The role of Visualization in the Teaching and Learning of Multivariate Calculus and Systems of Ordinary Differential Equations

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List of Acronyms

2D: refers to objects in the *xy*-cartesian or the r-θ polar coordinate plane

3D: refers to objects in the x-y-z, the r-θ-z or the ρ-ϕ-θ coordinate systems

APOS: Dubinsky's Action-Process-Object-schemas framework to gauge students' level of understanding of maths concepts.

CalcPlot 3D : Applet used in illustrating 3D calculus concepts by P. Seeburger

CSTR: Controlled Stirred Tank Reactor

DE/ODE: Ordinary differential equation

DS: Dynamical Systems defined by systems of differential equations

LV: Lotka-Volterra

MVC : Multivariate Calculus

MVT : Maths Visualization Toolbox - Software program for maths visualization

ODE solver: Software for plotting phase portraits of ODE systems

SAS : Statistical Analysis Software

VA : Visualization-Analysis framework

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Declaration

I hereby declare that I am the sole author of this dissertation and I have not submitted it to any other university for a qualification. This is a true copy of the dissertation, including the required final revisions, as recommended by my supervisors and examiners. All sources I have used or quoted have been indicated and acknowledged as complete references. I understand that my dissertation may be made electronically available to the public.

Signed, T. Sheikh,

15 March 2016

Abstract

The purpose of this study was to investigate the role of visualization in the conceptualisation and solution of problems in multivariate calculus and dynamical systems. The theoretical basis, and the visual and analytical aspects of evaluating multiple integrals, and the stability analysis of dynamical systems, were established. To address the research questions, a teaching experiment with activities to facilitate visualization of 3D objects and phase portraits of non-linear dynamic systems was conducted with an experimental class (n = 24) which received six activity sessions in the computer Laboratory in addition to traditional lectures. The control class (n = 26) received traditional lectures and tutorial instruction. Both groups were lectured by the researcher using the same set of class notes, assignments, worksheets and tutorials. Additional support materials were posted on the Blackboard on Web-City. The activities included tasks in the computer laboratory that reinforced visualization and spatial ability factors such as surface features, nets, projections, cross-sections and rotation of 3D objects as well as phase portraits of systems of differential equations.

The students were tested at several time points, and over both the short and long term to assess the impact on their visual and analytical solutions to problems in the two study domains. The pre-test on prior knowledge indicated no significant differences between the means of the experimental and control groups.

Results indicate that there were no significant differences between the achievement of the two groups in Test 1 and Test 2 while the activities were ongoing, but towards the end of the semester significant differences in favour of the experimental group were recorded. A multiple linear regression analysis confirmed that in addition to prior knowledge as measured by the pre-test, two of the spatial factors were significant predictors of achievement for the domains under investigation. Students had difficulties in visualising 3D regions of integration and in

switching the order of triple integrals. Very few (18%) recognised the need for split integrals to span the required area or volume.

While students could find analytical solutions to systems of differential equations and describe the stability of individual equilibrium points using eigenvalues, they struggled with translating rates of change into slopes on the phase portraits, with the interpretation of the solutions and in describing the global behaviour of the system.

Students had difficulties in visualizing the region of integration in R^3, the stability of equilibrium points in the phase portraits, and in coordinating the treatments and conversions between the geometric, numerical, symbolic and algebraic registers. The tendency to work in the algebraic register to determine the limits of the integral was noted, and students opted to use analytic methods in conducting a stability analysis of the given dynamic system rather than the geometric method.

This study adds to research on visualization in mathematics by examining how exposure to technologically enhanced representations complement and promote the conceptualisation of solutions to problems involving multiple integrals and systems of differential equations.

Chapter 1: Introduction - Visualization in Mathematics

1.0 Overview

In this chapter, we define visualization, and look at the role it plays in the conception and solution of problems in multivariate calculus and dynamical systems. We explain the purpose, the rationale and significance of the study. The strategies proposed to facilitate visualization in the learning and teaching of multiple integrals, and phase portrait analysis of dynamical systems are outlined. We end the chapter by defining the limitations of the study and give an outline of the chapters in the dissertation.

1.1 Defining visualization and analytical thinking in mathematics

Depending on the field of study, visualization has been defined in diverse and multiple ways. A comprehensive and all-embracing definition, that draws together the various aspects of visualization, and serves our purpose in mathematics education, is by Arcavi (2003, p. 217):

> 'Visualization is the ability, the process and the product of creation, interpretation, use of , and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings'.

In mathematics education, to 'visualize' means to construct, create, or make connections between an external mathematical object or its representation (a diagram, a table, or a picture) and a mental or internal construct or image and apply analytical methods to develop and advance understanding. Interaction with the mental image can be through physical models, manipulatives, sketches, computer-based static outputs or animations such as simulations. The ability to draw a simple figure to represent a mathematical problem, to interpret such figures with understanding, and to use such figures as an aid in problem solving are fundamental visualization skills.

The ‘thinking about and developing’ in Arcavi’s (2003, p. 217) definition is an important feature of visualization that distinguishes it from visual perception. Vision or visual perception provides direct access to the mathematical object and involves exploration of its physical properties such as number and geometry of faces, vertices, edges, angles, symmetry and so on. Vision is not visualization; to see or perceive is not necessarily to understand. Duval (1999, p.7) refers to visualization as ‘operative apprehension’ which involves exploration and coordination of mental images that we construct and reconstruct as we reason or analyse what we see with the eyes.

Phillips, Norris and Macnab (2014 , p. 26) have made three important distinctions in types of visualization:

1. ‘Visualization objects’ include physical objects, 3D representations, sketches, and pictures that can be viewed on paper, computer displays, slides etc.
2. ‘Introspective visualization’: are mental images constructed through visual experiences. It does not necessarily involve physical objects
3. ‘Interpretative visualizations’ involve making meaning by interpreting information from the objects or introspections.

Guiterrez (1996, pp. 7 – 10) also has four main elements of visualization in his scheme, namely, external representations, mental images, processes of visualization and abilities of visualization. Mental images include verbal or linguistic symbols as well as picture images. They are a tool for our own individual cognitive (mental thinking) processes. Mental imagery is the act of forming mental pictures of objects or events and does not necessarily involve the eyes. It serves as a kind of ‘mental blackboard’ where images can be recalled from memory and can assist in active and dynamic information processing. The visual image or entity, also known as a cognitive object, can be mentally recreated, explored and manipulated consciously or unconsciously during reasoning. It is held in the working memory and can be recalled for comparison or manipulation or for creating new, simple or complex, visual images. This aspect of mental imagery in visualization is being actively researched by cognitive psychologists. In mathematics education, we are concerned with the interaction of mental images with external visual representations like diagrams, pictures, and sketches that assist in analytical thinking and solutions to problems.

Zazkis, Dautermann and Dubinsky (1996, p. 442) define an act of analysis or analytical thinking as ‘the mental manipulation of objects with or without the aid of symbols’. Here ‘object’ is defined in terms of the Action-Process-Object-Schemas (APOS) in Dubinsky’s (1991) framework. An action is a transformation of a mathematical object using explicit algorithms. As students repeat and reflect on actions, they interiorize them into a process. When they become aware of the process as a whole, or encapsulate it, an object conception is constructed. A collection of action, process and object conceptions, constitute a mathematical schema, which are then synthesized to form mathematical structures. The processes involve reflection, abstraction, coordination, reversal and encapsulation that are essential elements of analysis. For the purpose of this research, we consider visualization and analysis as two interacting modes of thinking that support each other in developing understanding and problem solving.

Visualization has an important role to play in the problem representation process (Kosslyn & Koenig, 1995). The value of visual representations, on paper or in the head, lies in its potential to facilitate and generate analytical thinking (mathematical thought) and can be an important aid to solving all sorts of problems, including problems in which nothing geometric is evident (Zimmermann & Cunningham, 1991).

Visualization is based on the production of semiotic representations, which could be 1D or 2D geometrical shapes, Cartesian graphs, sketches, propositions or words and using them to solve problems. Semiotics refers to the signs and symbols used in writing and communicating mathematics. We use a wide range of semiotic systems such as ‘natural’ language, numeric and algebraic notation, graphs and diagrams in mathematics. Semiotic systems allow different kinds of operations and have different potentials for meaning making (Duval, 2000 ; O'Halloran, 2005).

Stylianou (2002) investigated characteristics of visual representations that underlie problem solving across ages and levels of mathematics knowledge and concluded that successful understanding is related to selecting what needs to be visualized and verbalised and the oscillation between the visualized and verbalised components. We illustrate these ideas with two examples, adapted from Stylianou (2002, p. 308 & p. 314).

Example 1: Will the given net fold into a closed cube?

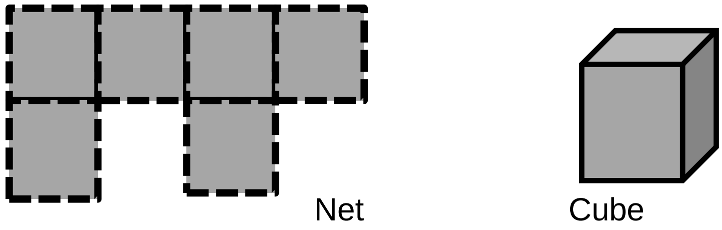


Figure 1.1 Visualizing the folding and unfolding of a cube and its net.
Adapted from Stylianou (2002, p.308)

This question involves seeing if the net in 2D, shown in Figure 1.1, will fold into the cube in 3D. The initial perceived figure in 2D, at first glance, perceptual apprehension, on Duval's (1999) framework, shows six squares, essential figural units for the six faces of the cube.

Next we look for various ways in which the arrangement of the units can be modified. By folding the net, we visualise a cube in 3D, as it forms, making sure we have no overlaps between the faces and that the faces are at right angles to each other. Mentally we construct and de-construct the cube by folding and unfolding, rotating, and matching the sides and marking them, if necessary. We may fold the four squares at the top to form a square tube and then the squares at the bottom. We conclude that the net will not give us a closed cube as the squares at the bottom overlap. This task involves spatial ability, an aspect of visualization dealing with objects in space. It also involves analytical thinking; matching sides, checking angles, folding, unfolding, marking sides that match. The visualization is much easier if, in the past, we have done activities such as drawing, cutting and folding 2D nets and the 3D solids we get from them. The process is reversible in that we could unfold the 3D solid to explore its net.

Example 2: The second example is similar, but we now visualise unfolding a truncated cylinder into its net (See Figure 1.2). The question: Which of the three nets A, B or C is the net for the truncated cylinder?

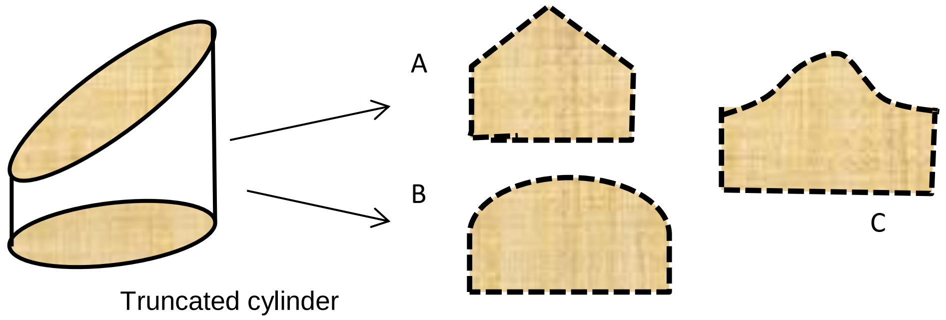
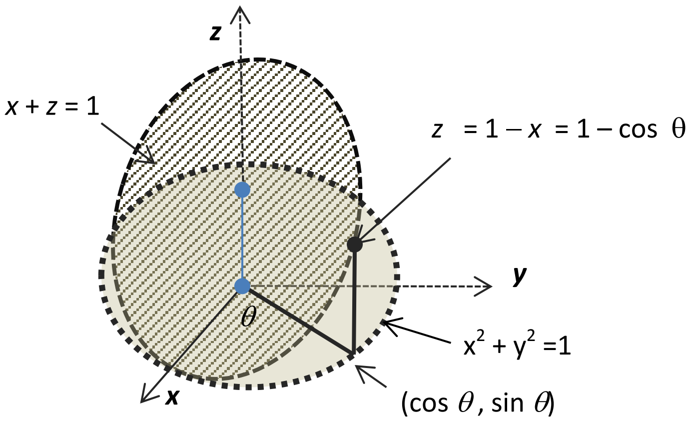


Figure 1.2 Which of A, B, C is the net for the truncated cylinder? Adapted from Stylianou (2002, p.314)

The initial perceived figure (perceptual apprehension) reveals a circular base, an elliptical top, and a curved surface that make the sides of the truncated cylinder. Mathematically speaking, we have an oblique plane intersecting a cylinder. The intersection looks elliptical. Alternative A looks appealing but, will the ellipse at the top of the truncated cylinder give us the triangle, with straight edges, at the top of the net in A? What about B? Will folding/unfolding activities help? One way is to unfold a truncated cylinder and check its net. That has obvious limitations of accuracy. How do we show (analytically) that C is the correct solution? The analytical thinking is deeper and if we are to avoid guesswork, it involves changing from geometric to algebraic representation.

This problem was posed to mathematics educators by Stylianou (2002, p. 314). After several visual-analytic steps, one mathematician in her research sample, arrived at the sketch similar to that shown in Figure 1.3(a), in which the cutting plane is represented by $x + z = 1$ and the cylinder by $x^2 + y^2 = 1$. The cutting plane was moved down to intersect with the circular base, whose equation is $x^2 + y^2 = 1$. Any point on the base has xy-coordinates ($\cos\theta$, $\sin\theta$). By moving around the base and checking how this movement affects the height, z, on the cutting edge of the cylinder, we arrive at the trigonometric expression $z = 1 - \cos\theta$. The final step is to plot a graph of z with θ and match points on the graph with corresponding points on the cutting edge. Note θ is in radians, is measured anticlockwise from the x-axis, and runs from 0 to 2π.



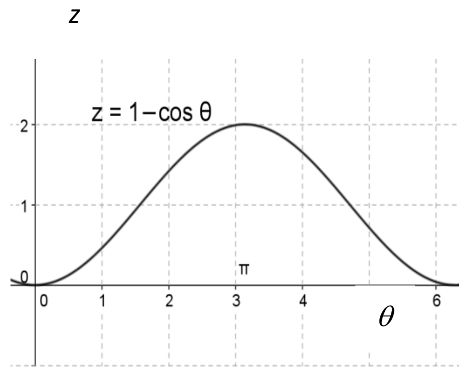


Figure 1.3 (a) Graphical representation of the truncated net problem b) graph z vs θ
Adapted from Stylianou (2002, p. 314).

A conversion from the geometric register to the algebraic enables an environment in which we can work with the critical variables more easily. Looking back at our definition of visualization, we note perceptual apprehension, abstraction, reflection, coordination of mental and external images, and acts of analysis. We note conversion from the geometric register to the algebraic, and finally, to the graphic register. We see treatments within algebraic and geometric registers. Prior learning plays an important part as sketches need to be drawn, variables selected, equations and formulae have to be recalled, manipulated and solved, and graphs need to drawn.

On page 22, we look at the role of visualization and analysis in finding the volume of a truncated cylinder. This research is concerned about the role of visualization in the solution of problems such as these.

In summary, in this section, we attempted to define visualization in mathematics education. We note that the interaction, connection or reflection that a person makes between the mental construct (cognitive object) and the mathematical object (physical or virtual), or its representation (internal or external), constitutes an act of visualization. Such a connection can be made in either of two directions. An act of visualization may consist of any mental construction of objects or processes that an individual associates with objects or events perceived by her or him as external. Alternatively, an act of visualization may consist of the construction, on some external medium such as paper, chalkboard or computer screen, of objects or events which the individual identifies with object(s) or process(es) in her or his mind. We stress that visualization is not just vision or visual perception or perceptual apprehension or visual representation. Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for understanding, problem solving and mathematical discovery. It involves what Duval (1999, p. 12), calls 'operative apprehension'.

1.2 Visualization in mathematics education

Visualization has played an important role in the development of mathematics throughout history. Early Pythagoreans, who developed mathematics in the modern sense, used visual representations to study the properties of geometric figures and relationships among numbers. Euclid's elements and Book of Fallacies have numerous geometrical representations that punctuate the reasoning elaborated through the accompanying text. Descartes (as reported in Massironi, 2010, p. 8) used numerous images and figures in geometrical thinking. The calculus of the seventeenth century had strong visual elements that interacted with geometrical and physical problems. However, paradoxes in the foundations of calculus created mistrust leading mathematicians to aggressively abandon visualization. This situation persisted into the 1980s and mathematics curricula were fairly devoid of visuals (Kaput, 1993). While visualization was considered an integral part of geometry, only recently have educators and researchers begun to explore its potential in calculus, algebra and statistics.

During the last two decades, there has been a renaissance in visualization driven by technological advances (Zimmermann and Cunningham, 1991a). Technology now provides greater access to multiple representations of mathematical concepts.

Arcavi (2003, p.221) highlights three powerful and complimentary roles of visualization:

a) as a support for and illustration of symbolic results;

b) as a way of resolving conflict between symbolic solutions and incorrect intuitions, and

c) as a way to recover conceptual underpinnings which may be overlooked by formal solutions. Several mathematics education researchers have highlighted the need for interaction and active engagement of the learner and the importance of translation among mathematical representations. Duval (1999) emphasises that we learn about mathematical objects by transforming their representations. When we calculate, prove, sketch, solve and work with representations such as equations, functions, groups, fields, etc. and transform them from one register to another, for example, from algebraic to geometric or numerical, we learn and get to know the mathematical object better.

We discuss representations in greater depth under Duval's theory of semiotic representations under the research framework in Chapter 3, section 3.2.

According to Piaget (1964) to know a mathematical object is to act on it, to modify it, to transform it and understand how it is constructed. Artigue (2002, p.248) notes that we work with mathematical objects 'through ostensive representations which can be very diverse in nature' and include: discourse in natural language, schemas, drawings, symbolic representations, gestures, manipulatives. We also work with non-ostensive objects that we bring to mind when doing mathematics.

The field of visualization is wide with specialists in psychology, radiography, geology, computer science and mathematics education using terms like spatial ability, visual reasoning, visual images, mental images and visual representations interchangeably. The proliferation of digital technologies such as the internet, the smart phone, I-pads, tablets, computers, programmable calculators and online learning systems have pushed the boundaries of visual learning and visual mathematics to new levels by making available powerful representation tools. While curriculum developers, teachers and textbook authors are paying more attention to drawings, pictures, and images in their publications and, psychologists have developed detailed frameworks for their research, there is far less research in the field of mathematics education on visualization. Tertiary students use visual tools and technology, visual arguments and visual representations more, but there is little in the literature that informs when, why and for what purpose and how they interact with other modes of representation and thinking. Given that technology increasingly influences and impacts on workplace practices and the teaching and learning of mathematics in higher education, there is a need to investigate ways in which the potential of technology and media can be used to enhance and reinforce the conception and solution of problems in mathematics. This also opens the possibility of tackling more complex problems that are a feature of the workplace. Kozma (1994) points out, that instead of asking questions about whether technology impacts learning, we should be looking at ways in which the new media impact and influence future learning.

In the 1990's, the debate over the role of visual versus analytical methods, intensified with some educational researchers (Gutierrez, 2012 ; Owens & Clements, 1998), claiming that visual representations (pictures, diagrams, graphs and images), whether they are imagined, drawn on paper or created using a technological tool, can aid conceptual understanding and be a resource for intuition and discovery. They see visualization as a key component of mathematical reasoning, modelling and problem solving. A good visual representation can serve to 'concretize the referent' (Presmeg , 1986, p.44) and provide the mental scaffolding needed in establish the meaning of a problem, channel problem-solving approaches, and influence cognitive constructions (Owens & Clements, 1998). Visual representations can condense information, and suggest new results or potential approaches to essentially symbolic results.

However, visualization in mathematics education also has its critics, who point out that the same objects can mean very different things, even to experts in the field. Learners may focus on aspects of a visual representation that experts consider distractions, irrelevant and dismissible. Other educators, for example, Tall (1994), claim that visualization has subtle aspects that can deceive and that we should adhere to formal or analytical methods of solving problems in mathematics. Hoz in Presmeg (1985, p.295) describes the ‘rigidity’ that results when student conceptions are limited by the use of diagrams or mental images. Presmeg (1986, p. 52) points out that a diagram may tie thought to irrelevant detail or may even introduce false data and induce inflexible thinking. Magidson (1989) noted that when Grade 7 students were asked to sketch graphs of $y = 2x + 1$, $y = 3x + 1$, $y = 4x + 1$ using software, few noticed that they all pass through (0, 1) and made mistakes in the intercept for the graph of $y = 5x + 1$.

Presmeg (1986) reported that high achievers were almost always non-visualizers. Eisenbeg and Dreyfus (1991) and Vinner and Dreyfus (1989), report that students often face difficulties and are reluctant to use visual representations in solving problems.

Eisenberg and Dreyfus (1991, p. 2) identified three reasons to explain the observed reluctance of some students to visualize:

a) cognitive (visual is more difficult),

b) sociological (visual is harder to teach), and one related to

c) beliefs about the nature of mathematics (visual is not mathematical).

According to the cognitive load theory (Sweller, 1999), splitting attention between visual representations and text, can overload working memory capacity. The cognitive demand is high when learners face conceptually rich images or when there are intervening conceptual structures (Fishbein, 1993).

1.3 Mathematical Representations

Mathematics, by its very nature is abstract, and we can only access mathematical concepts through their representations. We learn about mathematical objects through their representations (for example, functions, direction fields, graphs, equations, 3D space figures, and groups) which undergo transformations such as calculating, proving, and solving.

Representations are useful tools that support mathematical reasoning, enable mathematical communication and convey mathematical thought (Kilpatrick, Swafford & Findel, 2001). Calculus reform efforts have stressed the importance of using multiple representations to include analytic, numerical, graphic and symbolic in the solution of mathematics problems.

Lesh, Landau, & Hamilton (1983, p. 265) identify five distinct types of representations (See Figure 1.4) that students use to solve problems. These include (a) manipulative models such as nets (b) graphics, pictures, and diagrams (c) experiences that serve as context to describe and solve other problems (d) specialized forms of spoken languages, such as used by mathematicians (e) spoken or written language as used in context by non-mathematicians and (f) written symbols and phrases—including algebraic symbols. Translations and transformations between the representations occur through student activities such as simplifying, generalizing and draw upon spatial, logical, linguistic, and numerical competences.

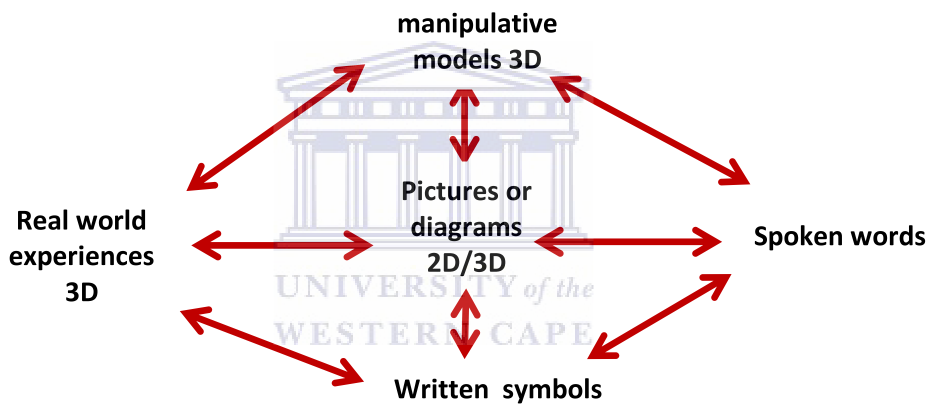


Figure 1.4 Representations useful for understanding mathematics, Lesh, Landau, and Hamilton (1983, p. 265)

The Lesh et al. (1983) model (See Figure 1.4), has particular relevance to our study and many similarities with Duval's theory of semiotic representations with its symbolic, verbal, geometric, algebraic and numerical registers. Connections between the representations are made by translations and transformations in the Lesh, Landau and Hamilton model, whereas in Duval's (1995) model we talk about treatments and conversions within and between the registers. We are interested in the difficulties and type of errors students make as they use representations and translate between representations when they solve problems in Multivariable Calculus and Dynamical systems.

While different semiotic systems may appear to be used to refer to similar mathematical objects, it is important to recognise that a particular representation may have different meanings. For example, the verbal description, 'paraboloid' can be represented by an equation, $z = x^2 + y^2$, or $z = r^2$ (in cylindrical coordinates), a graph in 3D or a table of numerical values.

We distinguish between external and internal representations. External representations such as symbols, graphs, sketches, textbook illustrations, diagrams, and geometric drawings may be examined, analysed, and processed by the perceptual system, as means of amplifying cognition. They include interactive learning tools such as Geogebra, Matlab and CalcPlot3D that are useful for static diagrams as well as animations. Internal representations or mental models represent knowledge and structures include schemas and propositions.

In summary, while we take cognisance of the debate around the merits, or otherwise, of mathematical representations, there is consensus among educators that visual representations like 2D and 3D sketches and diagrams, direction fields and phase portraits, can facilitate communication, conceptualisation, and function as a tool for thinking and intuition, as well as provide the inspiration for the operations necessary to solve problems in calculus and dynamical systems. The question we should be posing is not whether we should use visual representations, but how best we can facilitate learning and teaching of mathematics, analytical thinking and problem solving, through their use.

There is paucity of empirical research done in understanding how transitions between different representations, or as Duval (1999), calls them, registers of representation, occur during problem solving in multivariable calculus and dynamical systems. The important research questions are: What factors assist in constructing, interpreting, transforming, and coordinating visual representations? What factors determine successful transition between multiple representations? What strategies facilitate the connections between visual representations and analysis or analytical thinking? These are important questions to address if we want to enhance operative apprehension, which Duval (1999) identifies, is a key element of visualization.

In this study we draw on practical strategies such as drawing, cutting and folding nets of 3D solids and computer generated images to engage the learner in the exploration and sketching of 3D space figures (sphere, cylinder, cone, polyhedron) and their cross-sections and intersections. We also use phase portraits in order to facilitate, reinforce and strengthen the connections between visual and analytical thinking in the conception and solution of problems in dynamical systems.

1.4 Visualization and Spatial ability

Spatial ability is the ability to visualize, that is, picture or mentally construct and manipulate 2D or 3D representations. The solution to problems in multiple integrals and systems of differential equations using phase portrait analysis depend to a large extent on spatial visualization and so in this section we define spatial visualization and spatial ability.

Clements and Battista (1992, p. 423), defined spatial visualization as the 'process of understanding and performing imagined movements of objects in two- and three- dimensional space'.

Lohman (1999) defined spatial ability as the ability to generate, retain, and manipulate abstract visual images.

Sherman (1979) reported that the spatial ability factor was one of the main factors significantly affecting student performance in mathematics and that the correlation between spatial ability and performance increases with the complexity of mathematical task.

Linn and Petersen (1985) define spatial ability as the mental process used to perceive, store, recall, create, edit, and communicate spatial images.

Spatial ability is not a single construct but a collection of attributes. McGee (1979) lists four attributes of spatial ability that are relevant to this study. These include the ability to:

1. imagine the rotation of an object eg *xy* projection onto 3D object
2. fold a net and unfold an object eg sector of a circle in 2D into a cone in 3D

3. imagine movements such as translations, rotations, enlargements of 3D objects
4. transform or manipulate spatial patterns into other arrangements eg object in rectangular to spherical or cylindrical coordinate systems.

Several studies (Sorby, 2001; Piburn, Reynolds, McAuliffe, Leedy, and Johnson, 2005), have shown that practice tends to improve students' spatial ability. Participation in courses, with occasional exposure to spatial exercises, seems to improve spatial ability. More directed interventions using software, hand-held objects, and mental imagery practice with computer generated images, also improve students' spatial ability.

Students entering tertiary institutions have difficulties in predicting the intersection of a cylinder, a sphere or a cone with a plane. They need to develop skills in spatial ability including constructing, interpreting, transforming, coordinating and sketching representations.

In summary, in this section we looked at several attributes of spatial ability. Some of these have a direct bearing and inform the activities designed to focus on visualization in the subject domains selected for this research. These include: knowing surface features of 3D objects, folding and unfolding nets of 3D objects; identifying cross-sections and projections and translating and rotating 3D objects. It is our contention that these attributes of spatial ability have relevance to the conceptualisation and solution of problems involving multiple integrals and dynamical systems.

1.5 The problem and its motivation

The focus of this research is visualization enhanced by technology in the teaching and learning of multiple integrals, and dynamical systems to Mathematics 3 (Calculus 3) students in a university of technology. Researchers (Orton ,1983 ; Mahir , 2009 ; Nguyen and Rebello, 2011), have indicated that visualising and sketching space figures and the transitions between graphical and algebraic representations in 3D is often the most difficult part of the solution to problems involving multiple integrals.

Multiple integrals have a wide range of applications in tertiary mathematics, including, finding plane areas, the mass, and centre of gravity of lamina, finding volumes, moments of inertia and surface areas of objects in 3D space. Developing competencies in finding multiple integrals is both necessary and important. Setting up and transforming the integrals from rectangular to cylindrical or spherical coordinate systems involves visualization and, in particular, several attributes of spatial ability identified earlier.

Students can typically evaluate a given integral using heuristics and techniques from earlier mathematics courses but struggle to visualize and set up the integral and transform it from rectangular to other coordinate systems. In R^2 switching a double integral from $\int\int dx\, dy$ to $\int\int dy\, dx$ or transforming it to the polar coordinates system is often necessary. In R^3 there are 6 possible orders for writing down the rectangular integral $\int\int\int dx\, dy\, dz$ and visualising the Riemann sum for the volume of the 3D solid is conceptually challenging. To evaluate the triple integral, one often needs to move from rectangular to an appropriate coordinate system such as the polar, spherical or cylindrical system. Both these depend on students' ability to visualize regions in 2D and the space objects in 3D space. This study aims to engage students in their interactions with 3D mathematical objects and their 2D nets, cross-sections and projections and seeks to answer the main question: How can the transformations, treatments and conversions in Duval's (1995) framework, be facilitated to enable students to solve problems involving double and triple integrals? What strategies will help students to visualise 3D objects? What role does technology play in developing skills that enable students to make connections and transitions between the registers of representations?

The second important domain of study that this research addresses is the solution to problems in dynamical systems, expressed in a set of first order differential equations. We explore the phase space, whose graphical depiction, the phase portrait, is a powerful tool for visualising the behaviour of the dynamical system. The phase space is the space of points that completely specify the state of the system. We focus on applications in population dynamics and simple chemical reactions. An understanding of dynamical systems is dependent on students' ability to assimilate the dynamic and static visualization involving rates of change as represented by slopes of trajectories or solution curves on phase portraits. Students have met a slope as the ratio between 'change in *y* over the change in *x*' or as 'rise over run'. Slope fields and phase portraits give a concise visual summary of the dynamics of a system.

In this study, we begin with solutions to single ODEs. Students are familiar with several analytical methods of solution including solutions by separation of variable, Laplace transform methods, and linear integrating factors. Laplace transform methods are used by students to solve the numerous differential equations that arise in thermodynamics, chemical kinetics as well as process control. The direction field is a useful tool to visualize the general solution of an ODE. It helps us read the long term behaviour of the quantity represented by the ODE. We then look at how two 'quantities' vary and interact with respect to each other in time. The relation could be linear or non-linear and the two 'quantities' could be two chemicals involved in a reaction, predator prey populations, the interaction of glucose and the hormone, insulin, in the body or the amplitude of the oscillations of a pendulum under a varying force. Most non-linear systems of ODEs do not have analytical solutions. We can get some idea of how non-linear systems behave by linearization near the critical or equilibrium points. Using technology and software we can plot phase portraits and solution curves and predict the long term behaviour of the system.

In this study, the Lotka-Volterra model was used to introduce the basic concepts of non-linear dynamical systems. It is the simplest known two-state model that exhibits sustained oscillations and has wide applications in fields ranging from chemistry, ecological systems, financial markets to power systems. For example, in population dynamics, we look at the interaction between number of predators and their prey by constructing visual representations called phase portraits. In marketing we look at the sales of a new product competing with an older product it is trying to replace. In chemical kinetics, we look at the interactions between the reactants and products in a reaction as measured by their concentrations or pH. Under several simplifying assumptions, such as spatial homogeneity and a sufficiently large number of interacting species or molecules, the concentrations of the species can be modelled mathematically by a set of ordinary differential equations. It is known that complex balanced systems possess within each invariant space of the system a unique positive equilibrium concentration and that concentration is locally asymptotically stable. Although the models are simple, most cannot generally be solved analytically due to their non-linearity, and the potential behaviours are surprisingly robust—they can exhibit multi-stability, periodic behaviour, as well as oscillatory and chaotic behaviour. The Lotka-Volterra equation can be solved analytically and gives students an opportunity to compare the analytical solution with the graphical solution

Many real world phenomena are modelled by ODEs. ODEs represent phenomena that cannot otherwise be seen, touched, or sensed. In chemistry, which are the research students' majors, the reactions and processes involved demand a much more integrated understanding of dynamical systems and chemistry. This is complicated by the fact that much of chemical kinetics exists at the sub-microscopic level, well beyond the level of students' experience and senses and, therefore, we depend on representations of reaction rates such as differential equations and phase portraits, to make sense of these environments. In addition, we have available technology and software which can easily give graphical solutions to non-linear systems of equations.

To summarise: This study seeks to contribute to the teaching and learning of multiple integrals and dynamical systems by designing activities that facilitate visualization. The focus is the complementary role of analytical thinking and visualization in the solution of the problems. We attempt to define the visual and the analytical steps in the solution using the Visualization-Analysis (VA) framework by Zazkis et al. (1996) in their study of dihedral groups. A growing number of researchers (Presmeg, 2001; Haciomeroglu, Aspinwall, Shaw & Presmeg, 2010 ; Stylianou & Silver, 2004) are shifting from Krutetski's earlier model that categorised learners as visual, analytic or harmonic, towards the VA-model on the basis of the finding that most learners are harmonic and use both visual and analytical methods in problem solving.

This study addresses the paucity of research on visualization in multiple integrals and dynamical systems. We note that constructing static representations in R^2 and R^3 is an essential first step in visualising and setting up the multiple integrals. Likewise, direction fields and phase portraits for systems of nonlinear differential equations are essential components of the graphical representations of solutions to problems involving dynamical systems.

1.6. Rationale for the study

Researchers (Eisenberg and Dreyfus, 1991 ; Sweller, 1999 ; Winslow, 2000) have highlighted difficulties experienced by students in the use of visual representations and in particular in the transition between modes of representation. Among other causes, they have attributed the difficulties to the increased cognitive load that representations demand, to cultural beliefs where algebraic is preferred as opposed to visual, and to sociological difficulties.

An important question that researchers (Presmeg, 1996 ; Habre, 2000; Zazkis and Dubinsky, 1996) in the past have asked is: Given a choice, which approach the analytic (verbal-logical) or graphical (visual-pictorial) do students prefer? The study by Habre (2004) found that analytical approach was preferred by students in solution of ODEs. Zazkis and Dubinsky (1996) found that a combination of visual and analytical approaches were used on tasks requiring students to list and find the products of the elements of the dihedral group D_4. Zandieh (2000) extended the visual analytical framework to include the kinaesthetic mode. Zazkis (2013) refined the Visual-Analytical model to include physical aspects as most problems are based in some real world context. This study extends the application of the VA model to multiple integrals and dynamical systems represented by systems of ordinary differential equations.

Other research (Goldin , 2003 ; Fennel and Rowan, 2001) suggests that students often have difficulty in understanding, manipulating, and translating between various representational forms. When used in conjunction with 2D sketches, concrete and virtual models have been shown to increase understanding of 2D and 3D representations and promote representational and diagrammatic competence. Hence, the need to identify and develop visualization tools and strategies so that students can explore interaction between surfaces, space curves, vector fields and 3D space figures.

In the solution of systems of differential equations, a conceptual understanding of gradients and the qualitative theory for analysis of nonlinear systems is essential. Students' understandings of the solutions of systems of ODEs, including analytical or algebraic, visual and graphic forms needs further exploration. In particular, the difficulties students experience in visualizing and interpreting phase portraits and solution curves and in the transition between visual-analytic modes, need further research.

Mathematics education literature currently provides a sparse treatment of systems of ODEs, but there is a growing demand in higher education to understand and interpret the solutions and the graphs that accompany them. Educators need to explore alternative teaching strategies and provide students with more meaningful activities geared toward developing their mathematical understanding and reasoning skills (Rasmussen & Keynes, 2003).

1.7 Illustrative examples

The following examples illustrate the type of the problems this study will address. The focus is the conversions and transformations between registers of semiotic representation and the role of visualization and analysis or analytical thinking to support the solution of the problems.

1.7.1 Problems involving double integrals

Example 1: Find $\iint_D dA$, where D is defined by $1 \leq x \leq 5;\ 0 \leq y \leq \sqrt{(x-1)}$.

An essential first step is to sketch the region of integration. (See Figure 1.5). On Duval's (1995) framework, this requires conversion from the algebraic to the graphical register. Some students may need to go through a table (a numerical register), to find points for the curves and plot them. This has been practised repeatedly in Mathematics 1 and 2, and is routine. Next, we visualize the Riemann sum for the region of integration by slicing horizontally or vertically depending on the choice of the order ∬d*x*d*y* or ∬d*y*d*x*. For ∬ d*x*d*y*, we slice horizontally. The left *x*-limit can be found by making *x* the subject of the equation, $y = \sqrt{x-1}$, a treatment within the algebraic register, to give $x = y^2 + 1$ and the right *x* limit is constant at $x = 5$ (See Figure 1.5).

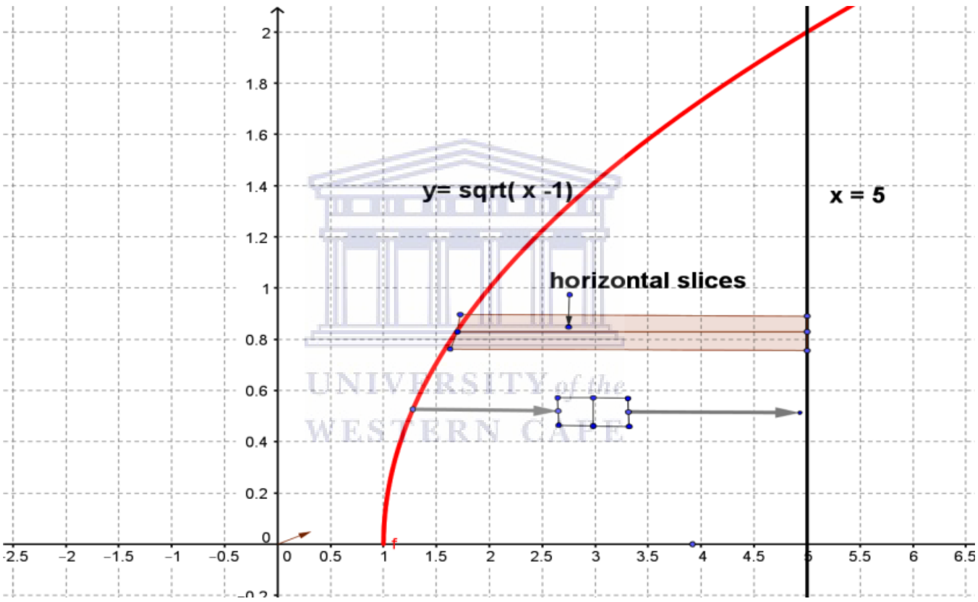


Figure 1.5 Traversing the region with horizontal slices in the *x*-direction followed by stacking in the *y*-direction

Next, we imagine the slices filling up (stacking) the region of integration. Our slices must start on the *x*-axis and go all the way to $y = 2$. The outer *y* limits run from $y = 0$ to $y = 2$. The double integral we set up is $\int_0^2 \int_{y^2+1}^5 dx\, dy$. For ∬d*y* d*x*, we slice vertically traversing the region in the *y*-direction, with limits $y = 0$ to $y = \sqrt{x-1}$ and then sideways from $x = 1$ to $x = 5$. The ∬ d*y*d*x* is $\int_1^5 \int_0^{\sqrt{(x-1)}} dydx$. Both give answers of 5.33.

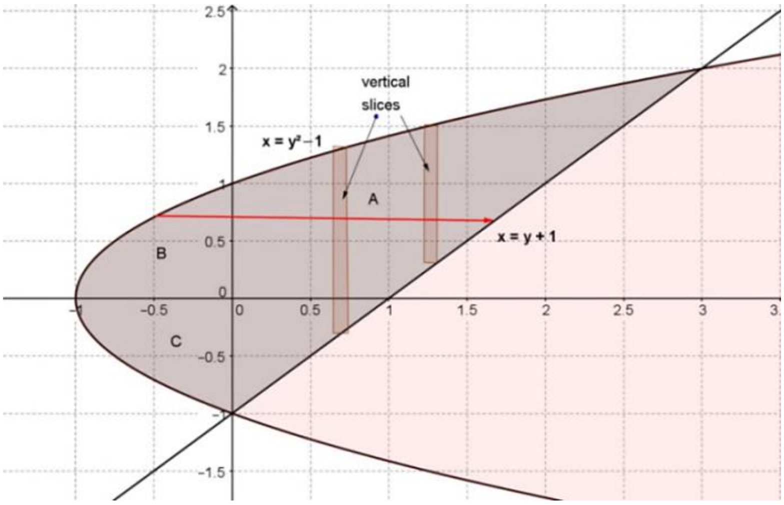


Figure 1.6 Visualizing slicing as we traverse the region to find ∬d*y*d*x*

Example 2: To find the area bounded by $y = x - 1$ and $y = \sqrt{x+1}$, using double integrals, we start with a sketch (See Figure 1.6). We select the order, ∬d*x*d*y* or ∬d*y*d*x*. Of the two, ∬d*x*d*y* is simpler and gives: $\int_{-1}^{2}\int_{y^2-1}^{y+1} dx\,dy = \frac{9}{2}$. The horizontal slices we make are always between the two curves, $x = y^2 - 1$ and $x = y + 1$, and we stack the slices up from –1 to 2. Integration in the order d*y*d*x* is visually challenging, even for capable students. The vertical slices are between the two curves only for positive *x*-values. Hence, we split the region into three (labelled A, B and C in Figure 1.6). A is the region in the first and fourth quadrants between $x = y^2 - 1$ and $x = y + 1$, B is the region in the second quadrant and C, is a region in the third quadrant. We could use symmetry for B and C. Several conversions from the geometrical register to the algebraic register, explained more fully in Chapter 5, finally yields the area integral as:

$$\int_{0}^{3}\int_{x-1}^{\sqrt{x+1}} dydx + \int_{-1}^{0}\int_{0}^{\sqrt{x+1}} dydx + \int_{-1}^{0}\int_{-\sqrt{x+1}}^{0} dydx$$

$$= \frac{19}{6} + \frac{2}{3} + \frac{2}{3} = \frac{9}{2} \quad or$$

$$\int_{0}^{3}\int_{x-1}^{\sqrt{x+1}} dydx \quad + \quad 2\int_{-1}^{0}\int_{0}^{\sqrt{x+1}} dydx = \frac{9}{2}$$

Note: Using single integration, the same area can be found by:

$$\int (x_2 - x_1)dy = \int_{-1}^{2} (y+1) - (y^2 - 1)dy$$

Example 3: A third type of problem, involving double integrals, is over circular regions (See Figure 1.7). Here a conversion from rectangular coordinates to polar coordinates makes the computation easier. For example, the integration of the region in the first quadrant defined by $1 \le x^2 + y^2 \le 4$ is easier to visualise and set up in polar coordinates as $\int_0^{\pi/2} \int_1^2 r\, dr\, d\theta$ than in rectangular coordinates. The slicing is now in sectors between radii r_1=1 and r_2 = 2 and angles θ_1= 0 (*x*-axis) and θ_2 = $\pi/2$ radians (*y*-axis). The Riemann sum applies.

In Chapter 5, we shall look at the difficulties students experience in evaluating these integrals in detail using Duval's (1995) and the Zazkis et al. (1996) theoretical frameworks.

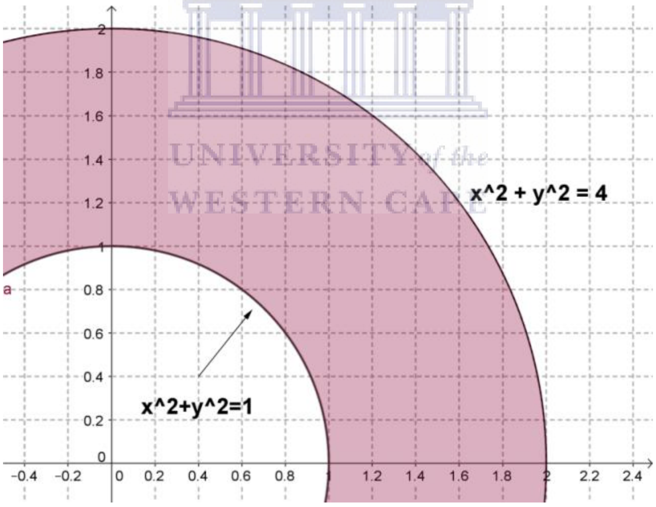


Figure 1.7 Region defined by $1 \le x^2 + y^2 \le 4$, $x \ge 0$, $y \ge 0$

1.7.2 Problems involving triple integrals

For a triple integral, the region of integration is a three-dimensional shape, which is usually not so easy to visualize. We can reduce the 'hard' parts of setting up the limits of integration from three-dimensions (3D) to two-dimensions (2D) by projecting the space figure in the *xy*, *y*z and z*x* planes. The main steps are:

- Use the given equations to sketch projections in the *xy*, *y*z and the *x*z planes.

• Use the projections to sketch the 3D object

• Find the limits of integration and,

• Set up and evaluate the triple integral.

We illustrate by finding the volume of the truncated cylinder we discussed earlier, see page 4.

Example 4 : Find the volume of the truncated cylinder, defined by $x^2 + y^2 = 1$, and the planes, $z = 0$ and $z = 2 - y$.

Solution

In this research, we explore the main proposition that encouraging students to use technology and software will help in the conceptualization of the problem. Here, we have an object bounded by the three planes, $x^2 + y^2 = 1$, $z = 0$ and $z = 2 - y$. To help students 'see' the object, we use Matlab, and offer different viewpoints, surface features, intersections and projections (See Figure 1.8). Students can rotate, zoom in and out, check out the intersections, project and look at cross-sections in order to see clearly the region of integration.

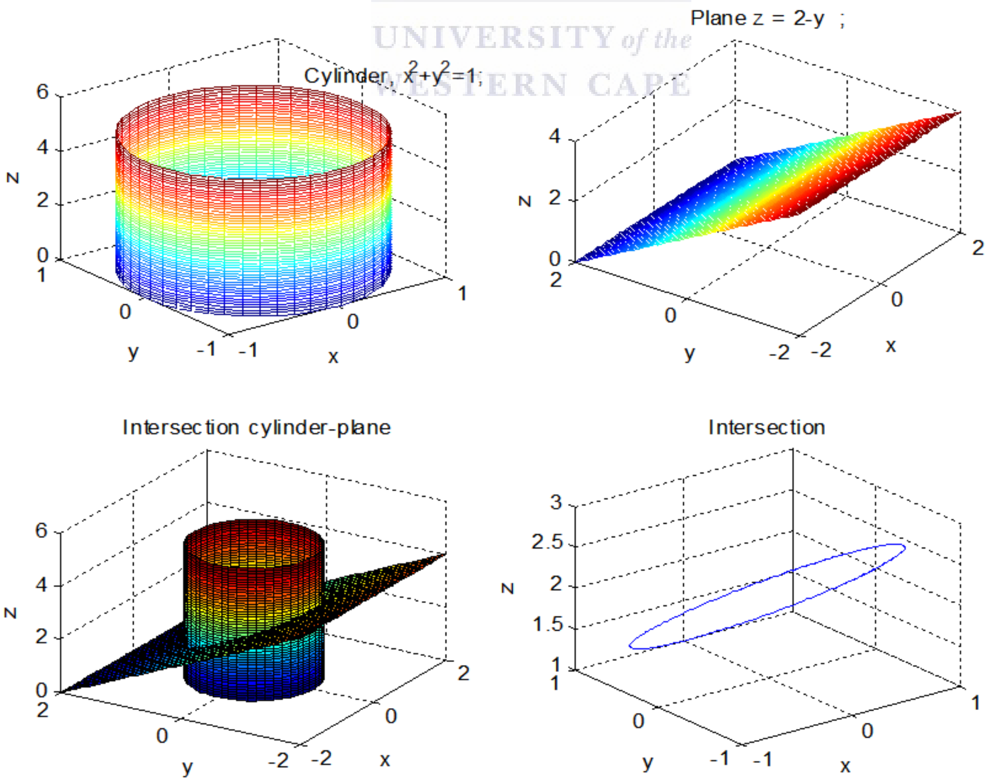


Figure.1.8 Space figure enclosed by the surfaces represented by $x^2 + y^2 = 1$ and $z = 2 - y$

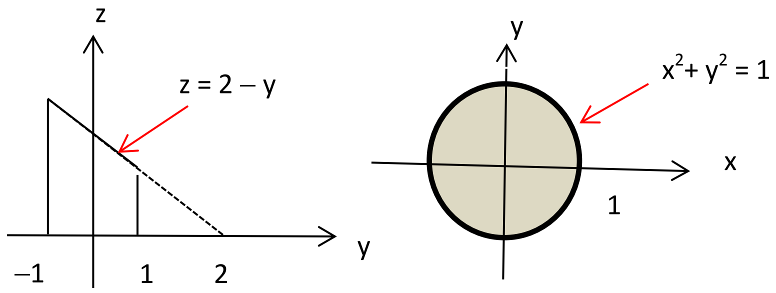


Figure. 1.9 Projections in the *yz* and *xy* planes

To find the limits of integration we visualize movements up and down for dz and as before between polar radii r_1 and r_2 and angles $\theta_1 = 0$ (x-axis) and $\theta_2 = 2\pi$ radians. Drawing the projections in the *yz* and *xz* planes helps to 'see' the limits clearly and easily. The limits in cylindrical coordinates are: r : the polar radius, from 0 to 1, θ : the polar angle, from 0 to 2π and z : the height of the surface from 0 to $2 - r\sin\theta$. Setting up the integral in cylindrical coordinates gives : $\int_0^{2\pi}\int_0^1\int_0^{2-r\sin\theta} dz\, rdr\, d\theta = \frac{32\pi}{3}$.

The integral and the limits in rectangular coordinates are: $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=0}^{z=2-y} dz\,dy\,dx$. It is difficult to do this integration manually in rectangular coordinates.

Acts of visualization interspersed with analytical thinking are essential features of the solution. We discuss the details of the visual-analytic steps fully in Chapter 4. Here we highlight the fact that visualization and analytical thinking are an essential part of the solution to the problem. We expect students to have built-in schemas for the mathematical objects that represent the algebraic functions: $f(x, y) = x^2 + y^2$ and $g(x, y) = 2 - y$. These students can proceed directly to the conversions from the algebraic register to the geometric register.

1.7.3 Problems in dynamical systems

The second area of research is non-linear dynamical systems. Details and worked examples are given in Chapter 5. We will be looking at the role of visualization in application domains such as population dynamics and chemical reactions. Here we briefly look at an example of non-linear differential equations (the Lotka-Volterra equations, LV) applied to Predator-Prey interactions.

Question: Given the predator-prey non- linear differential equations:

$$\frac{dx}{dt} = 0.08\,x - 0.001\,xy$$
$$\frac{dy}{dt} = -0.02\,y + 0.00002\,xy$$

Source: Lia Vas, Maths 320, l.vas@usciences.edu

where x represents the prey and y represents predators, we need to determine the number of equilibrium (fixed) points, their locations, and discuss their stability. The LV assumptions apply. We need to draw a phase-portrait which includes the trajectories that clearly indicate the possible outcomes as time evolves and discuss the coexistence of the species.

Solution: We can address this problem in three ways. The analytical solution is found by setting the right hand side of both DEs equal to 0 and solving simultaneously to give us the equilibrium points (0 , 0) and (1000 , 80). Next, we use linearization by taking partial derivatives, and writing down the Jacobian matrix at each equilibrium point.

The Jacobian matrix, whose entries are partial derivatives of each of the differential equations is: $J(x\,;y) = \begin{bmatrix} 0.08 - 0.001y & -0.001x \\ 0.00002y & -0.02 + 0.00002x \end{bmatrix}$.

Evaluating at (1000, 80) gives $J(1000,80) = \begin{bmatrix} 0 & -1 \\ 0.0016 & 0 \end{bmatrix}$ and the eigenvalues are 0.04i and – 0.04i. As the eigenvalues are purely imaginary, we conclude that the fixed point is the centre of an elliptical orbit (for details, see Chapter 5). Thus the predator prey populations move periodically about this equilibrium point and there is a balance or coexistence between the two populations near these population figures.

The second approach is geometric or graphical and highly visual. We determine (analytically) and plot the equilibrium points. We work out the slopes dx/dt and dy/dt at various points in the coordinate plane, preferably in a table, and we sketch the trajectories and their flow directions. We notice the same features that we found analytically by linearization at the fixed points. i.e an unstable saddle point at (0,0) and an elliptical centre at (1000, 80). We notice that all solutions are closed and circle the equilibrium point (1000, 80) regardless of the initial condition. We notice that both populations oscillate with time (which is a hidden variable, a parameter). This process is laborious and messy but gives students good practice in plotting slopes as vectors and finding their resultants. However, it is the only way students can sketch their phase portraits in examinations (where they do not have access to software).

The third approach is to use software. We simply type in the equations, tweak the scales and, lo and behold, we have the phase portrait. Right clicking with the mouse at convenient points gives us the trajectories passing through the points (See Figure 1.10).

The interpretation of the phase portrait is important. It tests visual and analytical skills. We 'see' that the equilibrium point (1000, 80), is the centre of an elliptical orbit. We notice the predator prey populations move periodically about this equilibrium point, and we deduce that there is a balance or coexistence between the two populations near these population figures.

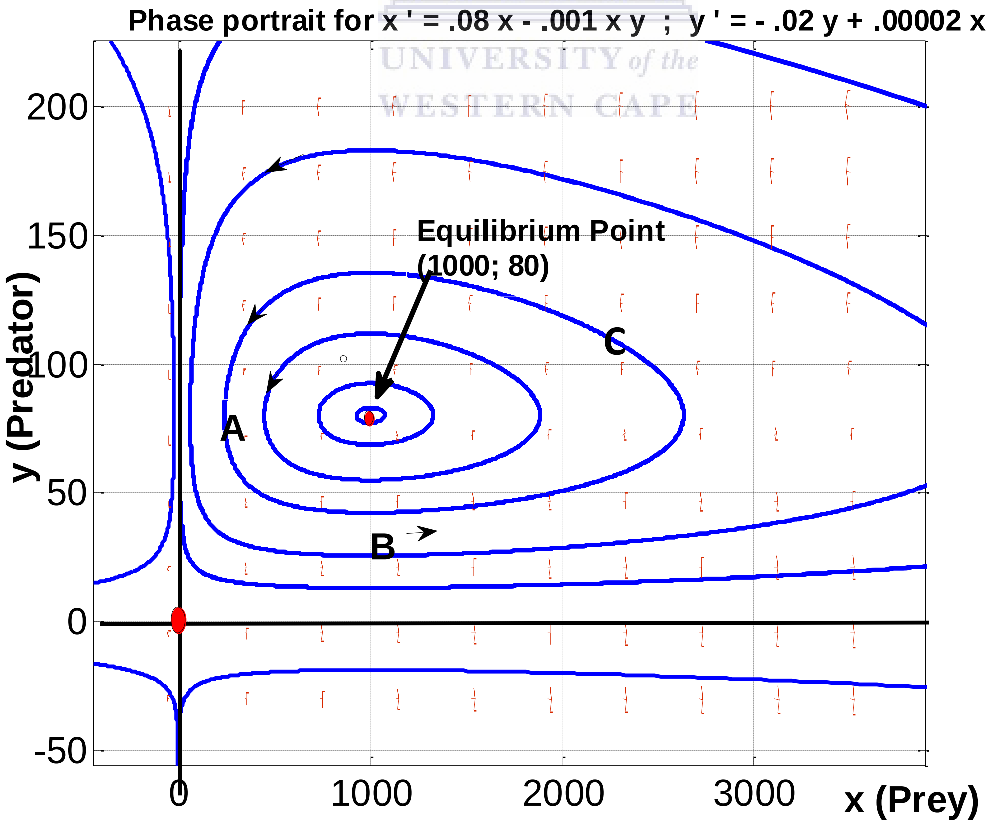


Figure 1.10 Phase portrait for the Predator- Prey equations

Similarly, we ‘see’ that (0 , 0) is an unstable saddle point. As the trajectories approach (0 , 0) they veer away sharply from (0 , 0). The extinction of the predator prey population is not likely, unless the prey population all die, in which case the predators will starve to death.

Thus, near A on the phase portrait, the prey population is at its least and the predator population is falling. Near B, as there are few predators, the prey population is recovering. Near C the predator population begins to recover as there is abundant prey.

To draw the *x*-*t* and *y*-*t* graphs we need the time for one cycle or oscillation (about 7 years in this case). Figure 1.11(b).

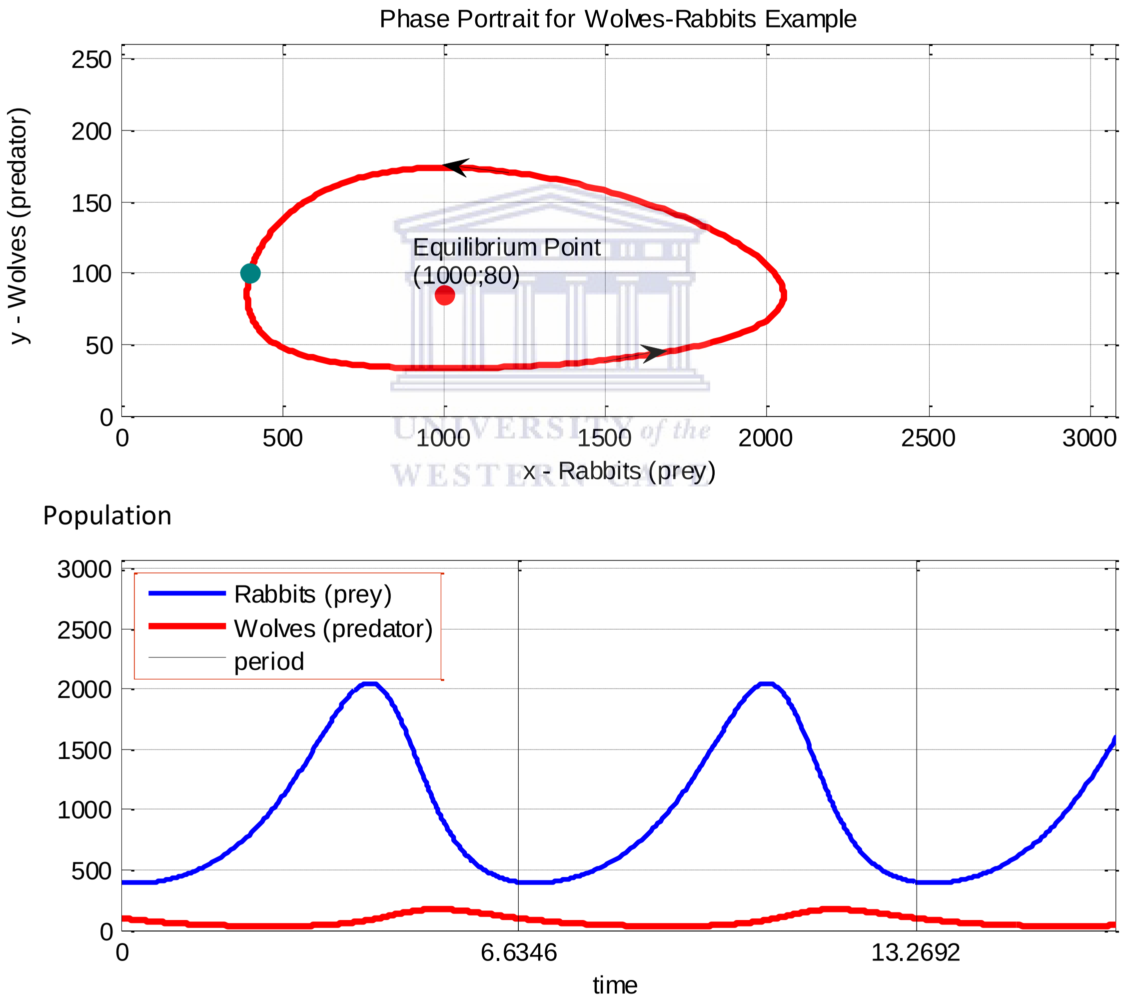


Figure 1.11 a) The orbit for $x = 400$, $y = 100$ (see green dot) for predator prey and b) the *x*-*t* and *y*-*t* graphs showing time variation of predator prey populations

Software can also churn out the predator-time and the prey- time graph (See Figure 1.11) shows the trajectory for initial values (400 , 100). See green dot. We can state the limits of the oscillations in a cycle: For *x*, the limits are 460 to 2100 and for *y* the limits are between 40 and 180.

In conclusion, in this section we looked at four illustrative examples of the type of problems this research will attempt to answer. Next, we outline the aim, research objectives and research questions for this study.

1.8 Aim of the study

The aim of this study is to investigate practical strategies, using technology, that can engage the learner in the exploration, visualization of 3D space figures (polyhedrons, sphere, cylinder, cones) and phase portraits in order to facilitate, reinforce and strengthen the connections between visual and analytical thinking in the conception and solution of problems in multivariate calculus and dynamical systems.

1.9 Research objectives

The objectives of the study are:

1. To identify students' needs and difficulties in visualization of 3D space figures, multiple integrals and phase portraits of non-linear dynamical systems.
2. Design and conduct activities to address the needs identified in Objective 1. Some activities are planned with manipulatives eg. folding and unfolding nets, rotations with 3D objects, while other activities will be in the computer laboratory using software. During the activities the key elements of visualization (perception, mental rotation, cross-sections, projections, sketching) are emphasised. The activities will include:
    a) Constructing nets and 3D wire or straw models of common mathematical objects eg pyramids, cylinder,
    b) Sketching 2D and 3D space figures given their algebraic representations in rectangular, cylindrical and spherical coordinate systems
    c) Sketching cross-sections and projections of 3D space figures

d) Setting up and evaluating volume integrals.

Activities for dynamical systems include:

a) Sketching and interpreting slope fields for ODE's

b) Sketching and interpreting phase portraits for systems of ODE's

c) Conceptualising and interpreting phase portraits, trajectories, equilibria and stability of the system

3. Use Duval's (1995) semiotic representation theory and Zazkis et al.(1996) Visualization-Analysis (VA) frameworks to analyse students' solutions to problems.

4. Highlight Teaching and Learning strategies educators can use to enhance visualization and analytical solutions to problems in multiple integrals and dynamical systems

1.10 Research Questions

The guiding research questions are:

1. What are students' needs and difficulties in conception and solution of problems in multiple integration and dynamical systems?

2. Do the activities facilitate visualization and solution of problems in the two domains?

3. What factors influence the effectiveness of the visualization?

4. What Teaching and Learning strategies help in the conceptualization and solution of problems in multiple integrals and dynamical systems?

1.11 Limitations of the study

Multivariate calculus is a broad area of study. It includes the study of vectors, lines, curves and space figures as well as differentiation and integration of functions of several variables. In this study we restrict ourselves to the study of multiple integrals applied to finding volumes of space figures. Likewise, in dynamical systems, we restrict ourselves to systems involving two variables with applications in population dynamics (predator- prey equations) and chemical reactions. Our focus will be the role of visualization and analytical thinking in the solution of area and volume integrals.

Many factors including socio-cultural, linguistic and affective and personality factors such as attitudes and motivation, as well as gender are known to impact on achievement in mathematics. In this study we restrict ourselves to six predictor variables and study their impact on achievement using multiple linear regression.

1.12 Dissertation Outline

In the first chapter we define visualization in mathematics and describe the purpose and the motivation for the study. We also situate the study with respect to current literature on visual and analytical thinking. The chapter concludes with the aims and objectives of the research and problems that the study seeks to address.

The second chapter provides a literature review on the role of visualization in multivariate calculus and systems of differential equations. It ends with a review of studies where technology was used to enhance teaching and learning of multivariate calculus and dynamical systems.

The third chapter discusses the research methodology, the theoretical and pedagogical frameworks, the design of the study, and development of the analytical tools. Duval's (1995) semiotic representation theory, the Zazkis' et al. (1996) VA framework and the teaching experiment, with the type of activities students engage in, are explained.

The fourth chapter looks at common 3D mathematical objects and the role of visualization in setting up single, double and triple integrals. Conceptual difficulties coordinating the treatments and conversions between rectangular, cylindrical and spherical coordinate systems are highlighted.

The fifth chapter discusses the use of qualitative and analytic methods in the solution of dynamical systems. We look at plotting and interpreting direction fields and phase portraits with a focus on the Lotka-Volterra system, which has wide applications in business, economics, science and chemistry.

Chapter six presents the results of the comparisons with t-tests and ANOVA, and the findings from the multiple regression analysis. Using Duval's(1996) and Zazkis' et al. (1996) frameworks, we analyse difficulties students experience in multiple integration and dynamical systems.

The final chapter presents a summary of the findings, discusses the implications of this research and recommends future directions for research on visualization in Mathematics.

1.13 Chapter Summary

In this chapter, we noted that the definition and characteristics of visualization vary widely from field to field, and we began with a definition of visualization in mathematics education as used in this dissertation. Visualization involves making connections between external or internal representations in order to communicate information and advance mathematical thinking and understanding. We note that analytical skills such as abstracting surface features, sketching projections, cross-sections and 3D solids, coordinating rates of change, and covariational reasoning are interwoven with visualization and are associated with successful performance in multiple integrals and dynamical systems. We note that computer technology enables static and interactive visualization and we gave illustrative examples of problems in multiple integration and dynamical systems involving visualization that this study seeks to address. Finally, we stated the aims, objectives and research questions and mentioned briefly the frameworks within which we operate.

In chapter 2, we conduct a review of literature that informs us about the status of research in visualization and we situate our study in the work of other researchers with the aim of extending the boundaries of our knowledge.

Chapter 2: Review of literature on visualization in mathematics

2.0 Overview

This chapter begins with a review of literature on research on visualization in mathematics education, its classifications, its strengths and weaknesses. The second section outlines literature on visualization in multivariate calculus, followed by a review of visualization in the conception of solutions to systems of differential equations. The final section reports on the role of technology in enhancing visualization and sets the scene for the theoretical framework for the research.

2.1 Visualization in mathematics

Visualization in mathematics has a long history – pre-dating the use of diagrams in geometry by Greek, Indian and Chinese mathematicians – and has played an important role in the development of mathematics. For the Pythagoreans, visualization was an integral part of mathematics. Descartes (1596 – 1650), as reported in Massironi (2010), used a wide range of graphical representations including geometric patterns to explain dioptrics, and abstract images to explain his theory of vortices and magnetism. He strongly emphasised the different roles of sketches, figures and images in mathematical thinking and reasoning. Calculus in the 17th Century had a very strong visual element with constant interaction with geometrical and physical problems. Visualization research started slowly in the 1980s with mathematics educators probing the difficulties and strengths associated with visual mathematical thinking. Sommer (1978, p. 149) saw visual thinking as a kind of code switching whose goal should be 'development of capacity to switch back and forth between different modes of thinking as needed' and that our educational system is to blame for the lack of emphasis on visualization and visual thinking skills. He asserts:

‘School more than any other institution, is responsible for the downgrading of visual thinking. Most educators are not only disinterested in visualization, they are hostile toward it. They regard it as childish, primitive, and prelogical. Classes in mechanical drawing and the arts, in which spatial thinking still plays a role, are considered second-rate intellectual activities’.

In the 1990’s technology and in particular computer software, began to influence teaching and learning, pedagogy and curricula in school mathematics. The importance of visual processing in mathematics, which has diagrams, tables and graphs, spatial arrangements of symbols and representations, was increasingly recognized. More recently digital media including computers, the internet and Ipads, tablets and cellphones have had a profound influence on the learning styles of students. The scope of the research widened in the 2000’s to semiotic aspects of visualization and currently is focused on pedagogy to enhance the use and power of visualization.

Visualization in mathematics is being actively researched.

- Arcavi and Nachmias (1989) investigated the use of linear functions in getting adults to visualize the notion of slopes.
- Mariottii and Pesci (1994, p.22) investigated inverse problems and acknowledge *visualization* occurring when 'thinking is spontaneously accompanied and supported by images' and *imagery* as 'the power to imagine the possible and the impossible'.
- Gutierrez (1996), Kwon, Kim and Kim (2001) and Pinkernell (2000) investigated the role of visualization in space geometry.
- Owens (1999) and Lawrie, Pegg and Guitierrez (2002) looked at reasoning processes in visualization and Murray (2001) and Littler (2002) investigated the relation between visualization and students’ knowledge and ways of learning.

Other researchers are investigating the relationship between visualization and the use of software (Kwon, Kim and Kim, 2001), problem solving (Lampen and Murray, 2001; Stylianou, Leikin and Silver, 1999), and theories framing research and curriculum development (Gutierrez, 1996; Owens, 1999).

Ferguson (1992) claims that the engineering education of today has diverged too much from its artistic, visual beginnings, and that our curriculum relies too heavily on analytical methods, with little attention paid to tactile and visual perception.

Dreyfus(1991, p. 37) suggested in a Plenary paper that the status of visualization be upgraded from that of a helping learning aid to that of a fully recognised tool for learning. He calls for integration across algebraic, visual and verbal abilities.

Nixon (2002) conducted a teaching experiment in sequences and series with six higher grade matriculants of mixed gender. The main emphasis was visualization, exploring patterns and generalization. She reported that the strategies made a positive contribution towards progress through the van Hiele's geometric levels.

Kosslyn (1983, p.191), highlighted five important aspects of visualization. These are:

a) **image generation** which is the ability to form visual images, requires activating stored visual information and using it to create a pattern in a spatial short-term memory structure called the "visual buffer"

b) **image maintenance**, is the ability to retain images over time; such processing is necessary because the visual buffer retains information very briefly, and images are maintained only by continual effort.

c) **image inspection** is the ability to interpret a pattern in the visual buffer as depicting an object or part of the object.

d) **Image scanning** the systematic shifting of attention over an imaged pattern, plays a critical role in this ability.

e) **image transformation** is the ability to rotate or otherwise alter an imaged pattern.

Kosslyn also noted that students are relatively poor at scanning, rotating, and generating objects in images.

In 1985, Presmeg conducted her doctoral investigation on the role of visually mediated processes in high school mathematics. Her findings (in Thornton 2000, p.254) were that five types of visual-spatial representations were used by high school students while solving mathematical word problems. These were:

1. **concrete imagery** - having a clear picture in the mind of the problem.
2. **pattern imagery** – looking purely at relations stripped of concrete details.
3. **kinaesthetic imagery** - involving physical movements.
4. **memory images** eg of quadratic formula, complex number forms, and
5. **dynamic imagery,** involving transformation or movement of image.

Concrete imagery (pictures in the mind) was most prevalent followed by memory images of formulae, pattern imagery and kinaesthetic imagery. Dynamic imagery was rarely used and all students in her sample experienced problems with abstraction and generalization of information from the visual representations they constructed.

She concluded that most gifted math students are non-visualizers and that the practice of procedures and formulas in mathematics leads to habituation, which takes a learner away from the visual method. Presmeg (1997) cautioned that concrete imagery needs to be coupled with analytical thought processes.

Gutiérrez (1996) explored the role that geometry software plays in the development of visualization and spatial ability. Based on a literature survey of relevant psychological and educational literature, Gutiérrez concluded that there is no general agreement about the terminology used and defines visualization as 'the kind of reasoning activity based on the use of visual and spatial elements, either mental or physical, performed to solve problems or prove properties' (p.9). He reconciles the varying theoretical approaches to understanding visualization, and finds that many of these seemingly different perspectives actually share a lot of common ground. He suggests the following main elements unify visualization:

- **Mental Images** – any kind of cognitive representation of a mathematical concept or property perceived by means of visual or spatial elements. Mental images include kinaesthetic images, which are created, transformed or communicated with the help of physical movements, and dynamic images – those images with movement in the mind.
- **External Representations** – any kind of verbal or graphical representation of concepts or properties including pictures, drawings, diagrams, etc. that helps to create or transform mental images and to do visual reasoning.
- **Process of Visualization** – a mental or physical action where mental images are involved. There are two central processes of visualization – visual interpretation of information (used in creating mental images), and interpretation of mental images (used to generate information).

The nature of the specific mental images necessary for a given problem is dependent on the specific characteristics of the problem being solved, but Gutiérrez (1996, p.10) identifies the main abilities as being:

1. **Figure-ground perception** – the ability to identify and isolate a specific figure out of a complex background
2. **Perceptual constancy** – the ability to realize that some characteristics of an object are independent of 'size, colour, texture, or position'.
3. **Mental rotation** – the ability to produce dynamic mental images and to visualize a configuration in movement.
4. **Perception of spatial positions** – the ability to relate figures (object, picture, or mental image) to oneself.
5. **Perception of spatial relationships** – the ability to relate several figures (as above) to "each other, or simultaneously to oneself
6. **Visual Discrimination** – the ability to compare several figures and to determine how they are similar or different.

Visualization is a significant aspect of all branches of mathematics and not merely of obviously visual branches, such as geometry. Symbolism may, in and of itself, entail spatial characteristics, thereby implicating visualization.

Researchers' Comments

This brief overview has established that visualization is being actively researched in various domains in mathematics. Our attempts at identifying the characteristic components or attributes of visualization have resulted in a wide range of abilities including images (generation, maintenance, inspection, transformation), imagery (concrete, pattern, mental, dynamic), perception (figure-ground, visual, surface features, rotations, projections, cross-sections), and spatial ability (spatial relations, positions). Many of these attributes apply to our problems in multiple integrals and dynamical systems. There is a clear need to define and delimit the relevant components for the current research to align with the topic or domain under consideration. Next, we look at visualization in calculus.

2.2 Research on visualization in Multivariate Calculus

The role of visualization and visual thinking in the teaching and learning of calculus has been recognised during the past two decades and given further emphasis by reform calculus textbooks. Hughes-Hallett (1991, p.125) advocates a balance between the graphical, the numerical, and the analytical: 'A balance is required because it's seeing the links between various approaches that constitutes understanding' .

Sevimli and Delice (2010) did a case study on 45 mathematics teachers to investigate the influence of spatial visualization ability in representations used in definite integrals. Tests, document analysis and semi-structured interviews were the research instruments. The proportion of teachers using algebraic representation was 46% and the graphical representation was 17%. The least preferred type of representation was numerical representation. Analysis of the data showed that focussing on spatial visualization ability of the teacher candidates improved the performance on solving definite integral problems.

Martínez-Planell and Trigueros (2009) investigated students' understanding of functions of two variables and identified difficulties students have in the transition from one variable to two-variable functions. Using APOS theory, they related these difficulties to specific coordination that students need to make among the one variable function, and R^3 schemata.

In a study about geometric aspects of two variable functions, Trigueros and Martínez-Planell (2010) found that students had difficulties intersecting fundamental planes (that is, planes of the form $x = c$, $y = c$, or $z = c$ where c is a constant) with surfaces given in different representational formats. Hence, their difficulties with transversal sections, contour curves, and projections.

Trigueros and Martínez- Planell (2011) designed and used four activity sets to help students make those constructions found to be needed to understand functions of two variables: (a) fundamental planes and surfaces, (b) cylinders, (c) graphs of functions, and (d) contour maps. Constructions and coordinations found to be missing in studies of students' construction of graphs of two-variable functions can be addressed with activities specifically designed to foster those constructions..

In a large longitudinal study involving senior high school students, Sherman (1979) after careful analysis, in which mathematical performance was related with a number of other cognitive and affective variables controlled, reported that the spatial ability factor was one of the main factors which significantly affected mathematical performance.

Meadows (2008) conducted a qualitative case study of a calculus III class in order to obtain descriptive data on students' visual and analytical understanding of surface areas of familiar shapes of spheres, cylinders, prisms, and pyramids in the context of multivariate calculus. Specifically, her research focused on application of the surface area formula of surfaces described by a function of two variables. The best demonstrated understanding was observed in the case of students with mathematical visualization preference and above average mathematical accuracy. Analytical thinkers struggled with graphing and geometric thinkers, with below-average mathematical accuracy, showed deviations from traditional understanding of basic shapes.

Schlatter (1999) used MatLab to help students in visualizing objects and surfaces encountered in the multivariate course and observed positive students' responses on post- tests.

Schlatter (2002) used multiple choice, thought-provoking 'concept tests' with questions involving visualization to generate discussions during lectures. He reported greater active student involvement and positive results on the visual enhancement and concept tests.

Habre (1999) exposed 26 students in a university calculus class to both analytical and visual methods of solving problems using computer software and noted that even with instructor emphasis on visualization, most students preferred the analytical approach. He concluded that this was likely because many students came from traditional schools of mathematical instruction where the view of mathematics was entirely algebraic.

In a second study, Habre (2001), students used computers software to explore quadric surfaces, parametric equations, surfaces in spherical coordinates, vector fields and differential equations. He points to the void in understanding students' learning of visual concepts and calls for more attention to research in multivariable calculus.

Stylianou (2002) was interested in the interplay between visualization and analytical thinking as mathematics educators solved problems using the VA model. The model assumes that visualization and analysis, although distinct forms of thinking, inform one another and work together in the process of mathematical problem solving. The study suggests that mathematicians build the visual representation in steps which are clearly separated by a few moments, during which they attempt to analyse the visual representation with respect to the problem situation. This analysis consists of some well-structured processes which involve four types of actions: inferring additional consequences, elaborating on the new mathematical information, stating a new goal, and monitoring their problem-solving process. Each time a mathematician either constructed a new diagram or modified a previously constructed one, he/she took a few seconds to 'extract' any additional information and to understand any possible implications. Davis (1984, p.35) calls the phenomenon a 'visually moderated sequence' which is more or less 'look, ponder, write, look, ponder…'.

In conclusion, most of the research findings on the role of visualization are promising. Activities designed to focus on spatial visualization factors like projections, rotations, contours as well as cross-sections seem to yield positive results on performance and attitudes towards mathematics. Several studies (Sherman, 1979 ; Sorby, 2001; Cohen & Hegarty, 2012) reported strong correlations between spatial ability factors and achievement in mathematics and that spatial ability can be improved by training.

2.3 Research on visualization of solutions to systems of ODE's

The analysis of nonlinear systems is a subject of much recent interest. Some dynamical systems, such as are found in biology or chemical kinetics, are dominated by nonlinear behaviour. In this study we use linearization, a well-established approach to analysis, which gives results that are always, at best, local.

Research on students' understandings of solutions to differential equations is also scarce with fewer than 10 studies reported during the last decade. The proliferation of differential equations software has made it possible to switch attention from laborious algebraic solutions to interpretation of solutions presented in multiple forms. Research on investigations of understandings of solutions to differential equations has been reported by the following:

Habre (2000) explored strategies that students use to solve ordinary differential equations. Using a research sample of 9 students who were interviewed, he found that most students attempted to solve the differential equations using quantitative methods. Only two students were successful in drawing the solutions on slope fields and reading graphical information from the solutions. None of the students were successful in switching between the visual and algebraic aspects of the solution. Habre (2000, p.14) concludes that ‘idea of solving has remained purely algebraic in the minds of all students’ and ‘integrating software programs is not always a complete success’ as students have first got to learn the syntax that will produce the graphical solution. Habre recommends that the analytic and graphic-visual approaches go hand in hand and students be given time to develop visualization.

Trigueros (2000) interviewed 18 university students about their solutions to ordinary differential equations. Students' understandings of parametric functions and variation were found to conflict with their understanding of phase space representations and notions of solutions and equilibrium. Analysis of the interviews revealed that some students had problems interpreting the meaning of equilibrium, interpreting the meaning of a point in phase space, and seeing the dependence of time in the phase space. Students in her study also showed a tendency to focus on just part of the information provided by phase portraits. Only a few students analyzed long-term behaviour of solutions in relation to equilibrium solutions.

Trigeuros (2004) studied 12 students’ understanding of straight line solutions to a linear system of differential equations. She reported that only one had a complete understanding of straight line solutions as analysed using a framework that categorized the solutions as inter, intra, and trans modes of understanding. Her primary conclusion was that few students exhibit a strong understanding of solutions to differential equations.

Klein (1993) compared the effectiveness of instructions in differential equations with and without a computer algebra system (CAS). Two classes worked with CAS and the other two without CAS. On a common post-test, Klein found no significant differences in student ability to find the analytical solution to the differential equations. However, CAS classes showed a significantly more positive attitude.

Artigue (1992) explored the teaching of qualitative solutions of ordinary differential equations. Lecture sessions were supplemented with exercises using computers. She found that students used criteria such as signs of dy/dx, monotonicity, zeroes of f, and slopes at various points to correctly match several ordinary differential equations with their solution curves.

Rasmussen (1999) found that students often had incorrect conceptions of equilibrium solutions arising from the difficulty of treating the solution as a function. For example, given the non autonomous equations $dy/dt = t + 1$, students tended to reason that $t = -1$ was an equilibrium solution.

Rasmussen (2001) conducted semi-structured interviews with one student (Amy B.) to explore her understanding of qualitative solutions of first order differential equations. He found that Amy drew strongly on her work on modelling physical phenomenon, her conceptual understanding of the derivative as slope, and her work in Mathematica to infer properties of the differential equation, such as direction fields, slopes, solution curves, equilibrium solutions and rates of change in populations.

Zandieh and McDonald (1999) also found that 7 of the 23 students in their study generalized incorrectly the idea of equilibrium solutions to non- autonomous differential equations.

Rasmussen (2001) examined the connections students make between graphical and algebraic representations by providing students with the autonomous differential equation $dN/dt = -4N(1 - N/3)(1 - N/6)$ and the corresponding graph of dN/dt vs. N. While students could state the equilibrium solutions and the stability values they were unable to find the limiting populations N(2) , N(7). Questioning revealed that the graphs did not carry the intended conceptual meaning.

Allen (2006) investigated how students develop and use parametric reasoning as one basis for understanding dynamical systems of differential equations in an inquiry-oriented differential equations class. She found that students already have understandings of time and rate from earlier experience and from their instruction covering solutions to single ordinary differential equations and they use this to build their conceptions and understandings of solutions to systems of differential equations. The study also provides case studies of two students' mathematical activity as they learn systems of differential equations. Finally, the study uses a new construct of "advancing mathematical activity" and the mathematical practices of symbolizing, algorithmatizing, justifying, and experimenting to document how students enculturate into the larger mathematical community.

In summary, a definite shift in the orientation of the ODE courses towards multiple representation of solutions is noticeable. There appears to be more emphasis on general principles and concepts, on the use of computer tools, on graphical representations, and on numerical approximations. There is less emphasis on analytical solutions through use of algebraic algorithms that took up almost all the time and effort of the students in traditional courses on differential equations. However, although more emphasis has been placed on the graphical solutions graphing may not necessarily develop better conceptualizations.

2.4 Role of Technology in visualization

Technology offers mathematics educators a unique opportunity to generate, manipulate and present visual images in order to understand and address the problem. The power of technology helps transcend the limitations of the mind in thinking, learning and problem solving activities and facilitate the visualization of three-dimensional objects.

Pea (1987) distinguishes between the use of digital technology as an amplifier, doing tedious calculations, and as a reorganiser by producing dynamic interactive novel representations.

Gutiérrez (1996) reported that not much research has been done into the role of visualization in the learning and teaching of 3D geometry. While some research highlights students' difficulties in moving between 3D objects and their 2D representations, Gutiérrez (1996) claims that research needs to look into the potential of computer software to enhance students' visualization skills. Gutiérrez believes that the plethora of different representational positions possible with computer software create a rich spatial experience for the teaching and learning of visualization.

Seeburg (2005) investigated the role of software (CalcPlot3D) for the teaching of multivariable calculus. In his program student activities are designed to be intuitive and allow various geometric interactions between surfaces, space curves, vectors, vector fields, points, and other calculus -related objects to be dynamically and visually explored and manipulated. Surfaces and space curves can be easily rotated to gain a 3D perspective. Intersections between surfaces can be verified visually. The motion of a particle can be animated along a space curve, showing position, velocity, and acceleration vectors at each point. Contour plots can be displayed and then rotated into three-dimensions to see how they fit on the corresponding surface. A progression of level surfaces can be shown as a ‘movie’ by varying the value of a constant over a specified range and number of steps (or frames). Complex, even discontinuous, surfaces can be investigated easily and intuitively. It is easy to zoom in or out, make surfaces transparent or opaque, hide edges or faces, change viewpoint, focus, window size, and rescale the illustration.

Cretchley, Harman, Ellerton, and Fogarty (2004) showed that with the use of software (MATLAB), students were able to compare, classify, analyze errors, and support the students who struggled with solving problems. The study showed that the software improved students’ attitude and confidence in mathematics.

Operating under the Realistic Mathematics Education instructional design heuristics of emergent models, Rasmussen and Bloomenfeld (2007), analysed student reasoning with analytic expressions, as they reinvented solutions to systems of two linear, homogeneous, differential equations with constant coefficients. The data comes from the inquiry-oriented differential equations project. The study offers teachers insight into student thinking by highlighting qualitatively different ways that students reason proportionally.

Palais (1999) discussed and encouraged mathematical visualization as well as the integration of computer graphics in the math classroom. He found that ‘applied mathematicians find that the highly interactive nature of the images produced by recent mathematical visualization software allows them to do mathematical experiments with an ease never before possible' Palais (1999, p.652).

Hennessy, Fung and Scanlon (2001) argue that over-reliance on computer programs can be a problem for the students. Computer-aided software programs can be used mechanically, and student' understanding might prove superficial in even simple mathematical domains.

Camacho-Machin, Perdomo Diaz and Santos-Trigo (2012) investigated types of mathematical concepts and representations students use in dealing with ODE's. Their findings were similar to Habre's; that students do not use graphical representations or concept of derivative to explore meanings and mathematical relations inherent in the ODE's. To check if a function is a solution of a DE, they substitute the function or solve directly the given equation. Also they tend to search for an algorithm to solve particular groups of equations.

Researcher's comments

We note the potential benefits of using technology to enhance visualization of 3D objects and solutions to systems of DE's. We also note the impact of visualization on the motivation, interest, engagement and enthusiasm of the students. Some researchers (for example, Hennessy et al. 2001 and Rasmussen, 2001) noted the lack of conceptual understanding of graphical solutions and we find that few studies indicate which features of CAS provide the most leverage for enhancing understanding.

2.5 Chapter Summary

In summary, the review of literature reported on studies that conducted research in visualization and its impact on performance in MVC and solutions to dynamical systems. We found that visualization is being actively researched in all branches and domains of mathematics. We identified classification schemes being used for attributes or characteristics of visualization and spatial ability and the range of methods used including case studies, think aloud sessions, pre and post-tests, experiments, as well as assessments. We note that there has been a large push to incorporate more visualization into mathematics curricula (Hughes-Hallet ,1991 ; NCTM , 2004) and that fluency and the ability to translate between representations is an important aspect of problem solving in calculus. We also note some research (Aspinwall, Shaw and Presmeg ,1997) has shown that the effects of visualization on student reasoning are not always positive.

We note that previous research on differential equations has focussed mainly on first order ODE’s (Klein, 1993; Trigeuros, 2004 ; Camacho-Machin et al., 2012) and that attention is slowly shifting to non-linear systems of ODE’s (Allen , 2006 ; Rassmussen, 2006 ; Habre, 2002). There is a paucity of research on teaching and learning of multiple integrals as well as dynamical systems, which have been recognised as areas with a wide range of applications to real world phenomenon. Several studies (Palais, 1999 ; Hennessy et al., 2001) have examined the role of technology, in shifting the focus from analytical solutions to numerical and geometric solutions of systems of differential equations.

In Chapter 3 we focus on the theoretical framework, the research methodology , the instructional design and the choice of activities for the computer laboratory sessions.

Chapter 3 : Research framework, Design and Methodology

3.0 Overview

This chapter presents the theoretical and pedagogical frameworks used in the teaching and learning of multiple integrals and dynamical systems and relates them to the design and methodology deployed in the research. We begin with a recapitulation of the aims and objectives of the study.

The main research problem this study seeks to address is to facilitate visualization in the solution of problems in multiple integration and dynamical systems. The focus of the research is volume integrals in rectangular, cylindrical and spherical coordinate systems. The topics are rich in visual representations and the solution of problems need frequent conversions between registers on Duval's (1995, 1996) framework, outlined in section 3.2.1 of this chapter. The mathematics laboratory activities engage students in the exploration of 3D objects through investigations of surface properties, rotations, zooming in and out, planar projections, cross-sections, and sketching. The theoretical foundations of this are set out in section 3.6.

The second important area of the research is to identify strategies that could be used to enhance the conception and visualization of solutions to systems of ODEs. The interpretation of phase portrait solutions to systems of nonlinear differential equations is complex. It involves the integration of knowledge from different domains of mathematics and the use of multiple representation tools. This complexity makes the analysis of students' conceptions of solutions a challenging task. The theoretical foundations are laid out in Chapter 5.

In Chapter 2, section 2.4, we saw that computers offer educators an opportunity to generate, manipulate and present visual images of all kinds in two or three dimensions including curves and surfaces, direction fields, and contour plots. The images may be dynamic or interactive. In particular, we want to take advantage of the power of digital technologies to facilitate the visualization of 3D mathematical objects and phase portraits. The ability to draw a figure to represent a mathematical problem, to interpret the figure with understanding, and to use the figure as an aid in problem solving are fundamental visualization skills. Pea (1987) highlights several applications of digital technologies including their use as an amplifier, in doing tedious calculations, and as a reorganiser by producing dynamic interactive novel representations. We exploit these applications of digital technology in the teaching experiment.

3.1 Rationale for choice of MVC and dynamical systems

Multivariate calculus (MVC) depends to a large extent on the visual representational skills that students bring with them from single variable calculus. Problems involving double and triple integration have a strong visual element that involves projecting, seeing cross-sections and sketching. These are necessary to identify and sketch the bounding surfaces and find the limits of integration for the volume integrals. The analytical solution often follows a set of routine procedures.

Traditional instructional methods offer MVC concepts through the use of transparencies, three-dimensional models, and demonstrations using freehand sketches on whiteboards or chalkboards. Attempts are made to develop or enhance students' visualization skills through a series of drawing exercises. Two- and three-coordinate drawing, rotation of objects, and cross-sections of solids are highlighted using paper and pencil sketching.

Digital technologies offer several aids to visualization in MVC and dynamical systems. Concepts involving surfaces and solids of revolution and the intersection of solids can be developed through the use of software. The first basic concepts of projection are explained and practiced using simple, solid objects with surfaces such as rectangles, triangles, cylinders, and cones.

Advances in technology, together with an increased interest in dynamical systems, and modelling with nonlinear differential equations, have shifted the emphasis from analytical solutions to qualitative, numerical and graphical solutions. Analytical methods of solution are important but they are no longer the sole focus. There is a need for students to move flexibly between algebraic, graphical and numerical solutions and interpret and predict the long term behaviour of the system. Habre (2000), points out that students conceptions of solutions to DEs are analytic and highly resistant to change and the move to a graphical setting is extremely difficult. Students in his study were able to draw trajectories given the phase portrait to the equations: $\dot{x} = -x + 4y$, $\dot{y} = -3x - y$ and also plot the *x*-t and *y*-t graphs. Trigueros (2000) found that students in her study had problems interpreting equilibrium solutions and showed a tendency to focus on part of the phase portrait neglecting the long term behaviour of the solutions. In the Rassmussen (2001) study, students had great difficulties in interpreting graphs they had generated using Mathematica for solutions to the undamped linear model $\ddot{x} + x = 0$ and the damped nonlinear model $\ddot{x} + \sin x = 0$ of an oscillating pendulum. These studies emphasise a need for using a computer algebra system to reinforce student's visual understanding of phase portraits, slope fields and solution trajectories of DEs.

3.2 Theoretical framework for the study

In this section we discuss the theoretical framework that will be used for the analysis of student solutions to problems in multiple integrals and dynamical systems. Two conceptual frameworks inform the theoretical basis used in this study. Duval's (1996) semiotic representation theory provides the conceptual tools to analyse flexibility in the use of representations. The Visualization-Analysis framework, by Zazkis et al.(1996), highlights the visual and analytical steps used in the solution.

3.2.1 Duval's semiotic representation theory

According to Duval (1995, 1996, 2006), thinking processes in mathematics require not only the use of representation systems, but also their cognitive coordination. Duval (2006, p.106) maintains that 'semiotic representations are not only a means to externalise mental representations in order to communicate, but they are also essential for the cognitive activity of thinking'. He elaborates further that 'mathematical processing always involves substituting some semiotic representation for another' (p.107). Semiotics is the study of human sign systems. Signs culturally mediate activity and direct the individual's attention towards the mathematical object. The cognitive activities that play a role in representations are:

1. **Formation** of representations in a particular *semiotic register* either to express a mental representation or to recall a 'real' object.

2. **Treatment** - a transformation within the *register*. Treatments are transformations inside a semiotic system such as writing the equation of a sphere as $x^2 + y^2 + (z - 2)^2 = 4$ which is in rectangular coordinates as $\rho = 2 \sin \phi$ in spherical coordinates.

3. Conversion - a transformation that results in a representation in another register.

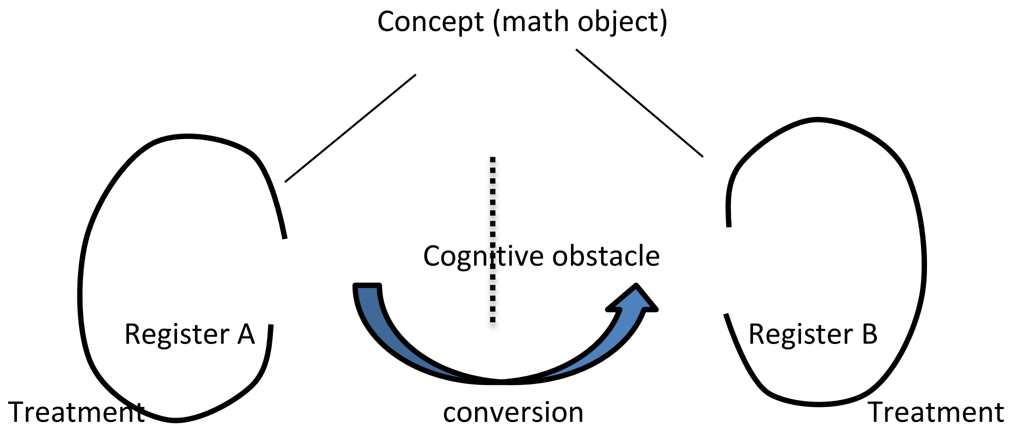


Figure 3.1 : Transformative processes: treatments are transformations within the same register; conversions occur across registers without changing the mathematical object. Adapted from Duval (2006).

As we noted in the studies reported in the rationale (section 3.2), students often have problems with conversions, particularly if this change of representational form does not include a set of steps or algorithms for translating parts of representation in the starting register to parts of the representation in the target register. For example, a change from a plot of a function (a graph) to an algebraic formula is difficult, whereas the other way (from formula to graph) is conceptually simple, since creating a table of (x, f(x)) values in principle constitutes a set procedure. Although a transformation of semiotic representations can be difficult much of the creative potential in mathematics stems from these transformations. Digital technology has an important role to play in these conversions. Computer Algebra Systems (CAS) support multiple forms of representations, where a number of mathematical registers can be activated ρsimultaneously using visual and diagrammatic types of semiotic representations. These enable conversion from one representation to another and facilitate accessibility of the mathematical object (See Figure 3.1).

In this research, the computer laboratory activities promote the use of several systems of representations and the reflective use of technology that allows the student to find meaning for the mathematical concepts and notions he/she is learning. For example, in the algebraic register, the equation of a sphere, radius 2, can be expressed as : $x^2 + y^2 + z^2 = 4$ in Rectangular Coordinates, as $r^2 + z^2 = 4$, in Cylindrical Coordinates and as $\rho = 2$ in Spherical Coordinates. The symbolic register (software code) has its own semiotic system. The three semiotic representations (Figure 3.2) can be obtained from each other by transformations that preserve their (common) object, the sphere. However, given the rectangular representation in the algebraic register, conversion to the geometric register requires several treatments.

The activities are used to reinforce the analytical processes that students will use in solving the problems. Computer software provides immediate feedback by giving students the solution to a complex triple integral or a system of differential equations. This can take a student at least 10 minutes to work out by hand.

Duval's semiotic registers have been used by McGee and (2014), for studying the development of the definite integral of 2 and 3 variables. His main findings were that 3 semiotic registers, namely, the geometric, the numerical and the expanded sum notation, (a sum in the sigma notation) and a definite integral were in use in teaching. He noted that it was rare for textbooks as well as tutors to use the numerical or the expanded sum representations.

Four types of apprehension of a representation were proposed by Duval (1999). These are

1. perceptive apprehension, which enables recognition of the form of the mathematical object.
    a. discursive apprehension, the representation is seen according to a verbal description.
2. sequential apprehension: we look at the steps, and their order, according to which a representation must be constructed. Finally*y*,
3. the operative apprehension is the most complicated. It is supposed to show the 'idea' of the solution of a problem.

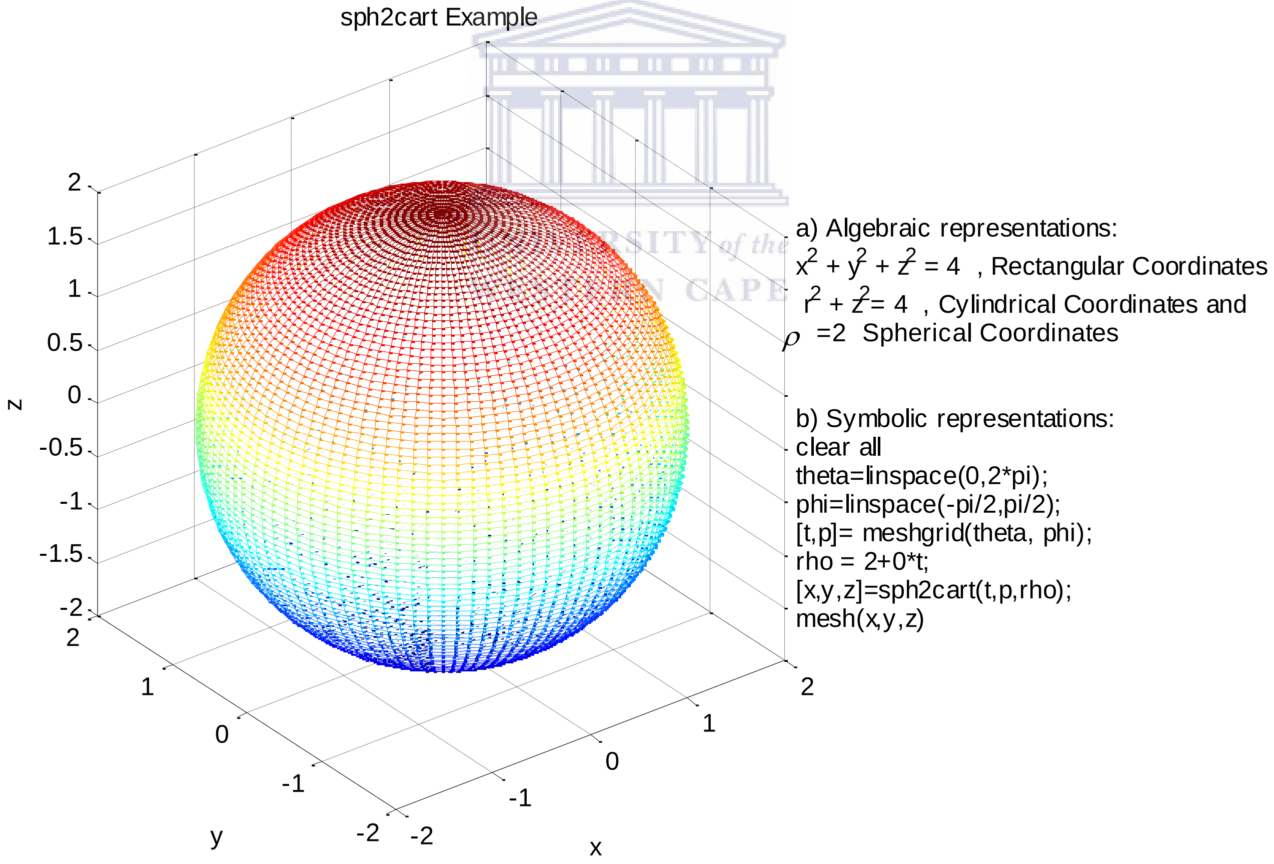


Figure 3.2 .A sphere centre (0 , 0, 0) , radius 2 units a) Algebraic representations in rectangular, cylindrical and spherical coordinates b) Symbolic representation in Matlab and c) the geometric representations

3.2.2 Visualization-Analysis (VA) framework

The second theoretical framework is the visualization–analysis (V-A) model proposed by Zazkis' et al.'(1996) to account for flexibility and links between acts of visualization and analysis or analytical thinking in mathematical performance. The model views visual and analytical reasoning as complementing each other in the solution of mathematical problems. It has been used by Nilsson and Juter (2011), to account for processes and flexibility and links between acts of visualization and acts of analysis in 3D pattern generalization. It was extended and used by Zazkis (2013) in her doctoral thesis to the V-A-P model, where P refers to the physical situation of the problem. The model assumes that visualization and analysis are distinct forms of thinking and work together and inform each other during problem solving. Stylianou (2002, p. 306) clarifies this:

> "The thinking, as it is described by the V–A model, begins with an act of visualization, V_1 which is defined as an act in which the individual establishes a strong connection between an internal construct and the actual drawing of a picture, or the expression of a mental image. This is followed by an act of analysis, A_1, in which the person does logical analysis and reasons about what was visualized in V_1. Thus analysis involves mental manipulation of the objects or processes with or without the aid of symbols. It includes logical reasoning and naming of parts or processes and reflections on the mathematical process. It may lead to a revised visual representation. Then follows a second visualization step V_2, enriched as a result of A_1…."

The model proposes a series of switches between repeated acts of visualization V_1, V_2, V_3, interspersed with acts of analysis A_1, A_2, A_3…. following a spiral of steps, Each act of analysis leads to a better and richer visual representation followed by more sophisticated analyses. Figure 3.3, shows the V-A model as presented by Zaskiz, (1996, p447). The top of the V-A spiral represents a solution which may be an algebraic or numerical expression. The relevance of the V−A model to solutions of problems in 2D and 3D will be established when we look at the type of problems the study will engage in (sections 3.7 and 3.8).

The V−A model has been refined by Stylianou (2002), following the coding of mathematical problem solving processes, and elaborates on the nature of activities during the analysis steps. These include:

a) inferring additional consequences from the visual representation,

b) engaging in elaboration and further investigation for additional consequences,

c) setting new goals with respect to the visual representation and

d) monitoring the outcomes of earlier analysis.

Thus, logical analysis and reasoning, and reflections are an integral part of the V-A model.

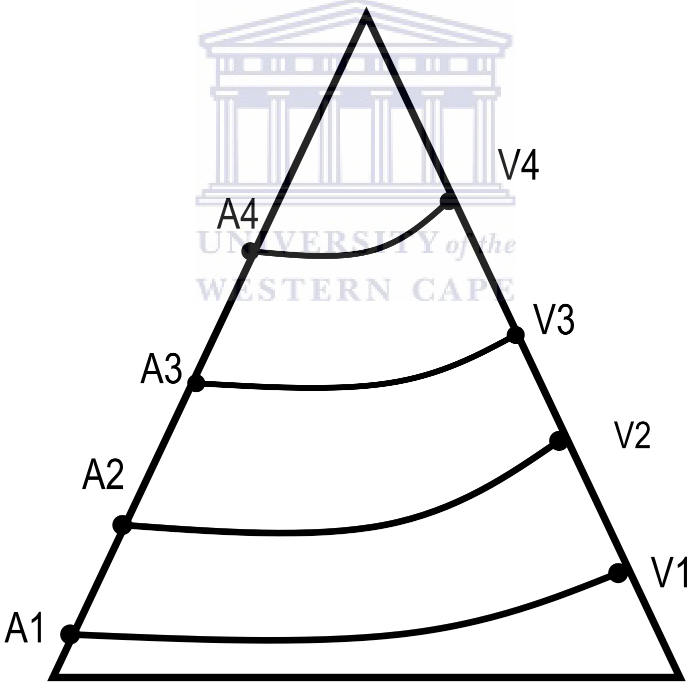


Figure 3.3 The Visualization-Analysis model for probing student thinking Source: Zazkis' et al. (1996, p. 447)

3.3 Pedagogical framework

Inquiry oriented teaching (IOT) using constructivism, provides a framework for researchers probing the construction of mental representations, and addresses the learner's role in the learning process (Ashcraft, 1989). IOT has been implemented in reform calculus curricula as a pedagogical framework and is informed by Hughes-Hallets' (1994) rule of three, (often called the rule of four, if we include physical/kinaesthetic considerations). Three sub-processes have been highlighted within the constructive process. These are organisation of concepts within the existing prior knowledge, the selection of concepts based on their importance and relevance to the activity and the connection of concepts through inferences or elaborations. The learner uses multiple senses (vision, hearing, haptic, olfactory) to participate in the learning experiences. IOT uses critical thinking and involves the students by questioning, reflecting, predicting and explaining ideas. It has been used by Allen (2007) in the student understanding of ODEs.

The review of literature, Chapter 2, informed the design of the present study. We chose activities and maths laboratory sessions with manipulatives (wire models and manila cardboard) to concretise 3D models, their nets, cross-sections etc. We used software to generate and manipulate visual images of 3D objects and phase portraits for DE systems. We conducted computer laboratory sessions using worksheets, assignments, hand-outs, and sketches.

3.4 Methodology

3.4.1 Pilot Study

In the first semester of 2013, a pilot study was conducted with a sample of 31 students enrolled for a semester course in Calculus 3 (MAT300S). The students had a similar background (completed Calculus 1 and 2) and followed the same course as in the main study. The topics covered included MVC and dynamical systems and in addition we tried out the lab activities. Data gathered from students' assignments, lab worksheets and tests was used to formulate the hypothesis and finalize the activities and test items as well as decide on the variables in the regression study. We found that students had difficulties with solving equations, integration and differentiation and errors stemming from poor mastery of fundamental concepts in univariate calculus were common. One implication was that mastery of basic mathematics and univariate calculus (solving equations, sketching 2D graphs of functions and differentiating and integrating) were an essential pre-requisite before delving into multivariate calculus. We drew up review problems and a pre-test to check on prior knowledge for the main study. The second part of the pilot study focussed on modelling a non- linear pollution problem and a system of ODE equations. In the pollution problem students found the analytical solutions of the ODE using Laplace transform methods. The coupled non-linear equations were solved analytically using linearization as well as phase portrait analysis. Students found the critical (equilibrium) points and decided on the stability of the system by setting up the Jacobian matrix and working out the eigenvalues.

3.4.2 The main study

The main study was conducted with 2 groups of students at a university of technology following the Calculus 3 (Mathematics 3) syllabus in the first semester of 2014. The students could not be assigned randomly to treatments and therefore, the non-equivalent control group design was chosen from the quasi-experimental design choices.

All of the participants had taken and passed semester courses equivalent to calculus 1 and calculus 2 in the previous years. Lectures for both groups were recitation and included chalkboard, overhead projectors and transparencies. All students had access to the Webcity where supplementary materials, assignments and tutorials were posted.

3.5 Research and instructional design

The experimental group (n = 22) attended full time, receiving their lectures on Wednesday and Thursday (1.45 pm to 3.30 pm.) and in addition participated in computer laboratory activities on Fridays (11.30 am to 1.00 pm), working in pairs on computers and worksheets. The control group (n=26) were part time students who attended on Wednesday evening (5 pm to 8 pm) and Saturday (10 am to 12.00 noon). Most of these students were working in the petro-chemical industry as operatives or controllers. The amount of class time dedicated to integration using four coordinate systems (rectangular, polar, cylindrical and spherical) and dynamical systems was the same for the experimental and control groups. They used the same notes and the assignments were the same for the two groups. The only differences were the additional computer laboratory activities for the experimental group supplemented by suitable problems on worksheets.

To reduce the effects of selection bias, the researcher administered pre-tests to both the control and the experimental group. Small tests and assignments were administered to both the groups at regular intervals during the semester course and major examination-like assessments labelled T1, T2, T3 on dates determined by the Department of Mathematics and Physics. The design of the study is shown in Figure 3.4.

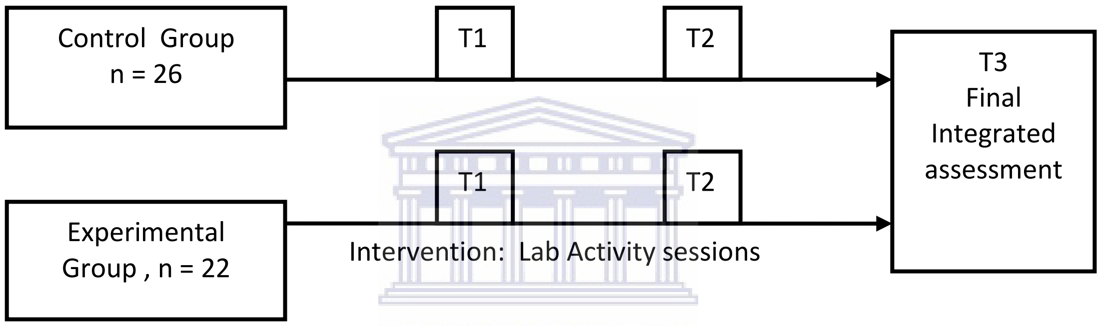


Figure 3.4 Design of the study

The first week was spent on reviewing work from Mathematics 100 and 200, which are semester courses equivalent to Calculus 1 and 2 at a university (see Appendix 3). A pre-test, assessing prior knowledge was administered at the beginning of the study (week 2) to both groups (Appendix 4.1). The pre-test scores were taken as a measure of students' prior knowledge. To check for differences between the two groups on prior knowledge the following hypotheses were formulated:

Null hypothesis: H_o – There are no differences between the scores of the control and experimental groups on the pre-tests.

Alternative: H_1 – There are significant differences between the scores of the control and experimental groups on the pre-tests.

During the examinations and tests, students did not have access to software and all 3D sketches as well as phase portraits were sketched by hand. The Laboratory sessions ran for six weeks during February and March 2014. The performance of classes was compared through post-intervention tests (T1 in March, T2 in April and T3 in June). These tests covered a range of topics in multivariate calculus including vector analysis, and partial differential equations, but for this research, only the performance on integration and dynamical systems is reported.

3.6 Design of mathematics laboratory activities: Interventions

The laboratory activities involved the use of software (Matlab) to construct, explore and sketch 2D and 3D representations of various mathematical objects and phase portraits for differential equations. Students could use MESH (to produce a transparent 3D object) or SURF (to get an opaque shaded surface of the 3D object). They could change the view angle to see the 3D object in different perspectives, rotate the solid to examine surface features, view intersections and cross-sections and sketch the projections in the *xy*, *yz*, and *xz* planes (See Figures 3.5 and 3.6). The worksheets had spaces for drawing projections and cross-sections and the 3D solid. See completed worksheets in Chapter 7.

Using the symbolic toolbox in Matlab, students could check the results of their integration for each set of triple integrals they set up and change the order of integration. Students were also encouraged to do hand calculations. Immediate feedback came from the software as answers could be checked and changes made to correct errors.

Lab Session 1: Visual exploration and sketching of 3D objects (ellipsoid, sphere, cylinder, cones) given in the algebraic registers: Rotation, translation, Zooming in Out, from different viewpoints (See Figures. 3.5 and 3.6).

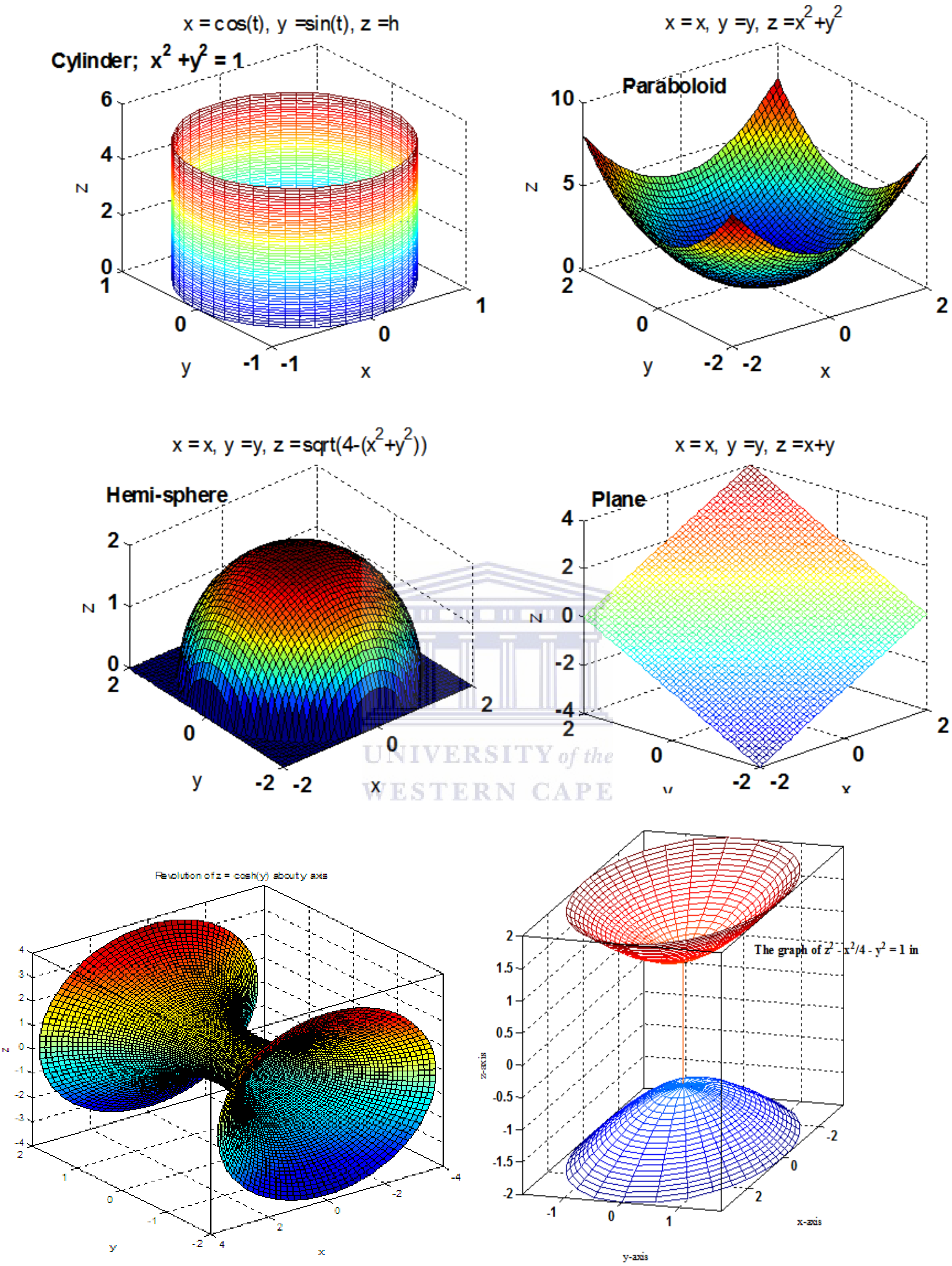


Figure 3.5 Exploring 3D objects in MatLab

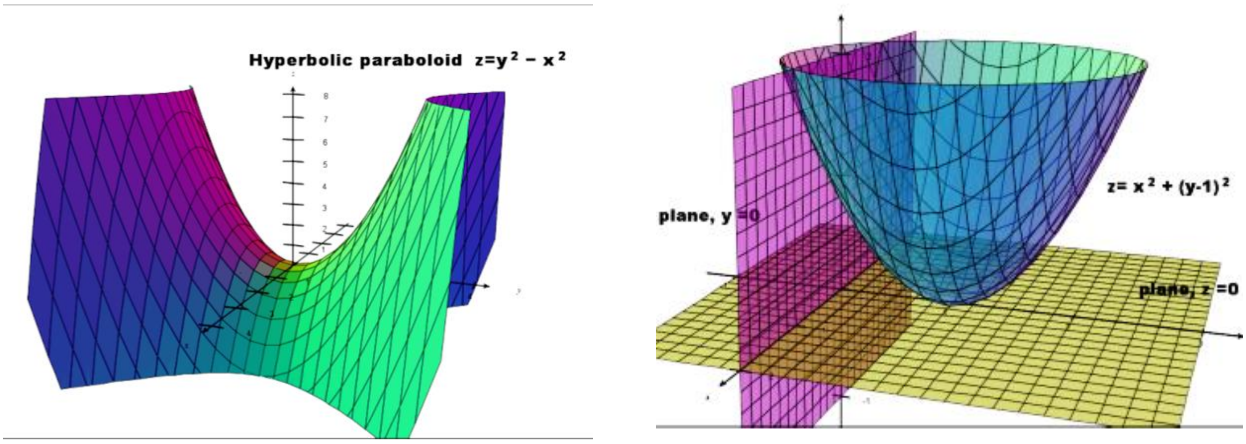


Figure 3.6 Exploring 3D objects in CalcPlot3D an applet available on the Internet

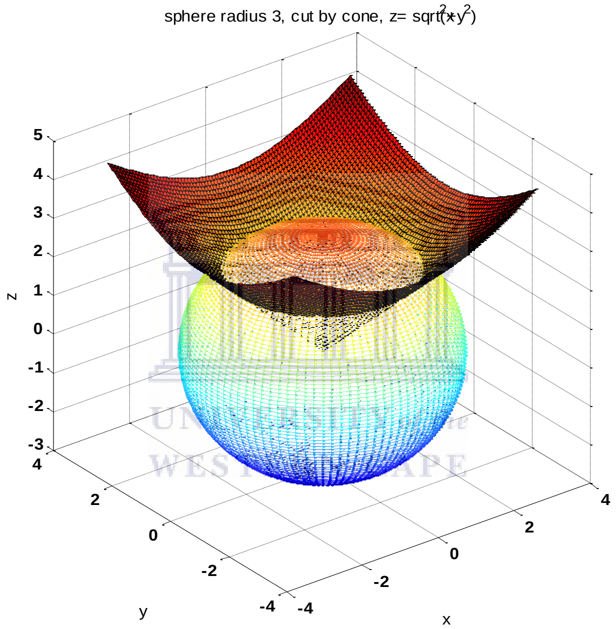
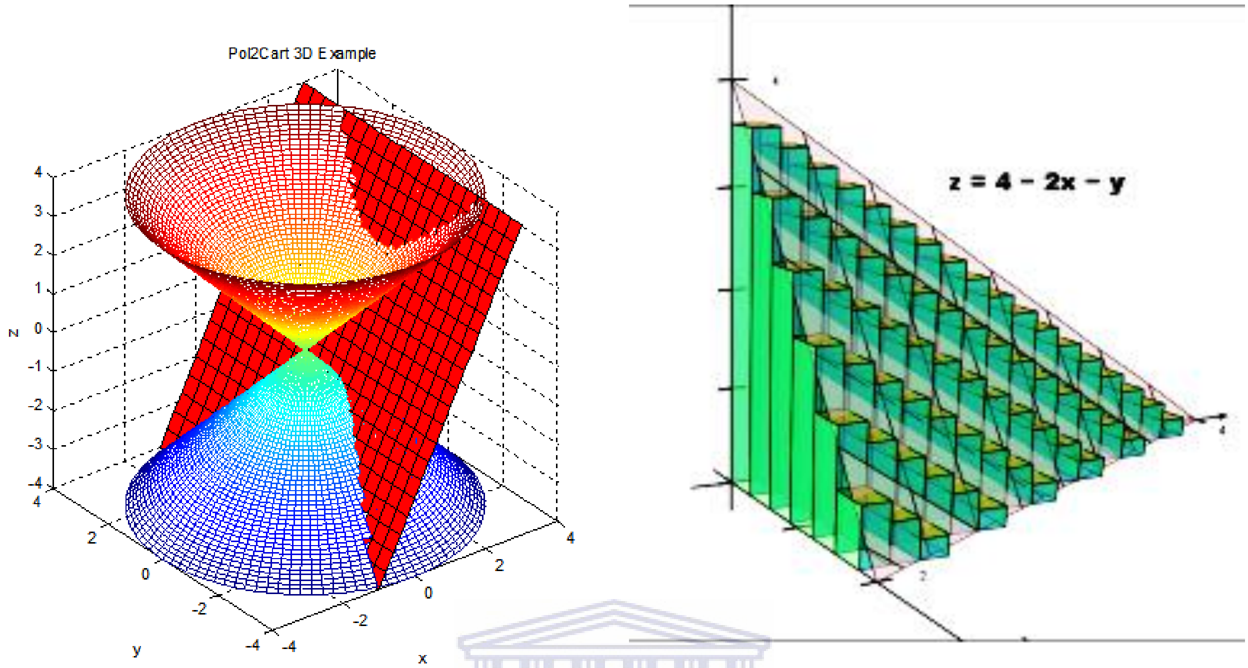


Figure 3.7 Intersections : Cone and sphere

Lab Session 2 : Identifying and sketching intersections between planes and 3D objects

The goal of this activity was for students to explore the varied intersections of 3D shape with a plane and to describe the attributes of the intersection with algebraic equations. (See Figure 3.7 and Figure 3.8a)

Lab Session 3: Riemann Sums by increasing the number of subintervals, the difference between the lower and upper sums can be made to decrease, suggesting that the lower and upper limits eventually coincide with the value of the definite integral**.** Single and double integrals using Matlab.(See Figure 3.8 b)



(a) (b)

Figure 3.8 a) Plane cone intersections b) Riemann Sum: Using rectangular prisms to approximate the volume of the solid region between f(x,y) = 4−2x + y and the xy-plane in the first octant

Lab Session 4 Triple integrals. Sketching level curves and projections in *xy*, *yz* and *xz* planes. Sketching contours (See Figures 3.9 for examples).

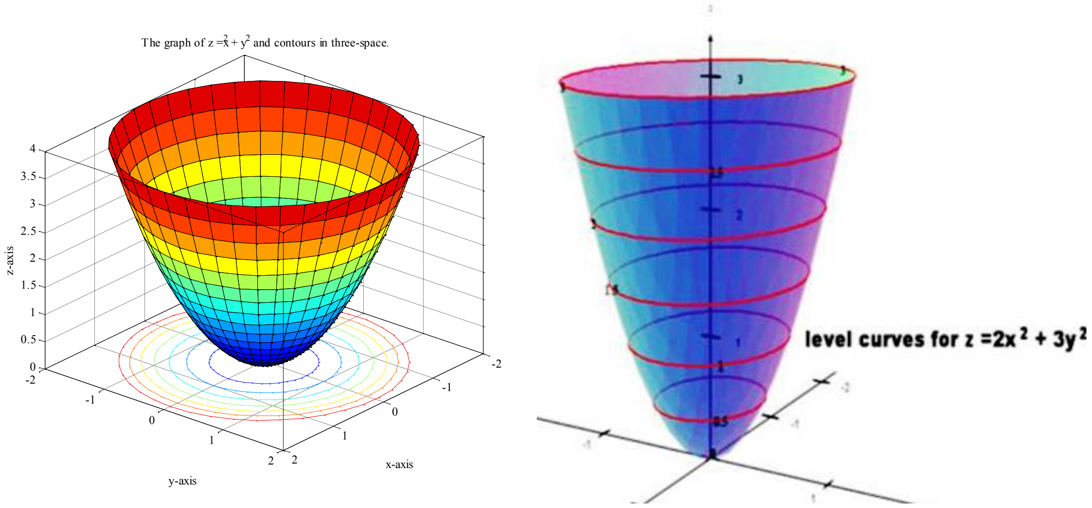
Lab session 5: Plotting direction fields. Identif*y*ing equilibrium points, solution curves isoclines, nullclines and interpreting long term trends

a) Direction fields: Type in the Matlab codes. See Figure 3.10.

b) Phase portraits – Lotka-Volterra Equation

Lab session 6: Sketching phase portraits and identifying equilibrium points. Stability analysis. Using eigenvalues. Describing the stability. Figure 3.11 shows a completed worksheet.

We hypothesize that the visualization of the solution to problems involving 2D and 3D objects and directions fields and phase portraits can be enhanced by the Laboratory activities and also that the interaction with the mathematical objects and their representations in the algebraic and geometric registers can promote the perceptual, sequential, discursive and operative apprehension in



(a) (b)

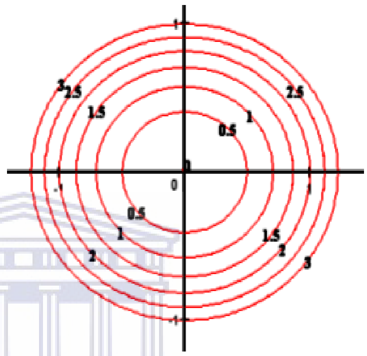


Figure 3.9 a) Paraboloid
$z = 2x^2 + 3y^2$
(b) level sets for $z = 2x^2 + 3y^2$
(c) contour diagram

(c)

Duval's (1996) framework. By changing the values of the parameter a student can see the effect on the 2D as well as 3D drawings. An example of this is the introduction of small perturbations in the Lotka Volterra equations discussed on pages 119- 120. Encouraging students to solve the problem analytically and at the same time verifying the solution on the software provides immediate feedback.

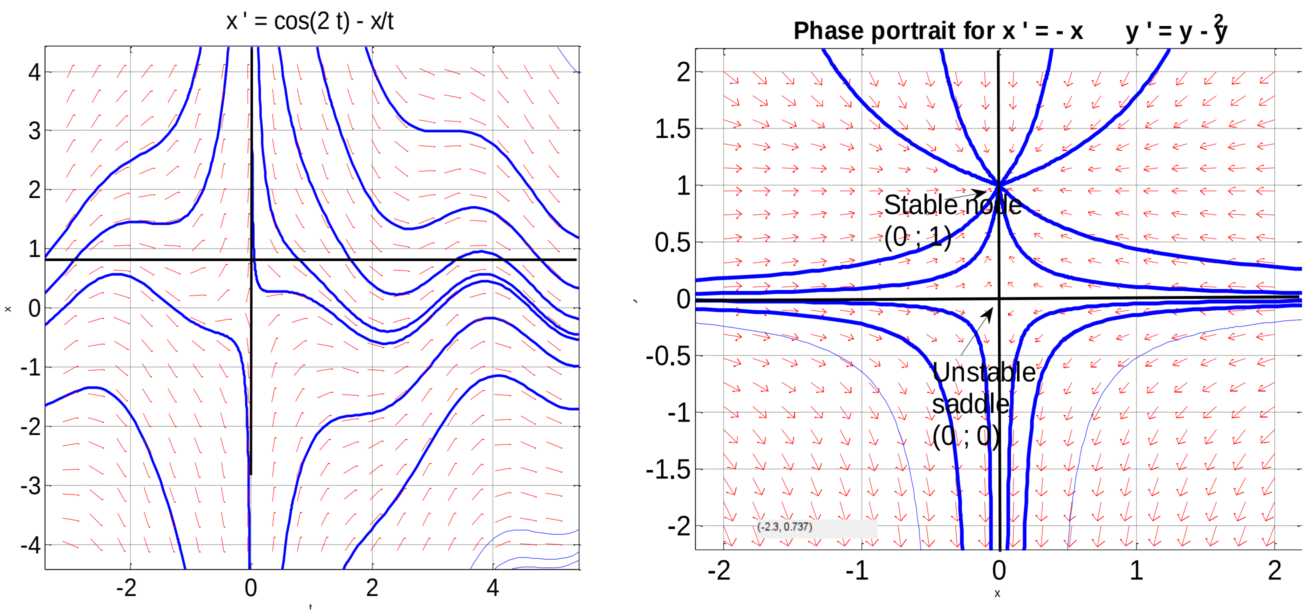
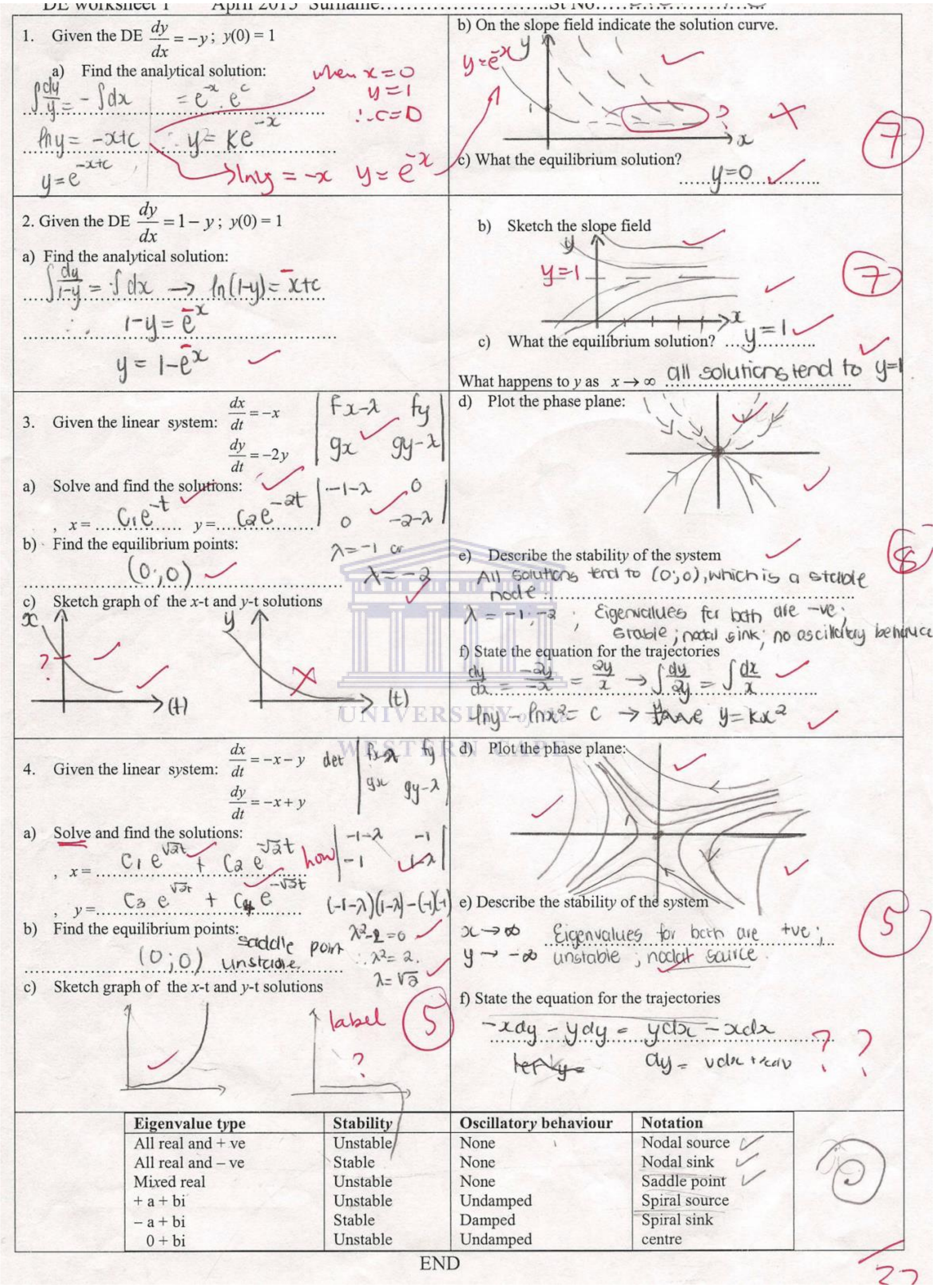


Figure 3.10 a) Direction field for $dy/dt = \cos(2t) - y/t$ b) Phase portrait for $x' = -x$ and $y' = y - y^2$



DE worksheet 1 April 2015 Surname……………………St No……………

1. Given the DE $\frac{dy}{dx} = -y$; $y(0) = 1$

a) Find the analytical solution:

b) On the slope field indicate the solution curve.

c) What the equilibrium solution?

2. Given the DE $\frac{dy}{dx} = 1 - y$; $y(0) = 1$

a) Find the analytical solution:

b) Sketch the slope field

c) What the equilibrium solution?

What happens to y as $x \to \infty$

3. Given the linear system: $\frac{dx}{dt} = -x$, $\frac{dy}{dt} = -2y$

a) Solve and find the solutions: $x =$ …… $y =$ ……

b) Find the equilibrium points:

c) Sketch graph of the x-t and y-t solutions

d) Plot the phase plane:

e) Describe the stability of the system

f) State the equation for the trajectories

4. Given the linear system: $\frac{dx}{dt} = -x - y$, $\frac{dy}{dt} = -x + y$

a) Solve and find the solutions: $x =$ …… $y =$ ……

b) Find the equilibrium points:

c) Sketch graph of the x-t and y-t solutions

d) Plot the phase plane:

e) Describe the stability of the system

f) State the equation for the trajectories

Eigenvalue type	Stability	Oscillatory behaviour	Notation
All real and + ve	Unstable	None	Nodal source
All real and – ve	Stable	None	Nodal sink
Mixed real	Unstable	None	Saddle point
+ a + bi	Unstable	Undamped	Spiral source
– a + bi	Stable	Damped	Spiral sink
0 + bi	Unstable	Undamped	centre

END

Figure 3.11 Example of a differential equations Lab worksheet used in the computer sessions

3.7 Ethical Issues

Before the beginning of the research study, ethical approval was obtained from the Cape Peninsula University of Technology and the University of Western Cape Higher degrees Research committees (See Appendix 1, p.190). Ethics were approved based on the researcher following particular conditions as were laid out in the ethics application form. These included:

a) Providing participants with a participant information sheet (See Appendix 2) which included what the purpose of the investigation was, what their participation involved and how they would benefit from the investigation.

b) Providing participants with a participant consent form, which confirmed they were willing to participate in the study.

c) Confidentiality of the participants was guaranteed by not releasing names or student numbers of the participants and by only allowing the investigator access to the data.

3.8 Data Sources

Data was collected from two groups (Control and Experimental) who were registered for a Bachelor of Technology degree in chemical Engineering following Mathematics 3 in the first semester of 2014. In addition to theoretical work the experimental group attended six laboratory sessions where they were exposed to activities in Matlab as described in section 3.6. Instructions and questions were given on worksheets. At the end of the Lab sessions students submitted the completed worksheets. Data was also collected through student Tests (T1, T2 and T3), assignments and interviews with selected students.

a. Pretest This consisted of 9 items and its purpose was to see if there were any differences between the control and experimental groups and to identify difficulties and obstacles students experienced with functions, 2D curves, differentiation, integration and DEs.

b. Assignments Two major assignments were set, one on double and triple integration and another on solutions to systems of non- linear Differential Equations.

c. Laboratory worksheets: Designed mainly to reinforce visualization of 3D objects and phase portraits. See Chapter 7. Students plotted the 3D shape and viewed the projections. These were then used to find the limits of integration.

d. Observations and oral interviews: Oral interviews were conducted at the end of the semester with four students from the experimental group selected on the basis of their work as presented in test scripts. The goal of these interviews was twofold : first to clarify the visualization of the required region and second to follow up on errors they made in finding the limits of integration and their interpretation of the phase portraits. The interviews were audiotaped, semi-structured and lasted between 10 minutes for the short questions on double integrals to half an hour for the phase portraits and accompanying graphs. During the interview, the participants were given their scripts and asked to go over their solutions explaining how they arrived at the limits . If participants said something that was unclear or interesting, they were asked to clarify expand on these issues. The interviews were conducted with students after class and audio recorded. Transcripts of the interviews are presented in Chapter 6 as interview excerpts.

e. End of semester evaluation: An end of semester lecturer evaluation where students ranked variables such as clarity of notes, visuals, lectures, appropriateness of laboratory sessions, worked examples, and use of software on a scale of 1 to 5 was conducted at the end of the semester. This provided feedback on different aspects of the research.

3.9 Research Validity

A key criterion in research is validity. Research validity is concerned with construct validity, internal and external validity as well as statistical conclusions validity. Our concern is the appropriateness of the inferences drawn from the data.

The researcher conducted a thorough literature search to identify the constructs defining visualization as they apply to multiple integration and dynamical systems. The preoperational explication of the constructs was based on the results of a conceptual analysis of visualization using Duval's (1995, 1996) semiotic representation theory and Zazkis' et al. (1996) V-A frameworks. In addition to the semiotic registers the perceptual, sequential, discursive, and operational apprehension stages in Duval's framework adequately captured student thinking and understandings in the two subject domains. There were no confounding levels of constructs or need for additional levels.

Internal validity deals with issues such as dropouts, absentees, diffusion of treatment between control and experimental groups and so on. Student dropouts occurred for various reasons including reasons to do with employment. Some had night time shifts and found it difficult to attend all sessions. Other reasons included financial and personal reasons (pregnancy) and inability to cope with the demands of work and study. However, as this affected both groups it is unlikely to confound the findings or bias the results of the study in any way. Students who missed a test were given an equivalent test on the same topics. Students who missed the computer laboratory sessions had an opportunity to schedule another session when the laboratory was free but this was discouraged. This was not possible for the regression study as the researcher did not have alternative versions of the tests. These students were excluded from the regression study leaving 21 students out of 28 with complete data for the regression analysis.

It was not possible to prevent communication between the control and experimental groups and as such diffusion of treatment between the groups may be a confounding factor affecting the results of the study. In particular, assignments and worksheets were completed collaboratively outside lecture time.

As far as external validity is concerned, the researcher worked with two classes doing Mathematics 3 (Calculus 3) in a university of technology and the findings can only be generalised to tertiary students doing similar courses at this level after having completed Calculus 1 and 2.

Random assignment to experimental and control groups was not possible. The choice of the topics was based on what was in the syllabus and the findings of the study do not necessarily apply to other content domains or branches of mathematics.

The violations of assumptions for the ANOVA and multiple regression analysis are discussed with the results of the data analysis in Chapter 6. Preliminary data analysis was done using Excel followed by a more thorough analysis in the statistical software package, SAS.

The choice of the variables and the reliability of the test items for the dependent and independent variables for the regression study and pretest are discussed in the following section. The pilot study, conducted in the first semester of 2013 and the reliability coefficients of the testing instruments determined using the split half method, were found to lie between 0.58 and 0.67. The surface features test was found too easy ($\bar{X} = 68.57;\ SD = 26$) while the Nets most difficult ($\bar{X} = 51.86\ ;\ SD = 21.57$).

3.10 Data Analysis

As this study compared the groups, each with two levels, men and women, for the characteristics identified, a two way unbalanced analysis of variance was run to check for significant differences between the control and experimental groups as well gender differences. The distribution of the number of students in the control and experimental groups by gender are shown in Table 1.

Table 1 Distribution of students in the study by group and gender

	CONTROL	EXPERIMENTAL	Totals
Male	18	10	28
Female	8	14	22
Totals	26	24	50

In the regression analysis, complete data on all independent variables was available for only 21 students in the experimental group. The control group did not participate in the laboratory activities as they attended part-time after hours when the laboratories were out of bounds.

3.11 Multiple Linear Regression Analysis

Multiple regression analysis is a statistical technique used to predict or explain the variation in the dependent variable $\hat{y}$ in terms of independent variables, $x_1, x_2, \ldots, x_k$. We applied multiple linear regression of y on $x_1, x_2, \ldots, x_k$ based on the equation:

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2 + \ldots\ldots\ldots + b_k x_k + \varepsilon \text{ (linear)}$$

Where $\hat{y}$ stands for the predicted y value, b_o is the intercept, the value of y when all the x's are zero. Here $b_1, b_2, b_3, \ldots, b_k$ are analogous to the slope in linear regression equation and are called regression coefficients. They can be interpreted the same way as slope. Thus $b_i = 2.5$, indicates that y will increase by 2.5 units if x_i increases by 1 unit. The conditions for estimation and inference in a multiple linear regression are:

1. The errors are normally distributed
2. The mean of the errors is zeroe
3. Errors have a constant variance
4. The model errors are independent

Violations of the assumptions are checked using standardised residual plots, i.e residuals divided by their standard deviations.

The appropriateness of the multiple regression model as a whole can be verified by the F-test in the ANOVA table. A significant F indicates a linear relationship between y and at least one of the independent variables.

Once a multiple regression equation has been constructed, one can check how good it is (in terms of predictive ability) by examining the coefficient of determination (R^2). R^2 always lies between 0 and 1 and is often expressed as a percentage.

A related question is whether the independent variables individually influence the dependent variable significantly. Statistically, it is equivalent to testing the null hypothesis that the relevant regression coefficient is zero.

When two variables are highly correlated, they are basically measuring the same phenomenon. When one enters into the regression equation, it tends to explain most of the variance in the dependent variable that is related to that phenomenon. If a correlation coefficient matrix with all the independent variables indicates correlations of 0.75 or higher, then there may be a problem with multi-collinearity in the model. If multi-collinearity is discovered, the researcher may drop one of the two variables that are highly correlated with the dependent variable.

A dependent variable is defined as 'the measured outcome of interest'. The measured characteristics of the dependent variable were mathematics achievement scores on Test 2 (on multiple integrals and ODE systems), for each student. A treatment variable is one whose value defines group membership (i.e Experimental or Control Groups) . The review of literature helped to identify six independent variables. One of these was Pre-test scores (prior knowledge) and the other five are described in Table 2.

Table 2 Description of variables used in the regression study

Variable	Description
1.Surface features of 3D objects (SURF)	Students count edges, faces and vertices of 3D objects Perceive and describe the figures' properties by their similarities and differences, recognize the regularity or irregularity of the shapes
2. Net and solid matching (NET)	Students match net to solid obtained on folding net Students match edges on folding nets to edges on 3D sketch of solid and draw their nets.
3. Projections (PROJ)	Students sketch projections of 3D object in the *xy*, *yz* and *xz* planes
4. Cross-sections (XSECT)	Students sketch the cross-sections when solid is intersected by vertical, oblique or horizontal cutting planes
5. Solids and Rotations (ROTNS)	Students match solid after rotation

Test items for each of the independent variables in the regression model were selected from various standardised tests. Examples of items used for each independent variable follow.

1. **Surface features**

Given the 3D solid situated in the first octant (Figure 3.12) :

a) How many plane faces are there?

b) How many vertices does the 3D solid have?

c) How many edges does the 3D solid have?

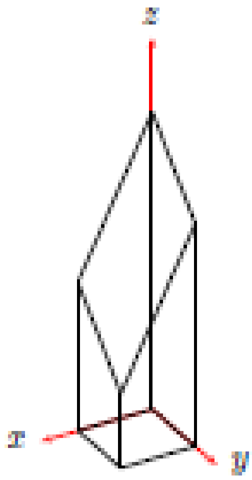


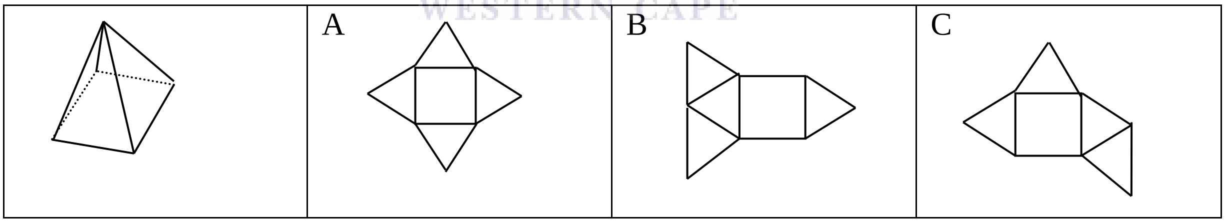
Figure.3.12 3D solid

2. **Identifying and sketching Projections**
Given the 3D solid (Figure. 3.12) , sketch

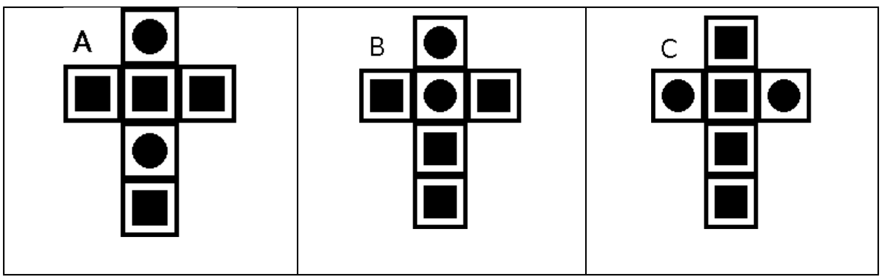
a) *xy* projection b) *yz* projection c) *xz* projection

3. **Nets and solids:**

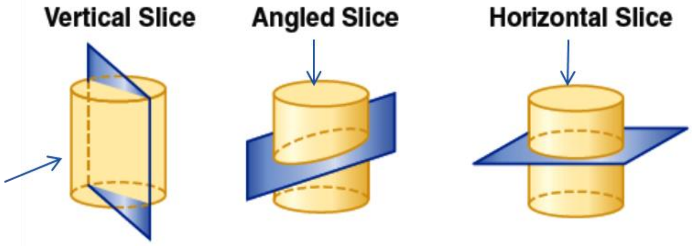
a) Which of the nets A, B, C shown fold into the pyramid on the left?



b) Which two of the nets shown fold into the same box?

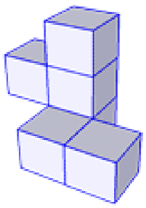
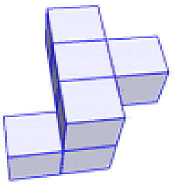
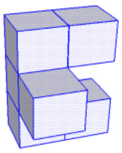
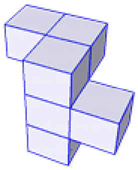
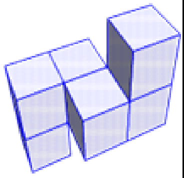
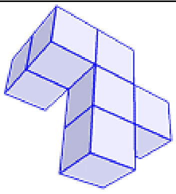


4. **Drawing cross-sections of given solids**. If you look in the direction of the arrow what do you see in the cutting plane.



Source: Math Online, Glenco.com, p.3

5. **Identifying solid after rotations**: Which of the solids A,B,C, D, E will *you* get after rotating the solid in the first block on the left.

					
	A	B	C	D	E

Source: Source: Alaskan Spatial Test Battery

3.12 Chapter Summary

In chapter 3, we discussed the research methodology and the theoretical and pedagogical frameworks for the study. Duval's (1995, 1996) semiotic representation framework, the Zazkis' et al. (1996) visualization-analysis framework and the teaching experiment with the type of activities students engaged in were outlined. We also looked at the design of the study, development of the analytical tools, and testing their reliability and validity. The teaching experiment and the activities in the computer laboratory were outlined. The data sources were identified and methods of analysis include a 2 by 2 unbalanced ANOVA and multiple regression analysis. The chapter ended by giving examples of test items for each of the independent variables used in the multiple regression.

Chapter 4 focusses on the theoretical foundations of multiple integration with a focus on the role of visualization in solving problems involving single, double and triple integrals.

Chapter 4 : The role of visualization in evaluating multiple integrals

4.0 Overview

The motivation for research in the teaching and learning of multiple integrals is the numerous real world applications of integration in science and engineering. We use single integrals to find the planar area under a curve $\int f(x)dx$, to find the work done when a force $\boldsymbol{F}$ displaces an object $\int \boldsymbol{F.dr}$, to find the energy changes when a gas expands $\int \boldsymbol{p\ dV}$ and so on. We use double integrals when we want to find the planar area, moments of inertia and centre of gravity of laminae, volume and mass enclosed under a surface. Triple integrals are useful to compute volumes, masses, centres of gravity, moments of inertia of space figures and solid objects in R^3. They are fundamental to the study of fluid flow and heat and mass transfer.

In this chapter, we review single integrals in section 4.11 and introduce the double integral $\iint_D f(x, y)\, dx\, dy$ where D is the plane area in R^2. We look at areas in rectangular coordinates using the Riemann sum $\Sigma dxdy$ and extend these to polar coordinates (sections 4.2 and 4.3). In section 4.7, we look at triple integrals $\iiint_R f(x,y,z)dV$ where R is in 3D. We discuss setting up the integral in rectangular coordinates in different orders and we extend this to cylindrical coordinates for regions which are circular. Finally, in section 4.8 we look at using spherical coordinates for conical and spherical regions. We unpack the visual elements in each of these domains.

4.1 Visualising integration of single variable functions

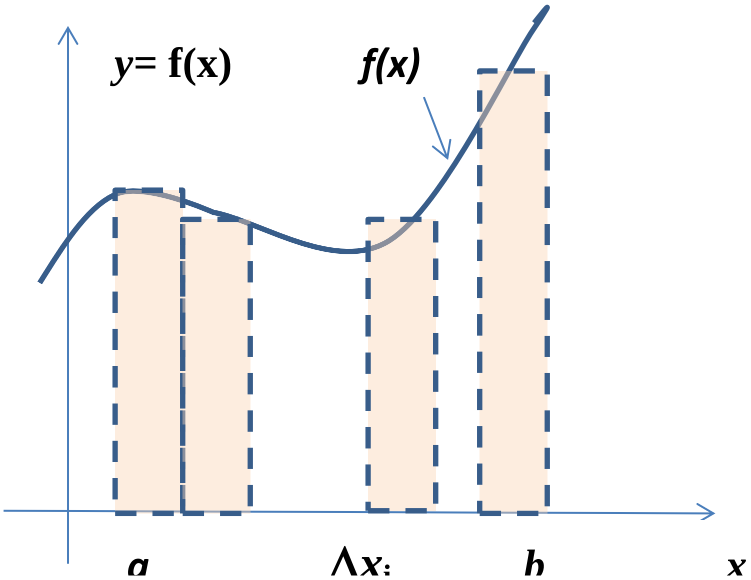


Figure 4.1. Single integral: Planar area under *f(x)* by slicing

Consider a single variable function $f(x) \geq 0$ plotted against *x* as shown in Figure 4.1. Given *f(x)* is continuous over [a ; b], we can divide [a : b] into rectangular subintervals of equal width, Δx and height, *f(x)*. The height is taken as the value of *f(x)* in the middle of Δx_i. We can approximate the area of the region under *f(x)* and the *x*-axis by summing the areas of the rectangles $f(x_i)\Delta x_i$ over all the rectangles taking the limit as $\Delta x_i \to 0$. The definite integral of this function *f*(*x*) over a range [*a*, *b*] is:

$$\int_a^b f(x)dx = \lim_{\Delta x_i \to 0} \sum_i f(x_i)\Delta x_i = [g(x)]_{x=b} - [g(x)]_{x=a}$$

where x_i and Δx_i is the position and the width of the *i*-th rectangle respectively. The function *f*(*x*) is called the integrating function or the integrand and the range [*a*, *b*] is called the integrating region with limits *a* and *b*. The definite integral is equal to the area bounded by the curve *y* = *f*(*x*) and the *x*-axis.

The indefinite integral of this function is defined as the inverse of the derivative, i.e.

$$\int f(x)dx = g(x) + C \quad \text{where } \frac{dg}{dx} = f(x) \text{ and } C \text{ is an arbitrary constant.}$$

The physical interpretation depends on what $f(x)$ and x represent. If the integrand $f(x)$ is the linear density $\rho(x)$ of a wire (i.e. mass per unit length) at position x, then the definite integral: $\int_a^b \rho(x)dx = \lim_{\Delta x_i \to 0} \sum_i \rho(x_i)\Delta x_i = \lim_{\Delta mi \to 0} \sum_i \Delta m_i$ represents the total mass of the wire as $\rho(x_i)\Delta x_i = \Delta m_i$ is the mass of the i-th partition of the wire.

Similarly, if the integrand is the velocity function $v(t)$ of a particle at time t, then the definite integral is the total distance travelled between the time interval $t = a$ and $t = b$, that is, $\int_a^b v(t)dt = \lim_{\Delta t_i \to 0} \sum_i v(t_i)\Delta t_i$ as $v(t_i)\Delta t_i$ is the distance travelled in the i-th time interval.

4.2 Visualizing double integrals in rectangular coordinates

In the case of double integral of a two variable function, the integrating function $f(x, y)$, is a two variable function and the integrating region D, is in the xy plane. If the function $f(x, y)$ is plotted as the z-axis, then $z = f(x, y)$ is a surface over the integrating region R. Figure 4,2.

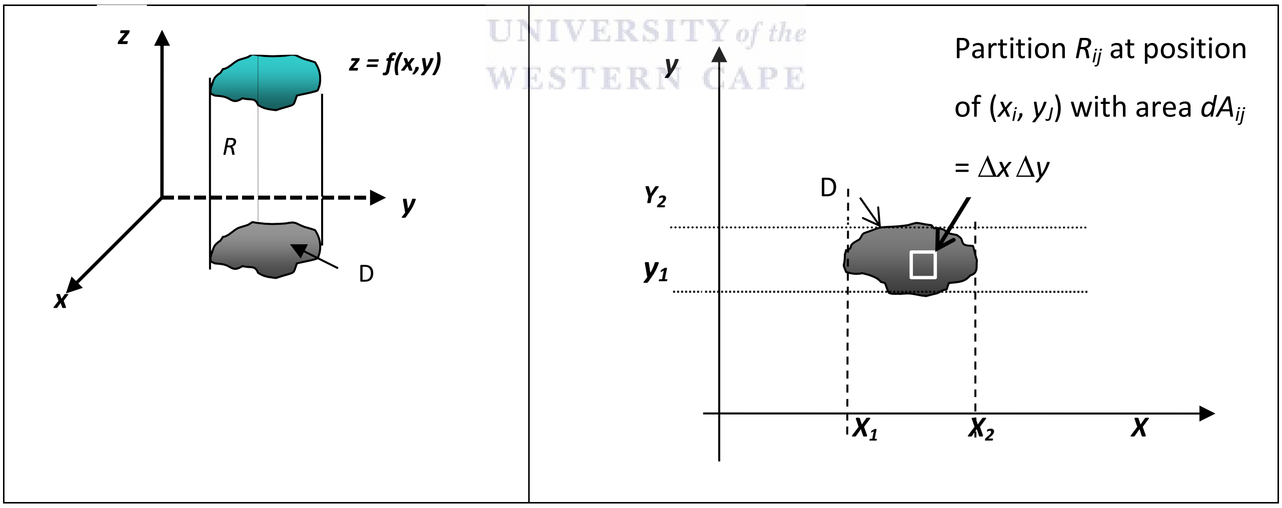


Figure 4.2 Double integrals for areas and volumes

The integrating region D is partitioned into small rectangles R_{ij} centered at (x_i, y_j) with area of ΔA_{ij}. The double integral of the function $f(x, y)$ over the region R is defined as:

$$\iint_D f(x, y)dxdy = \iint_D f(x, y)dA = \lim_{\Delta A_{ij} \to 0} \sum_{i,j} f(x_i, y_j)\Delta A_{ij}$$

For each of the small rectangle R_{ij} centered at (x_i, y_j), the term $\Delta A_{ij} f(x_i, y_j)$ represents the volume bounded by the surface $z = f(x, y)$ and the small rectangle R_{ij}. Thus, the double integral, defined by the summation of the $\Delta A_{ij} f(x_i, y_j)$ terms over the integrating region R, is the volume bounded by the surface $z = f(x, y)$ and the integrating region R.

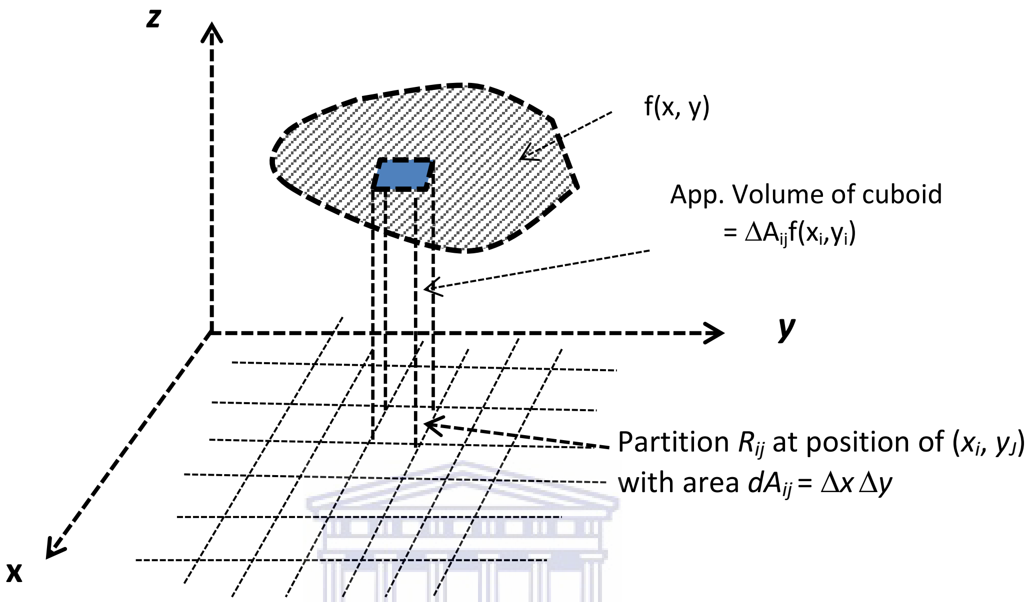


Figure 4.3 Finding the volume between $f(x, y)$ and the xy-plane

Physical interpretations of the double integral depend on the functions $f(x, y)$. If the integrating function $f(x, y)$ is unity i.e. $f(x, y) = 1$, in the whole integrating region, then the term $f(x_i, y_j)\Delta A_{ij} = \Delta A_{ij}$, is the area of the partition R_{ij} and the double integral $\iint_R f(x,y)dA = \iint_R dA = \lim_{\Delta A_{ij}\to 0} \sum_{i,j} \Delta A_{ij}$ is the total area of the region R.

If $f(x, y)$ is the height from the (x, y)-plane, then $\iint_R f(x,y)dA = \lim_{\Delta A_{ij}\to 0} \sum_{i,j} f(x_i, y_j)\Delta A_{ij}$ is the volume between $f(x, y)$ and the xy-plane, bounded by region D.

Note that we can span the region by fixing x_i and summing over all possible y_i to give the volume: $dV_i = \sum_j f(x_i, y_j)\Delta x \Delta y = \left(\int_{y_1}^{y_2} f(x_i, y)dy\right)\Delta x$. The total volume bounded by the surface $z = f(x, y)$ and the region R can be obtained by summing dV_i over all possible x_i, i.e.

$$V = \sum_i dV_i = \sum_i \left[\int_{y_1}^{y_2} f(x_i, y)dy \right] \Delta x = \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} f(x, y)dy \right] dx .$$

If y is temporarily fixed at y_j first, then the volume is given by: $V = \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} f(x, y)dx \right] dy.$

In general: $\iint_R f(x, y)dxdy = \int_{x_1}^{x_2} \left[\int_{y_1}^{y_2} f(x, y)dy \right] dx = \int_{y_1}^{y_2} \left[\int_{x_1}^{x_2} f(x, y)dx \right] dy.$

4.3 Double integrals in polar coordinates

If the integrating region R is given in polar coordinates (r , θ), the whole region is partitioned into some small area A_{ij} as shown in Figure 4.4. The area of the small partition A_{ij} is $r\Delta\theta\,\Delta r$ and the volume enclosed by the surface $z = f(x, y)$ and the small partition A_{ij} is $dV = f(x, y)\, r\Delta\theta\,\Delta r$. Therefore, the volume bounded by the surface $z = f(x, y)$ and the region A_{ij} is given by the double integral: $\iint_R f(x, y)\, dx\, dy = \iint_\Omega f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$.

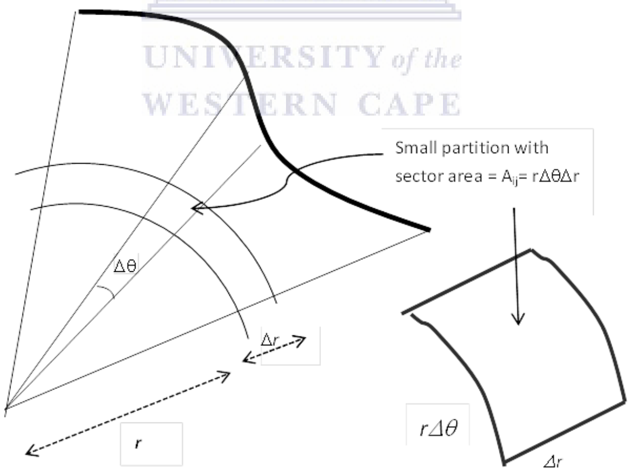


Figure 4.4 Area element in polar coordinates

4.4 Illustrative examples on double integrals

To find the area of the region in R^2 , bounded by the curves $y = \ln(x)$, $y = 2$ and the coordinate axes, we draw a sketch and identify the region (See Figure. 4.5). To find the limits of the integrals we slice the region in the x- or y- directions into rectangles. In this example, the required region can be sliced (traversed or spanned) more easily in the x-direction and then the y-direction. We find left x-limit is $x = 0$. To find the right x-limit we need to make x the subject of the equation $y = \ell n\,(x)$. This gives $x = e^y$, and our inner limits for x run from $x = 0$ to $x = e^y$.

The y limits run from $y = 0$ to $y = 2$. And the double integral set up is $\int_0^2 \int_0^{e \wedge y} dxdy$.

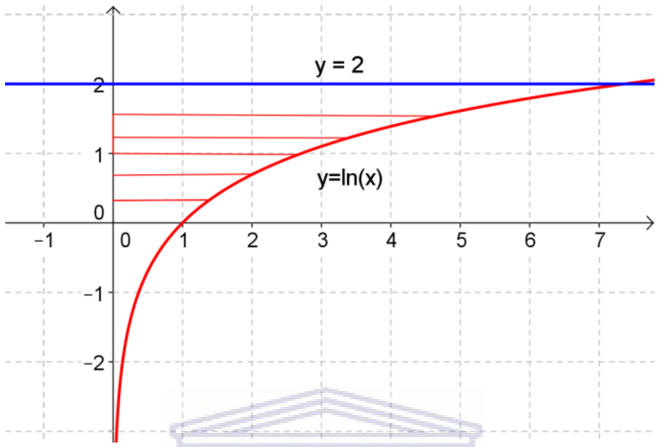


Figure. 4.5 Horizontal slicing of required area defined by $y = \ln(x)$, $y = 2$ and the x and y axes

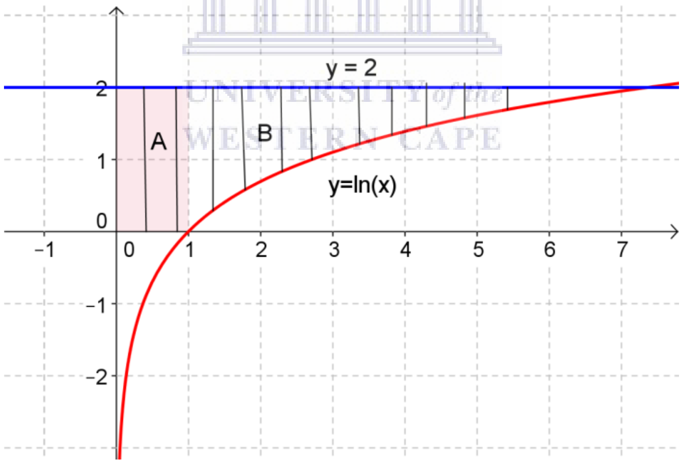


Figure 4.6 Vertical slicing of required area defined by $y = \ln(x)$, $y = 2$ and the x and y axes

Slicing (spanning or traversing) the region in the y-direction we need a split integral: to give the area of the rectangle A, between $x = 0$, $y = 2$ and $x = 1$ and then the area B, between $x = 1$ and the curve $x = e^y$. The split integrals we set up are: $\int_0^1 \int_0^2 dydx + \int_1^{e \wedge 2} \int_{\ln x}^2 dydx$. While the first part of the split integral is easy to set up, for most students visualizing the second part is challenging. Evaluating the second integral without software is also difficult. In Chapter 6, we look more closely at the difficulties students experience in evaluating these integrals using the theoretical frameworks.

4.5 Visualizing triple integrals

If $f(x;y;z)$ is an integrable function over a region *R*, then the triple integral of $f(x;y;z)$ over *R* is $\iiint_R f(x;y;z)\,dV$. If $f(x;y;z)=1$ then $\iiint_R dV$ gives the volume.

Definition: For a region *R*, a partition into *n* pieces, is a list of disjoint rectangular boxes inside R, where the *k* th rectangle contains the point $(x_i; y_j; z_k)$, has length Δx_i, width Δy_j, height Δz_k, and volume $\Delta V_{ijk} = \Delta z_k\, \Delta y_j\, \Delta x_i$.

Definition: For f(*x*; *y*; z) a continuous function and P a partition of the region D, we define the Riemann sum of f(*x*; *y*; z) on D corresponding to P to be $\Sigma\, f(x_i; y_j; z_k)\Delta V_{ijk}$.

Consider the integrating region *R*, in this case a parallelepiped as shown in Figure 4.7. It is partitioned equally into a number of small cubes. We label the cubes so that the cube R_{ijk} is located at (x_i, y_j, z_k) with lengths of Δx_i, Δy_j and Δz_k in the *x*, *y* and *z* directions respectively. The volume of each small partition R_{ijk} is $\Delta x_i \Delta y_j \Delta z_k$. Therefore, the triple integral of the function *f*(*x*, *y*, *z*) over the integrating region *R* is defined as:

$$\iiint_R f(x,y,z)dxdydz = \iiint_R f(x,y,z)dV = \lim_{\Delta V_{ijk}\to 0} \sum_{i,j,k} f(x_i, y_j, z_k)\Delta V_{ijk}$$

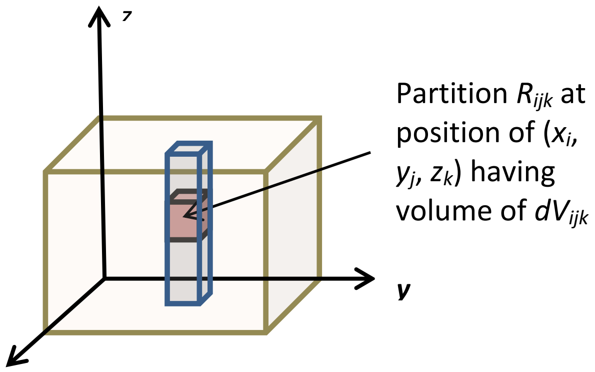


Figure 4.7 Conceptualising Riemann sum for triple integrals

If the integrating function is the constant function whose value is one, i.e. $f(x, y, z) = 1$, then the term $f(x_i, y_j, z_k)\, \Delta V_{ijk} = \Delta V_{ijk}$ will be the volume of the partition R_{ijk}. Thus the limit of the summation over all the partitions will give the total volume of the integrating region R, which is the triple integral in this case.

4.5.1 Triple Integrals over Rectangular Integrating Regions

Suppose the integrating region R is a rectangular box. (See Figure 4.7). If R is enclosed by the planes $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$, $z = z_1$ and $z = z_2$, i.e. $R = \{(x, y, z): x_1 \le x \le x_2, y_1 \le y \le y_2 \text{ and } z_1 \le z \le z_2\}$, the triple integral over R is given by the expression:

$$\iiint_R f(x,y,z)dV = \int_{x_1}^{x_2}\left\{\int_{y_1}^{y_2}\left[\int_{z_1}^{z_2} f(x,y,z)dz\right]dy\right\}dx = \int_{x_1}^{x_2}\left\{\int_{z_1}^{z_2}\left[\int_{y_1}^{y_2} f(x,y,z)dy\right]dz\right\}.$$

4.5.2 Triple Integrals over Non-rectangular Integrating Regions

To evaluate a triple integral over a non-rectangular region, we can first integrate with respect to z, then with respect to y, and finally with x. Thus the triple integral over the region $R = \{(x, y, z): x_1 \le x \le x_2, g_1(x) \le y \le g_2(x) \text{ and } h_1(x, y) \le z \le h_2(x, y)\}$ is given by the expression:

$$\iiint_R f(x,y,z)dV = \int_{x_1}^{x_2}\left\{\int_{g_1(x)}^{g_2(x)}\left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dz\right]dy\right\}dx$$

If the order of integration is switched, the integral would be given by a similar expression

Using Fubini’s theorem, multi-dimensional integrals may be reduced to iterated integrals. Depending on the integrand and the shape of the region it may be necessary to change the order of the variables or use a different coordinate system most commonly polar, cylindrical or spherical coordinates.

The difficulty in setting up triple integrals is converting the description of the region into explicit bounds of integration. To do this, we choose an order of integration, and then slice up the region of integration accordingly. Two methods for determining bounds are the shadow or projection method and the cross-section method

Fubini's Theorem guarantees that the order of integration does not matter as long as the function is continuous.

Theorem (Fubini's Theorem 2). Given f(x; y; z) is continuous on $R = f(x; y; z) : a \leq x \leq b; g_1(x) \leq y \leq g_2(x); h_1(x; y) \leq z \leq h_2(x; ,y)\}$, then the triple integral can be computed using iterated integrals, $\iiint_R f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$. And the iteration can be done in any one of six orders. All orders of integration will also yield the same value.

We note that the bounds for the outer variable must be constants, the bounds for the middle variable can only depend on the outer variable, and the bounds for the inner variable can depend on both of the others.

In effect, we have simplified calculating a triple integral to calculating three single integrals, where each of the integrand is a one-dimensional integral in the x, y and z directions. In this case, we need first to calculate the inner integral with respect to the variable z, and then the integrals with respect to the variables y and x.

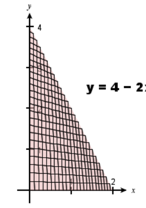
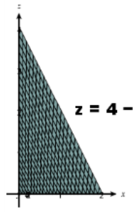
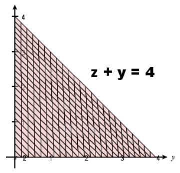
Physical interpretations of the triple integral depend on what the integrating function f(x,y,z) represents. If the integrating function is density $\rho(x, y, z)$ (i.e. mass per unit volume) of a solid and the integrating region R is the solid, then the term $\rho(x_i, y_j, z_k)\, \Delta V_{ijk}$ represents the mass of the cube R_{ijk}. The limit of the summation over all the partitions will give the total mass of the solid.

4.6 Triple integrals in rectangular coordinates - Illustrative Examples

Example 1 : Use a triple integral to find the volume of a tetrahedron enclosed by the plane $z = 4 - 2x - y$ in the first octant i.e $x, y, z \geq 0$.

To sketch the solid we find the traces or projections in the xy, yz and xz planes. As discussed earlier, in our theoretical framework analysis, treatments and conversions are necessary. For example to get the trace in the xy plane we let $z = 0$ in $z = 4 - 2x - y$ to give $y + 2x = 4$, which is the equation of a straight line. We record the traces as shown in Table 3 for each plane.

Table 3 Projection of $z = 4 – 2x – y$ in the xy, yz and zx planes

Plane	Equation	Description	Sketch of trace
Let z = 0, *xy* plane	$2x + y = 4$	Straight line	
Let *y* =0, *x*z plane	$z = 4 – 2x$	Straight line	
Let *x* =0, z*y* plane	$z = 4 – y$	Straight line	

Combining the traces gives us the sketch shown in Figure 4.8. Next we set up the volume integral. We find the limits of triple integration. We work on the innermost limit first which corresponds with the variable 'z'. Think of standing anywhere in R vertically with the feet on the lower limit and head touching the higher limit. The lower limit, anywhere in the *xy*-plane, is $z = 0$. The upper limit is the plane $z = 4 – 2x – y$.

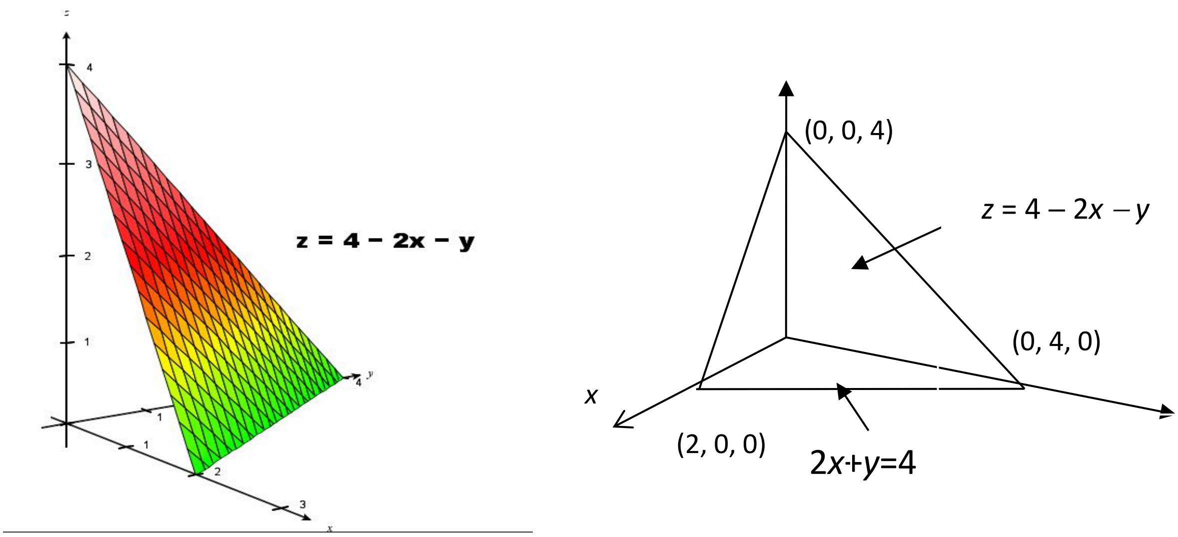
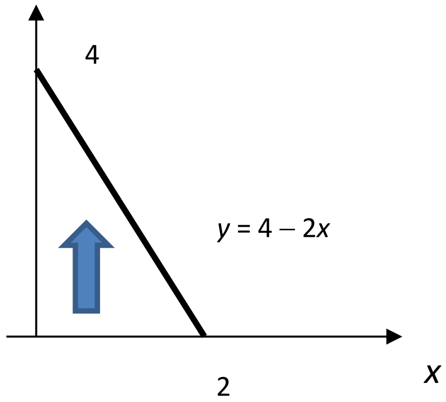


Figure 4.8 The plane $z = 4 – 2x – y$ in Matlab and its sketch

Solving for z, we get $z = 4 - 2x - y$. We let *z* run from z = 0 to the plane $z = 4 - 2x - y$. The *x* and *y* coordinates are in the projection of $z = 4 - 2x - y$ on the *xy*-plane, bounded by the *x* and *y* axes and the line $2x + y = 4$. See Fig 4.2 (b).

Figure. 4.9 Finding limits for *y* using the *xy*-projection



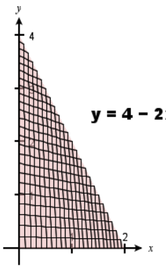
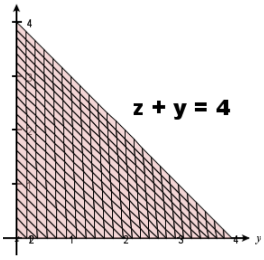
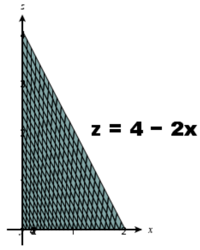
Now we work on the middle limits that correspond to the variable '*y*'. We look at the projection of the surface in the *xy*-plane (see Figure 4.9). The lower limit is just $y = 0$ and the upper limit is found by setting $z = 0$ and solving for *y*. We get for the upper limit as $y = 4 - 2x$.

Finally, we find the outer limits, corresponding to the variable '*x*'. The lowest *x* gets is 0 and highest *x* gets is 2. Hence $0 < x < 2$. The triple integral is thus $\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx$.

An alternative approach used by some authors is to obtain the limits of integration in the order *dz dy dx* is to begin on the outer integral and find the range of *x*: we see *x* runs from 0 to 2. We then slice the region into cross-sections parallel to the *x*-axis. Each cross-sections is a triangle with length in *y* direction equal to $4 - 2x$. Finally we fix *x* and *y* and determine the bounds on *z*. These run from 0 to $4 - 2x - y$.

The volume integral could have been set up in 6 different ways, depending on the coordinate plane where $z = 4 - 2x - y$ is projected. We looked at the projection in the *xy* plane. The same projection can be used to set up the integral with d*y*d*x* interchanged. The limits along *z* remain unchanged but we need to solve for *x* in $y = 4 - 2x$. So the *x* limits run from $x = 0$ to $x = (4-y)/2$ and the *y* limits run from 0 to 4. The projections and the 6 permutations are shown in the Table. 4.

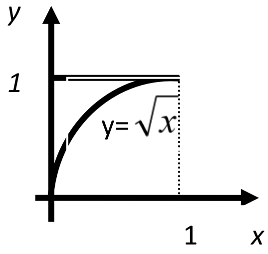
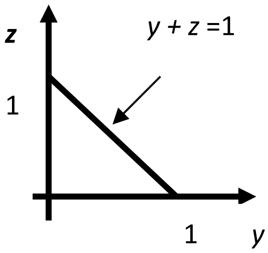
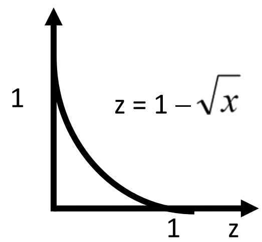
Table 4 Projections and the six permutations of dxdydz for volume of tetrahedron

Projection *xy* plane	Projection *zy* plane	Projection *xz* plane
		
$\int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\,dy\,dx$	$\int_0^4\int_0^{4-z}\int_0^{(4-z-y)/2} dx\,dy\,dz$	$\int_0^2\int_0^{4-2x}\int_0^{4-2x-z} dy\,dz\,dx$
$\int_0^4\int_0^{(4-y)/2}\int_0^{4-2x-y} dz\,dx\,dy$	$\int_0^4\int_0^{4-y}\int_0^{(4-z-y)/2} dx\,dz\,dy$	$\int_0^4\int_0^{2-z/2}\int_0^{4-2x-z} dy\,dx\,dz$

We discuss another example. We require the volume in the first octant bounded by the surfaces, $x = 0$, $z = 1 - y$ and $y = \sqrt{x}$. We sketch the region, find the limits of integration and set up an appropriate integral. Computer software can help us sketch the figure and identify the projections and the required volume. See Figure 4.10.

We place our volume element *dz dy dx* in the required volume and visualise the element growing into a column from the floor to the roof in the *z* direction, then a wall parallel to the *y* axis, and finally look at the limits for *x* as the wall is stretched in the x direction. See Table 5.

Table 5 Projections and the limits of dxdydz for volume enclosed by $z = 1 - y$ and $y = \sqrt{x}$.

Projection in *xy* plane	Projection in *yz* plane	Projection in *xz* plane
 Projection in *xy* plane a) *y* runs from $y = \sqrt{x}$ to 1 b) *x* runs from 0 to $x = y^2$	 Projection in *yz* plane a) z runs from 0 to $1 - y$ b) *y* runs from 0 to 1 – z	 Projection in *xz* plane a) z runs from 0 to $1 - \sqrt{x}$ b) *x* runs from 0 to $(1 - z)^2$

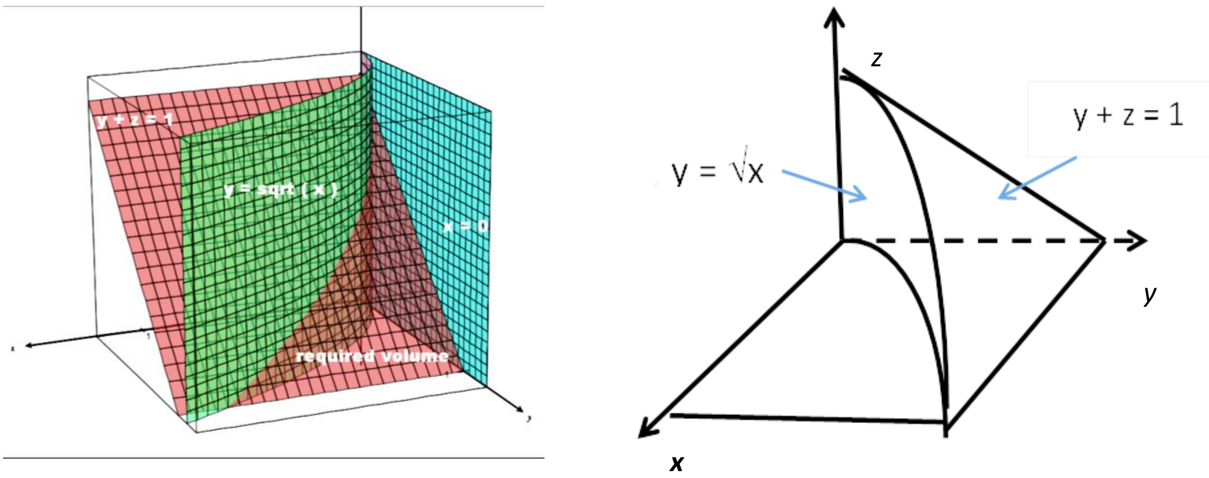


Figure 4.10 The R^3 solid bound by the surfaces $x = 0,\; z = 1 - y$ and $y = \sqrt{x}$.

The limits for z are: from $z = 0$ to $z = 1 - y$, the limits for y are from $y = \sqrt{x}$ to $y = 1$ and the limits for x run from 0 to 1. Our integral is $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dz\, dy\, dx$. Table 6 shows the rest of the permutations for *dxdydz*

Table 6. The six permutations of dxdydz for volume bound by $x = 0,\; z = 1 - y$ and $y = \sqrt{x}$.

$\int_0^1 \int_0^{y^2} \int_0^{1-y} dz\, dx\, dy$	$\int_0^1 \int_0^{1-z} \int_0^{y^2} dxdydz$	$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^1 dydxdz$
$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dz\, dy\, dx$	$\int_0^1 \int_0^{1-y} \int_0^{y^2} dx\, dz\, dy$	$\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^1 dy\, dz\, dx$

Another approach to this problem is to use cross-sections. In determining the limits for integration in the order: $\int_0^1 \int_0^{y^2} \int_0^{1-y} dz\, dx\, dy$, the outer limits for y range from 0 to 1. Slicing the region into cross-sections along the y-axis gives triangular cross-sections with length in the x direction ranging from 0 to y^2. Fixing x and y, gives the bounds on z which are 0 to $1 - y$.

We end this section, with a final example on integration in rectangular coordinates. We need the volume of the solid polar cap bounded by the sphere $x^2 + y^2 + z^2 = 9$.

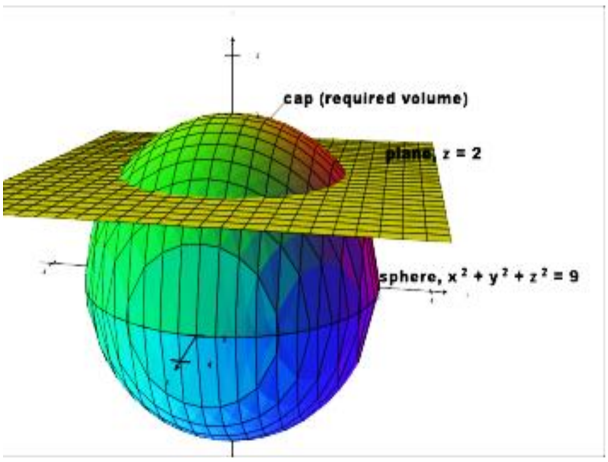
Example: Find the volume of the cap cut by the plane z =2 from the sphere given by $r^2 + z^2 = 9$.

The lower boundary of the region is the plane $z = 2$ and the upper boundary is the portion of the sphere on which $z = \sqrt{9 - x^2 - y^2}$ (See Figure 4.11). These are the inner limits of the triple integral $\iint \int_{z=2}^{\sqrt{9-x^2-y^2}} dzdydx$. For the middle and outer limits we look at the projection of R in the *xy* plane. To find the intersection of the sphere $x^2 + y^2 + z^2 = 9$ and the plane $z = 2$ we substitute $z = 2$ in $x^2 + y^2 + z^2 = 9$. This gives the intersection, $x^2 + y^2 = 5$. This is a circle of radius √5. The projection on the xy plane is also the circle, $x^2 + y^2 = 5$. Thus the limits for the integration are :

$$\int_{-\sqrt{5}}^{\sqrt{5}} \int_{y=-\sqrt{5-x^2}}^{y=\sqrt{5-x^2}} \int_{z=2}^{\sqrt{9-x^2-y^2}} dz\, dy\, dx .$$

If the inner integration is done with respect to *y* the left boundary of the cap is given by $y = -\sqrt{9 - x^2 - z^2}$ and the right boundary by $y = \sqrt{9 - x^2 - z^2}$. The projection of the cap in the *xz* plane is bounded by $z = 2$ and the circle $x^2 + z^2 = 9$. Therefore the integrals are:

$$\int_{-\sqrt{5}}^{\sqrt{5}} \int_{z=2}^{z=\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-z^2}}^{\sqrt{9-x^2-z^2}} dy\, dz\, dx \quad \text{and} \quad \int_{2}^{3} \int_{x=-\sqrt{5-z^2}}^{x=\sqrt{5-z^2}} \int_{-\sqrt{9-x^2-z^2}}^{\sqrt{9-x^2-z^2}} dy\, dx\, dz .$$



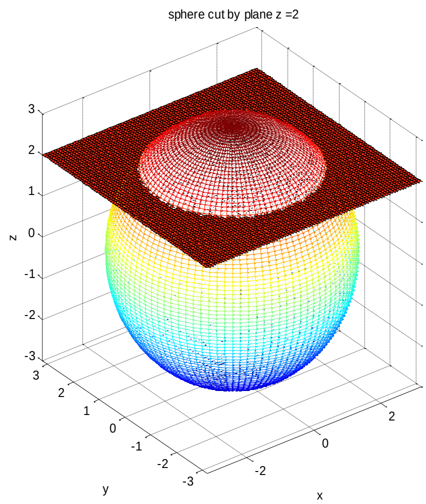


Figure 4.11 Plane z = 2 cutting sphere
$x^2 + y^2 + z^2 = 9$

Finally the integration with the inner integral as dx gives $\int_{-\sqrt{5}}^{\sqrt{5}} \int_{z=2}^{z=\sqrt{9-y^2}} \int_{-\sqrt{9-y^2-z^2}}^{\sqrt{9-y^2-z^2}} dx\,dz\,dy$ and

$$\int_{z=2}^{z=3} \int_{y=-\sqrt{5-z^2}}^{y=\sqrt{5-z^2}} \int_{-\sqrt{9-y^2-z^2}}^{\sqrt{9-y^2-z^2}} dx\,dy\,dz\,.$$

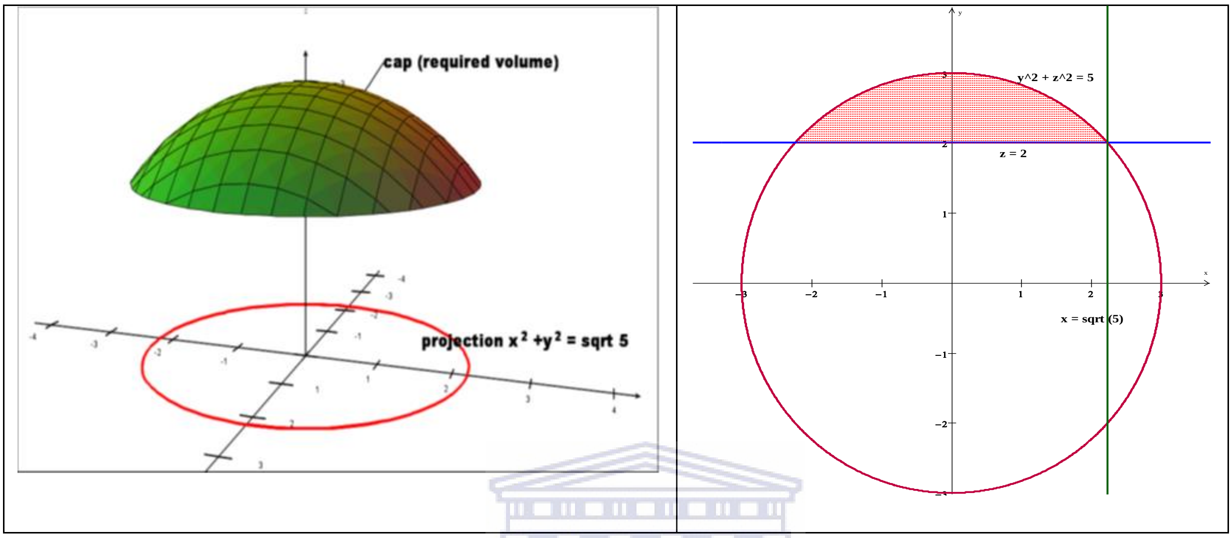


Figure 4.12 Projections of the cap in the *xy* and *yz* planes

These integral expressions are difficult to compute without parametrization and in the following section we consider cylindrical and spherical coordinates systems.

4.7 Visualising space figures in cylindrical coordinates

In this section we look at visualising space figures in the cylindrical coordinate systems, useful for space figures like cones, spheres and cylinders that have line and plane symmetry. It is coordinate systems (rectangular, cylindrical, and spherical) as working in the appropriate coordinate system, reduces the number of variables in expressions from three to one or two, eliminating much of the computational complexity.

Cylindrical coordinates (r, θ, z) and the Jacobian

To define cylindrical coordinates of a point *P* we need : the polar coordinates (r, θ) of the projection of O*P* on the *xy* plane, and the *z* coordinate of the point, that is, we need the 3 coordinates (r, θ, z). The dimensions of r and z are units of length and θ is measured in radians anticlockwise from the positive *x*-axis (See Figure 4.14).To transform from rectangular to cylindrical coordinates we use the relationships: $x = r \cos \theta$; $y = r \sin \theta$, z = z.

In cylindrical coordinates $r = c$, traces vertical cylinders of radius c, centred in the z axis. Also θ = c generates vertical planes about the z-axis, and z = c generates a horizontal plane (See Figure 4.13 (a), (b) and (c) respectively.

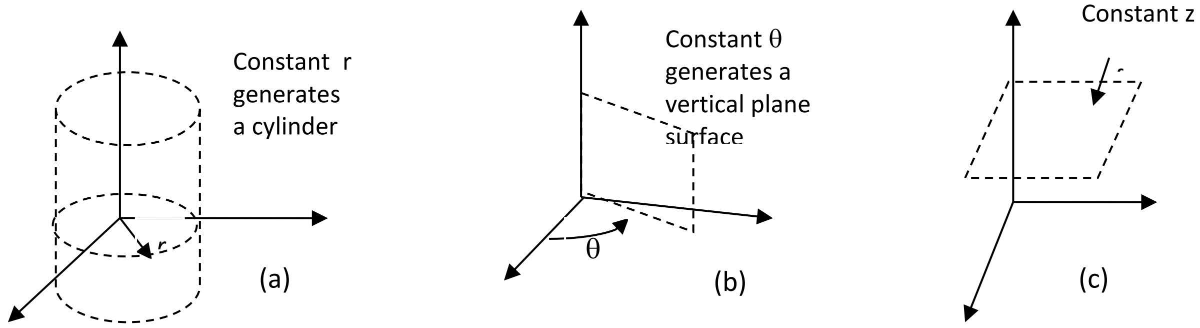


Figure 4.13 Surfaces in cylindrical coordinates

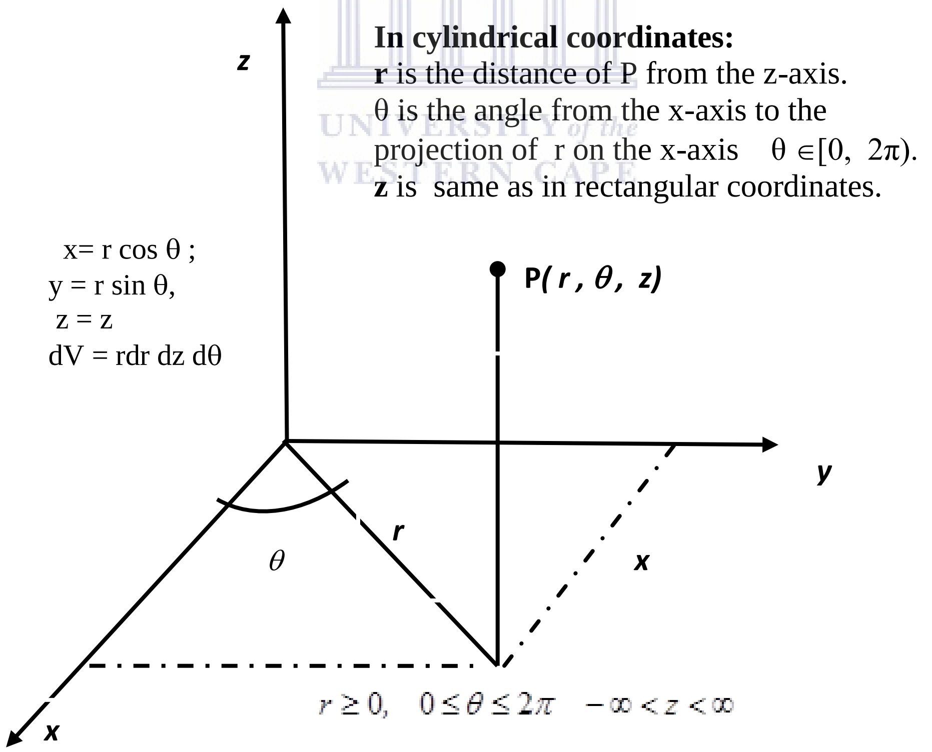


Figure. 4.14 Coordinates of a point in cylindrical coordinates

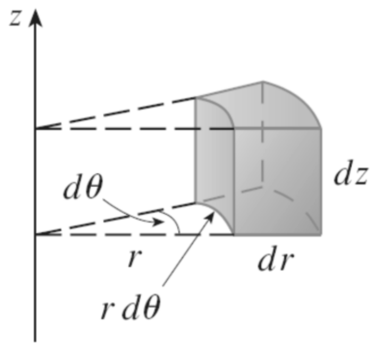


Figure 4.15 Volume element $r\,dr\,d\theta\,dz$

Consider the partitioned small volume as shown in Figure.4.15. The volume of this small partition is $dV = (rd\theta)(dr)(dz)$. We again have 6 ways of ordering our variables *i.e* $r\ dr\ d\theta\ dz$, $r\ dr\ dz\ d\theta$, $dz\ r\ dr\ d\theta$, $dz\ d\theta\ r\ dr$, $d\theta\ dz\ r\ dr$ *and* $d\theta\ r\ dr\ dz$.

Therefore, the triple integral of the function $f(x, y, z)$ over the region R is equal to:

$\iiint\limits_R f(x,y,z)dV = \iiint\limits_{r,\theta,z} f(r\cos\theta, r\sin\theta, z)r dr d\theta dz$ and the Jacobian of transformation from Cartesian to cylindrical coordinates is

$$J(r,\theta,z) = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/dr & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \quad \geq 0$$

If the integrating function is already given in cylindrical coordinate, then

$$\iiint_R f dV = \iiint_{r,\theta,z} f(r,\theta,z) r dr d\theta dz$$

Example

We will integrate over the solid S formed under the paraboloid $z = 4 - x^2 - y^2$, $z \geq 0$, using cylindrical coordinates.

Integration using the order dz dθ dr

To determine the limits of integration, we take the outermost variable and work inward. The integral will have the general form

$$\int\limits_{r_1}^{r_2} \int\limits_{\theta_1(r)}^{\theta_1(r)} \int\limits_{z_1=f(r,\theta)}^{z_2=f(r,\theta)} f(r,\theta,z)\, r\, dz\, d\theta\, dr$$

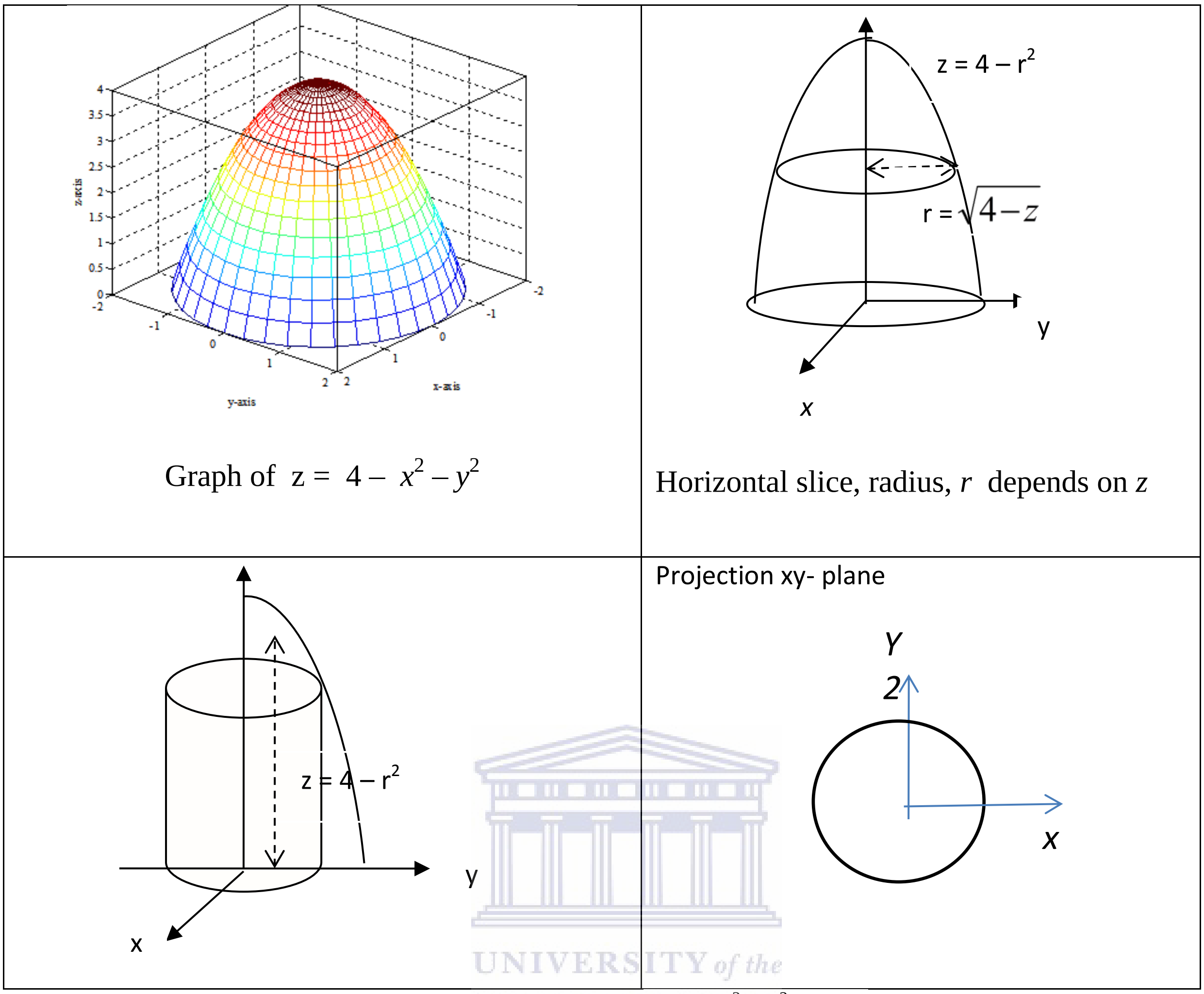


Figure 4.16 Paraboloid $z = 4 - x^2 - y^2$, $z \geq 0$,

The outermost variable is *r*. We determine the maximum and minimum values of the outermost variable, *r*. These will be the limits of integration on the first (outer) integral. In our projection on the *xy* plane we see *r* runs from 0 to 2 and enter these as limits for *r*.

$$\int_{r=0}^{r=2} \int_{\theta_1(r)}^{\theta_1(r)} \int_{z_1=f(r,\theta)}^{z_2=f(r,\theta)} f(r,\theta,z)\, r\, dz\, d\theta\, dr$$

Next we take a slice formed by keeping the outermost variable r, constant. Now determine the maximum and minimum values of the middle variable, θ. This gives the limits of integration for the middle integral. θ is between 0 and 2π and doesn't depend on z. Thus the limits of the middle integral are 0 and 2π.

$$\int_{r=0}^{r=2} \int_{\theta_1=0}^{\theta_2=2\pi} \int_{z_1=f(r,\theta)}^{z_2=f(r,\theta)} f(r,\theta,z)\, r\, dz\, d\theta\, dr$$

Finally, we determine the range of the innermost variable in terms of the other two variables. z runs between 0 and 4 – r^2. The final integral is as follows:

$$\int_{r=0}^{r=2}\int_{\theta_1=0}^{\theta_2=2\pi}\int_{z_1=0}^{z_2=4-r^2} f(r,\theta,z)\, r\, dz\, d\theta\, dr$$. To find the volume we set $f(r, \theta, z) = 1$.

As in the case of rectangular coordinates there are 6 permutations of $drdzd\theta$ for integration in cylindrical coordinates. We have discussed the integration in the order $dz\, d\theta\, dr$ using the projection of z = 4 – x^2– y^2 on the r -θ or xy plane. Using the same projection we can write the integral for $dz\ dr\ d\theta$. The limits for the outer variable θ are independent of r and z, so θ runs from 0 to 2π, as before. The limits for the middle variable r run are the same as before; r runs from 0 to 2π : z depends on r and runs from 0 to 4 – r^2. Hence the integral is

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{z_1=0}^{z_2=4-r^2} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$. To set up the integration $\int_{z_1}^{z_2}\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} f(r,\theta,z)\, r\, d\theta\, dr\, dz$ we start with the outermost variable, dz. The minimum value of z is 0, maximum value of z is 4, so the outer limits are 0 to 4. Next we keep z constant and look at r.

We notice that r runs from 0 to $\sqrt{4-z}$. We enter this in our integral: Finally keeping variables z and r constant we find the range for θ is 0 to 2π.

$$\int_{z_1=0}^{z_2=4}\int_{r_1=0}^{r_2=\sqrt{4-z}}\int_{\theta_1=0}^{\theta_2=2\pi} f(r,\theta,z)\, r\, d\theta\, dr\, dz$$

Problem : Find the volume of the space figure enclosed by the surfaces represented by

$f(x , y) = x^2 + y^2$ and $g(x , y) = 6 - \sqrt{x^2 + y^2}$.

Next we set up the volume integral. By reference to the projections, limits in the x, y and z directions are established. However, the limits for x and y are in the plane of intersection of the space figures. Not everyone can visualise the plane of intersection of the cone and the paraboloid easily or see the cross-sections clearly.

The algebraic equation for the intersection can be found by analysis: Thus :

$z = 6 - \sqrt{x^2 + y^2} = x^2 + y^2$. An easy way to solve this is to use the polar relation $x^2 + y^2 = r^2$.

Therefore, $6 - r = r^2 \rightarrow (r + 3)(r - 2) = 0$ and $r = 2$ or -3.

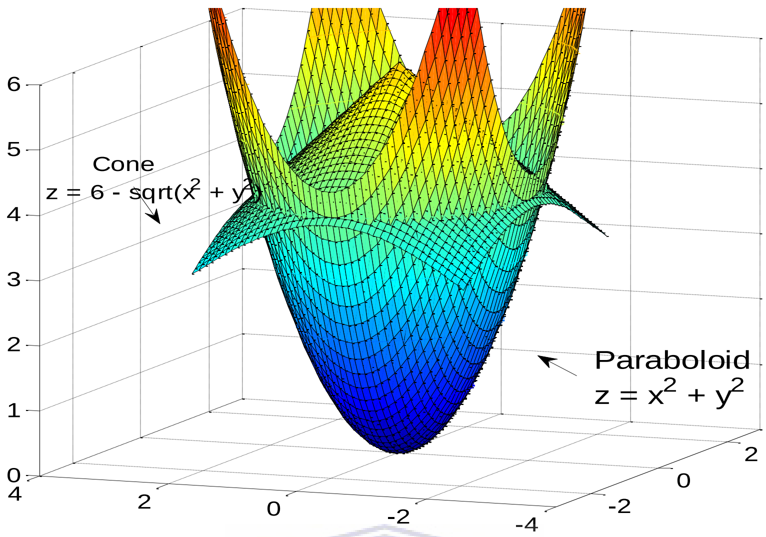


Figure.4.17 The space figure $f(x;y) = x^2 + y^2$ and $g(x;y) = 6 - \sqrt{x^2 + y^2}$

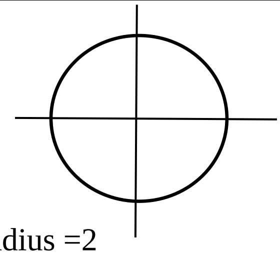
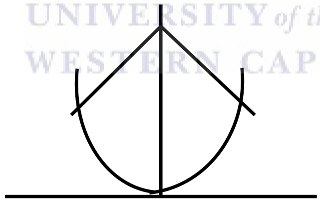
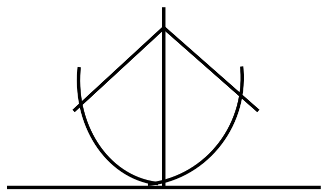
xy projection	*yz* projection	*xz* projection
 Radius =2		

Figure 4.18. Projections in the *xy*, *yz* and *zx* planes

The integral and the limits in rectangular coordinates drawn by reference to the sketches and projections are: $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{6-\sqrt{x^2+y^2}} dzdydx$

It is difficult to do this integration in rectangular coordinates manually and so we transform. Inspection of the 3D representation shows line symmetry and so the limits in cylindrical coordinates are: r : from 0 to 2, θ : from 0 to 2π and z : from r^2 to $6 - r$.

Setting the integral in cylindrical coordinates gives: $\int_{0}^{2} \int_{0}^{2\pi} \int_{r^2}^{6-r} r\,dz\,d\theta\,dr = \ldots = \frac{32\pi}{3}$

Looking back at the solution, and the VA framework, we find acts of visualization interspersed with analytical thinking. We list these acts in Table 7. We do not expect all students to take each step in the same order. We expect some students to already have well-built schemas for the mathematical objects. These students would proceed directly to the visual representations of the functions without going through the process of detailed projections.

The difficulty of visualizing the projections of 3D solids is one of the reasons that students find multiple integrals challenging. Close scrutiny of the solution indicates a need for constant transition between analysis (finding equations of projections, intersections) and visualization (sketching the space figures).

Table 7 Acts of visualization and analytical thinking used to find the volume enclosed by $f(x ; y) = x^2 + y^2$ and $g(x ; y) = 6 - \sqrt{x^2 + y^2}$.

Analysis	**Visualization**
A1: Algebraic register: Finding equations for traces/projections eg Let x =0, $f(x ; y) = y^2$ A2: Coordinating projections mentally A3: Assembling solid in 3D mentally	Geometric register V1: Sketching projections and traces V2: Sketching cross-sections V3: Sketching the solid $z = f(x ; y) = x^2 + y^2$
A3: Finding traces/projections algebraically for $z = f(x ; y) = 6 - \sqrt{x^2 + y^2}$ A4: Putting together the projections	V4: Sketching $z = f(x ; y) = 6 - \sqrt{x^2 + y^2}$ V5: Drawing projections and traces V6: Drawing cross-sections
A5: Determining plane of intersection: substitution $f(x ; y) = x^2 + y^2 = 6 - \sqrt{x^2 + y^2}$. A6: Solving to give $x^2 + y^2 = 4$.	V7: Visualizing need for intersection V8: Reassembling sketch
A7. Selecting appropriate coordinate system A8. Choosing an order of integration A9: Evaluating the integral	V9. Visualizing rectangular, cylindrical , spherical coordinate systems V10. Finding limits by slicing/spanning

4.8 Triple integrals using spherical coordinates

We have looked at integration in rectangular and cylindrical coordinates. Next we discuss integration in spherical coordinates. We use spherical coordinates when we are dealing with objects that are parts of spheres and cones i.e when there is point/line symmetry. We establish that $\rho^2 \sin\phi$ as the Jacobian for integration in spherical coordinates.

To define spherical coordinates, we choose a ray *OP* originating at the origin, *O* and terminating in *P(x,y,z)*. The spherical coordinates of the point *P* are: the distance ρ from *P* to the origin; the angle ϕ between the line *OP* and the positive Z- axis; and the angle θ between the initial ray and the projection of *OP* on the *xy* plane (See Figure 4.19).

In the spherical coordinate system:

ρ is the distance from the point to the origin. ρ cannot be negative.

θ same as in cylindrical coordinates , θ must be in the interval [0,2π).

ϕ is the angle between the vector OP and the z-axis. ϕ must be in the interval [0,π].

Geometrically we can establish the following relations between rectangular coordinates, *x*, *y* and z and the spherical coordinates ρ, θ, ϕ:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$$
$$\tan\theta = \frac{y}{x} \quad \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

The volume element is

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta .$$

Note: $\rho \geq 0$, $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$

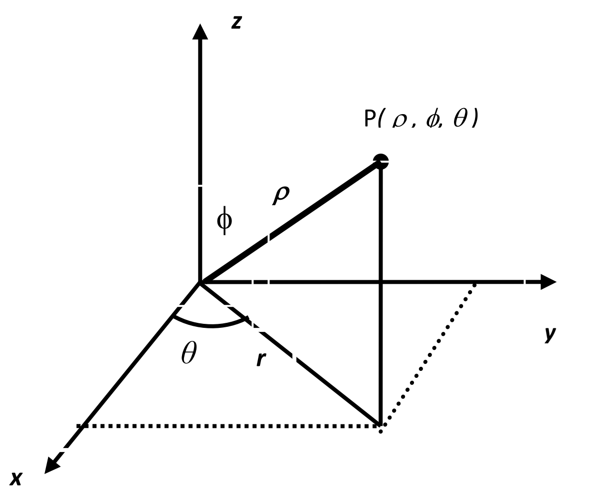


Figure 4.19 Variables in spherical coordinate system

The Jacobian of transformation from Cartesian to spherical coordinates is found as follows:

$$J(\rho,\phi,\theta)=\begin{vmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{vmatrix}=\begin{vmatrix} \sin\phi\cos\theta & -\rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}=\rho^2\sin\phi$$

Hence, the formula of change of variables for this transformation is

$$\iiint_V f(x,y,z)\,dx\,dy\,dz=\iiint_V f(\rho cos\phi\, sin\theta,\rho sin\phi\, sin\theta,\rho cos\phi)\rho^2\, sin\theta\, d\rho\, d\phi\, d\theta$$

Or if the integrating function is already given in the spherical coordinate, then

$$\iiint_R f\ dV=\iiint_{\rho,\theta,\varphi} f(\rho,\varphi,\theta)\ \rho^2\sin\varphi\ d\rho d\varphi\ d\theta$$

It is easier to calculate triple integrals in spherical coordinates when the region of integration *U* is a sphere (or some portion of it) and/or when the integrand is a kind of $f(x^2+y^2+z^2)$.

Again there are 6 possible permutations of the variables dρ dθ dϕ. We look at an example.

Example

Find the volume of the solid bound by $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=4$.

Solution: See sketches Figure 4.18. We begin with the outermost variable and work inward. The integral will have the general form

$$\int_{\theta 1}^{\theta 2}\int_{\phi 1}^{\phi 2}\int_{\rho=\rho 1}^{\rho=\rho 2}\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

1. We find the maximum and minimum values of the outermost variable, θ. θ runs from 0 to 2π radians and we enter these in the outer integral:

$$\int_{0}^{2\pi}\int_{\phi 1}^{\phi 2}\int_{\rho=\rho 1}^{\rho=\rho 2}\rho^2\, sin\phi\ d\rho\ d\phi\ d\theta$$

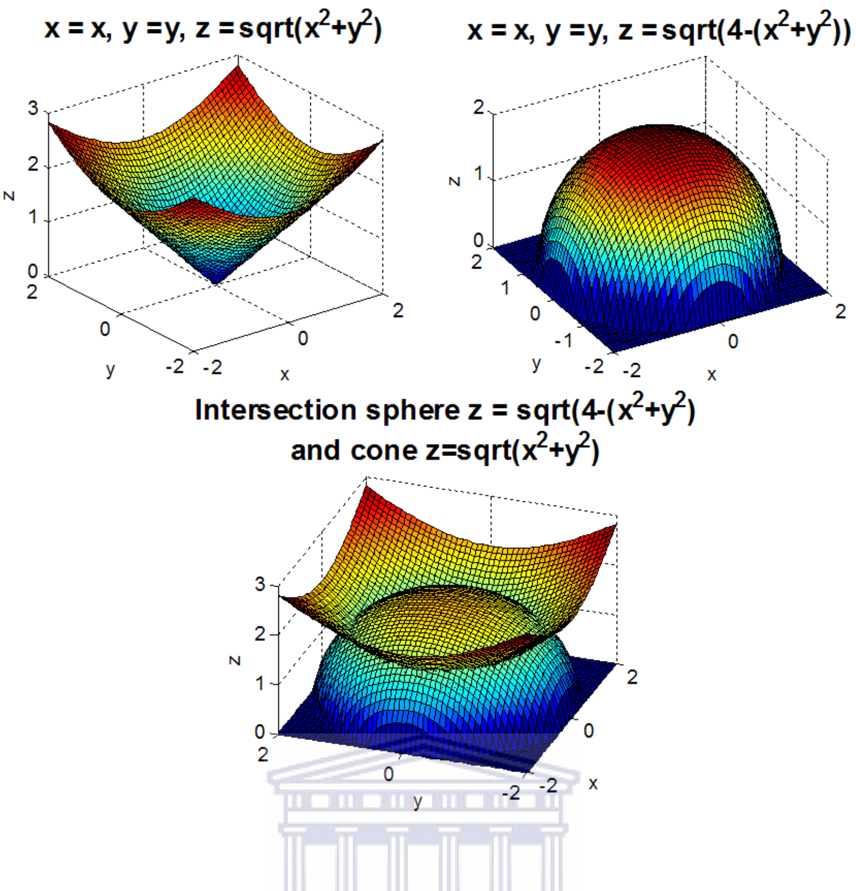


Figure. 4.20 Intersection $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 4$

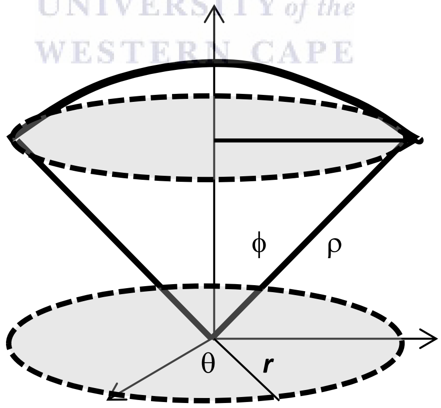


Figure 4.21 Sketch of intersection $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 4$

2. We keep θ constant and determine the maximum and minimum values of the middle variable ϕ in terms of the outermost variable. This will give the limits of integration for the middle integral. ϕ runs from 0 to π/2: $\int_0^{2\pi} \int_0^{\pi/2} \int_{\rho=\rho 1}^{\rho=\rho 2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

3. Finally we substitute the limits for ρ: ρ runs from 0 to 2. And we integrate:

$$\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{\rho=2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \quad = \int_0^{2\pi}\int_0^{\pi/4}\frac{\rho^3}{3}\Big|_0^2 d\phi d\theta$$

$$= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\,d\phi\,d\theta \quad = \frac{8}{3}(-\cos\phi)\Big|_0^{\pi/4}\int_0^{2\pi}d\theta \quad = \quad \frac{8\sqrt{2}}{3}\pi(\sqrt{2}-1)$$

We outline the general process for setting up and evaluating triple integrals:

Step 1: Sketch and determine the region of integration.

Step 2: Determine the projection in the *xy*, *yz* and *xz* coordinate planes. Sketch and label the curves on the projections. If necessary determine the intersections of the bounding surfaces. Slice up the region according to the chosen order: first slice into 2-dimensional cross-sections perpendicular to the outer variable. Then further slice up each 2-dimensional cross-section using the procedure for double integrals.

Step 3: Determine the limits of integration one at a time, starting with the outer variable, then the middle variable, then the inner variable.

Step 4: Evaluate each iterated integral as a single-variable integral in the appropriate variable. Remember that all variables except the current variable of integration are to be treated as constants.

4.9 Chapter Summary

This chapter provides the theoretical background and examples of problems in integration that students are expected to solve. The approach to single and double integrals for finding areas was outlined and the strategies for visualizing the switch of variables highlighted. The need for polar coordinates was explained. This was extended to triple integrals in cylindrical and spherical coordinates where the problems of switching are compounded. The teaching method emphasized using projections and cross-sections for setting up the volume integrals. The VA framework was applied to look at the analytical and visual aspects of solutions.

Chapter 5 : Stability analysis of dynamical systems

5.0 Overview

In this chapter, we look at the role of visualization in the solution of dynamical systems defined by systems of ODES. Our focus is the use of visual representations such as direction fields and phase portraits to graphically display the solutions. We seek answers to questions such as what difficulties do students encounter in visualising, constructing and interpreting solutions to ODEs? What is the relationship between visual and analytical strategies in solving systems of DEs?

A dynamical system describes how two or more quantities evolve over time. The quantities we look at are interdependent entities, such as the population sizes of plant and animal species and their interactions in an ecosystem, the positions and velocities of celestial bodies in the solar system, and the concentrations of reactants in a chemical reaction. There are many other examples of phenomena or processes that can be described by dynamical systems in science, business and engineering.

We begin by defining terms used in the study of dynamical systems and distinguish between systems represented first by linear and then by non-linear ODEs. Dynamical systems are deterministically causal, in that a given initial condition, determines the state of the system at every future time. In general, the stability of linear systems can be described completely in terms of the eigenvalues and eigenvectors of the system. We extract the characteristic trajectories around equilibrium points and critical elements such as limit cycles and describe the long term behaviour of the solution curves. We then extend our analysis to nonlinear systems.

It is usually not possible to find a closed form explicit solution for most nonlinear differential equations. The behaviour of nonlinear systems may be studied using phase plane analysis. The goal will not be that of finding analytical solutions to the equations, but rather that of determining the possible geometric configurations of the solution curves. The solution curves will be studied in the *xy*-plane with the independent variable *t* treated as a parameter.

5.1 Visualizing ODE solutions using direction fields

A differential equation is simply an equation that relates quantities with their rates of change. For example, given d*x*/dt = *g*, we see that the amount by which *x* changes, *dx*, in some small amount of time, *dt*, is equal to a constant, *g*. The solution to the differential equation is a function *x(t)* = *gt* + *c*. It is a general solution. We can find particular solutions by substituting initial or boundary conditions.

A direction or slope field is a graphical representation of a differential equation. It is a graph of short line segments whose slope is determined by evaluating the derivative at the mid-point of the line segments. Figure 5.1 shows the slope field for d*x*/d*t* = 0.5. The slopes of all the line segments are 0.5. Analytically, the solution is *x*(*t*) = 0.5 *t* + c.

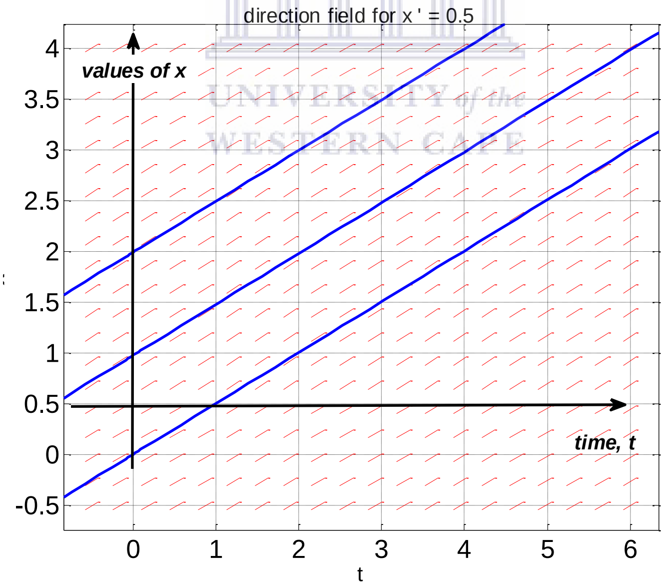


Figure 5.1 Direction field for dx/d*t* = 0.5 – All solution curves have slope 0,5. Chosen initial values are (0, 0) (0,1) and (0,2)

Slope fields provide two types of information about the Differential Equation

1. **Sketch of solutions**. The line segments in the direction fields are tangents to the actual solution of the differential equation. We can use these as guides to sketch the graphs of solutions to the differential equation. In Fig. 5.1 the solution curves are all straight lines with slope 0,5

2. **Long Term Behaviour**. Slope fields can tell us how the solution behaves as time, t increases. This can be used to predict the long term behaviour of the solution.

An equilibrium solution of a dynamical system is a solution value that does not change with time. This means if the system starts at equilibrium, the state will remain at the equilibrium. In a continuous dynamical system, such as dx/dt = f(x), we can find the equilibrium solution by setting dx/dt = 0, that is, we solve the equation $f(x_e) = 0$.

A fixed point is either stable or unstable depending on the behaviour of the trajectories in a neighbourhood of the fixed point. If all the trajectories remain near the fixed point, then the point is considered stable, and if any of these trajectories do not remain in a neighbourhood of the fixed point, the fixed point is considered unstable.

Example 2. Sketch the direction field and a set of solution curves for the autonomous DE: $x' = (x-1)(x-3)$. Determine how the solutions behave as $t \to \infty$ and if this behaviour depends on the value of $x(0)$.

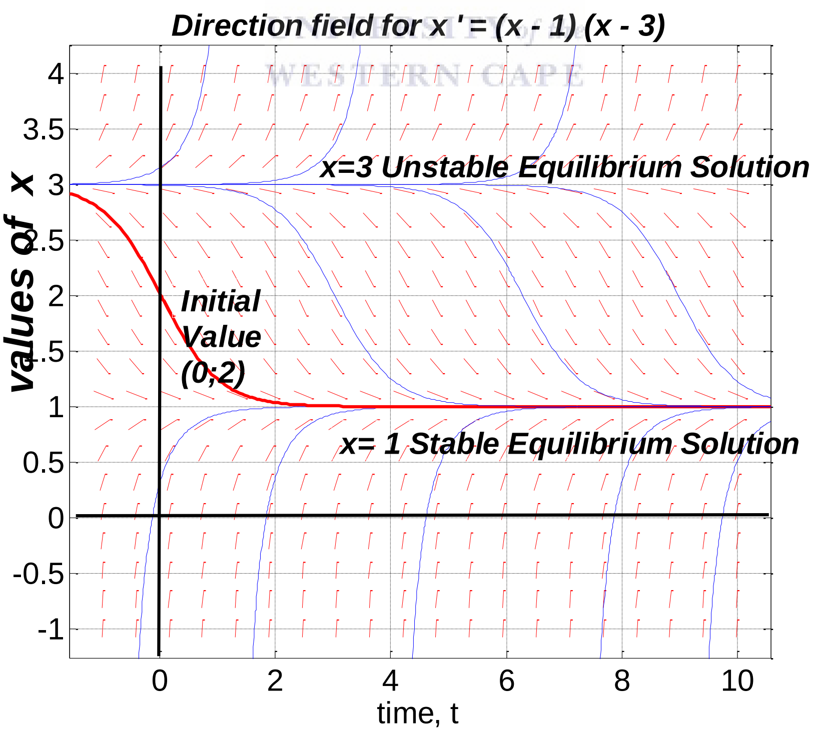


Figure 5.2 **(a)** Direction field for the DE**:** $x' = (x-1)(x-3)$ b) equilibrium solutions (nullclines) $x = 1$ and $x = 3$ c) some trajectories (solution curves) for initial values are shown

The equilibrium solutions can be found by setting the function on the right hand side of the differential equation equal to 0. Thus, $(x-1)(x-3) = 0$ gives the solutions $x = 1$, $x = 3$. These are the equilibrium solutions (slope = 0). The long term behaviour depends on x. The direction field is shown in Figure 5.2. It confirms our analytical solution, obtained using MATLAB or by separation of variables. Using Matlab the solutions are:

```
>> dsolve('Dx = (x–1)*(x –3)','t')

ans =

x =   3  ,  x =  1,     x = 1 – 2/(exp(C1 + 2*t) – 1)
```

The equilibrium solutions are $x_{e1} = 1$, $x_{e2} = 3$. The long term behaviour in this case depends on the value of x at $t = 0$. As $t \to \infty$, for $x(0) < 1$, $x \to 1$, for $1 < x(0) < 3$, $x \to 1$ and for $x(0) > 3$, $x \to \infty$. Therefore $x_e = 1$ is a stable solution, whereas $x_e = 3$ is an unstable solution.

For an ODE system, nullclines are the geometric shape in the x-t plane for which $dx/dt = 0$ for any t. They are boundaries for determining the direction of the motion along the trajectories. They split the phase plane into regions of similar flow. The intersection point of all the nullclines is an equilibrium point of the system. The analytical solution can be found easily by separating the variables. The initial values determine the unique solution trajectories. For example, the solution through $x(0) = 2$ is $x(t) = \dfrac{3+e^{2t}}{1+e^{2t}}$. This solution curve is shown on the direction field in red (See Figure 5.2).

5.2 Linear dynamical systems

In this section we discuss some of the terms relevant to dynamical system analysis. We also look at some methods for analysing the behaviour of dynamical systems. We consider parameters, variables, equilibrium points, stability, and other key concepts in understanding the dynamical behaviour of the system.

Linear systems obey two properties, superposition and homogeneity. The principle of superposition states that for two different inputs, *x* and *y*, in the domain of the function *f*, $f(x+y) = f(x) + f(y)$. This principle enables us to split a system of equations into parts that are easier to solve. The whole solution is the sum of the partial solutions. By applying the principle of superposition, we can find exact, predictive solutions for most linear systems.

The property of homogeneity states that for a given input, *x*, in the domain of the function *f*, and for any real number *k*, $f(kx) = k\,f(x)$.

A function that does not satisfy superposition and homogeneity is nonlinear.

For most linear systems, it is relatively easy to find exact solutions that we can use to predict the future behaviour within the system. In general, the variables that describe the state of linear systems can:

1. **Grow or decay exponentially**. An example is where the rate of growth or decay dN/dt is proportional to the number present at any instant, N i.e $dN/dt \propto N$. Inserting the constant of proportionality *k* gives $dN/dt = k\,N$. The solution $N = c\,e^{kt}$ where *c* represents the initial number N_o and the sign of *k* determines whether there is growth or decay.
   a) When $k > 0$ there is growth. An example of this occurs when bacteria are allowed to grow with unlimited resources.
   b) When $k < 0$ there is **exponential decay**, with N heading toward zero. A common example is the decay of radioactive materials.
   c) When $k = 0$, *N* is constant. (We met this case in section 5.1).
2. **Cycle periodically**, forever oscillating between values. An example is a harmonic oscillator such as a pendulum oscillating in the absence of friction.
3. Exhibit any combination of the above behaviours. The pendulum in air (with friction) oscillates, but each cycle is shorter than the preceding one until the mass stops moving. All of these behaviours are nice and predictable in the linear view.

The **analytical approach to stability** relies on analysing the effects of small perturbations. We say that the equilibrium point (x_e, y_e) is locally stable if the system returns to (x_e; y_e) after a small perturbation, and unstable otherwise. Mathematically, we can analyse this by linearizing the right-hand side of each of the differential equations about the equilibrium point. Consider the linear system:

$x' = ax + by$

$y' = cx + dy$

By letting f(x , y) = ax + by and g(x , y) = cx + dy, the system can be written in matrix form

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

where the right-hand side is a vector-valued function that maps a point in $\mathbf{R}^2$ (the two-dimensional real plane) into a point in $\mathbf{R}^2$. To linearize a vector-valued function, we need to linearize each component separately. Linearizing a function of two variables about a specific point is equivalent to finding the tangent plane at this point. The equation of a tangent plane of $f(x, y)$ about (x_e , y_e) is given by

$$\alpha(x,y) = f(x_e,y_e) + \frac{\partial f(x_e,y_e)}{\partial x}(x - x_e) + \frac{\partial f(x_e,y_e)}{\partial y}(y - y_e)$$

We thus find for the linearization of the vector-valued function $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$

$$\begin{bmatrix} \alpha(x,y) \\ \beta(x,y) \end{bmatrix} = \begin{bmatrix} f(\hat{x},\hat{y}) \\ g(\hat{x},\hat{y}) \end{bmatrix} + \begin{bmatrix} \frac{\partial f(\hat{x},\hat{y})}{\partial x} & \frac{\partial f(\hat{x},\hat{y})}{\partial y} \\ \frac{\partial g(\hat{x},\hat{y})}{\partial x} & \frac{\partial g(\hat{x},\hat{y})}{\partial y} \end{bmatrix} \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix}$$

Letting the perturbations, $\xi = x - \hat{x}$ and $\eta = y - \hat{y}$, with $f(\hat{x},\hat{y}) = 0$ and $g(\hat{x},\hat{y}) = 0$, we find

$$\begin{bmatrix} \frac{d\xi}{dt} \\ \frac{d\eta}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial f(\hat{x},\hat{y})}{\partial x} & \frac{\partial f(\hat{x},\hat{y})}{\partial y} \\ \frac{\partial g(\hat{x},\hat{y})}{\partial x} & \frac{\partial g(\hat{x},\hat{y})}{\partial y} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

The matrix $J(x,y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} & \frac{\partial f(x,y)}{\partial y} \\ \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \end{bmatrix}$ is called the *Jacobian*. The system is a linear system of two equations, and we can use the results from linear systems of two differential equations to determine the stability of the equilibria, namely:

- The equilibrium is a node if both eigenvalues of the Jacobian evaluated at the equilibrium are real, distinct, nonzero, and are of the same sign. The node is locally stable if the eigenvalues are negative, and unstable if the eigenvalues are positive.
- The equilibrium is a *saddle* if both eigenvalues of the Jacobian evaluated at the equilibrium are real and nonzero but have opposite signs. A saddle is an unstable equilibrium point..
- The equilibrium is a *spiral* if both eigenvalues of the Jacobian evaluated at the equilibrium are complex conjugates with nonzero real parts. The spiral is locally stable if the real parts of the eigenvalues are negative, and unstable if the real parts of the eigenvalues are positive.

To summarise, analytically, we can solve a linear system defined by the two ODEs $dx/dt = ax + by$ and $dy/dt = cx + dy$ where *a, b, c* and *d* are constants by determining the equilibrium points, finding the general Jacobian *J*(*x . y*), the eigenvalues at the equilibrium points and eigen-vectors. These provide enough information about the stability of the equilibrium points.

Consider a pair of coupled linear homogenous 2D system of ordinary differential equations: $x' = ax + by$, $y' = cx + dy$, where the differentials x' and y' are with respect to time, *t* and *a, b, c, d* are constants. We can find a formula for the general solutions using eigenvalues. We write the system in matrix form as : $AX = \lambda X$ where lambdas are the eigenvalues for which non-zero solutions exist and $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and A is the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Rewriting the equation as

$AX - \lambda I = (A - \lambda I)X = 0$ we have $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} X = 0$ where , ***I*** is the 2 × 2 identity matrix.

For non-zero solutions the determinant $|A - \lambda I|$ must equal 0. Therefore,

$$|A - \lambda I| = \lambda^2 - \lambda(a + d) - (ad - bc) = 0.$$

This equation, also called the Characteristic Equation, is central to the theory that establishes that trajectory behaviour is dependent on the eigenvalues. Each eigenvalue has associated eigenvectors, *V*, that specify lines which are invariant under the transformation. The way such a line itself is transformed is given by the corresponding eigenvalue. Knowledge of eigenvalues and eigenvectors is sufficient to describe the qualitative behaviour of the linear system.

Thus, if V_1 and V_2 are the eigenvectors associated with the eigenvalues then the general solution can be expressed as : $X(t) = c_1 V_1 e^{\lambda 1t} + c_2 V_2 e^{\lambda 2t}$, where c_1 and c_2 are constants of integration.

Next, we look at some cases of qualitative solutions of linear dynamical systems and use trajectories to describe their stability.

5.3 Visualizing phase portraits of linear systems

A phase portrait is a graphical tool to visualize the solutions of a given system of differential equations. We can use it to interpret the behaviour of the system in the long run. It has a representative sample of trajectories. A trajectory is a directed path traced by a solution. The *xy*-plane is called the phase plane because a point in it represents the state or phase of the system. The phase portrait method is often the only method available for nonlinear ODEs.

Case 1: Distinct real roots: System Equations: Example: $dx/dt = x$ and $dy/dt = 2y$

Characteristics Equations: $\lambda^2 - 3\lambda + 2 = 0$;

Eigenvalues $\lambda = 1$; $\lambda = 2$; Stability: Unstable node

System $dx/dt = x$ and $dy/dt = 2y$ has solutions $x = c_1e^t$ and $y = c_2e^{2t}$ easily found by separation of variables. Dividing the second equation, $dy/dt = 2y$ by the first, $dx/dt = x$, and solving for *y* in terms of x gives dy/dx = 2y/x. Separate the variables and integrate to give the solution $y = cx^2$. Figure 5.3 shows the phase portrait. On the phase portrait the trajectories have the opposite direction depending on c. Thus, if c is positive, $x \to 1$ and $y \to 1$ when $t \to 1$. In this case (0 , 0) is an **unstable node.** The variation of x and y against t are also of interest. These are shown in Figure 5.4. As $t \to \infty$ $x \to \infty$ *and* $y \to \infty$.

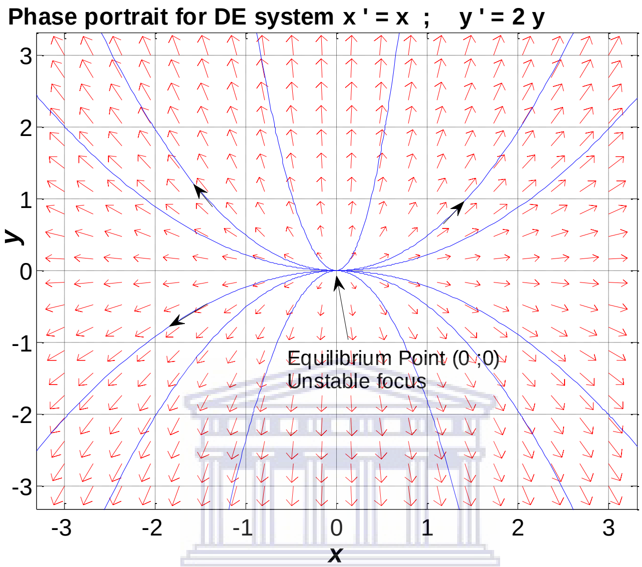


Figure 5.3 Phase portrait showing y Vs x. (0 ; 0)is an unstable node. All solutions move away from (0 ; 0). All trajectories are parabolas.

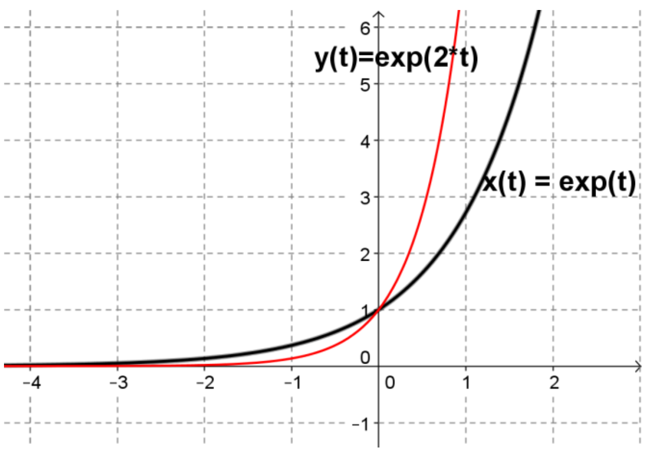


Figure 5.4 All solutions tend to ∞; (0;0) is an unstable node

Case 2 : Distinct real roots . Both eigenvalues are negative

Differential Equations: dx/dt = $-x$, dy/dt = $-2y$

Characteristic equation: $\lambda^2 + 3\lambda + 2 = 0$ has distinct real roots.

Eigenvalues $\lambda = -2$; $\lambda = -1$; Solution : $x = c_1 e^{-t}$, $y = c_2 e^{-2t}$

Equilibrium Point: (0 ; 0) is a stable node Analytical Solution $y = cx^2$.

Trajectories: $dy/dx = -y/x$: All trajectories tend to (0 ; 0) Thus, the node (0,0) is asymptotically stable.

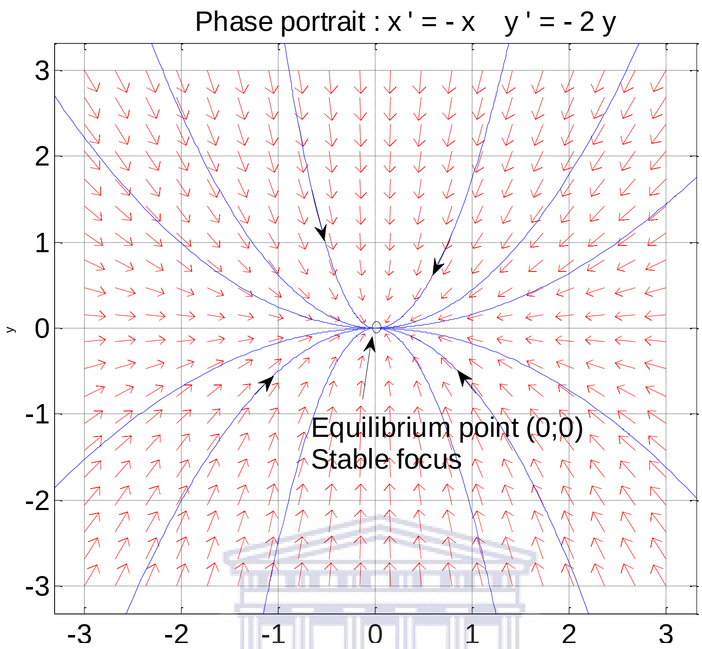


Figure 5.5 Phase portrait for the linear DE system, dx/dt = – x ; dy/dt = – 2y

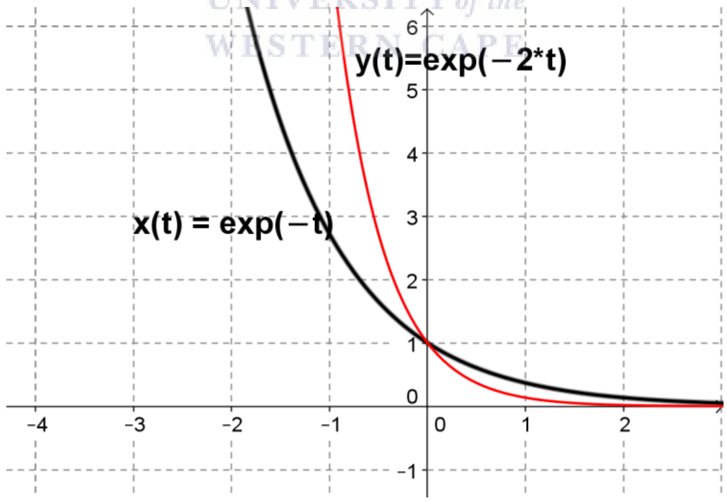


Figure 5.6 All solutions tend to 0. (0 , 0) is a stable node

Case 3 : Consider the system : $dx/dt = -x + y$ and $dy/dt = -x - y$. The system has just one equilibrium point (0 , 0). Eigenvalues are 0 and –2. The solutions using dsolve in Matlab are $x = c_1e^{-t}\cos t + c_2e^{-t}\sin t$ and $y = -c_1e^{-t}\sin t + c_2e^{-t}\cos t$. The spiral point (0,0) is asymptotically stable. All trajectories go clockwise, in spirals to (0,0).

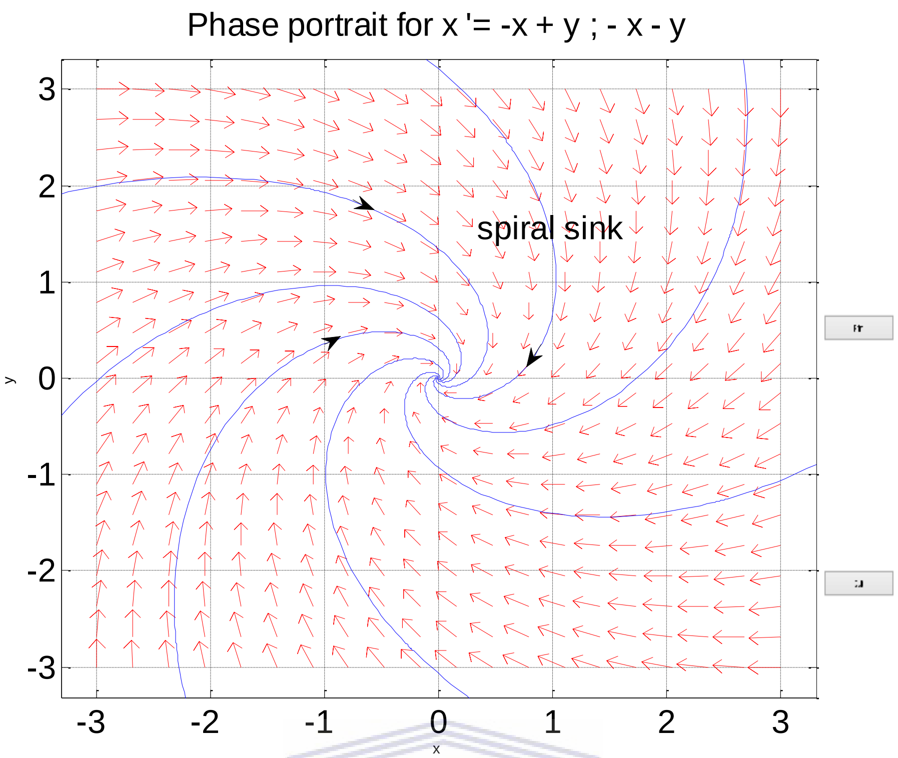


Figure 5.7 Dynamical system has one equilibrium point (0 , 0) which is a spiral sink and is asymptotically stable.

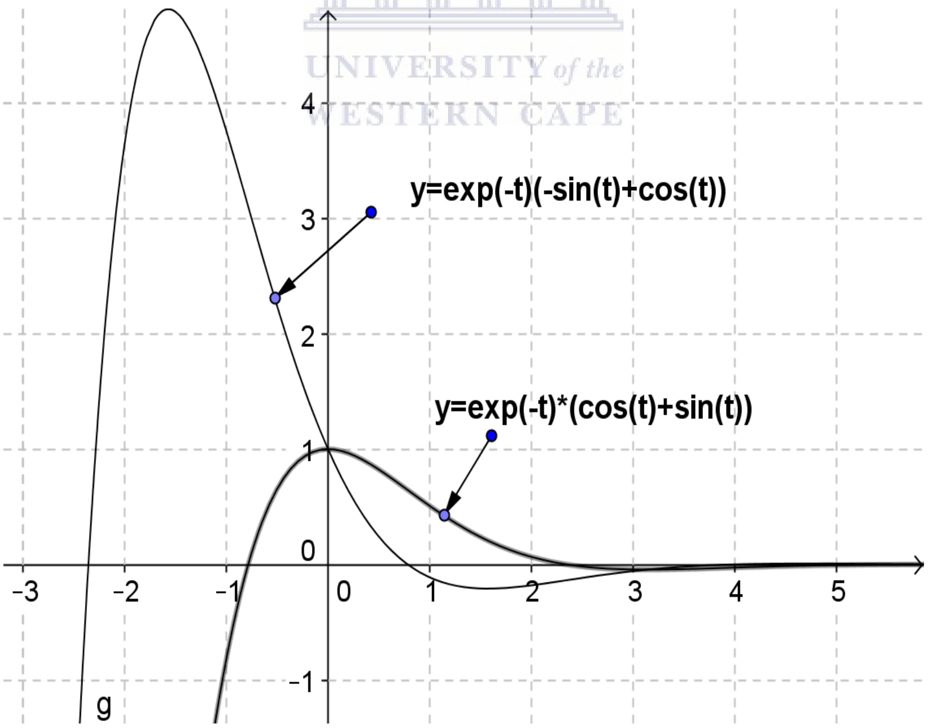


Figure 5.8 Spiral sink. All solutions tend to 0 as t → ∞

Case 4 : Differential Equations: $x' = -x - y$, $y' = x - y$

Characteristic Equation*:* $\lambda^2 + 2\lambda + 2 = 0$ has complex roots, $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$

The critical point (0 ; 0) is a stable focus. Spiral sink counterclockwise.

Case 5: Differential Equations: $x' = -x - y$, $y' = x + y$

CE *:* $\lambda^2 + 2\lambda + 2 = 0$ has complex roots $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$

The critical point (0 , 0) is a stable focus (spiral sink)

Case 6: The system $dx/dt = -x - y$ and $dy/dt = -x + y$ has one equilibrium point (0 , 0). The solutions are linear combinations of e^{2t} and e^{-2t} . The graphs of solutions are hyperbolas. The point (0 ; 0) is called a saddle point. See Figure. 5.9

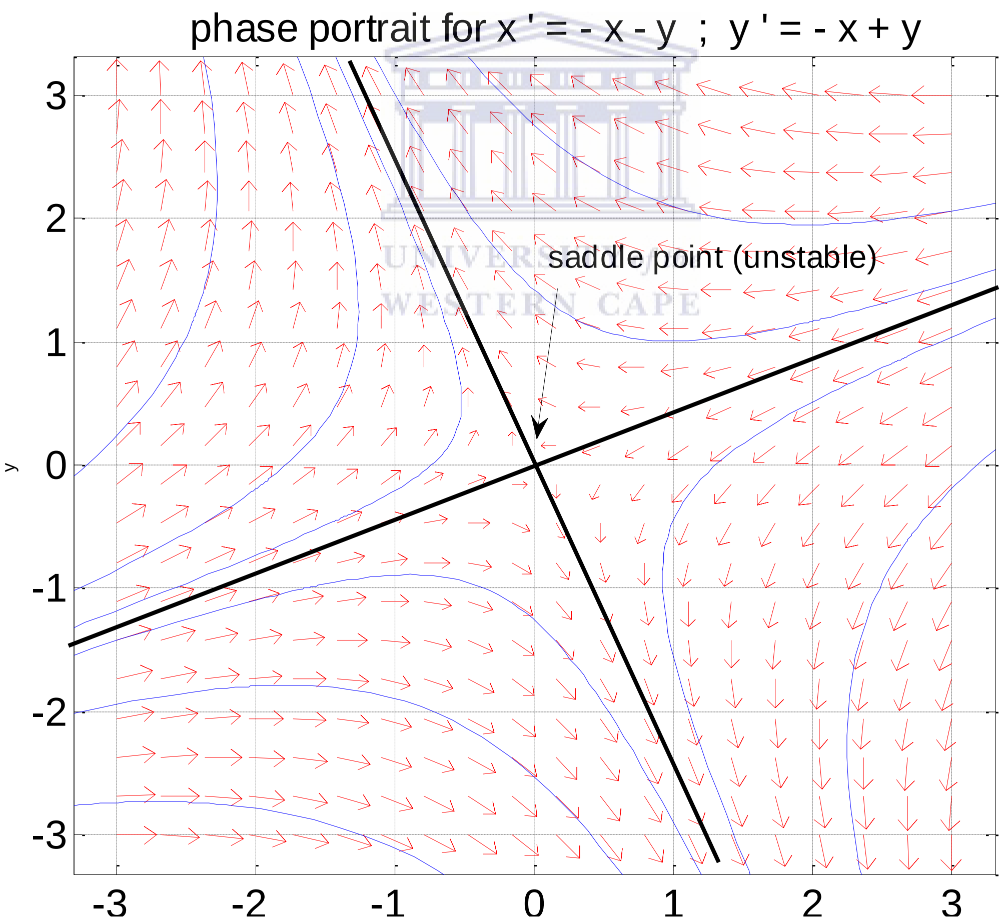


Figure 5.9 Phase portrait for the system: $dx/dt = -x - y$ and $dy/dt = -x + y$. Equilibrium point (0 , 0) is a saddle point.

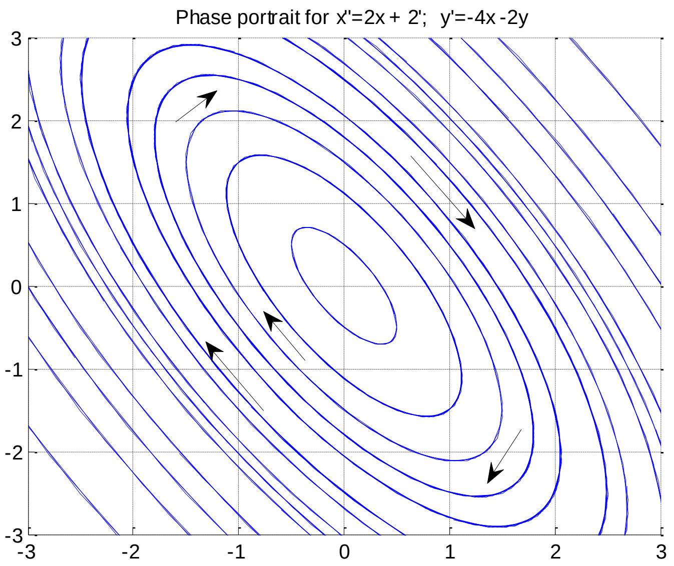


Figure 5.10 Phase portrait shows a stable equilibrium point, a centre

Differential Equations: $x'=2x+2y$; $y'=-4x-2y$

Characteristic Equation: $\lambda^2 + 4 = 0$ has complex roots $\lambda_1 = +2$ i and $\lambda_2 = -2$ i

The critical point (0 , 0) is a stable centre.

Consider now the system dx/dt = $-y$ and dy/dt = x: Its only equilibrium point is (0,0). The solutions are linear combinations of cos t and sin t. So, the solutions are circles with the centre at (0,0). In this case (or similar case when the solutions are ellipses), the equilibrium point is called a **centre.**

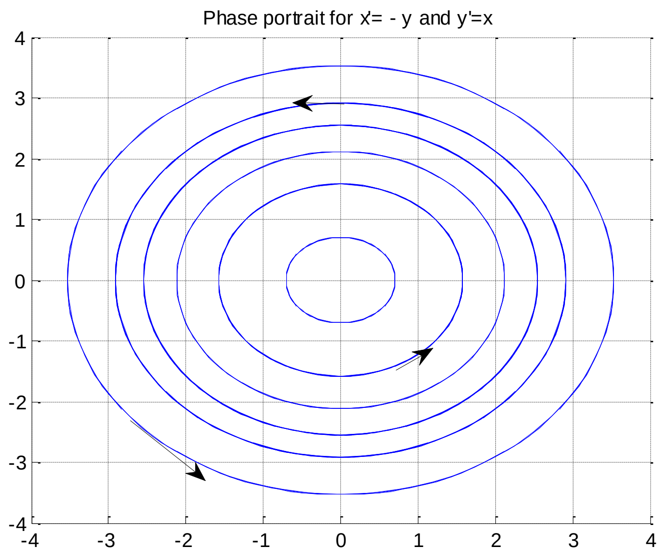


Figure 5.11 Critical point (0;0), a stable centre

We summarise the behaviour of linear dynamical systems:

1. The system is asymptotically stable if all solutions converge to (0;0) as t $\rightarrow\infty$. The eigen values are real and both negative (nodal sink) or have negative real parts (spiral sink)
2. The system is unstable if the solutions near the origin stay near the origin for all times and the eigen values are purely complex or one is 0 and the other negative (ellipse)
3. The system is unstable if it is neither of the above two cases : At least one trajectory leaves the vicinity of the origin.

Table 8 gives a summary of the stability corresponding to each type of eigenvalue.

Table 8. Summary of stability that the eigenvalues represent.

Eigenvalue Type	**Stability**	**Oscillatory Behaviour**	**Notation**
All Real and +	Unstable	None	Unstable Node; source or repeller
All Real and – ve	Stable	None	Stable Node; sink or attractor
Mixed + & – Real	Unstable	None	Unstable saddle point
+a + b*i*	Unstable	Undamped	Unstable spiral
–a + b*i*	Stable	Damped	Stable spiral
0 + b*i*	Unstable	Undamped	Circle : Focus or centre
Repeated values	Depends on orthogonality of eigenvectors		

The Determinant-Trace method diagram is often used to establish the stability of a linear dynamic system. In the Characteristic Equation $\lambda^2 - \lambda(a + d) + (ad - bc) = 0$ the sum of the diagonal elements, $(a + d)$ is called the trace, T and the difference of the products of the main diagonal and the off diagonal elements, $ad - bc$ is the determinant, D of matrix A.

Figure 5.12 shows the equilibrium point can be one of six types: A stable node, an unstable node, a stable focus, an unstable focus, a centre, and a saddle point.

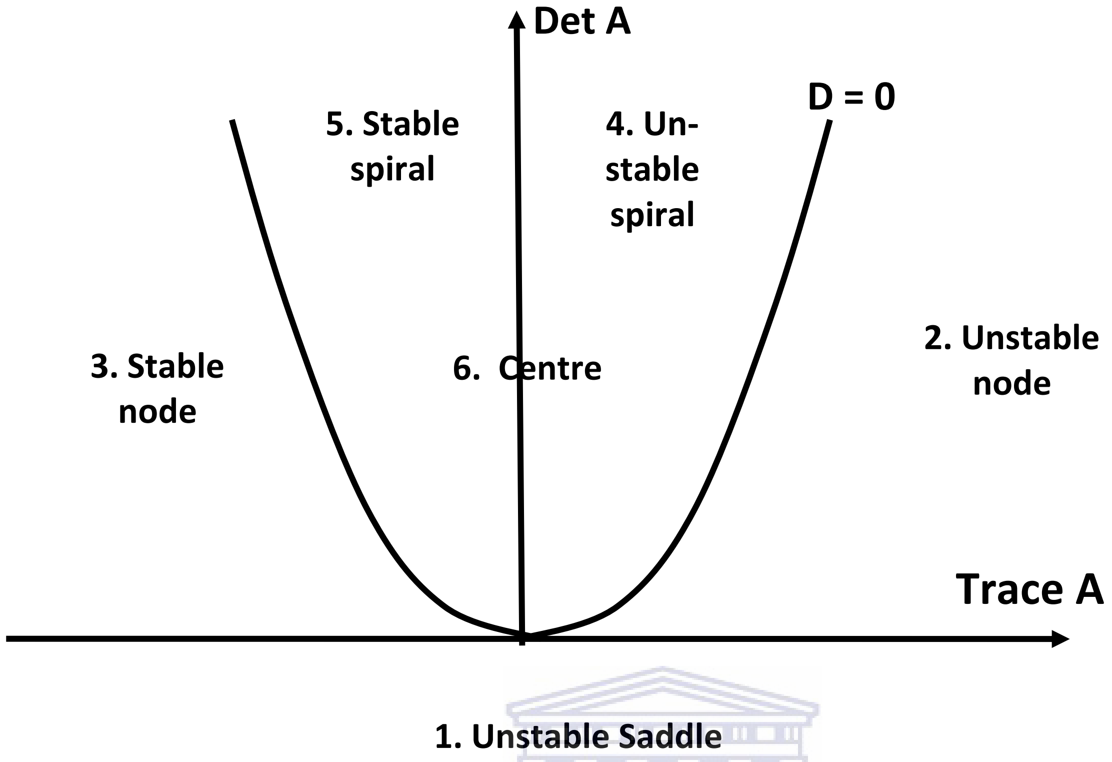


Figure 5.12 Determinant-Trace diagram to establish nature of Equilibrium points

If the determinant is negative we have exponential growth in one direction and exponential decay in another direction giving rise to a saddle point. If the determinant is positive and the trace is positive we have exponential growth with or without oscillations. If the determinant is positive and the trace is negative we have exponential decay with or without oscillations.

These findings are summarised here pointwise:

1. If D is negative then the fixed point is a saddle point.

2. If D is positive and T is positive then the fixed point is unstable.

3. If D is positive and T is negative then the fixed point is stable.

4. If D is positive and T = 0 then the fixed point is a centre.

5. If D is positive and $T^2 - 4D$ is positive then the fixed point is a node.

6. If D is positive and $T^2 - 4D$ is negative then the fixed point is a focus.

5.4 Non-linear dynamical systems

Non-linear systems are common in science and engineering. Unlike linear differential equations, there are few analytic techniques available for solving non-linear systems. We approximate these systems with a linear system in a neighbourhood of each equilibrium solution. We then use matrices to solve for the eigenvalues of the linearized system at each equilibrium point. These indicate the type of stability and characteristics of the steady states near equilibrium points. The stability of an equilibrium point depends on what happens to solutions near the equilibrium. The linear approximation is a good approximation to the function f for points close to the equilibrium. The linearized ODEs indicate exactly how far from steady state a given process deviates. Nonlinear, autonomous systems of ordinary differential equations are of the form

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n) \\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

where each of the functions f_i on the right-hand side are real-valued functions in n variables.

Examples of non-linear equations include: $\left[\frac{dy}{dt}\right]^2 - 3y = 4$ which has a derivative raised to power 2 and $\frac{dy}{dt} - 3ln\, y = 4$ which has $\ln y$ a non- linear function of y.

The solution of non linear systems can be approximated using our knowledge of linear systems. Often the behaviour of the full non-linear system is like that of the linearised system. We start by finding the equilibrium or fixed points. We linearize the system using the Jacobian at each fixed point, determine the eigenvalues and the nature of stability of the fixed points and we deduce the rest of the phase portrait from this information.

The principle of superposition does not apply for solutions to nonlinear differential equations and analytical methods such as Fourier transform and Laplace transform cannot be applied.

The Lotka-Volterra (LV) Model

The LV predator-prey model has been used to represent changes in variables in a wide range of fields. In a simple ecological model, one species, the predator, feeds on the other one, the prey, while the prey feeds on something else (vegetation) already in the environment. One example, would be lynx and hare in a forest, where the lynx (predator) eat the hare (prey) and the hare eat natural vegetation. In a study by Ahmadian (2008), the Lotka-Volterra competition equations were used to describe how a new technology (such as new transportation fuel or solar electricity run vehicle) grows in a system dominated by an old technology (such as internal combustion engine). In a model in economics, employment rate was used as the prey, the wage bill was predator. It has important dynamical concepts that can be useful to illustrate the stability of chemical reactions in which x is concentration of one reactant and y is the concentration of a product. The LV system is given by:

$$dx / dt = f(x, y) = a\ x - b\ xy,$$

$$dy / dt = g(x, y) = -c\ y + d\ xy$$

where x represent the prey (old technology, employment, hare, etc.) and y represent the predator (new technology, reactant, foxes, lynx, etc.). In this set of equations a, b, represent the growth constants and proportionality constants for prey and c and d represent the growth and proportionality constant for predators. In the absence of predator, $y = 0$, the prey population grows exponentially as $dx/dt = ax$. In the absence of prey, $x = 0$, the predator population will decay or perish (due for example, to unavailability of food) and $dy/dt = -cy$. When both predator and prey are present, the intensity of interaction is proportional to population sizes. The proportionality constants d and b, increase the predator population ($+ dxy$) and decrease the prey population ($-bxy$) respectively. The equilibrium points satisfy:

$$f(x, y) = a\,x - b\,x\;\;y = 0,$$

$$g(x, y) = -\,c\,y + d\,x\,y = 0.$$

Solving simultaneously, gives the equilibrium points of the system as (0, 0), and (c / d , a / b).

The Jacobian Matrix is $J(x; y) = \begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}$

The trivial fixed point (0 ; 0) is a hyperbolic saddle point. At the second equilibrium point (c / d, a / b), the eigenvalues are purely imaginary, nonzero, and complex conjugate. The fixed point is neutrally stable. An example and phase portraits follow in section 5.6.

Second order nonlinear differential equations can be studied in the phase plane by setting up two ordinary differential equations and analysis of the system singular points using the eigenvalue approach. In many cases the behaviour of the nonlinear system is similar to an approximating linear system near the singular points.

5.5 Illustrative examples on dynamical systems

The following system of non-linear ODEs model a simple chemical reaction:

$$\frac{dx}{dt} = 4x - x^2 - xy$$
$$\frac{dy}{dt} = 6y - 2xy - y^2$$

where x and y represent the concentrations of two reactants.

We need to find the equilibria of the model, evaluate the Jacobian at each equilibrium point and determine the stability of the system.

Solution

As outlined above, two approaches are available, a purely analytical approach and a graphical approach. We present and discuss these separately.

Analytical Solution: Given the nonlinear system of differential equations; $x' = 4x - x^2 - xy$ and $y' = 6y - 2xy - y^2$ we introduce functions f(x ; y) and g(x ; y) to represent the right hand side of the differential equations.

$x' = f(x, y) = 4x - x^2 - xy$

$y' = g(x, y) = 6y - 2xy - y^2$

a) To find the critical (equilibrium or fixed points) we set $f(x, y)$ and $g(x, y)$ to 0 and solve simultaneously for x and y. The point (x, y) is in equilibrium if and only if $4x - x^2 - xy = 0$ and $6y - y^2 - 2xy = 0$. We factorise and solve simultaneously:

$x(4 - x - y) = 0 \rightarrow x = 0, \quad x + y = 4$

$y(6 - y - 2x) = 0 \rightarrow y = 0, \quad 2x + y = 6$

to give the 4 critical fixed points : (0 , 0) (4 , 0) (0 , 6) (2 , 2)

b) Next we find the Jacobian Matrix $J(x, y)$ and use it to find the eigen-values and eigen-vectors at each of the critical points. $J(x; y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 4-2x-y & -x \\ -2y & 6-2y-2x \end{bmatrix}$

Table 9 presents a summary of the results for each equilibrium point.

Table 9. Stability of equilibrium points for $x' = 4x - x^2 - xy$; $y' = 6y - 2xy - y^2$

Equilibrium Points	A	Eigen Value	Eigen vector	Stability type
(0;0)	$\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$	$\lambda_1 = 4$ and $\lambda_2 = 6$	$V_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$	λs both real and +ve ∴ unstable nodal point
(0;6)	$\begin{bmatrix} -2 & 0 \\ -8 & -2 \end{bmatrix}$	$\lambda_1 = -2$ and $\lambda_2 = -6$	$V_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	λs both real and –ve ∴stable nodal point
(4;0)	$\begin{bmatrix} -4 & -3 \\ 0 & -2 \end{bmatrix}$	$\lambda_1 = -2$ and $\lambda_2 = -4$	$V_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$	λs both real and –ve ∴stable nodal point
(2;2)	$\begin{bmatrix} -2 & 2 \\ -4 & -2 \end{bmatrix}$	$\lambda_{1,} = -2 + \sqrt{8}$ $= 1.82$ $\lambda_{1,} = -2 - \sqrt{8}$ $= -4.82$	$V_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $V_2 = \begin{bmatrix} -2 \\ 2{,}8 \end{bmatrix}$	λs both real and with opposite signs ∴saddle point

We draw in the eigenvectors at the critical points with arrowheads indicating direction of motion into the critical point if λ < 0, away from critical point if λ > 0 and add in nearby trajectories. We then guess at some other trajectories compatible with these. Further information can be obtained by considering the associated first-order ODE in *x* and *y*.

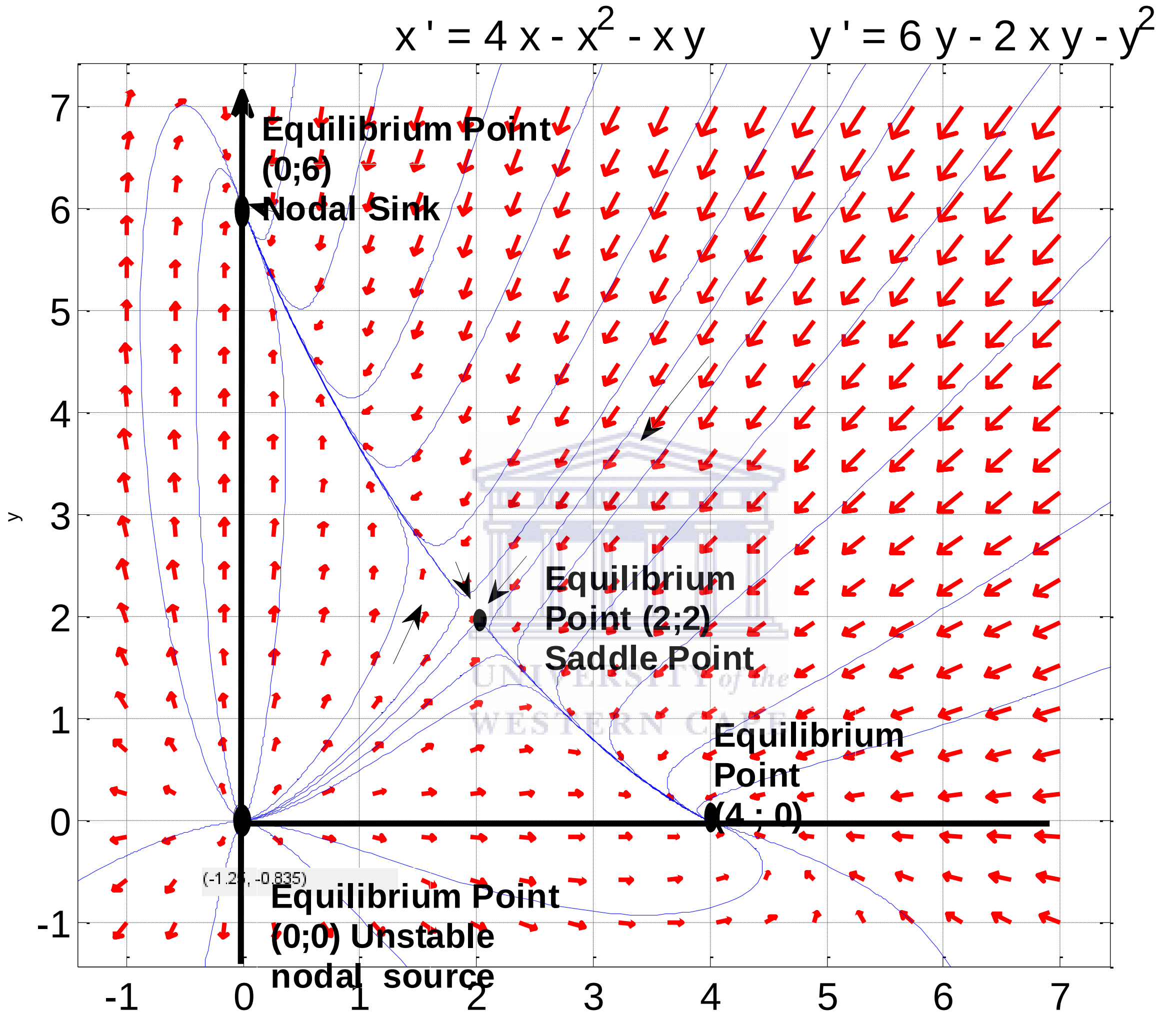


Figure 5.13. Phase portrait for $x' = 4x - x^2 - xy$; $y' = 6y - 2xy - y^2$ shows the four equilibrium points

Graphical Method

The alternative to the analytical method is the graphical method. It relies on the fact that in the phase plane there is a unique solution curve (trajectory) and no two trajectories can intersect. As before, the equilibrium solutions can be found by setting d*x*/dt and d*y*/dt to 0. The process is outlined in the following steps.

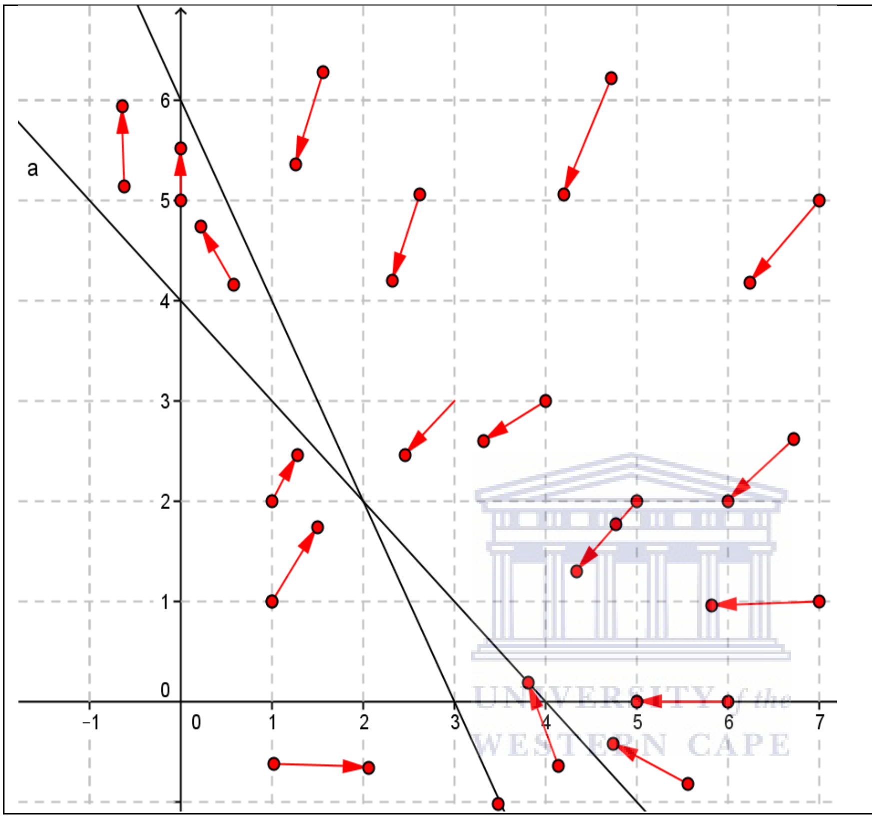
1. We identify and draw d*x*/dt = 0 null-clines using vertical hashes

2. We identify and draw dy/dt = 0 null-clines using horizontal hashes

3. The intersection in 1 and 2 gives equilibrium or fixed points.

4. We determine the direction of the trajectories on either side of the equilibrium points

5. We orient all nullclines - showing the direction of the vector field on the nullclines



	dx/dt	dy/dt
(3, 3)	−6	−9
(4, 3)	−12	−15
(1, 2)	1	4
(1, 1)	2	3
(0, 5)	0	5
(6, 0)	−8	0

Figure 5.14 Plotting a phase portrait by hand

6. We orient regions by looking at points and evaluating x' and y' . We can find the directions using the following rule: if f $(x,y) > 0$ the x component is directed as $\rightarrow$, if f $(x,y) < 0$ it is directed as $\leftarrow$; if g$(x,y) > 0$ the y-component is directed as $\uparrow$, g$(x,y) < 0$ it is directed as $\downarrow$.

7. In the xy-plane, we mark the critical points and sketch the trajectories in the immediate neighbourhood of the equilibrium points, including the direction of motion.

8. Finally, we sketch in a few more trajectories to fill out the phase portrait, making them compatible with the behaviour of the trajectories already sketched near the fixed points. Mark with an arrowhead the direction of motion of each trajectory.

The graphical approach is highly visual and depends to a large extent on an understanding of slopes d*x*/dt and d*y*/dt. For example the slopes at (1 , 1) are d*x*/dt = 2; d*y*/dt = 3;   We draw vectors of length 2 and 3 respectively in the *x* and *y* directions and find their resultant. The resultant gives the direction of the tangent to the trajectory at (1 , 1). Figure 5.15.

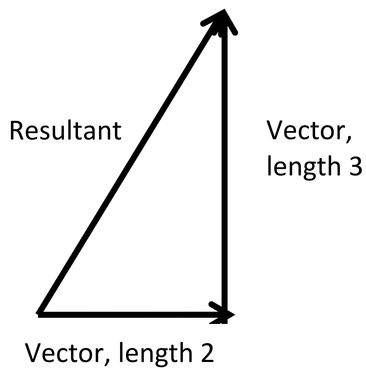


Figure 5.15 Finding the direction of the trajectory

To summarise, we began with analytical reasoning which gave us the equilibrium points. We calculated the Jacobian matrix at each equilibrium point. We solved to find the eigen-values and eigen-vectors using algebraic methods. Further analysis was used to determine the nature of stability of each of the equilibrium points and the system as a whole.

In the graphical approach we determined the fixed points by plotting the equilibrium solutions and used the slopes d*x*/dt and d*y*/dt to plot the trajectories and their flow directions. Both analytical and visual thinking are necessary in order to sketch the phase portrait.

It is equally important to be able to interpret the phase portrait analysis in terms of the physical situation it represents. In this case the negative real eigenvalues for (0 , 6) and (4 , 0) are indicative of desirable concentrations for the two variables and imply stable reactions in the Continuous Stirred Tank Reactor (CSTR). Any perturbations around these points will bring the system back into stability.

Example 2: Lotka-Volterra (LV) model

In this example we arbitrarily choose $a = 3$, $b = 2$, $c = 3$, and $d = 2$, for the LV nonlinear system and introduce a parameter k to study the effect of small perturbations in the system. The system equations are:

$$\mathrm{d}x / \mathrm{d}t = \mathrm{f}(x, y) = 3\,x - 2\,x\,y,$$

$$\mathrm{d}y / \mathrm{d}t = \mathrm{g}(x, y) = -3\,y + 2\,x * y - k\,y^2.$$

Taking partial derivatives, the Jacobian matrix is: $J(x;y) = \begin{pmatrix} 2-2y & -2x \\ 2y & -3+2x-2ky \end{pmatrix}$.

With $k = 0$ we have real positive eigenvalues giving rise to a stable centre at (1.5, 1.5) and represented by the phase portrait in Figure 5.16. We also have an unstable saddle at (0 , 0). Trajectories approaching (0 , 0) veer away sharply. The populations are unlikely to be driven to extinction.

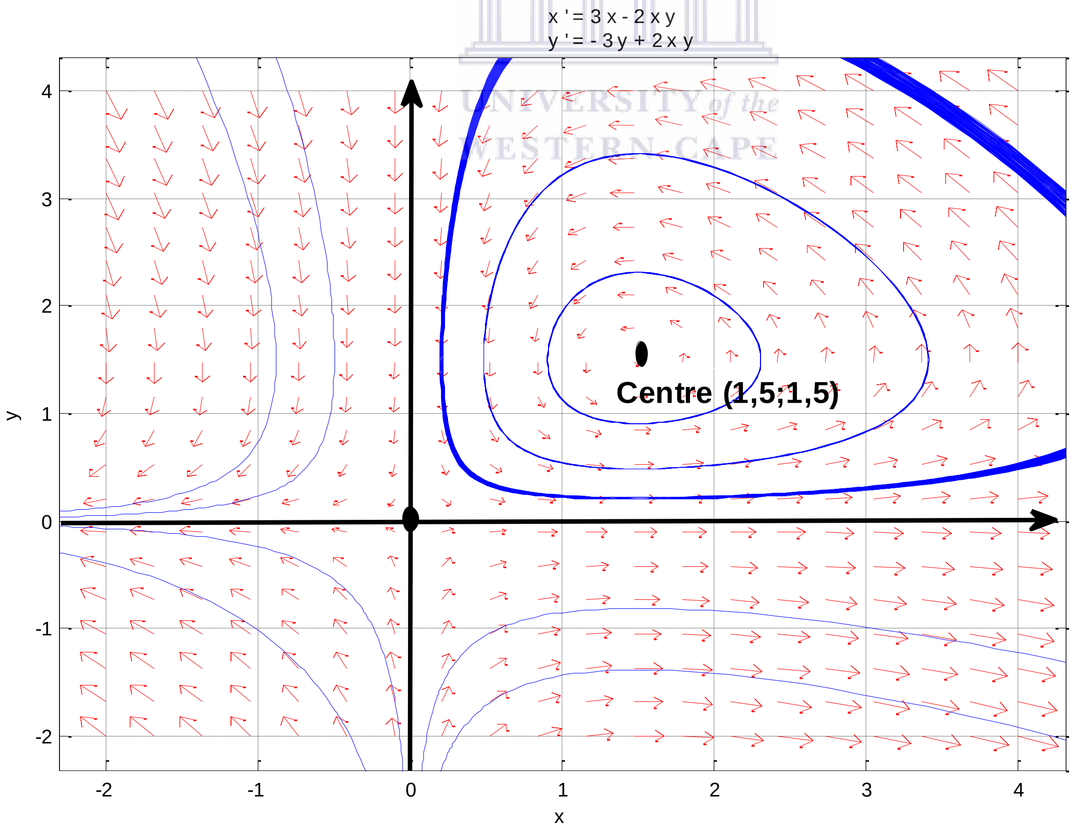


Figure 5.16 Phase portrait for $\mathrm{d}x / \mathrm{d}t = \mathrm{f}(x, y) = 3\,x - 2\,x\,y$, $\mathrm{d}y / \mathrm{d}t = \mathrm{g}(x, y) = -3\,y + 2\,x * y - k\,y^2$, $k = 0$.

When parameter k > 0, say +0.5, the equilibrium point is a **hyperbolic, repeller**. All trajectories spiral out from the new unstable equilibrium point. (See Figure 5.17).

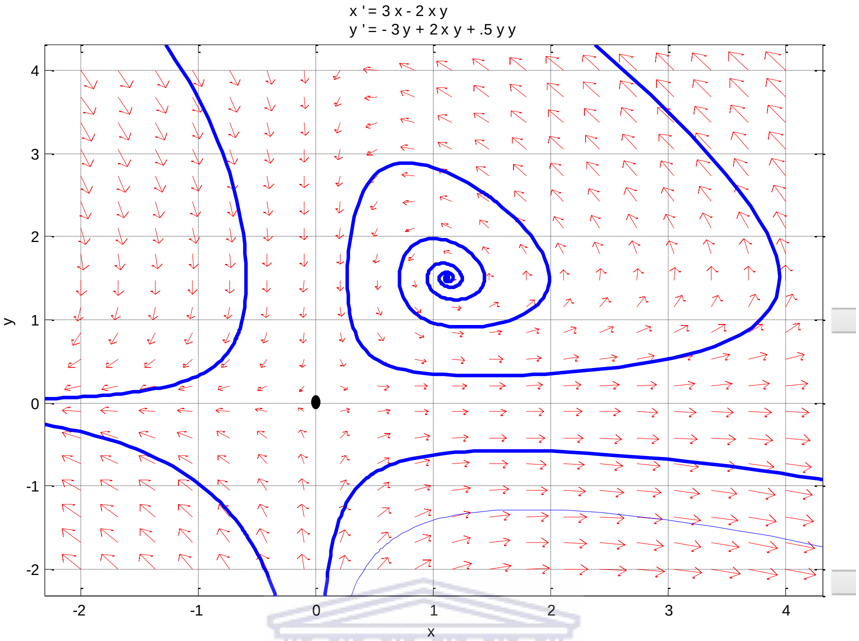


Figure 5.17 Phase portrait for dx / dt = f(*x*, *y*) = 3 *x* – 2 *x y*, dy / dt = g(*x*, *y*) = –3 *y* + 2 *x* * *y* – *k* y^2, *k* = 0.5.The point (1.2 , 1.6) is an unstable spiral point and (0 , 0) is an unstable saddle

With *k* < 0, say –0,5 , we have a a **hyperbolic, attractor fixed point** (See Figure 5.18). Both populations spiral in towards the fixed point.

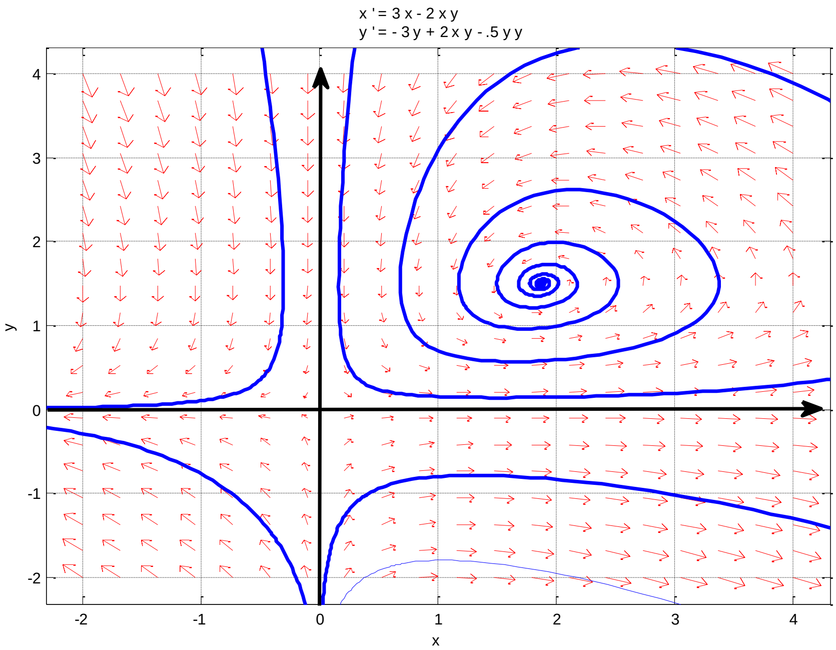


Figure 5.18 Phase portrait for dx / dt = f(*x*, *y*) = 3 *x* – 2 *x y*, dy / dt = g(*x*, *y*) = –3 *y* + 2 *x* * *y* – *k* y^2, *k* = –0.5. The point (1.8 , 1.5) is a stable spiral point and (0;0) is an unstable saddle

Note: For the Lotka Volterra equations, explicit solutions cannot be found. Implicit solution, by separation of variables, gives an expression of the form: $3 \ln y + 3 \ln x = 2y + 2x + c$. The graphs are like Figure 5.16 when c = –4, –5,.. –6 .

5.6 Chapter summary

In this chapter we looked at the theoretical background to differential equations and gave examples of problems in ODES and systems of differential equations that students are expected to solve. The approach to finding analytical solutions and direction fields was outlined and the problems with visualization of equilibrium solutions, and stability were highlighted. This was extended to systems of linear differential equations and the nature and stability of the equilibrium points discussed. We used the slopes dx/dt and dy/dt to plot the trajectories and their flow directions on the phase portrait. Finally we looked at non-linear systems using linearization of the differential equations near the equilibrium points. By performing a local analysis of the system near the equilibrium points and the Jacobian matrix we were able to predict the global behaviour of system.

Chapter 6 : Data Analysis. Impact of visualization

6.0 : Overview

This chapter presents the findings of the study based on the analysis of data collected from the tests, assignments, interviews and worksheets administered during the first semester when the course Math300S (Calc3) was taught. The chapter is organised in the order of the research questions, reproduced in section 6.1 for ease of reference.

The aim of the study is to investigate practical strategies that can engage the learner in the exploration of 3D space figures and phase portraits in order to facilitate, reinforce and strengthen the connections between visual and analytical thinking in the conception and solution of problems in multiple integrals and dynamical systems. We conducted a teaching experiment using Matlab to enhance visualization and support student conceptualisation of double and triple integrals and phase portraits of dynamical systems.

In Chapter 2, the review of literature, we identified important attributes of visualization that we expect to relate to achievement in problems involving 3D objects and dynamical systems. We use multiple linear regression to explore this relationship and determine which of the selected factors are significant predictors of achievement in the domains under study and mathematics achievement at calculus 3 level.

We use Duval's (1995, 1996) semiotic representation framework, outlined in section 3.2.1, and Zazkis et al. (1996) visualization-analysis (VA) framework, outlined in section 3.2.2, to evaluate the impact of the laboratory activities on students' solutions to problems in multiple integration and dynamical systems.

Knowing the rules, conventions and symbols of each type of representation allows understanding the meaning of a representation. Under Duval's (1995, 1996) framework, these representations may undergo two types of transformations: a treatment within the semiotic system and a conversion from one system to another. Perceptual, sequential and discursive apprehension impact on operative apprehension, a higher level of visual processing, that occurs when a geometric figure is modified.

6.1 Research questions

The guiding research questions of this study were:

1. What are students' needs and difficulties in conception and solution of problems in multiple integrals and dynamical systems?
2. Do the activities facilitate visualization and solution of problems in the two domains?
3. What factors influence the effectiveness of the visualization?
4. What Teaching and Learning strategies help in the conceptualization and solution of problems in multiple integrals and dynamical systems?

6.2 The teaching experiment

Two groups of students, the experimental (n = 24) and the control (n = 26) participated in the study. The experimental group were registered as full time students, received four 50 minute lectures from the researcher each week and in addition, participated in computer laboratory activities once a week, working in pairs on worksheets designed by the researcher. Tutorial sessions were held as the need arose. The control group were part time students who attended on Wednesday evening (5 pm to 8 pm) and Saturday (10 am to 12 noon). Spot tests and assignments were given to both the groups at regular intervals during the semester course and major examination-like assessments labelled T1, T2, T3 on dates determined by the Department of Mathematics and Physics.

In both groups, the first week was spent on reviewing work from Mathematics 1 and 2, which are semester courses equivalent to Calculus 1 and 2 at a university. The review worksheet is attached in Appendix 3. A pre-test (see Appendix 4.1) assessing prior knowledge from

Mathematics 1 and 2 was administered at the beginning of the study (week 2) to both groups. The aim of the pre-test was to check if there were significant differences between the two groups initially. The pre-test scores were taken as a measure of students' prior knowledge. In order to check if there were significant differences between the control and experimental groups on prior knowledge, the following hypotheses were tested:

Null hypothesis: H_o: There are no differences between the mean scores of the control and experimental groups on the pre-tests.

Alternative, H_1: There are significant differences between the mean scores of control and experimental groups on the pre-tests.

Table 10 gives a summary of the pre-tests results:

Table 10 Summary of the pre-tests results

	Experimental	Control
Number	22	26
Mean	54.636	51.296
SD	18.293	18.563
t-test, calc	0.657	df = 46

Findings: On the day the pretest was given, 22 students from the experimental group were present, while all 26 students in the control group sat the pretest. The critical t-value is 1.645 and we reject the alternative hypothesis at 5% level of significance (t = 0.657 at p = 0.05, 2 tail test, df = 46). We conclude that the mean pre-test score of the experimental group ($\bar{x}$ = 54.64, SD =18.29, n = 22) is not significantly different from that of control group ($\bar{x}$ = 51.30, SD = 18.56, n = 26).

Note: The part-time students were working mostly in the chemical industry and often on shifts which were not easy to move. The numbers sitting any test varies. The regression analysis was carried out with 21 students out of 24 in the experimental group who had complete data for all variables.

6.3 Pre-test Item Analysis

During the first week of the course we reviewed Maths 1 and 2 topics including functions, matrices, differentiation, integration and differential equations. This was necessary as most of the students, especially the part-time students who were working in the chemical industry after graduating with their National Diplomas, had been out of the education system for some time. This review helped them to revise some of the earlier work.

Problem areas were identified and an attempt was made to correct them. Overall, most students (67%) could match equations like $xy = 4$, $y = \sqrt{x+1}$, $y = (x-1)^2$ to their graphs while fewer (59%) could match $y = \frac{2}{x-2}$ and $9x^2 + 4y^2 = 36$ to their graphs. Nearly 30% of the students did not recognize $y = 2 + x - x^2$ as a quadratic equation or the equation representing a parabola. Several looked for points to plot the graph and ended up with a straight line. Errors included finding turning points, lines of symmetry, and intercepts.

Nearly half of the students struggled with the questions on quadratic equation (question 1), finding gradients using differentiation (question 2(a), solving the differential equation $\frac{dy}{dx} = y + 1$ (question 6) and finding areas using single integration (question 8). Question 8 required students to find the area under the graph of $y = x^2 + 1$ for $1 \le x \le 3$. Overall, 42% of the students could tackle this. Supplementary practice worksheets were posted on the webCT and tutorial support was made available to those in need. The pre-test served to make students aware of their knowledge gaps and misconceptions and also informed the researcher about the type of background knowledge to expect during the laboratory sessions.

6.4 Comparison of Overall achievement between control and experimental groups

During the semester, three major tests, T1, T2 and T3 were given to each group. The content distribution of the tests is shown in Table 11. Test 1 had questions on double and triple integrals in rectangular coordinates and Test 2 covered questions on multiple integrals (in rectangular coordinates, cylindrical coordinates and spherical coordinates) as well as systems of differential equations. The tests were marked and results converted to percentages. It was hypothesised that the treatment (Lab interventions) would enhance the performance of the experimental group on all tests.

Table 11 Content of the tests students sat during the semester

Test	Content
T1	Vector analysis, Lines, planes, gradient function, Directional derivatives, Rectangular coordinate systems, Polar coordinates, and Double integration
T2	Integration in rectangular, cylindrical and spherical coordinate systems, Dynamical Systems. Fourier Series
T3	Summative Test includes vectors, coordinate systems, Fourier Series and Partial Differential Equations

Two way ANOVA were conducted in SAS to see if there were significant differences in the performance between the control and experimental groups on each of the tests. Details of the SAS analysis are presented in Appendix 5. Here we present and discuss the main findings.

Results

Test 1: A two-way ANOVA of test results with treatment group (control Vs experimental) and gender (male vs female) revealed no significant differences in student performance. The main effect of treatments was not significant ($F(1, 46) = 1.10$, $p > 0.30$) , while gender differences were also not significant ($F(1,46) = 0.17$, $p > 0.87$). Treatments vs gender interactions also showed the same lack of effect, [$F(1,46) = 0.006$, $p > 0.811$].

Test 2: A two-way ANOVA of test results with treatment group (control Vs experimental) and gender (male vs female) also revealed no significant differences in student performance. The main effect of treatments was not significant ($F(1, 46) = 0.01$, $p > 0.925$) , gender differences were also not significant ($F(1,46) = 0.92$, $p > 0.344$) and treatment-gender interactions were not significant [$F(1,46) = 1.45$, $p < 0.235$] .

Test 3: There were significant differences between the two groups on test 3, [F(1, 46) = 4.1, p = 0.048)] in favour of the experimental group. However, the gender differences were not significant, [F(1, 46) = 1.04, $p < 0.313$] and treatment-gender interactions were not significant [F(1, 46) = 2.47, $p < 0.1226$].

In conclusion, the laboratory activities did not translate into significant gains for the experimental group while the activities were on-going (Test 1 and Test 2), but towards the end of the semester there were gains in favour of the experimental group on Test 3. A detailed analysis by questions and topics relevant to this study follows.

6.5 Difficulties in visualising double integrals

In this section, we look at students' solutions to questions on double integration with the aim of highlighting the underlying difficulties in visualization and analytical thinking. We apply Duval's (1995, 1996) semiotic representation framework and the Visualization-Analysis (VA) framework to analyse the solutions. According to Duval (1996), we can only try to gain access to concepts through semiotic representations used to deal with them. Students' use of these representations provide useful information about the difficulties they are facing. However, students' internal representations are not directly accessible but the way in which 'a student generates or relates to an external representation reveals information about how he or she has represented the information internally' Camacho-Machin, Perdomo-Diaz and Santos-Trigo (2012, p. 5). The students' solutions presented here represent their answers to questions under examination conditions without the aid of computers or assistance from the tutor or peers. After the solutions were marked, four students, (two from each group) were interviewed to clarify their responses with a focus on underlying thinking especially for answers that were incorrect. These were recorded and transcribed and four of these are presented here as interview excerpts. The analysis enabled the researcher to pinpoint the source of the difficulty in the solution of the problem. The students are named ST1, ST2, etc. and the interviewer is INT.

Question 3: T1 (Test 1). This question looks at students' understanding of single and double integrals in terms of the VA framework. The question is reproduced first and then the student's solution, followed by the VA analysis.

Question

The graph shows the region between the curves

$y = \sqrt{x} + 1$, $y = 0$, $x = 0$ and $x = 3$.

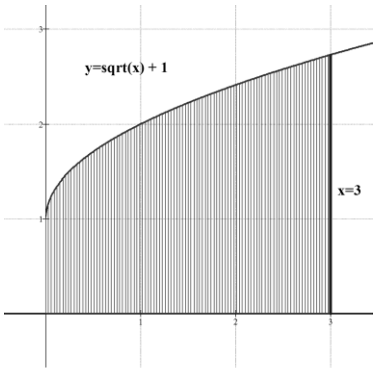
3.1 Write down a single integral for the area of the region. Evaluate the integral.

3.2 Write down a double integral for the area in the order:

a) $\iint dydx$ b) $\iint dxdy$

3.3 Sketch and write down a double integral for the area enclosed by $y = 4 - x^2$; $y = 0$ and $x = 0$ in the order

a) $\iint dydx$ b) $\iint dxdy$



The work presented by student, ST1 is shown in Figure. 6.1. The interview excerpt was a follow up exercise to seek clarity about the thinking and steps used in the solution. We use V for visualization or visual steps and A refers to analytical thinking or analysis.

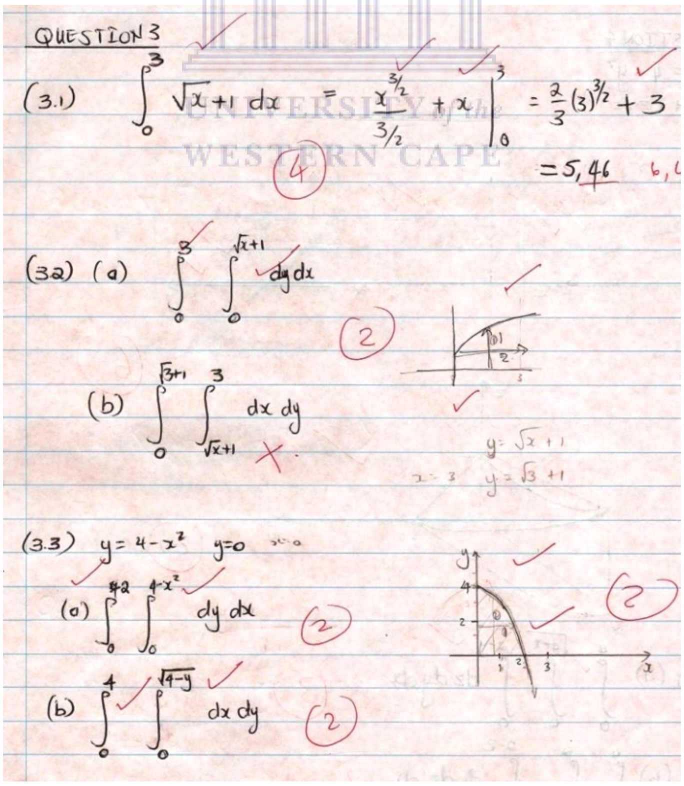


Figure 6.1 ST1's solution to question on single and double integrals

We note that visualization and analysis serve as two mutually supporting and interacting modes of thinking. The student, ST1, starts with visual steps V1 and V2 which gives the limits of integration. With the exception of Question 3.2 (b), where ST1 has been unable to convert from the geometric register to the algebraic register, all other treatments and conversions were successful. There is evidence of perceptual apprehension (notes function, reads limits) as well as sequential apprehension (correct steps in calculations, sketches). On the VA framework our analysis of the solution presented by ST1 is shown in Table 12.

Table 12 Visual Analytical steps in the solution by ST1 to question on single and double integrals

Q 3,1 Single integral **Visual Steps**	**Analysis**
Conversion: From the 2D graphical register to algebraic register V1. Checks figure. Notes functions. V2: Reads limits V3: Writes single integral. V4: Vertical slicing-	A1 Read and record limits x =0 and x = 3 on integral A2: Integrates $\sqrt{x}$ A3 Integrates 1 dx A4: Substitutes upper and lower limits A5: Calculation error. A6 : Incorrect final answer
Q 3.2 (a) Double integral order dy dx	
Visual steps Sketch shows movement along arrows. No vertical slices. V5: Reads limits for dy from sketch V6 Reads and enters correct limits for dx	**Analysis** **A7:** Records y limits from 0 to $\sqrt{x}$ +1 A8 Records x-limits from 0 to 3
Q3,2 (b) Double integral order *dxdy*	
Visual Steps Sketch shows arrows indicating movements No indication or attempt at slicing. V7: Working only in the upper region of the integral under the curve V8 Error reading lower x limit. Unable to perform conversion.	**Analysis** Does not see need for split integral A9: Error. Takes lower x limit as $\sqrt{x}$ +1 A10: Reads correct upper limit for x as $x = 3$ A11: Uses $y = \sqrt{x} + 1$ to find upper y-limit. A12: Incorrect double integral in order dxdy
Question 3.3 (a) Double integral order dy dx	
Visual steps V9:. Numerical register: Finds coordinate points for drawing the graph of $y = 4 - x^2$. Correct sketch $y = 4 - x^2$ V10: Geometric register: correct graph. V11 Shades required region. V12: Vertical Slicing and reading limits	Algebraic register: A13 Double integral., limits for dy then dx. A14: Reads and enters correct limits for y : 0 to $4 - x^2$ A15: Reads and enters correct limits for x : 0 to 2
Question 3.3 (b) Double integral order dx dy	Algebraic register.
V13: Same graph. V14: Horizontal slicing	A16 Makes x the subject of the equation. A17: Records x limits as 0 to $\sqrt{(4 - y)}$ A18: Reads correct y limits A19: Correct switch

However, visualizing the slices and using a split integral is an important step that the student ignored or missed and so was unable to switch the double integral.

Interview Excerpt 1 (ST1)

In Q3.2a the student successfully found the limits for ∫∫dydx. The purpose of the interview was to find out what student thinking about ∫∫dxdy was. The student was shown his/her solution, given time to go over the solution, and asked questions:

INT: In Q3.2 b what were you thinking when you wrote down the limits for ∫∫dxdy. How did you arrive at these limits?

ST1: I moved horizontally first. I started at $x = 0$, and moved to $x = 3$. I couldn't turn the integral round.

INT: In terms of movements on the sketch, what would you be doing? What did you do?

ST1: I moved from the line at 0 (the *y*-axis) to the line $x = 3$. (*Points at lines*)

INT: If you are on the curve and you go horizontally would you still start at $x = 0$. What should you be doing?

ST1: I ..I am not sure. Start at the curve?

INT: Did you think about slicing horizontally using thin rectangles? Did you slice the region horizontally.

ST1: I didn't think about slicing .. I forgot about slicing.

INT: So if you slice horizontally what would your slices look like. Remember you must stay within the region of integration

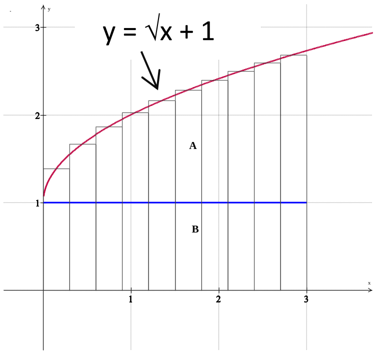
ST1: *Draws slices on the sketch.*

INT: Watch the slices. Are your slices always from 0 to 3?

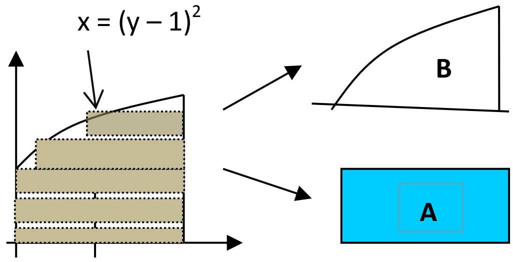
ST1: No, only up to y =1. Then they are from the curve to 3. Does that mean we take an x on the curve to $x = 3$ for this part.

INT : Your lower x limit for the upper part should start x =…. And you need to split the integral into two.

Slicing the region of integration was necessary to define the limits of integration in Question 3.2 (b). We find the student worked with given equations, turned them around, and used these as limits without checking the diagram. Although the student could set up single and double integrals, switching was incorrect. We note that the student has problems with visualizing the region of integration as a Riemann Sum, and with the switch, which requires a split integral to span the sliced area. This difficulty is highlighted in Figure 6.2. In Figure 6.2(a) the vertical slices (rectangles) have height $y = \sqrt{x} + 1$ and width δx. So the area of each rectangle is $\delta A = (\sqrt{x} + 1)\delta x$. Taking limits and summing up gives the total area expressed as single integral as $\int_0^3 (\sqrt{x} + 1)dx$ or as a double integral as $\int_0^3 \int_0^{\sqrt{x}+1} dydx$.



a) Slicing the region parallel to the y-axis first and then in the x direction gives the integral: $\int_0^3 \int_0^{\sqrt{x}+1} dydx$



b) Slicing parallel to the x-axis first and then in the y direction needs a split integral.:

For A: $\int_0^1 \int_0^3 dxdy$ For B: $\int_1^{\sqrt{3}+1} \int_{(y-1)^2}^3 dxdy$

Figure 6.2 a) Vertical slicing and b) horizontal slicing of the region under $y = \sqrt{x} + 1, \quad 0 \le x \le 3$

Figure 6.2 (b) shows horizontal slicing. The region of integration is split into A and B, each requiring a separate area integral. On Duval's framework we note that conceptualising the Riemann sum through slicing/spanning is missing in the work presented by ST1. Operational apprehension is lacking and we have a breakdown in visualization.

In Question 3.3, students had to set up a double integral to find the area under $y = 4 - x^2$, $x > 0$, $y > 0$. We note once again the absence of horizontal or vertical slicing. The student identified the region of integration by cross-hatched shading. Figure 6.3 shows sloping slices and it is clear that iteration in the reversed order dxdy was not done with horizontal slices. ST2 does not see a need to find x in terms of y i.e $x = \sqrt{4-y}$. On Duval's (1995) framework, the difficulty experienced by the students can be attributed to a lack of coordination among representational registers. An important visual step was missed as the student did not see that the lower limit in the x direction changes from $x = 0$ to $x = \sqrt{4-y}$.

In the next example, we see that the coordination of the visual (geometric) and algebraic registers by ST3, led to a better understanding and solution of the problem. The mobilization of both registers is accompanied by reflection, slicing and spanning the required region with little squares. Duval (2006, p.126) proposes that comprehension in mathematics assumes

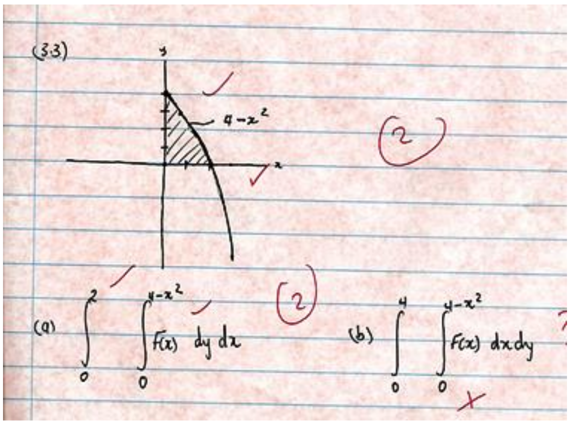


Figure 6.3 Student ST2's solution to find the area under $y = 4 - x^2$ for $0 \le x \le 2$

the simultaneous awareness and coordination of at least two registers of semiotic representation and further that this evolves into a synergy of the registers of representation. The VA analysis of the student's solution follows in Table 13.

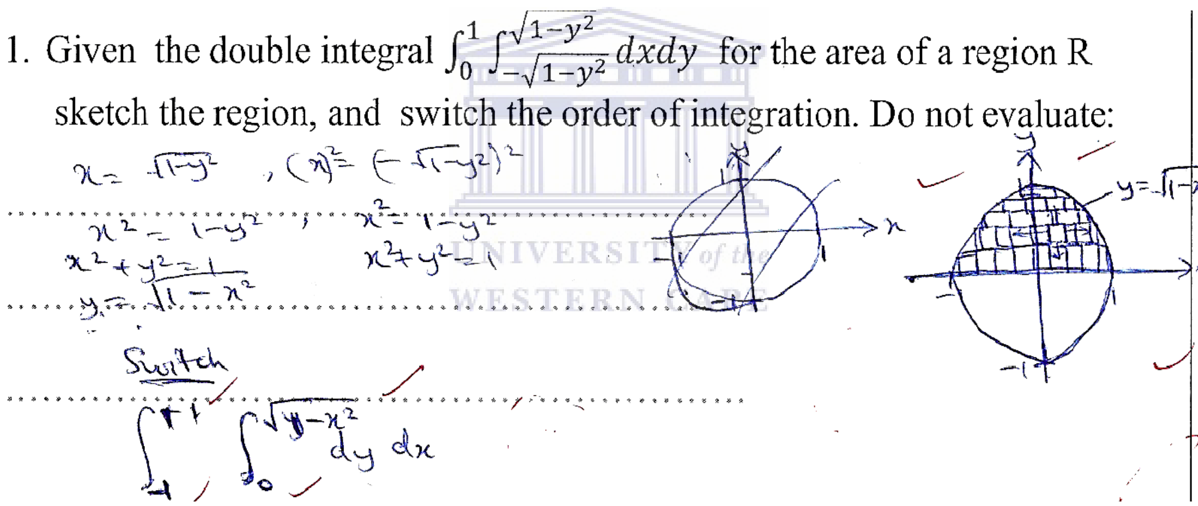


Figure 6.4. Solution by ST3. Slicing horizontally with squares first to fill the region of integration.

The Visual-analytic steps are shown in the Table. 13. Switching the region of integration horizontally along y gives the limits $y = 0$ to $y = \sqrt{1-x^2}$ and then in the x- direction to give the limits -1 to 1. The switched integral is $\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} dy\, dx$. ST3 shows clear understanding of the process involved.

Each little square has area *dxdy*. Reading the limits of the given integral helped to identify the limits of integration. Switching for this student was fairly simple. Several simultaneous conversions between the geometric and algebraic registers are evident. Operational apprehension and visualization as well as analysis were evident.

Table 13 Visual Analytical steps for switching the double integral $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy\, dx$

Q1. Switch the order of integration $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy\, dx$. See ST3's solution in Figure 6.4	
Visual Steps V1: Conversion from algebraic to geometrical register: V2: ST3 recognises equation of circle $x^2 + y^2 =1$, and draws circle, radius 1 centre (0 ; 0). V3: ST3 identifies and fills region of integration with squares. Spanning horizontally parallel to *x* axis. V4: Reads and enters correct limits for *y* V5 Reads and enters correct limits for *x*	Analysis A1: Treatment in algebraic register. Squares both sides of $y = \sqrt{1-x^2}$ A2: Simplifies to give $x^2 + y^2 =1$. A3: Records y limits from 0 to $y = \sqrt{1-x^2}$ A4: Records x-limits from 0 to 1

6.6 Difficulties in visualising triple integrals in rectangular coordinates

The question was set in Test 1, in rectangular coordinates and required students to sketch and find the volume of the region in the first octant bounded by $y + z = 2$, $x = 4 - y^2$. We look at the work of two students ST4 in Figure 6.5, ST5 in Figure 6.6 and list the VA steps in Table 14 for ST4.

Discussion

On the Visualization-Analysis framework, we see numerous instances of visual and analytical thinking in the solutions by the two students ST4 and ST5. The first few steps involve analysis of the given equations. Both students work in the *xz, yz* and *yx* planes and draw correct projections in 2D. An important next step is to put the projections together to assemble the solid. ST4 draws the projections and assembles them into the 3D object easily. This requires coordination of the projections, and moving and positioning them on the 3D sketch. The intersection is a curve sitting in 3D space. See Figure 6.5.

ST5 couldn't figure out where the planes $y + z = 2$ and $x = 4 - y^2$ intersect. We see an incomplete 3D solid (See Figure 6.6). Next they need to set up the limits of the triple integrals. Whereas ST5 got the correct limits in the order *dzdydx* both students had problems finding the

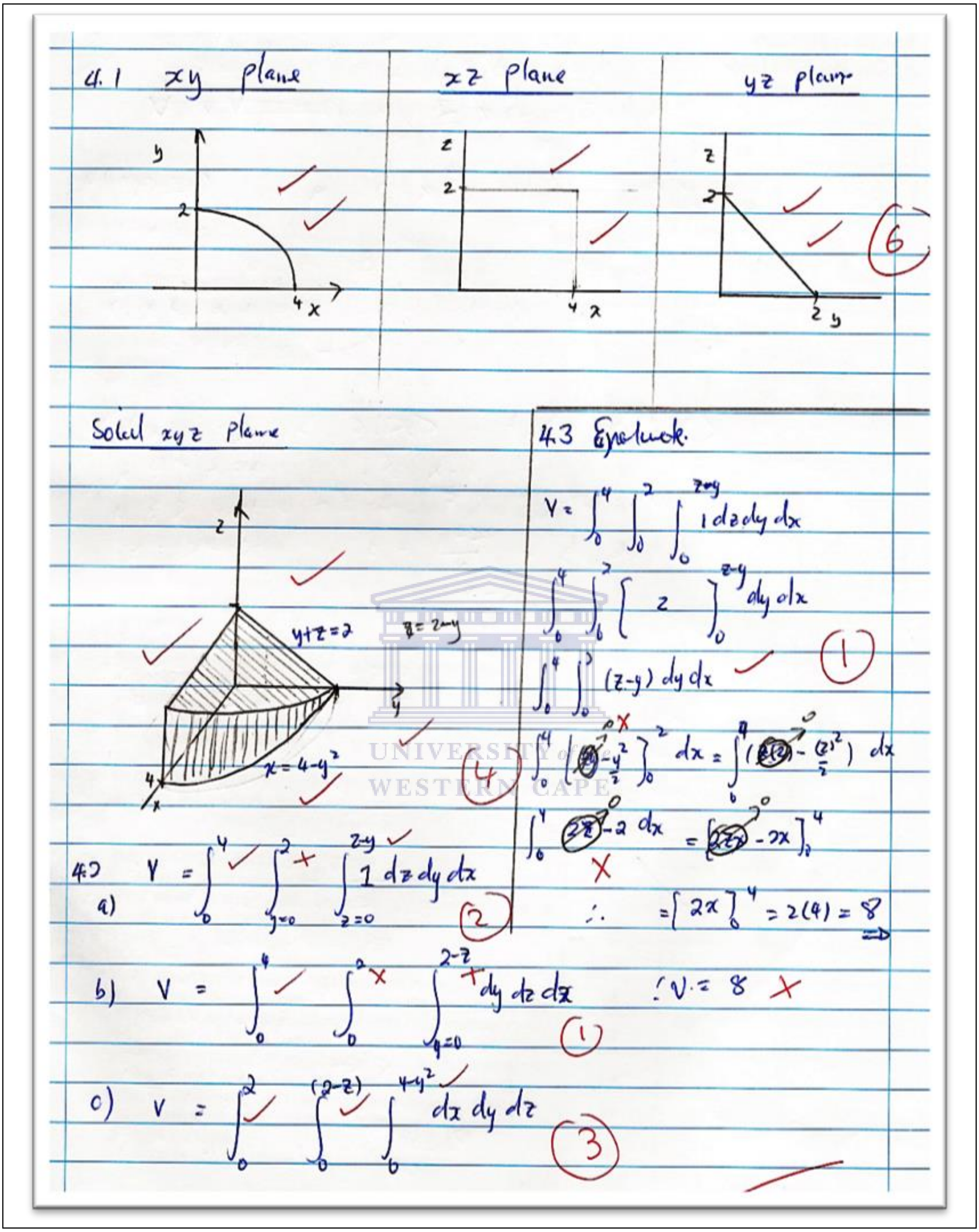


Figure 6.5 Solution by ST4 region of integration bounded by $y + z = 2$, $x = 4 - y^2$ in the first octant

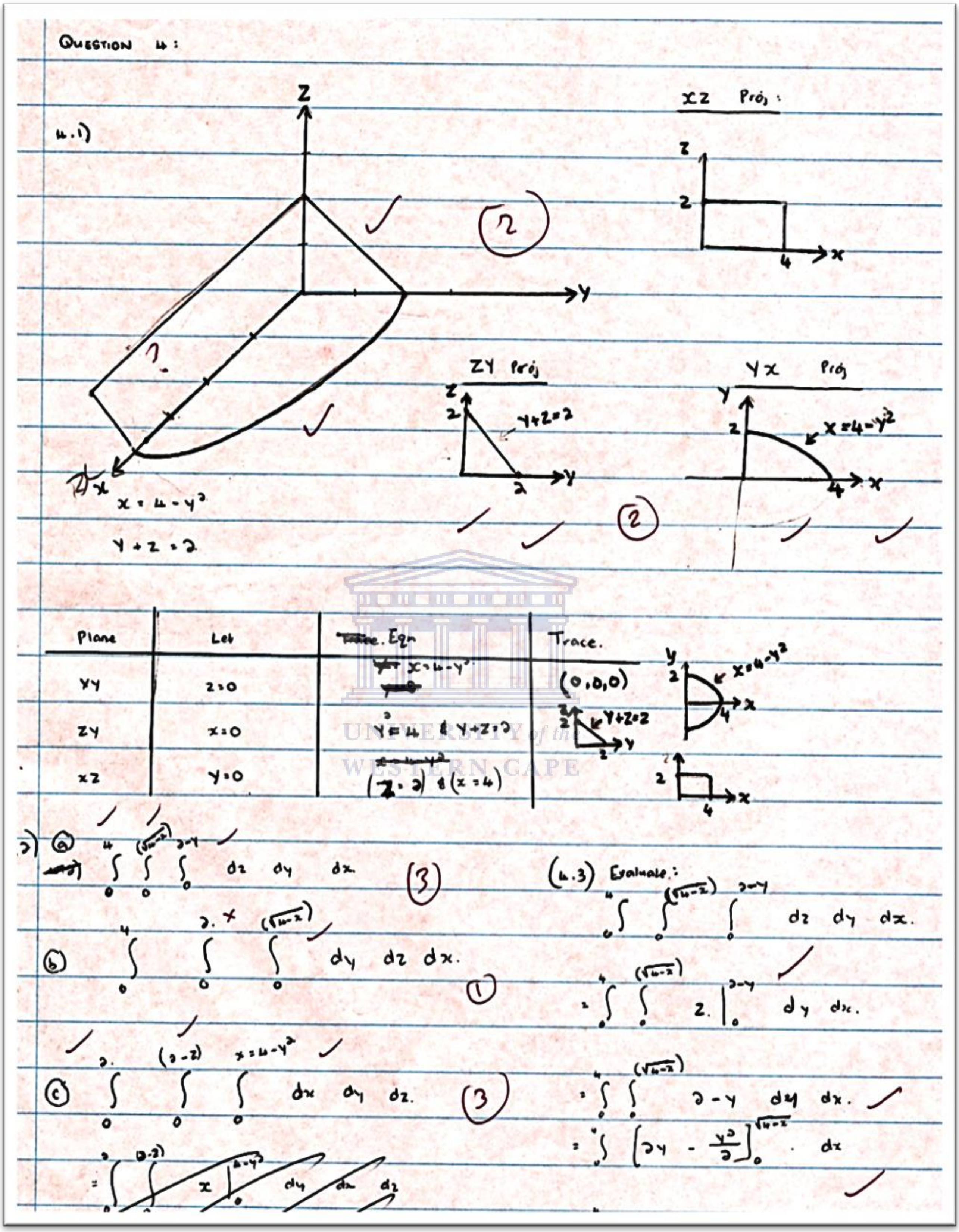


Figure 6.6 Solution to finding the volume bounded by $y + z = 2$ and $x = 4 - y^2$ in the first octant by ST5. Although the student has drawn the projections he/she is unable to visualise the intersection of $y + z = 2$ and $x = 4 - y^2$

Table 14 Visual Analytical steps in finding volume of solid bounded by $y + z = 2$, $x = 4 - y^2$ for ST4

Q 4.1 Volume of solid defined by $y + z = 2$, $x = 4 - y^2$ in the first octant. (See Figure 6.5)	
Visual Steps Plotting the projections V1: xy, projection Correct plot of $x = 4 - y^2$. V2: Plots xz projections using *yz* planes. V3: Plots *yz* projection using $y + z = 2$ V4: Assembles the projections into solid	**Analysis:** Conversion from algebraic to geometric registers. A1: For xy projection selects $x = 4 - y^2$. A2: For yz projection selects y + z = 2 A3. The xz projection at *x* =4 and z = 2
Q4.2 (a)Triple integral in the order dzdydx	
Visual steps V4: Vertical span from z = 0 to z = 2 - *y* V5 Uses *xy* projection, slicing up to find limits for *y* . Incorrect *y*-limits V6 Uses xy projection to find limits for *x*	**Analysis** **A1**: Records z limits from 0 to 2 - y A2 Records *y*-limits from 0 to 2. Error in this step A3: Records correct *x* limits : 0 to 4
Q4.2 (b) Triple integral in the order *dydzdx*	
Visual Steps V1: Spans across in y direction but stays only in the $y + z = 2$. Does not see that moving parallel to axis may also take him to $x = 4 - y^2$. V2 Working with xz projection. Avoids using $y + z = 2$ giving him incorrect limit for *z* V3 Working with *xz* projection: Correct limit for *x*: 0 to 4	Analysis A1: Does not see need for split integral A2: Reads incorrect upper limit for z from the *xz* projection
Q4.2 (c) Triple integral in the order dxdydz	Analysis
Visual steps: All three limits read correctly using the 3D solid and the yz projections	Reads and records correct limits Performs the first integration correctly.
4.3 Evaluating the integral	Appears confused. Not sure how to integrate (*z - y)dy*. Circles and cancels *zdy*. Incorrect final answer.

limits for the integral in the order *dydzdx*. Here, again we need to split the region of integration as the integral along *dy* is partly under $x = 4 - y^2$ and also under y = 2 – z, an extremely difficult visual step under a solid whose shape and form is unclear to ST5. The intersection of $y + z = 2$ and $x = 4 - y^2$ is $z = 2 - \sqrt{4-x}$. It forms the boundary between the two split integrals as shown in Figure 6.7. The final split integrals are:

$$\int_0^4 \int_0^{2-\sqrt{4-x}} \int_0^{\sqrt{4-x}} dy\ dz\ dx + \int_0^4 \int_{2-\sqrt{4-x}}^{2} \int_0^{\sqrt{4-x}} dy\ dz\ dx = 2{,}667 + 4 = 6{,}67$$

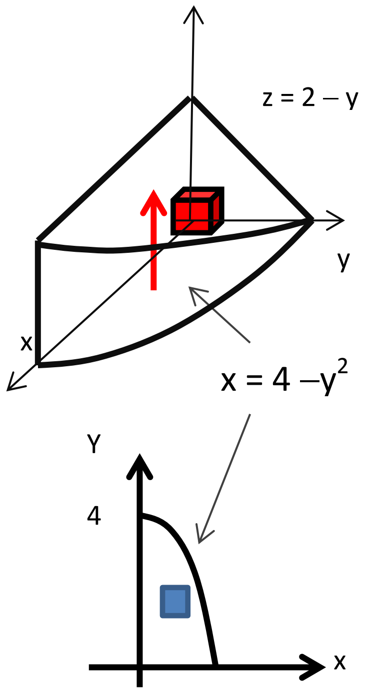
Both students missed to see the need for a split integral. Only 3 out of 21 students got this right. The integrals are not easy to do without Matlab.

Interview Excerpt 2 (student ST5)
ST5 was shown a copy of her solution (See Figure 6.6). .
INT: This is your solution to find the volume of the region in the first octant bounded by $y + z = 2$, $x = 4 - y^2$. Take us through the solution. What did you do?
ST5: I did the projections but I wasn't sure what this drawing was like. (*Points to 3D sketch. Figure 6.6*). I couldn't see the object.
INT: After you draw the projections, do you go to the projection or the 3D solid for the limits?
ST4: Sir, I go to the equations. I just check the equation with z in it and find z.
INT: You obviously couldn't use a 3D sketch that is incomplete. So you went to the equations.
ST4: Yes Sir. But if I have a 3D sketch, I start at the bottom (xy plane) and I go up.
INT: How did you get the limits for the order dydzdx? You got the first *y* limits right!
ST4: I used the xy projection. I had to turn $x = 4 - y^2$ round. That gave me $y = \sqrt{(4 - x)}$.
For the *y*-limits I used the ….I think something is wrong. We need more examples, like this in tuts (*tutorials*), where we need to split.
INT: Once you have done the inner *x* limits, you should go to the *yz* projections. So what would the *y*-limits be?
ST4: Looks in the *yz* projection. Going *y* first gives me the limits $y = 0$ to $y = 2 - z$.
INT: And the *z* limits.
ST4: Then *z* runs from z = 0 to z =2

Discussion: It is interesting to note from the interview that even when the students have drawn projections and a 3D sketch they are reluctant to use them. Students are more comfortable working with the equations to find limits of integration. We also note that selecting, moving and coordination between several projections is problematic. Several research studies, Habre (2002) and Trigueros (2004), for example, mention reluctance by students to work in the geometric register, and others eg Sweller (1999), have mentioned the cognitive overload that students face when they are dealing with visual representations. We note that the three dimensional solid in R^3 is represented by algebraic equations in two-dimensions, R^2. These need conversion to the geometric register and are then depicted by two-dimensional projections which are reassembled to give us the three dimensional solid. This is a complex task, a conversion between registers as well as two types of representations requiring the spatial orientation, rotation and translation of geometric figures using mental visualization. Finally, the solid must be represented by a triple integral in the algebraic register – a task that requires visualization, spanning and slicing within the 3D solid in order to arrive at the limits of integration.

On Duval's (1995, 1996) framework, the initial conversion from the algebraic to the geometric register was successful for ST5, following a chain of treatments and conversions, but we find that students have not moved beyond perceptual apprehension, where the surface features are recognised in the form of 2D representations that the students have drawn. There is evidence of sequential apprehension as students systematically follow the heuristic steps in the construction of the projections and the 3D objects, but operative apprehension is lacking as students abandon their drawings and go back to the equations to get their limits for the integrals. While this works for one or two orders of integration with inner variables dz, they run into problems with the other orders as the Riemann process of summing over the entire region of integration is incomplete. We note that there are 6 permutations of *dydxdz,* each requiring its own manoeuvres within the object to determine the limits, and the easier of these is integration in the order *dzdydx*.

We note that the region of integration may need to be split and visualization is necessary to keep track of the surfaces and their equations. Most students (75% in the Experimental, 90% in the Control groups) did not recognise the need for a split integral. In class demonstrations, students were shown that to find the innermost limits they should use the 3D solid they had drawn and span upwards for *dz*, parallel to the *y* axis for *dy* and parallel to the *x* axis for the inner *x*-limits. Figure 6.7 shows the spanning process for $\iiint dzdydx$ and for $\iiint dydzdx$. Moving in the *y*-direction takes us to different surfaces and hence the need for splitting the integral. After spanning in the 3D sketch we move to the xz-projections and use these for the remaining limits ie, the dzdx limits. Needless to say, if students cannot split a planar region of integration in R^2 for double integrals, they are going to struggle with splitting volumes in R^3 for triple integrals. Figure 6.8 shows the slicing and stacking for *dxdydz* . We only need a single triple integral to span the volume. In conclusion, inability to visualize the Riemann process makes it difficult to move between the geometrical and algebraic registers as students cannot identify and transfer limits from their 3D sketch to the triple integrals.

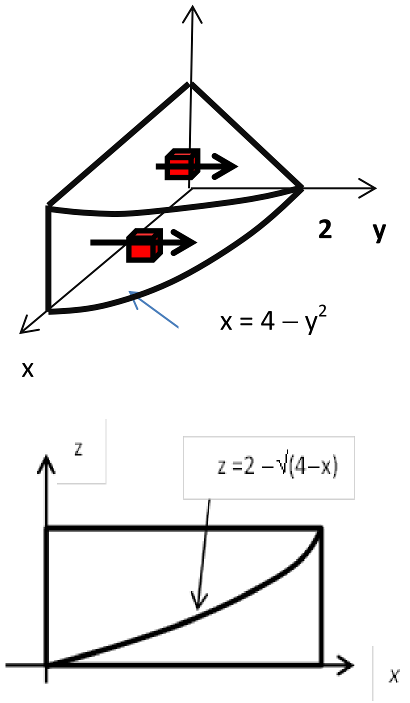


(a) To find the volume enclosed by $x = 4 - y^2$ and $y + z = 2$ in the first octant in the order $\int\int\int dz\, dx\, dy$ we only need a single triple integral as the inner z always runs from $z = 0$ to $z = 2 - y$. The *dxdy* limits come from the *xy* projections.

$$\int_0^2 \int_0^{4-y^2} \int_0^{2-y} dz\, dx\, dy = \frac{20}{3} \qquad \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\, dy\, dx = \frac{20}{3}$$

Check in Matlab:

```
syms x y z f
f = 1
int(int(int(f,z,0,2-y),y,0,sqrt(4-x))
,x,0,4)   =  20/3
```



b)The volume integral in the order $\int\int\int dy\, dx\, dz$ needs to split into two, an upper and lower integral. The *dxdz* limits are taken from the *xz* projection.

i) Top inner integral for *dy* runs from $y = 0$ to $y = 2 - z$.

$$\int_0^2 \int_0^{4-(2-z)^2} \int_0^{2-z} dy\, dx\, dz = 4$$

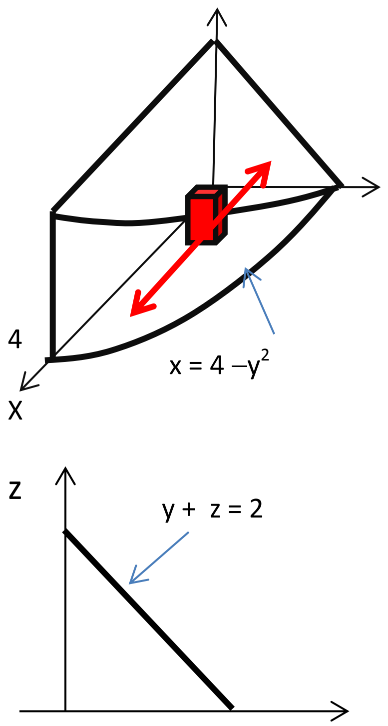
ii) Lower volume integral for *dy* runs from y = 0 to y = sqrt (4 – x).

$$\int_0^2 \int_{4-(2-z)^2}^{4} \int_0^{\sqrt{4-x}} dy\, dx\, dz = \frac{8}{3}$$

The lower integral (ii) is too complicated to evaluate. So you may need to turn dxdy around

```
int(int(int(f,y,0,2-z),x,0,4-(2-
z)^2),z,0,2)   = 4
int(int(f,y,0,sqrt(4-x)),z,0,2-
sqrt(4-x)),x,0,4)    =  8/3
```

Figure 6.7 Visualizing the slicing and stacking for a) dzdxdy b) dydxdz



The volume integral $\int\int\int dx\,dy\,dz$ is the easier of the

integrals to set up. We only need a single triple integral as the inner z always runs from $x = 0$ to $x = 4 - y^2$. The dydz limits are taken from the yz projections.

$$\int_0^2 \int_0^{2-z} \int_0^{4-y^2} dx\,dy\,dz = \frac{20}{3}$$

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\,dz\,dy = \frac{20}{3}$$

Check in Matlab:

```
int(int(int(f,x,0,4-y^2),y,0,2-z),z,0,2)
=20/3
int(int(int(f,x,0,4-y^2),z,0,2-y),y,0,2)
=20/3
```

Figure 6.8 Visualizing the slicing and stacking for dxdydz

In the end of term test (T3) the question involving triple integrals in rectangular coordinates asked students to find the volume of the region in the first quadrant enclosed by the surfaces represented by the equations: $z = 4 - y^2$ and $x + y = 3$. Only 6 students out of 48 (about 13%) could sketch the 3D solid. The problems with identifying the intersection and recognising when to split the integral persisted throughout the semester.

Figure 6.9 shows the attempt by ST5 was nearly successful. ST5 has sketched the 3D solid and recognized the need to split the integral. Earlier in Test 2, ST5 struggled with the visualization and could not find the intersection of the surfaces (see Figure 6.6). However, she realized the need for a split integral and with the exception of the sign error in the limits for y most of the solution is correct.

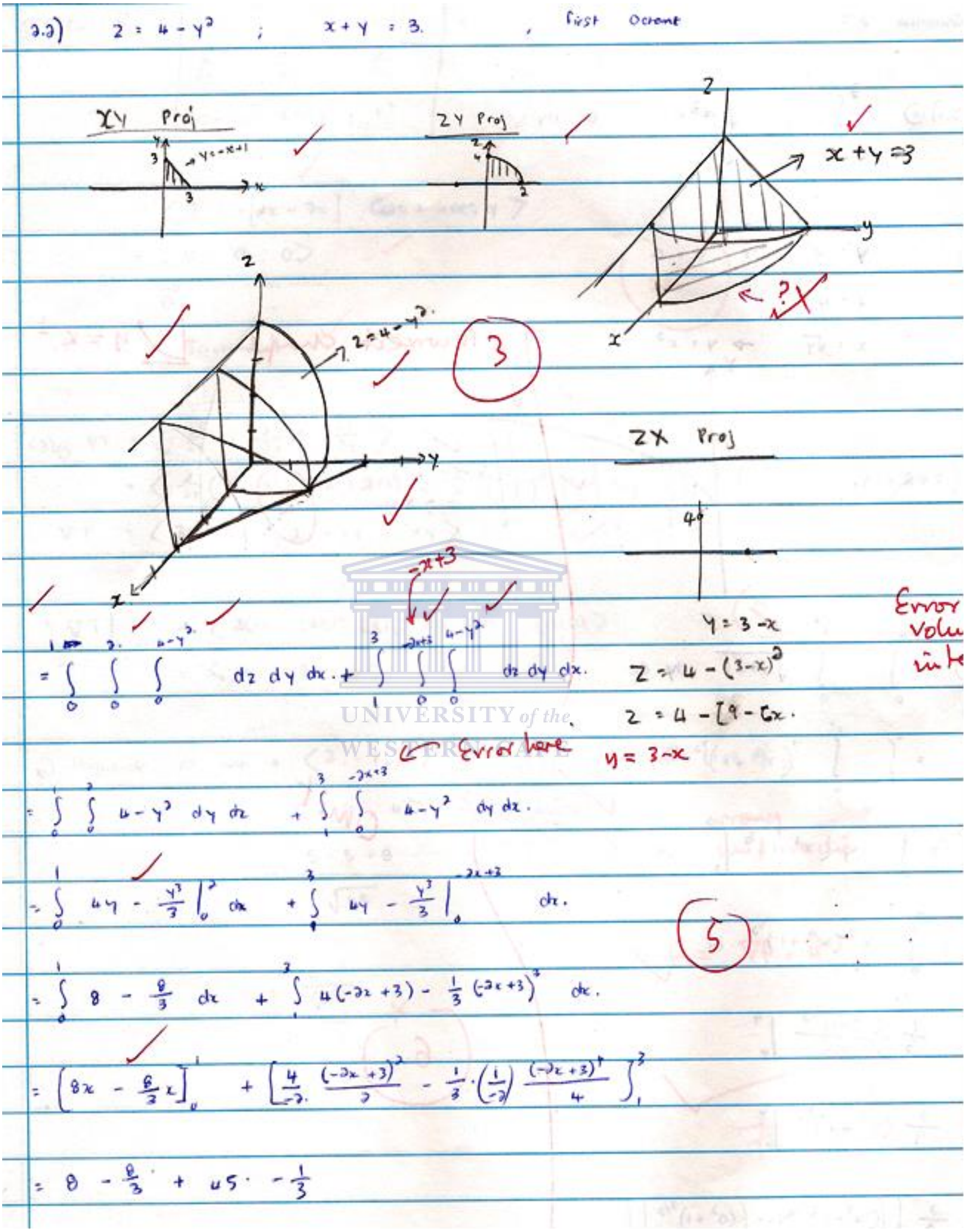


Figure 6.9 ST5’s solution to finding volume enclosed by : z = 4 – y^2 and $x + y = 3$

6.7 Volume integrals in spherical and cylindrical coordinates

This question asked students to calculate the volume of the region bound by the cone

$z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 6$. In answer to this question, ST6, realised the need to change the coordinate system: We analyse the student ST6's solution, presented in Figure 6.10, using the V-A framework (See Table 15).

Table 15. Visual Analytical steps in finding volume of solid bound by $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 6$ by ST6

Q 4.1 Volume of solid defined by $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 6$. See Figure 6.10	
Analysis Works from a table of traces: A1: For xy projection, lets z = 0, gets point (0;0). A2: For yz projection., lets x = 0, gets trace z = y A3. For the xz projection lets y = 0 gets z = x A4: Calculates radius of circle but makes an error, writes r = 3 instead of r = √3	**Visual Steps** V1: Correct 3D representation of solid. V2 : Clearly labelled diagram. V3: Plots trace in the xy plane. Circle but incorrect radius. V4: Plots incomplete yz and xz projections V5: Correct sketch of solid in 3D
For integral in cylindrical coordinates: A5: Enters correct limits for z: A6: Enters correct limits for r A7: Enters correct limits for θ A8. Enters the Jacobian $dV = rdzdrd\theta$	V6: Reads correct limits for z: V7: Reads correct limits for r V8: Reads correct limits for θ V9. Reads the Jacobian $dV = rdzdrd\theta$ V10: All limits read from 3D representation
A9: For spherical coordinates: A10: Calculates ρ, the radius of sphere A11: Calculates ϕ, A12: Checks θ = 2π A13: Correct triple integral in spherical coordinates and Jacobian.	V11: Refers to 3D representation for ρ, ϕ, θ

ST6 has used projections to sketch the 3D solid. See Figure 6.10. Some traces were not necessary. ST6 recognises the circular intersection and calculates its radius r = √3 correctly. The projections in the *xz* and *yz* planes are incomplete (missing the spherical top), both integrals in the cylindrical and the spherical coordinate systems are correct. The only error is in the limits for the inner integral, dz, in cylindrical coordinates, which should run from the slanting edge of the cone $z = r$ to the sphere $z = \sqrt{6 - r^2}$. Several treatments within registers were necessary and ST6 has shown the working for ϕ, θ and ρ clearly. The limits for the spherical coordinates integrals were determined using algebraic equations for conversions between the coordinate systems. The same comment applies to limits for ϕ and ρ. Overall, treatments and conversions were efficiently coordinated and performed, and the triple integrals in cylindrical and spherical coordinates were established with correct limits.

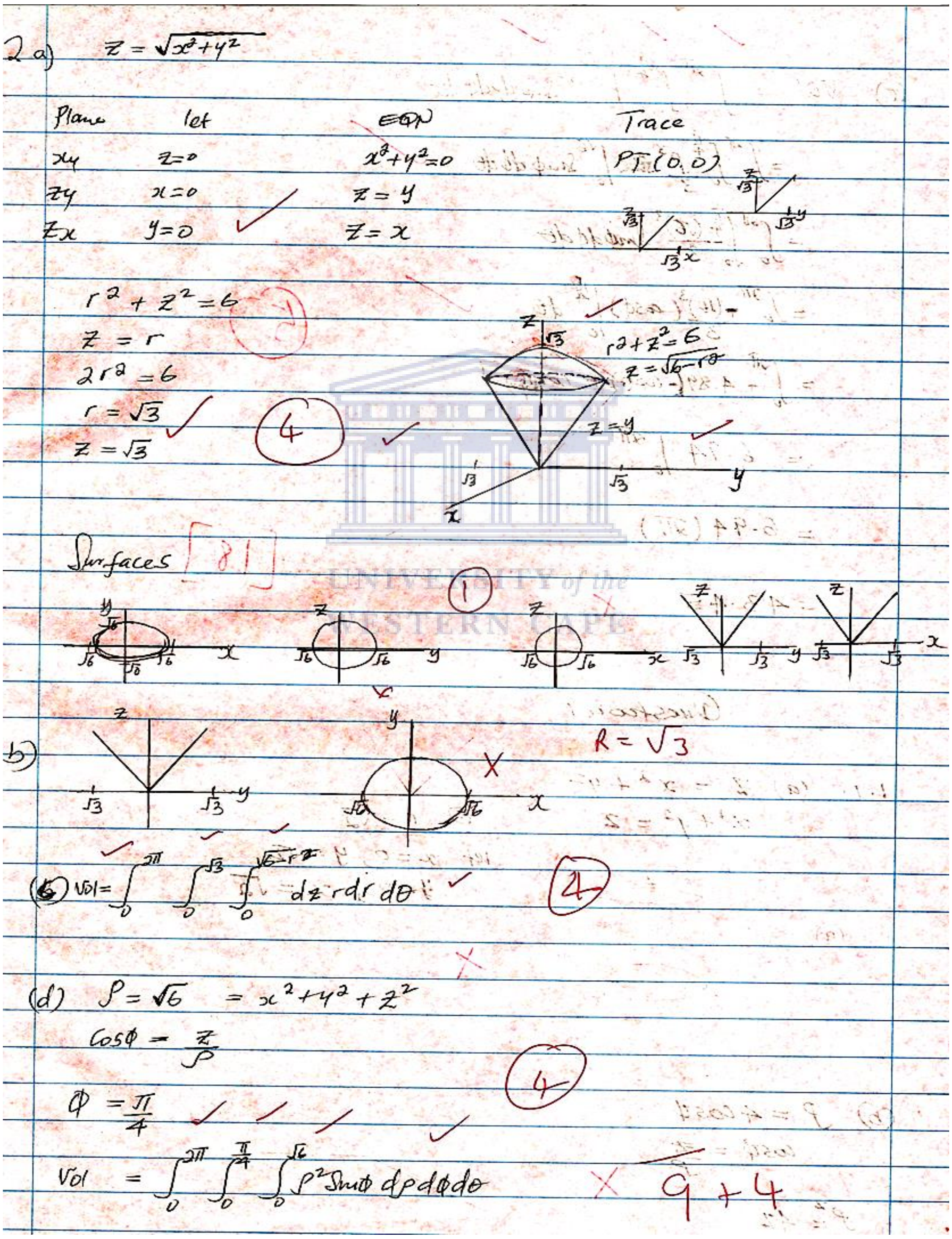


Figure 6.10 Solution by ST6 to find the volume of solid defined by $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 6$.

6.8 Visualizing direction fields

Figure 6.11 shows the analytical solution and the direction field presented by ST7 for the ordinary differential equation $\frac{dy}{dt} = y - t$.

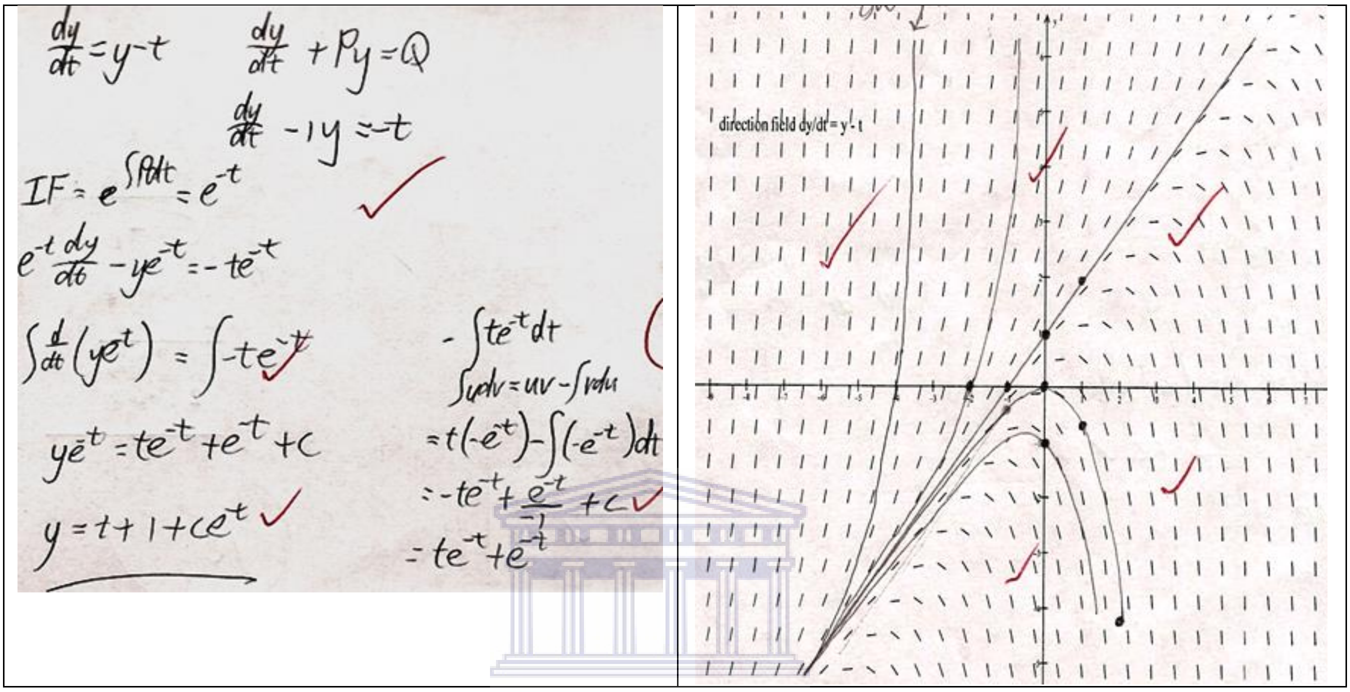


Figure 6.11 a) Analytical solution for d*y*/dt = *y* – t and b) the direction field with trajectories through (–1;0); (0;0) and (0 ;–1) by ST7

Table 16. Visual analytical steps in the solution by ST7 for the DE d*y*/dt = *y* – t (See Figure 6.11).

Analytical solution	**Visual steps**
A1: Recognises the equation as a linear DE. Recalls the general form: $\frac{dy}{dt} + py = Q$. A2: In the algebraic register. Rearranges the DE to fit the general form. A3: Algebraic register: Finds the Integrating factor, *IF*: Multiplies both sides of the DE b*y* the IF. Collapses the left hand side into a single expression and integrates. A4: Uses integration by parts on the Right hand side. A5: Divides both sides by e^{-t} and writes down the general solution, $y = t + 1 + ce^{-t}$. A perfect analytical solution without any errors.	On the direction field which was supplied: V1: Draws solution curves through initial values (0;0), (0;–1) and (0 ; 1). V2: To find the solution through (0 ; 0) he finds c b*y* substitution and writes *y* = t +1. V3: Verifies on the direction field that this is correct.

Discussion:

ST7 has shown operational apprehension of the analytical solution process, drawing on prior knowledge and resources like integration, differentiation and sketching graphs where necessary. ST7 moves between the algebraic and the geometric registers easily. On the other hand, the direction field presented by ST8 in Figure 6.12(a) is far from satisfactory. The solution curves do not follow the slope segments. ST8 is aware that the solution curves cannot cross and has gone out of the way to prevent that from happening in the second and third quadrants. The correct solution with $c = 1$ is $y = t + 1 + e^t$ shown in Fig 6.12(b). Solution for c = −2 is also incorrect.

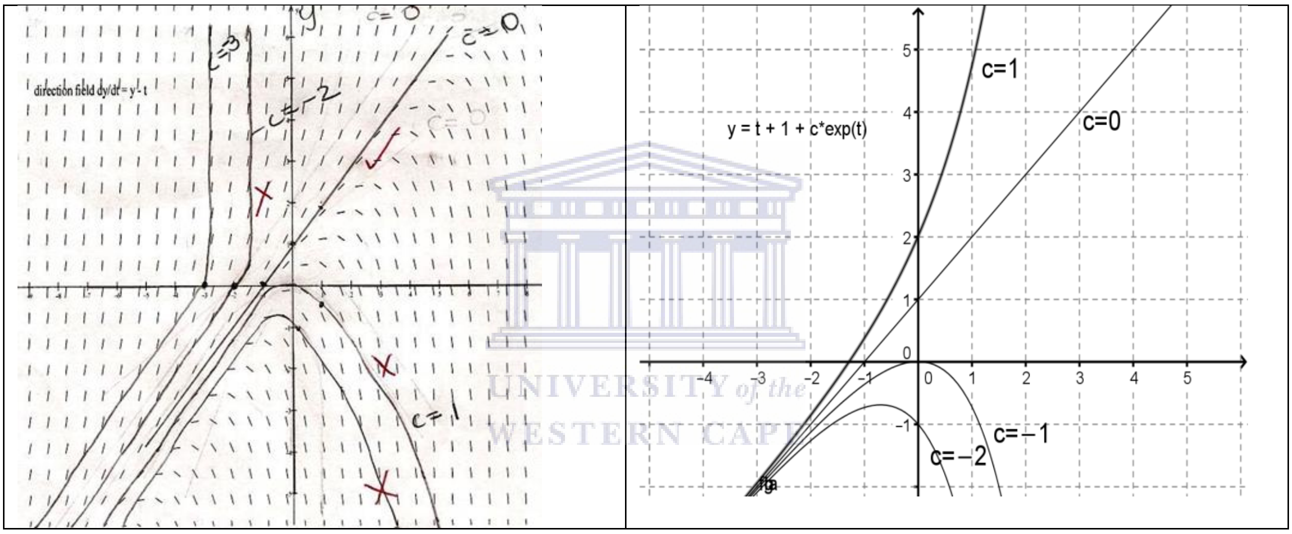


Figure 6.12 (a) Direction field for dy/dt= y –t with solution curves by ST8 and (b) a plot of the correct solutions

6.9 Visualizing phase portraits

In this section, we look at the difficulties students have in visualising and sketching solutions to dynamical systems given by the set of non-linear differential equations: $x' = x(2 - y)$ and $y' = y(x - 3)$. This is a set of predator prey Lotka-Volterra (LV) equations. Students were asked to sketch the phase portrait and describe the long term behaviour of the system.

In lectures, two methods were presented: a qualitative method using slopes to plot the phase portrait near the equilibrium points and an analytical method using eigen-values to identify the stability of the equilibrium points.

The work presented by ST9 is shown in Figures 6.12 and 6.13. The eigen-pair calculations for the other equilibrium points are similar and have not been shown in Figure 6.12

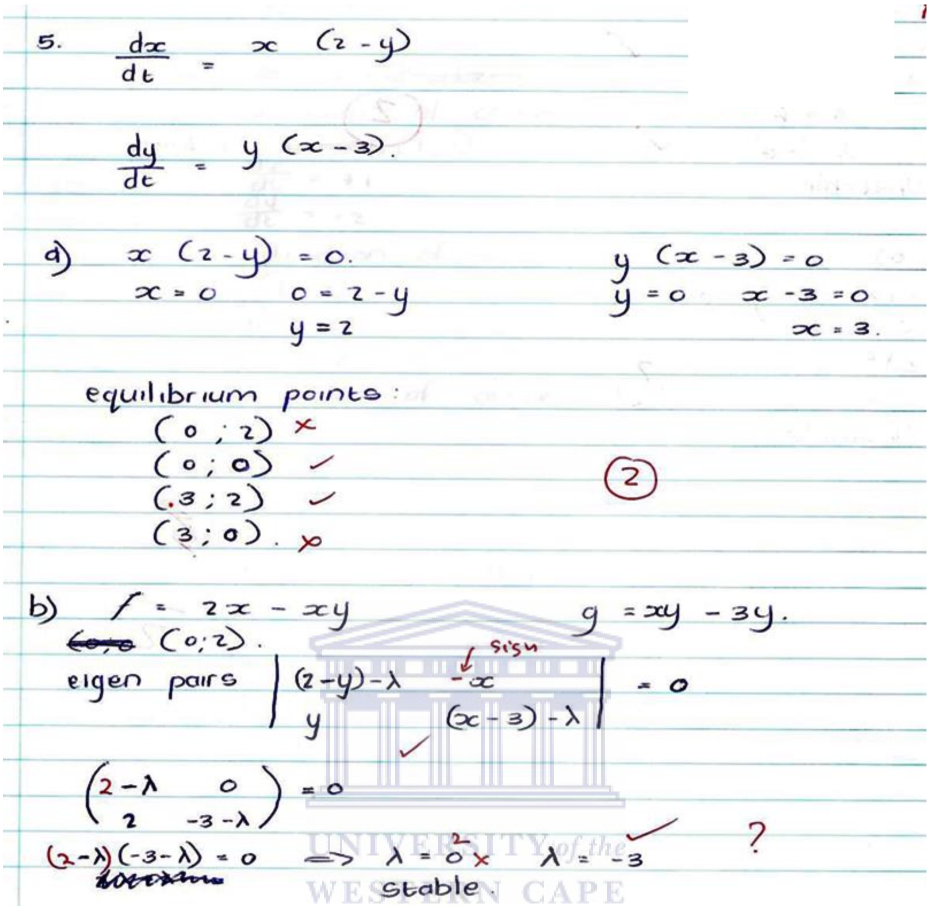


Figure 6.13 Solution to the LV system $x' = x(2-y)$; $y' = y(x-3)$ by ST9

Table 17. Visual Analytical steps in solution of $x' = x(2-y)$; $y' = y(x-3)$.

Q5: Phase portrait and Analytical solution to $x' = x(2-y)$; $y' = y(x-3)$. See Figure 6.13	
Analytical Steps	**Visual steps**
A1:Starts by finding the equilibrium points. The points (0 ; 2) and (3 ; 0) do not satisfy the equations simultaneously and cannot be equilibrium points	V1: Uses vector addition to find the resultants. For example at (2 ; 1) the slopes are x'=2 and y'= −1. The resultant vector diagram gives the direction of the trajectory at (2 ; 1). This is clearly shown on the phase portrait.
A2 Finds the Jacobian matrix and the eigen-value pairs.	V2. In the qualitative solution, appears to have corrected the error in the equilibrium points as (0 ; 2) and (3 ; 0) are not shown on the phase portrait.
A3: Finds the partial derivatives. f_y introduces another error with the consequence that the eigen-pairs have an error.	V3: Draws the phase portrait
A3: Calculates the slopes at 4 points. Fig 6.13	

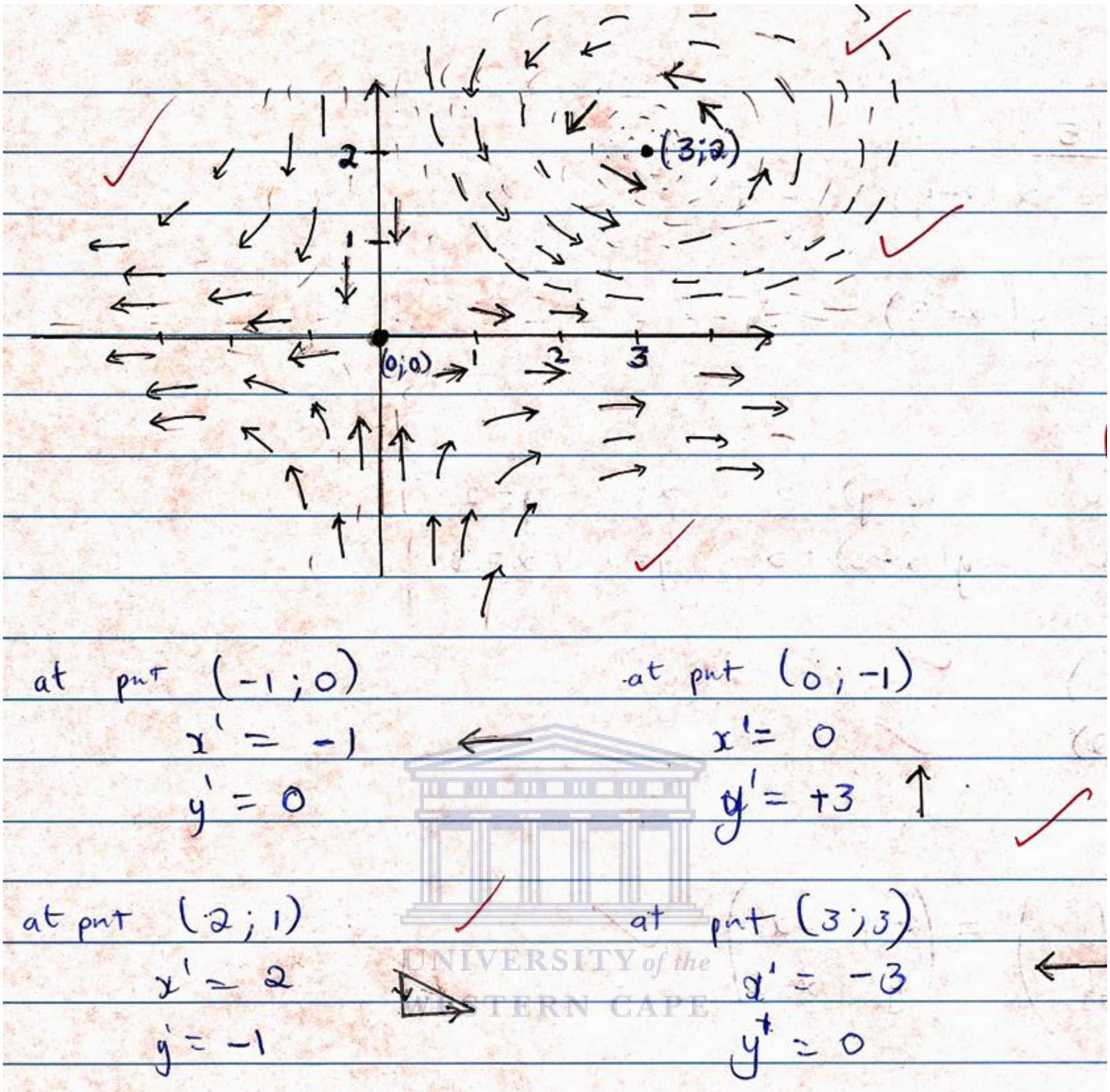


Figure 6.14 Hand drawn phase portrait for $x' = x(2-y)\ ;\ y' = y(x-3)$. by ST9

Discussion

We note that ST9, worked entirely in the algebraic register using the analytical method to find the equilibrium points and the eigen-pair values at each point. Errors were made in finding he equilibrium points and calculating eigenvalues. In the graphical register, ST9 used slopes at various points to sketch the phase portrait. Calculation errors in the analytical solutions were corrected in the phase portrait. (See Figure 6.14). Overall, the student has a good grasp of the analytical method and has visualised the phase portrait correctly. In fact, the phase portrait served as a check on the analytical calculations, as the 'extra' equilibrium points are not shown.

Figures 6.15, shows ST10's interpretation of the Predator-Prey phase portrait. The species involved are shark (Predators) and fish (prey). The student has made an error in the eigenvalue calculations giving a spiral source at (0, 0). In the interpretation, the student was not specific as to which section of the phase portrait shows growth of prey or predators. The student refers to the populations oscillating without reference to the diagram or to the populations of predators and prey. There is no indication of cohabitation populations and numbers.

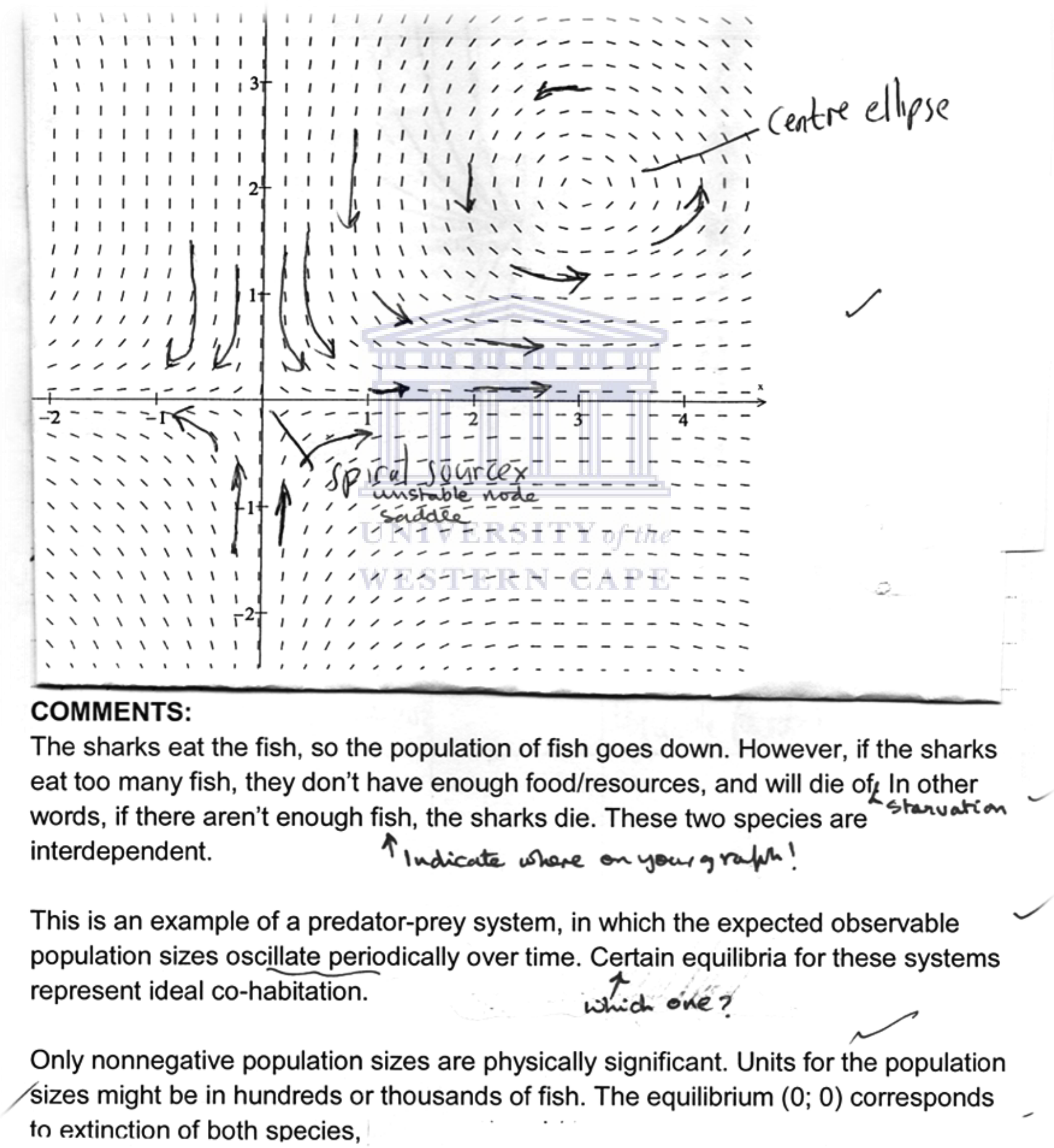


Figure 6.15   Student, ST10's interpretation of the Lotka-Volterra phase portrait.

The student ST10 was interviewed and the interview excerpt follows. ST10 was shown the marked solution and the system of equations: $x' = x(2 - y)\ ;\ y' = y(x - 3)$.

Interview Excerpts 3 (Student ST10)

INT: Look at the phase portrait that you drew. We started with two equations. The first, $dx/dt = 2x - xy$ is the equation for prey, (fish in thousands) and the second $dy/dt = xy - 3y$ is the equation for predators, (sharks in hundreds). The usual assumptions apply i.e Sharks depend only on fish. Fish have unlimited food supply. There are no other threats to both. You found the equilibrium points (0 , 0) and (3 , 2). Can you take us through what you did to draw the phase portrait.

ST9: What did I do? I took points in the plane like (1, 1) and found the slopes dx/dt =1 and dy/dt =–2 by substitution. I used the slopes to draw vectors in the x and y directions and I found the resultant. I drew the resultants on the phase portrait at each point. (Illustrates by drawing → ↓ and resultant ↘).

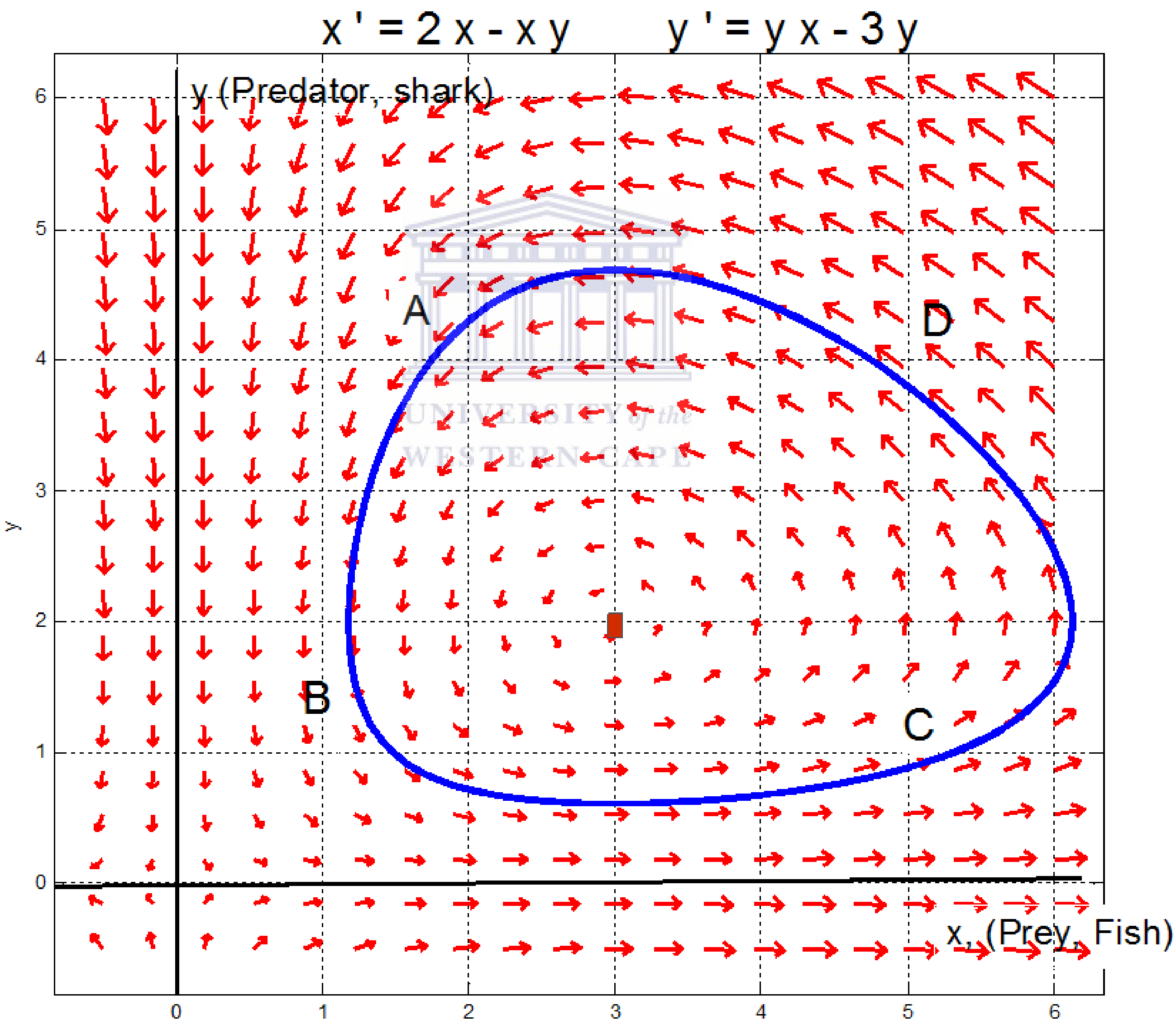


Figure 6.16 Phase portrait for Fish (in '000s) and sharks (in '100s)

INT: Take the phase portrait you have drawn (Figure 6.15). I have done another sketch for you with labels A,B, C and D. (INT shows Figure 6.16). I want you to go along the solution curve (the blue curve) and tell me what is happening. Let's follow this trajectory along AB. What is happening? Think in terms of number of fish, x and number of sharks, y.

ST10: Along AB, shark population is dropping. Fish population also drops a little.

INT: Why would shark population drop?

ST10: There are few fish around. No food for sharks.

INT: What is happening at B?

ST10: Yes fish population in the cycle is smallest along AB. Near B shark population has also dropped to low numbers.

INT: So what is the effect on fish?

ST10: Population of fish starts to grow. Shark numbers are constant. Plenty of fish near C

INT: Nearly constant. Carry on, what is happening between CD?

ST10: Plenty of fish . Shark have more food. Shark population is growing. (recovering)

INT: In your answer you mention oscillations. Between what values do shark numbers oscillate?

ST10: *Reads figures from Figure 6.15 by extrapolating to the y-axis*. Between 0.8 and 4.5.

INT: Remember the axes are scaled 1: 100 for sharks and 1: 1000 for fish.

ST10: Between 80 and 4500

INT: And the fish?

ST10. Looks along the x-axis. Between 1200 and 6000.

INT: You also mention cohabitation. You said certain equilibria are ideal for cohabitation. On this phase portrait which equilibrium point is that?

ST10: There is (0 ,0) but at (0 , 0) both populations have died. Then there is (3, 2). So 3000 shark and 200 fish.

INT: Look at the vector arrows (trajectories) around (0 ; 0). Are there arrows (trajectories) going into (0, 0)?

ST10: *Puzzled*. No arrows enter (0, 0).

INT: So how can you say the shark and fish die?

Draws sketch graphs labelled Fish and shark Vs time. See Figure.6.16.

INT: This graph shows how the fish and shark populations change with time. What is happening to the fish population?

ST10: Fish population is small to begin with, increases and then drops back to same values.

INT : So which points on the phase portrait (*Points at Figure. 6.15*) correspond to points on the time graph in. Figure 6.16?

ST10: Mmm… I really can't see. How we get this graph from the phase portrait. There is no time on the phase portrait. Also what is the time scale?

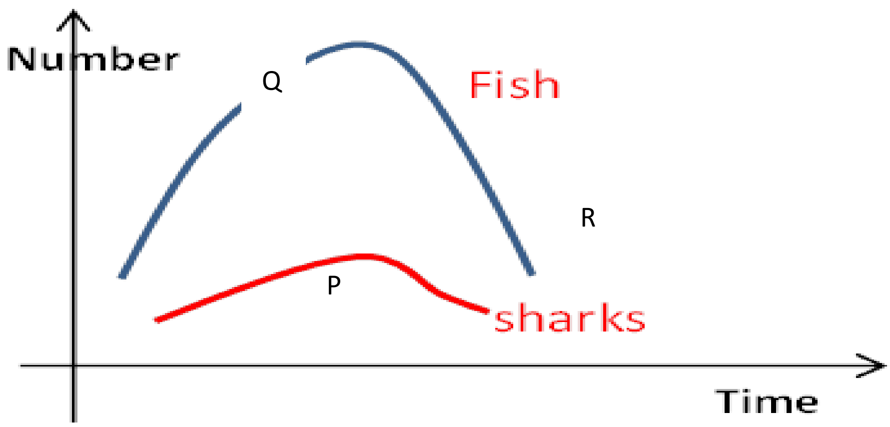


Figure 6.17 Variation in predator prey numbers Vs time

INT: Both graphs are showing one cycle. On the phase portrait, when are the fish numbers least?
ST10: During AB. Least at B. So AB must correspond to these points (*Indicates R*).
INT: And the sharks. Which points correspond? When is shark population least?
ST10: Midway between B and C.
INT: And on the time graph?
ST10: Shows bottom of sharks graph near R.
INT: And the peaks for fish and Shark?
ST110: Peak for fish is between C and D. Peaks for shark is the same.
INT: Are you sure?
ST10: Yes
INT: Let us look at the predator-time and the prey–time graphs more closely. *Points to Figure 6.18*. The predator is in green and the scale is 1:100 while the prey graph is in red on a scale of 1:1000.. What point on the phase-portrait (Figure 6.16) corresponds to P on this graph?
ST10: Ok We comparing the blue graph and the red graph. *Points to A on the phase portrait. A is near the peak.*
INT: Are you sure. Check again? What about Q? Which points correspond?
ST10: Q and B.
INT: How long in time is a cycle?
ST10: A cycle is from P to R. About 11 minus 4 = 7 years
INT: At P the prey numbers are high. What happens to predators?
ST10: Predator population is dropping.
INT. Look again.
END

Several points are noteworthy.

1. The student ST9 (See Figure 6. 13), found more equilibrium points than exist. This could have been checked by substitution. ST9 could find the slopes in *x*- and *y*-directions. Plotting the resultants was arbitrary. Sketching the trajectories was messy.(See Figure 6.14). With software readily available, not many students see the point of the exercise and often express annoyance at having to do this manually.

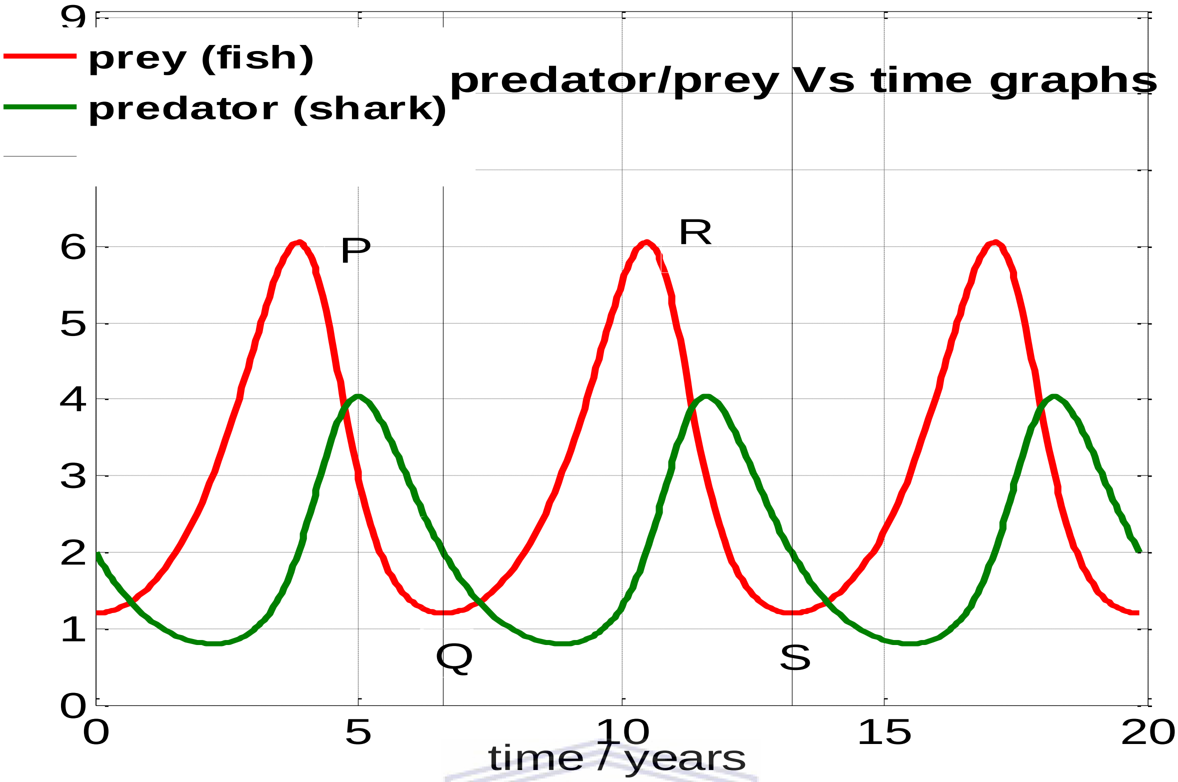


Figure 6.18 Predator and prey vs time graphs for the fish and shark dynamic system

2. Interpreting and transforming the phase portrait was not easy. The need to constantly switch between the axes of two variables (shark and fish) was a challenge. The student's interpretation of the equilibrium point (0 , 0) was ambiguous and was fixed on extinction and cohabitation. The direction of trajectories around (0 , 0) was misinterpreted. On Duval's framework discursive apprehension was lacking. The description of events around the blue curve and the description of the equilibrium points is vague.
3. Converting the phase portrait to the time graph was conceptually challenging. What would have helped is to make the transformation through a numerical register. We found that many number (13 out of 21 , 63%) students have difficulty extracting information from these representations. Time is a hidden parameter and the connections between the phase portrait and the time graphs were hard to see. Getting students to relate the rates of change in the interacting species is conceptually challenging.
4. The different scales used in the graph for fish and shark added to the confusion.

6.10 Comparison between treatment groups by questions

As we have seen in section 6.4, the results of the ANOVA showed that there were significant differences on test 3 between the Treatments (experimental and control) but not on gender (Male Vs female) or the treatment-gender interactions. The differences between the achievement scores of the groups in Tests 1 and Test 2 were not significant. Here, we investigate if there are differences by type of questions. The three types of questions we addressed in integration were

a. single integrals
b. double integrals and
c. triple integrals.

Table 18 shows the percentage of students categorised by groups who had correct answers for each type of questions:

Table 18 Percentage of correct responses by type of question and groups

Integral	**Experimental n=24**	**Control n =26**
1. single	75%	73%
2 Double a) *dydx* b) limits for *dxdy*	91% (no split) 23%(split) 63 (no split) 6% (split)	88% (no split) 0 %split 56% 18%
3.Triple Integrals a) Projections + sketch	61,7%	46,1%
b)limits for *dzdydx* (split)	25%	11,1%
c)limits for *dydzdx* (no split)	52%	32%
d)limits for *dxdydz* (no split)	75%	58,3%

We observe that there were no differences between the groups on a) single integrals and b) double integrals where the integral did not need splitting. There are significant differences between the control and experimental groups on questions requiring the splitting of double or triple integrals. Overall students in the experimental group did significantly better on the triple integral questions with better performance (61.7 % vs 46.1%) on the projections and split integrals (25% Vs 11.1%).

6. 11 Factors impacting on visualization - Regression analysis

As we reported in Chapter 3, several research studies (Hegarty, 1999; Battista,1990, McGee, 1979) have reported that the spatial ability factor was one of the main factors significantly affecting mathematical performance. Kaufmann (1990) noted that the correlation increases with the complexity of the mathematical task. Among the factors identified are previous experiences with 3D objects, prior knowledge, attitudes, motivation, social and cultural factors. Rather than meander along on an unfocussed path, testing all possible variables, for this research, we decided to narrow the focus to a subset of the factors identified by Kosslyn (1995). These do not exhaust the whole spectrum that one encounters in the literature, but we argue that it sufficiently captures the rationale of the research which deals specifically with multiple integrals and solutions to systems of differential equations. For an overview, see Hegarty and Waller (2005). The variables we identified were the pretest scores (Prior knowledge), surface features of 3D objects, rotations, cross-sections, projections, and nets of 3D objects as independent variables. These were variables that we could reinforce through the activities in the computer laboratory sessions and, that we could measure with reliable tests. Next we performed a multiple regression analysis using Test 2 marks (on multiple integration and dynamic systems) as the dependent variable and scores on surface features (SURF), projections (PROJ), nets (NETS), cross-sections (XSECT), rotations (ROTN) and prior knowledge scores (PRIOR) as the independent variables. See chapter 3, page 69, for a description of each of the variables.

Test items for each of the variables were selected from standard tests used in research. See page 70 for the sample test items. For example, the test items on rotation were obtained from the Alaskan Rotation Test inventory.

The proposed multiple regression model was:

$$\hat{y} = \beta o + \beta 1 * SURF + \beta 2 * PROJ + \beta 3 * NETS + \beta 4 * XSECT + \beta 5 * ROTN + \beta 6 * PRIOR$$

where $\hat{y}$ is the Test 2 mark (Multiple Integration and Dynamic systems) , the variables SURF, NETS etc. are as defined in the table on page and the β_1, β_2 ...βs are the multiple regression coefficients.

The null and alternative hypotheses for the model are:

Ho: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$

H_1: One or more of the parameters $\beta \neq 0$. At least one independent variable is a significant predictor of mathematics achievement.

The following assumptions were validated:

a) The relationship between the dependent variable (BT2 score) and the independent variables is linear. See Appendix 5.4. Table of Results from SAS.
b) The independent variables are not overly correlated (see correlation table, Table 19).
c) The variances are equal. (Variance inflation factor less than 5 (See Table 20).
d) The variables are normally distributed. (See graphs in Appendix 5.4, page 212).

The correlation matrix reveals that except for the pretest scores, all the independent variables have a correlation coefficient below 0.7.

Correlation and multiple regression analyses were conducted to examine the relationship between achievement and the potential predictor factors. Tables 19 and 20 summarize the descriptive statistics and regression analysis results obtained from SAS. A detailed output with graphs is attached in Appendix 5.

The findings:

1) Overall the model is valid and there is a significant relation between the dependent variable (maths achievement scores on multiple integration and dynamical systems) and the dependent variables at 0.05 sig level, $F = 9.67$. The six predictor model was able to account for 72,2% of

the variance in maths achievement, $F(6, 14) = 9{,}67$, $p < .001$, $R^2 = 0.806$, 90% CI [0.35,0 .72].

Table 19. Means, S.D. and correlation coefficients for the six predictor variables

	Surface	*Proj*	*Nets*	*xsect*	*Rotns*	*Pretest*	*BT2*
Surface	1						
Proj	–0.141	1.000					
Nets	0.473	0.109	1.000				
xsect	0.270	0.071	0.387	1.000			
Rotns	0.518	–0.029	0.575	0.516	1.000		
Pretest	0.437	0.213	0.459	0.677	0.416	1.000	
BT2	0.653	–0.058	0.538	0.693	0.535	0.781	1
Mean	68.57	57.52	51.86	59.90	54.90	54.86	58.38
SD	26.00	17.03	21.57	10.37	22.70	14.62	17.18

Table 20 Output of the Results of the Multiple Regression analysis in SAS.

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	6	4753.565	792.261	9.67	0.0003
Error	14	1147.387	81.956		
Corrected Total	20	5900.952			

Root MSE	9.05297	**R-Square**	0.8056
Dependent Mean	58.38095	**Adj R-Sq**	0.7222
Coeff Var	15.50672		

Parameter Estimates						
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**	**Variance Inflation**
Intercept	1	–9.11499	14.45827	–0.63	0.5386	0
Surface	1	0.22226	0.10262	2.17	0.0481	1.73784
Proj	1	–0.13638	0.12826	–1.06	0.3056	1.16443
Nets	1	0.08021	0.12320	0.65	0.5255	1.72356
xsect	1	0.49775	0.29200	1.70	0.1103	2.23864
Rotns	1	–0.02538	0.12655	–0.20	0.8439	2.01359
Pretest	1	0.50161	0.21530	2.33	0.0353	2.41776

2. The model passed the heteroscedasticity (constant variance) test. The residuals do not really look normally distributed, but linear regression models are not very sensitive to violations of the normality assumption unless the sample size is extremely small. Also multi-collinearity does not seem to be a problem since all the variance inflation factors are less than 5.

3. The adjusted R-square value of 0.722 tells us that 72.2% of the variation in the scores on multiple integration and dynamical systems is accounted for by the independent variables.

4. The regression analysis shows that prior knowledge (Pretest) had the largest influence in predicting the achievement scores.

5. The pretest (Prior knowledge) and surface feature scores had significant positive regression weights, indicating students with higher scores on these variables were expected to have higher achievement scores on Multiple integrals and Dynamical Systems after controlling for the other variables in the model. The projections, and rotations have negative regression weights, indicating that after accounting for Prior knowledge and surface features, those students with higher achievement scores in Proj, Nets and Rotation were expected to have lower scores in Multiple integration and Dynamical Systems (a suppressor effect).

6. A step wise regression analysis with inclusion exclusion significance levels set at 0.15 confirmed that pretest, surface features, as well as cross-sections are significant predictors. Note the *p* level is different from the main regression analysis.

7. Based on the regression results the best model for predicting achievement on multiple integration appears to be: $\hat{Y} = -9.115 + 0.222 * SURF + 0.498 * XSECT + 0.5016 * PRIOR$

8. The regression analysis was repeated without the Pretest scores. We find the model as a whole is significant , $F(5, 15) = 8{,}812$, $p < 0.0007$, adj $R^2 = 0{,}73$ with surface features ($p < 0{,}013$) and cross-sections ($p < 0{,}0033$) as significant predictors.

6.12 Chapter summary

In this chapter we reported the findings of the analysis of the data from the teaching experiment. We successfully applied Duval's (1995, 1996) semiotic representation theory and Zazkis's (1996) VA frameworks to analyse students' solutions. Overall we found no differences between the achievement of students in the control and the experimental groups on test 1 and 2. However a question by question analysis showed improved performance by the experimental group, which also performed significantly better on Test 3. The main problems experienced in dynamical systems were the interpretation of the phase portraits to predict the long term behaviour of the solutions. A multiple regression analysis was performed utilizing mathematics achievement as the dependent variable and six predictor variables. The regression analysis was found to be statistically significant $F(20) = 9{,}67$, $p < .001$. The multiple regression accounted for 72,2% of the variability, as indexed by the adjusted R^2 statistic. At $p = 0{,}05$ level of significance, the regression equation for predicting mathematics achievement was found to be Y = .5061* Prior + 0.498* SURF – 9.115. The variable of prior knowledge, as indexed by its β value of 0.5016, was shown to have the strongest relationship to achievement. At $p = 0{,}15$ and using inclusion-exclusion stepwise regression, of the six independent variables identified, three were found to be significant predictors of achievement scores on multiple integration and dynamical systems. These were prior knowledge, knowledge of surface features and cross-sections of 3D objects.

Chapter 7: Conclusions

7.0 Overview

In this chapter, we discuss the results of the research and its implications for mathematics education. This research study was an attempt at using technology and computers to enhance the teaching and learning of multiple integrals and dynamical systems with emphasis on analytical thinking and visualization of 3D space figures, direction fields and phase portraits. The emphasis in the experimental class was facilitating simultaneous connections between the algebraic, numerical and geometric or graphic registers. Duval's semiotic representation theory (1996) Duval's cognitive apprehension levels for geometric thinking (1996) and the Zazkís' et al. (1996) Visualization-Analysis frameworks were used to examine student solutions in the two domains under study. We revisit the teaching experiment and look at the role of enhanced visualization, using software in the Teaching and Learning of multiple integrals and dynamical systems. We look at some of the limitations of the study and discuss contributions in the field of mathematics education. Finally, we conclude with remarks about pedagogical considerations in teaching these domains and give directions for future research.

7.1 Research questions

The research questions and strategies that guided this work are:

1) What are students' needs and difficulties in visualization and solution of problems in multiple integration and dynamical systems?
2) Do the activities and laboratory sessions facilitate visualization and solution of problems in the two domains?
3) What factors influence the effectiveness of the visualization?
4) What Teaching and Learning strategies help in the conceptualization and solution of problems in multiple integrals and dynamical systems?

7.2. The teaching experiment

This comprised two parts, the activities and the computer laboratory sessions conducted with the experimental class (n = 24). The activities were described in Chapter 3. They were designed to target skills such as recognising surface features and properties of 3D objects, generating, rotating and sketching 3D objects and their projections, and folding and unfolding nets.

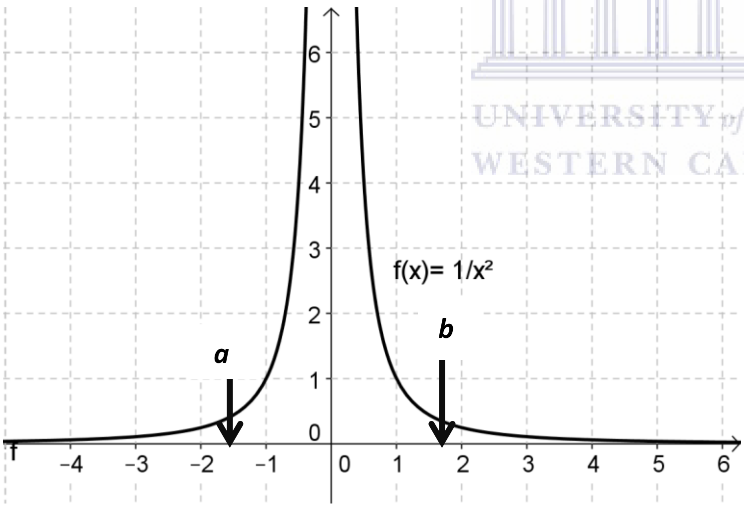
The laboratory sessions targeted visualization of 2D and 3D space figures and phase portraits of systems of the Lotka Volterra differential equations. The students in the experimental group had sessions in Matlab. All students had access to CalcPlot 3D and Mathematics Visualization Toolbox (MVT) on the Internet. Data was collected through written tests, interviews, assignments, and laboratory worksheets.

7.3 Summary of findings

7.3.1 Difficulties with integrals

The first research question was addressed using Duval's (1995, 1996) semiotic representation theory and Visualization-Analysis (VA) framework. which is is a tool for diagramming the transitions between the two interacting and complimentary, visual and analytical modes of thinking. After marking the solutions to the test items, the steps in students' solutions were categorised as visual or analytical. It was noted that the dichotomy between visual and analytic thinking was not always clear cut in all cases especially at a micro-level. For example, in determining or reading the limits of the integration from the projections or 3D sketches, several actions and processes occur simultaneously. One has the visual representation in mind, or on the 'mental blackboard' and you imagine spanning between the surfaces. You then record the limits. The mental action is hard to see even with the students verbalising their thinking but the outcome, writing down the limits, is visible. These steps were categorised as largely visual as they required students to refer to the visual representation (mentally or externally on paper) and move between the curves and surfaces on their sketches to find the limits of integration. The mental actions happen more rapidly than the oral articulation. According to our definition in chapter 1, section 1.2, these actions classify as visualization. Duval's semiotic representation framework provided us with a tool to look at the representational mode, namely, tabular or numerical, algebraic, geometric and symbolic. In this research symbolic is used to distinguish between the software command syntax (codes) from the algebraic symbols in mathematics.

For problems in single integration, we found that the majority of students were able to move between registers and do treatments in just about any order that was requested. However, the tendency to rush into the analytical solution, skipping visual aspects, lead to errors in single integrals such as $\int_{-1}^{1}\frac{1}{x^2}dx = \left[-x^{-1}\right]_{-1}^{1} = -2$. In the symbolic toolbox, Matlab gives the solutions as 'int (1/(*x*^2), *x*, −1,1) = Inf'. A simple sketch shows that the function is undefined at *x* = 0, and we need to split the integral −1 to *a* and *b* to 1 and take limits as *a* and $b \to 0$. (See Figure 7.1)



$$\int_{-1}^{1}\frac{1}{x^2}dx = \ell im_{a\to 0}\int_{-1}^{a}\frac{1}{x^2}dx + \ell im_{b\to 0-}\int_{b}^{1}\frac{1}{x^2}dx$$

$$\ell im_{a\to 0-}\left[-\frac{1}{x}\right]_{-1}^{a} + \ell im_{b\to 0+}\left[-\frac{1}{x}\right]_{b}^{1}$$

$$\ell im_{a\to 0}\left[-\frac{1}{a}+\frac{1}{1}\right] + \ell im_{b\to 0}\left[-\frac{1}{1}-\frac{1}{b}\right] = \text{inf}$$

which is divergent

Figure 7.1 Integrating the discontinuous function f(*x*) = $1/x^2$

We used the 'regular' definition of Riemann sum by introducing *a* , *b* , to give us a finite interval. Then taking the limit as *a, b* tend to 0, and integrating, we find that the integral from −1 to *a* and *b* to 1 are divergent. Therefore, the integral is divergent.

Likewise, we found that students could move between the registers and do the necessary treatments and evaluate double integrals easily when they were set up in the order ∬ dy dx. Difficulties in visualization became apparent when the order of integration needed switching from∬ dy dx to ∬ dx dy, and, in particular, when two split integrals were required to span the region of integration.

In 3D, sketching and translating among different representations of the same mathematical object proved to be difficult. While most students were able to recognise and sketch the projections in the *xy, xz* and *yz* planes, they found it difficult to assemble the projections into a coherent object. The mathematical object was difficult to visualise when it sat between two or more surfaces and the intersecting planes were hard to identify. This compounded the difficulties of setting the limits of integration in multiple integrals. We note that there are 6 permutations of *dxdydz* and the *dzdydx* triple integral was easier to set up but changing the order of integration was not easily accomplished and most students did not know how and when to split the integral into two. We hypothesise that this can be attributed to a poor conceptualisation of the Riemann process and spatial visualization in filling up the required area or volume. In her interview one student (ST2) admitted she forgot about slicing and she went straight to the equations to find the limits of integration. Somehow, even at this late and final stage of their study of calculus in a university of technology, the tendency to stay in the algebraic register predominates and there is reluctance to use information that is sitting in the visual representations students have sketched correctly. The Riemann Sum process certainly needs reinforcement when dealing with double and triple integrals and views of the object from all angles, cross-sections and projections need to be reinforced.

Students met the 3D rectangular, spherical and cylindrical coordinate systems for the first time in this course and conversions and treatments within these systems were hard to visualize. There was often confusion between ϕ, the angle between the z axis and ρ, the distance of the point from the origin. Similarly θ, the polar angle in the *xy*- plane between the *x*-axis and the polar radius, *r*, were confused. Part of reason for the confusion between ϕ and θ is that different notations are used in textbooks: Some authors give spherical coordinates of a point in the order (ρ, θ, ϕ) while others use (ρ, ϕ, θ). There was also confusion between the polar radius, r and ρ. Often r was difficult to find as treatments within the algebraic registers were problematic. For example, students found it difficult to find r needed for cylindrical integrals given the bounding surfaces z = $\sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 7$. Similarly, objects such as ϕ = π/3, ρ = 2 , given in cylindrical and spherical coordinates, were not easily recognised, except through the lengthy process of conversion to rectangular coordinates, sketching projections and assembling the 3D solid (See Figure 7.7).

7.3.2 Difficulties with differential equations

In this section we summarise our findings on the second domain of the study. The purpose of the differential equations course for the students, who were registered for a Bachelor in Technology in Chemical engineering, is to solve the numerous DEs that they meet in diverse areas such as chemical kinetics, thermodynamics, process control and design. Often the teaching of DEs is exclusively reduced to finding algebraic/analytical solutions to the differential equations and graphing the solution. The current research attempted to extend this to simple applications such as mixing problems and modelling chemical reactions using the Lotka Volterra equations which have wide applications in chemistry, economics, population dynamics, etc. The teaching experiment stressed qualitative solutions of systems of DE's. The approach covered direction fields, and phase portraits of systems of differential equations.

A direction field is a visual display of gradients or slope vectors at different points in space drawn with Cartesian axes as the state variables. It has an infinite number of solution curves or trajectories. Depending on what initial values we take, the qualitative behaviour of the DE can be determined by focusing on a few selected trajectories. In the geometric approach, the students found the trajectories by selecting and plotting the gradients at various points in the field.

A phase portrait is a visual display of a sampling of the trajectories of the systems of differential equations. It shows the behaviour of the system as the state variables vary with time.

In this study, students sketched phase portraits manually, by finding the equilibrium points by setting the Right Hand Sides of the DEs to zero and solving the resulting simultaneous equations. Equilibrium points are an important feature of a dynamical system since they define the states corresponding to constant operating conditions. For linear systems, the equilibrium point is the origin (0 , 0). There are two gradients dx/dt and dy/dt to consider in determining the trajectory at any space point in the phase portrait. Vector addition gave them the resultant of the two gradients. This is plotted on the phase portrait and a few selected trajectories then indicate the nature and stability of the equilibrium point. This was the qualitative or geometric approach used by the students in plotting the phase portrait.

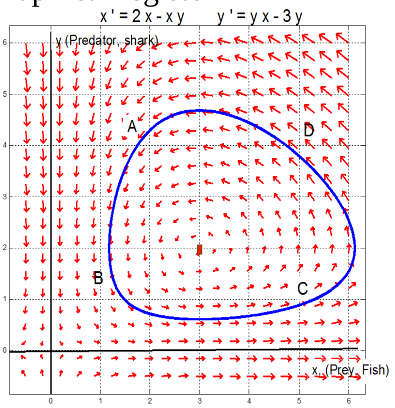
In the analytic method, the equilibrium points were found by solving the simultaneous equations with their right hand side set to zero. The method of linearization using the Jacobian matrix at each equilibrium point yields the eigen-values. Using these students were able to determine the nature of stability of the equilibrium points.

Of the two approaches, the geometric approach was found tedious and, students expressed annoyance when required to go through the intensive chore of plotting slopes. We note limitations in some students' understanding and plotting of slopes and the connection between the slopes and the differential equation was not always clear. Also, the focus in the chemistry lectures was mainly analytical solutions and this influenced the importance students attached to qualitative solutions.

Surprisingly, students experienced difficulties in solving the simultaneous equations that they set up and often missed one or two equilibrium points in the system. At times they found more equilibrium points than were there especially with the non-linear systems.

Once they had the phase portrait, in terms of Duval's framework, discursive and global apprehension did not happen for most students, and as such the long term behaviour of the system was often missed. Interpretation of the phase portraits in terms of the real world situation, the cyclical oscillations and time, the hidden parameter, also proved conceptually difficult and elusive. A typical example is the Lotka-Volterra phase portrait where the interacting species were shark and fish. Visualizing covariation, that is, two quantities changing in relation to each other was not easy. Working simultaneously in two graphical registers was difficult. The phase portrait is one graphical representation and the time graph, showing numbers of fish and shark is another graphical representation of the same information. Both show one cycle. Students found that the connections within geometric or graphical registers between the two graphs were easier to understand through the numerical register used as an intervening medium (See Figure 7.2). This supports the dual emphasis between visualization and analytical thinking and flexibility in the coordination with other representations in the same or different registers.

Graphical register



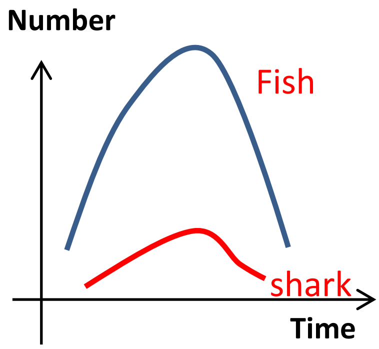
a) Phase portrait: Fish Vs Shark

Numerical register

	Fish '000	Shark '00	Comment
AB	1,5→ 1,2	4,6→ 1,5	Shark down
BC	1,2 →6	1,5 →0,5	Fish up
CD	6→5	1 → 4	Sharks up
DA	5→3	4→ 5	Fish drop

b) Table showing how the number of fish and shark fluctuate

Graphical register



c) Fish and shark vs time Graph

Figure 7.2 Using the numerical register as an intervening medium to foster connections between two graphs the graphical register

The second research question is whether the activities enhance visualization and solution of problems in the two domains. In seeking answers to this question, a teaching experiment was conducted using two groups of students following the same mathematics course at the university of technology. The control group comprised 26 students, 8 female and 18 male, and the experimental group comprised 24 students, 14 female and 10 male. The differences in average age of students, on group and gender comparisons, were considered to be of insufficient size to bias the study in any way. Pre-tests covering Mathematics (Calculus) 1 and 2 work, were administered to both groups and the mean group scores of 47.4 (SD = 5.6) and 45.7 (SD = 4.8) confirmed insignificant differences in mathematics ability of the two groups.

The lecturer presented his lectures with the aid of a chalkboard and an Overhead Projector using the same set of slides to both the groups. The control group received take home assignments whereas the experimental group worked on similar assignments called worksheet activities in a computer laboratory (See Figures 7.3, 7.4 and 7.6 for examples of students' work in the mathematics laboratory). The software package, Matlab, used in the laboratory for activities was chosen, principally, because the department of mathematics had installed it in the laboratory in 2010. Matlab has a large library of visualization tools and can do computations of complex double and triple integrals as well as solve differential equations and display their solutions. Other software that students could use were Maths Visualization Toolbox (MVT) and CalcPlot3D, an applet by Seeburger (2007) available on the internet. Prior to this study, the students had some exposure to Matlab but none with the computer software, MVT or the applet CalcPlot3D. A second factor in selecting the software packages was access. Students had free access to CalcPlot3D and MVT outside laboratory time on the internet. Structured laboratory orientation sessions were conducted to familiarise everyone with the basics of the software packages needed for the activities. The software provided access to 3D mathematical objects and phase portraits, the subject of this research.

Six activity sessions were designed with questions, instructions and software codes to help students see and manipulate the 3D solids and phase portraits, and record their observations in the spaces provided on the worksheets. In the Mathematics laboratory session, students typed in the Matlab code and viewed, orientated, and sketched the 3D solid, and evaluated the integral using the software as well as manually. Figure 7.3 shows an example.

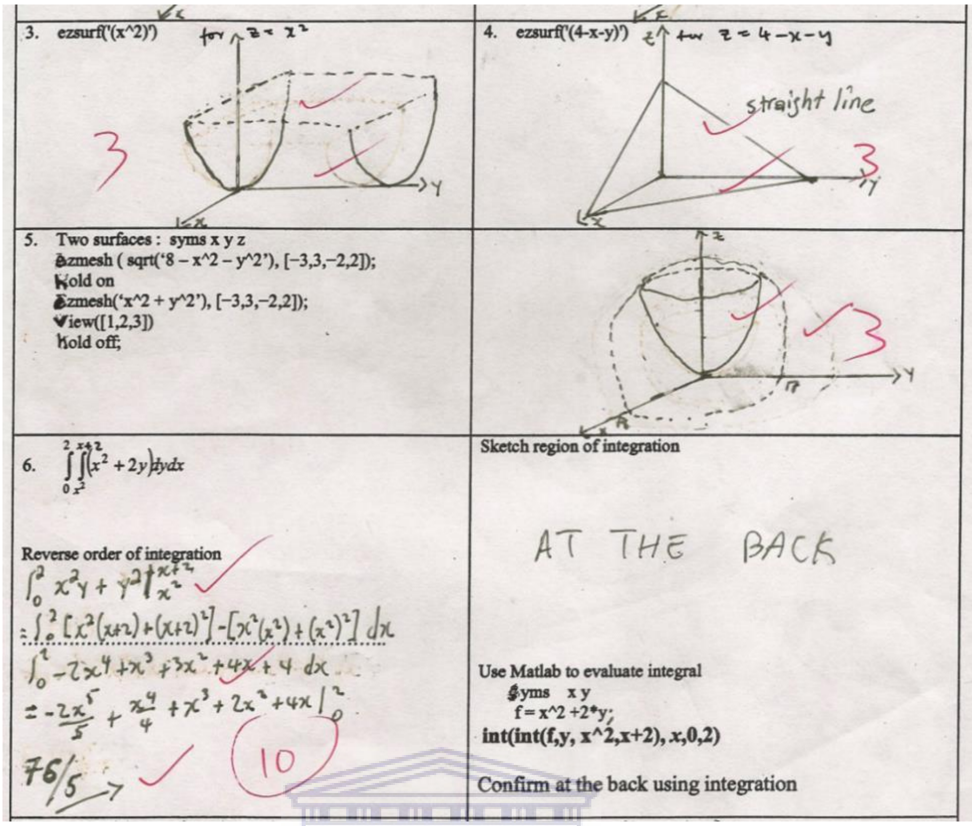


Figure 7.3 Worksheet activity: Sketching3D solids and double and triple integration using Matlab

The second worksheet focused on intersections between planes and 3D objects. Figure 7.4 is an example of a student's work showing the intersection of a plane with a cone. This was extended to other 3D solids like spheres and cylinders. The top left corner shows the Matlab codes with a built in parameter to change the angle of the plane.

Figure 7.6 is a worksheet on sketching 3D solids with their projections, and evaluating double and triple integrals manually as well as using the software.

The last two activity sets focussed on differential equations emphasising the link between the analytical and qualitative solutions including direction fields and phase portraits. Figure 3.10, chapter 3, is an example of a DE worksheet.

Our findings were that there are no significant differences between the achievement of students who used computers and went through the activities and the achievement of students using the traditional approach with pen and paper on tests T1 and T2. A two-way ANOVA of test

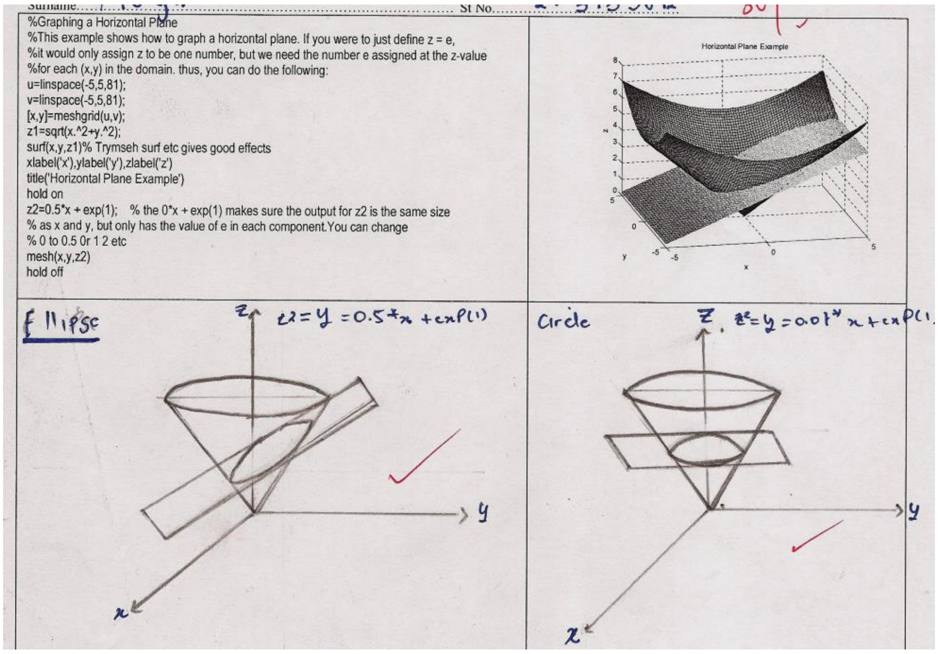


Figure 7.4 Intersections of planes and solids: conic sections

results with group (Control Vs experimental) and gender (male vs female) revealed no significant differences in student performance. The main effect of treatments was not significant while gender differences were also not significant. However, significant differences between treatments were found on Test 3 at the end of the semester in favour of the experimental group.

An end of semester questionnaire was completed by the experimental group (n= 24) comprised seven questions answered on a scale of 1 to 5, where 1 was poor and 5 was excellent. It revealed that the computer-enriched sessions provided a positive learning experience for the students, who recommended more sessions and time allocation in future. The Laboratory sessions and worksheets were rated 3.8 , and the work on double and triple integrals was rated 4.2. The preference for computer-assisted learning is not surprising, considering the generally positive attitudes towards computers. The visuals used in lectures were rated ‘clear’ by 15 students (63%) while 9 (37%) said visuals needed improvement. Everyone found the worksheets helpful.

There were some negative comments as well. A number of students who did the activities suggested that they were actually learning about two things, the activities and how to use the software program. The codes had to be typed in, the display adjusted, and the sketches drawn.

Also the way the mathematical objects were sketched and presented in lectures was not the same as on the computer. Some adjustment, rescalings and reorientation was necessary to get the axes aligned to give a suitable display.

Our expectations and intention were that the lab activities would provide a higher degree of semiotic flexibility, by facilitating processing and plotting, which is a conversion. On the contrary, as Winslow (2000; p. 278) points out we have an extra medium (the computer), an additional special code (Matlab) for semiotic activity, and a kind of automatic semiotic agent that presents a final output whether it is the result of a computation or a 3D graph. The software tends to leave out several intermediate steps that may, or may not, be made explicit, an effect known as the ‘black box effect’ (opus cit., p281). For example, one does not see the intermediate steps in the computation of a double integrals or the analytical solution of a differential equation. At times the outputs were difficult to decipher. To illustrate, Figure 7.5 shows the code required to solve the ODEs: dx/dt = 2x – y and dy/dt = 3x – 2y and plot the solutions.


```
syms x y t
dsolve('Dx=2*x-y','Dy=3*x-2*y';
x = exp(-t)*(3*exp(2*t)/2 +
3/2)/3;
y = exp(-t)*(exp(2*t)/2 + 3/2);
ezplot(x, [-1,5]);
hold on
ezplot(y, [-1,5]);
hold off
```

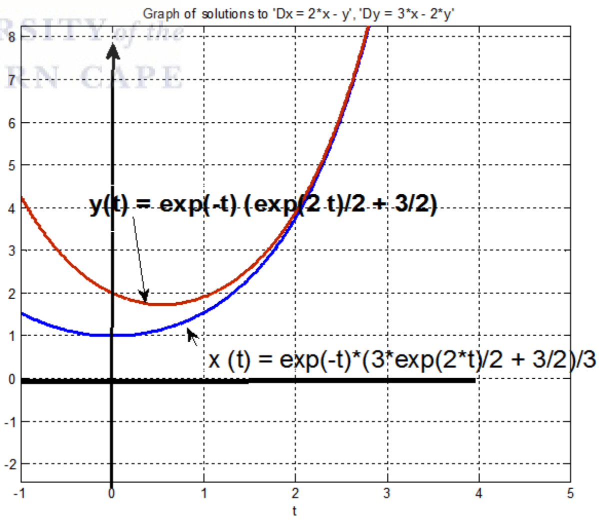
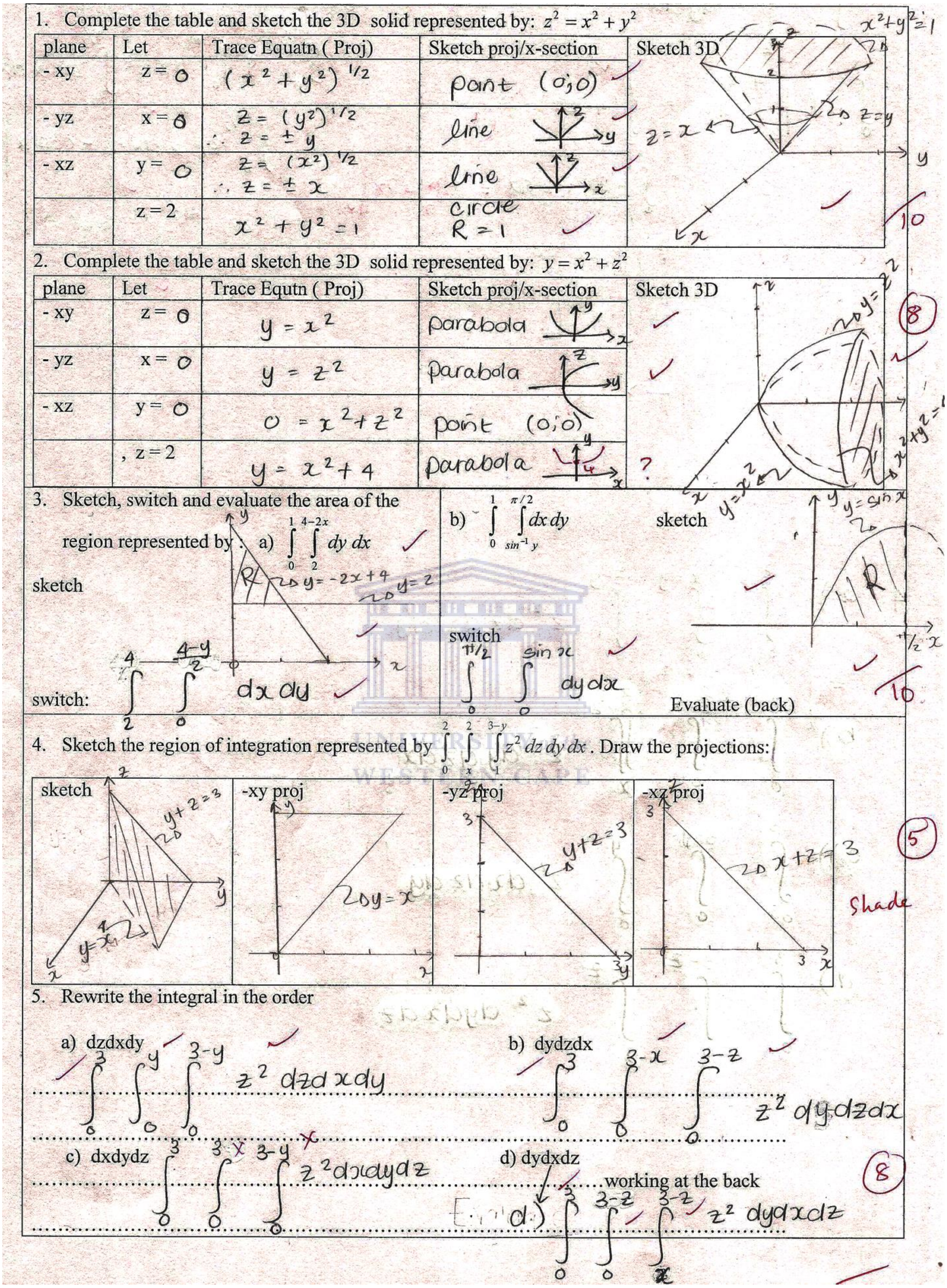


Figure 7.5 Code required to solve and plot graph of solutions to dx/dt = 2x – y and dy/dt = 3x – 2y in Matlab

CalcPlot 3D was found easier to work with and gave excellent visuals that could be easily manipulated. However, CalcPlot 3D does not have any facilities for differential equations or for computation.



1. Complete the table and sketch the 3D solid represented by: $z^2 = x^2 + y^2$

plane	Let	Trace Equatn (Proj)	Sketch proj/x-section	Sketch 3D
- xy	z = 0	$(x^2+y^2)^{1/2}$	point (0;0)	
- yz	x = 0	$z = (y^2)^{1/2}$ ∴ $z = \pm y$	line	
- xz	y = 0	$z = (x^2)^{1/2}$ ∴ $z = \pm x$	line	
	z = 2	$x^2 + y^2 = 1$	circle R = 1	

2. Complete the table and sketch the 3D solid represented by: $y = x^2 + z^2$

plane	Let	Trace Equtn (Proj)	Sketch proj/x-section	Sketch 3D
- xy	z = 0	$y = x^2$	parabola	
- yz	x = 0	$y = z^2$	parabola	
- xz	y = 0	$0 = x^2 + z^2$	point (0;0)	
	, z = 2	$y = x^2 + 4$	parabola	

3. Sketch, switch and evaluate the area of the region represented by : a) $\int_0^1 \int_2^{4-2x} dy\,dx$

sketch

switch: $\int_2^4 \int_0^{\frac{4-y}{2}} dx\,dy$

b) $\int_0^1 \int_{\sin^{-1} y}^{\pi/2} dx\,dy$ sketch

switch $\int_0^{\pi/2} \int_0^{\sin x} dy\,dx$

Evaluate (back)

4. Sketch the region of integration represented by $\int_0^2 \int_x^2 \int_1^{3-y} z^2\,dz\,dy\,dx$. Draw the projections:

sketch | -xy proj | -yz proj | -xz proj

5. Rewrite the integral in the order

a) dzdxdy $\int_0^3 \int_0^y \int_0^{3-y} z^2\,dz\,dx\,dy$

b) dydzdx $\int_0^3 \int_0^{3-x} \int_0^{3-z} z^2\,dy\,dz\,dx$

c) dxdydz $\int_0^3 \int_0^3 \int_0^{3-y} z^2\,dx\,dy\,dz$

d) dydxdz ....working at the back $\int_0^3 \int_0^{3-z} \int_x^{3-z} z^2\,dy\,dx\,dz$

Figure 7.6 Sketching projections, 3D solids and switching the order in double and triple integrals

What is clear is that transition from 2D to 3D and understanding of graphs of functions of two variables is not easy for students. The difficulties stem mainly in generalising from 2D to 3D. One assumes thorough familiarity with graphs in 2D such as $y = 3x + 4$, $z = 4 - x^2$ that define a line and a parabola. In 3D, the same equations define a plane and a parabolic cylinder respectively. In spherical coordinates, $\rho^2 - 2\rho\cos\phi = 8$ defines a sphere, radius 3, centre (0,0,3). In cylindrical coordinates, $r^2 + (z - 1)^2 = 9$ defines the same sphere. In rectangular coordinates the equation is: $x^2 + y^2 + z^2 - 2z = 8$. Often students converted from spherical to rectangular coordinates in order to "see" the object and to sketch it. Figure 7.7 shows an attempt by a student to sketch $\rho = 4\cos\phi$, in Test 3. The intersection of surfaces with planes, and predicting the result of this intersection plays a fundamental role in setting up the integrals and was particularly difficult for students.

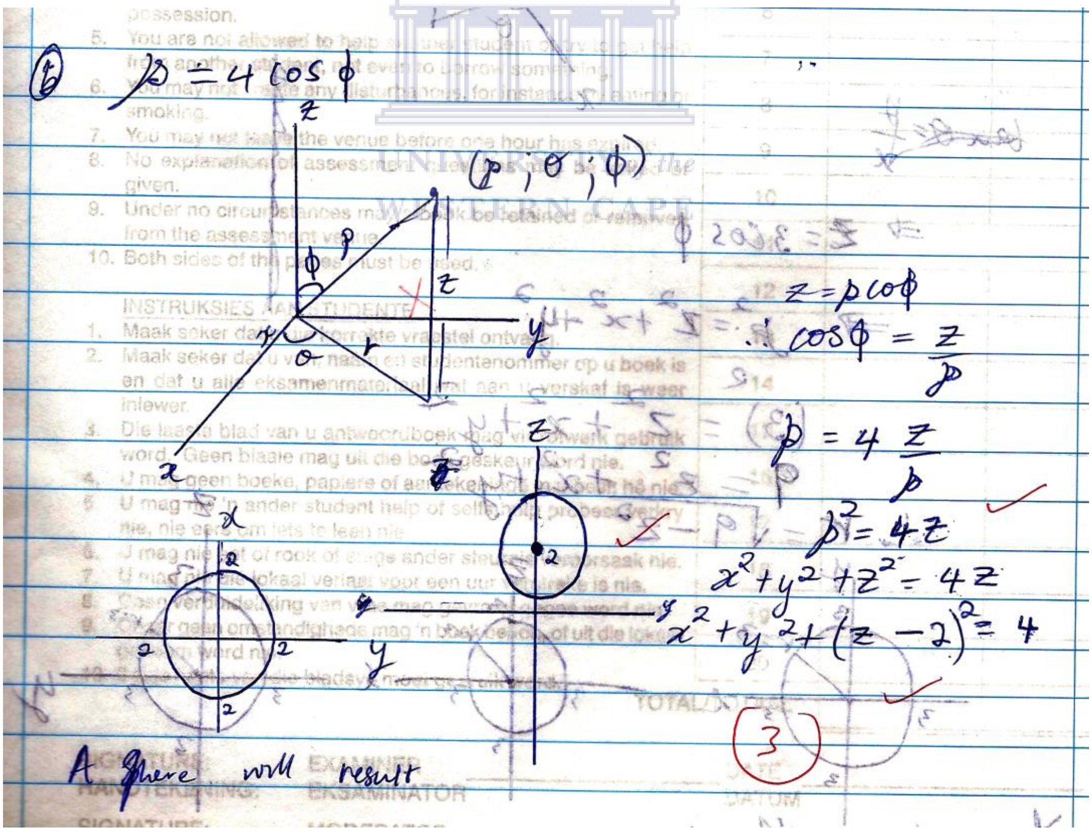


Figure 7. 7 Student's attempt to sketch ρ= 4 cos ϕ, Student converts to rectangular coordinates, recognises the mathematical object as a sphere, centre (0, 0, 2), radius 2. The 2D sketch in the yz plane is not accurate and the 3D sketch is missing

The third research question was what factors influence visualization and spatial ability in particular. The review of literature identified a large number of factors impacting on spatial ability such as spatial orientation, surface features, recognising rotated figures, opening and folding nets, mental manipulation of objects, reflection and symmetry , etc. The 3D objects that our students would be dealing with were cylinders, spheres, cones, pyramids, cuboids, and hyperboloids. We made manila cardboard and wire models of some of these so students could look at the surface features (circles, triangles, sectors of cones, vertices and edges) and examine their nets by folding or unfolding. The 3D objects could be viewed, projected and sketched from different perspectives and orientations. The objects could be combined, eg cone on cylinder. We chose 6 factors that we felt would have most impact on achievement in the type of problems we were attempting in multiple integration. These factors were surface features (names, faces, vertices), rotation (turning an object about a point or axis), cross sections (with horizontal and vertical or oblique cutting planes), Nets (folding/unfolding to give 3D object) and prior knowledge. Prior knowledge tested their knowledge of functions, roots, differentiation, integration and differential equations that were covered in Maths 1 and 2. The test for prior knowledge had been constructed and tested over several years. Reliable test items were available for each of the other factors. We then performed a multiple linear regression with achievement scores in Test 2 as the dependent variable and the six factors as independent variables.

The regression analysis was found to be statistically significant $F(20) = 9{,}67$, $p < .01$. The multiple regression accounted for 72,2% of the variability, as indexed by the adjusted R^2 statistic.

The regression equation for predicting maths achievement was found to be Y = .5061* Prior + 0.498* SURF – 9.115. The variable of prior knowledge, as indexed by its β value of 0.5016, was shown to have the strongest relationship with achievement in test 2.

A step wise regression analysis with inclusion exclusion significance levels set at 0,15 confirmed that pretest, surface features, as well as cross-sections are significant predictors of achievement on multiple integrals and dynamical systems.

We decided to repeat the regression analysis using the examination scores at the end of the semester as the dependent variable. We found that none of the variables were significant predictors of achievement on the examination.

Our conclusion is in line with the findings of other researchers (eg Hegarty and Cohen, 2012) that training in specific spatial skills (projections, cross-sections, nets) correlates and transfers well with problems in multiple integrals that utilise those skills. Prior knowledge had the largest influence on achievement.

The fourth research question looks at Teaching and Learning strategies that help in the conceptualization and solution of problems in multiple integrals and dynamical systems.

The analysis of students' solutions highlight several challenges in the Teaching and Learning of multiple integration and dynamical systems. We identified visualising mathematical objects in three dimensions as one of the difficulties. Students faced difficulties in setting up and switching the order of double and triple integrals.

The software promoted a combination of visual and non-visual reasoning essential to problem solving and promoting visual and analytical thinking. From a didactical point of view the software acted as a pedagogically appropriate mathematics construction and exploration tool, reducing the cognitive load that hampers mathematical activities. It embraces all four components of visual thinking that Koslyn (1983) proposed, namely: image generation, inspection, manipulation and maintenance. Visualization is therefore, in many cases, facilitated by computer rendered images and relies on the power of the human visual sense to analyse the content of images.

In addition the software enabled quick and easy conversions between the geometric and algebraic registers. The outcome was improved simultaneous coordination (synergy) between the numerical, algebraic, graphic and symbolic registers. At the same time, we could realise some benefits of using visualization in the instructional process listed by Dwyer (1988), among them :

1. visualization increases learner interest and concentration
2. visualization illustrates, emphasises and reinforces oral and printed instructions and
3. visualization develops discrimination and clarity of thought.

Important implications for Teaching and Learning emerge from this study. Mathematics educators need to pay careful attention to the use of visual representations, space figures in 3D and phase portraits in 2D, their construction, sketching, and interpretation. They need to encourage students to use their visualization skills to better conceptualise, represent and solve problems. Students should be encouraged to imagine and visualise the solutions to differential equations by sketching graphs. Often visual representations drawn by students in answer to problems are viewed as scratch work and ignored by lecturers. Tutors can stress their importance and include them in the marking rubric.

Given the increasing availability and use of technology, more research is needed on technology-enhanced techniques that aid with visualization and representations, and the conditions under which they are effective. Integration of technology into mathematics, by modifying existing curricula, is a viable and effective method for curriculum development at the tertiary level. To avoid the 'black box' effect students should check that the solutions which software offers are reasonable. Do the functions satisfy the DEs? What steps did the program use to give the answer to the double integral and to solve the DE?

Individual differences became apparent in spatial visualization of objects in 3D. While there was some familiarity with mathematical objects like planes, lines and curves, cones, cylinders, pyramids, and spheres, most students found it difficult to visualise their projections, intersections and cross-sections, that are necessary in triple integration. The study found that the use of manipulatives is not out of place even at this stage. Models of cones, spheres, cuboids, ellipsoids help in the concept development, visualization and problem solving. The power of manipulatives is in helping to move between the concrete representations to abstract ideas, and helping students to visualise and internalise abstract concepts. Drawing nets for solids and folding to assemble the 3D object, is an important strategy to draw attention to the surface properties like faces, edges and vertices that sit together or are parallel or perpendicular to each other. Well-designed activities can strengthen spatial visualization which, as other researchers, eg McGee (1979) have pointed out, is a potential indicator of success in mathematics. The use of manipulatives is similar to modelling and simulation that mathematicians apply to conceptualise and solve problems.

In dynamical systems, the concept of slope or gradient challenged many students who struggled with their direction fields and in sketching and interpreting phase portraits. These concepts need to be linked to their definitions, change in *y* over change in *x*, rise over run, and rates of change. The implications are that mathematics educators need to expand their repertoire of skills and tools beyond chalk talk and challenge students to make connections between physical, geometric, algebraic, numerical, symbolic and verbal representations. They need to incorporate activities including nets and folding activities, composition and decomposition, rotation of 3D figures, sketching cross-sections, projections of 3D objects.

However, at the same time, we need to be aware of the shortcomings. According to cognitive load theory (Sweller, 1999), when learners split their attention between visually presented text and graphics, it overloads working memory capacity. Time used in typing in the codes, getting to know the programme and symbolic codes for the software, and sketching projections increases the cognitive load at times to a point where students switch to the algebraic equations and use them to write the limits of the integrals, thus defeating the purpose of the activities.

7.4 Recommendations

This study showed that students have difficulty constructing, using, interpreting and applying 3D geometric representations, direction fields, and phase portraits and can benefit from technologically enhanced activities focussed on visualization and analytical thiking. The majority of students were at the perceptual apprehension level when it came to setting up triple integrals in orders other than dzdxdy. We observed that students have difficulty relating and translating among different representations of the same entity within a register eg between cylindrical and spherical or rectangular systems as well as conversions from algebraic to graphical and verbal modes. The scope of the activities and interventions using worksheets was limited. Additional support for students struggling with spatial relations and 3D visualization must be accommodated. The use of computers and technology needs to be timetabled, and should form an integral part of the mathematics courses.

In dynamical systems extracting information from graphical representations, and describing the long term behaviour of solution curves hinges on visualization and students' facility with language. More attention needs to be paid to discursive, sequential and operational apprehension, that is, moving beyond the representation of dynamical systems in phase portraits to interpretation and prediction of the long term behaviour of solutions. In particular, the hidden parameter 'time' in phase portraits needs to be unpacked carefully.

The results of the study show that with computer enhanced static images, non-linear dynamical systems can be handled easily at this level. However the curriculum, the examination structure and assessments and evaluations need to be revised to accommodate a qualitative approach to systems of differential equations. This research focussed mainly on student interactions with static visualization. Further research is necessary to assess the effect of dynamic and interactive visualization.

7.5 Contributions

Business, science and industry increasingly use computer visualization to solve complex problems and to optimise and monitor processes and production in the workplace. Many disciplines and professions such as data science, engineering, radiography, geographical information systems and digital animation and simulation depend on visualization.

This research provides empirically based evidence that guides the design of activities that can enhance visualization of functions of two variables. It looks at the role of visualization complementing analytical thinking in two main areas in mathematics; namely multiple integrals and systems of differential equations. It reinforces the need to shift the emphasis from a purely analytical and algebraic approach towards a technologically enhanced setting where the emphasis is on building connections between algebraic, graphic and numerical representations, on visualization as well as analytical thinking.

Duval's semiotic representation framework and the VA-framework gave useful insight into students' cognitive difficulties as students moved back and forth between various registers. This research showed that it is possible to go beyond the semiotic representation approach and incorporate software, and activities in the collaborative learning and teaching of multiple integrals and dynamical systems.

Finally, the research made useful recommendations for the teaching and learning of the topics through visual mediation by focussing on pedagogical principles, learning strategies and mathematical achievement into a unified whole. On a personal level, the teaching experiment helped to break down the barrier between the lecturer and the students enabling one on one interaction. The lecturer could listen to the students' utterances and note their actions, scaffolding and providing support where necessary. It endorsed the findings of other researchers that the use of technologically enhanced visualization of multiple representations can facilitate and promote learning.

7.6 Extending the scope of the study

This study followed two small groups of students, one receiving traditional lectures, and the other had their lectures supplemented by computer Laboratory activities. The findings are specific to the small group of students from mainly disadvantaged backgrounds. The teaching experiment needs to be replicated with larger groups, and in a broader range of mathematical domains using students with diverse backgrounds.

The regression model used six predictor variables in assessing their relation with achievement on multiple integrals and two were significantly related to achievement. Other important variables such as affective factors (attitudes and motivation), gender, study styles, socio-cultural factors need to be investigated and included in the model.

Visualization creates a bridge between the real world and the abstract world of reasoning and thinking in mathematics. Mathematical objects such as a point, a line, a plane, a number are abstract entities that can only be 'seen' through their representations. Differential equations represent abstract phenomenon and processes in the real world involving rates of change. In the differential equations course, the problems tackled were based on applications in population dynamics and chemistry. In multiple integrals the area of application was volumes of familiar objects such as a cone, a cylinder, a sphere in three coordinate systems. We need to look at the role of visualization in other areas and applications of mathematics such as surface areas of 3D objects, topology etc.

7.7 Future directions

The field of visualization is on an ever changing landscape with new horizons opening up as technology develops. Even the definition of visualization has seen several changes. Linn (2010, p. 732.) defined visualization as 'interactive computer based animations such as models', simulations, and virtual experiments of phenomenon". Simulations are computer models of real phenomenon that allow users to change parameters and explore the implications on the state variables. Visual mathematics refers to learning of mathematics using a variety of representations (diagrams, graphs, 3D sketches) that aid in the exploration and visualization of mathematics concepts usually in cyber-learning environments such as networks, WebCity (Howse and Stapleton, 2008). The ubiquity and accessibility of the internet and Web and increasing adoption of portable devices (smart phones, tablets, computers, iPads, etc), have contributed to and are driving the demand for visual tools. Visual calculus, visual group theory, PDEs; a visual approach, and data visualization are some of the domains that are turning to visualization to improve pedagogical practices.

Tools such as MVT and CalcPlot3D enable integration of visualization with instructional scaffolds. While there has been much research devoted to the successful design of pedagogically impactful learning environments in which to embed visualizations, the body of literature addressing the design and evaluation of didactic visualizations is relatively underdeveloped. This needs further research.

There has been a tremendous growth in the availability of tools for teaching, learning, and research in visualization. The learning styles and behaviours of students in the current generation (also called the Net Generation) are influenced by the digital media and technologies and are highly visual and perceptual. Their thinking and learning is influenced by technologies such as the computer, the internet and smart phones and games, simulations and applets. Educators need to look at the potential of these for enhanced teaching and learning. Games have the advantage of immediate feedback and help the learners monitor their own progress towards their goals. As Zimmermann and Cunningham (1991) note we have a visualization renaissance driven by technology on our hands. The early days of the renaissance were static visuals with hand held scientific and graphic calculators. The emphasis now is on dynamic and interactive visualization, internet connectivity, communication and information-sharing resources, and photo and video capture capabilities. The effective adoption and successful integration of digitals tools in teaching and learning of mathematics need further research.

Figure 7.8 Students inside the Mathematics Laboratory, CPUT

References

Ahmadian, A. (2008) System dynamics and technological innovation system, Model of multi-technological substitution processes, M.Sc. Thesis, Chalmers University of Technology, Report No. 2008:15. Goteborg, Sweden.

Allen, K. S. (2006). Students' Participation in a Differential Equations Class: Parametric Reasoning to Understand Systems, Ph.D dissertation. Purdue University, West Lafayette, Indiana

Arcavi, A. N. (1989). ‘Re-exploring familiar concepts with a new representation’, Proceedings of the 13th International Conference on the Psychology of Mathematics Education (PME 13), 1, 77–84.

Arcavi, A. N. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52: 215-241

Artigue, M. (1992). Cognitive difficulties and teaching practices. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* , pp. 109-132. Washington, DC: The Mathematical Association of America.

Artigue, M. (2002). Learning mathematics in a CAS environment : The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*. 7 (3), 245-274

Ashcraft, M.H. (1989). Human memory and cognition. Glenview, IL: Scot, Foresman and Company.

Aspinwall, L., Shaw, K. L. & Presmeg, N. C. (1997). Uncontrollable mental imagery; Graphical connections between a function and its derivative. *Educational Studies in Mathematics*, 33 (3), 301 - 317

Battista, M.T. (1990). Spatial visualization and gender differences in high school geometry. *Journal of Research in Mathematics Education*, 21(1), 47 – 60.

Camacho-Machin, M., Perdomo-Diaz, J., & Santos-Trigo, M. (2012a) An exploration of students’ conceptual knowledge built in a first ordinary differential equations course (Part I), *The Teaching of Mathematics*, Vol. XV, 2, 1–20

Camacho-Machin, M., Perdomo-Diaz, J., & Santos-Trigo, M. (2012b) An exploration of students’ conceptual knowledge built in a first ordinary differential equations course (Part II), *The Teaching of Mathematics*, Vol. XV, 2, pp. 63–84

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420– 464). New York: Macmillan

Cretchley P., Harman C., Ellerton N., & Fogarty G. (2004). Computation, Exploration, Visualization: Reaction to MATLAB in First-Year Mathematics. Toowoomba, QL, Australia: In Proceedings of Delta '99 Symposium on Undergraduate Mathematics (ERIC Document Reproduction Service No. ED477688).

Davis, R.B. (1984) Learning Mathematics. The cognitive science approach to Mathematics Education.,Ablex, New Jersey.

Dreyfus, T. (1991). On the status of visual reasoning in Mathematics and Mathematics education. Paper presented at the 15th Conference of the International Group for the Psychology of Mathematics Education, (Univ. de Genova: Genova, Italy), vol. 1, 33-48..

Dubinsky, E. (1991) Reflective abstraction in advanced mathematical thinking. In D.Tall (Ed.), Advanced mathematical thinking, pp. 95-123. Boston: Kluwer.

Dubinsky, E., & MacDonald, M. (2001). APOS: a constructivist theory of learning in undergraduate mathematics education research. In: D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 273–280). Dordrecht: Kluwer Academic Publishers.

Duval, R. (1995). Geometrical Pictures: Kinds of Representation and Specific Processes, in R. Sutherland & J. Mason (eds.), *Exploiting mental imagery with computers in mathematical education*, Berlin, Springer, pp. 142- 157.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In Hitt, Fernando & Santos, Manuel (Eds*.), Proceedings of the Twenty First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH

Duval, R. (2000) 'Basic issues for research in mathematics education', *Proceedings of the 24th International Conference, Psychology of Mathematics Education*, Vol. 1,Hiroshima, Japan, pp. 59–69**.**

Duval, R. (2006). A cognitive analysis of problems of comprehension in the learning of mathematics. *Educational Studies in Mathematics*, 61:103-131.

Dwyer, F.M (1988) Adapting visual illustrations for effective learning, *Harvard Educational Review*. 37, 250 - 263

Eisenberg, L. & Dreyfus, T. (1991) .On understanding how students learn to visualize function transformations. *CBMS Issues in Mathematics Education*, 4, 45–68.

Fennell, F., & Rowan, T. (2001). Representation: An important process for teaching and learning mathematics. *Teaching Children Mathematics, 7*, 288-292.

Ferguson, E.S. (1992) *Engineering and the Mind's Eye*. Cambridge, MA, USA: MIT Press, 1992.

Fishbein, E. (1993) The theory of figural concepts. *Educational Studies in Mathematics*, V24 (2) p 139 – 162

Owens, K. D., & Clements, M. A. K. (1998). Representations in spatial problem solving in the classroom. *The Journal of Mathematical Behavior*, 17, 197–218

Goldin, G. A. (1998). The PME working group on representations. *Journal of Mathematical Behavior, 17*, 283-301.

Goldin, G. A. (2003). Representation in school mathematics: A unifying research perspective. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics*, (pp. 275-285). Reston, VA: NCTM.

Goldin, G. A. **& Kaput, J. J.** (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe & P. Nesher (Eds.), *Theories of mathematical learning* (pp. 397-430). Mahwah, NJ: Erlbaum Associates.

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: in search of a framework. In L. Puig & A.Gutiérrez (Eds.), Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education (Vol. 1,, pp. 3–19). Valencia: Universidad de Valencia.

Gutiérrez, R. (2012). Embracing "Nepantla:" Rethinking knowledge and its use in teaching. *REDIMAT-Journal of Research in Mathematics Education, 1(1), 29-56*.

Habre (1999) Visualization Enhanced by Technology in the Learning of Multivariable Calculus. Proceedings of the ICTMT4, Plymouth, 9 – 13 August 1999

Habre, S. (2000). Exploring students' strategies to solve ordinary differential equations in a reformed setting. *Journal of Mathematical Behavior, 18*, 455-472.

Habre, S. (2001a). Visualization Enhanced by Technology in the Learning of Multivariable Calculus. *The International Journal of Computer Algebra in Mathematics Education*,Vol. 8, No. 2, pp. 115 – 130.

Habre, S. (2001b). Visualization in Multivariable Calculus; The Case of 3D Surfaces. *Focus on Learning problems in Mathematics*, Vl. 23, N. 1, pp. 30 – 47.

Habre, S. (2003). Investigating Students' Approval of a Geometrical Approach to Differential Equations and their Solutions. *International Journal of Mathematical Education in Science and Technology*, Vol. 34, No. 5, pp. 651 – 662

Habre, S., &   Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *The Journal of Mathematical Behavior*, 25(1), 57-72.

Haciomeroglu, E. S., Aspinwall, **L., & Presmeg, N. C**. (2010). Contrasting cases of calculus students' understanding of derivative graphs. Mathematical thinking and learning, 12(2), 152-176.

Hegarty, M., and Cohen C.A. (2012) Inferring cross-sections of 3D objects; A new spatial thinking test, learning & individual dfferences, 22, pp 868 - 874

Hennessy S., Fung P., & Scanlon, E. (2001). The Role of the Graphic Calculator in Mediating Graphing Activity. *International Journal of Mathematical Education in Science and Technology*, 32(2), 267-29

Hoz, R. (1979). The use of heuristic models in mathematics teaching. *International Journal of Mathematical Education in Science and Technology, 10*, 137-151.

Howse, J. & Stapleton, G. (2008). Visual mathematics: Digrammatic formalisation and proof. Intelligent computer mathematics, Proceeding of the 9th international conference, AISC, Birmingham, U.K.

Hughes-Hallett, D. (1991). Visualization and calculus reform. In W. Zimmermann, & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 127 – 138).Washington, DC: MAA.

Hughes-Hallett, D. & Gleason, A (1994) Calculus . New York: John Wiley and Sons.(1994

Kaput, J.J. (1989). Supporting concrete visual thinking in multiplicative reasoning: Difficulties and Opportunities. Focus on Learning Problems in Mathematics. 11 (1), 35- 47

Kaput, J.J. (1993) The urgent need for proleptic research in the representation of quantitative relationshionships. Integrating research on graphical representation of functions, pp. 279 – 312.

Kaput, J. & Dubinsky, E. (Eds.), (1994). Research issues in undergraduate mathematics learning: Preliminary analysis and results. MAA Notes #33.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up.* Washington, DC: National Academies Press

Klein, T. J. (1993). A comparative study on the effectiveness of differential equations instruction with and without a computer algebra system. *Dissertation Abstracts International, 54*(11), (University Microfilms No. AAC94-12432).

Kosslyn, S. M. (1980). Image and mind. London: Harvard University Press.

Kosslyn, S.M. (1983). Ghosts in the mind's machine: Creating and using images in the brain. W. W. Norton and Company. London.
http://wwwgth.die.upm.es/~macias/doc/pubs/aircenter99/www.aircenter.net/tk.h.

Kosslyn, S.M. (1995) Mental Imagery. In S.M. Kosslyn & D.N. Osherson (Eds.), Visual Cognition: An invitation to cognitive Science (Vol 2, pp 267 – 296) Cambridge, MA: MIT Press.

Kosslyn, S. M., & Koenig, O. (1995). Wet mind: The new cognitive neuroscience (First published 1992). New York: Free Press.

Kozma, R. (1994). Will media influence learning: Reframing the debate. *Educational Technology Research and Development*, 42(2), 7-19.

Krutetski, V. A. (1976). The psychology of mathematical abilities in school children, Chicago,IL University of Chicago Press.

Kwon, O.N.; Kim, S.H.; Kim, Y. (2001). Enhancing spatial visualization through virtual reality on the web: Software design and impact analysis, *Proceedings of the 25th PME Conference 3,* 265-272.

Lampen, E. & Murray, H. (2001) Children's intuitive knowledge of the shape and structure of 3D containers, Proceedings of the 25th Psychology of Mathematics Education conference, 3, 265 -272

Larkin and Simon (1987) J. H. Larkin and F. Reif, "Understanding and teaching problem solving in physics," *Eur. J. Sci. Educ.* **1** (2), 191-203 (1979).

Lawrie, C., Pegg, J., Gutierrez, A. (2002). Unpacking students meaning of cross-sections: A frame for curriculum development, *Proceedings of the 26th PME Conference 3,* 289-296

Lesh, R., Landau, M., & Hamilton, E. (1983). Conceptual models and applied mathematical problem solving research. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 263–343). NY: Academic Press.

Linn M.C (2010) Technology and Science education: Starting points, research programs and trends , International Journal of Science Education, 25 (6), 725 - 758

Littler G.H (2002), Investigating cognitive and communicative processes throughchildren's handling of solids, Proceedings of the 26th PME conference, 3, 145 – 152.

Lohman, D. F., Ackerman, P. L., Kyllonen, P. C., & Roberts, R. D. (1999). Minding our p's and q's: On finding relationships between learning and intelligence Learning and individual differences: Process, trait, and content determinants (pp. 55–76). Washington, DC: American Psychological Association.

Martinez-Planell, R. & Trigueros, M. (2009) Student ideas on functions of two variables. Domain range and representations. In S. Swars, D., W. Stinson & Lems-Smith (Eds.). Proceedings of the 31st annual meeting of Psychology of Mathematics Education (pp – 73 – 77) Atlanta, G&A, Georgia State University.

Magidson, S. (1989), Revolving Lines: Naive Theory Building in a Guided Discovery Setting, Unpublished Manuscript, School of Education, University of California, Berkeley, USA

Mahir, N., (2009) Conceptual and procedural performance of undergraduate students in integration. *Inter. J. of Mathematical Educ. in Science and Technol.*, 40, **2**, 201-211 (2009).

Mariotti, M. A. and **Pesci, A. (1994)** Visualization in Teaching - Learning Situations. *Proceedings of* Psychology of Mathematics Education 18, 1, p 22.

Massironi, M. (2010) Translated by N Bruno, The psychology of graphic images: Seeing, drawing and communicating,

McGee, M. G. (1979). Human spatial abilities: Psychometric studies and environmental, genetic, hormonal, and neurological influences. Psychological Bulletin, 86, 889–918.

McGee, D.L and Martinez-Plannel, R. (2014) A study of semiotic registers in the development of the definite integral of functions of two and three variables, *International journal of Science and Mathematics*, August 2014, Volume 12, Issue 4, pp 883-916

Meadows, Y. (2008) Calculus III students' analytic and visual understanding of surface areas of spheres, cylinders, pyramids and prisms, P.hD dissertation, Florida State University.

Murray, H.(2001) Children's intuitive knowledge of the shape and structure of three dimensional containers, Proceedings of the 25th annual meeting of Psychology of Mathematics Education 3, 273 – 280.

NCTM (2004) National Council of Teachers of Mathematics.(2004). The teaching principle. Retrieved November 3, 2007, from http://standards.nctm.org/document/chapter2/teach.htm

Nemirovsky, R. and Noble, T. (1997) On mathematical visualization and the place where we live, *Educational Studies in Mathematics*. 33(2), 99–131.

Nilsson, P. & Juter, K. (2011) Flexibility and coordination among acts of visualization and analysis in a pattern generalization activity, *Journal of Mathematical Behavior*, *30*, 194 – 205

Nixon E.G (2002) Influence of visualization, exploring patterns and generalisation on thinking levels in the formation of concepts of sequences and series, Master of Arts Dissertation., Dept of Maths, Applied Maths and Astronomy, UNISA

Nguyen D.H. and Rebello N.S. (2011) Students' difficulties with integration in electricity, Phys. Rev. ST Phys. Educ. Res. **7**, 010113

O'Halloran, K. L. (2005) Multimodal Discourse Analysis. In K. Hyland and B. Paltridge (eds) Companion to Discourse. London and New York: Continuum. Multimodal Discourse Analysis

Orton, A. (1983). Students' Understanding of Integration. *Educational Studies in Mathematics*, Vol. 14, pp. 1 – 18

Owens (1999) The role of visualization in young students' learning, Proceedings of the 23rd International Group for the Psychology of Mathematics Education, Volume 1 (pp. 220-230).

Owens, K. D., & Clements, M. A. K. (1998). Representations in spatial problem solving in the classroom. *The Journal of Mathematical Behavior*, 17, 197–218

Palais R S (1999) The visualization of Mathematics; Towards a mathematical expliratium, Notices of Ameican Mathemtical society, 46(6). 647 -658

Pea R. (1987) 'Cognitive technologies for mathematics education', in A.H. Schoenfeld (ed.), Cognitive Science and Mathematics Education, Erlbaum, Hillsdale, NJ, pp. 89– 122.

Phillips, L.M. , Norris, S.P.,& Macnab, J.S. (2014) Visualization in mathematics, reading and science education, Dordrecht, Netherlands: Springer

Piaget, J. (1964). Development and learning. Journal of Research in Science Teaching, 2, 176-186.

Piburn, M.D., Reynolds, J.S.,MvAuliffe, C., Leedy, D.E.,Birk J.P, and Johnson K.J.(2005) The role of visualization in in learning from computer-based images, International journal of science education, 27(5). 513 – 527

Pinkernell, G. & Meissner, H.(2000) Spatial abilities in primary schools, Proceedings of the 24th PME Conference 3, 287-294. Owens, K. (1996). Responsiveness: A key aspect of spatial problem solving, Proceedings of the 20th PME Conference 4, 99-106.

Polya, G. (1957). How to solve it: Anew aspect of mathematical method (2nd ed.). Garden City, NY: Doubleday.

Presmeg, N. C. (1985). *Visually mediated processes in high school mathematics: A classroom investigation*. Unpublished Ph.D. dissertation, University of Cambridge.

Presmeg, N. C. (1986a). Visualization and mathematical giftedness. Educational Studies in Mathematics, 17, 297–311.

Presmeg, N. C. (1986b). Visualization in high school mathematics. *For the Learning of Mathematics*, 6, 42–46.

Presmeg, N. C. **(1986c).** Visualization and mathematical giftedness. *Educational Studies in Mathematics*, *17,* 297-311.

Presmeg, N. C. (1997). Generalization using imagery in mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies,metaphors, and images* (pp. 299–312). Mahwah, NJ: Erlbaum.

Presmeg, N. C. (2001). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 205-235). Rotterdam, The Netherlands: Sense Publishers

Rasmussen, C. (1999) . *Qualitative and numerical methods for analyzing differential equations: A case study of students' understandings and difficulties*. Unpublished doctoral dissertation, University of Maryland, College Park.

Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*, 55-87.

Rasmussen, C. & Keynes, K. (2003). Lines of eigenvectors and solutions to systems of differential equations. PRIMUS, Volume XIII(4), pp 308-320

Rasmussen, C and Bloomfield, H. (2007) Reinventing solutions to systems of linear differential equations. A case of emergent models involving analytical expressions. *Journal of Mathematical Behavior, 26,* 195-210.

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. Journal for Research in Mathematics Education, 37(5), 388-420 p.45

Schlatter, M.D.(1999). Using Matlab in a Multivariable Calculus Course. San Francisco, CA: the Annual International Conference on Technology in Collegiate Mathematics. (ERIC Document Reproduction Service No. ED460005)

Schlatter, M.D.(2002). Writing concepts for a multivariate calculus class. PRIMUS, Vol. XII, N4, pp 305 - 314

Seeburger (2007). Dynamic visualization for multivariate calculus, available on National Science Foundation website http://web.monroecc.edu/calcNSF

Sevimle, E and Delice, A (2010). The influence of teacher candidates' spatial visualization ability on the use of multiple representations in problem solving of definite integrals: A qualitative analysis. Joubert, M. (Ed.) Proceedings of the British Society for Research into Learning Mathematics 30(2) June 2010 From Informal Proceedings 30-2 (BSRLM) available at bsrlm.org.uk © the author - 53

Sherman, J.A. (1979) Predicting mathematical performance in high school girls and boys. Journal of educational psychology, 71. 242 - 249

Sommer, R (1978) The mind''s eye. Imagery in everyday life. Delacorte Press, New York

Sorby, S. A. (2001). A course in spatial visualization and its impact on the retention of female engineering students. *Journal of Women and Minorities in Science and Engineering, 7*, 153-172.

Stylianou, D.A., Leikin, R., & Silver, E.A. (1999). Exploring students' solution strategies in solving a spatial visualization problem involving nets. In O. Zaslavsky (Ed.), Proceedings of the 23rd PME International Conference, 4, 241-248.

Stylianou, D. A. (2000). Expert and novice use of visual representations in advanced mathematical problem solving. (Doctoral Dissertation, The University of Pittsburgh, 2000). Dissertation Abstracts International, 61(12), 9998584.

Stylianou, D.A. (2002) On the interaction of visualization and analysis: the negotiation of a visual representation in expert problem solving, *Journal of Mathematics Behaviour*, 21 (2002), p. 303 - 317

Stylianou, D.A. and Silver E.A. (2004) The role of visual representations in advanced mathematical problem solving. An examination of expert novice similarities and differences. *Mathematical thinking and learning*, 6(4), p. 353 – 387.

Sweller, J. (1999), Instructional Design in Technical Areas, (Camberwell, Victoria, Australia: Australian Council for Educational Research (1999).

Tall, D (1994) *Intuition and rigour: The role of visualisation in the calculus.*Visualisation in Teaching and Learning Mathematics, *W. Zimmerman and* S.*Cunningham (Eds.), MAA Notes Series, 19.*

Trigueros, M. (2000). Students' conceptions of solution curves and equilibrium in systems of differential equations. In Fernandez, M. L. (Ed.), *Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 93-97). Columbus, OH: ERIC.

Trigueros, M. (2004). Understanding the meaning and representation of straight line solutions of systems of differential equations. In D. McDougall and J. Ross (Eds.) Proceedings of PMENA-24, Ontario, Canada, ERIC.

Trigueros, M., & Martínez-Planell, R. (2010). Geometrical representations in the learning of two variable functions, *Educational Studies in Mathematics*, *73*(1), 3-19.

Trigueros M., & Martínez-Planell, R. (2011). How are graphs of two variable functions taught? In L.R. Wiest & T. Lamberg (Eds.), *Proceedings of the 33rd Annual Meeting of the North America Chapter of the International Group for the Psychology of Mathematics Education.* Reno, Nevada: University of Nevada at Reno

Vinner, S.(1989) The avoidance of visual considerations in Calculus students. Focus on Learning in Mathematics, V11, pp 149 - 154

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20(4), 356–366.

Winslow C. (2000) Semiotic and discursive variables in CAS based didactical engineering. Educational Studies in mathematics, V. 52, pp 271 – 288.

Zandieh, M. & McDonald, M. (1999). Student understanding of equilibrium solution in differential equations. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 253-258). Columbus, OH: ERIC.

Zaskis, D. and Dubinsky, E. (1996) Dihedral groups: A tale of two interpretations. Research in Collegiate Mathematics Education, VII, pp 61 – 82.

Zazkis, R., Dautermann, J. & Dubinsky, E. (1996). Coordinating visual and analytic strategies : a study of students' understanding. *Journal for Research in Mathematics Education*, 27(4), 435–437

Zaskis, D. (2013) Calculus Students' Representation Use in Group-Work and Individual Settings. A dissertation submitted for the degree Doctor of Philosophy in Mathematics and Science Education University of California, San Diegos State University.

Zimmermann, W. & Cunningham, S. (1991) in W. Zimmermann, & S. Cunningham (Eds.), Visualization in teaching and learning mathematics (MAA Notes, 19, pp. 26–37). Washington, DC: The Mathematical Association of America.

Zimmerman, W., & Cunningham, S. (1991). Editors' introduction: What is mathematical visualization? In W. Zimmerman & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 1–7). Washington, DC: Mathematical Association of America.

Appendices

Appendix 1 Ethical Clearance

Enquiries:
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Faculty of Applied Sciences
Chair: Ethics Committee
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oppermanm@cput.ac.za

Cape Peninsula University of Technology

29 August 2013
Mr Omar Sheikh
Department of Mathematics and Physics
Cape Peninsula University of Technology

Dear Mr Sheikh

The role of visualization in the teaching and learning of multivariate calculus and systems of ordinary differential equations (Ref. 04/2013)

The Ethics Committee has considered your application for Ethics approval for the above project and would like to advise that approval for the project is hereby granted.

We wish you every success with your research.

Kind regards

Signed

Dr Maretha Opperman (RD (SA))

Cape Peninsula
University of Technology

Appendix 2

PARTICIPANT INFORMATION SHEET AND CONSENT FORM

TITLE OF THE RESEARCH STUDY

The Role of Visualization in the Teaching and Learning of Multivariate Calculus and Systems of Ordinary Differential Equations

NAME OF RESEARCHER: T. Omar Sheikh

ADDRESS: Maths and Physics Department, Faculty of Science, Cape Peninsula University of Technology, Bellville Campus, Symphony Way, Bellville, P.O. Box 1906, Cape Town

CONTACT NUMBER AND EMAIL 072 251 3441 , sheikht@cput.ac.za

RESEARCH STUDY SUPERVISORS: PROF R.L. FRAY , PROF F. BENYAH , UNIVERSITY OF WESTERN CAPE, CAPE TOWN

INVITATION

You are invited to take part in a research study. Please take time to read the following information. Your participation is entirely voluntary. You are free not to participate if you so wish. Ask if there is anything that is not clear or if you would like more information. Before you decide it is important that you understand why the study is being done and what it involves.

1. What is the purpose of the study?

The purpose of this study is to find out if the use of enhanced visual representations using computer programs such as the Maths Visualisation Toolbox and Calc Plot 3D will help you see 3D solids and their surfaces clearly from different view points. Once you have a clear view of the solid you can sketch its projection or image in different planes. This will help you to find the limits of integrals which you need to find the volume, surface area etc of the solid. Another application is to plot the solution of simultaneous differential equations so that you can see clearly how the solution curves change with time. This is used in stability analysis of chemical reactions.

2. What will happen to me if I take part?

You will be shown how to use the programs and apply them to sketch the solids and their projections as well as phase portraits for systems of differential equations. You then participate in some lab activities with similar problems. These should help you answer multivariate questions and questions on phase portraits better than without these aids. At the end of the semester you will be asked questions about the program, the visuals and their usefulness in solving problems in Calculus or differential equations.

3. Are there any risks / benefits involved?

There are no risks involved. The programs are small (5 MB) and will fit on your flashdrive. You can work with them on any computer anywhere. The use of the program should help you solve problems in 3D calculus and systems of differential equations. The benefits are mainly to students who do maths. l

4. Do I have to take part?

It is up to you to decide whether or not to take part. You are free to withdraw at any time and without giving a reason. A decision to withdraw will not affect you in any way.

5. Will I be paid for taking part?

No. You will not be paid to take part in this study and there are no costs involved in the participation.

6. Will my taking part in the study be kept confidential?

All the information obtained will be kept confidential. You will receive a participant number and remain anonymous. The information obtained will be summarised and used in a dissertation and for publishing a research article.

7. What will your responsibilities be?

You should attend the lab sessions explaining the programs. You then try problems similar to the ones you did in the lab sessions on your own. Extra help sessions will be held if required. At the end of the semester you will be asked to answer 2 problems using the programs and whether you found the programs useful.

8. When do I start?

You indicate your willingness to take part by signing the declaration to participate. The researcher will obtain clearance from the Applied Sciences Research Ethics Committee and the first computer lab session will be held at a time convenient to everyone participating after the clearance has been granted.

Declaration by the Participant

By signing below, I …………………………………………………………………………… agree to take part in a research project entitled "The Role of Visualization in the Teaching and Learning of Multivariate Calculus and systems of Ordinary Differential Equations".

I declare that :

- I have read this participant information sheet and consent form and it is written in a language I understand
- I had a chance to ask questions and all my questions have been adequately answered
- I understand that taking part in this study is voluntary and that I have not been pressured to take part
- I may choose to leave the study at any time and will not be penalised for doing so.

Signed at (place) ……………………………………………………………………..on

(date)………………………………………..2013

……………………………………………………………………

Signature of participant

Declaration by researcher

I , …………………………………………………………..… declare that :

- I explained the information in this leaflet to the participant

- I encouraged her/him to ask questions and took time to answer them

- I am satisfied that he/she understands the study, as discussed above

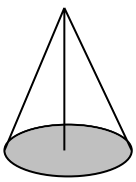
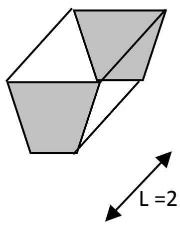
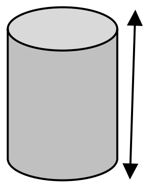
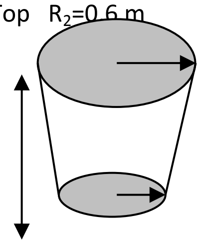
Signed at (place)…………………………………………………………………on

(date)…………………2013

…………………………………………………………………………

Signature of researcher

Appendix 3 Review of Mathematics 1 and 2

Topics	Questions for *you* to try
1. Algebra: Identities Eg: $(x \pm y)^2 = x^2 \pm 2xy + y^2$ $(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$ $\frac{f(x)}{(x-a)(x-b)^2(x^2+c)} = \frac{A}{x-a} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2} + \frac{Dx+E}{(x^2+c)}$	1) Solve: $2x^2 + 5x - 4 = 0$ 2) Solve: $x^3 + x^2 - 4x - 4 = 0$ 3) Split : $\frac{x^2+3}{x^2-3x+2}$ (partial fractions): 4) Simplify: $\frac{x^3-y^3}{(x-y)}$
2. Functions : graphs Curves **(a) straight lines y=mx+c** **b) quadratic –parabolas $y=ax^2+bx+c$** **c) hyperbolas xy=12** **d) circle: $x^2+y^2=9$ (e) ellipse: $ax^2+by^2=c$**	Sketch the graphs of 1) $y = 3 - 2x$ 2) $y = x^2 + 3x - 2$ 3) $xy=8$ 4) $y = 2^{-x}$ 5) $x^2+y^2 = 6$ 6) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ 7) $4x^2 + 9y^2 = 36$ 8) $4x^2 - 9y^2 = 36$ 9) $x/2+y/3=1$
3. Limits of functions Find the limits if they exist (a) substitution (b) cancel common factor (c) rationalize (d) L`hopital (e)graph	(a) $\lim_{x\to\infty} \frac{2x}{x^2+1}$ (b) $\lim_{x\to 2} \frac{4x}{x-2}$ (c) $f(x) = \frac{x^2-1}{x^2-x}$ Find $\lim f(x)$ as (i) $x\to 0$, (ii) $x \to 1$ (iii) $x\to\infty$ (d) $\lim_{x\to 0} \frac{\ell n(1+x)}{x}$ (e) $\lim_{x\to 0} \frac{2\,\ell n\, x^2}{x^2}$ (f) $\lim_{x\to\pi/2} \tan x \cos 3x$ (g) $\lim_{x\to 0} \frac{\ell n\, 1+x^2)}{e^x - \cos x}$
4. Continuity*:* The function *f*(*x*) is continuous at *a* if $\lim_{x\to a}(x) = f(a)$. If a function has a derivative at a point then it is also continuous at the point. However the converse is not necessarily true.	4.Discuss the continuity of $f(x) = \begin{cases} x & if\ x<1 \\ 4 & if\ x=1 \\ 5x^3-4 & if\ x>1 \end{cases}$ b. What values of *a* and *b* make f(x) differentiable everywhere? $f(x) = \begin{cases} 3x^2 & if\ x \le 1 \\ ax+b & if\ x>1 \end{cases}$
5. Mensuration Formulae **Cylinder:** $A_{cross} = \pi r^2$ $V_{cyl} = \pi r^2 h$ Surface Area (SA) $=2\pi r h$ **Sphere:** SA= $4\pi r^2$ V = $^4/_3 \pi r^3$ 	5. How many m^3 will each hold?  L =2 **Trough**  h =3,5 m **Cylinder** Top R_2=0,6 m 

6. Differentiation: (a) Product (b) Quotient (c) Chain (d) Log (e) implicit	**6.** Differentiate the following: (a) $y = x^3 + 5\sqrt{x-1}$ (b) $y = \sqrt[5]{(2x^3-1)}$ (c) $y = \sin(\tan^{-1}(2t))$ (d) $5^{2t} \ln(3t)$ (e) $y = \left(\dfrac{e^x \tan 2x}{3x-1}\right)$ (f) $e^{xy} + y^2 = \cos x$
7. Functions: Partial Differentiation Notation: $f_x \equiv \dfrac{\partial f}{\partial x}$ $f_{xy} \equiv \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ Total differential: $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$ eg: $P(V;R) = V^2/R$ then $P_{VV} = \dfrac{\partial^2 P}{\partial V^2} = \dfrac{2}{R}$	**7.** If $f(x;y) = 3x^2 + 4xy - 2y^2$ find (a) $f(2;-3)$ (b) $f_x(2;-3)$ (c) $f_y(2;-3)$ (d) $f_{xx}(2;-3)$ (e) $f_{yx}(2;-3)$ 7.1 If $P(V;R) = \dfrac{V^2}{R}$ what is (a) P_R (b) P_{VR} (c) P_{RV} 7.2. If $f(x;y) = e^y \cosh 2x$ what is the differential df ? 7.3 Find the rate of change with respect to y of $x^2 + y^2 + z^2 = 1$ at $P(2/3, 1/3, 2/3)$.
8. Total differential: a) Suppose $f(x,y) = \sqrt{x^2+y^2}$. Use the total differential to approximate the change of f as (x, y) varies from (3, 4) to (3,04, 3,98). b) Suppose the radius of a right cylinder is measured with a 2% error, while the height is measured with an error of 4%. What is the maximum relative error in V, the volume of that cylinder.	
9. Integration Techniques : (a) direct integration (b) Substitution (c) Parts (d) Partial fractions (e) Completing the square	**9.** (a) $\int (x^3 + 2\sqrt{x} + \frac{1}{x})dx$ (b) $\int 2x(3x^2-5)^4 dx$ (c) $\int e^{2x} \sin x \, dx$ (d) $\int_1^3 \dfrac{x}{x^2+2}dx$ (e) $\int \sqrt{1-4x^2}$ (f) $\int_2^3 \dfrac{2}{x^2+3x-4}dx$ (g) $\int \dfrac{dx}{x^2+2x+5}$
10. Integration applications Areas Volumes	**10.** Sketch the curve given by $y = \ell n\ x$ and find (a) the plane area enclosed between the curve and the x-axis for $1 < x < 4$ (b) the volume of the solid of rotation as the curve rotates about the x-axis
11. Matrices. operations. Determinant inverse cramers rule	**11.** Solve: $\begin{array}{l} x - 2y + 3z = 9 \\ 3x + y - 2z = -2 \\ 2x + 3y + z = 1 \end{array}$ for x, y and z
12. Differential Equations: Solve by: a) Separation b) Homogeneous c) Linear IF	**12.** Solve the following ODEs : (a) $(5x - 2y)\,dx = (2x - y)dy$ (b) $x\,dy + y\,dx = e^x\,dx$ (c) $2x^2y' - xy = y$ (d) $x\,dy = (x + 2y)dx$

Appendix 4 Tests

4.1 Pretest Math 3

1. Solve : $2x^2 + 3x - 2 = 0$

2. Sketch the graph of $y = 2 + x - x^2$

a) What is the gradient of the curve at $x = 1$?

b) What is the equation of the line of symmetry?

c) For what values of x is $2 + x - x^2 \leq 0$?

3. You are given 6 equations 1,2…6 and 6 graphs, A, B, C….. F. Match the equation to its graph

1. $xy = 4$

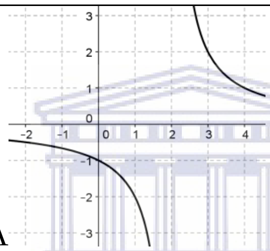
2. $y = \sqrt{x+1}$

3 $\frac{x}{2} + \frac{y}{3} = 1$

4 $\frac{x^2}{4} + \frac{y^2}{9} = 1$

5 $(x - 1)^2 + y^2 = 1$

6 $y = \frac{2}{x-2}$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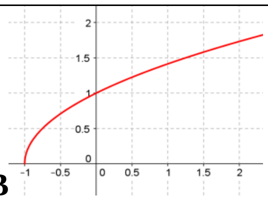
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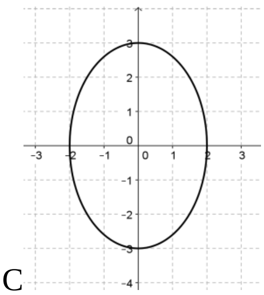
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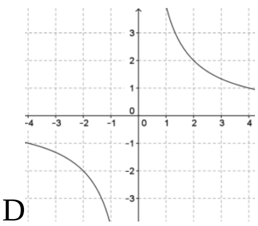
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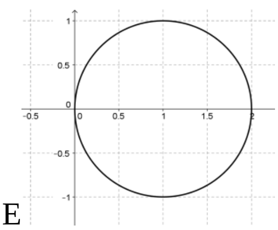

E


F

5. a) Given $P\ (V;R) = \dfrac{V}{R^2}$ what is $\dfrac{\partial P}{\partial R}$? (3)

b) Find $\dfrac{dy}{dx}$ given: a) $y = \dfrac{3}{x-2}$ b) $y = \sqrt{x^2+3}$ (3+3)

6. Solve the differential equation: $\dfrac{dy}{dx} = y + 1$ given $y = 1$ when x = 0

7. Integrate: a) $\int \sqrt{4x+1}\, dx$ b) $\int \dfrac{x}{x^2+1} dx$

8. What is the area in the first quadrant enclosed by $y = x^2 + 1$ and $0 \le x \le 2$?

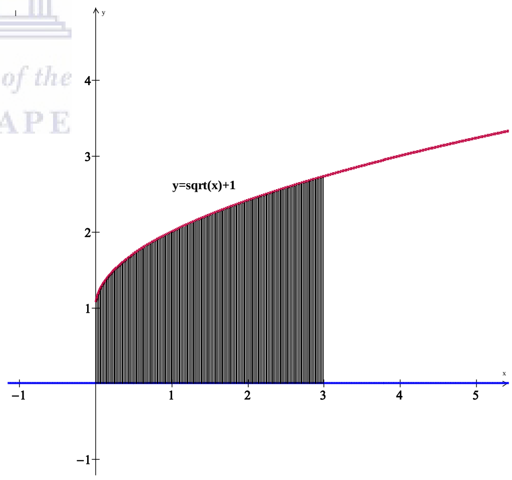
4.2 Test 1 (BT1)

Question 3 Double Triple Integration in rectangular coordinates

The shaded region on the graph shows the area between the curves:

$$y = \sqrt{x} + 1\ ;\ x = 3\ ;\ x = 0\ ;\ y = 0$$

3.1 Write down the single integral for the area of the region $\int y\, dx$. Do not evaluate **(4)**



3.2 Write down a double integral for the area in the order

a) $\int \int dy\, dx$

b) $\int \int dx\, dy$ **(6)**

3.3. Sketch and write down a double integral for the area enclosed by y = 4 – x^2 ; y = 0 and x = 0. in the order:

a) $\int \int dy\, dx$ **b)** $\int \int dx\, dy$ **(8)**

Question 4

Given the solid in the first octant bounded by the surface x = 4 – y^2 and the plane y + z = 2:

4.1 Sketch the solid and its projections in the xy-, xz and yz planes;

4.2 Write down the triple integral for the volume R in the order:

a) ∭ dzdydx

b) ∭ dydzdx

c) ∭ dxdydz

4.3 Evaluate one of the integrals**.**

4.3 Test 2 (BT2)

1.1 Sketch the following 3D solids given by the algebraic equation in the coordinate system stated.

a) $z = x^2 + y^2$, $0 < z < 2$ Rectangular coordinates

b) $\rho = 4\cos\phi$ Spherical coordinates

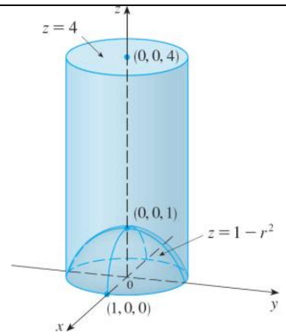
c) $r = 3\sin\theta$ Cylindrical coordinates (10)

1.2. A solid D, lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$ and above the paraboloid $z = 1 - x^2 - y^2$. See diagram.

a) Set up a triple integral in cylindrical coordinates for the volume of the solid. (4)

b) Write down an equivalent volume integral for D in rectangular coordinates is: $\int\int\int dz\, dy\, dx$ (4)

c) Evaluate one of the integrals you set up. (6) [20]



2. You are required to find the volume of the solid enclosed above by the surface represented by $x^2 + y^2 + z^2 = 8$ and below by the surface $z = \sqrt{x^2 + y^2}$.

a) Sketch the surfaces and the solid (4)

b) Sketch the xy and the yz projections of the solid (4)

c) Set up a triple integral in cylindrical coordinates for the volume of the solid. (4)

d) Set up the triple integral in spherical coordinates for the volume of the solid. (4)

e) Evaluate one of the integrals that you set up. (6) **[22]**

TEST 2 (BT2 Continued) Dynamical Systems

4.1 Given the linear system of DEs:

$$\frac{dx}{dt} = -2x \qquad x(0) = 3$$
$$\frac{dy}{dt} = -y \qquad y(0) = 2$$

a) Find the equilibrium point (2)

b) Find the particular solutions x(t) and y(t) of the DEs (4)

c) Sketch x-t and y-t graphs of the solutions (4)

d) Use eigen-pairs to determine the stability of the system (4)

e) Sketch the phase plane (4)

f) Find the y(x) solution of the system (4)
[22]

5. Given the non-linear system of DEs:

$$\frac{dx}{dt} = x\,(2 - y)$$
$$\frac{dy}{dt} = y(x - 3)$$

a) Find the equilibrium points (2)

b) Use eigen-values at each equilibrium point to confirm the stability of the equilibrium points in the system. (5)

c) Sketch the phase plane and trajectories (with arrows) on the phase plane to show the type of stability of each equilibrium point in the system. (4)

d) Find the y(x) analytical solution for the system (5) **[14]**

Test 3 (BT3)

Question 2 Double and Triple Integration

2.1 a) Sketch the region represented by the double integral : $\int_{0}^{16}\int_{\sqrt{y}}^{4}\sqrt{x^3+1}\,dx\,dy$ (3)

b) Switch the order of integration and evaluate the integral. (7)

2.2 Sketch and find the volume of the solid in the **first octant** bounded by the surfaces represented by $z = 4 - y^2$, $x + y = 3$ and the coordinate axes. (10)

Assignment

1. You are required to find the volume of the solid bounded above by the surface represented by $x^2 + y^2 + z = 6$ and below by the surface $z = \sqrt{x^2 + y^2}$.

a) Sketch the surfaces and the solid (4)

b) Sketch the xy and the yz projections of the solid (4)

c) Set up a triple integral in cylindrical coordinates for the volume of the solid. (4)

d) Set up the triple integral in spherical coordinates for the volume of the solid. (4)

e) Evaluate one of the integrals that you set up. (4) **[20]**

Appendix 5 Statistical Analysis

5.1 Two Way ANOVA for BT1 with Gender and Group as Factors

The GLM Procedure

Dependent Variable: BT1

Class Level Information		
Class	**Levels**	**Values**
Group	2	Control Experimental
Gender	2	Female Male

Number of Observations Read	50
Number of Observations Used	50

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	379.16532	126.38844	0.56	0.6435
Error	46	10364.51468	225.31554		
Corrected Total	49	10743.68000			

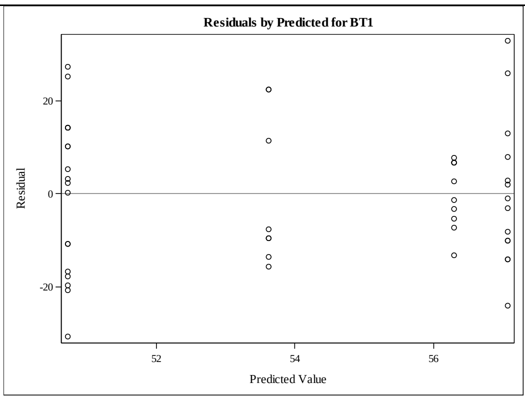
R-Square	Coeff Var	Root MSE	BT1 Mean
0.035292	27.75613	15.01051	54.08000

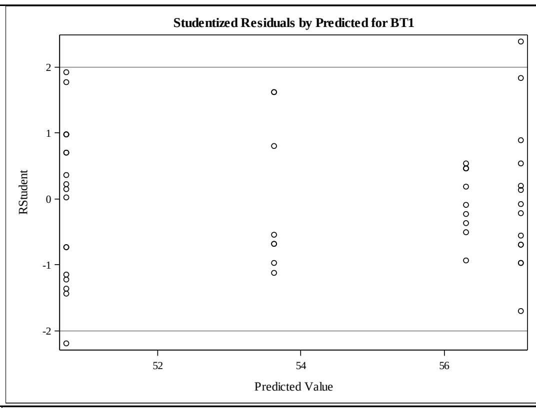
Source	DF	Type II SS	Mean Square	F Value	Pr > F
Group	1	247.5664944	247.5664944	1.10	0.3000
Gender	1	37.2333276	37.2333276	0.17	0.6863
Group*Gender	1	12.9058361	12.9058361	0.06	0.8119

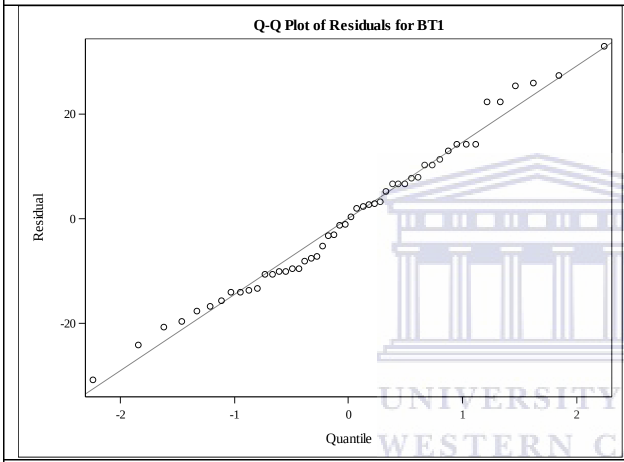
Two Way ANOVA for BT1 with Gender and Group as Factors

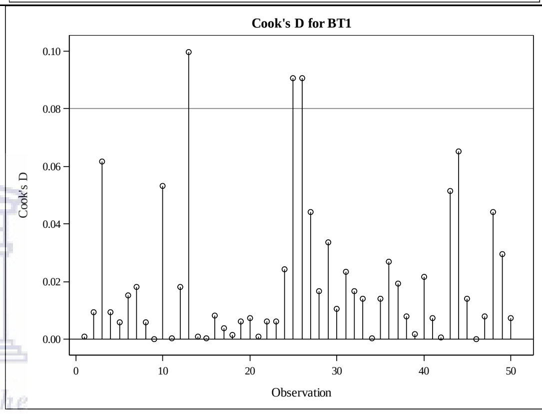
The GLM Procedure

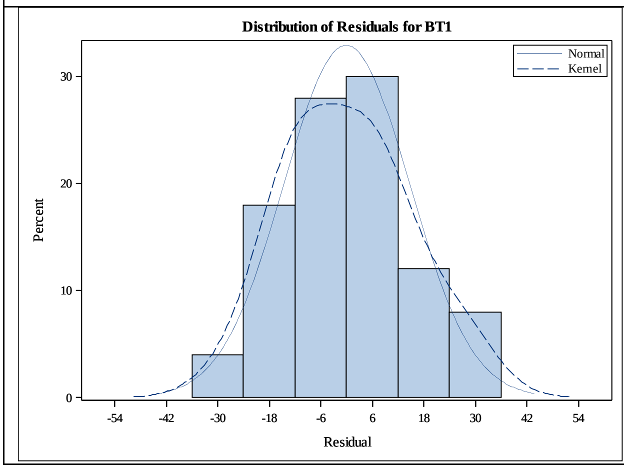
Dependent Variable: BT1

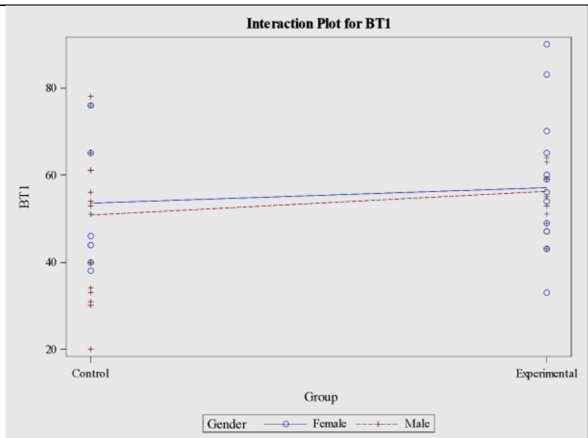












5.2 Two Way ANOVA for BT2 with Gender and Group as Factors

The GLM Procedure

Dependent Variable: BT2

Class Level Information		
Class	**Levels**	**Values**
Group	2	Control Experimental
Gender	2	Female Male

Number of Observations Read	50
Number of Observations Used	50

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	548.66571	182.88857	0.80	0.5008
Error	46	10529.81429	228.90901		
Corrected Total	49	11078.48000			

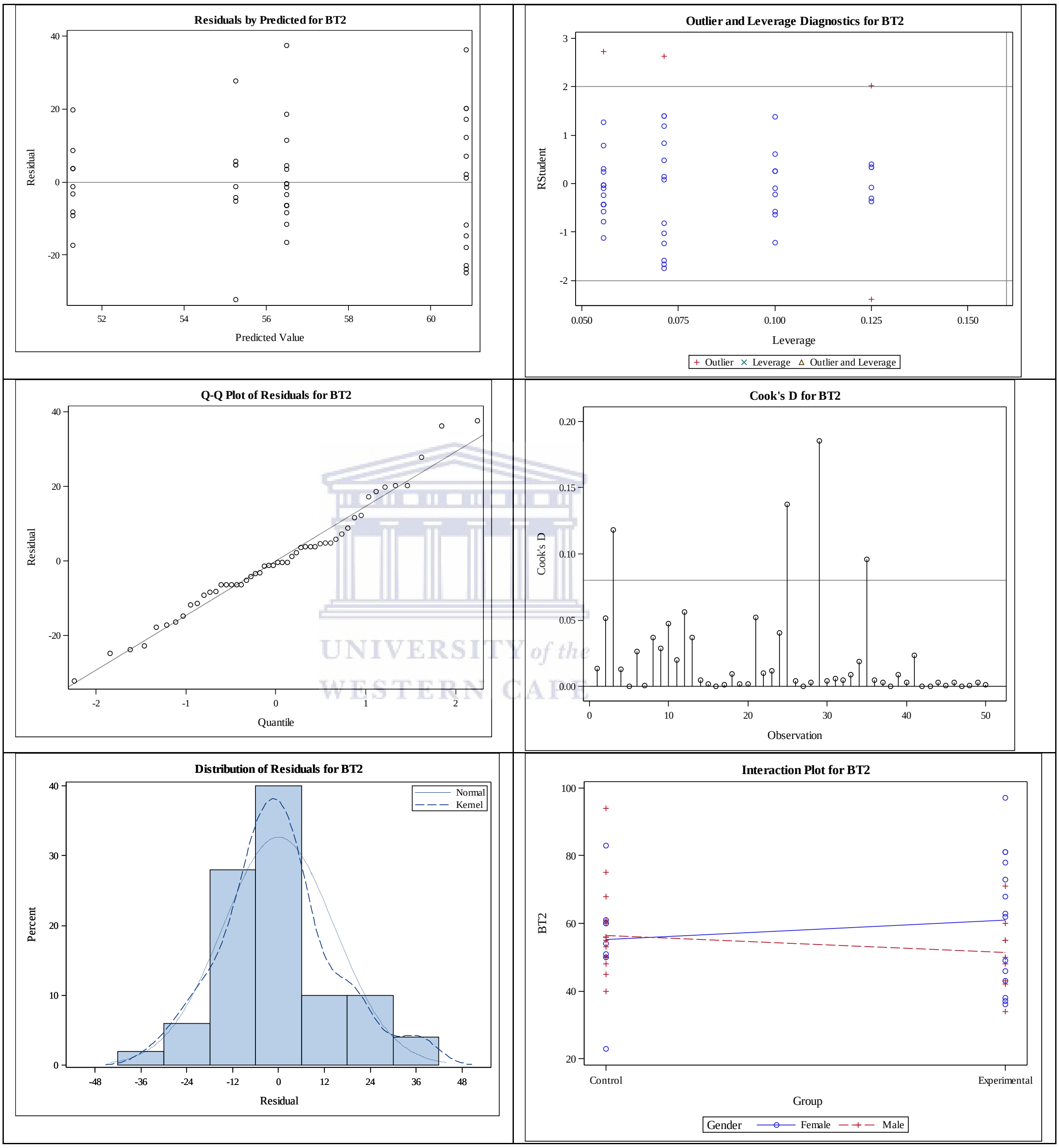
R-Square	Coeff Var	Root MSE	BT2 Mean
0.049525	26.78778	15.12974	56.48000

Source	DF	Type II SS	Mean Square	F Value	Pr > F
Group	1	2.0699571	2.0699571	0.01	0.9247
Gender	1	209.6475046	209.6475046	0.92	0.3436
Group*Gender	1	331.8170559	331.8170559	1.45	0.2348

Two Way ANOVA for BT2 with Gender and Group as Factors

The GLM Procedure

Dependent Variable: BT2



5.3 Two Way ANOVA for BT3 with Gender and Group as Factors

The GLM Procedure
Least Squares Means
Adjustment for Multiple Comparisons: Bonferroni

Class Level Information		
Class	**Levels**	**Values**
Group	2	Control Experimental
Gender	2	Female Male

Number of Observations Read	50
Number of Observations Used	50

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	1256.035556	418.678519	3.09	0.0360
Error	46	6224.044444	135.305314		
Corrected Total	49	7480.080000			

R-Square	Coeff Var	Root MSE	BT3 Mean
0.167917	20.50790	11.63208	56.72000

Source	DF	Type II SS	Mean Square	F Value	Pr > F
Group	1	555.1297420	555.1297420	4.10	0.0486
Gender	1	140.3384500	140.3384500	1.04	0.3138
Group*Gender	1	334.7965928	334.7965928	2.47	0.1226

Group	BT3 LSMEAN	H0:LSMean1=LSMean2 Pr > |t|
Control	52.5277778	0.0333
Experimental	60.1000000	

Two Way ANOVA for BT3 with Gender and Group as Factors

The GLM Procedure
Least Squares Means
Adjustment for Multiple Comparisons: Bonferroni

Group	BT3 LSMEAN	95% Confidence Limits	
Control	52.527778	47.553224	57.502331
Experimental	60.100000	55.252807	64.947193

Least Squares Means for Effect Group				
i	j	Difference Between Means	Simultaneous 95% Confidence Limits for LSMean(i)-LSMean(j)	
1	**2**	-7.572222	-14.517829	-0.626615

Gender	BT3 LSMEAN	H0:LSMean1=LSMean2	
			Pr > |t|
Female	58.0000000		0.3335
Male	54.6277778		

Gender	BT3 LSMEAN	95% Confidence Limits	
Female	58.000000	52.811390	63.188610
Male	54.627778	50.010442	59.245113

Least Squares Means for Effect Gender				
i	j	Difference Between Means	Simultaneous 95% Confidence Limits for LSMean(i)-LSMean(j)	
1	**2**	3.372222	-3.573385	10.317829

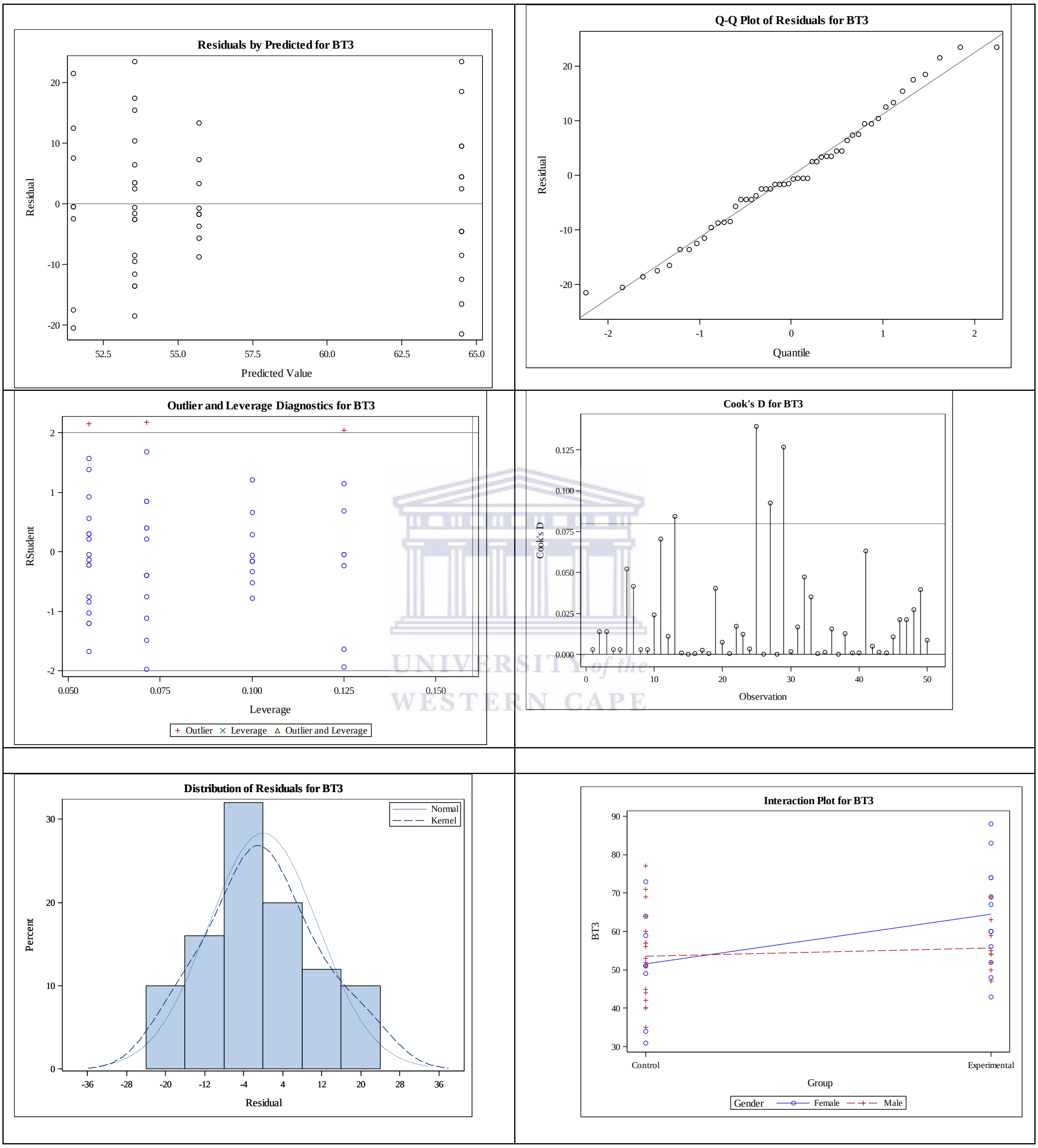
Two Way ANOVA for BT3 with Gender and Group as Factors

The GLM Procedure
Least Squares Means
Adjustment for Multiple Comparisons: Bonferroni

Group	Gender	BT3 LSMEAN	95% Confidence Limits	
Control	Female	51.500000	43.221843	59.778157
Control	Male	53.555556	48.036784	59.074327
Experimental	Female	64.500000	58.242301	70.757699
Experimental	Male	55.700000	48.295791	63.104209

Least Squares Means for Effect Group*Gender				
i	j	Difference Between Means	Simultaneous 95% Confidence Limits for LSMean(i)-LSMean(j)	
1	2	-2.055556	-15.683399	11.572288
1	3	-13.000000	-27.214253	1.214253
1	4	-4.200000	-19.412934	11.012934
2	3	-10.944444	-22.373134	0.484245
2	4	-2.144444	-14.793686	10.504797
3	4	8.800000	-4.478937	22.078937

Two Way ANOVA for BT3 with Gender and Group as Factors



5.4 Linear Regression Results

The REG Procedure
Model: Linear_Regression_Model
Dependent Variable: BT2

Number of Observations Read	21
Number of Observations Used	21

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	6	4753.56478	792.26080	9.67	0.0003
Error	14	1147.38761	81.95626		
Corrected Total	20	5900.95238			

Root MSE	9.05297	**R-Square**	0.8056
Dependent Mean	58.38095	**Adj R-Sq**	0.7222
Coeff Var	15.50672		

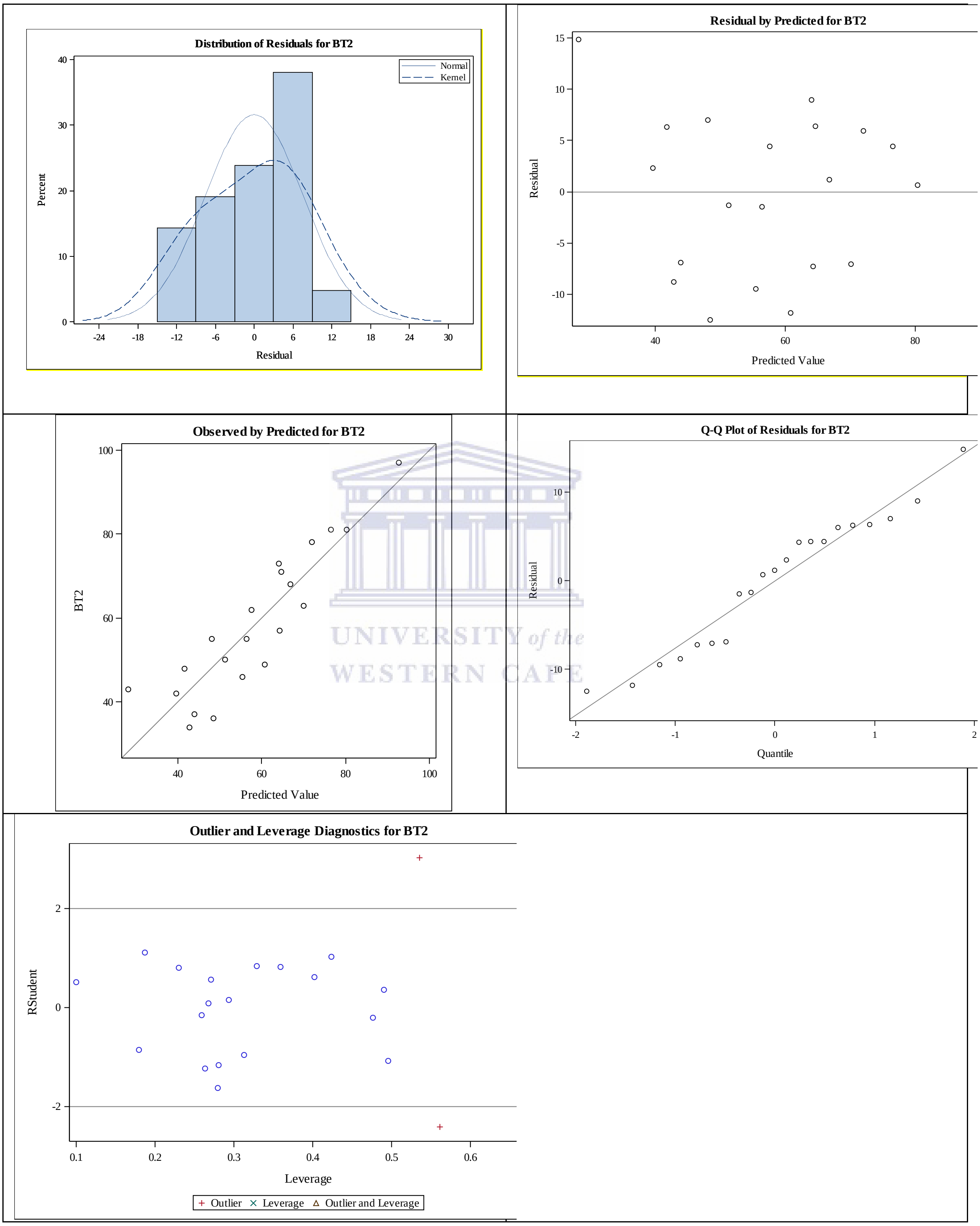
Parameter Estimates								
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**	**Variance Inflation**	**95% Confidence Limits**	
Intercept	1	-9.11499	14.45827	-0.63	0.5386	0	-40.12489	21.89491
Surface	1	0.22226	0.10262	2.17	0.0481	1.73784	0.00217	0.44236
Proj	1	-0.13638	0.12826	-1.06	0.3056	1.16443	-0.41146	0.13871
Nets	1	0.08021	0.12320	0.65	0.5255	1.72356	-0.18403	0.34445
xsect	1	0.49775	0.29200	1.70	0.1103	2.23864	-0.12853	1.12402
Rotns	1	-0.02538	0.12655	-0.20	0.8439	2.01359	-0.29682	0.24605
Pretest	1	0.50161	0.21530	2.33	0.0353	2.41776	0.03983	0.96339

Test of First and Second Moment Specification		
DF	**Chi-Square**	**Pr > ChiSq**
20	16.71	0.6716

The REG Procedure

Model: Linear_Regression_Model

Dependent Variable: BT2



5.5 Stepwise Linear Regression Results

The REG Procedure
Model: Linear_Regression_Model
Dependent Variable: BT2

Number of Observations Read	21
Number of Observations Used	21

Stepwise Selection: Step 1
Variable Pretest Entered: R-Square = 0.6099 and C(p) = 11.0904

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	1	3598.77121	3598.77121	29.70	<.0001
Error	19	2302.18117	121.16743		
Corrected Total	20	5900.95238			

Variable	Parameter Estimate	Standard Error	Type II SS	F Value	Pr > F
Intercept	8.04665	9.54316	86.14536	0.71	0.4096
Pretest	0.91755	0.16836	3598.77121	29.70	<.0001

Bounds on condition number: 1, 1

Stepwise Selection: Step 2
Variable Surface Entered: R-Square = 0.7303 and C(p) = 4.4187

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	2	4309.46549	2154.73274	24.37	<.0001
Error	18	1591.48689	88.41594		
Corrected Total	20	5900.95238			

Variable	Parameter Estimate	Standard Error	Type II SS	F Value	Pr > F
Intercept	1.43841	8.47868	2.54472	0.03	0.8672
Surface	0.25485	0.08989	710.69428	8.04	0.0110
Pretest	0.71945	0.15990	1790.04128	20.25	0.0003

Bounds on condition number: 1.236, 4.9442

Stepwise Selection: Step 3

Variable xsect Entered: R-Square = 0.7862 and C(p) = 2.3950

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	3	4639.23898	1546.41299	20.84	<.0001
Error	17	1261.71340	74.21844		
Corrected Total	20	5900.95238			

Variable	Parameter Estimate	Standard Error	Type II SS	F Value	Pr > F
Intercept	-16.60226	11.55827	153.13029	2.06	0.1690
Surface	0.26172	0.08242	748.37161	10.08	0.0055
xsect	0.53243	0.25259	329.77349	4.44	0.0502
Pretest	0.45830	0.19186	423.49556	5.71	0.0288

Bounds on condition number: 2.12, 15.623

All variables left in the model are significant at the 0.1500 level.

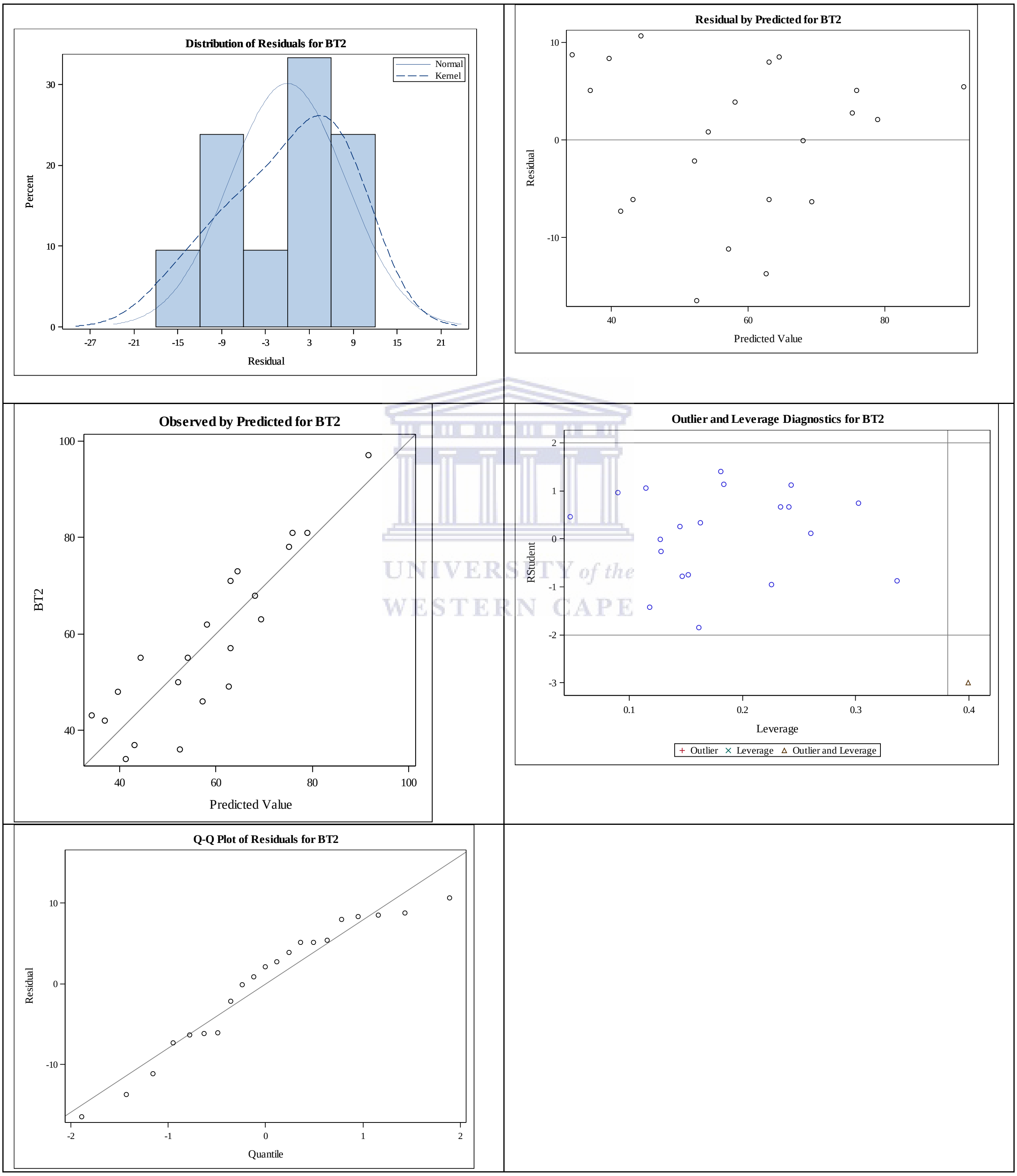
No other variable met the 0.1500 significance level for entry into the model.

Summary of Stepwise Selection								
Step	**Variable Entered**	**Variable Removed**	**Number Vars In**	**Partial R-Square**	**Model R-Square**	**C(p)**	**F Value**	**Pr > F**
1	Pretest		1	0.6099	0.6099	11.0904	29.70	<.0001
2	Surface		2	0.1204	0.7303	4.4187	8.04	0.0110
3	xsect		3	0.0559	0.7862	2.3950	4.44	0.0502

Number of Observations Read	21
Number of Observations Used	21

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	3	4639.23898	1546.41299	20.84	<.0001
Error	17	1261.71340	74.21844		
Corrected Total	20	5900.95238			

Root MSE	8.61501	**R-Square**	0.7862
Dependent Mean	58.38095	**Adj R-Sq**	0.7485
Coeff Var	14.75655		



Parameter Estimates								
Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**	**Variance Inflation**	**95% Confidence Limits**	
Intercept	1	-16.60226	11.55827	-1.44	0.1690	0	-40.98807	7.78354
Surface	1	0.26172	0.08242	3.18	0.0055	1.23798	0.08783	0.43562
xsect	1	0.53243	0.25259	2.11	0.0502	1.84978	-0.00048144	1.06535
Pretest	1	0.45830	0.19186	2.39	0.0288	2.12005	0.05351	0.86309

BT2	predicted_BT2	residual_BT2	student_BT2	rstudent_BT2	lcl_BT2	lclm_BT2	ucl_BT2	uclm_BT2
73	64.5234	8.4766	1.04570	1.04877	45.3337	58.3693	83.713	70.677
37	43.1417	-6.1417	-0.87522	-0.86888	22.1287	32.5978	64.155	53.686
97	91.5788	5.4212	0.75362	0.74364	70.8329	81.5777	112.325	101.580
49	62.6857	-13.6857	-1.73458	-1.85493	43.0988	55.3868	82.273	69.985
62	58.0859	3.9141	0.46562	0.45463	39.4797	54.1085	76.692	62.063
78	75.2552	2.7448	0.34822	0.33903	55.6550	67.9205	94.855	82.590
63	69.3108	-6.3108	-0.79292	-0.78388	49.8487	62.3536	88.773	76.268
81	75.8888	5.1112	0.68095	0.66982	55.6415	66.9680	96.136	84.810
46	57.1828	-11.1828	-1.38215	-1.42323	37.9643	50.9395	76.401	63.426
36	52.4830	-16.4830	-2.46907	-2.99094	30.9803	40.9942	73.986	63.972
81	78.9216	2.0784	0.26086	0.25358	59.4749	72.0075	98.368	85.836
68	68.0977	-0.0977	-0.01214	-0.01177	48.7964	61.6039	87.399	74.591
34	41.3040	-7.3040	-0.96329	-0.96113	21.1836	32.6751	61.424	49.933
50	52.1279	-2.1279	-0.26448	-0.25711	32.8255	45.6307	71.430	58.625
48	39.6461	8.3539	1.11456	1.12310	19.3811	30.6850	59.911	48.607
43	34.2386	8.7614	1.12559	1.13510	14.4637	26.4493	54.013	42.028
55	54.1847	0.8153	0.11005	0.10680	33.7787	44.9091	74.591	63.460
55	44.3201	10.6799	1.36955	1.40864	24.5703	36.5947	64.070	52.045
71	63.0407	7.9593	0.96842	0.96655	44.0655	57.5923	82.016	68.489
42	36.8974	5.1026	0.67667	0.66549	16.7077	28.1081	57.087	45.687
57	63.0856	-6.0856	-0.76720	-0.75752	43.5750	55.9938	82.596	70.177

END OF APPENDICES