Topogenous Structures on Categories

A research project submitted in partial fulfilment of the requirements for the degree of Masters of Science in the Department of Mathematics and Applied Mathematics, University of the Western Cape.

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Dedication

Dedicated with much appreciation and love to my mom. Without your advice and encouragement I would have never been here.

Abstract

Although the interior operators correspond to a special class of neighbourhood operators, the closure operators are not nicely related to the latter. We introduce and study the notion of topogenous orders on a category which provides a basis for categorical study of topology. We show that they are equivalent to the categorical neighbourhood operators and house the closure and interior operators. The natural notion of strict morphism with respect to a topogenous order is shown to capture the known ones in the settings of closure, interior and neighbourhood operators.

November 2016.

Declaration

I declare that *Topogenous structures on categories* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Minani Iragi November 2016

Signed:

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Finally, I thank the Lord for his blessings and amazing grace.

Contents

Chapter 1

Introduction

In his book [Csá63] (preceded by a French edition [Csá60]) Á. Császár has introduced the syntopogenous structures to provide a common generalization of topological, proximity and uniform structures. The fundamental concept of this theory is the notion of order relation between subsets of a set. He termed topogenous order $<$ on a set X, a binary relation on 2^X which satisfies the following:

(01) $\emptyset < \emptyset$ and $X < X$,

(02) $A < B \Rightarrow A \subseteq B$,

(03) $A \subseteq B < C \subseteq D \Rightarrow A < D$ and

(04) $A_i < B_i \ (i \in I) \ \Rightarrow \bigcup_{i\in I} A_i < \bigcup_{i\in I} B_i$ and $\bigcap_{i\in I} A_i < \bigcap_{i\in I} B_i$ if I is finite.

A pair $(X, \mathcal{S})$ consisting of a set X and a non empty family of topogenous orders on X is called syntopogenous space provided the following hold:

$(S1)$ for every $<, <' \in \mathcal{S}$ there is $<'' \in \mathcal{S}$ with the property that $< \subseteq <''$ and $< \subseteq <''$

$(S2)$ for $< \in \mathcal{S}$ there is $<' \in \mathcal{S}$ with the property that $< \subseteq <' \circ <'$

A syntopogenous space $(X, \mathcal{S})$ is a proximity provided $\mathcal{S} = \{<\}$ and $<$ satisfies the axiom below

$$A < B \Rightarrow X \setminus B < X \setminus A \qquad (*).$$

It is topological if $\mathcal{S} = \{<\}$ and it holds that

$$A_i < B_i \ (i \in I) \ \Rightarrow \bigcup_{i\in I} A_i < \bigcup_{i\in I} B_i$$

and a uniform if (04) hold for any indexed set I and each $<\in \mathcal{S}$ satisfies $(*)$.

Many researchers have studied syntopogenous structures (see.e.g.[AL97, Tan76, SP64, Csá83, CM81]). Although the above study was from the set theoretic point of view, Sadr [Sad68] attemped to introduce the notion of topogenous structures on a category. However, instead of basing his study on a subobject lattice as one does when studying closure operators on categories, he departed from an ordered class $(C, \leq)$ then studied the topogenous structures induced by the pair $(\mathcal{C}^*, J)$ where $\mathcal{C}^*$ is a category associated to the class $(C, \leq)$ and J is an ideal of $\mathcal{C}^*$. Sadr obtained a number of elementary properties that Á. Császár proved in [Csá63].

Closure operators themselves were introduced first in analysis by Riez ([Rie09]) and Moore ([Moo09]). Thereafter, they have then been intensively studied in other branches of mathematics: Birkhoff ([Bir40]) in lattice theory and Birkhoff ([Bir37]) in Algebra, Kuratowski ([Kur22]) and Cech([Čec37]) in topology, Hertz ([Her22]) and Tarski ([Tar29])in logic. It was Dikran Dikranjan and Eraldo Giuli who, in their fundamental article ([DG87]), introduced the categorical closure operators. This has unified different important notions and has led to interesting examples and applications in various areas of Mathematics. Closure operators has allowed a categorical study of many topological concepts such as separation, compactness, connectedness, perfectness, closedness, regularity (see e.g.[CGT96, CT97, Cas01, Cle01, GT00, Šla09]) studied in general topology with the classical Kuratowski closure operator.

In general topology, associated closure and interior operators provide an equivalent description of the topology. A categorical notion of interior operator was introduced by Vorster in [Vor00]. However, it is not categorical dual to the notion of closure operator. One does not obtain a categorical interior operator when taking a closure operator in the opposite category. The recent paper [DT15] introduces dual closure operators which do not lead to categorical interior operators. A number of recent papers have been devoted to the investigation of categorical interior operators: [Cas15, RH14, CM13, HŠ11, HŠ10, CR10].

One often defines many topological concepts using the Kuratowski closure operator, but neighbourhoods are intuitive for introducing the notion of convergence on spaces ([Šla12]). This seems to have been the principal motivation for introducing a concept of neighbourhood with respect to a closure operator in [GŠ09] and [Šla11]. Categorical neighbourhood operators were introduced in [HŠ11] and subsequently studied in [Raz12a, RH14]. This has motivated a further investigation of the interaction between closure, interior and neighbourhood operators. The paper ([HŠ10]) presents a systematic analysis of some interactions as well as a framework in which a number of other investigations can be understood.

In a category with a suitable factorization system, we first define the closure, interior and neighbourhood operators. Then following [Raz12a] and [HŠ10] we investigate some interactions between the three operators. We show that interior operators are special neighbourhood operators and take a closer look at three correspondances between neighbourhood or interior and closure operators. These are not Galois connections in general unless some restrictions are considered on the subobject lattices on which the operators interact. The equivalent description of the continuity condition of neighbourhood operators leads us to four different notions of morphisms.

We next introduce the notion of topogenous orders on a category which provides a unified treatement of closure, interior and neighbourhood operators. We show that topogenous orders are equivalent to categorical neighbourhood operators. Then we proceed by proving that interior and closure operators correspond to special kind of topogenous orders. To demonstrate the advantages of working in this more general topogenous order setting, we define strict subobjects and morphisms with respect to a topogenous order and show that they agree with the generalization of both closed and open maps. Looking closely at the equivalence between topogenous orders and neighbourhood operators, we naturally define the notions of open, closed, initial and final morphisms relative to a topogenous order and study their basic properties.

Our thesis contains this introduction and five chapters with two to three sections each, organised as follows:

The notion of subobjects on a category is presented in chapter 2. We then slowly move to the factorization system on the category $\mathcal{C}$ which enables us to efficiently deal with images and pre-images of subobjects. Some definitions and results on closure and interior operators which will be of use throughout this work are also presented. The theory of categorical neighbourhood operators is investigated in chapter 3. We explore the relationship between neighbourhood or interior and closure operators. Then follows some basic properties of four classes of morphisms with respect to a neighbourhood operator. Chapter 4 which constitutes the core of this work introduces the theory of topogenous structures on categories. We demonstrate the equivalence between topogenous orders and neighbourhood operators. The interior and closure operators are shown to be nicely embedded in the topogenous orders. In chapter 5 the strict morphisms with respect to the topogenous orders are studied. We show that they capture the notions of i-open, c-closed and ν-open morphisms which were studied separately for closure, interior and neighbourhood operators. The pulback stability of such morphisms in $\mathcal{M}$ is also discussed. Using the equivalence between neighbourhood operators and topogenous orders we present in chapter 6 an analogue of the notions of open, closed, initial and final morphisms considered in chapter 3 for neighbourhood operators.

This work is meant for a reader with some knowledge of general topology and category theory with little more presupposed from algebra, order and lattices ([Fuc73, DP02, Eng89]). However, we have recalled the necessary tools which can help the reader to go smoothly through the work. Our chapters are numbered according to their order of appearance in the text. The same rule holds for sections in chapters and for propositions, lemmas, definitions in sections. The main results of this thesis were published in [HIR16]

Chapter 2

Preliminaries

Throughout the text, we consider a category $\mathcal{C}$ and a fixed class $\mathcal{M}$ of $\mathcal{C}$-monomorphisms. The class $\mathcal{M}$ is assumed to be closed under composition and contain all isomorphisms. Our categorical terminology follows [AHS06] as well as [ML98]. We discuss the notions of subobjects, inverse images and image factorization as they will be needed for the whole of the thesis. From there we provide some defintions and results on closure and interior operators that are needed in the remaining chapters.

2.1 $\mathcal{M}$-subobjects, Images and Pre-images

A subobject of an object X in a category $\mathcal{C}$ provides a generalisation of notions such as subsets, subrings, subgroups, subspaces and a lot of other classical sub-structures notions that appear in mathematics. The subobjects shall be described by special morphisms $m : M \longrightarrow X$ that we think of as generalized inclusion maps $m : M \hookrightarrow X$.

Definition 2.1.1. [DT95] For every object X in $\mathcal{C}$, the class sub$X$ of all $\mathcal{M}$-morphisms with codomain X is called the subobjects of X.

The relation on sub$X$ given by $m \leq n$ if and only if there exists j such that $n \circ j = m$

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "m"'] & & N \arrow[dl, "n"] \\ & X & \end{tikzcd}$$

is reflexive and transitive, hence sub$X$ is a preordered class. Such j is uniquely determined since n is monic. Furthermore, $m \leq n$ and $n \leq m$ if and only if there exists an isomorphism j with $n \circ j = m$. We write $m \cong n$ in this case. $\cong$ is an equivalence relation, hence sub$X$ modulo $\cong$ is is a partially ordered class for which lattice notations such as $\bigwedge$, $\bigvee$, $\wedge$, $\vee$ can be used. We shall use these notations for elements of sub$X$ rather than their $\cong$ equivalence classes. Indeed for $n, m \in$ sub$X$, if the meet exists, $n \wedge m$ denotes a representative in sub$X$ of the meet of the $\cong$ equivalence classes of n and m. If $[m]$ denotes the $\cong$-equivalence class of m, we have that $m \leq n \Leftrightarrow [m] \subseteq [n]$ and $m \cong n \Leftrightarrow [m] = [n]$.

For the rest of the thesis, if $m, n \in \mathcal{M}$ and $n \cong m$, we shall simply write $n = m$.

Definition 2.1.2. [DT95] We will say that $\mathcal{C}$ has $\mathcal{M}$-pullbacks, if for every morphism $f : X \longrightarrow Y$ and every $n \in$ sub$Y$ a pullback diagram

$$\begin{tikzcd}
D \arrow[dr, dotted, "q"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "g"'] & & \\
& M \arrow[r, "h"] \arrow[d, "m"'] & N \arrow[d, "n"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

exists in $\mathcal{C}$ with $m \in$ sub$X$. This means that $n \circ h = f \circ m$, and whenever $f \circ g = n \circ p$ holds in $\mathcal{C}$, then there is a unique morphism q with $h \circ q = p$ and $m \circ q = g$.

The morphism m is uniquely determined up to isomorphism, it is called the inverse image of n under f and denoted by $f^{-1}(n) : f^{-1}(N) \longrightarrow X$.

The pullback property of the previous definition yields the following

Proposition 2.1.3. *[DT95] If $\mathcal{C}$ has $\mathcal{M}$-pullbacks, then for each $f : X \longrightarrow Y$ the map $f^{-1}(-) : subY \longrightarrow subX$ is an order preserving map.*

Proof. Let $m : M \longrightarrow Y$ and $n : N \longrightarrow Y$ be two subobjects of Y such that $m \leq n$. Then there is a morphism j with $m = n \circ j$. We have the following diagram.

$$\begin{tikzcd}
f^{-1}(M) \arrow[rrr, "h"] \arrow[dr, dotted, "k"] \arrow[dd, "f^{-1}(m)"'] & & & M \arrow[dl, "j"'] \arrow[dd, "m"] \\
& f^{-1}(N) \arrow[r, "l"] \arrow[dl, "f^{-1}(n)"] & N \arrow[dr, "n"'] & \\
X \arrow[rrr, "f"] & & & Y
\end{tikzcd}$$

□

Since $n \circ (j \circ h) = f \circ f^{-1}(m)$, the universal property of pullbacks yields a unique morphism $k : f^{-1}(M) \longrightarrow f^{-1}(N)$ such that $f^{-1}(n) \circ k = f^{-1}(m)$; that is

$f^{-1}(m) \leq f^{-1}(n)$.

Remark 2.1.4. Although we have included the proof of the monotonicity of $f^{-1}(-)$, we shall see later (cf. Proposition 2.1.12) that this follows by adjointness.

The notion of adjointness in the context of preordered classes is important for us in this work.

Definition 2.1.5. [DT95] A pair of mappings $\varphi : P \longrightarrow Q, \psi : Q \longrightarrow P$ between preordered classes is called adjoint if

$$m \leq \psi(n) \Leftrightarrow \varphi(m) \leq n \qquad (*)$$

holds for all $m \in P$ and $n \in Q$.

One says that φ is left-adjoint to ψ or ψ is right-adjoint to φ and writes $\varphi \dashv \psi$

Proposition 2.1.6. *[DT95] Let $\varphi : P \longrightarrow Q, \psi : Q \longrightarrow P$ be a pair of mappings between preordered classes, the following are equivalent:*

(1) $\varphi \dashv \psi$*;*

(2) ψ *is order preserving, and* $\varphi(m) \cong \min\{n \in Q \mid m \leq \psi(n)\}$ *holds for all* $m \in P$*;*

(3) φ *is order preserving, and* $\psi(n) \cong \max\{m \in P \mid \varphi(m) \leq n\}$ *holds for all* $n \in Q$*;*

(4) φ *and* ψ *are order preserving, and* $m \leq \psi(\varphi(m))$ *and* $\varphi(\psi(n)) \leq n$ *holds for all* $m \in P, n \in Q$*.*

Proof. [DT95] (1) $\Rightarrow$ (2) and (3), put $n = \varphi(m)$ in $(*)$, then $m \leq \psi(\varphi(m))$. Now let

$$Q_m = \{n \in Q \mid m \leq \psi(n)\}$$

we get that $\varphi \in Q_m$. Furthermore, for all $n \in Q_m$, $(*)$ gives $\varphi(m) \leq n$; hence

$$\varphi(m) \cong \min Q_m.$$

This formula implies that φ is an order preserving map since if $m \leq n$ in P, we have that $m \leq \psi(\varphi(n))$. Thus, by $(*)$ we get that $\varphi(m) \leq \varphi(n)$. The fact that ψ is order-preserving and its formula in (3) are obtained dually.

(2) $\Rightarrow$ (4) If $m \leq n$ in P then

$$\{p \in Q \mid n \leq \psi(p)\} \subseteq \{q \in Q \mid m \leq \psi(q)\}.$$

by taking the minimum we get that

$$\varphi(m) \leq \varphi(n)$$

Thus φ is monotone. Moreover since $\varphi(m) \in Q_m$, one has $m \leq \psi(\varphi(m))$ for all $m \in P$. Likewise since $n \in Q_{\psi(n)}$, we get that $\varphi(\psi(n)) \leq n$ for all $n \in Q$.

(3) $\Rightarrow$ (4) follows dually.

(4) $\Rightarrow$ (1) $m \leq \psi(n) \Rightarrow \varphi(m) \leq \varphi(\psi(n)) \leq n$ and $\varphi(m) \leq n \Rightarrow m \leq \psi(\varphi(m)) \leq \psi(n)$. $\square$

Proposition 2.1.7. *[DT95] Let $\varphi : P \longrightarrow Q, \psi : Q \longrightarrow P$ be a pair of mappings between preordered classes. If $\varphi \dashv \psi$ then*

$$\varphi(\bigvee_{i \in I} m_i) = \bigvee_{i \in I} \varphi(m_i) \text{ and } \psi(\bigwedge_{i \in I} n_i) = \bigwedge_{i \in I} \psi(n_i)$$

Proof. [DT95] Let $\{m_i : i \in I\} \subseteq P$ be a subclass and $m = \bigvee_{i \in I} m_i$. Then $\varphi(m)$ is an upper bound of $\{\varphi(m_i) : i \in I\}$ by monotonocity of φ.

If n is any other upper bound, one has that $m_i \leq \psi(n)$ for all $i \in I$; hence $m \leq \psi(n)$ and so $\varphi(m) \leq n$. This proves that φ preserves joins. The proof that ψ preserves meets follows dually. $\square$

We are now ready to define the image of a subobject m of X under a $\mathcal{C}$-morphism.

Definition 2.1.8. [DT95] Let $\mathcal{C}$ have $\mathcal{M}$-pullbacks and for every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, let $f^{-1}(-) : \text{sub}Y \longrightarrow \text{sub}X$ have a left adjoint $f(-) : \text{sub}X \longrightarrow \text{sub}Y$. The image of $m : M \longrightarrow X$ in sub$X$ under f is the morphism $f(m) : f(M) \longrightarrow Y$ in sub$Y$. It is uniquely determined, up to isomorphism, by the following property:

$$m \leq f^{-1}(n) \Leftrightarrow f(m) \leq n$$

for all $n \in \text{sub}Y$.

We are interested in right $\mathcal{M}$-factorization to be able to smoothly handle images and inverse images of subobjects.

Proposition 2.1.9. *[DT95] Let $\mathcal{C}$ have $\mathcal{M}$-pullbacks and for every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, let $f^{-1}(-) : subY \longrightarrow subX$ have a left adjoint $f(-) : subX \longrightarrow subY$ then there are morphisms e, m in $\mathcal{C}$ such that*

(1) *$f = m \circ e$ with $m : M \longrightarrow Y$ in $\mathcal{M}$, and*

(1) *(Diagonalization) whenever one has a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "e"'] & N \arrow[dd, "n"] \\
M \arrow[ur, dotted, "w"] \arrow[d, "m"'] & \\
Y \arrow[r, "v"'] & Z
\end{tikzcd}$$

in $\mathcal{C}$ with $n \in \mathcal{M}$, then there is a uniquely determined morphism $w : M \longrightarrow N$ with $n \circ w = v \circ m$ and $w \circ e = u$.

Proof. Let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism and $f(1_X) : f(X) \longrightarrow Y$. We obtain the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "l"] \arrow[d, "1_X"'] & f^{-1}(f(X)) \arrow[r, "k"] \arrow[d, "f^{-1}(f(1_X))"'] & f(X) \arrow[d, "f(1_X)"] \\
X \arrow[r, "1_X"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

□

The morphism l exists since $1_X \leq f^{-1}(f(1_X))$. Now, let $e = k \circ l$ with $m = f(1_X)$, we obtain (1). Let $f = m \circ e$ and consider the commutative diagram in (2) with $n \in \mathcal{M}$, one obtains the following pullback diagram for morphisms $v : Y \longrightarrow Z$ and $n : N \longrightarrow Z$

$$\begin{tikzcd}
X \arrow[dr, dotted, "t"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "f"'] & & \\
& v^{-1}(N) \arrow[r, "c"] \arrow[d, "v^{-1}(n)"] & N \arrow[d, "n"] \\
& Y \arrow[r, "v"'] & Z
\end{tikzcd}$$

thus the morphism $t : X \longrightarrow v^{-1}(N)$ with $f = v^{-1}(n) \circ t$ by pullback property. Also the same pullback property implies the existence of a unique morphism $s : X \longrightarrow f^{-1}(v^{-1}(N))$ as is seen in the following commutative diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "s"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "1_X"'] & & \\
& f^{-1}(v^{-1}(N)) \arrow[r] \arrow[d, "f^{-1}(v^{-1}(n))"] & v^{-1}(N) \arrow[d, "v^{-1}(n)"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

with $f^{-1}(v^{-1}(n)) \circ s = 1_X$ i.e $1_X \leq f^{-1}(v^{-1}(n))$. Hence $m = f(1_X) \leq v^{-1}(n)$ by adjointness. We now have the following

$$\begin{tikzcd}
M = f(X) \arrow[r, "h"] \arrow[d, "m"'] & v^{-1}(N) \arrow[r, "c"] \arrow[d, "v^{-1}(n)"'] & N \arrow[d, "n"] \\
Y \arrow[r, "1_Y"'] & X \arrow[r, "v"'] & Z
\end{tikzcd}$$

take $w = c \circ h$ so that $n \circ w = v \circ m$. Since n is monic w is uniquely determined by $w = c \circ h$ and $w \circ e = u$ follows from $n \circ w \circ e = v \circ m \circ e = n \circ u$.

Definition 2.1.10. [DT95] A right $\mathcal{M}$-factorization of morphism $f : X \longrightarrow Y$ is any factorization $f = m \circ e$ such that properties (1) and (2) of Proposition 2.1.9. hold.

With the right $\mathcal{M}$-factorization, the image of a subobject of an object X under a $\mathcal{C}$-morphism is given as follows.

Definition 2.1.11. [DT95] Let every morphism in $\mathcal{C}$ have right $\mathcal{M}$-factorization. For a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ and $m : M \longrightarrow X$, one defines $f(m) : (M) \longrightarrow Y$ to be the $\mathcal{M}$-part of a right $\mathcal{M}$-factorization of the composite $f \circ m$.

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "m"'] & f(M) \arrow[d, "f(m)"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Then $f(-) : \text{sub}X \longrightarrow \text{sub}Y$ is an order preserving map.

We can now prove the following:

Proposition 2.1.12. *[DT95] Let $\mathcal{C}$ have $\mathcal{M}$-pullbacks and every morphism in $\mathcal{C}$ have right $\mathcal{M}$-factorization. For every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, $f(-)$ and $f^{-1}(-)$ are adjoint to each other with $f(-)$ being the left adjoint.*

Proof. We need to show that, $f(m) \leq n \Leftrightarrow m \leq f^{-1}(n)$ for all $m \in \text{sub}X$ and $n \in \text{sub}Y$. Assume that $f(m) \leq n$, then there is $j : f(M) \longrightarrow N$ such that the diagram below commutes.

$$\begin{tikzcd}
M \arrow[r, "e"] \arrow[d, "m"'] & f(M) \arrow[r, "j"] \arrow[d, "f(m)"'] & N \arrow[d, "n"] \\
X \arrow[r, "f"'] & Y \arrow[r, "1_Y"'] & Y
\end{tikzcd}$$

This implies that $f \circ m = n \circ j \circ e$ and we have the commutative diagram below

$$\begin{tikzcd}
X \arrow[drr, bend left, "m"] \arrow[dr, dotted, "j_1"] \arrow[ddr, bend right, "j \circ e"'] & & \\
& f^{-1}(N) \arrow[r, "f^{-1}(n)"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\
& N \arrow[r, "n"'] & Y
\end{tikzcd}$$

The arrow j_1 exists by the pullback property of the diagram. So $m = f^{-1}(n) \circ j_1$ and $j \circ e = g \circ j_1$. Hence $m \leq f^{-1}(n)$.

On the other hand if $m \leq f^{-1}(n)$, then there is $k : M \longrightarrow f^{-1}(N)$ such that $m = f^{-1}(n) \circ k$. Now consider the diagram below

$$\begin{tikzcd}
M \arrow[r, "k"] \arrow[dr, "m"'] & f^{-1}(N) \arrow[r, "t"] \arrow[d, "f^{-1}(n)"] & N \arrow[d, "n"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Using the diagram in Defintion 2.1.11, we get that $f(m) \circ e = f \circ m = f \circ f^{-1}(n) \circ k = n \circ t \circ k$.

This gives the diagram below

$$\begin{tikzcd}
M \arrow[r, "t \circ k"] \arrow[d, "e"'] & N \arrow[dd, "n"] \\
f(M) \arrow[ur, dotted, "h"] \arrow[d, "f(m)"'] & \\
Y \arrow[r, "1_Y"'] & Y
\end{tikzcd}$$

Since $n \in \mathcal{M}$, by right $\mathcal{M}$-factorization there is a unique $h : f(M) \longrightarrow N$ such that $f(m) = n \circ h$ and $t \circ k = h \circ e$. This implies that $f(m) \leq n$.

Thus $m \leq f^{-1}(n) \Leftrightarrow f(m) \leq n$. $\square$

Remark 2.1.13. The fact that $f(-)$ and $f^{-1}(-)$ are monotone maps is not used here. It follows from adjointness.

One obtains the following formulas from Proposition 2.1.6

(1) $m \leq k \Rightarrow f(m) \leq f(k)$;

(2) $m \leq f^{-1}(f(m))$ and $f(f^{-1}(n)) \leq n$;

(3) $f(\bigvee_{i \in I} m_i) \cong \bigvee_{i \in I} f(m_i)$;

(4) $f^{-1}(\bigwedge_{i\in I} n_i) \cong \bigwedge_{i\in I} f^{-1}(n_i)$.

Proposition 2.1.14. *[DT95] The following assertions are equivalent:*

(1) *$\mathcal{C}$ has $\mathcal{M}$-pullbacks, and every morphism has a right $\mathcal{M}$-factorization;*

(2) *$\mathcal{C}$ has $\mathcal{M}$- pullbacks, and $f^{-1}(-)$ has a left adjoint for every morphism f ;*

(3) *every morphism has a right $\mathcal{M}$-factorization, and $f(-)$ has a right adjoint for every morphism f.*

Proof. [DT95] (1) $\Rightarrow$ (2) follows Proposition 2.1.12 and for (1) $\Rightarrow$ (3), let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism and $m \in \mathrm{sub}X$. Then by Definition 2.1.2, one obtains the following pullback diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "t"] \arrow[drr, bend left, "e"] \arrow[ddr, bend right, "m"'] & & \\
& f^{-1}(f(M)) \arrow[r] \arrow[d, "f^{-1}(f(m))"] & f(M) \arrow[d, "f(m)"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

So $m \leq f^{-1}(f(m))$. Let $n \in \mathrm{sub}Y$, from the diagonalization property, we get that $f(f^{-1}(n)) \leq n$. Since $f^{-1}(-)$ and $f(-)$ are order preserving maps (cf. Propostion 2.1.3 and Definition 2.1.8), Proposition 2.1.6 gives the adjointness.

(2) $\Rightarrow$ (1) by Proposition 2.1.9.

(3) $\Rightarrow$ (1) Let $f : X \longrightarrow Y$ have a right $\mathcal{M}$-factorization and $f^{-1}(-)$ be the right adjoint of $f(-)$. Then for every $n : N \longrightarrow Y$ in $\mathcal{M}$, one has the following commutative daigram

$$\begin{tikzcd}
f^{-1}(N) \arrow[r, "c"] \arrow[d, "f^{-1}(n)"'] & f(f^{-1}(N)) \arrow[r, "d"] \arrow[d, "f(f^{-1}(n))"'] & N \arrow[d, "n"] \\
X \arrow[r, "f"'] & Y \arrow[r, "1_Y"'] & Y
\end{tikzcd}$$

Now consider the following commutative diagram

$$\begin{tikzcd}
Z \arrow[dr, dotted] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& f^{-1}(N) \arrow[r] \arrow[d, "f^{-1}(n)"] & N \arrow[d, "n"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

and let $g = k \circ e$ with $k : K \longrightarrow X \in \mathcal{M}$ be the right $\mathcal{M}$-factorization of g.

By the diagonalization property, there is a morphism $w : K \longrightarrow N$ which makes the following diagram commute

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "e"'] & N \arrow[dd, "n"] \\ K \arrow[ur, dotted, "w"] \arrow[d, "k"'] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Again by the same property, $f(k) \leq n$, so $k \leq f^{-1}(n)$ by adjointness. Thus there is a morphism $j : K \longrightarrow f^{-1}(N)$ with $f^{-1}(n) \circ j = k$.

If $t = j \circ e : Z \longrightarrow f^{-1}(N)$, then we get $f^{-1}(n) \circ t = k \circ e = g$. Since n and $f^{-1}(n)$ are monic, t is uniquely determined and $d \circ c \circ t = h$. □

Definition 2.1.15. [DT95] We shall say that $\mathcal{C}$ is finitely $\mathcal{M}$-complete if one and then all of the assertions of the Proposition 2.1.14 hold.

The existence of $\mathcal{M}$-pullbacks in $\mathcal{C}$ implies that the preordered class sub$X$ has binary meets for all $X \in \mathcal{C}$. Indeed, if $m : M \longrightarrow X$ and $n : N \longrightarrow X$ are subobjects of X the binary meet is given by the diagonal of the following pullback diagram

$$\begin{tikzcd} M \wedge N \arrow[r] \arrow[d] & N \arrow[d, "n"] \\ M \arrow[r, "m"'] & X \end{tikzcd}$$

This means that $m \wedge n = m \circ m^{-1}(n) = n \circ n^{-1}(m)$.

We are interested in the existence of arbitrary meets in sub$X$ as we need sub$X$ to be a complete lattice for each $X \in \mathcal{C}$.

Definition 2.1.16. [DT95] We shall say that $\mathcal{C}$ has $\mathcal{M}$-intersections if for every family $(m_i)_{i \in I}$ in sub$X$ (I may be infinite class or empty), if a multiple pullback diagram

$$\begin{tikzcd} M \arrow[rr, "\gamma_i"] \arrow[dr, "m"'] & & M_i \arrow[dl, "m_i"] \\ & X & \end{tikzcd}$$

exists in $\mathcal{C}$ with $m \in$ sub$X$.

In this case m plays the role of the meet of $(m_i)_{i \in I}$ in sub$X$, that is $m = \bigwedge\{m_i \mid i \in I\}$.

This also implies the existence of the join $\bigvee$ of subobjects and in particular the least subobject $o_X : 0_X \longrightarrow X$ exists for every $X \in \mathcal{C}$.

Definition 2.1.17. We shall say that $\mathcal{C}$ is $\mathcal{M}$-complete if it has $\mathcal{M}$-pullbacks and $\mathcal{M}$-intersections.

We see from the dual of Theorem 15.14 in [AHS06] that $\mathcal{M}$-completeness implies finite $\mathcal{M}$-completeness.

The closedness of the class $\mathcal{M}$ under composition is of importance. It makes the right $\mathcal{M}$-factorization symmetric in both factors.

Theorem 2.1.18. *[DT95] The following statements are equivalent*

(i) *every morphism has a right $\mathcal{M}$-factorization, and $\mathcal{M}$ is closed under composition;*

(ii) *there is a class $\mathcal{E}$ of morphisms in $\mathcal{C}$ such that*

(1) *every morphism $f \in \mathcal{C}$ has a factorization $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, and*

(2) *for every commutative diagram*

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "u"'] & Y \arrow[d, "v"] \arrow[dl, dotted, "w"'] \\ M \arrow[r, "m"'] & Z \end{tikzcd}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there is a uniquely determined morphism $w : Y \longrightarrow M$ with $w \circ e = u$ and $m \circ w = v$.

Proof. . [DT95] $(i) \Rightarrow (ii)$ We will say that e is orthogonal to m and write $e \perp m$ if for the commutative diagram in $(ii)(2)$, there is a unique w with $w \circ e = u$ and $m \circ w = v$. Let $\mathcal{E}$ be the class $M^{\perp} = \{e \in \mathcal{C} : \text{ for all } m \in \mathcal{M}\ e \perp m\}$. We just need to show that in the right-$\mathcal{M}$ factorization $f = m \circ e$ of f, $e \in \mathcal{E}$. So let $e = n \circ d$ be the right $\mathcal{M}$-factorization and consider the diagram below

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d, "e"'] & N \arrow[dd, "m \circ n"] \\ M \arrow[ur, dotted, "t"] \arrow[d, "m"'] & \\ Y \arrow[r, "1_Y"'] & Y \end{tikzcd}$$

Since $m \circ n \in \mathcal{M}$, by diagonalization property of the right $\mathcal{M}$-factorization there is a morphism $t : M \longrightarrow N$ with $m \circ n \circ t = m$. Since m is mono, $n \circ t = 1$ and since n is mono, it is an isomorphism and so is t. Now consider the following diagram

$$\begin{tikzcd}
\cdot \arrow[r, "d"] \arrow[d, "d"'] & \cdot \arrow[dd, "p"] \\
\cdot \arrow[ur, dotted, "k"] \arrow[d, "n"'] & \\
\cdot \arrow[r, "v"'] & \cdot
\end{tikzcd}$$

with u, v arbitrary morphisms and $p \in \mathcal{M}$. By the diagonalization property there is k such that $p \circ k = v \circ n$ and $k \circ d = u$. This gives the following diagram

$$\begin{tikzcd}
\cdot \arrow[r, "d"] \arrow[d, "u"'] & \cdot \arrow[d, "v \circ n"] \arrow[dl, dotted, "j"'] \\
\cdot \arrow[r, "p"'] & \cdot
\end{tikzcd}$$

where $j = k \circ n^{-1}$. Since n is an isomorphism, we get that j is a unique morphism such that $p \circ j = v \circ n$ and $u = j \circ d = j \circ e$. Hence $e \in \mathcal{E}$.

$(ii) \Rightarrow (i)$ We first show that $\mathcal{M}$ coincides with the class $\mathcal{E}_{\perp} = \{m \in \mathcal{C} :$ for all $e \in \mathcal{E}\ e \perp m\}$. $\mathcal{M} \subseteq \mathcal{E}_{\perp}$ by property (2). To show that $\mathcal{E}_{\perp} \subseteq \mathcal{M}$, consider the a factorization $m = k \circ e$ with $k \in \mathcal{M}$ and $e \in \mathcal{E}$ for $m \in \mathcal{E}_{\perp}$. One obtains the following diagram

$$\begin{tikzcd}
\cdot \arrow[r, "1"] \arrow[d, "e"'] & \cdot \arrow[d, "m"] \\
\cdot \arrow[r, "k"'] & \cdot
\end{tikzcd}$$

Since $m \in \mathcal{E}_{\perp}$, there is a morphism w with $w \circ e = 1$ and $m \circ w = k$. Now w is a monic since k is monic and so w is an isomorphism. Hence e is an isomorphism. Thus $k = m$ and the classes $\mathcal{E}$ and $\mathcal{M}$ determine each other by (ii). It also implies that $\mathcal{E}_{\perp} = \mathcal{M}$ is stable under composition. $\square$

Definition 2.1.19. [DT95] A pair $(\mathcal{E}, \mathcal{M})$ satisfying conditions (i) and (ii) of Theorem 2.1.16 is called a $(\mathcal{E}, \mathcal{M})$-factorization system or simply a factorization system. Condition (ii) is refered to as the diagonalization property of the $(\mathcal{E}, \mathcal{M})$ factorization system.

One can easily see from condition (ii) of Theorem 2.1.14 that a $(\mathcal{E}, \mathcal{M})$ factorization for a morphism is essentially unique.

Corollary 2.1.20. *Let every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ have an $(\mathcal{E}, \mathcal{M})$-factorization system. Then $f \in \mathcal{E}$ if and only if $f(1_X) = 1_Y$*

Proof. Consider the following commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "f"'] & f(X) \arrow[d, "f(1_X)"] \\
Y \arrow[r, "1_Y"'] & Y
\end{tikzcd}
$$

Since $f \in \mathcal{E}$ and $f(1_X) \in \mathcal{M}$, by the diagonalization property of $(\mathcal{E}, \mathcal{M})$ factorizations, there is a morphism $t : Y \longrightarrow f(X)$ such that $f(1_X) \circ t = 1_Y$, that is $1_Y \leq f(1_X)$.

Conversely if $1_Y = f(1_X)$, then the commutativity of the above diagram gives $f = f(1_X) \circ e = 1_Y \circ e = e$. Hence, $f \in \mathcal{E}$ □

We now give sufficient conditions for the image and pre-image of subobjects to be partially inverse to each other.

Proposition 2.1.21. *[DT95] Let $\mathcal{C}$ have $\mathcal{M}$-pullbacks and every morphism in $\mathcal{C}$ have an $(\mathcal{E}, \mathcal{M})$-factorization. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{C}$.*

(1) *If $f \in \mathcal{M}$, then $f^{-1}(f(m)) = m$ for all $m \in subX$.*

(2) *If $f \in \mathcal{E}$ and $\mathcal{E}$ is stable under pullback then $f(f^{-1}(n)) = n$ for all $n \in subY$*

Proof. (1) Consider the diagram

$$
\begin{tikzcd}
M \arrow[r, "e"] \arrow[d, "m"'] & f(M) \arrow[d, "f(M)"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Since $f \in \mathcal{M}$. Then by taking $e = 1_M$, the diagram becomes a pullback. This implies by Definition 2.1.2 that m is the inverse image of $f(m)$ under f . Thus, $f^{-1}(f(m)) = m$.

(2) Consider the diagram

$$
\begin{tikzcd}
f^{-1}(N) \arrow[r, "f'"] \arrow[d, "f^{-1}(n)"'] & f(X) \arrow[d, "n"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

with $f \in \mathcal{E}$ and $\mathcal{E}$ is stable under pullback, then $f' \in \mathcal{E}$. This implies by Definition 2.1.11 that n is the image of $f^{-1}(n)$ under f . Hence $f(f^{-1}(n)) = n$. □

Definition 2.1.22. Let $\mathcal{C}$ have $\mathcal{M}$-pullbacks and every morphism in $\mathcal{C}$ have an $(\mathcal{E}, \mathcal{M})$-factorization. A morphism $f : X \longrightarrow Y$ reflects o if $f^{-1}(o_Y) = o_X$ (equivalently $f(m) = o_Y \Leftrightarrow m = o_X$).

The following proposition will also be usefull

Proposition 2.1.23. *[DT95] Let $\mathcal{C}$ have $\mathcal{M}$-pullbacks and every morphism in $\mathcal{C}$ have an $(\mathcal{E},\mathcal{M})$-factorization. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{C}$. For any morphism $g : Y \longrightarrow Z$ in $\mathcal{C}$, one has that $(g\circ f)(-) = g(f(-))$ and $(g\circ f)^{-1}(-) = f^{-1}(g^{-1}(-))$.*

Proof. We first show that $(g \circ f)(m) = g(f(m))$ for any $m \in \mathrm{sub}X$. By Definition 2.1.11, the following two diagrams commute

$$\begin{tikzcd}
M \arrow[r, "e_1"] \arrow[d, "m"'] & f(M) \arrow[r, "e_2"] \arrow[d, "f(m)"'] & g(f(M)) \arrow[d, "g(f(m))"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "e_3"] \arrow[d, "m"'] & (f\circ g)(M) \arrow[d, "(g\circ f)(m)"] \\
X \arrow[r, "f\circ g"'] & Y
\end{tikzcd}$$

So $g\circ f\circ m = (g\circ f)(m)\circ e_3$ and $f(g(m))\circ e_2\circ e_1 = g\circ f\circ m$

Since $(\mathcal{E},\mathcal{M})$ factorizations are unique up to isomorphism (see Theorem 2.1.18), there is an isomorphism h which makes the following square commutes.

$$\begin{tikzcd}
M \arrow[r, "e_3"] \arrow[d, "e_2\circ e_1"'] & (f\circ g)(M) \arrow[d, "(g\circ f)(m)"] \arrow[dl, dotted, "h"'] \\
g(f(M)) \arrow[r, "g(f(m))"'] & Z
\end{tikzcd}$$

Thus $(g\circ f)(m) = g(f(m))$.

We next show that $(g\circ f)^{-1}(m) = f^{-1}(g^{-1}(m))$ for any $m \in \mathrm{sub}Z$.

Consider the following diagram

$$\begin{tikzcd}
f^{-1}(g^{-1}(M)) \arrow[r, "h_1"] \arrow[d, "f^{-1}(g^{-1}(m))"] & g^{-1}(M) \arrow[r, "h_2"] \arrow[d, "g^{-1}(m)"] & M \arrow[d, "m"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Since the two squares are pullbacks, the large square is a pullback and by Definition 2.1.2 $(g\circ f)^{-1}(m)$ is a pullback of m along $g\circ f$. Hence, $(g\circ f)^{-1}(m) = f^{-1}(g^{-1}(m))$ by the uniqueness of pullbacks.

□

In the sequel, we shall assume that our category $\mathcal{C}$ is $\mathcal{M}$-complete and that it is endowed with $(\mathcal{E},\mathcal{M})$ factorization system.

2.2 Closure and Interior operators

This section recalls the basic material on closure and interior operators that will be useful in the rest of this work.

2.2.1 Closure operators

We shall first recall the definition of closure operator.

Definition 2.2.1. [DT95] A closure operator c on $\mathcal{C}$ with respect to $\mathcal{M}$ is given by a family of maps $\{c_X\colon \mathrm{sub}X \longrightarrow \mathrm{sub}X \mid X \in \mathcal{C}\}$ such that:

$(C1)$ $m \leq c_X(m)$ for all $m \in \mathrm{sub}X$;

$(C2)$ $m \leq n \Rightarrow c_X(m) \leq c_X(n)$ for all $m, n \in \mathrm{sub}X$;

$(C3)$ every morphism $f : X \longrightarrow Y$ is c-continuous, that is: $f(c_X(m)) \leq c_Y(f(m))$ for all $m \in \mathrm{sub}X$.

Because of $(C1)$, one has the following commutative diagram for all $m \in \mathrm{sub}X$

$$\begin{tikzcd} M \arrow[rr, "j_m"] \arrow[dr, "m"'] & & c_X(M) \arrow[dl, "c_X(m)"] \\ & X & \end{tikzcd}$$

An equivalent description of the continuity condition is provided by the proposition below.

Proposition 2.2.2. *[DT95] Let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism. Then under $(C2)$, $(C3)$ is equivalent to $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$ for all $n \in subY$.*

Proof. If $(C3)$ holds, then by putting $m = f^{-1}(n)$, one gets $f(c_X(f^{-1}(n)) \leq c_Y(f(f^{-1}(n)))$. Now $f(f^{-1}(n)) \leq n$ and $(C2)$ implies that $c_Y(f(f^{-1}(n))) \leq c_Y(n)$.

Consequently $c_X(f^{-1}(n)) \leq f^{-1}(c_X(n))$ by adjointness. Conversely if $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$ for all $n \in \mathrm{sub}Y$, by taking $n = f(m)$, we obtain that $c_X(f^{-1}(f(m)) \leq f^{-1}(c_Y(f(m))$. Since $m \leq f^{-1}(f(m)), (C2)$ implies that $c_X(m) \leq c_X(f^{-1}(f(m))$. Thus, $f(c_X(m)) \leq c_Y(f(m))$ by adjointness. $\square$

We note that the class $\mathcal{M}$ can be seen as a full subcategory of the category $\mathcal{C}^2$ of morphisms in $\mathcal{C}$. Its objects are the elements of the class $\mathcal{M}$ and a morphism (g, f) :

$m \longrightarrow n$ is given by a pair of morphisms in $\mathcal{C}$ such that

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Now, if g factors as $g = p \circ e$ with $p \in \mathcal{M}$ and $e \in \mathcal{E}$, then $f \circ m = n \circ p \circ e$ which by definition 2.1.11 implies that $f(m) = n \circ p$; that is $f(m) \leq n$. On the other hand if f is a morphism in $\mathcal{C}$ such that $f(m) \leq n$, then there is an arrow (g, f) in $\mathcal{M}$. In particular, for any $f : X \longrightarrow Y$ in $\mathcal{C}$ and $m \in \mathrm{sub}X$, $n \in \mathrm{sub}Y$, $m \longrightarrow f(m)$ and $f^{-1}(n) \longrightarrow n$ always exist by Definitions 2.1.11 and 2.1.2.

The conglomerate of all closure operators on $\mathcal{C}$ with respect to $\mathcal{M}$ is denoted by $CLOS$. It is ordered as follows: $c \leq c'$ if $c_X(m) \leq c'_X(m)$ for all $m \in \mathrm{sub}X$ and $X \in \mathcal{C}$.

Definition 2.2.3. [DT95] A subobject m of X is called c-closed if it is isomorphic to its closure. m is called c-dense if its closure is isomorphic to 1_X.

Some stability properties of these subobjects are listed below.

Proposition 2.2.4. *[DT95] Let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism and c a closure operator.*

(1) *If n is c-closed in Y, then $f^{-1}(n)$ is c-closed in X.*

(2) *If m is c-dense in X and $f \in \mathcal{E}$, then $f(m)$ is c-dense in Y .*

(3) *If for monomorphisms m and n, $n \circ m$ is a c-closed $\mathcal{M}$-subobject, then m is a c-closed $\mathcal{M}$-subobject*

(4) *if $m_i : M_i \longrightarrow X$ is a family of c-closed subobjects, then the infinimum $\bigwedge m_i$ is c-closed*

Proof. (1)

$$c_Y(n) = n \Rightarrow c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n)) = f^{-1}(n)$$
$$\Rightarrow c_X(f^{-1}(n)) \leq f^{-1}(n)$$

(2) If $c_X(m) = 1_X$ and $f \in \mathcal{E}$, then $1_Y = f(1_X) = f(c_X(m)) \leq c_Y(f(m)) \Rightarrow 1_Y \leq c_Y(f(m))$.

(3) The following diagram is a pullback since n is a monomorphism.

$$\begin{tikzcd} M \arrow[r, "1_M"] \arrow[d, "m"'] & M \arrow[d, "n\circ m"] \\ X \arrow[r, "n"'] & Y \end{tikzcd}$$

That is $n^{-1}(n \circ m) = m$. Thus $c_X(m) = c_X(n^{-1}n \circ m) \leq n^{-1}(c_Y(n \circ m)) = n^{-1}(n \circ m) = m$

(4) If m_i is c-closed for all i, then

$$\begin{aligned} \bigwedge m_i \leq m_i \Rightarrow c_X(\bigwedge m_i) &\leq c_X(m_i) = m_i && \text{for all } i \\ \Rightarrow c_X(\bigwedge m_i) &\leq m_i && \text{for all } i \\ \Rightarrow c_X(\bigwedge m_i) &\leq \bigwedge m_i = \bigwedge c_X(m_i) && \text{for all } i \end{aligned}$$

□

The following lemma will be important in proving Proposition 2.2.11.

Lemma 2.2.5. *[DT95] For a commutative diagram*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with $m, n \in \mathcal{M}$, there is a uniquely determined morphism w making the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "j_m"'] & N \arrow[d, "j_n"] \\ c_X(M) \arrow[r, "w"] \arrow[d, "c_X(m)"'] & c_X(N) \arrow[d, "c_X(n)"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commute

Proof. [DT95] By the diagonalization property of right M-factorizations, there is a morphism $j : f(M) \longrightarrow N$ which makes the following diagram commute.

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d] & N \arrow[dd, "n"] \\ f(M) \arrow[ur, dotted, "j"] \arrow[d, "f(m)"'] & \\ Y \arrow[r, "1_Y"'] & Y \end{tikzcd}$$

$$
\begin{aligned}
\text{So } f(m) \leq n &\Rightarrow c(f(m)) \leq c(n) \\
&\Rightarrow f(c(m)) \leq c(f(m)) \leq c(n) \\
&\Rightarrow f(c(m)) \leq c(n)
\end{aligned}
$$

This gives the following commutative diagram

$$
\begin{tikzcd}
c(M) \arrow[r, "e"] \arrow[d, "c(m)"'] & f(c(M)) \arrow[r, "t"] \arrow[d, "f(c(m))"'] & c(N) \arrow[d, "c(n)"] \\
X \arrow[r, "f"'] & Y \arrow[r, "1_Y"'] & Y
\end{tikzcd}
$$

Thus $t \circ e$ is the desired morphism. Its uniqueness follows from the fact that $c(n)$ is a monomorphism. □

We will refer to the above lemma as the functorial property of closure operators.

Corollary 2.2.6. *[DT95] If m is c-dense and n is c-closed, then there is a unique p for which the following diagram commutes:*

$$
\begin{tikzcd}
M \arrow[r, "m"] \arrow[d, "u"'] & X \arrow[d, "v"] \arrow[dl, dotted, "p"'] \\
N \arrow[r, "n"'] & Y
\end{tikzcd}
$$

We next present two important properties of closure operators: idempotency and weakly hereditariness.

Definition 2.2.7. [DT95] A closure operator c is idempotent if $c_X(m)$ is c-closed for every $m \in \text{sub}X$. It is weakly hereditary if j_m is c-dense for every $m \in \text{sub}X$ and $X \in \mathcal{C}$

Proposition 2.2.8. *[Cas03] If c is weakly hereditary, the class of c-closed suobjects is closed under composition.*

Proof. [Cas03] Assume c is weakly hereditary and consider the following diagram with n and m c-closed $\mathcal{M}$-subobjects

$$
\begin{tikzcd}
N \arrow[r, "j_{m \circ n}"] \arrow[d, "n"'] & c(N) \arrow[d, "c(m \circ n)"] \arrow[dl, dotted, "d"'] \\
M \arrow[r, "m"'] & X
\end{tikzcd}
$$

Then $j_{m\circ n}$ is c-dense and so by Corollary 2.2.6 there is a unique morphism $d : c_X(N) \longrightarrow M$ such that $d \circ j_{m\circ n} = n$ and $m \circ d = c(m \circ n)$. Consequently, one obtains the following commutative diagram

$$\begin{tikzcd} N \arrow[r, "j_{m\circ n}"] \arrow[d, "id_N"'] & c(N) \arrow[d, "d"] \arrow[dl, dotted, "t"'] \\ N \arrow[r, "n"'] & X \end{tikzcd}$$

So $n \circ t = d$ and $t \circ j_{m\circ n} = id_N$. Thus t is a monomorphism and retraction and so an isomorphism. Hence $j_{m\circ n}$ is also an isomorphism, that is $m \circ n$ is c-closed. □

We now turn to c-closed morphisms.

Definition 2.2.9. [DT95] Let c be a closure operator with respect to $\mathcal{M}$. A $\mathcal{C}$-morphism $f : X \longrightarrow Y$ is said to be c-closed (c-preserving) if $f(c_X(m)) = c_Y(f(m))$ for all $m \in$ sub$X$.

Clearly, if f is c-closed then $f(m)$ is c-closed in subY whenever m is c-closed in sub$X$, the converse holds if c is idempotent.

The c-closed morphisms behave as follows:

Proposition 2.2.10. *[DT95] Let c be a closure operator. The following statements are true:*

(1) *c-closed morphisms are closed under composition.*

(2) *If $g \circ f$ is c-closed and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullbback, then g is c-closed.*

(3) *If $g \circ f$ is closed and $g \in \mathcal{M}$, then f is c-closed.*

Proof. (1) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are c-closed morphisms. Then for all $m \in$ sub$X$,

$$\begin{aligned}(g \circ f)(c_X(m)) &= g(f(c_X(m)) \\ &= g(c_Y(f(m)) \\ &= c_Z(g(f(m))\end{aligned}$$

(2) If $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback and $g \circ f$ is c-closed, then for all $m \in \mathrm{sub}Y$,

$$\begin{aligned} c_Z(g(n)) &= c_Z(g(f(f^{-1}(n)))) \\ &= g(f(c_X(f^{-1}(n)))) \\ &\leq g(f(f^{-1}(c_Y(n)))) \\ &= g(c_Y(n)) \end{aligned}$$

(3) If $g \in \mathcal{M}$ and $g \circ f$ is c-closed, then for all $m \in \mathrm{sub}X$,

$$\begin{aligned} c_Y(f(m)) &= c_Y(g^{-1}(g(f(m))) \\ &= g^{-1}(c_Z(g(f(m))) \\ &= g^{-1}(g(f(c_X(m)))) \\ &= f(c_X(m)) \end{aligned}$$

□

The fact that every isomorphism is c-closed follows from Proposition 2.2.10(3). The following establishes a relationship between c-closed subobjects and c-closed morphisms.

Proposition 2.2.11. *[CGT96] Let c be a closure operator. The following statements are true:*

(1) *Every c-closed morphism in $\mathcal{M}$ is c-closed subobject.*

(2) *If c is weakly hereditary, then every c-closed subobject is a c-closed morphism.*

Proof. [CGT96]

(1) If $m : M \longrightarrow X$ is a c-closed morphism in $\mathcal{M}$, then it preserves in particular the closure of 1_M , that is $m = c_X(m)$.

(2) Let c be weakly hereditary and $m : M \longrightarrow X$ be a c-closed subobject. Then for every $k : K \longrightarrow M$, $m \circ k \leq m \Rightarrow c_X(m \circ k) \leq c_X(m) = m$. By the functorial property of closure operators, there is a morphism d making the following diagram

commute

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "j_{m\circ k}"'] & M \arrow[d, "j_n"] \\
c_X(K) \arrow[r, "d"] \arrow[d, "c_X(m\circ k)"'] & c_X(M) \arrow[d, "c_X(m)"] \\
X \arrow[r, "1_X"] & X
\end{tikzcd}$$

Since $m = c_X(m), j_m = 1_M$ so $d \circ j_{m\circ k} = k$. Consequently, we have the following commutative diagram

$$\begin{tikzcd}
K \arrow[r, "1_K"] \arrow[d, "j_{m\circ k}"'] & K \arrow[d, "j_k"] \\
c_X(K) \arrow[r, "h"] \arrow[d, "c_{c_X(K)}(j_{m\circ k})=1_{c_X(K)}"'] & c_M(K) \arrow[d, "c_M(k)"] \\
c_X(K) \arrow[r, "d"] & M
\end{tikzcd}$$

$j_k = h \circ j_{m\circ k}$ and $c_M(k) \circ h = d \circ 1_{c_X(K)} = d$.

This implies that

$$\begin{aligned}
c_X(m \circ k) &= c_X(m) \circ d \\
&= m \circ d \\
&= m \circ c_M(k) \circ h
\end{aligned}$$

Hence $c_X(m \circ k) \leq m \circ c_M(k)$ that is m is a c-closed morphism

□

2.2.2 Interior operators

We first give the definition.

Definition 2.2.12. [Cas15] An interior operator i on $\mathcal{C}$ with respect to $\mathcal{M}$ is given by a family of maps $\{i_X : \text{sub}X \longrightarrow \text{sub}X \mid X \in \mathcal{C}\}$ such that

$(I1)$ $i_X(m) \leq m$ for every $m \in \text{sub}X$ and $X \in \mathcal{C}$;

$(I2)$ $m \leq n \Rightarrow i_X(m) \leq i_X(n)$ for every $m, n \in \text{sub}X, X \in \mathcal{C}$;

$(I3)$ every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is i-continuous, $f^{-1}(i_Y(n)) \leq i_X(f^{-1}(n))$ for each $n \in \text{sub}Y$.

We denote by INT the conglomerate of all interior operators on $\mathcal{C}$ with respect to $\mathcal{M}$. It is ordered as follows: $i \leq i'$ if $i_X(m) \leq i'_X(m)$ for all $m \in \text{sub}X, X \in \mathcal{C}$.

Concerning the arbitrary joins and meets in INT, we offer the following.

Proposition 2.2.13. *[Raz12a] Let $\{i_k \mid k \in K\} \subseteq INT$. Then*

$$i^{\star}_X(m) = \bigwedge\{(i_k)_X(m) \mid k \in K\}$$

for all $m \in subX, X \in \mathcal{C}$ is an interior operator and if any join of subobjects commutes with the pullback, then

$$i^{\diamond}_X(m) = \bigvee\{(i_k)_X(m) \mid k \in K\},$$

for all $m \in subX, X \in \mathcal{C}$ is also an interior operator.

Proof. [Raz12a] $(I1)$ and $(I2)$ are satisfied by construction of $i^{\diamond}$ and $i^{\star}$. To prove $(I3)$, we take any $\mathcal{C}$-morphism $f : X \longrightarrow Y$ and $m \in \text{sub}Y$. Then

$$\begin{aligned} f^{-1}(i^{\star}_Y(m)) &= f^{-1}(\bigwedge\{(i_k)_Y(m) \mid k \in K\}) \\ &= \bigwedge\{f^{-1}((i_k)_Y(m)) \mid k \in K\} \\ &\leq i^{\star}_X(f^{-1}(m)) \end{aligned}$$

Similarly if pre-images commute with joins of subobjects, we have

$$\begin{aligned} f^{-1}(i^{\diamond}_Y(m)) &= f^{-1}(\bigvee\{(i_k)_Y(m) \mid k \in K\}) \\ &= \bigvee\{f^{-1}((i_k)_Y(m)) \mid k \in K\} \\ &\leq i^{\diamond}_X(f^{-1}(m)) \end{aligned}$$

□

As for c-closed subobjects and c-closed morphisms, i-open subobjects and i-open morphisms deserve our attention.

Definition 2.2.14. [Cas15] A subobject m of X is said to be i-open if $m = i_X(m)$. We shall say that i is idempotent provided that $i_X(m)$ is i-open for every $m \in \mathcal{M}$ and $X \in \mathcal{C}$.

Proposition 2.2.15. *[Cas15] Let i be an interior operator. The following holds true:*

(1) *The pre-image of an i-open subobject is i-open.*

(2) *If n and m are monomorphisms such that $n \circ m$ is i-open subobject then m is i-open.*

(3) *If $m_i : M_i \longrightarrow X$ is a family of i-open subobjects, then the supremum $\bigvee m_i$ of m_i is i-open.*

Proof. Similar to $(1), (3)$ and (4) of Proposition 2.2.4 □

Definition 2.2.16. [Cas15] An interior operator is grounded if $i_X(1_X) = 1_X$ for all $X \in \mathcal{C}$. It is additive if for any $X \in \mathcal{C}, i_X(m \wedge n) = i_X(m) \wedge i_X(n)$.

Definition 2.2.17. [Cas15] A morphism $f : X \longrightarrow Y$ is i-open if $f^{-1}(i_Y(n)) = i_X(f^{-1}(n))$ for any $n \in \text{sub}Y$ or equivalently $f(i_X(m)) \leq i_Y(f(m))$ for any $m \in \text{sub}X$.

If f is an i-open morphism, $f(-)$ sends i-open subobjects to i-open subobjects. The converse holds if i is idempotent. The i-open morphisms behave in a similar way to the c-closed.

Proposition 2.2.18. *[Cas15] Let i be an interior operator. The following statements are true:*

(1) *Every isomorphism in $\mathcal{C}$ is i-open.*

(2) *i-open morphisms are closed under composition.*

(3) *If $g \circ f$ is i-open and $g \in \mathcal{M}$, then f is i-open.*

(4) *If $g \circ f$ is i-open and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullbback, then g is i-open*

Proof. (1) Let $f : X \longrightarrow Y$ be a $\mathcal{C}$-isomorphism with inverse $g : Y \longrightarrow X$ and $n \in \text{sub}Y$, then

$$\begin{aligned}
i_X(f^{-1}(n)) &= (g \circ f)^{-1}(i_X(f^{-1}(n))) \\
&= f^{-1}(g^{-1}(i_X(f^{-1}(n)))) \\
&\leq f^{-1}(i_Y(g^{-1}(f^{-1}(n)))) \\
&= f^{-1}(i_Y(n))
\end{aligned}$$

(2) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are i-open morphisms, then for all $m \in \text{sub}Z$;

$$\begin{aligned}(g \circ f)^{-1}(i_Z(m)) &= f^{-1}(g^{-1}(i_Z(m)))\\ &= f^{-1}(i_Y(g^{-1}(m)))\\ &= i_X(f^{-1}(g^{-1}(m)))\\ &= i_X((g \circ f)^{-1}(m))\end{aligned}$$

(3) If $g \in \mathcal{M}$ and $g \circ f$ is i-open then for all $n \in \text{sub}Y$,

$$\begin{aligned}i_X(f^{-1}(n)) &= i_X(f^{-1}(g^{-1}(g(n)))\\ &= (g \circ f)^{-1}(i_Z(g(n)))\\ &= f^{-1}(g^{-1}(i_Z(g(n))))\\ &\leq f^{-1}(i_Y(g^{-1}(g(n))))\\ &= f^{-1}(i_Y(n))\end{aligned}$$

(4) If $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback and $g \circ f$ is i-open, then for all $n \in \text{sub}Z$;

$$\begin{aligned}i_Y(g^{-1}(n)) &= f(f^{-1}(i_Y(g^{-1}(n))))\\ &\leq f(i_X(f^{-1}(g^{-1}(n))))\\ &= f((g \circ f)^{-1}(i_Z(m))))\\ &= f(f^{-1}(g^{-1}(i_Z(m))))\\ &= g^{-1}(i_Z(m))\end{aligned}$$

□

Definition 2.2.19. [Cas15] An interior operator i is called weakly hereditary if for every pair of $\mathcal{M}$-subobjects $n \leq m$ with m i-open, one has that $i_X(n) = m \circ i_M(n_m)$ where n_m is the unique morphism such that $m \circ n_m = n$. It is easy to see that $n_m = m^{-1}(n)$.

We have the following:

Proposition 2.2.20. *[Cas15] Given an interior operator i, the following statements are true:*

(1) *If i is standard, then every i-open morphism in $\mathcal{M}$ is an i-open subobject.*

(2) *If i is weakly hereditary then every i-open subobject is an i-open morphism.*

Chapter 3

Neighbourhood operators

In 1902 Hilbert published his paper, *Foundations of geometry*, which perhaps deserves to be considered as the beginning of ideas of neighbourhoods in topology. These ideas continued through the work of Frechet(1906). While Frechet defined abstract spaces in terms of convergent sequences and Riez in terms of accumulations points(1907); Weyl, in *Die Idee der Riemannschen Fläche*, proposed a study in terms of neighbourhood systems ([Wil70]. A satisfactory axiomatization of Weyls neighbourhoods was proposed by Hausdorff in 1914 in his book *Grundzüge der Mengenlehre* which according to Bourbaki ([AL97]) commences general topology as it is today. This chapter presents the theory of categorical neighbourhood operators. Introduced in ([HŠ11]) only half a decade ago, the categorical neighbourhood operator has been developed in ([HŠ10, Raz12b, RH14]. We shall define the operator at the subobject level and show that the interior operators are special neigbhourhoods. This leads us to the correspondence between neighbourhood (or interior) and closure operators. The chapter ends with the description of four classes of morphisms with respect to a neighbourhood operator.

3.1 Neighbourhood or Interior and Closure operators

We start with the following definition given in [Raz12b], which is equivalent to the ones in [HŠ10, HŠ11]).

Definition 3.1.1. A neighbourhood operator ν on $\mathcal{C}$ with respect to $\mathcal{M}$ is a family of maps $\{\nu_X : \mathrm{sub}X \longrightarrow P(\mathrm{sub}X) \mid X \in \mathcal{C}\}$ such that

$(N1)$ $n \in \nu_X(m) \Rightarrow m \leq n$ for every $m \in \text{sub}X$ and $X \in \mathcal{C}$;

$(N2)$ $m \leq n \Rightarrow \nu_X(n) \subseteq \nu_X(m)$ for every $m, n \in \text{sub}X$ and $X \in \mathcal{C}$;

$(N3)$ $p \in \nu_X(m)$ and $p \leq q$ then $q \in \nu_X(m)$ for every $m, p, q \in \text{sub}X$ and $X \in \mathcal{C}$;

$(N4)$ every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is ν-continuous, $n \in \nu_Y(f(m)) \Rightarrow f^{-1}(n) \in \nu_X(m)$ for every $m \in \text{sub}X$ and $n \in \text{sub}Y$.

The congolomerate of all neighbourhood operators on $\mathcal{C}$ with respect to $\mathcal{M}$ is denoted by NBH. It is ordered as follows: $\nu \leq \nu \Rightarrow \nu(m) \subseteq \nu'(m)$ for all $m \in \mathcal{M}$ and $X \in \mathcal{C}$. As observed in the previous chapter, for any morphism $f : X \longrightarrow Y$ in $\mathcal{C}, m \in \text{sub}X$ and $n \in \text{sub}Y$; if there is an arrow $m \longrightarrow n$ in $\mathcal{M}$ which involves f, then $f(m) \leq n$. This implies that $\nu(n) \subseteq f(\nu(m))$ by $(N2)$ or equivalently $f^{-1}(\nu(n)) \subseteq \nu(m)$ by adjointness and since the arrows $m \longrightarrow f(m)$ and $f^{-1}(n) \longrightarrow n$ always exist, the ν-continuity can equivalently be expressed by the proposition below.

Proposition 3.1.2. *[Raz12a] Let ν be a neighbourhood and $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism. Let $n \in subY$ and $m \in subX$. The following are equivalent in expressing the ν-continuity.*

(i) $f^{-1}(\nu(n)) \subseteq \nu(f^{-1}(n))$;

(ii) $f^{-1}[\nu(f(m))] \subseteq \nu(m)$;

(ii) $\nu(n) \subseteq f(\nu(f^{-1}(n)))$.

Since its introduction in [HŠ11], the categorical neighbourhood operator has been shown to be strongly related to the interior operator. In [RH14], the relationship was established in the form of an adjunction. We next establish the interaction between the two operators, both acting on the subobject lattice.

Definition 3.1.3. [Raz12a] A neighbourhood operator ν is said to be left-adjoint if it satisfies the following axiom.

(L) For any $X \in \mathcal{C}$ and $\mathcal{G} \subseteq \text{sub}X$, if $m \in \nu_X(g)$ for every $g \in \mathcal{G}$, then $m \in \nu_X(\bigvee G)$ for $m \in \text{sub}X$.

We shall denote by $LNBH$ the class of all left-adjoint neighbouhood operators, ordered pointwise, on $\mathcal{C}$ with respect to $\mathcal{M}$.

NBH is a large complete lattice, that is, it has arbitrary joins and arbitrary meets. Indeed, if $\{\nu_i \mid i \in I\} \subseteq NBH$, the supremum is given by

$$\nu^*(m) = \bigcup\{\nu_i(m) \mid i \in I\},$$

for all m sub$X, X \in \mathcal{C}$ and the infinimum is provided by

$$\nu^\star(m) = \bigcap\{\nu_i(m) \mid i \in I\},$$

for all m sub$X, X \in \mathcal{C}$

If each ν_i satisfies (L), then $\nu^\star$ satisfies also (L).

Proposition 3.1.4. *[Raz12a, HŠ10] $LNBH$ is order isomorphic to INT . For a left-adjoint neighbourhood operator ν and an interior operator i, the inverse assignments are given by*

$$\nu^i(m) = \{n \mid m \leq i(n)\} \text{ and } i^\nu(m) = \bigvee\{n \mid m \in \nu(n)\}$$

Proof. Clearly by construction i^ν satisfies $(I1)$ and $(I2)$. To show continuity consider any $\mathcal{C}$-morphism $f : X \longrightarrow Y$ and $n \in$ sub$X$. Then $n \in \nu_Y(i^\nu_Y(n))$ by Definition 3.1.3 and by $(N4)$, $f^{-1}(n) \in \nu_X(f^{-1}(i^{\nu_Y}(n))$ that is $f^{-1}(i^\nu_Y(n)) \leq i^\nu_X(f^{-1}(n))$. On the other hand for any $m \in$ sub$X$, ν^i is left-adjoint since for any $\mathcal{G} \subseteq$ sub$X$, $g \leq i(m)$ for all $g \in \mathcal{G} \Rightarrow \bigvee \mathcal{G} \leq i(m) \Rightarrow n \in \nu^i(\bigvee \mathcal{G})$ and

$(N1)$ $n \in \nu^i(m) \Rightarrow m \leq n$ by $(I1)$.

$(N2)$ If $m \leq n$ and $n \leq i(p)$ then $m \leq i_X(p)$. Hence $\nu^i(n) \subseteq \nu^i(m)$.

$(N3)$ If $p \in \nu^i(m)$ and $p \leq q$ then $m \leq i(p) \leq i(q) \Rightarrow m \leq i(q) \Rightarrow q \in \nu^i(m)$

$(N4)$ Let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism and $n \in$ sub$Y$. Then $n \in \nu^i(f(m)) \Rightarrow f(m) \leq i(n) \Rightarrow m \leq f^{-1}(f(m)) \leq f^{-1}(i(n)) \leq i(f^{-1}(n))$. Thus $f^{-1}(n) \in \nu^i(m)$.

The assignments preserve order since,

If $\nu \subseteq \nu'$ in $LNBH$ then

$\{p \mid m \in \nu(p)\} \subseteq \{p \mid m \in \nu'(p)\} \Rightarrow \bigvee\{p \mid m \in \nu(p)\} \leq \bigvee\{p \mid m \in \nu'(p)\}$.

Thus $i^\nu(m) \leq i^\nu(m)$. On the other hand if $i \leq i'$ in INT then $\{p \mid m \leq i(p)\} \subseteq \{p \mid m \leq i'(p)\}$. Hence $\nu^i(m) \subseteq \nu^i(m)$.

Finally, they are inverse to each other:

If $i \in INT$ and $m \in$ sub$X$, then

$$i^{\nu^i}(m) = \bigvee\{n \mid m \in \nu^i(n)\} = \bigvee\{n \mid n \leq i(m)\} = i(m).$$

Conversely if $n \in \nu^{i^\nu}(m)$ then $m \leq i^\nu(n)$ and since $\nu \in LNBH$, $n \in \nu(i^\nu(n)) \subseteq \nu(m)$. Also if $n \in \nu(m)$ then $m \leq i(n)$ by definition and so $n \in \nu^{i^\nu}(m)$. □

Closure and neighbourhood are naturally related in topology, a point x in a space X belongs to the closure of a subset A of X if and only if there is an neighbourhood of x which meets A. This relationship between closure and neigbhourhood aquire another meaning once points are no longer present. Due to the fact that they are intuitive in introducing the notion of convergence, neighbourhoods were introduced with respect to a closure. It turns out that there is a closure operator on $\mathcal{C}$, which emerged from the study of convergence, naturally associated to a neighbourhood operator ν.

[HŠ11] Let m be a subobject of $X, X \in \mathcal{C}$. For a neighbouhood operator put

$$c^\nu_X(m) = \bigvee\{n \mid (\forall n' \leq^+ n),\ m \wedge \nu_X(n') > o_X\}$$

where the relations $m \wedge \nu_X(n') > o_X$ means that for any $k \in \nu_X(n')$ we have that $m \wedge k > o_X$ and the relation $n' \leq^+ n$ means $o_X < n' \leq n$.

Proposition 3.1.5. *[Raz12a, HŠ11] $(c_X)_{X \in \mathcal{C}}$ is a closure operator on $\mathcal{C}$ and the assignment $\nu \longrightarrow c^\nu$ is order reversing.*

Proof. It is easily seen that c^ν satisfies $(C1)$.

$(C2)$ If $m \leq p$ then $\{n \mid (\forall n' \leq^+ n), m \wedge \nu(n') > o_X\} \subseteq \{q \mid (\forall q' \leq^+ q), p \wedge \nu(q') > o_X\} \Rightarrow \bigvee\{n \mid (\forall n' \leq^+ n), m \wedge \nu(n') > o_X\} \leq \bigvee\{q \mid (\forall q' \leq^+ q), p \wedge \nu(q') > o_X\}$

that is $c^\nu_X(m) \leq c^\nu_X(p)$.

$(C3)$ Let $f : X \longrightarrow Y, C_n = \{f(n) \mid (\forall n' \leq^+ n), m \wedge \nu(n') > o_X\}$, and $C_p = \{p \mid (\forall p' \leq^+ p), f(m) \wedge \nu(p') > o_Y\}$. If $f(n) \in \mathcal{C}_n, \forall n' \leq^+ n, f(n') \leq^+ f(n)$ and

$$\begin{aligned} m \wedge \nu(n') > o_X &\Rightarrow o_Y < f(m \wedge \nu(n')) \leq f(m) \wedge f(\nu(n')) \\ &\Rightarrow o_Y < f(m) \wedge f(\nu(n')) \\ &\Rightarrow o_Y < f(m) \wedge \nu(f(n')) && \text{by } \nu\text{-continuity} \end{aligned}$$

Thus $f(n) \in C_p$. This implies that $\bigvee C_n \leq \bigvee C_p$; that is $f(c^\nu_X(m)) \leq c^\nu_Y(f(m))$.

Clearly, if $\nu \leq \nu'$ then, $c^\nu \leq c^{\nu'}$

 □

In point-set settings, one moves from closure to interior and back via complements. With this intuition in mind, the restriction to the class of $LNBH$ or INT suggests other possibilities of getting a notion of closedness from a neighbhourhood or interior operator.

Definition 3.1.6. [HŠ10] Let i be an interior operator and $m \in \text{sub}X$. We shall say that m is:

(1) A^i-closed if $i(m \vee n) \leq m \vee i_X(n)$ for all $n \in \text{sub}X$.

(2) B^i-closed if $m \vee n = 1_X \Rightarrow m \vee i_X(n) = 1_X$ for all $n \in \text{sub}X$.

(3) C^i-closed if m is pseudocomplemented and $m^* = i_X(m^*)$ where the pseudocomplement of m is a subobject m^* such that for any $n \in \text{sub}X$, $n \leq m^* \Leftrightarrow m \wedge n = o_X$

Clearly if $\mathcal{C} = \textbf{Top}$, category of topological spaces and continuous maps, and i the usual interior then the three notions coincide. Our next proposition establishes the relationship between the above three notions. As one can notice, the validity of the proposition depends on the existence of (pseudo)complements.

Proposition 3.1.7. *[HŠ10] Let i be an interior operator and $m \in subX$.*

(1) *m is A^i-closed $\Rightarrow$ m is B^i-closed if i is grounded.*

(2) *m is B^i-closed $\Rightarrow$ m is C^i-closed if $subX$ is Boolean algebra.*

(3) *m is C^i-closed $\Rightarrow$ m is A^i-closed if $subX$ is Boolean algebra and i is additive.*

Proof. (1) [HŠ10]

$$\begin{aligned} 1_X = m \vee n = i_X(m \vee n) & & i \text{ is grounded} \\ \leq m \vee i_X(n) & & \text{Definition 3.1.6(1)} \\ \Rightarrow m \vee i_X(n) = 1_X & & \end{aligned}$$

(2) Let sub$X$ be a boolean algebra. Then $\overline{m} = m^*$ with $\overline{m} \in \text{sub}X$ the complement of m; that is $m \wedge \overline{m} = o_X$ and $m \vee \overline{m} = 1_X$

Now

$$
\begin{aligned}
m \vee m^* = 1_X \Rightarrow m \vee i_X(m^*) &= 1_X \\
&\Rightarrow m^* \leq i_X(m^*) \\
&\Rightarrow m^* = i_X(m^*)
\end{aligned}
$$

(3) For any $n \in \text{sub}X$,

$$
\begin{aligned}
i_X(m \vee n) \wedge \overline{m} &= i_X(m \vee n) \wedge i_X(\overline{m}) && \text{sub}X \text{ is Boolean} \\
&= i_X((m \vee n) \wedge \overline{m}) && i \text{ is additive} \\
&= i_X((m \wedge \overline{m}) \vee (n \wedge \overline{m})) \\
&= i_X(n \wedge \overline{m}) \\
&= i_X(n) \wedge i_X(\overline{m}) && \text{additivity of } i \\
&= i_X(n) \wedge \overline{m} && \text{Definition 3.1.6(3)}
\end{aligned}
$$

$$
\begin{aligned}
\text{Now}, i_X(m \vee n) &= i_X(m \vee n) \wedge 1_X \\
&= i_X(m \vee n) \wedge (m \vee \overline{m}) \\
&= (i_X(m \vee n) \wedge m) \vee (i_X(m \vee n) \wedge \overline{m}) \\
&\leq m \vee (i_X(n)) \wedge \overline{m}) \\
&= (m \vee i_X(n)) \wedge (m \vee \overline{m}) \\
&= (m \vee i_X(n)) \wedge 1_X \\
&= (m \vee i_X(n))
\end{aligned}
$$

□

One perfoms the following technique, the so-called construction of closure operator depending on the parameter $\mathcal{F}$, and apply Definition 3.1.6 to obtain three different types of closure operators.

[HŠ10] For a class $\mathcal{F} \subseteq \mathcal{M}$, we form $\mathcal{F}^* = \{f^{-1}(m) \mid m \in \mathcal{F}, f \in \mathcal{C}\}$, the smallest pullback stable class containing $\mathcal{F}$. Given a subobject m of X, the assignment

$$c_X^{\mathcal{F}}(m) = \bigwedge\{n \in \mathcal{F}^* \mid m \leq n\}$$

defines a closure operator. $(C1)$ and $(C2)$ are trivially satisfied by construction of $c_{\mathcal{F}}$. For $(C3)$, let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism. Then

$$\{f^{-1}(n) : n \in \mathcal{F}^* \mid m \leq n\} \subseteq \{n \in \mathcal{F}^* \mid f^{-1}(m) \leq n\}$$

This implies that

$$\bigwedge\{n \in \mathcal{F}^* \mid f^{-1}(m) \leq n\} \leq \bigwedge\{f^{-1}(n) : n \in \mathcal{F}^* \mid m \leq n\}$$

that is $c^{\mathcal{F}}(f^{-1}(m)) \leq f^{-1}(c^{\mathcal{F}}(m))$.

Definition 3.1.8. [Raz12a, HŠ10] Let i be an interior operator, we shall denote by α^i, β^i, γ^i the closure operator obtained by considering for the class $\mathcal{F}$, the classes of A^i -closed, B^i-closed, C^i -closed subobjects, respectively.

The assignments $i \longmapsto \beta^i$ and $i \longmapsto \gamma^i$ define order reversing maps from INT (or $LNBH$) to NBH. In fact, if $i(m) \leq i'(m)$, then $\{n \mid n$ is B^i-closed and $m \leq n\} \subseteq \{n \mid n$ is $B^{i'}$-closed and $m \leq n\}$ since for all $p \in \text{sub}X$, if $n \vee p = 1_X \Rightarrow n \vee i_X(p) = 1_X$ then $n \vee i'_X(p) = 1_X$. This implies that $\bigwedge\{n \mid n$ is $B^{i'}$-closed and $m \leq n\} \leq \bigwedge\{n \mid n$ is B^i-closed and $m \leq n\}$, that is $\beta^{i'}(m) \leq \beta^i(m)$.

Similarly, for all $m \in \text{sub}X$, if $i(m) \leq i'(m)$, then $\{n \mid n$ is C^i-closed and $m \leq n\} \subseteq \{n \mid n$ is $C^{i'}$-closed and $m \leq n\}$ since for all $p \in \text{sub}X$, if n is pseudocomplemented and $n^* = i_X(n^*)$, then $n^* \leq i'_X(n^*)$ which gives $n^* = i_X(n^*)$. This implies that $\bigwedge\{n \mid n$ is $C^{i'}$-closed and $m \leq n\} \leq \bigwedge\{n \mid n$ is C^i-closed and $m \leq n\}$, that is $\gamma^{i'}(m) \leq \gamma^i(m)$.

The assignment $i \longmapsto \alpha^i$ does not respect order in general. Indeed, if any morphism in $\mathcal{C}$ reflects o and i an interior operator such that $i(m) = o_X$, i' an interior operator with $i'(m) = m$ for all m, then for any $n \in \text{sub}X$, n is A^i-closed and $A^{i'}$-closed and so $\alpha^i(m) = m$ and $\alpha^{i'}(m) = m$. This behaviour explains why α^i is not part of a Galois connection.

Now given a closure operator c on $\mathcal{C}$ with respect to $\mathcal{M}$, one associates a neighbourhood operator. Guided by the intuition from topological spaces, we get the following definition.

Definition 3.1.9. [HŠ10] Let c be a closure operator and $m \in \text{sub}X$, $X \in \mathcal{C}$. We shall say that m is:

(1) $\mathfrak{A}^c$-open if $m \wedge c_X(n) \leq c_X(m \wedge n)$ for all $n \in \text{sub}X$.

(2) $\mathfrak{B}^c$-open if $m \wedge n = o_X \Rightarrow m \wedge c_X(n) = o_X$ for all $n \in \text{sub}X$.

(3) $\mathfrak{C}^c$-open if m is pseudocomplemented and $m^* = c_X(m^*)$.

A similar operation as the one for interior operator can also be performed for a closure operator c and get three different neighbourhood operators by applying Defintion 3.1.9.

Given a class $\mathcal{F} \subseteq \mathcal{M}$,

$$\nu^{\mathcal{F}}(m) = \{n \in \mathcal{M} \mid (\exists\, p \in \mathcal{F}^*), m \leq p \leq n\}$$

is clearly a neighbourhood operator. $\nu^{\mathcal{F}}$ is regular if it holds that for any $\mathcal{G} \subseteq \mathcal{F}^*, \bigvee \mathcal{G} \in \mathcal{F}^*$.

Definition 3.1.10. [Raz12a, HŠ10] Given a closure operator c, we denote by $\mathfrak{a}^c, \mathfrak{b}^c, \mathfrak{c}^c$, the neighbourhood operators obtained by considering respectively for the class $\mathcal{F}$ the class of $\mathfrak{A}^c$, $\mathfrak{B}^c$, $\mathfrak{C}^c$-open subobjects.

Analogously to α^i, β^i and γ^i, the assignments $c \longmapsto \mathfrak{b}^c$ and $c \longmapsto \mathfrak{c}^c$ are clearly order-reversing while the assignment $c \longmapsto \mathfrak{a}^c$ behave like α^i, that is, it does not respect any order.

The correspondances we have looked at in this section between INT(or NBH) and $CLOS$ do not offer a natural way of moving from interior or neighbourhood to closure and back in the sense that they are not Galois connections in general. However, if the subobject lattices on which the operators interact are boolean algebras, then the pairs $(\alpha, \mathfrak{a})$, $(\beta, \mathfrak{b})$, $(\gamma, \mathfrak{c})$ are Galois connection between NBH or INT and $CLOS$. A rather natural way of treating the three operators in one setting will be later introduced.

3.2 The classes of Morphisms

Although considered implicitly for a long time in topos theory, c-open morphisms were explicity introduced in ([DT95]) as morphisms whose inverse image commutes with the closure (cf.[GT00]). In this paper (see also [Cle01, CGT01, CT01]) the behaviour of these morphisms together with the c-initial morphisms and their duals, namely the c-closed and c-final morphisms are studied. This section investigates four types of morphisms with respect to a neighbourhood operator. These are crucial to understanding topological constructions such as formation of subspaces, quotients etc. In contrast to our approach which considers neighbourhood operators to act on subobject lattices, a lax approach to neighbourhood operators is used in [RH16] to study these classes of morphisms. We shall mainly define the morphisms and study some of their basic properties which will be useful in the remaining chapters.

Definition 3.2.1. [Raz12a] Let ν be a neighbourhood operator. We say that a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is:

(i) ν-closed if for any $n \in \text{sub}Y$,

$$\nu_X(f^{-1}(n)) = f^{-1}(\nu_Y(n))$$

(ii) ν-initial if for any $m \in \text{sub}X$,

$$\nu_X(m) = f^{-1}(\nu_X(f(m)))$$

One uses Theorem 1.4.12 in [Eng89] to see that in the category **Top** of topological spaces and continuous maps if we consider the neighbourhood operator

$$\mathcal{N}_X(A) = \{B \mid A \subseteq O \subseteq B \text{ for some open } O \subseteq X\}$$

for all $A \subseteq X$ and X a topological space, a ν-closed morphism coincides with a closed map and a ν-initial coincides with an initial continous map, that is a continuous map whose domain carries the initial topology induced by the map itself from its domain.

Our next two propostions describe the behaviour of ν-closed and ν-initial morphisms respectively.

Proposition 3.2.2. *[Raz12a] Let ν be a neighbourhood operator. The following statements are true.*

(1) *Every isomorphism in $\mathcal{C}$ is ν-closed.*

(2) *ν-closed morphisms are closed under composition.*

(3) *If $g \circ f$ is ν-closed and $g \in \mathcal{M}$, then f is ν-closed.*

(4) *If $g \circ f$ is ν-closed and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback, then g is ν-closed.*

Proof. (1) Let $f : X \longrightarrow Y$ be an isomorphism in $\mathcal{C}$ with inverse $g : Y \longrightarrow X$ then for all $n \in \text{sub}Y$,

$$\nu(f^{-1}(n)) = \nu(g(n)) \subseteq g(\nu(n)) = f^{-1}(\nu(n))$$

(2) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be ν-closed morphisms. Then for all $n \in \text{sub}Y$,

$$\begin{aligned}\nu((g \circ f)^{-1}(n)) &= \nu(f^{-1}(g^{-1}(n))) \\ &= f^{-1}(\nu(g^{-1}(n))) \\ &= f^{-1}(g^{-1}(\nu(n))) \\ &= (g \circ f)^{-1}(\nu(n)))\end{aligned}$$

(3) If $g \circ f$ is ν-closed and $g \in \mathcal{M}$, for all $n \in \text{sub}Y$,

$$\begin{aligned}\nu(f^{-1}(n)) &= \nu(f^{-1}(g^{-1}(g(n)))) \\ &= (g \circ f)^{-1}(\nu(g(n)) \\ &\subseteq f^{-1}(\nu(g^{-1}(g(n))) = f^{-1}((n))\end{aligned}$$

(4) Let $g \circ f$ be ν-closed and f be in $\mathcal{E}$ with $\mathcal{E}$ stable under pullback. Then

$$\begin{aligned}\nu(g^{-1}(n)) &= (f(f^{-1}\nu(g^{-1}(n)))) \\ &\subseteq f(\nu(f^{-1}(g^{-1}(n)))) \\ &= f(\nu(g \circ f)^{-1}(n)) \\ &= f(f^{-1}(g^{-1}(\nu(n))) = g^{-1}(\nu(n))\end{aligned}$$

While the notion of c-closed morphisms require the closure to commute with the images of subobjects (see Definition 2.2.9), the ν-closed require the pre-images of subobjects to commute with the neighbourhood operator. Despite this duality the behaviour of the two notions is quite similar as this can be seen from Propositions 2.2.10 and 3.2.2. The strategies in the proofs of the two Propositions are the same although it seems easier to manipulate ν-closed morphisms than the c-closed.

□

Proposition 3.2.3. *[Raz12a] The following statements hold true for a neighbourhood operator ν.*

(1) *Every isomorphism in $\mathcal{C}$ is ν-initial.*

(2) *ν-initial morphisms are closed under composition.*

(3) *If $g \circ f$ is ν-initial then then f is ν-initial.*

(4) *If $g \circ f$ is ν-initial and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback, then g is ν-initial*

Proof. (1) Let $f : X \longrightarrow Y$ be a $\mathcal{C}$-isomorphism and $g : Y \longrightarrow Y$ its inverse. Then

$$\nu(m) = \nu(g(f(m)) \subseteq g(\nu(f(m)) = f^{-1}(\nu(f(m))).$$

(2) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be ν-initial morphisms. Then for all $n \in \text{sub}Y$

$$\nu(m) = f^{-1}(\nu(f(m)) = f^{-1}(g^{-1}(\nu(g(f(m))))).$$

(3) If $g \circ f$ is ν-initial, then for all $n \in \text{sub}X$,

$$\nu(m) = f^{-1}(g^{-1}(\nu(g(f(m))))) \subseteq f^{-1}(\nu(f(m))) \subseteq \nu(m).$$

(4) If $g \circ f$ is ν-initial and f is in $\mathcal{E}$ with $\mathcal{E}$ stable under pullback,

$$\begin{aligned} f^{-1}((n)) &\subseteq \nu(f^{-1}(n)) \\ &\subseteq (g \circ f)^{-1}(\nu((g \circ f)(f^{-1}(n)))) \\ &= f^{-1}(g^{-1}(\nu(g(f(f^{-1}(n)))))) \\ &= f^{-1}(g^{-1}(\nu(g(n)))) \end{aligned}$$

which implies that $\nu(n) \subseteq g^{-1}(\nu(g(n)))$ □

The relationship between the ν-initial and ν-closed morphisms is described in the next proposition.

Proposition 3.2.4. *Let ν be a neighbourhood operator, then*

(1) *every section (or split monomorphism) is ν-initial.*

(2) *every ν-closed monomorphism is ν-initial.*

(3) *every ν-initial morphism in $\mathcal{E}$ is ν-closed provided $\mathcal{E}$ is pullback stable.*

Proof. (1) Clear from 3.2.4(3). (2) If $f : X \longrightarrow Y$ is a ν-closed mono, then

$$\nu(m) = \nu(f^{-1}(f(m)) = f^{-1}(\nu(f(m))$$

for all $m \in \text{sub}X$.

(3) If $\mathcal{E}$ is pullback stable then $f(f^{-1}(m)) = m$. Thus

$$\nu(f^{-1}(n)) = f^{-1}(\nu(f(f^{-1}(n))) = f^{-1}(\nu(m))$$

□

We next consider the dual notions to ν-closed and ν-initial namely the ν-open and ν-final. These are obtained by assuming that the neighbourhoods commute with the images.

Definition 3.2.5. [Raz12a] Let ν be an neighbourhood operator. A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is said to be: (i) ν-open if

$$\nu_Y(f(m)) = f(\nu_X(m))$$

for all $m \in \text{sub}X$. (ii) ν-final if for any $n \in \text{sub}Y$,

$$\nu_X(n) = \{k \geq n \mid f^{-1}(k) \in \nu_X(f^{-1}(n))\}$$

While the ν-closed, ν-initial and ν-open morphisms were obtained by replacing in the continuity condition "$\subseteq$ " with "$=$ ", this does not hold for ν-final, unless the morphism is in $\mathcal{E}$ and $\mathcal{E}$ is pullback stable.

We have already encountered the notion of open morphism (see Definition 2.2.17). Here the operator needs to commute with the images of subobjects while previously we needed the interior to commute with the pre-mages of subobjects. However, these two notions behave in a similar way (see Propositions 2.2.18 and 3.2.6).

The basic properties of ν-open and ν-final morphisms are provided by the following two propostions.

Proposition 3.2.6. *[Raz12a] Let ν be a neighbourhood operator. The following statements are true:*

(1) *Every isomorphism in $\mathcal{C}$ is ν-open.*

(2) *ν-open morphisms are closed under composition.*

(3) *If $g \circ f$ is ν-open and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback, then g is ν-open.*

(4) *If $g \circ f$ is ν-open and $g \in \mathcal{M}$, then f is ν-open.*

Proof. . (1) Let $f : X \longrightarrow Y$ be a $\mathcal{C}$-isomorphism with inverse $g : Y \longrightarrow X$ then for all $m \in \text{sub}X$, $f(\nu_X(m)) = g^{-1}(\nu_X(m)) \subseteq \nu_Y(g^{-1}(m)) = \nu_Y(f(m))$

(2) If $f : X \longrightarrow Y$ and $Y \longrightarrow Z$ are ν-open morphisms, then for all $m \in \text{sub}X$,

$$\begin{aligned} \nu_Z((g \circ f)(m)) &= \nu_Z(g(f(m)) \\ &= g(\nu_Y(f(m)) \\ &= g(f(\nu_X(m)) \end{aligned}$$

(3) If $g \circ f$ is ν-open and $g \in \mathcal{M}$ for all $m \in \text{sub}X$,

$$\begin{aligned} f(\nu_X(m)) &= g^{-1}(g(f(\nu_X(m))) \\ &= g^{-1}((g \circ f)(\nu_X(m))) \\ &= g^{-1}(\nu_Z(g \circ f)(m)) \\ &\subseteq \nu_Y(g^{-1}(g(f(m))) \\ &= \nu_Y(f(m)) \end{aligned}$$

(4) If $g \circ f$ is ν-open and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback, then for all $m \in \text{sub}Y$,

$$\begin{aligned} g(\nu_Y(m)) &= g(\nu_Y(f(f^{-1}(n)))) \\ &\subseteq g(f(\nu_X(f^{-1}(m))) \\ &= \nu_Z(g(f(f^{-1}(m)))) \\ &= \nu_Z(g(m)) \end{aligned}$$

□

Proposition 3.2.7. *[Raz12a] The following statements hold true for a neighbourhood operator ν.*

(1) *Every isomorphism in $\mathcal{C}$ is ν-final.*

(2) *ν-final morphisms are closed under composition.*

(3) *If $g \circ f$ is ν-final then then g is ν-final.*

(4) *If $g \circ f$ is ν-final and g is mono, then f is ν-final.*

Proof. (1) Let $f : X \longrightarrow Y$ be a $\mathcal{C}$-isomorphism with inverse $g : Y \longrightarrow X$ and $n \leq k$ in sub$Y$. Then

$$\begin{aligned} f^{-1}(k) \in \nu(f^{-1}(n)) &\Rightarrow g(k) \in \nu(g(n)) \\ &\Rightarrow g^{-1}(g(k)) \in \nu(n) \qquad \text{by continuity of } g \\ &\Rightarrow k \in \nu(n) \end{aligned}$$

(2) If $f : X \longrightarrow Y$ and $Y \longrightarrow Z$ are ν-final morphisms, then for all $k \geq n$ in subZ, $g^{-1}(k) \geq g^{-1}(n)$, Hence,

$$\begin{aligned} f^{-1}(g^{-1}(k)) \in \nu(f^{-1}((g^{-1}(n)) &\Rightarrow g^{-1}(k) \in \nu(g^{-1}(n)) \\ &\Rightarrow k \in \nu(n)) \end{aligned}$$

(3) Assume $g \circ f$ is ν-final, $k \geq n$ and $g^{-1}(k) \in \nu(g^{-1}(n))$ for $n \in \text{sub}Z$,

then $(g \circ f)^{-1} \in \nu((g \circ f)^{-1}(n))$ by ν-continuity of f. Thus $k \in \nu(n)$ by finality of $g \circ f$

(4) Let g be mono and $k \geq n$. Then $k = g^{-1}(g(k)), n = g^{-1}(g(n))$ for some $n, k \in \text{sub}Y$.

If $f^{-1}(k) \in \nu(f^{-1}(n))$ then $(g \circ f)^{-1}(g(k)) \in \nu((g \circ f)^{-1}(g(n))$. Since $g(n) \leq g(k)$, $g(k) \in \nu(g(n))$ by ν-finality of $g \circ f$ and hence $k \in \nu(n)$ by ν-continuity of g. $\square$

The following relates the ν-open, ν-final and ν-closed morphisms.

Proposition 3.2.8. *[Raz12a] Let ν be a neighbourhood operator, then*

(1) *every retraction (or split epimorphism) is a ν-final.*

(2) *every ν-open morphism in $\mathcal{E}$ is ν-final morphism in $\mathcal{E}$ provided $\mathcal{E}$ is pullback stable.*

(3) *every ν-final monomorphism is ν-open.*

(4) *every ν-closed monomorphism is ν-final.*

Proof. (1) Clear from Proposition 3.2.7(3).

(2) Let $f : X \longrightarrow Y$ be ν-open and $k \geq n$ then

$$\begin{aligned} f^{-1}(k) \in \nu(f^{-1}(n)) &\Rightarrow k \in \nu(f(f^{-1}(n)) \\ &\Rightarrow k \in \nu(n) \qquad\qquad \text{since } f \in \mathcal{E} \text{ and } \mathcal{E} \text{ is pullback stable} \end{aligned}$$

(3) If f is ν-final mono, the

$$\begin{aligned} f^{-1}(n) \in \nu(m) &\Rightarrow f^{-1}(n) \in \nu(f^{-1}(f(m))) \\ &\Rightarrow n \in \nu(f(m)) \end{aligned}$$

(4) Let $f : X \longrightarrow Y$ be ν-closed in $\mathcal{E}$ with $k \geq n$ then

$$\begin{aligned} f^{-1}(k) \in \nu(f^{-1}(n) &\Rightarrow f(f^{-1}(k)) \in \nu(n) \\ &\Rightarrow k \in \nu(n) \end{aligned}$$

□

In the previous chapter, we have studied the c-closed and i-open morphisms. As observed, their behaviour is quite similar to those of ν-open morphisms described in this chapter. These three classes of morphisms will later be considered as as being essentially the same. This fact will be seen using the notion of topogenous order on a category that we introduce in the next chapter.

Chapter 4

Topogenous structures

In the previous chapter we have looked at different ways of moving between closure, interior or neighbourhood operator. We showed that none of them provides a suitable way to switch back and forth between closure and interior or neighbourhood operators. In this chapter we wish to present a framework in which the three operators can be treated in one setting. We introduce the notion of topogenous order on a category and show that it is equivalent to the categorical neighbourhood operator. Then we proceed by showing that the closure and interior operators are special topogenous orders. We end the chapter by providing a few examples for the developed theory.

4.1 Topogenous orders or neighbourhood operators

In this section we define the topogenous order on a category . We show that $TORD$, the congolomerate of all topogenous orders on $\mathcal{C}$, is a complete lattice. We end the section by proving the equivalence between the topogenous orders and the neighbourhood operators.

Definition 4.1.1. A topogenous order $\sqsubseteq$ on $\mathcal{C}$ is a family $\{\sqsubseteq_X \mid X \in \mathcal{C}\}$ of relations, each $\sqsubseteq_X$ on sub$X$, such that:

$(T1)$ $m \sqsubseteq_X n \Rightarrow m \leq n$ for every $m, n \in \text{sub}X$,

$(T2)$ $m \leq n \sqsubseteq_X p \leq q \Rightarrow m \sqsubseteq_X q$ for every $m, n, p, q \in \text{sub}X$, and

$(T3)$ every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is $\sqsubseteq$-continuous, $m \sqsubseteq_Y n \Rightarrow f^{-1}(m) \sqsubseteq_X f^{-1}(n)$ for every $m, n \in \text{sub}Y$.

An equivalent formulation of the $\sqsubseteq$-continuity is provided by the following proposition:

Proposition 4.1.2. *A morphism $f : X \longrightarrow Y$ is in $\mathcal{C}$ is $\sqsubseteq$-continuous iff $f(m) \sqsubseteq_Y n \Rightarrow m \sqsubseteq_X f^{-1}(n)$ for any $m \in subX$ and $m \in subY$.*

Proof. Let f be $\sqsubseteq$-continuous and $f(m) \sqsubseteq_Y n$, then $m \leq f^{-1}(f(m)) \sqsubseteq_X f^{-1}(n)$ and so $m \sqsubseteq_X f^{-1}(n)$ by $(T2)$. Conversely since $f(f^{-1}(m)) \leq m$ for any $m \in \mathrm{sub}Y$, again by $(T2)$, $m \sqsubseteq_Y n \Rightarrow f(f^{-1}(m)) \sqsubseteq_Y n \Rightarrow f^{-1}(m) \sqsubseteq_X f^{-1}(n)$. □

At times we omit the subscripts if there is no danger of confusion. $TORD$ is ordered as follows: $\sqsubseteq \subseteq \sqsubseteq'$ in $TORD$ if and only if for all $m, n \in \mathrm{sub}X$, $m \sqsubseteq_X n \Rightarrow m \sqsubseteq'_X n$. It is a complete lattice with set theoretic union and intersection giving the suprema and infinima as shown by the following proposition.

Proposition 4.1.3. *Let $\{\sqsubseteq^i_X \mid i \in I\} \subseteq TORD$ for all $X \in \mathcal{C}$.*

(i) $\sqsubseteq^*_X = \bigcup\{\sqsubseteq^i_X \mid i \in I\}$ for all $X \in \mathcal{C}$, and

(ii) $\sqsubseteq^\diamond_X = \bigcap\{\sqsubseteq^i_X \mid i \in I\}$ for all $X \in \mathcal{C}$

are topogenous orders on $\mathcal{C}$.

Proof. Let m, n, p and q be in sub$X$:

(i) Since $m \sqsubseteq^*_X n \Rightarrow m \sqsubseteq^i_X n$ for some $i \in I$, we have that $m \leq n$. If $m \leq n \sqsubseteq^*_X p \leq q$, then $m \leq n \sqsubseteq^i_X p \leq q$ for some $i \in I$. Thus $m \sqsubseteq^i_X q$ for some $i \in I$ and so $m \sqsubseteq^*_X n$. Now, let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism and $m \sqsubseteq^*_Y n$ for all $m, n \in$sub$Y$. Then $m \sqsubseteq^i_Y n$ for some $i \in I$. This implies that $f^{-1}(m) \sqsubseteq^i_X f^{-1}(n)$ for some i and so $f^{-1}(m) \sqsubseteq^*_X f^{-1}(n)$.

(ii) Since $m \sqsubseteq^\diamond_X n \Rightarrow m \sqsubseteq^i_X n$ for all $i \in I$, we have that $m \leq n$. If $m \leq n \sqsubseteq^\diamond_X p \leq q$, then $m \leq n \sqsubseteq^i_X p \leq q$ for all $i \in I$. Thus $m \sqsubseteq^i_X q$ for all $i \in I$ and so $m \sqsubseteq^\diamond_X n$. Now, let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism and $m \sqsubseteq^\diamond_X n$ for all $m, n \in \mathrm{sub}Y$. Then $m \sqsubseteq^i_Y n$ for all $i \in I$. This implies that $f^{-1}(m) \sqsubseteq^i_X f^{-1}(n)$ for all i and so $f^{-1}(m) \sqsubseteq^\diamond_X f^{-1}(n)$.

□

The study of topogenous orders on the category of toplogical spaces was motivated by the following order $A \sqsubseteq B \Leftrightarrow A \subseteq B^o$. This order is responsible for an adjunction between neighbourhood and interior namely U is a neighbourhood of x iff $\{x\} \subseteq U^o$ (cf. [RH14]) and indeed topogenous orders are equivalent to neighbourhood operators.

Proposition 4.1.4. *$TORD$ and NBH are order isomorphic with the inverse assignments $\sqsubseteq \longrightarrow \nu^{\sqsubseteq}$ and $\nu \longrightarrow \sqsubseteq^{\nu}$ given by*

$$\nu^{\sqsubseteq}_X(m) = \{n \mid m \sqsubseteq_X n\} \text{ and } m \sqsubseteq^{\nu}_X n \Leftrightarrow n \in \nu_X(m) \text{ for all } X \in \mathcal{C}$$

Proof. Let $\sqsubseteq \in TORD$ and $m, n \in \text{sub}X$, we have that

$(N1)$ $n \in \nu^{\sqsubseteq}_X(m) \Rightarrow m \sqsubseteq_X n \Rightarrow m \leq n$.

$(N2)$ If $m \leq n$ and $p \in \nu^{\sqsubseteq}_X(n)$ then $m \leq n \sqsubseteq_X p \Rightarrow m \sqsubseteq_X p \Rightarrow p \in \nu^{\sqsubseteq}_X(m)$ and $\nu^{\sqsubseteq}_X(n) \subseteq \nu^{\sqsubseteq}_X(m)$.

$(N3)$ If $n \in \nu^{\sqsubseteq}_X(m)$ and $n \leq q$, then $m \sqsubseteq_X n \leq q \Rightarrow m \sqsubseteq_X q \Rightarrow q \in \nu^{\sqsubseteq}_X(m)$.

$(N4)$ Let $f : X \longrightarrow Y$, then $p \in \nu^{\sqsubseteq}_Y(f(m)) \Rightarrow f(m) \sqsubseteq_X p \Rightarrow m \sqsubseteq f^{-1}(p)$ by Proposition 4.1.2; so $f^{-1}(p) \in \nu^{\sqsubseteq}_X(m)$.

On the other hand given if $\nu \in NBH$ and $m, n \in \text{sub}X$, we see that:

$(T1)$ $n \sqsubseteq^{\nu}_X m \Rightarrow m \leq n$.

$(T2)$ If $m \leq n \sqsubseteq^{\nu}_X p \leq q$, then $p \in \nu_X(n)$ and by $(N3)$ $q \in \nu_X(n)$. Since $m \leq n \Rightarrow \nu_X(n) \subseteq \nu_X(m)$ it follows that $q \in \nu_X(m)$ and so $m \sqsubseteq^{\nu}_X q$.

$(T3)$ Let $f : X \longrightarrow Y$ and $p \sqsubseteq^{\nu}_X n$ for n and $p \in \text{sub}X$. Then $n \in \nu_Y(f(f^{-1}(p))) \Rightarrow f^{-1}(n) \in \nu_Y(f^{-1}(p)) \Rightarrow f^{-1}(p) \sqsubseteq^{\nu}_Y f^{-1}(n)$

If $\sqsubseteq \subseteq \sqsubseteq'$ in $TORD$, then $\{p \mid m \sqsubseteq_X p\} \subseteq \{q \mid m \sqsubseteq'_X q\}$. Thus $\nu^{\sqsubseteq}_X(m) \subseteq \nu^{\sqsubseteq'}_X(m)$. On the other hand if $\nu \leq \nu'$ in NBH, then $n \in \nu(m) \Rightarrow n \in \nu'(m)$. So $\sqsubseteq^{\nu} \subseteq \sqsubseteq^{\nu'}$.

Finally they are inverse to each other since for $m, n \in \text{sub}X$:

(1) $m \sqsubseteq^{\nu^{\sqsubseteq}}_X n \Leftrightarrow n \in \nu^{\sqsubseteq}(m) \Leftrightarrow m \sqsubseteq_X n$.

(2) $n \in \nu^{\sqsubseteq^{\nu}}_X(m) \Leftrightarrow m \sqsubseteq^{\nu}_X n \Leftrightarrow n \in \nu_X(m)$.

 $\square$

4.2 Topogenous orders which respect suprema and infima

We know from previous section that topogenous orders are equivalent to neighbourhood operators. In this section, we prove that closure and interior operators are nicely embedded (in the sense that they can be seen as reflections of sub-quasicategories) in topogenous orders. Consider the following conditions :

(S) $(\forall i \in I : m_i \sqsubseteq_X n) \Rightarrow \bigvee m_i \sqsubseteq_X n$ for all $X \in \mathcal{C}$.

(I) $\forall i \in I : m \sqsubseteq_X n_i) \Rightarrow m \sqsubseteq_X \bigwedge n_i$ for all $X \in \mathcal{C}$.

(P) If $m \sqsubseteq_X n$ then there is p such that $m \sqsubseteq_X p \sqsubseteq_X n$ for all $X \in \mathcal{C}$

One obtains different types of topogenous orders and consequently different types of sub-quasicategories of $TORD$:

- $\bigvee -TORD$: the class of all topogenous orders which respect suprema, those satisfying (S).
- $\bigwedge -TORD$: the class of all topogenous orders which respect infima, those satisfying (I).
- $INTORD$: the class of all interpolative topogenous orders, those satisfying (P).

Lemma 4.2.1. *Let $\{\sqsubseteq_X^i \mid i \in I\} \subseteq TORD$ for all $X \in \mathcal{C}$.*

(i) If each $\sqsubseteq_i$ satisfies condition (S) (resp. (I)) then so does the topogenous order defined by

$$\sqsubseteq_X^{\diamond} = \bigcap \{\sqsubseteq_X^i \mid i \in I\}$$

for all $X \in \mathcal{C}$;

(ii) If each $\sqsubseteq_i$ satisfies condition (P) then so does the topogenous order defined by

$$\sqsubseteq_X^{*} = \bigcup \{\sqsubseteq_X^i \mid i \in I\}$$

for all $X \in \mathcal{C}$.

Proof. (i) $\sqsubseteq^{\diamond}$ is a topogenous order by Proposition 1.1.2 and (S) (resp. (I)) is trivially satisfied if each $\sqsubseteq^i$ satisfies condition (S) (resp. (I))

(*ii*) If $m \sqsubseteq^* n$ then $m \sqsubseteq_X^i n$ for some $i \in I$. This implies that there is p such that $m \sqsubseteq_X^i p \sqsubseteq_X^i n$ and so $m \sqsubseteq_X^* p \sqsubseteq_X^* n$.

□

The above lemma shows that $\bigvee -TORD$ and $\bigwedge -TORD$ are closed under arbitrary intersection in $TORD$ and that $INTORD$ is closed under arbitrary union in $TORD$. Embeddings that preserve infima (resp. suprema) are reflections (resp. coreflection). Hence $\bigvee -TORD$ and $\bigwedge -TORD$ are reflective sub-quasicategories of $TORD$ and $INTORD$ is a coreflective sub-quasicategories of $TORD$. Our next two propositions show that $\bigvee -TORD$ and $\bigwedge -TORD$ are precisely the interior and closure operators.

Proposition 4.2.2. *$\bigvee -TORD$ is order isomorphic to INT with the inverse assignments given by*

$$i_X^{\sqsubseteq}(m) = \bigvee\{p \mid p \sqsubseteq_X m\} \text{ and } m \sqsubseteq_X^i n \Leftrightarrow m \leq i_X(n) \text{ for all } X \in \mathcal{C}$$

Proof. Let $m \in \text{sub}X$ and $I_m = \{p \mid p \sqsubseteq_X m\}$. Since $p \sqsubseteq m \Rightarrow p \leq n, i^{\sqsubseteq}(m) \leq m$. If $m \leq n$ and $p \in I_m$, then $p \sqsubseteq m \leq n$. Thus $p \sqsubseteq n$ and $I_m \subseteq I_n$ giving $\bigvee I_m \leq \bigvee I_n$ and so ${}_X^{\sqsubseteq}(m) \leq i_X^{\sqsubseteq}(n)$. Now, let $f : X \longrightarrow$Y be a $\mathcal{C}$-morphism. Since $\sqsubseteq \in \bigvee -TORD$, $i_Y^{\sqsubseteq}(m) = \max\{p \mid p \sqsubseteq_Y m\}$ and so by $(T3)$ we have that $f^{-1}(i_Y^{\sqsubseteq}(m)) \in \{q \mid q \sqsubseteq_X f^{-1}(m)\} \Rightarrow f^{-1}(i_Y^{\sqsubseteq}(m)) \leq \max\{q \mid q \sqsubseteq_X f^{-1}(m)\}$. Hence $f^{-1}(i_Y^{\sqsubseteq}(m)) \leq i_X^{\sqsubseteq}(f^{-1}(m))$.

Likewise, since $i_X(n) \leq n$, we have $m \sqsubseteq_X^i n \Rightarrow m \leq n$. If $m \leq n \sqsubseteq_X^i p \leq q$, then $m \leq n \leq i_X(p) \leq i_X(q)$. As a result, $m \leq i_X(p)$ and so $m \sqsubseteq_X^i q$ for n, p and q in sub$X$. Now let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism and $m \sqsubseteq_Y^i n$ for $n, m \in \text{sub}Y$. Then $m \leq i_Y(n)$. Thus, $f^{-1}(m) \leq f^{-1}(i_Y(n)) \leq i_X(f^{-1}(m)) \Rightarrow f^{-1}(m) \leq i_X(f^{-1}(n))$ and so $f^{-1}(m) \sqsubseteq_X^i f^{-1}(n)$.

Assume that $\sqsubseteq \subseteq \sqsubseteq'$ in $\bigvee -TORD$. Then $\{p \mid p \sqsubseteq_X m\} \subseteq \{q \mid q \sqsubseteq'_X m\} \Rightarrow \bigvee\{p \mid p \sqsubseteq_X m\} \leq \bigvee\{q \mid q \sqsubseteq'_X m\}$. So $i_X^{\sqsubseteq}(m) \leq i_X^{\sqsubseteq'}(m)$ for all $X \in \mathcal{C}$. On the other hand if $i \leq i'$ in INT then, $m \leq i_X(n) \Rightarrow m \leq i_X(n) \leq i'_X(n) \Rightarrow m \leq i'_X(n)$. Therefore $\sqsubseteq^i \subseteq \sqsubseteq^{i'}$.

Finally let $i \in INT$ and $m \in \text{sub}X$, then $i_X^{\sqsubseteq^i}(m) = \bigvee\{n \mid n \sqsubseteq_X^i m\} = \bigvee\{n \mid n \leq i_X(m)\} = i_X(m)$. Conversely if $m \sqsubseteq_X^{i^{\sqsubseteq}} n$ then $m \leq i^{\sqsubseteq}(n) = \bigvee\{p \mid p \sqsubseteq n\} \Rightarrow m \sqsubseteq_X n$.

On the other hand, if $m \sqsubseteq_X n \Rightarrow m \leq \{p \mid p \sqsubseteq_X n\}$ by definition. Hence $m \sqsubseteq_X^{i^{\sqsubseteq}} n$.

□

The condition that the joins of subobjects commute with pullbacks is very important for interior operators. If f^{-1} commutes with joins, as is the case for example for functions on sets then $i^{\sqsubseteq}$ is an interior operator for any $\sqsubseteq \in TORD$. Since $\{f^{-1}(p) \mid p \sqsubseteq_Y m\} \subseteq \{q \mid q \sqsubseteq_X f^{-1}(m)\}$ by $(T3)$, we have

$$\begin{aligned} f^{-1}(i_Y^{\sqsubseteq}(m)) &= f^{-1}(\bigvee\{p \mid p \sqsubseteq_Y m\} \\ &= \bigvee\{f^{-1}(p) \mid p \sqsubseteq_Y m\} \\ &\leq \bigvee\{q \mid q \sqsubseteq_X f^{-1}(m)\} = i_X(f^{-1}(m)) \end{aligned}$$

Corollary 4.2.3. *Interpolative topogenous orders in $\bigvee -TORD$ are equivalent to idempotent interior operators*

Proof. Assume that i is idempotent. Then

$$\begin{aligned} m \sqsubseteq_X^i n &\Rightarrow m \leq i_X(n) \\ &\Rightarrow m \leq i_X(i_X(n)) \leq i_X(n) \\ &\Rightarrow m \sqsubseteq_X^i i_X(n) \sqsubseteq_X^i n \end{aligned}$$

Conversely if $\sqsubseteq$ is interpolative then $\{n \mid n \sqsubseteq_X m\} \subseteq \{p \mid p \sqsubseteq_X i_X^{\sqsubseteq}(m)\}$ for all $X \in \mathcal{C}$ since

$$\begin{aligned} m \sqsubseteq_X n &\Rightarrow \exists\, q : m \sqsubseteq_X q \sqsubseteq_X n \\ &\Rightarrow m \sqsubseteq_X q \leq i_X^{\sqsubseteq}(n) && \text{definition of } i^{\sqsubseteq}(m) \\ &\Rightarrow m \sqsubseteq_X i_X^{\sqsubseteq}(n) && \text{(T2)} \end{aligned}$$

This implies that $\bigvee\{n \mid n \sqsubseteq_X m\} \leq \bigvee\{p \mid p \sqsubseteq_X i_X^{\sqsubseteq}(m)\}$ and so $i_X^{\sqsubseteq}(m) \leq i_X^{\sqsubseteq}(i_X^{\sqsubseteq}(m))$. $\square$

Proposition 4.2.4. *$\bigwedge -TORD$ is order isomorphic to $CLOS$ with the inverse assignments given by*

$$c_X^{\sqsubseteq}(m) = \bigwedge\{p \mid m \sqsubseteq_X p\} \textit{ and } m \sqsubseteq_X^c n \Leftrightarrow c_X(m) \leq n \textit{ for all } X \in \mathcal{C}$$

Proof. Let $m \in \text{sub}X$ and $C_m = \{p \mid m \sqsubseteq_X p\}$. Since $m \sqsubseteq_X p \Rightarrow m \leq p$, we have $m \leq c_X^{\sqsubseteq}(m)$. If $m \leq n$ and $q \in C_n$; then $m \leq n \sqsubseteq_X q$. Hence, $m \sqsubseteq_X q$ and $C_n \subseteq C_m \Rightarrow \bigwedge C_m \leq \bigwedge C_n$ and so $c_X^{\sqsubseteq}(m) \leq c_X^{\sqsubseteq}(n)$. Now let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism and $m \sqsubseteq_Y^c n$ for $n, m \in \text{sub}Y$. Since $\sqsubseteq \in \bigwedge -TORD$, $c^{\sqsubseteq}(m) = \min\{p \mid m \sqsubseteq p\}$ and so by

$(T3)$, we have that $f^{-1}(c_Y^{\sqsubseteq}(m)) \in \{q \mid f^{-1}(m) \sqsubseteq q\} \Rightarrow \bigwedge\{q \mid f^{-1}(m) \sqsubseteq q\} \leq f^{-1}(c_Y^{\sqsubseteq}(m))$. Hence, $c_X^{\sqsubseteq}(f^{-1}(n)) \leq f^{-1}(c_Y^{\sqsubseteq}(m))$.

Similarly, since $m \leq c_X(m)$, we have that $m \sqsubseteq_X^i n \Rightarrow m \leq n$. If $m \leq n \sqsubseteq^c p \leq q$ then $c_X(m) \leq c_X(n) \leq p \leq q$. Thus, $c_X(m) \leq q$ and so $m \sqsubseteq_X^c q$. Now, let $f : X \longrightarrow Y$ be a $\mathcal{C}$-morphism and $m \sqsubseteq_Y^c n$ for $n, m \in \text{sub}Y$. Then $c_X(f^{-1}(m)) \leq f^{-1}(c_Y(m)) \leq f^{-1}(n) \Rightarrow c_X(f^{-1}(m)) \leq f^{-1}(n)$. So $f^{-1}(m) \sqsubseteq_X^c f^{-1}(n)$.

Assume $\sqsubseteq \subseteq \sqsubseteq'$ in $\bigwedge TORD$. Then $\{p \mid m \sqsubseteq_X p\} \subseteq \{q \mid m \sqsubseteq'_X q\} \Rightarrow \bigwedge\{q \mid m \sqsubseteq'_X q\} \leq \bigwedge\{p \mid m \sqsubseteq_X p\}$. So $c^{\sqsubseteq'} \leq c^{\sqsubseteq}$. On the other hand if $c \leq c'$ in $\bigwedge TORD$, then $c_X(m) \leq n \Rightarrow c_X(m) \leq c'_X(m) \leq n \Rightarrow c_X(m) \leq n$ for all $X \in \mathcal{C}$. Therefore, $\sqsubseteq^{c'} \subseteq \sqsubseteq^c$.

Lastly, let $c \in CLOS$ and m sub$X$, then $c_X^{\sqsubseteq^c}(m) = \bigwedge\{n \mid m \sqsubseteq_X^c n\} = \bigwedge\{n \mid c_X(m) \leq n\} = c_X(m)$ for all $X \in \mathcal{C}$. Conversely, if $m \sqsubseteq_X^{c^{\sqsubseteq}} n$ then $c_X^{\sqsubseteq}(m) = \bigwedge\{p \mid m \sqsubseteq_X p\} \leq n \Rightarrow m \sqsubseteq_X n$. On the other hand if $m \sqsubseteq_X n$, then $c_X(m) \leq n$ by definition, So $\bigwedge\{p \mid m \sqsubseteq_X p\} \leq n$. Hence, $m \sqsubseteq_X^{c^{\sqsubseteq}} n$. $\square$

One can see that $c^{\sqsubseteq}$ is a closure operator for any $\sqsubseteq \in TORD$. In fact if $f : X \longrightarrow Y$ is a morphism in $\mathcal{C}$ and $\sqsubseteq \in TORD$, then by $(T3)$, $\{f^{-1}(p) \mid m \sqsubseteq_Y p\} \subseteq \{q \mid f^{-1}(m) \sqsubseteq_X q\}$. So for any $m \in \text{sub}Y$;

$$\begin{aligned} f^{-1}(c_Y^{\sqsubseteq}(m)) &= f^{-1}(\bigwedge\{p \mid m \sqsubseteq_Y p\} \\ &= \bigwedge\{f^{-1}(p) \mid m \sqsubseteq_Y p\} && f^{-1} \text{ is right adjoint} \\ &\geq \bigwedge\{q \mid f^{-1}(m) \sqsubseteq_Y q\} = c_X^{\sqsubseteq}(f^{-1}(m)) \end{aligned}$$

Corollary 4.2.5. *Interpolative topogenous orders in $\bigwedge -TORD$ are equivalent to idempotent closure operators.*

Proof. Let c be idempotent. Then

$$\begin{aligned} m \sqsubseteq_X^c n &\Leftrightarrow c_X(m) \leq n \\ &\Rightarrow c_X(c_X(m)) \leq c_X(m) \leq n \\ &\Rightarrow c_X(m) \sqsubseteq_X^c c_X(m) \sqsubseteq_X^c n \\ &\Rightarrow m \sqsubseteq_X^c c_X(m) \sqsubseteq_X^c n \end{aligned}$$

On the other hand if $\sqsubseteq$ interpolates, the $\{n \mid m \sqsubseteq_X n\} \subseteq \{p \mid c^{\sqsubseteq}_X(m) \sqsubseteq_X p\}$ since for all $X \in \mathcal{C}$

$$\begin{aligned} m \sqsubseteq_X n &\Rightarrow \exists\, q : m \sqsubseteq_X q \sqsubseteq_X n \\ &\Rightarrow c^{\sqsubseteq}_X(m) \leq q \sqsubseteq_X n && \text{definition of } c^{\sqsubseteq} \\ &\Rightarrow c^{\sqsubseteq}_X(m) \sqsubseteq_X n && (T2) \end{aligned}$$

This implies that $\bigwedge\{p \mid c^{\sqsubseteq}_X(m) \sqsubseteq_X p\} \leq \bigwedge\{n \mid m \sqsubseteq_X n\}$, that is $c^{\sqsubseteq}_X(c^{\sqsubseteq}_X(m)) \leq c^{\sqsubseteq}_X(m)$. $\square$

Remark 4.2.6. The best way to caracterize the relationship between Propositions 4.2.2 and 4.2.4 is given by the following observation:

- $m \sqsubseteq n \Leftrightarrow m \leq i^{\sqsubseteq}(n)$ in $\bigvee - TORD$,
- $m \sqsubseteq n \Leftrightarrow c^{\sqsubseteq}(m) \leq n$ in $\bigwedge - TORD$.

As indicated earlier, categorical closure and interior operators are not dual. However, there is a notational symetry between the two operators. This symmetry is clarified by the notion of topogenous orders. Furthermore, this observation that they are situated in the same category explains how many concepts and definitions which have been studied separatly for closure and interior are essentially the same. Many results in the next chapter will make this more clear.

4.3 A few Examples

We present in this section a number of examples constructed from those of closure, interior and neighbourhood operators. This does not come as a supprise since we have proved that the topogenous orders allow us to treat these three operators in one setting. Example 4.3.1(a) is induced by the usual interior operator while Example 4.3.1(b) is induced by the Kuratowski closure operator. Examples 4.3.1(c) and 4.3.1(d) are essentially from interior operstors while 3.4.1(e) comes from the sequential closure operator.

Examples 4.3.1. Let $\mathcal{C}$ be the category **Top** of topological spaces and continuous maps with the $(\mathcal{E}, \mathcal{M})$ factorisation formed by continuous surjections and embeddings. For any topological space X and $A \subseteq X$, $B \subseteq X$,

(a) $A \sqsubseteq B$ iff $A \subseteq C \subseteq B$ for some open $C \subseteq X$ is a topogenous order on **Top**. Conditions $(T1)$ and $(T2)$ are easily satisfied. To check $(T3)$, let $f : X \longrightarrow Y$ be a continuous map and A, B subsets of the topological space Y. If $A \subseteq C \subseteq B$ for some open Cin Y, then $f^{-1}(A) \subseteq f^{-1}(C) \subseteq f^{-1}(B)$. Since the inverse image of an open set by a continuous map is open, we get that $f^{-1}(A) \sqsubseteq f^{-1}(B)$.

(b) $A \sqsubseteq B$ iff $A \sqsubseteq C \sqsubseteq B$ for some closed $C \sqsubseteq X$ is a topogenous order on **Top**. We just need to check condition $(T3)$ since conditions $(T1)$ and $(T2)$ are trivially satisfied. So for $f : X \longrightarrow Y$ a continuous map and A, B subsets of Y, we have that, if $A \subseteq C \subseteq B$ for some closed $\mathcal{C}$ in Y then $f^{-1}(A) \subseteq f^{-1}(C) \subseteq f^{-1}(B)$. Since the inverse image of a closed set by a continuous map is closed, we get that $f^{-1}(A) \sqsubseteq f^{-1}(B)$.

(c) $A \sqsubseteq B$ if $A \subseteq C \subseteq B$ for some clopen $C \subseteq X$ is a topogenous order on **Top**. The axioms $(T1)$ and $(T2)$ are easily satisfied. To check $(T3)$, let $f : X \longrightarrow Y$ be a continuous function and $A, B \subseteq Y$, if $A \subseteq C \subseteq B$ for some clopen $C \subseteq Y$, then $f^{-1}(A) \subseteq f^{-1}(C) \subseteq f^{-1}(B)$. Since the inverse image of a clopen set by a continuous function is clopen, we have that $f^{-1}(A) \sqsubseteq f^{-1}(B)$.

(d) $A \sqsubseteq B$ if for all $x \in A$ there is an open neighbourhood U_x of x such that $\overline{U_x} \subseteq B$ is a topogenous order on **Top**. We only need to verify condition $(T3)$. So let $f : X \longrightarrow Y$ be a continuous function and A, B subsets of the topological space Y . If $A \sqsubseteq B$ and $x \in f^{-1}(A)$ then $f(x) \in A$. This implies that there is an open neighbourhood $U_{f(x)}$ of $f(x)$ such that $\overline{U}_{f(x)} \subseteq B$. Consequently, $x \in f^{-1}(\overline{U}_{f(x)}) \subseteq f^{-1}(B)$. Since $f^{-1}(U_{f(x)})$ is a neighbourhood of x and $f^{-1}(\overline{U}_{f(x)}) \supseteq \overline{f^{-1}(U_{f(x)})}$ by continuity of f, we get that $f^{-1}(A) \sqsubseteq f^{-1}(B)$.

(e) $A \sqsubseteq B$ iff for all x such that there is a sequence (x_n) in A converging to x, $x \in B$ is a topogenous order on **Top**. We see that $(T1)$ is satisfied by just taking a constant sequence $(x, x, x, ...)$ for each $x \in A$. $(T2)$ is easily seen to be satisfied. For $(T3)$, let $f : X \longrightarrow Y$ be a continuous map and $A \subseteq X, B \subseteq Y$. Let $f(A) \sqsubseteq B$ and assume that for all $x \in X$ with is a sequence (x_n) in A converging to x. By continuity of f , $f(x_n)$ converges to $f(x)$. Since $f(A) \sqsubseteq B$, $f(x) \in B \Rightarrow x \in f^{-1}(B)$. Hence $A \sqsubseteq f^{-1}(B)$.

The next two examples are constructed from neighbourhood operators. We can also get Example 4.3.2 from the normal closure operator.

Examples 4.3.2. Let $\mathcal{C}$ be the category **Grp** of groups and group homomorphisms with the $(\mathcal{E}, \mathcal{M})$ factorisation formed by surjective and injective homomorphisms. For any group G and A, B subgroups of $G, A \sqsubseteq B$ if $A \leq N \leq B$ for some $N \lhd G$ is a topogenous order on **Grp** with respect to $\mathcal{M}$. The axioms $(T1)$ and $(T2)$ are trivial. To check $(T3)$, let $f : G \longrightarrow H$ be a group homomorphism and A, B subgroups of H, if $A \leq N \leq B$ for some $N \lhd H$, then $f^{-1}(A) \leq f^{-1}(N) \leq f^{-1}(B)$. Since the inverse image of a normal subgroup by a group homomorphism is normal, we get that $f^{-1}(A) \sqsubseteq f^{-1}(B)$.

Examples 4.3.3. A (directed) graph is a set X together with a binary relation $\rightarrow$. The elements of X are called vertices and edges of X are pairs $(x, y) \in X \times Y$ where $x \rightarrow y$ in X. A morphism of graphs is a function $f : X \longrightarrow Y$ preserving the $\rightarrow$ that is, if $x \rightarrow y$ in X then $f(x) \rightarrow f(y)$. We denote by **Gph** the category of (directed) graphs and graph homomorphisms ([DT95, Raz12a]). The $(\mathcal{E}, \mathcal{M})$ is formed by embeddings and surjective graph homomorphisms. For graph $X, A, B \subseteq X$, $A \subseteq B \Leftrightarrow (\forall x \in A)(\forall y \in X \setminus B)$ there is no edge $x \rightarrow y$ is a topogenous order on **Gph**. We just need to show the continuity condition. Let $f : X \rightarrow Y$ be a graph homomorphism and $A, B \subseteq Y$ with $A \sqsubseteq B$. Assume there is a $x \in f^{-1}(A)$ and $y \in X \setminus f^{-1}(B)$ such that $x \longrightarrow y$. Then $f(x) \rightarrow f(y)$ but $f(x) \in A$ and $f(y) \in X \setminus B$. Hence, $f^{-1}(A) \sqsubseteq f^{-1}(B)$.

Chapter 5

Strict Morphisms

Looking at the equivalent formulation of the $\sqsubseteq$-continuity in Proposition 1.1.2, one would ask when do $\sqsubseteq$-continuous morphisms fulfil the other implication. In such case, the notion of strict morphisms with respect to the topogenous order is obtained. In this chapter we study the basic properties of these morphisms, define a few notions related to them and show that they capture the notions of c-closed, i-open, and ν-open morphisms. Moreover, the pullback stability of these morphisms is also discussed.

5.1 Strict subobjects

We present in this section the notion of $\sqsubseteq$-strict subobject which captures the known ones of c-closed and i-open subobjects and some of its stability properties. The notion of $\sqsubseteq$-dence subobject is also presented.

Definition 5.1.1. A subobject m of X is called $\sqsubseteq$-strict if

$$m \sqsubseteq_X m$$

m is said to $\sqsubseteq$-dense if

$$m \sqsubseteq_X n \Leftrightarrow n \cong 1_X$$

for any subobject n of X.

Using Propositions 4.2.4 and 4.2.2, one can rewrite the definition of $\sqsubseteq$-strict subobject in the notation of closure and interior as: $m \sqsubseteq m \Leftrightarrow c(m) \leq m$ and $m \sqsubseteq m \Leftrightarrow m \leq i(m)$. These are respectively the c-closed and i-open subobjects.

In the category **Top** of topological spaces, if we consider the topogenous orders in Examples 4.3.1$(a), (b)$, the $\sqsubseteq$-strict subobjects coincident with the open, closed subsets respectively. For the $\sqsubseteq$-dense subobject with the topogenous order in Examples 4.3.1(a), it coincides with a dense subset. In the category **Grp** of groups and group homomorphisms, if the the topogenous order is the one in example 4.3.2, a $\sqsubseteq$-strict subobject coincides with a normal subgroup.

We are interested in stability properties of $\sqsubseteq$-strict and $\sqsubseteq$-dense subobjects. Already the condition $(T3)$ implies that the inverse image of a $\sqsubseteq$-strict subobject is $\sqsubseteq$-strict.

Proposition 5.1.2. *Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{C}$.*

(1) *If n is $\sqsubseteq$-strict in Y, then $f^{-1}(n)$ is $\sqsubseteq$-strict in X.*

(2) *If n is $\sqsubseteq$-dense in X, $f \in \mathcal{E}$ and $\mathcal{E}$ is stable under pullback, then $f(m)$ is $\sqsubseteq$-dense in Y .*

(3) *If m and n are monomorphisms and $m \circ n$ is a $\sqsubseteq$-strict $\mathcal{M}$-subobject, then m is a $\sqsubseteq$-strict $\mathcal{M}$-subobject.*

Proof. (1) If $n \sqsubseteq_Y n$ then $f^{-1}(n) \sqsubseteq_X f^{-1}(n)$ by $(T3)$.

(2) If $f(n) \sqsubseteq_Y q$ and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback, then $n \sqsubseteq_Y f^{-1}(q) \Rightarrow f^{-1}(q) = 1_X \Rightarrow q = f(f^{-1}(q)) = f(1_X) = 1_Y$.

(3) The square

$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "m"'] & \cdot \arrow[d, "n\circ m"] \\ \cdot \arrow[r, "n"'] & \cdot \end{tikzcd}$$

is a pullback since n is a mono, so $n \circ m \sqsubseteq n \circ m \Rightarrow m = n^{-1}(n \circ m) \sqsubseteq n^{-1}(n \circ m) = m$. $\square$

Proposition 5.1.3. *For a commutative diagram below with $m, n \in \mathcal{M}$, if n $\sqsubseteq$-dense and m is $\sqsubseteq$-strict, then there is a unique p for which the following diagram commutes:*

$$\begin{tikzcd} \cdot \arrow[r, "n"] \arrow[d, "a"'] & \cdot \arrow[d, "b"] \arrow[dl, dotted, "p"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

Proof. Let m' be a pullback of m and consider the diagram below

$$\begin{tikzcd} \cdot \arrow[dr, dotted, "k"] \arrow[drr, bend left, "n"] \arrow[ddr, bend right, "a"'] & & \\ & \cdot \arrow[r, "m'"] \arrow[d, "q"'] & \cdot \arrow[d, "b"] \\ & \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

The morphism k exists by the pullback property. So $n \leq m$ and $m \sqsubseteq_X m'$ for all $X \in \mathcal{C}$ by Proposition 5.1.2(1). This gives $n \sqsubseteq_X m' \Rightarrow m' = 1_X$ since n is $\sqsubseteq$-dense. Thus, q is the desired diagonal. Its uniqeness follows from the fact that m is a monomorphism. $\square$

Proposition 5.1.4. *Let $m_i : M_i \longrightarrow X$ be a family of $\sqsubseteq$-strict subobjects. Then*

(i) If $\sqsubseteq \in \bigvee -TORD$, then the supremum $\bigvee m_i$ of m_i is a $\sqsubseteq$-strict subobject.

(ii) If $\sqsubseteq \in \bigwedge -TORD$, then the infinimum $\bigwedge m_i$ is a $\sqsubseteq$-strict subobject.

Proof. (i) For all $i \in I$;

$$\begin{aligned} m_i \leq \bigvee m_i &\Rightarrow m_i \sqsubseteq m_i \leq \bigvee m_i && m_i \text{ is } \sqsubseteq\text{-strict} \\ &\Rightarrow m_i \sqsubseteq \bigvee m_i && T2 \\ &\Rightarrow \bigvee m_i \sqsubseteq \bigvee m_i && \sqsubseteq \in \bigvee TORD \end{aligned}$$

(ii) for all $i \in I$;

$$\begin{aligned} \bigwedge m_i \leq m_i &\Rightarrow \bigwedge m_i \leq m_i \sqsubseteq m_i && m_i \text{ is } \sqsubseteq\text{-strict} \\ &\Rightarrow \bigwedge m_i \sqsubseteq m_i && T2 \\ &\Rightarrow \bigwedge m_i \sqsubseteq \bigwedge m_i && \sqsubseteq \in \bigwedge TORD \end{aligned}$$

$\square$

5.2 Description of Strict Morphisms

We have, in the previous chapter, provided a framework in which both the closure and interior operartors fit nicely in a categorical way. This more general topogenous order setting has among other benefits, seeing a number of results and defintions on closure and interior as the same. We present in this section the notion strict morphism with respect to a topogenous order which constitute an illustration of these benefits.

Definition 5.2.1. A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is $\sqsubseteq$-strict if

$$m \sqsubseteq_X f^{-1}(n) \Leftrightarrow f(m) \sqsubseteq_Y n$$

for any $m \in \text{sub}X$ and $n \in \text{sub}Y$.

We note that the use of "strict" in two different context is due to the fact that $\sqsubseteq$-strict subobjects capture the i-open and the c-closed subobjects as explained in the first section of this chapter while the $\sqsubseteq$-strict morphisms capture the i-open and the c-closed morphisms (see Propositions 5.2.5 and 5.2.6). Under the condition that $1_X \sqsubseteq_X 1_X$ for all $X \in \mathcal{C}$, every $\sqsubseteq$-strict morphism in $\mathcal{M}$ is a $\sqsubseteq$-subobject (see Proposition 5.3.1).

$\sqsubseteq$-strict morphisms behave as follows:

Proposition 5.2.2. *Let $\sqsubseteq$ be a topogenous order. The following statements hold.*

(1) *The class of $\sqsubseteq$-strict morphisms contains all isomorphisms,*

(2) *The class of $\sqsubseteq$-strict morphisms is closed under composition,*

(3) *If $g \circ f$ is $\sqsubseteq$-strict and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback, then g is $\sqsubseteq$-strict,*

(4) *If $g \circ f$ is $\sqsubseteq$-strict and g is a monomorphism, then f is $\sqsubseteq$-strict.*

Proof. (1) If $f : X \longrightarrow Y$ is an isomorphism with inverse $g : Y \longrightarrow X$, then

$$\begin{aligned} m \sqsubseteq_X f^{-1}(m) &\Rightarrow f(m) = g^{-1}(m) \sqsubseteq_Y g^{-1}(f^{-1}(n)) = (f \circ g)^{-1}(n) \\ &\Rightarrow f(m) \sqsubseteq_Y 1_Y^{-1}(n) = n \end{aligned}$$

(2) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are $\sqsubseteq$-strict, then

$$\begin{aligned} m \sqsubseteq_X (g \circ f)^{-1}(n) = f^{-1}(g^{-1}(n) &\Leftrightarrow f(m) \sqsubseteq_Y g^{-1}(n) \\ &\Leftrightarrow (g \circ f)(m) = g(f(m)) \sqsubseteq_Z n \end{aligned}$$

(3) If $\mathcal{E}$ is stable under pullbacks and $f \in \mathcal{E}$, then $f(f^{-1}(m)) = m$. Hence

$$\begin{aligned} m \sqsubseteq_Y g^{-1}(n) &\Rightarrow f^{-1}(m) \sqsubseteq_X f^{-1}(g^{-1}(n) = (g \circ f)^{-1}(n) \\ &\Rightarrow (g \circ f)(f^{-1}(m)) \sqsubseteq_Z n \\ &\Rightarrow g(m) \sqsubseteq_Z n \end{aligned}$$

(4) If $g : Y \longrightarrow Z$ is a monomorphism, then $g^{-1}(g(n)) = n$

$$\begin{aligned} m \sqsubseteq_X f^{-1}(n) = f^{-1}(g^{-1}(g(n))) = (g \circ f)^{-1}(g(n)) &\Rightarrow g(f(m)) = (g \circ f)(m) \sqsubseteq g(n) \\ &\Rightarrow f(m) \sqsubseteq_Y g^{-1}(g(n)) \\ &\Rightarrow f(m) \sqsubseteq_Y n \end{aligned}$$

$\square$

Proposition 5.2.3. *Consider the following properties of $f : X \longrightarrow Y$ in $\mathcal{C}$.*

(1) *f is a $\sqsubseteq$-strict morphism,*

(2) *f preserves $\sqsubseteq$,*

(3) *$m \sqsubseteq_X n \Leftrightarrow f(m) \sqsubseteq_Y f(n)$ for any $m, n \in subX$,*

(4) *$m \sqsubseteq_Y n \Leftrightarrow f^{-1}(m) \sqsubseteq_X f^{-1}(n)$ for any $m, n \in subY$,*

Then (1) $\Leftrightarrow$ (2). If $f \in \mathcal{M}$ then (1) $\Leftrightarrow$ (3). If $\mathcal{E}$ is pullback stable, then for $f \in \mathcal{E}$, (1) $\Rightarrow$ (4).

Proof. • If f is $\sqsubseteq$-strict, then

$$\begin{aligned} m \sqsubseteq_X n \leq f^{-1}(f(n)) &\Rightarrow m \sqsubseteq_X f^{-1}(f(n)) \\ &\Rightarrow f(m) \sqsubseteq_Y f(n) \end{aligned}$$

On the other hand if f preserves $\sqsubseteq$, then

$$\begin{aligned} m \sqsubseteq_X f^{-1}(n) &\Rightarrow f(m) \sqsubseteq_Y f(f^{-1}(n)) \leq n \\ &\Rightarrow f(m) \sqsubseteq_Y n \end{aligned}$$

- If $f \in \mathcal{M}$ then $f^{-1}(f(m)) = m$ for $m \in \text{sub}X$. Thus

$$\begin{aligned} m \sqsubseteq_X n &\Leftrightarrow m \sqsubseteq_X f^{-1}(f(n)) \\ &\Leftrightarrow f(m) \sqsubseteq_Y f(n) \end{aligned}$$

Conversely if $m \sqsubseteq_X n \Leftrightarrow f(m) \sqsubseteq_Y f(n)$ then f is $\sqsubseteq$-strict since (2) $\Rightarrow$ (1)

- If $\mathcal{E}$ is pullback stable, then for $f \in \mathcal{E}, f(f^{-1}(n) = n$ for $n \in \text{sub}X$. Hence

$$\begin{aligned} m \sqsubseteq_Y n &\Leftrightarrow f(f^{-1}(m)) \sqsubseteq_Y n \\ &\Leftrightarrow f^{-1}(m) \sqsubseteq_X f^{-1}(n) \end{aligned}$$

□

Corollary 5.2.4. *If f is a $\sqsubseteq$-strict morphism, then $f(-)$ takes $\sqsubseteq$-strict subobjects to $\sqsubseteq$-strict subobjects.*

Proof. Let $f : X \longrightarrow Y$ be a $\sqsubseteq$-strict morphism, then

$f(m) \sqsubseteq_Y f(m)$ by Proposition 5.2.3(2)

□

We now turn our attention to strict morphisms in $CLOS$ and INT. These are exactly the c-closed and i-open morphisms as it can be seen from the next two propositons.

Proposition 5.2.5. *If $\sqsubseteq$ is in $\bigwedge -TORD$ then $f : X \longrightarrow Y$ is $\sqsubseteq$-strict iff for any $m \in subX$, $f(c^{\sqsubseteq}_X(m)) = c^{\sqsubseteq}_Y(f(m))$.*

Proof. . Since in $\bigwedge -TORD, m \sqsubseteq n \Leftrightarrow c^{\sqsubseteq}(m) \leq n$ and f^{-1} is right adjoint to f, then

$$\begin{aligned} f(c^{\sqsubseteq}_X(m)) \leq n &\Leftrightarrow c^{\sqsubseteq}_X(m) \leq f^{-1}(n) \\ &\Leftrightarrow m \sqsubseteq_X f^{-1}(n) \\ &\Leftrightarrow f(m) \sqsubseteq_Y n \\ &\Leftrightarrow c^{\sqsubseteq}_Y(f(m)) \leq n \end{aligned}$$

Conversely if $f(c^{\sqsubseteq}_X(m)) = c^{\sqsubseteq}_Y(f(m))$ then,

$$\begin{aligned} m \sqsubseteq_X f^{-1}(n) &\Leftrightarrow c^{\sqsubseteq}_X(m) \leq f^{-1}(n) \\ &\Leftrightarrow f(c^{\sqsubseteq}_X)(m) \leq n \\ &\Leftrightarrow c^{\sqsubseteq}_Y(f(m)) \leq n \\ &\Leftrightarrow f(m) \sqsubseteq_Y n \end{aligned}$$

□

Proposition 5.2.6. *If $\sqsubseteq$ is in $\bigvee -TORD$ then $f : X \longrightarrow Y$ is $\sqsubseteq$-strict iff for any $m \in subY, f^{-1}(i^{\sqsubseteq}_Y(m)) = i^{\sqsubseteq}_X(f^{-1}(m))$*

Proof. Since in $\bigvee -TORD$ $m \sqsubseteq n \Leftrightarrow m \leq i(n)$ and f^{-1} is right adjoint to f, if f is strict then,

$$\begin{aligned} m \leq f^{-1}(i_Y(n)) &\Leftrightarrow f(m) \leq i_Y(n) \\ &\Leftrightarrow f(m) \sqsubseteq_Y n \\ &\Leftrightarrow m \sqsubseteq_X f^{-1}(n) \\ &\Leftrightarrow m \leq i_X(f^{-1}(n)) \end{aligned}$$

On the other hand if $f^{-1}(i^{\sqsubseteq}_Y(m)) = i^{\sqsubseteq}_X(f^{-1}(m))$ then,

$$\begin{aligned} m \sqsubseteq_X f^{-1}(n) &\Leftrightarrow m \leq i^{\sqsubseteq}_X(f^{-1}(n)) \\ &\Leftrightarrow m \leq f^{-1}(i^{\sqsubseteq}_Y(n)) \\ &\Leftrightarrow f(m) \leq i^{\sqsubseteq}_Y(n)) \\ &\Leftrightarrow f(m) \sqsubseteq_Y n \end{aligned}$$

□

With the help of Proposition 4.1.4, the definition of $\sqsubseteq$-strict morphisms can be rewritten in the notation of neighbourhoods as: $n \in \nu(f(m)) \Leftrightarrow f^{-1}(n) \in \nu(m)$. Thus f is $\sqsubseteq$-strict iff $\nu(f(m)) = f(\nu(m))$. This is the $\sqsubseteq$-open morphism (see Definition 3.2.5). One uses Propositions 4.1.4, 4.2.2 and 4.2.4 to see that Proposition 5.2.2 captures Propositions

2.2.10, 2.2.18 and 3.2.6 which were studied separatly for closure, interior and neighbourhood operators.

In the category **Top** of topological spaces, if we consider the topogenous orders in Examples 4.3.1$(a), (b)$ and (c), the strict morphisms relative to them are those continuous functions which preserve respectively the open, closed and clopen subsets. For the category **Grp** of groups and group homomorphisms, the strict morphisms relative to the topogenous order in Examples 4.3.2 are those group homomorphisms preserving normal subgroups.

5.3 Pullback stability

In [CGT04] Clementino, Giuli and Tholen have developed the functional approach to general topology. They depart from a class of morphisms satisfying (1), (2), (3) of Proposition 5.2.2, as well as the pullback stability of $\mathcal{M}$-morphisms in the class. We discuss in this section the pullback stability of strict morphisms in $\mathcal{M}$.

Proposition 5.3.1. *Assume that $1_X \sqsubseteq_X 1_X$ for each $X \in \mathcal{C}$, then*

$$\{f \in \mathcal{M} \mid f \sqsubseteq -\textit{strict}\} \subseteq \{f \in \mathcal{M} \mid f \sqsubseteq f\}$$

Proof. If f is a $\sqsubseteq$-morphism such that $f \in \mathcal{M}$ and $1_X \sqsubseteq 1_X$, then by Proposition 4.2.3, $f = f(1_X) \sqsubseteq f(1_X) = f$ □

The class $\{f \in \mathcal{M} \mid f \sqsubseteq f\}$ is pullback stable by Proposition 5.1.2(1). In general we don't know whether there is a condition on the topogenous order which would make the class $\{f \in \mathcal{M} \mid f \sqsubseteq -\text{strict}\} = \{f \in \mathcal{M} \mid f \sqsubseteq f\}$ so that one can make a general result on the pullback stability of strict morphisms in $\mathcal{M}$. However, this holds true in particular cases:

- If the topogenous order is induced by a weakly hereditary closure operator, then by Proposition 2.2.11 the two classes are equal and so the strict morphisms in $\mathcal{M}$ are pullback stable.

- The same holds true by Proposition 2.2.20 if the topogenous order is induced by a weakly hereditary interior operator.

Chapter 6

Classes of morphisms with respect to a topogenous order

Four classes of morphisms with respect to a neighbourhood operator were studied in [$Raz12a$]. We have summarized the basic properties of these morphisms in the second section of the third chapter of this thesis. Following Proposition 4.1.4 which establishes the equivalence between neighbourhood operators and topogenous orders, we present in this chapter the basic properties of ν-open, ν-closed, ν-initial and ν-final in the notation of topogenous orders and make a few observations on how they are related to the strict morphisms studied in the previous chapter. We shall see that this approach offers easy proofs comparing to the one via neighbourhood operators.

6.1 Closedness and initiality

A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is ν-closed if for any $n \in \text{sub}Y$, $\nu_X(f^{-1}(n)) = f^{-1}(\nu_Y(n))$. This definition corresponds to $\{q \mid f^{-1}(n) \sqsubseteq q\} = \{f^{-1}(p) \mid n \sqsubseteq p\}$ by Proposition 4.1.4. Since f is $\sqsubseteq$-continuous, the crucial inclusion is $\{q \mid f^{-1}(n) \sqsubseteq_X q\} \subseteq \{f^{-1}(p) \mid n \sqsubseteq p\}$. This leads us to the definition below.

Definition 6.1.1. A $\mathcal{C}$-morphism $f : X \longrightarrow Y$ is said to be $\sqsubseteq$-closed if for any $n \in \text{sub}X$, $m \in$ sub$Y$,

$$f^{-1}(m) \sqsubseteq n \Rightarrow \exists\, p \mid m \sqsubseteq p \text{ and } f^{-1}(p) \leq n.$$

A similar argument to the above produces the following definition.

Definition 6.1.2. A $\mathcal{C}$-morphism $f : X \longrightarrow Y$ is said to be $\sqsubseteq$-initial if for any $n, m \in \text{sub}X$,

$$m \sqsubseteq n \Rightarrow \exists\, p \mid f(m) \sqsubseteq p \text{ and } f^{-1}(p) \leq n$$

The next two propositions are the counterpart of Propostions 3.2.2 and 3.2.3.

Proposition 6.1.3. *Let $\sqsubseteq$ be a topogenous order. The class of $\sqsubseteq$-initial morphisms in $\mathcal{C}$*

(1) *is closed under composition,*

(2) *is left cancellable that is, $g \circ f$ $\sqsubseteq$-initial $\Rightarrow$ f is $\sqsubseteq$-initial,*

(3) *is right cancellable with respect to $\mathcal{E}$ that is, $g \circ f$ $\sqsubseteq$-initial $\Rightarrow g$ is $\sqsubseteq$-initial provided $\mathcal{E}$ is stable under pullback.*

Proof. (1) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are $\sqsubseteq$-initial then

$$\begin{aligned} m \sqsubseteq n &\Rightarrow \exists\, p \mid f(m) \sqsubseteq p \text{ and } f^{-1}(p) \leq n \\ &\Rightarrow \exists\, q \mid g(f(m)) \sqsubseteq q \text{ and } g^{-1}(q) \leq p \\ &\Leftrightarrow g(f(m)) \sqsubseteq q \text{ and } f^{-1}(g^{-1}(q)) \leq n \end{aligned}$$

(2) If $g \circ f$ is $\sqsubseteq$-initial then

$$\begin{aligned} m \sqsubseteq n &\Rightarrow \exists\, q \mid g(f(m)) \sqsubseteq q \text{ and } f^{-1}(g^{-1}(q)) \leq n \\ &\Rightarrow f(m) \sqsubseteq g^{-1}(q) \text{ and } f^{-1}(g^{-1}(q)) \leq n \qquad \text{by } (T3) \end{aligned}$$

(3) If $g \circ f$ is $\sqsubseteq$-initial and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback then

$$\begin{aligned} m \sqsubseteq n &\Rightarrow f^{-1}(m) \sqsubseteq f^{-1}(n) \qquad \text{by } (T3) \\ &\Rightarrow \exists\, p \mid (g \circ f)(f^{-1}(m)) \sqsubseteq p \text{ and } f^{-1}(g^{-1}(p)) \leq f^{-1}(n) \\ &\Rightarrow g(m) \sqsubseteq p \text{ and } g^{-1}(p) \leq n \end{aligned}$$

□

The fact that the class of $\sqsubseteq$-initial morphisms contains all isomorphisms follows from Proposition 6.1.3(2). Indeed if $f : X \longrightarrow Y$ is an isomorphism with $g : Y \longrightarrow X$, then $g \circ f = 1_X$ is $\sqsubseteq$-initial, implies f is $\sqsubseteq$-initial.

Proposition 6.1.4. *Let $\sqsubseteq$ be a topogenous order. The class of $\sqsubseteq$-closed morphisms in $\mathcal{C}$*

(1) *is closed under composition,*

(2) *is left cancellable with respect to $\mathcal{M}$ that is, $g \circ f$ $\sqsubseteq$-closed and $g \in \mathcal{M} \Rightarrow f$ is $\sqsubseteq$-closed,*

(3) *is right cancellable with respect to $\mathcal{E}$ that is, $g \circ f$ $\sqsubseteq$-closed $\Rightarrow g$ is $\sqsubseteq$-closed provided $\mathcal{E}$ is stable under pullback.*

Proof. (1) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are $\sqsubseteq$-closed then,

$$\begin{aligned} f^{-1}(g^{-1}(m)) \sqsubseteq n &\Rightarrow \exists\, p \mid g^{-1}(m) \sqsubseteq p \text{ and } f^{-1}(p) \leq n \\ &\Rightarrow \exists\, q \mid m \sqsubseteq q \text{ and } g^{-1}(q) \leq p \\ &\Rightarrow m \sqsubseteq q \text{ and } f^{-1}(g^{-1}(q)) \leq f^{-1}(p) \leq n \end{aligned}$$

(2) If $g \circ f$ $\sqsubseteq$-closed and $g \in \mathcal{M}$ then

$$\begin{aligned} f^{-1}(m) \sqsubseteq n &\Leftrightarrow f^{-1}(g^{-1}(g(m))) \sqsubseteq n \\ &\Rightarrow \exists\, p \mid g(m) \sqsubseteq p \text{ and } f^{-1}(g^{-1}(p)) \leq n \\ &\Rightarrow m \sqsubseteq g^{-1}(p) \text{ and } f^{-1}(g^{-1}(p)) \leq n \end{aligned}$$

(3) If $g \circ f$ $\sqsubseteq$-closed and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback, then

$$\begin{aligned} g^{-1}(m) \sqsubseteq n &\Leftrightarrow f(f^{-1}(g^{-1}(m))) \sqsubseteq n \\ &\Rightarrow f^{-1}(g^{-1}(m)) \sqsubseteq f^{-1}(n) && \text{by } (T3) \\ &\Rightarrow \exists\, p \mid m \sqsubseteq p \text{ and } f^{-1}(g^{-1}(p)) \leq f^{-1}(n) \\ &\Rightarrow m \sqsubseteq p \text{ and } g^{-1}(p) \leq n \end{aligned}$$

□

It is important to observe from the above two propositions that $\sqsubseteq$-closed and $\sqsubseteq$-initial morphisms behave in a similar way as their conterpart in neighbourhood notations.

We next provide an immediate connection between the notions of $\sqsubseteq$-initiality and $\sqsubseteq$-closedness.

Proposition 6.1.5. *For a topogenous order $\sqsubseteq$;*

(1) *Every $\sqsubseteq$-closed morphism in $\mathcal{M}$ is $\sqsubseteq$-initial.*

(2) *Every $\sqsubseteq$-initial morphism in $\mathcal{E}$ is $\sqsubseteq$-closed provided $\mathcal{E}$ is pullback stable.*

Proof. (1) If $f : X \longrightarrow Y$ is $\sqsubseteq$-closed in $\mathcal{M}$, then for any $m, n \in \text{sub}X$, $m \sqsubseteq n \Leftrightarrow f^{-1}(f(m)) \sqsubseteq n \Rightarrow \exists\, p \mid f(m) \sqsubseteq p$ and $f^{-1}(p) \leq n$.

(2) If $f : X \longrightarrow Y$ is $\sqsubseteq$-initial in $\mathcal{E}$ with $\mathcal{E}$ stable under pullback, then

$$\begin{aligned} f^{-1}(m) \sqsubseteq n &\Rightarrow \exists\, p \mid f(f^{-1}(m) \sqsubseteq p \text{ and } f^{-1}(p) \leq n \\ &\Rightarrow m \sqsubseteq p \text{ and } f^{-1}(p) \leq n \end{aligned}$$

□

Our next proposition links $\sqsubseteq$-initiality and $\sqsubseteq$-strictness.

Proposition 6.1.6. (1) *Any $\sqsubseteq$-initial morphism in $\mathcal{E}$ maps $\sqsubseteq$-strict subobjects to $\sqsubseteq$-strict subobjects.*

(2) *Any $\sqsubseteq$-strict morphism in $\mathcal{M}$ is $\sqsubseteq$-initial.*

Proof. (1) If $f : X \longrightarrow Y$ is $\sqsubseteq$-initial in $\mathcal{E}$ with $\mathcal{E}$ stable under pullback, then

$$\begin{aligned} m \sqsubseteq m &\Rightarrow \exists\, p \mid f(m) \sqsubseteq p \text{ and } f^{-1}(p) \leq m \\ &\Rightarrow f(m) \sqsubseteq p \text{ and } p = f(f^{-1}(p)) \leq f(m) \\ &\Rightarrow f(m) \sqsubseteq f(m) \end{aligned}$$

(2) If $f : X \longrightarrow Y$ is $\sqsubseteq$-strict in $\mathcal{M}$, then $m \sqsubseteq n \Rightarrow f(m) \sqsubseteq f(n)$. Put $p = f(n)$ to get $f^{-1}(p) = f^{-1}(f(n)) = n$

□

6.2 Finality and openness

Final and open morphisms with respect to a topogenous order are also obtained by the same process used to get the $\sqsubseteq$-closed and $\sqsubseteq$-initial.

Definition 6.2.1. A $\mathcal{C}$-morphism $f : X \longrightarrow Y$ is said to be $\sqsubseteq$-final if for any $n \in \text{sub}Y$ and $k \geq n$,

$$f^{-1}(n) \sqsubseteq f^{-1}(k) \Rightarrow n \sqsubseteq k$$

Definition 6.2.2. A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is said to be $\sqsubseteq$-open if

$$(\exists\, p \mid m \sqsubseteq p \text{ and } f(p) \leq n) \Rightarrow f(m) \sqsubseteq n$$

for all $m \in \text{sub}X$ and $n \in \text{sub}Y$.

The notion of $\sqsubseteq$-open morphism that we have obtained is just the $\sqsubseteq$-strict morphism as observed in the previous chapter. In fact if f is $\sqsubseteq$-strict and $(\exists\, p \mid m \sqsubseteq p$ and $f(p) \leq n)$, then $p \leq f^{-1}(n)$ by adjointness. So $m \sqsubseteq f^{-1}(n)$ by $(T2)$ and $f(m) \sqsubseteq n$. Conversely f is $\sqsubseteq$-open, one puts $p = f^{-1}(n)$ in Definition 2.2.2 to see that f is $\sqsubseteq$-strict.

We shall now provide some basic properties of the $\sqsubseteq$-final and $\sqsubseteq$-strict morphisms

Proposition 6.2.3. *Let $\sqsubseteq$ be a topogenous order. The following statements hold true*

(1) *$\sqsubseteq$-final morphisms are closed under composition.*

(2) *If $g \circ f$ is $\sqsubseteq$-final then then g is $\sqsubseteq$-final.*

(3) *If $g \circ f$ is $\sqsubseteq$-final and g is mono, then f is $\sqsubseteq$-final.*

Proof. (1) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are $\sqsubseteq$-final then,

$$\begin{aligned} f^{-1}(g^{-1}(n)) \sqsubseteq f^{-1}(g^{-1}(k)) &\Rightarrow g^{-1}(n) \sqsubseteq g^{-1}(k) \\ &\Rightarrow n \sqsubseteq k \end{aligned}$$

(2) If $g \circ f$ is $\sqsubseteq$-final and $k \geq n$ for any $n \in \text{sub}Z$, then

$$\begin{aligned} g^{-1}(n) \sqsubseteq (g^{-1}(k)) &\Rightarrow f^{-1}(g^{-1}(n)) \sqsubseteq f^{-1}(g^{-1}(k)) \\ &\Rightarrow n \sqsubseteq k \end{aligned}$$

(3) If g is mono and $k \geq n$, then $k = g^{-1}(g(k))$ and $n = g^{-1}(g(n))$ for any $n \in \mathrm{sub}Y$. Hence,

$$\begin{aligned} f^{-1}(n) \sqsubseteq (f^{-1}(k)) &\Leftrightarrow f^{-1}(g^{-1}(g(n))) \sqsubseteq f^{-1}(g^{-1}(g(k))) \\ &\Rightarrow g(n) \sqsubseteq g(k) \\ &\Rightarrow n \sqsubseteq k \end{aligned}$$

□

The following proposition is the same as Proposition 5.2.2. We shall leave the proof as it was already given in the previous chapter.

Proposition 6.2.4. *Let $\sqsubseteq$ be a topogenous order. The class of $\sqsubseteq$-strict morphisms in $\mathcal{C}$*

(1) *is closed under composition,*

(2) *is left cancellable with respect to $\mathcal{M}$ that is, $g \circ f$ $\sqsubseteq$-strict and $g \in \mathcal{M} \Rightarrow f$ is $\sqsubseteq$-strict,*

(3) *is right cancellable with respect to $\mathcal{E}$ that is, $g \circ f$ $\sqsubseteq$-strict $\Rightarrow g$ is $\sqsubseteq$-strict provided $\mathcal{E}$ is stable under pullback.*

The following is a further relationship between the types of morphisms with respect to a topogenous order.

Proposition 6.2.5. *Let $\sqsubseteq$ a topogenous order;*

(1) *Every $\sqsubseteq$-strict morphism in $\mathcal{E}$ is $\sqsubseteq$-final provided $\mathcal{E}$ is pullback stable.*

(2) *If $g \circ f = 1$ in $\mathcal{C}$ then f is a $\sqsubseteq$-initial morphism and g is a $\sqsubseteq$-final morphism in $\mathcal{E}$.*

(3) *Every $\sqsubseteq$-closed morphism in $\mathcal{E}$ is $\sqsubseteq$-final.*

Proof. (1) If $f : X \longrightarrow Y$ is $\sqsubseteq$-strict and $k \geq n$ for any $n \in \mathrm{sub}Y$, then

$$\begin{aligned} f^{-1}(n) \sqsubseteq f^{-1}(k) &\Rightarrow f(f^{-1}(n)) \sqsubseteq f(f^{-1}(k)) \\ &\Rightarrow n \sqsubseteq k \end{aligned}$$

(2) Follows from 6.1.3(2) and 6.2.2(2) respectively.

(3) If $k \leq n$ and $f : X \longrightarrow Y$ in $\mathcal{E}$ with $\mathcal{E}$ pullback stable, then

$$\begin{aligned} f^{-1}(n) \sqsubseteq f^{-1}(k) &\Rightarrow \exists\, p \mid n \sqsubseteq p \text{ and } f^{-1}(p) = f^{-1}(k) \\ &\Rightarrow n \sqsubseteq p \text{ and } p = f(f^{-1}(p)) = f(f^{-1}(k)) = k \end{aligned}$$

$\square$

References

[AHS06] J. Adámek, H. Herrlich, and G. E. Strecker. Abstract and concrete categories: the joy of cats. *Repr. Theory Appl. Categ.*, (17):1–507, 2006. Reprint of the 1990 original [Wiley, New York; MR1051419].

[AL97] C. E. Aull and R. Lowen, editors. *Handbook of the history of general topology. Vol. 1.* Kluwer Academic Publishers, Dordrecht, 1997.

[Bir37] G. Birkhoff. The meaning of completeness. *Ann. of Math. (2)*, 38(1):57–60, 1937.

[Bir40] G. Birkhoff. *Lattice Theory.* American Mathematical Society, New York, 1940.

[Cas01] G. Castellini. Connectedness with respect to a closure operator. *Appl. Categ. Structures*, 9(3):285–302, 2001.

[Cas03] G. Castellini. *Categorical closure operators.* Mathematics: Theory and Applications. Birkhäuser Boston, Inc., Boston, MA, 2003.

[Cas15] G. Castellini. Interior operators, open morphisms and the preservation property. *Appl. Categ. Structures*, 23(3):311–322, 2015.

[Čec37] E. Čech. On bicompact spaces. *Ann. of Math. (2)*, 38(4):823–844, 1937.

[CGT96] M.M. Clementino, E. Giuli, and W. Tholen. Topology in a category: compactnes. *Portugal. Math..*, 53(4):397–433, 1996.

[CGT01] M.M. Clementino, E. Giuli, and W. Tholen. What is a quotient map with respect to a closure operator?. *Applied Categorical Structures.*, 9(2):139–151, 2001.

[CGT04] M. M. Clementino, E. Giuli, and W. Tholen. A functional approach to general topology. In *Categorical foundations*, volume 97 of *Encyclopedia Math. Appl.*, pages 103–163. Cambridge Univ. Press, Cambridge, 2004.

[Cle01] M. M. Clementino. On connectedness via closure operators. *Appl. Categ. Structures*, 9(6):539–556, 2001.

[CM81] Á. Császár and K. Matolcsy. Syntopogenous extensions for prescribed topologies. *Acta Math. Acad. Sci. Hungar.*, 37(1-3):59–75, 1981.

[CM13] G. Castellini and E. Murcia. Interior operators and topological separation. *Topology Appl.*, 160(12):1476–1485, 2013.

[CR10] G. Castellini and J. Ramos. Interior operators and topological connectedness. *Quaest. Math.*, 33(3):290–304, 2010.

[Csá60] Á. Császár. *Fondements de la topologie générale.* Akadémiai Kiadó, Budapest, 1960.

[Csá63] Á. Császár. *Foundations of general topology.* A Pergamon Press Book. The Macmillan Co., New York, 1963.

[Csá83] Á. Császár. Complete extensions of quasiuniform spaces. In *General topology and its relations to modern analysis and algebra, V (Prague, 1981)*, volume 3 of *Sigma Ser. Pure Math.*, pages 104–113. Heldermann, Berlin, 1983.

[CT97] M. M. Clementino and W. Tholen. Separation versus connectedness. *Topology Appl.*, 75(2):143–181, 1997.

[CT01] M. M. Clementino and Tholen. T. A note on local compactness. *In Proceedings of the II Italian-Spanish Congress on General Topology and its Applications(Italian) (Trieste, 1999).*, 32:17–31(2002), 2001.

[DG87] D. Dikranjan and E. Giuli. Closure operators. I. In *Proceedings of the 8th international conference on categorical topology (L'Aquila, 1986)*, volume 27, pages 129–143, 1987.

[DP02] B.A. Davey and H.A Priestley. *Introduction to lattices and order.* Cambridge University Press, New York, second edition, 2002.

[DT95] D. Dikrajan and W. Tholen. *Categorical structure of closure operators With Applications to Topology, Algebra and Discrete Mathematics.* volume 346 of Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht,, 1995.

[DT15] D. Dikranjan and W. Tholen. Dual closure operators and their applications. *J. Algebra*, 439:373–416, 2015.

[Eng89] R. Engelking. *General topology.* volume 6 of Sigma Series in Pure Mathematics. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[Fuc73] L. Fuchs. *Infinite abelian groups. Vol. II.* Pure and Applied Mathematics., Series, 36. Academic Press, New York-London, 1973.

[GŠ09] E. Giuli and J. Šlapal. Neighborhoods with respect to a categorical closure operator. *Math. Slovaca.*, 61(5):717–732, 2009.

[GT00] E. Giuli and W. Tholen. Openness with respect to a closure operator. *Appl. Categ. Structures*, 8(3):487–502, 2000. Dedicated to Professor H. Herrlich on the occasion of his sixtieth birthday.

[Her22] P. Hertz. Über Axiomensysteme für beliebige Satzsysteme. *Math. Ann.*, 87(3-4):246–269, 1922.

[HIR16] D. Holgate, M. Iragi, and A Razafindrakoto. Topogenous and nearness structures on categories. *Appl. categor. struct.*, (doi:10.1007/s10485-016-9455-x), 2016.

[HŠ10] D. Holgate and J. Šlapal. Closure, interior and neighbourhood in a category. *Preprint.*, 2010.

[HŠ11] D. Holgate and J. Šlapal. Categorical neighborhood operators. *Topology Appl.*, 158(17):2356–2365, 2011.

[Kur22] K. Kuratowski. Sur lopération j de lanalyse situs. *Fund. Math.*, (3):182–199, 1922.

[ML98] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Moo09] E.H. Moore. On a form of general analysis, with applications to linear differential and integral equations. *In Atti del IV congrsso interazional dei Matematici(Roma),6-11 Aprile 1909).*,, II(Roma):98–114, 1909.

[Raz12a] A. Razafindrakoto. Neighbourhood operators on categories. *PhD thesis, University of Stellemboch.*, 2012.

[Raz12b] A. Razafindrakoto. On coarse and fine neighbourhood operators. *Topology and its Applications.*, 159(13):3067–3079, 2012.

[RH14] A. Razafindrakoto and D. Holgate. Interior and neighbourhood. *Topology Appl.*, 168:144–152, 2014.

[RH16] A. Razafindrakoto and D. Holgate. A lax approach to neighbourhood operators. *Appl. categor. struct.*, (doi:10.1007/s10485-016-9441-3, 2016.

[Rie09] F. Riesz. Sietigkeiisbegriff und absirakte mengenlehre. *Atti del IV cong.Int.di Mat.*, II(Roma):18–19, 1909.

[Sad68] M. T. Sadr. Structures topogènes sur une catégorie. *Cahiers Topologie Géom. Différentielle*, 10:475–505, 1968.

[Šla09] J. Šlapal. Another approach to connectedness with respect to a closure operator. *Appl. Categ. Structures*, 17(6):603–612, 2009.

[Šla11] J. Šlapal. Neighborhoods and convergence with respect to a closure operator. *Collectanea Mathematica.*, 63(2):123–137, 2011.

[Šla12] J. Šlapal. Compacteness and convergence with respect to a neighborhood operator. *Collectanea Mathematica.*, 63(2):123–137, 2012.

[SP64] J. L. Sieber and W. J. Pervin. Connectedness in syntopogenous spaces. *Proc. Amer. Math. Soc.*, 15:590–595, 1964.

[Tan76] D. Tanré. Espaces fonctionnels et structures syntopogènes. In *Categorical topology (Proc. Conf., Mannheim, 1975)*, pages 624–634. Lecture Notes in Math., Vol. 540. Springer, Berlin, 1976.

[Tar29] P. Tarski. Remarques sur les notions fondamentales de la methodologie des mathematiques. *Fund. Polon.*, (7):270–270, 1929.

[Vor00] S. J. R. Vorster. Interior operators in general categories. *Quaest. Math.*, 23(4):405–416, 2000.

[Wil70] S. Willard. *General topology.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970.