(t, C) = \sup_{\phi_2, \phi_3} \mathbb{E}[U(C(T))|C(t) = C],$$

where $0 < t < T$. The marginal utility of the value function is a constant, and the marginal utility of the original utility function $U(\cdot)$ decreases to zero as $C \to \infty$ [42]. The value function inherits the convexity of the utility function and is strictly convex for $t < T$, even if $U(\cdot)$ is not [37]. Our choice of utility function that satisfies the above conditions allows us to obtain explicit solutions for the optimal strategies. Since it measures a bank's preference towards risk, the choice was made carefully. However, in case one of these conditions are not met, we need then to consider a different methodology to obtain explicit solutions, if it exists.

The Hamilton-Jacobi-Bellman (HJB) equation arising from the maximum principle is

$$\begin{aligned} H_t &+ \max_{\phi_2, \phi_3} \Bigg\{ \Big[\phi_1 r + \phi_2 (r + m_1) + \phi_3 (r + m_2) + c - \mu_B - \mu_D \Big] H_C \\ &+ \frac{1}{2} \Big[(\phi_2 \sigma_1)^2 + (\phi_3 \sigma_2)^2 + \sigma_B^2 + \sigma_D^2 \Big] H_{CC} \Bigg\} = 0, \end{aligned} \tag{5.1}$$

where the time variable t has been suppressed and the variables H_t, H_C and H_{CC} denote the first and second order partial derivatives with respect to time and bank capital.

We first derive differentiate with respect to ϕ_2 in Eq.(5.1), which yields

$$(r + m_1) H_C + \phi_2 \sigma_1^2 H_{CC} = 0.$$

Similarly, differentiating with respect to ϕ_3 gives

$$(r + m_2) H_C + \phi_3 \sigma_2^2 H_{CC} = 0.$$

We therefore obtain the first-order maximizing conditions for the optimal investment strategies ϕ_2 and ϕ_3 in, respectively, the marketable security and loan as

$$\phi_2 = -\frac{(r + m_1) H_C}{\sigma_1^2 H_{CC}} \tag{5.2}$$

and

$$\phi_3 = -\frac{(r+m_2)H_C}{\sigma_2^2 H_{CC}}. \tag{5.3}$$

By substituting Eq.(5.2) and Eq.(5.3) into Eq.(5.1), we obtain the following PDE for the value function:

$$\begin{aligned} H_t &+ \left[\phi_1 r - \frac{(r+m_1)^2 H_C}{\sigma_1^2 H_{CC}} - \frac{(r+m_2)^2 H_C}{\sigma_2^2 H_{CC}} + c - \mu_B - \mu_D\right] H_C \\ &+ \frac{1}{2}\left[\frac{(r+m_1)^2 H_C^2}{\sigma_1^2 H_{CC}^2} + \frac{(r+m_2)^2 H_C^2}{\sigma_2^2 H_{CC}^2} + \sigma_B^2 + \sigma_D^2\right] H_{CC} = 0 \end{aligned} \tag{5.4}$$

The problem is now to solve the PDE in Eq.(5.4) for the value function H and then substituting it back into Eq.(5.2) and Eq.(5.3) to obtain the optimal investment strategies ϕ_2 and ϕ_3. The PDE in Eq.(5.4) admits an explicit solution for the utility function of the form

$$U(C) = -\frac{1}{g}e^{-gC},$$

where $g > 0$ is a positive constant for which

$$-\frac{U''(C)}{U'(C)} = g$$

(see [19] and [52] for instance). We try to find an explicit solution for the PDE in Eq.(5.4) with the structure

$$H(t, C) = -\frac{1}{g}e^{-gC+b(t)} \tag{5.5}$$

with $b(T) = 1$. Computing the partial derivatives for Eq.(5.5), we obtain:

$$\begin{aligned} H_t &= -\frac{b'(t)}{g}e^{-gC+b(t)} \\ H_C &= e^{-gC+b(t)} \\ H_{CC} &= -ge^{-gC+b(t)} \end{aligned}$$

Substitution of the above derivatives into Eq.(5.4) yields

$$- \frac{b'(t)}{g}e^{-gC+b(t)} + \left[\phi_1 r - \frac{(r+m_1)^2 e^{-gC+b(t)}}{\sigma_1^2(-ge^{-gC+b(t)})} - \frac{(r+m_2)^2 e^{-gC+b(t)}}{\sigma_2^2(-ge^{-gC+b(t)})} + c - \mu_B\right.$$

$$
\begin{aligned}
- \quad \mu_D\Big] e^{-gC+b(t)} + \frac{1}{2}\Bigg[\frac{(r+m_1)^2(e^{-gC+b(t)})^2}{\sigma_1^2(-ge^{-gC+b(t)})^2} + \frac{(r+m_2)^2(e^{-gC+b(t)})^2}{\sigma_2^2(-ge^{-gC+b(t)})^2} + \sigma_B^2 \\
+ \quad \sigma_D^2\Bigg](-ge^{-gC+b(t)}) = 0,
\end{aligned}
$$

or

$$
\begin{aligned}
- \quad & \frac{b'(t)}{g}e^{-gC+b(t)} + \Bigg[\phi_1 r + \frac{(r+m_1)^2}{\sigma_1^2 g} + \frac{(r+m_2)^2}{\sigma_2^2 g} + c - \mu_B - \mu_D\Bigg]e^{-gC+b(t)} \\
- \quad & \frac{1}{2}\Bigg[\frac{(r+m_1)^2}{\sigma_1^2 g^2} + \frac{(r+m_2)^2}{\sigma_2^2 g^2} + \sigma_B^2 + \sigma_D^2\Bigg]ge^{-gC+b(t)} = 0. \qquad (5.6)
\end{aligned}
$$

By simultaneously multiplying by $-g$ and dividing by $e^{-gC+b(t)}$ above, we obtain

$$
\begin{aligned}
b'(t) \quad - \quad & g\Bigg[\phi_1 r + \frac{(r+m_1)^2}{\sigma_1^2 g} + \frac{(r+m_2)^2}{\sigma_2^2 g} + c - \mu_B - \mu_D\Bigg] \\
+ \quad & \frac{1}{2}g^2\Bigg[\frac{(r+m_1)^2}{\sigma_1^2 g^2} + \frac{(r+m_2)^2}{\sigma_2^2 g^2} + \sigma_B^2 + \sigma_D^2\Bigg] = 0. \qquad (5.7)
\end{aligned}
$$

Hence if we let

$$
\begin{aligned}
\lambda \quad = \quad & g\Bigg[\phi_1 r + \frac{(r+m_1)^2}{\sigma_1^2 g} + \frac{(r+m_2)^2}{\sigma_2^2 g} + c - \mu_B - \mu_D\Bigg] \\
- \quad & \frac{1}{2}g^2\Bigg[\frac{(r+m_1)^2}{\sigma_1^2 g^2} + \frac{(r+m_2)^2}{\sigma_2^2 g^2} + \sigma_B^2 + \sigma_D^2\Bigg],
\end{aligned}
$$

then Eq.(5.7) takes the form

$$
b'(t) - \lambda = 0, \qquad (5.8)
$$

which is a separable ODE.

We proceed to solve Eq.(5.8) as follows for b:

$$
b'(t) - \lambda = 0
$$

or

$$
b'(t) = \lambda. \qquad (5.9)
$$

By integrating both sides of Eq.(5.9) we find that the solution of Eq.(5.9) is

$$
b(t) = \lambda t + \tau. \qquad (5.10)
$$

By imposing the condition $b(T) = 1$, we can solve for τ in Eq.(5.10) as follows:

$$\begin{aligned} b(T) = 1 \Rightarrow \lambda T + \tau &= 1 \\ \Rightarrow \tau &= 1 - \lambda T. \end{aligned} \tag{5.11}$$

Substitution of Eq.(5.11) into Eq.(5.10), yields

$$\begin{aligned} b(t) &= \lambda t + 1 - \lambda T \\ &= \lambda(t - T) + 1. \end{aligned} \tag{5.12}$$

By substituting Eq.(5.12) into Eq.(5.5), the value function takes the form

$$H(t, C) = -\frac{1}{g} e^{-gC + \lambda(t-T)+1}.$$

We note that the second-order conditions are satisfied as

$$\begin{aligned} \sigma_1^2 H_{CC} &= \sigma_1^2(-ge^{-gC+\lambda(t-T)+1}) \\ &= -g\sigma_1^2 e^{-gC+\lambda(t-T)+1} < 0, \end{aligned}$$

and

$$\begin{aligned} \sigma_2^2 H_{CC} &= \sigma_2^2(-ge^{-gC+\lambda(t-T)+1}) \\ &= -g\sigma_2^2 e^{-gC+\lambda(t-T)+1} < 0. \end{aligned}$$

From Eq.(5.2) and Eq.(5.3) we can derive the optimal amounts of capital for investment in the marketable security and loan, respectively, as

$$\begin{aligned} \phi_2 &= -\frac{(r + m_1)e^{-gC+b(t)}}{\sigma_1^2(-ge^{-gC+b(t)})} \\ &= \frac{(r + m_1)}{\sigma_1^2 g}, \end{aligned}$$

and

$$\begin{aligned} \phi_3 &= -\frac{(r + m_2)e^{-gC+b(t)}}{\sigma_2^2(-ge^{-gC+b(t)})} \\ &= \frac{(r + m_2)}{\sigma_2^2 g}. \end{aligned}$$

The optimal amount of capital to invest in the treasury is therefore

$$\phi_1 = A - \phi_2 - \phi_3 = A - \frac{(r+m_1)}{\sigma_1^2 g} - \frac{(r+m_2)}{\sigma_2^2 g}.$$

The optimal proportions of capital for investment in the marketable security, loan and treasury, are, respectively:

$$\begin{aligned} f_2 &= \frac{\phi_2}{A} = \frac{(r+m_1)}{\sigma_1^2 g A}, \\ f_3 &= \frac{\phi_3}{A} = \frac{(r+m_2)}{\sigma_2^2 g A}, \\ f_1 &= \frac{\phi_1}{A} = 1 - \frac{(r+m_1)}{\sigma_1^2 g A} - \frac{(r+m_2)}{\sigma_2^2 g A} = 1 - f_2 - f_3. \end{aligned}$$

We now provide a simulation study to characterize the behaviour of the optimal investment strategy, as well as the behaviour of the asset portfolio and bank capital under the optimal investment strategy. We consider an investment horizon of $T = 10$ years and assume that $c = 0.0145$. The other parameter values considered in the simulation study are

$$\begin{aligned} r &= 0.065, m_1 = 0.035, \sigma_1 = 0.08, m_2 = 0.045, \sigma_2 = 0.095, \mu_B = 0.07, \sigma_B = 0.14, \mu_D = 0.12, \\ \sigma_D &= 0.15, g = 15, \end{aligned}$$

with the initial conditions being

$$A(0) = 2, C(0) = 0.4, B(0) = 0.7, \text{ and } D(0) = 0.9.$$

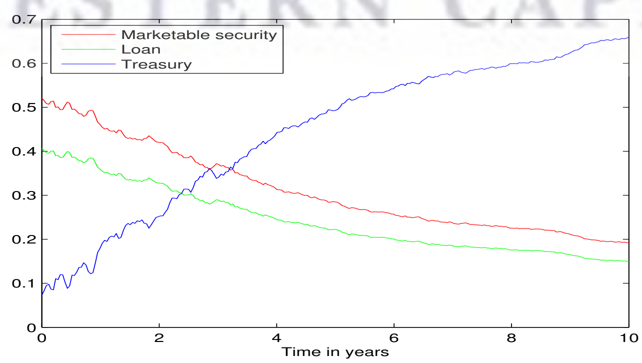

Figure 5.1: A simulation of the evolution of the optimal proportions $f_2(t), f_3(t)$, and $f_1(t)$ of capital invested, respectively, in the marketable security, loan and treasury.

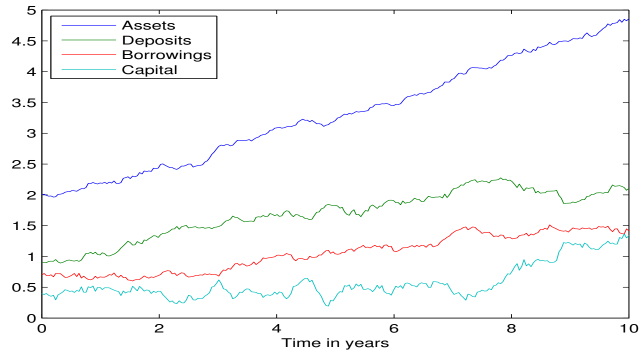

Figure 5.2: A simulation of the evolution of the bank capital $C(t)$ and asset portfolio $A(t)$ under the optimal investment strategy, together with the borrowings $B(t)$ and deposits $D(t)$.

In Figure 5.1 we present a numerical simulation of the evolution of the optimal investment strategy by way of the optimal proportions of capital invested in the treasury, marketable security and loan. This simulation is performed over a period of 10 years. We consider parameter values similar to those of Muller [52]. From Figure 5.1 it can be observed that the bank initially invests most of its capital in the marketable security and loan. Over time, the amounts of capital invested in these two assets decrease, but the amount of capital invested in the treasury increases. Thus, the optimal investment strategy is to diversify the bank's asset portfolio away from the risky assets (the marketable security and loan), and towards the riskless treasury. This is consistent with Witbooi et al. [65], Muller and Witbooi [50], Muller [52, 53], Danjuma et al. [17] and Danjuma [18]. In Witbooi et al. [65], the proportions of capital invested in the marketable security stays constant over time, while the proportion invested in the loan progressively decreases over time. On the other hand, the optimal proportion invested in the treasury increases in [65]. In Muller and Witbooi [50], Muller [52, 53] and Danjuma et al. [17] and Danjuma [18], the proportions of capital invested in the marketable security and loan decrease, while the proportions invested in the treasuries increase. Chakroun and Abid [12] found that the optimal proportions invested in the securities and loans increases over time, while the riskless bank account decreases over time. This is because at the beginning of the investment horizon, the need for a conservative investment strategy for generating an increase in wealth and a lower risk results in a higher proportion being invested in the bank account, while the investment in securities and loans are low. Over time, there is a shift from the investment in the bank account to the securities and loans. This investment strategy's riskiness leads to a higher investment in the securities and loans, and lower investments in the riskless bank account. Hence, the bank in [12] also maintains a diversified portfolio. In Figure 5.2 we present a numerical simulation of the evolution of the bank capital and asset portfolio under the optimal investment strategy. We also simulate the borrowings and deposits. We observe that the bank capital, asset portfolio, borrowings and deposits all exhibit upward trends. Muller [52, 53], made a similar observation. The optimized asset portfolio of [50] also showed an upward trend.

Let us now also characterize the evolution of the CAR and NSFR. The parameter values used in these simulations are the same as those used to characterize the optimal investment strategy. In addition, we consider the parameter values

$$\mu_O = 0.11 \text{ and } \sigma_O = 0.2$$

with the initial conditions

$$Y(0) = 1.7, X_C(0) = 0.08, F_A(0) = 2.50, F_R(0) = 2.50, \text{ and } X_N(0) = 1.$$

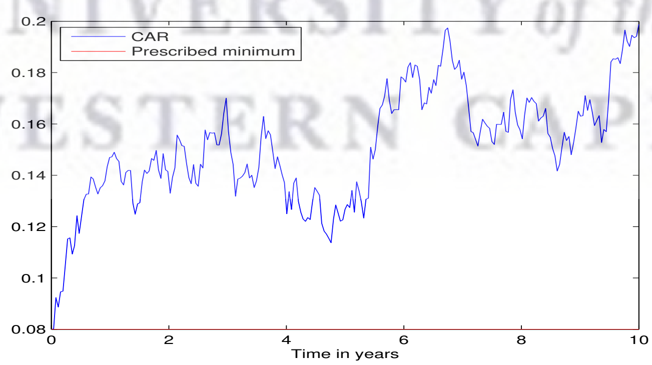

Figure 5.3: A simulation of the evolution of the capital adequacy ratio, $X_C(t)$, under the optimal investment strategy.

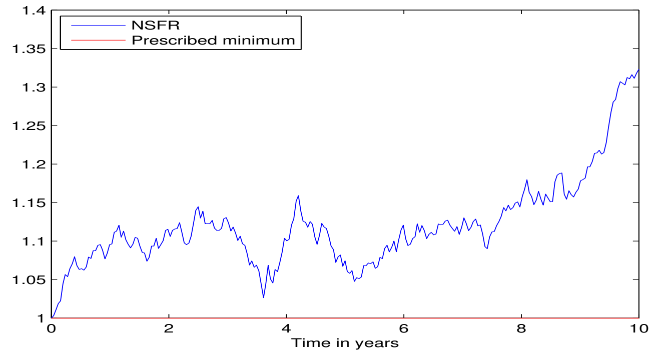

Figure 5.4: A simulation of the evolution of the net stable funding ratio, $X_N(t)$, under the optimal investment strategy.

In Figure 5.3 we present a numerical simulation of the evolution of the CAR, which we derived in Chapter 4. By following the optimal investment strategy depicted in Figure 5.1, and for the parameters considered, the underlying bank maintains its CAR above the minimum Basel III prescribed level of 8% over the 10 year period. Since the bank meets the minimum CAR requirement, it is guaranteed the ability to absorb reasonable levels of losses before becoming insolvent. We also note that the CAR remains is in the range of 8% and 20% over the entire 10-year period. If the CAR was below 8%, the bank in question would not have had enough capital on hand to cover the risk associated with its assets. On the other hand, if the CAR was above 20%, the bank would not have utilized its capital efficiently. Hence by maintaining its CAR between 8% and 20% the bank utilizes its capital efficiently and is covered against the risk associated with its assets. This is consistent with the observation made of Muller and Witbooi [50]. Witbooi et al. [65] observed that the CAR modelled in their paper resembles a mean-reverting process subject to the optimal allocation strategy. In the paper by Chakroun and Abid [12], the authors modelled a CAR that maintained its level above the threholds and observed that the bank on which their study is based is adequately capitalized to maintain the relevant amount of capital commensurate with the aggregate risk. In the paper [17], Danjuma et al. modelled a CAR that maintained its level above the threholds and observed that the higher the percentage of the CAR, the more capital is needed to maintain the prescribed CAR for the financial institution.

In Figure 5.4 we present a numerical simulation of the evolution of the NSFR which we derived in Chapter 4. By following the optimal investment strategy depicted in Figure 5.1, the underlying bank maintains its NSFR well above the minimum Basel III prescribed level of 100% over the 10 year period. Since the underlying bank meet its minimum NSFR requirement of Basel III, it is able to withstand medium to long term stress periods as it has adequate funding to support its investment practices over the 10 year period. This result is consistent with that of Muller [51] who made a similar observation in a setting where the NSFR followed a jump diffusion process.

Chapter 6

The deposit insurance pricing problem

This chapter is devoted to the problem of estimating the fair price for DI coverage for the underlying bank under different levels of volatilities in its asset portfolio. We present a numerical simulation of the bank's insured deposits and a numerical simulation of the asset portfolio (for a specific volatility level) under the asset value reset rule towards the end of the chapter. Throughout this chapter, we reference the sources Merton [47], Duan and Yu [21], and Muller [52, 53].

We model DI as a European put option, where the underlying bank's asset portfolio is taken to be the stock price and its insured deposits plus accrued interest is taken to be the strike price. We assume that the bank does not pay any dividends to its shareholder over the interval $[0, T]$ on which the bank is covered. We further assume that the bank is audited at the times $t(k)$, $k = 1, 2, 3, \ldots, n-1, n$, where $t(k)$ are positive integers such that $0 = t(1) < t(2) < t(3), \ldots, t(n-1) < t(n) = T$.

We denote the bank's total insured deposits by $\hat{D}(t)$ and assume, like Muller [52, 53], that its dynamics is given by

$$\hat{D}(t) = \rho D(t).$$

Here the quantity ρ satisfies the condition $0 \leq \rho \leq 1$. The SDE governing $\hat{D}(t)$ is therefore

$$d\hat{D}(t) = \rho dD(t),$$

or

$$d\hat{D}(t) = \rho[\mu_D dt + \sigma_D dW_D(t)]. \tag{6.1}$$

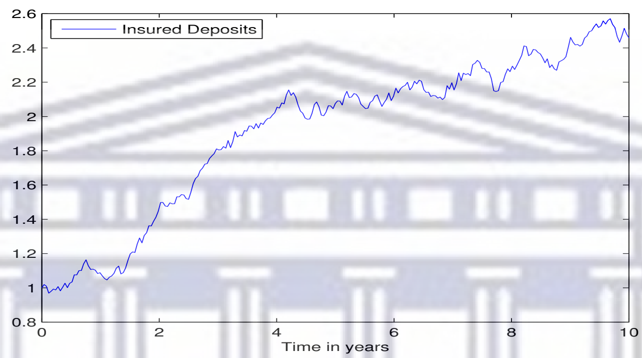

Figure 6.1: A simulation of the value of the bank's insured deposits, $\hat{D}(t)$, over a 10-year period with ρ=0.95, $\mu_D = 0.12$, $\sigma_D = 0.15$ and $\hat{D}(0) = 1$.

We assume, like Duan and Yu [21] and Muller [52, 53], that the bank's total asset value is subject to reset at the auditing time of the bank. In particular, as a means to conduct insolvency resolution, we assume that the insuring agent adopts a government-assisted merger or purchase-and-assumption. In doing so, the insuring agent provides a lump sum transfer to the acquirer of the failing bank. The lump sum is an amount sufficient to cover the face value of the insured deposits plus accrued interest. The total asset value of the bank at time $t(k)$ is calculated according to the rule.

- If $A(t(k)) < e^{rt(k)}\hat{D}(t(k))$, then the bank's total asset value will reset to a value of $e^{rt(k)}\hat{D}(0)$, which is the face value of the total insured deposits plus accrued interest.
- If on the other hand, the bank is found to be solvent at time $t(k)$, then the total asset value will follow the SDE in Eq.(4.5).

At time $t(k)$, for $k = 1, 2, 3, \ldots, n-1, n$, the value of the DI can be described as a put option with the underlying asset price given by $A(t(k))$, and with strike price $e^{rt(k)}\hat{D}(t(k))$ (see [21]). The insuring agent is therefore faced with a stream of put option like liabilities, each leading to a payment denoted by $Q(t(k))$. The payment $Q(t(k))$ is determined by the rule

$$Q(t(k)) = \begin{cases} e^{rt(i)}\hat{D}(t(k)) - A(t(k)), & \text{if } A(t(k)) < e^{rt(k)}\hat{D}(t(k)), \\ 0, & \text{if otherwise.} \end{cases}$$

The payment, $Q(t(k))$, at time $t(k)$ can be generalized to the expression

$$\begin{aligned} Q(t(k)) &= [e^{rt(k)}\hat{D}(t(k)) - A(t(k))]^{+} \\ &= \max[0, e^{rt(k)}\hat{D}(t(k)) - A(t(k))]. \end{aligned} \tag{6.2}$$

Since the bank's asset portfolio does not follow a geometric Brownian motion, we can not use the Black-Scholes model (see [10]) to price the option-like liabilities faced by the insuring agent. Instead, we use a Monte Carlo simulation method to estimate the price of these liabilities.

We assume, as was done by the authors of [21] and [52, 53], that the fairly-priced premium for the bank can be calculated via the formula

$$\hat{\omega} = \frac{1}{n\hat{D}(0)} \sum_{k=1}^{n} e^{rt(k)} \mathbb{E}[Q(t(k))]. \tag{6.3}$$

By substituting Eq.(6.2) into Eq.(6.3), we obtain

$$\hat{\omega} = \frac{1}{n\hat{D}(0)} \sum_{k=1}^{n} e^{rt(k)} \mathbb{E}[e^{rt(k)}\hat{D}(t(k)) - A(t(k))]^{+}. \tag{6.4}$$

In Algorithm 1 below, we present the Monte Carlo simulation algorithm of Muller [52, 53] for calculating $\hat{\omega}$. Given the similarity between our pricing problem and that of [52, 53], we employ this algorithm to estimate the fairly-priced premiums for the underlying bank of our study.

Muller's algorithm for the Monte Carlo simulation method used to estimate $\hat{\omega}$:

While generating 1,000,000 sets, each consisting of a pair for $\hat{D}$ and A on the time interval $[0, T]$,

DO

At each $t(k)$, where $k = 1, 2, 3, \ldots, n-1, n$ and $t(1) < t(2) < t(3), \ldots, t(n-1) < t(n) = T$ are positive integers:

Calculate the payoff $[e^{rt(k)}\hat{D}(t(k)) - A(t(k))]^+$ for each set consisting of the sample paths of $\hat{D}$ and A.

Using all the sets of sample paths of $\hat{D}$ and A, calculate the average of the payoffs $[e^{rt(k)}\hat{D}(t(k) - A(t(k))]^+$ as a proxy to $\mathbb{E}[e^{rt(k)}\hat{D}(t(k)) - A(t(k))]^+$.

Discount the proxy to time zero by multiplying it by $e^{-rt(k)}$.

END

Sum the values of all the discounted proxies calculated at times $t(k), k = 1, 2, 3, \ldots, n-1, n$.

Divide the sum of the discounted proxies by $n\hat{D}(0)$.

Algorithm 1: Muller's [52] Algorithm for the Monte Carlo simulation used to estimate $\hat{\omega}$.

In Table 6.1 we used Algorithm 1 to estimate the fairly-priced premium that the underlying bank should be charged for entering into a DI. In estimating the DI premium, we used the same parameter values $r = 0.065, m_1 = 0.035, \sigma_1 = 0.08, m_2 = 0.045, \sigma_2 = 0.095, \mu_D = 0.12, \sigma_D = 0.15, g = 15$ that we used earlier in the simulation study of the optimal investment strategy in Chapter 5. However, for σ_1 we consider values ranging from 0.08 to 0.16 with increments of 0.02, as this allows us to calculate $\hat{\omega}$ for different volatility levels. We assume that 95% of the deposits are insured so that in Eq.(6.1), ρ=0.95.

Table 6.1: Estimating the fairly-priced deposit insurance premium, $\hat{\omega}$.

	σ_1=**0.08**	σ_1=**0.1**	σ_1=**0.12**	σ_1=**0.14**	σ_1=**0.16**
$\hat{\omega}$	0.00490	0.00582	0.00640	0.00679	0.00704

Clearly the estimated price of the DI contract increases as the volatility coefficient σ_1 increases.

This means that the bank will pay higher premiums for DI coverage as the volatility in the asset portfolio is increased. This observation is consistent with Duan and Yu [21], Muller [52] and Hariati et al. [30]. The aforementioned authors found that the fairly-priced premium will increase as the volatility of the asset portfolio increases. Muller [53], on the other hand, found that the fairly-priced premium will decrease as the volatility of the asset portfolio increases.

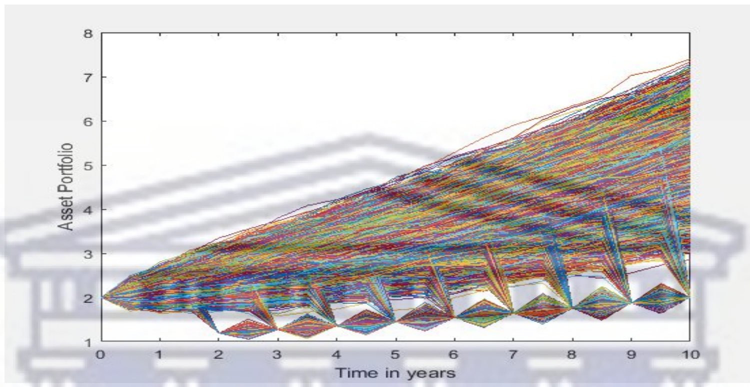

Figure 6.2: A simulation of 10^6 sample paths of the asset portfolio $A(t)$ under the asset value reset rule, with $\sigma_1 = 0.08$ and $A(0) = 2$.

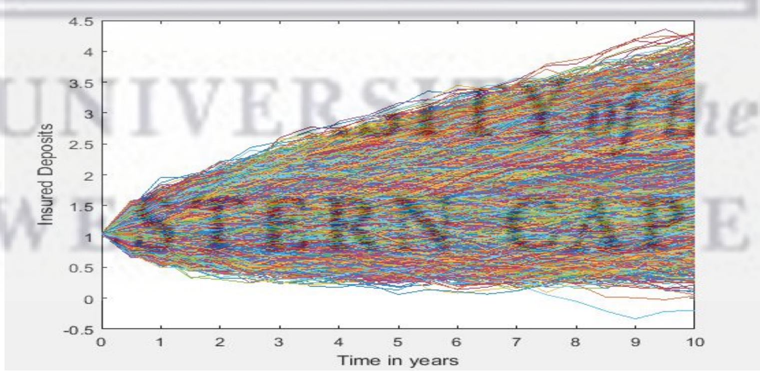

Figure 6.3: A simulation of 10^6 sample paths of the insured deposits, $\hat{D}(t)$, with ρ=0.95, $\mu_D = 0.12$, $\sigma_D = 0.15$ and $\hat{D}(0) = 1.045$.

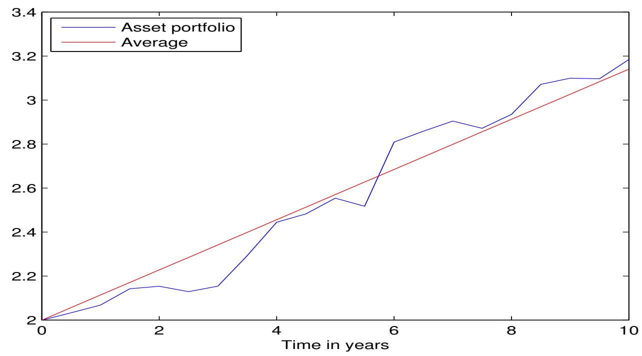

Figure 6.4: A simulation of the evolution of the average of 10^6 sample paths of the asset portfolio, $A(t)$, under the asset value reset rule with $\sigma_1 = 0.08$ and $A(0) = 2$.

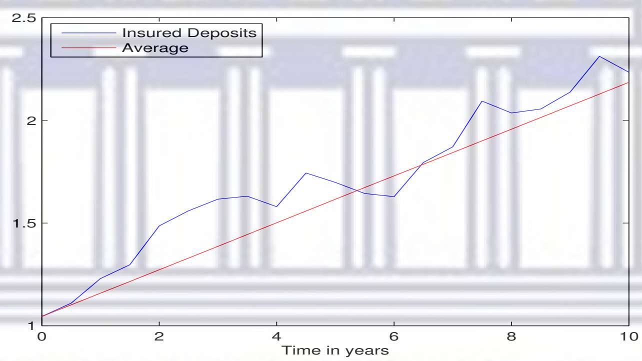

Figure 6.5: A simulation of the evolution of the average of 10^6 sample paths of the insured deposits, $\hat{D}(t)$, with ρ=0.95, $\mu_D = 0.12$, $\sigma_D = 0.15$ and $\hat{D}(0) = 1.1.045$.

In Figure 6.4, we present a simulation of the average of 10^6 sample paths of the asset portfolio along with one sample path of the asset portfolio under the asset value reset rule. This simulation is based on the same model parameters and initial conditions specified for the DI pricing simulation. For the parameter σ_1 we have assigned a value of 0.08. We consider an initial leverage condition of 1.045. In Figure 6.5, we present a simulation of the average of 10^6 sample paths of the insured deposits, $\hat{D}(t)$, along with one sample path of the insured deposits under the asset value reset rule. We observe from Figures 6.4 and 6.5 that the quantities simulated exhibit upward trends, which is consistent with [53].

Chapter 7

Conclusion

We model a commercial bank that invests its capital in a financial market consisting of three different assets, namely a treasury security, a marketable security and a loan. The interest rate in the market is assumed to remain constant. The bank's liabilities come in the form of borrowings and deposits. We introduce models, by means of differential equations, that describe the evolution of the aforementioned bank items and the bank's off-balance sheet activities. We derive a stochastic differential equation (SDE) for the bank's capital, which is the value of its assets minus its liabilities. Furthermore, we study a variety of related commercial banking problems in continuous and discrete time settings. These problems can be summarized as follows. In the first problem we derive models for the underlying bank's capital adequacy ratio (CAR) and net stable funding ratio (NSFR). Both these derivations involve using Itô's Lemma and Itô's Product Rule. Since the CAR is computed from the capital and total risk-weighted assets (TRWAs), we also derive an SDE for the TRWAs. The NSFR, on the other hand, is computed from the Available Amount of Stable Funding (AASR) and Required Amount of Stable Funding (RASF), hence SDEs are derived for the latter quantities as well. In the second problem we study an optimal control problem involving the bank's capital. More specifically, we use the stochastic optimal control technique to derive a strategy for investing the capital in the assets so as to maximize an expected exponential utility of the bank's capital at a future date $T > 0$. In the third and last problem we study a DI pricing problem that involves using a Monte Carlo simulation method to estimate the price that the underlying bank should be charged for entering into a DI contract for a coverage horizon of T years. Here T is the date at which the expected utility of the bank's capital is to be maximized. This pricing model incorporates an asset value reset rule similar to that of Duan and Yu [21], and Muller [52, 53], which allows us to apply the DI pricing algorithm of Muller [52] to estimate our DI premium. We apply the aforementioned algorithm for different levels of volatility in the bank's asset portfolio so that we can see how changes in the volatility affect the DI premium.

The main results of the thesis can be summarized as follows. The optimal investment strategy for the bank is to diversify its asset portfolio away from the two risky assets (marketable security and loan), and towards the riskless treasury. That is, initially the bank should invest more of its capital in the two risky assets than in the riskless asset. However, over time, the bank should invest less of its capital in the two risky assets and more in the riskless asset. This finding is consistent with those of Muller and Witbooi [50], Muller [52, 53] and Danjuma et al. [17] and Danjuma [18]. Under the optimal investment strategy, and for the parameters considered, the bank maintains its CAR and NSFR levels above the minimum prescribed Basel III requirements. Since the bank meets the minimum CAR and NSFR requirements as prescribed by Basel III, it is guaranteed the ability to absorb reasonable levels of losses before becoming insolvent while at the same time being able to withstand medium to long term stress periods due to having adequate funding to support its investment practices. The latter finding is similar to that of Muller [51]. We also note that by following the optimal investment strategy, the price of the DI premium increases as the volatility of the asset portfolio increases. This is consistent with the results of Duan and Yu [21], Muller [52] and Hariati et al. [30]. Muller [53] on the other hand found that the fairly-priced premium will decrease as the volatility of the asset portfolio increases. In the paper [53], the author considers a constant rate of capital influx, an asset portfolio and a capital model that would maintain the CAR at its Basel III prescribed level.

We wish to note that a research article titled "Capital optimization for a commercial bank" by Richard S. Hercules, Garth J. Van Schalkwyk and Grant E. Muller is in progress. In the aforementioned paper we derive two different investment capital maximization strategies and compare them by way of numerical simulations. We also observe graphically the level of the bank's CAR under the two strategies.

Possible future research on the topics studied in this thesis could be the characterization of the behaviours of the Tier 1, Leverage and Liquidity coverage ratios for the bank modelled in this thesis. It would be interesting to observe whether the underlying bank also meets the aforementioned minimum Basel III requirements while following the optimal investment strategy derived in this thesis. One could possibly also derive, for the bank modelled here, a multiperiod DI pricing model by using a variance reduction method and compare it to the multiperiod DI model assumed in this thesis. Furthermore, since DI can be priced as an American put option that could be exercised any time that a financial crisis may occur during the lifetime of the contract, we could try to apply this approach to our bank model.

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