Pricing Methods for Asian options

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Abstract

We present various methods of pricing Asian options. The methods include Monte Carlo simulations designed using control and antithetic variates, numerical solution of partial differential equation and using lower bounds.

The price of the Asian option is known to be a certain risk-neutral expectation. Using the Feynman-Kac theorem, we deduce that the problem of determining the expectation implies solving a linear parabolic partial differential equation. This partial differential equation does not admit explicit solutions due to the fact that the distribution of a sum of lognormal variables is not explicit. We then solve the partial differential equation numerically using finite difference and Monte Carlo methods.

Our Monte Carlo approach is based on the pseudo random numbers and not deterministic sequence of numbers on which Quasi-Monte Carlo methods are designed. To make the Monte Carlo method more effective, two variance reduction techniques are discussed.

Under the finite difference method, we consider explicit and the Crank-Nicholson's schemes. We demonstrate that the explicit method gives rise to extraneous solutions because the stability conditions are difficult to satisfy. On the other hand, the Crank-Nicholson method is unconditionally stable and provides correct solutions.

Finally, we apply the pricing methods to a similar problem of determining the price of a European-style arithmetic basket option under the Black-Scholes framework. We find the optimal lower bound, calculate it numerically and compare this with those obtained by the Monte Carlo and Moment Matching methods.

Our presentation here includes some of the most recent advances on Asian options, and we contribute in particular by adding detail to the proofs and explanations. We also contribute some novel numerical methods. Most significantly, we include an original

contribution on the use of very sharp lower bounds towards pricing European basket options.

Declaration

I declare that *Pricing Methods for Asian Options* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Walter Mudzimbabwe   February 2010

Signature:...

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for my teacher, Mr Dongijena (DJ).

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List of Publications

Part of this thesis has already been submitted in the form of the following research paper:

1. Walter Mudzimbabwe, Kailash C. Patidar and Peter J. Witbooi, European basket option pricing by maximizing over a subset of lower bounds, submitted for publication.

Another paper that is being generated from Chapter 6 is:

1. Walter Mudzimbabwe, Kailash C. Patidar and Peter J. Witbooi, An efficient numerical method to price arithmetic Asian options, in preparation.
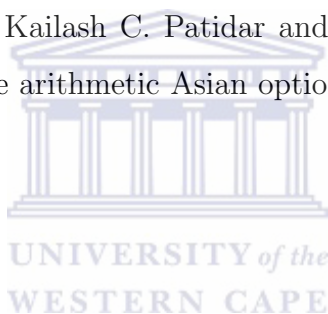
Introduction

Among the most commonly traded exotic derivatives on today's foreign exchange, interest rate and commodity markets are the Asian options (see, e.g., Carr [14]). Asian options are options whose payoff depends on the average of the underlying asset price for part or all the duration of the contract (see Alziary et al. [1], Briys et al. [12], Joshi [35], Musiela et al. [46]). These options have been traded since the late 1980's when the employees of the Bankers Trust in Asia, priced the option in connection with average price of crude oil. These workers then coined the name as Asian options ([14]). Since their inception, Asian options have attracted interest from practitioners and scholars alike [12].

There are reasons for this surge in popularity of Asian option on the markets. Asian options are generally cheaper than plain vanilla European options. We will confirm this observation later in the thesis. A possible explanation is that the volatility of the average value of the underlying tends to be lower than that of the individual assets. Asian options are also less prone to price manipulation near the date of maturity as compared to European options. Manipulating the average price of the underlying is clearly difficult but if only the price at maturity was considered then this would be possible.

Usually, two ways of taking the average are considered (see [1], [12] and Kemna et al. [36]). We can have a geometric average or arithmetic average. Thus we have the two payoffs

$$\left(\frac{1}{N}\sum_{i=1}^{N}S_{t_i}-K\right)^{+} \quad \text{and} \quad \left(\left(\prod_{i=1}^{N}S_{t_i}\right)^{\frac{1}{N}}-K\right)^{+}$$

where S_{t_i} is the price of the underlying asset at time t_i, K is the strike price, N is the total number of trading days and $x^{+}=\max\{x,0\}$.

The geometric average gives an explicit formula for the price of the geometric option [6, 36]. The reason being that the product of lognormal variables is also lognormal. On

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the other hand, there is no explicit formulae for the price of the arithmetic option because the distribution of the sum of lognormal variables is not explicit [1, 14]. The inclusion of the geometric average in the study of the arithmetic average option is twofold. Firstly, it gives insight into pricing the arithmetic average option [19] and secondly, it can be used as a control variate in the design of Monte Carlo simulations [36].

Our focus will be on pricing arithmetic Asian options. The average will be in the continuous sense, i.e, the payoff structure takes the form

$$\left(\frac{1}{T}\int_0^T S_u\, du - K\right)^+$$

where T denotes the exercise time, S_t is the price of the underlying asset and K is the strike price.

Although in practise the asset prices are taken at discrete times [14], t_i, such that $0 < t_1 < t_2 \ldots < t_N = T$, the continuous average enables us to characterize the price of the option as a solution of a partial differential equation. This partial differential equation is similar to the classical Black-Scholes partial differential equation (see Hull [32] and [60] for details).

The literature shows various attempts to price arithmetic Asian options analytically using closed form solutions. Kemna and Vorst [36] priced the geometric Asian option. Turnbull and Wakeman [57] used the idea that the arithmetic average is approximately lognormal. Levy [41] also worked along the same line of thought, and managed to improve the results of Turnbull and Wakeman. He confirmed his claims using Monte Carlo procedures. Henderson and Wojakowski [28] showed that there is a symmetry between the price of a fixed strike and a floating strike Asian option.

The other analytical solution approaches were based on the idea of conditioning. Rogers and Shi [52] introduced the conditioning in their method and since then, it has gained popularity [21, 56]. In their paper, they proposed a general conditioning variable, although it turned out that the normally distributed one is the best. Curran [18, 19] used the

geometric average as the conditioning variable. The work of Kuan [15] and Thompson [56] is very similar and is an extension of Rogers and Shi's ideas. The method of conditioning is usually followed to get bounds.

On the numerical side, a very popular method that has been used for Asian options and option valuations in general is the Monte Carlo method [36]. It has gained considerable attention since its introduction to option pricing by Boyle [10]. For example, not only are pseudo random numbers used, it can also accommodate deterministic sequences (see, e.g., Corwin [17] and Lamiex [40] for further discussion). The only problem of the Monte Carlo approach is that of the propagation of error. Although the error incurred in this process is inversely proportional to the number of Monte Carlo loops, it is known that the approach becomes progressively impractical in view of the computational complexities, in particular, the CPU time. As a remedy to this problem, researchers tried to improve its efficiency, for example, the variance reduction procedures given in Higham [31] and [36, 40]. Some of these techniques, the antithetic and the control variate methods, will also be used in this thesis. The strength of the control variate method lies in the ability to identify the right candidate for the control variate.

The price of an Asian option can be determined as a solution of a partial differential equation (PDE). Various authors have used this method, see for instance, [1], Benhamou [8], Ingerson [34], Rogers and Shi [52], and [58]. Barraquand [4] and Ingerson [34] showed that the price of an Asian option satisfies the two state PDEs:

$$C_t + rSC_s + \frac{1}{2}\sigma^2 S^2 C_{ss} + \frac{1}{t}(S - A)C_A - rC = 0,$$
$$V_t + rSV_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + SV_I - rV = 0,$$

respectively, where $I(T) = \int_0^T S(\tau) \, d\tau$ and $A(T) = I(T)/T$ and the states are the variables s, A, I. Ingerson [34] and Alziary et al. [1] showed that the two state PDEs can be reduced to one state PDEs by using a homogeneity argument [35]. The homogeneity property holds for a collective class of models known as the log-type. An example is the Black-Scholes

model. Benhamou [8] also applied the same idea.

The other approaches include finite difference, finite element and finite volume methods. While we will consider the finite difference methods in this thesis, it is worth mentioning here that Foufas et al. [24] priced Asian options by a finite element method whereas Zvan et al. [62] used the finite volume method (initially developed to model fluid flow in computational fluid dynamics). Of all the PDE methods, the Vecer's method [58] is the easiest one to implement even for low volatilities. We will not follow Vecer line of thought as the underlying theory of stochastic control is rather out of context in this work.

We will firstly apply the explicit finite difference methods which gives us the oscillatory solutions. This is largely due to the fact that for the volatility values of interest the diffusion term in the PDE is very small and hence the PDE tends to be convection dominant, causing the PDE to be predominantly hyperbolic. Rogers and Shi [52] proposed a brute force method to overcome this problem. This has resulted in the authors preference for the Crank-Nicholson's method over the classical explicit method(s) [8].

Furthermore, we have extended some of our proposed methods to price a European-style basket option. This we do in the Black-Scholes framework. It is to be noted that the basket options are similar to Asian options and it is not a surprise that the methods used to price these derivatives overlap [20, 50]. In both cases their payoffs involve taking sum of lognormal variables. It cannot be overemphasized that the lack of explicit formulae for the distribution of this sum, just as for Asian options, has hampered the derivation of closed form expressions for the price of the basket options. The difference between the two options is that an Asian option is path dependent whereas a European-style basket option is path-independent [21]. For this reason, the Monte Carlo method for basket options should be less complex. We will show how we can adapt Monte Carlo methods for Asian options to basket options. We assume that the prices of the assets in the basket are correlated. Of course, it makes the implementation of Monte Carlo a bit more

difficult but following [27] we find a way to generate these correlated prices. We will do so by making effective use of some techniques from linear algebra.

We also derive an optimal lower bound for the price. The determination of this Lower bound is our original work being submitted for a publication [45]. In fact, for a particular set of lower bounds of the Asian option price, we find the maximum. The lower bound turns out to be an excellent approximation and so can be taken as the real price. We suitably determine a random variable then use the method of conditioning [19, 52]. By assuming that the basket of lognormally distributed asset is also lognormal, we investigate a moment matching procedure. We accordingly determine the parameters of the lognormal distribution. In other words, we assume the basket to be a synthetic asset which follows a geometric Brownian motion. The results of the optimal lower bound and the moment matching procedure are benchmarked using the Monte Carlo simulations.

The rest of the thesis is organised as follows. The chapters are grouped into three parts. Part I is devoted to the underlying theory of Asian options. We also study qualitative aspects of Asian options. Based on the stochastic calculus, we determine the associated PDEs. Some lower bounds are also derived. In Part II, we infer the price of the option by numeric means. Lastly in Part III, we apply our methods to basket options.

Part I

Theoretical considerations

1. Preliminaries and Important Concepts

In this chapter, we lay the foundation for all the work that will follow. We will give brief discussions of some of the relevant facts, details of which can be found in standard texts, for example, Doob [22], Nielsen [47], Øksendal [48], etc.

Definition 1.1. *A stochastic process $\{W_t\}_{t\geq 0}$ is called a Wiener process if*

(i) $W_0 = 0$

(ii) W_t *is continuous*

(iii) *for* $0 = t_0 \leq t_1 \leq \cdots \leq t_n$, $W_1, W_2 - W_1, W_3 - W_2, \cdots, W_n - W_{n-1}$ *are independent*

(iv) *for* $0 \leq s \leq t$, $W_t - W_s \sim N(0, t - s)$.
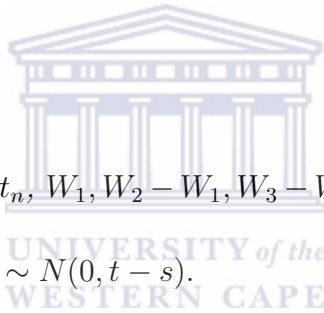
A Brownian motion is a stochastic process of the form $B_t = a + \sigma W_t$ for constants a and σ, a Wiener process W_t. Figure 1.1 shows some simulated Wiener process paths. In the first part (a), we have a single path and in (b) there are 100 paths.

1.1 Conditional Expectation

Definition 1.2. *Consider a probability space $(\Omega, P, \mathcal{F})$ on a set Ω. Let $\mathcal{G}$ be a sub-sigma field of $\mathcal{F}$, and let X be a random variable. A $\mathcal{G}$-measurable random variable $E(X|\mathcal{G})$ is called the conditional expectation of X relative to the subfield $\mathcal{G}$ if*

$$\int_G E(X|\mathcal{G})dP = \int_G X dP, \quad \forall G \in \mathcal{G}.$$

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(a) Single Wiener process path   (b) 100 Wiener process paths
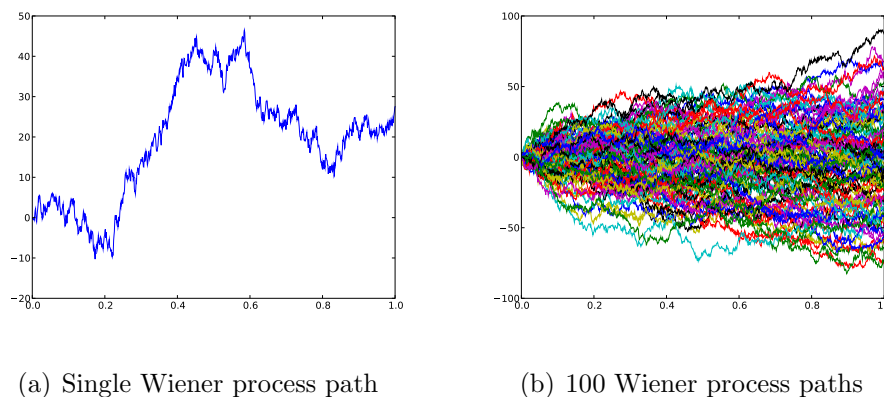
Figure 1.1: Wiener process path Simulations

We write $\mathbb{E}_{\mathcal{G}}(X)$ to mean $\mathbb{E}(X|\mathcal{G})$. The following are some properties of the conditional expectation (see, e.g., Capinski et al. [13], [22], Shreve [53] for detailed discussion), for subfields $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{F}$ and random variables X and Y:

(i) $\mathbb{E}(\mathbb{E}_{\mathcal{G}}(X)) = \mathbb{E}(X)$,

(ii) if X is $\mathcal{G}$-measurable, then $\mathbb{E}_{\mathcal{G}}(X) = X$,

(iii) if X is independent of $\mathcal{G}$, then $\mathbb{E}_{\mathcal{G}}(X) = \mathbb{E}(X)$,

(iv) the tower property or law of iterating expectations: if $\mathcal{H} \subset \mathcal{G}$ then $\mathbb{E}_{\mathcal{H}}(\mathbb{E}_{\mathcal{G}}(X)) = \mathbb{E}_{\mathcal{H}}(X)$, (generalising (i) above)

(v) linearity: $\mathbb{E}_{\mathcal{G}}(aX + bY) = a\mathbb{E}_{\mathcal{G}}(X) + b\mathbb{E}_{\mathcal{G}}(Y)$, for any $a, b \in \mathbb{R}$.

The conditional expectation is an important concept in mathematical finance. It enables us to study derivatives by taking into account the flow of information, which we express in terms of subfields $\mathcal{F}_t$ ([13]). We are usually interested in knowing the behaviour of a random variable X and how the information at time t can help us study the random variable X. Thus the problem would be to find the best estimate of the random variable given the information we have. This random variable is $\mathbb{E}(X|\mathcal{F}_t)$.

Suppose we have a probability space $(\Omega, P, \mathcal{F})$. Then we can define a new measure Q through the Radon-Nikodym derivative ([13])

$$\frac{dQ}{dP} := Z.$$

The following result shows that if the Radon-Nikodym derivative is restricted to a subfield $\mathcal{G}$, then $\mathbb{E}^Q(Z|\mathcal{G})$ denoted by $\left(\frac{dQ}{dP}\right)_{|\mathcal{G}}$ can be expressed as the restriction of the measures P and Q on $\mathcal{G}$ denoted by $dP_{|\mathcal{G}}$ and $dQ_{|\mathcal{G}}$ respectively.

Proposition 1.3. *Let two subfields $\mathcal{G}$ and $\mathcal{F}$ be such that $\mathcal{G} \subseteq \mathcal{F}$, then*

$$\frac{dQ_{|\mathcal{G}}}{dP_{|\mathcal{G}}} = \left(\frac{dQ}{dP}\right)_{|\mathcal{G}}.$$

We omit the proof of the very straightforward result. The Bayes theorem is handy when we change from one measure to the other.

Theorem 1.4. *Bayes Theorem: For any random variable X, if $\mathcal{G} \subseteq \mathcal{F}$ then*

$$\mathbb{E}(Z|\mathcal{G})\mathbb{E}^Q(X|\mathcal{G}) = \mathbb{E}(ZX|\mathcal{G}).$$

Proof. The proof can be found in [47]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

1.2   Moment Generating Function

Definition 1.5. *The moment generating function (mgf) of a random variable X is defined as*

$$M_X(\theta) = \mathbb{E}(e^{\theta X}), \ \ for \ \ \theta \in \mathbb{Z}^+,$$

where $\mathbb{Z}^+$ is the set of positive integers.

Example 1.6. If X follows a normal distribution (we write $X \sim N(\mu, \sigma)$), then

$$
\begin{aligned}
M_X(\theta) &= \mathbb{E}(e^{\theta X}) \\
&= \int_{-\infty}^{\infty} e^{\theta x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx,
\end{aligned}
$$

which will be simplified further. Note the factor $\exp(\omega)$ in the integrand, where

$$\omega = \theta x - \frac{(x-\mu)^2}{2\sigma^2}.$$

Completing the square we get

$$\omega = \mu\theta + \frac{(\sigma\theta)^2}{2} - \frac{u^2}{2},$$

with $u = \frac{x-(\mu+\sigma^2\theta)}{\sigma}$, and consequently $du = \frac{1}{\sigma}dx$. Therefore

$$M_X(\theta) = e^{\mu\theta + \frac{(\sigma\theta)^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2\theta))^2}{2\sigma^2}} \, dx$$

$$= e^{\mu\theta + \frac{(\sigma\theta)^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du$$

$$= e^{\mu\theta + \frac{(\sigma\theta)^2}{2}},$$

where $\int_{\infty}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du = 1$.

Now let us consider an example of where the mgf is important.

Example 1.7. Suppose the random variable S_T is such that

$$S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma W_{T-t}},$$

where $W_t \sim N(0,t)$. The stochastic process $\{W_t\}_{t\geq 0}$ is called a Wiener process. The expectation of S_T is

$$\mathbb{E}(S_T) = S_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t)} \mathbb{E}(e^{\sigma W_{T-t}})$$

$$= S_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t)} e^{\frac{1}{2}\sigma^2(T-t)}$$

$$= S_t e^{\mu(T-t)}.$$

1.3   Itô Lemma

When dealing with stochastic integrals, i.e, integrals w.r.t. a Wiener process, it is worth mentioning that it will be the Itô integral. For further discussion on the Itô integral, we refer to Øksendal [48].

Lemma 1.8. *Suppose $f(t,x)$ is twice continuously differentiable and $\frac{\partial f}{\partial x}(t,W_t)$ is $\mathcal{F}_t$ measurable $\forall\ 0 \leq t \leq T$. Then*

$$f(t,W_t) - f(0,W_0) = \int_0^t \frac{\partial}{\partial s}f(s,W_s)ds + \int_0^t \frac{\partial}{\partial x}f(s,W_s)dW_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2}f(s,W_s)ds.$$

For the proof see any standard text, for example [48]. We usually want to write the above in a convenient way as

$$df(t,W_t) = \dot{f}(t,W_t)dt + f'(t,W_t)dW_t + \frac{1}{2}f''(t,W_t)dt,$$

where $\dot{f}(t,W_t)$ denotes differentiating w.r.t. time and $f'(t,W_t)$ denotes differentiating w.r.t. W_t. In the literature the phrases Itô Theorem, Itô formula and Itô Lemma are used to mean the same thing; we are not going to be exceptional. Among other uses, we can solve some stochastic differential equations using Itô Theorem [23].

Example 1.9. Suppose the price S_t of an asset follows a *geometric Browian Motion*, i.e., a process satisfying the Stochastic differential equation, (SDE, for brevity)

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ is the drift and σ is the volatility. We can also regard μ as being the average returns and σ is the standard deviation of returns. We first write these dynamics as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

Now let us write $f(t,W_t) = \log S_t$. Then

$$\dot{f}(t,W_t) = 0, \qquad f'(t,W_t) = \frac{1}{S_t} \quad \text{and} \quad f''(t,W_t) = -\frac{1}{S_t^2}.$$

By Itô Lemma, we can find the differential of $f(t,W_t)$:

$$d\left(\log S_t\right) = \frac{1}{S_t}dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2$$

$$= \mu dt + \sigma dW_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\sigma^2 S_t^2 dt$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t.$$

Integrating both sides of the equation we have

$$\log S_T - \log S_t = \int_t^T \left(\mu - \frac{1}{2}\sigma^2 \right) ds + \int_t^T \sigma dW_s.$$

Finally

$$S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)}. \qquad \blacksquare^1$$

We may consider the Itô lemma for the multidimensional case [47]. Suppose now

$$\mathbf{X}_t = (X^1, X^2, \cdots, X^n)$$

and consider a function $f(t, \mathbf{X}_t)$ then

$$df(t, \mathbf{X}_t) = \frac{\partial f}{\partial t}(t, \mathbf{X}_t)dt + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(t, \mathbf{X}_t)dX_t^i + \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x^i x^j}(t, \mathbf{X}_t) \, dX_t^i dX_t^j.$$

If the coordinates $(X^1, X^2, \cdots, X^n)$ are independent Brownian motions, then the last term simplifies according to

$$dX_t^i dX_t^j = \begin{cases} dt, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$

and $dt dX_t^i = dt dX_t^j = dt dt = 0$. If there is correlation between them, we have $dX_t^i dX_t^j = \sigma_{ij} dt$, where σ_{ij} is the correlation between X_i and X_j.

1.4 Lognormal Distribution

The lognormal distribution [6, 37] is important for price determination in the Black-Scholes economy. In this economy, we assume that the price of an asset follows a geometric Brownian motion. We have already shown that by application of the Itô Lemma we are

---

[1]This indicates the end of the example

able to see that the price follows a log-normal distribution. At this stage it is important to describe the lognormal distribution.

A random variable X is said to follow a log-normal distribution, written $X \sim \mathrm{Log}N(\mu_x, \sigma_x^2)$ if its probability density function $g(x)$ is given by

$$g(x) = \frac{1}{x\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{\ln x - \mu_x}{\sigma_x})^2}.$$

There is also an equivalence between normal variates and log-normal variates. If Y is a normal variate, $Y \sim N(\mu_y, \sigma_y^2)$, then $e^Y \sim \mathrm{Log}N(\mu_y, \sigma_y^2)$, that is

$$Y \sim N(\mu_y, \sigma_y^2) \Leftrightarrow e^Y \sim \mathrm{Log}N(\mu_y, \sigma_y^2).$$

In the Moment matching method, which we will use in connection with pricing basket options, the moments of the Log-normal distribution form the core of the concept [20] and [11]. Therefore an understanding of them turns out to be invaluable. The first moment of a random variable X is the mean and the second moment is the expectation of X^2. Now we want to characterise these moments. For the random variable $X \sim \mathrm{Log}N(\mu_x, \sigma_x^2)$, we have

$$e^{\ln X} = X \sim \mathrm{Log}N(\mu_x, \sigma_x^2) \Leftrightarrow \ln X \sim N(\mu_x, \sigma_x^2).$$

Therefore, the moments of X, generalised as expectation of X^θ for $\theta \in \mathbb{Z}^+$ are given by

$$E(X^\theta) = E(e^{\theta \ln X}) = e^{\theta\mu_x + \frac{1}{2}\theta^2\sigma_x^2},$$

since $\ln X \sim N(\mu_x, \sigma_x^2)$. Usually, the first and second moments are important. These are

$$E(X) = e^{\mu_x + \frac{1}{2}\sigma_x^2} \quad \text{and} \quad E(X^2) = e^{2(\mu_x + \sigma_x^2)}. \tag{1.1}$$

Consequently, the variance is

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = e^{2\mu_x + \sigma_x^2}(e^{\sigma_x^2} - 1). \tag{1.2}$$

1.5 The Bivariate Normal Distribution

The material presented here can be found in Renyi [51] or other books on Probability Theory. Suppose that two random variables X and Y are normally distributed with means μ_x and μ_y and variances σ_x^2 and σ_y^2. Suppose further that the correlation between these random variables is ρ. Then the pair (X, Y) is said to be a bivariate normal random variable, and we write $(X, Y) \sim \mathrm{BiN}(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$. The probability density function is given by

$$
f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}} \exp\left\{ -\frac{1}{2(1 - \rho^2)} \left(\left(\frac{x - \mu_x}{\sigma_x}\right)^2 - \right.\right.
$$
$$
\left.\left. 2\rho\frac{x - \mu_x}{\sigma_x}\frac{y - \mu_y}{\sigma_y} + \left(\frac{y - \mu_y}{\sigma_y}\right)^2 \right) \right\},
$$
(1.3)

provided that neither σ_x nor σ_y is zero. If we define the marginal density of X to be

$$
f(x) = \int_{-\infty}^{\infty} f(x, y)\, dy,
$$

then it is easy to show that

$$
f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - \mu_x}{\sigma_x}\right)^2}.
$$
(1.4)

Likewise the marginal density function for Y is given by

$$
f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx
$$

which entails

$$
f(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y - \mu_y}{\sigma_y}\right)^2\right).
$$

So, to say that two random variables follow a bivariate normal distribution implies that both of these random variables are normally distributed. The Figure 1.2 further clarifies this. Whichever section we look at the diagram, we see the bell shape of the normal
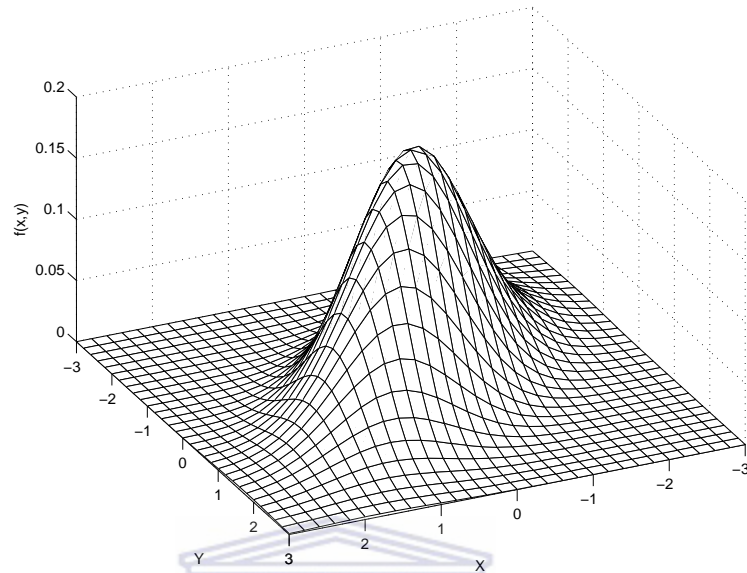
Figure 1.2: Bivariate Normal density function

density function. We would also want to determine the conditional density function. We define the conditional density function of X on Y to be

$$f(x|y) := \frac{f(x, y)}{f(y)},$$

provided that $f(y) \neq 0$. Using (1.3) and (1.4) we get

$$f(x|y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \right.\right.$$
$$\left.\left. 2\rho\frac{x-\mu_x}{\sigma_x}\frac{y-\mu_y}{\sigma_y} + \rho^2\left(\frac{y-\mu_y}{\sigma_y}\right)^2 \right) \right\},$$

There is a nice result about bivariate normal variables. Their conditional distributions are also normal.

Proposition 1.10. *Suppose X and Y are bivariate normally distributed, i.e., $X, Y \sim BiN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$ then the conditional density of X given Y is normal; specifically*

$$X|Y = y \sim N\left(\mu_x + \rho\sigma_x\left(\frac{y-\mu_y}{\sigma_y}\right), \sigma_x^2(1-\rho^2)\right).$$

Proof. We are going to use the moment generating function (mgf) for the normal distribution. We know that the mgf of a random variable uniquely identifies the distribution. Before we proceed further, let us simplify our notation by making the following substitutions $u = \frac{x - \mu_x}{\sigma_x}$ and $v = \frac{y - \mu_y}{\sigma_y}$. We can write the following immediately

$$\mathbb{E}(e^{\theta X} | Y = y) = \int_{-\infty}^{\infty} e^{\theta x} f(x|y) \, dx$$

$$= \frac{1}{\sigma_x \sqrt{2\pi} \sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} e^{\theta(\sigma_x u + \mu_x)} e^{-\frac{1}{2(1-\rho^2)}(u^2 + \rho^2 v^2 - 2\rho v u)} \sigma_x du$$

$$= \frac{1}{\sqrt{2\pi} \sqrt{1 - \rho^2}} e^{\theta \mu_x} e^{-\frac{\rho^2 v^2}{2(1-\rho^2)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(u^2 - 2(\rho v + (1-\rho^2)\theta \sigma_x)u)} \, du.$$

Completing the square in the exponent of the integrand results in the expectation being written as

$$= \frac{1}{\sqrt{2\pi} \sqrt{1 - \rho^2}} \exp\left\{ \theta \mu_x - \frac{\rho^2 v^2}{2(1 - \rho^2)} + \frac{(\rho v + (1 - \rho^2)\theta \sigma_x)^2}{2(1 - \rho^2)} \right\} \times$$

$$\int_{-\infty}^{\infty} \exp\left\{ -\frac{(u - (\rho v + (1 - \rho^2)\theta \sigma_x))^2}{2(1 - \rho^2)} \right\} \, du$$

The final step is then to normalize the normal density; that is make it be $N(0, 1)$. To do this, we let $z = \frac{u - (\rho v + (1 - \rho^2)\theta \sigma_x)}{\sqrt{1 - \rho^2}}$. Consequently

$$\mathbb{E}(e^{\theta X} | Y = y) = \frac{1}{\sqrt{2\pi} \sqrt{1 - \rho^2}} \exp\left\{ \theta \mu_x - \frac{\rho^2 v^2}{2(1 - \rho^2)} + \frac{(\rho v + (1 - \rho^2)\theta \sigma_x)^2}{2(1 - \rho^2)} \right\} \times$$

$$\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \sqrt{1 - \rho^2} dz$$

$$= \exp\left\{ \theta \mu_x - \frac{\rho^2 v^2}{2(1 - \rho^2)} + \frac{(\rho v + (1 - \rho^2)\theta \sigma_x)^2}{2(1 - \rho^2)} \right\}$$

$$= e^{\theta(\mu_x + \sigma_x \rho v) + \frac{1}{2}\theta^2(1 - \rho^2)\sigma_x^2}.$$

But this is the mgf of a normal variable with mean being the coefficient of θ and variance

is the multiplicand of $\frac{1}{2}\theta^2$. Therefore

$$(X|Y=y) \sim N\left(\mu_x + \rho\sigma_x \left(\frac{y-\mu_y}{\sigma_y}\right), \sigma_x^2(1-\rho^2)\right). \qquad \square$$

1.6 Feynman-Kac Theorem

The Feynman-Kac Theorem is used to represent certain types of partial differential equations as an expectation of a functional of a given diffusion process (see [23, 39, 53]).

Theorem 1.11. *(Feynman-Kac). If $V(x,t)$ solves the partial differential equation*

$$\frac{\partial V}{\partial t}(x,t) + \mu(x,t)\frac{\partial V}{\partial x}(x,t) + \frac{1}{2}\sigma(x,t)^2\frac{\partial^2 V}{\partial x^2}(x,t) = 0, \tag{1.5}$$
$$subject\ to: \quad V(x,T) = G(x),$$

and X_t is defined by the stochastic differential equation

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t,$$

where W_t is a Wiener process. Furthermore we assume that $G(x)$ is a continuous function which is either non-negative or satisfies the condition (see e.g. [39]):

$$|G(x)| \leq L(1+|x|^\lambda), \quad L > 0, \lambda \geq 1.$$

Then

$$V(x,t) = \mathbb{E}^P(G(X_T)|X_t = x).$$

Actually, we must say that X_t's dynamics are so under some probability measure, in this case P. Then W_t is a P-Wiener process. We may also write

$$V(x,t) = \mathbb{E}^P(G(X_T)|\mathcal{F}_t)$$

There is also a technical requirement that

$$\int_0^T \mathbb{E}\left[\left(\sigma(X_t, t)\frac{\partial V}{\partial x}(X_t, t)\right)^2\right] dt < \infty,$$

for the theorem to hold.

Example 1.12. The following problem is exercise 18 of Chapter 4 from Etheridge [23].
Suppose we want to find $f(x, t)$ which satisfies the partial differential equation:

$$\frac{\partial f}{\partial t}(x, t) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(x, t) = 0,$$

$$\text{subject to:} \quad f(x, T) = x^2,$$

We note that X_t follows the sde:

$$dX_t = \sigma dW_t.$$

Then applying the Feynman-Kac theorem we have

$$f(x, t) = \mathbb{E}(X_T^2 | X_t = x)$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{x^2}{2\sigma^2 t}} \, dx.$$

1.7 Girsanov Theorem

This theorem helps us identify a Brownian Motion if we appropriately change measures
(see, e.g., [39, 53]). Typically, we are interested in finding the corresponding Brownian
Motion under an equivalent measure to the real world probability.

Theorem 1.13. *(Girsanov Theorem). Let W_t is a P-Brownian Motion and consider an*
$\mathcal{F}_t$ *measurable function $\lambda(t)$. For $0 \leq t \leq T$, define*

$$\widetilde{W_t} := \int_0^t \lambda(s)\, ds + W_t$$

and

$$Z_t := e^{-\frac{1}{2}\int_0^t \lambda^2(s)\, ds - \int_0^t \lambda(s)\, dW_s}$$

also define a measure $\tilde{P}$ by

$$\frac{d\tilde{P}}{dP} = Z_t,$$

the Radon-Nikodym derivative, then under $\tilde{P}$, $\widetilde{W}_t$ is a Brownian Motion for $0 \le t \le T$.

Remark 1.14. We note that $\lambda(t)$ can be constant. In that case,

$$Z_t := e^{-\frac{1}{2}\lambda^2 t - \lambda W_t}.$$

Suppose $\theta = -\lambda$, then $Z_t = e^{-\frac{1}{2}\theta^2 t + \theta W_t}$ and $\widetilde{W}_t = W_t - \theta t$ is a Brownian Motion under $\tilde{P}$. This is a variant of the Girsanov theorem [33].

2. The basics of Asian Options

In this chapter, we present a detailed discussion about Asian options (see also Alziary [1], Carr [14], Chacko, [6], Higham [31]). In particular, we discuss the terminology associated with these options as well as their characteristics. We also establish some properties, for example parity between Asian calls and puts, the concept of martingales in option valuation. We close the chapter by deriving the price of a geometric Asian option which we will use later in the study of numerical solutions for arithmetic Asian options. We motivate the presentation here by discussing first the European options.

A *European Call Option* is a financial contract that gives its *holder* the right to buy an *asset* for a prescribed price at a prescribed future date [31, 60]. On the contrary, a *European Put Option* gives the holder the right to sell the asset for a prescribed price at a prescribed future date. When an option is being traded, it involves two parties, namely the *writer* and the *holder*. The writer of a call option must sell the asset if the holder chooses to exercise the option. Similarly, the writer of the put option is obliged to buy the asset if the holder of the put chooses to exercise the right to sell the asset.

If the prescribed time for the European call option is T, the prescribed price is K and the price of the asset is S_T, then the holder will buy the asset if $S_T > K$ otherwise they do not exercise the option. The holder realises a profit of $S_T - K$ by buying the asset for K and selling it on the market for S_T. Therefore the holder's profit or *payoff* is

$$\max\{S_T - K, 0\}.$$

On the other hand if $K > S_T$, the holder of the put gains a profit of $K - S_T$ by buying the asset for S_T on the market and exercising the right of selling the asset for K. The payoff in that case is

$$\max\{K - S_T, 0\}.$$

Unlike the case of European Options where the payoff depends only on the price of the

underlying at the last day of holding the option, an Asian Option is an option for which the payoff depends on the average of the price of the underlying asset, over some period of holding the option (for more examples see also Carr [14], Joshi [35],Rogers and Shi [52], Shreve [53], Wilmot [60], etc).

Now there are different ways of taking the average, resulting in different kinds of payoff structures and hence different types of options (see also Fusai[26], Higham [31]). We can have an Asian option written on a stock with price S_t at time t which can be exercised at time T with strike price K by taking the arithmetic average [1] for the period $[t_0, T]$. In such a case we can define a *fixed strike* Asian call option payoff [60] as

(a)
$$\left(\frac{1}{T - t_0} \int_{t_0}^{T} S_u \, du - K \right)^+ .$$

The fixed strike put payoff is therefore
$$\left(K - \frac{1}{T - t_0} \int_{t_0}^{T} S_u \, du \right)^+ .$$

A *floating strike* Asian call option has the following payoff

(b)
$$\left(S_T - \frac{1}{T - t_0} \int_{t_0}^{T} S_u \, du \right)^+ .$$

Specifically, these are S_t-values continuously averaged. We can also consider the continuous geometric average [6] where the payoff is

(c)
$$\left(\exp \left(\frac{1}{T - t_0} \int_{t_0}^{T} \log S_u \, du \right) - K \right)^+ .$$

We may replace continuous average by discrete sampling and have payoffs, in the case of a fixed strike K, taking the form

(d)

$$\left(\frac{1}{N} \sum_{i=1}^{N} S_{t_i} - K \right)^{+},$$

and

(e)

$$\left(\left(\prod_{i=1}^{N} S_{t_i} \right)^{\frac{1}{N}} - K \right)^{+},$$

where $0 < t_1 < t_2 \ldots < t_N = T$. The put payoffs for (b),(c),(d) and (e) can be easily written from the call payoffs.

In practise discrete sampling ([8]) is common but in this work we are going to consider continuously averaged options. In fact our solution to the pricing problem might then be considered an approximation to the discrete sampled one [14]. There are of course explicit pricing formulae for some options, for example, (c) (see Chacko [6] for derivation) and (e) (Higham [31]). In section 2.4, we shall derive a closed formula for the prices in these cases.

2.1 Mathematical Setting

Before we into detail, we need to specify the modelling assumptions. This involves understanding the price dynamics. We will assume the standard settings as in the Black-Scholes model. In this model, the market has two assets, a bond with price B_t and a risky asset with price S_t. There is a riskless interest rate r, so that if B_0 is put into a bank account then at time t it is worth $B_0 e^{rt}$. The expected return on the risky asset μ (also called drift) and the volatility σ (standard deviation) are constant, i.e.,

$$\mathbb{E}^P \left(\frac{dS_t}{S_t} \right) = \mu dt \ \text{ and } \ \text{Var} \left(\frac{dS_t}{S_t} \right) = \sigma^2 t.$$

The real world measure is denoted by P. The stochastic differential equation which governs the price of the risky asset is

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P,$$

where W_t^P is a P-Brownian motion. By applying the Fundamental theorem of asset pricing ([5]): To ensure that there are no arbitrage opportunities, we must find an equivalent martingale measure (EEM) Q (ie $P(A) = 0 \Leftrightarrow Q(A) = 0$), where P is the real world probability. The measure Q is called the risk neutral measure [23]. Under Q, the discounted price of the stock $e^{-r(T-t)}S_T$, is a Q-martingale. We can define the measure Q equivalent to P through the the Radon-Nikodym derivative of the Girsanov theorem [5], i.e,

$$\frac{dQ}{dP} = e^{-\frac{1}{2}\lambda^2 t - \lambda W_t}.$$

We will now determine the risk neutral dynamics of S_t. We split the drift μ into two components, a risky part $\mu - r$ and riskless part r [32] and write

$$\frac{dS_t}{S_t} = rdt + \mu dt - rdt + \sigma dW_t^P,$$
$$= rdt + \sigma \left(\frac{\mu - r}{\sigma} dt + dW_t^P \right). \tag{2.1}$$

By the Girsanov theorem [33]

$$W_t^Q := \lambda t + W_t^P, \quad \text{where} \quad \lambda = \frac{\mu - r}{\sigma}$$

is a Q-Brownian motion. The variable λ is called the *market price of risk* [5]. Consequently (2.1) becomes

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^Q. \tag{2.2}$$

Proposition 2.1. *Under Q, the discounted price of the asset $e^{-rt}S_t$ is a martingale.*

Proof. We will prove the martingale property by showing that the stochastic differential equation which describes the process $e^{-rt}S_t$ has no drift term. Let $\widetilde{S}_t = e^{-rt}S_t$. Then by

Itô's lemma

$$
\begin{aligned}
d\widetilde{S}_t &= -re^{-rt}S_t dt + e^{-rt}dS_t \\
&= -re^{-rt}S_t dt + re^{-rt}S_t dt + \sigma e^{-rt}S_t dW_t^Q \\
&= \sigma e^{-rt}S_t dW_t^Q \\
&= \sigma \widetilde{S}_t dW_t^Q. \qquad \qquad \qquad \square
\end{aligned}
$$

We can apply the Itô lemma to find S_t. Thus

$$
S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}Z},
$$

where Z is a standard normal variable, that is $Z_t \sim N(0,1)$.

2.2 Characteristics of the Prices

We have already found the measure Q, by the Fundamental theorem of Asset Pricing [5]. So the price of the option is found by taking the conditional expectation under Q. For the asset with payoff (a) the price for the fixed strike call option is given by

$$
C_{a,t} = \mathbb{E}^Q\left[e^{-r(T-t)}\left(\frac{1}{T-t_0}\int_{t_0}^T S_u\,du - K\right)^+\middle|\mathcal{F}_t\right]. \tag{2.3}
$$

On the other hand, the price of the fixed strike put option is

$$
P_{a,t} = \mathbb{E}^Q\left[e^{-r(T-t)}\left(K - \frac{1}{T-t_0}\int_{t_0}^T S_u\,du\right)^+\middle|\mathcal{F}_t\right]. \tag{2.4}
$$

Similarly, the floating strike call and put prices are

$$
C_{b,t} = \mathbb{E}^Q\left[e^{-r(T-t)}\left(S_T - \frac{1}{T-t_0}\int_{t_0}^T S_u\,du\right)^+\middle|\mathcal{F}_t\right],
$$

$$
P_{b,t} = \mathbb{E}^Q\left[e^{-r(T-t)}\left(\frac{1}{T-t_0}\int_{t_0}^T S_u\,du - S_T\right)^+\middle|\mathcal{F}_t\right]
$$

respectively. For simplicity, we shall take t_0 to be 0.

Likewise, we can characterize the price for any payoff structure as we have done for (a) and (b). The following result [8] is important as it ensures that the price is fair. We shall also make use of it in the derivation of the PDE whose solution is the price of the Asian option.

Lemma 2.2. *The processes $e^{r(T-t)}C_{a,t}$ and $e^{r(T-t)}C_{b,t}$ are Q-martingales* [8].

Proof. Let $Z_T := e^{r(T-t)}C_{a,t}$. Consider times t, s such that $0 < s < t < T$. Then

$$
\begin{aligned}
\mathbb{E}^Q\left[Z_t|\mathcal{F}_s\right] &= \mathbb{E}^Q\left[e^{r(T-t)}\mathbb{E}^Q\left(\left(e^{-r(T-t)}\frac{1}{T}\int_0^T S_u\,du - K\right)^+\middle|\mathcal{F}_t\right)\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}^Q\left[\mathbb{E}^Q\left(\left(\frac{1}{T}\int_0^T S_u\,du - K\right)^+\middle|\mathcal{F}_t\right)\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}^Q\left[\left(\frac{1}{T}\int_0^T S_u\,du - K\right)^+\middle|\mathcal{F}_s\right] \quad \text{by tower property} \\
&= e^{r(T-s)}\mathbb{E}^Q\left[\left(e^{-r(T-s)}\frac{1}{T}\int_0^T S_u\,du - K\right)^+\middle|\mathcal{F}_s\right] \\
&= e^{r(T-s)}C_{a,s} \\
&= Z_s.
\end{aligned}
$$

This proves the martingale property in the first case. The second part of the proof follows similarly as the foregoing by substituting the appropriate payoff. $\square$

From now onwards we shall write $\mathbb{E}_t^Q(.)$ for $\mathbb{E}^Q(.|\mathcal{F}_t)$.

2.3 Put-Call Parity

As the expression would suggest, the put-call parity gives a relationship between the value of the call and the put. The put-call parity helps us to determine the prices of the Asian

options. If we know the value of the call, say, then immediately we can find that of the put. Since our task is to find the price of an Asian call option, knowing that the buyer of the option exercises their right to buy, then in this case the put value is zero. This implies that from the expression of the put-call parity, we have the price of the call. The following result shall be important in establishing the parity:

Lemma 2.3.

$$\mathbb{E}_t^Q \left(\int_t^T S_u \, du \right) = \frac{S_t}{r} (e^{r(T-t)} - 1).$$

Proof.

$$\mathbb{E}_t^Q \left(\int_t^T S_u \, du \right) = \mathbb{E}_t^Q \left(\int_t^T e^{r(u-t)} (e^{-r(u-t)} S_u) \, du \right)$$

$$= \int_t^T e^{r(u-t)} \mathbb{E}_t^Q (e^{-r(u-t)} S_u) \, du, \quad \text{by Fubini Theorem [33]}$$

$$= \int_t^T e^{r(u-t)} S_t \, du, \quad \text{since } S_T \text{ is a martingale,}$$

$$= S_t \int_t^T e^{r(u-t)} \, du$$

$$= \frac{S_t}{r} (e^{r(T-t)} - 1). \qquad \square$$

The result serves the purpose of finding the mean of the integral of S_u. It confirms that the conditional expectation and integration can be interchanged.

Proposition 2.4. *(Put-Call Parity). Let $C_{a,t}$ and $P_{a,t}$ denote the price of the fixed strike Asian call and put options, respectively. Also let $C_{b,t}$ and $P_{b,t}$ denote the price of the floating strike Asian call and put options, respectively. Then*

$$(i) \ P_{a,t} = C_{a,t} - \frac{S_t}{Tr}(1 - e^{-r(T-t)}) + e^{-r(T-t)} \left(K - \frac{1}{T} \int_0^t S_u \, du \right),$$

(ii) $P_{b,t} = C_{b,t} - S_t(1 - \frac{1}{Tr}(1 - e^{-r(T-t)})) + e^{-r(T-t)}\frac{1}{T} \int_0^t S_u \, du.$

Proof. (i) Let $\chi_{(.)}$ be the characteristic function and define the sets:

$$A = \left\{ w \in \Omega : \frac{1}{T} \int_0^T S_u(w) \, du < K \right\} \quad \text{and} \quad B = \left\{ w \in \Omega : \frac{1}{T} \int_0^T S_u(w) \, du \geq K \right\}.$$

Then we can write the difference between the put (2.3) and the call (2.4) values as

$$P_{a,t} - C_{a,t} = \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(K - \frac{1}{T} \int_0^T S_u \, du \right)^+ \right] - \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u \, du - K \right)^+ \right]$$

$$= e^{-r(T-t)} \mathbb{E}_t^Q \left[\left(K - \frac{1}{T} \int_0^T S_u \, du \right) \chi_A - \left(\frac{1}{T} \int_0^T S_u \, du - K \right) \chi_B \right],$$

Therefore the above expression becomes

$$= e^{-r(T-t)} \mathbb{E}_t^Q \left[\left(K - \frac{1}{T} \int_0^T S_u \, du \right)(\chi_A + \chi_B) \right],$$

$$= e^{-r(T-t)} \mathbb{E}_t^Q \left(K - \frac{1}{T} \int_0^T S_u \, du \right)$$

$$= e^{-r(T-t)} \mathbb{E}_t^Q \left(K - \frac{1}{T} \int_0^t S_u \, du - \frac{1}{T} \int_t^T S_u \, du \right)$$

$$= e^{-r(T-t)} \left(K - \frac{1}{T} \int_0^t S_u \, du \right) - \frac{1}{T} \mathbb{E}_t^Q \left(\int_t^T S_u \, du \right).$$

Here we have used the fact that we can take out the $\mathcal{F}_t$ measurable part outside the expectation. Finally, by using Lemma 2.3, in the last expression in the above, we get

$$P_{a,t} = C_{a,t} - \frac{S_t}{Tr}(1 - e^{-r(T-t)}) + e^{-r(T-t)} \left(K - \frac{1}{T} \int_0^t S_u \, du \right).$$

(ii) We can mimic the above argument to deduce the second parity.

Similarly defining the sets A and B we have

$$P_{a,t} - C_{a,t} = e^{-r(T-t)} \mathbb{E}_t^Q \left[\left(\frac{1}{T} \int_0^T S_u \, du - S_T \right)^+ \right] - e^{-r(T-t)} \mathbb{E}_t^Q \left[\left(S_T - \frac{1}{T} \int_0^T S_u \, du \right)^+ \right]$$

$$= e^{-r(T-t)} \mathbb{E}_t^Q \left[\left(\frac{1}{T} \int_0^T S_u \, du - S_T \right) \chi_A - \left(S_T - \frac{1}{T} \int_0^T S_u \, du \right) \chi_B \right]$$

$$= e^{-r(T-t)} \mathbb{E}_t^Q \left(\frac{1}{T} \int_0^T S_u \, du - S_T \right).$$

We can as well split the integral to get

$$= e^{-r(T-t)}\frac{1}{T}\int_0^t S_u\,du + e^{-r(T-t)}\frac{1}{T}\mathbb{E}_t^Q\left(\int_t^T S_u\,du\right) - \mathbb{E}_t^Q(e^{-r(T-t)}S_T)$$

$$= e^{-r(T-t)}\frac{1}{T}\int_0^t S_u\,du + \frac{e^{-r(T-t)}}{T}\frac{S_t}{r}(e^{r(T-t)} - 1) - S_t$$

by Lemma 2.3 and since S_T is a Q martingale ie $\mathbb{E}_t^Q(e^{-r(T-t)}S_T) = S_t$. Finally, we have

$$P_{b,t} = C_{b,t} - S_t(1 - \frac{1}{Tr}(1 - e^{-r(T-t)})) + e^{-r(T-t)}\frac{1}{T}\int_0^t S_u\,du. \qquad \square$$

Remark 2.5. If at time t we know that the known part of the average $\frac{1}{T}\int_0^t S_u\,du$, is greater than K, then the option will surely be exercised at time T. In this case the put option is worthless and the price of the call option is

$$C_{a,t} = \frac{S_t}{Tr}(1 - e^{-r(T-t)}) - e^{-r(T-t)}\left(K - \frac{1}{T}\int_0^t S_u\,du\right), \qquad (2.5)$$

and

$$C_{b,t} = S_t(1 - \frac{1}{Tr}(1 - e^{-r(T-t)})) - e^{-r(T-t)}\frac{1}{T}\int_0^t S_u\,du. \qquad (2.6)$$

The last result can be found in Wilmot [60] where it was derived in a different way. In that method, the result is obtained from the PDE whose solution is the price of the option.

2.4   The Discrete Geometric Averaged Asian Option

In this section we are going to determine the explicit formula for the case of the discretely sampled geometric averaged Asian option. It turns out that the expression is like the Black-Scholes formula. One reason for finding this formula is to use it as a control variate in the Monte Carlo method. We shall expand on this later. We give an alternative derivation to the one given by Kemna [36]. To this end, we can immediately write down the expression for the price of this option as

$$C_{e,t} = \mathbb{E}_t^Q\left[e^{-r(T-t)}\left(\left(\prod_{i=1}^N S(t_i)\right)^{\frac{1}{N}} - K\right)^+\right]. \qquad (2.7)$$

Proposition 2.6. *(Closed form pricing formula for geometric Asian option) The price of the geometric Asian option $C_{e,t}$ satisfies*

$$C_{e,0} = S_0 e^{(\tilde{r}-r)T} \Phi(\tilde{d}_1) - K e^{-rT} \Phi(\tilde{d}_2), \tag{2.8}$$

where $\Phi(.)$ is the cumulative normal distribution function and

$$\tilde{d}_1 = \frac{\log \frac{S_0}{K} + \left(\tilde{r} + \frac{\tilde{\sigma}^2}{2}\right) T}{\tilde{\sigma}\sqrt{T}},$$

$$\tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}\sqrt{T}$$

$$\tilde{\sigma}^2 = \sigma^2 \frac{(N+1)(2N+1)}{6N^2}$$

$$\tilde{r} = \frac{1}{2}\tilde{\sigma}^2 + \left(r - \frac{1}{2}\sigma^2\right)\frac{N+1}{2N}.$$

Proof. As is pointed out by Higham ([31] exercise 19.6) the product can be split as

$$\prod_{i=1}^{N} S(t_i) = \frac{S(t_N)}{S(t_{N-1})} \left(\frac{S(t_{N-1})}{S(t_{N-2})}\right)^2 \left(\frac{S(t_{N-2})}{S(t_{N-3})}\right)^3 \cdots \left(\frac{S(t_3)}{S(t_2)}\right)^{N-2}$$
$$\cdot \left(\frac{S(t_2)}{S(t_1)}\right)^{N-1} \left(\frac{S(t_1)}{S(t_0)}\right)^N S^N(t_0).$$

Therefore

$$\log\left(\frac{\left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}}}{S_0}\right) = \frac{1}{N}\left[\log\left(\frac{S(t_N)}{S(t_{N-1})}\right) + 2\log\left(\frac{S(t_{N-1})}{S(t_{N-2})}\right) + 3\log\left(\frac{S(t_{N-2})}{S(t_{N-3})}\right)\right.$$

$$+ \ldots + (N-2)\log\left(\frac{S(t_3)}{S(t_2)}\right) + (N-1)\log\left(\frac{S(t_2)}{S(t_1)}\right)$$

$$\left. + N\log\left(\frac{S(t_1)}{S(t_0)}\right)\right].$$

Now

$$S(t_N) = S(t_{N-1})e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma W_{\Delta t}},$$

where $\Delta t = T/N$ and $W_{\Delta t} \sim N(0, \Delta t)$. Assuming uniform spacing on the interval $[0, T]$, we have

$$\log\left(\frac{S(t_N)}{S(t_{N-1})}\right) \overset{\mathcal{D}}{=} \log\left(\frac{S(t_{N-1})}{S(t_{N-2})}\right) \overset{\mathcal{D}}{=} \log\left(\frac{S(t_{N-2})}{S(t_{N-3})}\right) \overset{\mathcal{D}}{=} \cdots$$
$$\overset{\mathcal{D}}{=} \log\left(\frac{S(t_2)}{S(t_1)}\right) \overset{\mathcal{D}}{=} \log\left(\frac{S(t_1)}{S(t_0)}\right),$$

where $\overset{\mathcal{D}}{=}$ means having the same distribution and

$$\log\left(\frac{S(t_N)}{S(t_{N-1})}\right) \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t, \sigma^2 \Delta t\right).$$

Define

$$Z := \log\left(\frac{\left(\prod_{i=1}^{N} S(t_i)\right)^{\frac{1}{N}}}{S_0}\right).$$

We can find the expectation and variance of Z:

$$\mathbb{E}(Z) = \frac{1}{N}\left((r - \frac{1}{2}\sigma^2)\Delta t + 2(r - \frac{1}{2}\sigma^2)\Delta t + 3(r - \frac{1}{2}\sigma^2)\Delta t + \cdots + N(r - \frac{1}{2}\sigma^2)\Delta t\right)$$
$$= \frac{1}{N}(r - \frac{1}{2}\sigma^2)\Delta t \sum_{i=1}^{N} i$$
$$= \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)\frac{N+1}{N}T.$$

Likewise, the variance is

$$\text{Var}(Z) = \frac{1}{N^2}\left(\sigma^2 \Delta t + 4\sigma^2 \Delta t + 9\sigma^2 \Delta t + \cdots + N^2 \sigma^2 \Delta t\right)$$
$$= \frac{1}{N^2}\sigma^2 \Delta t \sum_{i=1}^{N} i^2$$
$$= \sigma^2 \frac{(N+1)(2N+1)}{6N^2}T.$$

We can now write down the distribution of the geometric sum. This is the key issue in describing the price of the option. If we can find the distribution of the sum then we can find the price by integrating a suitable function using the distribution function. This is

difficult for some payoffs like (a) or (b) because we cannot characterise the distribution function. In such cases we have to turn to other means of determining the price. We observe that

$$\log \left(\left(\prod_{i=1}^{N} S(t_i) \right)^{\frac{1}{N}} \right) \sim N \left(\log S_0 + \left(r - \frac{1}{2}\sigma^2 \right) \frac{N+1}{2N} T, \sigma^2 \frac{(N+1)(2N+1)}{6N^2} T \right).$$

We need to compare this random variable with the corresponding expression for the European call option, so that we can determine the new parameters $\tilde{\sigma}^2$ and $\tilde{r}$. For the European case we use

$$\log S_T \sim N \left(\log S_0 + \left(r - \frac{1}{2}\sigma^2 \right) T, \sigma^2 T \right).$$

This implies

$$\tilde{\sigma}^2 := \sigma^2 \frac{(N+1)(2N+1)}{6N^2}$$

and

$$\tilde{r} - \frac{1}{2}\tilde{\sigma}^2 = \left(r - \frac{1}{2}\sigma^2 \right) \frac{N+1}{2N} \Rightarrow \tilde{r} = \frac{1}{2}\tilde{\sigma}^2 + \left(r - \frac{1}{2}\sigma^2 \right) \frac{N+1}{2N}.$$

Writing

$$Y = \left(\prod_{i=1}^{N} S(t_i) \right)^{\frac{1}{N}},$$

we see that

$$Y \sim N \left(\log S_0 + \left(\tilde{r} - \frac{1}{2}\tilde{\sigma}^2 \right) T, \tilde{\sigma}^2 T \right).$$

From (2.7), the price of the option at time $t = 0$ becomes

$$C_{e,0} = e^{-rT} \mathbb{E}^Q \left(\left(\left(\prod_{i=1}^{N} S(t_i) \right)^{\frac{1}{N}} - K \right)^+ \right)$$

$$= e^{-rT} \mathbb{E}^Q \left(e^{\log Y} - K \right)^+.$$

$$= e^{-rT} \int_{\log K}^{\infty} \left(e^{\log Y} - K \right) \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 T}} e^{\frac{-\left(\log Y - \log S_0 - \left(\tilde{r} - \frac{1}{2}\tilde{\sigma}^2 \right) T \right)^2}{2\tilde{\sigma}^2 T}} \, d\log Y.$$

Letting $X = \log \frac{Y}{S_0}$, then $C_{e,0}$ becomes

$$C_{e,0} = e^{-rT} \int_{\log \frac{K}{S_0}}^{\infty} S_0 e^X \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 T}} e^{\frac{-\left(X - \left(\tilde{r} - \frac{1}{2}\tilde{\sigma}^2\right)T\right)^2}{2\tilde{\sigma}^2 T}} dX$$

$$- Ke^{-rT} \int_{\log \frac{K}{S_0}}^{\infty} S_0 \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 T}} e^{\frac{-\left(X - \left(\tilde{r} - \frac{1}{2}\tilde{\sigma}^2\right)T\right)^2}{2\tilde{\sigma}^2 T}} dX.$$

Completing the square in the first integral, we obtain

$$C_{e,0} = S_0 e^{\tilde{r}T} e^{-rT} \int_{\log \frac{K}{S_0}}^{\infty} \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 T}} e^{\frac{-\left(X - \left(\tilde{r} + \frac{\tilde{\sigma}^2}{2}\right)T\right)^2}{2\tilde{\sigma}^2 T}} dX$$

$$- S_0 K e^{-rT} \int_{\log \frac{K}{S_0}}^{\infty} \frac{1}{\sqrt{2\pi\tilde{\sigma}^2 T}} e^{\frac{-\left(X - \left(\tilde{r} - \frac{1}{2}\tilde{\sigma}^2\right)T\right)^2}{2\tilde{\sigma}^2 T}} dX.$$

We make two more substitutions (to standardise the normal distribution, that is make it a N(0,1) realisation)

$$U := \frac{X - \left(\tilde{r} + \frac{\tilde{\sigma}^2}{2}\right)T}{\tilde{\sigma}\sqrt{T}} \quad \text{and} \quad V := \frac{X - \left(\tilde{r} - \frac{\tilde{\sigma}^2}{2}\right)T}{\tilde{\sigma}\sqrt{T}}.$$

Simplifying further

$$C_{e,0} = S_0 e^{\tilde{r}T} e^{-rT} \int_{-\log \frac{S_0}{K} - \frac{(\tilde{r} + \tilde{\sigma}^2/2)T}{\tilde{\sigma}\sqrt{T}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-U^2}{2}} dU$$

$$- S_0 K e^{-rT} \int_{-\log \frac{S_0}{K} - \frac{(\tilde{r} - \tilde{\sigma}^2/2)T}{\tilde{\sigma}\sqrt{T}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-V^2}{2}} dV.$$

Finally, we can write down the price of the option as

$$C_{e,0} = S_0 e^{(\tilde{r} - r)T} \Phi(\tilde{d}_1) - Ke^{-rT} \Phi(\tilde{d}_2), \tag{2.9}$$

where $\Phi(.)$ is the cumulative normal distribution function and

$$\tilde{d}_1 = \frac{\log \frac{S_0}{K} + \left(\tilde{r} + \frac{\tilde{\sigma}^2}{2}\right)T}{\tilde{\sigma}\sqrt{T}},$$

$$\tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}\sqrt{T}. \qquad \square$$
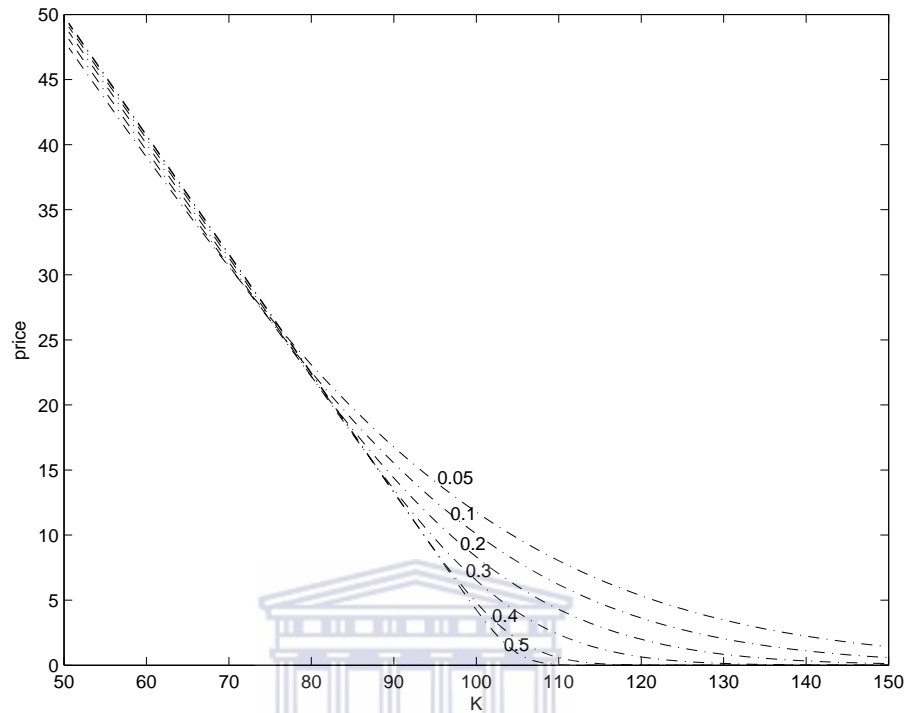
Figure 2.1: Prices of Geometric averaged Asian option using (2.9) for different volatilities

After what appears to be a plethora of processes, we have been able to characterise the price. The key was making use of the normal distribution. Figure 2.1 shows the price (at $t = 0$) of the Geometric averaged Asian option. It is a result of using $C_{e,0}$ for different values of K. Later in Part II of the thesis, we will explain in detail why it takes this particular shape and how it can be used to find the value of the arithmetic averaged Asian option.

3. Some Analytical results for Asian Options

In this chapter, we focus on how the price of the Asian option varies with the strike or the price of the underlying asset. Our intuition is that the price of a call option should decrease with strike since the right to exercise for low strikes should be surely more costly than for higher strikes. We also expect the price of the call to increase with the price of the underlying. We also investigate how the option price varies with duration-the time the option is held. The effect of volatility on the price of the option also needs to be addressed (see also Carr [14]).

3.1 The Effect of the Strike price

As usual, we assume a filtered probability space $(\Omega, P, \mathcal{F})$, S_t being the price of the underlying asset, for fixed $T > 0$. Define the random variable

$$\phi_T(\omega) = \frac{K - \frac{1}{T}\int_0^T S_u(\omega)\, du}{S_T},$$

where $\omega \in \Omega$.

Proposition 3.1. *Consider the set $D = \{\omega \in \Omega : \phi_T(\omega) < 0\}$, and let $Q(D)$ denote the probability that $\phi_T < 0$. Then the price of an Asian call $C_{a,t}$ is a decreasing function of the time zero-price of the underlying and*

$$\frac{\partial C_{a,t}}{\partial K} = -e^{-r(T-t)}Q(D). \tag{3.1}$$

Proof. Recall that

$$C_{a,t} = \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u \, du - K \right)^+ \right]$$

$$= \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u \, du - K \right) \mathbb{1}_D \right],$$

where $\mathbb{1}_{(.)}$ is the indicator function. We differentiate $C_{a,t}$ to get

$$\frac{\partial C_{a,t}}{\partial K} = -e^{-r(T-t)} \mathbb{E}_t^Q (\mathbb{1}_D)$$

$$= -e^{-r(T-t)} Q(D). \qquad \square$$

Lemma 3.2. *The price of the call $C_{a,t}$ can be written as*

$$C_{a,t} = \frac{S_t}{Tr} (1 - e^{-r(T-t)}) \tilde{Q}(D)$$

$$- e^{-r(T-t)} \left(K - \frac{1}{T} \int_0^t S_u \, du \right) Q(D), \tag{3.2}$$

where $\tilde{Q}$ is defined by $\tilde{Q}(A) = \int_A \tilde{Z}_t \, dQ$, for $A \in \mathcal{F}_T$ and $\tilde{Z}_t$ is the Radon-Nikodym derivative [1]

$$\tilde{Z}_t = \frac{\int_t^T S_u \, du}{\mathbb{E}^Q \left(\int_t^T S_u \, du \right)}.$$

Proof. We recall that $C_{a,t} = \mathbb{E}_t^Q \left(e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u \, du - K \right)^+ \right)$. We can split the integral to have

$$C_{a,t} = \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(\frac{1}{T} \int_0^t S_u \, du + \frac{1}{T} \int_t^T S_u \, du - K \right)^+ \right]$$

$$= \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(\frac{1}{T} \int_0^t S_u \, du + \frac{1}{T} \int_t^T S_u \, du - K \right) \mathbb{1}_{D\}} \right]$$

$$= e^{-r(T-t)} \left(\frac{1}{T} \int_0^t S_u \, du - K \right) \mathbb{E}_t^Q (\mathbb{1}_D)$$

$$+ \frac{1}{T} e^{-r(T-t)} \mathbb{E}_t^Q \left(\int_t^T S_u \, du \cdot \mathbb{1}_D \right). \tag{3.3}$$

The latter simplification is valid since the first integral and the constant K are $\mathcal{F}_t$ measurable (property (ii) of the conditional expectation in Chapter 1), and since the conditional expectation distributes over addition. At this stage we can make a change of measures in the second expectation by defining a new measure $\tilde{Q}$. Let

$$\widetilde{Z}_t = \frac{d\tilde{Q}}{dQ} = \frac{\int_t^T S_u \, du}{\mathbb{E}^Q \left(\int_t^T S_u \, du \right)}.$$

We observe that

$$\mathbb{E}_t^Q \left(\frac{\int_t^T S_u \, du}{\mathbb{E}^Q \left(\int_t^T S_u \, du \right)} \right) = \frac{\mathbb{E}_t^Q \left(\int_t^T S_u \, du \right)}{\mathbb{E}_t^Q \left(\mathbb{E}^Q \left(\int_t^T S_u \, du \right) \right)}, \quad \text{by Proposition (1.3)}$$

$$= \frac{\mathbb{E}_t^Q \left(\int_t^T S_u \, du \right)}{\mathbb{E}_t^Q \left(\int_t^T S_u \, du \right)}, \quad \text{by tower property of } \mathbb{E}_t^Q(.)$$

$$= 1. \tag{3.4}$$

Using Bayes theorem, we can write

$$\mathbb{E}_t^Q \left(\frac{\int_t^T S_u \, du}{\mathbb{E}^Q \left(\int_t^T S_u \, du \right)} \mathbb{1}_D \right) = \mathbb{E}_t^Q \left(\frac{\int_t^T S_u \, du}{\mathbb{E}^Q \left(\int_t^T S_u \, du \right)} \right) \mathbb{E}_t^{\tilde{Q}}(\mathbb{1}_D)$$

$$= \mathbb{E}_t^{\tilde{Q}} \left(\mathbb{1}_D \right), \quad \text{by (3.4)}$$

$$= \tilde{Q}(D). \tag{3.5}$$

Therefore

$$\mathbb{E}_t^Q \left(\int_t^T S_u \, du \cdot \mathbb{1}_D \right) = \mathbb{E}^Q \left(\int_t^T S_u \, du \right) \mathbb{E}_t^Q \left(\frac{\int_t^T S_u \, du}{\mathbb{E}^Q \left(\int_t^T S_u \, du \right)} \mathbb{1}_D \right)$$

$$= \mathbb{E}^Q \left(\int_t^T S_u \, du \right) \tilde{Q}(D), \quad \text{by (3.5)}$$

$$= \frac{S_t}{r} (e^{r(T-t)} - 1) \tilde{Q}(D), \quad \text{by Lemma (2.3).}$$

Finally (3.3) becomes

$$C_{a,t} = \frac{S_t}{Tr}(1 - e^{r(T-t)})\tilde{Q}(D)$$
$$- e^{-r(T-t)}\left(K - \frac{1}{T}\int_0^t S_u\,du\right)Q(D) \tag{3.6}$$

and this completes the proof.   □

3.2   The Effect of the price of the underlying asset

Proposition 3.3. *The price of an Asian call $C_{a,t}$, is an increasing function of the price of the underlying asset and*

$$\frac{\partial C_{a,t}}{\partial S_t} = \frac{1}{Tr}(1 - e^{-r(T-t)})\tilde{Q}(D) \tag{3.7}$$

Proof. The result follows at once by differentiating (3.6) w.r.t. S_t.   □

A study of the effect of the volatility on the price of the Asian option can be found in Carr [14]. The argument is based on the Maximum principle for parabolic PDEs (see, e.g., Williams [59]). As Carr [14] points out, it is in a Black-Scholes market setting where the price of the option increases as the volatility increases. In a Binomial model this is not true.

4. Bounds of Asian option values

We explore another method which is both easy to use as well as to derive. It is much more appealing to practitioners who would want to price the derivatives in the shortest possible times (for more detail, see Chen et al. [15], Deelstra et al. [21]). The method is to take bounds of the value of the option. The bound is found analytically. At face value it would appear as if this was not a very good idea but as we see from the results of calculations, the bounds are *staggeringly accurate*, quoting Rogers et al. [52].

We will now derive the bounds for both the fixed and floating strike.

4.1 A Lower Bound for a fixed strike price Asian option

From Chapter 2, we know that

$$C_{a,t} = \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u \, du - K \right)^+ \right].$$

We define $A := \left\{ w \in \Omega : \frac{1}{T} \int_0^T S_u(w) \, du > K \right\}$ and let $\mathbb{1}_A$ be the indicator function. The value of the option becomes

$$
\begin{aligned}
C_{a,0} &= \mathbb{E}^Q \left[e^{-rT} \left(\frac{1}{T} \int_0^T S_u \, du - K \right) \mathbb{1}_A \right], \\
&= e^{-rT} \mathbb{E}^Q \left(\frac{1}{T} \int_0^T (S_u - K) \mathbb{1}_A \, du \right), \\
&= \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q[(S_u - K) \mathbb{1}_A] \, du.
\end{aligned}
$$

Define a new set

$$\mathcal{A} := \left\{ w \in \Omega : \frac{1}{T} \int_0^T W_u(w) \, du > \gamma \right\}.$$

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From the properties of integration, we know that if two sets are such that $X \subset Y$, then $\int_X f \le \int_Y f$. Therefore $C_{a,0}$ becomes

$$\frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q[(S_u - K)\mathbb{1}_A]\, du \ge \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q[(S_u - K)\mathbb{1}_\mathcal{A}]\, du.$$

The lower bound $\check{C}_{a,0}(\gamma, S_t, K)$ of the option value is defined as

$$\check{C}_{a,0} = \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q[(S_u - K)\mathbb{1}_\mathcal{A}]\, du \tag{4.1}$$

$$= \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q \left(S_u - K, \frac{1}{T} \int_0^T W_u(w)\, du > \gamma \right) du.$$

We shall optimally find γ, that is the gamma must be one which maximises the lower bound $\check{C}_{a,0}(\gamma, S_t, K)$. To do this we differentiate $\check{C}_{a,0}(\gamma, S_t, K)$ with respect to γ and equate that to zero. So we have to solve the following equation:

$$\frac{\partial}{\partial \gamma} \int_0^T \mathbb{E}^Q \left(S_u - K, \frac{1}{T} \int_0^T W_u(w)\, du > \gamma \right) du = 0. \tag{4.2}$$

We now appeal to Theorem 6 in [15].

Theorem 4.1. *Suppose the random variables S_t and U are jointly distributed with density function $f(S_t, U)$, so that U has marginal density function $f_U(u)$. Then*

$$\frac{\partial}{\partial \gamma} \int_0^T \mathbb{E}^Q \left(S_u - K, U > \gamma \right) du = - \int_0^T \mathbb{E}^Q \left(S_u - K | U = \gamma \right) f_U(\gamma)\, du.$$

We omit this proof and only remark that it is a straight forward one. It is based on the Leibnitz's rule; a theorem which enables one to take derivatives of integrals.

Let $U = \frac{1}{T} \int_0^T W_u(w)\, du$. Then 4.2 becomes

$$\frac{\partial}{\partial \gamma} \int_0^T \mathbb{E}^Q \left(S_u - K, U > \gamma \right) du = - \int_0^T \mathbb{E}^Q \left(S_u - K | U = \gamma \right) f_U(\gamma)\, du$$

$$= -f_U(\gamma) \int_0^T \left(\mathbb{E}^Q \left(S_u | U = \gamma \right) - K \right) du$$

$$= -f_U(\gamma) \int_0^T \mathbb{E}^Q \left(S_u | U = \gamma \right) du + f_U(\gamma) TK.$$

Equating the above expression to zero and substituting γ by γ^* (an indication that this is optimal), we have

$$\frac{1}{T} \int_0^T \mathbb{E}^Q \left(S_u | U = \gamma^* \right) du = K. \tag{4.3}$$

In order to simplify (4.3), we infer the conditional distribution of W_t on $\frac{1}{T} \int_0^T W_u(w) \, du$. From now onwards we will drop the explicit dependence of W_t on ω. Let us denote $\min\{a, b\}$ by $a \wedge b$. First we find the covariance

$$\begin{aligned}
\mathrm{Cov}\left(W_t, \frac{1}{T} \int_0^T W_u \, du \right) &= \mathbb{E}^Q \left(W_t \frac{1}{T} \int_0^T W_u \, du \right) - \mathbb{E}^Q(W_t) \mathbb{E}^Q \left(\frac{1}{T} \int_0^T W_u \, du \right) \\
&= \frac{1}{T} \mathbb{E}^Q \left(W_t \int_0^T W_u \, du \right), \quad \text{since } \mathbb{E}^Q(W_t) = 0 \\
&= \frac{1}{T} \int_0^T \mathbb{E}^Q(W_t W_u) \, du \\
&= \frac{1}{T} \int_0^t u \wedge t \, du + \frac{1}{T} \int_t^T u \wedge t \, du \\
&= \frac{1}{T} \int_0^t u \, du + \frac{1}{T} \int_t^T t \, du \\
&= t \left(1 - \frac{t}{2T} \right).
\end{aligned} \tag{4.4}$$

From the definition of Brownian motion, $\mathrm{Var}(W_t) = t$. We now change the integral to be with respect to W_u by making use of a result in [23] (page 96):

$$\int_0^T W_u \, du = \int_0^T (T - u) \, dW_u. \tag{4.5}$$

Consequently, the variance of $\frac{1}{T} \int_0^T W_u \, du$ becomes

$$
\begin{aligned}
\mathrm{Var}\left(\frac{1}{T} \int_0^T W_u \, du\right) &= \frac{1}{T^2}\mathbb{E}^Q\left\{\left(\int_0^T W_u \, du\right)^2\right\} \\
&\quad - \frac{1}{T^2}\mathbb{E}^Q\left(\int_0^T W_u \, du\right)\mathbb{E}^Q\left(\int_0^T W_u \, du\right) \\
&= \frac{1}{T^2}\mathbb{E}^Q\left\{\left(\int_0^T W_u \, du\right)^2\right\}, \quad \text{since} \quad \mathbb{E}^Q(W_t) = 0 \\
&= \frac{1}{T^2}\mathbb{E}^Q\left\{\left(\int_0^T (T-u) \, dW_u\right)^2\right\}, \quad \text{by (4.5)} \\
&= \frac{1}{T^2}\int_0^T (T-u)^2 \, du, \quad \text{by Itô Isometry [48]} \\
&= \frac{T}{3}.
\end{aligned}
\tag{4.6}
$$

See [52] for the case where $T = 1$.

From Proposition 1.10 we have

$$
\mathbb{E}^Q\left(W_t \,\bigg|\, \frac{1}{T}\int_0^T W_u \, du = y\right) = \frac{3t}{T^2}\left(T - \frac{t}{2}\right)y
$$

$$
\mathrm{Var}\left(W_t \,\bigg|\, \frac{1}{T}\int_0^T W_u \, du = y\right) = t - \frac{3t^2}{T^3}\left(T - \frac{t}{2}\right)^2.
$$

We can now write this as

$$
\left(W_t \,\bigg|\, \frac{1}{T}\int_0^T W_u \, du = y\right) \sim N\left(\frac{3t}{T^2}\left(T - \frac{t}{2}\right)y, \, t - \frac{3t^2}{T^3}\left(T - \frac{t}{2}\right)^2\right).
\tag{4.7}
$$

Condition (4.3) involves finding the conditional distribution of $S_u = S_0 e^{(r - \frac{\sigma^2}{2})u + \sigma W_u}$ on

$\frac{1}{T} \int_0^T W_u \, du$. Letting $U = \frac{1}{T} \int_0^T W_u \, du$ and using the mgf of $N(.,.)$, we get

$$
\begin{aligned}
\mathbb{E}^Q(S_u | U = y) &= \mathbb{E}^Q(S_0 e^{(r - \frac{\sigma^2}{2})u + \sigma W_u} | U = y) \\
&= S_0 e^{(r - \frac{\sigma^2}{2})u} \mathbb{E}^Q(e^{\sigma W_u} | U = y) \\
&= S_0 e^{(r - \frac{\sigma^2}{2})u} \exp \left\{ \sigma \mathbb{E}^Q(W_u | U = y) + \frac{\sigma^2}{2} \text{Var}(W_u | U = y) \right\} \\
&= S_0 e^{(r - \frac{\sigma^2}{2})u} \exp \left\{ \sigma \frac{3u}{T^2} \left(T - \frac{u}{2} \right) y + \frac{\sigma^2}{2} \left(t - \frac{3u^2}{T^3} \left(T - \frac{u}{2} \right)^2 \right) \right\} \\
&= S_0 \exp \left\{ ru + \frac{3\sigma u}{T^2} \left(T - \frac{u}{2} \right) y - \frac{3\sigma^2 u}{2T^3} \left(T - \frac{u}{2} \right)^2 \right\}.
\end{aligned}
$$

Finally, equation (4.3) simplifies to

$$
\frac{S_0}{T} \int_0^T \exp \left\{ ru + \frac{3\sigma u}{T^2} \left(T - \frac{u}{2} \right) \gamma^* - \frac{3\sigma^2 u}{2T^3} \left(T - \frac{u}{2} \right)^2 \right\} du = K. \tag{4.8}
$$

Before we can derive the lower bound, we need to make use of yet another result in [15]:

Proposition 4.2. If $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ and $c = \text{Cov}(X, Y)$ then

$$
\mathbb{E}^Q(e^X \mathbb{1}_{\{Y > 0\}}) = e^{\mu_x + \frac{\sigma_x^2}{2}} \Phi \left(\frac{\mu_y + c}{\sigma_y} \right).
$$

We are now in a position to compute the bound $\check{C}_{a,0}$. Proposition 4.2 implies

$$
\begin{aligned}
\check{C}_{a,0} &= \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q[(S_u - K) \mathbb{1}_{\mathcal{A}}] \, du. \\
&= \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q(e^{\ln S_u} \mathbb{1}_{\mathcal{A}} - e^{\ln K} \mathbb{1}_{\mathcal{A}}) \, du \\
&= \frac{e^{-rT}}{T} \int_0^T \left\{ e^{\ln S_0 + (r - \frac{\sigma^2}{2})u + \frac{\sigma^2 u}{2}} \Phi \left(\frac{-\gamma^* + \sigma u \left(1 - \frac{u}{2T} \right)}{\sqrt{T/3}} \right) - e^{\ln K} \Phi \left(\frac{-\gamma^*}{\sqrt{T/3}} \right) \right\} du \\
&= \frac{e^{-rT}}{T} \int_0^T \left\{ S_0 e^{ru} \Phi \left(\frac{-\gamma^* + \sigma u \left(1 - \frac{u}{2T} \right)}{\sqrt{T/3}} \right) - K \Phi \left(\frac{-\gamma^*}{\sqrt{T/3}} \right) \right\} du. \tag{4.9}
\end{aligned}
$$

The same ideas can be used to derive the bound for the floating strike Asian option.

4.2   A lower bound for a floating strike Asian option

We will now derive the lower bound for the floating strike option. The ideas from the case of the fixed strike are largely unchanged. Let us recall from Chapter 2 that

$$C_{b,t} = \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(S_T - \frac{1}{T} \int_0^T S_u \, du \right)^+ \right].$$

Define $B := \left\{ w \in \Omega : \frac{1}{T} \int_0^T S_u(w) \, du < S_T \right\}$ and $\mathbb{1}_B$ to be the indicator function. The value of the option becomes

$$
\begin{aligned}
C_{b,0} &= \mathbb{E}^Q \left[e^{-rT} \left(S_T - \frac{1}{T} \int_0^T S_u \, du \right) \mathbb{1}_B \right], \\
&= e^{-rT} \mathbb{E}^Q \left(\frac{1}{T} \int_0^T (S_T - S_u) \mathbb{1}_B \, du \right), \\
&= \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q[(S_T - S_u)\mathbb{1}_B] \, du.
\end{aligned}
$$

Let us define a new set $\mathcal{B} := \left\{ w \in \Omega : \frac{1}{T} \int_0^T W_u(w) \, du - W_T < \gamma \right\}$.

Since $X \subset Y$ implies $\int_X f \leq \int_Y f$, therefore

$$\frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q[(S_T - S_u)\mathbb{1}_B] \, du \geq \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q[(S_T - S_u)\mathbb{1}_{\mathcal{B}}] \, du.$$

The lower bound of the option value $\check{C}_{b,0}(\gamma, S_t, K)$ is

$$
\begin{aligned}
\check{C}_{b,0} &= \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q[(S_T - S_u)\mathbb{1}_{\mathcal{B}}] \, du \tag{4.10} \\
&= \frac{e^{-rT}}{T} \int_0^T \mathbb{E}^Q \left(S_T - S_u, \frac{1}{T} \int_0^T W_u(w) \, du - S_T < \gamma \right) du.
\end{aligned}
$$

As before, γ will be determined optimally. To do that, we solve the following equation

$$\frac{\partial}{\partial \gamma} \int_0^T \mathbb{E}^Q \left(S_T - S_u, \frac{1}{T} \int_0^T W_u(w) \, du - W_T < \gamma \right) du = 0. \tag{4.11}$$

Proposition 4.3. *Suppose the random variables S_t and U are jointly distributed with density function $f(S_t, U)$, so that U has marginal density function $f_U(u)$. Then*

$$\frac{\partial}{\partial \gamma} \int_0^T \mathbb{E}^Q (S_T - S_u, U < \gamma) \, du = \int_0^T \mathbb{E}^Q (S_T - S_u | U = \gamma) f_U(\gamma) \, du.$$

Due to the slight change to the proof found in [15], we will prove this proposition. By and large the arguments are unchanged, but our proof is more in detail.

Proof. By the definition of expectation,

$$\mathbb{E}^Q\left(S_T - S_u, U < \gamma\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\gamma} (S_T - S_u)\, dU\, dB_u. \tag{4.12}$$

The Leibnitz rule implies

$$\frac{\partial}{\partial \gamma} \int_{-\infty}^{\gamma} (S_T - S_u)\, dU = (S_T - S_u) f_{B_u, U}(B_u, \gamma). \tag{4.13}$$

We now interchange differentiation and integration and use (4.12) together with (4.13) to get

$$
\begin{aligned}
\frac{\partial}{\partial \gamma} \int_0^T \mathbb{E}^Q\left(S_T - S_u, U < \gamma\right) du &= \frac{\partial}{\partial \gamma} \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\gamma} (S_T - S_u)\, dU\, dB_t\, dt \\
&= \int_0^T \int_{-\infty}^{\infty} \frac{\partial}{\partial \gamma} \int_{-\infty}^{\gamma} (S_T - S_u)\, dU\, dB_u\, du \\
&= \int_0^T \int_{-\infty}^{\infty} (S_T - S_u) f_{B_u, U}(B_u, \gamma)\, dB_u\, du \\
&= \int_0^T \int_{-\infty}^{\infty} (S_T - S_u) f_{B_u, U}(B_u, \gamma) \frac{f_U(\gamma)}{f_U(\gamma)}\, dB_u\, du \\
&= \int_0^T \int_{-\infty}^{\infty} (S_T - S_u) f_{B_u, U}(B_u | \gamma) f_U(\gamma)\, dB_u\, du \\
&= \int_0^T \mathbb{E}^Q(S_T - S_u | U = \gamma) f_U(\gamma)\, du. \qquad \square
\end{aligned}
$$

Therefore if we let

$$U = \frac{1}{T} \int_0^T W_u(w)\, du - W_T$$

and apply Proposition 4.3 to (4.11), it becomes

$$\mathbb{E}^Q(S_T | U = \gamma^*) = \frac{1}{T} \int_0^T \mathbb{E}^Q(S_u | U = \gamma^*)\, du. \tag{4.14}$$

Note:

$$
\begin{aligned}
\text{Cov}\left(W_t, \frac{1}{T}\int_0^T W_u\,du - W_T\right) &= \mathbb{E}^Q\left(W_t\frac{1}{T}\int_0^T W_u\,du - W_tW_T\right) \\
&= \frac{1}{T}\mathbb{E}^Q\left(W_t\int_0^T W_u\,du\right) - \mathbb{E}^Q(W_tW_T) \\
&= t\left(1 - \frac{t}{2T}\right) - t\wedge T \\
&= t\left(1 - \frac{t}{2T}\right) - t \\
&= -\frac{t^2}{2T}
\end{aligned}
\tag{4.15}
$$

and

$$
\begin{aligned}
\text{Var}\left(\frac{1}{T}\int_0^T W_u\,du - W_T\right) &= \text{Var}\left(\frac{1}{T}\int_0^T W_u\,du\right) + \text{Var}(W_T) \\
&\quad - 2\text{Cov}\left(W_T, \frac{1}{T}\int_0^T W_u\,du,\right) \\
&= \frac{T}{3} + T - 2T\left(1 - \frac{T}{2T}\right), \quad \text{from (4.4) and (4.6)} \\
&= \frac{T}{3}.
\end{aligned}
$$

Again evoking Proposition 1.10, we can write

$$
\mathbb{E}^Q\left(W_t \,\middle|\, \frac{1}{T}\int_0^T W_u\,du - W_T = z\right) = -\frac{3t^2}{2T^2}z.
$$

$$
\text{Var}\left(W_t \,\middle|\, \frac{1}{T}\int_0^T W_u\,du - W_T = z\right) = t - \frac{3t^4}{4T^3}.
$$

Using the mgf of a normal random variable, the conditional distribution of the price S_u

takes the form

$$
\begin{aligned}
\mathbb{E}^Q(S_u|U=y) &= \mathbb{E}^Q(S_0 e^{(r-\frac{\sigma^2}{2})u+\sigma W_u}|U=y) \\
&= S_0 e^{(r-\frac{\sigma^2}{2})u}\mathbb{E}^Q(e^{\sigma W_u}|U=y) \\
&= S_0 e^{(r-\frac{\sigma^2}{2})u}\exp\left\{\sigma\mathbb{E}^Q(W_u|U=z)+\frac{\sigma^2}{2}\mathrm{Var}(W_u|U=z)\right\} \\
&= S_0 e^{(r-\frac{\sigma^2}{2})u}\exp\left\{-\sigma\frac{3u^2}{2T^2}z+\frac{\sigma^2}{2}\left(u-\frac{3u^4}{4T^3}\right)\right\} \\
&= S_0 \exp\left\{ru-\sigma\frac{3u^2}{2T^2}z-\sigma^2\frac{3u^4}{8T^3}\right\}.
\end{aligned}
$$

The equation (4.14) is now written as

$$
S_0\exp\left\{rT-\frac{3}{2}\sigma\gamma^*-\frac{3}{8}\sigma^2 T\right\}=\frac{1}{T}\int_0^T S_0\exp\left\{ru-\sigma\frac{3u^2}{2T^2}\gamma^*-\sigma^2\frac{3u^4}{8T^3}\right\}\,du. \tag{4.16}
$$

To derive the lower bound, we make use of the following result:

Proposition 4.4. *If $X\sim N(\mu_x,\sigma_x^2)$ and $Y\sim N(\mu_y,\sigma_y^2)$ and $c=\mathrm{Cov}(X,Y)$ then*

$$
\mathbb{E}^Q(e^X\mathbb{1}_{\{Y<0\}})=e^{\mu_x+\frac{\sigma_x^2}{2}}\Phi\left(-\frac{\mu_y+c}{\sigma_y}\right).
$$

We omit the proof which we get by mimicking the steps of the proof in [15]. Finally we determine $\check{C}_{b,0}$. Applying Proposition 4.4 we get

$$
\begin{aligned}
\check{C}_{b,0} &= \frac{e^{-rT}}{T}\int_0^T \mathbb{E}^Q[(S_T-S_u)\mathbb{1}_{\mathcal{B}}]\,du. \\
&= \frac{e^{-rT}}{T}\int_0^T \mathbb{E}^Q(e^{\ln S_T}\mathbb{1}_{\mathcal{B}}-e^{\ln S_u}\mathbb{1}_{\mathcal{B}})\,du \\
&= \frac{e^{-rT}}{T}\int_0^T\left\{S_0 e^{rT}\Phi\left(\frac{\gamma^*+\sigma\frac{T}{2}}{\sqrt{T/3}}\right)-S_0 e^{ru}\Phi\left(\frac{\gamma^*+\sigma\frac{u^2}{2T}}{\sqrt{T/3}}\right)\right\}\,du \\
&= \frac{S_0}{T}\int_0^T\left\{\Phi\left(\frac{\gamma^*+\sigma\frac{T}{2}}{\sqrt{T/3}}\right)-e^{-r(T-u)}\Phi\left(\frac{\gamma^*+\sigma\frac{u^2}{2T}}{\sqrt{T/3}}\right)\right\}\,du. \tag{4.17}
\end{aligned}
$$

4.3   Computational aspects

The worrisome thing about our formulas (4.8) and (4.9) is that since $\Phi(.)$ is the standard normal distribution function, an integral itself, we are essentially in a scenario where we have to do some double integration in both (4.8) and (4.9). In most computing software, there are predefined functions to perform such tasks. They are based on quadrature methods (Atkinson [3]). These can be used to perform definite integrals. The problem that arises when we try these built-in functions is that in (4.9) the argument of $\Phi(.)$ is a variable. Had it been a constant then we would use these built-in functions.

The workaround for this problem is to do numerical integration. We can do this by any suitable numerical integration techniques (Kincaid [16]). We have used the trapezoidal rule in our case. To get the optimal γ^*, we 'shoot' to get K. This is done by taking guesses of γ^* or more efficiently by taking a list of them then doing the numerical integration. After that we plot the list of γ^*s against the corresponding values of the integral, i.e., left-handside of (4.8). From that graph we find the γ^* that corresponds to K.

Once we have this γ^* we need to define a new function **phi** to implement the integration $\Phi(.)$ again by the trapezoidal rule. Then we define one more function **Intphi**, where **phi** will be called. The new function **Intphi** will do the Trapezoidal rule over $[0, T]$. Note that from the definition of $\Phi(x)$, the integration is from $-\infty$ to x. In the computations, we have used -5 as being sufficient to serve as $-\infty$.

5. Partial Differential Equation Approach

The partial differential approach is one of the commonly used methods to price Asian options. It has been used by Alziary et al. [1], Benhamou [8], Ingerson [34], Vecer [58] and Zvan et al. [62], Foufas and Larson [24], to mention but a few.

We have seen how in some cases we are able to explicitly describe the price of an Asian option by making use of the distribution function. In this section, we focus on other payoff structures. The determination of an explicit formula for the price of this option is not easy. The reason is that the distribution function of a sum of lognormal variables is not explicit (see Rogers and Shi [52]), it is a mixture of lognormal distributions [1]. Let us consider

$$C_{a,t} = \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u \, du - K \right)^+ \right]$$

and

$$C_{b,t} = \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(S_T - \frac{1}{T} \int_0^T S_u \, du \right)^+ \right].$$

From the price dynamics, we know that S_t is lognormally distributed (since $\log S_t$ is normally distributed). Hence it is clear from the above expressions that we have an integral of the lognormally distributed random variables which poses a problem. The distribution of this integral is not explicit (see also Stuart [57] for explanation). Thus we have to turn to other methods to find the price and the one that can resolve this problem (among the numerous methods) is the PDE approach.

5.1   Change of the Numeraire

We are going to find the price relative to the price of the underlying asset. To do this we must change our numeraire (see, e.g., [5] for more examples). At the same time we introduce a new measure, equivalent to Q. Now

$$C_{a,t} = e^{-r(T-t)}\mathbb{E}_t^Q\left[S_T\left(\frac{K - \frac{1}{T}\int_0^T S_u\, du}{S_T}\right)^-\right],$$

since $(-x)^- = \max\{x, 0\} = x^+$. Let

$$\phi_T = \frac{K - \frac{1}{T}\int_0^T S_u\, du}{S_T}, \tag{5.1}$$

and define the new measure Q^* through it's Radon-Nikodym derivative as

$$\frac{dQ^*}{dQ} = \frac{e^{-rT}S_T}{S_0}.$$

Then our price can be expressed equivalently as

$$C_{a,t} = e^{-r(T-t)}\frac{S_0}{e^{-rT}}\,\mathbb{E}_t^Q\left(\frac{e^{-rT}}{S_0}S_T\phi_T^-\right).$$

Alziary et al. [1] use the Radon Nikodym derivative

$$\frac{dQ^*}{dQ} = \frac{S_T}{\mathbb{E}^Q(S_T)}.$$

This is the same as our measure above since $\mathbb{E}^Q(S_T) = S_0 e^{rT}$.

Proposition 5.1. *Under the new measure Q^*, the price of the option is*

$$C_{a,t} = S_t\,\mathbb{E}_t^{Q^*}(\phi_T^-).$$

Proof. We apply the conditional Bayes Theorem (Theorem 3.5):

$$\mathbb{E}^Q(Z|\mathcal{G})\mathbb{E}^{Q^*}(X|\mathcal{G}) = \mathbb{E}^Q(ZX|\mathcal{G}),$$

where $\mathcal{G} \subseteq \mathcal{F}_T$ and Z is the Radon Nikodym derivative, together with Proposition 1.3:

$$\mathbb{E}^Q(Z|\mathcal{F}_t) := \left(\frac{dQ^*}{dQ}\right)_{|\mathcal{F}_t} = \frac{(dQ^*)_{|\mathcal{F}_t}}{(dQ)_{|\mathcal{F}_t}}.$$

Then the expression for $C_{a,t}$ becomes

$$\begin{aligned}
C_{a,t} &= e^{-r(T-t)}\frac{S_0}{e^{-rT}}\,\mathbb{E}_t^Q\left(\frac{e^{-rT}}{S_0}S_T\phi_T^-\right) \\
&= e^{-r(T-t)}\frac{S_0}{e^{-rT}}\frac{\mathbb{E}_t^Q(e^{-rT}S_T)}{\mathbb{E}_t^Q(S_0)}\,\mathbb{E}_t^{Q^*}(\phi_T^-) \\
&= e^{-rT}\frac{S_0}{e^{-rT}}\frac{\mathbb{E}_t^Q\left(e^{-r(T-t)}S_T\right)}{S_0}\,\mathbb{E}_t^{Q^*}(\phi_T^-), \quad \text{but } S_T \text{ is a } Q\text{-martingale.} \\
&= S_t\,\mathbb{E}_t^{Q^*}(\phi_T^-). \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square
\end{aligned}$$

Remark 5.2. If we define $\tilde{C}_{a,t} := C_{a,t}/S_t$, then the relative price of the option to the price of the asset is

$$\tilde{C}_{a,t} = \mathbb{E}_t^{Q^*}(\phi_T^-). \tag{5.2}$$

By similarly defining

$$\varphi_T = \frac{S_T - \frac{1}{T}\int_0^T S_u\,du}{S_T} \tag{5.3}$$

and repeating the same process, the relative price for the floating strike option is found to be

$$\tilde{C}_{b,t} = \mathbb{E}_t^{Q^*}(\varphi_T^+). \tag{5.4}$$

Now the change of measure is justified; through it we are able to find the prices relative to S_t.

The two state PDE

We are going to determine a two state PDE whose solution is the value of the Asian option [53]. The method of PDE has been used in option valuation, see, e.g., [14, 60].

Now let y_t be such that $dy_t = S_t \, dt$ so

$$y_T = \int_0^t S_u \, du + \int_t^T S_u \, du \Rightarrow y_T = y_t + \int_t^T S_u \, du.$$

If we generalise our payoff to be a function $h(y_T)$, the value $v(t, x, y)$ of the option is

$$v(t, x_t, y_t) = \mathbb{E}_t^Q \left(e^{-r(T-t)} h \left(\int_0^T S_u \, du \right) \right). \tag{5.5}$$

Typically, $h(y_T) = (y_T - K)^+$, for a constant K. As we have already seen that

$$S_T = x_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}Z_t},$$

where $x_t = S_t$. Without loss of generality, we may drop the subscripts while keeping in mind that both x and y depend on t. Let us write down the undiscounted price

$$u(t, x, y) = \mathbb{E}_t^Q h(y_T),$$

so that $v(t, x, y) = e^{-r(T-t)} u(t, x, y)$. The time t is such that $0 \le t \le T$, $y \in \mathbb{R}$ and $x \ge 0$. Immediately we see that u is a Q-martingale: let $0 \le s \le t \le T$, then

$$\mathbb{E}_s^Q \left(u(t, x, y) \right) = \mathbb{E}_s^Q (\mathbb{E}_t^Q h(y_T)) = \mathbb{E}_s^Q h(y_T) = u(s, x, y),$$

The implication of this is that the dt terms in the differential form of u must be zero. Applying Itô's lemma to u we get

$$
\begin{aligned}
du(t, x, y) &= u_t dt + u_x dS_t + u_y dy_t + \frac{1}{2} u_{xx} (dS_t)^2 + \frac{1}{2} u_{yy} (dy_t)^2 \\
&= u_t dt + u_x (rS_t dt + \sigma S_t dW_t) + u_y S_t \, dt + \frac{1}{2} u_{xx} \sigma^2 S_t^2 dt + u_{yy} (S_t \, dt)^2 \\
&= (u_t + rx u_x + x u_y + \frac{1}{2} \sigma^2 x^2 u_{xx}) dt + \sigma x u_x dW_t.
\end{aligned}
$$

Equating the coefficient of dt to zero, we have

$$u_t + rx u_x + \frac{1}{2}\sigma^2 x^2 u_{xx} + x u_y = 0,$$

$$\text{subject to:} \quad u(T, x, y) = h(y), \quad x \ge 0, y \in \mathbb{R}, \tag{5.6}$$

$$u(t, 0, y) = h(y), \quad 0 \le t \le T, y \in \mathbb{R}.$$

Noting that $v(t, x, y) = e^{-r(T-t)}u(t, x, y)$, we can transcribe the PDE (5.6) using:

$$u_t = -re^{-r(T-t)}v + e^{-r(T-t)}v_t$$

$$u_x = e^{-r(T-t)}v_x, \quad u_{xx} = e^{-r(T-t)}v_{xx}$$

$$u_y = e^{-r(T-t)}v_y.$$

Finally, (5.6) becomes

$$-rv + v_t + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} + xv_y = 0,$$

$$\text{subject to:} \quad v(T, x, y) = h(y), \quad x \geq 0, y \in \mathbb{R}, \tag{5.7}$$

$$v(t, 0, y) = e^{-r(T-t)}h(y), \quad 0 \leq t \leq T, y \in \mathbb{R}.$$

5.2   Reduction to a PDE with one state variable

The Black-Scholes model falls into a broad family of models called Log-type models [35]. Under the Black-Scholes model the difference of the log of the final price S_T and log of S_t does not depend on either S_t or S_T. Following [35], we write the distribution as $\Theta(\frac{S_T}{S_t})d\log S_T = \Theta(\frac{S_T}{S_t})\frac{dS_T}{S_T}$, to show that the distribution is with respect to $\log S_T$.

Suppose $C(S_t, \frac{1}{T}\int_0^t S_u\,du - K, t)$ is the price of an Asian option. Then in the Black-Scholes price $C(S_t, \frac{1}{T}\int_0^t S_u\,du - K, t)$ is homogenous in S_t and $\frac{1}{T}\int_0^t S_u\,du - K$. For this reason, the PDE which we have found can be reduced Alziary [1], Wilmot [60], Zvan [62] to a one state PDE which is much easier to implement. This reduction can be generalised for a $(n+1)$ state PDE to a PDE with n state variables [8].

The processes ϕ_T and φ_T, defined by (5.1) and (5.3), should take the following forms for $T = t$ [1, 62]:

$$\phi_t = \frac{K - \frac{1}{T}\int_0^t S_u\,du}{S_t} \quad \text{and} \quad \varphi_t = \frac{S_T - \frac{1}{T}\int_0^t S_u\,du}{S_t}. \tag{5.8}$$

Lemma 5.3. *Under Q^* the dynamics of ϕ_t and φ_t are governed by the stochastic differential equations (SDEs)*

$$d\phi_t = \left(-\frac{1}{T} - r\phi_t\right)dt - \sigma\phi_t d\widetilde{W}_t, \quad \phi_0 = \phi, \tag{5.9}$$

$$d\varphi_t = \left(-\frac{1}{T} - r(\varphi_t - 1)\right)dt - \sigma(\varphi_t - 1)d\widetilde{W}_t, \quad \varphi_0 = \varphi, \tag{5.10}$$

respectively.

Proof. (i) Applying Itô's Lemma (Integration by parts [23]) to ϕ_t, we obtain

$$d\,\phi_t = \frac{1}{S_t}d\left(K - \frac{1}{T}\int_0^t S_u\,du\right) + \left(K - \frac{1}{T}\int_0^t S_u\,du\right)d\left(\frac{1}{S_t}\right)$$
$$+ d\left(K - \frac{1}{T}\int_0^t S_u\,du\right)d\left(\frac{1}{S_t}\right)$$
$$= -\frac{1}{S_t}\frac{1}{T}S_t\,dt + \left(K - \frac{1}{T}\int_0^t S_u\,du\right)d\left(\frac{1}{S_t}\right) + \left(-\frac{1}{T}S_t dt\right)d\left(\frac{1}{S_t}\right). \tag{5.11}$$

But by Itô's lemma

$$d\left(\frac{1}{S_t}\right) = -\frac{1}{S_t^2}dS_t + \frac{1}{2}\frac{2}{S_t^3}(dS_t)^2$$
$$= -\frac{1}{S_t^2}(rS_t dt + \sigma S_t dW_t) + \frac{1}{S_t^3}\sigma^2 S_t^2 dt$$
$$= \frac{1}{S_t}\left((\sigma^2 - r)dt - \sigma dW_t\right).$$

Clearly,

$$\left(-\frac{1}{T}S_t dt\right)d\left(\frac{1}{S_t}\right) = \left(-\frac{1}{T}S_t dt\right)\left(\frac{1}{S_t}\left((\sigma^2 - r)dt - \sigma dW_t\right)\right) = 0,$$

since $dtdW_t = 0$, $dtdt = 0$, (for derivation see, e.g., Etheridge [23]). Then (5.11) becomes

$$d\phi_t = -\frac{1}{T}dt + \left(K - \frac{1}{T}\int_0^t S_u\,du\right)\left(\frac{1}{S_t}\left((\sigma^2 - r)dt - \sigma dW_t\right)\right)$$
$$= -\frac{1}{T}dt + \phi_t\left((\sigma^2 - r)dt - \sigma dW_t\right)$$
$$= \left(-\frac{1}{T} - r\phi_t\right)dt - \sigma\phi_t(dW_t - \sigma dt). \tag{5.12}$$

Now we see why particularly we chose our Radon Nikodym derivative to be $\frac{dQ^*}{dQ} = \frac{e^{-rT} S_T}{S_0}$ and not any other. This is because $\frac{e^{-rT} S_T}{S_0}$ can be written as

$$e^{-rT} \frac{S_0}{S_0} e^{(r-\frac{\sigma^2}{2})T + \sigma W_T} = e^{-\frac{\sigma^2}{2}T + \sigma W_T},$$

and then

$$Q^*(A) = \int_A e^{-\frac{\sigma^2}{2}T + \sigma W_T} dQ, \quad \text{for } A \in \mathcal{F}_T.$$

The Girsanov Theorem [23, 33] ensures that under Q^*, $\widetilde{W}_t = W_t - \sigma t$ is a Brownian Motion. The equation (5.12) becomes

$$d\phi_t = \left(-\frac{1}{T} - r\phi_t\right) dt - \sigma \phi_t d\widetilde{W}_t. \tag{5.13}$$

(ii) We proceed in a similar manner.

$$
\begin{aligned}
d\varphi_t &= \frac{1}{S_t} d\left(S_t - \frac{1}{T}\int_0^t S_u\,du\right) + \left(S_t - \frac{1}{T}\int_0^t S_u\,du\right) d\left(\frac{1}{S_t}\right) \\
&\quad + d\left(S_t - \frac{1}{T}\int_0^t S_u\,du\right) d\left(\frac{1}{S_t}\right) \\
&= \frac{1}{S_t}\left(dS_t - \frac{1}{T}S_t\,dt\right) + \left(S_t - \frac{1}{T}\int_0^t S_u\,du\right)\left(\frac{1}{S_t}\left((\sigma^2 - r)dt - \sigma dW_t\right)\right) \\
&\quad + \left(dS_t - \frac{1}{T}S_t dt\right)\left(\frac{1}{S_t}\left((\sigma^2 - r)dt - \sigma dW_t\right)\right) \\
&= \frac{1}{S_t}\left(rS_t dt + \sigma S_t dW_t - \frac{1}{T}S_t\,dt\right) + \varphi_t\left((\sigma^2 - r)dt - \sigma dW_t\right) \\
&\quad + \left(rS_t dt + \sigma S_t dW_t - \frac{1}{T}S_t dt\right)\left(\frac{1}{S_t}\left((\sigma^2 - r)dt - \sigma dW_t\right)\right) \\
&= (r - \frac{1}{T})dt + \sigma dW_t + \varphi_t\left((\sigma^2 - r)dt - \sigma dW_t\right) - \sigma^2 dt \\
&= (-\frac{1}{T} - r(\varphi_t - 1))dt - \sigma\varphi_t(dW_t - \sigma dt) + \sigma(dW_t - \sigma dt) \\
&= (-\frac{1}{T} - r(\varphi_t - 1))dt - \sigma(\varphi_t - 1)d\widetilde{W}_t.
\end{aligned}
$$

This completes the proof. $\square$

5.3   Analytical solution of SDEs

We would like to solve the SDEs (5.9) and (5.10). We will use the Itô Lemma ([47]) to confirm the analytic solution of (5.9). Sometimes it is easy to realise the analytic solution of an SDE by relating it to known ones [6, 38]. We apply that technique here to get the solution of (5.9).

Proposition 5.4. *The solution of* (5.9) *is given by*

$$\phi_t = \phi_0 e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t} - \frac{1}{T}\int_0^t e^{(-\frac{\sigma^2}{2}-r)(t-s)-\sigma(W_t-W_s)}ds. \tag{5.14}$$

Proof. If we split the integrand of ϕ_t (taking out what is s-independent) and find the differential of ϕ_t, then

$$d\phi_t = \phi_0 d\left(e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\right) - \frac{1}{T}d\left(e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\int_0^t e^{(\frac{\sigma^2}{2}+r)s+\sigma W_s}ds\right). \tag{5.15}$$

Now by Itô's Lemma

$$\begin{aligned}
d\left(e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\right) &= -\sigma e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}dW_t - \left(\frac{\sigma^2}{2}+r\right)e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}dt \\
&\quad + \frac{\sigma^2}{2}e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}(dW_t)^2 \\
&= e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\left(-\sigma dW_t - \left(\frac{\sigma^2}{2}+r\right)dt + \frac{\sigma^2}{2}dt\right) \\
&= e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\left(-rdt - \sigma dW_t\right).
\end{aligned} \tag{5.16}$$

We are going to make use of the Itô product rule [48] (Integration by parts):

$$d(Y_t Z_t) = Y_t\,dZ_t + Z_t\,dY_t + dY_t\,dZ_t.$$

Therefore

$$\begin{aligned}
d\left(e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\int_0^t e^{(\frac{\sigma^2}{2}+r)s+\sigma W_s}ds\right) &= d\left(e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\right)\int_0^t e^{(\frac{\sigma^2}{2}+r)s+\sigma W_s}ds \\
&\quad + e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}d\left(\int_0^t e^{(\frac{\sigma^2}{2}+r)s+\sigma W_s}ds\right) \\
+ d\left(e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\right)&d\left(\int_0^t e^{(\frac{\sigma^2}{2}+r)s+\sigma W_s}ds\right).
\end{aligned} \tag{5.17}$$

Clearly, using (5.16)

$$d\left(e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\right)d\left(\int_0^t e^{(\frac{\sigma^2}{2}+r)s+\sigma W_s}ds\right) = -e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\left(rdt+\sigma dW_t\right)\cdot$$

$$e^{(\frac{\sigma^2}{2}+r)t+\sigma W_t}dt$$

$$= 0,$$

since $(dt)^2 = 0$ and $dtdW_t = 0$. Consequently, (5.17) becomes,

$$d\left(e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\int_0^t e^{(\frac{\sigma^2}{2}+r)s+\sigma W_s}ds\right) = e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\left(-rdt-\sigma dW_t\right)\cdot$$

$$\int_0^t e^{(\frac{\sigma^2}{2}+r)s+\sigma W_s}ds + e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}\cdot$$

$$e^{(\frac{\sigma^2}{2}+r)t+\sigma W_t}dt$$

$$= (-rdt - \sigma dW_t)\int_0^t e^{(-\frac{\sigma^2}{2}-r)(t-s)-\sigma(W_t-W_s)}ds$$

$$+ dt. \tag{5.18}$$

Finally combining (5.18) and (5.16) in (5.15), we get

$$d\phi_t = \phi_0\left(e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t}(-rdt-\sigma dW_t)\right)$$

$$- \frac{1}{T}(-rdt-\sigma dW_t)\int_0^t e^{(-\frac{\sigma^2}{2}-r)(t-s)-\sigma(W_t-W_s)}ds - \frac{dt}{T}$$

$$= (-rdt-\sigma dW_t)\left(\phi_0 e^{-(\frac{\sigma^2}{2}+r)t-\sigma W_t} - \frac{1}{T}\int_0^t e^{(-\frac{\sigma^2}{2}-r)(t-s)-\sigma(W_t-W_s)}ds\right) - \frac{dt}{T}$$

$$= (-rdt-\sigma dW_t)\phi_t - \frac{dt}{T}$$

$$= \left(-\frac{1}{T}-r\phi_t\right)dt - \sigma\phi_t dW_t. \qquad \square$$

We will visualize this analytical solution at a later stage in the thesis.

5.4   Numerical Solutions of SDEs

Even if we had an analytic solution of (5.9), at a glance it does not seem very revealing. We need to see its path to realise its significance. More often than not, we get SDEs that do not have analytic solutions. In these scenarios we use numerical methods to approximate the solutions (see, e.g., Higham [30] for more cases). For the sake of simplicity, we will explore a simple method, the Euler method (Kloeden [38]). Other methods can be used for example, Milstein's method, Euler-Maruyama's method, etc. [30, 38].

Suppose we have an SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t,$$

where as usual W_t is a Brownian motion. To begin with, we partition the interval $[0, T]$ into n equal parts. Let $h = T/n$ so that $t_j = jh$ for $j = 0, 1, \cdots, n$. The discretization of the SDE by the Euler method then reads as

$$X_{jh} = X_{(j-1)h} + \mu(X_{(j-1)h}, (j-1)h)h + \sigma(X_{(j-1)h}, (j-1)h)\sqrt{h}\,Z,$$

where $Z \sim N(0, 1)$. Consequently (5.9) and (5.10), respectively, take the forms:

$$\phi_{jh} = \phi_{(j-1)h} + \left(-\frac{1}{T} - r\phi_{(j-1)h}\right)h - \sigma\phi_{(j-1)h}\sqrt{h}\,Z, \tag{5.19}$$

$$\varphi_{jh} = \varphi_{(j-1)h} + \left(-\frac{1}{T} - r(\varphi_{(j-1)h} - 1)\right)h - \sigma(\varphi_{(j-1)h} - 1)\sqrt{h}\,Z, \tag{5.20}$$

The pseudo code can be written as

Algorithm 5.4.1: Euler Scheme(n)

$\phi \leftarrow \phi_0$

for j **from** 1 **to** n

$\quad$ **do** $\begin{cases} \text{generate} \quad Z \sim N(0, 1) \\ \phi \leftarrow \phi + \left(-\frac{1}{T} - r\phi\right)h - \sigma\phi\sqrt{h}\,Z. \end{cases}$

The scheme for φ_t can also be written immediately.

The variables ϕ_t and φ_t indicate whether the call option is in the money, at the money or out of the money. When the call is in the money then the strike price is less than the price of the underlying asset and it is at the money if the strike price equals the price of the underlying asset. If the strike is more than the price of the underlying then the call is said to be out of the money.

In the case of Asian options the price at time t is the sum of prices up to time t. If we were considering an American Asian option, in the money entails that the option would be exercised as one would not wait for the maturity to exercise. If at time t, $\phi \leq 0$ then the option is in the money, when $\phi = 0$ the option is at the money and when $\phi \geq 0$ it is out of the money. We will consider these two cases. We will simulate the paths for ϕ_t by the Euler method.

The Figure 5.1 shows 100 simulated paths for ϕ_t. Subfigure 5.1(a) shows that if we start at the money then the option will surely be exercised at time T. Again, if the option is in the money then in its future it can never be out of the money or at the money.

For floating strikes (shown by the paths of φ), we cannot make these conclusions as in the case of fixed strike (shown by the paths of ϕ). Figure 5.2 shows we can start at the money and end in the money or at the money or out of the money.

Theorem 5.5. *$\tilde{C}_{a,t}$ and $\tilde{C}_{b,t}$ are solutions of the following partial differential equations:*

$$\frac{\partial \tilde{C}_{a,t}}{\partial t} + \left(-\frac{1}{T} - r\phi_t \right) \frac{\partial \tilde{C}_{a,t}}{\partial \phi_t} + \frac{1}{2}\sigma^2 \phi_t^2 \frac{\partial^2 \tilde{C}_{a,t}}{\partial \phi_t^2} = 0,$$

$$subject\ to \tag{5.21}$$

$$\tilde{C}_{a,T} = \phi_T^-.$$
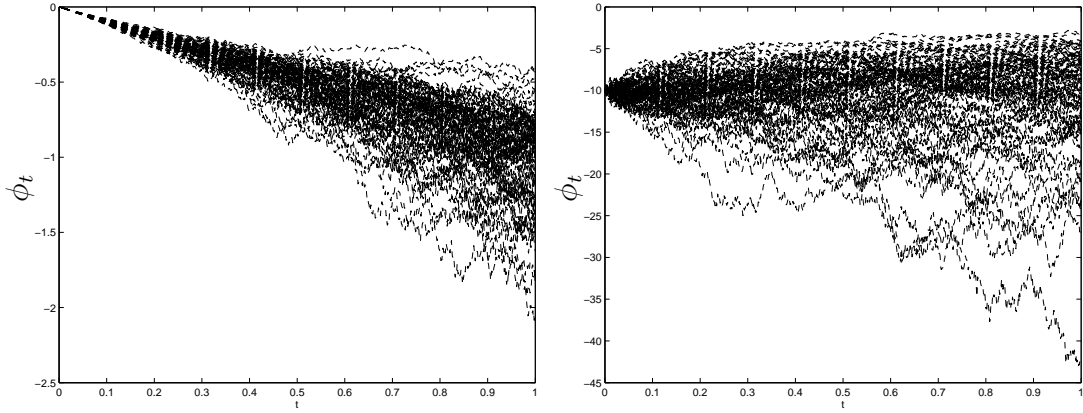
(a) At the money   (b) In the money

Figure 5.1: The ϕ paths using Euler scheme
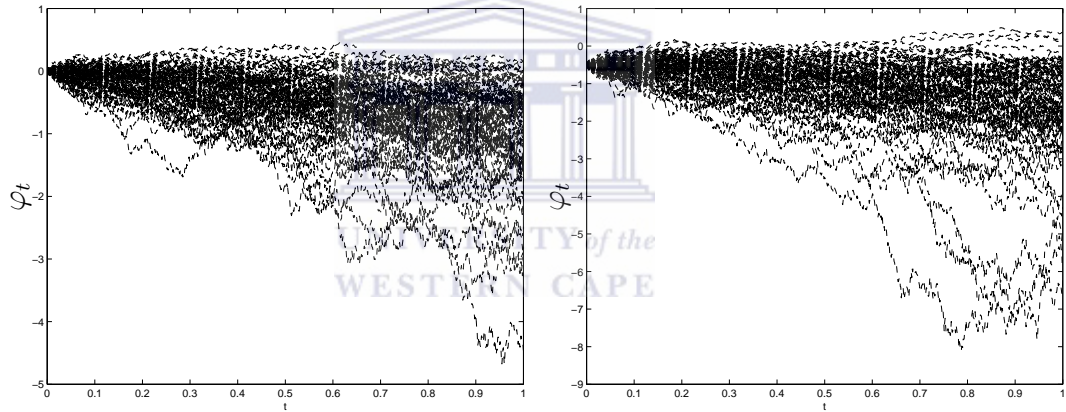
(a) At the money   (b) In the money

Figure 5.2: The φ paths using Euler scheme

$$\frac{\partial \tilde{C}_{b,t}}{\partial t} + \left(-\frac{1}{T} - r(\varphi_t - 1) \right) \frac{\partial \tilde{C}_{b,t}}{\partial \varphi_t} + \frac{1}{2}\sigma^2(\varphi_t - 1)^2 \frac{\partial^2 \tilde{C}_{b,t}}{\partial \varphi_t^2} = 0,$$

$$\text{subject to} \tag{5.22}$$

$$\tilde{C}_{b,T} = \varphi_T^+.$$

Proof. From Lemma 5.3, we have the dynamics of ϕ_t and φ_t. The proof follows from the Feynman-Kac Theorem, which implies

$$\tilde{C}_{a,t} = \mathbb{E}_t^{Q^*}(\phi_T^-).$$

The second result follows similarly. $\square$

5.5   Comparisons with European Options

We are interested in how our change of numeraire in Section 5.1 and our partial differential equation can be used to find the prices of European options. Although the Black-Scholes PDE [32] already exists, we would want to check if our change of measure is consistent. Our motivation is that there is already a formula to characterise the value of a European Option [35]. We derive the analogy of the partial differential equations which we found in Theorem 5.5, but for European options.

We show how $C_{e,t}$ can be written as an expectation with respect to the measure Q^*, which is defined as

$$Q^*(A) = \int_A e^{-\frac{1}{2}\sigma^2 T + \sigma W_T}\, dQ, \quad \text{for } A \in \mathcal{F}_T,$$

as before. The value of a European Call is given by

$$
\begin{aligned}
C_{e,t} &= \mathbb{E}_t^Q \left(e^{-r(T-t)}(S_T - K)^+ \right) \\
&= e^{-r(T-t)} \mathbb{E}_t^Q \left(S_T \left(1 - \frac{K}{S_T} \right)^+ \right) \\
&= e^{-r(T-t)} \frac{S_0}{e^{-rT}} \mathbb{E}_t^Q \left(\frac{e^{-rT} S_T}{S_0}(1 - \psi_T)^+ \right), \qquad \text{by definition of } \psi_t \\
&= e^{-r(T-t)} \frac{S_0}{e^{-rT}} \mathbb{E}_t^Q \left(\frac{e^{-rT} S_T}{S_0} \right) \mathbb{E}_t^{Q^*}\!\left((1 - \psi_T)^+ \right), \qquad \text{by Bayes Theorem} \\
&= e^{-rT} \frac{S_0}{e^{-rT}} \mathbb{E}_t^Q \left(\frac{e^{-r(T-t)} S_T}{S_0} \right) \mathbb{E}_t^{Q^*}\!\left((1 - \psi_T)^+ \right) \\
&= S_0 \frac{\mathbb{E}_t^Q (e^{-r(T-t)} S_T)}{\mathbb{E}_t^Q (S_0)} \mathbb{E}_t^{Q^*}\!\left((1 - \psi_T)^+ \right), \qquad \text{by Proposition 1.3} \\
&= S_0 \frac{S_t}{S_0} \mathbb{E}_t^{Q^*}\!\left((1 - \psi_T)^+ \right), \qquad \text{since } S_T \text{ is } Q\text{-martingale} \\
&= S_t \, \mathbb{E}_t^{Q^*}\!\left((1 - \psi_T)^+ \right).
\end{aligned}
\tag{5.23}
$$

Under the new measure Q^*, $\tilde{W}_t = W_t - \sigma t$ is a Brownian Motion by Girsanov Theorem. We can write (5.23) as

$$\tilde{C}_{e,t} = \frac{C_{e,t}}{S_t} = \mathbb{E}_t^{Q^*}\left((1 - \psi_T)^+\right).$$

The dynamics of ψ_t are given by

$$\begin{aligned}
d\psi_t &= Kd\left(\frac{1}{S_t}\right) \\
&= \frac{K}{S_t}\left((\sigma^2 - r)dt - \sigma dW_t\right), \qquad \text{from proof of Lemma 5.3} \\
&= -r\psi_t\, dt - \sigma\psi_t(dW_t - \sigma dt), \qquad \text{definition of } \psi_t \\
&= -r\psi_t\, dt - \sigma\psi_t\, d\tilde{W}_t.
\end{aligned} \tag{5.24}$$

Proposition 5.6. *Denote the value of the European call by $C_{e,t}$ and let us write*

$$\tilde{C}_{e,t} = \frac{C_{e,t}(\psi, t)}{S_t},$$

where $\psi_t = \frac{K}{S_t}$. Then $\tilde{C}_{e,t}$ satisfies

$$\frac{\partial \tilde{C}_{e,t}}{\partial t} - r\psi\frac{\partial \tilde{C}_{e,t}}{\partial \psi} + \frac{1}{2}\sigma^2\psi^2\frac{\partial^2 \tilde{C}_{e,t}}{\partial \psi^2} = 0,$$

$$\text{subject to:} \quad \tilde{C}_{e,T} = (1 - \psi_T)^+. \tag{5.25}$$

Proof. The direct application of the Feynman-Kac Theorem implies

$$\tilde{C}_{e,t} = \mathbb{E}_t^{Q^*}\left((1 - \psi_T)^+\right). \qquad \square$$

Just as we did in Chapter 2 we can evaluate this expectation. It is important to note that unlike the SDEs of ϕ_t and φ_t, the dynamics of ψ_t is clearly geometric Brownian motion, very much like the dynamics of S_t.

Proposition 5.7. *The solution of* (5.25) *is given by*

$$\tilde{C}_{e,t}(\psi_t, t) = N(d_1) - \psi_t e^{-r(T-t)}N(d_1 - \sigma\sqrt{T-t}), \tag{5.26}$$

where

$$d_1 = \frac{\log\frac{1}{\psi_t} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

Proof. Applying the Itô Lemma to (5.24) we get

$$\psi_T = \psi_t e^{-(r+\frac{1}{2}\sigma^2)(T-t)-\sigma\tilde{W}_{T-t}}.$$

This implies

$$\log \psi_T \sim N\left(\log \psi_t - (r + \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)\right).$$

Now

$$\tilde{C}_{e,0} = \mathbb{E}^{Q^*}\left((1 - \psi_T)^+\right)$$

$$= \int_{-\infty}^{0} (1 - e^{\log \psi_T}) \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{(\log \psi_T - \log \psi_t + (r+\frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)}} d\log \psi_T$$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{(\log \psi_T - \log \psi_t + (r+\frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)}} d\log \psi_T$$

$$- \int_{-\infty}^{0} e^{\log \psi_T} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{(\log \psi_T - \log \psi_t + (r+\frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)}} d\log \psi_T.$$

Now let

$$x = \log \frac{\psi_T}{\psi_t}.$$

Then $dx = d\log \psi_T$ and our calculation becomes

$$\tilde{C}_{e,0} = \int_{-\infty}^{-\log \psi_t} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{(x+(r+\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}} dx$$

$$- \int_{-\infty}^{-\log \psi_t} \psi_t e^x \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{(x+(r+\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}} dx$$

$$= \int_{-\infty}^{-\log \psi_t} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{(x+(r+\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}} dx$$

$$- \psi_t e^{-r(T-t)} \int_{-\infty}^{-\log \psi_t} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} e^{-\frac{(x+(r-\frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}} dx$$

$$= N(d_1) - \psi_t e^{-r(T-t)} N(d_2), \tag{5.27}$$

where

$$d_1 = \frac{-\log \psi_t + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$
\begin{aligned}
d_2 &= \frac{-\log \psi_t + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \\
&= \frac{-\log \psi_t + (r + \frac{1}{2}\sigma^2 - \sigma^2)(T - t)}{\sigma\sqrt{T - t}} \\
&= d_1 - \sigma\sqrt{T - t}.
\end{aligned}
$$

Finally we have

$$
\tilde{C}_{e,t}(\psi_t, t) = N(d_1) - \psi_t e^{-r(T-t)} N(d_1 - \sigma\sqrt{T - t}). \qquad \square
$$

Remark 5.8. $\tilde{C}_{e,t}$ is the Black-Scholes formula [9]. Our derivation is consistent with the formula due to Black and Scholes.

Part II

Numerical Methods for pricing

Asian options

6. Monte Carlo Method

The Monte Carlo method is a traditional method of pricing options Boyle [10], et al. [17] etc. It is based on simulating many paths according to the underlying assumptions either deterministically (Corwin et al. [17], Lamieux and L'Ecuyer [40]) or by pseudo-random numbers [10]. In this thesis, only the use of the pseudo-random numbers is considered. We assume that the price of the underlying follows the geometric Brownian motion. In order to get the price of the option we simply generate many paths and take an average of them.

6.1   The General Monte Carlo Method

Consider the problem of evaluating

$$C_{a,t} = e^{-r(T-t)} \mathbb{E}_t^Q \left[\left(\frac{1}{T} \int_0^T S_u \, du - K \right)^+ \right].$$

Let us estimate $C_{a,t}$ by $\hat{C}_{a,t}$. To determine $\hat{C}_{a,t}$, we simulate m paths of S_u where $S_u = S_0 e^{(r-\frac{1}{2}\sigma^2)u + \sigma\sqrt{u}Z}$, for which $Z \sim N(0,1)$. It is assumed that the price of the underlying follows a geometric Brownian motion. We partition $[0, T]$ into n equal parts so that $t_i = i\Delta t$ for $i = 0, 1, 2, \cdots, n$. Consequently $\Delta t = T/n$. Also, we approximate the integral by its Riemann sum that is

$$\int_0^T S_u \, du \approx \Delta t \sum_{i=1}^n S_{t_i}.$$

Consequently,

$$\frac{1}{T} \int_0^T S_u \, du \approx \frac{1}{n} \sum_{i=1}^n S_{t_i},$$

where

$$S_{t_i} = S_0 e^{(r-\frac{1}{2}\sigma^2)i\Delta t + \sigma\sqrt{\Delta t}Z}.$$

The following algorithm [31] can be used to calculate $\hat{C}_{a,t}$

Algorithm 6.1.1: General Monte Carlo(m, n)

$\Delta t \leftarrow T/n$

for j **from** 1 **to** m

$\quad$**do** $\begin{cases} \textbf{for } i \textbf{ from } 1 \textbf{ to } n \\ \quad \textbf{do} \begin{cases} \text{generate} \quad Z \sim N(0,1) \\ S_{t_i} \leftarrow S_0 e^{(r-\frac{1}{2}\sigma^2)i\Delta t + \sigma\sqrt{\Delta t}Z} \end{cases} \\ \omega_j \leftarrow \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \\ C_j \leftarrow e^{-rT}\max\left(\omega_j - K, 0\right) \end{cases}$

$\hat{C}_{a,t} \leftarrow \frac{1}{m}\sum_{j=1}^{m} C_j$

To implement the algorithm, we generate m paths. From the algorithm we estimate $C_{a,t}$ by $\hat{C}_{a,t}$ which is the mean:

$$\hat{C}_{a,t} = \frac{1}{m}\sum_{j=1}^{m} C_j.$$

Clearly $\mathbb{E}(\hat{C}_{a,t}) = C_{a,t}$. Such estimates are called *unbiased estimates*. Also the variance of $\hat{C}_{a,t}$ is $\mathrm{Var}(\hat{C}_{a,t}) = \sigma^2/n$, where σ is the variance of $C_{a,t}$. Since σ is not known, we approximate it by $\hat{\sigma}$ given by

$$\hat{\sigma}^2 = \frac{1}{m-1}\sum_{j=1}^{m}(C_j - \hat{C}_{a,t})^2.$$

Again it can be shown that $\hat{\sigma}$ is an unbiased estimate of σ. The central limit theorem tells us that

$$\frac{\hat{C}_{a,t} - C_{a,t}}{\sqrt{\frac{\hat{\sigma}^2}{m}}} \longrightarrow N(0,1). \tag{6.1}$$

The difference $\hat{C}_{a,t} - C_{a,t}$ is the Monte Carlo error. We denote the absolute value of this error, i.e., $E_{mc} := |\hat{C}_{a,t} - C_{a,t}|$ by E_{mc}. We see immediately from (6.1) that the Monte Carlo error is proportional to $\frac{\hat{\sigma}}{\sqrt{m}}$, that is

$$E_{mc} \propto \frac{\hat{\sigma}}{\sqrt{m}}.$$

This means we need 100 paths to reduce the error by 10. We could decrease the error by having many paths but this would then require more computational time and computer memory. However, by reducing $\hat{\sigma}$ we can minimise E_{mc}. This introduces the *variance reduction methods* [10, 50], two of which are described below.

6.2   Variance reduction using antithetic variates

The antithetic method uses the idea that if $Z_i \sim N(0,1)$, then $-Z_i \sim N(0,1)$. This method works by introducing negative correlation to counter the error introduced by using only Z_i in the calculation. In this method we simulate new paths using $-Z_i$ for the price process $\bar{S}_{t_i}$ given by

$$\bar{S}_{t_i} = S_0 e^{(r-\frac{1}{2}\sigma^2)i\Delta t - \sigma\sqrt{\Delta t}Z}.$$

We then apply Monte Carlo method to the average of

$$C_j = e^{-rT}\max\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K, 0\right)$$

and

$$\bar{C}_j = e^{-rT}\max\left(\frac{1}{n}\sum_{i=1}^{n} \bar{S}_{t_i} - K, 0\right)$$

which is

$$C_j^* = \frac{1}{2}(C_j + \bar{C}_j).$$

Then we estimate $C_{a,t}$ by $\check{C}_{a,t}$, that is

$$\check{C}_{a,t} = \frac{1}{m}\sum_{j=1}^{m} C_j^*.$$

The Algorithm 6.1.1 can accordingly be improved to the following algorithm

---

Algorithm 6.2.1: Antithetic Monte Carlo(m, n)

$\Delta t \leftarrow T/n$

for j **from** 1 **to** m

$\quad$ **do** $\begin{cases} \textbf{for } i \textbf{ from } 1 \textbf{ to } n \\ \qquad \textbf{do} \begin{cases} \text{generate} \quad Z \sim N(0,1) \\ S_{t_i} \leftarrow S_0 e^{(r-\frac{1}{2}\sigma^2)i\Delta t + \sigma\sqrt{\Delta t}Z} \\ \bar{S}_{t_i} \leftarrow S_0 e^{(r-\frac{1}{2}\sigma^2)i\Delta t - \sigma\sqrt{\Delta t}Z} \end{cases} \\ \omega_j \leftarrow \dfrac{1}{n}\sum_{i=1}^{n} S_{t_i} \\ \bar{\omega}_j \leftarrow \dfrac{1}{n}\sum_{i=1}^{n} \bar{S}_{t_i} \\ C_j \leftarrow e^{-rT}\max\left(\omega_j - K, 0\right) \\ \bar{C}_j \leftarrow e^{-rT}\max\left(\bar{\omega}_j - K, 0\right) \\ C_j^* \leftarrow \dfrac{1}{2}(C_j + \bar{C}_j) \end{cases}$

$\check{C}_{a,t} \leftarrow \dfrac{1}{m}\sum_{j=1}^{m} C_j^*$

The variance of C_j^* is given by

$$\mathrm{Var}(C_j^*) = \frac{1}{4}\left(\mathrm{Var}(C_j) + \mathrm{Var}(\bar{C}_j) + 2\mathrm{Cov}(C_j, \bar{C}_j)\right)$$
$$= \frac{1}{2}\left(\mathrm{Var}(C_j) + 2\mathrm{Cov}(C_j, \bar{C}_j)\right), \quad \text{since} \quad C_j \overset{\mathcal{D}}{=} \bar{C}_j$$
$$< \frac{1}{2}\mathrm{Var}(C_j)$$

if $\mathrm{Cov}(C_j, \bar{C}_j) < 0$. We have $\mathrm{Cov}(C_j, \bar{C}_j) < 0$ if C_j is monotonic increasing (consequently, $\bar{C}_j$ is monotonic decreasing).

6.3 Variance reduction using control variates

We can also reduce the variance by introducing a second random variable. Suppose we wanted to estimate the mean of the random variable X and that there was another random variable Y which mimics X or is close to X. This closeness means that when X is small then Y is also small. Suppose we also knew the mean of Y. Let Z be such that

$$Z := X + \mathbb{E}(Y) - Y. \tag{6.2}$$

Notice that $\mathbb{E}(Z) = \mathbb{E}(X)$. So we could use Z to estimate the mean of X. We call Y the *control variate*.

In the problem that we must solve, the random variable X is the arithmetic average payoff and Y is the geometric average payoff, that is

$$X = \left(\frac{1}{N} \sum_{i=1}^{N} S_{t_i} - K \right)^{+} \tag{6.3}$$

and

$$Y = \left(\left(\prod_{i=1}^{N} S_{t_i} \right)^{\frac{1}{N}} - K \right)^{+}.$$

As we already know the expectation of X is not known but we have an explicit formula for the expectation of Y. From Chapter 2 Section 2.4,

$$\mathbb{E}(Y) = \mathbb{E}^{Q}(e^{\ln Y})$$
$$= S_0 e^{(\tilde{r}-r)T} \Phi(\tilde{d}_1) - K e^{-rT} \Phi(\tilde{d}_2),$$

where $\Phi(.)$ is the cumulative normal distribution function and

$$\tilde{d}_1 = \frac{\log \frac{S_0}{K} + \left(\tilde{r} + \frac{\tilde{\sigma}^2}{2} \right) T}{\tilde{\sigma}\sqrt{T}},$$
$$\tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}\sqrt{T},$$

where

$$\tilde{\sigma}^2 := \sigma^2 \frac{(N+1)(2N+1)}{6N^2},$$

$$\tilde{r} = \frac{1}{2}\tilde{\sigma}^2 + \left(r - \frac{1}{2}\sigma^2\right)\frac{N+1}{2N}.$$

The improved Monte Carlo algorithm in this case would then be as shown in Algorithm 6.3.1

Algorithm 6.3.1: Control Variate Monte Carlo(m,n)

$\Delta t \leftarrow T/n$

for j **from** 1 **to** m

$\quad$**do** $\begin{cases} \textbf{for } i \textbf{ from } 1 \textbf{ to } n \\ \quad \textbf{do } \begin{cases} \text{generate} \quad Z \sim N(0,1) \\ S_{t_i} \leftarrow S_0 e^{(r-\frac{1}{2}\sigma^2)i\Delta t + \sigma\sqrt{\Delta t}Z} \end{cases} \\ \omega_j \leftarrow \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \\ \bar{\omega}_j \leftarrow \exp\left(\frac{1}{n}\sum_{i=1}^{n} \ln S_{t_i}\right) \\ C_j \leftarrow e^{-rT}\max\left(\omega_j - K, 0\right) \\ \bar{C}_j \leftarrow e^{-rT}\max\left(\bar{\omega}_j - K, 0\right) \end{cases}$

$\hat{C}_{a,t} \leftarrow \frac{1}{m}\sum_{j=1}^{m} C_j + S_0 e^{(\tilde{r}-r)T}\Phi(\tilde{d}_1) - Ke^{-rT}\Phi(\tilde{d}_2) - \frac{1}{m}\sum_{j=1}^{m} \bar{C}_j$

Now the variance of Z is given by

$$\text{Var}(Z) = \text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - \text{Cov}(X, Y)$$

$$< \text{Var}(X),$$

if $\frac{1}{2}\text{Var}(Y) < \text{Cov}(X, Y)$. This condition can be controlled during a Monte Carlo simulation.

Figure 6.1 shows the comparison of these two variance reduction methods. We observe that the control variate method performs better than the antithetic technique to price Asian options. The price via antithetic approach keeps oscillating around the price obtained by the control variate technique.
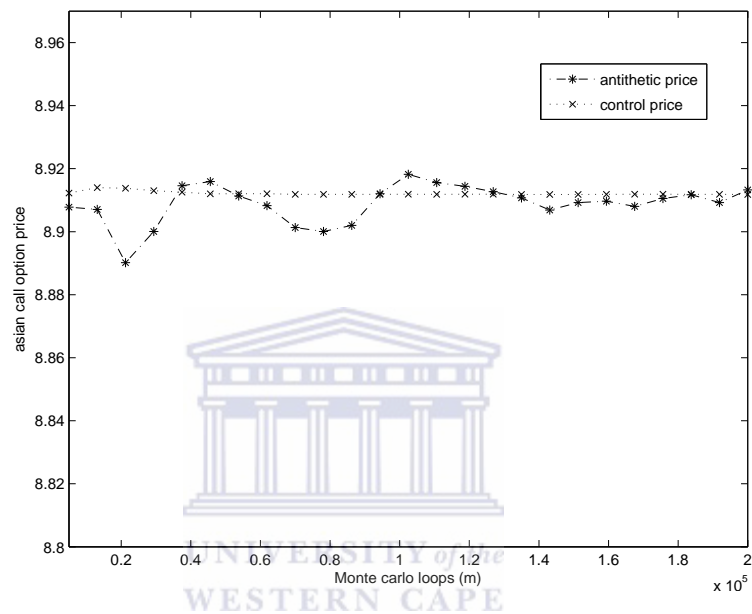

Figure 6.1: Comparison of the control and antithetic methods

Although the basic Monte Carlo method and its improved versions seem easy to implement, computationally they are very expensive and time consuming. One typically needs a large number of paths to come up with a reasonable solution. To overcome this, in the next chapter we propose the finite difference methods.

7. Finite Difference Methods

We have been able to deduce the partial differential equations whose solutions are the prices of the options. At this stage, we will solve these PDEs using the finite difference methods, decribed in Ames [2], Fox [25], Morton [44], Zvan et al [62] etc. These are also some of the traditional methods for the numerical solution of PDEs [31, 32, 43].

We know that (5.21) and (5.22) do not admit explicit solutions. In this case, we should approximate the solutions. As we shall see shortly (and generally in option pricing see, e.g., Korn [39], Vecer [58], Wilmot [60] etc), the PDEs that mainly arise in mathematical finance are of the parabolic type (see Smith [54], Strauss [55], Williams [59] etc for details). An example of a parabolic PDE is the heat equation, $u_t = ku_{xx}$.

7.1   Boundary Conditions

Numerical simulation via finite difference methods requires the discretization of a finite domain. So the first thing that we have to do is have a bounded region on which we will design a suitable grid (Morton [44]). One approach, as in Higham [31] is to impose a large value L, say for ϕ in the ϕ, t plane. This is because we are solving the pde for ϕ in the interval $[0, \infty)$. Instead of this approach, we use the transformation $x = e^{-\phi_t}$ [1], which maps $[0, \infty)$ to $(0, 1)$. At this stage let us drop a and t in the subscripts and keep in mind that we are referring to (5.21). The transformation implies

$$\widetilde{C}_\phi = -x\widetilde{C}_x,$$
$$\widetilde{C}_{\phi\phi} = x^2\widetilde{C}_{xx} + x\widetilde{C}_x.$$

We can write (5.21) as

$$\frac{\partial \tilde{C}}{\partial t} + \left[\left(\frac{1}{T} - r\ln x\right)x + \frac{\sigma^2}{2}x(\ln x)^2\right]\frac{\partial \tilde{C}}{\partial x} + \frac{1}{2}\sigma^2 x^2(\ln x)^2\frac{\partial^2 \tilde{C}}{\partial x^2} = 0,$$

$$\text{subject to:} \quad \tilde{C}(x,T) = 0. \tag{7.1}$$

The boundary condition is so because $\tilde{C}(x,T) = x^- = (e^{-\phi_t})^- = 0$.

From the remark of the put-call parity if the known part of the average is greater than the strike, equivalently $\phi_t \leq 0$ then

$$C_{a,t} = \frac{S_t}{Tr}(1 - e^{-r(T-t)}) - e^{-r(T-t)}\left(K - \frac{1}{T}\int_0^t S_u\,du\right).$$

So

$$\tilde{C}_{a,t} = \frac{1}{Tr}(1 - e^{-r(T-t)}) - e^{-r(T-t)}\phi_t.$$

We are going to use this as the boundary condition at $\phi_t = 0$ (which implies $x = 1$), therefore

$$\tilde{C}(1,t) = \frac{1}{Tr}(1 - e^{-r(T-t)}).$$

The other boundary at infinity comes naturally from the definition of the call; if the strike becomes very large then the call is useless. Since the price of the call is a decreasing function of the strike, Proposition 3.1, then if the strike becomes too large then the call price is 0. Thus

$$\lim_{\phi \to \infty} \tilde{C}(\phi, t) = 0.$$

Finally (7.1) becomes the boundary value problem

$$\tilde{C}_t + \left[\left(\frac{1}{T} - r\ln x\right)x + \frac{\sigma^2}{2}x(\ln x)^2\right]\tilde{C}_x + \frac{1}{2}\sigma^2 x^2(\ln x)^2\tilde{C}_{xx} = 0,$$

$$\text{subject to:} \quad \tilde{C}(x,T) = 0$$

$$\tilde{C}(1,t) = \frac{1}{Tr}(1 - e^{-r(T-t)}) \tag{7.2}$$

$$\tilde{C}(0,t) = 0.$$

Remark 7.1. (i)   The partial differential equation (7.2) is not directly an initial bound-
ary value problem (IBVP) in the strict sense of the phrase (Chacko [6]). We need to
transform it into an IBVP by making use of the time to maturity τ given by $\tau = T - t$.
Consequently,

$$\widetilde{C}_t = -\widetilde{C}_\tau.$$

Therefore the PDE (7.2) becomes

$$\tilde{C}_\tau = \frac{1}{2}\sigma^2 x^2 (\ln x)^2 \tilde{C}_{xx} + \left[\left(\frac{1}{T} - r \ln x \right) x + \frac{\sigma^2}{2} x (\ln x)^2 \right] \tilde{C}_x,$$

$$\text{subject to:} \quad \tilde{C}(x,0) = 0$$

$$\tilde{C}(1,\tau) = \frac{1}{Tr}(1 - e^{-r\tau}) \tag{7.3}$$

$$\tilde{C}(0,\tau) = 0$$

(ii)The PDE (7.3) is parabolic and linear [7, 54, 55].

In some cases, for example the Black-Scholes PDE we can transform the PDE to a simple
one, like the heat equation as in Wilmot [60]. The literature (Hull [32], Higham [31], Zvan
et al.[62] etc) suggests a logarithmic transformation. The Asian option PDEs cannot be
tranformed to the heat equation, as is shown in Mahomed [42]. Nevertheless, we will go
ahead with discretisation.

7.2   Discretisation

The domain $(0,1] \times [0,T]$ is partitioned as follows: We define $\Delta\tau = \frac{T}{N_\tau}$, where N_τ is the
number of points in $[0,T]$ and $\Delta x = \frac{1}{N_x}$, where N_x is the number of points in the interval
$[0,1]$. A typical grid point is then denoted by (x_j, τ_i) where $x_j = j\Delta x$ and $\tau_i = i\Delta\tau$,
where $i = 0,1,\cdots,N_\tau$ and $j = 0,1,\cdots,N_x$. We shall approximate $\tilde{C}(x_j,\tau_i)$ by C_j^i

7.3 Explicit Method

This method will be applied to the problem (7.3). We will approximate the partial derivative C_y by a forward difference, C_{yy} by central difference and C_t by a forward difference [1] as follows

$$
\begin{aligned}
(\widetilde{C}_x)_j^i &\approx \frac{C_{j+1}^i - C_j^i}{\Delta x}, \\
(\widetilde{C}_{xx})_j^i &\approx \frac{C_{j+1}^i - 2C_j^i + C_{j-1}^i}{(\Delta x)^2}, \\
(\widetilde{C}_t)_j^i &\approx \frac{C_j^{i+1} - C_j^i}{\Delta \tau}.
\end{aligned}
\tag{7.4}
$$

Substituting these approximations in (7.3) we have the finite difference scheme

$$
C_j^{i+1} = \lambda_j C_{j-1}^i + (1 - 2\lambda_j - \mu_j)C_j^i + (\lambda_j + \mu_j)C_{j+1}^i
\tag{7.5}
$$

$$
C_j^0 = 0 \qquad \forall \, 0 \le j \le N_x
$$

$$
C_{N_x}^{i-1} = \frac{1}{Tr}(1 - e^{-ri\Delta\tau})
$$

$$
C_0^{i-1} = 0,
$$

where

$$
\lambda_j = \frac{\Delta\tau}{(\Delta x)^2} \frac{\sigma^2}{2} x_j^2 (\ln x_j)^2 = \frac{\sigma^2}{2} j^2 \Delta\tau (\ln x_j)^2,
$$

$$
\mu_j = \left[\left(\frac{1}{T} - r \ln x \right) x_j + \frac{\sigma^2}{2} x_j (\ln x_j)^2 \right] \frac{\Delta\tau}{\Delta x}
$$

$$
= \left(\frac{1}{T} - r \ln x + \frac{\sigma^2}{2} (\ln x_j)^2 \right) j\Delta\tau.
$$

From equation (7.5) and Figure 8.1 we see how C_{j-1}^i, C_j^i and C_{j+1}^i (indicated by shaded circles) are used to get C_j^{i+1} (indicated by unshaded circle). We then iterate for $i = 1, 2, \cdots, N_\tau - 1$, in (7.5) to get the price of the option.
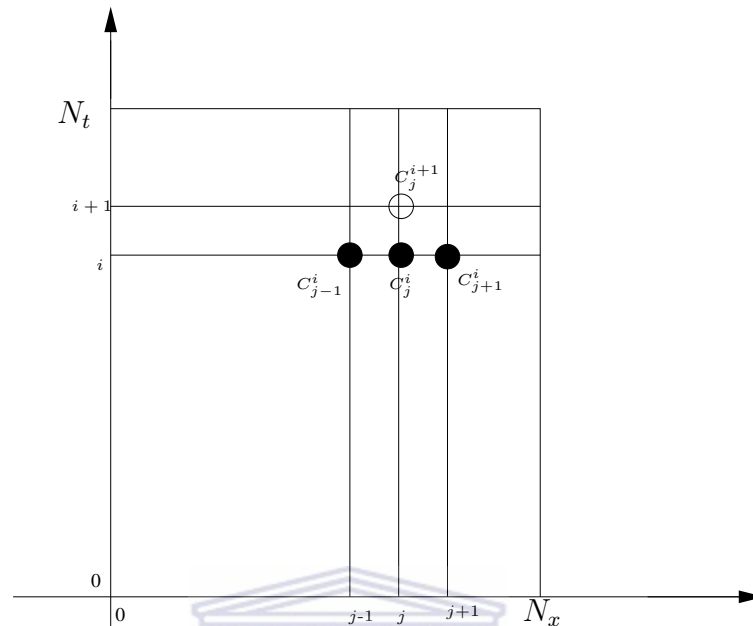
Figure 7.1: A typical grid on which the PDE is discretised

7.4 Convergence analysis of the explicit method

In this section, we discuss the consistency and the stability of the explicit method to analyse the error which we make by the approximations to (7.3). The truncation error [2] is the error incurred when we substitute $\tilde{C}(x, \tau)$ by C_j^i. It is the difference between the two sides of (7.3) [44] when we have replaced the derivatives by the approximations (7.4). We define the truncation error $T(x, \tau)$ as

$$T(x,\tau) := \frac{\Delta_{+\tau}\tilde{C}(x,\tau)}{\Delta\tau} - b(x)\frac{\delta_x^2\tilde{C}(x,\tau)}{(\Delta x)^2} - a(x)\frac{\Delta_{+x}\tilde{C}(x,\tau)}{\Delta x}, \tag{7.6}$$

where $b(x) = \frac{\sigma^2}{2}x^2(\ln x)^2$, $a(x) = \left(\left(\frac{1}{T} - r\ln x\right)x + \frac{\sigma^2}{2}x(\ln x)^2\right)$ and the difference operators [44] $\Delta_{+\tau}$, Δ_{+x} and δ_x^2 are defined as

$$\Delta_{+\tau}\tilde{C}(x,\tau) = \tilde{C}(x, \tau + \Delta\tau) - \tilde{C}(x,\tau)$$

$$\Delta_{+x}\tilde{C}(x,\tau) = \tilde{C}(x + \Delta x, \tau) - \tilde{C}(x,\tau)$$

$$\delta_x^2\tilde{C}(x,\tau) = \tilde{C}(x + \Delta x, \tau) - 2\tilde{C}(x,\tau) + \tilde{C}(x - \Delta x, \tau).$$

We now expand the operators using Taylor series expansion about the point (x,τ) so that (7.6) becomes

$$
\begin{aligned}
T(x,\tau) = &\frac{1}{\Delta\tau}\left(\tilde{C} + \Delta\tau\tilde{C}_\tau + \frac{(\Delta\tau)^2}{2}\tilde{C}_{\tau\tau} + \frac{(\Delta\tau)^3}{6}\tilde{C}_\tau^{(iv)}(x,\tau^*) - \tilde{C}\right) - \\
&\frac{b(x)}{(\Delta x)^2}\left(\tilde{C} + \Delta x\tilde{C}_x + \frac{(\Delta x)^2}{2}\tilde{C}_{xx} + \frac{(\Delta x)^3}{6}\tilde{C}_{xxx} + \frac{(\Delta x)^4}{24}\tilde{C}_x^{(iv)}(x^*,\tau) - 2\tilde{C}\right. \\
&\left. + \tilde{C} - \Delta x\tilde{C}_x + \frac{(\Delta x)^2}{2}\tilde{C}_{xx} - \frac{(\Delta x)^3}{6}\tilde{C}_{xxx} + \frac{(\Delta x)^4}{24}\tilde{C}_x^{(iv)}(x^*,\tau)\right) \\
&- \frac{a(x)}{\Delta x}\left(\tilde{C} + \Delta x\tilde{C}_x + \frac{(\Delta x)^2}{2}\tilde{C}_{xx} + \frac{(\Delta x)^3}{6}\tilde{C}_{xxx} + \frac{(\Delta x)^4}{24}\tilde{C}_x^{(iv)}(x^*,\tau) - \tilde{C}\right),
\end{aligned}
$$

where $\tau^* \in [\tau, \tau + \Delta\tau]$ and $x^* \in [x, x + \Delta x]$. This simplifies to

$$
\begin{aligned}
&(\tilde{C}_\tau - b(x)\tilde{C}_{xx} - a(x)\tilde{C}_x) + \frac{\Delta\tau}{2}\tilde{C}_{\tau\tau} + \frac{(\Delta\tau)^2}{6}\tilde{C}_\tau^{(iv)}(x,\tau^*) - \frac{a(x)}{2}\Delta x\tilde{C}_{xx} \\
&- \frac{a(x)}{6}(\Delta x)^2\tilde{C}_{xxx} - \frac{a(x)}{24}(\Delta x)^3\tilde{C}_x^{(iv)}(x^*,\tau) - \frac{b(x)}{12}(\Delta x)^2\tilde{C}_x^{(iv)}(x^*,\tau).
\end{aligned}
$$

We can simplify further by noting that $\tilde{C}$ satisfies the PDE (7.3). This implies the first bracketed term above is zero. Finally the truncation error can be expressed as

$$T(x,\tau) = \frac{\Delta\tau}{2}\tilde{C}_{\tau\tau} - \frac{\Delta x}{2}a(x)\tilde{C}_{xx} + \mathcal{O}((\Delta\tau)^2) + \mathcal{O}((\Delta x)^2). \tag{7.7}$$

Thus

$$T(x,\tau) \to 0 \quad \text{as} \quad \Delta\tau \to 0 \quad \text{and} \quad \Delta x \to 0, \tag{7.8}$$

which implies that the explicit method is indeed consistent.

Let us define the error at each point by $e_j^i = C_j^i - \tilde{C}(x_j, t_i)$. Then substituting e_j^i into (7.5) we have

$$
\begin{aligned}
e_j^{i+1} &= \lambda_j e_{j-1}^i + (1 - 2\lambda_j - \mu_j)e_j^i + (\lambda_j + \mu_j)e_{j+1}^i \\
&\quad - (C_j^{i+1} - C_j^i) + \lambda_j(C_{j+1}^i - 2C_j^i + C_{j-1}^i) + \mu_j(C_{j+1}^i - C_j^i) \\
&= \lambda_j e_{j-1}^i + (1 - 2\lambda_j - \mu_j)e_j^i + (\lambda_j + \mu_j)e_{j+1}^i - \Delta\tau T_j^i \qquad (7.9)
\end{aligned}
$$

Define the error at each time step as

$$
E^i := \max_{0 \leq j \leq N_x} |e_j^i|.
$$

Assume that the truncation error is bounded by $|T|$, then (7.9) becomes $E^{i+1} \leq E^i + \Delta\tau|T|$. Since $E^0 = 0$ from the initial condition, and using as inductive argument we have

$$
E^i \leq i\Delta\tau|T|.
$$

This shows that, the truncation error approaches zero as the time step-size becomes smaller. However, we need to investigate the stability as well to determine convergence because it is possible for a scheme to have its truncation error approaching zero, but converging to a wrong solution [54]. To this end, we will use the Fourier method [25]. The Fourier method assumes that the PDE solution has a fourier representation. In other words the solution of the PDE can be written in form of sines and cosines. We then investigate the growth of the error made a each time node by considering the function $e^{\Delta\tau}$. We are interested in finding out whether or not rounding errors made at each time step blow or at least stays the same. For the finite difference scheme to be stable, $e^{\Delta\tau}$ should be bounded by one [2]. Let

$$
C_j^i = e^{\alpha i \Delta\tau} e^{\sqrt{-1}\beta j \Delta x}, \qquad (7.10)
$$

where $\alpha, \beta \in \mathbb{R}^+$.

We substitute (7.10) into (7.5) and obtain

$$
e^{\alpha\Delta\tau} = 1 - 2\lambda_j \sin^2\left(\frac{\beta\Delta x}{2}\right) + \mu_j(e^{\sqrt{-1}\beta\Delta x} - 1). \qquad (7.11)
$$

From our stability criterion, $|e^{\alpha\Delta\tau}| \leq 1$, (7.11) suggests that the explicit method is unstable. Since stability is a necessary condition for convergence, by the Lax equivalence theorem [54], we cannot say whether the explicit scheme will converge to the real solution of the PDE. As an alternative, we therefore design the following implicit method, namely, the Crank-Nicholson method. Due to the deficiencies of the explicit method, we will tabulate results found using the Crank-Nicholson's method.

7.5 Implicit Method: Crank-Nicholson's scheme

As we see from the previous section, the explicit method is conditionally stable. On the other hand, the Crank-Nicholson Method [2, 8, 44, 60] is implicit and unconditionally stable. We shall demonstrate this by the Von Neumann method (Fourier method) [54].

Using the same uniform grid as above, we approximate $\widetilde{C}_x$ by a an average of centered difference, $\widetilde{C}_{xx}$, by an average of centered second difference and $\widetilde{C}_t$ by a forward difference [8]. We could also use the average of an fully-implicit and explicit scheme to the get the Crank-Nicholson method [31, 60]. Thus

$$(\widetilde{C}_x)_j^i \approx \frac{1}{2}\left(\frac{C_{j+1}^{i+1} - C_{j-1}^{i+1}}{2\Delta x} + \frac{C_{j+1}^i - C_{j-1}^i}{2\Delta x}\right),$$

$$(\widetilde{C}_{xx})_j^i \approx \frac{1}{2}\left(\frac{C_{j+1}^{i+1} - 2C_j^{i+1} + C_{j-1}^{i+1}}{(\Delta x)^2} + \frac{C_{j+1}^i - 2C_j^i + C_{j-1}^i}{(\Delta x)^2}\right),$$

$$(\widetilde{C}_t)_j^i \approx \frac{C_j^{i+1} - C_j^i}{\Delta\tau}.$$

Similarly substituting these expressions into (7.3), we get the following finite difference

equation

$$-(\lambda_j - \mu_j)C_{j-1}^{i+1} + (1 + 2\lambda_j)C_j^{i+1} - (\lambda_j + \mu_j)C_{j+1}^{i+1} = (\lambda_j - \mu_j)C_{j-1}^i$$
$$+ (1 - 2\lambda_j)C_j^i + (\lambda_j + \mu_j)C_{j+1}^i \qquad (7.12)$$

$$C_j^0 = 0. \qquad \forall\ 0 \leq j \leq N_x$$

$$C_{N_x}^{i+1} = \frac{1}{Tr}(1 - e^{-ri\Delta\tau)})$$

$$C_0^{i+1} = 0,$$

where

$$\lambda_j = \frac{\sigma^2}{2}x_j^2(\ln x_j)^2\frac{\Delta\tau}{2(\Delta x)^2}$$

$$= \frac{\sigma^2}{4}j^2\Delta\tau(\ln x_j)^2,$$

$$\mu_j = \left(\left(\frac{1}{T} - r\ln x\right)x_j + \frac{\sigma^2}{2}x_j(\ln x_j)^2\right)\frac{\Delta\tau}{4\Delta x}$$

$$= \frac{j\Delta\tau}{4}\left(\frac{1}{T} - r\ln x_j + \frac{\sigma^2}{2}(\ln x_j)^2\right).$$

Letting

$$\alpha_j^+ = -(\lambda_j - \mu_j) \qquad\qquad \alpha_j^- = \lambda_j - \mu_j$$

$$\beta_j^+ = 1 + 2\lambda_j \qquad\qquad \beta_j^- = 1 - 2\lambda_j$$

$$\gamma_j^+ = -(\lambda_j + \mu_j) \qquad\qquad \gamma_j^- = \lambda_j + \mu_j.$$

The equation (7.12) can now be written as

$$\alpha_j^+ C_{j-1}^{i+1} + \beta_j^+ C_j^{i+1} + \gamma_j^+ C_{j+1}^{i+1} = \alpha_j^- C_{j-1}^i + \beta_j^- C_j^i + \gamma_j^- C_{j+1}^i. \qquad (7.13)$$

Figure 7.2 shows why the Crank-Nicholson is an implicit scheme. To get C_j^{i+1} not only do we require C_{j-1}^i, C_j^i and C_{j+1}^i but also C_{j-1}^{i+1} and C_{j+1}^{i+1}. This method requires to solve a system of linear equations at each time step. We can represent the difference equation by matrices. The equation (7.13) can be represented as

$$\mathbf{AC}^{i+1} = \mathbf{BC}^i + \mathbf{q}^i - \mathbf{p}^{i-1}, \qquad (7.14)$$
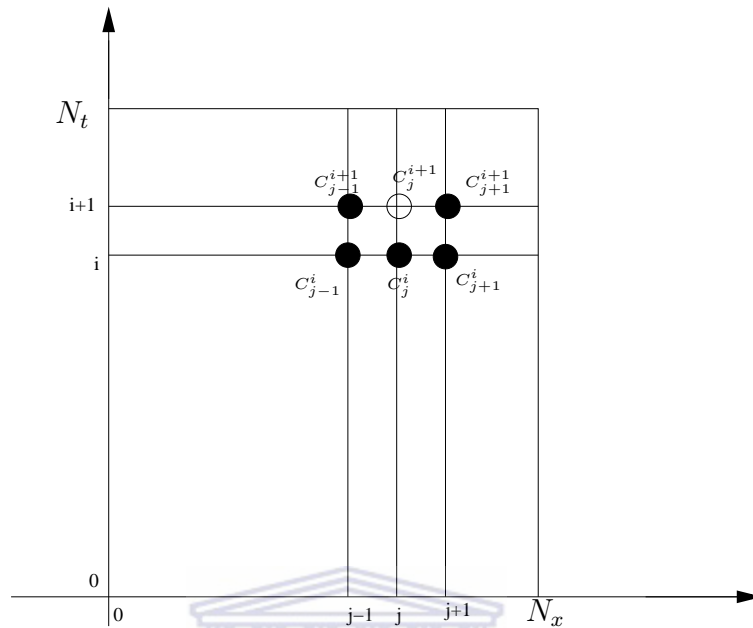
Figure 7.2: Typical grid for the Crank-Nicholson discretisation.

where $\mathbf{A}$ is the $(N_x - 1) \times (N_x - 1)$ matrix

$$\mathbf{A} = \begin{bmatrix} \beta_1^+ & \gamma_1^+ & & & & 0 \\ \alpha_2^+ & \beta_2^+ & \gamma_2^+ & & & \\ & \alpha_3^+ & \beta_3^+ & \cdot & & \\ & & \cdot & \cdot & \gamma_{N_x-2}^+ \\ 0 & & & \alpha_{N_x-1}^+ & \beta_{N_x-1}^+ \end{bmatrix}$$

and $\mathbf{B}$ is the $(N_x - 1) \times (N_x - 1)$ matrix

$$\mathbf{B} = \begin{bmatrix} \beta_1^- & \gamma_1^- & & & & 0 \\ \alpha_2^- & \beta_2^- & \gamma_2^- & & & \\ & \alpha_3^- & \beta_3^- & \cdot & & \\ & & \cdot & \cdot & \gamma_{N_x-2}^- \\ 0 & & & \alpha_{N_x-1}^- & \beta_{N_x-1}^- \end{bmatrix}.$$

The rest of the terms are the following vectors:

$$\mathbf{C}^{i+1} = [\, C_1^{i+1}, \, C_2^{i+1}, \cdots, \, C_{N_x-1}^{i+1} \,]^T,$$

$$\mathbf{C}^{i} = [\, C_1^{i}, \, C_2^{i}, \cdots, \, C_{N_x-1}^{i} \,]^T,$$

$$\mathbf{p}^{i+1} = [\, \alpha_1^+ C_0^{i+1}, 0, \cdots, 0, \, \alpha_{N_x-1}^+ C_{N_x}^{i+1} \,]^T,$$

$$\mathbf{q}^{i} = [\, \alpha_1^- C_0^{i}, \, 0, \cdots, 0, \, \alpha_{N_x-1}^- C_{N_x}^{i} \,]^T.$$

We simplify $\mathbf{q}^i$ and $\mathbf{p}^{i+1}$ at $i = 0$ by the initial condition and then at all points $i = 1, 2, \cdots, N_\tau - 1$ using the boundary conditions. To solve (7.14), we iterate for all $i = 1, 2, \cdots, N_\tau - 1$. To get $\mathbf{C}^{i+1}$, we solve an $(N_x - 1) \times (N_x - 1)$ system of equations at each time step. The two matrices on both sides of the system are tridiagonal which reduces the calculations significantly since algorithms adapted to tridiagonal systems [16, 44] can be used. For completeness we describe the algorithm.

Solving the system of equations

Let d_j^+ be the right handside of (7.14). The for $j = 2, 3, \cdots, N_x - 1$ we change

$$\beta_j = \beta_j^+ - \frac{\alpha_j^+}{\beta_{j-1}^+} \gamma_{j-1}^+,$$

$$d_j = d_j^+ - \frac{\alpha_j^+}{\beta_{j-1}^+} d_{j-1}^+,$$

This makes the matrix A to be upper diagonal. To find $C_{N_x-1}^{i+1}, C_{N_x-2}^{i+1}, \cdots, C_1^{i+1}$, we perform a backward substitution as follows

$$C_{N_x-1}^{i+1} = \frac{d_{N_x-1}}{\beta_{N_x-1}}.$$

The the rest are found by

$$C_j^{i+1} = \frac{d_j - \gamma_j^+ C_{j+1}^{i+1}}{\beta_{j-1}},$$

for $j = N_x-2, N_x-3, \cdots, 1$. We can now summarize the above in the following algorithm:

Algorithm 7.5.1: Tridiagonal Solver$(\alpha_j^+, \beta_j^+, \gamma_j^+, d_j^+, N_x - 1)$

for j **from** 2 **to** $N_x - 1$

   do $\begin{cases} \beta_j^+ & \leftarrow \beta_j^+ - \frac{\alpha_j^+}{\beta_{j-1}^+}\gamma_{j-1}^+ \\ d_j^+ & \leftarrow d_j^+ - \frac{\alpha_j^+}{\beta_{j-1}^+}d_{j-1}^+ \end{cases}$

$C_{N_x-1}^{i+1} \leftarrow \frac{d_{j-1}}{\beta_{j-1}}$

for j **from** $N_x - 2$ **to** 1

   do $\begin{cases} C_j^{i+1} & \leftarrow \frac{d_j^+ - \gamma_j^+ C_{j+1}^{i+1}}{\beta_{j-1}^+} \end{cases}$

The solution of PDE will now be found by implementing the algorithm above at each time node i. Figure 7.3 shows the surface generated by solving (7.12) by the Crank-Nicholson's method.
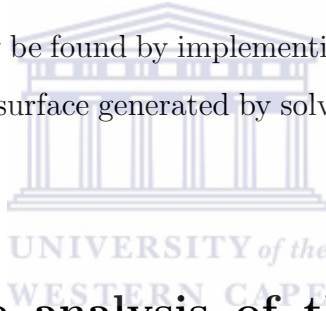
7.6 Convergence analysis of the Crank-Nicholson's method

Following the similar procedure as we had for the explicit method, we can show that the Crank-Nicholson's method is consistent. In fact, the truncation error in this case is

$$T^c(x, \tau) = \mathcal{O}((\Delta\tau)^2) + \mathcal{O}((\Delta x)^2).$$

This implies that

$$T^c(x, \tau) \to 0 \ \text{ as } \ \Delta\tau \to 0 \ \text{ and } \ \Delta x \to 0.$$

To discuss its stability, we substitute C_j^i into (7.12) and obtain

$$e^{\alpha\Delta\tau} = \frac{1 + 2\mu_j \sin(\beta\Delta x) - 4\lambda_j \sin^2(\frac{\beta\Delta x}{2})}{1 - 2\mu_j \sin(\beta\Delta x) + 4\lambda_j \sin^2(\frac{\beta\Delta x}{2})}. \tag{7.15}$$

The stability criterion $|e^{\alpha\Delta\tau}| \leq 1$ implies two cases $e^{\alpha\Delta\tau} \leq 1$ and $e^{\alpha\Delta\tau} \geq -1$. If $e^{\alpha\Delta\tau} \leq 1$, then

$$1 + 2\mu_j \sin(\beta\Delta x) - 4\lambda_j \sin^2(\frac{\beta\Delta x}{2}) \leq 1 - 2\mu_j \sin(\beta\Delta x) + 4\lambda_j \sin^2(\frac{\beta\Delta x}{2}).$$

This implies $\lambda_j \sin(\frac{\beta\Delta x}{2}) \geq \mu_j \cos(\frac{\beta\Delta x}{2}) \Rightarrow \lambda_j \geq \mu_j$. For values of interest: $T = 1$, $r = 0.09$, $\sigma = 0.05$ and $\Delta x = \Delta\tau = 0.005$, λ_j is always greater that μ_j.

If $e^{\alpha\Delta\tau} \geq -1$, then

$$1 + 2\mu_j \sin(\beta\Delta x) - 4\lambda_j \sin^2(\frac{\beta\Delta x}{2}) \geq -1 + 2\mu_j \sin(\beta\Delta x) - 4\lambda_j \sin^2(\frac{\beta\Delta x}{2})$$

$$\Leftrightarrow 1 \geq -1.$$

From above discussion, it is clear that both of the above inequalities are satisfied without any restrictions on the step-sizes and hence this method is unconditionally stable.

Combining the two aspects (consistency and stability) above, by the Lax equivalence theorem [2], the Crank-Nicholson's method is convergent.
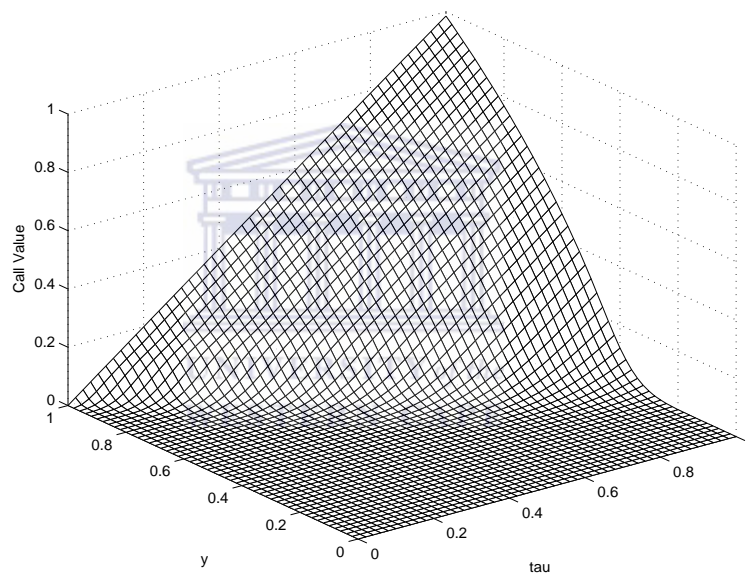
Figure 7.3: The Call option surface obtained by the Crank-Nicholson's method

8. Numerical Results

In this chapter, we provide various results obtained by the methods described in the preceding chapters. Some of our numerical results will be confirmed by the analytical results. We tabulate the results for evaluating the Asian call option prices by the Crank-Nicholson's method, the improved Monte Carlo method and compare them with the lower bound of the price obtained using the formula derived in chapter 4. Furthermore, as we explain below, our numerical results confirm theoretical investigations done by many other researchers.

Figure 8.1 is obtained by the Crank-Nicholson's method. The parameters used for the simulation are as follows ([1, 15]): σ for both the European Call option and the Asian option takes the values 0.05, 0.1, 0.2, 0.3, 0.4 and 0.5; the interest rate is $r = 0.09$ and the time $t = 0$. Each curve respresents the price of either a European call or Asian call at time $t = 0$ for varying strike price K in the interval $[50, 150]$.

Figure 8.1 confirms what we have indicated in the introduction of this thesis about the important feature of an Asian option that it is cheaper than a European option for any value of strike price K. The figure confirms the result that the Asian option price is decreasing function of the strike price. The prices of the European call option can be easily found by the Black-Scholes formula [9, 31, 35]. In this case, the price of a European call option $C_{e,t}$ is given by

$$C_{e,t}(S_t, K, t) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2),$$

where

$$d_1 = \frac{\log \frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$
$$d_2 = d_1 - \sigma\sqrt{T - t}.$$

Figure 8.1 also shows that the price of the Asian option is a increasing function of the

volatility σ. This has been proved analytically by Carr [14] using arguments based on the maximum principle of the parabolic PDEs. Carr also demonstrates that the result holds in the Black-Scholes model and not in a general model like for instance the binomial model. From Figure 8.1 , we see that due to the averaging nature of the Asian options, they are less sensitive to volatility as compared to their European counterparts [1]. Small changes in volatility will not change Asian option price as they would do if they were European options.
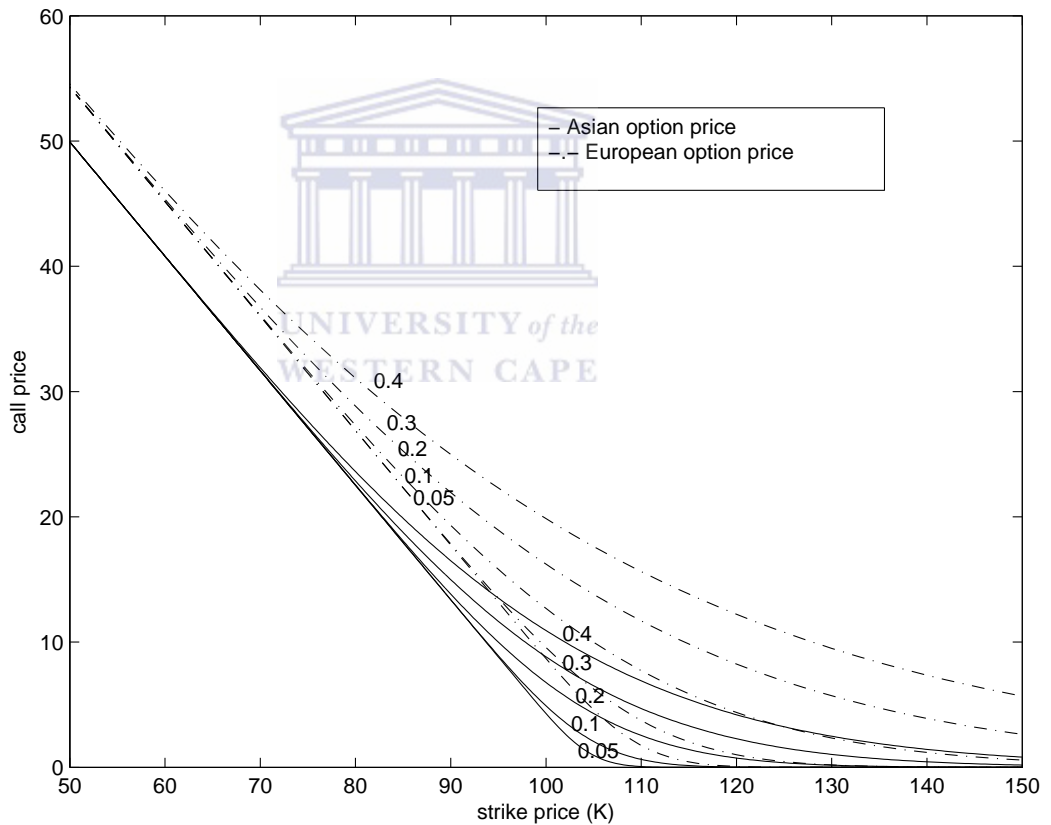

Figure 8.1: Comparison of European and Asian option prices obtained by the Black-Scholes formula and the Crank-Nicholson's method, respectively.

In figure 8.2, we set our parameters as follows: σ takes values in the list $[0.05, 0.1, 0.2]$.

The strike price K is varied in the interval $[50, 150]$, while the time t is kept constant at 0. Each curve represents the price of either an arithmetic Asian call or a geometric Asian call.

We see why we said the price of the geometric Asian option is used as a control variate of the arithmetic Asian option. In our motivation of the control variate Monte Carlo method, we said that the geometric Asian option was close to or resembles the arithmetic Asian option. However, the diagram shows that we cannot differentiate between these prices which demonstrate the effectiveness of the Crank-Nicholson's method.

Table 8.1 shows the results of calculations based on the three methods Monte Carlo method, Crank-Nicholson's method and the evaluation of the lower bound (4.9). The tabular results show that the error incurred from the Crank-Nicholson's method is approximately 0.0005. The lower bound is also very accurate. We also tabulate the times of computation (CPU). As we can see, the Monte Carlo takes the largest time. The Crank-Nicholson is faster than the Monte Carlo method. For low volatilities, e.g., $\sigma = 0.05$ the Crank-Nicholson's method takes very long time but since in those cases, the PDE is convection dominant and hence it is acceptable as we still achieve a high degree of accuracy. It should be noted that we have added the CPU times for evaluating the lower bound also because we do use some numerical integration techniques there but as such it is merely for the comparison purpose as this is more an analytical formula.
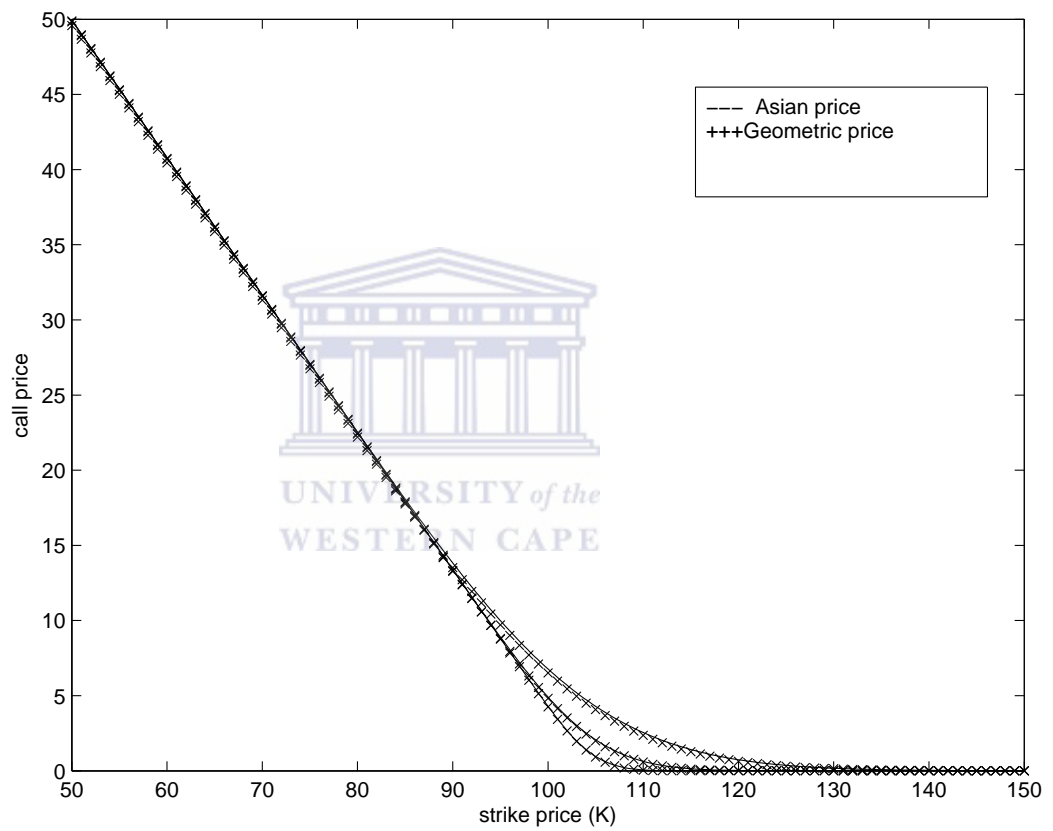
Figure 8.2: Comparison of Geometric and Arithmetic option prices obtained by using (2.9) and the Crank-Nicholson's method, respectively.

Table 8.1: Comparison of the Finite difference and Monte Carlo method with the lower bound obtained by (4.9)

σ	K	Finite Difference method		Monte Carlo method		Lower Bound	
		Crank-Nicholson	CPU	Control variate	CPU	Equation (4.9)	CPU
	95	8.8088	2747.1	8.8092	9713.4	8.8088	1.4966
0.05	100	4.3081	4671.5	4.3086	9710.4	4.3081	1.4846
	110	0.0524	2644.8	0.0522	9629.0	0.0521	1.4505
	95	8.9115	76.607	8.9119	9708.3	8.9115	1.5440
0.1	100	4.9146	77.876	4.9155	9981.61	4.9145	1.6814
	110	0.6307	79.640	0.6304	9999.79	0.6299	1.9514
	95	11.6558	121.26	11.6560	9656.4	11.6530	1.5960
0.3	100	8.8287	122.52	8.8289	9673.2	8.8259	1.3244
	110	4.6967	121.72	4.6968	9668.5	4.6938	1.1737
	95	15.4427	75.029	15.4427	10968.9	15.4342	1.1317
0.5	100	13.0282	74.547	13.0279	10344.7	13.0200	1.5438
	110	9.1243	75.075	9.1242	10023.0	9.1157	1.2696

Part III

Application to European basket options

9. Extension of pricing approaches of Asian options to pricing a European basket option (EBO)

Most of the methods which we have used in previous chapters can be extended to determining the price of European basket options. The structure and problems faced in finding the price of EBO and an Asian option are largely similar.

The European basket option (EBO) is a popularly traded option (see Briys et al. [12], Deelstra et al. [21]). It is an option which depends on the value of a portfolio (or basket) of assets. We will consider a basket option written on n assets which are all drawn from the same economy. More precisely, we form a basket consisting of a_i units of ith asset, for $i \in \{1, 2, \cdots, n\}$. Let Q be an equivalent martingale measure (EMM), then the price of an EBO is

$$C(K, T) = e^{-rT} \mathbb{E}^Q \left[\left(\sum_{i=1}^n a_i S_i(T) - K \right)^+ \right],$$

where $S_i(T)$ is the price of asset i and K is the strike price.

Writing a basket on n assets is comparable to having n European options, but there are advantages of buying a basket option. The EBO takes the correlation of the assets into account. basket options are cost-effective. There is an obvious advantage of reduction of transactional costs in buying a EBO rather than buying several European options [21]. basket options are usually cheaper than the corresponding European options [12].

The basket option is similar to the Asian options which we analysed in the preceding chapters. It takes the sum of the assets prices. The difference is that whereas the Asian option is path-dependent the basket option is not. We recall that the Asian option takes

the sum of the asset price over some period of its existence and compares this with the strike price K. The EBO only considers the prices of the assets at maturity and compares this with the strike price K.

Under these circumstances it is not a surprise that the problems that we face in pricing Asian options are again encountered in pricing the European basket options. As in the Asian option scenario, the distribution of the sum of lognormally distributed random variables is not explicit or tractable. This is the major drawback in formulating a closed form expression for the price of the EBO. We again restrict ourselves to the Black-Scholes market where the price processes follow lognormal distributions.

9.1   Setting

We consider a basket option written on n assets with prices $S_i(t)$. More precisely, we form a basket consisting of a_i units of i th asset, for $i \in \{1, 2, \cdots, n\}$. We assume a Black-Scholes economy, the return for each asset is μ_i and volatility σ_i, both being constant. There is also a riskless interest rate r such that if an investment of B_0 is put in a bank account then after time t its value is $B_0 e^{rt}$. The dynamics of the prices of the assets are

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)d\widetilde{W}_i(t),$$

where $\{\widetilde{W}_i(t), t > 0\}$ are Brownian motions under the real world probability (see, e.g., Baxter [5], Shreve [53]). As we have shown in Chapter 2, under a risk neutral measure Q the dynamics of the price process are

$$dS_i(t) = r S_i(t)dt + \sigma_i S_i(t)dW_i(t),$$

where $\{W_i(t), t > 0\}$ are Q-Brownian motions. We know that the explicit formulae of the price processes are given by

$$S_i(T) = S_i(0)e^{\left(r - \frac{\sigma_i^2}{2}\right)T + \sigma W_i(T)}.$$

At each time t, the Brownian motions are assumed to be constantly correlated, i.e.,

$$\mathrm{Cov}^Q(dW_i(t), dW_j(t)) = \rho_{ij}\, dt.$$

The strike price for the basket option is K. The payoff structure for the arithmetic basket option (so named because of the summation involved) is given by

$$\left(\sum_{i=1}^{n} a_i S_i(T) - K \right)^+,$$

where a_i are the weights for each asset. Just as we priced the Asian options, the price at time T of a basket call option is given by the expectation of the discounted payoff under the risk neutral measure Q. Therefore

$$C(K,T) = e^{-rT}\mathbb{E}^Q\left[\left(\sum_{i=1}^{n} a_i S_i(T) - K \right)^+ \right].$$

As has been highlighted before the problem in getting a closed form expression for $C(K,T)$ is that the distribution for $\sum_{i=1}^{n} a_i S_i(T)$ is not known. The basket put option $P(K,T)$ can likewise be given by

$$P(K,T) = e^{-rT}\mathbb{E}^Q\left[\left(K - \sum_{i=1}^{n} a_i S_i(T) \right)^+ \right].$$

9.2   A call-put parity

A parity also exists between the basket call and the basket put. Since we consider the option to be of European nature, in which case it is exercised on maturity, the derivation is not so difficult.

Proposition 9.1. *The basket call option and basket put option written on n assets whose prices are $S_i(t)$ with maturity T with strike price K satisfies*

$$C(K,T) + Ke^{-rT} = P(K,T) + \sum_{i=1}^{n} a_i S_i(0).$$

Proof. Let us define the sets $X := \{\omega \in \Omega : \sum_{i=1}^{n} a_i S_i(\omega, T) > K\}$ and
$Y := \{\omega \in \Omega : \sum_{i=1}^{n} a_i S_i(\omega, T) \leq K\}$. Then subtracting the put from the call we get

$$C(K, T) - P(K, T) = e^{-rT} \mathbb{E}^Q \left[\left(\sum_{i=1}^{n} a_i S_i(T) - K \right)^+ \right]$$

$$- e^{-rT} \mathbb{E}^Q \left[\left(K - \sum_{i=1}^{n} a_i S_i(T) \right)^+ \right]$$

$$= e^{-rT} \mathbb{E}^Q \left\{ \left(\sum_{i=1}^{n} a_i S_i(T) - K \right) (\mathbb{1}_X + \mathbb{1}_Y) \right\}$$

$$= e^{-rT} \mathbb{E}^Q \left(\sum_{i=1}^{n} a_i S_i(T) - K \right)$$

$$= \sum_{i=1}^{n} a_i \mathbb{E}^Q \left(e^{-rT} S_i(T) \right) - K e^{-rT}$$

$$= \sum_{i=1}^{n} a_i S_i(0) - K e^{-rT}. \qquad \square$$

9.3   A lower bound by Conditioning

We are going to derive a lower bound for the basket option in a similar way that we used to get bounds for the Asian options. Our motivation for these bounds is that they are easy to deduce, and when implemented, they are computionally less expensive (also noted by Chen et al. [15], Rogers and Shi [52], Thompson [56], etc). First let us define the sets D and $E = E(\gamma)$ below, where γ is any real number, i.e,

$$D := \left\{ \omega \in \Omega : \sum_{i=1}^{n} a_i S_i(\omega, T) > K \right\}$$

$$E(\gamma) := \left\{ \omega \in \Omega : \sum_{i=1}^{n} a_i \sigma_i S_i(0) e^{(r - \frac{1}{2}\sigma_i^2)T} W_i(\omega, T) > \gamma \right\}.$$

The variable γ can be any number but for the bound which we will determine, it must be the one that optimises the bound. From now onwards we drop the explicit dependency of $W_i(\omega, T)$ and $S_i(\omega, T)$ on ω.

Proposition 9.2. *For any $\gamma \in \mathbb{R}$, the following inequality holds.*

$$C(K,T) \geq e^{-rT}\mathbb{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)\mathbb{1}_E\right].$$

Proof. Let us write $E = E_1 \cup E_2$ where $E_1 = E \cap D$ and $E_2 = E \setminus E_1$. Then $E_1 \subseteq D$ and for $\omega \in E_1$ we have $(\sum_{i=1}^n a_i S_i(T) - K) > 0$. Therefore

$$\mathbb{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)\mathbb{1}_{E_1}\right] \leq \mathbb{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)\mathbb{1}_D\right].$$

On the other hand, for $\omega \in E_2$ we have $(\sum_{i=1}^n a_i S_i(T) - K) \leq 0$ and therefore

$$\mathbb{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)\mathbb{1}_{E_2}\right] \leq 0.$$

Therefore,

$$\begin{aligned}
C(K,T) &= e^{-rT}\mathbb{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)\mathbb{1}_D\right] \\
&\geq e^{-rT}\mathbb{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)\mathbb{1}_{E_1}\right] \\
&\geq e^{-rT}\mathbb{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)\mathbb{1}_{E_1}\right] + e^{-rT}\mathbb{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)\mathbb{1}_{E_2}\right] \\
&= e^{-rT}\mathbb{E}^Q\left[\left(\sum_{i=1}^n a_i S_i(T) - K\right)\mathbb{1}_E\right]. \qquad \square
\end{aligned}$$

We are going to optimally get γ. We want a γ^* that maximises the bound. To this end we have to solve the problem

$$\frac{\partial}{\partial\gamma}\sum_{i=1}^n \mathbb{E}^Q\left(a_i S_i(T) - \frac{K}{n}, \sum_{i=1}^n a_i \sigma_i S_i(0)e^{(r-\frac{1}{2}\sigma_i^2)T}W_i(T) > \gamma\right) = 0. \qquad (9.1)$$

Proposition 9.3. *Let U and W_t be jointly distributed with density function $f(W_t, U)$. Suppose U has marginal density function $f_U(u)$ and let $v_i := g(W_t)$ be a function of W_t. Then*

$$\frac{\partial}{\partial\gamma}\mathbb{E}^Q\left(v_i, U > \gamma\right) = -f_U(\gamma)\mathbb{E}^Q\left(v_i | U = \gamma\right).$$

Proof. By the Leibnitz rule

$$\frac{\partial}{\partial \gamma} \int_{\gamma}^{\infty} v_i f_{W_t,U}(W_t, U) \, dU = -v_i f_{W_t,U}(W_t, \gamma). \tag{9.2}$$

Applying (9.2) together with the definition of expectation, we can write

$$\frac{\partial}{\partial \gamma} \mathbb{E}^Q (v_i, U > \gamma) = \frac{\partial}{\partial \gamma} \int_{-\infty}^{\infty} \int_{\gamma}^{\infty} v_i f_{W_t,U}(W_t, U) \, dU \, dW_t$$

$$= \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial \gamma} \int_{\gamma}^{\infty} v_i f_{W_t,U}(W_t, U) \, dU \right) dW_t$$

$$= \int_{-\infty}^{\infty} -v_i f_{W_t,U}(W_t, \gamma) dW_t$$

$$= \int_{-\infty}^{\infty} -v_i f_{W_t,U}(W_t | U = \gamma) f_U(\gamma) dW_t$$

$$= -f_U(\gamma) \mathbb{E}^Q (v_i | U = \gamma). \qquad \square$$

Our evaluation involve summations. To incorporate this we also have the following result.

Corollary 9.4.

$$\frac{\partial}{\partial \gamma} \sum_{i=1}^{n} \mathbb{E}^Q (v_i, U > \gamma) = -f_U(\gamma) \sum_{i=1}^{n} \mathbb{E}^Q (v_i | U = \gamma).$$

The proof of the above is omitted since it is immediate. We now evoke Corollary 9.4 so that (9.1) becomes

$$\sum_{i=1}^{n} \mathbb{E}^Q \left(a_i S_i(T) - \frac{K}{n} \bigg| U = \gamma \right) = 0, \tag{9.3}$$

with $U = \sum_{i=1}^{n} a_i \sigma_i S_i(0) e^{(r-\frac{1}{2}\sigma_i^2)T} W_i(T)$. Then (9.3) can be written equivalently as

$$\sum_{i=1}^{n} \mathbb{E}^Q(a_i S_i(T) | U = \gamma^*) = K. \tag{9.4}$$

So the γ that maximises the bound is uniquely determined by (9.4). To be able to find the distribution of $S_i(T)|U$, we need to find that of $W_i(T)|U$. To this end we first find the covariance of $W_i(T)$ and U:

$$\text{Cov}^Q\left(W_i(T), \sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}W_j(T)\right) = \sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}.$$

$$\text{Cov}^Q(W_i(T), W_j(T))$$

$$= \sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{ij}T.$$

We also need the variance σ_U^2 of U:

$$\text{Var}^Q\left(\sum_{i=1}^n a_i\sigma_i S_i(0)e^{(r-\frac{1}{2}\sigma_i^2)T}W_i(T)\right) = \text{Cov}^Q\left(\sum_{i=1}^n a_i\sigma_i S_i(0)e^{(r-\frac{1}{2}\sigma_i^2)T}W_i(T),\right.$$

$$\left.\sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}W_j(T)\right)$$

$$= \sum_{i=1}^n\sum_{j=1}^n a_ia_j\sigma_i\sigma_j S_i(0)S_j(0)e^{(r-\frac{1}{2}\sigma_i^2)T}e^{(r-\frac{1}{2}\sigma_j^2)T}.$$

$$\text{Cov}^Q(W_i(T), W_j(T))$$

$$= \sum_{i=1}^n\sum_{j=1}^n a_ia_j\sigma_i\sigma_j S_i(0)S_j(0)e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}T.$$

The correlation $\rho_{W_i,U}$ of $W_i(T)$ and U is

$$\rho_{W_i,U} = \frac{\text{Cov}^Q(W_i(T), U)}{\sigma_{W(T)}\sigma_U} = \frac{\sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{ij}T}{\sqrt{T}\sqrt{\sum_{i=1}^n\sum_{j=1}^n a_ia_j\sigma_i\sigma_j S_i(0)S_j(0)e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}T}}$$

$$= \frac{\sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{ij}}{\sqrt{\sum_{i=1}^n\sum_{j=1}^n a_ia_j\sigma_i\sigma_j S_i(0)S_j(0)e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}}.$$

Clearly $W_i(T)$ and U are normally distributed random variables (U is a sum of normal variables and so is normal [51]). The pair $(W_i(T), U)$ is bivariate normal, we write $(W_i(T), U) \sim BiN(\mu_{W_i(T)}, \mu_u, \sigma_{W_i(T)}^2, \sigma_u^2, \rho_{W_i,U})$, where $\mu_{W_i(T)}$ and μ_u are the means and

$\sigma^2_{W_i(T)}, \sigma^2_u$ are variances. Then $W_i(T)|U$ is normally distributed with expectation and the variance being

$$\mathbb{E}^Q(W_i(T)|U = \gamma) = \mu_{W_i(T)} + \frac{\rho_{W_i,U}\sigma_{W_i(T)}}{\sigma_u}(\gamma - \mu_{W_i(T)})$$

$$= \gamma \frac{\sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{ij}}{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0)S_j(0)e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}$$

$$\text{Var}^Q(W_i(T)|U = \gamma) = \sigma^2_{W_i(T)}(1 - \rho^2_{W_i,U})$$

$$= T\left(1 - \frac{\left(\sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{ij}\right)^2}{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0)S_j(0)e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}\right)$$

respectively. Now we will determine the expectation of $a_k S_k(T)$ conditional on $U = \gamma$:

$$\mathbb{E}^Q(a_k S_k(T)|U = \gamma) = a_k S_k(0)e^{(r-\frac{\sigma_k^2}{2})T}\mathbb{E}^Q(e^{\sigma_k W_k(T)}|U = \gamma)$$

$$= a_k S_k(0)e^{(r-\frac{\sigma_k^2}{2})T}\exp\left\{\sigma_k\mathbb{E}^Q(W_k(T)|U = \gamma) + \frac{\sigma_k^2}{2}\text{Var}^Q(W_k(T)|U = \gamma)\right\}$$

$$= a_k S_k(0)\exp\left(rT + \frac{\sigma_k\gamma\sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{kj}}{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0)S_j(0)e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}\right.$$

$$\left. - \frac{T\left(\sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{kj}\right)^2}{2\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0)S_j(0)e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}\right).$$

Consequently, (9.4) becomes

$$\sum_{k=1}^n a_k S_k(0)\exp\left(rT + \frac{\sigma_k\gamma\sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{kj}}{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0)S_j(0)e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}\right.$$

$$\left. - \frac{T\left(\sum_{j=1}^n a_j\sigma_j S_j(0)e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{kj}\right)^2}{2\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0)S_j(0)e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}\right) = K. \qquad (9.5)$$

Theorem 9.5. *Let γ satisfy (9.5), then the optimal lower bound for a EBO is given by*

$$\check{C}(K,T) = \sum_{k=1}^{n} \left\{ a_k S_k(0) \Phi \left(\frac{-\gamma^* + T\sigma_k \sum_{j=1}^{n} a_j \sigma_j S_j(0) e^{(r-\frac{1}{2}\sigma_j^2)T} \rho_{kj}}{\sqrt{T \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_i \sigma_j S_i(0) S_j(0) e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T} \rho_{ij}}} \right) \right.$$
$$\left. - \frac{K}{n} e^{-rT} \Phi \left(\frac{-\gamma^*}{\sqrt{T \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_i \sigma_j S_i(0) S_j(0) e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T} \rho_{ij}}} \right) \right\},$$

(9.6)

where $\Phi(.)$ is the cumulative normal distribution function.

Proof. Denoting the lower bound by $\check{C}(K,T)$, from Proposition 9.2 we have:

$$\check{C}(K,T) = e^{-rT} \sum_{i=1}^{n} \left[\mathbb{E}^Q \left(a_i S_i(T) - \frac{K}{n} \right) \mathbb{1}_E \right]$$
$$= e^{-rT} \sum_{i=1}^{n} \mathbb{E}^Q \left(e^{\ln(a_i S_i(T))} \mathbb{1}_E - \frac{K}{n} \mathbb{1}_E \right).$$

From Chapter 4, Proposition (4.2): If $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ then

$$\mathbb{E}^Q(e^X \mathbb{1}_{\{Y>0\}}) = e^{\mu_x + \frac{\sigma_x^2}{2}} \Phi \left(\frac{\mu_y + c}{\sigma_y} \right),$$

where c is the covariance between X and Y. We can easily determine this value as

$$c = \text{Cov}^Q \left(\ln a_i S_i(T), \sum_{j=1}^{n} a_j \sigma_j S_j(0) e^{(r-\frac{1}{2}\sigma_j^2)T} W_j(T) - \gamma \right)$$
$$= \text{Cov}^Q \left(\ln a_i S_i(0) + \left(r - \frac{\sigma_i^2}{2} \right) T + \sigma_i W_i(T), \sum_{j=1}^{n} a_j \sigma_j S_j(0) e^{(r-\frac{1}{2}\sigma_j^2)T} W_j(T) - \gamma \right)$$
$$= \sigma_i \text{Cov}^Q \left(W_i(T), \sum_{j=1}^{n} a_j \sigma_j S_j(0) e^{(r-\frac{1}{2}\sigma_j^2)T} W_j(T) \right)$$
$$= T\sigma_i \sum_{j=1}^{n} a_j \sigma_j S_j(0) e^{(r-\frac{1}{2}\sigma_j^2)T} \rho_{ij}.$$

Therefore

$$\mathbb{E}^Q(e^{\ln a_k S_k(T)} \mathbb{1}_E) = \exp\left\{\ln a_k S_k(0) + \left(r - \frac{\sigma_k^2}{2}\right)T + \frac{\sigma_k^2}{2}T\right\} \cdot$$

$$\Phi\left(\frac{-\gamma + T\sigma_k \sum_{j=1}^n a_j \sigma_j S_j(0) e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{kj}}{\sqrt{T\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0) S_j(0) e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}}\right)$$

$$= a_k S_k(0) e^{rT} \Phi\left(\frac{-\gamma + T\sigma_k \sum_{j=1}^n a_j \sigma_j S_j(0) e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{kj}}{\sqrt{T\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0) S_j(0) e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}}\right)$$

and

$$\mathbb{E}^Q(\mathbb{1}_E) = \Phi\left(\frac{-\gamma}{\sqrt{T\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0) S_j(0) e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}}\right).$$

Consequently the bound becomes

$$\check{C}(K,T) = \sum_{k=1}^n \left\{ a_k S_k(0) \Phi\left(\frac{-\gamma^* + T\sigma_k \sum_{j=1}^n a_j \sigma_j S_j(0) e^{(r-\frac{1}{2}\sigma_j^2)T}\rho_{kj}}{\sqrt{T\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0) S_j(0) e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}}\right)\right.$$

$$\left. - \frac{K}{n} e^{-rT} \Phi\left(\frac{-\gamma^*}{\sqrt{T\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_i \sigma_j S_i(0) S_j(0) e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T}\rho_{ij}}}\right)\right\}.$$

This completes the proof. $\square$

9.4 Moment Matching Method

The moment matching method is used when the distribution of a random variable is not known (Joshi [35], Brigo et al. [11], Deelstra et al. [20]). In this method, we assume a particular distribution based on some assumption or observations of the random variable. When we have settled for the distribution, we then calibrate it. The analytical pricing of European basket options rests entirely on discovering the distribution for the sum of lognormal variables. In order to use the moment matching technique, we approximate

the sum of the lognormal variables by a lognormal variable [11]. This is intuitive and the success of the method depends to some extent on how we calibrate the lognormal distribution, i.e, determine the mean and the variance of the lognormal distribution.

For convenience, let us define

$$\mathbb{S}_T = \sum_{i=1}^{n} a_i S_i(T)$$

and

$$Y_i(T) = (r - \frac{1}{2}\sigma_i^2)T + \sigma_i W_i(T).$$

The variable $Y_i(T)$ is normally distributed with mean $(r - \frac{1}{2}\sigma_i^2)T$ and variance $\sigma_i^2 T$. The mean of $\mathbb{S}_T$ is given by

$$
\begin{aligned}
\mathbb{E}(\mathbb{S}_T) &= \sum_{i=1}^{n} a_i S_i(0)\mathbb{E}(e^{Y_i(T)}) \\
&= \sum_{i=1}^{n} a_i S_i(0)e^{rT}.
\end{aligned}
\tag{9.7}
$$

The random variable $e^{Y_i(T)}$ follows a lognormal distribution. We write

$$e^{Y_i(T)} \sim \text{Log}N((r - \frac{1}{2}\sigma_i^2)T, \sigma_i^2 T).$$

We have assumed that the underlying Brownian motions are correlated. Of interest is the variable $Y_i(T) + Y_j(T)$. The mean is directly found to be $(2r - \frac{1}{2}(\sigma_i^2 + \sigma_j^2))T$. We proceed to determine the variance as

$$
\begin{aligned}
\text{Var}(Y_i(T) + Y_j(T)) &= \text{Var}(Y_i(T)) + \text{Var}(Y_j(T)) + 2\text{Cov}(Y_i(T), Y_j(T)) \\
&= \sigma_i^2 T + \sigma_j^2 T + 2\text{Cov}(\sigma_i W_i(T), \sigma_j W_j(T)) \\
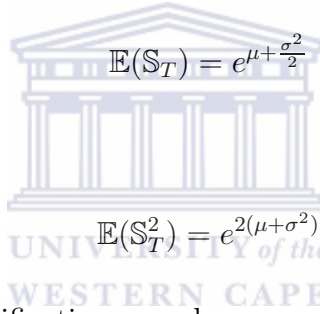&= (\sigma_i^2 + \sigma_j^2 + 2\sigma_i\sigma_j\rho_{ij})T.
\end{aligned}
$$

Therefore

$$e^{Y_i(T)+Y_j(T)} \sim \text{Log}N((2r - \frac{1}{2}(\sigma_i^2 + \sigma_j^2))T, (\sigma_i^2 + \sigma_j^2 + 2\sigma_i\sigma_j\rho_{ij})T).$$

Since we have already found the first moment (mean) of $\mathbb{S}_T$, we now determine its second moment.

$$\mathbb{E}(\mathbb{S}_T^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j S_i(0) S_j(0) \mathbb{E}(e^{Y_i(T)+Y_j(T)})$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j S_i(0) S_j(0) e^{(2r-\frac{1}{2}(\sigma_i^2+\sigma_j^2))T+\frac{1}{2}(\sigma_i^2+\sigma_j^2+2\sigma_i\sigma_j\rho_{ij})T)}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j S_i(0) S_j(0) e^{(2r+\sigma_i\sigma_j\rho_{ij})T}. \tag{9.8}$$

Let us now assume that $\mathbb{S}_T \sim \mathrm{Log}N(\mu, \sigma)$. In that case the first and second moments are given by

$$\mathbb{E}(\mathbb{S}_T) = e^{\mu+\frac{\sigma^2}{2}}$$

and

$$\mathbb{E}(\mathbb{S}_T^2) = e^{2(\mu+\sigma^2)}$$

respectively. For further clarifications, readers are referred to the discussion on the log-normal distribution in Chapter 1. The mean μ and variance σ^2 are given by

$$\sigma^2 = \ln\mathbb{E}(\mathbb{S}_T^2) - 2\ln\mathbb{E}(\mathbb{S}_T)$$
$$\mu = \ln\mathbb{E}(\mathbb{S}_T) - \frac{\sigma^2}{2}.$$

The explicit expressions for $\mathbb{E}(\mathbb{S}_T)$ and $\mathbb{E}(\mathbb{S}_T^2)$ are given by (9.7) and (9.8) respectively.

The problem of finding the price of a basket option has now been reduced to that of a European option. We just need to modify the Black-Scholes formula. This is due to the fact that in the European option case the price of the underlying asset at maturity is lognormally distributed hence the availability of the explicit formula. Let us write the price of a basket option as

$$C(n, K, T) = e^{-rT}\mathbb{E}(\mathbb{S}_T - K)^+.$$

Therefore

$$C(n, K, T) = e^{-rT}(e^{\mu + \frac{1}{2}\sigma^2}\Phi(d_1) - K\Phi(d_2)),$$

where

$$d_1 = \frac{\mu + \sigma^2 - \ln K}{\sigma},$$

$$d_2 = d_1 - \sigma.$$

The above expressions are derived in a similar manner as those for the geometric Asian options in Chapter 2 Section 2.4. The details are therefore omitted.

9.5 Monte Carlo method for pricing European basket options

The Monte Carlo method will be used again as another pricing method (see, e.g., Deelstra et al. [21], Glasserman [27]) for European basket options. As we saw in Chapter 6, the Monte Carlo is easy to use. The problem with it is that it takes much computer resources, that is, it needs lots of computer memory. Consequently, the use of variance reduction techniques to improve the Monte Carlo simulations cannot be over-emphasised.

The second idea which needs to be addressed is generating correlated prices for each asset in the basket option (see Glasserman [27] for more). We assumed that the returns of the assets are correlated, with correlation ρ_{ij} for pairs of assets. We will use the following idea of *Cholesky decomposition* from linear algebra to simulate correlated prices.

Let $\mathbf{C}$ and $\mathbf{Z}$ be $n \times 1$ vectors, that is $\mathbf{C} = (c_1, c_2, \cdots, c_n)^T$, $\mathbf{Z} = (z_1, z_2, \cdots, z_n)^T$, where $z_i \sim N(0, 1)$. We know that

$$c_1 z_1 + c_2 z_2 + \cdots + c_n z_n \sim N(0, c_1^2 + c_2^2 + \cdots + c_n^2). \tag{9.9}$$

We note that $c_1^2 + c_2^2 + \cdots + c_n^2 = \mathbf{C}^T\mathbf{C}$. In general C can be $n \times m$. In this case, the matrix $\mathbf{C}^T\mathbf{C}$ is referred to as the *covariance matrix*. It is usually denoted by $\mathbf{\Sigma}$. The ij element of the covariance matrix $\mathbf{\Sigma}_{ij}$ gives the covariance of random variables X_i and X_j, i.e, $\mathbf{\Sigma}_{ij} = \mathrm{Cov}(X_i, X_j)$. The above equation (9.9) may be written as

$$\mathbf{C^T Z} \sim \mathbf{N(0, \Sigma)},$$

where $\mathbf{N}(,)$ indicates that its multidimensional.

Suppose a vector of random variables is such that $\mathbf{U} \sim \mathbf{N(0, \Sigma)}$. Then we can write $\mathbf{U}$ as $\mathbf{U} = \mathbf{C^T Z}$. To simulate correlated observations of $\mathbf{U}$, the problem reduces to finding $\mathbf{C}$ such that $\mathbf{C}^T\mathbf{C} = \mathbf{\Sigma}$. Some of the properties of the covariance matrix $\mathbf{\Sigma}$ are

  (i) positive semidefinite

 (ii) $\mathbf{\Sigma}_{ii} \geq 0$.

The diagonal entries give the variance of each of the vectors since $\mathbf{\Sigma}_{ii} = \mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i)$. The positive semidefiniteness of $\mathbf{\Sigma}$ ensures that it can be factorised by the *Cholesky decomposition*.

We will now demonstrate how to generate correlated prices for a basket option with two assets. The process can easily be generalised for a basket of n assets. We know that

$$S_1(T) = S_1(0)e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 W_1(T)}$$
$$S_2(T) = S_2(0)e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 W_2(T)}.$$

Let $X_i = \sigma_i W_i(T)$, for $i = 1, 2$. We can write

$$\mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i) = \sigma_i^2 T$$
$$\mathrm{Cov}(X_i, X_j) = \sigma_i \sigma_j \rho_{ij} T.$$

Consequently, the covariance matrix for $\mathbf{X} = (X_1, X_2)^T$ is

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 T & \sigma_1\sigma_2\rho_{12}T \\ \sigma_1\sigma_2\rho_{12}T & \sigma_2^2 T \end{pmatrix}.$$

As we have explained, by using Cholesky decomposition, we can find a matrix $\mathbf{C}$ such that $\mathbf{C}^T\mathbf{C} = \mathbf{\Sigma}$. The correlated $\mathbf{X}$ denoted by $\mathbf{X}^c$, is then found by $\mathbf{X}^c = \mathbf{C}^T\mathbf{Z}$.

The last considerations pertain to the choice of the variance reduction technique. If the results of Chapter 6 are anything to go by, then we would prefer a control variate method. An example of a control variate is the geometric basket option [21]

$$C^G(n, K, T) = e^{-rT}\mathbb{E}\left(\prod_{i=1}^{n} S_i(T)^{a_i} - K\right)^+.$$

The prices we obtained by using this control variate procedure, show that this is not a good procedure. However, the antithetic method gave satisfactory results. The values obtained were essentially the ones obtained by [21]. Our tabulated results are a product of the antithetic method. Here is the algorithm that we will use for our Monte Carlo method for the two assets:

Algorithm 9.5.1: Antithetic Monte Carlo(m)

$\mathbf{C} \leftarrow$ Cholesky decomposition of $\mathbf{\Sigma}$

for j **from** 1 **to** m

do $\begin{cases} \text{generate} \quad \mathbf{Z} = (z_1, z_2)^T \sim N(0,1) \\[4pt] \mathbf{X} \leftarrow \mathbf{C}^T \mathbf{Z} \\[4pt] S_1(T) \leftarrow S_1(0)e^{(r-\frac{1}{2}\sigma_1^2)T + X_1} \\[4pt] S_1^*(T) \leftarrow S_1(0)e^{(r-\frac{1}{2}\sigma_1^2)T - X_1} \quad \text{(antithetic price)} \\[12pt] S_2(T) \leftarrow S_2(0)e^{(r-\frac{1}{2}\sigma_2^2)T + X_2} \\[4pt] S_2^*(T) \leftarrow S_2(0)e^{(r-\frac{1}{2}\sigma_2^2)T - X_2} \quad \text{(antithetic price)} \\[16pt] C_j \leftarrow e^{-rT}\max\left(a_1 S_1(T) + a_2 S_2(T) - K, 0\right) \\[4pt] C_j^* \leftarrow e^{-rT}\max\left(a_1 S_1^*(T) + a_2 S_2^*(T) - K, 0\right) \\[12pt] \bar{C}_j = \frac{1}{2}(C_j + C_j^*) \end{cases}$

$\hat{C} \leftarrow \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} \bar{C}_j$

Algorithm 9.5.1 shows that the Monte Carlo method for European basket options is less complicated than the Asian option Monte Carlo method. The reason is that the basket option which we are considering is a vanilla type; the price of the asset at maturity is what matters to us. One may recall that the Asian options are path-dependent.

9.6   European basket option results

In this section we perform calculations with the three methods; Optimal lower bound, Monte Carlo and Moment Matching.

In Table 9.1 we have used equally weighted, i.e, $a_1 = a_2 = 0.5$, the volatilities of the two assets σ_1 and σ_2 are equal and take the values 0.1 or 0.2. Spot prices are equal, i.e, $S_1(0) = S_2(0) = 100$. The correlation between the two asset return is 0.2 or 0.8. The time of holding the option is T with take either 1 or 3 and $r = 0.05$. We see that our lower bound is very close to the Monte Carlo price. The maximum error incurred is 0.08 and the least is 0.0002. The moment matching method is even more accurate. The maximum error incurred is 0.004 and the least is 0.0006.

In Table 9.2 we consider unequally weighted assets in the basket, i.e, $a_1 \neq a_2$. We take two asset $n = 2$, with different spot prices $S_1(0)=130$, $S_2(0) = 70$. The parameters T, $r = 0.05$ and σ_2 are taken analogous to the previous case. The lower bound is also very close to the Monte Carlo values. The maximum error incurred is 0.07 and the least is 0.0006. The Moment Matching method is more accurate than the lower bound. The maximum error incurred is 0.008 and the least is 0.0006.

Table 9.3 shows the results of taking unequally weighted assets in the basket, i.e, $a_1 \neq a_2$ unequal spot prices $S_1(0) \neq S_2(0)$. This time the volatilities σ_1 and σ_2 are unequal. The time T is 5 years and $r = 0.09$ and there are 2 assets in the basket. Moreover, the strike price is 35. The lower bound generally performs better the Moment matching method. In particular, for low volatilities. The lower bound incurs a maximum error of 0.05 and the least error of 0.0001. On the other hand the Moment matching method incurs an maximum error of 0.16 and a least error of 0.0009. The Moment matching method does not perform well for high volatilities where the error can be as big as 0.16. We conclude that the Moment matching method does not always give better results.

Table 9.1: Comparison of Lower bound obtained by using (9.6), Monte Carlo simulations and Moment Matching for $n = 2$, $r = 0.05$, $a_1 = a_2 = 0.5$, $S_1(0) = S_2(0) = 100$.

K	T	Correlation	Volatility	Monte Carlo	Lower bound	Moment Match
		ρ	$\sigma_1 = \sigma_2$			
117	1	0.2	0.1	0.31216	0.31195	0.31279
			0.2	2.37865	2.36810	2.38175
		0.8	0.1	0.65066	0.64921	0.64927
			0.2	3.57133	3.56994	3.57069
123	3	0.2	0.1	3.07084	3.06057	3.07060
			0.2	8.38480	8.29776	8.38278
		0.8	0.1	4.20915	4.20713	4.20766
			0.2	10.70939	10.70096	10.70533

Table 9.2: Comparison of Lower bound obtained by using (9.6), Monte Carlo simulations and Moment Matching for $n = 2$, $r = 0.05$, $a_1 = 0.3$, $a_2 = 0.7$, $S_1(0) = 130$, $S_2(0) = 70$.

K	T	Correlation ρ	Volatility $\sigma_1 = \sigma_2$	Monte Carlo	Lower bound	Moment Match
97	1	0.2	0.1	1.16689	1.16500	1.16625
			0.2	2.72347	2.71138	3.72491
		0.8	0.1	1.69408	1.69373	1.69381
			0.2	4.89570	4.89311	4.89382
108	3	0.2	0.1	2.78607	2.77739	2.78616
			0.2	7.48480	7.41850	7.49250
		0.8	0.1	3.77637	3.77579	3.77624
			0.2	9.51068	9.50135	9.50519

Table 9.3: Comparison of Lower bound obtained by using (9.6), Monte Carlo simulations and Moment Matching for $n = 2$, $r = 0.09$, $a_1 = 0.52$, $a_2 = 0.48$, $S_1(0) = 29$, $S_2(0) = 43$.

K	T	Correlation	Volatility		Monte Carlo	Lower bound	Moment Match
		ρ	σ_1	σ_2	ρ		
35	5	0.2	0.05	0.1	13.40373	13.40327	13.40377
		0.6			13.40468	13.40454	13.40553
		0.9			13.40679	13.40664	13.40793
		0.2	0.1	0.2	13.53949	13.53021	13.60738
		0.6			13.65207	13.64820	13.71300
		0.9			13.74663	13.74547	13.80073
		0.2	0.2	0.3	14.38692	14.33540	14.55644
		0.6			14.79805	14.78219	14.91392
		0.9			15.10367	15.10240	15.19208

Conclusion

In this thesis, we have discussed how one can price arithmetic Asian options numerically. Firstly, we considered a Monte Carlo procedure where two variants where explored to make the process more efficient and feasible. The control variate method turned out to be more accurate as compared to the antithetic variates method. Even with variance reduction, the Monte Carlo method was time consuming and uneconomic as far as computer resources are concerned.

The price of the Asian option was characterised by a linear parabolic partial differential equation which was solved by finite difference schemes, namely, the explicit method and the implicit (Crank-Nicholson's) method. Of the two, the Crank-Nicholson method was found to be unconditionally stable and converged to the true solution. There was a trade-off between speed and accuracy with regards to the Crank-Nicholson and the Monte Carlo methods. Though the Monte Carlo method was more accurate it was slow whereas the Crank-Nicholson method was less accurate but faster. The worst case scenario was when the volatility of the underlying asset was very low (0.05). Currently we are investigating possible improvements in our methods.

As far as the analytical approximations for the price of the Asian option are concerned, a lower bound was explored. The lower bound was very close to the Monte Carlo price, that we could literally take it to be the price of the Asian option. Our results also confirm some other facts about Asian options studied from different pespectives by various researchers, e.g. we found that the Asian option is cost-effective, i.e., cheaper than plain vanilla European options. Also due to its averaging nature, an arithmetic Asian option is less sensitive to changes of volatility of the underlying asset.

We extended our methods to price a European basket option (EBO). We have derived an optimal lower bound for this option and evaluated it. Then we compare the results

with other two methods, namely, the Monte Carlo method and the Moment matching method. With the exception of generating correlated prices, the Monte Carlo method was less complicated and hence faster than the corresponding Monte Carlo method for Asian options. Structural differences accounted for this observation. The EBO is path independent whereas the Asian option is path dependent. We were also successfully able to adapt variance reduction techniques for Asian options to the EBO.

We then derived a lower bound based on the conditioning method. Various calculations confirmed that the bound was indeed optimal. We also approximated the basket, i.e., sum of lognormal variables by a lognormal distribution and got reasonable accuracy.

There are possible extensions to the thesis that can be considered. As a starting point, the assumptions of the Black-Scholes model could be relaxed. The volatility of the underlying asset could be taken as a function of time or it could be driven by a stochastic differential equation. This gives rise to stochastic volatility models an example being the Heston model ([29]). There is a lot of empirical evidence from actual market data to suggest that the log of returns on assets is not logonormally distributed as in the Black-Scholes model. We could model the prices of assets with general stochastic processes like the Lévy processes which take the possibility of jumps in price processes into account (see e.g., [49]).

We could also improve the numerical solutions. The low volatility problem where the PDE approach suffered most could be looked from a perturbation theory perspective and asymptotic solutions could be explored (see [61]).

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