Robust Computational Methods for Two-Parameter Singular Perturbation Problems

David Elago

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Supervisor: Prof. Kailash C. Patidar
Co-supervisor: Dr. Justin B. Munyakazi

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KEYWORDS

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Richardson extrapolation

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Shishkin mesh

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Two-point boundary value problems

Convergence analysis.

Abstract

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This thesis is concerned with singularly perturbed two-parameter problems. We study a fitted finite difference method as applied on two different meshes namely a piecewise mesh (of Shishkin type) and a graded mesh (of Bakhvalov type) as well as a fitted operator finite difference method. We notice that results on Bakhvalov mesh are better than those on Shishkin mesh. However, piecewise uniform meshes provide a simpler platform for analysis and computations. Fitted operator methods are even simpler in these regards due to the ease of operating on uniform meshes. Richardson extrapolation is applied on one of the fitted mesh finite difference method (those based on Shishkin mesh) as well as on the fitted operator finite difference method in order to improve the accuracy and/or the order of convergence. This is our main contribution to this field and in fact we have achieved very good results after extrapolation on the fitted operator finite difference method. Extensive numerical computations are carried out on to confirm the theoretical results.

December 2010.

Declaration

I declare that **Robust computational methods for two-parameter singular perturbation problems** is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

David Elago December 2010

Signed:..

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DEDICATION

This thesis is dedicated to my parents tate na meme Elago.

Contents

List of Tables

List of Publications

Following technical reports are generated out of this thesis whose revised forms will be submitted as research papers for publications:

- D. Elago, J.B. Munyakazi and K.C. Patidar, A comparison of extrapolation techniques on Bakhvalov and Shishkin meshes when applied to a two-parameter singular perturbation problems, Report No. UWC-MRR 2010/09, University of the Western Cape, 2010.

- D. Elago, J.B. Munyakazi and K.C. Patidar, A convergence acceleration technique of a fitted operator method for a two-parameter singular perturbation problem, Report No. UWC-MRR 2010/10, University of the Western Cape, 2010.

Chapter 1

General Introduction

This chapter provides a general introduction to our work. We discuss some general information pertaining to singular perturbation problems (SPPs). We compare the behaviour of the solution to a one-parameter singular perturbation problem (SPP) to that of a two-parameter SPP. A short account of the areas where two-parameter SPPs occur is made. A review of the work done about two-parameter SPPs as well as the scope of this thesis are also provided.

The perturbation methods for ordinary and partial differential equations have become increasingly important in the world of science and technology. Perturbation problems are divided into two types: regular and singular perturbation problems. A problem P_ϵ is called regular if the smoothness of its solution u_ϵ depends on a parameter $0 < \epsilon << 1$, otherwise P_ϵ is a singular perturbation problem. The parameter ϵ, which is called the singular perturbation parameter, multiplies the highest derivative term of the differential equation underlying the problem P_ϵ.

This thesis is concerned with two-parameter SPPs. These are SPPs whose underlying differential equation characterised by the presence of small perturbation parameters μ and ϵ which multiply the first (convection) and second (diffusion) derivative terms respectively, namely

$$L_{\epsilon,\mu}u = \epsilon u''(x) + \mu a(x)u' - b(x)u(x) = f(x), \quad x \in \Omega = (0,1) \tag{1.0.1}$$

$$u(0) = \alpha_0, u(1) = \alpha_1; \alpha_0, \alpha_1 \in \mathbb{R}, \tag{1.0.2}$$

where $0 < \epsilon, \mu \leq 1$ are small parameters and $a(x)$, $b(x)$, $f(x)$ with $a(x) \geq \alpha > 0$, $b(x) \geq \beta > 0$ are sufficiently smooth to ensure that there exists a unique solution u to the problem (1.0.1)-(1.0.2). As it can be seen, this is one of classical example of two-parameter singular perturbation problems (TPSPPs). Generally speaking, a solution to such a problem contains boundary layers with distinct widths in the surrounding of the sides and the corners of the underlying interval.

Standard numerical methods fail to provide the required accuracy in the approximated solution, therefore establishing a need for appropriate numerical methods. As far as finite difference methods are concerned, two major classes of numerical methods can be of interest: Fitted operator methods and fitted mesh ones. The former are methods where the differential operator of the problem is approximated by a non-standard finite difference one on a uniform mesh while the later are methods where the differential operator is approximated by a standard finite difference one on a layer-adapted mesh.

Layer-adapted meshes include meshes of Shishkin type (S-meshes), of Bakhvalov type (B-meshes) and of Vulanovíc type (V-meshes). S-meshes are piecewise uniform meshes (they are fine in the layer region of the solution and coarse outside the layer region), while B-meshes and V-meshes are graded meshes finer in the layer region and coarser outside.

In next section we shall discuss one-parameter and multi-parameter singular perturbation problems.

1.1 One parameter vs multi-parameter singular perturbation problems

Suppose either one of the above-mentioned small perturbation parameters is zero, then the problem (1.0.1)-(1.0.2) becomes a one parameter problem. The solutions to both one and two parameter singular perturbation problems (TPSPPs) usually undergo rapid changes in narrow regions called boundary and/or interior layers. The situation becomes worse, when the problem has two perturbation parameters, due to presence of multiple scales, both which are small and multiplied to different derivative terms.

Moreover, what is more important is the relative ratio of ϵ and μ. To obtain a better solution, depends on the magnitude of that ratio, the behavior of solution changes from dispersive when μ^2/ϵ approaches zero as ϵ approaches zero and dissipative when ϵ^2/μ approaches zero as μ approaches zero, we refer the readers to [50] and [56].

In [40] Roos and Linss indicated that there is a vast literature dealing with both convection and reaction-diffusion dominated one parameter singular perturbation problems, but much less is known about uniform methods for two-parameter SPPs.

There is evidence that one-parameter singular perturbation problems has been attempted by many researchers, scientists and engineers, but only few have paid attention to two-parameter ones. Almost two decades ago many researchers have presented numerous convergence results for linear multi-step methods, Runge-Kutta, Rosenbrock, one-leg and general linear methods, for one parameter SPPs. [56], while Liu and Xiao pointed out in [42] that there are few convergence results of A(α)-stable linear multi-step methods for TPSPPs. It is not easy to get error analysis in case of two-parameter SPPs.

The number of good quality and quantity research papers concerning the study of asymptotic behaviour of the solution to one parameter SPPs is huge as compared to that of two-parameter SPPs. Only few researchers have studied the analytical and numerical solutions of these problems, while very little is done for their asymptotic solution [52].

1.2 Occurrence of multi-parameter singular perturbation problems

This section provides applications and one model concerning two parameter singular perturbation problems.

The multi-parameter SPPs arise in many areas of applied mathematics. The two small perturbation parameters ϵ and μ are associated with diffusion and convection terms respectively, its applications occur in chemical-reaction theory and lubrication theories [11, 27, 56]. Some applications of the singular perturbation method to the bending problems of thin plates and shells [10].

According to Kadalbajoo and Gupta [27] and O'Malley [52], the two-parameter SPPs play a major role in chemical flow reactor and dc-motor analysis. Kadalbajoo and Gupta [27] and Gupta [16] mentioned that multi-parameter SPPs occur in fluid mechanics, quantum mechanics and elasticity.

Bohl and Bigge[6] pointed out that some transport phenomena arising in biology and chemistry are governed by multi-parameter singular perturbation problems. It can be found in the case of boundary layers controlled by suction (or blowing) of some fluid.

In [5] Verma and Bhathawala mentioned that a hydrological situation of one dimensional vertical ground water recharged by spread is two-parameter singular perturbation problem. They formulated a problem's equation of continuity for an unsaturated porous media which is given by following model:

$$\frac{\delta}{\delta t}(\rho_r \theta) = \nabla . \overrightarrow{M}, \tag{1.2.3}$$

where ρ_r is the bulk density of the medium, $\overrightarrow{M}$ is moisture mass flux, θ and ∇ are moisture content on a dry weight basis and differential operator vector respectively.

They applied Darcey's law to get:

$$\overrightarrow{V} = -K\nabla\phi, \tag{1.2.4}$$

where ϕ and $\overrightarrow{V}$ are potentials, the volume flux of the moisture respectively, and K is aqueous conductivity coefficient. They took into account that

$$\rho_r \frac{\delta\theta}{\delta t} = \frac{\delta}{\delta z}\left(\rho K \frac{\delta\psi}{\delta z}\right) - \frac{\delta}{\delta z}\rho K g \tag{1.2.5}$$

where g is the gravitational constant, ρ the fluid density, ψ the capillary pressure potential and $\phi = \psi - zg$. The gravity and positive direction of Z-axis are similar. They rewrote (1.2.5) as the following equation

$$\frac{\delta\theta}{\delta t} = \frac{\delta}{\delta z}\left(D\frac{\delta\theta}{\delta z}\right) + \frac{\rho}{\rho_r} g \frac{\delta K}{\delta z}, \tag{1.2.6}$$

where D is a small diffusivity coefficient given by: $D = \frac{\rho}{\rho_r} K \frac{\delta\psi}{\delta\theta}$, By assuming that $K = \frac{K_0\theta}{\sqrt{t}}$ and replaced D by D_1 which they considered as an average value of D in (1.2.6) to get

$$\frac{\delta\theta}{\delta t} = D_1 \frac{\delta^2\theta}{\delta z^2} + \frac{\rho}{\rho_r} g K_0 \frac{\delta\theta}{\delta z}, \tag{1.2.7}$$

By considering L the level (depth) of water these authors came up with

$$\xi = \frac{z}{L}, \quad T = \frac{t}{L^2}$$

and making $\frac{\rho g K_0}{\rho_r} = M_1$ and rewrote (1.2.7) as the following patial differential equation of order two:

$$\frac{\delta\theta}{\delta T} = D_1 \frac{\delta^2\theta}{\delta\xi^2} + \frac{M_1}{\delta\xi} g K_0 \frac{\delta\theta}{\delta z}, \tag{1.2.8}$$

Imposing the set of boundary condition as:

$$\theta(0, T) = \theta_0, \quad \theta(1, T) = 1 \tag{1.2.9}$$

For an arbitrary constant C_2 and any integer ν the authors expressed a moisture content into separable variables form:

$$\theta(\xi, T) = H(T)F(\nu); \quad H(T) = C_2 T^{\nu}$$

and they further applied the Boltzman transformation $\nu = \frac{\xi}{2\sqrt{T}}$ on equations (1.2.8) and (1.2.9) yields two-parameter singular perturbation boundary value problem.

$$\epsilon F''(\nu) + 2\mu F'(\nu) - F(\nu) = 0, \qquad F(0) = \theta_0, \quad F\left(\frac{1}{2\sqrt{T}}\right) = 1,$$

where two perturbation parameters:

$$\epsilon = \frac{D_1}{4n}, \quad \mu = \frac{M_1 - \nu}{4n}.$$

In the next section the literature review, arranged in authors' name alphabetical order, is given. Due to the limited amount of information about TPSPPs, no separation between asymptotic and numerical methods is made.

1.3 Literature review on the two-parameter singular perturbation problems

A posteriori error estimates by Linss for two-parameter singular perturbation problems [41] used fixed number of Shishkin (S-type) and Bakhvalov (B-type) meshes points concentrated in the boundary layers region that produces the rate of convergence of second order. The accuracy is guaranteed, irrespective of the magnitude of the parameters and depends on *a priori* information of the solutions and its derivatives.

O'Riordan and Pickett in [75] studied a class of two-parameter elliptic PDEs singular perturbation problem on a rectangular domain. The solution to this kind of problem was decomposed into a sum of corner layer and boundary layer components. They further explicitly derived perturbation parameter bounds on the derivatives of each of these two components. The author's numerical method was based on monotone upwind finite difference operator. Analysis of tensor product of piecewise uniform meshes of S-type done and parameter uniform asymptotic error bounds to the approximated solution established.

The two-parameters SPPs tackled by Cai *et al.* [7] in which they decomposed its solutions into smooth and singular components, using piecewise unform meshes and three transition points scheme to capture the property of boundary layer very well. The results showed a small parameters uniform convergence of the first order, which was higher than traditional Shishkin scheme.

Kadalbajoo and Yadaw in [25] presented B-spline collocation method for a class of two-parameter linear singular perturbation boundary value problems of convection-diffusion type with two boundary layers. These methods were based on application of B-spline collocation method on piecewise-uniform Shishkin mesh with two transition parameters. They suitably formed piecewise-uniform grid so that more points can be generated within the boundary layer region. They divided underlying domain $[0, 1]$ into three sub-domains.

Teofanov and Ross considered an elliptic two parameter singular perturbation boundary value problem on a unit square [72]. They applied the finite element numerical method to solve the boundary value problem. This was done in such away that either piecewise linear or piecewise bilinear elements were used on piecewise uniform mesh of Shishkin type. In their case the method showed small parameters uniformly convergent in an energy norm. The numerical results confirmed their theoretical analysis.

Patidar in [56] studied two parameter boundary value problems by using fitted operator finite difference method (FOFMD). It followed that generally exponentially fitted finite difference methods (EFFMDs) are more effective inside the layers and give parameter uniform convergence. Unfortunately, they do not give fairly good approximations in the whole interval of interest. For this reason Patidar developed a non-standard finite difference methods (NSFDMs) in order to overcome this weakness in EFFMDs.

By consindering the fact that two step W-methods is one of the efficient class of numerical methods for stiff intial value problems of ODEs, Liu and Xiao applied one of these methods to two parameter SPPs. They concentrated their investigation on simpler case in article [42] and studied quantitative error behaviour of parallel two step W-methods (PTSW). This approach was an extension of non order reduction results described in Weiner *et al.*'s report in [63]. Their method has a high parallelization and its computational cost less than that of implcit Runge-Kutta methods with same stages. They indicated that computational results confirmed a theoretical results PTSW is of third order.

Roos and Teofanov in [71] considered an elliptic singularly perturbed problem with two parameters on a unit square domain. They pointed out that a solution to such problems might have exponential, parabolic and corner layers. The solution was decomposed into regular, layer components and derived pointwise bounds on the components and its derivatives. The estimates are obtained by the analysis of appropriate problems on unbounded domains.

In [74] Shishkin *et al.* investigated a parameter uniform numerical methods for a class of singular perturbation parabolic PDEs with two parameters on a rectangular domain. They constructed upwind finite difference operator on suitable piecewise uniform meshes of Shishkin type. Moreover a parameter explicit theoretical bounds on the derivatives of the solutions were derived. Finally, parameter unform error bounds for the numerical approximations are established.

Cheng studied SPPs for non-local reaction-diffusion equation involving two-parameters [9] by using formal asymptotic solution. This method contains two steps expansions and unform validity of the solutions that were proven using the differential inequalities. These ordinary differential equations would be able to be extended to those of partial differential equations.

Surla and Teofanov [69] studied two-parameter singular perturbation problems of self-adjoint with two boundary layers. They constructed a numerical method based on difference scheme obtained using quadratic spline function as an approximation function. This was done by choosing a suitable collection of points. They proved that their scheme has the inverse monotone matrix on the corresponding Shishkin mesh. Based on prior information of error estimation for the simple upwind finite difference scheme a proper chosen Shishkin mesh is given. Their problem satisfied the continuous minimum principle and the regular components of solution satisfied the bounds. They discretized the domain and defined piecewise uniform mesh with two mesh transition points. The solution to the problem was approximated with quadratic spline on each subinterval. They collected points and equations to obtain the system of equations. It was discovered that a standard collection method on the system of equations do not satisfy the discrete minimum principle because an inverse monotone matrix was not obtained. To achieve this, a suitable parameter which moved the collection points to satisfy the discrete minimum principle was chosen to get inverse monotone matrix. The numerical results indicated uniform convergence.

O'Riordan *et al.* [18] used the same method on two parameters singular perturbation problems of second order, but in this case fitted finite difference method is a combination of the central difference, mid-point and standard upwind difference operators. These three finite difference operators are monotone in various subdomains of the parameter space. It is a known fact that the standard upwind operator is always monotone and has a second order truncation error, when both two small parameters ϵ and μ are relatively small so that $\epsilon, \mu \leq CN^{-1}$, the central difference operator is monotone if ϵ is relatively large and μ is relatively small so that $N\epsilon \geq C_1\mu$ while the mid-point scheme is monotone for all ϵ and for μ relatively large so that $\mu N \geq C_2$.

Roos and Linss in journal [40] considered a class of linear reaction-convection-diffusion two parameters singular perturbation problem. The numerical solution to such a problem might display two exponential layers at both end of the interval of interest ($x = 0$ and $x = 1$) which depends on the size of two perturbation parameters. They applied a simple upwind finite difference scheme on piecewise meshes of Shishkin. The sharp bounds for the derivatives were derived with the aid of appropriate barrier function and comparison principle techniques. These bounds were applied in the analysis numerical method. This method indicated uniform convergence with almost one order and of the two perturbation parameters independent.

Koren *et al.* studied the boundary value problem for two-parameter singular perturbation parabolic equation in [32]. This ordinary differential equation (ODE) can be either convection-diffusion or reaction-diffusion SPPs and its boundary layers can be parabolic or regular all these depends on the relation between two parameters. They considered case of parabolic layer and constructed a monotone finite difference scheme on piecewise unform meshes. This numerical method showed the parameters uniform convergence and of the $\mathcal{O}(N^{-1}\ln N + N_0^{-1})$ where N and N_0 are numbers of mesh points and nodes in time mesh respectively.

In [73] O'Riordan and Pickett studied a two parameter singular perturbation problems of higher order, based on upwind finite difference operator and an appropriate piecewise uniform mesh of Shishkin. They noticed that a boundary layer of width $\mathcal{O}(\epsilon)$ appeared in the neighborhood of the end point $x = 0$. The relative ratios of two perturbation parameters were very important. Their analysis argument consists of establishment of maximum principle, decomposition of the solution into regular and layer components and derivation of sharp parameter-explicit bounds on these components and their derivatives.

Moreover, discrete solution was decomposed and, using discrete maximum principle, truncation error analysis and appropriate barrier function to analyze numerical error between continuous and discrete components. It has been pointed out that two parameter singular perturbation problems naturally should be split into two cases $\mu \leq c\sqrt{\epsilon}$ which was close to single parameter reaction-diffusion problems case and

$\mu \geq c\sqrt{\epsilon}$ which they considered to be more intricate. The numerical approximated solution showed a parameter unform convergence.

Furthermore, numerical results were obtained and extended to finite difference schemes without an assumption of maximum principle. It was shown that apart from the mesh which is fitted, finite element methods are not suitable for SPPs whose solutions have a parabolic boundary layers. In the nature of the solution in the neighborhood of the boundary layer, one should consider that the discrete maximum norm is appropriate than any other norm. So in this manner, one can obtained a parameters uniform convergence of the solution of the boundary value problem.

The SPPs involving several small parameters have attracted considerable attention and there are some notable and interesting convergence results of spline difference schemes for linear singular perturbation boundary value problems by combining a cubic spline and central difference schemes on Shishkin and Bakhavalov type meshes [68]. Although, in most known Bakhavalov mesh type (B-type) there are results better than Shishkin type mesh (S-type), which require a very simple use if the small loss of accuracy occurs, one should consider it as an irrelevant. The rate of convergence $\mathcal{O}(n^{-2})$ and $\mathcal{O}(n^{-2}\ln^2 n)$ are proved on the B-type and S-type respectively and truncation error less than or equal to 2. There was no parameter convergence of polynomial approximation or spline collocations on uniform meshes.

Clavero *et al.* [47] studied problem (1.0.1)-(1.0.2) by using a defect correction method for its numerical solution on one dimension. This method consist of a combination of the stable upwind scheme of the first order and unstable central difference to obtain not only higher order convergent results, but also a numerical solution with efficiency and accuracy on two dimensional, in which they got second-order. In this regards the rate of convergence were calculated and showed almost second order of convergence by using a double mesh principle.

The two parameters singular perturbation problems were tackled by Chen *et al.* [79] based on $A(\alpha)$-stable linear multistep method. The authors considered as special class of stiff problems on three dimensions. The classical applications of such a problems often arise on chemical reactor theory, fluid mechanics and combustion.

Li considered a bilinear finite element method for two parameter singular perturbation with elliptic layer in article [36]. They used Butozov asymptotic expansion. This method is small parameter uniformly convergent. Their numerical results was much better than a classical finite element method.

In a paper [64], Shishkin studied a two-parameter singularly perturbed boundary value problems for parabolic and elliptic equation by constructing a grid approximation which was based on fitted operator method. More specifically it dealt with a solution that contains a parabolic boundary layer. On the one hand, it shows that finite difference schemes, finite element techniques are included in the term grid approximation methods. Furthermore, it has also shown that no finite difference scheme from the natural class of fitted operation methods on a uniform mesh exists, whose numerical solutions would converge uniformly with respect to the parameter discrete maximum norm.

Sukon in a paper [66] tackled two parameters SPPs using alternating group explict(TAGE) method for singular perturbation problems. The variable coefficients singularly perturbed elliptic two points boundary value problem of convection-diffusion type were considered. The derivatives are approximated both by compact fourth order differences and central differences. The solution obtained using fourth order scheme are found to be both oscillation free and convergent for large cell Reynolds number. They concluded that TAGE method is flexible and very suitable for use on parallel computers.

Saydy in [60] studied a two-parameter singular perturbation problem by considering the stability of the families of matrices relatively to domains with a polynomial guarding map. They obtained sufficient and necessary condition for the stability of the new problem.

Abed [1] derived explicit two upper bounds on the singular perturbation parameters to ensure a uniform asymptotic stability of the general time-varying multi-parameter singularly perturbed problems. The upper bounds were obtained on the weighted norm of the vector of singular perturbation parameters by using Lyapunov function. These two upper bounds gave an indication that a uniform asymptotic stability and bounded decoupling transformation for fast subsystem do exist.

Verma and Bhathawala in [5] tackled and discussed two-parameter singular perturbation problems of one dimensional flow through unsaturated porous media. They mathematically formulated a non-linear diffusion type of equation. This equation was transformed by similar method into an ordinary differential equation containing two small parameters.

The research in this field is ongoing and one may have a longer list of works to this account than what we have mentioned above.

The major objective of this study is to investigate Richardson extrapolation effects on fitted operator finite difference methods (FOFDMs) for two-parameter singular perturbation problems. It is a postprocessing procedure where a linear combination of two computed solutions approximating a particular quantity to give a better third approximation [44]. It was implemented for the first order differential equations by Keller in [29], Stynes and Natividad [48] for linear diffusion-convection of dimension one, Munyakazi and Patidar [44], for fitted operator finite difference methods and for a high order fitted mesh method for self-adjoint singularly perturbed problems in [45].

1.4 Outline and scope of the work in this thesis

We give analytical results for two-parameter singular perturbation problems in general. The asymptotic analysis results and some error estimates on numerical approaches of upwinding scheme on Bakhvalov and Shishkin meshes are given in Chapter 2 after a succinct description of these meshes.

Chapter 3 deals with error analysis of the Fitted Mesh Finite Difference Methods (FMFDMs). Performance of Richardson extrapolation on these numerical methods is studied in this chapter.

The complexity of the use of B-meshes and S-meshes is mitigated by employing fitted operator finite difference method (FOFDM) in Chapter 4. We also proceed to investigate the impact of a Richardson extrapolation on the above-mentioned FOFDM.

Lastly Chapter 5 concludes this piece of work and provides some orientation for future research.

Chapter 2

Some analytical and numerical results

In this chapter some analytical results, which will be required for numerical analysis in subsequent chapters, and their proofs are provided. We shall give results from asymptotic analysis and conclude the chapter with some *a priori* error estimates.

2.1 Analytical results for two-parameter singular perturbation problems

In this section, we prove two lemmas and a theorem which will be needed in the next two chapters.

Lemma 2.1.1. [56] (Continuous minimum principle) *Assume that* $\Pi(x)$ *is any sufficiently smooth function satisfying* $\Pi(0) \geq 0$ *and* $\Pi(1) \geq 0$. *Then* $L_{\epsilon,\mu}\Pi(x) \leq 0, \forall x \in (0,1)$ *implies* $\Pi(x) \geq 0$, $\forall x \in [0,1]$.

Proof. Let x^* be such that $\Pi(x^*) = \min_{x\in[0,1]} \Pi(x)$ and assume that $\Pi(x^*) < 0$, Clearly $x^* \notin \{0,1\}$, $\Pi^{'}(x^*) = 0$ and $\Pi^{''}(x^*) \geq 0$. We have

$$L_{\epsilon,\mu}\Pi(x^*) = \epsilon\Pi^{''}(x^*) + \mu a(x^*)\Pi^{'}(x^*) - b(x^*)\Pi(x^*) \geq 0,$$

which is a contradiction. It follows that $\Pi(x^*) \geq 0$ and thus $\Pi(x) \geq 0, \forall x \in [0,1]$.

Lemma 2.1.2. [56] *Let u(x) be the solution of the problem (1.0.1), then*

$$||u|| \leq \beta^{-1}||f|| + \max(|\alpha_0|, |\alpha_1|).$$

Proof. Consider two barrier functions $\Pi^{\pm}$ defined by:

$$\Pi^{\pm}(x) = \beta^{-1}||f|| + \max(|\alpha_0|, |\alpha_1|) \pm u(x).$$

We have

$$\begin{aligned} \Pi^{\pm}(0) &= \beta^{-1}||f|| + \max(|\alpha_0|, |\alpha_1|) \pm u(0) \\ &= \beta^{-1}||f|| + \max(|\alpha_0|, |\alpha_1|) \pm \alpha_0 \geq 0 \end{aligned}$$

and

$$\begin{aligned} \Pi^{\pm}(1) &= \beta^{-1}||f|| + \max(|\alpha_0|, |\alpha_1|) \pm u(1) \\ &= \beta^{-1}||f|| + \max(|\alpha_0|, |\alpha_1|) \pm \alpha_1 \geq 0 \end{aligned}$$

Thus $\Pi^{\pm} \geq 0$ at $x = \{0, 1\}$. For all $x \in (0,1)$, we have

$$\begin{aligned} L_{\epsilon,\mu}\Pi^{\pm}(x) &= \epsilon(\Pi^{\pm}(x))'' + \mu a(x)(\Pi^{\pm}(x))' - b(x)\Pi^{\pm}(x) \\ &= -b(x)(\beta^{-1}||f|| + \max(|\alpha_0|, |\alpha_1|) \pm L_{\epsilon,\mu}u(x) \\ &= -b(x)(\beta^{-1}||f|| + \max(|\alpha_0|, |\alpha_1|) \pm f(x) \\ &= \frac{-b(x)}{\beta}||f|| \pm f(x) - b(x)\max(|\alpha_0|, |\alpha_1|) \\ &= -(||f|| \pm f(x)) - b(x)\max(|\alpha_0|, |\alpha_1|) \leq 0 \end{aligned}$$

Therefore by applying Lemma 2.1.1, $\Pi^{\pm} \geq 0, \forall x \in \bar{\Omega}$.

Theorem 2.1.1. *Assuming that $a(x)$, $b(x)$, $f(x)$ are sufficiently smooth, with $\frac{b(x)}{a(x)} \geq$*

$\delta \geq 0$, *then the solution* $u(x)$ *of the boundary value problem ((1.0.1) satisfies*

$$|u^{(k)}_{\epsilon,\mu}(x)| \leq M(1+\epsilon^{-k}e^{\frac{-\mu\alpha x}{\epsilon}}+\mu^{-k}e^{-\delta\frac{1-x}{\mu}}), \quad k=1(1)4. \tag{2.1.1}$$

Proof The result follows using the techniques for the proof analogous of the theorem in Miller et *al.* [43] , (pp 55-57).

In the next section we shall describe some results that were obtained through extensive asymptotic analysis by O'Malley in [50].

2.2 Results from asymptotic analysis

Consider the following two-parameter singular perturbation problem:

$$Y = \epsilon u^{''} + \mu a(x)u' - bxu = 0, u(0) = u_0, u(1) = u_1. \tag{2.2.2}$$

Case I. $\epsilon/\mu^2 \to 0$ as $\mu \to 0$.

In solving this problem the author in [50] considered the auxiliary polynomial given by:

$$\epsilon D^2 + \mu a(x)D - bx = 0, \tag{2.2.3}$$

with solutions

$$D = -\frac{\mu a}{2\epsilon}\left(1 \pm \left(1+\frac{4\epsilon b}{\mu^2 a^2}\right)^{1/2}\right).$$

The explicit solution of the constant coefficient boundary value problem (2.2.2) has the asymptotic expansion:

$$\begin{aligned} u(x) \cong \quad & u(1)\exp\left[-\frac{\mu a}{2\epsilon}\left(1 \pm \left(1+\frac{4\epsilon b}{\mu^2 a^2}\right)^{1/2}\right)(x-1)\right] \\ & +u(0)\exp\left[-\frac{\mu a}{2\epsilon}\left(1 \pm \left(1+\frac{4\epsilon b}{\mu^2 a^2}\right)^{1/2}\right)x\right] \end{aligned} \tag{2.2.4}$$

Clearly, $u(x) = v(x) \equiv 0$ as $\mu \to 0$ uniformly convergent on interval $[\delta, 1-\delta]$, $\delta > 0$, but it is not uniform convergent near $x = 0$ and $x = 1$ for $u(0) \neq 0$ and $u(1) \neq 0$ again $v(x) \equiv 0$ satisfies the reduced equation but, in general, it fails to satisfy the boundary conditions. Note that the boundary layer at $x = 0$ is due to the root

$$-\frac{\mu a}{2\epsilon}\left(1 + \left(1 + \frac{4\epsilon b}{\mu^2 a^2}\right)^{1/2}\right)$$

of the auxiliary equation (2.2.4) which approaches $-\infty$ as $\mu \to 0$, while the boundary layer at $x = 1$ can be associated with the root

$$-\frac{\mu a}{2\epsilon}\left(1 - \left(1 + \frac{4\epsilon b}{\mu^2 a^2}\right)^{1/2}\right)$$

which approaches $+\infty$ as $\mu \to 0$.

Case II. μ^2/ϵ as $\epsilon \to 0$, here the author in [50] considered the auxiliary polynomial equation (2.2.4) having one solution

$$-\left(\frac{b}{\epsilon}\right)^{1/2}\left(\left(\frac{\mu^2 a^2}{4\epsilon b}\right)^{1/2} - \left(1 + \frac{\mu^2 a^2}{4\epsilon b}\right)^{1/2}\right)$$

approaches $+\infty$ as ϵ approaches zero, while the other solution

$$-\left(\frac{b}{\epsilon}\right)^{1/2}\left(\left(\frac{\mu^2 a^2}{4\epsilon b}\right)^{1/2} + \left(1 + \frac{\mu^2 a^2}{4\epsilon b}\right)^{1/2}\right)$$

approaches $-\infty$ as ϵ approaches zero. The solution of the boundary value problem (2.2.2) has the asymptotic expansion:

$$u(x) \cong u(1)\exp\left[-\left(\frac{b}{\epsilon}\right)^{1/2}\left(\left(\frac{\mu^2 a^2}{4\epsilon b}\right)^{1/2} - \left(1 + \frac{\mu^2 a^2}{4\epsilon b}\right)^{1/2}\right)(x-1)\right]$$

$$+u(0)\exp\left[-\left(\frac{b}{\epsilon}\right)^{1/2}\left(\left(\frac{\mu^2 a^2}{4\epsilon b}\right)^{1/2} + \left(1 + \frac{\mu^2 a^2}{4\epsilon b}\right)^{1/2}\right)x\right]$$

Similar to above-mentioned case I, $u(x) = v(x) \equiv 0$ as $\mu \to 0$ uniformly on every closed interval $[\delta, 1-\delta]$, for all $\delta > 0$, but non-uniform convergence near $x = 0$ and $x = 1$ provided that $u(0) \neq 0$ and $u(1) \neq 0$. The boundary layer at $x = 0$ is associated with a root of the auxiliary polynomial which approaches $-\infty$ as ϵ approaches zero, while the boundary layer at $x = 1$ is associated with a root of 2.2.4 which approaches $+\infty$ as ϵ approaches zero.

To split off these terms which were not singular as $\epsilon \to 0$ by expanding the exponents in the powers of μ^2/ϵ. Moreover, if $\mu = \mathcal{O}(\epsilon)$ then

$$u(x) \cong u(1) \exp\left[\left(\frac{b}{\epsilon}\right)^{1/2}(x-1)\right] + u(0)\exp\left[\left(-\frac{b}{\epsilon}\right)^{1/2} x\right]$$

which is the solution of the semi-reduced boundary value problem

$$\epsilon z'' - bz = 0, \quad z(0) = z_0, \quad z(1) = z_1.$$

2.3 Some error estimates on numerical approaches used for two-parameters singular perturbation problems

In this section we describe the S- and B-meshes. Then we provide error estimates of some numerical methods used on these meshes.

2.3.1 Description of Bakhvalov meshes (B-mesh)

Bakhavalov meshes are non-uniform mesh which can be constructed to overcome the difficulties that arise in using uniform meshes to solve SPPs [68]. In 1969 the prominent Russian mathematician Nikolai Sergeevich Bakhvalov invented a graded mesh type which was more applicable to one dimensional problems, but very difficult to use on non-linear problems in several dimensions. He used an equidistant ξ-grid near $x = 0$, then to map this grid back onto $x-$axis by means of the boundary layer function $y = e^{-\upsilon x/2}$ where $\upsilon \geq 0$ and occur in the exact solution. In order to design these meshes successfully, a very complicated construction and theoretical techniques are required.

We follow the construction provided in [39]. An equidistant ξ-grid is considered near $x = 0$, then mapped back onto the x-axis by means of the (scaled) boundary layer function. That is, grid points x_i near $x = 0$ are defined by

$$q\left(1 - \exp\left(-\frac{\beta x_i}{\sigma\epsilon}\right)\right) = \xi_i = \frac{i}{N} \text{ for } i = 0, 1, 2, ...$$

where the scaling parameters $0 < q < 1$ and $\sigma > 0$ are user chosen in such away that q is the ratio of mesh points used to resolve the layer and σ determines the grading of the mesh inside the layer. Away from the layer a uniform mesh in x is used with transition point τ such that the resulting mesh generating function is $C^1[0, 1]$, algebraically we have:

$$\varphi(\xi) = \begin{cases} \chi(\xi) := -\frac{\sigma\epsilon}{\beta}\ln(1 - \frac{\xi}{q}), & for \quad \xi \in [0, \tau], \\ \pi(\xi) := \chi(\tau) + \chi^{'}(\tau)(\xi - \tau) & for \quad \xi \in [\tau, 1] \end{cases}$$

where $\chi(\tau) + \chi^{'}(\tau)(1 - \tau) = 1$.

2.3.2 An error estimate on Bakhvalov mesh

In [41], the equation of reaction-convection-diffusion two-parameter singular perturbation boundary value problem on $(0, 1)$ is

$$Lu = -\epsilon_d u'' - \epsilon_c u' + cu = f, u(0) = u(1) = 0 \tag{2.3.5}$$

A streamline diffusion finite element method on Bakhvalov mesh with chosen mesh parameters $\kappa_0, \kappa_1 > 0$ and $0 < \sigma_0, \sigma_1$. The author uses and analyze the method by generating Bakhvalov meshes for equation (2.3.5) used the following equidistributing monitor function:

$$N\int_{x_{i-1}}^{x_i} M_{Ba}(s)ds = \int_0^1 M_{Ba}(s)ds$$

to produce mesh points x_i for $i = 0(1)n$ this method yields order 2 and error bound:

$$||u - U^N|| \leq CN^{-2}.$$

2.3.3 Description of Shishkin meshes (S-meshes)

These meshes are known as piecewise equidistant and are most frequently studied, can be for the following reasons:

- They are very simple to construct.
- The ability to solve numerous singular perturbation problems.
- They can make it possible to study perturbation parameters uniformly convergent grid methods on non-uniform meshes.
- It encourages a very significant progression in techniques for obtaining a priori estimates.
- It contributes widely to the development of the parameters uniformly convergent difference schemes to both ordinary differential equations (ODEs) and partial differential equations (PDEs).

We describe S-mesh for our problem (1.0.1) as follows [39]: Let two mesh parameters be $0 \leq q \leq 1$ and $\sigma > 0$. We define a mesh transition point λ by

$$\lambda = \min\left\{q, \frac{\sigma\epsilon}{\beta}\ln N\right\}.$$

Then intervals $[0, \lambda]$ and $[\lambda, 1]$ are divided into qN and $(1-q)N$ equidistant subintervals (assuming that qN is an integer). This mesh may be generated by the mesh generating function

$$\varphi(\xi) = \begin{cases} \frac{\sigma\epsilon}{\beta} \ln N \frac{\xi}{q}, & for \quad \xi \in [0, q], \\ \\ 1 - \left(1 - \frac{\sigma\epsilon}{\beta} \ln N\right) & for \quad \xi \in [q, 1] \end{cases}$$

The parameter q is the amount of mesh points used to resolve the layer. The mesh transition point λ has been chosen such that the layer term $\exp(-\beta x/\epsilon)$ is smaller than $N^{-\sigma}$ 0o $[\lambda, 1]$. Typically σ will be chosen equal to the formal order of the method or sufficiently large to accommodate the error analysis.

Note that unlike Bakhvalov the underlying mesh generating function is only piecewise $C^c[0, 1]$ and depends on N, the number of mesh points. For simplicity we assume throughout that $q \geq \lambda$ as otherwise N is exponentially large compared to $1/\epsilon$ and uniform mesh is sufficient to cope with the problem.

Although Shishkin meshes have a simple structure and numerical methods using them are easier to analyze than methods using Bakhvalov meshes, they give numerical results that are inferior to those obtained by B-type meshes.

The S-type mesh is piecewise uniform which finer near the layer(s) and coarse elsewhere [43]. For the problem 1.0.1, we assume that an error is locally generated in the boundary layer region near $x = 0$ and then transported throughout the domain of interest. Let denote a non-negative integer $N = 2^m$, $m \geq 2$ and we divide unit interval $[0, 1]$ into two subintervals: $[0, \delta]$ and $[\delta, 1]$ each of these ones has $N/2$ points and are equally spaced, where δ is a transition point defined by

$$\delta = \min\{1/2, \epsilon \ln N\}. \tag{2.3.6}$$

The mesh grids x_j is given by the following:

$$h_j = \{x_j : x_j = 2\delta/N, j \leq N/2; \quad x_i = x_{j-1} + 2(1-\delta)/N, N/2 < j\}$$

where $h_j = x_j - x_{j-1}$ and we denote this mesh by $\Omega_{n,\delta}$.

2.3.4 An error estimate on Shishkin mesh

On the above-mentioned method and problem 2.3.5 applied on a Shishkin mesh with fixed mesh parameters $0 < q_0, q_1$ and $0 < \sigma_0, \sigma_1$ with positive sum of q_0, q_1[41]. The author uses two transition parameters:

$$\tau_0 = \min\left\{q_0, \frac{\sigma_0}{|\mu_0|}\ln N\right\}$$

and

$$\tau_1 = \min\left\{q_1, \frac{\sigma_1}{|\mu_1|}\ln N\right\},$$

with $q_0, q_1 = 1/4$ and $\sigma_0, \sigma_1 = 3$. The quarter of the mesh points used to resolve both layers. The method produces a second order parameters uniform error bound of the form

$$||u - U^N|| \leq C(N^{-2}\ln^2 N).$$

By considering two parameters singular perturbation boundary value problem:

$$\epsilon u'' + \mu a u' - bu = f(x),$$

where $\epsilon \in (0,1]$, $\mu \in [0,1]$, $0 < \alpha \leq a(x)$, $0 < \beta \leq b(x)$ and $u(0), u(1)$ are given in[73]. The authors employ a monotone numerical method on suitable Shishkin mesh with two transition parameters σ_1 and σ_2 defined by:

$$\sigma_1 = \begin{cases} \min\{\frac{1}{4}, \frac{4\sqrt{\epsilon}}{\sqrt{\gamma\alpha}}\ln N\}, & if \quad \mu^2 \leq \frac{\gamma\epsilon}{\alpha}, \\ \min\{\frac{1}{4}, \frac{4\epsilon}{\mu\alpha}\ln N\}, & if \quad \mu^2 \leq \frac{\gamma\epsilon}{\alpha} \end{cases}$$

and

$$\sigma_2 = \begin{cases} \min\{\frac{1}{4}, \frac{4\sqrt{\epsilon}}{\sqrt{\gamma\alpha}} \ln N\}, & if \quad \mu^2 \leq \frac{\gamma\epsilon}{\alpha}, \\ \min\{\frac{1}{4}, \frac{4\mu}{\gamma} \ln N\}, & if \quad \mu^2 \leq \frac{\gamma\epsilon}{\alpha}. \end{cases}$$

Depends on the relative ratio of two parameters the error bound obtained

$$||u - U|| \leq \left\{ \begin{array}{lll} CN^2 \ln^3 N\}, & if & \gamma\epsilon \leq \alpha\mu^2, \\ C(N^{-1} \ln N)^2\}, & if & \gamma\epsilon \leq \alpha\mu^2. \end{array} \right\}$$

In paper [73], the authors considered a problem (1.0.1) and applied finite difference operator on piecewise uniform mesh consists of two transition points:

$$\sigma_1 = \min\left\{\frac{1}{4}, \frac{2}{\theta_1} \ln N\right\},$$

$$\sigma_2 = \min\left\{\frac{1}{4}, \frac{2}{\theta_2} \ln N\right\}.$$

This numerical method produces first order and second order in cases of $\mu > \sqrt{\epsilon}$ and $\mu < \sqrt{\epsilon}$ respectively and the parameters uniform error bound [18], of the form:

$$||u - U^N||_{\Omega^N} \leq C(N^{-1} \ln N)^2$$

In the case of class of singularly perturbed elliptic problems posed on the unit square $\Omega = (0,1)^2$ O'Riordan and Pickett discretized a problem and defined the tensor product of two piecewise uniform Shishkin meshes Ω^N and Ω^M in recent paper [75]. They used upwind finite difference and central difference operators on above-mentioned S-type meshes. These meshes contains two transition points given by:

$$\sigma_1^N = \min\left\{\frac{1}{4}, \frac{2\epsilon}{\mu\alpha} \ln N\right\},$$

and

$$\sigma_2^N = \min\left\{\frac{1}{4}, \frac{2\mu}{\gamma}\ln N\right\}.$$

This produces the order of convergence which tends towards $(N^{-1}\ln N)^2$ and is in the line with the theoretical order of convergence $N^{-1}(\ln N)^2$. For the reaction-convection-diffusion problem (1.0.1), Gracia et al [18], employed a finite difference scheme that uses upwind, midpoint schemes and central difference on Shishkin mesh.

Considering the reaction-convection-diffusion two parameter problem, Kadalbajoo and Yadaw [25] established a numerical method contains a B-spline collocation method and Shishkin mesh selection strategy of the second order. The interval $\Omega = [0, 1]$ is divided into three subinterval:

$$\Omega_0 = [0, \sigma_1], \Omega_c = [\sigma_1, 1 - \sigma_2], \Omega_1 = [1 - \sigma_2, 1],$$

with transition parameters of form:

$$\sigma_1 = \min\left\{\frac{1}{4}, \frac{2}{\mu_1}\ln N\right\},$$

$$\sigma_2 = \min\left\{\frac{1}{4}, \frac{2}{\mu_2}\ln N\right\}.$$

In the next chapter we will discuss the implementation of extrapolation techniques on both Bakhvalov and Shishkin meshes and then compare the numerical results obtained by using the FMFDM on these meshes.

Chapter 3

A comparison of extrapolation technique on Bakhvalov and Shishkin meshes

In this chapter, we describe an upwinding scheme on the Bakhvalov and Shishkin meshes. Some error estimates are given. The comparative numerical results are provided.

3.1 Decomposition of the solution

In [73] the solution u of a problem (1.0.1) can be decomposed into regular and singular components, in order to achieve a parameters-uniformly convergent. The existance a function v (regular component) with the boundary conditions such that

$$Lv = f$$

on $(0, 1)$ and $||v^{(k)}|| \leq C$ for $i \in \{0, 1, 2\}$.

Consider the following decomposition:

$$u = v + w_L + w_R$$

where

$$\begin{aligned} Lv &= f, \quad v(0), v(1) chosen \\ Lw_L &= 0, \quad w_L(0) = u(0) - v(0), w_L(1) = 0, \\ Lw_R &= 0, \quad w_R(0) = 0, w_R(1) = u(1) - v(1). \end{aligned}$$

3.2 An upwind numerical method on S-mesh and B-mesh

We adopt the notation $w_j = w(x_j)$. Also, let: $D^+w_j = \frac{w_{j+1}-w_j}{h_{j+1}}$, $D^-w_j = \frac{w_j-w_{j-1}}{h_j}$ and $D^+D^-w_j = \frac{2}{h_j h_{j+1}(h_j+h_{j+1})}\left[h_{j+1}u_{j-1} - (h_{j+1}+h_j)u_j + h_j u_{j+1}\right]$, where D^+w_j, D^-w_j and $D^+D^-w_j$ are first and second order finite differences respectively. Using the finite difference above, we discretize problem 1.0.1 as follows:

$$L^N w_j \equiv \epsilon D^+D^-w_j + \mu a_j D^+w_j = f_j$$

$$L^N w_j \equiv \frac{2\epsilon}{h_j+h_{j+1}}\left[\frac{h_{j+1}w_{j-1}-(h_j+h_{j+1})w_j+h_j w_{j+1}}{h_j h_{j+1}}\right] + \mu\tilde{a}(j)\frac{w_{j+1}-w_j}{h_j} - \tilde{b}(j)w_j = f_j$$

$$L^N w_j \equiv \frac{2\epsilon}{h_j(h_j+h_{j+1})}W_{j-1} - \frac{2\epsilon}{h_j h_{j+1}}W_j + \frac{2\epsilon}{h_{j+1}(h_j+h_{j+1})}W_{j+1} + \frac{\mu\tilde{a}(j)}{h_j}w_{j+1} - \frac{\mu\tilde{a}(j)}{h_j}w_j - \tilde{b}(j)w_j = f_j.$$

Now we have

$$r^-{}_j = \frac{2\epsilon}{h_j(h_j+h_{j+1})};$$

$$r^c{}_j = \frac{-2\epsilon}{h_j h_{j+1}} - \frac{\mu\tilde{a}_j}{h_j} - \tilde{b}(j);$$

$$r^+{}_j = \frac{2\epsilon}{h_{j+1}(h_j+h_{j+1})} + \frac{\mu\tilde{a}_j}{h_j},$$

where ${r^-}_j$ is sub-diagonal, ${r^c}_j$ and ${r^+}_j$ are main and super diagonals, respectively. Therefore

$$L^N w_j \equiv r_j^+ w_{j+1} + r_j^c w_j + r_j^- w_{j-1} = f_j, j = 1(1)n-1 \tag{3.2.1}$$

In the next section, we shall analyze the above-mentioned numerical method.

3.3 Error analysis of the numerical methods

We first prove some lemmas which are required in the analysis.

Lemma 3.3.1. [56] (Discrete minimum principle). *Let* Γ_i *be any mesh function that satisfies* $\Gamma_0 \geq 0$, $\Gamma_n \geq 0$ *and* $L^N_{\epsilon,\mu}\Gamma_i \leq 0$, *for* $i = 1(1)n-1$, *then* $\Gamma_i \geq 0 \quad \forall i = 0(1)n$.

Proof The proof is obtained by contradiction. Let Γ_i, $i = 0(1)n$, be a mesh function. Assume that $\Gamma_j < 0$ and $\Gamma_j = \min_i \Gamma_i$.

We have $\Gamma_{j+1} - \Gamma_j \geq 0$ and $\Gamma_j - \Gamma_{j-1} \leq 0$. Now

$$L^N_{\epsilon,\mu}\Gamma_j = \frac{\epsilon}{\bar{h}_j}[\frac{\Gamma_{j+1} - \Gamma_j}{h_{j+1}} - \frac{\Gamma_j - \Gamma_{j-1}}{h_j}] + \mu\tilde{a}_j(\frac{\Gamma_{j+1} - \Gamma_j}{h_j}) - \tilde{b_j}\Gamma_j \geq 0,$$

which is a contradiction.

By using the above-mentioned discrete minimum principle, we now show that the FMFDM (3.2.1) also satisfies the uniform stability result provided in the following lemma.

Lemma 3.3.2. [56] *The operator* $L^N_{\epsilon,\mu}$ *is uniformly stable, in the sense that if* Z_i *is any mesh function such that* $Z_0 = Z_n = 0$, *then*

$$|Z_i| \leq \frac{1}{\alpha} \max_{1 \leq j \leq n-1} |L^N_{\epsilon,\mu} Z_j|, \forall \quad 0 \leq i \leq n.$$

Proof See [56].

Consider the mesh $\Omega_{2n,\delta}$ where δ is given by (2.3.6), and $\Omega_{2n,\delta}$ is obtained from

$\Omega_{n,\delta}$ by bisecting each mesh sub-interval. Thus

$$\Omega_{n,\delta} = \{x_j\} \subset \Omega_{2n,\delta} = \{\tilde{x_j}\}$$

and $\tilde{x_j} - \tilde{x_{j-1}} = \tilde{h_j} = h_j/2$.[45].

Lemma 3.3.3. [73] *Let u be the solution of the differential equation (1.0.1) and v the numerical solution is obtained via (3.2.1). Then at each mesh point $x_j \in \Omega_{n,\delta}$ we have*

$$|(u-v)(x_j)| \leq Cn^{-1}(\ln n)^2. \tag{3.3.2}$$

Proof See [73].

3.3.1 Extrapolation formula

Let us denote by $\tilde{v}_j$ the numerical solution computed via (3.2.1). Estimate (3.3.2) implies that

$$u(x_j) - v_j^n = CN^{-1}(\ln n)^2 + R_n(N^{-1}(\ln n)^2), \quad \forall x_j \in \Omega_{n,\delta} \tag{3.3.3}$$

and

$$u(\tilde{x_j}) - \tilde{v_j} = C(2n)^{-1}(\ln n)^2 + R_{2n}((2n)^{-1}(\ln n)^2), \quad \forall \tilde{x_j} \in \Omega_{2n,\delta}, \tag{3.3.4}$$

where the remainders R_n and R_{2n} are $\mathcal{O}((2n)^{-1}(\ln n)^2)$.

We multiply equation (3.3.4) by factor 2 yields:

$$2[u(\tilde{x_j}) - \tilde{v_j}] = C(n)^{-1}(\ln n)^2 + 2R_{2n}((n)^{-1}(\ln n)^2) \quad \forall x_j \in \Omega_{n,\delta}. \tag{3.3.5}$$

Then the difference of equations (3.3.5) and (3.3.3) gives

$$2[u(x_j) - \tilde{v_j}] - (u(x_j) - v_j^n) = 2R_{2n}(n^{-1}/2(\ln n)^2) - (R_n(n^{-1}(\ln n)^2), \quad \forall x_j \in \Omega_{n,\delta},$$

Hence

$$[u(x_j) - (2\tilde{v}_j - v_j)] = \mathcal{O}(n^{-1}(\ln n)^2), \quad \forall \tilde{x}_j \in \Omega_{n,\delta}$$

and therefore, we shall use

$$v_j^{ext} = (2\tilde{v}_j - v_j), \quad \forall x_j \in \Omega_{n,\delta}$$

as the extrapolation formula in next section. Now using stability Lemma 3.3.2, on the mesh function $(u_j - v_j^{ext})_j$ to obtain

$$\max_{0<j\leq n} |u(x_j) - v^{ext}| \leq Cn^{-1}(\ln n)^2.$$

Therefore

$$\sup_{0<\epsilon,\mu\leq 1} \max_{0<j\leq n} |u(x_j) - v^{ext}| \leq Cn^{-1}(\ln n)^2.$$

3.4 Comparative numerical results

In this section, numerical results are presented for two test examples. Maximum errors on both Shishkin and Bakhvalov meshes are computed. Results before and after extrapolation are also provided.

Example 3.4.1. [56] Consider the problem:

$$\epsilon u''(x) + \mu u'(x) - u(x) = -x, x \in \Omega \tag{3.4.6}$$

whose exact solution is given by:

$$u(x) = (x + \mu) + \frac{((1-\mu) + (1+\mu)e^{-D_2})e^{D_{1x}} - ((1+\mu) + (1-\mu)e^{D_1})e^{-D_{2(1-x)}}}{1 - e^{\sqrt{(\mu^2+4\epsilon/\epsilon)}}} \tag{3.4.7}$$

where $D_{1,2} = (-\mu \pm \sqrt{\mu^2 + 4\epsilon})/2\epsilon$.

Maximum errors at all the mesh points are evaluated using the following formulae:

Before Extrapolation

$$E^B_{n,\epsilon,\mu} := \max_{0\le j\le n} |u(x_j) - v_j| \tag{3.4.8}$$

After Extrapolation

$$E^A_{n,\epsilon,\mu} := \max_{0\le j\le n} \left|u(x_j) - v^{ext}\right| \tag{3.4.9}$$

where v_j is the solution of (1.0.1) obtained using (3.2.1) and v^{ext} is the solution after extrapolation of v_j.

Example 3.4.2. [56] Consider the problem:

$$\epsilon u''(x) + \mu(1+x)u'(x) - u(x) = (1+x)^2, x \in \Omega; \quad u(0) = u(1) = 0 \tag{3.4.10}$$

For this problem the exact solution is not known, therefore we shall use double mesh principle [44]. From above example, let $v_j \equiv v^n_j$, then we denote maximum errors for different values of n, ϵ and μ at all the mesh points by $E^B_{n,\epsilon,\mu}$ and $E^A_{n,\epsilon,\mu}$ as follows:

Before Extrapolation

$$E^B_{n,\epsilon,\mu} := \max_{0\le j\le n} \left|v^n_j - v^{2n}_{2j}\right| \tag{3.4.11}$$

After Extrapolation

$$E^A_{n,\epsilon,\mu} := \max_{0\le j\le n} \left|v^{ext}_j - v^{ext}_{2j}\right| \tag{3.4.12}$$

where v^{2n}_{2j} is the numerical solution of (1.0.1) obtained using (3.2.1) on the mesh $\Omega_{2n,\delta}$ and v^{ext}_{2j} is the solution after extrapolation of v^{ext}_j on same mesh.

The numerical rates of convergence are calculated using the following formula [44]:

$$r_k \equiv r_{\epsilon,\mu,k} := \log_2(\tilde{E}_{n_k}/\tilde{E}_{2n_k}), \quad k = 1, 2, 3, ...$$

where $\tilde{E}$ stands for $E_{n,\epsilon,\mu}$ and $E^{ext}_{n,\epsilon,\mu}$ respectively. Moreover, we compute

$$E_n = \max_{0\leq j\leq 1} E_{n,\epsilon,\mu}$$

and

$$E_n = \max_{0\leq j\leq 1} E^{ext}_{n,\epsilon,\mu}$$

whereas the numerical rate of uniform convergence is computed as

$$R_n := \log_2(E_n/E_{2n})$$

and

$$R^{ext}_n := \log_2(E^{ext}_n/E^{ext}_{2n}).$$

3.5 Discussion

In this chapter we have investigated the performance of Richardson extrapolation on fitted mesh finite difference methods for two-parameter singular perturbation problems. Our observation that Richardson extrapolation improves accuracy is convincing, with the rates of convergence slightly increased. This was our as expected and confirms the assertion that Richardson extrapolation [44] improves accuracy of the lower order methods. However, the order of convergence is still not up to the level one would expect and therefore in next chapter, we perform the Richardson extrapolation on a fitted operator finite difference method where we find some wonderful results. In fact the improved results are perfectly of order two.

Table 3.1: Results for Example 3.4.1, Max. Errors using Shishkin mesh before extrapolation for $\mu = 1$

ϵ	n=32	n=64	n=128	n=258	n=512	n=1024
1	1.54E-003	7.79E-004	3.92E-004	1.96E-004	9.84E-005	4.92E-005
2^{-1}	4.75E-003	2.42E-003	1.22E-003	6.15E-004	3.08E-004	1.54E-004
2^{-2}	1.21E-002	6.24E-003	3.17E-003	1.60E-003	8.03E-004	4.02E-004
2^{-3}	2.57E-002	1.37E-002	7.02E-003	3.56E-003	1.80E-003	9.01E-004
2^{-5}	3.35E-002	2.29E-002	1.48E-002	9.02E-003	5.32E-003	3.05E-003
2^{-6}	3.26E-002	2.21E-002	1.42E-002	8.71E-003	5.16E-003	2.97E-003

Table 3.2: Results for Example 3.4.1, Max. Errors using Bakhvalov mesh for $\mu = 1$

ϵ	n=32	n=64	n=128	n=258	n=512	n=1024
1	2.66E-003	1.96E-003	1.56E-003	1.30E-003	1.05E-003	7.91E-004
2^{-1}	6.97E-003	3.96E-003	2.65E-003	2.03E-003	1.73E-003	1.57E-003
2^{-2}	1.88E-002	9.97E-003	5.39E-003	3.22E-003	2.30E-003	1.85E-003
2^{-3}	4.00E-002	2.21E-002	1.17E-002	6.25E-003	3.59E-003	2.41E-003
2^{-5}	1.02E-001	7.91E-002	4.56E-002	2.52E-002	1.33E-002	7.05E-003
2^{-6}	8.54E-002	1.07E-001	8.06E-002	4.65E-002	2.56E-002	1.35E-002

Table 3.3: Results for Example 3.4.1, Max. Errors using Shishkin mesh before extrapolation for $\mu = 2^{-3}$

ϵ	n=32	n=64	n=128	n=258	n=512	n=1024
2^{-3}	1.69E-003	8.49E-004	4.25E-004	2.13E-004	1.07E-004	5.33E-005
2^{-4}	4.62E-003	2.34E-003	1.17E-003	5.89E-004	2.95E-004	1.48E-004
2^{-6}	2.50E-002	1.33E-002	6.83E-003	3.46E-003	1.74E-003	8.74E-004
2^{-8}	4.78E-002	2.71E-002	1.67E-002	1.00E-002	5.81E-003	3.30E-003
2^{-10}	7.23E-002	3.86E-002	1.98E-002	1.12E-002	6.47E-003	3.66E-003
2^{-12}	8.24E-002	4.45E-002	2.32E-002	1.18E-002	6.75E-003	3.82E-003

Table 3.4: Results for Example 3.4.1, Max. Errors using Bakhvalov mesh for $\mu = 2^{-3}$

ϵ	n=32	n=64	n=128	n=258	n=512	n=1024
2^{-3}	2.99E-003	1.49E-003	7.53E-004	3.88E-004	2.11E-004	1.27E-004
2^{-4}	8.65E-003	4.34E-003	2.19E-003	1.10E-003	5.58E-004	2.88E-004
2^{-6}	4.55E-002	2.47E-002	1.30E-002	6.68E-003	3.39E-003	1.71E-003
2^{-8}	1.20E-001	9.64E-002	5.52E-002	3.05E-002	1.61E-002	8.25E-003
2^{-10}	7.56E-002	1.24E-001	1.50E-001	1.08E-001	6.26E-002	3.41E-002
2^{-12}	8.32E-002	4.51E-002	8.31E-002	1.44E-001	1.62E-001	1.13E-001

Table 3.5: Results for Example 3.4.1, Max. Errors using Shishkin mesh before extrapolation for $\mu = 2^{-6}$

ϵ	n=32	n=64	n=128	n=258	n=512	n=1024
2^{-6}	3.75E-003	1.68E-003	7.90E-004	3.82E-004	1.88E-004	9.31E-005
2^{-8}	1.51E-002	6.95E-003	3.34E-003	1.63E-003	8.05E-004	4.00E-004
2^{-10}	5.99E-002	2.42E-002	1.40E-002	7.01E-003	3.51E-003	1.76E-003
2^{-12}	1.46E-001	9.50E-002	4.69E-002	2.26E-002	1.09E-002	5.24E-003
2^{-14}	1.75E-001	1.60E-001	8.92E-002	4.95E-002	2.55E-002	1.29E-002
2^{-15}	1.80E-001	1.79E-001	1.08E-001	5.97E-002	3.19E-002	1.63E-002

Table 3.6: Results for Example 3.4.1, Max. Errors using Bakhvalov mesh for $\mu = 2^{-6}$

ϵ	n=32	n=64	n=128	n=258	n=512	n=1024
2^{-6}	9.52E-003	3.70E-003	1.63E-003	7.68E-004	3.73E-004	1.84E-004
2^{-8}	3.54E-002	1.51E-002	6.86E-003	3.30E-003	1.61E-003	7.99E-004
2^{-10}	9.00E-002	5.49E-002	2.72E-002	1.39E-002	6.98E-003	3.50E-003
2^{-12}	1.48E-001	1.06E-001	9.27E-002	5.37E-002	2.95E-002	1.56E-002
2^{-14}	1.74E-001	1.63E-001	9.61E-002	1.40E-001	1.11E-001	6.37E-002
2^{-15}	1.79E-001	1.79E-001	1.09E-001	1.22E-001	1.60E-001	1.18E-001

Table 3.7: Results for Example 3.4.2, Max. Errors using Shishkin mesh before extrapolation for $\mu = 1$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
1	2.44E-003	1.25E-003	6.32E-004	3.18E-004	1.59E-004	7.98E-005
10^{-1}	2.71E-002	1.47E-002	7.65E-003	3.91E-003	1.98E-003	9.94E-004
10^{-2}	1.12E-001	8.61E-002	6.83E-002	5.59E-002	4.71E-002	4.08E-002
10^{-3}	1.14E-001	8.78E-002	6.93E-002	5.66E-002	4.76E-002	4.12E-002
10^{-5}	1.15E-001	8.80E-002	6.95E-002	5.67E-002	4.77E-002	4.12E-002
10^{-6}	1.15E-001	8.80E-002	6.95E-002	5.67E-002	4.77E-002	4.12E-002

Table 3.8: Results for Example 3.4.2, Max. Errors using Bakhvalov mesh before extrapolation for $\mu = 1$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
1	1.74E-002	1.43E-002	1.75E-002	2.09E-002	2.53E-002	3.17E-002
10^{-1}	6.20E-002	5.61E-002	4.86E-002	4.21E-002	3.70E-002	3.31E-002
10^{-2}	7.73E-002	8.09E-002	7.38E-002	6.14E-002	5.28E-002	4.39E-002
10^{-4}	2.91E-002	3.08E-002	3.20E-002	3.35E-002	3.61E-002	4.08E-002
10^{-6}	2.88E-002	3.01E-002	3.06E-002	3.06E-002	3.03E-002	2.99E-002
10^{-8}	2.88E-002	3.01E-002	3.06E-002	3.05E-002	3.02E-002	2.98E-002
10^{-10}	2.88E-002	3.01E-002	3.06E-002	3.05E-002	3.02E-002	2.98E-002
10^{-11}	2.88E-002	3.01E-002	3.06E-002	3.05E-002	3.02E-002	2.98E-002

Table 3.9: Results for Example 3.4.2, Max. Errors using Shishkin mesh after extrapolation for $\mu = 1$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
1	5.68E-005	1.44E-005	3.64E-006	9.12E-007	2.28E-007	5.72E-008
10^{-1}	2.46E-003	7.09E-004	1.92E-004	4.98E-005	1.27E-005	3.21E-006
10^{-2}	6.69E-002	5.43E-002	4.55E-002	3.95E-002	3.52E-002	3.19E-002
10^{-3}	6.82E-002	5.48E-002	4.58E-002	3.98E-002	3.54E-002	3.21E-002
10^{-5}	6.83E-002	5.49E-002	4.59E-002	3.98E-002	3.55E-002	3.21E-002
10^{-6}	6.83E-002	5.49E-002	4.59E-002	3.98E-002	3.55E-002	3.21E-002

Table 3.10: Results for Example 3.4.2, Max. Errors using Bakhvalov mesh after extrapolation for $\mu = 1$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
1	1.84E-002	2.14E-002	2.55E-002	3.18E-002	4.09E-002	5.02E-002
10^{-1}	5.01E-002	4.10E-002	3.56E-002	3.20E-002	2.91E-002	2.67E-002
10^{-2}	7.05E-002	6.98E-002	5.80E-002	4.42E-002	3.50E-002	3.01E-002
10^{-3}	4.08E-002	4.86E-002	5.93E-002	6.84E-002	6.75E-002	5.50E-002
10^{-5}	3.15E-002	3.12E-002	3.10E-002	3.09E-002	3.13E-002	3.26E-002
10^{-6}	3.14E-002	3.10E-002	3.06E-002	3.00E-002	2.95E-002	2.90E-002

Table 3.11: Results for Example 3.4.2, Max. Errors using Shishkin mesh before extrapolation for$\mu = 2^{-3}$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
10^{-3}	1.33E-001	1.08E-001	8.89E-002	7.43E-002	6.35E-002	5.55E-002
10^{-5}	1.28E-001	1.04E-001	8.56E-002	7.17E-002	6.12E-002	5.34E-002
10^{-7}	1.28E-001	1.04E-001	8.56E-002	7.16E-002	6.12E-002	5.34E-002
10^{-9}	1.28E-001	1.04E-001	8.56E-002	7.16E-002	6.12E-002	5.34E-002
10^{-11}	1.28E-001	1.04E-001	8.56E-002	7.16E-002	6.12E-002	5.34E-002
10^{-13}	1.28E-001	1.04E-001	8.56E-002	7.16E-002	6.12E-002	5.34E-002
10^{-14}	1.28E-001	1.04E-001	8.56E-002	7.16E-002	6.12E-002	5.34E-002

Table 3.12: Results for Example 3.4.2, Max. Errors using Bakhvalov mesh before extrapolation for $\mu = 2^{-3}$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
10^{-3}	1.18E-001	5.05E-002	1.63E-002	5.48E-003	2.30E-003	2.32E-003
10^{-5}	1.10E-001	1.94E-001	2.64E-001	2.33E-001	1.24E-001	5.87E-002
10^{-7}	1.05E-001	1.88E-001	2.87E-001	3.55E-001	3.26E-001	2.11E-001
10^{-9}	1.05E-001	1.88E-001	2.88E-001	3.57E-001	3.31E-001	2.18E-001
10^{-11}	1.05E-001	1.88E-001	2.88E-001	3.57E-001	3.31E-001	2.18E-001
10^{-13}	1.05E-001	1.88E-001	2.88E-001	3.57E-001	3.31E-001	2.18E-001
10^{-14}	1.05E-001	1.88E-001	2.88E-001	3.57E-001	3.31E-001	2.18E-001

Table 3.13: Results for Example 3.4.2, Max. Errors using Shishkin mesh after extrapolation for $\mu = 2^{-3}$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
10^{-3}	8.85E-002	7.32E-002	6.19E-002	5.41E-002	4.83E-002	4.38E-002
10^{-5}	8.64E-002	7.08E-002	5.97E-002	5.20E-002	4.64E-002	4.21E-002
10^{-7}	8.63E-002	7.08E-002	5.97E-002	5.20E-002	4.64E-002	4.21E-002
10^{-9}	8.63E-002	7.08E-002	5.97E-002	5.20E-002	4.64E-002	4.21E-002
10^{-11}	8.63E-002	7.08E-002	5.97E-002	5.20E-002	4.64E-002	4.21E-002
10^{-13}	8.63E-002	7.08E-002	5.97E-002	5.20E-002	4.64E-002	4.21E-002
10^{-14}	8.63E-002	7.08E-002	5.97E-002	5.20E-002	4.64E-002	4.21E-002

Table 3.14: Results for Example 3.4.2, Max. Errors using Bakhvalov mesh after extrapolation for $\mu = 2^{-3}$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
10^{-3}	3.90E-002	1.79E-002	5.36E-003	2.37E-003	2.33E-003	2.32E-003
10^{-5}	7.83E-002	8.20E-002	3.73E-002	2.44E-002	1.10E-002	4.20E-003
10^{-7}	8.01E-002	9.96E-002	6.05E-002	4.24E-002	8.17E-002	4.99E-002
10^{-9}	8.01E-002	9.98E-002	6.11E-002	4.29E-002	8.66E-002	5.51E-002
10^{-11}	8.01E-002	9.98E-002	6.11E-002	4.29E-002	8.66E-002	5.51E-002
10^{-13}	8.01E-002	9.98E-002	6.11E-002	4.29E-002	8.66E-002	5.51E-002
10^{-14}	8.01E-002	9.98E-002	6.11E-002	4.29E-002	8.66E-002	5.51E-002

Table 3.15: Result for Example 3.4.2, Max. Errors: Using Shishkin mesh before extrapolation for $\mu = 2^{-6}$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
10^{-6}	3.43E-001	2.30E-001	1.44E-001	8.21E-002	4.79E-002	4.19E-002
10^{-8}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.79E-002	4.19E-002
10^{-10}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.79E-002	4.19E-002
10^{-12}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.79E-002	4.19E-002
10^{-14}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.79E-002	4.19E-002
10^{-16}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.79E-002	4.19E-002
10^{-17}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.79E-002	4.19E-002

Table 3.16: Results for Example 3.4.2, Max. Errors using Bakhvalov mesh before extrapolation for $\mu = 2^{-6}$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
10^{-6}	1.34E-002	5.30E-003	1.55E-002	5.07E-002	1.08E-001	1.02E-001
10^{-8}	1.37E-002	6.31E-003	2.99E-003	1.63E-003	4.60E-003	1.39E-002
10^{-10}	1.37E-002	6.32E-003	3.03E-003	1.48E-003	2.06E-003	4.17E-003
10^{-12}	1.37E-002	6.32E-003	3.03E-003	1.48E-003	2.03E-003	4.07E-003
10^{-14}	1.37E-002	6.32E-003	3.03E-003	1.48E-003	2.03E-003	4.07E-003
10^{-16}	1.37E-002	6.32E-003	3.03E-003	1.48E-003	2.03E-003	4.07E-003
10^{-17}	1.37E-002	6.32E-003	3.03E-003	1.48E-003	2.03E-003	4.07E-003

Table 3.17: Results for Example 3.4.2, Max. Errors using Shishkin mesh after extrapolation for$\mu = 2^{-6}$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
10^{-6}	8.94E-002	5.86E-002	4.71E-002	4.11E-002	3.66E-002	3.31E-002
10^{-8}	8.97E-002	5.90E-002	4.71E-002	4.10E-002	3.66E-002	3.31E-002
10^{-10}	8.97E-002	5.90E-002	4.71E-002	4.10E-002	3.66E-002	3.31E-002
10^{-12}	8.97E-002	5.90E-002	4.71E-002	4.10E-002	3.66E-002	3.31E-002
10^{-14}	8.97E-002	5.90E-002	4.71E-002	4.10E-002	3.66E-002	3.31E-002
10^{-16}	8.97E-002	5.90E-002	4.71E-002	4.10E-002	3.66E-002	3.31E-002
10^{-17}	8.97E-002	5.90E-002	4.71E-002	4.10E-002	3.66E-002	3.31E-002

Table 3.18: Results for Example 3.4.2, Max. Errors using Bakhvalov mesh after extrapolation for $\mu = 2^{-6}$

ϵ	n=32	n=64	n=128	n=256	n=512	n=1024
10^{-6}	7.75E-003	3.77E-003	1.36E-002	3.13E-002	4.04E-002	3.99E-002
10^{-8}	8.01E-003	3.79E-003	1.81E-003	1.64E-003	4.50E-003	1.27E-002
10^{-10}	8.01E-003	3.80E-003	1.85E-003	1.00E-003	2.05E-003	4.14E-003
10^{-12}	8.01E-003	3.80E-003	1.85E-003	9.96E-004	2.03E-003	4.04E-003
10^{-14}	8.01E-003	3.80E-003	1.85E-003	9.96E-004	2.03E-003	4.04E-003
10^{-16}	8.01E-003	3.80E-003	1.85E-003	9.96E-004	2.03E-003	4.04E-003
10^{-17}	8.01E-003	3.80E-003	1.85E-003	9.96E-004	2.03E-003	4.04E-003

Table 3.19: Results for Example 3.4.2, The rate of convergence of Shishkin mesh before extrapolation for $\mu = 1$

ϵ	r_1	r_2	r_3	r_4	r_5
1	0.98	0.99	1.00	1.00	1.00
2^{-1}	0.97	0.99	0.99	1.00	1.00
2^{-2}	0.95	0.98	0.99	0.99	1.00
2^{-3}	0.91	0.96	0.98	0.99	0.99
2^{-5}	0.55	0.64	0.71	0.76	0.80
2^{-6}	0.56	0.64	0.71	0.76	0.79

Table 3.20: Results for Example 3.4.2, The rate of convergence of Shishkin mesh after extrapolation for $\mu = 1$

ϵ	r_1	r_2	r_3	r_4	r_5
1	1.97	1.99	1.99	2.00	2.00
2^{-1}	1.95	1.98	1.99	1.99	2.00
2^{-2}	1.91	1.96	1.98	1.99	1.99
2^{-3}	1.83	1.92	1.96	1.98	1.99
2^{-5}	0.39	0.46	0.58	0.83	1.71
2^{-6}	0.19	0.14	0.16	0.17	0.17

Table 3.21: Results for Example 3.4.2, The rate of convergence of Bakhvalov mesh before extrapolation for $\mu = 1$

ϵ	r_1	r_2	r_3	r_4	r_5
1	0.44	0.33	0.27	0.31	0.40
2^{-1}	0.82	0.58	0.38	0.23	0.15
2^{-2}	0.91	0.89	0.74	0.49	0.31
2^{-3}	0.85	0.92	0.90	0.80	0.57
2^{-4}	0.78	0.85	0.92	0.91	0.83
2^{-5}	0.36	0.79	0.86	0.92	0.92

Table 3.22: Results for Example 3.4.1, The rate of convergence of Shishkin mesh before extrapolation for $\mu = 2^{-3}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-3}	1.00	1.00	1.00	1.00	1.00
2^{-4}	1.00	1.00	1.00	1.00	1.00
2^{-5}	1.00	1.00	1.00	1.00	1.00
2^{-6}	0.99	1.00	1.00	1.00	1.00
2^{-7}	0.98	0.99	1.00	1.00	1.00
2^{-8}	0.96	0.98	0.99	1.00	1.00
2^{-9}	0.90	0.97	0.98	0.99	1.00

Table 3.23: Results for Example 3.4.1, The rate of convergence of Shishkin mesh after extrapolation for $\mu = 2^{-3}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-3}	2.00	2.00	2.00	2.00	2.00
2^{-4}	1.99	2.00	2.00	2.00	2.00
2^{-5}	1.98	1.99	2.00	2.00	2.00
2^{-6}	1.97	1.99	1.99	2.00	2.00
2^{-7}	1.95	1.97	1.99	1.99	2.00
2^{-8}	1.92	1.96	1.98	1.99	1.99
2^{-9}	1.85	1.91	1.96	1.98	1.99

Table 3.24: Results for Example 3.4.1, The rate of convergence of Shishkin mesh before extrapolation for $\mu = 2^{-6}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-6}	1.17	1.09	1.05	1.02	1.01
2^{-7}	1.17	1.09	1.05	1.03	1.01
2^{-8}	1.17	1.09	1.05	1.03	1.01
2^{-9}	1.18	1.09	1.05	1.03	1.01
2^{-10}	1.20	1.09	1.05	1.03	1.01
2^{-11}	1.17	1.09	1.05	1.03	1.01
2^{-12}	1.16	1.09	1.05	1.02	1.01
2^{-13}	1.13	1.08	1.04	1.02	1.01
2^{-14}	1.12	1.06	1.03	1.02	1.01
2^{-15}	1.06	1.02	1.02	1.01	1.01
2^{-17}	1.06	1.09	1.14	1.13	0.83

Table 3.25: Results for Example 3.4.1, The rate of convergence of Shishkin mesh after extrapolation for $\mu = 2^{-6}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-6}	2.00	2.00	2.00	2.00	2.00
2^{-7}	2.00	2.00	2.00	2.00	2.00
2^{-8}	2.00	2.00	2.00	2.00	2.00
2^{-9}	1.99	2.00	2.00	2.00	2.00
2^{-10}	1.99	1.99	2.00	2.00	2.00
2^{-11}	1.98	1.99	2.00	2.00	2.00
2^{-12}	1.97	1.99	1.99	2.00	2.00
2^{-13}	1.94	1.98	1.99	2.00	2.00
2^{-14}	1.87	1.96	1.98	1.99	2.00
2^{-15}	1.79	1.91	1.96	1.98	1.99
2^{-17}	0.30	0.36	0.48	0.72	1.51

Chapter 4

A convergence acceleration technique on a fitted operator method

In this chapter, we introduce a convergence acceleration techniques, namely, the Defect-correction method and the Richardson extrapolation method which are two of the most important acceleration techniques seen in the literature. The extrapolation formula and some results on the fitted operator finite difference method before and after extrapolation are given. Then we supply some numerical results to confirm the theoretical estimates.

4.1 A brief introduction to convergence acceleration techniques

Higher order methods are preferred over their lower order counterparts in the sense that an expected degree of accuracy can be obtained via the former. However, it is not always easy to design direct higher order methods. To circumvent this difficulty, convergence acceleration techniques can be useful. The two major types of these techniques are the defect-correction method and Richardson extrapolation method.

The defect-correction technique proceeds in three steps:

- Compute the defect which is a quantity that shows how well the problem has been solved.
- Use this defect in a simplified version of the problem to obtain a correction quantity.
- Apply the correction to the approximate solution to obtain a new and better approximate solution.

This process can be repeated until satisfactory results are reached.

The Richardson extrapolation is a convergence acceleration technique in which a linear combination of two computed solutions approximating a particular quantity gives a third and better approximated solution. Typically these solutions are calculated on single and double nested meshes. This method is used to increase the accuracy of computed approximations of the solutions of boundary value differential equations and to improve the parameter-uniform rates of convergence of computed solutions for one parameter linear singular perturbation problems (see, e.g., [48] and some of the references therein). In this chapter, we are only going to use Richardson extrapolation.

4.2 A fitted operator finite difference method (FOFDM)

Let n be a non-negative integer and consider the division of closed unit interval $[0, 1]$ into equidistant subintervals with:

$$x_0 = 0, \quad x_n = 1, \quad x_j = x_0 + hj, \quad h = x_j - x_{j-1}$$

for all $j = 1(1)n$.

We denote the above mesh by Ω_n while the mesh Ω_{2n} is obtained by bisecting each mesh interval in Ω_n, that means

$\Omega_{2n} = \{\tilde{x_j}\}$ with $\tilde{x_0} = 0$, $\tilde{x_{2n}} = 1$ and $\tilde{x_j} - \tilde{x_{j-1}} = \tilde{h} = h/2, \quad j = 1(1)2n.$

These two meshes will be used to derive the extrapolation formula in subsection 4.3.2.

With an assumption that all coefficient functions in the differential equation (1.0.1) are constant everywhere in the above-mentioned interval of interest, Patidar in [56] considered equation (1.0.1) at fixed points x_j:

$$L_{\epsilon,\mu}u(x_j) = \epsilon u''(x_j) + \mu u(x_j)u'(x_j) - b(x_j)u(x_j) = f(x_j), j = 1(1)n-1. \quad (4.2.1)$$

$$L^h_{\epsilon,\mu}v_j \equiv \epsilon\frac{v_{j+1} - 2v_j + v_{j-1}}{\varphi_j^2} + \mu\breve{b}_j\frac{v_{j+1} - v_j}{h} - \breve{b}_j v_j = \breve{f}_j \quad (4.2.2)$$

where

$$\breve{a}_j = \frac{a_j + a_{j+1}}{2}, \quad \breve{b}_j = \frac{b_j + b_{j+1}}{2}, \quad \breve{f}_j = \frac{f_{j+1} + f_j + f_{j-1}}{3}$$

and a dummy variable denominator function:

$$\varphi_j^2(\mu, h, \epsilon) = \varphi_j^2 \equiv \left(\frac{\mu\breve{a}_j}{\epsilon h}\right)^{-1}\left[\exp\left(\frac{\epsilon}{\mu\breve{a}_j h}\right)^{-1} - 1\right] \equiv h^2 + \mathcal{O}\left(\frac{\mu h^3}{\epsilon}\right)$$

The equation (4.2.2) is referred to as fitted operator finite difference method (FOFDM)and applied the system of linear equation, it can be written as the following tridiagonal matrix:

$$Au = F$$

With corresponding entries of A and F as follows:

$$A_{ij} = r_j^+, i = j-1; j = 2(1)n-1,$$

$$A_{ij} = r_j^c, i = j; j = 1(1)n-1,$$

$$A_{ij} = r_j^-, i = j+1; j = 1(1)n-2,$$

$F_1 = f_1 - r_1^- \omega_0$ and $F_{n-1} = f_{n-1} - r_{n-1}^+ \omega_1$ where $r^+ = \frac{\epsilon}{\varphi_j^2} + \frac{\mu \breve{a}_j}{h}$, $\quad r^c = -\frac{2\epsilon}{\varphi_j^2} + \frac{\mu \breve{a}_j}{h} - \breve{b}_j$, $r^- = \frac{\epsilon}{\varphi_j^2}$.

4.3 Extrapolation techniques applied to FOFDM given in Section 4.2

The existence and uniqueness of continuous boundary value problem (1.0.1) does not imply the same for the corresponding discrete problems. Hence it is a basic requirement to determine whether this discrete solution has a unique solution. To this end, the method (4.2.2) should satisfy the discrete minimum principle in the next subsection.

4.3.1 The error before extrapolation

Lemma 4.3.1. [56] (Discrete minimum principle) *Let Φ_i be any mesh function that satisfies $\Phi_0 \geq 0$, $\Phi_n \geq 0$, and $L_{\epsilon,\mu}^h \Phi_i \leq 0$, for all for $i = 1, 2, 3, \ldots n-1$, then $\Phi_i \geq 0$ for all $i = 1, 1, 3, \ldots .n$.*

Proof. Let $\Phi_j = \min_i \Phi_i$ and assume that $\Phi_j < 0$. It can be seen that $j \notin \{0, 1\}$, $\Phi_j - \Phi_{j-1} \leq 0$ and $\Phi_{j+1} - \Phi_j \geq 0$
We have

$$L_{\epsilon,\mu}^h \Phi_j = \frac{\epsilon}{\varphi_j^2} (\Phi_{j+1} - 2\Phi_j + \Phi_{j-1}) + \frac{\mu \breve{a}_j}{h} (\Phi_{j+1} - \Phi_j) - b_j \Phi_j$$

$$= \frac{\epsilon}{\varphi_j^2} [(\Phi_{j+1} - \Phi_j) - (\Phi_j - \Phi_{j-1})] + \frac{\mu \breve{a}_j}{h} (\Phi_{j+1} - \Phi_j) - b_j \Phi_j > 0$$

which is a contradiction. Therefore $\Phi_j \geq 0, \quad \forall i$.

Theorem 4.3.1. *Assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth so that $u(x) \in C^4[0, 1]$. Then the FOFDM (4.2.2) is first order ϵ and μ-uniformly convergent* *in the sense that the numerical solution v of problem (1.0.1) obtained via (4.2.2) (with $v_0 = \alpha_0$, $v_n = \alpha_1$) satisfies the error estimate*

$$\sup_{0<\epsilon,\mu\leq 1} \max_{0\leq j\leq n} |u_j - v_j| \leq Mh \tag{4.3.3}$$

Proof. The truncation error of method is given by:

$$\begin{aligned}
\tau_j(u) &= L_{\epsilon,\mu}u_j - \tilde{f}_j, \qquad (4.3.4)\\
&= (\epsilon u_j'' + \mu a(x)u' - b(x)u) - (\epsilon\frac{u_{j+1} - 2u_j + u_{j-1}}{\xi_j^2} + \mu\tilde{a}_j\frac{u_{j+1} - u_j}{h} - \tilde{b}_j u_j)\\
&= \epsilon u_j'' + \mu a_j u_j' - b_j u_j - \frac{\epsilon}{\xi_j^2}[u_{j+1} - 2u_j + u_{j-1}] - \frac{\mu\tilde{a}_j}{h}[u_{j+1} - u_j] + \tilde{b}_j u_j\\
&= \epsilon u_j'' + \mu a_j u_j' - b_j u_j - \frac{\epsilon}{\xi_j^2}[u_j + hu_j + \frac{h^2}{2}u_j^{''} + \frac{h^3}{6}u_j^{'''} + \frac{h^4}{24}u_j^4\\
&\quad -2u_j + \frac{h^2}{2}u_j^{''} - \frac{h^3}{6}u_j^{'''} + \frac{h^4}{24}u_j^4 + ...] - \frac{\mu\tilde{a}_j}{h}[u_j + hu_j + \frac{h^2}{2}u_j^{''}\\
&\quad +\frac{h^3}{6}u_j^{'''} + \frac{h^4}{24}u_j^4 - u_j + ...] + \tilde{b}_j u_j\\
&= \epsilon u_j^{''} + \mu a_j u_j^{'} - b_j u_j - \frac{\epsilon}{\xi_j^2}[h^2 u_j^{''} + \frac{h^4}{12}u_j^4 + ..] - \frac{\mu\tilde{a}_j}{h}[hu_j^{'}\\
&\quad +\frac{h^2}{2}u_j^{''} + \frac{h^3}{6}u_j^{'''} + \frac{h^4}{24}u_j^4 + ...] + \tilde{b}_j u_j\\
&= \epsilon u_j^{''} + \mu a_j u_j^{'} - b_j u_j - \epsilon\frac{\mu a_j}{h\epsilon}\left[\frac{\epsilon}{\mu a_j h} - \frac{1}{2} + \frac{1}{12}\frac{\mu a_j h}{\epsilon}\right]\\
&\quad +\frac{1}{120}\left(\frac{\mu a_j h}{\epsilon}\right)^3 + ..][h^2 u_j^{''} + \frac{h^4}{12}u_j^4 + ..]\\
&\quad -\mu\tilde{a}_j[u_j^{'} + \frac{h}{2}u_j^{''} + \frac{h^2}{6}u_j^{'''} + \frac{h^3}{24}u_j^4 + ...] + \tilde{b}_j u_j\\
&= \epsilon u_j'' + \mu a_j u_j^{'} - b_j u_j - \left[\frac{\epsilon}{h^2} - \frac{a_j\mu}{2h} + \frac{a_j^3\mu^3}{12\epsilon}\right][h^2 u_j^{''}\\
&\quad +\frac{h^4}{12}u_j^4 + ...] - \mu\tilde{a}_j[u_j^{'} + \frac{h}{2}u_j^{''} + \frac{h^2}{6}u_j^{'''} + \frac{h^3}{24}u_j^4 + ...] + \tilde{b}_j u_j\\
&= \epsilon u_j'' + \mu a_j u_j^{'} - b_j u_j - \epsilon u_j^{''} - \frac{\epsilon h^2}{12}u_j^4 + \frac{1}{2}h\mu a_j u^{''} - \frac{a_j^2\mu^2 h^2}{\epsilon}u_j^{''}\\
&\quad -\mu\tilde{a}_j[u_j^{'} + \frac{h}{2}u_j^{''} + \frac{h^2}{6}u_j^{'''}] + \tilde{b}_j u_j
\end{aligned}$$

$$
\begin{aligned}
&= \mu a_j u_j' - b_j u_j - \frac{\epsilon h^2}{12} u_j^4 + \frac{1}{2} h\mu a_j u'' - \frac{a_j^2 \mu^2 h^2}{\epsilon} u_j'' \\
&\quad -\mu \frac{a_j + a_{j+1}}{2} [u_j' + \frac{h}{2} u_j'' + \frac{h^2}{6} u_j'''] + \frac{b_j + b_{j+1}}{2} u_j \\
&= \mu a_j u_j' - b_j u_j - \frac{\epsilon h^2}{12} u_j^4 + \frac{1}{2} h\mu a_j u'' - \frac{a_j^2 \mu^2 h^2}{\epsilon} u_j'' - \frac{\mu}{2}(2a_j + h a_j' + \frac{h^2}{2} a_j'' \\
&\quad + \frac{h^3}{6} a_j''')[u_j' + \frac{h}{2} u_j'' + \frac{h^2}{6} u_j'''] + \frac{1}{2}[2b_j + h b_j' + \frac{b^2}{2} b_j'' + \frac{h^3}{6} b_j''']u_j \\
&= \mu a_j u_j' - b_j u_j - \frac{\epsilon h^2}{12} u_j^4 + \frac{1}{2} h\mu a_j u'' - \frac{a_j^2 \mu^2 h^2}{\epsilon} u_j''(\mu a_j + \frac{\mu h}{2} a_j' + \frac{\mu h^2}{4} a_j'' \\
&\quad + \frac{\mu h^3}{12} a_j''')[u_j' + \frac{h}{2} u_j'' + \frac{h^2}{6} u_j''']b_j u_j + \frac{b_j' h}{2} u_j + \frac{b_j'' h^2}{4} u_j + \frac{b_j''' h^3}{12} u_j \\
&= \mu a_j u_j' - \frac{\epsilon h^2}{12} u_j^4 + \frac{h\mu a_j}{2} u_j'' - \frac{h^2 \mu^2 a_j^2}{12\epsilon} u_j'' - \mu a_j u_j' - \frac{h\mu a_j}{2} u_j'' \\
&\quad - \frac{h^2 \mu a_j}{6} u_j''' - \frac{h\mu a_j'}{2} u_j' + \frac{h^2 \mu a_j'}{4} u_j'' \\
&= \frac{h^2 \mu a_j''}{4} u_j' + \frac{h^2 b_j'}{2} u_j + \frac{h^4 b_j''}{4} u_j + \frac{h^3 b_j'''}{12} u_j \\
&= -\frac{h\mu a_j'}{2} u_j' + \frac{h b_j'}{2} u_j - \frac{\epsilon h^2}{12} u_j^4 - \frac{h^2 \mu^2 a_j^2}{12\epsilon} u_j'' - \frac{h^2 \mu a_j}{6} u_j''' \\
&\quad + \frac{h^2 \mu a_j'}{4} u_j'' + \frac{h^2 \mu a_j''}{4} u_j' + \frac{h^4 b_j''}{4} u_j + \mathcal{O}(h^3).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
L(u_j - v_j) &= -\frac{h\mu a_j'}{2} u_j' + \frac{h b_j'}{2} u_j - \frac{\epsilon h^2}{12} u_j^4 - \frac{h^2 \mu^2 a_j^2}{12\epsilon} u_j'' - \frac{h^2 \mu a_j}{6} u_j''' \\
&\quad + \frac{h^2 \mu a_j'}{4} u_j'' + \frac{h^2 \mu a_j''}{4} u_j' + \frac{h^4 b_j''}{4} u_j + \mathcal{O}(h^3) \\
&= \left(\frac{h b_j'}{2} + \frac{h^4 b_j''}{4} \right) u_j + \left(\frac{h^2 \mu a_j''}{4} - \frac{h\mu a_j'}{2} \right) u_j' \\
&\quad + \left(\frac{h^2 \mu a_j'}{4} - \frac{h^2 \mu^2 a_j^2}{12\epsilon} \right) u_j'' - \left(\frac{h^2 \mu a_j}{6} \right) u_j''' \\
&\quad - \left(\frac{\epsilon h^2}{12} \right) u_j^4 + \mathcal{O}(h^3).
\end{aligned} \tag{4.3.5}
$$

Now let

$$T_0 = \left(\frac{hb_j'}{2} + \frac{h^2 b_j''}{4}\right), \quad T_1 = \left(\frac{h^2 \mu a_j''}{4} - \frac{h\mu a_j'}{2}\right), \quad T_2 = \left(\frac{h^2 \mu a_j'}{4} - \frac{h^2 \mu^2 a_j^2}{12\epsilon}\right),$$

$$T_3 = \left(-\frac{h^2 \mu a_j}{6}\right), \quad T_4 = \left(-\frac{\epsilon h^2}{12}\right).$$

From 4.3.5 we have

$$L(u_j - v_j) = T_0 u_j + T_1 u_j^{'} + T_2 u_j^{''} + T_3 u_j^{'''} + T_4 u_j^{iv} + \mathcal{O}(h^3)$$

since

$$|T_0| \leq Mh, \quad |T_1| \leq Mh, \quad |T_2| \leq Mh^2, \quad |T_3| \leq Mh^2, \quad |T_4| \leq Mh^2.$$

Where M is a positive constant which independent of ϵ, μ and the mesh size h.

Hence

$$\begin{aligned} |L(u_j - v_j)| &= |T_0 u_j + T_1 u_j^{'} + T_2 u_j^{''} + T_3 u_j^{'''} + T_4 u_j^{iv} + \mathcal{O}(h^3)| \qquad (4.3.6) \\ &\leq |T_0||u_j| + |T_1||u_j^{'}| + |T_2||u_j^{''}| + |T_3||u_j^{'''}| + |T_4||u_j^{iv}| \\ &\leq Mh|u_j| + Mh|u_j^{'}| + Mh^2|u_j^{''}| + Mh^2|u_j^{'''}| + Mh^2|u_j^{iv}| \\ |L(u_j - vj)| &\leq Mh. \end{aligned}$$

4.3.2 Extrapolation formula

Again, we denote by $\tilde{v}_j$ the numerical solution computed on the mesh Ω_{2n}. From estimate (4.3.3) we have:

$$u(x_j) - v_j^n = Mh + R_n(x_j), \quad \forall x_j \in \Omega_n \qquad (4.3.7)$$

and

$$u(\tilde{x_j}) - \tilde{v_j} = M\left(\frac{h}{2}\right) + R_{2n}(\tilde{x_j}), \quad \forall \tilde{x_j} \in \Omega_{2n}, \tag{4.3.8}$$

where the remainders $R_n(x_j)$ and $R_{2n}(\tilde{x_j})$ are $\mathcal{O}(h)$. We multiply equation (3.3.4) by factor 2 yields

$$2[u(\tilde{x_j}) - \tilde{v_j}] = Mh + 2R_{2n}(\tilde{x_j}) \quad \forall \tilde{x_j} \in \Omega_{2n} \tag{4.3.9}$$

Then subtracting equation (4.3.7) from (4.3.9) we obtain

$$2[u(x_j) - \tilde{v_j}] - u(x_j) - v_j^n) = 2R_{2n}(x_j) - R_n(x_j), \quad \forall x_j \in \Omega_{2n}$$

which gives

$$u(\tilde{x_j}) - (2\tilde{v_{2j}} - v_j) = \mathcal{O}(h^2), \quad \forall \tilde{x_j} \in \Omega_{2n}$$

and therefore, we use

$$v_j^{ext} = 2\tilde{v_{2j}} - v_j, \quad \forall x_j \in \Omega_n$$

as the extrapolation formula in next section.

4.3.3 The error after extrapolation

From (4.3.6) make $\widehat{T_j} = T_j$ and bisecting h gives,

$$L(u_j - v_j) = \widehat{T}_0 u_j + \widehat{T}_1 u_j' + \widehat{T}_2 u_j'' + \widehat{T}_3 u_j''' + \widehat{T}_4 u_j^{iv} + \mathcal{O}(h^3)$$

where

$$\widehat{T}_0 = \left(\frac{hb_j'}{4} + \frac{h^2 b_j''}{16}\right), \quad \widehat{T}_1 = \left(\frac{h^2\mu a_j''}{16} - \frac{h\mu a_j'}{4}\right), \quad \widehat{T}_2 = \left(\frac{h^2\mu a_j'}{16} - \frac{h^2\mu^2 a_j^2}{48\epsilon}\right)$$

$$\widehat{T}_3 = \left(-\frac{h^2\mu a_j}{24}\right), \quad \widehat{T}_4 = \left(-\frac{\epsilon h^2}{48}\right)$$

$$\begin{aligned}
L(u_j - v_j^{ext}) &= 2[L(u_j - \tilde{v_j})] - [L(u_j - v_j)] \\
&= 2[\widehat{T}_0 u_j + \widehat{T}_1 u_j^{'} + \widehat{T}_2 u_j^{''} + \widehat{T}_3 u_j^{'''} + \widehat{T}_4 u_j^{iv}] \\
&\quad -[T_0 u_j + T_1 u_j^{'} + T_2 u_j^{''} + T_3 u_j^{'''} + T_4 u_j^{iv}] \\
&= \tilde{T}_0 u_j + \tilde{T}_1 u_j^{'} + \tilde{T}_2 u_j^{''} + \tilde{T}_3 u_j^{'''} + \tilde{T}_4 u_j^{iv}
\end{aligned}$$

where

$$\tilde{T}_0 = 2\widehat{T}_0 - T_0, \quad \tilde{T}_1 = 2\widehat{T}_1 - T_1, \quad \tilde{T}_2 = 2\widehat{T}_2 - T_2, \quad \tilde{T}_3 = 2\widehat{T}_3 - T_3, \quad \tilde{T}_4 = 2\widehat{T}_4 - T_4,$$

$$\begin{aligned}
\tilde{T}_0 &= 2\widehat{T}_0 - T_0 \\
&= 2\left(\frac{hb_j^{'}}{4} + \frac{h^2 b_j^{''}}{16}\right) - \left(\frac{hb_j^{'}}{2} + \frac{h^2 b_j^{''}}{4}\right) \\
\tilde{T}_0 &= \left(\frac{hb_j^{'}}{4} + \frac{h^2 b_j^{''}}{16}\right) - \frac{hb_j^{'}}{2} - \frac{h^2 b_j^{''}}{4} \\
\tilde{T}_0 &= -\frac{b_j^{''} h^2}{8} \\
|\tilde{T}_0| &= Mh^2. \\
\tilde{T}_1 &= 2\widehat{T}_1 - T_1 \\
&= 2\left(\frac{h^2 \mu a_j^{''}}{16} - \frac{h\mu a_j^{'}}{4}\right) - \left(\frac{h^2 \mu a_j^{''}}{4} - \frac{h\mu a_j^{'}}{2}\right) \\
&= \left(\frac{h^2 \mu a_j^{''}}{8} - \frac{\mu h a_j^{'}}{2}\right) - \left(\frac{h^2 \mu a_j^{''}}{4} + \frac{h\mu a_j^{'}}{2}\right) \\
\tilde{T}_1 &= -\left(\frac{h^2 \mu a_j^{''}}{8}\right) \\
|\tilde{T}_1| &= Mh^2. \\
\tilde{T}_2 &= 2\widehat{T}_2 - T_2 \\
&= 2\left(\frac{h^2 \mu a_j^{'}}{16} - \frac{h^2 \mu^2 a_j^2}{48\epsilon}\right) - \left(\frac{h^2 \mu a_j^{'}}{4} - \frac{h^2 \mu^2 a_j^2}{12\epsilon}\right) \\
&= \left(\frac{h^2 \mu a_j^{'}}{8} - \frac{h^2 \mu^2 a_j^2}{24}\right) - \left(\frac{h^2 \mu a_j^{'}}{4} - \frac{h^2 \mu^2 a_j^2}{12\epsilon}\right)
\end{aligned}$$

$$
\begin{aligned}
\tilde{T}_2 &= -\frac{h^2\mu a_j^{'}}{8} + \frac{h^2\mu^2 a_j^2}{24} \\
|\tilde{T}_2| &= Mh^2.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{T}_3 &= 2\widehat{T}_3 - T_3 \\
&= 2\left(\frac{-\mu a_j h^2}{24}\right) + \left(\frac{\mu a_j h^2}{6}\right) \\
&= \frac{-\mu a_j h^2}{12} + \frac{\mu a_j h^2}{6} \\
\tilde{T}_3 &= \frac{\mu a_j h^2}{12} \\
|\tilde{T}_3| &= Mh^2. \\
\tilde{T}_4 &= 2\widehat{T}_4 - T_4 \\
&= 2\left(-\frac{\epsilon h^2}{48}\right) - \left(-\frac{\epsilon h^2}{12}\right) \\
&= -\frac{\epsilon h^2}{24} + \frac{\epsilon h^2}{12} \\
\tilde{T}_3 &= \frac{\epsilon h^2}{24} \\
|\tilde{T}_4| &= Mh^2. \\
|L(u_j - v_j^{ext})| &= |\tilde{T}_0 u_j + \tilde{T}_1 u_j^{'} + \tilde{T}_2 u_j^{''} + \tilde{T}_3 u_j^{'''} + \tilde{T}_4 u_j^{iv}| \\
&\leq |\tilde{T}_0||u_j| + |\tilde{T}_1||u_j^{'}| + |\tilde{T}_2||u_j^{''}| + |\tilde{T}_3||u_j^{'''}| + |\tilde{T}_4||u_j^{iv}| \\
&\leq Mh^2|u_j| + Mh^2|u_j^{'}| + Mh^2|u_j^{''}| + Mh^2|u_j^{'''}| + Mh^2|u_j^{iv}|
\end{aligned}
$$

$$|L(u_j - v_j^{ext})| \leq Mh^2$$

Now applying stability Lemma 3.3.2, on the mesh function $(u_j - v_j^{ext})_j$ to obtain

$$\max_{1\leq j\leq n-1} |u_j - v_j^{ext}) \leq Mh^2.$$

Finally, we have

$$\sup_{1<\epsilon,\mu\leq 1} \max_{1\leq j\leq n-1} |u_j - v_j^{ext}) \leq Mh^2.$$

4.4 Comparative numerical results

In this section we present some numerical results of two problems from [56] to demonstrate the theoretical results.

Example 4.4.1. Consider the problem:

$$\epsilon u''(x) + \mu(1+x)u'(x) - u(x) = (1+x)^2, x \in \Omega; \quad u(0) = u(1) = 0 \tag{4.4.10}$$

For this problem the exact solution is not known, we shall use double mesh principle [56]. Let $v_j \equiv v_j^n$, then we denote maximum errors for different values of n, ϵ and μ at all the mesh points by $E^B_{n,\epsilon,\mu}$ and $E^A_{n,\epsilon,\mu}$ as follows:
Before Extrapolation

$$E^B_{n,\epsilon,\mu} := \max_{0 \le j \le n} \left| v_j^n - v_{2j}^{2n} \right| \tag{4.4.11}$$

After Extrapolation

$$E^A_{n,\epsilon,\mu} := \max_{0 \le j \le n} \left| v_j^{ext} - v_{2j}^{ext} \right| \tag{4.4.12}$$

where v_{2j}^{2n} is the numerical solution of (1.0.1) obtained using (4.2.2) on the mesh Ω_{2n} and v_{2j}^{ext} is the solution after extrapolation of v_j^{ext} on same mesh.

Example 4.4.2. Consider the problem:

$$\epsilon u''(x) + \mu u'(x) - u(x) = -x, x \in \Omega \tag{4.4.13}$$

whose exact solution is given by

$$u(x) = (x+\mu) + \frac{((1-\mu)+(1+\mu)e^{-D_2})e^{D_1 x} - ((1+\mu)+(1-\mu)e^{D_1})e^{-D_{2(1-x)}}}{1 - e^{\sqrt{(\mu^2+4\epsilon/\epsilon)}}} \tag{4.4.14}$$

where $D_{1,2} = (-\mu \pm \sqrt{\mu^2 + 4\epsilon})/2\epsilon$. The boundary conditions can be obtained by substituting $x = 0$ and 1 in (4.4.14).

Before Extrapolation

$$E^B_{n,\epsilon,\mu} := \max_{0\le j\le n} |u(x_j) - v_j| \tag{4.4.15}$$

After Extrapolation

$$E^A_{n,\epsilon,\mu} := \max_{0\le j\le n} \left|u(x_j) - v^{ext}\right| \tag{4.4.16}$$

where v_j is the solution of (1.0.1) obtained using (4.2.2) and v^{ext} is the solution after extrapolation of v_j. The numerical rates of convergence are calculated using the following formula [44]:

$$r_k \equiv r_{\epsilon,\mu,k} := \log_2(\tilde{E}_{n_k}/\tilde{E}_{2n_k}), \quad k = 1, 2, 3, ...$$

where $\tilde{E}$ stands for $E_{n,\epsilon,\mu}$ and $E^{ext}_{n,\epsilon,\mu}$ respectively. Moreover, we compute

$$E_n = \max_{0\le j\le 1} E_{n,\epsilon,\mu}$$

and

$$E_n = \max_{0\le j\le 1} E^{ext}_{n,\epsilon,\mu}$$

whereas the numerical rate of uniform convergence is computed as

$$R_n := \log_2(E_n/E_{2n})$$

and

$$R^{ext}_n := \log_2(E^{ext}_n/E^{ext}_{2n})$$

4.5 Discussion

As seen in the previous chapter, the numerical results based on the FMFDM on Shishkin mesh were found to be inferior as compared to those obtained on the

Bakhvalov mesh. Still the order of convergence of the FMFDM and the extrapolated scheme was not very good. The main aim of this chapter was therefore to investigate whether we can gain something by using FOFDM rather than FMFDM. We did so by performing it on two test problems. There are some discrepancies seen in the numerical results for the first example but the results for second example invariably confirm the theoretical estimates. One can also see that the results obtained via the FOFDM are very stable for much smaller values of μ where the other methods (seen in the literature) failed to give stable results. This is indeed a big achievement for us.

Table 4.1: Results for Example 4.4.1, Max. Errors: Before Extrapolation Using (4.2.2) and (4.4.11) for $\mu = 1$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
1	3.30E-004	1.32E-004	6.27E-005	3.30E-005	1.69E-005	8.58E-006
2^{-1}	9.64E-004	4.93E-004	2.50E-004	1.26E-004	6.31E-005	3.16E-005
2^{-2}	3.06E-003	1.42E-003	6.86E-004	3.36E-004	1.67E-004	8.29E-005
2^{-3}	6.36E-003	2.71E-003	1.23E-003	5.87E-004	2.86E-004	1.41E-004
2^{-4}	1.08E-002	4.18E-003	1.75E-003	7.91E-004	3.74E-004	1.82E-004
2^{-5}	1.68E-002	6.21E-003	2.36E-003	9.80E-004	4.39E-004	2.07E-004
2^{-6}	2.24E-002	9.08E-003	3.29E-003	1.24E-003	5.12E-004	2.29E-004

Table 4.2: Results for Example 4.4.1, Max. Errors: After Extrapolation Using (4.2.2) and (4.4.12) for $\mu = 1$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
1	8.86E-005	2.32E-005	5.87E-006	1.48E-006	3.70E-007	9.25E-008
2^{-1}	7.55E-005	2.14E-005	5.54E-006	1.40E-006	3.53E-007	8.86E-008
2^{-2}	2.17E-004	5.28E-005	1.30E-005	3.21E-006	7.98E-007	1.99E-007
2^{-3}	9.72E-004	2.47E-004	6.18E-005	1.54E-005	3.85E-006	9.60E-007
2^{-4}	2.52E-003	6.89E-004	1.76E-004	4.41E-005	1.10E-005	2.75E-006
2^{-5}	4.46E-003	1.51E-003	4.07E-004	1.04E-004	2.61E-005	6.52E-006
2^{-6}	4.31E-003	2.54E-003	8.24E-004	2.21E-004	5.63E-005	1.41E-005

Table 4.3: Results for Example 4.4.1, Before Extrapolation Rate of Convergence Using (4.2.2) and (4.4.11) for $\mu = 1$

ϵ	r_1	r_2	r_3	r_4	r_5
1	1.32	1.08	0.92	0.96	0.98
2^{-1}	0.97	0.98	0.99	0.99	1.00
2^{-2}	1.10	1.05	1.03	1.01	1.01
2^{-3}	1.23	1.13	1.07	1.04	1.02
2^{-4}	1.37	1.25	1.15	1.08	1.04
2^{-5}	1.44	1.39	1.27	1.16	1.09
2^{-6}	1.30	1.46	1.41	1.28	1.16

Table 4.4: Results for Example 4.4.1, After Extrapolation Rate of Convergence (4.2.2) and (4.4.12) for $\mu = 1$

ϵ	r_1	r_2	r_3	r_4	r_5
1	1.93	1.98	1.99	2.00	2.00
2^{-1}	1.82	1.95	1.98	1.99	2.00
2^{-2}	2.04	2.03	2.01	2.01	2.00
2^{-3}	1.97	2.00	2.00	2.00	2.00
2^{-4}	1.87	1.97	2.00	2.00	2.00
2^{-5}	1.57	1.89	1.97	1.99	2.00
2^{-6}	0.77	1.62	1.90	1.97	1.99

Table 4.5: Results for Example 4.4.1, Max. Errors: Before Extrapolation Using (4.2.2) and (4.4.11) for $\mu = 2^{-3}$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
2^{-3}	1.60E-003	6.60E-004	2.93E-004	1.37E-004	6.64E-005	3.26E-005
2^{-4}	4.92E-003	1.82E-003	7.50E-004	3.36E-004	1.58E-004	7.65E-005
2^{-5}	1.16E-002	3.95E-003	1.51E-003	6.40E-004	2.91E-004	1.38E-004
2^{-6}	2.40E-002	7.64E-003	2.67E-003	1.05E-003	4.51E-004	2.07E-004
2^{-8}	7.47E-002	2.53E-002	7.84E-003	2.53E-003	9.10E-004	3.65E-004
2^{-10}	9.95E-002	5.18E-002	2.14E-002	6.97E-003	2.13E-003	6.94E-004
2^{-12}	9.97E-002	5.42E-002	2.82E-002	1.39E-002	5.54E-003	1.79E-003
2^{-14}	9.97E-002	5.42E-002	2.83E-002	1.45E-002	7.31E-003	3.52E-003

Table 4.6: Results for Example 4.4.1, Max. Errors: After Extrapolation Using (4.2.2) and (4.4.12) for $\mu = 2^{-3}$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
2^{-3}	3.05E-004	7.53E-005	1.87E-005	4.66E-006	1.16E-006	2.90E-007
2^{-4}	1.29E-003	3.26E-004	8.15E-005	2.03E-005	5.08E-006	1.27E-006
2^{-5}	3.67E-003	9.42E-004	2.36E-004	5.90E-005	1.48E-005	3.69E-006
2^{-6}	8.70E-003	2.31E-003	5.85E-004	1.47E-004	3.67E-005	9.17E-006
2^{-8}	2.40E-002	9.75E-003	2.77E-003	7.17E-004	1.81E-004	4.53E-005
2^{-10}	5.22E-003	9.42E-003	7.45E-003	2.70E-003	7.48E-004	1.92E-004
2^{-12}	9.18E-003	2.52E-003	8.78E-004	2.81E-003	1.97E-003	6.94E-004
2^{-14}	9.18E-003	2.57E-003	6.82E-004	1.64E-004	2.85E-004	7.36E-004

Table 4.7: Results for Example 4.4.1, Before Extrapolation Rate of Convergence Using (4.2.2) and (4.4.11) for $\mu = 2^{-3}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-3}	1.23	1.13	1.07	1.04	1.02
2^{-4}	1.37	1.25	1.15	1.08	1.04
2^{-5}	1.44	1.39	1.27	1.16	1.09
2^{-6}	1.30	1.46	1.41	1.28	1.16
2^{-14}	0.92	0.96	0.98	0.99	1.00

Table 4.8: Results for Example 4.4.1, After Extrapolation Rate of Convergence (4.2.2) and (4.4.12) for $\mu = 2^{-3}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-3}	1.97	2.00	2.00	2.00	2.00
2^{-4}	1.87	1.97	2.00	2.00	2.00
2^{-5}	1.57	1.89	1.97	1.99	2.00
2^{-6}	0.77	1.62	1.90	1.97	1.99
2^{-14}	1.89	1.94	1.97	1.99	2.07

Table 4.9: Results for Example 4.4.1, Max. Errors: Before Extrapolation Using (4.2.2) and (4.4.11) for $\mu = 2^{-6}$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
2^{-12}	3.33E-001	1.79E-001	6.73E-002	1.99E-002	5.37E-003	1.48E-003
2^{-14}	3.43E-001	2.30E-001	1.40E-001	6.30E-002	2.05E-002	5.68E-003
2^{-16}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.27E-002	1.75E-002
2^{-18}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.43E-002	2.30E-002
2^{-20}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.43E-002	2.30E-002
2^{-22}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.43E-002	2.30E-002
2^{-23}	3.43E-001	2.30E-001	1.45E-001	8.24E-002	4.43E-002	2.30E-002

Table 4.10: Results for Example 4.4.1, Max. Errors: After Extrapolation Using (4.2.2) and (4.4.12) for $\mu = 2^{-6}$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
2^{-12}	8.86E-005	2.32E-005	5.87E-006	1.48E-006	3.70E-007	9.25E-008
2^{-14}	7.61E-002	2.54E-002	3.61E-002	1.67E-002	4.98E-003	1.30E-003
2^{-16}	8.97E-002	5.88E-002	1.52E-002	1.25E-002	1.25E-002	4.81E-003
2^{-18}	8.97E-002	5.90E-002	2.03E-002	6.52E-003	8.53E-004	4.31E-003
2^{-20}	8.97E-002	5.90E-002	2.03E-002	6.58E-003	1.84E-003	4.76E-004
2^{-22}	8.97E-002	5.90E-002	2.03E-002	6.58E-003	1.84E-003	4.91E-004
2^{-23}	8.97E-002	5.90E-002	2.03E-002	6.58E-003	1.84E-003	4.91E-004

Table 4.11: Results for Example4.4.1, Before Extrapolation Rate of Convergence Using (4.2.2) and (4.4.11) for $\mu = 2^{-6}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-6}	1.30	1.46	1.41	1.28	1.16
2^{-14}	0.92	0.96	0.98	0.99	1.00
2^{-15}	0.92	0.96	0.98	0.99	1.00
2^{-16}	0.92	0.96	0.98	0.99	1.00
2^{-17}	0.92	0.96	0.98	0.99	1.00

Table 4.12: Results for Example 4.4.1, After Extrapolation Rate of Convergence (4.2.2) and (4.4.12) for $\mu = 2^{-6}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-6}	0.77	1.62	1.90	1.97	1.99
2^{-14}	1.89	1.94	1.97	1.99	2.07
2^{-15}	1.89	1.94	1.97	1.99	1.99
2^{-16}	1.89	1.94	1.97	1.99	1.99
2^{-17}	1.89	1.94	1.97	1.99	1.99

Table 4.13: Results for Example 4.4.1, Max. Errors: Before Extrapolation Using (4.2.2) and (4.4.11) for $\mu = 2^{-20}$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
2^{-20}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-21}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-22}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-23}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-24}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-25}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-26}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-27}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-28}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-29}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-30}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-31}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-32}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-33}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-34}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003
2^{-35}	2.93E-003	7.32E-004	3.03E-004	9.28E-004	1.94E-003	3.88E-003

Table 4.14: Results for Example 4.4.1, Max. Errors: After Extrapolation Using (4.2.2) and (4.4.12) for $\mu = 2^{-20}$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
2^{-20}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-21}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-22}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-23}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-24}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-25}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-26}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-27}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-28}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-29}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-30}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-31}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-32}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-33}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-34}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003
2^{-35}	1.46E-003	3.66E-004	3.94E-004	9.49E-004	1.93E-003	3.85E-003

Table 4.15: Results for Example 4.4.1, Max. Errors: Before Extrapolation Using (4.2.2) and (4.4.11) for $\mu = 2^{-40}$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
2^{-81}	2.93E-003	7.32E-004	1.83E-004	4.58E-005	1.14E-005	2.86E-006
2^{-82}	2.93E-003	7.32E-004	1.83E-004	4.58E-005	1.14E-005	2.86E-006
2^{-85}	2.93E-003	7.32E-004	1.83E-004	4.58E-005	1.14E-005	2.86E-006
2^{-86}	2.93E-003	7.32E-004	1.83E-004	4.58E-005	1.14E-005	2.86E-006
2^{-89}	2.93E-003	7.32E-004	1.83E-004	4.58E-005	1.14E-005	2.86E-006
2^{-90}	2.93E-003	7.32E-004	1.83E-004	4.58E-005	1.14E-005	2.86E-006
2^{-93}	2.93E-003	7.32E-004	1.83E-004	4.58E-005	1.14E-005	2.86E-006
2^{-95}	2.93E-003	7.32E-004	1.83E-004	4.58E-005	1.14E-005	2.86E-006

Table 4.16: Results for Example 4.4.1, Max. Errors: After Extrapolation Using (4.2.2) and (4.4.12) for $\mu = 2^{-40}$

ϵ	n=16	n=32	n=64	n=128	n=256	n=512
2^{-81}	1.46E-003	3.66E-004	9.16E-005	2.29E-005	5.72E-006	1.43E-006
2^{-82}	1.46E-003	3.66E-004	9.16E-005	2.29E-005	5.72E-006	1.43E-006
2^{-85}	1.46E-003	3.66E-004	9.16E-005	2.29E-005	5.72E-006	1.43E-006
2^{-86}	1.46E-003	3.66E-004	9.16E-005	2.29E-005	5.72E-006	1.43E-006
2^{-89}	1.46E-003	3.66E-004	9.16E-005	2.29E-005	5.72E-006	1.43E-006
2^{-90}	1.46E-003	3.66E-004	9.16E-005	2.29E-005	5.72E-006	1.43E-006
2^{-93}	1.46E-003	3.66E-004	9.16E-005	2.29E-005	5.72E-006	1.43E-006
2^{-95}	1.46E-003	3.66E-004	9.16E-005	2.29E-005	5.72E-006	1.43E-006

Table 4.17: Results for Example 4.4.1, Before Extrapolation Rate of Convergence using (4.2.2) and (4.4.11) for $\mu = 2^{-40}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-80}	0.92	0.96	0.98	0.99	1.00
2^{-83}	0.92	0.96	0.98	0.99	1.00
2^{-84}	0.92	0.96	0.98	0.99	1.00
2^{-85}	0.92	0.96	0.98	0.99	1.00
2^{-89}	0.92	0.96	0.98	0.99	1.00
2^{-91}	0.92	0.96	0.98	0.99	1.00

Table 4.18: Results for Example 4.4.1 After Extrapolation Rate of Convergence (4.2.2) and (4.4.12))for $\mu = 2^{-40}$

ϵ	r_1	r_2	r_3	r_4	r_5
2^{-80}	1.89	1.94	1.97	1.99	1.99
2^{-83}	1.89	1.94	1.97	1.99	1.99
2^{-84}	1.89	1.94	1.97	1.99	1.99
2^{-85}	1.89	1.94	1.97	1.99	1.99
2^{-89}	1.89	1.94	1.97	1.99	1.99
2^{-91}	1.89	1.94	1.97	1.99	1.99

Table 4.19: Results for Example 4.4.2, Max. Errors: Before Extrapolation Using (4.2.2) and (4.4.15) for $\mu = 1$

ϵ	n=64	n=128	n=256	n=512	n=1024	n=2048
2^{-3}	8.98E-005	2.25E-005	5.62E-006	1.40E-006	3.51E-007	8.78E-008
2^{-4}	2.11E-004	5.28E-005	1.32E-005	3.30E-006	8.25E-007	2.06E-007
2^{-5}	4.51E-004	1.13E-004	2.83E-005	7.09E-006	1.77E-006	4.43E-007
2^{-6}	9.19E-004	2.33E-004	5.84E-005	1.46E-005	3.65E-006	9.14E-007
2^{-7}	1.78E-003	4.66E-004	1.18E-004	2.96E-005	7.40E-006	1.85E-006
2^{-8}	3.06E-003	8.94E-004	2.34E-004	5.93E-005	1.49E-005	3.72E-006
2^{-9}	4.28E-003	1.54E-003	4.49E-004	1.18E-004	2.97E-005	7.46E-006

Table 4.20: Results for Example 4.4.2, Max. Errors: Before Extrapolation Using (4.2.2) and (4.4.15) for $\mu = 2^{-3}$

ϵ	n=64	n=128	n=256	n=512	n=1024	n=2048
2^{-6}	3.89E-004	9.74E-005	2.44E-005	6.09E-006	1.52E-006	3.81E-007
2^{-7}	8.47E-004	2.12E-004	5.31E-005	1.33E-005	3.32E-006	8.30E-007
2^{-8}	1.82E-003	4.59E-004	1.15E-004	2.87E-005	7.19E-006	1.80E-006
2^{-9}	3.81E-003	9.72E-004	2.44E-004	6.12E-005	1.53E-005	3.82E-006
2^{-10}	7.51E-003	2.00E-003	5.08E-004	1.27E-004	3.19E-005	7.97E-006
2^{-11}	1.30E-002	3.89E-003	1.03E-003	2.60E-004	6.54E-005	1.64E-005
2^{-12}	1.83E-002	6.73E-003	1.98E-003	5.21E-004	1.32E-004	3.31E-005

Chapter 5

Concluding remarks and scope for future research

This thesis dealt with robust numerical methods for singularly perturbed two-parameter problems. The first two chapters are concerned with a general overview, including the literature review, of the two-parameter singular perturbation problems. In chapters 3 and 4, we studied the performance of convergence acceleration technique, firstly on a fitted mesh finite difference method (FMFDM) as applied on two different meshes namely a piecewise mesh (of Shishkin type) and a graded mesh (of Bakhvalov type) and then on a fitted operator finite difference method (FOFDM).

We notice that results obtained by the FMFDM on Bakhvalov mesh are better than those on Shishkin mesh. Though the accuracy of the lower order method in this case was slightly improved, the order of convergence was changed very little. To overcome this discrepancy, the Richardson extrapolation is also applied on an FOFDM to investigate whether it improves the accuracy and/or the order of convergence and indeed we found some wonderful results. In fact the improved results are perfectly of order two. This is the main contribution in this thesis and in fact we have achieved very good results after extrapolation.

It is also worth mentioning here that the fitted operator finite difference methods are found to be simpler to analyze and implement, partly due to the ease of operating on uniform meshes. Extensive numerical computations were carried out for comparison and to confirm the theoretical results.

As far as the scope for the future research is concerned, we are currently busy doing the analysis of the Richardson extrapolation technique on a fitted mesh method based on the Bakhvalov mesh. We are also developing some direct higher order numerical methods for these two-parameter singular perturbation problems to investigate whether to use convergence acceleration techniques or to use direct higher order methods for these class of problems. This issue will mostly be concerned with computational complexity.

Lastly, we have used MATLAB to do all the computations presented in this thesis.

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