A model for managing pension funds with benchmarking in an inflationary market

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Abstract

A MODEL FOR MANAGING PENSION FUNDS WITH BENCH-MARKING IN AN INFLATIONARY MARKET

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Aggressive fiscal and monetary policies by governments of countries and central banks in developed markets could somehow push inflation to some very high level in the long run. Due to the decreasing of pension fund benefits and increasing inflation rate, pension companies are selling inflation-linked products to hedge against inflation risk. Such companies are seriously considering the possible effects of inflation volatility on their investment, and some of them tend to include inflationary allowances in the pension payment plan. In this dissertation we study the management of pension funds of the defined contribution type in the presence of inflation-recession. We study how the fund manager maximizes his fund's wealth when the salaries and stocks are affected by inflation. In this regard, we consider the case of a pension company which invests in a stock, inflation-linked bonds and a money market account, while basing its investment on the contribution of the plan member. We use a benchmarking approach and martingale methods to compute an optimal strategy which maximizes the fund wealth. Under this approach the objective functional is an increasing function of the relative performance of the asset portfolio compared to

a benchmark. Central to this dissertation are the papers by Lim, Andrew and Wong, B., (2010); Zhang, A., Korn, R., and Edwald, C.O., (2007); Malliaris, A.G and Mullady, W.F., (1991); and Deelstra, G., Grasselli, M., and Koehl, P.F., (2002).
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Declaration

I declare that this is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Mozart Nsuami December, 2010

Signed: ....

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I first thank the almighty God for that he always holds me in his hands. And so no matter how hard life and challenges are, I can stand tall and prevail. To my parents, let you find here the expression of my sincere love to you.

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Contents

List of Acronyms

a.s, almost surely

SDE, Stochastic Differential Equation

CRRA, Constant Relative Risk Aversion

CPI, Consumer Price Index

RPI, Retail Price Index

List of Notations

$\mathbb{E}^Q$, the expectation under Q

$\frac{d\mathbb{P}^{(L)}}{d\mathbb{P}}$, the Radon-Nikodym derivative of $\mathbb{P}^{(L)}$with respect to $\mathbb{P}$

σ, Volatility

Q, a martingale measure equivalent to the market measure

$\mathbb{P}$, a probability measure, usually the market measure

$(\Omega, \mathcal{F}, \mathbb{P})$, Probability triple

$P[X|Y]$, Conditional probability of X given Y

N, Standard normal distribution

$\{\mathcal{F}_n\}_{n\geq 0}, \{\mathcal{F}_t\}_{t\geq 0}$, Filtration

$\mathbb{E}[X/\mathcal{F}]$, $\mathbb{E}[X_{n+1}/X_n]$, Conditional expectation

$\triangleq$, defined equal to

W, Brownian motion or Wiener process

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Chapter 1

General Introduction

1.1 Introduction

Pension funds are among the most important of institutions in financial markets due to their large capacity of investment. They also fulfill an important function in that pension companies complement the role of the government, allowing those workers who have reached retirement age to maintain a reasonable standard of living. From a general point of view, there are two principal alternatives in pension plan designs with regard to the assignment of risk: defined contribution and defined benefit. In the defined benefit plan, the pension fund guarantees to pay the employee in retirement a fixed monthly income for life. Defined contributions plans are those in which the employer agrees to contribute a fixed amount to the employee's pension fund each year in which the employee is employed. The income that the employee receives during retirement depends upon how much money the plan had accumulated and how much income that amount can sustain. From a historical perspective, defined benefit plans proved to be more popular than defined contribution plans. Nowadays, most of the plans created in pension funds have been based on defined contributions, such as Appropriate Personal Pension in UK and Individual Retirement Accounts in USA (see Boulier [7]). There is another pension fund scheme, called the pay-as-you-go (PAYG) system. The PAYG is a scheme where workers pay contributions to the fund while pensioners receive their pensions, which are out-flows from the fund wealth. In France, the pay-as-you-go system was efficient in the past, but actually it is limited by the demographic and the economic situation. Indeed, the age structure of the population and the ratio of the working-age population show the limits of this system. The ratio of retirees to workers is about 50% in 2010. In 2040, this number will arrive at 70% if the retired age does not change (see Fitoussi [20]). On 25 March 1997, the French government allowed the creation of retirement savings funds ("Loi Thomas"). However, no enabling legislation has been enacted. Nevertheless, the foundations for private pension funds have been laid, and the financial community must be ready to bring retirement products to the market at the appropriate time. Traditional optimal asset allocation problems in the investment management typically implies maximization of the expected utility of a terminal portfolio value, and where the utility is a concave function that satisfies some properties. This should not always be the case. We can also turn around and look at the effects or impacts of variables such as liabilities, inflation, recession, stagflation and even guarantee on pension investment. Some of these factors are generally known as uncertainties. Then, the idea of benchmarking is very convenient in this case. The use of benchmarking has become a common approach for enhancing the performance of companies. Thus, when applying benchmarks, firms compare their own activities and performance to those of other, appropriately selected, comparable organisations. The purpose of benchmarking varies from one company to another and it helps in determining the true source of performance. Some of the major problems associated with benchmarking are dealt with in this dissertation including the risks involved and measures taken to deal with the inherent risk, such as risk adjustment. In fund management, benchmarks are used as a guide to improvement, and fund managers are assessed in terms of their relative performance against the benchmark. Benchmarks are based on an objective consideration of the needs of fund managers. They can change with the environment.

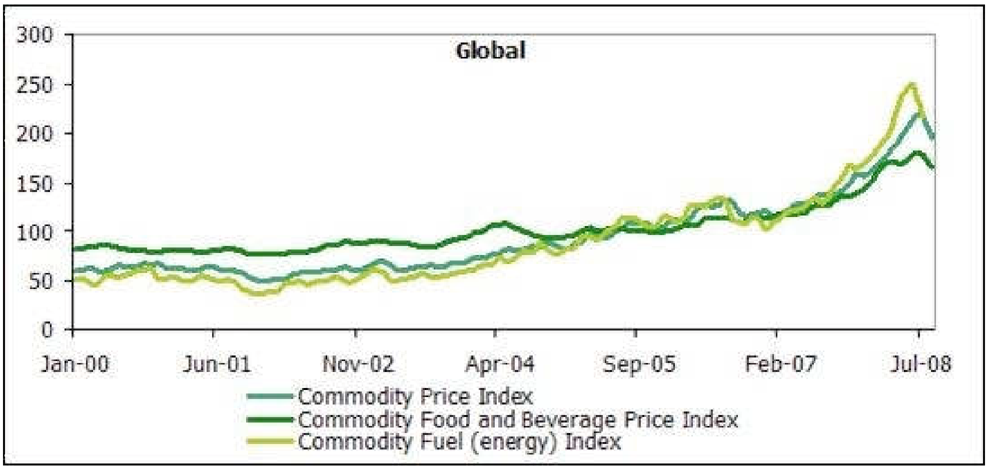

Figure 1.1: Graph of global inflation

Source: IMF Report 2009

The plan member in a classical defined contribution-pension plan experiences a risk linked to inflation which could amount to substantial losses. The pension manager must ensure that the benefit from non-inflation-linked pension will be sufficient to cover the future expenses as prices will have increased due to inflation. Figure 1.1 shows the graph of global inflation. It shows the evolution of global inflation rate over a certain period of time. The fact is that with a very simple computation, we can see how the real value of a certain amount can be reduced due to inflation. A given annual inflation rate of 3% over 35 years will certainly reduce the real value of \$200,000 to \$ 71,077 today. The second problem that the fund manager faces here is that the number of workers may decline due to recession. This is known also as the demographic risk. It is pertinent that the pension manager must link the fund to inflation in order to reduce his risk. Inflation-linked products include for instance inflations puts, calls, swaps, floors and caps (see Beletshi [4]). For more on these products, we refer to Korn and Korn [34].

This dissertation considers a model for managing pension funds with benchmarking in an inflationary market and we extend our framework to cover the following:

1. interest rates and inflation, and we follow a continuous time approach.

2. the financial market consisting of three assets:

   - a money market account
   - a classical stock
   - an inflation-linked bond.

3. the stochastic behavior of inflation in the financial market.

We assume that the money market account and the stock are specified as in the classical Black-Scholes model, while the inflation index $I(t)$ will be specified as a geometric Brownian motion. The time horizon of the pension fund management is denoted by $[0, T]$. We define by $c(t)$ the contribution rate of the plan member which depends on the salary $Y(t)$. This salary is assumed to be stochastic and follows a geometric Brownian motion. The pension company uses the defined-contribution process. We admit somehow a general correlation between stock $S(t)$, inflation index $I(t)$ and salary $Y(t)$. For simplicity, we adopt the assumption of completeness of markets and our setup includes models where stock real returns are driven by other underlying economic variables such as inflation and unemployment. For additional discussions on some implications of different asset return models for pension strategies, we can refer to the paper [6] of Blake et al.

1.2 Overview of research problem and background

The world economy is expected to grow bigger in different aspects everyday and investors have to be more concerned about movements in the economy. The inflation market is also expected to develop exponentially, and the number of participants is of course also growing. Cyclical economic events such as inflation, stagflation and recession can affect investments. Economists define inflation as a general and a progressive rise in prices. The recession is generally characterized by the unemployment when the state of the economy declines. Stagflation is an inflationary umployment situation or an inflation-recession event. The challenge for the future is to include more strategies, techniques and skills in our management that would allow participants to always be able to hedge against these cyclicalities. Several studies should consider models with jumps and other sources of incompleteness, models with inflation and recession. Another emerging challenge would be to create a model set of life-cycle pension funds, which can serve as benchmarks against which the performance of pension fund managers can be measured. Several works have been undertaken on inflation, recession, interest rates and portfolio management. These notions were studied extensively in economics and especially in macroeconomics. Nowadays these notions merge competences, skills and knowledge in the field of statistics, mathematics and finance. In macroeconomics, Fisher was a pioneer on the theory of interest rate (see Fisher [19]). He formed his famous hypothesis that the nominal interest rates should vary closely with the movements in expected inflation. This hypothesis connects two distinct parts of economy which are: inflation which expresses changes in the supply-demand conditions on the commodity spot market while nominal interest rates reflect differences in supply-demand conditions on the money market. Fisher's hypothesis was supported by several empirical studies either by using available survey data or by analyzing market data on inflation-linked bonds in the UK, the US or Canada. In their empirical studies Ang and Bekaert [1] investigate the connection between nominal and real interest rates and inflation. Changes in nominal interest rates must be due to either movements in real interest rates, expected inflation, or the inflation risk premium. Empirical research does not support the hypothesis that rates of inflation are constant over time or that there exists a long-run mean towards which current rates will regress over time. The variability of yearly measured inflation rates varied widely over the last century. There was some evidence that high variability can be associated with periods of high inflation. From an economic point of view, analyses of interest rate volatility can also be used to gauge inflation rate variability, since the level of interest rates provides an indicator of inflation expectations. Besides Fisher's hypothesis there exist a variety of different models describing the relation between nominal interest, real interest and inflation such as Taylor's rule and the forward rate rule (see Gerlach and Schnabel [25]). So far, from a series of critiques and discussions made by Tobin on the traditional Fisher equation for inflation, Malliaris and Mullady proposed an approach of interest rate, (see [39]). They presented two equations generalizing the traditional Fisher equation and an illustration using US long data from $1865 - 1972$.

1.3 Research Objectives

The main objective of the pension company is to increase the expected utility function of the relative performance of the asset portfolio compared to a given benchmark by investing strategically in inflation-linked bonds, the stock and the money market account. The benchmark can be either any target to attain or any ratio to be compared with the initial wealth or the final wealth of the pension company or against the pension company's asset allocation. The pension company would be required to compare his allocation or his wealth to the level of the benchmark at different time ticks during of the management process. The benchmark entity can also be a completely independent basket of commodities which is published monthly and which the pension company has adopted, in agreement with pension members. The main strategy to hedge against the risk associated with his investment or management, is by selling inflation-linked bonds and by also revising his investment strategy with the benchmarks. Thus, we can construct appropriate stochastic models which are subject to follow a restriction from a geometric Brownian motion. We then extend our study to interest rates and inflation in a continuous time approach as proposed by Malliaris and Mullady (see [39]), and to the pricing formula for inflation-linked bonds with the support of a European call option on inflation.

1.4 Research Problem

A model for managing pension funds with benchmarking in an inflationary market has not been extensively studied before. Related works can be found though and there is a series of publications which focus on modeling the inflation process using the Fisher equation. Our dissertation deals with the inflation modeling by Fisher and the revised

Fisher equation for inflation proposed by Malliaris and Mullady. Important contributions in continuous time are due to Deelstra et al. [15], Blake et al. [6] and Cairns [10]. In [6] and [15] the authors use stochastic dynamic programming to solve the optimization problem. The goal of the fund manager in these studies is to invest the accumulated wealth in order to optimize the expected terminal value using a suitable utility function. The classical model proposed initially by Merton [41] assumes a market structure with a constant interest rate. As the investment periods of pension funds are quite extensive, generally from 20 to 40 years, the idea of a constant interest rate will not fit our purpose. Similar models have recently been presented by Blake, Cairns, and Dowd [6], Boulier, Huang, and Taillard [7] and Deelstra, Grasselli, and Koehl [15]. [6] assume a stochastic process for salary including a nonhedgeable risk component and focus on the replacement ratio as the central measure for determining the pension flow. *A Replacement Ratio* is a person's gross income after retirement, divided by his or her gross income before retirement. For example, assume someone earns $120,000 per year before retirement. Further, assume he or she retires and receives $90,000 of Social Security and other retirement income. This person's replacement ratio is 75 percent ($90,000/$120,000).

The problem of optimal portfolio choice for a long-term investor in the presence of wage income is also treated by El Karoui et al. [16], Campbell and Viceira [12], and Franke, Peterson, and Stapleton [22]. In [16] the authors present under a complete market with a constant interest rate the solution of a portfolio optimization problem for an economic agent endowed with a stochastic insurable stream of labour income. Thus, they assume that the income process does not involve a new source of uncertainty. In [22] the authors focus on some aspects of labour income risk in discrete time. Franke et al. [22] analyze the impact of the resolution of the labour income uncertainty on portfolio choice. They show how the investors portfolio strategy changes when his labour income uncertainty is resolved earlier or later in life. Furthermore, the optimal management of pension funds has been studied by Zhang et al. (see [47]). The authors use inflation-linked bonds to hedge against inflation risk. Finally, the optimal asset allocation of pension funds using a benchmarking approach has been studied by Lim and Wong (see [36]). In their study, the objective functional is an increasing function of the relative performance of the insurance company's asset portfolio compared to a benchmark. Contrary to their study, our study consists of a model with benchmark in an inflationary market. We use an approach similar to this under which the objective of the pension company is an increasing function of the relative performance of its asset portfolio compared to a benchmark.

1.5 Structure of the thesis

This dissertation has been structured into eight chapters including the introduction. Chapter Two deals with mathematical preliminaries on stochastic calculus used throughout our dissertation. We present some stochastic concepts, such as Brownian motion, martingales and others. Chapter Three introduces the market structure under which the asset allocation problem is defined. We use the revised Fisher equation for inflation in a continuous time and stochastic approach. We present some remarks on the traditional Fisher equation for inflation. Chapter Four deals with management of pension funds. We define the salary process, the guarantee, asset allocation and portfolio process. Chapter Five generalizes the optimization problem. This chapter defines the notion of benchmark and derive the solution of the optimization problem in general. Chapter Six deals with pension porfolio against a benchmark. It focuses on the optimization problem where an option based portfolio pension strategy is used as a benchmark. Through this chapter, we are able to derive in closed form the wealth strategy and benchmarking asset allocation using martingale methods and measure transformation techniques. Chapter Seven provides a numerical application and shows the qualitative behavior of the benchmarking asset allocation strategy. We summarize the main results in Chapter Eight.

Chapter 2

Mathematical Preliminaries

In this present chapter, we record some useful mathematical background material, used throughout our dissertation. We define concepts such as random variables, stochastic processes, Martingales, Brownian motion, etc., and give some basic results. Our main references on such basics are Etheridge [18], Le gall [35], Dalang and Bernyk [14] and Hull [28].

2.1 The concepts of random variables and stochastic processes

To talk about a random variable in a formal way requires to specify a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is some set called the sample space, $\mathcal{F}$ is a collection of subsets of Ω, and $\mathbb{P}$ specifies the probability of each event $A \in \mathcal{F}$. The collection $\mathcal{F}$ is a σ-field, that is, $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under the operations of countable union and taking complements. The probability $\mathbb{P}$ must satisfy the usual axioms of probability [18, p29]

- $0 \leq \mathbb{P}[A] \leq 1$, for all $A \in \mathcal{F}$
- $\mathbb{P}[\Omega] = 1$
- $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$ for any disjoint A, $B \in \mathcal{F}$,

- If $A_n \in \mathcal{F}$ for all $n \in \mathrm{N}$ and $A_1 \subseteq A_2 \subseteq ...$, then $\mathbb{P}[A_n] \uparrow \mathbb{P}[\bigcup_n A_n]$ as $n \uparrow \infty$.

Definition 2.1.1. Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0,T]$ there is a σ-algebra $\mathcal{F}_t$. Assume further that $\mathcal{F}_s \subset \mathcal{F}_t$ and $\mathcal{F} = \bigcup_{t\geq 0} \mathcal{F}_t$ for all $0 \leq s < t < \infty$. Then we call the collection $\{\mathcal{F}_t\}$ of σ-algebras a *filtration* and then $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ is called a filtered probability space.

We consider $\mathcal{F}_t$ as the set of information available at time t. In other words we can consider $\{\mathcal{F}_t\}_{t>0}$ as describing the flow of information over time, where we suppose that we do not lose information as time passes (that is why we say $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$).

Definition 2.1.2. A real-valued stochastic process is an indexed family of real-valued functions, $\{X_t\}_{t\geq 0}$. We say that $\{X_t\}_{t\geq 0}$ is *adapted* to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if X_s is $\mathcal{F}_t$-measurable for each $t \geq s$ [18, 29].

In this condition one may think of $\mathcal{F}_t$ as all the information about the evolution of the stochastic process up until time t.

Definition 2.1.3. Suppose that X is an $\mathcal{F}$-measurable random variable with $\mathbb{E}[|X|] < \infty$, [18, p32]. Suppose that $\mathcal{G}$ is a σ-field. Then the *conditional expectation* of X given $\mathcal{G}$, written $\mathbb{E}[X|\mathcal{G}]$, is a $\mathcal{G}$-measurable random variable with the property that for any $A \in \mathcal{G}$,

$$\mathbb{E}[[X|\mathcal{G}]; A] \triangleq \int_A \mathbb{E}[X|\mathcal{G}]\, d\mathbb{P} = \int_A X d\mathbb{P} \triangleq \mathbb{E}[X; A].$$

Note that the conditional expectation exists, but is only unique up to the addition of a random variable that is zero with probability one a.s.

2.2 Brownian motion

In 1827 Robert Brown observed the complex and erratic motion of grains of pollen suspended in a liquid. It was later discovered that such irregular motion comes from the extremely large number of collisions of the suspended pollen grains with the molecules of the liquid. Norbert Wiener presented a mathematical model for this motion based on the theory of stochastic processes, [14, p20]. The position of a particle at each time $t \geq 0$ is a three dimensional random vector M_t .

Definition 2.2.1. A real-valued stochastic process $\{M_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion (or a $\mathbb{P}$-Wiener process) if for some real constant σ, under $\mathbb{P}$ we have that:

1. for each $s \geq 0$ and $t > 0$ the random variable $M_t - M_s$ has the normal distribution with mean zero and variance t,
2. for each $n \geq 1$ and any times $0 \leq t_0 \leq t_1 \leq ... \leq t_n$, the random variables $M_t - M_s$ are independent,
3. $M_0 = 0$,
4. M_t is continuous in $t \geq 0$.

Remark 2.2.2. Brownian motion is a Gaussian process. In fact, the probability distribution of a random vector $(M_{t_1}, ..., M_{t_n})$, for $0 < t_1 < ... < t_n$, is normal, because this vector is a linear transformation of the vector $(M_{t_1}, M_{t_2} - M_{t_1}, ..., M_{t_n} - M_{t_{n-1}})$ which has a joint normal distribution, because its components are independent and normal. The mean and auto covariance functions of the Brownian motion are:

$$\begin{aligned}
\mathbb{E}[M_t] &= 0 \\
\mathbb{E}[M_s M_t] &= \mathbb{E}[M_s(M_t - M_s + M_s)] \\
&= \mathbb{E}[M_s(M_t - M_s)] + \mathbb{E}[M_s^2] = \min(s, t).
\end{aligned}$$

2.3 Martingales

In probability theory, a martingale can be thought of as a stochastic process such that the conditional expected value of an observation at some time T, given all the observations up to some earlier time t, is equal to the observation at that earlier time t. A more formal and mathematical definition of this would be:

Definition 2.3.1. Suppose that $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ is a filtered probability space. The family of random variables $\{M_t\}_{t\geq 0}$ is a martingale with respect to $\mathbb{P}$ and $\{\mathcal{F}_t\}_{t\geq 0}$ if $\mathbb{E}[|M_s|] < \infty, \forall\ t$, and $\mathbb{E}[M_s|\mathcal{F}_t] = M_t, \forall\ s \geq t$.

2.3.1 Martingale Representation Theorem

The martingale representation theorem states that any martingale adapted with respect to a Brownian motion can be expressed as a stochastic integral with respect to the same Brownian motion.

Theorem 2.3.2. *Let B be a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\geq 0}$ be be its natural filtration.*
Then, every $\mathcal{F}_t$-local martingale M can be written as

$$M = M_0 + \int \xi dB \tag{2.1}$$

for a predictable, B-integrable, process ξ.

Definition 2.3.3. The stochastic process $\{X_t\}_{t\leq 0}$ with the natural filtration, $\{\mathcal{F}_t\}_{t\leq 0}$ is a *Markov process* if for any $s > t$ we have, $\mathbb{P}[X_s \in B|\mathcal{F}_t] = \mathbb{P}[X_s \in B|X_t]$ for all $B \in \mathcal{F}$.

Remark 2.3.4. This simple definition means that the probability that $X_s \in B$ given that we know the history of the process up to time t, is the same as the probability that $X_s \in B$ given only the value of X_t .

2.4 Stochastic integration

The theory of stochastic integration has a large spectrum of applications in almost every scientific area involving random functions. This topic requires a concise introduction to Itô calculus, constructions of Brownian motion and martingales and stochastic differential equations. Processes used to model stock prices are usually functions of one or more Brownian motions. Suppose that the stock price is of the form [18, p74]

$$S_t = f(t, W_t). \tag{2.2}$$

Using Taylor's Theorem, f can be written as

$$\begin{aligned} f(t+\delta t, W_{t+\delta} t) - f(t, W_t) &= \delta t \dot{f}(t, W_t) + 0(\delta t^2) + (W_{t+\delta t} - W_t) f'(t, W_t) \\ &+ \frac{1}{2}(W_{t+\delta t} - W_t)^2 f''(t, W_t) + ... \end{aligned} \tag{2.3}$$

where we have used the notation

$$\dot{f}(t,x) = \frac{\partial f}{\partial t}(t,x), \;\; f'(t,x) = \frac{\partial f}{\partial x}(t,x) \text{ and } f''(t,x) = \frac{\partial^2 f}{\partial x^2}\;.$$

A differential equation governing the stock price $S_t = f(t, W_t)$ takes this form, see [18, p75]:

$$dS_t = \dot{f}(t, W_t)dt + f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt. \tag{2.4}$$

It is convenient to write (2.4) in integrated form,

$$S_t = S_0 + \int_0^t \dot{f}(s, W_s)ds + \int_0^t f'(W_s)dW_s + \int_0^t \frac{1}{2}f''(W_s)ds. \tag{2.5}$$

2.4.1 Itô Process

A stochastic process $X = \{X_t, t \geq 0\}$ that solves the equation

$$X_t = X_0 + \int_0^t a(X_t, t)ds + \int_0^t b(X_t, t)dW_s \tag{2.6}$$

is called an Itô process. Then the stochastic differential equation relating to the above is

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t, \tag{2.7}$$

where $a(X_t, t)$ is the drift form, $b(X_t, t)$ is the diffusion form and W_s is a standard Wiener process.

2.4.2 Itô's Lemma

We formulate for easy reference the Itô Lemma. For proof we can refer to the book of Hull, [28, p287].

Lemma 2.4.1. *Suppose $F(S,t)$ is a twice differentiable function of t and S_t and also that S_t follows the Itô process*

$$dS_t = \alpha_t dt + \sigma_t dW_t, t \geq 0 \tag{2.8}$$

with well behaved drift and diffusion parameters α and σ_t. Then,

$$dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2 dt. \tag{2.9}$$

2.5 The Legendre-Fenchel transform

The Legendre-Fenchel transform (or conjugate) of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is a function defined on the topological dual space of X as (see [29])

$$p \in X^* \to f^*(p) := \sup_{x \in X} \left((p, x) - f(x) \right). \tag{2.10}$$

The new function f^* is automatically convex on X^*. In convex analysis, the transformation $f \to f^*$ plays a role similar to that of Fourier's or Laplace's transform in other areas of analysis. In particular, one cannot get away from it in analyzing the so-called dual versions of a given optimization problem. That explains why the Legendre-Fenchel transform occupies a key-place on convex analysis.

Chapter 3

The model and financial market

3.1 Introduction

In this chapter, we introduce a market structure under which the asset allocation problem is defined and we seek the optimal solution. We introduce a filtered probability space by $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the probability measure on a measure space Ω. We consider a filtration $\mathcal{F}_t$ of $\mathcal{F}$ generated by Brownian motion. It is common to view $\mathcal{F}_t$ as the information revealed by a given Brownian motion. We define and present the stochastic dynamics of interest rate, the assets value, and the risk of managing pension funds under inflation. The uncertainty in the financial market is modeled by the two dimensional standard Brownian motions

$$W_I(t) \text{ and } W_s(t),$$

respectively for the inflation index and the stock, with $t \in [0, T]$. We follow an approach in which the investment is compared with a certain porfolio, the latter being regarded as a benchmark. Based on a power utility function we aim to derive closed form solutions of associated stochastic optimal control problems and by applying the martingale methods and measure transformation techniques. Our main and basic references are Zhang [47], Battachio [3], Boulier [7] and Malliaris and Mullady [39].

3.1.1 Interest rates and inflation

We consider the approach proposed and tested by Malliaris and Mullady (see [39]) in the USA which followed a series of discussions and critiques made on the traditional Fisher equation for inflation. Fisher's derivation of the equation for inflation (see [19]) was based on macroeconomic principles, which relates the nominal interest rate r_N, the real interest rate r_R and the expected inflation rate $\mathbb{E}(i)$ over the planning horizon T using the following

$$r_N - r_R = \mathbb{E}(i). \tag{3.1}$$

In fact, such a view of equation (3.1) assumes constancy of the real interest rate. Therefore, to avoid this constancy, we assume in this dissertation that the nominal interest rate and the inflation rate follow Itô processes and derive an Itô equation that allows to express and compute the expected real interest rate and its volatility.

Assume that the inflation and the nominal interest rates follow these dynamics

$$\frac{dP}{P} = i(t, P)dt + \sigma_P(t, N)dW_P. \tag{3.2}$$

$$\frac{dQ}{Q} = r_N(t, N)dt + \sigma_N(t, N)dW_N \tag{3.3}$$

with

$$dW_N dW_p = \rho_{NP}dt.$$

Equation (3.3) describes the nominal return of an asset per unit of time an Itô process. r_N denotes the instantaneous nominal interest rate and σ_N denotes the instantaneous volatility. Equation (3.2) is expressed again as an Itô process with i denoting the instantaneous rate of expected inflation, that is $\mathbb{E}\left(\frac{dP}{dt}\frac{i}{P}\right) \equiv i$ and σ_P denoting its volatility.

From (3.2) and (3.3), it is easy to check that both the nominal interest and inflation rates are shocked by random forces denoted by dW_N and dW_p respectively.

Let us assume that $q = \frac{Q}{P}$ expresses the real value of an asset. We want to describe the behavior of q in view of the two processes in (3.2) and (3.3). We present this description in the following proposition, describing the proportional change in the real rate of interest as an Itô process. It is consistent with Fisher's view of the non constancy of the real rate of interest.

Proposition 3.1.1. *The real value q of an asset satisfies the following SDE:*

$$\begin{aligned} \frac{dq}{q} &= d\left(\frac{Q}{P}\right) / \left(\frac{Q}{P}\right) \\ &= \left(r_N - i - \sigma_N \sigma_P \rho_{Np} + \sigma_P^2\right) dt + \sigma_N dW_N - \sigma_P\left(t, P\right) dW_P \ . \end{aligned} \tag{3.4}$$

(3.4) describes the proportional change in the real rate of interest as an Itô's process and it is consistent with Fisher's view of the non constancy of the real rate of interest.

Proof. Let $X = \frac{Q}{P}$ and $f(P) = \frac{1}{P}$. Then by Itô we have

$$\begin{aligned} df(P) &= f'(t) + f'(P)dP(t) + \frac{1}{2} f''(P) \left[dP\right]^2 \\ &= 0dt - \frac{1}{P^2} dP(t) + \frac{1}{2}\left(\frac{2}{P^3}\right)\left[dP\right]^2 \ . \end{aligned} \tag{3.5}$$

Substituting (3.2) into (3.5) we have

$$\begin{aligned} df(P) &= -\frac{i}{P^2}\left[P\left(i(t,P)dt + \sigma_P(t,N)dW_P\right)\right] + \frac{1}{P^3}\left[\sigma^2 P^2 dt\right] \\ &= -\frac{i}{P}dt - \sigma_P \frac{1}{P} dW_P + \frac{1}{P}\sigma_P^2 dt \\ &= \left(\frac{-i}{P} + \frac{\sigma_P^2}{P}\right) dt - \frac{\sigma_P}{P} dW_P \ . \end{aligned} \tag{3.6}$$

It follows that $dX = d(Qf(P))$. Then by the product rule we have the following

$$\begin{aligned} dX &= Qdf(P) + f(P)dQ + dQdf(P) \\ &= Q\left(\frac{-i}{P} + \frac{\sigma_P^2}{P}\right) dt - \frac{Q}{P}\sigma_P dW_P + f(P)\left[Q(r_N(t,N)dt + \sigma_N(t,N)dW_N\right] \\ &+ \left[Q(r_N(t,N)dt + \sigma_N(t,N)dW_N\right]\left[\left(\frac{-i}{P} + \frac{\sigma_P^2}{P}\right) dt - \frac{\sigma_P}{P} dW_P\right] \ . \end{aligned} \tag{3.7}$$

The last term above reduces to the differential $X\sigma_N\sigma_P\rho_{NP}dt$ because all the product terms which have a dt as well as the differential of a brownian motion will vanish. Thus we have:

$$\begin{aligned} dX &= X\left(-i+\sigma_P^2\right)dt - X\sigma_P dW_P + X\left(r_N(t,N)dt+\sigma_N(t,N)dW_N\right) \\ &- X\sigma_N\sigma_P dW_N dW_P \ . \end{aligned} \quad (3.8)$$

In the following we divide (3.8) by X which is also equal to $\frac{Q}{P}$ in order to get $\frac{dq}{q}$.

$$\begin{aligned} \frac{dq}{q} &= \left(-i+\sigma_P^2\right)dt - \sigma_P dW_P + \left(r_N(t,N)dt+\sigma_N(t,N)dW_N\right) \\ &- \sigma_N\sigma_P dW_N dW_P \ . \end{aligned} \quad (3.9)$$

Rearranging terms one gets

$$\frac{dq}{q} = \left(r_N - i + \sigma_P^2\right)dt - \sigma_P dW_P + \sigma_N(t,N)dW_N - \sigma_N\sigma_P dW_N dW_P \ , \quad (3.10)$$

with $dW_N dW_p = \rho_{NP}dt$ the equation (3.11) becomes

$$\frac{dq}{q} = \left(r_N - i - \sigma_N\sigma_P\rho_{Np} + \sigma_P^2\right)dt + \sigma_N dW_N - \sigma_P\left(t,P\right)dW_P \ . \quad (3.11)$$

□

Taking the conditional expectation yieds

$$E\left(\frac{dq}{q}\right) = r_N - i - \sigma_N\sigma_p\rho_{Np} + \sigma_p^2 \ . \quad (3.12)$$

The equation (3.12) can be reduced to Fisher equation by assuming that $\sigma_N = \sigma_p = 0$ if we take that both the volatilities of nominal interest rates and inflation are zero.

3.2 More on inflation

Inflation is defined as an index measuring the economic evolution of prices. The inflation rate is calculated as a relative change of Consumer Price Index (**CPI**) or Relative Price Index (**RPI**). Therefore, the simple inflation rate $i_s(.)$ for the time interval $[t_0, t]$ is calculated as $i_s(t_0,t) = \frac{I(t)-I(t_0)}{I(t_0)}$, where $I(t)$ is Consumer Price Index at time t. One can easily see that by defining simple inflation rate in the way of the above, we have the following relation for $t_0 \leq s_1 \leq ... \leq s_n \leq t$:

$$\frac{I(t)}{I(t_0)} = (1 + i_s(t_0,t)) + ...(1 + i_s(s_n,t)). \tag{3.13}$$

Noticing the analogy between inflation rate and interest theory, continuously compounded inflation rate $i_c(t_0,t)$ on the time interval $[t_0,t]$ can be defined as the solution to the following equation:

$$\frac{I(t)}{I(t_0)} = e^{(t-t_0)i_c(t_0,t)}. \tag{3.14}$$

An application of logarithms to obtain the compounded interest rate yields:

$$i_c(t_0,t) = \frac{\ln(I(t)) - \ln(I(t_0))}{t - t_0}, \tag{3.15}$$

This way we can define the instantaneous inflation rate $i(t)$ at the time t similar to the way instantaneous short rate is defined in the interest rate theory:

$$i(t) = \lim_{s\to t} i_c(t,s) = \lim_{s\to t} \frac{\ln(I(t)) - \ln(I(t_0))}{s - t} = \frac{d\ln(I(t))}{dt} \tag{3.16}$$

Furthermore, we have the evolution of inflation of this form:

$$\frac{I(t)}{I(t_0)} = \exp \int_{t_0}^{t} i(s)ds. \tag{3.17}$$

Proposition 3.2.1. *Assume that the dynamics of Consumer Price Index under the risk neutral probability measure $\mathbb{P}$ follows the geometric Brownian motion according to the following*

$$dI(t) = I(t)\left((r_N(t) - r_R(t))dt + \sigma_I dW_I(t)\right),\ I(0) = i, \tag{3.18}$$

where the coefficients $r_N(t)$ and $r_R(t)$ are respectively the nominal and the real interest rates, which are assumed to be deterministic and σ_I is a constant volatility of the process. The following solves equation (3.5) by

$$I(t) = i\exp\left(\int_0^t (r_N(s) - r_R(s))ds - \frac{1}{2}\sigma_I^2 t + \sigma_I dW_I(t)\right). \tag{3.19}$$

Proof. The Consumer Price Index $I(t)$ can be described by the following:

$$\frac{dI(t)}{I(t)} = I(t)((r_N(t) - r_R(t))dt + \sigma_I dW_I(t)). \tag{3.20}$$

Letting $\log I(t) = f(t, I(t))$, we obtain

$$d\log I(t) = \frac{1}{I(t)} dI(t) + \frac{1}{2}[-\frac{1}{dI(t))^2}(dI(t))^2]. \tag{3.21}$$

Noting that $(dI(t))^2 = (I(t))^2(\sigma_I^2 dt)$, we obtain

$d\log I(t) = \frac{1}{I(t)} dI(t) + \frac{1}{2}[-\sigma_I^2 dt]$,

Replacing $dI(t)$ by its original expression, we find the following:

$$d\log I(t) = ((r_N(t) - r_R(t))dt + \sigma_I dW_I(t)) + \tfrac{1}{2}[-\sigma_I^2 dt],$$

We integrate over the interval $[0, t]$

$$\int_0^t d(\log I(t)) = \int_0^t \left((r_N(s) - r_R(s)) - \frac{1}{2}\sigma_I^2)ds + \sigma_I dW_I(s)\right), \tag{3.22}$$

The final expression is

$$I(t) = i \exp\left(\int_0^t (r_N(s) - r_R(s))ds - \frac{1}{2}\sigma_I^2 t + \sigma_I dW_I(t)\right). \tag{3.23}$$

□

In the next Figure 3.1 the Consumer Price Index $I(t)$ has been plotted over a period of 10 years.

Table 3.1: Simulated values for Consumer Price Index $I(t)$

$r_N(t)$	$r_R(t)$	σ_I	$i = I(0)$	T
0.05	0.03	0.03	100	10

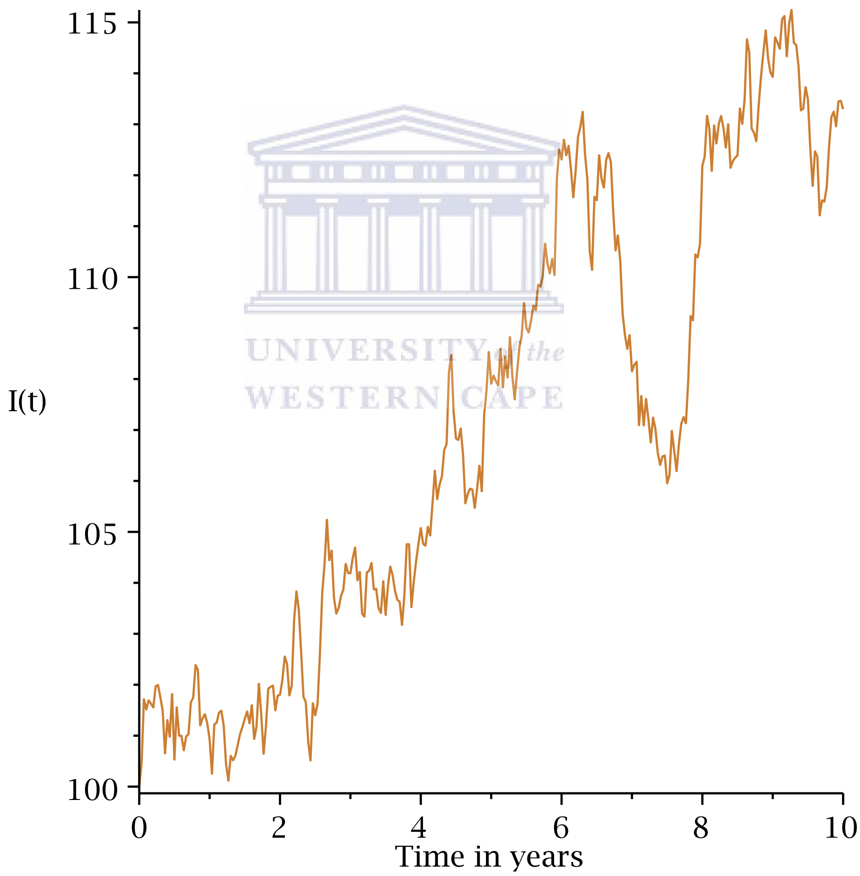

Figure 3.1: Consumer price index $I(t)$ as a geometric Brownian motion under the risk probability measure Q

3.3 Management of pension funds with inflation linked bonds

Three major categories of inflation-linked products could be clearly distinguished: Inflation-linked bonds, inflation swaps and inflation-structured products (see [4]).
Assume that the price of the inflation linked bond is derived with a Black-Scholes model using the real interest rate r_R seen in equation (3.11). Assume also that the price of the inflation-linked bonds and the inflation index are related to each other, then it would safisfy the following

$$\begin{aligned} \frac{dB(t,I(t))}{B(t,I(t)} &= r_N dt + \frac{dI(t)}{I(t)} \\ &= (r_R + \sigma_I \theta_I)dt + \sigma_I dW_I(t) \end{aligned} \tag{3.24}$$

with

$$\theta_I \text{ the market price of risk.}$$

Proposition 3.3.1. *The price process (3.16) is solved by*

$$B(t,I(t)) = B(0,I(0))\exp\left\{\left(r_R + \sigma_I\theta_I) - \frac{1}{2}\sigma_I^2\right) t + \sigma_I W_I(t)\right\}. \tag{3.25}$$

Proof. The inflation linked bonds follows this SDE

$$dB(t,I(t)) = B(t,I(t))\left((r_R + \sigma_I\theta_I)dt + \sigma_I dW_I(t)\right). \tag{3.26}$$

Letting $f(t,I(t)) = \log B(t,I(t))$,

$$dB(t,I(t)) = \frac{1}{B(t,I(t))}dB(t,I(t)) + \frac{1}{2}\left(\frac{-1}{B(t,I(t))^2}(dB(t,I(t)))^2\right). \tag{3.27}$$

Consider the following relation

$$(dB(t,I(t)))^2 = (t,I(t))^2(\sigma_I^2 dt).$$

This leads to

$$dB(t, I(t)) = \left[\left((r_R + \sigma_I \theta_I) - \frac{1}{2}\sigma_I^2 \right) dt + \sigma_I dW_I(t) \right]. \tag{3.28}$$

Integrating over the interval $[t, 0]$ the relation above becomes:

$$\int_0^t d[\log B(t, I(t))] = \int_0^t \left((r_R + \sigma_I \theta_I) - \frac{1}{2}\sigma_I^2 \right) dt + \int_0^t \sigma_I dW_I(t). \tag{3.29}$$

$$\log \left(\frac{B(t, I(t))}{B(0, I(0))} \right) = \left\{ \left((r_R + \sigma_I \theta_I) - \frac{1}{2}\sigma_I^2 \right) t + \sigma_I W_I(t) \right\}, \tag{3.30}$$

$$\left(\frac{B(t, I(t))}{B(0, I(0))} \right) = e^{\left\{ \left((r_R + \sigma_I \theta_I) - \frac{1}{2}\sigma_I^2 \right) t + \sigma_I W_I(t) \right\}}. \tag{3.31}$$

The following solves the SDE

$$B(t, I(t)) = B(0, I(0)) \exp \left\{ \left((r_R + \sigma_I \theta_I) - \frac{1}{2}\sigma_I^2 \right) t + \sigma_I W_I(t) \right\} \tag{3.32}$$

with $B(0, I(0)) = 1$ yields

$$B(t, I(t)) = \exp \left\{ \left((r_R + \sigma_I \theta_I) - \frac{1}{2}\sigma_I^2 \right) t + \sigma_I W_I(t) \right\}. \tag{3.33}$$

□

Taking the conditional expectation of $B(t, I(t))$ will result in

$$\begin{aligned}
\mathbb{E}[B(t, I(t))] &= \mathbb{E}\left[\exp\left\{\left((r_R + \sigma_I\theta_I) - \frac{1}{2}\sigma_I^2\right) t + \sigma_I W_I(t)\right\}\right] \\
&= \exp\left\{\left[(r_R + \sigma_I\theta_I) - \frac{1}{2}\sigma_I^2\right] t\right\} \mathbb{E}\left[\exp\left(\sigma_I W_I(t)\right)\right] \\
&= \exp\left\{\left[(r_R + \sigma_I\theta_I) - \frac{1}{2}\sigma_I^2\right] t\right\} \exp\left\{0 + \frac{1}{2}\sigma_I^2 t\right\} \\
&= \exp\left[\left(r_R + \sigma_I\theta_I\right) t\right]. \qquad (3.34)
\end{aligned}$$

In Figure 3.2 inflation-linked-bond has been plotted and simulated over a period of 10 years. It is quite interesting to investigate how the bond follows its expected value. The real return of such bond invested covers inflation risk.

Table 3.2: Parameter values for investment on inflation-linked bonds $B(t, I(t))$

$B(0, I(0))$	$r_R(t)$	σ_I	θ	T
1	0.07	0.08	0.04	10

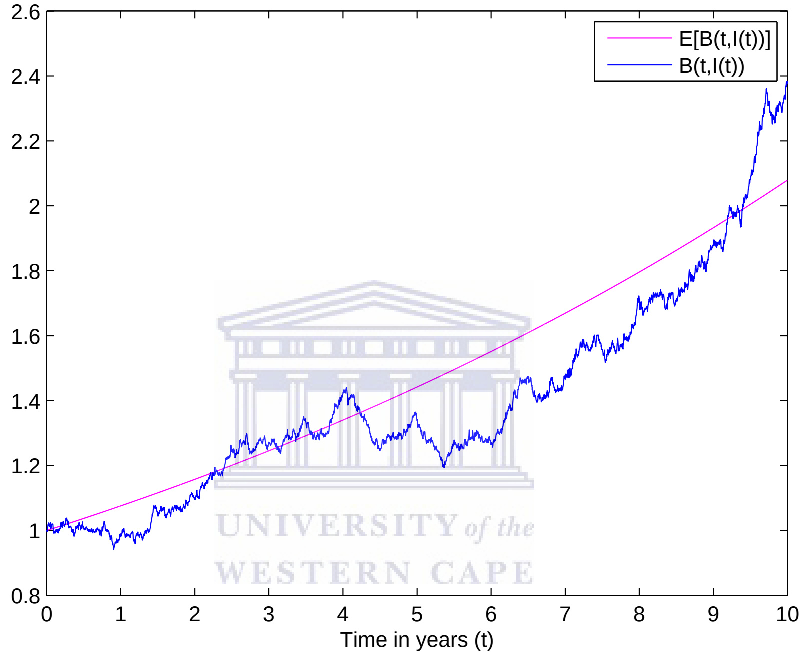

Figure 3.2: Simulation of inflation linked bond invested by the insurance company

3.3.1 Investing in money market account

In addition to inflation-linked bonds, the pension company invests also in a riskless money market account $S_0(t)$ which is nothing other than a bank account offering an interest rate which coincides with the real interest rate. The proof of the following proposition is straightforward and we omit it.

Proposition 3.3.2. *Suppose that the price process of $S_0(t)$ satisfies*

$$\frac{dS_0(t)}{S_0(t)} = r_R(t)dt, \;\; S_0(0) = 1. \tag{3.35}$$

Then the solution of this differential equation is given by

$$S_0(t) = \exp\left\{\int_0^t r_R(u)du\right\}. \tag{3.36}$$

3.3.2 Investing in a classical stock

Two different approaches which link the stock price to inflation (see Battachio [3]):

1. The stock price index can be considered as an inflation forecaster;

2. The stock price index can be considered as a variable following the inflation level.

For convenience, we view the stock price as a variable following the inflation index and therefore we set up the link between inflation index and stock price.

Proposition 3.3.3. *Suppose that stock price satisfies the following stochastic differential equation*

$$\frac{dS(t)}{S(t)} = \mu_1(t)dt + \sigma_s^I dW_I(t) + \sigma_s^S dW_s(t) \tag{3.37}$$

where $\mu_1(t)$ and $\sigma_s = \left(\sigma_s^I, \sigma_s^S\right)^T$ are assumed to be constants, while $W(t) = (W_I(t), W_S(t))^T$ is a two dimensional Brownian motion. We allow for correlation between inflation index and the stock price.

The stochastic differential equation (3.37) *is solved by*

$$S(t) = S(0)\exp\left\{\left[\mu_1(t) - \frac{1}{2}\left((\sigma_s^I)^2 + (\sigma_s^S)^2\right)\right]t + \sigma_s^I W_I(t) + \sigma_s^S W_S(t)\right\}. \tag{3.38}$$

Proof. The stock price satisfies:

$$\frac{dS(t)}{S(t)} = \mu_1(t)dt + \sigma_s^I dW_I(t) + \sigma_s^S dW_S(t), \tag{3.39}$$

$$dS(t) = S(t)\left(\mu_1(t)dt + \sigma_s^I dW_I(t) + \sigma_s^S dW_S(t)\right). \tag{3.40}$$

Letting $f(t, S(t)) = \log S(t)$,

$$d\log S(t) = \frac{1}{S(t)} dS(t) + \frac{1}{2}\left[\frac{-1}{(S(t))^2}\left(dS(t)\right)^2\right]. \quad (3.41)$$

Considering the relation below

$$(dS(t))^2 = (S(t))^2 \left(\sigma_s^I + \sigma_s^S\right) dt, \quad (3.42)$$

$$\frac{(dS(t))^2}{(S(t))^2} = \left(\sigma_s^I + \sigma_s^S\right) dt, \quad (3.43)$$

$$d\log S(t) = \left(\mu_1(t)dt + \sigma_s^I dW_I(t) + \sigma_s^S dW_s(t)\right) - \frac{1}{2}\left((\sigma_s^I)^2 + (\sigma_s^S)\right) dt, \quad (3.44)$$

$$d\log S(t) = \left(\mu_1(t) - \frac{1}{2}(\sigma_s^I)^2 - \frac{1}{2}(\sigma_s^S)^2\right) dt + \sigma_s^I dW_I(t) + \sigma_s^S dW_S(t). \quad (3.45)$$

Integrating over $[t, 0]$ results in

$$\int_0^t d\log S(t) = \int_0^t \left(\mu_1(u) - \frac{1}{2}(\sigma_s^I)^2 - \frac{1}{2}(\sigma_s^S)^2\right) dt + \int_0^t \left(\sigma_s^I dW_I(u) + \sigma_s^S dW_S(u)\right) . \quad (3.46)$$

$$\log\left(\frac{S(t)}{S(0)}\right) = \left(\mu_1(t) - \frac{1}{2}(\sigma_s^I)^2 - \frac{1}{2}(\sigma_s^S)^2\right) t + \sigma_s^I W_I(t) + \sigma_s^S W_S(t) \ , \quad (3.47)$$

$$\left(\frac{S(t)}{S(0)}\right) = \exp\left\{\left(\mu_1(t) - \frac{1}{2}(\sigma_s^I)^2 - \frac{1}{2}(\sigma_s^S)^2\right) t + \sigma_s^I W_I(t) + \sigma_s^S W_S(t)\right\} . \quad (3.48)$$

The Itô's expression for our SDE is:

$$S(t) = \exp\left\{\left(\mu_1(t) - \frac{1}{2}(\sigma_s^I)^2 - \frac{1}{2}(\sigma_s^S)^2\right)t + \sigma_s^I W_I(t) + \sigma_s^S W_S(t)\right\}. \quad (3.49)$$

□

This allows us to present the volatility matrix that corresponds to the two risky assets. The matrix is given by

$$\sigma : \begin{pmatrix} \sigma_I & 0 \\ \sigma_s^I & \sigma_s^S \end{pmatrix}. \quad (3.50)$$

A straight forward computation shows that the determinant of this matrix is different from zero, that is to say, det $(\sigma) = \sigma_I.\sigma_s^S \neq 0$.

We assume that there exists a stock market price of risk given by

$$\theta(t) = (\sigma(t))^-1(\mu_1(t) - r_R(t)). \quad (3.51)$$

An additional assumption to (3.51) is

$$\int_0^T \|\theta\|^2 \, dt < \infty. \quad (3.52)$$

This assumption is a very standard one in modelling which would imply that the stochastic exponential $Z_0(.)$ below to have suitable boundedness.

$$Z_0(t) = \exp\left\{-\int_0^t \theta'(u)dW(u) - \frac{1}{2}\int_0^t \|\theta(u)\|^2 \, du\right\}. \quad (3.53)$$

Let us define the stochastic process $H_0(t)$ as the quotient:

$$H_0(t) = \frac{Z_0(t)}{S_0(t)}$$

,

which expands to the following

$$H_0(t) = \frac{Z_0(t)}{S_0(t)} = \exp\left\{-\int_0^t \theta'(u)dW(u) - \frac{1}{2}\int_0^t \|\theta(u)\|^2 \, du - \int_0^t r_R(u)du\right\}. \quad (3.54)$$

Then $H_0(t)$ is also strictly positive on the interval $[0, T]$. It is worth to note that the assumption (3.56) does not automatically imply that $Z_0(t)$ will be a true martingale. If $Z_0(t)$ is only a strictly local martingale, then the variability of equilibrium and asset pricing will depend on the existence of credit constraints for economic agents (see Lim and Wong [36]).

In the next Figure 3.3 we simulate the evolution of the stock invested by the pension company. The uncertainty in the financial market is driven by the Brownian motion.

Table 3.3: Parameter values for investing in the classical stock $S(t)$

$S(0)$	$\mu_1(t)$	σ_s^I	σ_s^S	T
1	0.05	0.08	0.07	10

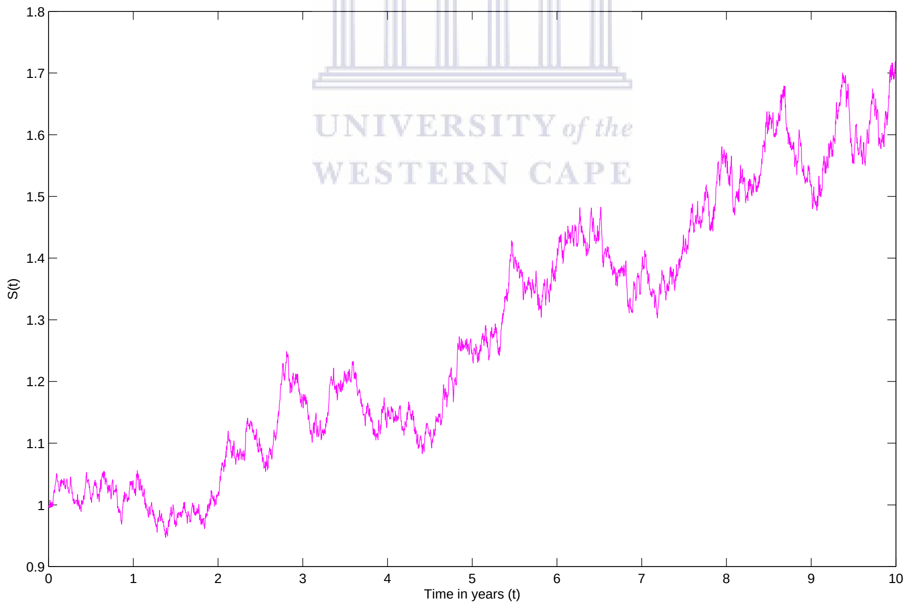

Figure 3.3: Simulation of the stock driven by geometric Brownian motion path.

Chapter 4

Management of pension funds

4.1 Introduction

As mentioned in chapter 1, there are two principal alternatives in pension plan designs with regard to the assignment of risk: defined-contribution and defined-benefit (see [7]). In this chapter we cover the defined-contribution scheme and the salary process, asset allocation, investment and portfolio process, the contribution rate, and the guarantee. Thus we focus on the following references: Menoncin [40], Boulier [7], Zhang et al. [47]. Additionally, our dissertation introduces the notion of inflationary allowances during the pension payment, the notion of elasticity in which the aim is to view the sensitivity reaction of asset allocation due to inflation and the guarantee.

4.2 The defined contribution and the salary process

A defined contribution plan is a scheme where only contributions are fixed and benefits depend on the returns on the assets of the funds. The risk derived from the fund management is borne by the beneficiary. This is unlike to the defined benefit plan where the benefits are normally related to the final salary level and the financial risk is assumed by the sponsor agent. The main objective of the shareholder in the defined contribution plan is to maximize the expected utility obtained from fund accumulation at a fixed date t.

The contribution rate $c(t)$ is exogenous to this optimization process, since it is generally determined by salary. In this dissertation we focus on the defined contribution plans where the guarantee is on the final salary when the employee retires, and therefore this guarantee depends on the behaviour of stochastic interest rate, inflation and recession. Moreover, we will assume in this section that the instantaneous mean of salaries is such that $\mu_Y(t) = (r_N - r_R + m)$, where m is also a constant. Note that the mean of salary $\mu_Y(t)$ is decomposed into two parts such that $(r_N - r_R)$ is set in order to adjust the workers' salaries for inflation and m is set in order to adjust the workers' salaries for economic growth and welfare, see Zhang [47].

Proposition 4.2.1. *We assume that the salary of a pension plan member follows the stochastic differential equation*

$$dY(t) = Y(t)\left(\mu_y(t)dt + \sigma_y^I dW_I(t) + \sigma_y^S dW_S(t)\right), \; Y(0) = y \tag{4.1}$$

for constant volatilities σ_y^I and σ_y^S.
The solution for this SDE is

$$Y(t) = Y(0)\exp\left\{\left(\mu_Y(t) - \frac{1}{2}(\sigma_y^I)^2 - \frac{1}{2}(\sigma_y^S)^2\right)t + \left(\sigma_y^I W_I(t) + \sigma_y^S W_S(t)\right)\right\}. \tag{4.2}$$

Proof. Let us introduce logarithms. Then, an application of Itô results to

$$d\log Y(t) = \frac{1}{Y(t)}d(Y(t)) + \frac{1}{2}\left[-\frac{1}{Y^2(t)}(dY(t))^2\right] \tag{4.3}$$

and

$$[dY(t)]^2 = Y^2(t)(\sigma_y^I + \sigma_y^S)^2 dt, \tag{4.4}$$

$$d\log Y(t) = \mu_y(t)dt + \sigma_y^I dW_I(t) + \sigma_y^S dW_s(t) - \frac{1}{2}\left((\sigma_y^1)^2 + (\sigma_y^2)^2\right)dt. \tag{4.5}$$

Rearranging the above leads to

$$d\log Y(t) = \left(\mu_y(t) - \frac{1}{2}\left((\sigma_y^I)^2 + (\sigma_y^S)^2\right)\right)dt + \sigma_y^I dW_I(t) + \sigma_y^S dW_S(t). \tag{4.6}$$

Integrating within the interval $[0, t]$ will result to

$$\int_0^t (\log Y(t)) = \int_0^t \left(\mu_y(u) - \frac{1}{2}\left((\sigma_y^1)^2 + (\sigma_y^2)^2\right)\right) dt + \int_0^t \sigma_y^1 dW_I(u) + \int_0^t \sigma_y^2 dW_S(u) \ . \tag{4.7}$$

$$\log\left(\frac{Y(t)}{Y(0)}\right) = \left(\mu_y(t) - \frac{1}{2}(\sigma_y^I)^2 - \frac{1}{2}(\sigma_y^S)^2\right) t + \sigma_y^I W_I(t) + \sigma_y^S W_S(t) \ , \tag{4.8}$$

$$\frac{Y(t)}{Y(0)} = \exp\left\{\left(\mu_y(t) - \frac{1}{2}(\sigma_y^I)^2 - \frac{1}{2}(\sigma_y^S)^2\right) t + \left(\sigma_y^I W_I(t) + \sigma_y^S W_S(t)\right)\right\}, \tag{4.9}$$

We finally obtain

$$Y(t) = Y(0) \exp\left\{\left(\mu_y(t) - \frac{1}{2}(\sigma_y^I)^2 - \frac{1}{2}(\sigma_y^S)^2\right) t + \left(\sigma_y^I W_I(t) + \sigma_y^S W_S(t)\right)\right\}. \tag{4.10}$$

□

The equation (4.10) presents two Brownian motions W_I and W_S and three stochastic key-variables: inflation index $I(t)$, stock price $S(t)$ and salary $Y(t)$. We can possibly express any of these as a function of others, multiplied by a deterministic function.

In the following we derive cross correlation matrices between:

inflation index $I(t)$ and salary $Y(t)$

$$\sum\nolimits^{I,Y} = \begin{pmatrix} \sigma_I & \sigma_y^I \\ 0 & \sigma_y^S \end{pmatrix}, \tag{4.11}$$

stock prices $S(t)$ and inflation index $I(t)$

$$\sum\nolimits^{S,I} = \begin{pmatrix} \sigma_s^I & \sigma_I \\ \sigma_s^S & 0 \end{pmatrix}, \tag{4.12}$$

stock prices $S(t)$ and salary $Y(t)$

$$\sum\nolimits^{S,I} = \begin{pmatrix} \sigma_s^I & \sigma_y^I \\ \sigma_s^S & \sigma_y^S \end{pmatrix}. \tag{4.13}$$

One expresses the cross correlation cofactors as

$$\eta_1 = -\frac{\left|\sum^{I,Y}\right|}{\left|\sum^{S,I}\right|} = -\frac{\sigma_y^S}{\sigma_s^S} \ , \tag{4.14}$$

$$\eta_2 = -\frac{\left|\sum^{S,Y}\right|}{\left|\sum^{S,I}\right|} = \frac{\left|\sum^{S,Y}\right|}{\sigma_s^S.\sigma_I}. \tag{4.15}$$

Let us define another term η_0 as below by dependent upon as below η_1 and η_2:

$$\eta_0 = \left(\mu_y(t) - \frac{(\sigma_y^I)^2 + (\sigma_y^S)^2}{2}\right) - \left(\mu_1(t) - \frac{(\sigma_s^I)^2 + (\sigma_s^S)^2}{2}\right)\eta_1 - \left(r_R + \theta_I \sigma_I - \frac{1}{2}\sigma_I^2\right)\eta_2 \tag{4.16}$$

The following proposition is as appears together with proof in the paper of Zhang et al. [47].

Proposition 4.2.2. *The correlated stochastic variables can be defined through the following equation:*

$$\frac{Y(t)}{Y(0)} = e^{\eta_0 t}\left(\frac{S(t)}{S(0)}\right)^{\eta_1}\left(\frac{I(t)}{I(0)}\right)^{\eta_2}. \tag{4.17}$$

In the first equation, we express the salary as a function of the stock price and inflation index. In a more formal way, we think on how much the salary process is being affected by the stock price and the inflation index. Economists define the real salary as the nominal salary divided by consumer price index.

Furthermore, we are able to express these terms as one in connection with the two others. From (4.17) we can derive inflation as a function of stock price and salary by

$$\left(\frac{I(t)}{I(0)}\right) = e^{-\frac{\eta_0}{\eta_2}t}\left(\frac{S(t)}{S(0)}\right)^{-\frac{\eta_1}{\eta_2}}\left(\frac{Y(t)}{Y(0)}\right)^{\frac{1}{\eta_2}}. \tag{4.18}$$

In (4.18) it is easy to check how higher prices or unfixed prices will necessary result in high inflation.

Definition 4.2.3. The degree to which a factor reacts to changes in others is referred to as *elasticity* (see Klein [33]).

This is more exciting if we could be interested in knowing the impact of the recession or inflation on the asset allocation.

Proposition 4.2.4. *Suppose that* $\epsilon_{y,s}$ *is the elasticity of salary process with respect to stock price or inflation index, and we note*

$$\epsilon_{y,s} = \frac{\partial\left(\frac{Y(t)}{Y(0)}\right)}{\partial\left(\frac{S(t)}{S(0)}\right)}\frac{\left(\frac{S(t)}{S(0)}\right)}{\left(\frac{Y(t)}{Y(0)}\right)} \tag{4.19}$$

Then

$$\epsilon_{y,s} = \eta_1 .$$

Proof.

$$\left(\partial\left(\frac{Y(t)}{Y(0)}\right)\Big/\partial\left(\frac{S(t)}{S(0)}\right)\right) = \eta_1 e^{\eta_0 t}\left(\frac{S(t)}{S(0)}\right)^{\eta_1 - 1}\left(\frac{I(t)}{I(0)}\right)^{\eta_2} \tag{4.20}$$

Then the coefficient of elasticity is given by

$$\epsilon_{y,s} = \eta_1 e^{\eta_0 t}\left(\frac{S(t)}{S(0)}\right)^{\eta_1 - 1}\left(\frac{I(t)}{I(0)}\right)^{\eta_2}\frac{\left(\frac{S(t)}{S(0)}\right)}{\left(\frac{Y(t)}{Y(0)}\right)}, \tag{4.21}$$

$$\epsilon_{y,s} = \eta_1 e^{\eta_0 t} \frac{\left(\frac{S(t)}{S(0)}\right)^{\eta_1}}{\left(\frac{S(t)}{S(0)}\right)} \left(\frac{I(t)}{I(0)}\right)^{\eta_2} \frac{\left(\frac{S(t)}{S(0)}\right)}{\left(\frac{Y(t)}{Y(0)}\right)} . \tag{4.22}$$

Substituting (4.17) in (4.22) yields

$$\epsilon_{y,s} = \eta_1 e^{\eta_0 t} \frac{\left(\frac{S(t)}{S(0)}\right)^{\eta_1} \left(\frac{I(t)}{I(0)}\right)^{\eta_2}}{e^{\eta_0 t} \left(\frac{S(t)}{S(0)}\right)^{\eta_1} \left(\frac{I(t)}{I(0)}\right)^{\eta_2}} . \tag{4.23}$$

This final expression shows that $\epsilon_{y,s}$ is equal to η_1 after simplification. □

Therefore, from (4.17) it follows that

$$\left(\frac{S(t)}{S(0)}\right) = e^{-\frac{\eta_0}{\eta_1} t} \left(\frac{I(t)}{I(0)}\right)^{-\frac{\eta_2}{\eta_1} t} \left(\frac{Y(t)}{Y(0)}\right)^{\frac{1}{\eta_1}} . \tag{4.24}$$

In (4.24) the stock price is viewed as something which partly measures the state of production of the economy. In this regard and due to the assumption of completeness of markets, by the introduction of some other derivatives in the financial market, our study includes a model where the stock real return is driven by other underlying economic variables such as unemployment and inflation.

4.3 Asset allocation, investment and portfolio process

In this pension asset allocation process, we also show that the fund manager faces some randomness due to:

- the stochastic interest rate due to the long run investment,
- the stochastic behavior of inflation-recession in the economy.

We denote by $\pi_1(t)$ and $\pi_2(t)$, the proportion of pension funds invested respectively in the money market account and the stock. The remainder $1-\pi_1(t)-\pi_2(t)$ goes to the inflation linked bonds. We suppose that $\{(\pi_1(t), \pi_2(t)) : t > 0\}$ is a Markovian control adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfying (see [30])

$$\mathbb{E}\int_0^t (\pi_1^2(t) + \pi_2^2(t))\, dt < \infty.$$

Proposition 4.3.1. *The portfolio process that corresponds to this investment, which is denoted by $X(t)$, will be governed by the following equation*

$$dX(t) = r_R(t)X(t)dt + \pi'(t)(\mu_i(t) - r_R(t))dt + \sigma'(t)\pi(t)dW(t) \tag{4.25}$$

we write also

$$dX(t) = \sum_{i=0}^{K} \frac{\pi_i(t)}{S_i(t)} dS_i(t), \text{ with } X(0) = x \tag{4.26}$$

and to ensure that this is self-financing we will require that (see [36])

$$X(t) = \pi_0(t) + \pi'(t).$$

Definition 4.3.2. A progressively measurable, self-financing portfolio strategy $\pi(.)$ with initial wealth $x > 0$ is called *admissible* if $X(t) \geq 0$, almost surely, for $0 \leq t \leq T$ (see [36]). The class of admissible portfolio strategies $\pi(.)$ with initial value x will be denoted by $\mathcal{A}(x)$.

For any portfolio $\pi(.)$ with terminal value $X(T) > 0$ and initial value x, we have the following budget constraint as a consequence of the supermartingale property of $H_0(.)X(.)$.

Definition 4.3.3. A *budget constraint* for a portfolio with terminal value $X(T)$ is

$$\mathbb{E}[H_0(T)X(T)] \geq x. \tag{4.27}$$

Consider a non-negative, $\mathcal{F}_T$ measurable random variable χ, and constraint $X > 0$ such that $\mathbb{E}[H_0(T)\chi] = x$. Then there exist a portfolio process $\pi(.) \in \mathcal{A}(t)$ with associated terminal value $\chi = X(T)$. Thus, with the aid of the martingale representation theorem (see Karatzas and Shreve [31]), we can find the exact form of $\pi(.)$. In practice $\pi(.)$ can be computed using the Malliavin calculus. Some related results can be seen in Broadie and Glasserman [8] and Glasserman [26].

4.3.1 The contribution rate $c(t)$, $t \in [0, t]$

Suppose that each employee decides to put a constant proportion of salary into the personal pension fund. Then the defined contribution level related to this last will be as follows (see [15])

$$c(t) = c(0)e^{it} \ \forall t \in [0, T] \tag{4.28}$$

where i is a constant.

Note that the equation (4.28) should solely mean the case of an employee who puts a constant part of his wage into the fund, wages which are subject to increase continuously at the inflation rate i. Suppose that the contributor pays a flow to the pension fund. This flow will consist in a lump of sum at date 0, denoted by X_0, and a continuous paid premium at the rate $c(t)$, $t \in [0, T]$. Then the flow of contributions is assumed to be a non-negative, progressive measurable process such that

$$\int_0^T C^2(t)dt < \infty, a.s \tag{4.29}$$

This simply means that the (time=0) value of the cash given by the worker or contributor is equal to

$$X_0' = X_0 + \mathbb{E}\left[\int_0^T H(s)c(s)ds\right]. \tag{4.30}$$

At the final date T, the fund manager will provide to the contributor in exchange a benefit which consists of two parts, as follows.

1. The first part $G(T)$ is guaranteed, which means that the benefit will be greater than $G(T)$ with probabilty one a.s. In particular, we require here the guarantee to be a positive random variable $\mathcal{F}_T$ measurable which is L^P integrable with $p \geq 2$ (see [15]). Note that this assumption allows for the case of a stochastic guarantee whose value will be known at time T, for instance, salary indexed.

2. The second part of the benefit is a fixed fraction of the surplus $Y_T(G(T))$, which is the

difference between the final wealth X_T of the managed portfolio and the guarantee $G(T)$.

4.3.2 The guarantee $G(T)$

The guarantee $G(T)$ is a contingent claim to the state of the markets at the date T, and therefore $\mathcal{F}(T)$ measurable random variable a.s. The most simple guarantee we may consider is the plan member's salary at the retirement date. Now assuming that the fund gives a life annuity to its retired members, then protection must be put on the annuity. Let

$$f(t)\ \forall t \in [T, T'] \tag{4.31}$$

be this minimal annuity where T' is the date of death. Then by hypothesis, the value of the guarantee is given by the following equation

$$G(T) = \int_T^{T'} f(s)B(T,s)ds \tag{4.32}$$

where $B(T,s)$ expresses the deflators.
Note that as $B(T,s)$ depends on the short interest rate observed at the retirement date, the guarantee $G(T)$ is also a function of this rate.

Chapter 5

Optimization problem and Solution

5.1 Introduction

In this chapter we provide an extensive explanation of the general optimization problem using a benchmarking approach. In insurance and pension applications, the benchmark can be a function of any target to achieve, or any contractual liability or again any ratio. Benchmarking is used in fund management and especially for asset allocation. In this dissertation some of the major problems associated with benchmarking are dealt with, including the risk involved and measures taken to deal with the inherent risk, such as risk adjustment. The use of benchmarking has become a common approach for enhancing the performance of companies. In applying those benchmarks, firms tend to compare their own activities and performance to that of others. Benchmarking is also the process of comparing a particular company with a group of benchmark companies. In fact, the benchmarking setup clearly reflects exactly where the company stands relative to its competitors. For additional discussion on benchmarking, we refer to the papers of Lim and Wong [36], Ansell et al. [2] and Hinz et al. [27].

5.2 Benchmark and benchmarking function

Due to the fact that uncertainty in our framework is driven by the Brownian motions, we can model our benchmark, denoted by ζ, with a known distribution. Particularly, ζ will have a gamma distribution very commonly used in pension modeling. For additional discussion on the construction of diffusions with specific distributions, we can refer to Wong [45], Madan and Yor [38] and Karlin and Taylor [32]. In some applications, the benchmark can be any deterministic or stochastic outcome. In this dissertation, we consider as benchmark the maximun of a random quantity (such as stock index, inflation-linked bonds) to compare with the pension company's asset allocation. The pension company's asset allocation depends on the contributions of the plan member. Thus, the pension company would revise its asset allocation periodically with respect to the benchmark level. Assume that the market has been characterised by inflation-recession. Then the pension company can tolerate shortfalls and possibly borrow the contributions for its asset allocation.

Definition 5.2.1. A *benchmark* ζ is a strictly positive, $\mathcal{F}_T$ measurable random variable satisfying

$$\mathbb{E}[H_0(T)\zeta] < \infty. \tag{5.1}$$

The above equation can be interpreted as a terminal wealth that can be attained with initial wealth x.

We can define a process $Y(.)$ by

$$Y(t) = \frac{1}{H_0(t)}\mathbb{E}[H_0(T)\zeta|\mathcal{F}_t]. \tag{5.2}$$

The starting and the terminal values of $Y(.)$ satisfies

$$Y(0) = y, \ Y(T) = \zeta. \tag{5.3}$$

Let us consider $F\left(\frac{X(T)}{Y(T)}\right)$, where $X(.)$ is the dollar value of our portfolio strategy and $Y(T)$ is the strictly positive result of the benchmark, (see [36]). Almost surely, we view $F(.)$ as a benchmarking function. Suppose that $F(.)$ represents a utility function. Then it would present a concave form, just to say a positive gradient. Note that $F(.)$ is concave when the marginal gains decrease as the benchmarking perfomance increases. This statement is in line with Gerber and Pafuni [24].

In a specific way, we define also a benchmarking function $F(.)$ on the benchmarking terminal wealth as a concave, non-decreasing, upper semicontinuous function $F\colon\ R \to [-\infty, \infty)$. We assume as for the case of standard utility functions, $F(.)$ would satisfy the following properties, [31]:

1. The half-line, dom $(F) \triangleq \{\alpha \in R; F(\alpha) > -\infty\}$ is a non-empty subset of $[0, \infty]$

2. F' is continuous, positive and strictly decreasing on the interior of dom (F), and $F'(\infty) = \lim_{\alpha\to\infty} F'(\alpha) = 0$.

3. We also set $\bar{\alpha} \triangleq \inf\{\alpha \in R; F(\alpha) > -\infty\}$, with $\bar{\alpha} \in [0, \infty)$

4. The strictly decreasing, continuous function $F'\colon\ (\bar{\alpha}, \infty) \to^{onto} (0, F'(\bar{\alpha}+))$ has a strictly decreasing, continuous inverse $\varPsi\colon\ (0, F'(\bar{\alpha}+)) \to^{onto} (\bar{\alpha}, \infty)$. We further set $\varPsi(\beta) = \bar{\alpha}$ for $F'(\bar{\alpha}+) \leq \beta \leq \infty$. This implies that $\varPsi(.)$ is well defined, finite and continuous on $(0, \infty]$, with

$$F'(\varPsi(\beta)) = \begin{cases} \beta;\ 0 < \beta < F'(\bar{\alpha}+) \\ F'(\bar{\alpha}+),\ \beta \geq F'(\bar{\alpha}+). \end{cases} \tag{5.4}$$

We note here that the above definition is general and we shall deal mostly with a benchmarking function of power type. This last can be presented as follows

$$F(\alpha) = \frac{\alpha^p}{p}, \tag{5.5}$$

with $\alpha > 0$ and for $p = (-\infty, 1)\backslash\{0\}$.

5.3 General optimization problem

Consider a given benchmarking function $F(.)$, the initial wealth x and the benchmark value $Y(.)$. Then the optimization problem can be presented as that of finding an optimal portfolio $\pi^*(.) \in \mathcal{A}'(x)$ for the problem

$$J(x) = \mathrm{Sup}_{\pi(.)\in\mathcal{A}'(x)}\mathbb{E}\left[F\left(\frac{X(T)}{Y(T)}\right]\right) \tag{5.6}$$

of maximizing the expected benckmarked utility from terminal wealth, with

$$\mathcal{A}'(x) = \left\{\pi(.) \in \mathcal{A}(x); \mathbb{E}\left[F^-\left(\frac{X(T)}{Y(T)}\right)\right] < \infty\right\} \tag{5.7}$$

and where

$$f^- \triangleq -\min\{f, 0\}.$$

In order to solve this optimization problem, we apply the martingale methods. The introduction of a benchmark in the optimization would require some adjustment to the standard approach. From a historical perspective, application of martingale techniques to standard asset-only traditional portfolio allocation problems without benchmarking was developed by Cox and Huang [13], Karatzas et al. [31] and Korn and Korn [34]. We provide a heuristic method of derivation of this optimal solution and provide mathematical proofs in details. Then, the martingale approach solves the optimization problem:

$$\max_{\xi\in\mathcal{F}_T}\mathbb{E}\left[F\left(\frac{\xi}{Y(T)}\right)\right] \tag{5.8}$$

$$\text{subject to: } \mathbb{E}\left[H_0(T)\xi\right] \leq x.$$

This problem can be easily understood as the optimal wealth ξ that can be achieved under our portfolio as described in equation (5.7) when our initial wealth is x. From this, one can see once the optimal terminal wealth for (5.8) is found, the optimal portfolio is the one which is achieved, ξ starting from x. Using the Lagragian duality and introducing a Lagrange multiplier $\beta > 0$, we note that

$$\mathbb{E}\left[F\left(\frac{\xi}{Y(T)}\right)\right] + \beta\left(x - \mathbb{E}\left[H_0(T)\frac{\xi}{Y(T)}\right]\right) \tag{5.9}$$

$$
\begin{aligned}
&= \beta x + \mathbb{E}\left[F\left(\frac{\xi}{Y(T)}\right) - \beta H_0(T)Y(T)\frac{\xi}{Y(T)}\right] \\
&\leq \beta x + \mathbb{E}\left[F\left(\frac{\xi}{Y(T)}\right) - \beta H_0(T)Y(T)\frac{\xi}{Y(T)}\right] \\
&= \beta x + \mathbb{E}\left[\text{Sup}_{\vartheta}\left\{F(\vartheta) - \beta H_0(T)Y(T)\vartheta\right\}\right] \\
&= \beta x + \mathbb{E}\left[\text{Sup}_{\vartheta}\left\{F(\vartheta)\beta H_0(T)Y(T) - H_0(T)Y(T)H_0(T)Y(T)\vartheta\right\}\right] \\
&= \beta x + \mathbb{E}\left[\tilde{F}\left(\beta H_0(T)Y(T)\right)\right] \qquad (5.10)
\end{aligned}
$$

where

$$\tilde{F}(\delta) = \text{Sup}_{\vartheta}(F(\vartheta) - \delta\vartheta) \qquad (5.11)$$

is the Legendre-Fenchel transform of $F(.)$ and $\vartheta = \frac{\xi}{Y(T)}$. We need to choose the Lagrange multiplier β that satisfies the budget constraint so that the above inequality holds with equality. Thus, the maximizer over ξ in such a case should be the optimal terminal wealth for the problem (5.8). Furthermore we can see that the conditions on $F(.)$ involve that the supremum in (5.11) is achieved by some $\delta^* = \Psi$ and that

$$\tilde{F}(\delta) = F(\Psi(\delta)) - \delta\Psi(\delta) \qquad (5.12)$$

In this case, we just recall that $\beta H_0(T)Y(T) = \delta$.

Therefore, the maximizer from the equation (5.10) satisties the following when $\beta H_0(T)Y(T)$ is strictly positive:

$$\vartheta^* = \frac{\xi}{Y(T)} = \Psi(\beta H_0(T)Y(T)) \qquad (5.13)$$

Note that from the above identity one can write

$$\xi = Y(T)\Psi(\beta H_0(T)Y(T)) \qquad (5.14)$$

where ξ represents the terminal wealth.

The terminal wealth ξ satisfies the following identity provided that the constant $\beta > 0$:

$$\mathbb{E}[H_0(T)\xi] = \mathbb{E}[H_0(T)Y(T)\Psi(\beta H_0(T)Y(T))] = x. \qquad (5.15)$$

Now let $\mathcal{X}(\beta) := \mathbb{E}[H_0(T)Y(T)\Psi(\beta H_0(T)Y(T))]$ and as $\beta > 0$, it can be defined through this interval $(0, \infty)$. Straightforward, we define its inverse by $\mathcal{X}(\eta(\alpha)) = \alpha$. The budget constraint (5.9) is satisfied provided that we choose $\beta = \eta(x)$.
Replacing $\beta = \eta(x)$ in (5.14), the optimal terminal wealth becomes

$$\xi = Y(T)\Psi(\eta(x)H_0(T)Y(T)) \tag{5.16}$$

with associated portfolio process $X(.)$ from our definition (4.3.3) .

Theorem 5.3.1. *Suppose*

$$\mathcal{X}(\beta) := \mathbb{E}\left[H_0(T)Y(T)\Psi(\beta H_0(T)Y(T))\right] < \infty; \forall \in (0, \infty) \tag{5.17}$$

and consider initial wealth $x \in (\mathcal{X}(\infty), \infty)$. The optimal benchmarked wealth problem has a unique terminal wealth $X^*(T) = \xi$ given by (5.16) with associated optimal portfolio $\pi^*(.)$, (see [36]).

Lemma 5.3.2. *Assume that the condition (5.17) is satisfied, then $\mathcal{X}(.)$ is non-decreasing and continuous on $(0, \infty)$, and strictly nondecreasing on $(0, \vartheta)$, where, (see [36])*

$$\mathcal{X}(0^+) := \lim_{\beta \downarrow 0} \mathcal{X}(\beta) = \infty \tag{5.18}$$

$$\mathcal{X}(\infty) := \lim_{\beta \to \infty} = \mathbb{E}\left[H_0(T)Y(T)\bar{\alpha}\right] \tag{5.19}$$

$$\vartheta = \sup\left\{\beta > 0 : \mathcal{X}(\beta) > \mathcal{X}(\infty)\right\}. \tag{5.20}$$

Proof. We recall here that $H_0(T)Y(T)$ is strictly positive. As $\Psi(.)$ is non-decreasing, it follows that $\mathcal{X}(.)$ is also decreasing. Continuity and (5.18-5.20) follows by the applications of the monotone convergence and dominated convergence theorems. □

Corollary 5.3.3. *For $\beta \in (0, \vartheta), \mathcal{X}(\beta)$ has a strictly decreasing inverse*

$$\eta : (\mathcal{X}(\infty), \infty) \to^{onto} (0, \vartheta) \tag{5.21}$$

such that

$$\mathcal{X}(\eta(\alpha)) = \alpha; \forall \alpha \in (\mathcal{X}(\infty), \infty\} \; . \tag{5.22}$$

5.4 Power benchmarking function

Consider a benchmarking function of power type with $p = (-\infty, 1) \setminus \{0\}$ and let

$$F(\alpha) = \frac{\alpha^p}{p} \; . \tag{5.23}$$

For our choice of constant relative risk aversion (CRRA) benchmarking function we have

$$\Psi(\beta) = (F')^{-1}(\alpha) = \beta^{\frac{1}{p-1}} \tag{5.24}$$

and then

$$\begin{aligned}
\mathcal{X}(\eta) &= \mathbb{E}\left[H_0(T)Y(T)(\eta H_0(T)Y(T))^{\frac{1}{p-1}}\right] \\
&= \eta^{\frac{1}{p-1}}\mathbb{E}\left[H_0(T)Y(T)H_0(T)^{\frac{1}{p-1}}Y(T)^{\frac{1}{p-1}}\right] \\
&= \eta^{\frac{1}{p-1}}\mathbb{E}\left[H_0(T)H_0(T)^{\frac{1}{p-1}}Y(T)Y(T)^{\frac{1}{p-1}}\right] \\
&= \eta^{\frac{1}{p-1}}\mathbb{E}\left[H_0(T)^{1+\frac{1}{p-1}}Y(T)^{1+\frac{1}{p-1}}\right] \\
&= \eta^{\frac{1}{p-1}}\mathbb{E}\left[H_0(T)^{\frac{p}{p-1}}Y(T)^{\frac{p}{p-1}}\right]
\end{aligned} \tag{5.25}$$

with

$$\frac{p}{p-1} < 1 \; . \tag{5.26}$$

By the definition of a benchmark and by the assumption of the financial market, we have

$$P(H_0(T)Y(T) > 0) = 1 \; . \tag{5.27}$$

This implies with the definition of a benchmark that

$$\mathbb{E}\left[H_0(T)^{\frac{p}{p-1}}Y(T)^{\frac{p}{p-1}}\right] \leq 1 + \mathbb{E}\left[H_0(T)Y(T)\right] < \infty \; , \tag{5.28}$$

From the above, we can see that the Theorem (5.17) is satisfied for $Y(.)$ and it follows from (5.25) that

$$\eta(x) = \frac{x^{p-1}}{\left(\mathbb{E}\left[H_0(T)^{\frac{p}{p-1}}Y(T)^{\frac{p}{p-1}}\right]\right)^{p-1}} \ . \tag{5.29}$$

Multiplying both sides by $H_0(T)Y(T)$ yields

$$\eta(x)H_0(T)Y(T) = \frac{x^{p-1}H_0(T)Y(T)}{\left(\mathbb{E}\left[H_0(T)^{\frac{p}{p-1}}Y(T)^{\frac{p}{p-1}}\right]\right)^{p-1}} \ . \tag{5.30}$$

Note that according to (5.25) we write

$$\Psi\eta(x)H_0(T)Y(T) \text{ as } \ \beta^{\frac{1}{p-1}}$$

We know that the optimal terminal wealth is

$$\begin{aligned} X(T) &= Y(T)\Psi(\eta(x)H_0(T)Y(T)) \\ &= Y(T)\left(\frac{x^{p-1}H_0(T)Y(T)}{\left(\mathbb{E}\left[H_0(T)^{\frac{p}{p-1}}Y(T)^{\frac{p}{p-1}}\right]\right)^{p-1}}\right)^{\frac{1}{p-1}} \\ &= \frac{xH_0(T)^{\frac{1}{p-1}}Y(T)^{\frac{p}{p-1}}Y(T)}{\mathbb{E}\left[H_0(T)^{\frac{p}{p-1}}Y(T)^{\frac{p}{p-1}}\right]} \ . \end{aligned} \tag{5.31}$$

We can now compare the optimal terminal wealth with the optimal terminal wealth in a standard optimal portfolio problem without benchmark using power utility function. The terminal wealth of the standard optimization problem is

$$\frac{x(H_0(T))^{\frac{1}{p-1}}}{\mathbb{E}\left[(H_0(T))^{\frac{1}{p-1}}\right]} \ . \tag{5.32}$$

When comparing the two (5.31) and (5.32), we can see that the introduction of benchmark has bent the optimal terminal wealth by a factor equal to $Y(T)^{\frac{p}{p-p}}$. We can now find the limit of the terminal wealth as $p \to -\infty$,

$$\tfrac{p}{p-1} \to 1$$

$\frac{1}{p-1} \to 0$

$$\lim_{p\to-\infty} X(T) = \frac{xY(T)}{\mathbb{E}\left[H_0(T)Y(T)\right]} = \frac{x}{y}Y(T) \tag{5.33}$$

In this regard one can see that the limit of $X(T)$, the optimal wealth when $p \to -\infty$, is nothing other than the benchmark $Y(T)$ scaled by $\frac{x}{y}$.

We recall while progressing that in the limit when $p \to 0$ the benchmarking function becomes this of log type.

Let us now consider a benchmarking function of log type

$$F(\alpha) = \ln \alpha \tag{5.34}$$

For our choice of constant relative risk aversion (CRRA) benchmarking function we have

$$\Psi(\beta) = (F')^{-1}(\alpha) = \frac{1}{\beta} \tag{5.35}$$

and then

$$\chi(\eta) = \mathbb{E}\left[\frac{H_0(T)Y(T)}{\eta H_0(T)Y(T)}\right] = \frac{1}{\eta}\ . \tag{5.36}$$

We write also

$$\eta(\chi) = \frac{1}{x}. \tag{5.37}$$

From the equation (5.14), we write

$$\begin{aligned} X(T) &= Y(T)\Psi(\eta(x)H_0(T)Y(T)) \\ &= \frac{Y(T)}{\frac{1}{x}H_0(T)Y(T)} \\ &= \frac{x}{H_0(T)}\ . \end{aligned} \tag{5.38}$$

In this case, the equation (5.38) totally ignores the benchmark and the optimal portfolio here is the growth optimal portfolio (See Platen [42] and Luenberger [37]).
In view of this result, we proceed as follows: Assume that the objective function is

$$J(x) = \sup_{\pi(.)\in\mathcal{A}'(x)} \mathbb{E}\left[\ln\left(\frac{X(T)}{Y(T)}\right)\right]$$

$$= \left(\sup_{\pi(.)\in\mathcal{A}'(x)} \mathbb{E}[\ln(X(T))]\right) - \mathbb{E}[\ln(Y(T))] \tag{5.39}$$

This is equivalent to a standard log-optimal problem without benchmarking as $Y(.)$ is not affected by the portfolio $\pi(.)$.

Chapter 6

Pension portfolio against a benchmark

6.1 Introduction

This chapter deals with an optimization problem where the objective is an increasing function of the relative performance of the pension company using a benchmark. We derive the terminal wealth strategy with the aid of change of measure techniques. Our main references in this regard are Lim and Wong [36] and Boulier et al. [7].

6.2 Portfolio pension strategy

We derive an option based portfolio pension strategy using a benchmark. We proceed in a similar way as in Black-Scholes pricing for a European option. Our objective is to maximize the expected utility of the pension company using a benchmark portfolio with pay-off

$$Y(T) = \rho \ \max(S(T), K), \tag{6.1}$$

where $S(T)$ and K are respectively the stock price and the strike price at time T, and with ρ a strictly positive constant.

This strategy as described above provides a floor guarantee of ρK. Assume that y is the initial cost of this portfolio. One can write y as

$$y = \mathbb{E}[H_0(T)\rho \ \max(S(T), K)] \ . \tag{6.2}$$

We can refer to $H_0(.)$ as the state price density, [see equation (3.54)]. We also note that y and x are not necessarily equal. The insurance can be characterised by the underfunded situation when $y > x$. This means that the benckmark can not be replicated by the initial wealth.

This application of benchmark in this dissertation can be interpreted as the problem of the relative performance of a portfolio insurance compared to any stochastic outcome. It is important to see that $S(T)$ performs well if it is geater than K. Otherwise, the measure of performance is bad.

Let us now express our pension portfolio benchmark as the power benchmarking function discussed in section (5.4). It follows that the optimal wealth is (see [36])

$$X(T) = \frac{xH_0(T)^{\frac{1}{p-1}}(\rho \ \max(S(T), K))^{\frac{p}{p-1}}}{\mathbb{E}\left[H_0(T)^{\frac{p}{p-1}}(\rho \ \max(S(T), K))^{\frac{p}{p-1}}\right]} \ , \tag{6.3}$$

with associated value process

$$X(T) = \frac{x\frac{1}{H_0(t)}\mathbb{E}\left[H_0(T)^{\frac{1}{p-1}}(\rho \ \max(S(T), K))^{\frac{p}{p-1}}|\mathcal{F}_t\right]}{\mathbb{E}\left[H_0(T)^{\frac{p}{p-1}}(\rho \ \max(S(T), K))^{\frac{p}{p-1}}\right]} \ . \tag{6.4}$$

Note that these formulas express the optimal terminal wealth for general financial market models. Therefore, with the aid of change of measure techniques or martingale techniques one derives specifically in a closed form, the wealth strategy and portfolio allocation with constant coefficients (see Geman et al.[23]) and Bismut-Elworthy formulas (see Elworthy and Li [17] and Qin [43]).

6.3 Derivation of the benchmarking asset allocation strategy

We can derive the benchmarking asset allocation strategy by using change of measure techniques. We decompose some identities (see Karatzas et al. [31]).

$$(H_0(t))^{\frac{p}{p-1}} = m_p(t) \wedge_p (t) \ , \tag{6.5}$$

where

$$m_p(T) = \exp\left\{\frac{p}{1-p} r_R T + \frac{p}{2(1-2)^2}\theta^2 T\right\} \tag{6.6}$$

and

$$\wedge_p(t) = \exp\left\{\frac{p}{(1-p)^2}\theta W(t) - \frac{1}{2}\frac{p}{(1-p)^2}\theta^2 t\right\} \ , \tag{6.7}$$

Note that $\wedge_p(.)$ is a strictly positive martingale. Therefore, one can define a measure $\mathbb{P}_1$ equivalent to $\mathbb{P}$ by

$$\left.\frac{d\mathbb{P}_1}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \wedge_p(T) \ . \tag{6.8}$$

By Girsanov's theorem and under $\mathbb{P}_1$ to define a Brownian motion $W_1(.)$ by

$$W_1(t) = W(t) - \left(\frac{p}{1-p}\theta\right) t \ . \tag{6.9}$$

Thus $S(T)$ is log-normal with parameters

$$\left(\ln(S(0)) + \left(\mu + \frac{p}{1-p}(\mu - r_R) - \frac{1}{2}\sigma^2\right) T, \sigma^2 T\right) \tag{6.10}$$

under $\mathbb{P}_1$.

Also in the same way, we decompose

$$(H_0(t))^{\frac{p}{p-1}}(S_0(t))^{\frac{p}{p-1}} = S^{\frac{p}{p-1}}(0)\varsigma_p(t)\gamma_p(t) \tag{6.11}$$

where

$$\varsigma_p(t) = \exp\left\{\frac{-p}{(1-p)^2}(\mu - r_R)T\frac{p}{2(1-p)^2}(\theta^2+\sigma^2)T\right\} \tag{6.12}$$

and

$$\gamma_p(t) = \exp\left\{\frac{p}{(1-p)^2}(\theta-\sigma)W(t)\frac{-1}{2}\left(\frac{p}{(1-p)^2}(\theta-\sigma)\right)^2 t\right\} \tag{6.13}$$

with $\gamma_p(.)$ a strictly positive martingale.

Hence, we can define a measure $\mathbb{P}_2$ equivalent to $\mathbb{P}$ by

$$\left.\frac{d\mathbb{P}_2}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \varsigma_p(T)\ . \tag{6.14}$$

Let us define now under $\mathbb{P}_2$ a Brownian motion $W_2(.)$ using Girsanov theorem

$$W_2(t) = W(t) - \left(\frac{p}{1-p}\right)(\theta-\sigma)t\ . \tag{6.15}$$

$S(T)$ is log-normal with parameters

$$\left(\ln(S(0)) + \left(\mu + \frac{p}{1-p}(\mu - r_R - \sigma^2) - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right) \tag{6.16}$$

under $\mathbb{P}_2$.

Let $\mathbb{Q}$ be the equivalent local martingale measure which we can also define by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = Z_0(T)\ . \tag{6.17}$$

with Brownian motion $W_Q(.)$ defined by

$$W_Q(t) = W(t) + \theta t. \tag{6.18}$$

Finally, we have defined four equivalent measures respectively $\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2$ and $\mathbb{Q}$. Possibly, we can now apply measure transformation techniques to evaluate some expectations in the following. This leads us to start our derivation by evaluating first the expected value in the denominator of the optimal wealth seen in (6.3). Consequently we have

$$\mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}(\rho\ \max(S(T),K))^{\frac{p}{p-1}}\right] \tag{6.19}$$

$$\begin{aligned}
&= \mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}\rho^{\frac{p}{p-1}}S(T)^{\frac{p}{p-1}}1_{S(T)>k}\right]+\mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}\rho^{\frac{p}{p-1}}K^{\frac{p}{p-1}}1_{S(T)\le k}\right]\\
&= \rho^{\frac{p}{p-1}}\left(S^{\frac{p}{p-1}}(0)\varsigma_p(T)\mathbb{E}_{p2}[1_{S(T)>K}]+K^{\frac{p}{p-1}}m_p(T)\mathbb{E}_{p1}[1_S(T)\le k]\right)\\
&= \rho^{\frac{p}{p-1}}\left(S^{\frac{p}{p-1}}(0)\varsigma_p(T)N(-c_{2.p}(0,T))+K^{\frac{p}{p-1}}m_p(T)N(c_{1.p}(0,T))\right) \qquad (6.20)
\end{aligned}$$

where

$$c_{1.p}(0,T)=\frac{\ln\left(\frac{K}{S(t)}\right)-\left(\mu+\frac{p}{1-p}(\mu-r_R)-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{6.21}$$

and

$$c_{2.p}(0,T)=c_{1.p}(0,T)+\frac{p}{1-p}\sigma\sqrt{T-t}\ . \tag{6.22}$$

Note that $N(.)$ is known as the distribution function of a standard normal random variable. We can also refer to $c_{1.p}(0,T)$ and $c_{2.p}(0,T)$ as the range of values of the integrand of Black-Scholes model in a European call option (see [20]).

In the following, we consider the portfolio allocation $\pi(.)$ where we make use of the martingale technique and Bismut-Elworthy formula which lead us to a closed form solution. Let $X(.)$ be the value process of the optimal terminal wealth seen in (6.4) defined within $0\le t\le T$. Then its associated optimal allocation in the stock or bond index of $\pi(.)$ is given by

$$\frac{x}{\mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}(Y(T))^{\frac{p}{p-1}}\right]}\Delta_p(t)S(t)\ , \tag{6.23}$$

where $\Delta_p(.)$ is the delta of a contingent claim with terminal payoff given by

$$(H_0(T))^{\frac{1}{p-1}}(Y(T))^{\frac{p}{p-1}}\ . \tag{6.24}$$

Let us turn back to the functions $W(T)$ and $S(T)$ seen in (6.9) and (6.10). We notice that

$$\mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}(Y(T))^{\frac{p}{p-1}}\right] \text{ is a function of both } W(T) \text{ and } S(T).$$

The value function for this claim can be presented as $V(S(.),.)$, with

$$V(S(t),t) = \mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}(Y(T))^{\frac{p}{p-1}}\right] \tag{6.25}$$

and

$$\begin{aligned} V(S(t),t) &= \frac{1}{H_0(t)}\mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}(Y(T))^{\frac{p}{p-1}}\middle|\mathcal{F}_t\right] \\ &= (H_0(t))^{\frac{p}{p-1}}\mathbb{E}\left[\left(\frac{H_0(T)}{H_0(t)}\right)^{\frac{p}{p-1}}(Y(T))^{\frac{p}{p-1}}\middle|\mathcal{F}_t\right] \end{aligned} \tag{6.26}$$

We can just think of the following to be the equivalent representation

$$V(S(t),t) = \mathbb{E}_Q\left[\frac{S_0(t)}{S_0(t)}(H_0(T))^{\frac{1}{p-1}}(Y(T))^{\frac{p}{p-1}}\middle|\mathcal{F}_t\right] . \tag{6.27}$$

Determine now $\Delta_p(0)$ using analogous techniques developed throughout our identities. It follows that

$$\mathbb{E}_Q = \left[\frac{1}{(S_0(T))^2}(H_0(T))^{\frac{2}{p-1}}(Y(T))^{\frac{2p}{p-1}}\right] < \infty \tag{6.28}$$

Therefore, we can now apply the Bismut-Elworthy formula, see Fournié et al. [21]. It follows from this formula that

$$\begin{aligned} \Delta_p(0) &= \left.\frac{\partial V(S,0)}{\partial S}\right|_{s=S(0)} \\ &= \mathbb{E}_Q\left[\frac{1}{S_0(T)}(H_0(T))^{\frac{1}{p-1}}(Y(T))^{\frac{p}{p-1}}\frac{W_Q(T)}{s\sigma T}\right]\Bigg|_{s=S(0)} \\ &= \mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}(Y(T))^{\frac{p}{p-1}}\frac{(W(T)+\theta T)}{s\sigma T}\right]\Bigg|_{s=S(0)} . \end{aligned} \tag{6.29}$$

This expression can be extended to

$$\begin{aligned} \Delta_p(0) &= \left[(H_0(T))^{\frac{p}{p-1}}\rho^{\frac{p}{p-1}}S(T)^{\frac{p}{p-1}}1_{S(T)>0}\frac{(W(T)+\theta T)}{s\sigma T}\right]\Bigg|_{s=S(0)} \\ &+ \left[(H_0(T))^{\frac{p}{p-1}}\rho^{\frac{p}{p-1}}K^{\frac{p}{p-1}}1_{S(T)\leq K0}\frac{(W(T)+\theta T)}{s\sigma T}\right]\Bigg|_{s=S(0)} . \end{aligned} \tag{6.30}$$

Now we can decompose (6.30) into two terms

$$\mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}\rho^{\frac{p}{p-1}}K^{\frac{p}{p-1}}1_{S(T)>K}\frac{(W(T)+\theta T)}{S(0)\sigma T}\right] \tag{6.31}$$

$$\begin{aligned} = \quad & \frac{\rho^{\frac{p}{p-1}}S^{\frac{p}{p-1}}(0)\varsigma_p(T)}{S(0)\sigma T}\left(\mathbb{E}_{p_2}[1_{S(T)>K}(W_2(T))]\right. \\ & \left. + \left(\left(\frac{\theta}{1-p}\right) - \left(\frac{p\sigma}{1-p}\right)\right)T\mathbb{E}_{p_2}[1_{S(T)>K}]\right) \end{aligned} \tag{6.32}$$

$$\begin{aligned} = \quad & \frac{\rho^{\frac{p}{p-1}}S^{\frac{p}{p-1}}(0)\varsigma_p(T)}{S(0)\sigma T}\left(\underline{n}(c_{2.p}(0,T))\sqrt{T}\right. \\ & \left. + \left(\left(\frac{\theta}{1-p}\right) - \left(\frac{p\sigma}{1-p}\right)\right)TN(-c_{2p}(0,T))\right) \ . \end{aligned} \tag{6.33}$$

where $\underline{n}(.)$ is the density function of a standard normal random variable Using transformation techniques, we derive the second term as follows.

$$\mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}\rho^{\frac{p}{p-1}}K^{\frac{p}{p-1}}1_{S(T)\leq K}\frac{(W(T)+\theta T)}{S(0)\sigma T}\right] \tag{6.34}$$

$$\begin{aligned} = \quad & \frac{\rho^{\frac{p}{p-1}}K^{\frac{p}{p-1}}(0)m_p(T)}{S(0)\sigma T}\left(\mathbb{E}_{p_1}[1_{S(T)\leq K}(W_1(T))]\right. \\ & \left. + \left(\frac{\theta}{1-p}T\right)\mathbb{E}_{p_1}[1_{S(T)\leq K}]\right) \end{aligned} \tag{6.35}$$

$$\begin{aligned} = \quad & \frac{\rho^{\frac{p}{p-1}}K^{\frac{p}{p-1}}(0)m_p(T)}{S(0)\sigma T}\left(-\underline{n}(c_{1.p}(0,T))\sqrt{T}\right. \\ & \left. + \left(\frac{\theta}{1-p}T\right)N(c_{1,p}(0,T))\right) \end{aligned} \tag{6.36}$$

Thus, from the normal density function's properties, it can be seen that

$$K^{\frac{p}{p-1}}m_p(T)\underline{n}(c_{1.p}(0,T)) = S^{\frac{p}{p-1}}(0)\varsigma_p(T)\underline{n}(c_{2.p}(0,T)), \tag{6.37}$$

Using this identity, we can now obtain $\Delta_p(0)S(0)$ by substitution

$$\begin{aligned}\Delta_p(0)S(0) &= \rho^{\frac{p}{p-1}}e^{\frac{p\theta^2 T}{2(1-p)^2}}\left(S^{\frac{p}{p-1}}(0)e^{\frac{p(r_R-u+\frac{1}{2}\sigma^2)T}{(1-p)^2}}\left(\frac{\theta}{(1-p)\sigma}-\frac{p}{1-p}\right)N(-c_{2,p}(0,T))\right.\\ &\left.+K^{\frac{p}{p-1}}e^{\frac{pr_RT}{(1-p)^2}}\left(\frac{\theta}{(1-p)\sigma}\right)N(c_{1,p}(0,T))\right). \quad (6.38)\end{aligned}$$

We can now derive the allocation by applying calculations analogous to the previous and to the representation of $V(S(t),t)$ seen in (6.26) and (6.27). More precisely, we consider the time horizon $T-t$ conditioning on $\mathcal{F}_t$. The expression becomes

$$\begin{aligned}\Delta_p(t)S(t) &= \rho^{\frac{p}{p-1}}e^{\frac{p\theta^2(T-t)}{2(1-p)^2}}(H_0(t))^{\frac{1}{p-1}}\left(S^{\frac{p}{p-1}}(t)e^{\frac{p(r_R-u+\frac{1}{2}\sigma^2)(T-t)}{(1-p)^2}}\left(\frac{\theta}{(1-p)\sigma}-\frac{p}{1-p}\right)\right.\\ &\left.\times N(-c_{2,p}(t,T))+K^{\frac{p}{p-1}}e^{\frac{pr_R(T-t)}{(1-p)^2}}\left(\frac{\theta}{(1-p)\sigma}\right)N(c_{1,p}(t,T))\right). \quad (6.39)\end{aligned}$$

It follows that the optimal portfolio $\pi(t)$ can be found by substituting (6.39) into (6.23) and also by replacing

$$\mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}(Y(T))^{\frac{p}{p-1}}\right]$$

by its final term by
$\rho^{\frac{p}{p-1}}\left(S^{\frac{p}{p-1}}(0)\varsigma_p(T)N(-c_{2.p}(0,T))+K^{\frac{p}{p-1}}m_p(T)N(c_{1.p}(0,T))\right)$, [see equation (6.20)].

$$\pi(t)=\frac{x}{\mathbb{E}\left[(H_0(T))^{\frac{p}{p-1}}(Y(T))^{\frac{p}{p-1}}\right]}\Delta_p(t)S(t), \quad (6.40)$$

$$\pi(t)=\frac{x(H_0(t))^{\frac{1}{p-1}}}{\left(S^{\frac{p}{p-1}}(0)\varsigma_p(T)N(-c_{2.p}(0,T))+K^{\frac{p}{p-1}}m_p(T)N(c_{1.p}(0,T))\right)}(\alpha+\beta) \quad (6.41)$$

where

$\alpha=S^{\frac{p}{p-1}}(t)e^{\frac{p(r_R-u+\frac{1}{2}\sigma^2)(T-t)}{(1-p)^2}}\left(\frac{\theta}{(1-p)\sigma}-\frac{p}{1-p}\right)N(-c_{2.p}(t,T))$

and

$$\beta = K^{\frac{p}{p-1}} e^{\frac{p r_R (T-t)}{(1-p)^2}} \left(\frac{\theta}{(1-p)\sigma}\right) N(c_{1.p}(t,T)) \quad .$$

Let us now define $\pi(t)$ in the limit as p tends to $-\infty$. We recall here that this limit was explained thoughout the definition of benchmark. In particular, as $p \to -\infty$, we will have

$$\pi(t) = \frac{x.S(t).N\left(-c_{1,\infty}(t,T) + \sigma\sqrt{T-t}\right)}{(S(0)N(-c_{2,\infty}(0,T)) + Ke^{-r_R T}N(c_{1,\infty}(0,T)))} \tag{6.42}$$

where $c_{i,\infty}(t,T) = \lim_{p\to}\infty\ c_{i,p}(t,T)$ for $i = 1,2$. Without loss of generality, one can see that in the limit when $p \to -\infty$ the pension company's portfolio strategy is just the allocation strategy of the benchmark. We also note that in the limit as $p \to 0$, the pension company's portfolio allocation strategy is just the log-optimal strategy,

$$\pi(t) = \left(\frac{x}{H_0(t)}\right)\left(\frac{\theta}{\sigma}\right). \tag{6.43}$$

Chapter 7

Numerical application

We use the benchmarking function of a power type with the parameters listed in the table 7.1. The time horizon T is 10 years. We consider a case of a pension company which invests in a money market account, a stock and an inflation-linked bond. Thus, the market has been characterized by inflation and further recession. Therefore, the pension company can hedge against the risk associated with inflation by investing in inflation-linked bond. We have constructed a stock real return compatible to hedge also against the risk of inflation. Therefore, with the aid of the benchmarking asset allocation approach, we propose the following parameter values and insist on the fact that we consider a market characterized by constants coefficients.

Table 7.1: Parameter values chosen for the benchmarking portfolio allocation strategy .

Parameters	Text references	Values
$r_R(t)$	Real interest rate	0.04
μ	Rate of real return of the risky asset	0.10
σ	Volatility of the risky asset	0.20
K	Final stock price	1
$S(0)$	Initial stock price	1
y	The price of the Benchmark	1.1255
x	Initial wealth	1.1255
p	Risk aversion parameter	$-\infty$
n	Number of points plotted	120
T	Time horizon	10

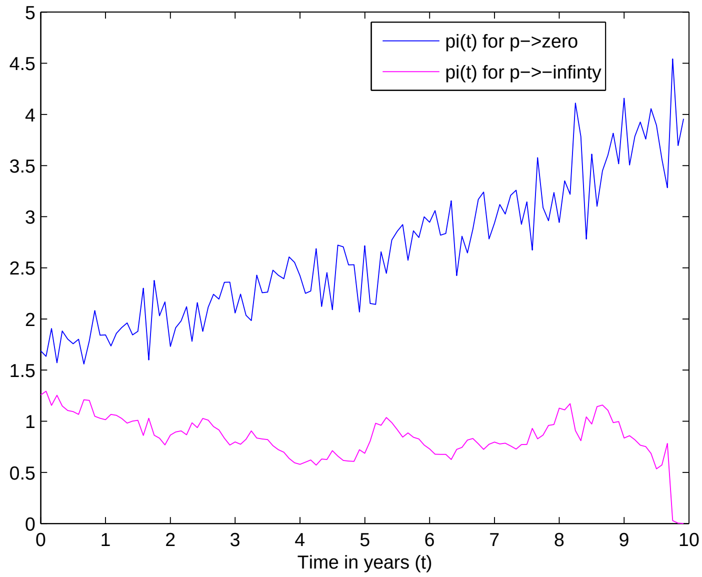

Figure 7.1: The benchmarking asset allocation strategy.

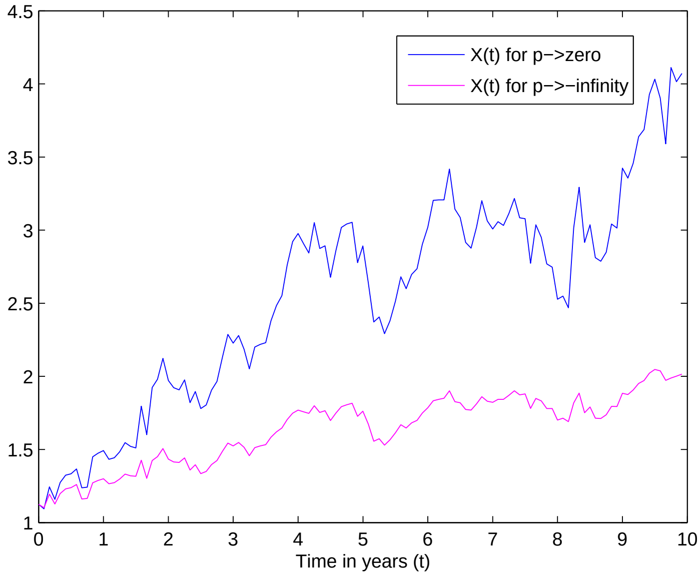

Figure 7.2: The wealth following the benchmarking asset allocation strategy over the period of 10 years.

Note that for the choice of parameters, we have chosen the initial portfolio value by $x(0) = 1.255$, and in the same way we consider the price of the benchmark $y = 1.255$. The first graph plotted shows clearly the evolution of the benchmark investor's asset allocation $\pi(t)$. From these plots, one may observe the benchmark investment strategy or asset allocation $\pi(t)$ with $p = -\infty$, $p = 0$. The pension company's asset allocation on the risky asset may be guaranteed at this stage. The pension company's investment strategy is just the benchmarking asset allocation in an inflationary market.

Then, the second graph plotted shows the wealth strategy $X(t)$ derived from the asset allocation $\pi(t)$ with $p = -\infty$ and $p = 0$. Thus, from this graph, one can observe the impact of the benchmark investor. This is also sure when the pension company's wealth is guaranteed by this outperformance. The pension company's asset allocation strategy is just the benchmarking. The pension company's asset allocation or portfolio allocation and wealth strategy are guaranteed by the benchmarking investment strategy.

In addition, Figure 1.1 shows the evolution of the global inflation. Figure 3.2 shows the evolution of the inflation-linked bonds invested by the insurance company over the period of 10 years. We also plotted the expected value of the bond. One can see that due to the fact that the bond is linked to inflation, the bond invested by the pension company follows the expectation value. Figure 3.3 shows the evolution of the stock invested by the pension company following geometric Brownian motion path. The stock presents a higher return over the period of 10 years invested. Note that in our definition of the stock return, we have said previously that we use a similar real stock return whose value is not affected by the inflation rate.

Chapter 8

Conclusion

In this dissertation, we study a model for managing pension funds with benchmarking in an inflationary market. Our main references in connection with this dissertation are Lim and Wong [36], Zhang et al. [47], Ansell et al. [2], Josa-Fombellida et al. [30], Malliaris and Mullaris [39], Belesti and Korn [4], Deelstra et al. [15] and Etheridge [18]. Pension funds are among the most important institutions in financial markets due to their large capacity of investment and also due to the fact that pension companies complement the role of the Government, allowing workers who have reached retirement age to maintain their standard of living. Pension fund benefits can be seen to decrease due to inflation rate and recession. Therefore, the possible effects of considering inflation volatility and even inflationary allowances on investment become very important. The plan member in a classical defined contribution pension plan experiences a risk linked to inflation which could amount to substantial losses. The pension manager must insure that the benefit from non-inflation-linked pension will be sufficient to cover the future expenses as prices will have increased due to inflation and maybe recession, that is to say stagflation.

Thus, we study how the fund manager maximizes his fund's wealth while investing in an inflationary market. In this regard, we have considered a case of a pension company which invests in stock, inflation linked bond and money market account while basing its investment on the contribution of members. Therefore to hedge against inflation, the pension company invests on inflation linked bonds. We use a benchmarking approach and martingale methods to derive an optimal strategy which maximizes the fund wealth. Under this approach the objective is an increasing function of the relative performance of the asset portfolio compared to a benchmark. We are able to derive in a closed form the benchmarking asset allocation strategy and the wealth strategy using martingale techniques, and more specifically by using change of measure techniques. Using the benchmark approach the pension company is able to increase the relative performance of his asset allocation and to hedge against some risks associated. As a process, benchmarking is an attempt to form a judgement based on objective criteria. It does, however, suffer from a number of drawbacks when used in fund management. First, a single benchmark is inadequate when considering funds with different objectives and of different maturity. Second, it should be recognised that benchmarks have an effect on the way fund managers behave. This may lead to distortion of the market and hence to lower returns on the funds under management. These factors make it necessary for the trustees to agree on appropriate criteria for judging performance with the managers of the funds, before any contract is agreed. The inclusion of risk adjustments for benchmarks adds a further layer of complexity. Whilst analysts can argue about the merit of the various adjustments, it has to be recognised that these are based on past behaviours and may not be a good indicator of the future behaviour. To deal with risk appropriately, it may be more sensible to examine the processes through which the return is achieved. There is a greater need for trustees to understand the risk-reduction and monitoring processes within an organisation. This calls for greater openness between trustees and fund managers. It suggests that trustees should, perhaps, be incorporating organisational features of fund managers into the selection criteria. Performance benchmarks are therefore important for three key reasons: they help to measure the investment performance of institutional fund managers, they provide investors with a reference point for monitoring that performance, and they can also have the effect of modifying the behaviour of fund managers.

Last but not the least, we have considered the role of the guarantee in the risk management of the fund. The guarantee $G(T)$ is a contingent claim on the state of the markets at the date T, and therefore a random variable. In a retirement plan, the most simple guarantee we may consider is the plan member's salary at the retirement date.

Finally, we plotted the benchmark asset allocation strategy and the wealth strategy as a function of the benchmark at the end and we see how the pension company is able to control its investment and to hedge against some of the risks associated. Therefore, we can see the impact of the benchmarking approach on investment under uncertainties. Future work should consider models with jumps and other sources of incompleteness, models with inflation and recession. Another way of addressing a challenge that emerges from the research would be to create a model set of life-cycle pension funds, which can serve as benchmarks against which the performance of pension fund managers can be measured, see Hinz, R. et al. [27, p6]. This would move the basis of competition from short-term returns, to trying to beat the benchmark on the model sets. The asset allocation would depend not only on age, but also on other parameters, including contribution rates, density of contributions, benefits from other social insurance programs, patterns of lifetime earnings, risk preferences, and correlations among these factors and asset returns.

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