A Teaching Strategy to Enhance Mathematical Competency of Pre-Service Teachers at UWC

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ABSTRACT

In this study a mixed methods approach was employed to investigate how exposure to a teaching strategy based on spiral revision, productive practice and a mainly direct expository instructional method would influence the mathematical competencies of procedural fluency and conceptual understanding of pre-service mathematics teachers at a South African university. A secondary concern of the study was how retention and transfer abilities of participants would be influenced if they experience mathematics through a teaching strategy underpinned by spiral revision and productive practice.

A revised version of the taxonomy table of Anderson et al (2001) was utilized to classify learning and instructional activities in the study in terms of mathematical reasoning and knowledge requirements. In this revised taxonomy the cognitive processes are understood to operate on knowledge structures during the process of cognition (i.e. reasoning categories based on knowledge categories.). The categories of the revised taxonomy table were the main measuring instrument for the study.

The findings of the study indicate that the competencies of procedural fluency and conceptual understanding were positively enhanced by the teaching strategy. Some categories however did not show the same level of positive enhancement. Arguments are presented as to why this might be the case and possible solutions are mooted. Findings also indicate that retention and near transfer abilities of participants were positively enhanced. Far transfer abilities were unchanged post intervention. Explanations are offered for this finding and possible resolutions are suggested.

DECLARATION

I declare that

'*A Teaching Strategy to Enhance Mathematical Competency of Pre-Service Teachers*'

Is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Bruce Mathew May June 2017

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CHAPTER 1: INTRODUCTION

1.1 The importance of mathematics for the individual and society

The importance of mathematics and mathematical ability in the 21st century for both the individual and society cannot be underestimated. The International Review Panel of Mathematical Sciences Research at South African higher education institutions states that there is an unprecedented worldwide demand for university graduates with mathematical skills. They contend that because the mathematical sciences form the basis of cross-disciplinary research, this inherent power is deployed in finding solutions to the most critical and complex contemporary problems in society (Department of Science and Technology, 2008).

Research has shown that a linear relationship exists between the quantitative demands of different occupations that require a degree in higher education and the wages associated with those occupations (Geary, 2000). Findings of that research indicate that the higher the mathematics requirements in the occupation the higher the entry-level is, at commensurate wages. Based on this the author argues that the development and maintenance of numerical and mathematical competencies is very important for individuals within these societies and for the society as a whole.

Brown (2009) affirms the importance of mathematics for the individual when he asserts that mathematics is an important enabling science. He points out that most disciplines benefit from a good foundation through the concepts and skills developed in mathematics and data sciences and that mathematics is both a prerequisite and a tool in many disciplines. Areas he cites include chemistry, physics, computer science, environmental sciences, meteorology, psychology, health sciences, geography, economics, finance, business and many others. It is also my view that for many disciplines like physics and engineering, mathematics should not be regarded as supplementary to the discipline but as a fundamental building block of knowledge in the discipline.

Also, many believe that the development of good mathematics and science teachers and students is a prerequisite for economic development on a national scale. As a prerequisite for economic growth the South African government identified mathematics, science and technology as areas in education that require investment. To this end it established the Dinaledi[1] Project to increase the quality and number of students who would take mathematics and science. The Accelerated and

[1] A crucial initiative arising from the National Strategy for Mathematics, Science and Technology Education in South Africa was the establishment of the Dinaledi Project in June 2001. As a result of this project, 102 secondary schools were selected to be centers of excellence for the development of mathematics, science and technology. The aim was to increase the participation rates, especially of those previously disadvantaged, and of girl learners. It was also to improve learner performance in these subjects.

Shared Growth Initiative for South Africa (Asgi-SA)[2] is a set of government interventions that aimed to increase economic growth to 6% per annum between 2010 and 2014 while also halving unemployment and poverty by 2014. Asgi-SA makes specific mention of the importance of mathematics and science in this endeavour.

1.2 Mathematics as a gatekeeper

Historically, in South African society mathematics has always been a gatekeeper to higher education and the world of work. The following quote makes this abundantly clear: "…What is the use of teaching the Bantu mathematics when he cannot use it in practice? The idea is quite absurd." (Verwoerd, 1953).

I am of the opinion that mathematics in post-apartheid South Africa still maintains its gatekeeping function. This is evidenced by vast disparities in the distribution of human and physical resources that still exist in schooling in South Africa. Similarly critical Vithal and Volmink (2005) argue that although a new curriculum was designed to change this state of affairs, the poor in South African society remain marginalized: for them opportunities provided by mathematical knowledge and skills are lacking. This lack of access to mathematical knowledge and skills is a two-edged sword for the marginalized since they also need mathematics as a tool by which to understand the social forces that contribute to their marginalization (Martin et al, 2010).

1.3 Rationale

It is widely agreed that prospective teachers should graduate with a command of the five kinds of mathematical competencies outlined by the National Research Council (2001) since these are essential for success in learning mathematics. The competencies are as follows: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and a productive disposition. Conceptual understanding refers to the comprehension of mathematical concepts, operations and relations. Procedural fluency is the skill that is required to carry out procedures flexibly, accurately, efficiently and appropriately. Strategic competence is understood to denote the ability to formulate, represent and solve mathematical problems. The capacity for logical thought, reflection, explanation and justification is defined as adaptive reasoning. The habitual predisposition to see mathematics as sensible, useful and worthwhile together with a personal belief in assiduousness is defined as productive disposition (National Research Council, 2001).

The authors argue that the five competencies are interwoven and interdependent.

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[2] AsgiSA is a set of government interventions which sought to achieve an average economic growth of 6% by 2010 and to halve poverty and unemployment by 2014.

Reinforcing this, other research has shown that two of these strands are particularly important to the development of mathematical proficiency – conceptual understanding and procedural fluency (Hiebert & Grouws, 2007). Thus, there is a need in pre- service mathematics education (in the South African context) for research that investigates which teaching strategies could lead to the enhancement of the competencies of conceptual understanding and procedural fluency of pre-service mathematics education students. The question is what is the current state of affairs regarding these competencies in the South African context?

1.3.1 South African learners' levels of mathematical content knowledge and reasoning

The teaching and learning of mathematics has always been a contentious issue both in South Africa and internationally. In the South African context the mathematics teaching and learning problems are exacerbated by socio-economic problems with historical roots. South African teachers of mathematics at school level have borne the brunt of severe criticism for how their perceived lack of requisite knowledge and competence prevent them from delivering quality teaching of mathematics. It might be that some of this criticism consists of unfair generalizations since it seems to include all South African teachers irrespective of their success in teaching mathematics. However, it cannot be denied that South Africa does have major problems in the teaching and learning of mathematics.

The foregoing critique seems to be corroborated by large scale international and national studies conducted to measure the mathematical proficiency of South African learners at school level. One such study is the Trends in International Mathematics and Science Study (TIMSS). TIMSS was conducted in South Africa for the first time in 1995 and subsequently in 1999, 2002 and 2011 (Mullis et al, 2011).

It is necessary to provide a short explanation of the term scale score since it will be used to describe proficiency of South African school learners as determined by TIMSS. Analyses of TIMSS data occurs in two phases namely scaling and estimation. TIMSS rely on item response theory (IRT) scaling to describe student achievement on assessments and to provide accurate measures of trends. IRT is used to determine the difficulty of each test item or item category. The difficulty of items is deduced using information about how likely it is for participating students to get some items correct versus other items. Once the parameters for each item are determined student ability can be estimated even when students have been presented with different items. Initially achievement scores are expressed in a standardized logit scale that ranges from -4 to +4. In order to make the scores more meaningful and to facilitate their interpretation, scores are transformed to a scale score with a mean of 500 and a standard deviation of 100.

The South African grade 8 achievement scale score for 1995 was 276 and for 1999 it was 275. In 2002 the scale score for grade 8 was 264 and for grade 9 it was 285. For 2011 the scale score for grade 9 was 352 (Reddy et al, 2012). The authors of the South African TIMSS report of 2011 argue that the South African mathematics scale score has increased by 67 points from 2002 to 2011 (Reddy et al, 2012). It may be the case that the scale score has improved from 2002 to 2011, but I think what is more significant is that the 352 scale score for 2011 is still below the Low International benchmark of 400. My expectation was that the South African learners would perform better since grade 9 had been assessed instead of grade 8. These statistics (as measured by the TIMMS instrument) show that South African learners have less than half of the mathematical content knowledge of their international peers.

A similarly bleak picture is painted for the cognitive domains. The international average percentage correct in the Knowing domain is 49% compared to 26% for South Africa. For the cognitive domain of Applying, the international average is 39% compared to 19% for South Africa. The international average for Reasoning is 30% and for South Africa it is 14%. These average achievement percentages show that South African students operate mostly in the cognitive domain of Knowing, but even here, at a low level.

Reddy et al (2012) also analysed the average achievement of learners with respect to the former racial categorisation of schools in South Africa. They found that the average achievement scores of the former House of Assembly (HOA – White) administered schools were the highest, whereas the former House of Representatives (HOR – Coloured) and Ex-African administered schools were the lowest performing schools. They found that although the former African-administered schools achieved the lowest scores for 2011, they also showed the greatest improvement between 2002 and 2011. South Africa also participated in TIMSS 2015. The results indicate that the grade 4 scale score is 376 which is second last of all the participating countries. The grade 8 scale score is 372 and is also second last on the list (Mullis, et al, 2016). My concern is that nearly twenty years after the advent of democracy in South Africa the mathematical competency divide is exactly where it was during the years of segregation. This suggests that the previously advantaged are still privileged in terms of mathematics and hence can expect better work and higher education opportunities in South Africa, whereas the previously disadvantaged remain disadvantaged.

In the South African education system the tradition has been to focus on the grade 12 examinations and the improvement of grade 12 results. This focus is directed at preparing students from the previously disadvantaged communities of South Africa for better work and higher education opportunities (DBE, 2011). However there is a growing realisation that perhaps this has not come to pass as many learners do not even make it to grade 12. Also, in many cases the help provided in grade 12 has not been beneficial since gaps in the knowledge base and learner misconceptions are too great to deal with in the limited time comprising the grade 12 year.

In 2008 and 2009 a new national assessment system called Annual National Assessments (ANAs) was piloted in South African primary schools. The ANAs were made compulsory for all public schools in South Africa and were conducted in all public schools for the first time in 2011. The purpose of the ANAs according to the Department of Education was to contribute to better learning in schools and to form part of a range of interventions for promoting quality teaching and learning especially in the poorest communities. In the pilot study approximately 6 million learners from grades 1 to 6 were assessed in languages and mathematics. In 2012 grade 9 was included in the assessment. For both 2011 and 2012 only grades 1 and 2 achieved average percentages higher than 50%. There was a general decline in average percentage from grade 3 to grade 9. The national average for grade 9 mathematics in 2012 was 12.7% (DBE, 2012), and in 2014 it was 10.9% (DBE, 2014). These results, together with the TIMSS results presented earlier, indicate that South Africa has a sizeable problem in terms of the mathematical proficiency of its learners. Both the ANAs and TIMSS indicate that the majority of South African learners of the General Education and Training (GET) band have low levels of content knowledge and that they operate mainly in the elementary cognitive domains of mathematical reasoning.

Results of the National Senior Certificate (NSC) indicate that learning and teaching problems in mathematics are not restricted to the GET band, but are also prevalent in the Further Education and Training (FET) band. This argument is corroborated by the following results: in 2011 52.6% of learners scored below 30%; in 2012 this applied to 46% of learners; in 2013 , 40.9% of learners and in 2014, 46.6% of learners scored below 30% for the NSC examination (DBE, 2014). One can therefore argue that the majority of South African learners at school level have not developed the requisite procedural fluency and conceptual understanding as envisioned by the curriculum documents.

1.3.2 South African mathematics teachers' mathematical understanding and content knowledge

It is commonly agreed that the demands of teaching require knowledge at the intersection of mathematical content knowledge and knowledge of teaching (Ball, Thames & Phelps, 2008; Shulman, 1986). Research has shown that both content knowledge and the skill to integrate content knowledge with pedagogical knowledge are required for effective teaching (Taylor & Taylor, 2013). Shulman (1986) distinguishes between three categories of teacher content knowledge namely, subject matter content knowledge, pedagogical content knowledge and curricula content knowledge. Although the organization, composition and characteristics of mathematical content knowledge for teaching have been extensively researched, there is no consensus among researchers concerning what mathematics teachers need to know in order to deliver effective teaching (Ball, Hill & Schilling, 2004). Furthermore, although research has shown that teachers' mathematical knowledge is significantly related to learner achievement, the nature and extent of that knowledge is not known (Ball, Hill & Rowan, 2005).

However, both subject matter knowledge and pedagogical content knowledge have been further refined. Ball and colleagues (Ball et al, 2008) have proposed that Subject matter knowledge be subdivided into the following categories: Common content knowledge (CCK), Specialized content knowledge (SCK) and Horizon content knowledge. They propose that pedagogical content knowledge consists of the sub-domains Knowledge of content and students (KCS), Knowledge of content and teaching (KCT) and Knowledge of content and curriculum (Ball et al, 2008). The focus of the present research however is subject matter knowledge and in particular Common content knowledge.

Common content knowledge is defined as the mathematical knowledge one has in common with others who know and use mathematics. This knowledge is mostly discipline specific but is also essential to teaching since pedagogical content knowledge is predicated on content knowledge (Ball et al, 2008; Shulman, 1986; Venkat & Spaull, 2015).

It is imperative that teachers have a thorough understanding of the mathematics they are teaching, since it is highly unlikely that teachers who do not have a good grasp of mathematical concepts will be able to teach such concepts to learners (Ball, et al, 2008; Hiebert et al, 2003; Venkat & Spaull, 2015; Lerman et al, 2010). It is essential therefore that one determine what research uncovered about the subject matter knowledge of teachers in the South African context. This would be done not to vilify teachers of mathematics, but to gain a better understanding of areas lacking, in order to design better teaching strategies and to improve the content knowledge and mathematical reasoning abilities of pre-service teachers. My specific interest therefore is to determine what the literature reveals regarding types of mathematical knowledge (procedural and conceptual) and attendant cognitive abilities of South African teachers.

The National Education Evaluation and Development Unit (NEEDU) was established in 2009 by the minister of Basic Education in South Africa. Their function is to investigate and report directly to the minister of education on the state of schools in South Africa and in particular on the status of teaching and learning. They measured the quality of teaching and learning by measuring the outcomes of learning evident in learner notebooks and through one-on-one assessment in learner reading. These measures were used in conjunction with the scores attained in the Annual National Assessment (ANA) tests (NEEDU, 2012).

The NEEDU report utilized some of the findings of SACMEQ III[3] of 2007. SACMEQ III assessed grade 6 teachers' subject knowledge in language and mathematics. Since some of the

[3] The Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) is an organisation consisting of 15 Ministries of Education in Southern and East Africa. They work together to apply scientific methods for monitoring and evaluating the conditions of schooling and the quality of education. Technical assistance is provided by UNESCO's International Institute for Educational Planning (IIEP). Between 1995 and 2005 SACMEQ completed two major education policy research projects: SACMEQ I and SACMEQ II. SACMEQ III commenced in 2007 and was completed in 2011.

test items utilized in this endeavour were common to teacher and learner tests they provided the opportunity for comparing teacher and learner scores directly (NEEDU, 2012). For example, on the item:

$$10 \times 2 + (6 - 4) \div 2 =$$

the number of teachers who computed it correctly was 54% whereas the number of learners who did the correct calculation was 22%.

The results of some of these tests are captured in the following summary: Teachers' average percentage score for Arithmetic was 67.15%; for Fractions, Ratio and Proportion it was 49.68%; for Algebraic Logic, it was 46.51%; for rate of change it was 42.3% and for Space and Shape it was 56.44%. The SACMEQ III authors found that teachers performed relatively well in questions requiring elementary cognitive functions, but struggled with items that required higher cognitive functions. The authors therefore concluded that South African primary school teachers demonstrate poor subject knowledge in language and mathematics and as a result are less capable of understanding the requirements of the curriculum and how to apply these in their classes. The authors have identified poor subject knowledge as a major barrier to quality teaching and learning at all levels of the teaching system.

Bansilal, Brijlall and Mkhwanazi (2014) investigated levels of common content knowledge of South African high school mathematics teachers. Data for this research was provided by in-service teachers' written responses to a shortened form of a grade 12 mathematics paper one examination. Their sample consisted of 253 teachers that were enrolled for an Advanced Certificate in Education (ACE). The ACE serves as an upgrading of high school mathematics qualifications of teachers. The researchers employed the APOS theory (action-process-object-schema) of Dubinsky (1999) to analyse the written responses of the participating teachers. Teachers obtained an average of 57% in the presented test. Findings of the study indicate that the majority of participants performed well with lower cognitive level questions, but struggled with the higher cognitive level questions. One cannot generalise these findings to all South African high school mathematics teachers. One cannot also conclude from this study that all South African high school mathematics teachers do not have adequate content knowledge for the level at which they are teaching. Consistent poor performance of grade 12 learners in the final mathematics examinations however is an indication that major problems exist in teaching and learning of mathematics at the high school level. It is therefore not unreasonable to argue that poor content knowledge of teachers is a contributing factor to the teaching and learning problems at the high school level.

Many others have also identified the poor content knowledge of teachers as a problem in mathematics education. For instance, Ball investigated the subject matter knowledge of prospective teachers at 5 different American tertiary institutions and found that the mathematical understanding of those students was heavily dependent on procedures and lacked insight (Ball, 1990). A further, very intriguing finding in that research is that majoring in mathematics does not always guarantee sufficient subject knowledge.

Stein and colleagues (Stein et al, 1990) investigated the relationship between subject matter knowledge and elementary instruction in the domain of functions. They found that limited content knowledge led to teaching that was lacking in several key areas. Meaningful connections between key concepts were not made as a result of this deficiency in content knowledge. Investigating the effect of teachers' mathematical knowledge for teaching on student achievement Hill and colleagues (Hill et al, 2005) found a strong relationship between these elements. Ball and others reiterate that adequate content knowledge remains an essential component of teacher competency (Ball, 1990; Shulman, 1987; Stein et al, 1990; Hill et al, 2005).

Based on their research into teacher mathematics content knowledge in the South African context, Venkat and Spaull (2015) argue that many teachers do not possess sufficient content knowledge to provide their learners with access to the disciplinary ideas of mathematics. In particular, it is argued that many teachers lack indispensable conceptual knowledge for the level at which they are teaching; they have a highly procedural orientation towards mathematics teaching and they manifest gaps in requisite content knowledge. Furthermore, corresponding analysis of learner and teacher performance indicates that only the most proficient teachers had a positive impact on learner performance.

Venkat and Spaull (2015) also analysed the South African mathematics teacher test data that had been gathered in the study conducted by the Southern and East African Consortium for Monitoring Educational Quality (SACMEQ) 2007. The test responses of 401 grade 6 mathematics teachers drawn from a nationally representative sample of South African primary schools forms the primary data. Their findings indicate that the majority of South African grade 6 teachers have mathematical content knowledge levels below grade 6. The authors argue that this lack of content knowledge prevents teachers from delivering quality teaching, hence the learning of mathematics is severely compromised. Teachers of primary school mathematics (indeed all mathematics teachers) need to have a well-connected, well developed deep understanding of fundamental mathematics (Ma, 1999). However, the pursuit of more advanced courses in mathematics does not necessarily translate into a deeper understanding of fundamental mathematics (Ma, 1999). In order to deal with the problem of inadequate teacher content knowledge Taylor and Taylor (2013) argue that both pre- and in-service teacher training should focus on providing teachers with a well-wrought conceptual understanding of mathematics.

There are however others who do not agree that content knowledge of teachers should always be the focus in mathematics education. Martin et al (2010) contend that research in mathematics education that is content-focused and that ignores or simplifies the social contexts in which the research participants are embedded risks perpetuating the power relations in the field. Although they agree that mathematics content is important in mathematics education research, they do not agree that studies which isolate content offer a sufficiently wide view of contemporary problems in mathematics education. They propose that multidimensional studies are needed to understand how and why learners interact with mathematics content in the ways that they do, as well as how and why they learn (Martin et al, 2010).

Julie (2015) is an example of a South African author who is critical of the studies that were used as a basis to argue that South African teachers' weak mathematical content knowledge is one of the main contributing factors to the poor mathematical proficiency of learners in South African schools. He is of the opinion that the instruments used to determine the levels of mathematical content knowledge of South African teachers in most instances do not comply with accepted criteria for good tests. The contention therefore is that the identified deficiency of mathematical content knowledge is more a function of the instruments used rather than teachers' lack of content knowledge. He argues that the tests lack in terms of the following: "ambiguity of item formulation; the retention problem – inability to recall information due to non-use of knowledge that was previously acquired; knowledge valued and legitimated through boundary objects in practice; disciplinary fidelity of the items with respect to mathematics and mathematics education; openness or not of items with respect to system 1 cognitive processing.

Julie (2015) argues that test items that are considered to be good test items should be unambiguously formulated such that there is no doubt as to the correct answer. He identified an item used in the Taylor (2011) study and another used in the Carnoy, Chisholm and Chilisa (2012) study as examples of test items that are ambiguously framed. A reason advanced for the weak performance of teachers in the tests that were used to determine their levels of mathematical content knowledge, is that they have forgotten the mathematics they have learnt during their studies.

Boundary objects are defined as: "objects which inhabit several social worlds and satisfy the informational requirements of each of them" (Star & Griesemer, 1989, pg 205). Julie (2016) maintains that teachers' normal exposure to school mathematics is through these boundary objects. Hence if a test item does not conform to knowledge as distributed by the boundary objects then the expectation is that such an item will have a low percentage of correct responses. Items based on the order of operations such as that of Carnoy, Chisholm and Chilisa (2012) were considered to be items that teachers are not normally exposed to and hence were answered poorly.

Disciplinary fidelity of test items is referent to its preciseness in terms of adherence to constructs constituting mathematics as a discipline. Julie (2015) argues that items with low mathematical fidelity can have multiple correct representations. A few such items were identified in the tests presented to teachers.

When people are presented with particular kinds of problems where analytic reasoning are required one of two systems of metacognitive processes can be applied (Frederick, 2005). System 1 ordinarily occur spontaneously and do not require much attention. When people are presented with test items in a particular format they tend to use system 1 cognitive processing since it requires less effort. Unfortunately, much of the salient features of such test items will be missed as a result of system 1 cognitive processing. Some of the test items presented to teachers can be classified as such items.

Despite his criticism of the studies used to determine levels of mathematical content knowledge of South African teachers Julie (2015) concedes that the content knowledge of teachers do require some form of improvement. He is of the opinion that the knowledge gaps of teachers should be identified at the sites where teachers are actively engaged in school mathematical work. In other words, the knowledge gaps identified should be directly related to the content they are teaching. The expectation is that, if the content knowledge that these gaps represent is attended to the practice of the teacher concerned will be improved.

Ball and Bass (2000) make the point that teachers need to hold and use mathematics differently to the way mathematicians do as a consequence of the requirements of teaching. Professional mathematicians commonly compress mathematical information into more abstract representations in order to convert it into more usable forms. Teachers of mathematics on the other hand work with mathematics as it is being learned and therefore need to 'unpack' mathematical ideas (Ball, Bass & Hill, 2004; Adler & Davis, 2006). Teaching that is geared towards the unpacking or decompressing of mathematical ideas is described as teachers' explicit coherent reasoning that mediates learner mathematical reasoning. Since it is essential that teachers of mathematics develop the ability to decompress mathematical ideas, Adler and Davis (2006) investigated to what extent decompression of mathematical knowledge forms part of assessment tasks in formal mathematics education courses for in-service teachers in South Africa. Their findings indicate that in the majority of assessment tasks compression of mathematical ideas was dominant.

1.3.3 Conclusion

From the foregoing discussion one can distil the following arguments regarding the teaching and learning of school mathematics in the current South African context.

Many learners have low levels of relevant content knowledge and operate mainly in the elementary cognitive domains in terms of mathematical reasoning (their reasoning is not developed to the appropriate school level). The majority of learners have not developed the requisite procedural fluency and conceptual understanding as envisioned by the curriculum documents.

In the previous section I have shown that the test items utilized to determine South African primary school teacher content knowledge is flawed in many ways. Sufficient evidence was provided to be sceptical about findings concerning levels of teacher content knowledge. Furthermore, although researchers such as Venkat and Spaull (2015) have identified a lack of content knowledge in teachers they do not specify how these deficiencies can be addressed. The preferred way to deal with these deficiencies in the South African context has been based on an approach by which content knowledge deficiencies is attended to in a general way. That is teachers have been presented with courses (for example ACE) or workshops where they are exposed to the same general mathematical content. In other words, a one-size fits all approach is mooted. These interventions have however achieved limited success since results of large scale testing such as ANAs indicate that many learner still have not achieved a desirable level of mathematical proficiency.

One cannot disregard all the research concerning levels of content knowledge of South African primary school teachers despite the fact that some of the research is flawed. This is especially the case with some of the linked analysis discussed in the previous section. Linked analysis of teachers and learners showed that learners struggle with the same mathematical content that teachers are struggling with, which is an indication that teachers' lack of proficiency with this content prevents them from delivering quality teaching and hence learning is compromised.

There can be no doubt that many South African teachers lack essential conceptual and procedural knowledge required for the level at which they are teaching; many also have a highly procedural orientation to mathematics teaching and have gaps in requisite content knowledge. Many of these teachers have completed mathematical courses at tertiary institutions. Exposure to these courses does not seem to have prepared the teachers adequately for their classroom teaching. Improved mathematical courses and teaching is required to adequately prepare pre-service teachers to teach at the school level so that new teachers that enter the system can improve on the current situation. There is therefore an urgent need to design and then implement and test teaching strategies for pre-service teachers that have as a goal development of conceptual and procedural knowledge based on the content they will ultimately teach. The question is what kind of teaching strategy will enhance pre-service teachers' conceptual and procedural knowledge and deal effectively with their content knowledge gaps? The question that this research is concerned with is what should be done in the mathematical training of pre-service teachers at the university level to prepare them adequately in terms of levels of content knowledge and conceptual and procedural understanding for the level they will eventually be teaching at?

Problems in the teaching of mathematics are not unique to South Africa but are prevalent in many other countries. For example, Hiebert and Grouws (2007) maintain that many teachers in the United States of America often focus on lower level skills using a tightly controlled and curtailed question-and-answer routine. This approach does not ordinarily develop deep mathematical understanding. It has been my experience that a similar situation exists in South Africa. To change this style of operating one has to change the way one teaches pre-service teachers. That is to say one has to make certain that in their teaching two fundamental learning outcomes are in place namely, conceptual understanding and procedural fluency.

1.4 Research questions

From the foregoing arguments, it is clear that many South African mathematics learners have major deficiencies in the requisite mathematical content knowledge. Also, many have not developed higher cognitive abilities in mathematical reasoning. South African teachers too, in many cases do not have the desired levels of mathematical content knowledge and cognitive competencies. This might contribute to learner deficiencies. So I agree with the argument that in order to increase the mathematical proficiency of learners first the mathematical proficiency of prospective teachers has to be increased. It is unrealistic to expect prospective teachers to learn to teach for mathematical proficiency without becoming proficient themselves (Hiebert et al, 2003; Ball et al, 2005; Shulman, 1986). As I see it, an increase in mathematical proficiency includes inter alia an enhancement of conceptual and procedural knowledge, improved ways of working with mathematics, augmented ways of reasoning and improved ways in which the knowledge is held.

Until 2011, at the University of the Western Cape (UWC) pre-service mathematics education students who were being prepared to teach in the senior phase (grades 7, 8 and 9) were required to do exactly the same mathematics course as mainstream mathematics students (students who would use mathematics as a tool in a discipline other than teaching.). These prospective mathematics teachers were required to complete at least the second year of pure university level mathematics. The expectation therefore was that these higher level mathematics courses would prepare the pre-service students for teaching in the senior phase. The completion of more advanced courses in mathematics however does not necessarily translate into a deeper comprehensive understanding of fundamental mathematics (Ma, 1999). As argued in the foregoing section, the struggles of some South African teachers are mute testimony to this contention. As I have argued previously teachers of primary school mathematics (indeed mathematics teachers at all levels) must have a well-connected deep understanding of fundamental mathematics (Ma, 1999).

Julie (2002) is of the view that school mathematics is a specific kind of Mathematics. He contends that school mathematics consists of various kinds of mathematics. His argument is that mathematics courses for prospective teachers should also be a blend of the different genres (Julie, 2011). He advocates that such mathematics courses should be based on links between higher level mathematics such as mainstream university mathematics and school level mathematics. Based on these arguments a new curriculum was developed for mathematics content courses for pre-service teachers in 2012. A major objective of these courses presented to pre-service teachers at UWC was to expose them to the mathematical content that they would eventually teach. The idea was to delve deeper into the school level content in order to enhance the procedural fluency and conceptual understanding of participating students.

The question that arises concerns the kind of teaching which may be likely to develop these abilities and knowledge in students. I am of the opinion that part of the reason why some school teachers struggle with school level mathematics is that they have simply forgotten the relevant procedures and concepts. Another possible reason for their struggle is that when they were initially exposed to mathematical content it was done in a compressed way. In other words, they were not exposed to explicit coherent reasoning that made visible the different facets of the concepts studied within the teaching and learning discourse. This approach limited their knowledge of concepts and did not always provide connections with other relevant concepts. It thus had a negative effect on the acquisition of conceptual knowledge, hence inhibiting conceptual understanding. The outcome of this is that the competencies of procedural fluency and conceptual understanding were not developed to the required level.

A possible way to counteract the forget problem and an underdeveloped procedural and conceptual understanding is to apply the strategy of spiral revision. The expectation is that this strategy would facilitate the retention of indispensable knowledge within long-term memory and increase the possibility of the transfer of knowledge. Productive practice is a strategy that can be used to uncover the different facets of mathematical concepts and to strengthen and deepen conceptual understanding.

My hypothesis is therefore: *If* South African mathematics education pre-service students are exposed to a teaching strategy that is premised on spiral revision and productive practice *then* their procedural fluency, conceptual understanding, knowledge retention and knowledge transfer abilities will be enhanced.

The research questions of the study are as follows:

Main Research question: How would exposure to a teaching strategy (based on spiral revision and productive practice) in requisite content areas of the specified curriculum influence the mathematical competencies of procedural fluency and conceptual understanding in pre-service mathematics teachers?

Sub question: Are retention and transfer abilities of pre-service mathematics teachers enhanced if they experience mathematics through a teaching strategy underpinned by spiral revision and productive practice?

1.5 Definitions of important constructs in this study

Automatization is defined as the practice of a skill or habit to the point of its having become routine so that little if any conscious effort or direction is required (Sweller, 1994).

Conceptual understanding refers to the comprehension of mathematical concepts, operations and relations.

Continuous review of previously taught and (possibly learned) content is revision that is done many times during a semester.

Deepening-thinking-like problems are mathematical problems that are designed to on the one hand determine depth of conceptual understanding and on the other hand to provide opportunities to deepen and strengthen conceptual understanding.

Direct instruction is defined as instruction that provides information that fully explains the concepts and procedures that students are required to learn (Kirschner et al, 2006).

Distributed (spaced) practice is a learning strategy where practice of specified knowledge and skills is distributed over multiple sessions (Rohrer & Taylor, 2006).

Flexible Procedural knowledge is defined as deep procedural knowledge that would allow a student that possess such knowledge to use mathematical procedures that would best fit a provided known or novel problem situation. A consequence of such flexibility is that students that possess such knowledge will have the ability to generate maximally efficient solutions for known and even sometimes unknown problem situations. Flexibility in problem solving requires knowledge of multiple strategies and efficiency in applying such strategies in problem situations (Star, 2005).

Intersession interval (ISI) is defined as the time between study or practice sessions (Rohrer & Pashler, 2007).

Mass (blocked) practice is a learning strategy where all practice is done in a single session and where usually all presented problems require for its solution the same procedure or strategy. It usually takes the form that upon the completion of a lesson, practice exercises are presented to students based on the concept or set of concepts that were covered in that lesson.

Overlearning (repetitive practice) is defined as the continuation of study immediately after error-free performance has been achieved. Research has shown that while overlearning often increases performance for a short period, the benefit diminishes sharply over time. (Rohrer & Pashler, 2007). This type of practice usually includes many problems of the same kind. The 'same kind' refer to cases such as where problems are presented where the same procedure are required or the same concept is dealt with.

Procedural fluency is the skill that is required to carry out procedures flexibly, accurately, efficiently and appropriately.

Productive practice is a didactic strategy where students are exposed to deepening thinking-like problems in order to enrich their conceptual knowledge in requisite content areas of the specified mathematics curriculum (Julie, 2013). The idea with this component of the teaching strategy therefore was to enhance conceptual knowledge of participating students.

Productive struggle is a strategy whereby students are allowed either during lessons or in tutorial class to struggle with mathematics that is deemed to be important. The word 'struggle' is referent to the amount of effort which students spend in making sense of mathematics (Hiebert & Grouws, 2007).

Retention interval (RI) is defined as the amount of time between study and test. It is usually measured from the second study or practice session (Rohrer & Pashler, 2007).

Schemata are cognitive constructs that organize elements of information according to the manner in which they will be utilized (Sweller, 1994).

Spiral revision (or repeated revision) is defined as the recurrent practising of previously covered mathematical work in specified content areas (Julie, 2013).

Spiral testing is defined as a type of assessment where each test presented to students during a semester contains test items based on current topics, but also on topics that were previously completed and that were also tested in previous tests.

Testing (retrieval) practice is defined as practice where students are required to recall information or knowledge (such as during tests; when working through assignments or old question papers; or supplying responses to verbal questions) as opposed to restudying.

The spacing effect is the finding that distributed practice yields better test scores than massed practice (Rohrer & Taylor, 2006).

Unguided or minimally guided instruction is defined as instruction in which students are required to discover or construct essential information for themselves as opposed to being presented with essential information (Kirschner et al, 2006).

CHAPTER 2: LITERATURE REVIEW

2.1 Introduction

Mathematics education has an ambiguous meaning since it can refer to the mathematical educational process or to a field of research or discipline (Niss, 1999; Freudenthal, 1981). The former has been taking place for millennia and forms an integral part of individual and societal development. The concern of this research however is with mathematics education as a field of research. In particular, this research is concerned with mathematics teacher education.

Mathematics education as a field of scientific research is relatively young. There is however not always consensus and clarity about what qualifies as mathematics education research and what exactly this research is (Sierpinska, Kilpatrick, Balacheff, Howson, Sfard & Steinbring, 1993; Niss, 1999). Many researchers active in the field of mathematics education research have attempted to define the field. I will however use the definition provided by Niss (1999, pg 5) when he states that mathematics education research is:

> …the scientific and scholarly field of research and development which aims at identifying, characterising and understanding phenomena and processes actually or potentially involved in the teaching and learning of mathematics at any educational level.

The fact that mathematics teacher education is such a broad field of research makes it difficult to select only relevant publications for a literature review. I therefore developed an elementary framework to assist in the selection of relevant literature. The framework serves to direct the exploration of literature in relation to the type of research, the object of the research; the identity of research participants, the specifics of the research question (or questions) and the findings.

Potential avenues of research in the field include theoretical research, empirical research, applied research, developmental research, etc. (Niss, 1999). My main focus is on empirical research, but where necessary theoretical, applied and developmental research is included.

The objects of study include inter alia the teaching of mathematics, the learning of mathematics, teaching and learning situations, the relations between teaching, learning and mathematical knowledge; societal perceptions of mathematics and its teaching and the system of education itself (Sierpinska et al, 1993).

The relations between teaching, learning and mathematical knowledge is very complex. The debates regarding knowledge requirements for mathematics teaching and teaching in general is not settled. Many researchers however are in agreement that teachers need different types of knowledge in order to deliver effective teaching. Shulman (1986) distinguishes between three categories of teacher content knowledge namely, subject matter content knowledge, pedagogical content knowledge and curricula content knowledge. Although the organization, composition and characteristics of mathematical content knowledge for teaching have been extensively researched, there is no consensus among researchers concerning what mathematics teachers need to know in order to deliver effective teaching (Ball et al, 2004). Furthermore, although research has shown that teachers' mathematical knowledge is related to learner achievement, the nature and extent of that knowledge is not known (Ball et al, 2005). By using a specified teaching strategy this study therefore aims to enhance prospective mathematics teachers' subject matter content knowledge and their approach to reasoning in mathematics. Since the object of study therefore broadly fits the category 'relations between teaching, learning and mathematical knowledge,' relevant literature will be reviewed.

While potential mathematics teacher education research participants might include in-service teachers or teacher educators working within the primary or secondary school curriculum, as the focus of this study I have selected participants active in *pre-service* teacher education. This, notwithstanding my acknowledgement that valuable insights might also be gained from related in-service teacher education research.

Interest in content knowledge issues in the international field of mathematics teacher education has increased recently (Adler, Moletsane, Pournara, Taylor & Thorne, 2009). This kind of research is however not entirely uncontested. Nonetheless, research questions that deal with these – and related – issues are also of interest to me and hence I will focus on studies that have attendant research questions.

I begin by first reviewing general trends in mathematics education research. I follow this with a comparison of selected teaching strategies or methods. After this I hone in on literature that deals with the following issues: mathematics teacher education research investigating teaching methods in teacher education courses and their influence on content knowledge; teaching strategies and their influence on conceptual and procedural knowledge; the link between features of teaching and students' learning in terms of skill efficiency and conceptual understanding.

2.2 Trends in mathematics teacher education research

A few researchers have attempted to provide a survey of mathematics teacher education studies conducted in specific periods. Some of these studies attempted to determine international trends while others focused on trends on the national scene.

Not much is known about the efficacy of teacher preparation in relation to the creation of opportunities for learning mathematics and how the form of instructional delivery affects teachers' mathematics knowledge (Tatto, et al, 2010). The International Commission on Mathematics Instruction (ICMI) study of 15 May 2005 focused on the organization of the mathematics preparation and development of teachers. One of the reasons advanced for the necessity of the study was that the education and development of pre- and in-service mathematics teachers is essential to learners' acquisition of mathematical knowledge.

Knowledge gained from such a study could potentially also be used to inform and strengthen mathematics teacher education policy and practices nationally or internationally (Tatto et al, 2010).

Data for the aforementioned study consisted of findings of research reports based on the organization of teacher education at national level in approximately 20 countries or regions submitted to ICMI -15. Tatto et al (2010) maintain that the findings indicate that in some countries mathematics teacher preparation is placed in universities in order to expose teachers to more rigorous training in mathematical content. Other participating countries were more disposed to placing teacher preparation in teacher colleges where presumably the emphasis is on mathematical pedagogy. The study findings indicate that in South Africa a high emphasis is placed on mathematical content knowledge and a low emphasis on pedagogical content knowledge in pre-service teacher education for prospective secondary school teachers whilst there is a low emphasis on content knowledge and a high emphasis on pedagogical content knowledge for prospective primary school teachers. Those researchers were perturbed that some primary school teachers are placed in preparation programs where there is a low emphasis on mathematical content knowledge. They maintain that it is essential that teachers of mathematics are well versed in the mathematical content that they teach.

Sfard (2005) presented a report on a survey that investigated the relations between mathematics education research and practice at ICME-10 in July 2004. The major focus of research that was done in the period 1970 to 2000 was on the student. However, in the period 2001 to 2004 the focus of the research turned to the teacher and teacher practice. Only one quarter of researchers in this period focused on the student.

Data for the Sfard report (2005) was collected by means of a questionnaire posted on the website of ICME 10 and by sending the questionnaire to colleagues known to be active researchers in the field. Sfard (2005) found that in the period under review (2001 – 2004) most of the empirical data was based on recordings of classroom interaction that focussed on the process of teaching and learning as compared to earlier studies which had tended to focus on the learning of the individual student. Most of the research was of a qualitative nature and emphasized the social context of learning.

Based on the aforementioned research and data types, Sfard (2005) contends that the bulk of the research in the period 2001 to 2004 conceptualizes learning as participationist as opposed to acquisitionist. If the learning process is conceptualized as the development of concepts and an acquisition of knowledge, then it can be referred to as acquisitionist. Alternatively, if the learning process is conceived of in terms of participation in certain kinds of activities rather than in accumulating or gaining permanent ownership of some commodity then the participationist metaphor applies. In this conceptualization learning a subject now implies becoming a member of a certain community, being able to communicate in the language of the community and subscribing to the norms of the community. Sfard (1998) maintains that in current discourses on learning the acquisitionist and participationist metaphors are central. She maintains that while both orientations are present in most research the acquisitionist concept of learning is more prominent in older studies whilst the participationist concept is more dominant in recent studies.

An international study conducted under the auspices of ICME 10 reports on a survey of mathematics teacher education conducted between 1999 and 2003 (Adler, Ball, Krainer, Lin & Novotna, 2005). International mathematics education journals, handbooks and conference proceedings formed the main sources of data for that survey. The researchers confined their investigation to research types that attempted to determine how, on what basis and under what conditions teachers learn. Studies that investigated how teachers' opportunities to learn can be improved were also included. The research participants in those studies were pre-service teachers, in-service teachers and teacher educators. Both primary and secondary teacher education was included. The authors contend that the focus of research in the 1970s was on curriculum issues, whereas in the 1980s and 1990s it was on learners. They observed that subsequently there has been another shift to a focus on teachers.

A finding of the aforementioned study was that a large component of the research consisted of small-scale qualitative research studies (Adler et al, 2005). The complexities of research that focuses on teachers are advanced as a rationale for the choice of small-scale studies. Small-scale studies might also contribute towards developing a theory of teacher learning.

They contend that, small-scale studies are limited since cross-case analyses are required to examine in what ways different instructional approaches, programs and settings affect the content knowledge teachers need to master in order to offer quality teaching. Another possible shortcoming of small-scale studies is that often these are done in the short term and consequently their analysis of the development of teacher knowledge may fail to provide a comprehensive picture. Studies that follow teachers over extended periods prove more informative since teachers' knowledge develops over time (Adler et al, 2005).

Another finding of this research is that the majority of teacher education research is conducted by teacher educators who rely on their own students to become participants. Presumably this is a consequence of teacher educators attempting to intervene and investigate so as to improve and understand their practice (Adler et al, 2005). In this regard, what typically happens is that a teacher educator designs a program and then attempts to illustrate the effectiveness (or not) of it by means of research.

The upshot of this is that some research questions have been studied extensively while others have not received as much attention. For instance, the question of teacher learning in a context where reform is not the objective has not been studied as extensively. Similarly, studies that investigate teachers' experiential learning have not been accorded much consideration. Questions such as *what* do teachers learn from their teaching experience, *whether* teachers learn from this experience and *what supports* learning from such experience have received little or no attention.

Finally Adler et al (2005) suggest that comparative studies on how one approach to teacher learning compares with another might greatly assist in an initiative to identify best practice, but this idea has not attracted much attention.

Mathematics teacher education in South Africa has always been a contentious issue, but even more so in the wake of post-apartheid educational reforms. Its current context is one of curriculum reform, social transformation, new policy frameworks and institutional change (Adler, et al, 2009). There is therefore a dire need for research and the dissemination of findings to generate debate that will lead to practices enhancing of the teaching and learning of mathematics in this very complex environment. A relatively recent survey of research into mathematics and science teacher education in South Africa (2000-2006) provides some insight into the current state of affairs (Adler et al, 2009). Data for that survey was gathered from peer-reviewed national and international journal articles and conference proceedings in the period 2000 to 2006.

The findings indicate that small-scale qualitative studies dominate. This concurs with the international study done by Adler et al (2005). In most cases the subjects of these studies were in-service teachers who were participating in upgrading programmes where high school mathematics was the focus. In the majority of cases the researchers were teacher educators who were also the presenters of the programmes and who were attempting to determine the efficacy of their courses. Teacher knowledge for teaching and teacher learning in a context of curriculum reform was the predominant focus of that research. In terms of content knowledge, the objective was to determine what it is that teachers need to know and be able to do mathematically in order to deliver quality teaching.

Teaching approaches and their potential effects were the focus of some research articles. Adler and colleagues (2009) contend that this focus on content knowledge is in agreement with international trends in the research field of mathematics teacher education. One has to bear in mind however that this 'content' is different from what is normally viewed as the accepted content by mathematicians. Included in the 'content' proposed by Adler et al (2009) are mathematical processes and learners' handling of content. They maintain that in South Africa both pre- and in-service teachers come to teacher education programmes with substantial gaps in their knowledge bases. To address these knowledge deficiencies the emphasis should not only be on more content but, instead should be about the kinds of content and pedagogic content knowledge teachers need to know in order to teach effectively (Adler et al, 2009).

Under-researched areas include pre-service mathematics teacher education, which is not very well represented in the literature (Adler et al, 2009). The investigators argue that given the recent introduction of the Bachelor of Education (B.Ed) degree, pre-service mathematics teacher research emerged as an area that needed urgent attention. They contend that the practice of mathematics teacher education in this new context should be researched thoroughly if we intend to produce effective teachers. One of the main tensions in this endeavour is between breadth and depth of discipline knowledge. The new curriculum requires a broadened domain knowledge base. We should however guard against sacrificing depth for increased domain knowledge.

Another prominent absence in the research field is studies on primary mathematics teacher education (Adler et al, 2009). Adler and her colleagues (2009) are of the opinion that primary teacher knowledge deficiencies contribute to poor learner achievement in mathematics at the primary level. They argue that this might be as a result of the fact that few primary teacher programmes include a serious focus on disciplinary knowledge. Despite this very little is known about primary mathematics teacher education programmes that are attempting to address primary teacher content knowledge issues. There is a particular lack of research about the kind and quantity of content knowledge primary teachers require and possible methods that may be used to develop such knowledge effectually (Adler et al, 2009).

Taylor (2014) did a study that deals specifically with pre-service teacher education at higher education institutions in South Africa. Five representative institutions were selected for the study. The purpose of the study was to describe curricula and practices of pre-service teacher education. The major aim of the investigation was to determine the range and depth of mathematics and English courses offered to B.Ed students specialising in Intermediate Phase teaching. The author argues that in-service teaching training in the South African context had limited success in improving learner proficiency. He is therefore of the opinion that in order to improve the quality of schooling pre-service teacher education needs to be improved.

Some of the pertinent findings of the Taylor (2014) research are:

- The content of modules and programmes varies widely among institutions.
- Pre-service teacher education programmes have low entrance requirements in comparison with other disciplines.
- A majority of the pre-service teacher education programmes lack a strong underlying logic and coherence.
- Students specialising in mathematics are required to take courses that deal specifically with mathematics content.
- The mathematics content courses offered deal mostly with school level topics taught to Intermediate and Senior Phase learners, but the content is presented at a deeper conceptual level and focus on the specialized mathematical knowledge a teacher would need to know.
- The courses align well with similar courses in the USA.

Taylor (2014) maintains that teacher proficiency in South Africa depends heavily on the quality of their university education. He is however of the opinion that the current system does not prepare pre-service teachers adequately to deliver quality teaching in mathematics.

Adler and Davis (2006) investigated mathematics for teaching in the teacher education courses of a range of tertiary institutions in South Africa. The focus of the study was on formal evaluative events in the education courses of selected institutions. Research sites were restricted to five of the nine provinces in South Africa and included both urban and non-urban contexts. The focus of the study was mathematics in-service courses. Sixteen such programs were identified across the provinces. These courses offered certification for teachers of the senior phase (grades 7 – 9) as well as the further education and training phase (grades 10 – 12). Some of these courses were taught by the mathematics faculty while others were taught by the mathematics teacher education faculty.

The researchers analysed formal assessment tasks which consisted primarily of written assignments and tests. The evaluation tasks were studied to determine the kinds of mathematical and pedagogical competencies that teachers in these courses were expected to display, and the kind of mathematical knowledge that was privileged. Assessment tasks were examined to determine whether mathematical reasoning or reasoning about mathematics teaching was required in the solution procedure. If either of these occurred in a task the researchers would investigate whether the task required unpacking or compression of mathematical ideas. Unpacking can be described as an elaboration of knowledge whereas compression refers to the compression of mathematical information into an abstract form that is highly usable. Since teachers work with mathematics as it is being learned, their work should mostly include unpacking of mathematical ideas. Conversely it is more convenient for research mathematicians to work with compressed mathematical ideas. The findings indicate that the majority of mathematics courses for teachers are premised on the compression of mathematical ideas rather than on the unpacking of mathematical ideas.

In what follows a summary of the previous two sections will be presented. This summary however will only focus on specific issues regarding pre-service mathematics teachers' content knowledge. The motivation for doing this is that the major focus of this study is pre-service teachers' subject matter knowledge:

- There is a paucity of research (both in South Africa and internationally) concerning how the form of instructional delivery affects pre-service mathematics teachers' mathematical knowledge.
- Of the few studies that researched teacher content knowledge in most cases the objective was to determine what it is that teachers need to know and be able to do mathematically in order to deliver quality teaching.
- It is argued that in South Africa both pre- and in-service teachers enter educational programmes with substantial gaps in their mathematical knowledge bases.
- It is claimed that in order to address these knowledge deficiencies depth of disciplinary knowledge should be the focus rather than breadth.

- The majority of research studies concerning teacher content knowledge are small-scale qualitative studies.

Based on the foregoing arguments a pertinent question is: What kind of teaching strategy (or strategies) would aid in developing content knowledge of pre-service teachers to requisite levels? In section 2.3, a few teaching strategies will be discussed in order to inform discussions regarding the question.

In the next sub-section examples of singular mathematics teacher education studies will be discussed.

2.2.1 Examples of singular mathematics teacher education research

How to design mathematics teacher education programs that influence the nature and quality of teachers' practice positively has to a large extent remained a mystery (Hiebert, Morris & Glass, 2003). Hiebert et al (2003) maintain this is in part due to the lack of a widely shared knowledge base for both teaching and teacher education. Very little research is done to determine the effectiveness of mathematics teacher education programs and very little information is shared about teacher education programs across tertiary institutions. Hence teacher educators mostly start anew when presenting teacher education courses. In the light of this the authors propose an 'experiment model' for teaching and teacher education as a possible way to address these issues. This model is premised on two learning aims: one is the development of the knowledge and disposition by which to learn to teach effectively over time, and the other is mathematical proficiency (Hiebert et al, 2003). As mentioned in section 1.3 mathematical proficiency is considered to consist of five kinds of mathematical competencies: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (National Research Council, 2001).

Hiebert et al (2003) believe that it is better to define learning goals in terms of students' thinking. Stated differently it means to assess goal-achievement by investigating changes in students' thinking about procedures and concepts. It is also worthwhile to note that learning is conceptualized as acquisitionist in this research. It is highly improbable that all the learning goals discussed in the foregoing section can be achieved in a relatively short period of time. I am therefore of the opinion that it is more important for prospective teachers to learn how to learn so that over time they themselves will continue to improve their mathematical proficiency.

In the experimental teaching model as proposed by Hiebert et al (2003), it is essential that each lesson be treated as an experiment. This entails setting clear learning goals, collecting data to evaluate the lesson efficacy and interpreting the data with a view to improving future lessons. It is a requirement of this model that lessons are designed in such a way as to expose student thinking; and that teachers develop the ability to listen with understanding and to assess student thinking in terms of cogency of reasoning, the connection of mathematical ideas, etc.

Rowland et al (2005) investigated elementary (school) teachers' subject knowledge in the United Kingdom by utilizing a case study. They used videotapes to analyse lessons presented by pre-service elementary teachers. Their goal was to determine in which ways the pre-service students drew on their knowledge of mathematics and mathematics pedagogy in teaching mathematics. The researchers used a grounded approach by which to analyse their data. They identified four broad mathematics-related knowledge requirements, which they refer to as the knowledge quartet. The knowledge quartet consists of foundation, transformation, connection and contingency. Foundational knowledge is defined as propositional knowledge and beliefs. This includes the meaning and description of relevant mathematical concepts, the factors that are significant in the teaching and learning of mathematics and the objective in or reasons for teaching mathematics. Transformational knowledge is described as knowledge required in the processes of deliberation and choice in planning and teaching. This knowledge is required for identifying the mathematical concepts and procedures in a lesson. Connection knowledge is knowledge required for making connections between different meanings and descriptions or alternative ways of representing concepts. Connection knowledge is also required in the sequencing of mathematical content. Contingency knowledge is described as the ability of the teacher to respond appropriately to unexpected developments in the presentation of a lesson.

Hill et al (2005) investigated whether and how teachers' mathematical knowledge for teaching contributes to students' mathematics achievement. Their study focused on the specialised knowledge and skills required for effective teaching. Unsurprisingly the findings of their study indicate that teachers' mathematics knowledge is strongly related to student achievement.

Stein et al (1990) explored the relationship between teachers' mathematical knowledge and their teaching practice. The authors videotaped an experienced fifth grade teacher teaching a sequence of lessons on functions and graphing. They also interviewed the teacher and presented the teacher with a card-sort task. They found that the teacher's knowledge of functions and graphing was missing several key mathematical concepts and that it was not organized in such a manner as would provide an easily accessible, cross-representational understanding of the domain. The authors found that these constraints gave rise to a narrowing of instruction in three different ways. One was a lack of making meaningful connections between key concepts and representations. Another was that no provision was made for future learning in the area of functions and graphing. The lessons also overemphasised limited truths.

Similar to the international situation there is a paucity of research that investigated the effectiveness of mathematics teacher education in the South African context. Paras (2001) concurs with this argument and claim that there is a lack of research in mathematics teacher education in South Africa, notwithstanding a perceived crisis in mathematics education. He devised a study to determine why students are failing Mathematics Education I in the School of

Educational Studies at the University of Durban Westville[4]. The research participants for this study were students enrolled for the course and lecturers who presented the course. The purpose of the study was to obtain students' and lecturers' explanations for the high failure rate of the Mathematics Education I course. The data collection method included interviews and questionnaires.

I will now elaborate on some salient findings of the Paras (2001) research.

One of the main findings was that the interactive strategies used by lecturers were inadequate for the advancement of effective learning. In terms of questioning techniques, it was found that lecturers did not direct questions at specific students, but directed questions at the class in general. Furthermore, lecturers frequently tended to answer their own questions and did not allow students time to complete responses to questions.

Participating lecturers generally agreed that students in this course had huge gaps in their requisite mathematical knowledge base. In many cases gaps in student knowledge existed simply because some fundamental content at school level had not been covered. Yet despite the fact that lecturers were aware of this, no concrete plans had been devised to deal with these knowledge gaps.

Another finding was that students were under-prepared for most lectures and tutorial sessions. Lecturers routinely gave students homework, but in the majority of cases those tasks were not done. Consequently, most students could not participate in task-based class discussions. Furthermore, students expected tutors to re-teach content that had been covered in lectures. A university enculturation program for first-year students was proposed as a possible solution, to help students to become independent learners. The author argues that this can be achieved by providing students with an intensive introductory program that delineates what university level studies require and in particular, what being a mathematics education student entails. Staff development programs were suggested as a possible means by which to enhance teaching and learning. These may then focus on teaching techniques such as questioning and so facilitate progress amongst less knowledgeable students.

Van Putten et al (2010) probed the attitudes of pre-service mathematics education students at the University of Pretoria in South Africa towards Euclidean geometry. Levels of understanding in terms of Euclidean geometry were also included in their investigation. Their approach entailed interviews with a group of pre-service students before and after they were taught a module in Euclidean geometry. A case study approach together with a pre-post-test procedure was followed in the study. The authors used the Van Hiele levels of thought as the theoretical framework of the study. They found that after instruction the students' attitudes towards geometry changed in

[4] The University of Durban Westville does not exist anymore, it was incorporated into University of Kwazulu-Natal

positive ways. However the instruction had still failed to bring about sufficient improvement in the students' levels of understanding to be able to teach geometry adequately.

Van der Sandt and Nieuwoudt (2003) investigated the geometry content knowledge of grade 7 teachers and pre-service teachers in the North West province of South Africa. They used the van Hiele theory and acquisition scales of Gutierrez et al (1991) to investigate their subjects' levels of content knowledge and geometric reasoning. Their findings indicate that both the teachers and the prospective teachers did not reach the level of geometric thinking and content knowledge that is required for the successful teaching of geometry at this level. The authors contend that these findings have significant implications for pre-service and in-service teacher education.

Van der Sandt (2007) did a two year study of teachers' and pre-service students' content knowledge of geometry using the Van Hiele theory and the acquisition scales of Gutierrez et al. (1991). Her subjects were drawn from five different cities in South Africa. The author found that both the teachers and the pre-service teachers failed to reach the expected level of geometric thinking and knowledge acquisition.

Long (2005) analysed the teaching and learning of a second-year university mathematics education course offered to pre-service teachers in South Africa by utilizing the distinction between conceptual and procedural knowledge. The topic to which this distinction was applied was number bases. Students in this course had a varied mathematical background; some only offered mathematics up to grade 9, while others offered mathematics up to grade 12.

Long (2005) advances a few reasons for why she thinks this distinction between conceptual and procedural knowledge is useful for this kind of analysis. She maintains that the constructs can provide the students with a scaffold both for learning mathematics and for thinking about the teaching of mathematics. She contends that these constructs are useful for uncovering the processes used in acquiring mathematical knowledge; and that they can also be a tool with which to tackle problematic areas of learning. She is also of the opinion that the distinction can be utilized by lecturers to analyse their own teaching and for students to analyse their learning (Long, 2005).

Long (2005) investigated students' ways of working when solving problems in the content domain of number bases, and subsequently advanced explanations as to their developmental levels in terms of their mathematical proficiency for this type of problem. Students who were able to use different strategies to solve the problems with which they were presented were deemed to have a conceptual understanding. It was found that such students grasped the relevant concepts very quickly and were able to work fluently with the procedures. A reason advanced for these abilities is that these students did mathematics at a higher level at school and resultantly were better able to process mathematical information.

A second group of students (who were also successful in solving the problems with which they were presented) tended to focus more on the application of the procedure while a conceptual understanding was not afforded the same level of focus. This group needed to be exposed to a number of examples in which the procedures involved are applied before they could fully understand the underlying concepts and hence develop a flexible comprehension. The teaching strategy the author proposes would work best for this group involves an integration of the concepts and the procedures so that the procedure becomes a scaffold for comprehending the concept.

A third group that was identified tended to focus on how to do the problem. In other words, these students were preoccupied by the steps involved in the procedure and hence did not form conceptual links between the procedure and the concepts. These students were able to apply the procedures only when presented with similar problems, and struggled when presented with problems where the context was changed. The teaching strategy applied in this group was one in which the links from the procedural to the conceptual and vice versa were made much more explicit.

Long (2005) were of the opinion that in the learning of mathematics, in some instances conceptual understanding precedes procedural competence whereas in others the opposite is true. Mathematical learning is viewed as a complex process in which conceptual understanding and the scaffolding function of procedures both play a role in establishing mathematical proficiency. Furthermore, the fluent execution of algorithms is considered to be an aspect of procedural fluency and the algorithms are understood to represent compressed conceptual understanding. Compressed conceptual understanding is assumed to refer to mathematical concepts that have been developed to a high degree of abstraction. The author was also of the opinion that if competency with algorithms is achieved within a content area then this frees the mind to focus on conceptual relationships.

Financial mathematics forms a small section of the latest South African high school mathematics curriculum. Despite the fact that financial mathematics is allocated only 7% of teaching time and forms approximately 6% of the mark allocation in the final grade 12 examination it is one of the rare school topics where real world applications is immediately obvious. Pournara (2014) designed a qualitative study to investigate teachers' knowledge requirements for the teaching of financial mathematics. His research participants were third and fourth year Bachelor of Educations students at the University of Witwatersrand. The aim of the study was to determine teacher knowledge requirements in the teaching of annuities.

Pournara (2014) used the term mathematics-for-teaching (MfT) to refer to teachers' knowledge for teaching mathematics in his study. The reasons he advances for not using the knowledge categories of pedagogical content knowledge (PCK) and subject matter knowledge (SMK) as posited by Shulman is that he found no empirical evidence for the existence of these categories of knowledge and that no productive and clear boundaries was established in the defining of the categories. He considers MfT to be an amalgam of PCK and SMK in which no fine distinctions is made between mathematical and pedagogical aspects of knowledge. The author maintains that knowledge for teaching annuities comprises knowledge of mathematical aspects, knowledge of pedagogical aspects and contextual knowledge of finance. Examples of typical annuity problems are used to unpack the knowledge requirements to teach annuities. Arguments based on this unpacking are subsequently used to show how knowledge of mathematics, knowledge of pedagogy and contextual knowledge of finance are intertwined.

The topic linear equations are one of the core content modules of the South African high school mathematics curriculum. Some concepts that are associated with this topic are fundamental to the understanding of topics that are dealt with later in school and tertiary mathematics education. The concept of a linear system is an example of such a concept. The type of linear system normally dealt with in high school is two equations in two unknowns. In higher mathematics, the system of two equations with two unknowns is generalized to n equations with n unknowns which forms part of the topic matrix algebra. Ndlovu and Brijlall (2015) devised a study to investigate the nature of mental constructions of matrix algebra concepts of 85 pre-service teachers at the University of KwaZulu-Natal. Study participants' responses to a structured activity sheet were the main data source for the study.

In order to determine the mental constructions research participants might make in their efforts to understand the presented matrix algebra concepts Ndlovu and Brijlal (2015) designed a genetic decomposition. A Genetic decomposition is defined as the structured set of mental constructs which describe how a given concept can develop cognitively. The authors used the APOS theory (action-process-object-schema) (Dubinsky & McDonald, 2002) to describe and analyse the pre-service teacher's constructed knowledge of matrix algebra concepts. Their findings indicate that most of the participants were operating at action and process stages while few were operating at the object stage. In addition, cognitive constructions made by the pre-service teachers in the majority of cases corresponded with the preliminary genetic decomposition.

2.3 A Comparison of a few teaching methods/strategies

The teaching and learning of mathematical concepts and procedures has always played an integral part in human and societal development. It is probable that only once our distant forebears had invented counting could they quantify possessions and then determine if their set of possessions had increased, decreased or remained the same. This ability to count could only have been passed on to the next generation by means of teaching. It would therefore not be an exaggeration to claim that mathematics education in its different guises has been with us for millennia. Swetz (1995) agrees that many of our current mathematical educational processes have evolved from practices employed in the distant past. He maintains that instructional techniques such as the use of discourse and the logical sequencing of mathematical problems and exercises from easy to complex are evident in historical mathematical texts. The creation of teaching strategies or methods that a majority of mathematics instructors find adequate and that satisfies the myriad educational needs of the contemporary mathematics classroom has however, remained elusive.

A synthesis of meta-analyses of effects on student achievement in schools was done by Hattie (2009). He designed an achievement continuum with a scale based on effect size to determine how six different factors affect student achievement at the school level. The six factors utilized were: the child; the home; the school; the curricula; the teacher; and the approaches to teaching.

Hattie (2009) identified an effect size of $d = 0.40$ as the average effect size for the majority of factors that affect student achievement. He refers to this effect size as the hinge point since it is the point on a continuum that provides the fulcrum around which all other effects are interpreted. Factors with an effect size lower than $d = 0.40$ is considered to have negligible effects on student achievement. Conversely factors with an effect size higher than $d = 0.40$ is considered to be factors that might influence student achievement in a noticeable way. Although Hattie's (2009) synthesis is mainly concerned with schools I am of the opinion that many of the identified factors that have a noticeable effect ($d > 0.40$) on student achievement might be applicable in higher education as well. I believe that this is especially so in the case of pre-service teacher education.

Over the years many teaching strategies or approaches has been designed and implemented in mathematics. Some features of these strategies were more successful than others in terms of enhancing student achievement. Teaching strategies has an average effect size of $d = 0.60$ (Hattie, 2009). The expectation therefore is that implementation of a successful teaching strategy will have a significant positive effect on student achievement in most cases. Donovan and Radosevich (1999) for example performed a meta-analytic review of the teaching strategy distributed practice and found an effect size of $d = .42$. Similarly meta-analytic studies done by Hattie (2003) found an effect size of $d = .82$ and $d = 1.00$ for the strategy direct instruction and instructional quality respectively. It is important therefore that educators (mathematics instructors in particular) are cognisant of those features of teaching strategies that would have a significant effect on student proficiency.

Despite the fact that this research is concerned with the development of effective teaching strategies for mathematics teacher education courses I did not limit investigation of the literature to teaching strategies employed in pre-service teacher mathematics content courses at the university level. My motivation for doing this is since I am of the opinion that the development of quality teaching strategies is not the sole preserve of any particular sector of mathematics teaching and learning but that it can be developed at any level where teaching and learning take place. I am also of the opinion that one can gain valuable knowledge on how to design effective teaching strategies from investigating weaknesses and strengths of teaching strategies discussed in the literature. Moreover, knowledge of different teaching strategies might also contribute to better designed and implemented teaching sessions. Consequently, any literature that contributes to better understanding of what constitute quality teaching strategies was included in the review.

In the next section I compare a few selected mathematics teaching methods/ strategies for their perceived strengths and weaknesses.

It is Wigley (1992) who distinguishes between two models for teaching mathematics. He points out that the 'path-smoothing model' is prevalent in mathematics teaching and textbooks, the main objective of which is to smooth the path for the student. The main features of this approach involve a teacher who is responsible for dividing the content into a number of categories and who will then present these to students one at a time. Students are then taught a method by which to solve problems in a series of steps. Teacher questions are geared towards leading students in a particular direction and towards checking that they follow the explanations. Students are presented with exercises by the teacher to enable them to practise the problem-solving methods they were taught. In an attempt to deal with student failure, revision involves returning to the same or similar work throughout the course. Thus Wigley (1992) maintains that this model emphasises repetitive rather than insightful activities.

Since many examinations (for example the South African grade 12 final mathematics examination) tend to follow a set pattern it is very susceptible to a path-smoothing approach. A limitation of the path smoothing model is that the teacher might offer explanations but rarely has the time to debate with students and hence to examine their perceptions of the explanations offered. Limitations of the model are exposed in cases when students struggle with unfamiliar problems or show a lack of insight into mathematical relationships.

Wigley (1992) offers the 'challenging model' as a better alternative. In this model the teacher presents students with a challenging problem based on a major area of the syllabus and then allows for time for engagement with the problem. Students' solution strategies form the basis for discussion and elicit an exposition of legitimate strategies. These strategies are then applied to a variety of problems. After this the different techniques are used to review previously covered content, to identify clearly what has been learnt and to show how it holds together and how it relates to other knowledge. Revision is done through encouragement and by exposing different points of view rather than by going over previously covered work in the same way. The instructional design is therefore based on discovery learning (Rittle-Johnson, 2006) alternatively referred to as unguided or minimally guided instruction (Kirschner, Sweller & Clark, 2006) (see chapter 3.8 for a discussion). An advantage of discovery learning is that it enhances both knowledge transfer and conceptual knowledge. Discovery learning however relies on extensive search through the problem text, which is very demanding on working-memory and therefore might not lead to better learning (Rittle-Johnson, 2006).

After having analysed the teaching model of Schoenfeld (1985), Wilson and Cole (1991) point out that the Schoenfeld teaching method includes the modelling of problem-solving strategies as well as a series of structured exercises performed as group or individual activities. Schoenfeld (1985) uses practice exercises to increase the skill levels of students. He also retraces the solution of previously completed problems in order to generalize the solution strategy. This method also includes a session in each period when students provide problems that the teacher attempts to solve. This serves to illustrate that problem-solving is not always a smooth error-free operation and that it is not a shame to struggle to find a solution. This model therefore includes both problem solving instruction as well as direct instruction.

Direct instruction frequently allows for students to learn correct mathematical procedures. It may even facilitate the invention of additional correct procedures and might lead to an improvement in conceptual knowledge. However, there is no evidence that this kind of instruction leads to enhanced transfer ability (Rittle-Johnson, 2006).

All of the teaching strategies thus far focus on the classroom teaching and to a lesser extent on practice strategies utilized to enhance learning in terms of long-term retention, procedural fluency and conceptual understanding. It is argued that the simplest principle of learning is based on the fact that practice of a skill improves the performance of that skill (Rohrer, Dedrick & Stershic, 2015). Many teaching strategies for mathematics learning are premised on this principle. These strategies are not all the same and differ in terms of when practice takes place, volume of practice exercises, temporal distribution of practice sessions, how problems in practice exercises are structured and if practice requires long-term retrieval of knowledge. Teaching strategies premised on some of these practice features are discussed next.

None of the teaching strategies discussed thus far in this section has both daily review and spiral testing as critical features of the strategy. The teaching strategy that will be discussed next is premised on a continuous review strategy and spiral testing where each of the tests presented to participating learners contained test items based on content that was recently covered as well as content that was covered in the previous five weeks.

Wineland and Stephens (1995) designed a study to determine if spiral testing and continuous review would improve mathematical achievement and retention of basic mathematical concepts of below average eighth and ninth grade learners. Their sample consisted of 48 below average eight grade learners that were placed in two classes and 42 below average ninth grade learners that were also randomly placed in two classes.

The continuous review strategy utilized in the research of Wineland and Stephens (1995) is the review-as-you-go method. In this method review problems from material covered in the previous five weeks was presented to the experimental group at the beginning of each class. For the control group review problems were taken exclusively from current content. The experimental group wrote a spiral test every Friday during a semester. Each of these tests consisted of five sections. Each section was based on one week's material and contained six to eight problems. Only one section was based on recently covered content, whereas the other four sections were based on content that was covered in the previous five weeks. The control classes on the other hand wrote chapter tests that covered only current content. Cumulative tests were presented to both control and experimental groups at mid-semester. No review was done for these tests. At the end of the semester both control and experimental groups wrote another cumulative test. Both control and experimental groups were given a two-day review for these tests.

Their findings indicate that the experimental groups of both the eighth and ninth grades scored significantly higher than the control groups on the cumulative mid-semester test. The experimental group of the ninth graders also scored significantly higher than the control group on the cumulative semester test. The results did not however show a significant difference between the control and experimental groups for the cumulative semester test of the eight graders.

The revision strategy of Wineland and Stephens (1995) is based on the premise that short daily reviews is more effective than once off review done immediately prior to a test or an examination. In addition, spiral testing is considered to aid retention better than review. I am of similar opinion and hence the teaching strategy employed in the present research also included a version of the review-as-you-go method as well as spiral testing. How these strategies were effected in the present research will be elaborated on at a later stage.

The incremental teaching approach of Saxon (1982) is based on cumulative continual practice and testing. In this approach topics are presented in increments to students during teaching sessions. After an increment has been presented to students it becomes part of their daily work for the remainder of the school year. In this approach it is not expected that students fully understand a concept immediately after it has been presented to them since they will be exposed to the concept many times. In the incremental approach a fundamental facet of a concept is presented and practiced for four or five problem sets before the next facet of the same concept is introduced. Subsequently both facets of the concept are practised before the next one is introduced and so on. Even after students have mastered a particular topic problems related to the topic continued to appear in every practice exercise and test. The incremental approach therefore is based on both distributed practice and interleaved practice.

Saxon (1982) contrasts his incremental approach with what he refers to as the spiral approach. In this spiral approach, review is at the end of each unit or topic where only content of that unit is reviewed. He argues that the spiral approach is disjointed since there is no connection with other completed topics. That is practice exercises following each unit includes only problems based on the content of that unit and does not include problems based on previously completed units. It should be noted that in the present study a different meaning is attached to spiral revision (see section 1.5 for a definition).

Another teaching approach that is premised on continuous practice is interleaved practice. In the interleaved approach each homework assignment consists of practice problems drawn from many previous lessons including the immediately preceding lesson. In the homework problem sets no two problems of the same kind appear consecutively. Therefore in the interleaved approach the practice of different skills is intermixed as opposed to grouped by type.

Taylor and Rohrer (2010) argue that proficiency in mathematics is a function of the ability to solve problems. In turn ability to solve problems is premised on the ability to distinguish between different kinds of problems. Once the type of problem is identified the problem must be connected with an appropriate procedure. Research has shown that interleaving mathematical practice improves both ability to discriminate between different types of problems and to associate each kind of problem with an appropriate strategy (Rohrer, Dedrick & Burgess, 2014).

The success of interleaved practice is ascribed to the fact that interleaved practice provides students with opportunities to practice associating each kind of problem with the appropriate procedure. It is argued that interleaved practice is based on two critical features (Rohrer et al, 2014). One of these features is that different kinds of problems are interleaved. This feature is thought to contribute to the development of ability to choose appropriate strategies based on problem features. The other feature is that problems of the same kind are spaced which is thought to improve retention abilities.

Rohrer et al (2014) maintain that mathematics students devote most of their practice time to blocked (massed) practice. They argue that this is a consequence of the fact that each lesson in most textbooks is followed immediately by a set of exercises based on that lesson. Problems within a blocked assignment are generally based on the same concept or procedure. Blocked practice therefore allows students to safely assume that each problem will require the same strategy as the previous problem and hence allow students to focus only on the execution of the strategy without having to associate the problem with its strategy. Research has shown that blocked or massed practice often increases performance for a short period, but this benefit sharply decreases over time (Pashler et al, 2007). Massed practice is therefore not very effective in terms of retarding forgetting of learned content. On the other hand research has shown that long-term retention is positively enhanced by distributed or spaced practice (Seabrook et al, 2005; Pashler et al, 2007; Rohrer &Taylor, 2006). Distributed practice also often result in better test scores than massed practice (Seabrook et al, 2005; Rohrer & Taylor, 2006).

In what follows I will compare features of the teaching strategies that were discussed in the previous sections in tabular form (see table 1). Since such a wide variety of features exist I will restrict the features that are compared to those that in my opinion have a high probability to enhance learning significantly. Some of the discussed teaching strategies did not pronounce on some features and hence these features are omitted for them. Note that some of the terms used in the comparison will only be discussed at length in chapter 6. See section 1.5 for an explication of the concept productive struggle.

			TEACHING STRATEGIES				
STRATEGY FEATURES		**Path smoothing strategy**	**Challenging strategy**	**Schoenfeld strategy**	**Spiral testing and Continuous review**	**Incremental Approach (Saxon)**	**Interleaved Mathematical practice**
Teaching Method		Mainly direct teaching	Mainly problem solving instruction	Both problem solving and direct instruction	Mainly direct teaching	Direct teaching	
Productive struggle		Not included	Productive struggle is a key component of strategy	included		included	included
Type(s) of practice employed		Mostly mass and repetitive practice	Practice of previously covered work not emphasized	Mostly mass and distributed practice	Mainly distributed practice	Mass, repetitive distributed, interleaved	Distributed and interleaved
Revision strategy:	**Frequency**	Regular revision	Seldom	Regular	Continuous review	Continuous review	Continuous review
	Format	Mostly teacher driven with presented tasks similar or the same. In other words, going over previously covered work in the same way	Discussion of student solution strategies is used to identify legitimate strategies which is subsequently applied to previously covered content to show how it relates to other knowledge	Solutions of previously completed problems is retraced in order to generalize solution strategies	Short daily review of completed work	Teacher driven , student seatwork and homework assignments	Homework assignments
	Is regular testing part of revision strategy?	No	No		Spiral testing	yes	No
	Does revision strategy include strategies to expose different facets of concepts in order to deepen understanding?	No	In some cases	Yes	It is not a focus of this strategy	yes	
Does the teacher seek feedback about efficacy of their teaching?		Seldom	Yes	Yes		yes	
Teacher questioning		Mostly lower level questions	Include both higher and lower order questions	Include both higher and lower order questions		Both higher and lower order	
Main focus of teaching: Procedural and/ or Conceptual knowledge		Procedural knowledge is afforded more attention	Conceptual knowledge is developed more than procedural knowledge	Focus is on the development of both procedural and conceptual knowledge	Both procedural and conceptual knowledge is developed, but procedural knowledge is focused on	Both	Both

Table 2.1: A comparison of features of teaching strategies

2.4 Linking particular features of teaching with students' learning

The main concern of my research is to determine how procedural fluency and conceptual understanding of pre-service teachers will be influenced by a teaching strategy based on spiral revision and productive practice. The aim therefore is to link particular teaching features (spiral revision and productive practice) with pre-service students' learning. Hiebert and Grouws (2007) analysed findings of a number of studies to investigate the effects of classroom mathematics teaching on students' learning. They contend that efforts to link particular teaching features with students' learning have not produced many results. Moreover, researchers who attempt to connect teaching with learning face a number of difficulties, such as a shortage of useful theories, and how to effectively document the effects of teaching on learning.

Thus, in order to document the effects of teaching on learning one has to decide which variables to study and how to measure the variables. This is not straightforward since many factors both inside and outside school influence what and how well students learn. Furthermore, since teaching methods consist of multiple features that intersect in many ways, in research it is difficult to isolate the effects of specific pedagogical features on students' learning. An additional complication is the fact that different teaching methods are required for different learning goals. For example, some teaching strategies might be better suited for retention of knowledge, whereas others might be more conducive for transfer of knowledge.

Hiebert and Grouws (2007) argue that research which focuses on connections between specific teaching features and student learning is better served by analysing relatively broad units of investigation such as the typical daily lesson or lessons. However, they maintain that each of these has advantages and disadvantages; thus researchers should think carefully about their own research before choosing the unit of study.

Also, it often happens that questions of how teaching affects learning become confused with questions of how *teachers* affect learning (Hiebert & Grouws, 2007). That is, it sometimes happens that methods of instruction are confused with teacher characteristics. The emphasis in my research falls upon how teaching methods influence learning. How years of experience and qualifications, or teaching characteristics might do so is not central to this discussion.

Hiebert and Grouws (2007) argue that one of the most firmly established links between teaching and learning is the notion of opportunity to learn. They maintain that 'opportunity to learn' is widely considered to be the single most important predictor of student achievement. Opportunity to learn is defined as circumstances that allow students to engage in and spend time on academic tasks. For the authors opportunity to learn includes inter alia consideration of students' entry knowledge, the nature and purpose of the tasks and activities, and the likelihood of engagement.

Research investigating which teaching features enhances the learning outcomes of skill efficiency and conceptual understanding has not received much attention from researchers (Hiebert & Grouws, 2007). Skill efficiency is defined as the accurate, smooth, and rapid execution of mathematical procedures. This definition does not include the flexible use of skills or the adaptation of skills to new situations.

Hiebert and Grouws (2007) in their review of the literature found no empirical studies that attempted to determine which features of teaching support skill efficiency, and which support conceptual understanding. (This is the gap in the literature I am attempting to address – at least partially). They argue that features of teaching which enhance skill efficiency include rapid pacing, many teacher-directed product-type questions, and teacher demonstration followed by substantial amounts of error-free practice. They also point out that the literature indicates that students can acquire conceptual understanding if teaching attends explicitly to conceptual underpinnings and to connections between mathematical facts, procedures and ideas. The kind of teaching that enhances conceptual understanding includes discussions about the mathematical meaning underlying procedures, and the probing of how different solution strategies are similar to and different from each other. Their findings also indicate that retention and transfer are aided by this kind of teaching.

Another teaching feature that appears to facilitate students' conceptual understanding is that students are allowed to struggle with important mathematics (Hiebert & Grouws, 2007). The word 'struggle' is used to refer to the amount of effort that is not immediately apparent, but which students expend in making sense of mathematics. Teaching strategies that facilitate productive struggle include posing problems that require making mathematical connections, allowing students to engage with the problem and subsequently working out these problems in ways that make the connections visible.

Hiebert and Grouws (2007) contend that it is not the case that only one set of teaching features aids skill learning while another set supports conceptual learning. They found better transfer of procedural skills is achieved if conceptual understanding has been the goal of teaching.

Hiebert and Grouws (2007) maintain that in the U.S.A. teaching strategies rarely include explicit attention to conceptual development and in many cases do not make allowance for students to struggle with key mathematical ideas. They contend that it is important that researchers are clear about the kinds of learning they intend to study and that teaching goals should be made explicit.

In the present research I will attempt to foster skill efficiency by applying the teaching strategy of spiral revision. I will attempt to enhance conceptual understanding by utilizing the teaching strategy of productive practice.

2.5 Conclusion

A comparison between international and national mathematics education studies reveals firstly, that current research tends to focus on the teacher and processes of teaching, and secondly, that small-scale qualitative studies predominate. The researchers in many cases were teacher educators who were attempting to determine the efficacy of the courses they offered. Small-scale-studies have the advantage that in-depth investigations of phenomena can be conducted and that individual progress can be monitored much closer. A shortcoming of these small-scale studies is that they are normally conducted over a short period, which does not allow for comprehensive study of proficiency development of pre-service teachers. Additionally, findings of qualitative studies normally are not generalizable which limit findings to the specific context and participants.

The literature indicates that there is a need for studies that investigate the mathematical knowledge and reasoning development of pre-service students over more extended periods. Furthermore, as has been indicated already, most studies have been small-scale qualitative studies. There is thus a gap in the literature in terms of large-scale quantitative or mixed methods studies. The present study is a mixed methods study done over two semesters, thus providing opportunities to study the knowledge and reasoning developments of prospective teachers a little more comprehensively than is done in the small-scale short term studies.

Curriculum reform is the preferred context for many of the South African studies. Furthermore, in pre-service teacher education for prospective *secondary* school teachers, great emphasis is placed on mathematical content knowledge and less on pedagogical content knowledge; conversely pedagogical content knowledge for prospective *primary* school teachers is emphasized while content knowledge is under-emphasized. This state of affairs might be one of the underlying reasons for the dismal mathematical performance of an overwhelming majority of South African primary school learners. There is general consensus in the mathematics education community that a well-connected deep understanding of fundamental mathematics is an absolute necessity for primary and high school teachers.

A major gap identified in South African research is that studies concerning pre-service mathematics teacher education are not well represented in the literature. This is despite the fact that many pre-service teachers in South Africa enter mathematics teacher education programmes with substantial gaps in their knowledge bases. There is thus a need for research that investigates which teaching methods are best suited to such pre-service students. Comparison studies on how one teaching approach compares to another in terms of enhancing mathematical content knowledge of teachers might assist in a major way for identifying best practice. Very few such studies however can be found in the literature.

It is the tendency of many of the mathematics courses for pre-service secondary school teachers in South Africa to teach these students more advanced mathematical content. However, research has shown that taking more advanced courses in mathematics does not necessarily translate into a deeper understanding of fundamental mathematics (Ma, 1999). An important question therefore is what kind of teaching would allow pre-service mathematics education students to develop the requisite procedural skills and a well-connected conceptual understanding. An analysis of the literature done by Hiebert and Grouws (2007) indicates that students can acquire conceptual understanding if teaching attends explicitly to conceptual underpinnings and to connections between mathematical facts, procedures and ideas. Moreover, I am of the opinion that prospective teachers would be better prepared for their teaching if courses designed for them would deal explicitly with the content they were to teach in schools.

Conclusions that might be drawn from the South African studies that performed investigations in the domain of Euclidean geometry, referred to in the foregoing section are twofold. On the one hand, it suggests that some teachers' content knowledge in the domain of Euclidean geometry is not sufficient and therefore might present a barrier to quality teaching and learning in this domain. The studies also suggest that the cognitive competencies of teachers in the domain of Euclidean geometry are below expected levels, which may in turn contribute to learner deficiencies in this area. There is thus a dire need for developing teaching strategies that broaden and deepen the content knowledge and cognitive abilities of prospective teachers in the domain of Euclidean geometry (and other school level content domains). This would facilitate the delivery of quality instruction and ultimately improve the learning of mathematics at the school level.

Based on findings of their study in the domain of matrix algebra Ndlovu and Brijlall (2015) advance similar arguments. Their findings indicate that the procedural development of the majority of participants were well ahead of their conceptual development in this domain. The authors maintain that a lack of basic school level algebra knowledge contributed to the retarded conceptual development of participants. This provides more evidence that in many instances pre-service teachers' content knowledge gaps impede progress in terms of their conceptual and procedural development.

The literature review indicates that there is a dearth of investigative studies into teaching strategies that could be employed to enhance procedural fluency and conceptual understanding of pre-service teachers. Also, very few studies examined which features of a teaching strategy would support skill efficiency and which would support conceptual understanding. This study attempts to address these gaps in the literature. It is the intention of this study to investigate whether a teaching strategy premised on spiral revision and productive practice would enhance procedural fluency and conceptual understanding of participating students. Moreover, none of the teaching strategies reviewed include a revision method that has the goal of enhancement of procedural flexibility and fluency, coupled with a strategy to cultivate conceptual knowledge and the embellishment of creative reasoning. My proposed teaching method seeks to include these design features.

Opinions in the literature are divided as to which of direct teaching method or discovery learning – is better suited for effective learning in mathematics. Researchers tend to favour one or the other, very rarely both, but in my view both teaching methods should be applied for effective learning. The intended learning outcome should determine which method would be more effective in a given teaching and learning situation. If knowledge transfer and the development of conceptual knowledge are the goals then discovery learning is best. Otherwise, if the aim is for students to learn correct mathematical procedures and to become flexible in their use of procedures then the direct method is better suited.

It is my contention that mathematics teaching strategies should include (both in instruction and student assignments) strategies to enhance long-term retention of covered content. Since research has shown that distributed or spaced practice is best suited to retard forgetting it was included as part of the teaching strategy of the present research. Mass practice on the other hand was utilized to enhance procedural mastery.

All of the teaching strategies reviewed suffered of some weakness in terms of effective student learning. Some of the teaching strategies (for example the Schoenfeld (1985) strategy) had fewer weaknesses than others. I am however of the opinion that each of the strategies offered some feature or features that has a high probability to contribute to effective learning. Moreover, I am of the opinion that many different teaching and learning contexts exist and that each of these require teaching strategies that will be suitable for it, but not necessarily for another context. In other words, I am of the opinion that there is no one size fits all teaching strategy but that features of available teaching strategies should be matched to the context. For example, in a context where there is a big difference in students' prior knowledge levels one could use direct teaching with low knowledge students and present high knowledge students with problem solving instruction. I therefore contend that for effective teaching and learning to take place teachers should be aware of the different teaching strategies available in order to employ a strategy or combine features from different teaching strategies to best suit their teaching context.

In the following chapter I will discuss the theories that inform and underpin this study.

CHAPTER 3: THEORETICAL UNDERPINNINGS

3.1 Introduction

In mathematics education, there is no consensus as to the definition of theory and its role and use in research. In this study, I will use the definitions and roles of theory as espoused by Assude, Boero, Herbst, Lerman and Radford (2008). They maintain that theory in mathematics education research when dealing with the teaching and learning of mathematics has a structural and functional perspective. Theory in the structural perspective is defined as an organized and coherent system of concepts and notions in the mathematics education field. In the functional perspective, it is considered as a system of tools that allow for speculation about some reality. When theory is used as a tool it can be used in the following ways (Assude et al,2008):

- To consider ways to enhance the teaching and learning environment including the curriculum
- Develop methodology
- Describe, interpret, explain, and justify classroom observations of student and teacher activities
- Convert practical problems into research problems
- Define different steps in the investigation of a research problem, and
- Generate knowledge

Theory can also function as an object. Aims in this perspective include the advancement of theory itself which can include testing a theory or as a means to produce new theoretical developments. In this study theory, will be used to describe, interpret and explain observed phenomena. To a lesser degree, it will also be used in theoretical developments.

Niss (2007) argues that the use of theory is essential in any discipline that perceives itself as scholarly or scientific. Lester (2005) contends that the role theory plays in the research should be situated within a research framework. A research framework is defined as a structure of the ideas that serve as a basis for a phenomenon that is to be investigated. Three types of frameworks are distinguished namely theoretical, practical and conceptual (Eisenhart, 1991). Lester (2005) contends that theory should also be considered as a framework, and as such theoretical frameworks guide research by relying on formal theories, such as cognitive structuralism, for example. Data gathered in this type of research is used to support, extend, or modify the theory.

However, the use of theoretical frameworks may be problematic in some instances. It is possible that strict adherence to a theoretical framework might cause the researcher to make the data rather than the evidence fit the theory. Another problematic use of theoretical frameworks is that in some cases data is stripped of its context and local meaning in order to serve the theory.

Practical frameworks are not informed by formal theories but by accumulated practice knowledge of practitioners, the findings of previous research or even public opinion (Scriven, 1986). Some limitations of this framework are that findings tend to be only locally generalizable and that too much emphasis is placed on local participants' perspectives (Lester, 2005). Researchers who adopt a conceptual framework normally use different theories and practitioner knowledge which are relevant to the study. This means relevant theories and perspectives are used to answer the research questions. In this framework both explanation and justification is important. Researchers utilizing this framework do not only explain phenomena, but also justify why they are doing the research in a particular way and why their explanations and interpretations are reasonable.

This study is concerned with the use of a teaching strategy to investigate whether it will increase the cognitive competencies and broaden and deepen mathematical knowledge of research participants in specified domains. This implies that a framework is required which will facilitate the achievement of such objectives – theoretically and pragmatically. I am of the opinion that in order to deal more effectively and perhaps more comprehensively with research in mathematics education, one has to utilize all tools available and therefore multiple perspectives will be employed in an attempt to address the research problem. In some cases, I refer to theories that are well established explanatory frameworks, whereas in other cases I rely on local theories or practitioner theories. Thus, the framework for this study is based on pragmatic concerns and can be described as a bricolage of theories and perspectives as espoused by Lester (2005) and Cobb (2007).

Descriptions of some of the theories used in the study are discussed next. However, not all the theories utilized will be discussed in this chapter since for practical reasons some theories are explored in other chapters where their relevance will be contextualized.

3.2 Human Memory System

Memory plays a crucial role not only in mathematical learning, but in almost all forms of learning. Knowledge of memory structures and processes would therefore greatly aid in understanding the learning process in mathematics. Theories of human memory usually include both the architecture of the memory system and the processes operating within that system. 'Architecture' refers to the components of the memory system and the organisation of these components whereas 'processes' refers to the activities occurring within the memory system.

A multi-store model of the memory system as proposed by Atkinson and Shiffrin (1968) enjoys widespread support among researchers. According to this model human memory is composed of three interconnected memory stores namely sensory memory (SM), short term memory (STM) store and long-term memory (LTM) store. Information from our senses is initially stored in sensory memory for a very short period in the same form that it was processed by our senses (for example sound or visual images). Subsequently a decision is made as to which parts of the information are important enough to pay attention to. The part to which attention is directed is transferred to the working memory where it is processed further.

Although the terms short-term memory and working memory (WM) are sometimes used interchangeably I will follow the distinction of Baddeley (2012). In this distinction, short-term memory refers to the simple temporary storage of information, while working memory refers to a limited capacity system that is capable of briefly storing and manipulating information required in the performance of complex cognition. The WM consists of four components namely central executive, phonological loop, visuo-spatial sketchpad and the episodic buffer (Repovs & Baddeley, 2006).

The central executive component is an attentional control system of limited processing capacity that has the role of controlling action. It is thought to have the capacity to focus attention on:

- relevant information and processes inhibiting irrelevant processes and information;
- switching (shifting) attention between tasks;
- planning, sub-tasks to achieve a goal;
- dividing attention between two or more tasks;
- updating and checking the contents of working memory;
- coding representations in working memory for time and place of appearance;
- controlling access to long-term memory (Baddeley, 2007; Smith & Jonides, 1999).

The phonological loop deals with auditory, primarily speech-based information and consists of two components namely, a temporary speech-related store and a sub-vocal articulatory rehearsal process.

The visuo-spatial sketchpad is responsible for the temporary storage and manipulation of spatial and visual information. As a consequence of the fact that the initial three-component model of working memory cannot account for the ways in which sub-systems work together and how these systems interface with long-term memory a fourth component was proposed (Baddeley, 2000). This fourth component known as ‘the episodic buffer’ is assumed to be a temporary store of limited capacity that is capable of combining a range of different storage dimensions thus allowing it to collate information from the visuo-spatial, phonological loop and the long-term memory.

Long-term memory refers to the component of memory where information is stored over extended periods. Two types of long-term memory can be distinguished namely declarative (explicit) and implicit memory. Declarative memories are memories that are available in consciousness. Declarative memory is further divided into episodic memory and semantic memory. Episodic memory is the memory of past experiences that occurred at a particular time and place (Tulving, 2002). Semantic memory on the other hand involves the storage and retrieval of factual knowledge about the world (Griffiths, Dickinson & Clayton, 1999). Implicit memories are those that are mostly unconscious and include memories of body movement and memories of procedures such as how to plait hair.

3.3 Worry and Mathematical Performance

We know that emotion can affect learning in both positive and negative ways (Kort et al, 2001; Goleman, 1995; Sylwester, 1994). Research has shown that mathematical anxiety has a negative effect on mathematical performance (Ashcraft & Krause, 2007). Anxiety is defined as an aversive emotional and motivational state that occurs in threatening circumstances. State anxiety is the currently-experienced level of anxiety (Eysenck, Derakshan, Santos & Calvo, 2007). State anxiety is a state in which a person is unable to initiate a clear pattern of behaviour to remove or alter the event or object that is threatening an existing goal (Power & Dalgleish, 1997). Worry is the component of state anxiety that is responsible for effects of anxiety on performance effectiveness and efficiency. Worry is activated in stressful situations such as in test conditions (Eysenck, 1992).

Processing efficiency theory which is a cognitive performance theory forms the basis of attentional control theory. The concepts of cognitive effectiveness and efficiency form the crux of processing efficiency theory (Eysenck & Calvo, 1992). Cognitive effectiveness is understood to refer to the quality of task performance as measured by response accuracy, for example. Cognitive efficiency on the other hand is said to refer to the relationship between the effectiveness of performance and the effort or mental resources spent in task performance. Efficiency is said to decrease as more resources are spent in task performance to realize a given performance level. According to this theory worry has a significantly greater negative effect on processing efficiency than on performance effectiveness.

According to processing efficiency theory worry has two effects on cognitive performance. On the one hand the worrisome thoughts are believed to preoccupy some of the limited attentional resources of working memory and hence fewer of the attentional resources are available for concurrent task processing. In other words, worry acts like a resource demanding secondary task (Ashcraft & Krause, 2007). On the other hand, worry might increase motivation to minimize the aversive anxiety state by promoting enhanced effort and use of auxiliary processing resources and strategies. It is an assumption of processing efficiency theory that the main effects of worry are on the central executive component of the working memory. The processing efficiency theory however has some major theoretical limitations. For example, the theory does not specify which of the central executive functions are impaired by anxiety and does not consider circumstances in which anxious persons might outperform non-anxious persons (Eysenck et al, 2007).

The attentional control theory attempts to build on the strengths of the efficiency theory and to address its limitations (Eysenck et al, 2007). This theory is more precise about the effects of anxiety on the functioning of the central executive. The revised theory emphasises that anxiety impairs processing efficiency more than it does performance effectiveness. Furthermore, anxiety decreases the influence of the goal-directed attentional system and increases the influence of the stimulus-driven attentional system. Attentional control is therefore reduced and consequently the inhibiting and shifting functions of the central executive are negatively affected (Eysenck et al, 2007). A person who suffers from the emotion of worry therefore will focus on the worrisome thoughts and hence less of the focus will be on the task at hand. Important information in the task might therefore be missed since full attention is not focused on the task. For example, a student who suffers from the emotion of worry and who is busy with a question in a test might focus so much on worrying about not achieving that less attention is given to the information provided in the task. Hence not all the relevant information is processed. The student may not be able to sufficiently inhibit the worrying thoughts that emerge in the working memory to be able to shift focus to the task at hand. The student may then have the sensation that the mind cannot access the relevant information in the long-term memory ('going blank') to deal with the task at hand since the attentional system is now attempting to focus on two tasks simultaneously.

3.4 Cognitive Load Theory

The learning and instructional framework for this study is based on a cognitive load perspective (Sweller, 1994). The assumption in this theory is that learning and problem-solving difficulty is artificial in the sense that it can be manipulated by instructional design. Cognitive load refers to the amount of mental effort being used in the working memory. Sweller (1988) contends that increased amounts of information in working memory increase cognitive load, and that learning and problem-solving is premised on both selective attention and cognitive load.

Sweller (1994) argues that schema acquisition and the automatization of learned procedures are two essential mechanisms in the learning process. In cognitive load theory mechanisms of learning determine which features of the material make it hard to learn. The theory is based on the assumption that gaining of knowledge and cognition *based* on this knowledge are heavily dependent on schema acquisition. The notion of schemata is thought to offer an explanation for a major part of learning-mediated cognitive performance. Sweller (1988) argues that experts in a domain have more domain specific knowledge (in the form of schemata) than novices in the domain.

Schemata are cognitive constructs that organize elements of information according to the manner in which they will be utilized. It is generally accepted that newly presented information is not internalized in the exact form that it is presented; instead, new knowledge is altered so that it fits in with current knowledge. It is argued that knowledge of subject matter is organized into schemata and it is these schemata which determine how new information is dealt with.

Sweller (1994) contends that people utilize schemata to deal with mathematical problems. These schemata allow for problems to be categorized according to the solution procedure. For example students who have been exposed to algebra will not only know how to solve a specific linear equation such as $2x + 5 = 7$, but will know the solution procedure for this category of problem and hence would be able to solve all problems of the form $ax \pm b = c, \quad \forall a, b, c \in \mathbb{R}$. These schemata therefore reduce the amount of mental effort required for solving such problems and allow people to potentially solve an infinite variety of such problems.

Schemata are not acquired in an all or nothing manner, but are assimilated gradually over a period of time. Consequently, when a student is exposed to new knowledge, the ability to use this knowledge is initially severely constrained since the schema has not been fully developed. Cognitive processing of information can either be controlled or automatic (Schneider & Shiffrin, 1977). Controlled cognition is said to occur when information is consciously attended to. In other words, the information is either not in the long-term memory or is not well-established in the long-term memory and therefore has to be processed in the working memory.

In Cognitive Psychology automatization is defined as the practice of a skill or habit to the point of its having become routine so that little if any conscious effort or direction is required. In other words, this occurs if the thought processes involved in the skill have been moved to the long-term memory. When a complex mental skill is first acquired it can only be utilized with considerable cognitive effort; however, over time and with enough practice the skill may become automated (Sweller, 1994). Consequently, if mundane procedural elements of a task have been practised to the extent that it have become automated, this would free cognitive capacity for more creative reasoning and the application of prior knowledge in unfamiliar situations. Moreover, if skills operate under automatic processing then cognitive load will be reduced. This I think is what teachers of mathematics would ideally want to achieve in mathematics instruction.

It is an accepted fact that a function of learning is to store information in long-term memory (Atkinson & Shiffrin, 1968). A function of learning according to cognitive load theory is to store automated schemata in long-term memory. As stated in the previous section, working memory has a limited capacity and duration; hence the amount of information that can be processed in the working memory is limited. This limitation can affect learning negatively so a function of instruction and learning should be to find ways of reducing working memory load. Schema acquisition and automatization have precisely this effect of reducing working memory load.

In cognitive load theory, an element is defined as any material that is to be learned. If in order to learn material it is required that mental connections have to be made between many other elements, then the material is said to have high element interactivity and is perceived to be harder to learn. Conversely, if elements of a task can be learned without making other mental connections then it is said to have low element interactivity and is perceived to be easier to learn. Mathematical tasks rarely have low element interactivity.

Further to this, total cognitive load is the sum of intrinsic and extraneous cognitive load (Sweller, 1994). Extraneous cognitive load is imposed by instructional methods whereas intrinsic cognitive load is determined by element interactivity. If a content area has a high number of interacting elements it is associated with a high intrinsic cognitive load. Conversely if material has a low element interactivity it is thought to have a low cognitive load. Instructors have no control over intrinsic cognition since it is dependent only on the element interactivity of the material to be learned. It is argued that if people have acquired schemata of a content area with high element interactivity, then the content is understood. If these schemata are automated, then the material is understood very well.

As noted earlier, learning and problem-solving is premised on selective attention and cognitive load. The knowledge levels of learners in a specified domain may also impact on cognitive load. If learners have low prior knowledge in a domain, then cognitive load will be high in the learning process. Conversely learners with high levels of prior knowledge will have a low cognitive load in the learning process. Based on cognitive load theory it is argued that instructional tasks that require the processing of many new elements of information simultaneously will not be very effective and the ability to use the information in related tasks will be severely constrained. Instructional designers should therefore always bear in mind the limitations of our cognitive architecture when designing tasks for instruction.

3.5 Procedural and Conceptual knowledge

Contributing to this exploration of cognitive mental patterning, De Jong and Ferguson-Hessler (1996) have devised a conceptual framework for examining the concept of knowledge. They contend that the concept of knowledge is dominant in research on teaching and learning. They maintain that the knowledge base of people is made up of different types of knowledge such as conceptual knowledge, procedural knowledge, domain specific knowledge, etc. Knowledge is also characterized by different qualities such as the level of knowledge, generality of knowledge, level of automatization of knowledge, structure of knowledge, etc. Those authors argue that the two dimensions of knowledge namely type of knowledge and quality of knowledge can be utilized to differentiate between novice and expert task performance.

In that framework, the description of knowledge is approached from the perspective of knowledge-in-use. This implies that task performance forms the basis for the identification of germane facets of knowledge. De Jong and Ferguson-Hessler (1996) argue that the knowledge-in-use perspective allows us to determine if a knowledge base is adequate for the solution of a given type of problem at a given level. They maintain that novices in a domain do not possess the requisite organized deep-level knowledge to deal effectively with tasks in the domain, whereas experts do indeed possess the indispensable knowledge components at the correct level.

Knowledge is referred to as deep-level knowledge if it is firmly entrenched in a person's knowledge base and external information has been transformed to basic concepts or procedures in a given domain (De Jong & Ferguson-Hessler, 1996). The authors argue that this implies that the knowledge has been comprehensively processed, organized and stored in memory in a way that makes it useful for application and task performance. Knowledge that is categorized as deep is, as a rule, associated with understanding and abstraction. Conversely, surface level knowledge is knowledge that is stored in the same way that it was received from an external source and is not connected in meaningful ways to other knowledge. Since the knowledge is not connected meaningfully to other knowledge it is less useful and is therefore normally associated with rote learning.

It is generally accepted in the mathematics education community that both procedural and conceptual knowledge is necessary for the effective learning of mathematics. There is however no consensus on the respective roles that procedural and conceptual knowledge play in student learning. It is argued that although an understanding of conceptual and procedural knowledge might provide significant insights into mathematical learning and performance, the relationship between these forms of knowledge is not yet well understood (Hiebert & Lefevre, 1986).

Much has been written about procedural and conceptual knowledge; but the ensuing discussion concerning these two knowledge types will be informed by the theoretical arguments of Hiebert and Lefevre (1986) (unless otherwise indicated). Conceptual knowledge is characterized as knowledge that is rich in relationships. Conceptual knowledge is developed by establishing cognitive links between different pieces of information. This linking process is possible between pieces of information that are already stored in the memory, or between an existing piece of knowledge and one that is newly learned. The result of the linking process is that the new knowledge is assimilated into appropriate knowledge structures and hence becomes part of an existing network. In other words, it becomes part of the schemata in long-term memory.

The term abstract is used to determine the level of connection between pieces of information; it refers to the degree to which a unit of knowledge is tied to a specific context. Abstraction is said to increase as knowledge is freed from a specific context. If the relationship is established at the same level of abstraction or at a less abstract level than the level at which the original information was presented, then the relationship is said to be at the primary level. If, however, the relationship between pieces of information is established at a higher abstract level than the pieces of information they connect then the relationship is said to be at the reflective level. Such relationships are less bonded to specific contexts. Building the cognitive connections in this case requires that one reflects on the information being connected and consequently more of the mathematical terrain becomes 'visible'. The reflective level is therefore perceived to be at a higher level than the primary level.

It is argued that procedural knowledge consists of two distinct parts (Hiebert & Lefevre, 1986). One part is said to consist of the formal language or symbol representation system of mathematics, while the other part consists of the algorithms for completing mathematical tasks. The first part requires that students and teachers of mathematics are familiar with the symbols used to represent mathematical ideas and of the syntactic rules for writing these symbols in an acceptable form. Knowledge of these symbols and syntax on its own however does not necessarily imply understanding of meaning, but only an awareness of surface features. For an understanding of meaning, conceptual knowledge is required.

Algorithms are said to be step-by-step instructions that prescribe how to solve mathematical tasks and are normally executed in a predetermined linear sequence. In some cases procedures are embedded in other procedures as sub-procedures. Together these sub-procedures form a super-procedure. Procedures operate on two types of objects namely standard written symbols (e.g., -5, $\div$, $\sqrt[3]{\ }$) or non-symbolic objects such as proof structure. At school level the objects in the majority of cases are symbols. Hiebert and Lefevre (1986) argue that in most school tasks the aim is to transform a given symbol form to an answer form by executing a sequence of symbol manipulation rules. They are therefore of the opinion that most of the procedural knowledge that school students possess are chains of prescriptions for manipulating mathematical symbols.

The description and definition of procedural and conceptual knowledge of Hiebert and Lefevre (1986) however has drawn some criticism. Star (2005) maintains that the definitions of procedural and conceptual knowledge of Hiebert and Lefevre (1986) do not fully account for knowledge type and knowledge quality. An assumption that can be drawn from their definition of conceptual knowledge is that conceptual knowledge is always deep and rich in relationships. Star (2005) however maintains that cognitive connections of conceptual knowledge may be limited and superficial or they may be extensive and deep. He maintains that a learner's initial conceptual knowledge is generally limited and superficial, but over time may develop, gain depth and establish more connections.

The definition for procedural knowledge (as proposed by Hiebert & Lefevre (1986)) seems to indicate that procedural knowledge is superficial and is not rich in connections (Star, 2005). This in turn seems to lead to a dichotomous state as far as students are concerned – that a student either knows how to do a procedure or does not know. However, this is not true in all cases as a procedure can be known on a very superficial level as a chronological list of steps or it can be known on a more abstract level; it may include planning knowledge in its cognitive representation. It is possible that a student possesses knowledge about the purpose of each part of a procedure and how these parts fit together. This is known as teleological semantics (Van Lehn & Brown, 1980). Planning knowledge of a procedure includes knowledge of the order of steps, the goals and sub-goals of steps, the type of situations in which the procedure might be used, possible constraints imposed upon the procedure by the given situation and any heuristics which might be inherent in the situation. Star (2005) contends that if a student possesses such procedural knowledge then the knowledge is both deep and connected.

Star (2005) argues that skilled problem-solvers in mathematics are also flexible in their use of known procedures. A student who does not possess such flexible procedural knowledge will not always be able to solve unfamiliar problems where the solution requires the student to use known procedural knowledge. The student will also not be able to produce a maximally efficient solution in the absence of such flexible procedural knowledge. One outcome of such flexibility is that students who possess such knowledge will have the ability to generate maximally efficient solutions for known and even sometimes unknown problem situations. Star (2005) contends that flexible procedural knowledge is deep procedural knowledge and will enable a student to use mathematical procedures which would best fit a provided known or novel problem situation.

Procedural knowledge on its own – even if it is well developed – does not always imply understanding. Meaningful mathematical learning has occurred if cognitive relationships between units of knowledge are created or recognized (Hiebert & Lefevre, 1986). Thus, the converse of meaningful learning is rote learning. Rote learning is defined as learning where cognitive relationships between units of knowledge are not established or are not well established. Since the knowledge is not linked with other knowledge, the holder of such knowledge will not be able to generalize to other situations where the knowledge is required. Knowledge gained through rote learning can only be accessed and applied in contexts that are very similar to the original (Hiebert & Lefevre, 1986).

It is argued that it is highly unlikely that conceptual and procedural knowledge are possessed as entirely independent knowledge systems (Hiebert & Lefevre, 1986). Even in cases where inefficient mathematical learning has taken place, some connections will be established between conceptual and procedural knowledge. Fully developed mathematical knowledge includes significant and fundamental cognitive connections between conceptual and procedural knowledge. Procedures are said to access and act on conceptual knowledge translating it into something observable. Procedures therefore inform on the state of conceptual knowledge.

The creation of appropriate cognitive connections between conceptual and procedural knowledge is thought to contribute to efficient memory storage and successful retrieval of procedures in applicable circumstances. Hiebert and Lefevre (1986) advance several reasons why procedures are stored and retrieved more successfully when they are connected to conceptual knowledge. If procedures are connected through conceptual knowledge they become part of a network of information that is held together by semantic relationships which do not deteriorate easily, since memory endures longer for meaningful relationships. As the procedures are part of a knowledge network, many alternative cognitive links can be used to access and retrieve them. It is thought that conceptual knowledge also has an executive control function in that it is utilized to monitor not only the selection and use of a procedure, but also to determine the reasonableness of the procedural outcome.

Conceptual knowledge only becomes useful in solving mathematical tasks when it is converted into an appropriate form. When a student is introduced to a new mathematical topic, initially the student does not know the procedures by which to solve mathematical problems that are based on the topic. In the absence of knowledge of procedures, known problems are solved by applying mathematical facts and concepts in an arduous way. However, as the student becomes exposed to similar problems and these are solved repeatedly, conceptual knowledge is gradually transformed into procedural knowledge. In this way knowledge that was initially conceptual can become procedural. Hiebert and Lefevre (1986) argue that this is a very important process in the learning of mathematics since well-known procedures can reduce the cognitive effort required and this cultivates a capacity to solve more complex problems. They maintain that automated procedures release cognitive resources which can, for example, be utilized to look for relationships between novel aspects of problems or where relevant conceptual knowledge can be applied.

Hiebert and Lefevre (1986) argue that sound mathematical knowledge includes significant and fundamental cognitive links between procedural and conceptual knowledge. They maintain that for students to be competent in mathematics, they need to possess both procedural and conceptual knowledge as well as cognitive links between these two essential knowledge components. Understanding in mathematics is said to occur when new mathematical information is appropriately connected to the existing knowledge networks.

3.6 Developmental relations between Conceptual and Procedural knowledge

The development of procedural and conceptual knowledge is central in the learning of mathematics. An enduring debate in the mathematics education community however is centered around which of these knowledge types should enjoy primacy during instruction. Rittle-Johnson et al (2015) maintain that the debate is based on the fact that people have different beliefs about the cognitive development of conceptual and procedural knowledge. Consequently, many of the theories concerning the development of conceptual and procedural knowledge in mathematics have focused on which type of knowledge develops first in a given mathematical domain (Rittle-Johnson, Siegler & Alibali, 2001).

It is generally accepted that conceptual knowledge underpins and in many instances, leads to the development of procedural knowledge (Rittle-Johnson, Schneider & Star, 2015). Conceptual understanding is defined as the comprehension and connection of concepts, operations and relations. An important question therefore is whether it is always the case that conceptual understanding should be achieved first to establish a foundation from which procedural skill can develop? In other words, if conceptual knowledge is focused on first during instruction does this lead to better learning of mathematics or is the converse true? Rittle-Johnson et al (2001) argue that the debate regarding which type of knowledge develops first might prevent us from gaining a better understanding of both the gradual development of each knowledge type and the relations between the two knowledge types.

There are four theoretical viewpoints concerning relations between procedural and conceptual knowledge (Schneider, Rittle-Johnson & Star, 2011). Each of these theories is supported by some empirical evidence, but is at the same time contradicted by other evidence. According to the concepts-first theory, students first acquire conceptual knowledge in a domain and subsequently use this conceptual knowledge to generate procedures to solve problems. In the procedures-first theory it is argued that students first learn procedures for solving problems in a domain and then gradually develop conceptual knowledge through repeated solving of problems (Rittle-Johnson et al, 2001).

My support lies with the iterative model as proposed by Rittle-Johnson, et al (2001). A major premise of this theory is that either procedural or conceptual knowledge may develop first, but one type of knowledge does not as a rule develop before the other. The authors argue that it is often the case that a particular type of knowledge is incomplete. More specifically, one type of knowledge might be better developed at a particular point in time, but this does not imply that the other type of knowledge is totally absent. Generally, initial knowledge in any domain is very limited. The contention is that levels of prior knowledge in a domain determine which type of knowledge will emerge first, and set the learning process in motion.

It is argued that conceptual knowledge is general and abstract and resultantly can be generalized to new problem types (Schneider et al, 2011). Procedural knowledge is thought to be more tied to a specific problem type because procedures are normally practised with a specific type of problem. Thus, it is argued that if students have some prior knowledge of material to be learned then conceptual knowledge might play a bigger role in the development of procedural knowledge than vice versa. Students with little prior knowledge in a domain tend to develop procedural knowledge first which subsequently facilitates the development of conceptual knowledge. For example, when students are introduced to the topic of subtraction of real numbers they often learn the procedure first and then through exposure to different types of problems in this domain develop the conceptual understanding. According to the iterative model therefore, improved conceptual knowledge results in improved procedural knowledge. Improved procedural knowledge in turn leads to improved conceptual knowledge, which then leads to improved procedural knowledge and so on.

As mentioned previously when students are first introduced to a mathematical topic their initial knowledge of the topic tends to be very limited. Consequently, it is often the case that students know a little about a topic, but do not fully understand the topic. Conceptual knowledge is indispensable for the construction, selection and correct application of procedures. Yet practising known procedures is thought to help students develop and deepen understanding of concepts. The main argument therefore is that both kinds of knowledge are required for effective mathematical learning and that over time each type of knowledge is required to strengthen the other. Additionally, if conceptual knowledge regarding a mathematical topic has not as yet reached a mature state, then a student who possesses such knowledge will tend to use the conceptual knowledge laboriously when constructing a solution procedure to a given mathematical task. Instruction in mathematics should therefore be directed towards developing both procedural and conceptual knowledge.

A student with low expertise in a mathematical domain usually has fragmented knowledge and is without the ability to 'see' how procedures and concepts relate to each other in the domain. As expertise in a domain increases so the ability to integrate the conceptual and procedural pieces of knowledge into a coherent knowledge structure increases (Baroody & Dowker, 2003; Linn, 2006; Schneider & Stern, 2009; Schneider et al, 2011).

In order to determine relations between conceptual and procedural knowledge, each must be assessed independently using multiple measures (Rittle-Johnson, Schneider & Star, 2015). Tasks used to measure conceptual knowledge are usually relatively unfamiliar to the research participants. This is so that participants are forced to produce a possible solution from their conceptual knowledge as opposed to applying a known procedure. Hence conceptual knowledge measures include tasks such as evaluating the correctness of a procedure or an example; and providing definitions and explanations of concepts (Crooks & Alibali, 2014; Rittle-Johnson & Schneider, 2015).

Tasks used to measure procedural knowledge are usually familiar to participants and involve problem types that participants have solved before. Measurement of procedural knowledge normally involves solving problems where the outcome measure is usually the correctness of the solutions provided. In some cases, procedural tasks include near transfer problems. Near transfer problems include problem types that have an unfamiliar feature and which require the student to recognise that a known procedure is relevant, or that a slight modification of a known procedure can be utilized to deal with the unfamiliar feature. Rittle-Johnson et al (2015) maintain that continuous knowledge measures should be used to measure both types of knowledge since these types of measures capture gradual changes in knowledge as well as depth and breadth of knowledge.

3.7 Students' level of expertise

In order to design instructional strategies that are effective and that satisfy the needs of students it is important to determine levels of student expertise rapidly (Kalyuga, 2007). Kalyuga (2007) argues that knowledge structures in long-term memory regulate our ability to deal with knowledge-based cognitive activities. Cognition is negatively affected if knowledge structures in long-term memory are not well developed for a specific class of tasks. It is therefore imperative to be able to determine levels of knowledge acquisition for a given class of tasks since these levels are an indicator of level of expertise for that class of tasks. Hence level of expertise can be determined by determining level of acquisition of knowledge structures.

In the present study levels of acquisition of procedural and conceptual knowledge (and concomitant reasoning abilities based on these knowledge structures) for a specific class of mathematical tasks are therefore perceived to be a measure of level of expertise.

The first step diagnostic method is a method that is utilized to determine students' level of expertise (Blessing & Anderson, 1996; Sweller et al, 1983). This method involves presenting learners with selected mathematical tasks for a limited time and then requesting them to indicate their first step towards a solution for each task. The method is based on the idea that well-learned higher-level solution procedures would allow more experienced learners to generate advance steps of the solution quickly and to skip some intermediate steps (Blessing & Anderson, 1996; Sweller et al, 1983). These well-learned procedures are thought to be automated in the majority of cases. Different (presented) first steps are therefore perceived to be indicators of different levels of expertise. The first-step diagnostic method was validated in a series of studies in algebra, coordinate geometry, and arithmetic word problems. Although the first-step diagnostic method was not employed in my study I am of the opinion that should participating students generate advance steps and skip some intermediate steps during testing situations that these can then be seen as indicators of expertise level. I am also of the view that if students perform some advance steps in a single step, then it is also an indication of an advanced expertise level.

It is generally accepted that problem-solving in mathematics requires both written steps and complex cognition. Cognition in turn requires the establishment of cognitive connections between different pieces of knowledge. The connections can be made between pieces of knowledge that are already established in memory or they can be made between information that is newly presented and prior knowledge. The written steps are the external manifestation of the internal cognitive processes.

The difference between '*the next step*' and a *cognitive connection* is that in order to write the next step in a mathematical argument (or solution procedure) you first need some connection with the previous step. This connection can take various forms such as a definition and a previously established result. I think it is commonly understood that the next step means the next written statement in a mathematical argument. The production of this next step in the majority of cases requires the bringing together of more than one idea (cognitive connections). It is a requirement in mathematical arguments that the written step is premised on a coherent chain of reasoning. In order for the argument to make sense step-by-step and as a coherent whole, connections need to be made in a specific order. Violation of this logical order will in most cases lead either to an inconclusive argument (or solution) or to an incoherent argument (or solution).

A next step that violates a mathematical principle or that does not follow logically from the previous step is perceived to be the result of incorrect reasoning or reasoning that is based on a misconception. The manner in which steps are presented in an argument therefore provides instructors of mathematics with insight into the level of development of mathematical reasoning in students, and into any misconceptions held by the student.

A distinction is made between errors and slips in terms of mathematical mistakes (Olivier, 1989; Brodie, 2014). Slips are defined to be mistakes that are easily corrected when they are pointed out. Slips are careless mistakes made by both experts and novices. Errors on the other hand are thought to arise from misconceptions. Misconceptions are defined to be conceptual structures constructed by learners that make sense in relation to their current knowledge, but that are not aligned with conventional mathematical knowledge (Brodie, 2014). The argument I want to present here however does not concern what happens when mistakes are presented in a solution procedure but rather what inferences can be drawn from the way steps are presented in a solution procedure or mathematical argument.

The majority of mathematical tasks require that more than one connection be made between concepts held in memory. As the different ideas are processed using conceptual knowledge, it becomes exceedingly difficult to hold the results of all processed thoughts in working memory (because of its limited capacity). Hence students who are newly introduced to a mathematical topic will tend to write down the conclusion of each of the constituent parts of their reasoning process. That is, they will tend to write many more steps than someone who has more experience with the topic.

A student who has had more exposure to and practice in a specific class of problem will have automated many of the reasoning processes involved in problem-solving. Certain parts of the reasoning process will not be brought into conscious focus (in working memory) since it is automated and will manifest itself in the solution procedure as redundant steps. That is, since it is no longer part of conscious reasoning it will not be a written step in the solution procedure. It is not written as a step since it is automated and so might be perceived to be obvious. Moreover, it is not included since it is not consciously attended to, and therefore it may be perceived to be included in the part that is consciously attended to. Steps that are normally performed separately in different lines – as a result of automatization – might also be performed simultaneously in one line in the argument. I posit therefore that if intermediate steps in a solution procedure are skipped or more than one step is performed simultaneously, then this is an indication of a higher level of expertise in a specific class of tasks. The more advanced the student's level of reasoning the more some intermediate steps in a solution become redundant. Conversely, one might infer a lower level of expertise if a student laboriously performs all steps of a solution procedure. I will utilize proof of a mathematical statement in the context of elementary number theory in order to further flesh out the above arguments.

A mathematical statement and its contra-positive are logically equivalent. Therefore, to prove a mathematical statement by contra-position one first writes the contra-positive of the given statement and then proves the contra-positive statement by a direct proof.

The method of proof by contraposition therefore consists of the following steps:

(i) Express the statement to be proved in the form: $\forall\, x \in D, if\ P(x) then\ Q(x)$;

(ii) Rewrite this statement in the contra-positive form:
$\forall\, x \in D, if\ \sim Q(x)\ then\ \sim P(x)$.

(iii) Prove the contra-positive by a direct proof:
(a) Suppose x is a particular but is an arbitrarily chosen element of D such that $Q(x)$ is false;
(b) Show that $P(x)$ is false.

If one wants to prove a statement by contraposition therefore, the first step is to write the contra-positive of the given statement. This requires one to *first* make the connection between contra-position and its definition. In other words, it requires one to know that one must take the negation of both parts of the universal statement and then write it in such a way that one starts with the consequent and ends with the antecedent part of the given statement. That is, the step requires one to write the contra-positive, but first one needs to make the connection between contraposition and its definition. This is the first connection. It is important to note that one does not write down what contra-positive means, but that one applies the meaning of contra-position to the given statement (this is the second connection: applying the meaning). Hence in order to write the next step one needs to make more than one connection in some cases.

The second step requires one to use the direct method of proof. Again, this requires one to make a connection between the contra-positive statement in the first step and what one is supposed to *do* next (the second step). In other words, one then has to use the antecedent of the contra-positive statement and to apply it in order to deduce the consequent of the statement by using prior knowledge like factorization, definitions, etc. This requires one to make another connection between the given statement and the required prior knowledge. That is, one needs to connect the statement with something one has learned in the past.

In the following sub-section I present an example to illustrate the foregoing arguments.

Prove the following by contraposition:

For all integers a, b and c, if $a \nmid bc$ *then* $a \nmid b$*.*

(Since step1 has already been done in the problem statement we start with step 2 in the proof)

Proof:

Steps	**Connections**
Step 2: Write the statement in the contra-positive: $For\ all\ integers\ a, b\ and\ c, if\ a/b\ then\ a/bc$ (i)	Connection 1: What is contraposition? Connection 2: Application of contraposition to the specific statement
Step 3: Prove the contra-positive by a direct proof: $Suppose\ a, b\ and\ c\ are\ arbitrarily\ chosen\ integers\ such\ that\ a/b,$ (ii) $then\ b = ak\ for\ some\ integer\ k$ (iii) $bc = akc$ – multiplying by c (iv) $bc = a(kc)$ – Now kc is an integer since the product of integers is an integer (v) Therefore a/bc (vi)	Connection 3: Start with the antecedent of the contra-positive statement i.e. make a connection between direct proof and its first step. Connection 4: Make a connection with definition of divisibility. Connection 5: Connecting the antecedent to the consequent Connection 6: Connection with properties of integers Connection 7: making a connection with definition of divisibility again, but this time in the reversed direction

In the process of proving, some parts of the above proof may be skipped by experts. For example, in (ii) the supposition that $a, b\ and\ c$ are arbitrarily chosen is skipped in many cases since it is assumed that this known. Similarly, in (v), the part that indicates that kc is an integer might sometimes be omitted since it is assumed obvious. It should be noted that much of the reasoning that takes place is not made visible but can be assumed on the basis of the written argument presented. This is also a common occurrence in problem solving in mathematics.

Hence it is my argument that level of expertise can be inferred from redundant steps and how some ideas are used together in presented steps.

3.8 Teaching strategy

It is generally agreed that both conceptual and procedural knowledge are required for mathematical competence. It was the intention of this research to determine whether a teaching strategy based on spiral revision and productive practice would improve the mathematical proficiency of research participants.

Mathematical proficiency is the term used by Kilpatrick and his colleagues (National Research Council, 2001) to capture what it means to learn mathematics successfully (see 1.3). Kilpatrick and colleagues (National Research Council, 2001) are of the opinion that the five strands of mathematical proficiency are interwoven and interdependent. This assertion has major implications for how successful learning of mathematics is viewed. The following quote provides more insight as to how important they view this statement to be:

> *The most important observation we make about these five strands is that they are interwoven and interdependent. This observation has implications for how students acquire mathematical proficiency, how teachers develop that proficiency in their students, and how teachers are educated to achieve that goal. (National Research Council, 2001, pg 5)*

The teaching strategy employed in this research is based on the above description of mathematical proficiency. This research however only focused on the strands conceptual understanding and procedural fluency. The motivation for this decision is based on the following arguments.

The argument has been presented that the most important aspect of the five strands of mathematical proficiency is that they are interwoven and interdependent. This implies that development of any strand will have a developmental effect on the other strands as well. Since the strands are believed to be interwoven it would be a very difficult exercise to investigate how each of these strands is influenced individually by a teaching strategy. Furthermore, research has shown that development of the strands conceptual understanding and procedural fluency are major contributors to the development of mathematical proficiency (Hiebert & Grouws, 2007). I therefore chose to focus only on the strands conceptual understanding and procedural fluency in order not to get distracted by too much data that is interconnected.

Spiral revision (or repeated revision) is defined as the recurrent practising of previously covered mathematical work in specified content areas (Julie, 2013) (see 1.5). When students are required to solve mathematical tasks using newly learned procedures or concepts, their working memories may easily be overloaded since they have to deal with many new elements of information at once. Conversely if students had had some practice in a mathematical content area, they would be able to use their existing knowledge structures to make inferences and to make connections between well-entrenched concepts in their long-term memories so as to solve mathematical problems presented to them (Cronbach & Snow, 1977; Kalyuga, 2007; Rittle-Johnson et al, 2009). The importance of practising procedural skills to the extent that these become part of long term memory (automatization) cannot be underestimated. This is a key reason why I advocate the need for spiral revision. That is, sometimes students struggle with mathematics because they have not practised lower level procedural skills to the extent that these have become part of their long-term memories. These lower level procedural skills in many instances are the building blocks of the higher order skills.

An important consideration in learning in general and for mathematics in particular is the problem of forgetting. Unless they consciously review learned material, humans forget approximately half of newly learned knowledge in a matter of days or weeks (Ebbinghaus, 1964; Rubin & Wenzel, 1996; Averell & Heathcote, 2010; Murre & Dros, 2015). The speed of forgetting is dependent on factors such as the difficulty of the learned material, how the material is presented, depth of learning and physiological factors such as stress and sleep. Very few studies have investigated retention of mathematical procedures and concepts. In particular, there is a dire lack of research into how the distribution of practice affects the retention of mathematical knowledge that requires more than verbatim recall.

Although Brownell's arguments (1956) concerning mathematical practice are not based on empirical studies and are confined to the domain of arithmetic, they nonetheless provide very useful suggestions on how to use practice to improve mathematical learning and retention of knowledge. Brownell (1956) argues that in mathematical learning understanding can happen in an instant, but to gain facility with mathematical procedures, practice is required. There are however different types of practice that can be utilized. Brownell (1956) is of the view that many different types of practice are possible and that these practice types can be placed on a continuum. For him, at one end of the continuum there is repetitive practice and at the other extreme, varied practice.

Varied practice is defined as practice which involves different approaches or procedures that are employed to deal with the same type of problem. This type of practice is normally used when the aim is to achieve or stimulate deeper understanding. Repetitive practice is defined as practice whereby the same kind of procedure is practiced over and over.

Repetitive practice is utilized in the following instances:

- when the learning goal is to achieve competence;
- to develop the ability to identify the most efficient and economical way to solve a mathematical problem;
- and to assimilate the learning into long-term memory.

Thus, it is between the extremes of repetitive and varied practice that every other type of practice can be placed. These other types of practice are a combination of varied and repetitive practice and differ in the extent to which either repetition or variation is employed. Brownell

(1956) maintains that in order to become proficient in arithmetic, both types of practice are required.

An empirical study done by Rohrer and Taylor (2006) examined how retention of mathematical knowledge is influenced by distributed practice and overlearning practice. Distributed practice is a strategy to promote learning and retention in which practice of specified knowledge and skills is distributed over multiple sessions. Overlearning is a strategy whereby a student first masters a skill and immediately continues to practice the same skill (Rohrer & Taylor, 2006). The term overlearning might however be perceived as negative since it can be compared to terms such as over-eating and also, the type of practice described by overlearning seems to be very similar to repetitive practice. I therefore prefer the term *repetitive practice* and will henceforth use this term instead.

The retention of knowledge in the learning of mathematics is absolutely crucial since in many cases prior knowledge is required to solve presented problems. It is thus important to know which type of practice would best enhance retention of mathematical procedures and concepts. A very popular teaching strategy in mathematics is to present students with an example of a specific type of problem and then require of the students that they practise solving many of the same type of problem immediately. This practice is usually done as a once-off exercise and can thus be classified as repetitive. Research done by Rohrer and Taylor (2006) has shown that long-term retention of a mathematical procedure was better aided by distributed practice. In this study, I utilized both distributed and repetitive practice strategies. I utilized repetitive practice mainly for mastery and distributed practice for retention purposes and for deeper understanding.

The strategy followed in the research was that completed work was revised in class on an on-going basis throughout the semester. Although the majority of revision problems were restricted to a specified content area or concept (the problems required knowledge from only one content area or concept), certain problems required integrated knowledge.

A major problem in the learning of mathematics is that students tend to compartmentalize mathematical knowledge. Ball (1988) argues that mathematics in the school curriculum is presented in compartments and mathematical content is treated as a collection of discrete bits of procedural knowledge. A consequence of this tendency to compartmentalize mathematical knowledge is that the cognitive load required in knowing and using mathematics is considerably increased. Students instructed in this manner will have a low level of knowledge integration and as a result knowledge accessibility will be negatively affected.

Consequently, the revision utilized in this study included problems whose solutions are based on a combination of two or more concepts from different, previously covered content areas. In addition, where possible, the solutions required the use of flexible procedural knowledge. The following is an example of such a problem based on concepts from different content areas (content areas of exponential equations and trigonometry) and which requires the use of flexible procedural knowledge:

$$\text{Solve for } x\text{: } 8^{2tanx} = 16, if \ \ x \in [0°; 360°]$$

Hiebert and Lefevre (1986) argue that sound mathematical knowledge includes significant and fundamental cognitive links between procedural and conceptual knowledge. They maintain that for students to be competent in mathematics they need to possess both procedural and conceptual knowledge as well as cognitive links between these two essential knowledge components. They describe rote learning as learning that produces knowledge which is closely tied to the context in which it was learned, and that is devoid of relationships with other knowledge. The consequence is that knowledge acquired through rote learning can only be accessed and applied in contexts that mirror the context in which it was learned.

Spiral revision was used in conjunction with productive practice in the study. Julie (2013) argues that productive practice is a didactic strategy in which students are exposed to deepening thinking-like problems with the aim of enriching their conceptual knowledge in requisite content areas of the specified mathematics curriculum. The idea in this teaching strategy therefore was to enhance the conceptual knowledge of participating students. As indicated earlier the enhanced conceptual knowledge will in turn support enhancement of procedural knowledge in an iterative process. The following is an example of a deepening thinking-like problem:

Problem-solving in mathematics requires students to view mathematical concepts from different angles. For example, a function can be viewed in terms of its operational character, as a process of co-variation (how the dependent variable co-varies with the dependent variable) or as a mathematical object (Doorman et al., 2012). The following is an example of such a question:

For what values of x will: $\{(2x; 2x-1); (x^2-3; 3x)\}$ not be a function?

This question requires the student to think of the function in terms of its operational character, but importantly, also to utilize the definition of the function concept to solve the problem and in so doing perhaps to enhance an understanding of the function definition. The solution strategy is also based on the exploitation of a known procedure. The student however has to make the connection between the concept and the procedure. The objective with deepening thinking-like questions is to enhance and to deepen conceptual knowledge of students while the student makes cognitive connections between procedural and conceptual knowledge.

The following solution illustrates the foregoing claims:
Functions can be defined in more ways than one. Here I will use the following definition:
A function f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain of the function and the set B is the range of f.

This definition implies the following:

1. *Each element in A must be matched with an element in B.*
2. *Some elements in B may not be matched with any element in A.*
3. *Two or more elements in A may be matched with the same element in B.*
4. *An element in A cannot be matched with two different elements in B.*

Since the question requires us to find those x values that will cause the ordered pairs not to represent a function we utilize Statement 4. That is, we want the first co-ordinates to be the same and the second co-ordinates to be different. What we are therefore attempting to achieve is to make explicit the fact that a first co-ordinate cannot be matched with two different second co-ordinates. The student would therefore be forced to think carefully about what the definition implies. To achieve this, we equate the first co-ordinates thus:

$$x^2 - 3 = 2x$$

One should then realize that a known procedure may be utilized to solve for x. In other words, one could use the solution of quadratic equations. In the process the student should make a connection between the procedure and the concept.

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

These provide two possible solutions:

$$x = -1 \; or \; x = 3$$

We have to check which x-value gives the desired result by substituting in the original co-ordinate pairs in the following way:

$x = -1$, yields $\{(-2, -3); (-2, -3)\}$. This is not the desired result.

If $x = 3, then\ we\ have\ \{(6,5); (6,9)\}$. This is the desired result, since we now have the first coordinates the same and the second coordinates different, which violates the 4th statement. This is to say an element in the domain cannot be matched with two different elements in the range.

Spiral revision is based on a number of premises or ideas. The first of these is that direct instruction is better suited to students with low levels of prior knowledge in a mathematical domain. The second premise is that continuous practice over an extended period will allow for better retention and retrieval of indispensable knowledge. A third premise is that repeated practice should include testing practice which in turn will better develop automaticity of the requisite procedural skills. Testing practice refer to the retrieval of prior knowledge when presented with test items (see 1.5). Another premise is that development and increase of procedural knowledge will aid in procedural flexibility and conceptual knowledge in a given mathematical domain. Each of these ideas will be discussed next.

There is an ongoing debate in the mathematics education community regarding which would best aid learning in mathematics: unguided instruction (or minimally guided) or direct instruction. As already stated learning is defined as a change in long-term memory. The aim of instruction therefore should be to increase the efficiency with which information is stored in and retrieved from long-term memory (Kirschner, Sweller & Clark, 2006). In mathematical learning, automated procedural skills are greatly beneficial in the solving of complex cognitive tasks. A lower level procedural skill that is automated would free cognitive capacity to deal with other features of the problem and would thus allow for cognitive connections to be made between other pieces of information provided.

Direct instruction is defined as instruction that provides information which comprehensively explains the concepts and procedures students are required to learn (Kirschner et al, 2006). Unguided or minimally-guided instruction, by comparison, is defined as instruction in which students are required to discover or construct essential information for themselves as opposed to being presented with essential information (Kirschner et al, 2006). Unguided instruction has been labelled variously as: discovery learning (Anthony, 1973; Bruner, 1961); problem-based learning (Barrows & Tamblyn, 1980; Schmidt, 1983); inquiry learning (Papert, 1980; Rutherford, 1964); experiential learning (Boud, Keogh & Walker, 1985; Kolb & Fry, 1975); and constructivist learning (Jonassen, 1991; Steffe & Gale, 1995).

Kirschner et al (2006) argue that instruction which does not take human cognitive architecture into consideration will not be very effective. Some of the disadvantages of unguided or minimally-guided instruction are ascribed to the fact that its design features do not include human cognitive architecture. Also, as indicated earlier, working memory is the cognitive structure wherein conscious processing of information occurs. When working, memory is required to process novel information, it is limited in terms of duration and capacity. Unguided or minimally-guided instruction requires that a student search a given problem for relevant information. The learning conditions therefore require a random search, which makes heavy demands on working memory. This heavy load on working memory does not allow for information to be moved to long-term memory and hence no effective learning can occur. Unguided or minimally guided instruction is however not totally without advantages. It is thought that students who discover or invent their own procedures in a domain have better transfer abilities and conceptual knowledge than students who have internalized taught concepts and procedures in that domain (Rittle-Johnson, 2006).

Students in the novice- to intermediate-knowledge levels with regards to a mathematical domain or content area are thought to be better served by direct instructional guidance in the concepts and procedures that are to be learnt (Kirschner et al, 2006). Experts in a mathematical content area are skilful because their long-term memory contains copious amounts of networked information concerning the content area. Conversely novices to a domain do not have properly connected prior knowledge and hence would have to make heavy use of their working memory in the learning process. Direct instructional guidance would circumvent the limitations of working memory since the student is presented with the procedures and concepts and hence no random searching through a problem space is required.

In this study participating students were very diverse in terms of prior knowledge of the content areas that were covered in the research. While some students could be classified as novices in many of the topics covered, others had prior exposure to some of the content that was taught. In most cases the teaching strategy was based on direct instructional guidance.

Long-term memory retrieval and storage play a key role in learning (Calderon-Tena & Caterino, 2016). It is thought that long-term memory storage and retrieval is comprised of associative memory skills and retrieval fluency. Associative memory skill is the ability to integrate new information with previously learned information and to store it in long-term memory. Retrieval fluency is defined as the ability to retrieve previously learned information efficiently. Calderon-Tena and Caterino (2016) conducted a study that assessed the relationship between long-term memory retrieval and mathematics, and mathematics problem solving achievement among elementary, middle and high school students in a national sample of American students. Their findings indicate that with an increase in age and grade, it is likely that an advanced capacity for long-term retrieval is a better predictor of both mathematics calculation and problem solving ability.

As mentioned previously learning is assumed to have taken place if there is a change in long-term memory**.** In other words, learning is assumed to have taken place if new information is appropriately connected and assimilated into long-term memory. It is normal practice to study content until one is able to recall it (that is, until it is learned) then to drop it from further study. In this study, it was my contention that even well learned material will be forgotten to some degree if it is not 'refreshed'. The term refresh here refers to knowledge in long-term memory that is either restudied or retrieved. Retrieval therefore is not viewed as a neutral event but is seen as a learning event. So retrieval during testing is seen as a learning event that fixes (retention) and expands learning. Retrieval is thought to expand knowledge. This means that if a test item prompts the executive control to bring information into consciousness but cognition suggests that the information has to be adapted in order to provide a solution procedure, then the information will not be retained in exactly the same way it was retrieved. In other words, the test item causes the retained information to be connected to other information or to be connected in a different way, and in the process the retrieved information itself is changed. For example, during instruction a student is presented with the two-variable linear equation in the standard form: $y = mx + c$ (for example: $y = 3x + 4$). Then in a subsequent test the student is presented with the equation in the form: $-3x + 5(y + 1) = 15$ and is requested to determine the gradient. It is highly probable that the student would retrieve the schema for the standard form, compare the two equations and realize that the equation provided has to be manipulated into standard form. As a consequence, a connection is made with the schema that contains previously learned manipulations of linear equations. This process changes the retained information to include connections to the specific type of manipulations of linear equations.

It is generally assumed that effective learning occurs when material is revised. It has however been found that repeated retrieval practice during testing enhanced long-term retention to a greater extent than repeated restudying (Karpicke & Roediger, 2008; Carpenter, Pashler, Wixted & Vul, 2008). This idea has been explored in this research. Students wrote 5 tests and an end of module examination for the first semester during which the research was conducted. Each test contained questions on content that had previously been covered. So, for example, four of the five tests contained questions on linear functions. This meant students were expected to restudy previously covered content and then, whilst writing the test, to retrieve information regarding linear functions. This thus offered repeated retrieval practice for the content area.

In the next section I discuss how the teaching strategy was implemented in the study.

3.8.1 How the teaching strategy was effected in the study

The teaching strategy that was implemented in the present research was premised on a direct instructional method, a version of continuous review and spiral testing. The continuous review strategy included the strategies of spiral revision and productive practice. In what follows I will elaborate on how each of the components of the teaching strategy was implemented in the study.

The preferred instructional method for the study was the direct instructional method. Minimally guided instruction was used very rarely. As explained in chapter 1 the objective of direct instruction is to provide information that fully explains the concepts and procedures that students are required to learn. An expository instructional method that attended explicitly to conceptual underpinnings and to connections between mathematical facts, procedures and ideas was employed (Hiebert & Grouws, 2007). On some teaching occasions students were allowed to struggle to master concepts and procedures. Productive struggle therefore formed part of the teaching strategy.

Revision of completed topics formed an important part of the teaching strategy. Various forms of revision were utilized in the study. One part of the spiral revision strategy was based on a version of the review-as-you-go method (Wineland & Stephens, 1995). In this method content from topics covered in a chapter was reviewed through teacher questioning and problem presentation at the beginning of each class. When a new chapter was started review of the previous chapter was slowly faded out.

On the completion of a topic in a chapter mass practice exercises (which included repetitive (overlearning) practice) was presented to students during lessons. Some of the problems presented to students during lessons were of the deepening-thinking-like kind. Tutorial assignments which were presented to students once a week included problems based on topics completed recently (in the week of the tutorial) as well as problems based on topics that were completed in the previous two to four weeks. Tutorial assignments also included deepening-thinking-like problems which are based on the learning strategy of productive practice. Tutorial assignments therefore were mostly based on the learning strategy of distributed practice.

Another component of the spiral revision strategy was based on the learning strategy spiral testing. The following delineation of how the spiral testing was employed only holds for the first semester since a slightly different format was employed in the second semester of the research. The basic principle of the method however holds for both semesters. Tests was written approximately every third week. The majority of the tests presented to students in the first semester of the research consisted of three sections each of which was based on different topics. Each section contained between four and seven questions. At least one and at most two of the sections was based on content that was not part of recently covered content. Usually only one of the sections was based on current topics. For example the fourth test contained a section based on content that was covered approximately three months earlier, a section that was covered approximately two months earlier and a section that was covered more recently. Testing practice therefore was premised on both retrieval and distributed practice.

The spiral revision component of the teaching strategy thus is comprised of various revision strategies:

- Review-as-you-go
- Mass practice
- Repetitive practice
- Distributed practice
- Spiral testing (retrieval practice)

The following diagram illustrates the basic design of the teaching strategy (see figure 1). It should be noted that this basic format was followed for the entire course and that productive practice was part of some lessons and tutorials. Moreover review of some of the earlier content would be slowly faded out from lesson 4 onwards.

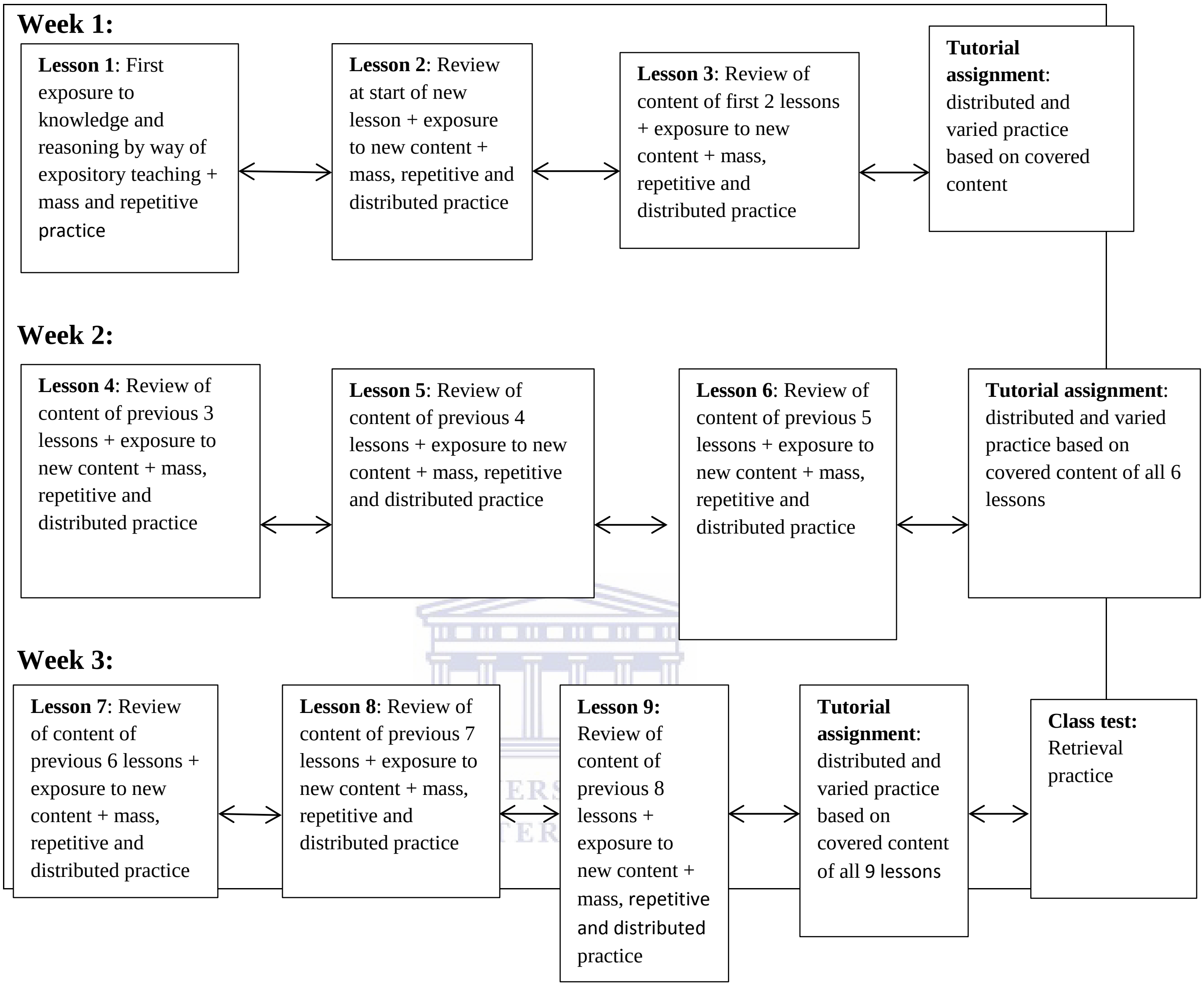

Figure 3.1: Basic design of teaching strategy

3.9 Conclusion

This chapter dealt with theories that I consider to be useful to describe phenomena, interpret results of the study, explain research findings, and to propose theories regarding the observed phenomena. Cognitive load theory in conjunction with theories regarding the human memory system and development of procedural and conceptual knowledge was used to describe, interpret and explain how the teaching strategy influenced participants in terms of their procedural fluency, conceptual understanding, levels of retention and transfer abilities. A theory regarding the effect of worry on mathematical performance was used to advance an explanation of the performance of participating students and a theory is proposed as to how level of expertise of participants can be determined.

It should be noted that not all of the theories discussed in this chapter will be afforded equal prominence in the explanatory framework in the study. Some theories will be central whilst others will be on the periphery of the proffered explanatory framework of the observed phenomena.

The diagram below (see figure 3.2) shows where and the degree to which the various theories and strategies was utilized in this study. The bigger and thicker arrows indicate that a theory was used substantially in the explanatory framework or that a theory regarding observed phenomena is proposed. The smaller and thinner arrows indicate that a theory was used less and were not as prominent in the framework. The arrows were also utilized to indicate the degree to which a strategy feature was used in the study. The bigger and thicker arrows indicate that a strategy feature was utilized extensively and is a prominent part of the study while the smaller and thinner arrows is an indication that a strategy feature was used to a lesser degree and is therefore not a prominent part of the study. Some of the lines have arrows on both ends while others only have arrows on one end. Lines with arrows on both ends are used to indicate that theory is used to inform findings and conversely findings are used to inform theory. Alternatively, lines with arrows on both ends indicate that a strategy feature was informed by the findings as well as vice versa. Lines with only one arrow indicate that the theory was used to inform findings but that the reverse did not happen.

Note that not all the strands of mathematical proficiency are shown in the diagram. Only the strands that formed part of the investigation are included.

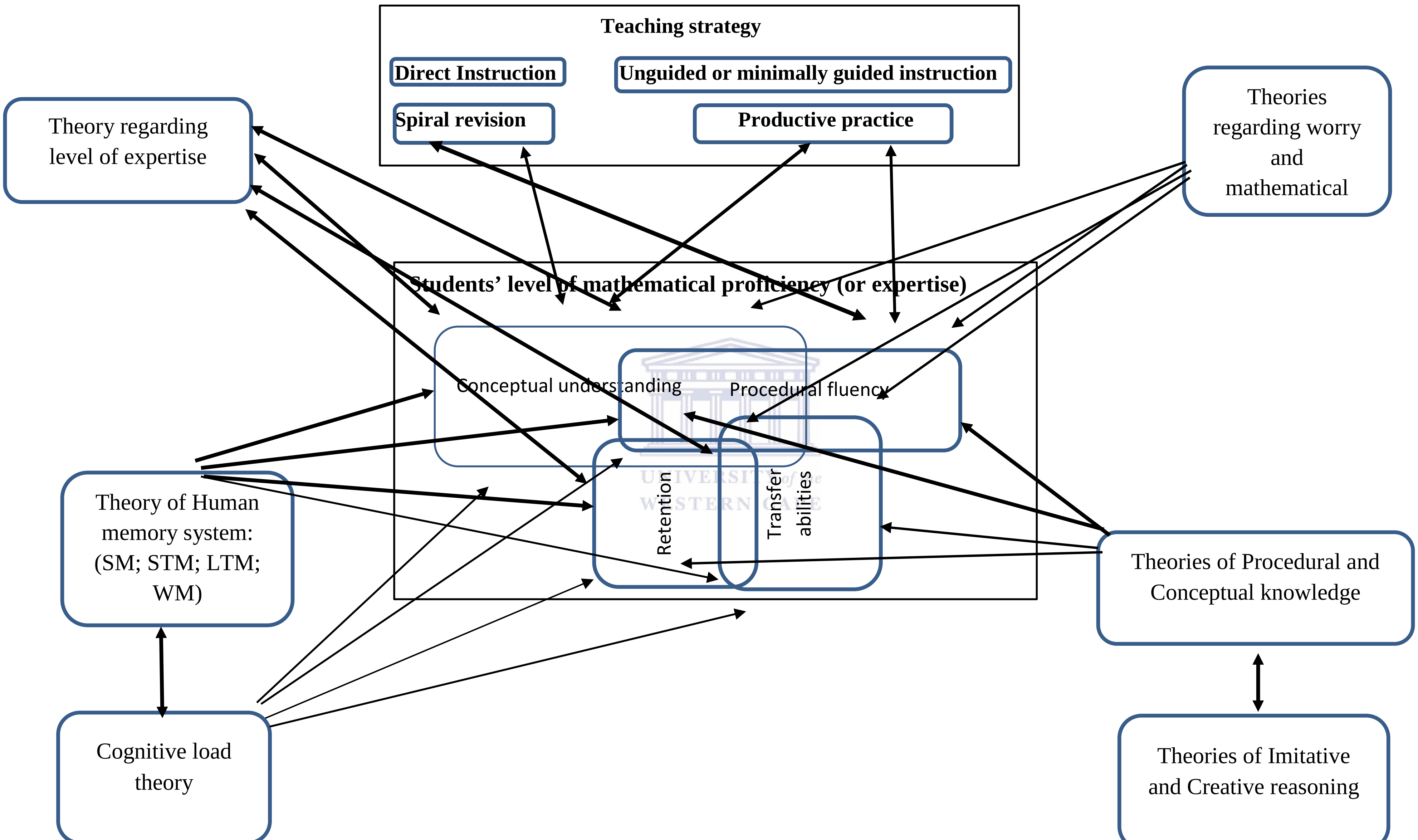

Figure 3.2: Theory and strategy utilization

Many studies in fields such as psychology, cognitive psychology, cognitive neuroscience, etc. have investigated phenomena that deal explicitly with the teaching and learning of mathematics. Many of the findings from such studies elucidate some of the features of learning that instructors of mathematics could use in the improvement of teaching and learning. I believe that without such research mathematics education would be much weaker in terms of understanding underlying phenomena in mathematics teaching and learning. It is therefore essential that research findings from these contributing fields of research are used in the advancement of the teaching and learning of mathematics.

In the next chapter I will discuss the research design of the study.

CHAPTER 4: RESEARCH DESIGN

4.1 Introduction

The process of research design requires researchers to make many multi-layered and interrelated decisions. Each decision is typically informed by a question or questions that the researcher attempts to provide answers to (Creswell, 2003). An initial question in educational research usually centres around the epistemology informing the research. The answer to the question as to which strategies of enquiry will be employed is ordinarily used by researchers to conceptualize a research approach or approaches. In turn these decisions inform the decision regarding methods of data collection and analysis. Furthermore, it is recommended that educational researchers base their choice of research approach on the research question or questions, the objective of the research, available populations, time and cost constraints, the possibility of the manipulation of an independent variable, and the availability of data (Johnson & Chistensen, 2012).

In this chapter I will delineate the decisions that informed strategies of inquiry, methods of data collection and analysis in my research design.

4.2 Research Paradigm

In educational research three major research paradigms or approaches are distinguished, namely quantitative research, qualitative research and mixed methods research. Quantitative research commonly employs the confirmatory scientific method to test theories and hypotheses. It uses quantitative data to determine whether the theories or hypotheses are confirmed or not. Quantitative researchers typically use a narrow-angle lens because they tend to focus on only one or a few causal factors at the same time while attempting to keep constant the factors that are not being studied. Quantitative researchers commonly attempt to use the data to identify cause-and-effect relationships that allow them to make probabilistic predictions and generalizations (Johnson & Chistensen, 2012). The term probabilistic is used to indicate that a phenomenon is considered likely to occur as opposed to there being certainty about such an occurrence. By contrast, qualitative researchers collect qualitative data (normally non-numerical), and ordinarily follow the exploratory scientific method. Qualitative research is customarily used to describe local phenomena and in some instances to generate hypotheses or to construct theories (Johnson & Christensen, 2012). Qualitative researchers use a wide-angle lens to investigate in full detail human behaviour as it occurs naturally. In this paradigm, the researcher is considered to be the instrument of data collection. Quantitative and qualitative research also differs in how human behaviour is perceived in each of these frames. The assumption in quantitative research is that human cognition and behaviour are highly predictable and explainable. Conversely, in qualitative research human behaviour is viewed as being dynamic and changeable over time and place.

In mixed methods research, quantitative and qualitative research approaches are used conjointly. Johnson and Christensen (2012) argue that the three research approaches can be regarded as being part of a research continuum with qualitative research on the left, quantitative research on the right side and mixed research in the centre of the continuum. Research can therefore be either fully quantitative, fully qualitative, mixed with an emphasis on qualitative, or mixed with an emphasis on quantitative. The degree to which the two methods are mixed is dependent on practical concerns and on the research questions.

Creswell (2003) describes a mixed methods research approach as one in which the researcher tends to base knowledge claims on pragmatic grounds. Pragmatism sees the problem as more important than the method of research and therefore in most cases relies on more than one approach in order to derive knowledge about the problem. The philosophical position adopted in the present study is based on pragmatism.

However, the pragmatist argument does not take into account the fact that quantitative and qualitative methods represent two different paradigms and are thus incommensurate (Sale, Lohfeld & Brazil, 2002). Fundamental to this argument is that the phenomenon under scrutiny is rendered differently by each of the two methods. This implies that one cannot use these two approaches to study different aspects of the same phenomenon. This in turn implies that one cannot use cross validation or triangulation to validate results of a mixed methods approach. However, it does not imply that one cannot use a mixed methods approach in a single study. Sale et al (2002) argue that if one distinguishes between phenomena by labelling the phenomenon investigated by each approach (and validating each separately) then a mixed method approach is appropriate.

In this study the quantitative paradigm provided the framework for measuring mathematical knowledge and cognition (in terms of presence, type and range), whereas the qualitative approach was employed to investigate whether retention and transfer abilities can be observed. Both retention and transfer are incorporeal and were inferred from written productions of participants. A quantitative approach which is based on positivism therefore would have been inappropriate in this case. For the qualitative part of the research, case studies were utilized to develop a narrative based on analyses of written productions of participants. Hence this part of the study can be classified as case study research. In this study test and examination results were used as quantitative data, and analysis of test and examination responses were used as qualitative data.

4.2.1 Conceptualization of the qualitative method employed in the study

In this study, qualitative methods were also used to gain deeper insight into how research participants were influenced by the teaching strategy implemented. Qualitative methods therefore were utilized to explain some of the quantitative results in more depth. This was not done for all the quantitative results but only for one type of question. As indicated in chapter 1, the main purpose of the qualitative method was to provide possible answers to the second research question. Findings of the quantitative and qualitative methods were combined to determine the effectiveness of the teaching strategy implemented. Moreover, the quantitative results took temporal precedence over the qualitative results. The design of this study can therefore be classified as a type of explanatory sequential design although it does not adhere in the strict sense to this design type.

Case study research is a form of qualitative research that is focused on providing a detailed account of one or more cases. A case is defined as a bounded system where it is incumbent on the researcher to identify and outline the boundaries of the system (Johnson & Christensen, 2012).

In general, there are three distinct types of case study: intrinsic case studies, instrumental case studies and collective case studies (Stake, 1995). The researcher's main objective with an intrinsic case study is to understand a specific case. The intrinsic case study is very popular in educational studies. Some of its uses include evaluation of educational programs like for example a study that evaluates the effectiveness of an implemented teaching strategy. In an instrumental case study the researcher's primary interest is in understanding something more general than the particulars of the case. In other words, the researcher is interested in drawing conclusions that apply beyond a particular case (Johnson & Christensen, 2012).

In a collective or multiple-case study several cases are studied. By studying more than one case the researcher typically attempts to gain greater insight into a research topic. Cases in a collective study are usually studied intrinsically. An advantage of studying more than one case is that the cases can be compared for similarities and differences. Another important advantage is that it is possible to generalize results as a consequence of replication logic. Replication logic refers to the idea that the more a research finding is shown to be true with different sets of people, the greater the confidence a researcher may invest in the finding and in generalizing beyond the original research participants (Johnson & Christensen, 2012).

This study utilized a collective case study design in which individual cases were studied instrumentally. Each of the six selected cases was first examined holistically. This was followed by a comparative cross-case analysis of similarities and differences. My rationale for choosing this method was founded on my desire to determine how the teaching strategy would affect retention and transfer abilities not only within selected cases, but in the whole group of participating students. In other words, the goal was to use the selected case studies to generalize findings. An eclectic approach was followed in a data analysis restricted to the data which contributed to understanding the cases reflected in the analysis.

4.2.2 Conceptualization of the quantitative method employed in the study

Quantitative research methods can broadly be categorized as experimental, quasi-experimental or non-experimental (Johnson & Christensen, 2012). The main purpose of experimental research is to determine cause-and-effect relationships. The experimental researcher typically creates experimental and control groups by means of random assignment. Random assignment is used to control for extraneous variables which might confound the results of the study. By manipulating one or more variables (normally the independent variables) and keeping the other extraneous variables constant the researcher strives to control for all confounding variables which may influence the results. Of all research methods, experimental research is said to provide the strongest evidence about the cause-and-effect relationships that emerge from the manipulation and control for extraneous variables (Johnson & Christensen, 2012). However, it is not always possible to do experimental research, either because the independent variable cannot be manipulated or because it might be unethical to manipulate the independent variable. In such cases the researcher can then opt for either the quasi-experimental or the non-experimental method.

In this research the non-experimental method was followed.

Non-experimental research is described as research in which the independent variable is not manipulated, and no random assignment to groups is effected (Johnson & Christensen, 2012). Since there is no random assignment to groups and no manipulation of the independent variable in the non-experimental method, it is perceived to provide weaker evidence of a causal relationship between variables. The non-experimental method is however very important in educational research since many research problems in education do not lend themselves to experimentation (Kerlinger, 1986). For example, if one attempts to determine how socio-economic status influences mathematical learning then manipulation of variables is neither ethical nor possible.

Johnson and Christensen (2012) argue that the dimensions of time and research objective should be utilized to classify non-experimental research. In order to do this, researchers should answer two questions namely: how is the data collected in relation to time; and what is the primary objective of the research? In this study data was collected from the same students at successive points over time. Thus, this research can be classified as longitudinal research and the *type* of longitudinal research as panel study.

It is argued that panel studies are superior to the other time dimension types for the following reasons (Johnson & Christensen, 2012). Firstly, since the same people are studied over a period of time the researcher is better able to determine cause and effect. Secondly, changes are measured at the level at which they occur, that is within the individuals who change; hence this type is more powerful in terms of providing causal evidence.

One of the research objectives was to establish whether there is evidence that the teaching strategy enhanced procedural fluency and conceptual understanding of research participants. Objectives therefore set out to determine whether there is evidence for cause-and-effect relationships, and to produce theories that explain how and why the teaching strategy was either effective or not effective. This determined whether the research would be classified as explanatory non-experimental research. Furthermore, a pre- and post-test 'repeated measures design' was utilized in the study.

4.3 Sampling

4.3.1 Quantitative sample

It is generally accepted that the aim of quantitative sampling is to draw a representative sample from the population in order to generalize findings back to the population. Moreover, the sample selection method ordinarily depends on the aim of the study (Marshal, 1996). The most common sampling method in quantitative studies is random or probability sampling. Random sampling however is not always possible. This is especially the case in mathematics education research where it is not always possible to randomly select a sample that have the majority of characteristics of the population. In addition, the circumstances of the study do not always allow for random sampling.

The population for this study are all pre-service mathematics education students in South Africa enrolled for a mathematics content course in the first or second year tertiary level. This population however is very diverse since students study to become teachers in different sectors of schooling. The different South African universities also have different acceptance criteria for the variety of mathematics education courses presented. For example, some universities would accept students who offered mathematical literacy at school level while others do not. An added difficulty is that universities have differing course content for similarly-named courses. It was therefore not possible to randomly select a strictly representative sample (which could be used to generalize results). Thus, the selected quantitative sample for this study was based on convenience.

The most common sampling method in educational studies is probably convenience sampling. Convenience sampling is normally applied in circumstances where the researcher has easy access to particular sites and participants. This method can suffer problems of bias given that the chosen sample might not be representative of the population. The researcher in this study engaged the students of a class he was teaching as research participants (sample size: sixty-three) since it was convenient.

4.3.2 Research participants

The research participants for this study were the 2014 second-year pre-service cohort of students enrolled for a mathematics course presented by the researcher. The research was done over both semesters of 2014. Originally eighty-eight students were registered, but eventually only sixty-three students formed part of the study. Students who missed certain tests and examinations were not used in the study.

Students enrolled for the course were diverse in terms of their school-leaving mathematical knowledge. The number of students who had completed mathematics up to grade 12 was forty-four; and the number of students who had done mathematical literacy in grade 12 was nineteen. The sample consisted of thirty males and thirty-three females of whom the average age was 23. Table 4.1 shows the demographics of the sample.

	Number of students	Average age	Mathematics	Mathematical Literacy
Female	33	23	24	11
Male	30	23	20	8
Total	63		44	19

Table 4.1: Demographics of study participants

4.3.3 Qualitative sample

In qualitative research sampling is normally purposeful. This implies that the study participants who are selected are best placed to facilitate an understanding of the phenomenon that is being studied. The purposeful sampling strategy in this study was maximal variation sampling. Thus, the same set of students was utilized for both the qualitative and quantitative parts of the study. The qualitative sample however consisted of only six of the total number of students who participated in the study. The selection of these was based on their achievement levels as determined by test and examination results of the quantitative part of the study. Two students were selected from the high achievers, two from the average achievers and two from the low achievers. This was done in order to gain diverse perspectives in terms of the phenomenon being investigated.

4.4 Integration of quantitative and qualitative data

Creswell (2015) argues that integration refers to how the researcher brings together the qualitative and quantitative results in a mixed methods investigation. In order to uphold research rigour it is important that as part of the mixed methods research design, a researcher indicates where and how integration of the quantitative and qualitative results will occur. This decision is influenced by the type of mixed method design that is utilized in the study.

Part of the intention of this study was to use qualitative methods to gain deeper insight into how and in which ways research participants were influenced by the teaching strategy implemented. As indicated earlier, qualitative methods were employed to explain some of the quantitative results at more depth. The two sets of data were brought together in the data analysis. The interpretation of results phases involved deploying the qualitative data set to explain some of the quantitative results and to help explicate the effectiveness of the teaching strategy.

4.5 Research validity of the qualitative research component

Validity, reliability and trustworthiness are central to the design of a research study. It is incumbent on a researcher to adequately address these issues in the design of a study. Validity is defined as the degree to which conclusions drawn by a researcher can be attributed to results of the study. The term validity however is generally associated with quantitative research. Trustworthiness on the other hand is the term that is usually associated with qualitative research (Boudah, 2011). Johnson and Christensen (2012) maintain that when qualitative researchers speak of research validity, they are usually referring to qualitative research that is plausible, credible, trustworthy, and therefore defendable. A study is said to be reliable if the results can be replicated if the study were conducted again. Research reliability and validity however are dependent on how the researcher deals with confounding variables. In the next sections I outline some of the strategies that were utilized in this research to deal with threats to validity and reliability.

A qualitative researcher must guard against a host of threats to validity. It is expected that as part of the research design qualitative researchers indicate how they intend to deal with issues such as researcher bias, descriptive validity, interpretive validity, theoretical validity, internal validity and external validity (Johnson & Christensen, 2012).

Researcher bias occurs when the researcher allows his/her personal views and perspectives to affect selective observation, recording of information and how data is interpreted. Researcher bias will result in the researcher obtaining results that are consistent with what the researcher wanted to find. Reflexivity is mooted as one of the strategies that can be utilized to counter the effects of researcher bias. Reflexivity refers to the idea that the researcher engages in critical self-reflection about his/her potential biases and predispositions so that s/he may decide on a strategy by which to deal with these threats.

Another strategy used to counter the threat of researcher bias is negative-case sampling. Negative-case sampling is employed when the researcher purposely seeks out examples that disconfirm expectations. This strategy was followed in some parts of this study. Since it was my expectation that the majority of the higher ability research participants would positively confirm the research question, I deliberately included low ability participants who represented the negative-case sample. It was my expectation that these low ability students would disconfirm my expectation.

Descriptive validity refers to how accurate the researcher is in reporting what they saw and heard (Johnson & Christensen, 2012). Since the data in this study consists of the actual written products of participants, this aspect reduced the threat to validity.

Interpretive validity refers to the extent to which research participants' perspectives, thoughts, intentions and experiences are correctly understood and reported by the researcher (Johnson & Christensen, 2012). It is argued that participant feedback constitutes the most important strategy by which to reduce threats to interpretive validity. The researcher shares his/her interpretations and conclusions of participants' subjective worlds with the participants for verification. Because of the way this study had been set up this strategy was employed inconsistently and in only a few instances. It was not employed across all cases. The qualitative analysis part of the study was done last and consequently participants were not always available for consultation.

Another available strategy for reducing threats to interpretive validity is the use of low-inference descriptors. Low-inference descriptors are descriptions that closely resemble the participants' accounts (Johnson & Christensen, 2012). The verbatim description that uses participants' exact words is viewed as the lowest inference descriptor. In this study the actual written renditions of participants were utilized. These therefore can be viewed as a form of verbatim submission.

Internal validity is defined as the degree to which a researcher is justified in concluding that an observed relationship is causal (Johnson & Christensen, 2012). It is then incumbent on the researcher to ensure that the observed change in the dependent variable can be ascribed to the independent variable and not to confounding extraneous variables. Data triangulation is proposed as a strategy by which to reduce the threat to internal validity. In this study, I used data triangulation in the following ways: data was collected at different times; data was collected with different students; multiple observations were utilized presenting more than one test to participating students over two semesters.

External validity is described as the degree to which an observed relationship between dependent and independent variables can be generalized to other populations, settings and conditions (Boudah, 2011). It is argued that qualitative research may be weak in terms of external validity because of a lack of random sampling and also since the majority of qualitative researchers are more interested in particularistic findings than universalistic findings (Johnson & Christensen, 2012).

Replication logic is advocated as a strategy for reducing threats to external validity. As stated earlier, if a finding is replicated with different people and in different circumstances then the finding can be generalized beyond the people in the study even if there was no random sampling. In this study six students with different abilities were treated as the case studies. Furthermore, all their written products over two semesters were utilized. These written products were produced during testing situations that were not all the same since some testing situations were high stakes (such as the examinations) whereas other tests (such as class tests) were not pitched at the same high stakes level. One can therefore argue that the circumstances for the different sets of data collected were not the same. Therefore, if a research finding is replicated in the six case studies one can argue that because of replication logic the findings can be generalized beyond the current population.

4.6 Research validity of the quantitative research component

In quantitative research the four major types of validity that are usually considered are internal, external, construct and statistical conclusion. In a one-group pre-test/post-test design (such as this study) the issues that might affect internal validity include history, maturation, testing, instrumentation and regression.

History refers to any event that occurs between the pre-test and post-test measurements of the dependent variable that influences the post measurement of the dependent variable. Such history events however only represent a threat to internal validity if they represent plausible rival explanations for the findings of the study. In this study no history event (known to the researcher) occurred that affected the findings.

Maturation refers to any physical or mental changes that may occur within research participants that might affect performance on the dependent variable. Since most of the participants of this study were mature individuals no threat was expected in terms of maturation. They might have matured in terms of the knowledge they hold and how they hold this knowledge due to engagements with knowledge that are not necessarily linked to understanding of their mathematics courses. For example, they might have matured in the way they engaged with academic texts.

Johnson and Christensen (2012) describe testing as changes that occur in participants' scores obtained on the second administration of a test as a result of their having taken the test before. In this study, more than one test formed the pre-test and post-test score and test items were different for each subsequent test. Hence the threat to internal validity was reduced.

Instrumentation pertains to any changes in the instrument that was used for measurement. One way in which an instrumentation threat may occur is when the pre-test and post-test measurement instrument is different. This would include instances when test items that are presented in pre-test and post-test are not equivalent. In this study, pre- and post-tests were equivalent.

As stated earlier external validity refers to the degree to which the results of a study can be generalized to other populations, settings and conditions. The two steps involved in achieving external validity in quantitative research are firstly, to identify the target population and secondly, to randomly select a sample from the target population. For various practical reasons it is however not always possible to implement these two steps. In many instances in educational research an accessible population is used instead of the target population. From this accessible population a random sample is then selected. In this study, students taking one of my classes were selected as the accessible population. No random selection from this group was made and instead the whole class participated in the study. The fact that the whole class participated increased the possibility that the accessible population is representative of the target population.

The inferential steps involved in generalizing from the study sample to the target population normally require the researcher to first generalize from the study sample to the accessible sample and then from the accessible population to the target population. However, one can very rarely generalize from the accessible population to the target population with any degree of confidence since the accessible population is seldom representative of the target population. Also, it is incumbent on researchers to specify if their intention is to generalize to a target population or to generalize across a target population (Johnson & Christensen, 2012).

When the intention is to generalize across a target population, we attempt to determine whether the finding holds for each of the subpopulations within the target population. In this research the target population would be all pre-service teachers taking a mathematics content course at a tertiary institution. This target population has various sub-populations such as students taking mathematics courses to teach at the foundation level of primary school or to teach at the further education and training level. For this study the sub-population would be students taking mathematics content courses in preparation for teaching at the senior phase (grades 7, 8 and 9). It is the intention of this study to generalize to this sub-population. As I have argued here, since a whole class constituted the research participants, this increased the likelihood that the status of the sample could be said to be representative of the sub-population.

4.7 The Quantitative research instrument

As already indicated a pre- and post-test repeated measures design was followed in this research. The teaching strategy implemented was underpinned by spiral revision which was likely to bear fruit only after many revision sessions (in our case the tests formed part of the revision process). For this reason, all the class tests together were considered to constitute a pre-test. The examinations which were written at the end of each semester (there were two examinations) were together considered to be a post-test. The research was conducted over both the first and second semester of 2014.

A taxonomy table based on Bloom's revised taxonomy was utilized to categorize each question for all tests and examinations. The taxonomy of educational objectives is defined as a framework for classifying statements of what instructors expect or intend students to learn as a result of instruction (Anderson & Krathwohl, 2001). Bloom's original taxonomy consisted of six major categories namely knowledge, comprehension, application, analysis, synthesis and evaluation. The categories were ordered from simple to complex and from concrete to abstract. It is also understood that mastery of the objectives was hierarchical. That is mastery of each simpler category was a prerequisite to comprehension of the next more complex category (Krathwohl, 2002).

Krathwohl (2002) argues that statements of objectives describing intended learning outcomes typically consist of a noun or noun phrase and a verb or verb phrase. Both noun and verb aspects are associated with the knowledge category in Bloom's original taxonomy. This implied that the knowledge category was dual in nature, which was different from the other categories. In Bloom's revised taxonomy this inconsistency was eliminated by allowing the noun and verb aspects to form separate dimensions. In this new taxonomy, the noun provided the basis for the knowledge dimension while the verb formed the basis for the cognitive Process dimension (Anderson, Krathwohl, Airasian, Cruikshank, Mayer, Pintrich, Raths & Wittrock, 2001). Using these two dimensions a two-dimensional table was constructed which the authors termed the taxonomy table. The knowledge dimension formed the vertical axis of the table whereas the cognitive process dimension formed the horizontal axis. The intersections of the knowledge and cognitive process categories form the cells of the table.

The knowledge dimension of the revised taxonomy contains four main categories: Factual knowledge, Conceptual knowledge, Procedural knowledge and Meta-cognitive knowledge. The main categories of the Cognitive Process dimension are: Remember, Understand, Apply, Analyse, Evaluate and Create.

The taxonomy table is shown in Table 4.2. It should be noted that in this revised taxonomy the cognitive processes are understood to operate on knowledge structures during the process of cognition. To reveal this way of thinking one would therefore refer to understanding *based* on conceptual knowledge, etc. It is recommended that the taxonomy table be used to classify the instructional and learning activities which target achievement of the learning objectives, as well as the assessment employed to determine how well the objectives were mastered by the students (Anderson et al, 2001).

Krathwohl (2002) argues that one of the most frequent uses of the original Bloom's taxonomy (and also the revised taxonomy table) has been to classify curricular objectives and test items in order to show the breadth, or lack thereof, of the objectives across the spectrum of categories. He argues that such analysis has routinely shown a predominant emphasis on learning objectives that require only recognition or straightforward recall of information; and a lack of objectives that require understanding and new use of knowledge. Analyses such as these have been used on a continuing basis in an attempt to improve on curricula and tests by including more of the cognitively demanding categories.

The Cognitive Process Dimension

The Knowledge Dimension	1. Remember	2. Understand	3. Apply	4. Analyze	5. Evaluate	6. Create
A. Factual Knowledge						
B. Conceptual Knowledge						
C. Procedural Knowledge						
D. Metacognitive Knowledge						

Table 4.2: The Taxonomy Table

4.7.1 The Revised Taxonomy Table

4.7.1.1 The knowledge categories

The categories of the Knowledge and Cognitive Process dimension of the revised taxonomy table however were not sufficient for my study. Thus, I have modified the table to suit the needs of my investigation. In my version of the taxonomy table I keep some of the Knowledge dimension categories, but I discard most of the Cognitive Process dimension categories. My reason for discarding some of the cognitive dimension categories is that I wanted to focus on the cognition categories prevalent in mathematical cognition. Similarly, the knowledge categories I include are those that are predominant in mathematics. I have also not utilized sub-categories in the Knowledge dimension, but have included subcategories in the cognitive Process dimension. Moreover, since this study is concerned with the teaching and learning of mathematics, the knowledge categories and cognitive processes categories are approached from the perspective of mathematics.

The categories of my revised Knowledge dimension are as follows: Factual Knowledge, Procedural Knowledge, Flexible Procedural Knowledge, and Conceptual Knowledge. It should be noted that it was not the intention to create an exhaustive list of the knowledge categories possible in mathematical learning, but rather that the categories were chosen based on the needs of the study. I now define and discuss each category briefly.

Factual Knowledge is comprised of the basic elements that constitute the discipline of mathematics which students must know in order to solve problems or communicate. This includes knowledge of terminology, format and syntax of symbol representation, allowable operations, etc. (Eisenhart, Borko, Underhill, Brown, Jones & Agard (1993).

Procedural knowledge is knowledge that consists of rules and procedures for solving mathematical problems. Procedural knowledge also consists of knowledge of mathematical symbols and the syntactic conventions for the manipulation of such symbols (Hiebert & Lefevre, 1986; Star, 2005).

Star (2005) argues that skilled problem solvers in mathematics are also flexible in their use of known procedures. A student who does not possess such flexible procedural knowledge will not always be able to solve unfamiliar problems where the solution requires the student to use known procedural knowledge. The student will also be unable to produce a maximally efficient solution in the absence of such flexible procedural knowledge. An outcome of such flexibility is that students who possess such knowledge will have the ability to generate maximally efficient solutions for known and even sometimes unknown problem situations. Star (2005) contends that flexible procedural knowledge is deep procedural knowledge that will allow a student who possesses it to use mathematical procedures best suited to a provided known or novel problem situation.

Hiebert and Lefevre (1986) contend that conceptual knowledge is knowledge that is rich in relationships. This connected web of knowledge is a network in which the linking relationships are as prominent as the discrete pieces of information.

4.7.1.2 The Cognitive Process categories

As already intimated I have adapted the Cognitive Process dimension provided by Anderson et al (2001) in order to deal more specifically with cognition in the mathematical context. The main categories of my version of the Cognitive Process dimension are Imitative Reasoning (IR) and Creative Reasoning (CR). Imitative reasoning is sub-divided into the categories Memorised Reasoning (MR) and Algorithmic Reasoning (AR). Algorithmic Reasoning in turn is sub-divided into the sub-categories Familiar Algorithmic Reasoning (FAR) and Delimiting Algorithmic Reasoning (DAR). Creative Reasoning is sub-divided into the sub-categories Local Creative Reasoning (LCR) and Global Creative Reasoning (GCR). The categories and sub-categories are discussed next.

Lithner (2008) has developed a conceptual framework that is concerned with reasoning in mathematics. In this framework reasoning is defined as the line of thought required to produce assertions and reach conclusions in solving mathematical tasks. Reasoning in this framework can either be the thinking processes or the product of thinking processes, or both. A line of thought may be classified as reasoning even if it is incorrect. The only proviso is that it makes sense to the thinker.

Lithner (2008) differentiates between two different types of reasoning in mathematics. Imitative reasoning (IR) occurs when a student produces a solution procedure that s/he memorized. Conversely creative reasoning (CR) is reasoning that is characterized by flexibility and novel approaches to mathematical problems (Bergqvist, 2007).

Lithner (2008) distinguishes between two main categories of imitative reasoning, namely memorized and algorithmic imitative (AR) reasoning. He asserts that for imitative reasoning to be classified as memorized reasoning it needs to fulfil two conditions. On one hand the reasoning should be based on recalling a complete answer. On the other hand, the implementation strategy should consist of only writing down the answer. A reasoning sequence is classified as algorithmic reasoning (AR) if the reasoning is based on the recall of an algorithm. An algorithm is described as a finite sequence of executable directives which allows one to find a result for a given class of problems. A reasoning sequence is classified as algorithmic reasoning if it satisfies two conditions.

On one hand the strategic choice for the reasoning should be to recall an algorithm as a solution. No other reasoning should be required except to implement the algorithm. On the other hand if, however, a task requires mostly creative reasoning then the reasoning involved is classified as global creative reasoning (GCR). Bergqvist (2007) claims that in some instances students use algorithmic reasoning to solve mathematical tasks, without any comprehensive understanding of the underlying mathematical concepts. Bergqvist (2007) describes creative reasoning (CR) as reasoning that is not hindered by fixation and is characterized by flexibility, novel approaches to mathematical problems and well-founded task solutions. If a task is very nearly solvable using imitative reasoning and creative reasoning is required only to modify an algorithm, then the reasoning required is local creative reasoning (LCR).

The revised taxonomy table based on the foregoing discussion is shown in Table 4.3. The possible categories based on the table are shown in Table 4.4.

The knowledge Dimension (KD)	The cognitive process dimension (CPD)				
	1. Imitative reasoning (IM)			2. Creative mathematically founded reasoning	
	(a.) Memorised reasoning (MR)	(b.) Algorithmic Reasoning (AR)		(a.) Local creative reasoning	(b.) Global creative reasoning
		(i) familiar AR	(ii) delimiting AR		
A.) Factual knowledge					
B.) Procedural knowledge					
C.) Flexible procedural knowledge					
D.) Conceptual knowledge					

Table 4.3: The Revised Taxonomy Table

1.	A1a – Memorized Reasoning based on Factual knowledge
2.	A1bi – not possible
3.	A1bii – not possible
4.	A2a – not possible
5.	A2b – not possible
6.	B1a – Memorized Reasoning based on Procedural knowledge
7.	B1bi – Familiar Algorithmic Reasoning based on Procedural Knowledge
8.	B1bii – Delimiting Algorithmic Reasoning based on Procedural Knowledge
9.	B2a – Local Creative Reasoning based on Procedural Knowledge
10.	B2b – Global Creative Reasoning based on Procedural Knowledge
11.	C1a – Memorised Reasoning based on Flexible Procedural Knowledge
12.	C1bi – Familiar Algorithmic Reasoning based on Flexible Procedural knowledge
13.	C1bii – Delimiting Algorithmic Reasoning based on Flexible Procedural knowledge
14.	C2a – Local Creative Reasoning based on Flexible Procedural knowledge
15.	C2b – Global Creative Reasoning based on Flexible Procedural knowledge
16.	D1a – Memorised Reasoning based on Conceptual knowledge
17.	D1bi – Familiar Algorithmic Reasoning based on Conceptual knowledge
18.	D1bii – Delimiting Algorithmic Reasoning based on Conceptual knowledge
19.	D2a – Local Creative Reasoning based on Conceptual knowledge
20.	D2b – Global Creative Reasoning based on Conceptual knowledge

Table 4.4: Possible question categories based on the Revised Taxonomy Table

The categories (as shown in Table 4.4) formed the measuring instrument that was used to measure knowledge and reasoning proficiency of the research participants. It should be noted that the classification of a problem was also based on how often the problem had been done before and how long ago similar problems had been done. The classification was dependent on number of prior practice sessions, temporality and perceived major knowledge and reasoning requirements of the problem. To illustrate this, a problem can initially be classified as Delimiting algorithmic reasoning based on flexible procedural knowledge (C1bii). But after students have been exposed to it for a number of times the knowledge requirements then become Familiar algorithmic, and the reasoning type becomes ordinary procedural knowledge. The new classification thus becomes Familiar Algorithmic Reasoning based on Procedural knowledge (B1bi), which is a category perceived to be less difficult.

4.8 Data collection and data analysis

4.8.1 The Quantitative procedure

In order to determine if the mathematical competencies of procedural fluency and conceptual understanding of pre-service mathematics students can be enhanced by exposure to a teaching strategy underpinned by spiral revision and productive practice I compared pre- and post-test scores. As previously indicated together all class tests were considered as the pre-test whereas examinations written at the end of each semester were together considered a post-test. Only students who had written all the tests and examinations were included. Data was collected in both the first and second semesters of 2014.

Test items for both the class tests and examination were categorized using the revised taxonomy table. These categories formed the main measuring instrument for the study. It should be noted that not all possible categories were utilized since some categories were not present in the assessments.

Each participating student received a score in each category for each individual class test and examination. Subsequently all the individual scores for each category were added and a total determined. This was done separately for class tests and examinations. For example, all individual scores for the category Familiar Algorithmic Reasoning based on Procedural knowledge (B1bi) were added to provide a total sum for this category per student. The sums for the categories Memorized Reasoning based on Factual knowledge (A1a), Memorized Reasoning based on Procedural knowledge (B1a) and Familiar Algorithmic Reasoning based on Procedural knowledge (B1bi); Familiar Algorithmic Reasoning based on Flexible Procedural knowledge (C1bi) and Delimiting Algorithmic Reasoning based on Flexible Procedural knowledge (C1bii) were added in turn to provide a sum total. This was done separately for pre- and post-tests. This sum total was considered to represent measures of the mathematical competency procedural fluency.

Similarly, the sum scores for the categories Memorised Reasoning based on Conceptual knowledge (D1a), Familiar Algorithmic based on Conceptual knowledge (D1bi), Delimiting Algorithmic Reasoning based on Conceptual knowledge (D1bii) and Local Creative reasoning based on Conceptual knowledge (D2a) were added to provide a new sum total. Subsequently these sums were added to provide a new sum total. This was done separately for pre- and post-tests. This sum total was considered to represent a measure of the mathematical competency Conceptual understanding.

Finally, the pre-test sum score for procedural fluency (SKILLPRE) and the pre-test sum score for conceptual understanding (CONCPRE) were added. This new sum total was termed (PRETOT); it was considered to be a measure of both procedural fluency and conceptual understanding. Similarly, the post-test sum score for procedural fluency (SKILLPOST) and the post-test sum score for conceptual understanding (CONCPOST) were added to provide the new sum total termed (POSTTOT).

Table 4.5 shows the terminology and the explanation used in the statistical analysis.

TERM	EXPLANATION
A1aPRE	Sum of all class test items; memorized reasoning based on factual knowledge expressed as %
A1aPOST	Sum of all exam test items; memorized reasoning based on factual knowledge expressed as %
B1aPRE	Sum of all class test items; memorized reasoning based on procedural knowledge expressed as %
B1aPOST	Sum of all exam test items; memorized reasoning based on procedural knowledge expressed as %
B1biPRE	Sum of all class test items; Familiar algorithmic reasoning based on Procedural knowledge expressed as %
B1biPOST	Sum of all exam test items; Familiar algorithmic reasoning based on Procedural knowledge expressed as %
C1biPRE	Sum of all class test items; Familiar Algorithmic reasoning based on Flexible Procedural knowledge
C1biPOST	Sum of all exam test items; Familiar Algorithmic reasoning based on Flexible Procedural knowledge expressed as %
C1biiPRE	Sum of all class test items; Delimiting Algorithmic Reasoning based on Flexible Procedural Knowledge expressed as %
C1biiPOST	Sum of all exam test items; Delimiting Algorithmic Reasoning based on Flexible Procedural Knowledge expressed as %
D1aPRE	Sum of all class test items; Memorized Reasoning based on Conceptual knowledge expressed as %
D1aPOST	Sum of all exam test items; Memorized Reasoning based on Conceptual knowledge expressed as %
D1biPRE	Sum of all class test items; Familiar algorithmic reasoning based on Conceptual knowledge
D1biPOST	Sum of all exam test items; Familiar algorithmic reasoning based on Conceptual knowledge expressed as %
D1biiPRE	Sum of all class test items; Delimiting Algorithmic Reasoning based on Conceptual knowledge expressed as %
D1biiPOST	Sum of all exam test items; Delimiting Algorithmic Reasoning based on Conceptual knowledge expressed as %
D2aPRE	Sum of all class test items; Local Creative Reasoning based on Conceptual knowledge expressed as %
D2aPOST	Sum of all exam test items;Local Creative Reasoning based on Conceptual knowledge expressed as %
SKILLPRE	Sum of A1aPRE, B1aPRE, B1biPRE, C1biPRE and C1biiPRE expressed as %
SKILLPOST	Sum of A1aPOST, B1aPOST, B1biPOST, C1biPOST and C1biiPOST expressed as %
CONCPRE	Sum of D1aPRE, D1biPRE, D1biiPRE and D2aPRE expressed as %
CONCPOST	Sum of D1aPOST, D1biPOST, D1biiPOST and D2aPOST expressed as %
PRETOT	Sum of SKILLPRE and CONCPRE expressed as %
POSTTOT	Sum of SKILLPOST and CONCPOST expressed as %

Table 4.5: Terminology and explanation used in the statistical analysis

4.8.2 The Qualitative procedure

Data for the qualitative method was composed of the written responses of students to test and examination questions. Only one type of question was utilized for this purpose since it would have been impractical to analyse all test and examination questions. The responses of only six students were taken into consideration since again it would have been impractical to analyse responses of all participating students. Data was collected in both the first and second semester of 2014.

The responses of the selected six students to questions that were based on linear functions were analysed. Written responses to all tests and examinations that contained questions on linear functions were included as data items. These written responses were analysed to determine if students exhibited retention and transfer abilities and if students showed progression in terms of these abilities. Analysis was also done with the intention of gaining deeper insight into some of the quantitative results. Subsequently a narrative was developed based on the findings of the analysis.

4.9 Conclusion

In this chapter the research design that was employed in this study was discussed. The discussion centred on how a mixed method approach was conceptualized in the study. Figure 2 is a synopsis of how the approach was effected in the study.

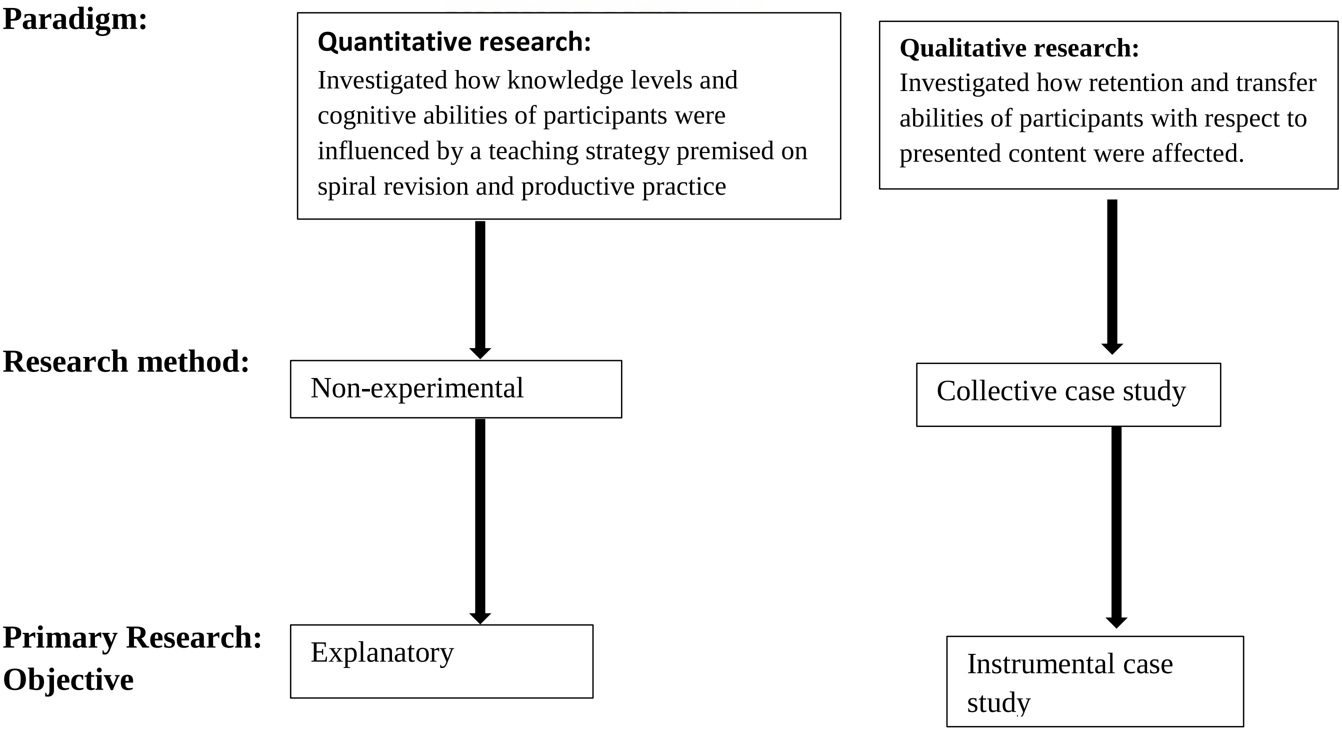

Figure 4.1: Research design

In the following chapter (chapter 5) the quantitative part of the study will be presented and discussed.

CHAPTER 5: STATISTICAL ANALYSIS, QUANTITATIVE RESULTS AND DISCUSSION

5.1 Introduction

Pre- and post-test scores were compared in order to determine whether the mathematical competencies of procedural fluency and conceptual understanding of pre-service mathematics teachers were enhanced by exposure to a teaching strategy underpinned by spiral revision and productive practice. As previously indicated all class tests were taken together to represent the pre-test, whereas examinations written at the end of each semester were taken together to represent the post-test. Test items for both the class tests and the examinations were categorized according to the revised taxonomy table. These categories formed the main measuring instrument in the study. The dependent variable in the study is the achievement scores (pre- and post-test) and the independent variable is the treatment 'teaching strategy underpinned by spiral revision and productive practice". The variables employed in the analysis have been discussed in Chapter 4.7.

In this chapter I show the results of the statistical analyses (IBM SPSS version 23 was used for this purpose), and present conclusions based on the findings. The statistical analysis was done in the following way: descriptive statistics were used initially to explore the data in order to check for violations of underlying assumptions of statistical tests. A paired samples test was performed next to determine whether there were significant differences between pre-test and post-test scores. Finally, a stratified analysis was performed in order to determine how the teaching intervention affected the research participants individually.

5.2 Descriptive Statistics for the univariate variables SKILLPRE, SKILLPOST, CONCPRE, CONCPOST, PRETOT and POSTTOT

It is common practice that prior to doing statistical analysis one explores the data by means of descriptive statistics and graphs as a thorough description is essential to understanding the data. Another important reason is that one needs to check for violation of underlying assumptions in statistical tests. For example, one needs to check if the data is normally distributed and if outliers exist, since both of these might influence correlation coefficients. The descriptive statistics of mean, standard deviation, range, skewness and kurtosis were obtained for the variables SKILLPRE, SKILLPOST, CONCPRE, CONCPOST, PRETOT and POSTTOT. The descriptive statistics for these variables are shown in Table 5.1.

	N	Range	Minimum	Maximum	Mean (%)	Std. Deviation	Skewness		Kurtosis	
	Statistic	Statistic	Statistic	Statistic	Statistic	Statistic	Statistic	Std. Error	Statistic	Std. Error
SKILL-PRE	63	76	23	99	65.71	21.093	-.140	.302	-.992	.595
CONC-PRE	63	77	14	90	48.99	18.692	.177	.302	-.896	.595
PRE-TOT	63	72	22	94	57.35	18.888	.134	.302	-1.088	.595
SKILL-POST	63	59	41	100	77.98	16.289	-.558	.302	-.645	.595
CONC-POST	63	74	20	94	59.94	17.850	.062	.302	-.855	.595
POST-TOT	63	57	39	96	68.96	15.960	-.106	.302	-1.115	.595
Valid N (listwise)	63									

Table 5.1: Descriptive statistics for variables SKILLPRE, SKILLPOST, CONCPRE, CONCPOST, PRETOT and POSTTOT

Skewness and kurtosis provide information concerning the distribution of scores on continuous variables (Gravetter & Walnau, 2002). The skewness value offers an indication of the symmetry of the distribution whereas the kurtosis provides information about the 'peakedness' of the distribution. Positive skewness values indicate scores clustered to the left at low values whereas negative skewness values indicate a clustering of scores at the high values. Positive kurtosis values indicate that the distribution is clustered in the centre (peaked) with long thin tails whereas negative kurtosis values indicate a distribution that is relatively flat. If the distribution is perfectly normal, both skewness and kurtosis will have a value of 0.

The skewness value for the variables SKILLPRE, SKILLPOST and POSTTOT is negative which indicates a distribution that is clustered at the high scores. The kurtosis values for these variables are also negative, which indicates a distribution that is relatively flat. This is an indication that too many scores are in the extremes. Skewness and kurtosis together therefore indicate that high scores are in the majority for all these variables.

The skewness value for the variables CONCPRE, PRETOT and CONCPOST is positive which indicates a distribution that is clustered at the low scores. The kurtosis values for these variables are negative, which indicates a distribution that is flat and thus most scores are in the extremes. Skewness and kurtosis taken together for these variables therefore indicates that most scores for these variables are low.

A normal distribution has a bell-shaped curve, which has the greatest number of scores in the middle with smaller frequencies towards the extremes. The skewness and kurtosis values for the variables SKILLPRE, SKILLPOST, CONCPRE, CONCPOST, PRETOT and POSTTOT indicate that scores are not normally distributed.

Another way to determine if scores are normally distributed is to use histograms. Histograms provide a visual picture of the distribution. Figure 5.1 shows the histogram for SKILLPRE; Figure 5.2 shows the histogram for CONCPRE; Figure 5.3 shows the histogram for PRETOT; Figure 5.4 shows the histogram for SKILLPOST; Figure 5.5 shows the histogram for CONCPOST and Figure 5.6 shows the histogram for POSTTOT. A normal distribution implies that most of the tallest columns or bars of the histogram should be concentrated in the middle with the shorter columns at either extreme. This is not the case for SKILLPRE, and thus one can conclude that the distribution for SKILLPRE is not normal. Although some of the tallest columns for CONCPRE are in the middle, this is not consistent which again indicates a non-normal distribution. PRETOT also has some of the tallest columns towards the middle, but once again it is not consistent and hence the conclusion is that it is not normally distributed. The tallest columns for SKILLPOST are on the left extreme and hence the conclusion is that the distribution is not normal. Although some of the tallest columns of CONCPOST are towards the middle, the majority are not concentrated there and hence the conclusion is that the distribution is not normal. The histogram of POSTTOT has an irregular spread of bars and therefore the conclusion is that the distribution is not normal. The conclusions derived from the histograms together with skewness and kurtosis findings therefore indicate that none of the distributions of the univariate variables SKILLPRE, SKILLPOST, CONCPRE, CONCPOST, PRETOT and POSTTOT are normal.

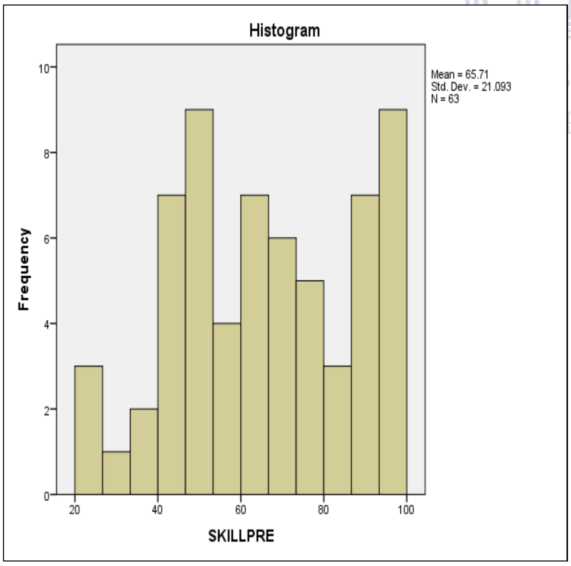

Figure 5.1: Histogram of SKILLPRE

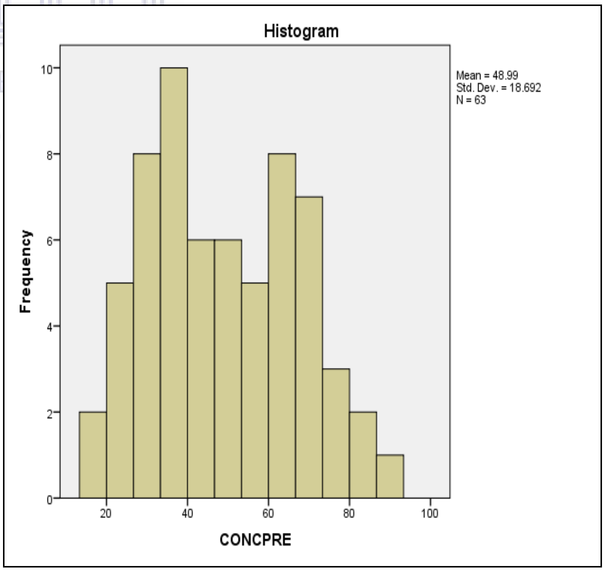

Figure 5.2: Histogram of CONCPRE

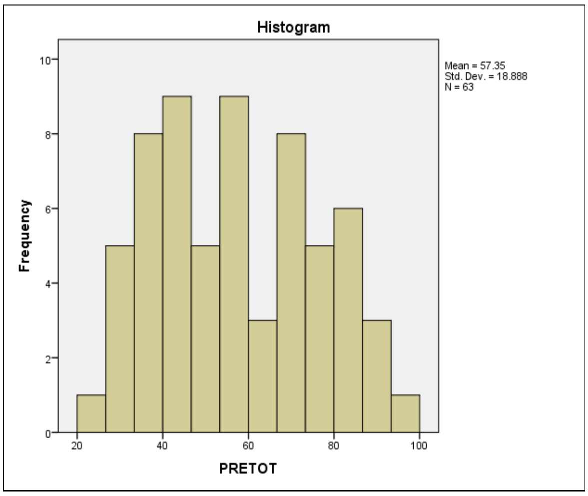

Figure 5.3: Histogram of PRETOT

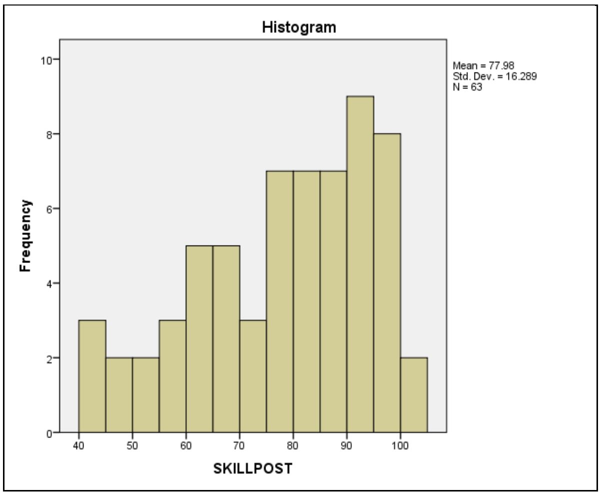

Figure 5.4: Histogram of SKILLPOST

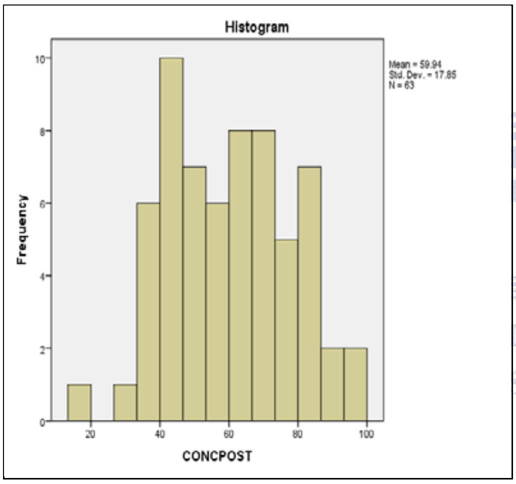

Figure 5.5: Histogram of CONCPOST

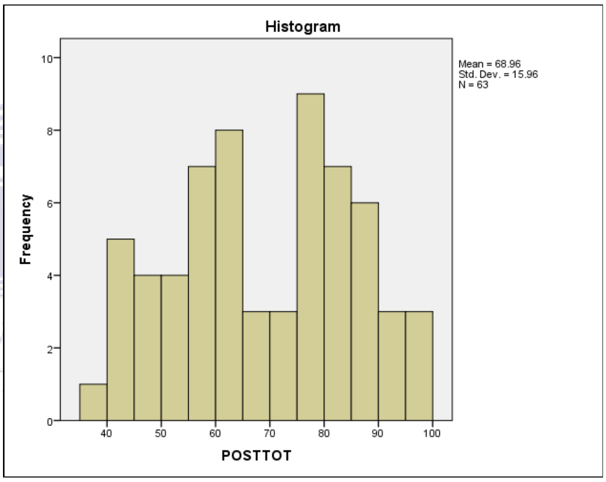

Figure 5.6: Histogram of POSTTOT

5.2.1 Outliers

The 5% trimmed mean will be utilized to determine how much the outliers are affecting the mean for the variables SKILLPRE, SKILLPOST, CONCPRE, CONCPOST, PRETOT and POSTTOT. To obtain the 5% trimmed mean the top and bottom 5 percent of cases are removed and a new mean value is calculated. One then compares the original and new means to determine what influence these extreme scores have on the mean, and whether they require further investigation. Tables 5.2 to 5.13 show the 5% trimmed mean and the extreme values for the variables. Figures 5.7 to 5.12 show the boxplots for the variables.

			Statistic	Std. Error
SKILLPRE	Mean		65.71	2.658
	95% Confidence Interval for Mean	Lower Bound	60.39	
		Upper Bound	71.02	
	5% Trimmed Mean		66.24	
	Median		65.52	
	Variance		444.931	
	Std. Deviation		21.093	
	Minimum		23	
	Maximum		99	

Table 5.2: SKILLPRE 5% trimmed mean

			Case Number	Student Number	Value
SKILLPRE	Highest	1	62	3043417	99
		2	33	3347907	97
		3	34	3375617	97
		4	8	3347834	96
		5	36	3347760	95
	Lowest	1	7	3300057	23
		2	25	3257801	24
		3	53	3102256	26
		4	3	3301443	29
		5	48	3347729	35

Table 5.3: SKILLPRE extreme values

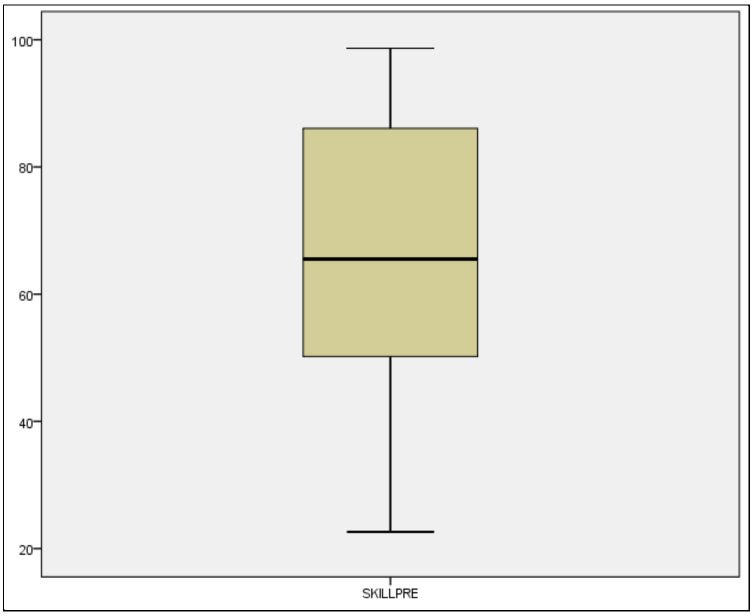

Figure 5.7: SKILLPRE boxplot

			Statistic	Std. Error
CONCPRE	Mean		48.99	2.355
	95% Confidence Interval for Mean	Lower Bound	44.29	
		Upper Bound	53.70	
	5% Trimmed Mean		48.77	
	Median		46.73	
	Variance		349.400	
	Std. Deviation		18.692	
	Minimum		14	
	Maximum		90	
	Range		77	
	Interquartile Range		30	
	Skewness		.177	.302
	Kurtosis		-.896	.595

Table 5.4: CONCPRE 5% trimmed mean

			Case Number	Student Number	Value
CONCPRE	Highest	1	62	3043417	90
		2	19	3347452	83
		3	8	3347834	82
		4	36	3347760	79
		5	5	3347786	78
	Lowest	1	2	3155417	14
		2	37	3300421	19
		3	57	3270745	20
		4	17	3213234	20
		5	7	3300057	21

Table 5.5: CONCPRE extreme values

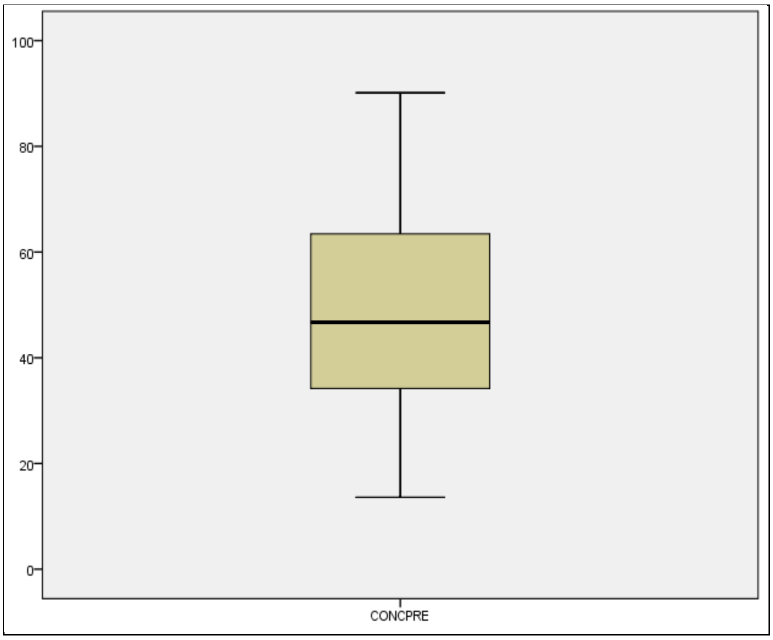

Figure 5.8: CONCPRE boxplot

			Statistic	Std. Error
SKILLPOST	Mean		77.98	2.052
	95% Confidence Interval for Mean	Lower Bound	73.88	
		Upper Bound	82.09	
	5% Trimmed Mean		78.73	
	Median		81.02	
	Variance		265.338	
	Std. Deviation		16.289	
	Minimum		41	
	Maximum		100	
	Range		59	
	Interquartile Range		26	
	Skewness		-.558	.302
	Kurtosis		-.645	.595

Table 5.6: SKILLPOST 5% trimmed mean

			Case Number	Student Number	Value
SKILLPOST	Highest	1	8	3347834	100
		2	27	3302287	100
		3	19	3347452	99
		4	36	3347760	99
		5	33	3347907	98[a]
	Lowest	1	15	3300909	41
		2	57	3270745	42
		3	29	3166125	44
		4	53	3102256	48
		5	4	3347494	49

Table 5.7: SKILLPOST extreme values

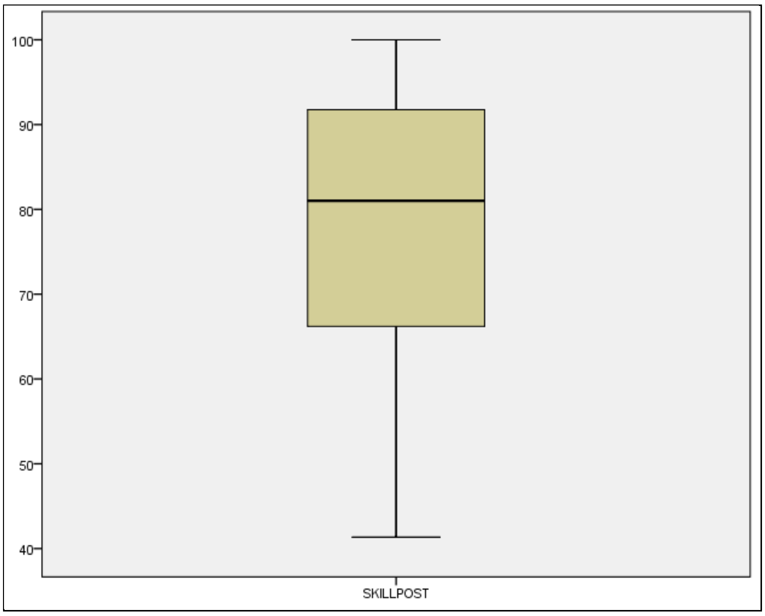

Figure 5.9: SKILLPOST boxplot

			Statistic	Std. Error
CONCPOST	Mean		59.94	2.249
	95% Confidence Interval for Mean	Lower Bound	55.45	
		Upper Bound	64.44	
	5% Trimmed Mean		59.89	
	Median		60.18	
	Variance		318.632	
	Std. Deviation		17.850	
	Minimum		20	
	Maximum		94	
	Range		74	
	Interquartile Range		30	
	Skewness		.062	.302
	Kurtosis		-.855	.595

Table 5.8: CONCPOST 5% trimmed mean

			Case Number	Student Number	Value
CONCPOST	Highest	1	5	3347786	94
		2	36	3347760	93
		3	27	3302287	90
		4	1	3347698	90
		5	20	3347615	86
	Lowest	1	7	3300057	20
		2	59	3213259	31
		3	13	3301450	34
		4	23	3301484	34
		5	15	3300909	36

Table 5.9: CONCPOST extreme values

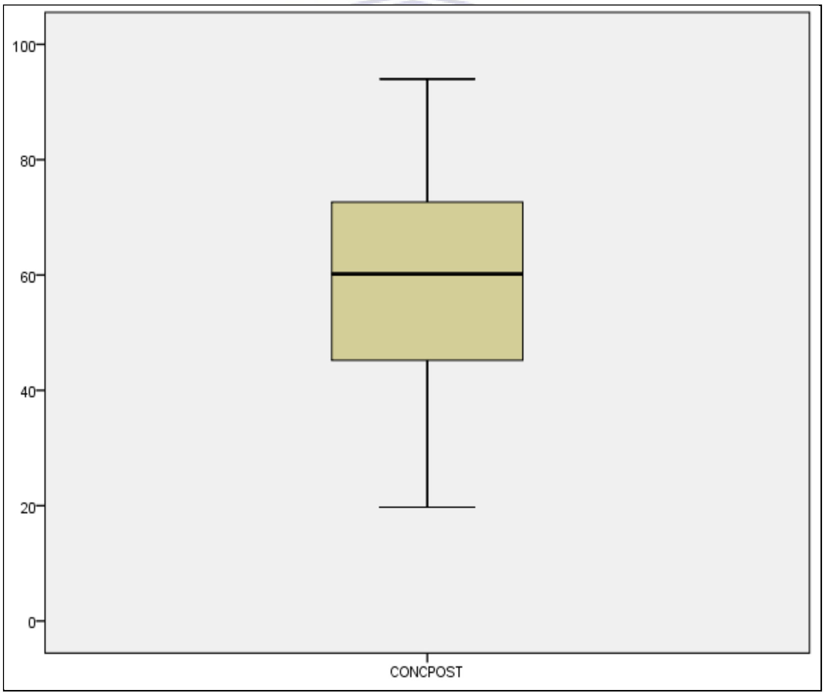

Figure 5.10: CONCPOST boxplot

			Statistic	Std. Error
PRETOT	Mean		57.35	2.380
	95% Confidence Interval for Mean	Lower Bound	52.59	
		Upper Bound	62.11	
	5% Trimmed Mean		57.26	
	Median		56.34	
	Variance		356.773	
	Std. Deviation		18.888	
	Minimum		22	
	Maximum		94	
	Range		72	
	Interquartile Range		31	
	Skewness		.134	.302
	Kurtosis		-1.088	.595

Table 5.10: PRETOT 5% trimmed mean

			Case Number	Student Number	Value
PRETOT	Highest	1	62	3043417	94
		2	8	3347834	89
		3	19	3347452	88
		4	36	3347760	87
		5	5	3347786	86
	Lowest	1	7	3300057	22
		2	53	3102256	27
		3	3	3301443	28
		4	37	3300421	29
		5	25	3257801	32

Table 5.11: PRETOT extreme values

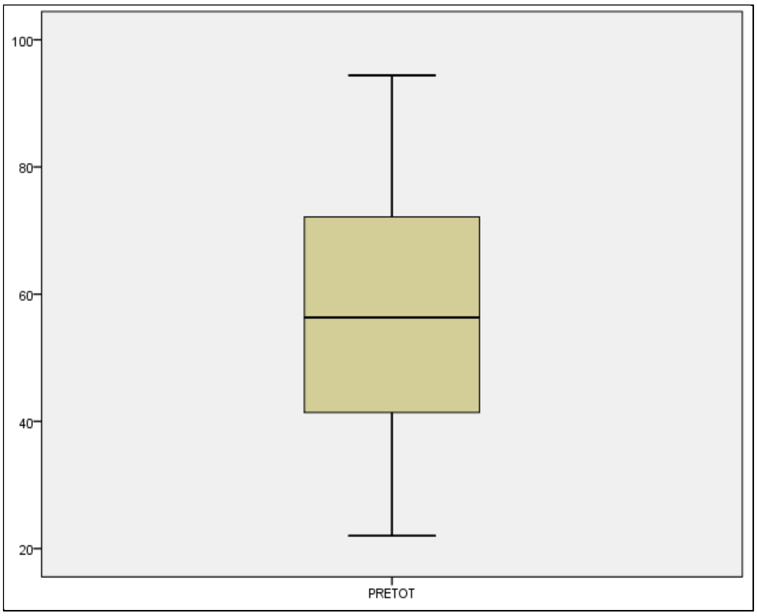

Figure 5.11: PRETOT boxplot

			Statistic	Std. Error
POSTTOT	Mean		68.96	2.011
	95% Confidence Interval for Mean	Lower Bound	64.94	
		Upper Bound	72.98	
	5% Trimmed Mean		69.03	
	Median		69.11	
	Variance		254.729	
	Std. Deviation		15.960	
	Minimum		39	
	Maximum		96	
	Range		57	
	Interquartile Range		26	
	Skewness		-.106	.302
	Kurtosis		-1.115	.595

Table 5.12: POSTTOT 5% trimmed mean

			Case Number	Student Number	Value
POSTTOT	Highest	1	36	3347760	96
		2	5	3347786	95
		3	27	3302287	95
		4	1	3347698	92
		5	8	3347834	92
	Lowest	1	15	3300909	39
		2	57	3270745	42
		3	4	3347494	43
		4	7	3300057	43
		5	53	3102256	45

Table 5.13: POSTTOT extreme values

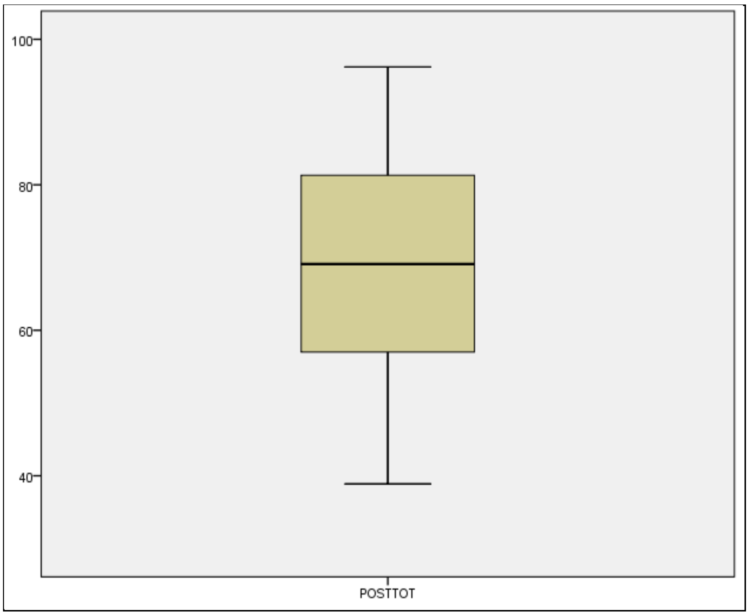

Figure 5.12: POSTTOT boxplot

An examination of the above trimmed means, extreme values and boxplots yield the following conclusions. Firstly, any scores that IMB SPSS (version 23) considers outliers appear as little circles with a number attached. Points are defined as outliers if they extend more than 1.5 box-lengths from the edge of the box and extreme points are those that extend more than three box-lengths from the edge of the box. No outlier or extreme outliers are identified by the boxplots. Secondly, the differences between the extreme values as indicated in the tables are not significant. Lastly, the differences between the original mean and the trimmed mean are not significant in all the cases (all differences are less than 1%) and hence the outliers do not have a major influence on the data.

5.3 Paired-samples tests

Comparisons of the means of the variables yield the following results. The mean for SKILLPRE was 65.71 whereas the mean for SKILLPOST was 77.98 which therefore imply that there was an increase of 12.27% in mean score for these variables. The mean for CONCPRE was 48.99 versus a mean of 59.94 for CONCPOST, which implies an increase of 10.95%. PRETOT had a mean of 57.35 and POSTTOT had a mean of 68.93 giving an increase of 11.61%. An important question arises concerning whether a significant difference exists between pre-test and post-test scores. Since this investigation is an example of a repeated-measures study I employed a paired-samples test (*t* statistic) that is based on the overall mean difference (μ_D) to answer this question.

In a paired-sample test the difference between pre- and post-test scores for each participant is calculated and then the overall mean difference is calculated by adding all the difference scores. The paired samples *t* statistic requires two basic assumptions: one, that observations within each treatment must be independent and two, that the population distribution of difference scores must be normal (Gravetter & Walnau, 2002). This study satisfies the first assumption since different scores were obtained from the same individual. Gravetter and Walnau (2002) maintain that if the sample is relatively large ($n > 30$) then the normality assumption can be ignored. Since $n = 63$ for this study the normality assumption is ignored. The related samples *t* statistic is therefore the preferred statistic. It is also important to report the effect size since it is difficult to evaluate the importance of differences between means or to compare such differences with other experiments (Myers et al, 2010). Gravetter and Walnau (2002) contend that the two commonly used measures of effect size are Cohen's d and r^2, the percentage of variance accounted for.

The null hypothesis is that there is no significant difference after exposure to the teaching strategy. In other words, the mean difference of the pre- and post-test score for the population is zero i.e.

$$H_0: \mu_D = 0$$

The alternative hypothesis is that the intervention caused the post-test scores to be higher or lower than the pre-test scores. In other words, the mean difference is not zero:

$$H_1: \mu_D \neq 0$$

The level of significance is set at $\alpha = .05$ for a two-tailed test. For this case $n = 63$, hence the *t* statistic will have $df = n - 1 = 63 - 1 = 62$. From the *t*-distribution table we find that the critical values for a two-tailed test are +2.000 and -2.000.

5.3.1 Paired-samples test for SKILLPRE and SKILLPOST

Table 5.14 shows the means, standard deviation and standard error for the individual variables SKILLPRE and SKILLPOST. Table 5.15 shows the statistics for the paired samples *t* test.

		Mean	N	Std. Deviation	Std. Error Mean
Pair 1	SKILLPRE	65.71	63	21.093	2.658
	SKILLPOST	77.98	63	16.289	2.052

Table 5.14: Descriptive statistics for SKILLPRE and SKILLPOST

	Paired Differences					t	df	Sig. (2-tailed)
	Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference Lower	Upper			
Pair 1 SKILLPRE - SKILLPOST	-12.278	15.455	1.947	-16.171	-8.386	-6.306	62	.000

Table 5.15: Paired samples test for SKILLPRE and SKILLPOST

In the calculations of effect size that follows below *D* represents the difference between SKILLPRE and SKILLPOST, *t* represents the *t* statistic and *df* refers to degrees of freedom. The calculations below show Cohen's d and r^2:

$$\bar{D} = \frac{\sum D}{n} = \frac{774}{63} = 12.286$$

$$SS = \sum D^2 - \frac{(\sum D)^2}{n} = 24\ 306 - \frac{(774)^2}{63} = 14\ 796.86$$

$$s^2 = \frac{SS}{n-1} = \frac{14\ 796.86}{63-1} = 238.659$$

$$Cohen's\ d = \frac{\bar{D}}{s} = \frac{12.286}{\sqrt{238.659}} = 0.795$$

This implies that we have a large effect since d is close to 0.8.

$$r^2 = \frac{t^2}{t^2+df} = \frac{(-6.306)^2}{(-6.306)^2+62} = 0.391$$

For these data 39% of the variance in the scores is accounted for by the difference before the intervention of the teaching strategy compared with what was the case after the intervention.

Since the obtained t value falls within the critical region that is, $t < -2.00$ the null hypothesis is rejected and we conclude that the teaching intervention did affect the post-score. Based on the above results one can therefore conclude that the teaching strategy resulted in an increase in SKILLPOST ($M = 12.28$, $SD = 15.46$). This increase was statistically significant, $t(62) = -6.306, p < 0.05$, two-tailed.

5.3.2 Paired-samples test for CONCPRE and CONCPOST

A paired-samples test was also performed for CONCPRE and CONCPOST. Tables 5.16 and 5.17 show the statistics for the paired-samples test.

		Mean	N	Std. Deviation	Std. Error Mean
Pair 1	CONCPRE	48.99	63	18.692	2.355
	CONCPOST	59.94	63	17.850	2.249

Table 5.16: Descriptive statistics for CONCPRE and CONCPOST

		Paired Differences					t	df	Sig. (2-tailed)
					95% Confidence Interval of the Difference				
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper			
Pair 1	CONCPRE - CONCPOST	-10.950	11.128	1.402	-13.752	-8.147	-7.810	62	.000

Table 5.17: Paired-samples test for CONCPRE and CONCPOST

Cohen's d and r^2 for CONCPRE and CONCPOST is shown below:

$$Cohen's\ d = 0.984$$

This implies that we have a large effect size since $d > 0.8$.

$$r^2 = 0.4959$$

For these data 50% of the variance in the scores is accounted for by the difference before the intervention of the teaching strategy compared with the case after the intervention.

Since the obtained *t* value falls in the critical region that is, $t < -2.00$ the null hypothesis is rejected and we conclude that the teaching intervention did affect the post-score. Based on the above results one can therefore conclude that the teaching strategy resulted in an increase in CONCPOST ($M = 10.95$, $SD = 11.128$). This increase was statistically significant, $t(62) = -7.81, p < 0.05$, two-tailed.

5.3.3 Paired-samples test for PRETOT and POSTTOT

The paired-samples test was also performed for PRETOT and POSTTOT. Tables 5.18 and 5.19 show the statistics for the paired-samples test.

		Mean	N	Std. Deviation	Std. Error Mean
Pair 1	PRETOT	57.35	63	18.888	2.380
	POSTTOT	68.96	63	15.960	2.011

Table 5.18: Descriptive statistics for PRETOT and POSTTOT

	Paired Differences					t	df	Sig. (2-tailed)
	Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference Lower	95% Confidence Interval of the Difference Upper			
Pair 1 PRETOT - POSTTOT	-11.614	9.623	1.212	-14.038	-9.190	-9.579	62	.000

Table 5.19: Paired-samples test for PRETOT and POSTTOT

Cohen's d and r^2 for PRETOT and POSTTOT is shown below:

$Cohen's\ d$ = 1.208
This implies that we have a large effect size since $d > 0.8$.

$$r^2 = 0.5968$$

For these data 60% of the variance in the scores is accounted for by the difference before the intervention of the teaching strategy compared with the case after the intervention.

Since the obtained *t* value falls in the critical region that is, $t < -2.00$ the null hypothesis is rejected and we conclude that the teaching intervention did affect the post-score. Based on the above results one can therefore conclude that the teaching strategy resulted in an increase in POSTTOT ($M = 10.95$, $SD = 11.128$). This increase was statistically significant, $t(62) = -9.579, p < 0.05$, two-tailed.

5.4 STRATIFIED ANALYSIS

In order to determine how the teaching intervention affected the research participants individually a stratified analysis was done by using the statistical technique known as cross-tabulation. What this essentially means is that the pre-test scores are compared to post-test scores to determine whether the score of individual participants improved, remained the same or regressed. Since it would be a very difficult task to use percentages for each individual, the scores were ranked as shown in Table 5.20. The aim of the stratified analysis was therefore to determine what the rank was in the post-test if for example a participant had had a rank of 1 in the pre-test. It was thought that such a comparison might provide more insight as to how the lower, middle and high-ranked individuals were affected by the teaching intervention.

RANK	PERCENTAGE BAND
1	0% - 30%
2	31% - 40%
3	41% - 50%
4	51% - 60%
5	61% - 70%
6	71% - 80%
7	81% - 90%
8	91% - 100%

Table 5.20: Percentage ranks

Using the ranks together with the two variables SKILLPRE and SKILLPOST, an 8×8 cross-tabulation table was constructed. A similar table was constructed for CONCPRE and CONCPOST. The statistical measures for SKILLPRE and SKILLPOST are shown in Tables 5.21 and 5.22. The chi-square test statistics (Table 5.21) are not valid for this investigation since 100% of cells have expected frequencies lower than 5. The symmetrical measures shown in Table 5.22 provide measures of the strength of the relationships or effect size of the variables involved. The assumptions and requirements for phi and Cramer's *V* are the same as for the chi-square tests, in other words, at least 80% of the expected frequencies should be 5 or larger. As already noted this condition is not met and hence these statistical measures will not be used.

Kendall's tau is a non-parametric measure of correlation between two ranked variables. It evaluates the degree of similarity between two sets of ranks given to the same set of objects (Abdi, 2007). Kendall's tau is different from Spearman's rho and Pearson's *r* in that it represents a probability. It can be interpreted as the difference between the probability that the observed data are in the same order and the probability that the observed data are not in the same order (Abdi, 2007). Kendall's tau is the probability of the difference of the concordant pairs and the discordant pairs. A concordant pair is when the rank of the second variable is greater than the rank of the former variable, whereas a discordant pair is when the rank is equal to or less than the rank of the first variable. The two variations of Kendall's tau namely tau-b and tau-c differ only in the way they handle rank ties. Since in our case we have two variables that are ranked and the interest is to determine movement in ranks, tau-b is the preferred statistic.

The analysis for SKILLPRE and SKILLPOST (Table 5.22) indicate a significant positive association between the two variables, tau-b $= .569,\ \rho < .001$. This tau is considered to be a large effect size (Cohen, 1992).

	Value	df	Asymptotic Significance (2-sided)	Exact Sig. (2-sided)	Exact Sig. (1-sided)
Pearson Chi-Square	59.628[a]	35	.006	.[b]	
Likelihood Ratio	68.205	35	.001	.[b]	
Fisher's Exact Test	.[c]			.[c]	
Linear-by-Linear Association	26.160	1	.000	.[b]	.[b]
N of Valid Cases	63				

Table 5.21: Chi-Square Tests for SKILLPRE and SKILLPOST

		Value	Asymptotic Standardized Error[a]	Approximate T[b]	Approximate Significance	Exact Significance
Nominal by Nominal	Phi	.973			.006	.[c]
	Cramer's V	.435			.006	.[c]
	Contingency Coefficient	.697			.006	.[c]
Ordinal by Ordinal	Kendall's tau-b	.569	.060	9.877	.000	.[c]
	Kendall's tau-c	.565	.057	9.877	.000	.[c]
	Gamma	.663	.069	9.877	.000	.[c]
	Spearman Correlation	.709	.058	7.848	.000[d]	.[c]
Interval by Interval	Pearson's R	.650	.068	6.673	.000[d]	.[c]
N of Valid Cases		63				

Table 5.22: Symmetric Measures for SKILLPRE and SKILLPOST

5.4.1 Stratified analysis for SKILLPRE and SKILLPOST

5.4.1.1 Controlling for a pre-test rank of 1

Table 5.23 reports on how participants moved within the ranks from pre-test to post-test. The table shows that four students had a pre-test rank of 1 which amounts to 6.3% of the total number of students. One of these students moved to rank 3, one moved to rank 5, one moved to rank 6 and another moved to rank 7. It is also worth noting that no students scored in the lowest two ranks in the post-test and that the lowest post-test rank is 3. There was thus 0% with rank 1 in the post-test and hence a reduction of 6% from pre- to post-test.

5.4.1.2 Controlling for a pre-test rank of 2

Two students had a pre-test rank of 2 which is 3.2% of the total number of students. One of these students moved to rank 4 and the other to rank 7. Again, we have 0% with rank 2 in the post-test and hence a reduction of 3% from pre- to post-test.

5.4.1.3 Controlling for a pre-test rank of 3

Eleven students had a pre-test rank of 3, which is 17.5% of the total number of students. After intervention one student remained ranked 3, three moved to rank 4, four moved to rank 5, two moved to rank 6 and one moved to rank 7. After intervention, 7.9% of students had a rank of 3, which is a decline of approximately 10%.

5.4.1.4 Controlling for a pre-test rank of 4

Nine students had a pre-test rank of 4, which is 14.3% of the total number of students. In the post-test, three of these students regressed to rank 3; one remained in rank 4; two moved to rank 5 and three moved to rank 6. Post intervention 11.1% of students had a rank of 4 which shows a decline of approximately 3%.

5.4.1.5 Controlling for a pre-test rank of 5

Eleven students had a pre-test rank of 5, which constitutes 17.5% of the total number of students. In the post-test one of these regressed to rank 4, one remained in rank 5, two progressed to rank 6, five moved to rank 7 and two moved to 8. After the intervention 12.7% of students had a rank of 5, which shows a decline of approximately 5%.

5.4.1.6 Controlling for a pre-test rank of 6

Eight students had a pre-test rank of 6, which is 12.7% of the total number of students. Post-test scores show that one of these students regressed to rank 4, one remained in rank 6, two moved to rank 7 and four moved to rank 8. In the post-test 15.9% of students had a rank of 6, which shows an increase of about 3%.

5.4.1.7 Controlling for a pre-test rank of 7

Eight students had a pre-test rank of 7, which is 12.7% of the total number of students. Post-test scores show that one of these students regressed to rank 6, three remained in rank 7 and four progressed to rank 8. After intervention 23.8% of students had a rank of 7, which is equal to an increase of 11%.

5.4.1.8 Controlling for a pre-test rank of 8

Ten students had a pre-test rank of 8, which constitutes 15.9% of the total number of students. Post-test ranks show that two students regressed to rank 7 and eight remained in rank 8. Post intervention 28.6% of students had a rank 8, which is equal to an increase of approximately 13%.

			SKILL POST RANK						
			3	4	5	6	7	8	Total
SKILLPRE RANK	1	Count	1	0	1	1	1	0	4
		% within RANK PRE	25.0%	0.0%	25.0%	25.0%	25.0%	0.0%	100.0%
		% of Total	1.6%	0.0%	1.6%	1.6%	1.6%	0.0%	6.3%
	2	Count	0	1	0	0	1	0	2
		% within RANK PRE	0.0%	50.0%	0.0%	0.0%	50.0%	0.0%	100.0%
		% of Total	0.0%	1.6%	0.0%	0.0%	1.6%	0.0%	3.2%
	3	Count	1	3	4	2	1	0	11
		% within RANK PRE	9.1%	27.3%	36.4%	18.2%	9.1%	0.0%	100.0%
		% of Total	1.6%	4.8%	6.3%	3.2%	1.6%	0.0%	17.5%
	4	Count	3	1	2	3	0	0	9
		% within RANK PRE	33.3%	11.1%	22.2%	33.3%	0.0%	0.0%	100.0%
		% of Total	4.8%	1.6%	3.2%	4.8%	0.0%	0.0%	14.3%
	5	Count	0	1	1	2	5	2	11
		% within RANK PRE	0.0%	9.1%	9.1%	18.2%	45.5%	18.2%	100.0%
		% of Total	0.0%	1.6%	1.6%	3.2%	7.9%	3.2%	17.5%
	6	Count	0	1	0	1	2	4	8
		% within RANK PRE	0.0%	12.5%	0.0%	12.5%	25.0%	50.0%	100.0%
		% of Total	0.0%	1.6%	0.0%	1.6%	3.2%	6.3%	12.7%
	7	Count	0	0	0	1	3	4	8
		% within RANK PRE	0.0%	0.0%	0.0%	12.5%	37.5%	50.0%	100.0%
		% of Total	0.0%	0.0%	0.0%	1.6%	4.8%	6.3%	12.7%
	8	Count	0	0	0	0	2	8	10
		% within RANK PRE	0.0%	0.0%	0.0%	0.0%	20.0%	80.0%	100.0%
		% of Total	0.0%	0.0%	0.0%	0.0%	3.2%	12.7%	15.9%
Total		Count	5	7	8	10	15	18	63
		% within RANK PRE	7.9%	11.1%	12.7%	15.9%	23.8%	28.6%	100.0%
		% of Total	7.9%	11.1%	12.7%	15.9%	23.8%	28.6%	100.0%

Table 5.23: RANK PRE and RANK POST Cross-tabulation for SKILLPRE and SKILLPOST

5.4.1.9 Conclusion for Stratified analysis of SKILLPRE and SKILLPOST

Table 5.24 summarizes the increase or decrease in the number of students for each rank. A negative value indicates a decrease whereas a positive indicates an increase. It should be noted that rank 6, 7 and 8 are the only ones that show an increase and also that no students were rank 1 and 2 in the post-test. In the pre-test 9.5% of students were in rank 1 and 2 and hence it implies that all of these students moved to higher ranks. Post-test only 7.9% of students were in rank 3 which ordinarily is the pass/fail cut-off score. It is also worthwhile noting that the biggest increase was in rank 8. This is perhaps an indication that the teaching strategy premised on repeated revision helped to increase the procedural fluency of participating students. Figure 5.13 shows the increase decrease graph.

RANK	INCREASE/DECREASE (PERC POST – PERC PRE)
1	-6%
2	-3.2%
3	-9.6%
4	-3.2%
5	-4.8%
6	3.2%
7	11.1%
8	12.7%

Table 5.24: Percentage increase/decrease for SKILLPRE and SKILLPOST

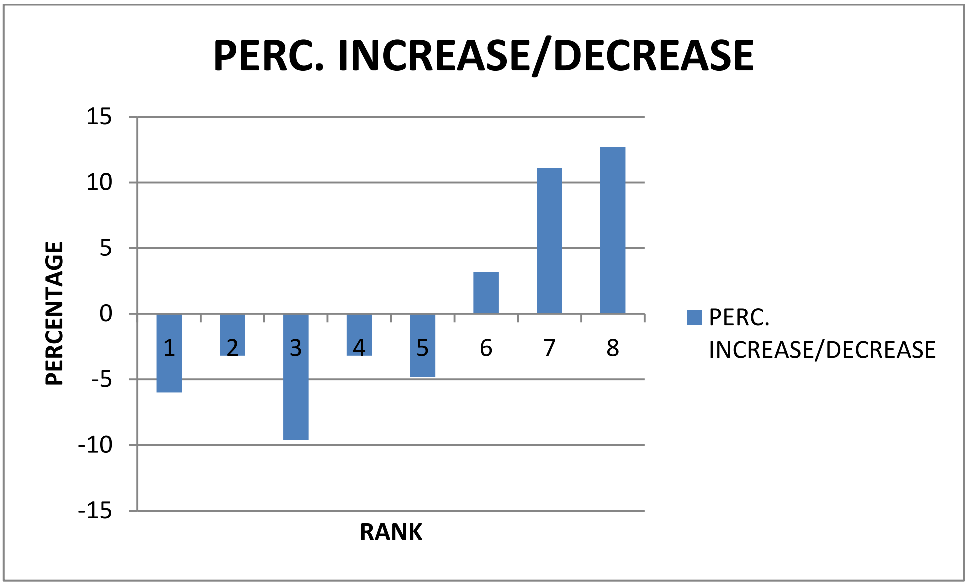

Figure 5.13: SKILLPRE and SKILLPOST percentage increase/decrease in number of students

5.4.2 Stratified analysis for CONCPRE and CONCPOST

A stratified analysis was also done for CONCPRE and CONCPOST the results of which are shown in Tables 5.25 and 5.26.

The analysis shows a significant positive association between CONCPRE and CONCPOST, tau-b (63) = $.661,\ \rho < .001$. This tau is considered to be a large effect size (Cohen, 1992).

		Value	Asymptotic Standardized Error[a]	Approximate T[b]	Approximate Significance	Exact Significance
Nominal by Nominal	Phi	1.102			.001	.[c]
	Cramer's V	.450			.001	.[c]
	Contingency Coefficient	.740			.001	.[c]
Ordinal by Ordinal	Kendall's tau-b	.661	.040	15.455	.000	.[d]
	Kendall's tau-c	.647	.042	15.455	.000	.[d]
	Gamma	.763	.043	15.455	.000	.[d]
Measure of Agreement	Kappa	.160	.062	3.359	.001	.[c]
N of Valid Cases		63				

Table 5.25: Symmetric Measures for CONCPRE and CONCPOST

5.4.2.1 Controlling for a pre-test rank of 1

Table 5.26 shows that eleven students had a pre-test rank of 1 which amounts to 17.5% of the total number of students. One of these students remained in rank 1, three moved to rank 2, six moved to rank 3 and one moved to rank 4. Post intervention only one student had rank 1, which amounts to 1.6% of the total number of students. This amounts to a decrease of approximately 16%.

5.4.2.2 Controlling for a pre-test rank of 2

Fifteen students were rank 2 before intervention, which amounts to 23.8% of the total number of students. Five of these students remained rank 2, four students moved to rank 3, three students moved to rank 4 and three students moved to rank 5. After intervention eight students were rank 2, which is 12.7% of the total number of students. There was therefore a decrease of about 11% in rank 2.

5.4.2.3 Controlling for a pre-test rank of 3

Seven students had a pre-test rank of 3 which amounts to 11.1% of the total number of students. After intervention two of these students remained rank 3, two moved to rank 4 and three moved to rank 5. Post intervention thirteen students were rank 3, which is 20.6% of the total number of students. There was therefore an increase of approximately 10% in rank 3.

5.4.2.4 Controlling for a pre-test rank of 4

Ten students had a pre-test rank of 4, which amounts to 15.9% of the total number of students. One of these students regressed to rank 3, one remained rank 4, three moved to rank 5, three moved to rank 6 and two progressed to rank 7. After intervention ten students were rank 4 and hence there was no net increase or decrease for this rank.

5.4.2.5 Controlling for a pre-test rank of 5

Nine students had a pre-test rank of 5, which amounts to 14.3% of the total number of students. Two of these students regressed to rank 4, three students remained in rank 5, one student moved to rank 6 and three students moved to rank 7. Post-test twelve students were in rank 5, which amounts to 19% of the total number of students. This is equivalent to an increase of 5%

5.4.2.6 Controlling for a pre-test rank of 6

Eight students had a pre-test rank of 6, which amounts to 12.7% of the total number of students. Post-test one of these students regressed to rank 4, three remained in rank 6, two moved to rank 7 and two progressed to rank 8. Post intervention eight students had a rank of 6, which amounts to 12.7% of the total number of students. There was thus no net movement in rank 6.

5.4.2.7 Controlling for a pre-test rank of 7

Three students had a pre-test rank of 7, which amounts to 4.8% of the total number of students. Post-test one of these students regressed to rank 6 and two remained rank 7. Post intervention nine students were in rank 7, which amounts to 14.3% of the total number of students. There was therefore an increase of approximately 10%.

5.4.2.8 Controlling for a pre-test rank of 8

No student had a pre-test rank of 8. Post intervention two students had a rank of 8, which amounts to an increase of 3.2% of the total number of students.

			CRANKPOST							
			1	2	3	4	5	6	7	8	Total
CRANKPRE	1	Count	1	3	6	1	0	0	0	0	11
		% within CRANKPRE	9.1%	27.3%	54.5%	9.1%	0.0%	0.0%	0.0%	0.0%	100.0%
		% of Total	1.6%	4.8%	9.5%	1.6%	0.0%	0.0%	0.0%	0.0%	17.5%
	2	Count	0	5	4	3	3	0	0	0	15
		% within CRANKPRE	0.0%	33.3%	26.7%	20.0%	20.0%	0.0%	0.0%	0.0%	100.0%
		% of Total	0.0%	7.9%	6.3%	4.8%	4.8%	0.0%	0.0%	0.0%	23.8%
	3	Count	0	0	2	2	3	0	0	0	7
		% within CRANKPRE	0.0%	0.0%	28.6%	28.6%	42.9%	0.0%	0.0%	0.0%	100.0%
		% of Total	0.0%	0.0%	3.2%	3.2%	4.8%	0.0%	0.0%	0.0%	11.1%
	4	Count	0	0	1	1	3	3	2	0	10
		% within CRANKPRE	0.0%	0.0%	10.0%	10.0%	30.0%	30.0%	20.0%	0.0%	100.0%
		% of Total	0.0%	0.0%	1.6%	1.6%	4.8%	4.8%	3.2%	0.0%	15.9%
	5	Count	0	0	0	2	3	1	3	0	9
		% within CRANKPRE	0.0%	0.0%	0.0%	22.2%	33.3%	11.1%	33.3%	0.0%	100.0%
		% of Total	0.0%	0.0%	0.0%	3.2%	4.8%	1.6%	4.8%	0.0%	14.3%
	6	Count	0	0	0	1	0	3	2	2	8
		% within CRANKPRE	0.0%	0.0%	0.0%	12.5%	0.0%	37.5%	25.0%	25.0%	100.0%
		% of Total	0.0%	0.0%	0.0%	1.6%	0.0%	4.8%	3.2%	3.2%	12.7%
	7	Count	0	0	0	0	0	1	2	0	3
		% within CRANKPRE	0.0%	0.0%	0.0%	0.0%	0.0%	33.3%	66.7%	0.0%	100.0%
		% of Total	0.0%	0.0%	0.0%	0.0%	0.0%	1.6%	3.2%	0.0%	4.8%
Total		Count	1	8	13	10	12	8	9	2	63
		% within CRANKPRE	1.6%	12.7%	20.6%	15.9%	19.0%	12.7%	14.3%	3.2%	100.0%
		% of Total	1.6%	12.7%	20.6%	15.9%	19.0%	12.7%	14.3%	3.2%	100.0%

Table 5.26: RANKPRE and RANKPOST Cross-tabulation for CONCPRE and CONCPOST

5.4.2.9 Conclusion for Stratified analysis of CONCPRE and CONCPOST

Table 5.27 summarizes the increase or decrease in the number of students for each rank. A negative value indicates a decrease whereas a positive indicates an increase. Ranks 1 and 2 showed a decrease and ranks 3, 5, 7 and 8 showed an increase. The lowest two ranks together showed a decrease of 27% whereas the top two ranks together showed an increase of 13%. Although twenty-two students (34.92%) were still in ranks 1, 2 and 3 (the below 50% ranks), thirty-one students (49.21%) were in ranks 5 to 8 (the above 60% ranks) post intervention. In the pre-test thirty-three students (52.38%) were in ranks 1, 2 and 3, whereas twenty students (31.74%) were in ranks 5 to 8. There was thus a net movement from the lower ranks to the higher ranks which perhaps can be attributed to the teaching strategy implemented. In particular, the productive practice component of the teaching strategy which emphasised conceptual understanding can most probably be one of the factors that contributed to this net increase. Figure 5.14 shows the increase/decrease in graph form.

RANK	INCREASE/DECREASE (PERC POST – PERC PRE)
1	-15.9%
2	-11.1%
3	9.5%
4	0%
5	4.7%
6	0%
7	9.5%
8	3.2%

Table 5.27: Percentage increase/decrease for CONCPRE and CONCPOST

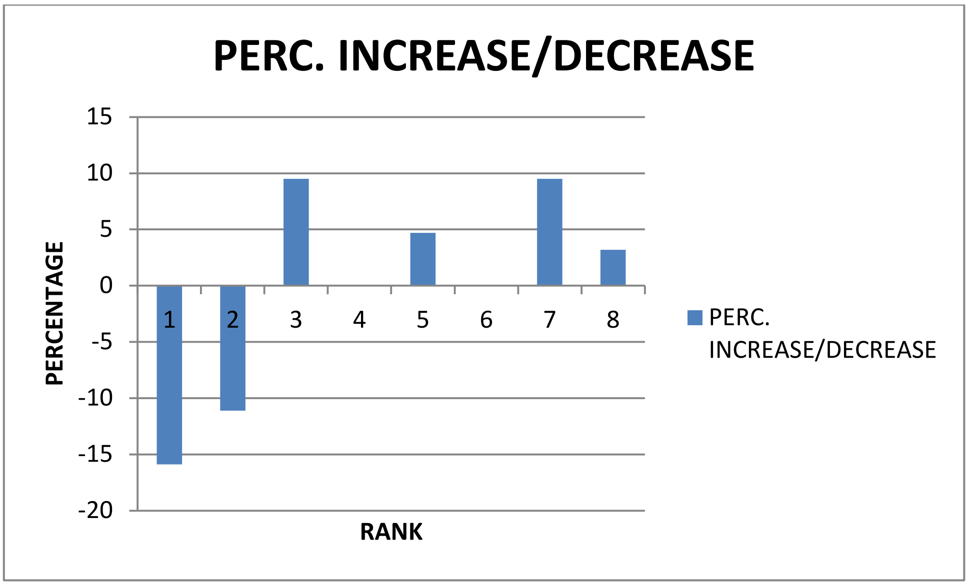

Figure 5.14: CONCPRE and CONCPOST percentage increase/decrease in number of students

5.4.3 Stratified analysis of categories constituting procedural fluency and conceptual understanding

Table 5.28 shows a comparison of the variables that constitute the pre-and post-test variables of SKILL (procedural fluency) and CONC (conceptual understanding). As indicated previously SKILL is constituted by A1a, B1a, B1bi, C1bi and C1bii whereas CONC is constituted by D1a, D1bi, D1bii and D2a. Tables 5.29 and 5.30 show the difference between the means pre- and post- for SKILL and CONC variables respectively. The difference for the variables A1a, B1a and B1bi is relatively high, whereas in the case of C1bii it is low and for C1bi it is negative. In the case of CONC the difference for both the variables D1bi and D1bii is above 15, but D1a and D2a each have a negative difference. Stratified analysis was performed in order to gain insight as to why the differences for C1bi, D1a and D2a are negative. Table 5.31 shows the stratified analysis for C1bi.

	N	Range	Minimum	Maximum	Mean (%)	Std. Deviation
A1aPRE	63	94	6	100	64.58	29.805
A1aPOST	63	100	0	100	90.48	29.590
B1aPRE	63	100	0	100	57.94	45.086
B1aPOST	63	100	0	100	68.25	40.213
B1biPRE	63	62	37	99	69.84	14.987
B1biPOST	63	28	72	100	90.42	7.922
C1biPRE	63	83	17	100	71.67	17.895
C1biPOST	63	100	0	100	69.44	28.379
C1biiPRE	63	98	2	100	64.49	23.964
C1biiPOST	63	66	34	100	71.32	16.811
D1aPRE	63	85	10	95	57.89	21.407
D1aPOST	63	95	5	100	57.88	24.661
D1biPRE	63	80	18	99	55.68	20.179
D1biPOST	63	79	21	100	72.75	20.771
D1biiPRE	63	92	5	97	45.35	25.671
D1biiPOST	63	67	32	99	72.98	18.183
D2aPRE	63	98	0	98	37.05	25.948
D2aPOST	63	94	0	94	36.17	25.068
Valid N (listwise)	63					

Table 5.28: Descriptive Statistics of categories constituting procedural fluency and conceptual understanding

difference = post – pre
A1aPOST - A1aPRE = 90.48 - 64.58 = 25.9
B1aPOST – B1aPRE = 68.25 – 57.94 = 10.31
B1biPOST – B1biPRE = 90.42 – 69.84 = 20.58
C1biPOST – C1biPRE = 69.44 – 71.67 = -2.23
C1biiPOST – C1biiPRE = 71.32 – 64.49 = 6.83

Table 5.29: Difference between SKILL means

difference = post – pre
D1aPOST – D1aPRE = 57.88 – 57.89 = -0.01
D1biPOST – D1biPRE = 72.75 – 55.68 = 17.07
D1biiPOST – D1biiPRE = 72.98 – 45.35 = 27.63
D2aPOST – D2aPRE = 36.17 – 37.05 = -0.88

Table 5.30: Difference between CONC means

5.4.3.1 Stratified analysis of C1biPRE and C1biPOST

5.4.3.1.1 Controlling for a pre-test rank of 1

Table 5.31 shows that two students had a pre-test rank of 1 which amounts to 3.2% of the total number of students. One of these students moved to rank 5 and the other to rank 6. Post intervention eight students had rank 1, which amounts to 12.7% of the total number of students. This amounts to an increase of 9.5% of the total number of students.

5.4.3.1.2 Controlling for a pre-test rank of 2

Only one student had a pre-score of 2, which amounts to 1.6% of the total number of students. This student regressed to rank 1. After intervention six students had rank 2 which amounts to 9.5% of the number of students. This amounts to an increase of approximately 8% of the total number of students.

5.4.3.1.3 Controlling for a pre-test rank of 3

Five students had a pre-test rank of 3, which is equal to 7.9% of the total number of students. One of these students regressed to rank 1, one progressed to rank 4, another one to rank 5 and two progressed to rank 7. After intervention three students had rank 3, which is equivalent to 5% of the total number of students. The difference between pre-and post is equivalent to a decrease of about 3%.

5.4.3.1.4 Controlling for a pre-test rank of 4

Seven students had a pre- rank of 4, which is equal to 11.1% of the total number of students. One of these students regressed to rank 1, while another regressed to rank 2. One student remained in rank 4; one progressed to rank 6 and three progressed to rank 8. After intervention two students were in rank 4, which is equivalent to 3.2% of the total number of students. This implies the number of students in this rank decreased by 7.9%.

5.4.3.1.5 Controlling for a pre-test rank of 5

Eleven students had a pre-rank of 5, which is equal to 17.5% of the total number of students. Four of these students regressed to rank 1, one regressed to rank 2 and one regressed to rank 3. One student progressed to rank 5 and four improved to rank 7. After intervention six students were in rank 5, which is equivalent to 9.5% of students. Thus, the number of students in this rank decreased by 8%.

5.4.3.1.6 Controlling for a pre-test rank of 6

Seventeen students had a pre-rank of 6, which is equal to 27% of the total number of students. One of these regressed to rank 1, three to rank 2, two to rank 3 and two to rank 5. Two students remained in rank 6, while three progressed to rank 7 and four progressed to rank 8. Post intervention six students were rank 6, which is equal to 9.5% of total number of students. Hence the number of students in this rank decreased by 17.5%.

5.4.3.1.7 Controlling for a pre-test rank of 7

Ten students had pre-rank of 7, which is equal to 15.9% of the total number of students. One of these students regressed to rank 2, one to rank 5 and one to rank 6. Two students remained in rank 7 and five progressed to rank 8. After intervention twelve students were in rank 7, which is equal to 19% of the total number of students. Hence the number of students in this rank increased by about 3%.

5.4.3.1.8 Controlling for a pre-test rank of 8

Ten students had a pre-rank of 8, which is equal to 15.9% of the total number of students. One of these students regressed to rank 6 while another regressed to 7 and eight remained in rank 8. After intervention twenty students were in rank 8, which is equal to 31.7% of the total number of students. The number of students in this rank had therefore increased by 15.8%.

		C1biPOST-RANK								Total
		1	2	3	4	5	6	7	8	
C1biPRERANK	1 Count	0	0	0	0	1	1	0	0	2
	% within C1biPRERANK	0.0%	0.0%	0.0%	0.0%	50.0%	50.0%	0.0%	0.0%	100.0%
	% of Total	0.0%	0.0%	0.0%	0.0%	1.6%	1.6%	0.0%	0.0%	3.2%
	2 Count	1	0	0	0	0	0	0	0	1
	% within C1biPRERANK	100.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	100.0%
	% of Total	1.6%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	1.6%
	3 Count	1	0	0	1	1	0	2	0	5
	% within C1biPRERANK	20.0%	0.0%	0.0%	20.0%	20.0%	0.0%	40.0%	0.0%	100.0%
	% of Total	1.6%	0.0%	0.0%	1.6%	1.6%	0.0%	3.2%	0.0%	7.9%
	4 Count	1	1	0	1	0	1	0	3	7
	% within C1biPRERANK	14.3%	14.3%	0.0%	14.3%	0.0%	14.3%	0.0%	42.9%	100.0%
	% of Total	1.6%	1.6%	0.0%	1.6%	0.0%	1.6%	0.0%	4.8%	11.1%
	5 Count	4	1	1	0	1	0	4	0	11
	% within C1biPRERANK	36.4%	9.1%	9.1%	0.0%	9.1%	0.0%	36.4%	0.0%	100.0%
	% of Total	6.3%	1.6%	1.6%	0.0%	1.6%	0.0%	6.3%	0.0%	17.5%
	6 Count	1	3	2	0	2	2	3	4	17
	% within C1biPRERANK	5.9%	17.6%	11.8%	0.0%	11.8%	11.8%	17.6%	23.5%	100.0%
	% of Total	1.6%	4.8%	3.2%	0.0%	3.2%	3.2%	4.8%	6.3%	27.0%
	7 Count	0	1	0	0	1	1	2	5	10
	% within C1biPRERANK	0.0%	10.0%	0.0%	0.0%	10.0%	10.0%	20.0%	50.0%	100.0%
	% of Total	0.0%	1.6%	0.0%	0.0%	1.6%	1.6%	3.2%	7.9%	15.9%
	8 Count	0	0	0	0	0	1	1	8	10
	% within C1biPRERANK	0.0%	0.0%	0.0%	0.0%	0.0%	10.0%	10.0%	80.0%	100.0%
	% of Total	0.0%	0.0%	0.0%	0.0%	0.0%	1.6%	1.6%	12.7%	15.9%
Total	Count	8	6	3	2	6	6	12	20	63
	% within C1biPRERANK	12.7%	9.5%	4.8%	3.2%	9.5%	9.5%	19.0%	31.7%	100.0%
	% of Total	12.7%	9.5%	4.8%	3.2%	9.5%	9.5%	19.0%	31.7%	100.0%

Table 5.31: PRERANK and POSTRANK Cross-tabulation for C1biPRe and C1biPOST

5.4.3.1.9 Conclusion for Stratified analysis of C1biPRE and C1biPOST

Table 5.32 summarizes the increase or decrease in the number of students for each rank. A negative value indicates a decrease whereas a positive indicates an increase. The table shows that after intervention ranks 1 and 2 increased by 17.5% and rank 7 and 8 increased by 18.5% whereas ranks 3, 4, 5 and 6 together show a decrease of 36.4%. Figure 5.15 shows the increase/decrease in graph form.

A more informative way of analysing the results of the stratified analysis is to look at the actual number of students who moved between ranks. Ranks 1, 2 and 3 are below 50% which is the pass cut-off point. Before intervention eight students were registered in these 3 ranks, whilst after intervention seventeen students were in these ranks. This implies that these ranks increased by 112.5%. This also implies that 26.98% of the total number of students had fail ranks for this category of question after intervention. Ranks 4, 5 and 6 range between 50% and 80%. Before intervention thirty-five students were in rank 4, 5 and 6, whilst after intervention only fourteen students were in these ranks. This implies that these ranks decreased by 60%. Nine of these students (14% of the total number of students) who had been in these ranks initially had regressed to ranks 1, 2 and 3.

C1bi is the category Familiar Algorithmic Reasoning based on Flexible Procedural Knowledge. The fact that approximately a quarter of students occupy the fail grades and that 14% of students regressed to the fail ranks for this category of question implies that the teaching strategy was not very effective in improving student ability for this category of question. It would seem therefore that a possible weakness in the teaching strategy was that it did not effectively assist all students to acquire the ability to use known procedural knowledge flexibly in novel contexts.

RANK	INCREASE/DECREASE (PERC POST – PERC PRE)
1	9.5%
2	8%
3	-3%
4	-7.9%
5	-8%
6	-17.5%
7	3%
8	15.8%

Table 5.32: Percentage increase/decrease for C1bi

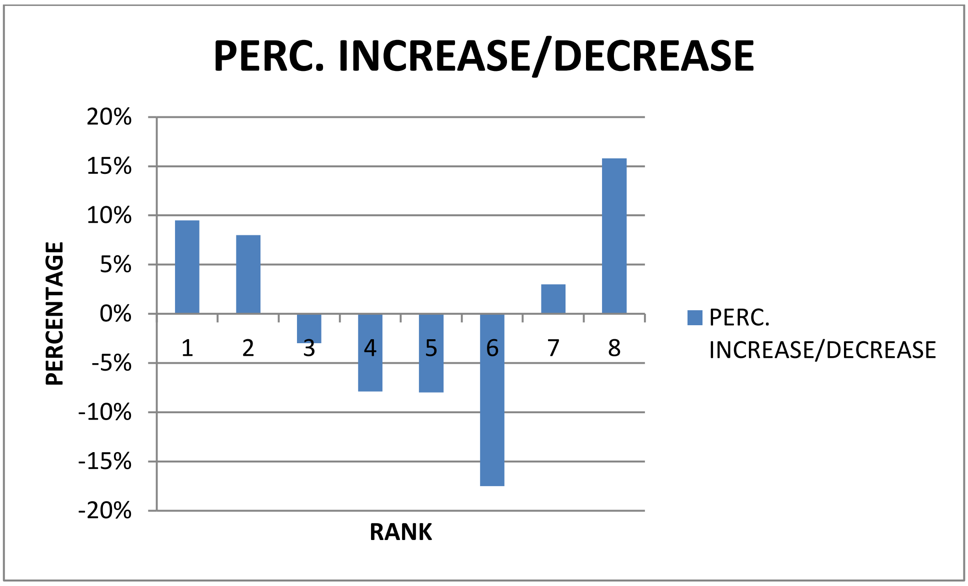

Figure 5.15: C1biPRE and C1biPOST percentage increase/decrease in number of students

5.4.4 Stratified analysis of D1aPRE and D1aPOST

Table 5.33 shows the stratified analysis for D1aPRE and D1aPOST.

			D1aPOSTRANK								Total
			1	2	3	4	5	6	7	8	
D1a PRERANK	1	Count	1	0	1	2	1	1	0	0	6
		% within D1aPRERANK	16.7%	0.0%	16.7%	33.3%	16.7%	16.7%	0.0%	0.0%	100.0%
		% of Total	1.6%	0.0%	1.6%	3.2%	1.6%	1.6%	0.0%	0.0%	9.5%
	2	Count	3	0	1	0	4	1	0	0	9
		% within D1aPRERANK	33.3%	0.0%	11.1%	0.0%	44.4%	11.1%	0.0%	0.0%	100.0%
		% of Total	4.8%	0.0%	1.6%	0.0%	6.3%	1.6%	0.0%	0.0%	14.3%
	3	Count	0	0	0	1	0	2	0	1	4
		% within D1aPRERANK	0.0%	0.0%	0.0%	25.0%	0.0%	50.0%	0.0%	25.0%	100.0%
		% of Total	0.0%	0.0%	0.0%	1.6%	0.0%	3.2%	0.0%	1.6%	6.3%
	4	Count	2	1	0	0	6	0	2	0	11
		% within D1aPRERANK	18.2%	9.1%	0.0%	0.0%	54.5%	0.0%	18.2%	0.0%	100.0%
		% of Total	3.2%	1.6%	0.0%	0.0%	9.5%	0.0%	3.2%	0.0%	17.5%
	5	Count	3	1	1	1	4	4	2	0	16
		% within D1aPRERANK	18.8%	6.3%	6.3%	6.3%	25.0%	25.0%	12.5%	0.0%	100.0%
		% of Total	4.8%	1.6%	1.6%	1.6%	6.3%	6.3%	3.2%	0.0%	25.4%
	6	Count	2	0	0	0	3	1	0	0	6
		% within D1aPRERANK	33.3%	0.0%	0.0%	0.0%	50.0%	16.7%	0.0%	0.0%	100.0%
		% of Total	3.2%	0.0%	0.0%	0.0%	4.8%	1.6%	0.0%	0.0%	9.5%
	7	Count	1	1	1	1	1	0	3	1	9
		% within D1aPRERANK	11.1%	11.1%	11.1%	11.1%	11.1%	0.0%	33.3%	11.1%	100.0%
		% of Total	1.6%	1.6%	1.6%	1.6%	1.6%	0.0%	4.8%	1.6%	14.3%
	8	Count	0	0	0	0	0	0	1	1	2
		% within D1aPRERANK	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	50.0%	50.0%	100.0%
		% of Total	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	1.6%	1.6%	3.2%
Total		Count	12	3	4	5	19	9	8	3	63
		% within D1aPRERANK	19.0%	4.8%	6.3%	7.9%	30.2%	14.3%	12.7%	4.8%	100.0%
		% of Total	19.0%	4.8%	6.3%	7.9%	30.2%	14.3%	12.7%	4.8%	100.0%

Table 5.33: PRERANK and POSTRANK Cross-tabulation for D1aPre and D1aPOST

5.4.4.1 Controlling for a pre-test rank of 1

Table 5.33 shows that six students had a pre-test rank of 1 which amounts to 9.5% of the total number of students. One of these students remained in rank 1, one moved to rank 3, two moved to rank 4, one moved to rank 5 and one moved to rank 6. Post intervention twelve students had rank 1, which amounts to 19% of the total number of students. This amounts to an increase of 9.5% of the total number of students.

5.4.4.2 Controlling for a pre-test rank of 2

Nine students had a pre-score of 2, which amounts to 14.3% of the total number of students. Three of these students regressed to rank I, one moved to rank 3, four moved to rank 5 and one moved to rank 6. After intervention three students had rank 2 which amounts to 4.8% of the total number of students. This amounts to a decrease of 9.5% for the total number of students.

5.4.4.3 Controlling for a pre-test rank of 3

Four students had a pre-test rank of 3, which is equal to 6.3% of the total number of students. One of these students moved to rank 4, two moved to rank 6 and one moved to rank 8. After intervention four students had rank 3, which is equivalent to 6.3% of the total number of students. There is therefore no net difference between pre-and post-rank.

5.4.4.4 Controlling for a pre-test rank of 4

Eleven students had a pre-test rank of 4, which is equal to 17.5% of the total number of students. After intervention two of these students regressed to rank 1, one regressed to rank 2, six moved to rank 5 and two moved to rank 7. The number of students who had a rank of 4 after intervention is five; this is equivalent to 7.9% of the total number of students and amounts to a decrease of 9.6%.

5.4.4.5 Controlling for a pre-test rank of 5

Sixteen students had a pre-rank of 5, which is equal to 25.4% of the total number of students. Three of these students regressed to rank 1, one regressed to rank 2, one regressed to rank 3, one regressed to rank 4, four remained rank 5, four moved to rank 6 and two moved to rank 7. After intervention nineteen students were in rank 5, which is equivalent to 30.2% of students. Thus, the number of students in this rank increased by 4.8%.

5.4.4.6 Controlling for a pre-test rank of 6

Six students had a pre-rank of 6, which is equal to 9.5% of the total number of students. After intervention two of these students regressed to rank 1, three regressed to rank 5 and one remained in rank 6. The number of students who had a rank of 6 after intervention was nine, which amounts to 14.3% of the total number of students. This amounts to an increase of 4.8%.

5.4.4.7 Controlling for a pre-test rank of 7

Nine students had a pre-rank of 7, which is equal to 14.3% of the total number of students. After intervention one student regressed to each of the ranks 1, 2, 3, 4 and 5, while three students remained in rank 7 and one student moved to rank 8. The number of students who had a rank of 7 after intervention was eight, which amounts to 12.7% of the total number of students. This signifies a decrease of 1.6%.

5.4.4.8 Controlling for a pre-test rank of 8

Two students had a pre-rank of 8, which is equal to 3.2% of the total number of students. After intervention one student regressed to the rank of 7, while the other student remained in rank 8. The number of students who had a rank of 8 after intervention was three, which amounts to 4.8% of the total number of students – an increase of 1.6%.

5.4.4.9 Conclusion for Stratified analysis of D1aPRE and D1aPOST

Table 5.34 summarizes the increase or decrease in the number of students for each rank. A negative value indicates a decrease whereas a positive indicates an increase. The table shows that after intervention ranks 1, 2 and 3 had a 0% increase. Similarly ranks 4, 5 and 6 and ranks 7 and 8 show a zero percent increase. Figure 5.16 shows the increase/decrease in graph form.

Actual number of students shows that there was no net movement of students between ranks. Before intervention ranks 1, 2 and 3 had nineteen students, which remained the same post intervention. Ranks 4, 5 and 6 had thirty-three students before and after intervention, while ranks 7 and 8 similarly showed no movement.

D1a indicates the category Memorised reasoning based on conceptual knowledge. Based on the foregoing analysis, one may conclude that the teaching strategy had no major impact on the improvement of students' ability to use the conceptual knowledge they had been exposed to in class in problem situations.

RANK	INCREASE/DECREASE (PERC POST – PERC PRE)
1	9.5%
2	-9.5%
3	0%
4	-9.6%
5	4.8%
6	4.8%
7	-1.6%
8	1.6%

Table 5.34: Percentage increase/decrease for D1a

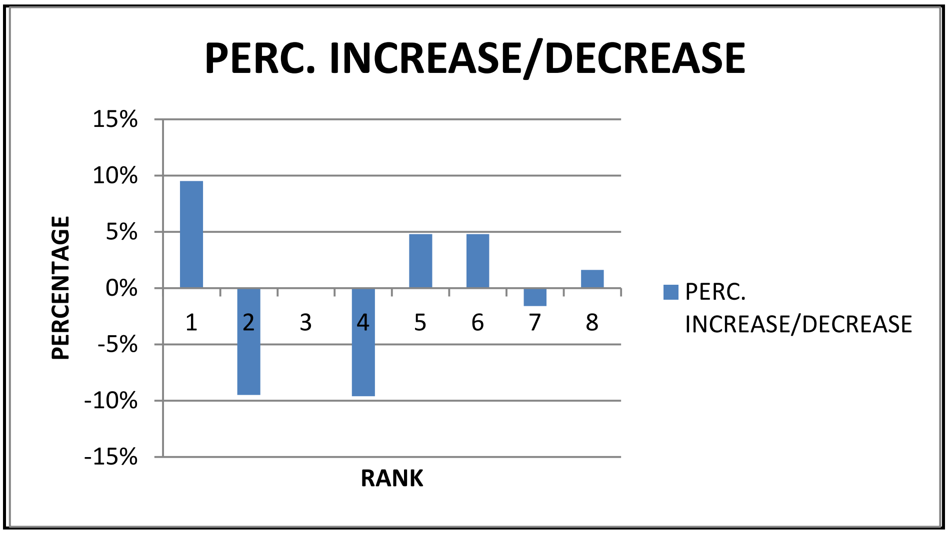

Figure 5.16: D1aPRE and D1aPOST percentage increase/decrease in number of students

5.4.5 Stratified analysis of D2aPRE and D2aPOST

Table 5.35 shows the stratified analysis for D2aPRE and D2aPOST.

			D2aPOSTRANK								Total
			1	2	3	4	5	6	7	8	
D2a PRERANK	1	Count	25	3	0	2	0	0	0	0	30
		% within D2aPRERANK	83.3%	10.0%	0.0%	6.7%	0.0%	0.0%	0.0%	0.0%	100.0%
		% of Total	39.7%	4.8%	0.0%	3.2%	0.0%	0.0%	0.0%	0.0%	47.6%
	2	Count	2	2	3	0	0	0	1	0	8
		% within D2aPRERANK	25.0%	25.0%	37.5%	0.0%	0.0%	0.0%	12.5%	0.0%	100.0%
		% of Total	3.2%	3.2%	4.8%	0.0%	0.0%	0.0%	1.6%	0.0%	12.7%
	3	Count	2	0	0	2	1	0	0	0	5
		% within D2aPRERANK	40.0%	0.0%	0.0%	40.0%	20.0%	0.0%	0.0%	0.0%	100.0%
		% of Total	3.2%	0.0%	0.0%	3.2%	1.6%	0.0%	0.0%	0.0%	7.9%
	4	Count	0	1	2	0	1	1	1	0	6
		% within D2aPRERANK	0.0%	16.7%	33.3%	0.0%	16.7%	16.7%	16.7%	0.0%	100.0%
		% of Total	0.0%	1.6%	3.2%	0.0%	1.6%	1.6%	1.6%	0.0%	9.5%
	5	Count	0	2	1	1	0	1	1	1	7
		% within D2aPRERANK	0.0%	28.6%	14.3%	14.3%	0.0%	14.3%	14.3%	14.3%	100.0%
		% of Total	0.0%	3.2%	1.6%	1.6%	0.0%	1.6%	1.6%	1.6%	11.1%
	6	Count	0	1	0	0	0	0	1	0	2
		% within D2aPRERANK	0.0%	50.0%	0.0%	0.0%	0.0%	0.0%	50.0%	0.0%	100.0%
		% of Total	0.0%	1.6%	0.0%	0.0%	0.0%	0.0%	1.6%	0.0%	3.2%
	7	Count	0	1	0	0	0	2	0	0	3
		% within D2aPRERANK	0.0%	33.3%	0.0%	0.0%	0.0%	66.7%	0.0%	0.0%	100.0%
		% of Total	0.0%	1.6%	0.0%	0.0%	0.0%	3.2%	0.0%	0.0%	4.8%
	8	Count	0	0	1	0	1	0	0	0	2
		% within D2aPRERANK	0.0%	0.0%	50.0%	0.0%	50.0%	0.0%	0.0%	0.0%	100.0%
		% of Total	0.0%	0.0%	1.6%	0.0%	1.6%	0.0%	0.0%	0.0%	3.2%
Total		Count	29	10	7	5	3	4	4	1	63
		% within D2aPRERANK	46.0%	15.9%	11.1%	7.9%	4.8%	6.3%	6.3%	1.6%	100.0%
		% of Total	46.0%	15.9%	11.1%	7.9%	4.8%	6.3%	6.3%	1.6%	100.0%

Table 5.35: PRERANK and POSTRANK Cross-tabulation for D2aPRE and D2aPOST

5.4.5.1 Controlling for a pre-test rank of 1

Table 5.35 shows that thirty students had a pre-test rank of 1 which amounts to 47.5% of the total number of students. Twenty-five of these students remained in rank 1, three moved to rank 2 and two moved to rank 4. Post intervention twenty-nine students had rank 1, which amounts to 46% of the total number of students. This amounts to a decrease of 1.5% of the total number of students.

5.4.5.2 Controlling for a pre-test rank of 2

Eight students had a pre-score rank of 2, which amounts to 12.7% of the total number of students. Two of these students regressed to rank 1, two remained in rank 2, three moved to rank 3 and one moved to rank 7. After intervention ten students had rank 2 which accounts for 15.9% of the total number of students. This amounts to an increase of 3.2% of the total number of students.

5.4.5.3 Controlling for a pre-test rank of 3

Five students had a pre-score rank of 3, which amounts to 7.9% of the total number of students. Two of these students regressed to rank 1, two moved to rank 4, and one progressed to rank 5. After intervention seven students had rank 3 which amounts to 11.1% of the total number of students. This amounts to an increase of 3.2% of the total number of students.

5.4.5.4 Controlling for a pre-test rank of 4

Six students had a pre-score rank of 4, which amounts to 9.5% of the total number of students. One of these students regressed to rank 2, two regressed to rank 3, one moved to rank 5, one moved to rank 6 and one moved to rank 7. After intervention five students had rank 4 which amounts to 7.9% of the total number of students. This amounts to a decrease of 1.6% in the total number of students.

5.4.5.5 Controlling for a pre-test rank of 5

Seven students had a pre-score rank of 5, which amounts to 11.1% of the total number of students. Two of these students regressed to rank 2 and one each regressed to rank 3 and 4 respectively. Three students progressed to ranks 6, 7 and 8 respectively. After intervention three students were in rank 5 which amounts to 4.8% of the total number of students. This amounts to a decrease of 6.3% in the total number of students.

5.4.5.6 Controlling for a pre-test rank of 6

Two students had a pre-score rank of 6, which amounts to 3.2% of the total number of students. One of these students regressed to rank 2 and the other progressed to rank 7. After intervention four students were in rank 6 which accounts for 6.3% of the total number of students. This amounts to an increase of 3.1% of the total number of students.

5.4.5.7 Controlling for a pre-test rank of 7

Three students had a pre-score rank of 7, which amounts to 4.8% of the total number of students. One of these students regressed to rank 2 and the other two regressed to rank 6. After intervention four students were in rank 7 which amounts to 6.3% of the total number of students. This amounts to an increase of 1.5% in the total number of students.

5.4.5.8 Controlling for a pre-test rank of 8

Two students had a pre-score rank of 8, which amounts to 3.2% of the total number of students. One of these students regressed to rank 3 while the other student regressed to rank 5. After intervention, only one student was in rank 8 which amounts to 1.6% of the total number of students. This amounts to a decrease of 1.6% of the total number of students.

5.4.5.9 Conclusion for Stratified analysis of D2aPRE and D2aPOST

Table 5.36 summarizes the increase or decrease in the number of students for each rank. A negative value indicates a decrease whereas a positive indicates an increase. The table shows that after intervention ranks 1, 2 and 3 – the below pass cut-off – had a 4.9% increase. Ranks 4, 5 and 6 decreased by 7.9% and ranks 7 and 8 showed only a minor change. Figure 5.17 shows the increase/decrease in graph form.

Student numbers indicate that in total three students moved from ranks 4, 5 and 6 to ranks 1, 2 and 3 while ranks 7 and 8 showed zero movement. D2a is the category Local Creative Reasoning based on Conceptual Knowledge which is the highest category in terms of achievement difficulty. This is also the category that I believe teachers should have in abundance for school level mathematics. I believe that in order for teachers of mathematics to explain concepts adequately they must be able to view and utilize concepts from different angles.

Before intervention the number of students within ranks 1, 2 and 3 was forty-three. This is equal to 68% of the total number of students. After intervention forty-six students were in ranks 1, 2 and 3, which is equal to 73% of the total number of students. Ranks 4, 5 and 6 contained fifteen students (23.8% of the total number of students) before intervention and twelve students (19% of the total number of students) after intervention. Ranks 7 and 8 had five students (7.9% of the total number of students) before intervention and five students after intervention.

The question is what do these statistics reveal?

The fact that 68% of students were in ranks 1, 2 and 3 before intervention implies that the majority of students entered with weak abilities in terms of Local Creative Reasoning based on Conceptual Knowledge. The fact that the number of students in these ranks increased after intervention is a probable indication that the teaching strategy did not have the desired effect of increasing ability in this category. Furthermore, the fact that the movement between ranks was very low suggests that for all categories of student, their ability in the domain of Local Creative Reasoning based on Conceptual Reasoning was not affected in a major way by the teaching strategy. What this means is that the majority of students was not moved to either higher or lower ranks and hence student ability for this category remained mostly unchanged by the teaching strategy.

RANK	INCREASE/DECREASE (PERC POST – PERC PRE)
1	-1.5%
2	3.2%
3	3.2%
4	-1.6%
5	-6.3%
6	3.1%
7	1.5%
8	-1.6%

Table 5.36: Percentage increase/decrease for D2a

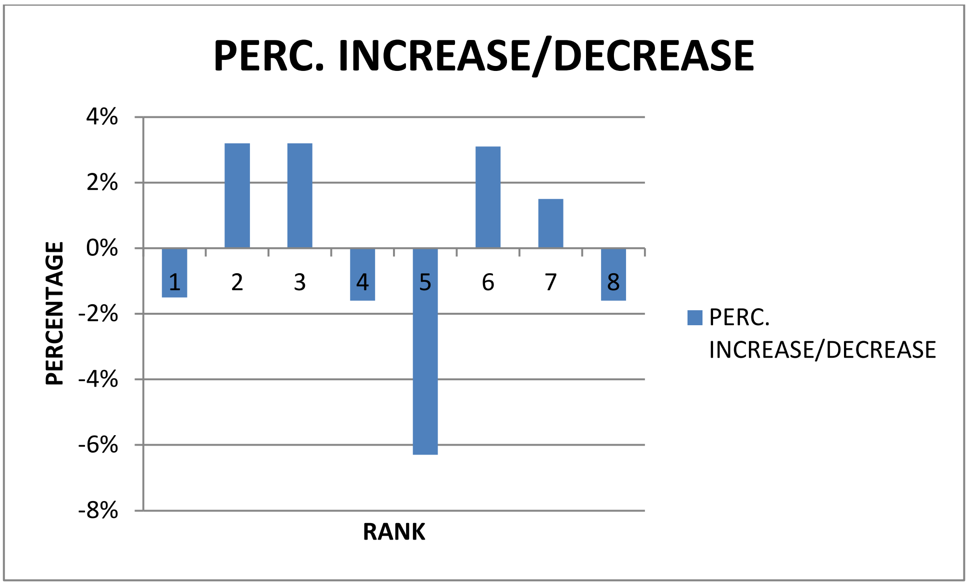

Figure 5.17: D2aPRE and D2aPOST percentage increase/decrease in number of students

5.5 Error variance (Nuisance variables)

In research the possibility always exists that factors other than those specified by the researcher might be responsible for observed effects. These factors are labelled 'nuisance variables'. Error variance consists of those chance fluctuations in scores that are attributable to the effects of nuisance variables. In order to draw valid conclusions a researcher must be able to rule out the possibility that nuisance variables could explain the apparent effect of the independent variable (Myers et al, 2010). One way to reduce error variance is to make certain that uniform conditions exist for all research participants.

In this study, all research participants received the same instruction in the same environment and hence control by uniform conditions was utilized to reduce error variance. Furthermore, since students participating in the study wrote tests and examinations independently without any consultation, each observation is independent.

5.6 Summary of Conclusions of statistical analysis

In this research, total score for an examination was 100 whereas total score for a class test was 50. Examinations included questions from all topics covered during the semester, whereas class tests included questions from only those topics covered prior to the test. Knowledge retention requirements for examinations therefore were greater and similarly the retention interval was greater for examinations than for tests. Research has shown that frequent classroom testing can improve examination scores, but improvement diminishes as test frequency increases (Bangert-Drowns, Kulik, Kulik, & Morgan, 1991). However, the deleterious effects of testing on students can influence this improvement. Some of these effects are test anxiety, lower intrinsic motivation, a decrease in student learning, etc. (Amrein & Berliner, 2002). In a study with elementary school students it was found that the students experienced significantly greater test anxiety about high-stakes assessment than about classroom tests (Segool, Carlson, Goforth, von der Embse & Barterian, 2013). In the current study, the class tests afforded students the opportunity to perform better in the next test or the one following that, etc. and hence their test anxiety might have been alleviated. By contrast, the examination was a once-off event and carried the attendant high stakes consequences such as non-promotion in the case of failure to pass it. One could thus argue that students are likely to be more anxious about examinations. The expectation therefore was that post-test scores (examinations) would be lower than pre-test (class tests). However, should statistical analysis show otherwise (like higher post-test scores) it would be reasonable to argue that the intervention played a role in increasing post-test scores.

The aim of this investigation was to determine how exposure to a teaching strategy (based on spiral revision and productive practice) would influence the mathematical competencies of procedural fluency and conceptual understanding of participating pre-service students. The following categories were considered to be measures of procedural fluency: Memorized Reasoning based on Factual knowledge (A1a), Memorized Reasoning based on Procedural knowledge (B1a) and Familiar Algorithmic Reasoning based on Procedural knowledge (B1bi); Familiar Algorithmic Reasoning based on Flexible Procedural knowledge (C1bi) and Delimiting Algorithmic Reasoning based on Flexible Procedural knowledge (C1bii) as determined according to the Revised Taxonomy table. Procedural fluency was represented by the variables SKILLPRE and SKILLPOST. A related samples *t* test based on differences between pre- and post-test scores was performed on these two variables.

The findings indicate a significant difference between SKILLPRE and SKILLPOST. A stratified analysis was also performed to compare students' pre- and post-test rankings. The findings indicate that the majority of students moved to a higher ranking for SKILLPOST. No students were in the lowest two ranks and only 7.9% of students were below the pass cut-off score post-test. If one considers all these statistical findings together a highly plausible conclusion is that procedural fluency was positively enhanced by the teaching strategy.

However, an investigation into the categories which constitute procedural fluency and conceptual understanding indicates that the teaching strategy has some weaknesses which should be addressed in order to improve effectiveness. The analysis of student performance in the category Familiar Algorithmic Reasoning based on Flexible Procedural Knowledge (C1bi) – which forms part of procedural fluency – indicates that approximately a quarter of the students are not at the required level of competence for this category. This category of question was used to determine whether students had developed the ability to use familiar procedural knowledge in a new way.

The following categories were considered to be measures of conceptual understanding: Memorised Reasoning based on Conceptual knowledge (D1a), Familiar Algorithmic Reasoning based on Conceptual knowledge (D1bi), Delimiting Algorithmic Reasoning based on Conceptual knowledge (D1bii) and Local Creative reasoning based on Conceptual knowledge (D2a). Conceptual understanding was represented by the variables CONCPRE and CONCPOST. A related samples *t* test was also performed on these two variables. The findings indicate a significant difference between pre- and post-test scores. The findings based on the stratified analysis indicate that the lowest two ranks decreased by 27% whereas the top two ranks together showed an increase of 13%. Pre-test, scores indicate 52.4% of students were below the cut-off score, whereas 34.9% of students were below the cut-off score according to post-test scores.

As discussed in the foregoing section, if all findings concerning CONCPRE and CONCPOST are considered, then a highly plausible conclusion is that conceptual understanding was improved by the teaching strategy.

Although the overall analysis of the categories constituting conceptual understanding show that the majority of students have improved abilities after intervention, an investigation into the individual categories indicates that weaknesses exist in some categories. Student abilities remained largely unchanged pre-and post-intervention for the category Memorized Reasoning based on Conceptual Knowledge (D1a). Similarly, the category Local Creative Reasoning based on Conceptual Knowledge (D2a) did not show a significant positive change after intervention. These two categories of questions were utilized to determine if students had developed the ability to use conceptual knowledge creatively in order to solve problems with which they were presented.

The qualitative part of the study will be discussed in the following chapter.

CHAPTER 6: QUALITATIVE RESULTS: DATA ANALYSIS AND DISCUSSION

6.1 Introduction

Two essential objectives in teaching mathematics are the enhancement of the retention and transfer of knowledge.

Mayer (2002) defines retention as the ability to recall learnt material at some future point in the same way in which it had been presented. Transfer is defined as the capacity to utilize prior knowledge in order to solve new problems or to learn new subject matter. Two of the major objectives of the teaching strategy were to enhance participating students' retention and transfer abilities. Spiral revision and productive practice provided the instructional strategy through which it was hoped to achieve this.

Retention of prior knowledge is an essential part of learning in general. Revision of previously learnt material is one of the main strategies employed to enhance retention in education. It was argued in chapter 3.8 that humans forget approximately half of newly learned knowledge in a matter of days or weeks (Ebbinghaus, 1964; Rubin & Wenzel, 1996; Averell & Heathcote, 2010; Murre & Dros, 2015) unless they consciously review the learned material. The 'forget rate' is dependent on factors such as the difficulty of the learned material, how the material is presented, the depth of learning and physiological factors such as stress and sleep.

Most of the research concerning knowledge retention has utilized tasks that require verbatim recall of, for example, nonsense words. A review of the literature shows that there is a paucity of studies which have investigated retention of mathematical procedures and concepts. More precisely there is a dire lack of research that has examined how the distribution of practice across learning sessions affects the retention of mathematical knowledge – a domain which requires more than verbatim recall.

Different types of practice are described and defined in the literature. Distributed practice is defined as a learning strategy in which practice of specified knowledge and skills is distributed over a differing number of practice sessions. Distributed practice may be massed into a single session, may be distributed across two sessions or more sessions (Rohrer & Taylor, 2006). The type of distributed practice favoured in this study is practice distributed across many sessions. An important issue in the determination of the effectiveness of such practice is the retention interval. Retention interval is defined as time elapsed between the most recent learning session and the test (Rohrer & Taylor, 2006).

Overlearning is a strategy in which a student first masters a skill and then immediately proceeds to practice the same skill (Rohrer & Taylor, 2006). A very popular teaching strategy in mathematics is to present students with an example of a specific type of problem and then immediately require the students to practice solving many examples of the same kind of problem. This type of practice is usually done as a once-off exercise and can thus be classified as overlearning. As I have stated earlier the term overlearning might be perceived as negative since it can be compared to terms such as overeating and in addition the type of practice described by overlearning appear to be very similar to repetitive practice. I therefore prefer the term repetitive practice and will use this term henceforth.

The retention of knowledge in the learning of mathematics is absolutely crucial since in the majority of cases prior knowledge is required for solving problems presented. It is thus important to know which type of practice would best enhance the retention of mathematical procedures and concepts. Research done by Rohrer and Taylor (2006) has shown that long-term retention of a mathematical procedure is better aided by distributed practice. In this research I utilized both distributed and repetitive practice. I utilized repetitive practice mainly for mastery, and distributed practice for retention purposes and deeper understanding.

Some researchers argue that retrieving information during a test is not a neutral event and that the retrieval process can change knowledge, and in itself can produce learning (Karpicke & Roediger, 2008). Furthermore research has shown that testing can be employed as a strategy to enhance retention of knowledge (Roediger & Karpicke, 2006; Carpenter, Pashler, Wixted & Vul, 2008). Findings in empirical research have shown that repeated testing of the same content boosted retention more than repeated study (Roediger & Karpicke, 2006). Also as indicated earlier, a meta-analysis of the effects of test frequency has shown that frequent classroom testing can improve examination scores, but that improvement reduces as test frequency increases (Bangert-Drowns, Kulik, Kulik, & Morgan, 1991). In this study I have employed repeated testing on the same content as a strategy to enhance retention of indispensable mathematical knowledge. Indispensable mathematical knowledge is the mathematical knowledge that allows students to have fruitful interaction in the mathematics classroom and with the broader mathematical community. It enables them to cope with future activities in mathematics (Linchevski et al, 2000).

The debates have not been settled regarding which conditions allow for transfer of knowledge and which teaching strategies are best suited to enhance transfer. However, I view transfer, practice and conceptual understanding as being closely linked in the learning of mathematics.

As argued in chapter 3.4 automatization is defined as the practice of a skill or habit to the point that it becomes routine and requires little if any conscious effort or direction. In other words it refers to the point at which the thought processes involved in the skill have been moved to the long-term memory. Kalyuga (2007) maintains that if mundane procedural elements of a task have been practised to the extent that it became automated it would free cognitive capacity to engage in more creative reasoning and to applying prior knowledge in unfamiliar situations. In other words it would increase the possibility of transfer taking place.

Since I wanted the students to become proficient in the content areas which form part of our curriculum I had to devise teaching strategies that would help them develop competency in the specified content areas. Thus amongst its goals, the instruction aimed to develop task specific proficiency. Kalyuga (2007) argues that the development of task-specific expertise is a prerequisite for becoming a higher-level expert in a broader domain of learning. Task-specific expertise is the ability to perform fluently in a specific class of tasks.

Ericsson (2000) argues that expert performance in a domain of learning is viewed as an extreme case of skill attainment and is the result of incremental improvement in performance during extensive experience in the domain. Experience is gained through deliberate practice on tasks designed by an instructor with the goal of improving aspects of a student's performance. Ericsson (2000) maintains that the amount of time spent in solitary practice influences the level of expertise attained. He contends that the greater the accumulated amount of practice the higher the level of expertise achieved.

As stated defined previously spiral revision is the recurrent practising of previously covered mathematical work in specified content areas (Julie, 2013). Spiral revision includes review-as-you-go, mass practice, distributed practice, repetitive practice, productive practice and spiral testing as part of the revision process. One of the mathematical competencies that the spiral revision strategy is premised on is procedural fluency.

In this study practice (either in class, in tutorial class, as homework or in class tests) consisted of working through selected mathematical tasks that were similar to examples encountered earlier, were reversal-type problems or deepening-thinking-like problems. Each successive class test included questions on content previously covered and therefore students were required to revise topics previously covered in their individual studies as well. Class tests therefore formed part of the revision process.

In most cases in teaching situations in mathematics revision is done as a once-off exercise and is presented in the same way as the original teaching (Wigley, 1992). However, as indicated earlier revision in this study was done on an ongoing basis throughout the semester and it included discussion of deepening thinking-like problems (May & Julie, 2014). It is generally accepted that mathematical proficiency is a function of both procedural and conceptual knowledge. I therefore considered it prudent to include productive practice in the teaching strategy. Julie (2013) contends that productive practice is a didactic strategy by which students are exposed to deepening thinking-like problems. Deepening thinking-like problems were utilized to enhance and deepen conceptual knowledge and to increase flexible procedural knowledge of students in requisite content areas of the specified mathematics curriculum (May & Julie, 2014).

A central idea of the teaching strategy is that tasks that are conceptually and procedurally more demanding are presented to students within each subsequent cycle of the revision process. I am of the opinion that problems requiring more complex solution strategies require more practice more often in order to develop skill in dealing with such problems, to deepen understanding and also to make it part of the long-term memory.

Retention of procedural knowledge and mastery of imitative reasoning was inferred if a participating student produced a correct procedure for an appropriate task. It was also assumed that procedural knowledge was retained if a correct formula (for example slope-intercept form of the two-variable linear equation) had been produced correctly and used appropriately. Tasks used to determine levels of conceptual knowledge were usually unfamiliar to participants and required participants to make cognitive connections between known pieces of information or between known and new pieces of information. Some of these tasks required creative reasoning.

6.1.1 A Deepening-thinking-like example

Productive practice forms an important part of my version of spiral revision. Deepening-thinking-like problems is the type of problem utilized for discussion purposes or for students to engage with in this type of practice. To make it clear what kind of problem I perceive to be of this kind a typical example is discussed next.

The following is an example of a deepening-thinking-like example. If in a lesson on rationalisation of denominators examples such as $\frac{12}{\sqrt{18}}, \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ and $\frac{1}{\sqrt[5]{6}}$ were discussed, then the following presented problem can be classified as a deepening-thinking-like problem:

Simplify $\frac{a}{\sqrt[n]{b}}$ *by rationalising the denominator.*

The following is an exposition of my motivation for this classification. In order to rationalize denominators the notion of 1 in a different form is utilized as well as the fact that 1 is the identity element for multiplication in the real number system. Since all the examples discussed in the lesson utilized integer roots this question extends the discussion into a new direction, prompting the further question, 'What would happen in the case of a general or variable root?' This case would therefore require the student to realize that after multiplication is effected the exponent under the root should be equal to the order of the root and hence that this is the general underlying principle being utilized in all such cases. Moreover they should also be aware that in the exponential form the exponent of the base of the denominator should be 1. The intention with this question therefore would be to expose the general conceptual underpinnings involved in the solution procedure for this type of question. The following solution serves to exemplify the above discussion:

$$\frac{a}{\sqrt[n]{b}} \times \frac{\sqrt[n]{b^{n-1}}}{\sqrt[n]{b^{n-1}}}$$

$$= \frac{a.\sqrt[n]{b^{n-1}}}{\sqrt[n]{b^{n-1}.b}}$$

6.2 Research participants

The second research question is concerned with retention and transfer abilities of the participating pre-service students. As mentioned previously the objective (regarding the topics taught) was to determine how retention and transfer abilities of participating students would be affected if they are exposed to a teaching strategy underpinned by spiral revision and productive practice. This question was investigated by means of case studies. As indicated earlier the students selected to participate in the case studies were chosen from different ability groups. In total six students formed the contingent for the case studies; two students were selected from the lower achieving group (Students A and B); two from the average achieving group (Students C and D) and two from the high achieving group (Students E and F). The student demographics and results are shown in Table 6.1. It should be noted that students tend to score higher in mathematical literacy than in mathematics but that these higher scores do not usually translate into higher scores in mathematics courses at university level.

Student	Male / Female	Grade 12 Mathematics/ Mathematical Literacy	Grade 12 results	MAE 121 results (first year university mathematics module for pre-service teachers)
Student A	Female	Mathematical Literacy	75%	48%
Student B	Female	Mathematical Literacy	86%	55%
Student C	Female	Mathematics	47%	64%
Student D	Female	Mathematics	65%	47%
Student E	Female	Mathematics	60%	95%
Student F	Male	Mathematics	61%	97%

Table 6.1: Case study student Demographics and Results

6.3 Presentation of results and data analysis

Data for this part of the study – the case studies – consisted of the written responses of the six participating students to either class tests or end of module examinations. Since it would have been a very difficult and time-consuming exercise to include responses on all topics, only responses to test and examination items based on the algebraic and graphical versions of the two-variable linear equation ($y = mx + c$) and related items in different contexts of the curriculum were considered. Such problems were included in the first four class tests of the first semester, the end of module examinations of the first semester, and they also formed part of the chapter on Analytic Geometry that was covered in the second semester.

6.3.1 Question 1.1 of class test 1 of semester 1

The first class test was written soon after the topic of linear functions (as part of the broader topic of analytic geometry) had been completed. Responses to the following question in Class test 1 were investigated: "*Determine the equation of the straight line through* $(2; 1)$ *and perpendicular to* $3y + 2x = 6$". This question was classified as Familiar Algorithmic Reasoning based on Procedural knowledge (B1bi). The statistical analysis has shown that the majority of students do not have a problem with this type of question (see chapter 5.4.3) and hence the expectation was that the case studies would corroborate this. Student responses are presented in Figures 6.1 to 6.6.

Question 1.1 of Test 1 was similar to a question that was discussed in class and hence the expectation was that the majority of students would provide a correct solution. Finding the solution required students to transform the two-variable linear equation to the form: $y = mx + c$ in order to determine m, the gradient of the line. The fact that the product of the gradients of perpendicular lines is equal to -1 is then utilized to determine the gradient of the required line. This new gradient together with the coordinates $(2;1)$ is then substituted in the point-slope form of the line: $y - y_1 = m(x - x_1)$ to obtain the required equation. Students B, D, E and F produced a correct solution. Both students A and C made an error in the calculation of the perpendicular gradient, but otherwise followed the correct procedure for this type of problem. The case studies therefore supports the evidence from the statistical analysis that the majority of students performed relatively well with this type of problem.

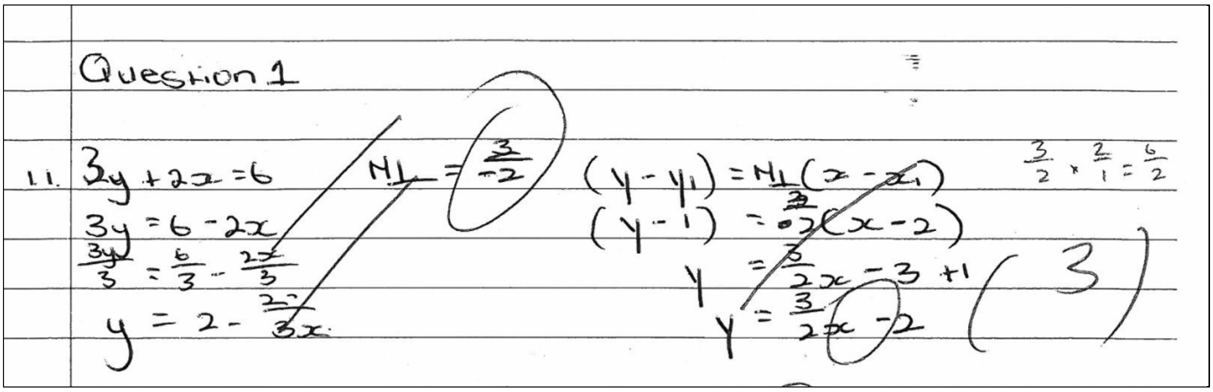

Figure 6.1: Student A's response to Question 1.1 of Class test 1

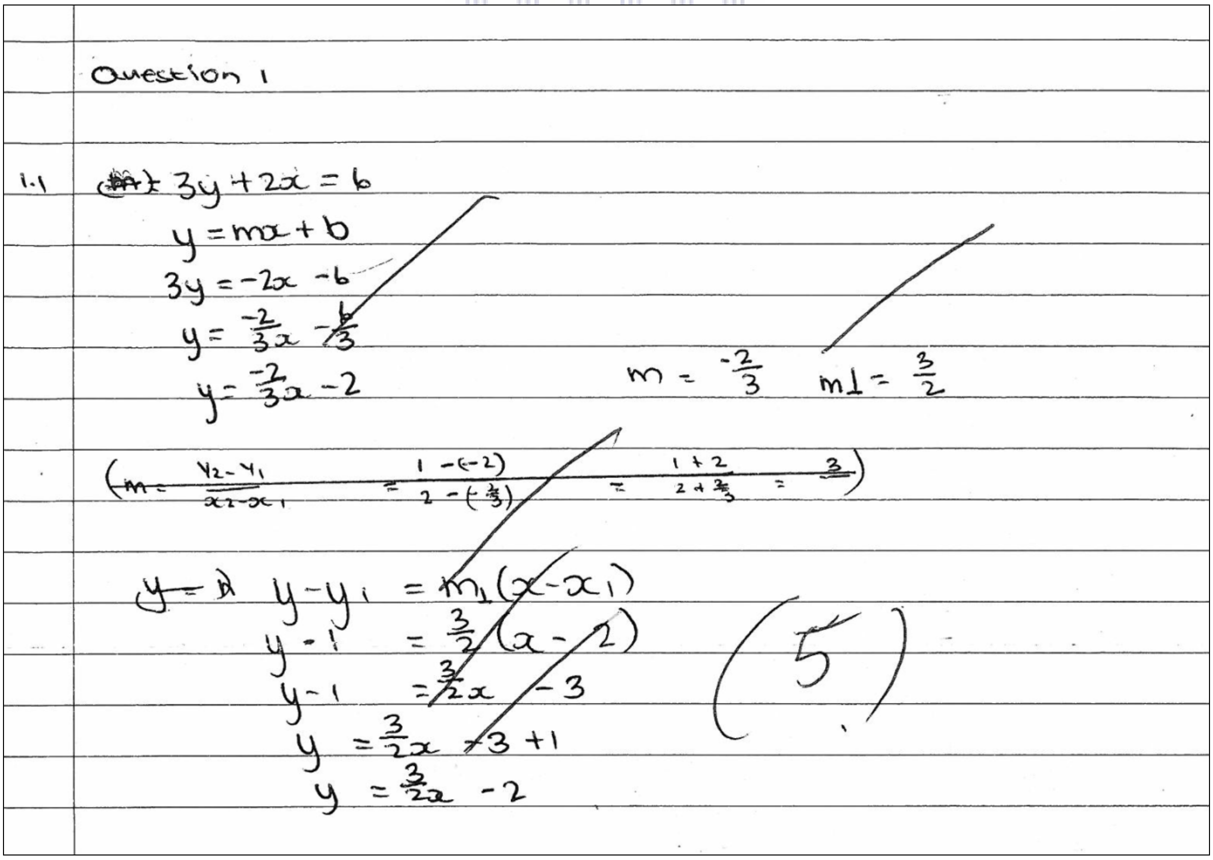

Figure 6.2: Student B's response to Question 1.1 of Class test 1

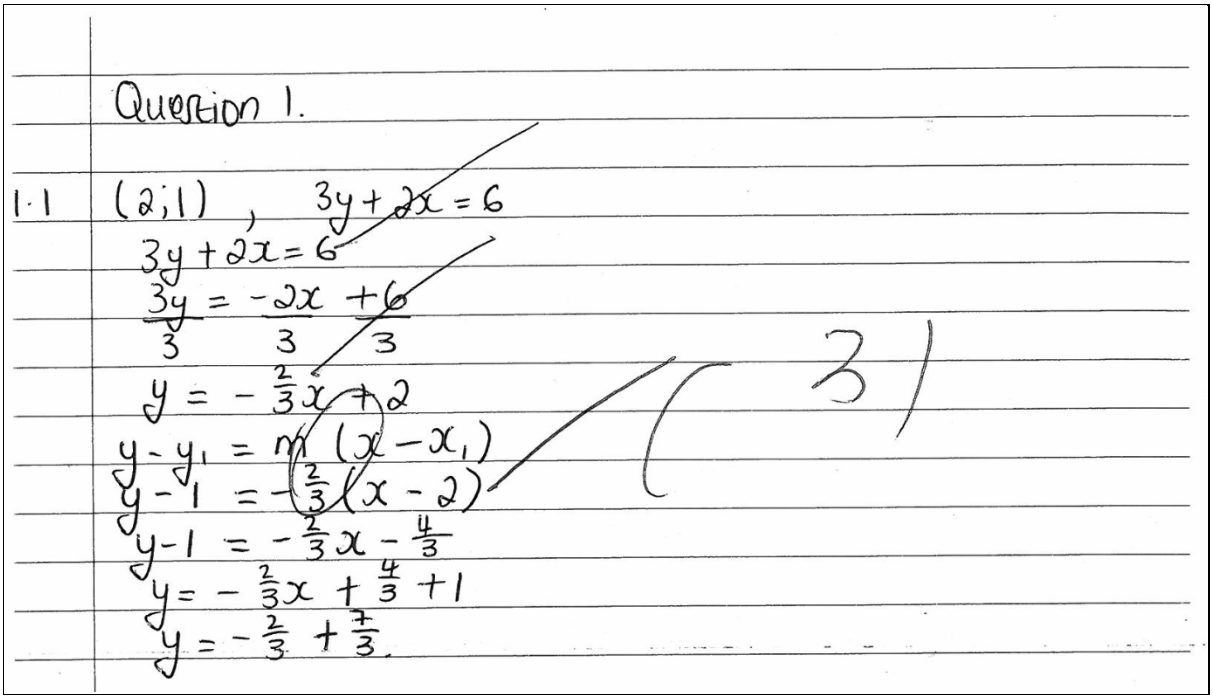

Figure 6.3: Student C's response to Question 1.1 of Class test 1

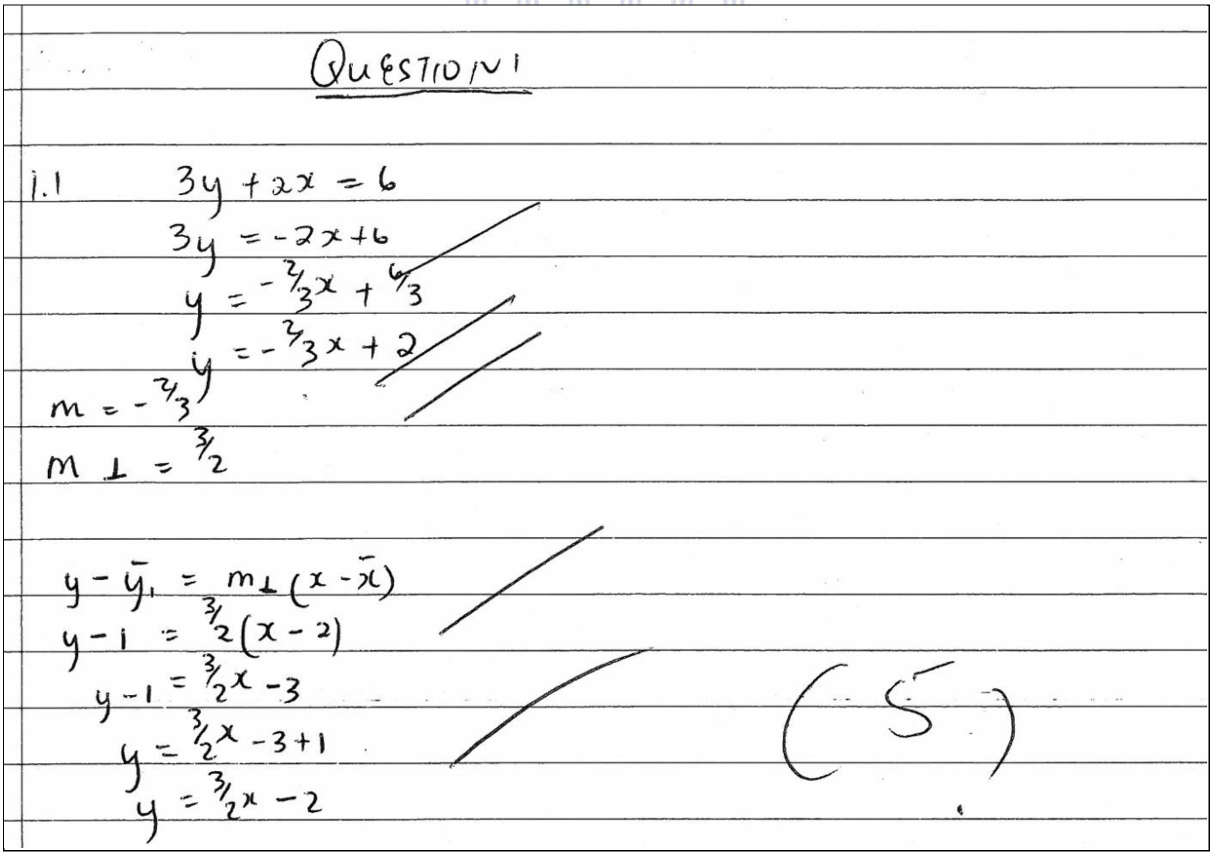

Figure 6.4: Student D's response to Question 1.1 of Class test 1

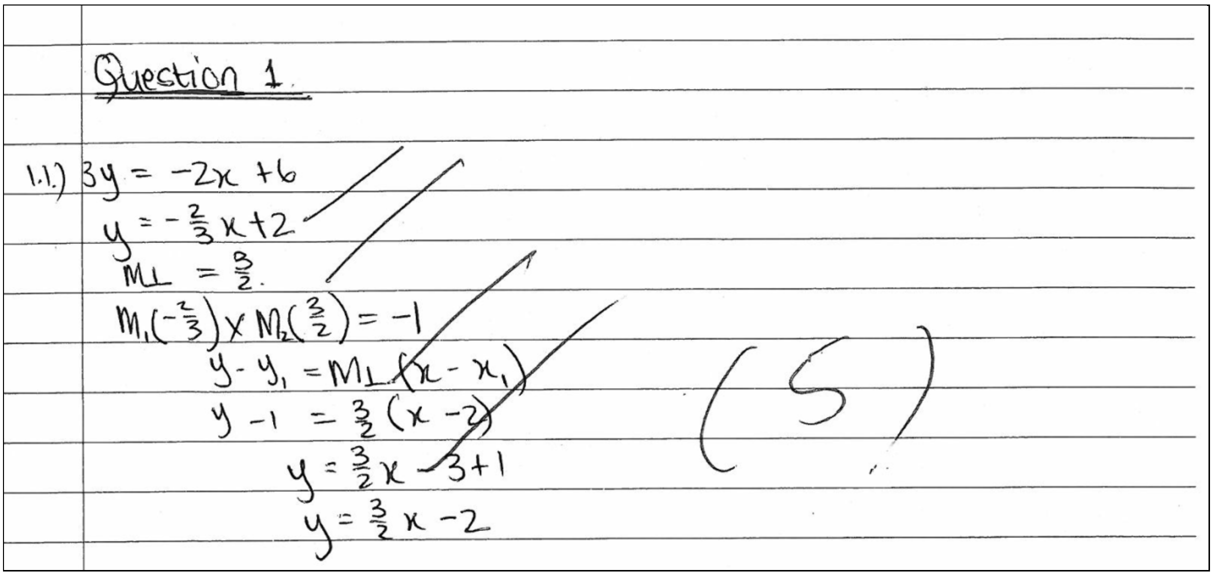

Figure 6.5: Student E's response to Question 1.1 of Class test 1

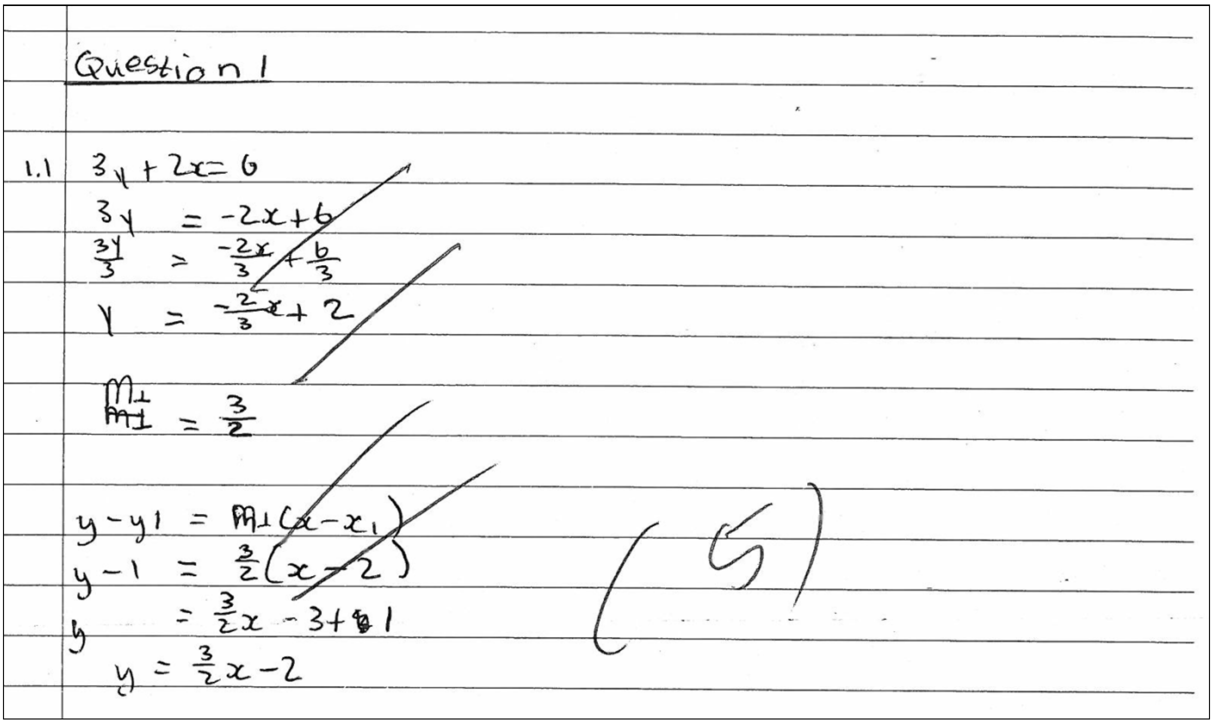

Figure 6.6: Student F's response to Question 1.1 of Class test 1

6.3.2 Question 1.1 of Class test 2 of semester 1

The second class test was written three weeks after the first test. The question for this test was as follows: *Determine the equation of the line with y-intercept -2 and passing through the point (2;1).* This question was classified as Familiar Algorithmic Reasoning based on Conceptual Knowledge (D1bi). The statistical analysis indicated that the majority of students developed the ability to deal with such problems in the context of linear functions, and hence the expectation was that the case studies would reflect this. Student responses are presented in Figures 6.7 to 6.12.

Student A correctly provided the coordinates for the *y*-intercept and then attempted to use the coordinate pairs to determine the gradient. However the student had the change in *y* as the denominator and hence calculated the gradient incorrectly. She then proceeded to use this incorrect gradient correctly in the point slope form of the two-variable linear equation. Student A was aware of the requirements for solving such a problem since she first attempted to determine the gradient and subsequently she substituted this value into the point slope form of the equation of a line.

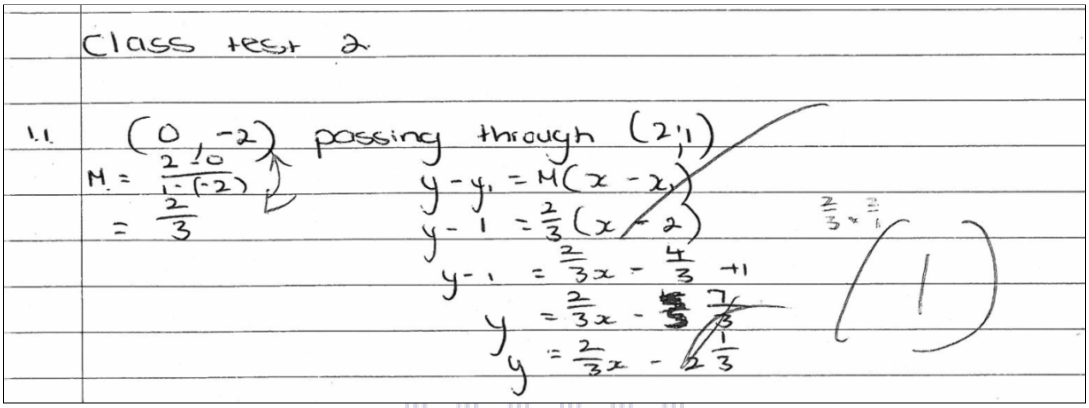

Figure 6.7: Student A's response to Question 1.1 of Class test 2

Student B produced a correct solution for this problem. In her solution she first calculated the gradient and subsequently used the slope and *y*-intercept form of the equation of a line to determine the required equation.

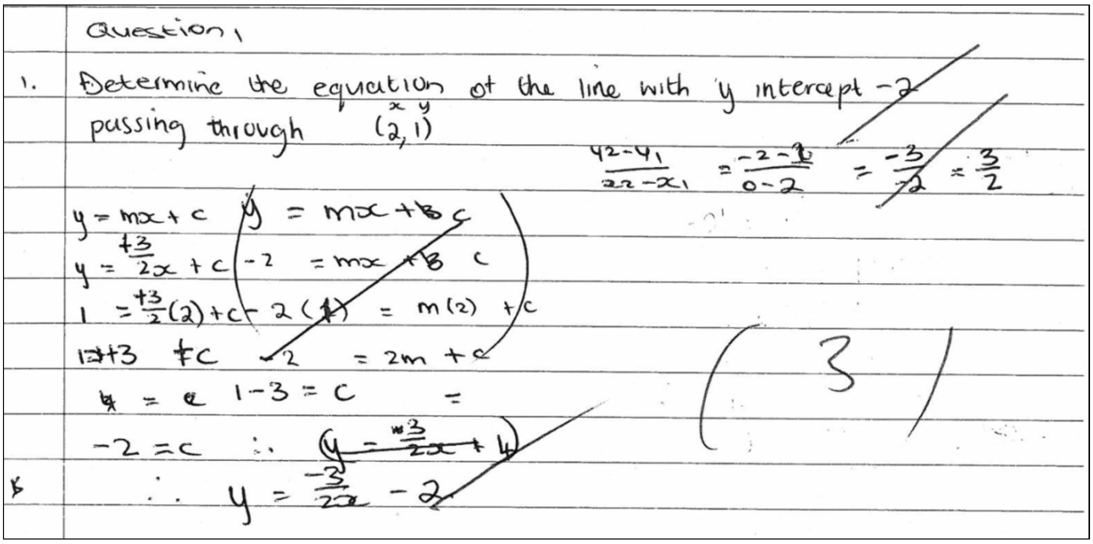

Figure 6.8: Student B's response to Question 1.1 of Class test 2

It seems that Student C was aware of the procedures required for solving the problem. However she made an error in the calculation of the gradient: she did not use the same order in the denominator as in the numerator. She proceeded to use the incorrect gradient correctly in the slope and *y*-intercept form of the equation of a line.

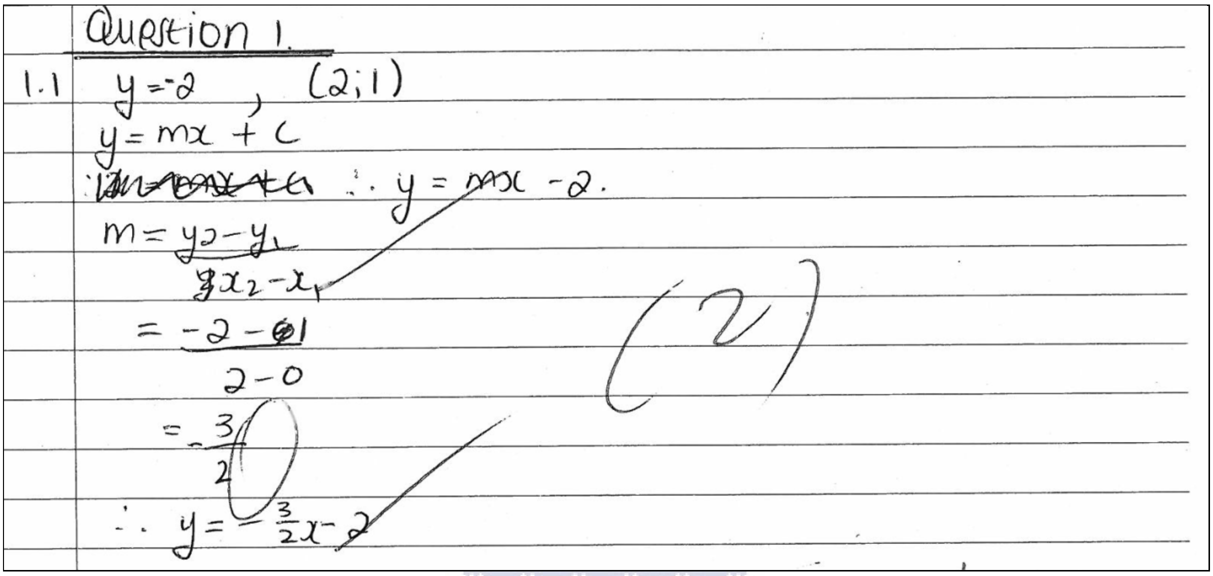

Figure 6.9: Student C’s response to Question 1.1 of Class test 2

Students D, E and F produced correct solutions for the problem. Student D used the point-slope form of the line whilst Student E utilized the slope and *y*-intercept form.

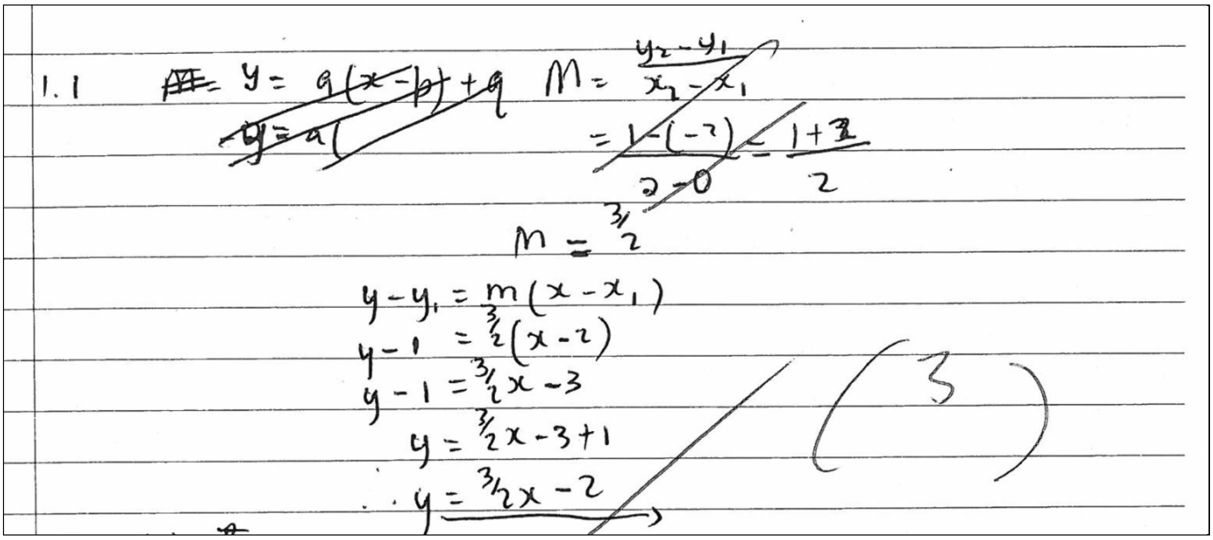

Figure 6.10: Student D’s response to Question 1.1 of Class test 2

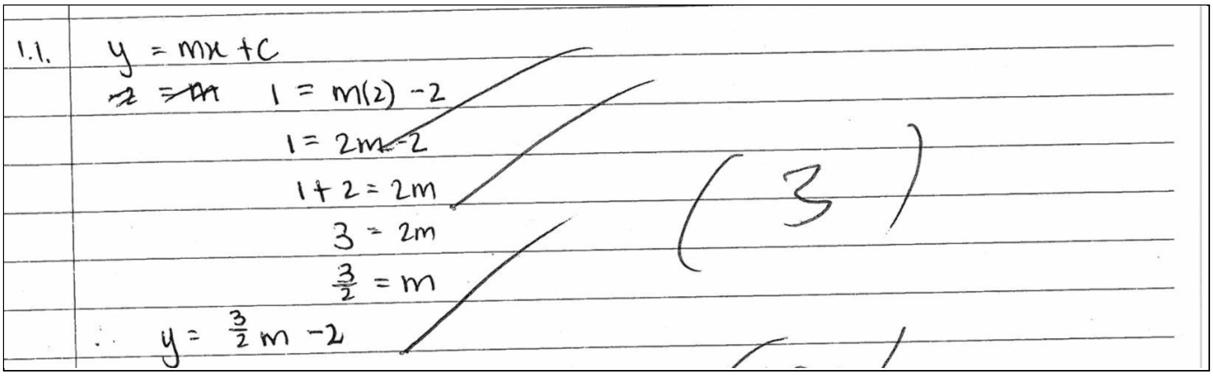

Figure 6.11: Student E's response to Question 1.1 of Class test 2

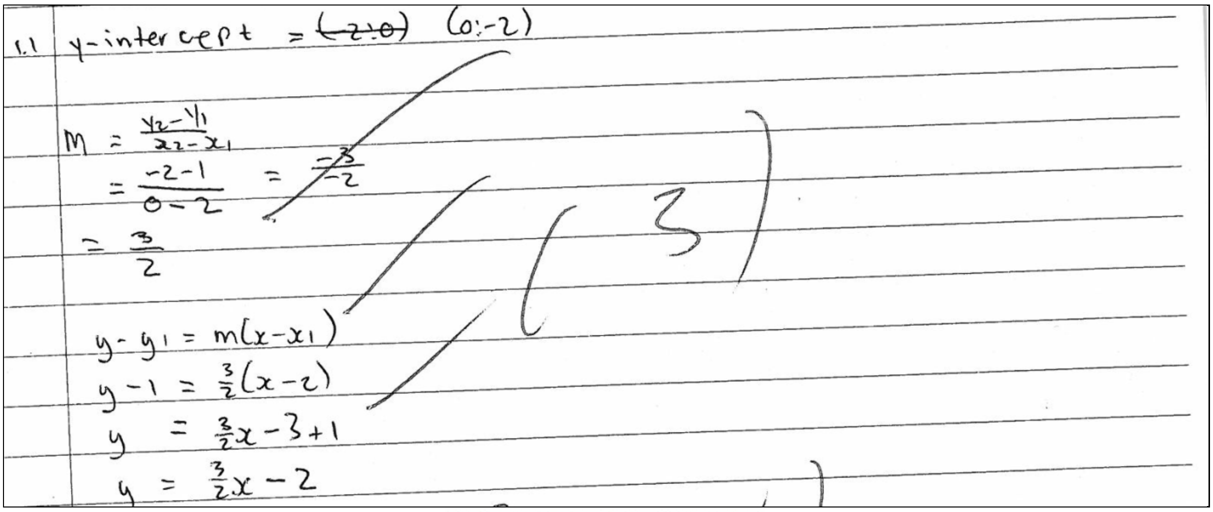

Figure 6.12: Student F's response to Question 1.1 of Class test 2

The foregoing analyses of student responses seem to indicate that these students had retained the relevant indispensable knowledge for linear functions at this time. Although Students A and C committed conceptual errors in their efforts to determine the gradient they used the results obtained correctly in appropriate equations. This question required students to use conceptual knowledge to solve the problem with which they were presented and since no similar problem had been done in class it implied that prior knowledge was required to solve this new type of problem. In other words the problem required transfer of knowledge. The fact that the majority of students could apply their prior knowledge in the new context can be perceived to be an instance of knowledge transfer.

6.3.3 Question 1.1 of Class test 3 of semester 1

Class test 3 was written approximately 7 weeks after Class test 1. The first question for Class test 3 is shown in Figure 6.13 below. Only Question 1.1 was utilized for our purposes however. This question was categorized as Familiar Algorithmic Reasoning based on

Flexible Procedural Knowledge (C1bi). The statistical analysis shows that approximately a quarter of students had average scores below 50% for this category of question which is an indication that this is a category that some students struggled with. Student responses are shown in Figures 6.14 to 6.19.

QUESTION 1

1. Given line DF with equation: $x + 2y - 2 = 0$ (as shown in the sketch)

1.1	Find the slope of line DF.	(2)
1.2	Find the length of OA.	(2)
1.3	Determine the length of OB.	(2)
1.4	The $y - coordinate$ of E is -1. Find the length of EF if EF is parallel to the $x - axis$.	(3)
1.5	Write the coordinates of A and F.	(2)
1.6	Determine the midpoint of line AF.	(2)
1.7	Determine the length of AF.	(2)

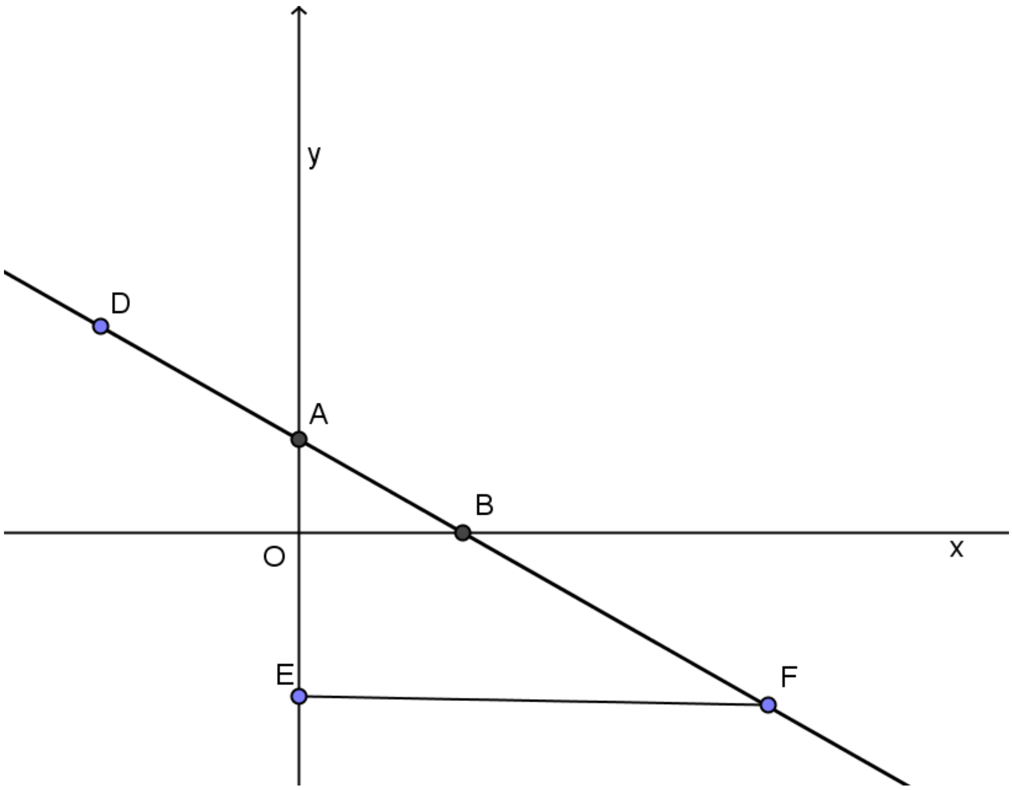

Figure 6.13: Question 1 of Class test 3 of semester 1

The solution procedure for this question required that the equation be manipulated in order to write it in standard form. In other words a known procedure (manipulation of linear equations) should be utilized to achieve a known form ($y = mx + c$) of the linear equation. The student should be aware that once the equation is in this form the coefficient of the

$x - term$ is the required gradient. The indispensable knowledge in this case is the comprehension that the equation of the linear function should be brought to standard form in order for the gradient to be determined. It seems as if Student A attempted to manipulate the equation, but was not certain how to go about it. However the fact that she attempted to manipulate the equation might indicate that she knew that the equation needed to be in standard form but forgot the manipulation procedure and what the standard form looks like. Her responses to Question 1.1 of Class test 1 and Class test 2 (see Figures 1 and 7) shows that in these cases she was aware of the standard form and of the position of the gradient in this form. A possible conclusion therefore is that she had forgotten, and hence retention of knowledge had not been achieved.

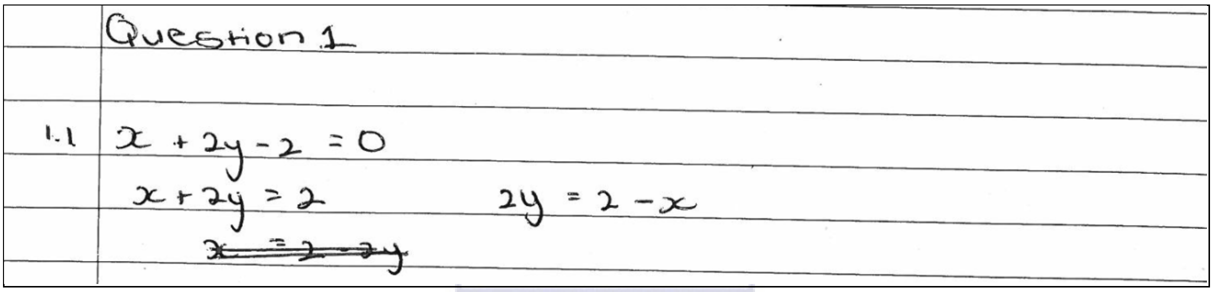

Figure 6.14: Student A's response to Question 1.1 of Class test 3

The first three deleted attempts of Student B are possibly an indication that she was not certain about how to proceed. In the last line however, it seems she attempted to manipulate the equation. The fact that she did not continue the manipulation might indicate that she was not certain that this was the correct way to proceed. Her responses to Question 1.1 of Class test 1 and Class test 2 however show that in both these cases she was aware of how to manipulate the linear equation to obtain standard form, and that once the equation was in standard form the coefficient of the $x - term$ is the gradient. The fact that she was not certain about how to proceed is perhaps an indication that she forgot the required procedure and hence one could argue that the requisite knowledge had not been retained.

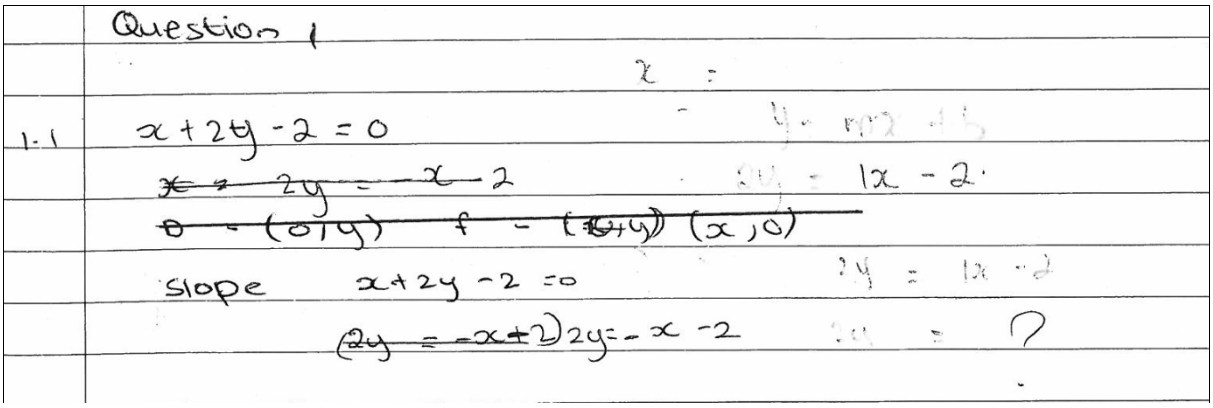

Figure 6.15: Student B's response to Question 1.1 of Class test 3

Student C manipulated the equation and arrived at standard form, but did not continue to write down the gradient. The fact that the gradient was not written down appears to be an oversight rather than not knowing what to do. One could therefore argue that this student had retained the requisite procedural and conceptual knowledge required to solve this type of problem.

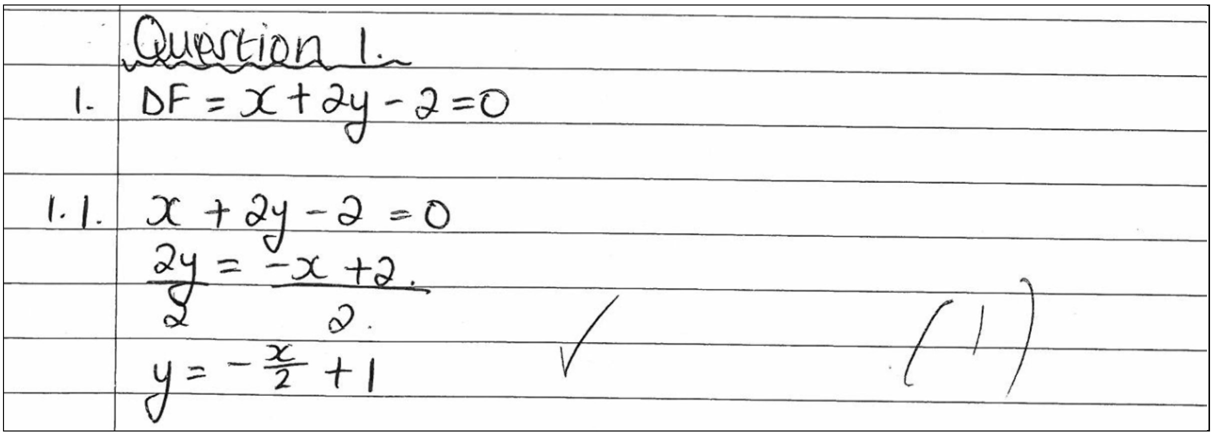

Figure 6.16: Student C's response to Question 1.1 of Class test 3

Student D attempted to manipulate the equation, but committed an error in this process. It does appear however that the student was aware that the equation had to be transformed to standard form before the gradient could be determined. One can therefore conclude that this student had retained requisite knowledge.

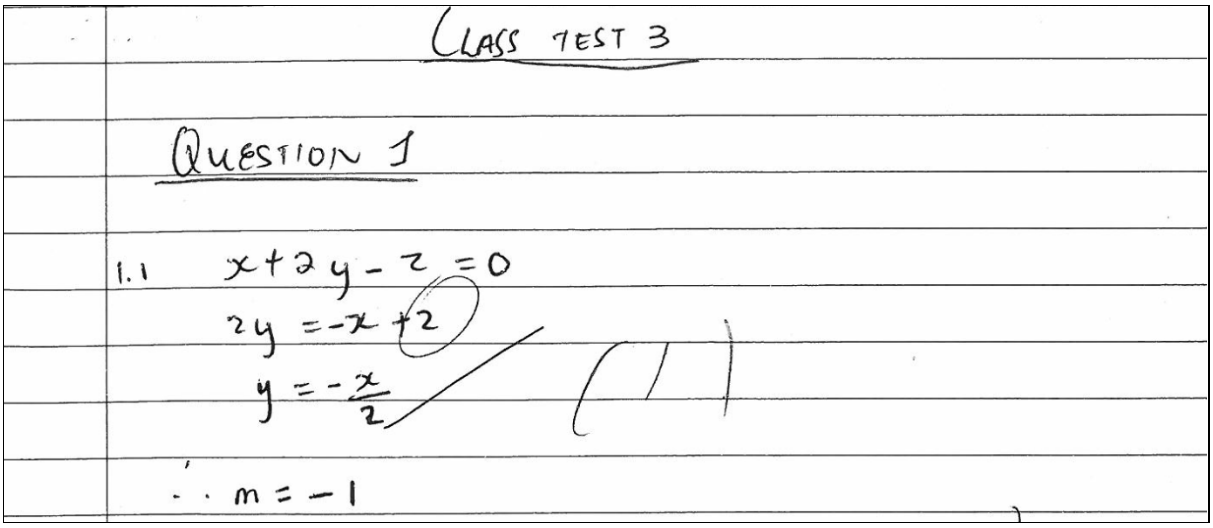

Figure 6.17: Student D's response to Question 1.1 of Class test 3

Students E and F produced correct solutions and hence the conclusion is that the indispensable knowledge was retained.

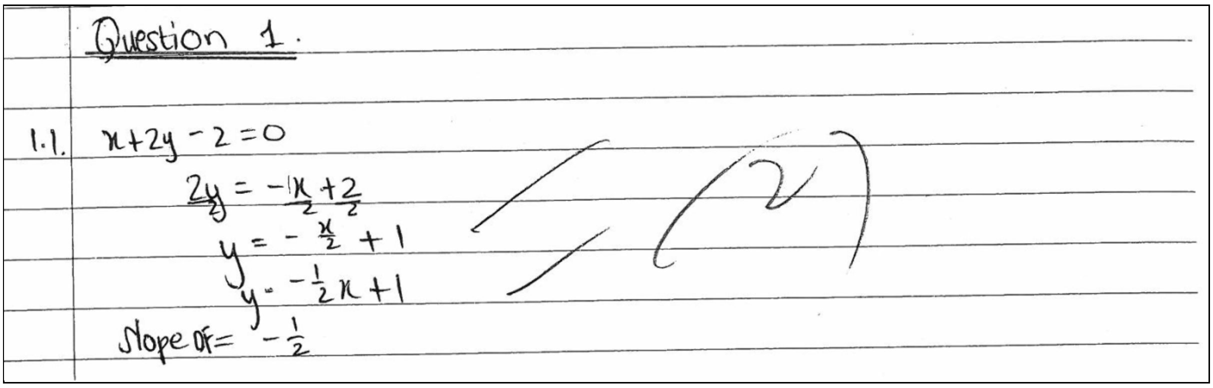

Figure 6.18: Student E’s response to Question 1.1 of Class test 3

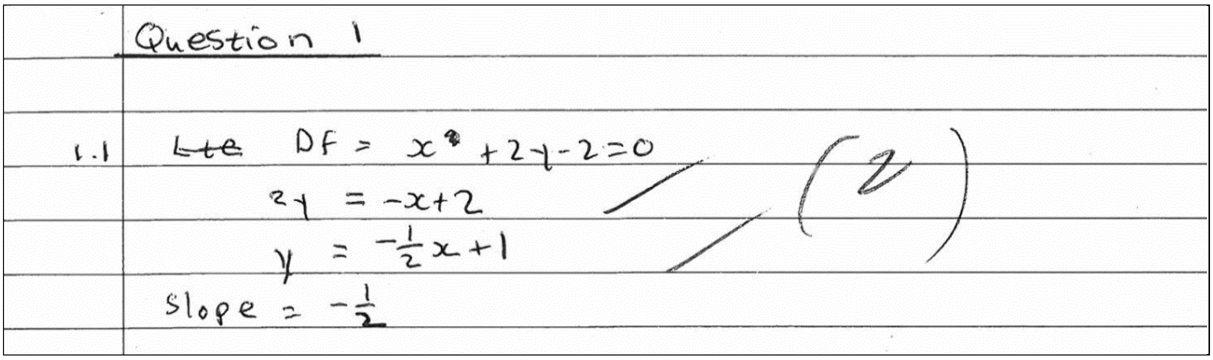

Figure 6.19: Student F’s response to Question 1.1 of Class test 3

6.3.4 Question 1.1 of Class test 4 of semester 1

Class test 4 was written approximately eleven weeks after Test 1. The first question for Class test 4 is shown in Figure 6.20. Only Question 1.1 was utilized for our purposes. This question was classified as Familiar Algorithmic Reasoning based on Procedural knowledge (B1bi). In order to answer the question, one had to realize that the numbers provided on the Cartesian axis in the sketch should be written as coordinate pairs. Thereafter one would determine the gradient using the coordinate pairs. One could then either use the slope-intercept form or the slope and *y*-intercept form of a line to determine the equation of the line. The most efficient method however is to use the slope and *y*-intercept form since it involves fewer calculations. As argued previously skilled problem-solvers in mathematics are also flexible in their use of known procedures (Star, 2005). A result of such flexibility is that students who possess such knowledge will thus have the ability to generate maximally efficient solutions for known and even sometimes unknown problem situations. If a student therefore utilized the more efficient way (slope and *y*-intercept method) of solving the problem, then one might argue that this is an indication of the presence of flexible procedural knowledge. Student responses are shown in Figures 6.21 to 6.26.

QUESTION 1

A sketch of line AD is shown below. Use the sketch to answer the following questions:

1.1 Find the equation of line AD. (3)

1.2 Determine the length of BC. (3)

1.3 Determine the equation of the line perpendicular to line AD and through the point (4;-2). (4)

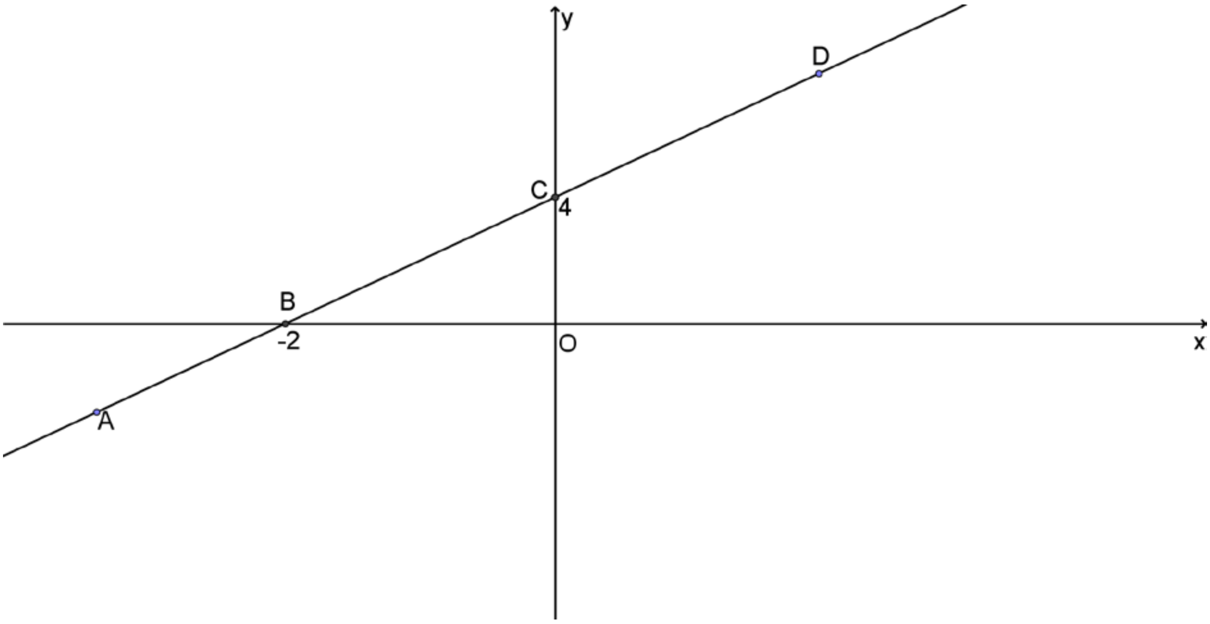

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Figure 6.20: Question 1 of Class test 4 of semester 1

It should be noted however that in this question a sketch with single numbers on the axes were provided whereas in the majority of class examples and exercises either coordinate pairs or equations were provided. During classroom discussions however students had been exposed to the idea that on an axis one of the coordinate pairs is zero. The question therefore requires students to transfer the aforementioned knowledge of coordinate pairs to this problem.

Student A calculated the gradient correctly, but did not proceed to determine the required equation. There are many possible reasons why the student did not determine the equation. One may therefore not make any judgement concerning her knowledge of how to determine the equation of the line. One could however conclude that knowledge of coordinate pairs and how to use the coordinate pairs to determine gradients had been retained.

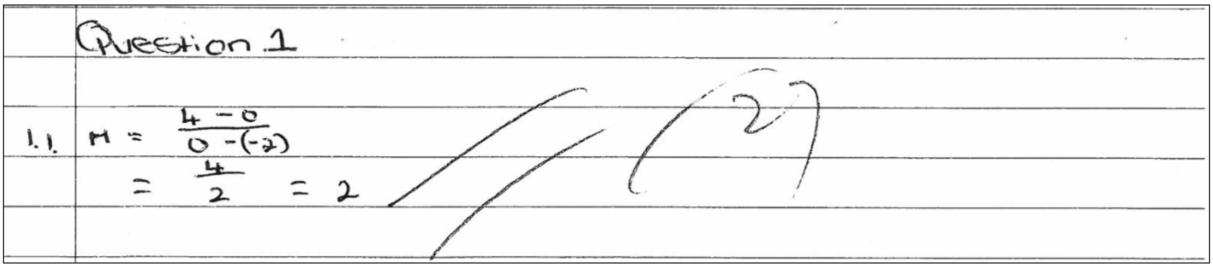


Figure 6.21: Student A's response to Question 1.1 of Class test 4

Although Student B correctly wrote down the formula to calculate the gradient, she substituted variables in both numerator and denominator. It seems as though she was not aware that the points provided in the sketch should be used to determine the gradient. One might therefore argue that Student B had not made the connection that the numbers provided on the axis in the sketch may be written as coordinate pairs which could then be used to calculate the gradient.

Student B wrote the correct formula for the point-slope form of the line, but again substituted variables where numbers were supposed to have been placed in the formula. This is further confirmation that she did not make the connection that one may write the numbers provided on the axis as coordinate pairs. As a result she substituted variables since she was under the impression that no coordinate pairs existed in the problem. Student B was aware that the solution required that first the gradient should be determined, and subsequently that the gradient together with a point should be utilized to determine the linear equation. One can therefore argue that indispensable knowledge had been retained regarding the gradient and point-slope formula of the line. However, what was lacking was knowledge transfer regarding how to write coordinate pairs if single numbers are provided on the axis.

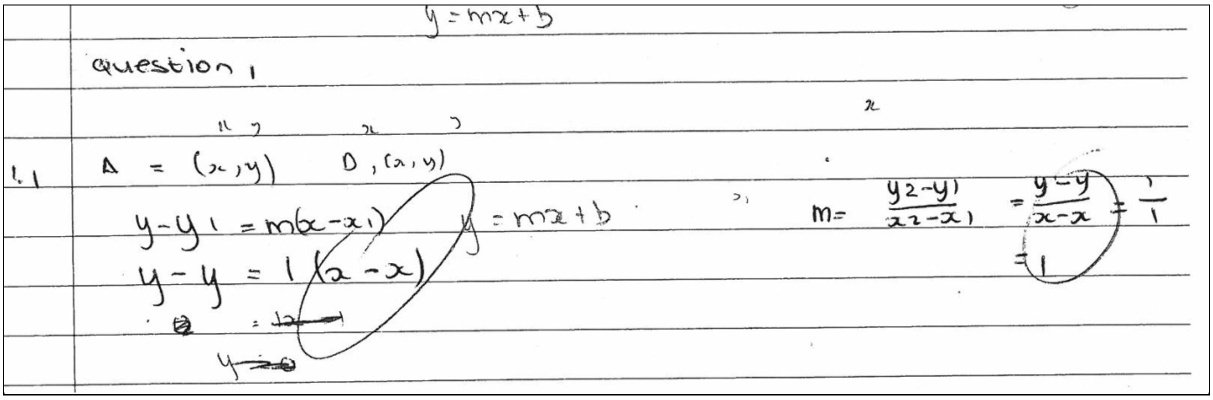


Figure 6.22: Student B's response to Question 1.1 of Class test 4

Student C provided a completely correct solution. Furthermore the student used the most efficient way to derive the answer and hence a possible conclusion is that the student possesses flexible procedural knowledge for this type of problem. Student C therefore exhibited both retention and transfer of the indispensable knowledge for this class of problem.

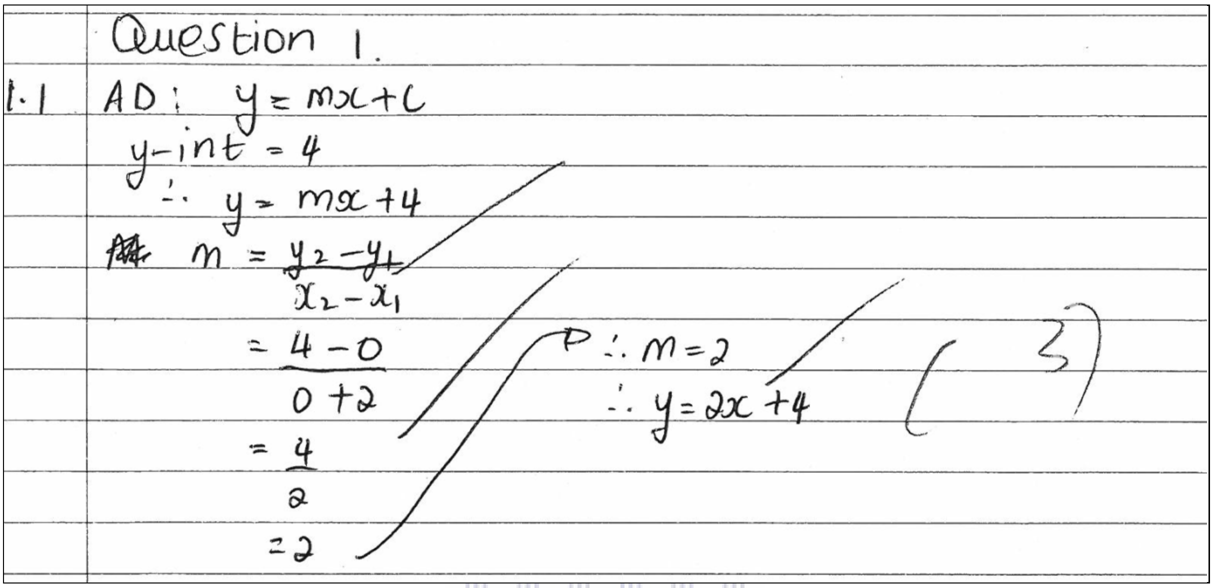


Figure 6.23: Student C's response to Question 1.1 of Class test 4

The solutions provided by Students D, E and F were also entirely correct. However these students did not use the most efficient way to solve the problem and hence one can argue that they did not use available knowledge flexibly. Solutions provided by these students however show signs of retention and transfer of requisite knowledge.

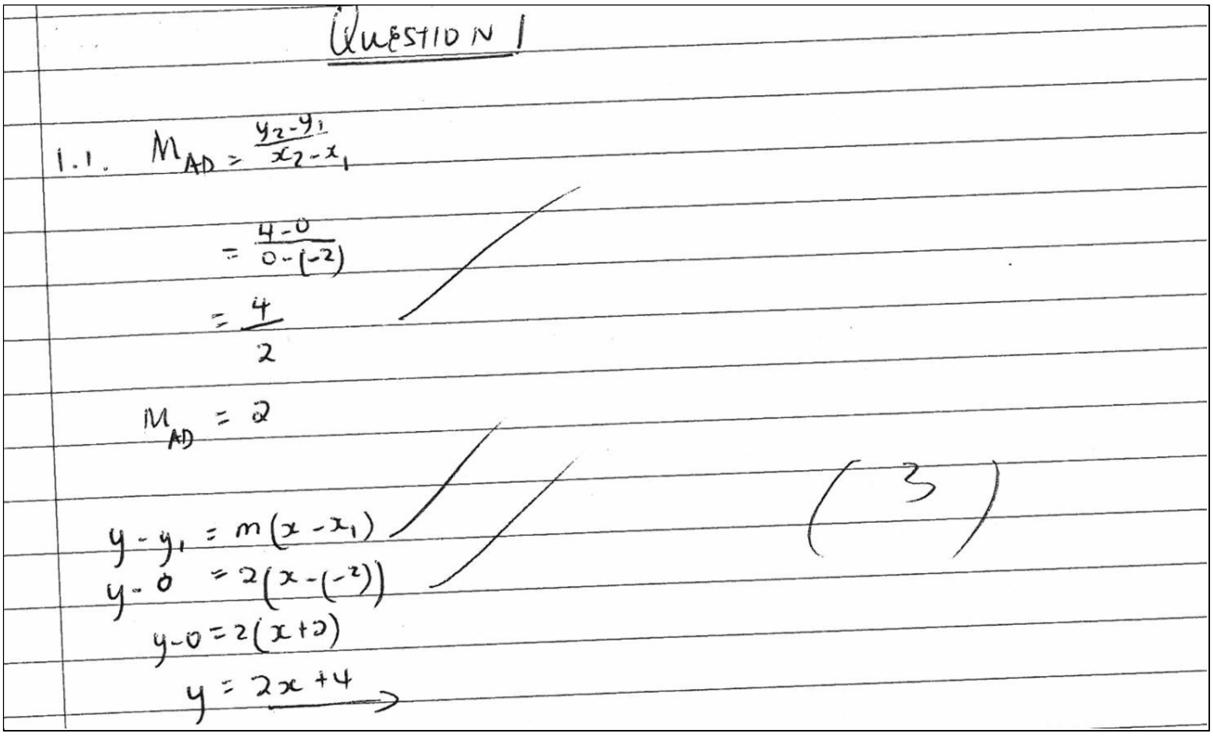


Figure 6.24: Student D's response to Question 1.1 of Class test 4

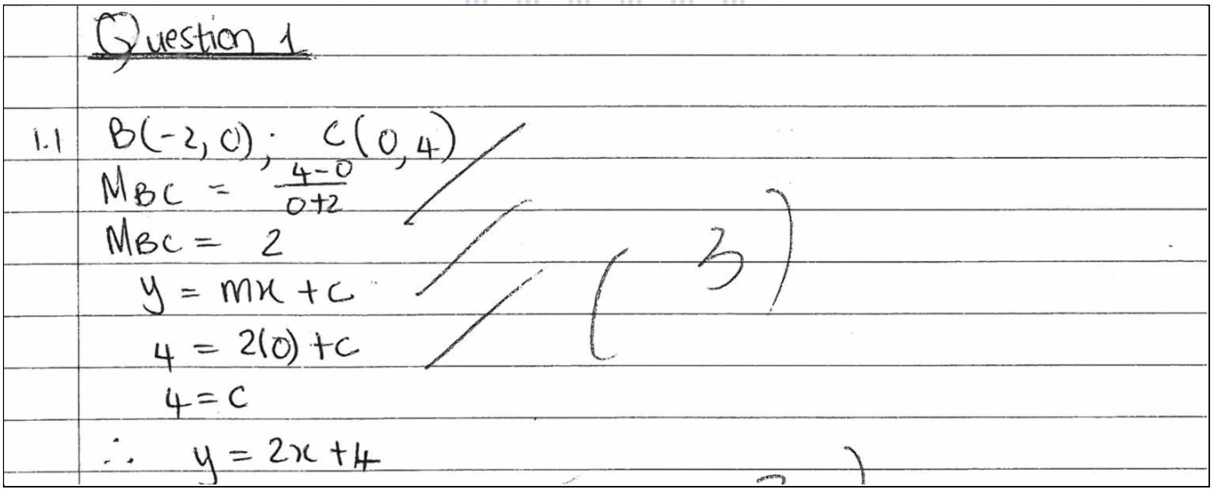


Figure 6.25: Student E's response to Question 1.1 of Class test 4

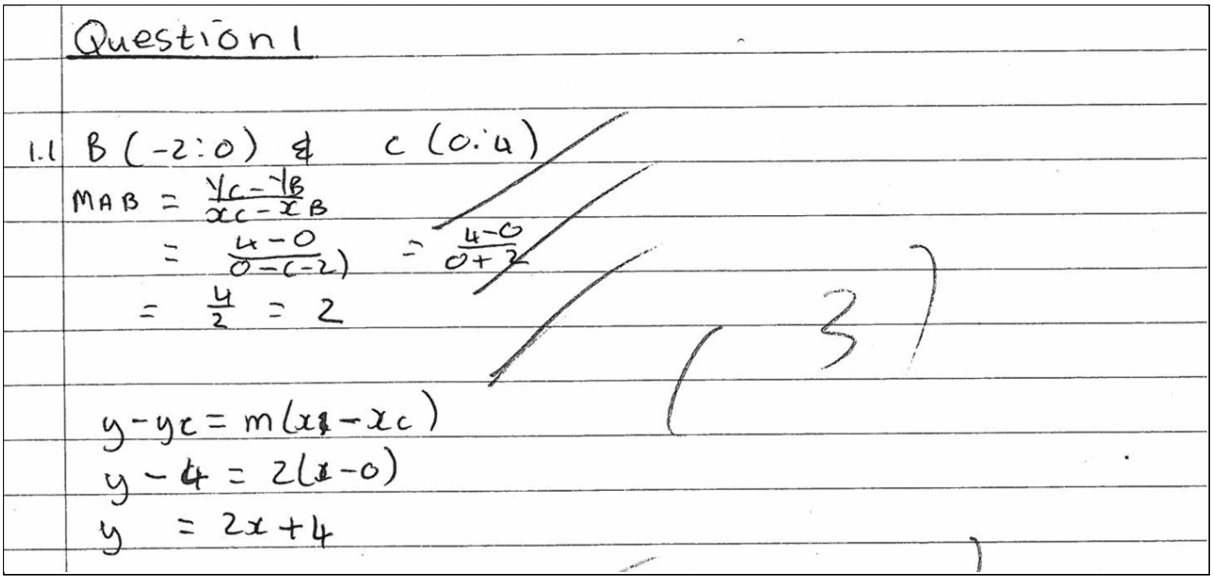


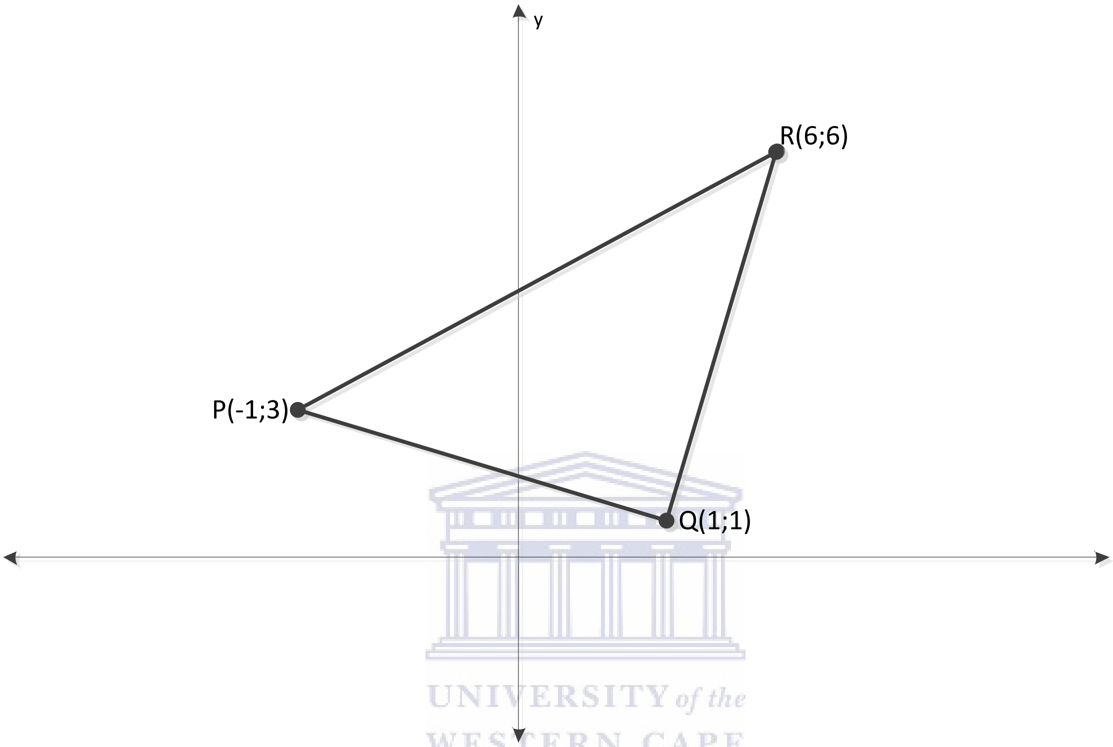
Figure 6.26: Student F's response to Question 1.1 of Class test 4

6.3.5 Question 1.5 of the first semester examination

The end of module examination was written approximately 14 weeks after Class test 1 had been written. The first question of the examination is shown in Figure 6.27. Only Question 1.5 was considered for the purposes of this analysis. This question was classified as Familiar Algorithmic Reasoning based on Conceptual knowledge (D1bi). In order to solve the problem one had to access the fact that gradients of parallel lines are equal. To determine the required equation the gradient of line PQ – which had been determined in the previous question – should therefore be used together with point R in the point-slope equation of the line (or alternatively the point-*y*-intercept form of the line). Student responses are shown in Figures 6.28 to 6.33.

QUESTION 1

PQR is a triangle with vertices $P(-1;3)$, $Q(1;1)$ and $R(6;6)$.



1.1	Determine the gradient of PQ.	(2)
1.2	Determine the gradient of QR.	(2)
1.3	Show that triangle PQR is right-angled at Q.	(2)
1.4	Determine the equation of line PQ.	(2)
1.5	Determine the equation of the line parallel to PQ and through the point R.	(3)
1.6	Show that the point (-3;5) lies on the line PQ.	(2)
1.7	Determine the coordinates of the midpoint M of PR.	(2)
1.8	Determine the equation of the perpendicular bisector of PR.	(4)

/19/

Figure 6.27: Question 1 of first semester examination

Students A, B, D and F produced correct responses to Question 1.5. All of these students correctly utilized the gradient of PQ in the point-slope form of the line to determine the required equation. Student E also produced a correct response. However this student utilized the point-*y*-intercept form of the line to determine the required equation. One can therefore argue that all these students have retained the requisite indispensable procedural and conceptual knowledge for this type of problem.

Student C was aware that parallel lines have equal gradients, and correctly substituted the gradient and coordinates of point R into the point-slope form of the line. The student transposed the -6 incorrectly and hence did not produce the correct equation. Nonetheless the student's solution shows that the indispensable knowledge had been retained.

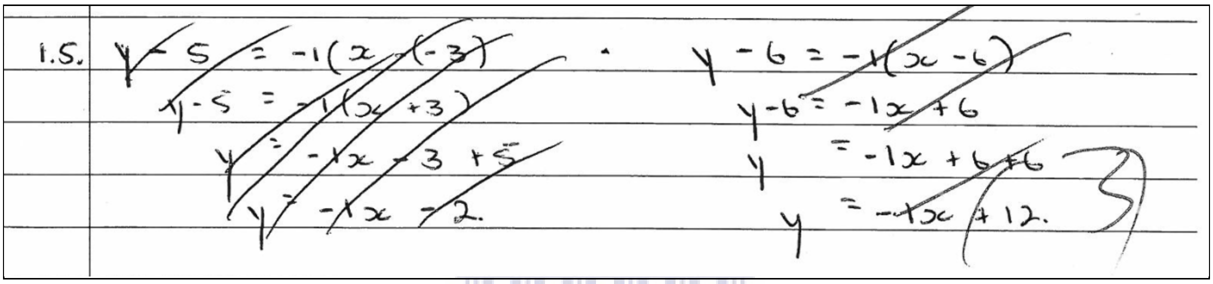


Figure 6.28: Student A's response to Question 1.5 of first semester examination

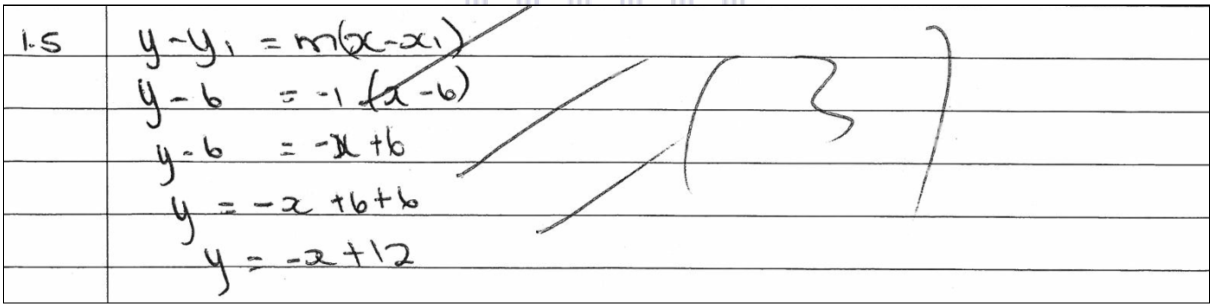


Figure 6.29: Student B's response to Question 1.5 of first semester examination

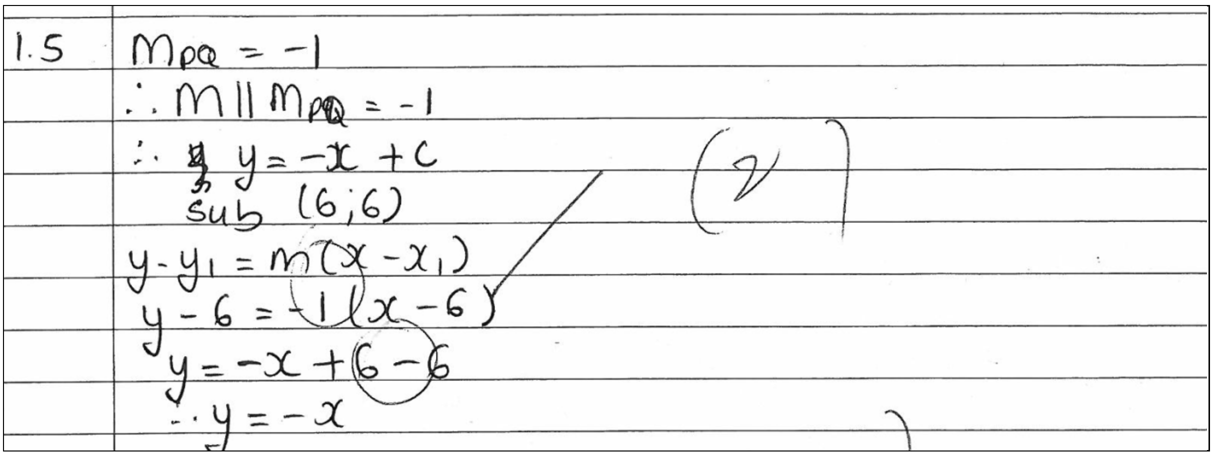
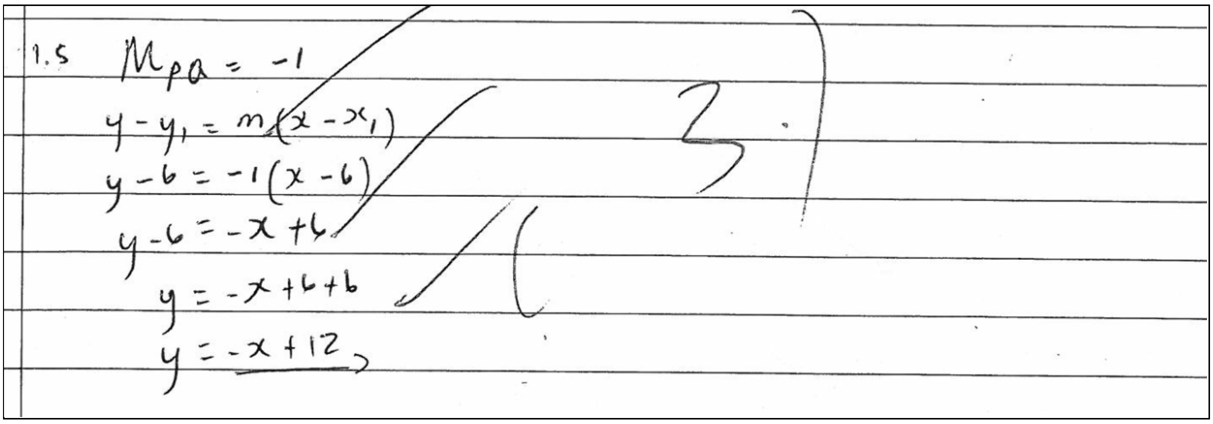


Figure 6.30: Student C's response to Question 1.5 of first semester examination



1.5 $M_{PQ} = -1$

$y - y_1 = m(x - x_1)$

$y - 6 = -1(x - 6)$

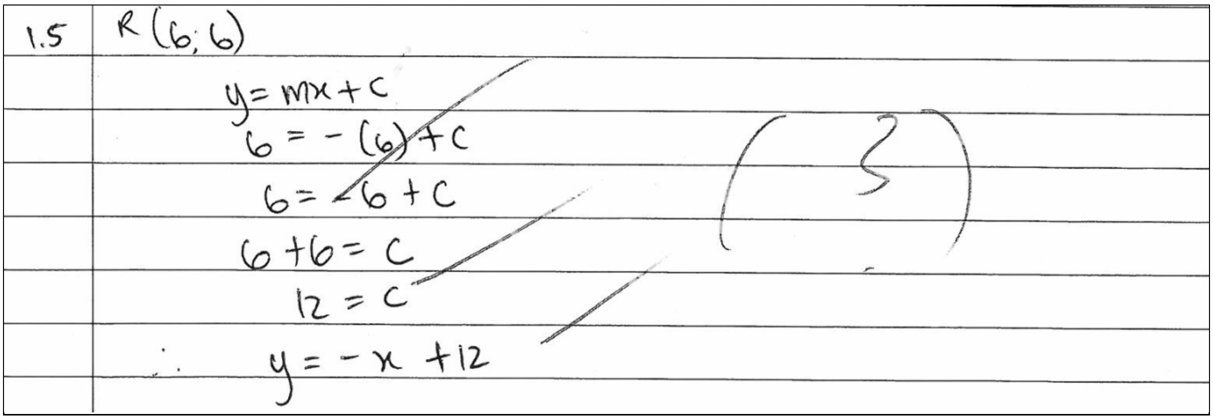
$y - 6 = -x + 6$

$y = -x + 6 + 6$

$y = -x + 12$

(3)

Figure 6.31: Student D's response to Question 1.5 of first semester examination



1.5 $R(6;6)$

$y = mx + c$

$6 = -(6) + c$

$6 = -6 + c$

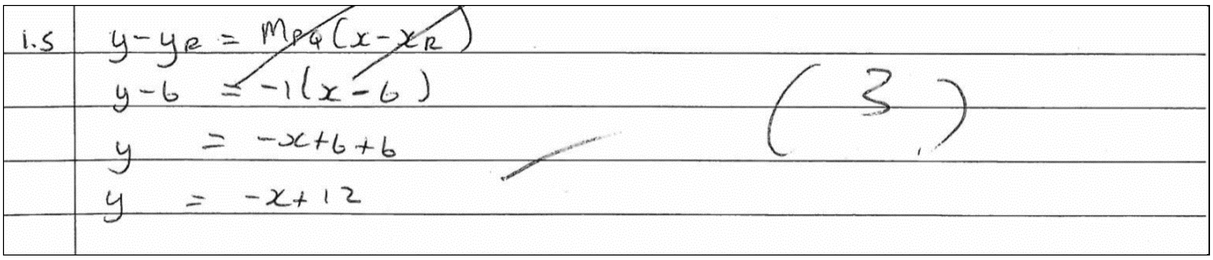
$6 + 6 = c$

$12 = c$

$\therefore y = -x + 12$

(3)

Figure 6.32: Student E's response to Question 1.5 of first semester examination



1.5 $y - y_R = m_{PQ}(x - x_R)$

$y - 6 = -1(x - 6)$

$y = -x + 6 + 6$

$y = -x + 12$

(3)

Figure 6.33: Student F's response to Question 1.5 of first semester examination

6.3.6 An example of a two-variable linear equation problem that is embedded in a more complex problem

Ordinarily complex mathematical problems contain many pieces of seemingly disparate information. In solving such problems, while it may be easy to make mental connections between some parts of the information provided and relevant prior knowledge, it may not be easy to make similar connections with other parts of the information provided. Furthermore, some of the information that is provided is not obvious and one has to make conceptual connections to 'see' this information. Should the problem have sub-questions or sub-goals then a further complication is that one has to decide between solution-relevant and solution-irrelevant information.

It is my contention that if a student can recognise instances where prior mathematical knowledge can be applied in a complex environment, and if they then proceed to apply the knowledge correctly, then such knowledge is well ensconced in the long-term memory and is most probably connected appropriately with relevant conceptual knowledge.

Students' perception of the degree of difficulty of mathematical problems varies. Some students from the same class group may find a particular problem very difficult to solve whereas others might find the same problem easier to solve. The complexity of a mathematical problem is dependent upon many factors. Examples of such factors are levels of student prior knowledge concerning the problem domain, number of past experiences with similar problems and student ability. Students were presented with more complex problems from the domain of analytic geometry that dealt with the topic of circles, but which simultaneously tested knowledge of linear functions. The following instruction with sketch, and its solution (see Figure 54) were utilized for class discussion purposes:

"Determine the equations of the tangents to the circle $x^2 + y^2 = 13$ *from the point* $A(-1;8)$ *outside the circle."*

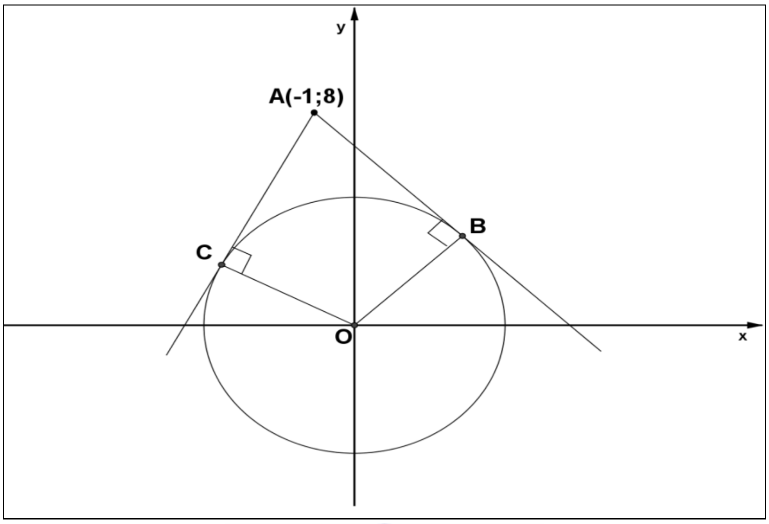


Figure 6.34: Sketch of an example of a more complex problem

Possible Solution

We write the equation $x^2 + y^2 = 13$ in terms of y to obtain: $y = \sqrt{13 - x^2}$

Therefore $B\left(x; \sqrt{13 - x^2}\right)$

The gradient of normal OB is: $\frac{\sqrt{13-x^2}}{x}$

$\therefore$ The gradient of tangent AB is: $\frac{-x}{\sqrt{13-x^2}}$

But the gradient of AB is also: $\frac{\sqrt{13-x^2}-8}{x+1}$

$$\therefore \frac{-x}{\sqrt{13 - x^2}} = \frac{\sqrt{13 - x^2} - 8}{x + 1}$$

$\therefore -x^2 - x = 13 - x^2 - 8\sqrt{13 - x^2}$

$\therefore 65x^2 + 26x - 663 = 0$

$\therefore 5x^2 + 2x - 51 = 0$

$\therefore (5x + 17)(x - 3) = 0$

$\therefore x = -\frac{17}{5} \quad or \quad x = 3$

$\therefore At\ B: \ x = 3 \quad and \quad y = 2$

$At\ C:\ x = -\frac{17}{5} \quad and \quad y = \frac{6}{5}$

Gradient of tangent AB is: $-\frac{3}{2}$

Equation of tangent AB is: $y = -\frac{3}{2}x + \frac{13}{2}$

Gradient of tangent AC is: $\frac{17}{6}$

Equation of tangent AC is: $y = \frac{17}{6}x + \frac{65}{6}$

In the foregoing example the coordinates of points B and C are written in a generalized form that is, they are the coordinates for any point anywhere on the circumference of this circle. For all participating students this was their first encounter with this way of writing coordinates. The remainder of the solution was dependent on knowledge of gradients and the equation of a linear function although this is in the context of circle geometry.

A second, similar example was discussed in order to consolidate the main ideas involved in the solution of such problems. This example was as follows:

"*Find the equations of the tangents to* $(x+1)^2 + y^2 = 20$ *which are parallel* to $2y - x = 0$."

In this case no sketch was provided, but a sketch was drawn during the class debate in order to provide a visual picture to aid the discussion.

The concepts involved in the solution of such problems are coordinates, gradients, equations of lines, parallelism of lines, perpendicularity of lines, general equation of circle, tangents to circles and angle between radius and tangent. A similar question was presented to students in the third class test of the second semester. The question was:

"*Find the equations of the tangents* to $x^2 + (y+1)^2 = 20$ *which are parallel to* $y + 2x = 0$."

The solution procedure for this problem is similar to the foregoing example. That is, first one has to write the equation of the parallel line in standard form in order to find the gradient. The equation of the circle is used to find the centre of the circle. The circle equation should then be written in terms of either *x* or *y* in order to write generalized coordinates for points of contact on the two tangents (in this case it is more convenient to write the equation in terms of *x*). These points are then utilized together with the coordinates of the centre to determine the gradient of the radius of the circle. The product of this gradient and the tangent gradient is put equal to -1, since the tangent and the radius are perpendicular. This equation is then manipulated to determine two values for either *x* or *y*. Back substitution yields the corresponding *x* or *y* values which are subsequently used to write the two coordinate pairs for the points of contact on the tangents. These points are then used together with the parallel gradient to determine the required equations. Student solutions are shown in Figures 6.35 to 6.40.

This test was written approximately 32 weeks after Test 1. Part of the objective of this test was to assess retention and transfer of knowledge of linear functions. In particular the aim was to determine whether students would – in a complex environment – be able to recognise those instances where they could apply their prior knowledge of linear functions. Elements of various categories form part of this question; however the question was classified based on the perceived major reasoning and knowledge requirements and hence it was categorised as Delimiting Algorithmic Reasoning based on Conceptual knowledge (D1bii).

Student A realized that the centre of the circle was (0;-1), but could not use this information to determine the required gradients. It seemed however as if Student A was aware that the gradient needed to be calculated, but could not determine the required coordinates. It seemed also that the student was aware that the fact that the product of the gradients of perpendicular lines is equal to -1, should be utilized. The student was able to recall the point-slope formula of the line, but substituted incorrect values. Since no coordinates were determined one cannot judge whether the student had retained requisite conceptual knowledge of coordinates and gradients of linear functions. One can however infer from the student's solution strategy that she struggled to transfer her knowledge to the new context.

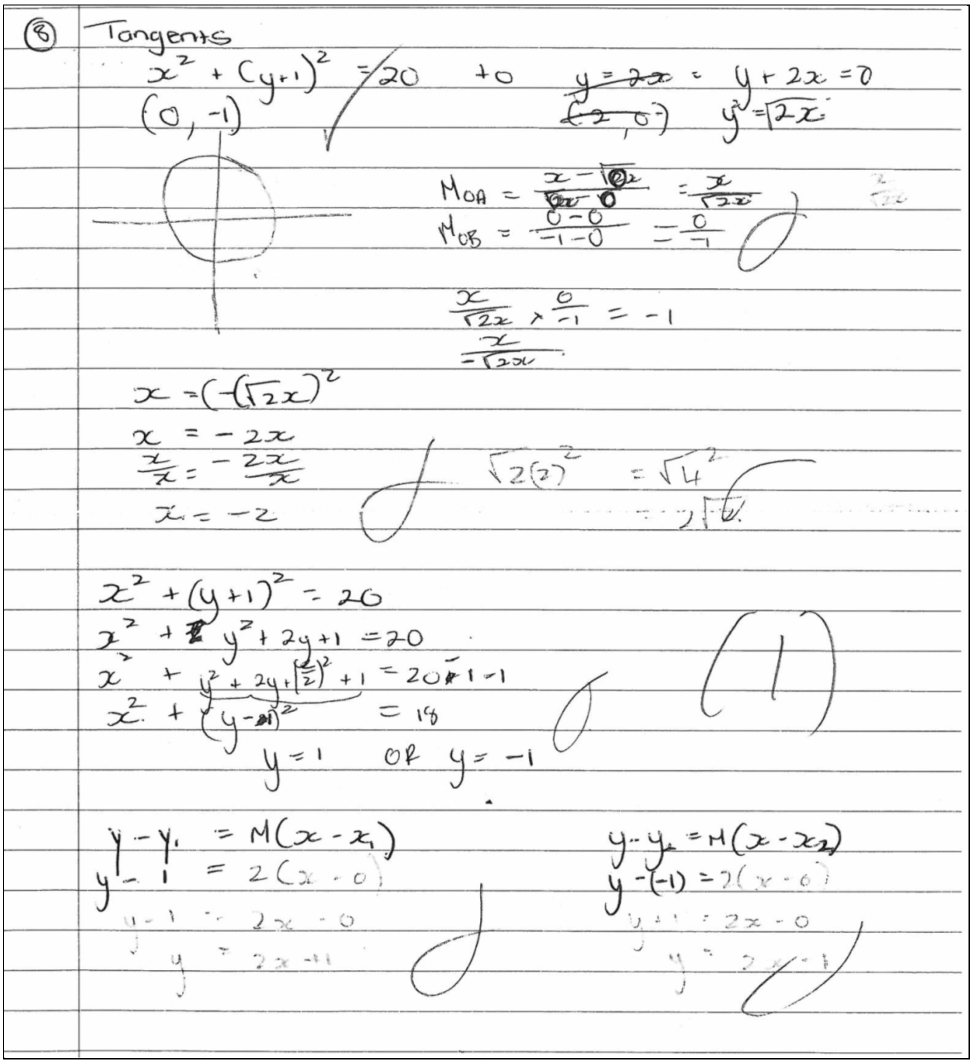
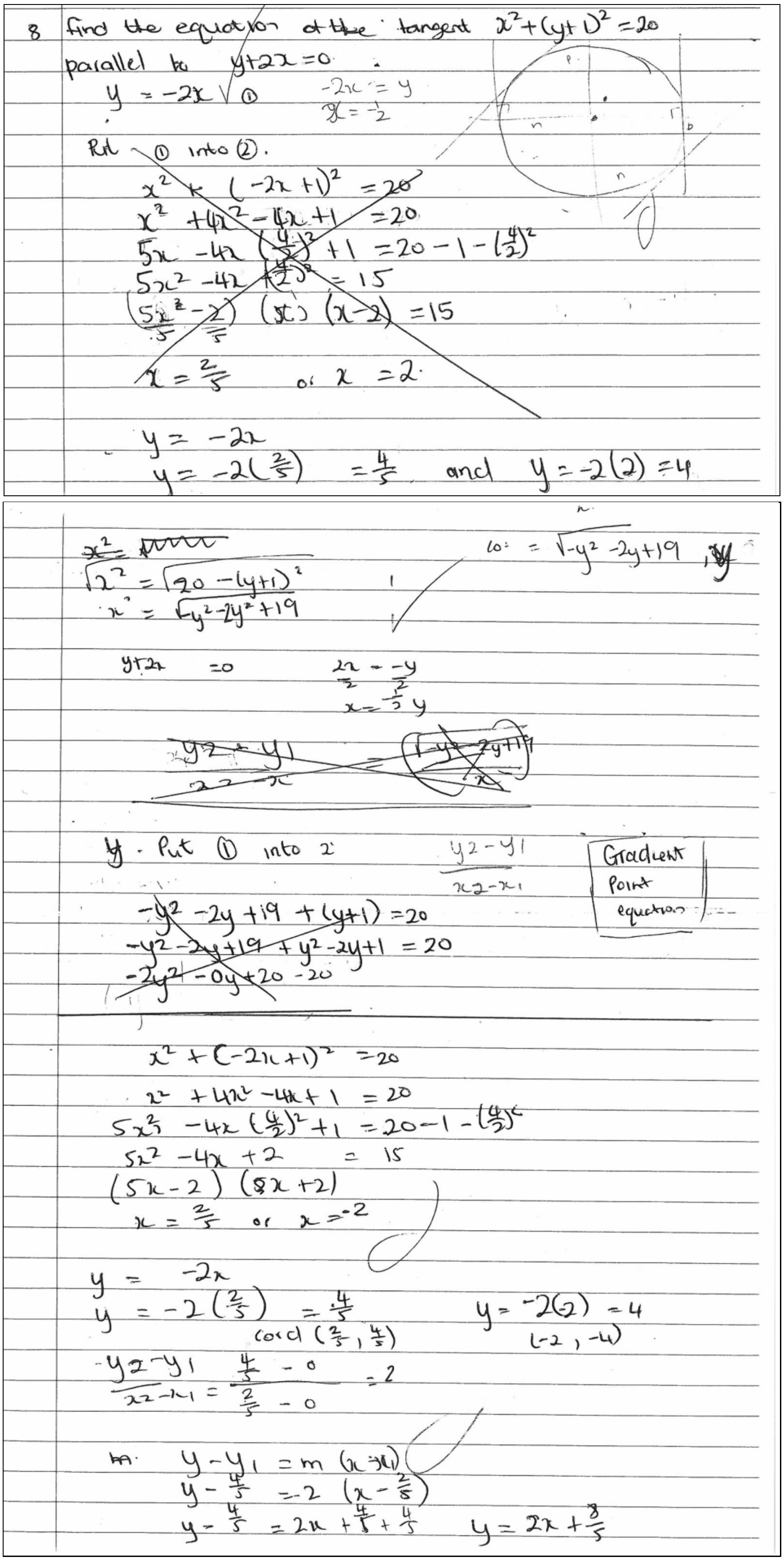


Figure 6.35: Student A’s solution to Question 8 of Class test 3 of semester 2

Student B wrote the equation of the parallel line in standard form in order to determine the gradient. However she did not substitute this gradient into the point-slope equation of a line that was written correctly. The student attempted to manipulate the provided equation of the circle in order to write *y* in terms of *x*. This information however was not used to write generalized coordinates. The student then proceeded to write the formula for gradient, but substituted no value or variables. This is perhaps an indication that she did not know how to determine the generalized coordinates and consequently did not substitute into the formula.

On the side of the page Student B wrote the words ‘gradient,’ ‘point’ and ‘equation.’ A strong possibility is that the student wrote these words as a reminder that these are the things she was to determine in order to solve the problem. One can therefore infer from her solution strategy that she had retained general knowledge regarding gradients and equations of linear functions, but could not transfer this knowledge to the context with which she was presented.



8 find the equation of the tangent $x^2+(y+1)^2=20$ parallel to $y+2x=0$.

$y=-2x$ ①

Put ① into ②.

$x^2+(-2x+1)^2=20$

$x^2+4x^2-4x+1=20$

$5x^2-4x+2=15$

$(5x-2)(x-2)=15$

$x=\frac{2}{5}$ or $x=2$

$y=-2x$

$y=-2(\frac{2}{5})=\frac{4}{5}$ and $y=-2(2)=4$

$x^2=\sqrt{20-(y+1)^2}$

$x^2=\sqrt{-y^2-2y^2+19}$

$y+2x=0$

$x=-\frac{1}{2}y$

Put ① into 2

Gradient Point equation

$\frac{y_2-y_1}{x_2-x_1}$

$-y^2-2y+19+(y+1)=20$

$-y^2-2y+19+y^2-2y+1=20$

$-2y^2-0y+20-20$

$x^2+(-2x+1)^2=20$

$x^2+4x^2-4x+1=20$

$5x^2-4x+2=15$

$(5x-2)(x+2)$

$x=\frac{2}{5}$ or $x=-2$

$y=-2x$

$y=-2(\frac{2}{5})=\frac{4}{5}$ coord $(\frac{2}{5},\frac{4}{5})$

$y=-2(2)=4$ $(-2,-4)$

$\frac{y_2-y_1}{x_2-x_1}=\frac{\frac{4}{5}-0}{\frac{2}{5}-0}=2$

$y-y_1=m(x-x_1)$

$y-\frac{4}{5}=-2(x-\frac{2}{5})$

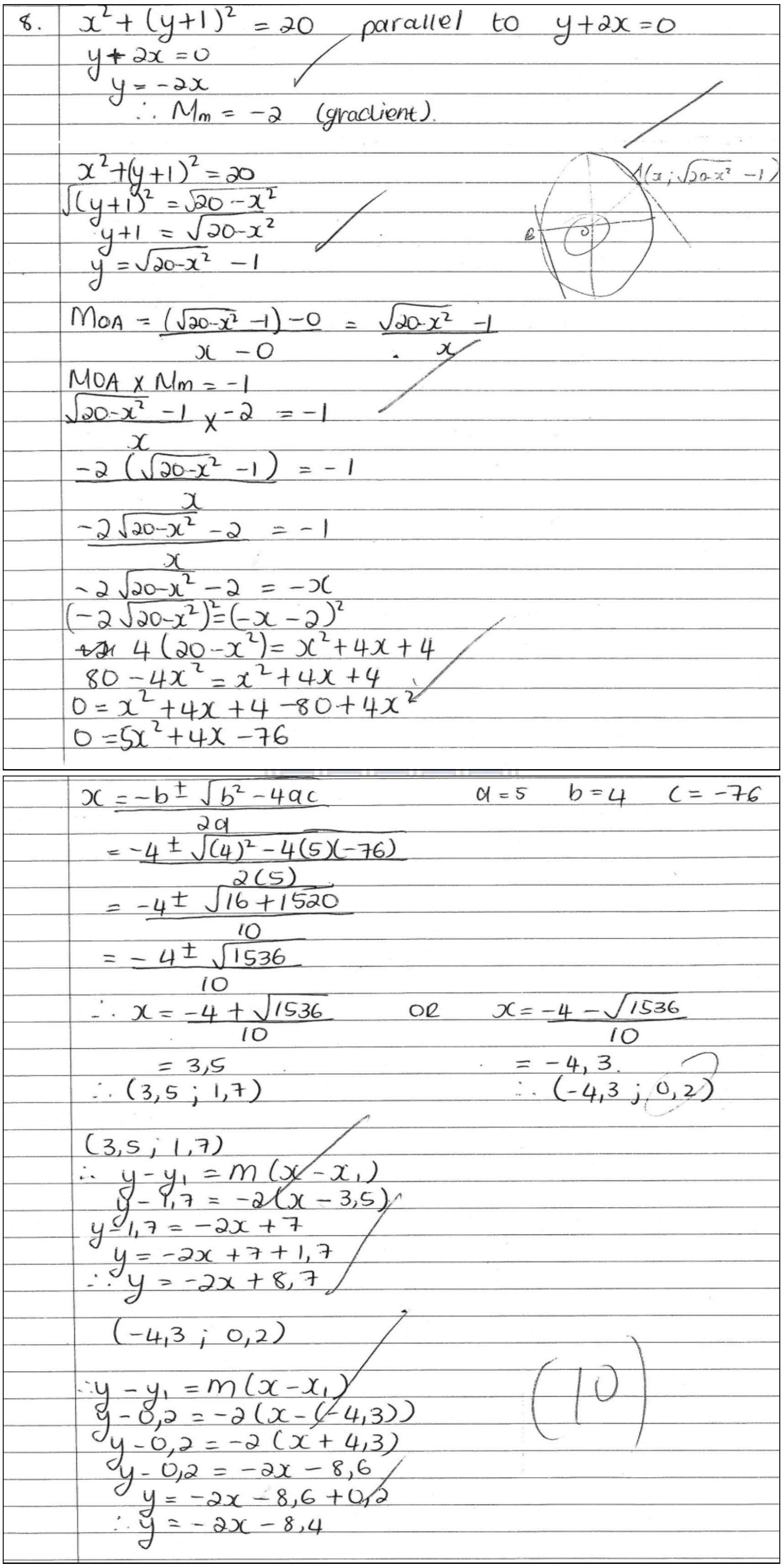
$y-\frac{4}{5}=2x+\frac{4}{5}+\frac{4}{5}$

$y=2x+\frac{8}{5}$

Figure 6.36: Student B’s solution to Question 8 of Class test 3 of semester 2

Student C wrote the equation of the parallel line in standard form in order to determine the gradient. The sketch was incorrectly drawn with centre (0;0). The student correctly manipulated the equation of the circle to make y the subject of the formula. However, a better option would have been to write x in terms of y. This is perhaps an indication that the student was more reliant on imitative reasoning (the class example made y the subject of the formula, so the inference is that the student was imitating this), and consequently did not transfer prior knowledge in the most effective way. In other words the student did not utilize prior procedural knowledge flexibly.

Using the incorrect centre the student proceeded to determine the gradient of the radius correctly. Consequently the student equated the product of this gradient and the parallel gradient with -1. This equation was then used to determine values for x. These x-values were then back substituted to find the y-coordinates. These two points were then utilized to determine the required equations. One can thus infer from the student's solution that indispensable knowledge regarding linear functions was retained and was transferred appropriately to the new context.



8. $x^2+(y+1)^2=20$ parallel to $y+2x=0$

$y+2x=0$

$y=-2x$

$\therefore M_m=-2$ (gradient).

$x^2+(y+1)^2=20$

$\sqrt{(y+1)^2}=\sqrt{20-x^2}$

$y+1=\sqrt{20-x^2}$

$y=\sqrt{20-x^2}-1$

$M_{OA}=\frac{(\sqrt{20-x^2}-1)-0}{x-0}=\frac{\sqrt{20-x^2}-1}{x}$

$M_{OA}\times M_m=-1$

$\frac{\sqrt{20-x^2}-1}{x}\times -2=-1$

$\frac{-2(\sqrt{20-x^2}-1)}{x}=-1$

$\frac{-2\sqrt{20-x^2}-2}{x}=-1$

$-2\sqrt{20-x^2}-2=-x$

$(-2\sqrt{20-x^2})^2=(-x-2)^2$

$4(20-x^2)=x^2+4x+4$

$80-4x^2=x^2+4x+4$

$0=x^2+4x+4-80+4x^2$

$0=5x^2+4x-76$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $a=5$ $b=4$ $c=-76$

$=\frac{-4\pm\sqrt{(4)^2-4(5)(-76)}}{2(5)}$

$=\frac{-4\pm\sqrt{16+1520}}{10}$

$=\frac{-4\pm\sqrt{1536}}{10}$

$\therefore x=\frac{-4+\sqrt{1536}}{10}$ or $x=\frac{-4-\sqrt{1536}}{10}$

$=3,5$ $=-4,3$

$\therefore(3,5;1,7)$ $\therefore(-4,3;0,2)$

$(3,5;1,7)$

$\therefore y-y_1=m(x-x_1)$

$y-1,7=-2(x-3,5)$

$y-1,7=-2x+7$

$y=-2x+7+1,7$

$\therefore y=-2x+8,7$

$(-4,3;0,2)$

$\therefore y-y_1=m(x-x_1)$

$y-0,2=-2(x-(-4,3))$

$y-0,2=-2(x+4,3)$

$y-0,2=-2x-8,6$

$y=-2x-8,6+0,2$

$\therefore y=-2x-8,4$

(10)

Figure 6.37: Student C's solution to Question 8 of Class test 3 of semester 2

Student D identified the coordinates of the centre of the circle correctly. This student attempted to write *y* in terms of *x*, but made a mistake in the process. Subsequently the student attempted to determine the gradient of the radius. Two gradients were calculated, presumably one for the radius towards a tangent on one side of the circle and the other to the tangent on the opposing side. The product of these gradients was then equated with -1.

The student's solution seems to indicate that she was aware that the equation of the circle should be written in terms of the variable *y* and that subsequently this result should be utilized to write generalized coordinates for points on the intersection of the circle and tangents. She was also aware that the coordinates of the centre of the circle and the generalized coordinates should be used to determine the gradient of the radius. One can infer from her solution that knowledge of gradients, coordinates and the product of perpendicular lines had been retained and transferred to the new context. One cannot make a judgment about her knowledge of equations of lines since her solution did not include these. She did not however make the connection between parallel lines and equality of gradients. Also, she did not choose to use the better option in the process of writing the equation of the circle in terms of one variable. This is perhaps an indication that flexible procedural knowledge was not well developed for this type of problem. Furthermore it seems as if she was relying more on imitative reasoning and hence utilized recall to apply her solution strategy. I am of the opinion that she was attempting to recall features of the example and relied less on reasoning through the problem before her and consequently committed the errors.

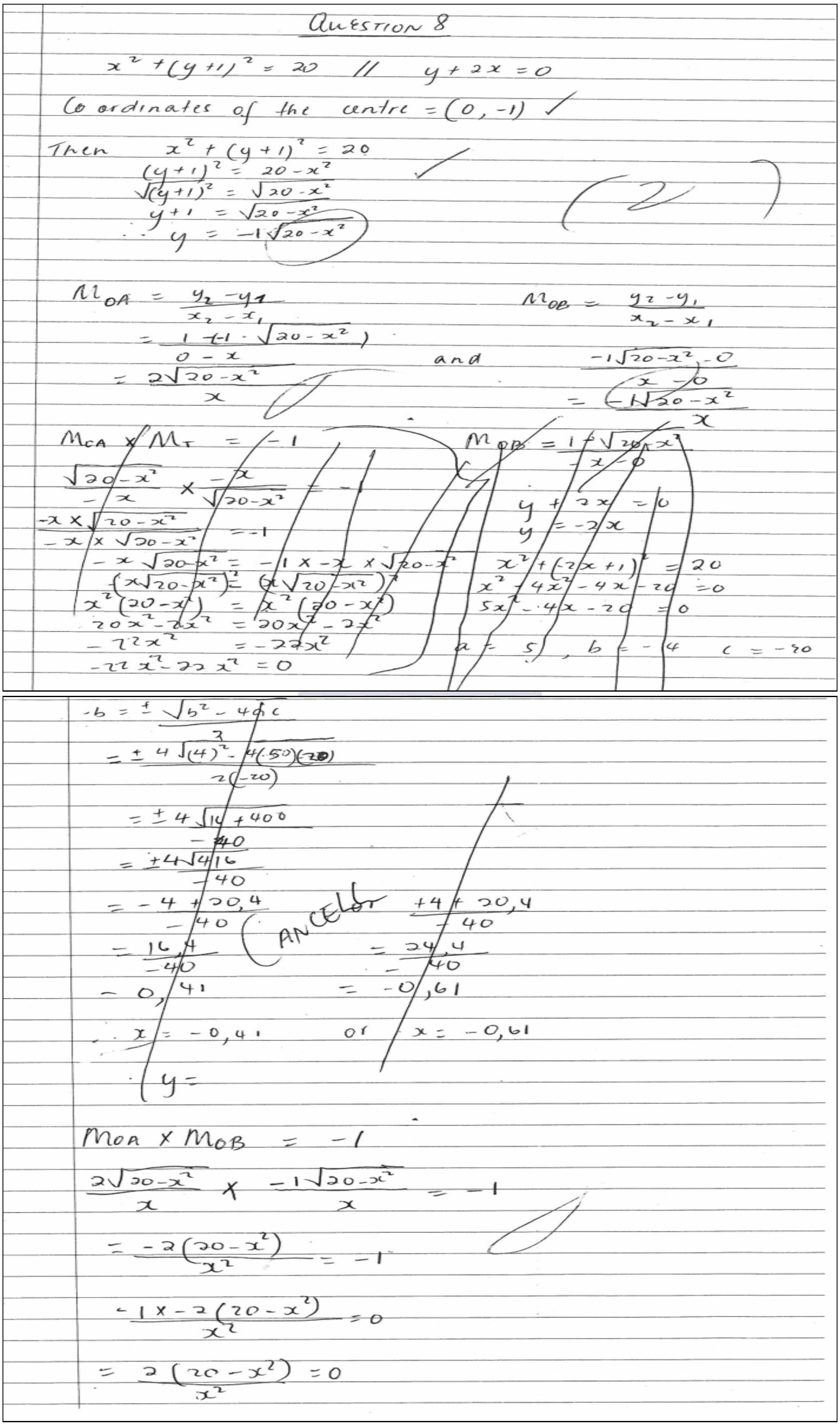
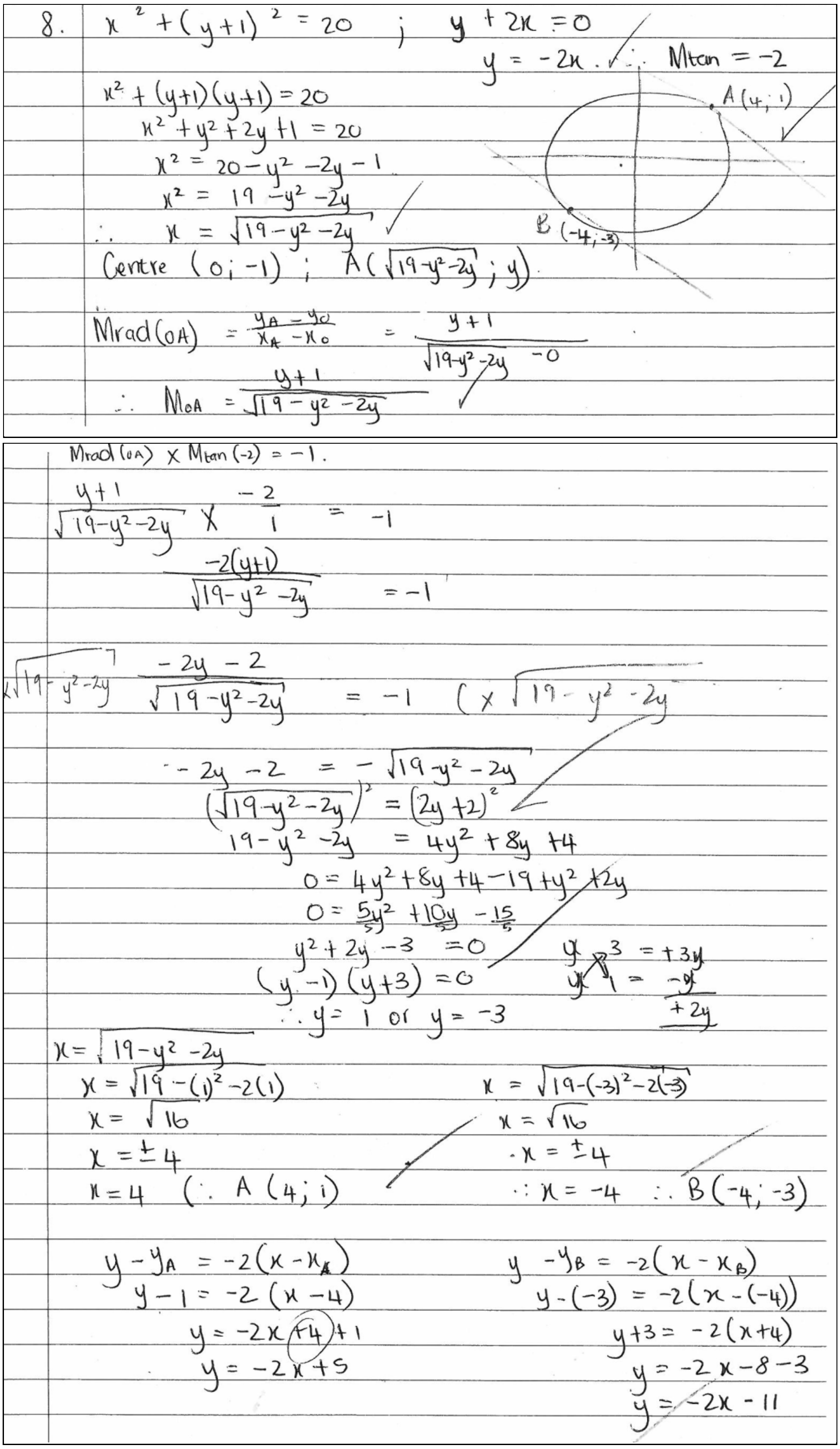


Figure 6.38: Student D’s solution to Question 8 of Class test 3 of semester 2

Student E provided an entirely correct solution except for a calculation error in the final part of her solution. One of the prominent features of her solution was that she manipulated the equation of the circle to write *x* in terms of *y*. This is contrary to the class example where *y* was written in terms of *x*. This is an indication that the student used creative reasoning in order to use prior procedural knowledge flexibly. This student is one of only two students in the entire class who used flexible procedural knowledge in their solution procedure. One can infer from her solution that Student E had retained indispensable knowledge regarding linear functions and had transferred relevant knowledge to the new context.

Student F made only one mistake in his attempt. He did not make the connection that one needs to take the negative value of the square root in the case of the coordinates in the third quadrant. This is perhaps an indication that his conceptual understanding of coordinates, quadrants and roots was not yet at the desired level. Also this student did not realize that it would have been more convenient to manipulate the circle equation provided in order to write *x* in terms of *y*. This demonstrates his reliance on imitative reasoning and perhaps an under-developed creative reasoning ability. An inference one can make however is that the student had retained the indispensable knowledge regarding linear functions and had transferred relevant knowledge to the new context.


8. $x^2 + (y+1)^2 = 20$; $y + 2x = 0$

$y = -2x$ ∴ $M_{tan} = -2$

$x^2 + (y+1)(y+1) = 20$

$x^2 + y^2 + 2y + 1 = 20$

$x^2 = 20 - y^2 - 2y - 1$

$x^2 = 19 - y^2 - 2y$

∴ $x = \sqrt{19 - y^2 - 2y}$

Centre $(0; -1)$; $A(\sqrt{19-y^2-2y}; y)$

$M_{rad(OA)} = \frac{y_A - y_O}{x_A - x_O} = \frac{y+1}{\sqrt{19-y^2-2y} - 0}$

∴ $M_{OA} = \frac{y+1}{\sqrt{19-y^2-2y}}$

$M_{rad(OA)} \times M_{tan}(-2) = -1$.

$\frac{y+1}{\sqrt{19-y^2-2y}} \times \frac{-2}{1} = -1$

$\frac{-2(y+1)}{\sqrt{19-y^2-2y}} = -1$

$\frac{-2y-2}{\sqrt{19-y^2-2y}} = -1 \quad (\times\sqrt{19-y^2-2y}$

$-2y - 2 = -\sqrt{19-y^2-2y}$

$(\sqrt{19-y^2-2y})^2 = (2y+2)^2$

$19 - y^2 - 2y = 4y^2 + 8y + 4$

$0 = 4y^2 + 8y + 4 - 19 + y^2 + 2y$

$0 = \frac{5y^2}{5} + \frac{10y}{5} - \frac{15}{5}$

$y^2 + 2y - 3 = 0$

$(y-1)(y+3) = 0$

∴ $y = 1$ or $y = -3$

$x = \sqrt{19 - y^2 - 2y}$

$x = \sqrt{19 - (1)^2 - 2(1)}$; $x = \sqrt{19-(-3)^2-2(-3)}$

$x = \sqrt{16}$; $x = \sqrt{16}$

$x = \pm 4$; $x = \pm 4$

$x = 4$ ∴ $A(4; 1)$; ∴ $x = -4$ ∴ $B(-4; -3)$

$y - y_A = -2(x - x_A)$; $y - y_B = -2(x - x_B)$

$y - 1 = -2(x - 4)$; $y - (-3) = -2(x - (-4))$

$y = -2x + 4 + 1$; $y + 3 = -2(x + 4)$

$y = -2x + 5$; $y = -2x - 8 - 3$

$y = -2x - 11$

Figure 6.39: Student E’s solution to Question 8 of Class test 3 of semester 2

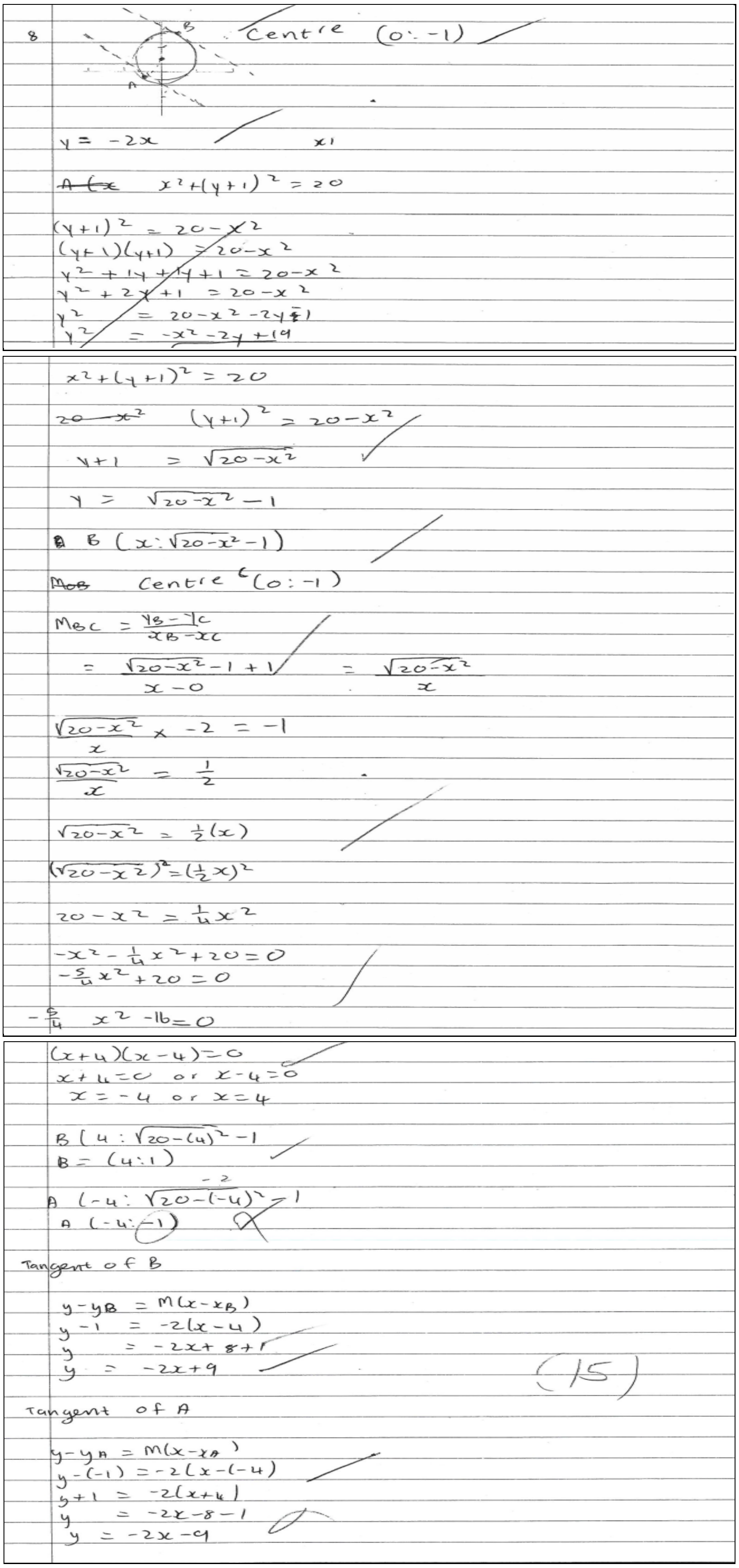


Figure 6.40: Student F's solution to Question 8 of Class test 3 of semester 2

6.3.7 Question 4 of the end of module examination of the second semester

This section deals with the end of module examination of the second semester. The focus will only be on questions that deal with problems that are similar to those discussed in section 6.3.6. Question 4 of this examination therefore is the only question that will be discussed as it is the only question that is similar. This question was classified as Creative reasoning based on conceptual knowledge (D2a).

The end of module examination was written approximately a month after Test 3 and approximately one and a half months after participating students had been taught problems similar to Question 4. Question 4 of the end of module second semester examination is shown in Figure 6.41, and a sketch (Figure 6.42) with a possible solution is shown thereafter.

Question 4

4.1 Determine the centre and radius of the following circle:

$x^2 - 2x + y^2 - 7 = 0$

4.2 Find the equation of the tangents to the circle in 4.1 which are parallel to $y - x + 3 = 0$

4.3 Do the tangents that you determined in 4.2 intersect at some point (yes or no)? Explain your reasoning.

Figure 6.41: Question 4 of the end of module second semester examination

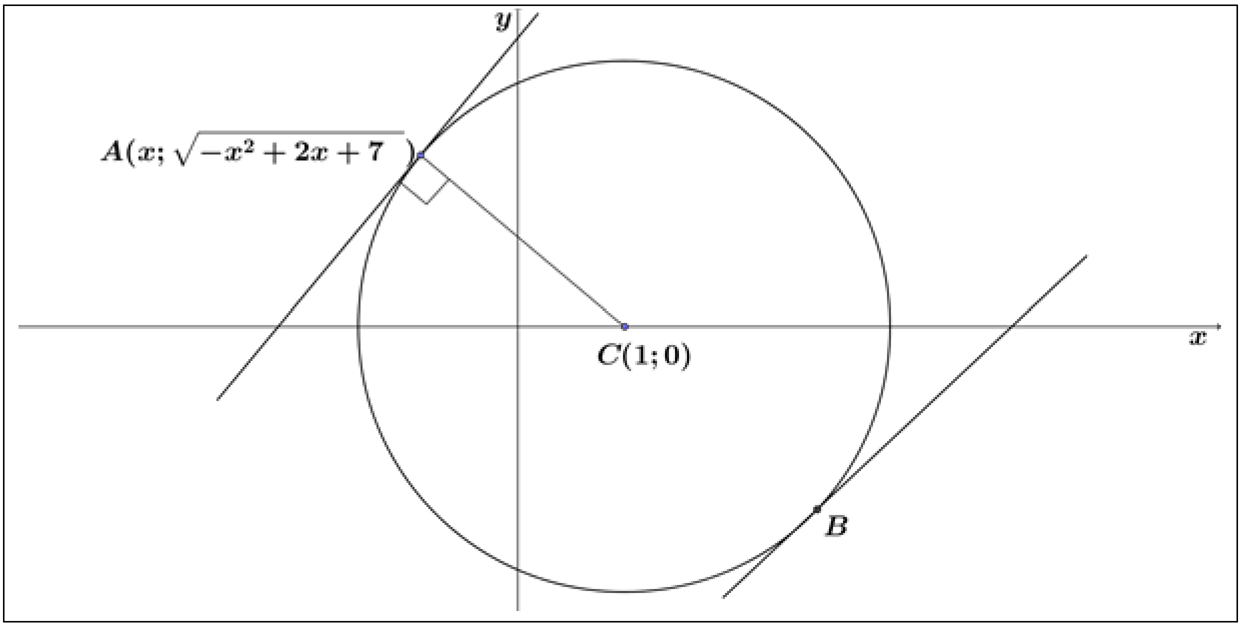


Figure 6.42: Sketch for Question 4

Possible solution to question 4

4.1 $x^2 - 2x + y^2 - 7 = 0$

$x^2 - 2x + \left(-\frac{2}{2}\right)^2 + y^2 = 7 + 1$

$(x-1)^2 + y^2 = 8$

$\therefore centre\ (1;0) \quad and \quad radius = \sqrt{8}$

4.2 First we write $y - x + 3 = 0$ in standard form: $y = x - 3$. The gradient of the two lines we are seeking should therefore be: $m_{\parallel} = 1$

Next we use the provided circle equation to determine generalized coordinates for point *A* which is a point of contact of one of the tangents to the circle:

$(x-1)^2 + y^2 = 8$

$\therefore y = \sqrt{-x^2 + 2x + 7}$

$\therefore A\left(x; \sqrt{-x^2 + 2x + 7}\right)$

The gradient of radius *CA* is: $m_{CA} = \frac{\sqrt{-x^2+2x+7}-0}{x-1}$

$m_{CA} \times m_{\parallel} = -1$ (radius and tangent are perpendicular to each other)

$\frac{\sqrt{-x^2+2x+7}}{x-1} \times 1 = -1$

$\sqrt{-x^2 + 2x + 7} = -(x-1)$

$-x^2 + 2x + 7 = x^2 - 2x + 1$

$-2x^2 + 4x + 6 = 0$

$\therefore x = -1 \quad or \quad x = 3$

$\therefore A(-1;2) \quad and \quad B(3;-2)$

The equations of the two parallel tangents therefore are:

$y_A - 2 = 1\big(x - (-1)\big) \qquad and \qquad y_B - (-2) = 1(x - 3)$

$\therefore y_A = x + 3 \qquad and \qquad y_B = x - 5$

6.3.8 Analysis of student responses to Question 4 of the end of module examination of the second semester

In the first sub-question, completion of the square is required in order to determine the center and radius of the circle. For Question 4.2 the equation of the line to which the tangents must be parallel is written in standard form in order to determine its gradient.

Next the equation of the circle is written in terms of *y*. Subsequently this new equation is used to write generalized coordinates for *A* (one of the points of contact of the tangents to the circle). Using this generalized coordinate the gradient of the radius to point *A* is determined. After this the product of the gradient of the radius and the gradient of the line to which the tangents must be parallel are equated to -1. The resulting equation is used to determine numerical values for the *x*-coordinates of points *A* and *B*. These numerical values are then substituted in the generlized coordinate to determine the numerical *y*-values for points *A* and *B*. Finally, these numerical coordinates are used together with the parallel gradient to determine the equations of the tangents.

Student A attempted to complete the square in Question 4.1 and provided a correct answer for the radius, but committed an error in completing the square (see Figure 6.43). For Question 4.2 Student A started correctly by writing the equation of the line in standard form and then proceeded (incorrectly) to substitute the *y*-value of the equation of the circle. In these calculations the student made another error and then stopped abruptly midway through the problem. It seems as if the student did not know how to proceed and so gave up. It would seem therefore that retention and transfer of relevant knowledge in terms of linear functions, and reasoning with these constructs, was negatively affected by the complexity of the presented problem.

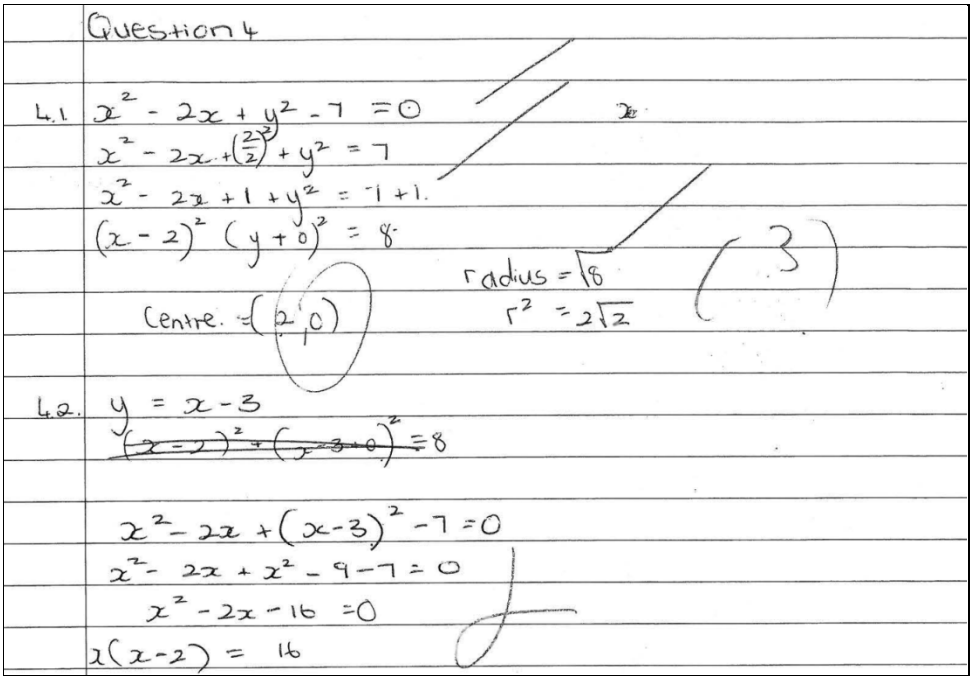


Figure 6.43: Student A's response to Question 4 of the end of module examination of the second semester

Student B's response is shown in Figure 6.44. Student B completed the square, determined the radius and centre of the circle correctly and proceeded to write the equation of the provided line in standard form. The student then substituted the algebraic *y*-value of this line into the equation of the circle. From this equation numerical values for the coordinates of A and B were calculated. The coordinates of point A and the coordinates of the origin were then used to calculate a gradient. The negative reciprocal of this number was determined and substitued for the value of the gradient. It would seem therefore that the student was aware that the radius and the tangent are perpendicular and hence they calculated the reciprocal (although the incorrect circle centre was used). A similar method was followed in the case of the line through point B.

Student B used the correct formula to calculate gradient (but utilized incorrect points) and used the correct method to determine the perpendicular gradient at the appropriate part of the problem. Furthermore the correct formula for the point slope form of the line was used. The fact that it is stated in the problem that the two tangents should be parallel to the line provided implies that the tangents are also parallel to each other and hence would not intersect. The student did not make this connection and it would seem therefore that conceptual understanding of parallelism is lacking in some respects since the student did not make the connection that parallel lines do not intersect. One could however argue that the indispensable knowledge regarding linear functions had been retained although it was not transferred into the new context in an entirely correct way.

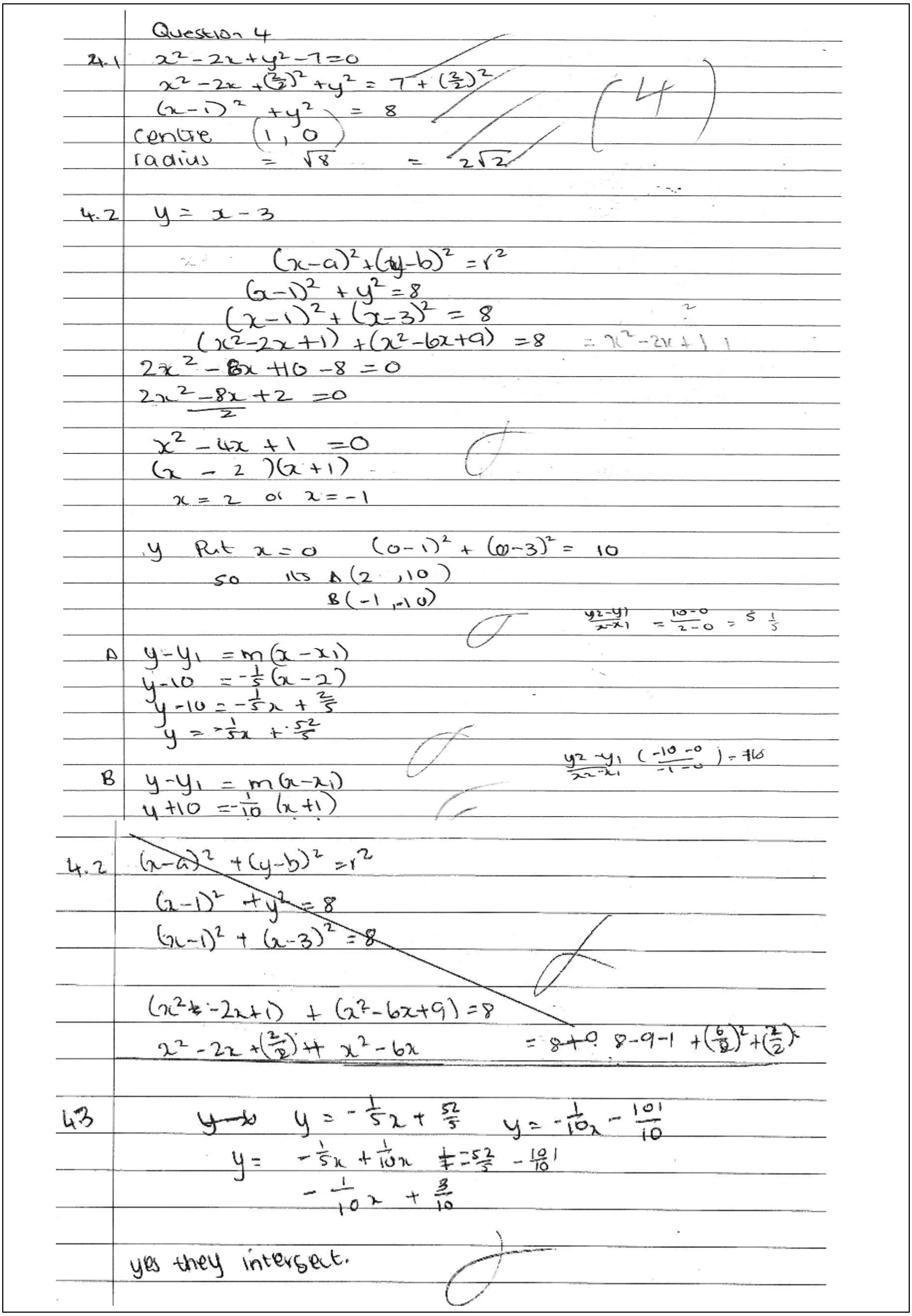


Figure 6.44: Student B's response to Question 4 of the end of module examination of the second semester

Student C's response is shown in Figure 6.45. The student attempted to complete the square for both *x* and *y* and produced an incorrect solution. For Question 4.2 Student C started correctly by writing the linear equation in standard form and subsequently by identfying the correct gradient. The equation of a circle with centre at the origin was then utilized to write generalized coordinates for a point on the circle. This is despite the fact that the student determined a centre that was not at the origin. The circle centre that was determined in the first part of the question together with the generalized coordinate was then utlized to determine the gradient of the radius. Next the product of the two gradients was equated to -1 and an attempt was made to simplify the resulting equation. However Student C stopped before calculating values for the *x*-coordinate.

Student C exhibits confused reasoning regarding completing the square and determining centres of circles that are not at the origin. Knowledge regarding linear equations in standard form and perpendicular gradients however seems to have been retained correctly. Although incorrect equations were used to determine the generalized coordinates, this was done correctly and hence one can argue that determining of generalized coordinates was also retained. Student C did not attempt to determine the linear equations and hence one cannot draw any conclusions regarding this knowledge.

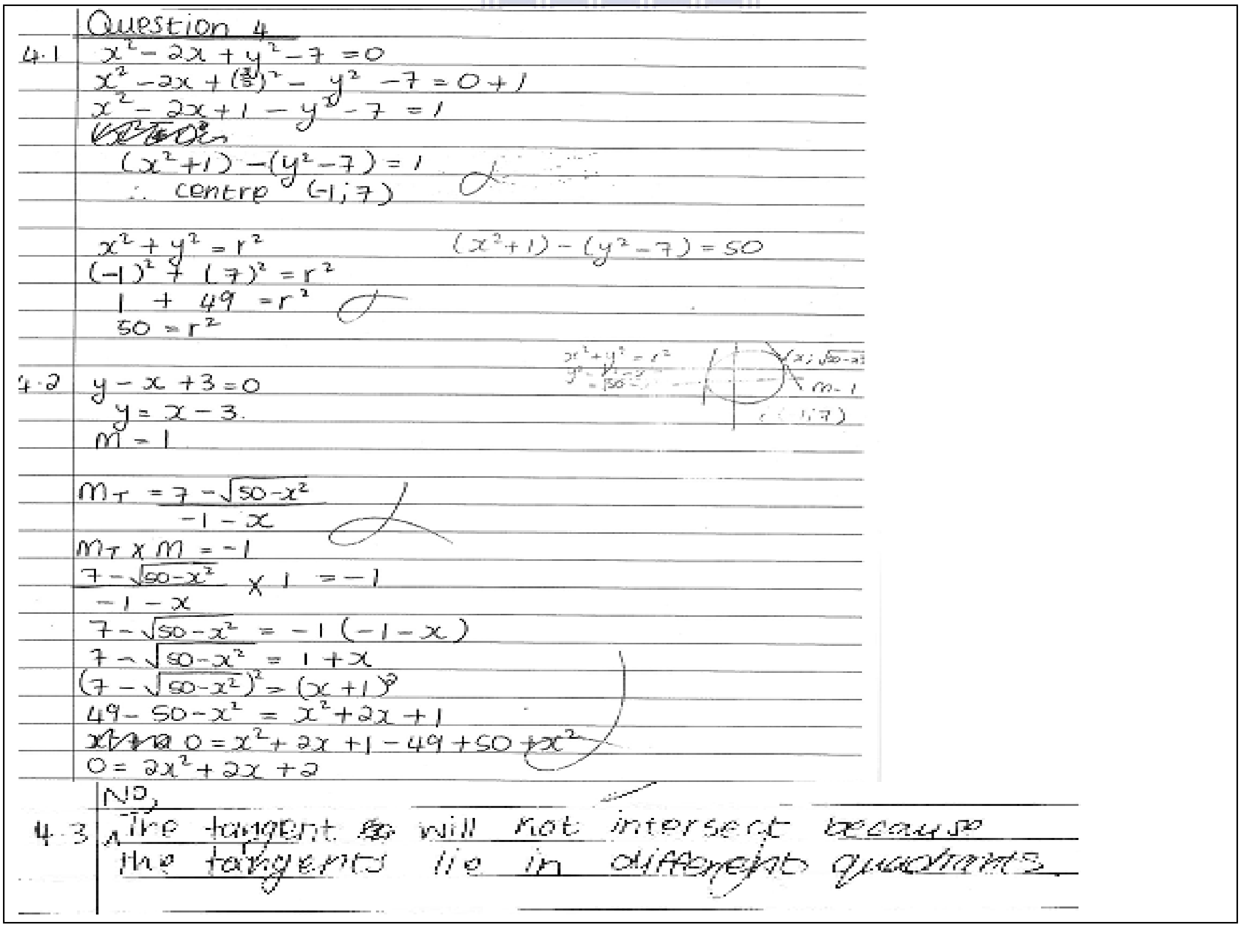
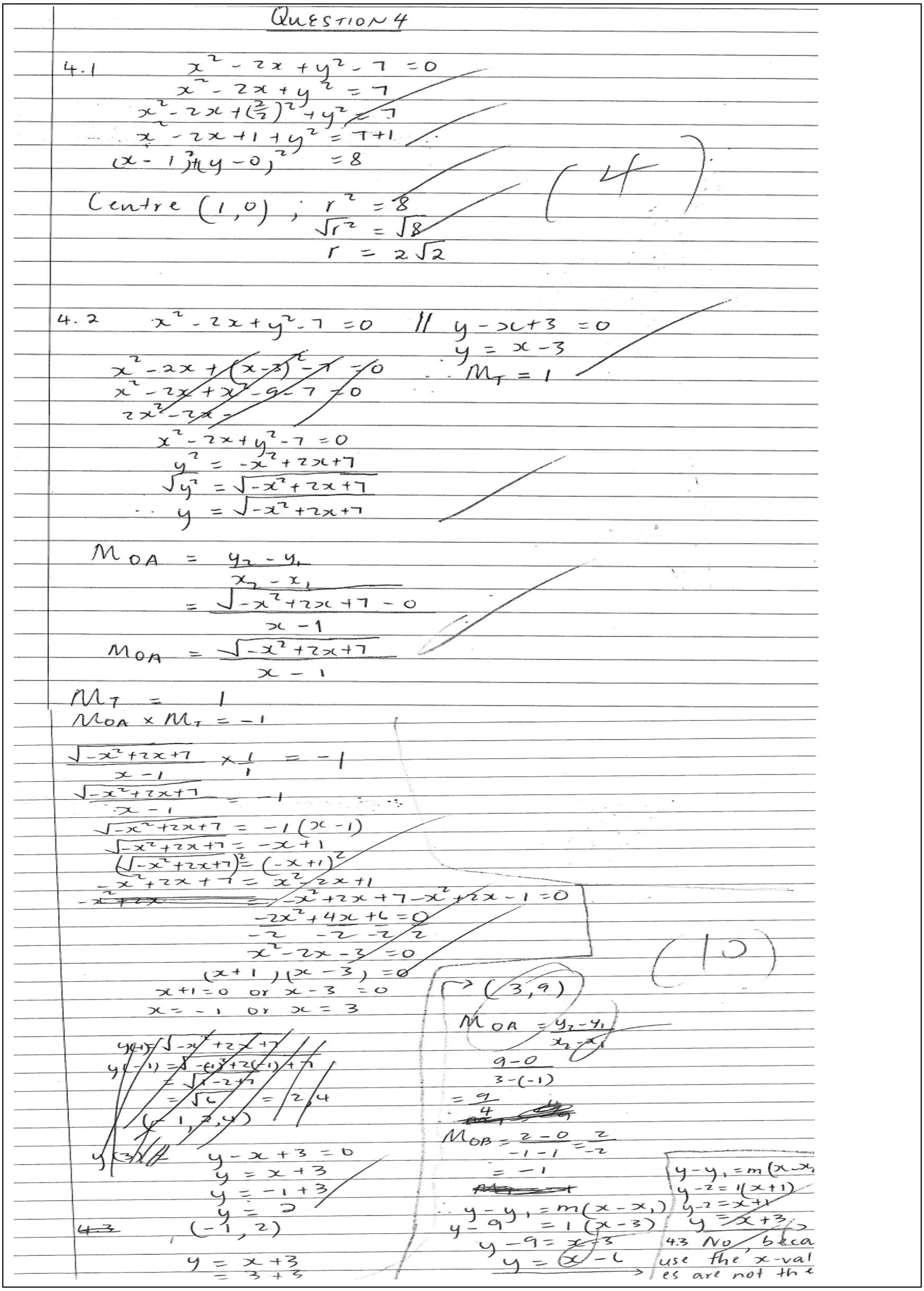


Figure 6.45: Student C's response to Question 4 of the end of module examination of the second semester

Student D's response is shown in Figure 6.46. The response of student D was correct up to the point where the determined *x*-coordinates were supposed to be substituted into the generalized coordinates to determine the *y*-coordinates. The student intially substituted into generalized coordinates, but made a mistake in the calculation and then drew lines through this apparently deciding that this was not the correct way and then substituted into the equation of the parallel line. Despite the mistake the student used the point-slope equation of the line correctly to determine linear equations. Knowledge regarding paralellism however seems to be lacking in some respects since the student did not make the connection that parallel lines do not intersect. One can therefore argue that the student had retained the indispensable knowledge regarding linear functions and that the knowledge was transferred correctly into the new context.


QUESTION 4

4.1 $x^2 - 2x + y^2 - 7 = 0$
$x^2 - 2x + y^2 = 7$
$x^2 - 2x + (\frac{2}{2})^2 + y^2 = 7$
$x^2 - 2x + 1 + y^2 = 7 + 1$
$(x-1)^2 + (y-0)^2 = 8$
Centre $(1, 0)$; $r^2 = 8$
$\sqrt{r^2} = \sqrt{8}$
$r = 2\sqrt{2}$
(4)

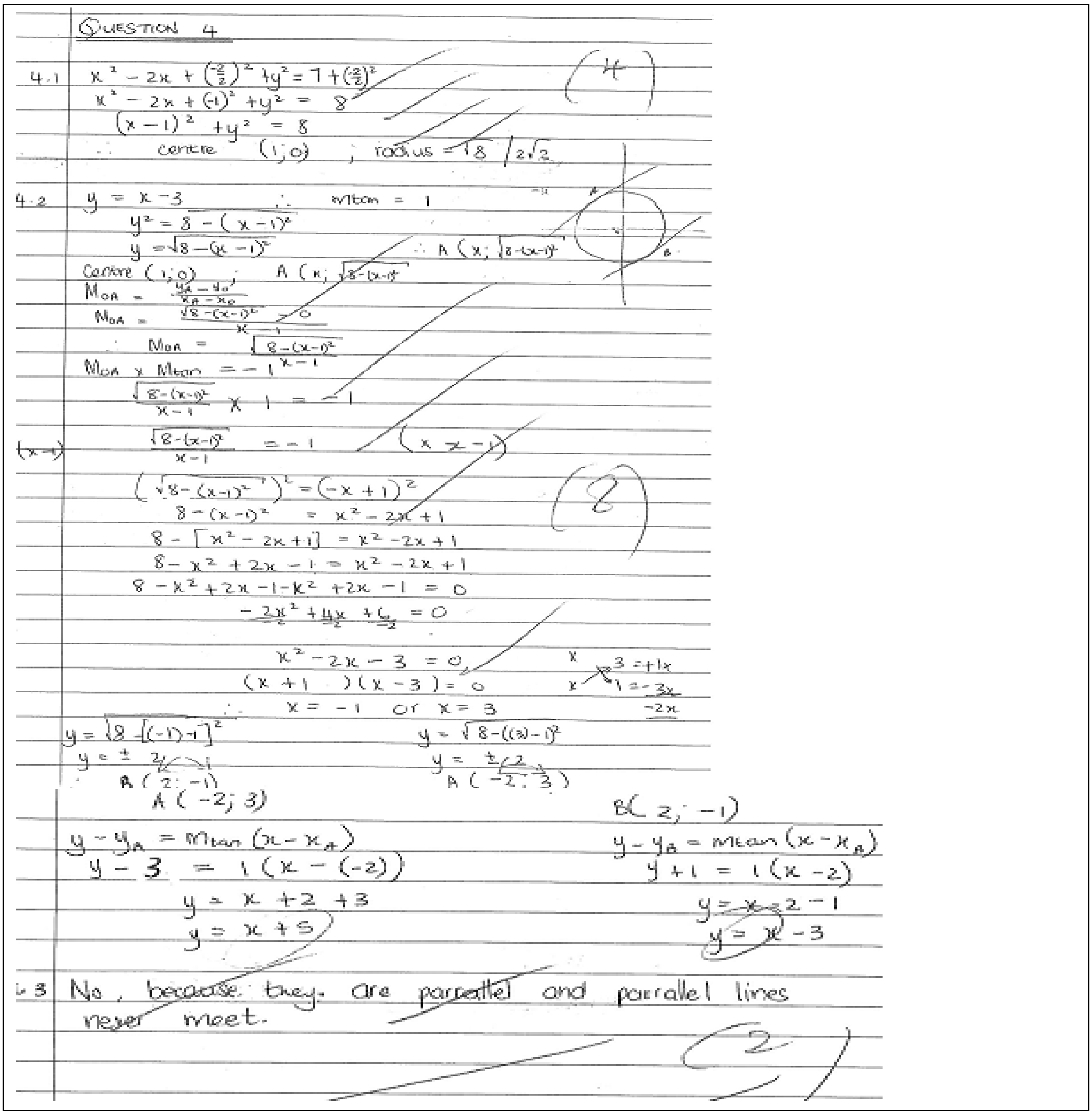
4.2 $x^2 - 2x + y^2 - 7 = 0$ // $y - x + 3 = 0$
$y = x - 3$
$\therefore M_T = 1$
$x^2 - 2x + y^2 - 7 = 0$
$y^2 = -x^2 + 2x + 7$
$\sqrt{y^2} = \sqrt{-x^2 + 2x + 7}$
$\therefore y = \sqrt{-x^2 + 2x + 7}$
$M_{OA} = \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{\sqrt{-x^2 + 2x + 7} - 0}{x - 1}$
$M_{OA} = \frac{\sqrt{-x^2 + 2x + 7}}{x - 1}$
$M_T = 1$
$M_{OA} \times M_T = -1$
$\frac{\sqrt{-x^2+2x+7}}{x-1} \times \frac{1}{1} = -1$
$\sqrt{-x^2+2x+7} = -1(x-1)$
$\sqrt{-x^2+2x+7} = -x+1$
$(\sqrt{-x^2+2x+7})^2 = (-x+1)^2$
$-x^2 + 2x + 7 = x^2 - 2x + 1$
$-x^2 + 2x + 7 - x^2 + 2x - 1 = 0$
$-2x^2 + 4x + 6 = 0$
$x^2 - 2x - 3 = 0$
$(x+1)(x-3) = 0$
$x + 1 = 0$ or $x - 3 = 0$
$x = -1$ or $x = 3$
$y - x + 3 = 0$
$y = x + 3$
$y = -1 + 3$
$y = 2$
(10)
$(3, 9)$
$M_{OA} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 0}{3 - (-1)} = \frac{9}{4}$
$M_{OB} = \frac{2 - 0}{-1 - 1} = \frac{2}{-2} = -1$
$\therefore y - y_1 = m(x - x_1)$
$y - 9 = 1(x - 3)$
$y - 9 = x - 3$
$y = x - 6$
$y - y_1 = m(x - x_1)$
$y - 2 = 1(x + 1)$
$y - 2 = x + 1$
$y = x + 3$

4.3 $(-1, 2)$
$y = x + 3$
$= 3 + 3$
4.3 No, because the x-values are not the

Figure 6.46: Student D's response to Question 4 of the end of module examination of the second semester

Student E's response is shown in Figure 6.47. This student produced an entirely correct response save for a minor error in the writing of the coordinate pairs. It would seem therefore that the indispensable knowledge had been retained and correctly transferred to the new context.

Student F's response is shown in Figure 6.48. This student produced an entirely correct solution with no errors. The conclusion therefore is that the student had retained the indispensable knowledge and had transferred it correctly to the new context.



QUESTION 4

4.1 $x^2 - 2x + (\frac{-2}{2})^2 + y^2 = 7 + (\frac{-2}{2})^2$

$x^2 - 2x + (-1)^2 + y^2 = 8$

$(x-1)^2 + y^2 = 8$

$\therefore$ centre $(1;0)$; radius $= \sqrt{8}$ / $2\sqrt{2}$

(4)

4.2 $y = x - 3$ $\therefore m_{tan} = 1$

$y^2 = 8 - (x-1)^2$

$y = \sqrt{8 - (x-1)^2}$ $\therefore A(x; \sqrt{8-(x-1)^2})$

Centre $(1;0)$; $A(x; \sqrt{8-(x-1)^2})$

$M_{OA} = \frac{y_A - y_O}{x_A - x_O}$

$M_{OA} = \frac{\sqrt{8-(x-1)^2} - 0}{x - 1}$

$\therefore M_{OA} = \frac{\sqrt{8-(x-1)^2}}{x-1}$

$M_{OA} \times M_{tan} = -1$

$\frac{\sqrt{8-(x-1)^2}}{x-1} \times 1 = -1$

$(x-1)$ $\frac{\sqrt{8-(x-1)^2}}{x-1} = -1$ $(x-1)$

$(\sqrt{8-(x-1)^2})^2 = (-x+1)^2$

$8 - (x-1)^2 = x^2 - 2x + 1$

$8 - [x^2 - 2x + 1] = x^2 - 2x + 1$

$8 - x^2 + 2x - 1 = x^2 - 2x + 1$

$8 - x^2 + 2x - 1 - x^2 + 2x - 1 = 0$

$-2x^2 + 4x + 6 = 0$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$\therefore x = -1$ or $x = 3$

$y = \sqrt{8 - [(-1)-1]^2}$ $y = \sqrt{8 - ((3)-1)^2}$

$y = \pm 2$ $y = \pm 2$

$A(2; -1)$ $A(-2; 3)$

$A(-2; 3)$ $B(2; -1)$

$y - y_A = m_{tan}(x - x_A)$ $y - y_B = m_{tan}(x - x_B)$

$y - 3 = 1(x - (-2))$ $y + 1 = 1(x - 2)$

$y = x + 2 + 3$ $y = x - 2 - 1$

$y = x + 5$ $y = x - 3$

(8)

4.3 No, because they are parallel and parallel lines never meet.

(2)

Figure 6.47: Student E's response to Question 4 of the end of module examination of the second semester

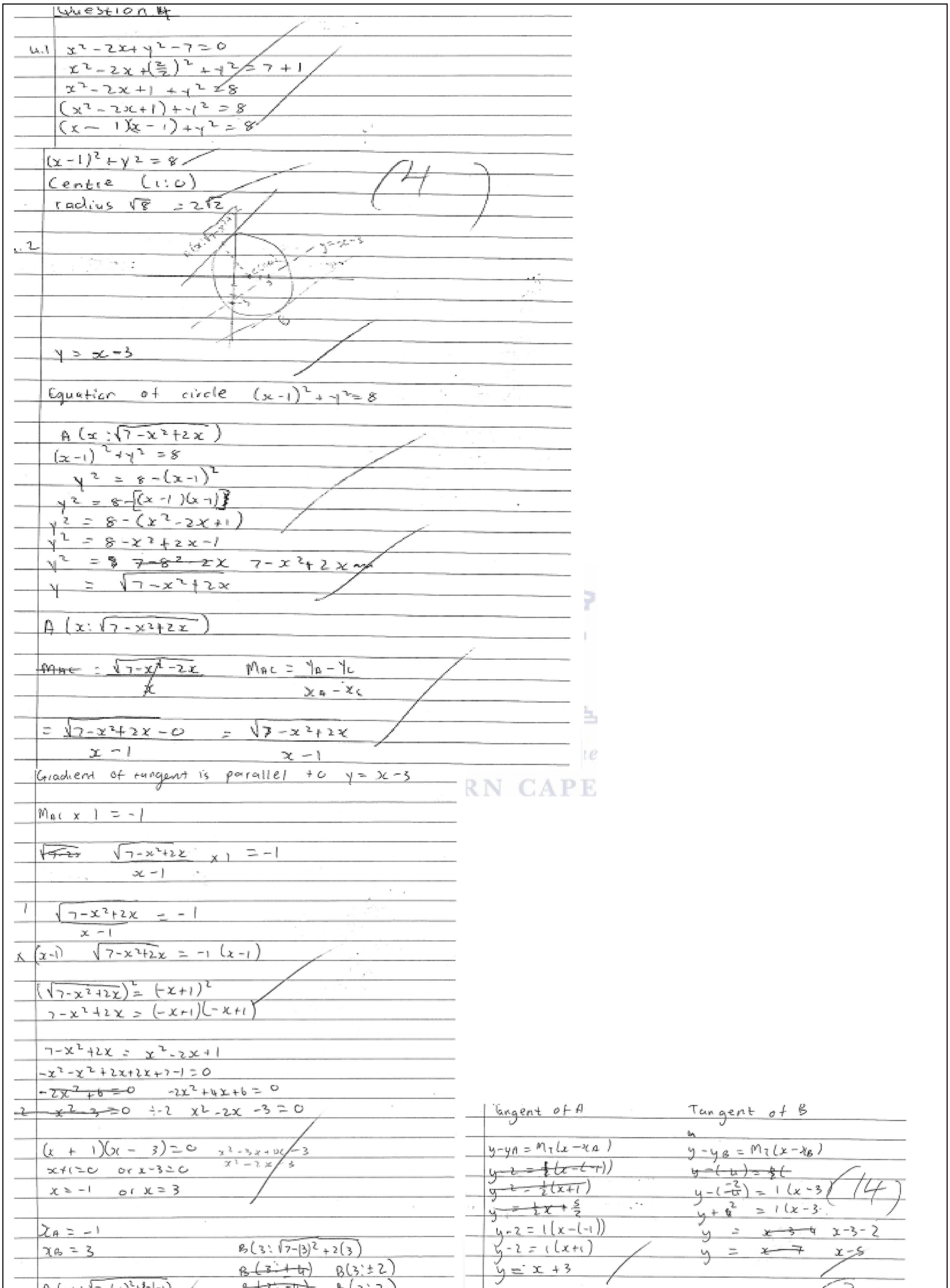


Figure 6.48: Student F's response to Question 4 of the end of module examination of the second semester

6.4 The teaching strategy adapted

In this section an incident which gave rise to a decision to adapt the implemented teaching strategy will be discussed.

This study was started at the beginning of February 2014 with the MAE 211 students. When the research had been underway for approximately two and a half months (in mid-April), a participating student (Student B) came to see me after having written Test 3 and complained bitterly that she "went blank" in the test (See Appendix E for Test 3).

In order to determine what she meant by "going blank" I entered into a discussion with her. During our discussion I presented the student with the following problem based on linear functions (this formed part of Question 1 of the test that had been written: see Figure 6.49): "*Find the equation of the line that is parallel to the line* $y = 2x + 4$ *and through the point*$(1; 2)$"

However the student could not produce a written solution despite the fact that this work had been revised more than once before then (see line 1 in Figure 6.49). Since she could not recall the slope-intercept form for the two variable linear equation $(y = mx + c)$, I wrote it down for her (this is the part that is circled in Figure 6.49) and requested that she use it to solve the problem. Still she could not proceed and so I asked her to write down the point-slope form for the equation of a line $[y - y_1 = m(x - x_1)]$ and to use this in her reasoning. She then substituted numbers for all the variables (see lines 2 and 3 in Figure 6.49). Although I asked her to think about her response and to think about how one goes about finding an equation of a line, again she substituted numbers for all the variables (see line 4 of Figure 6.49).

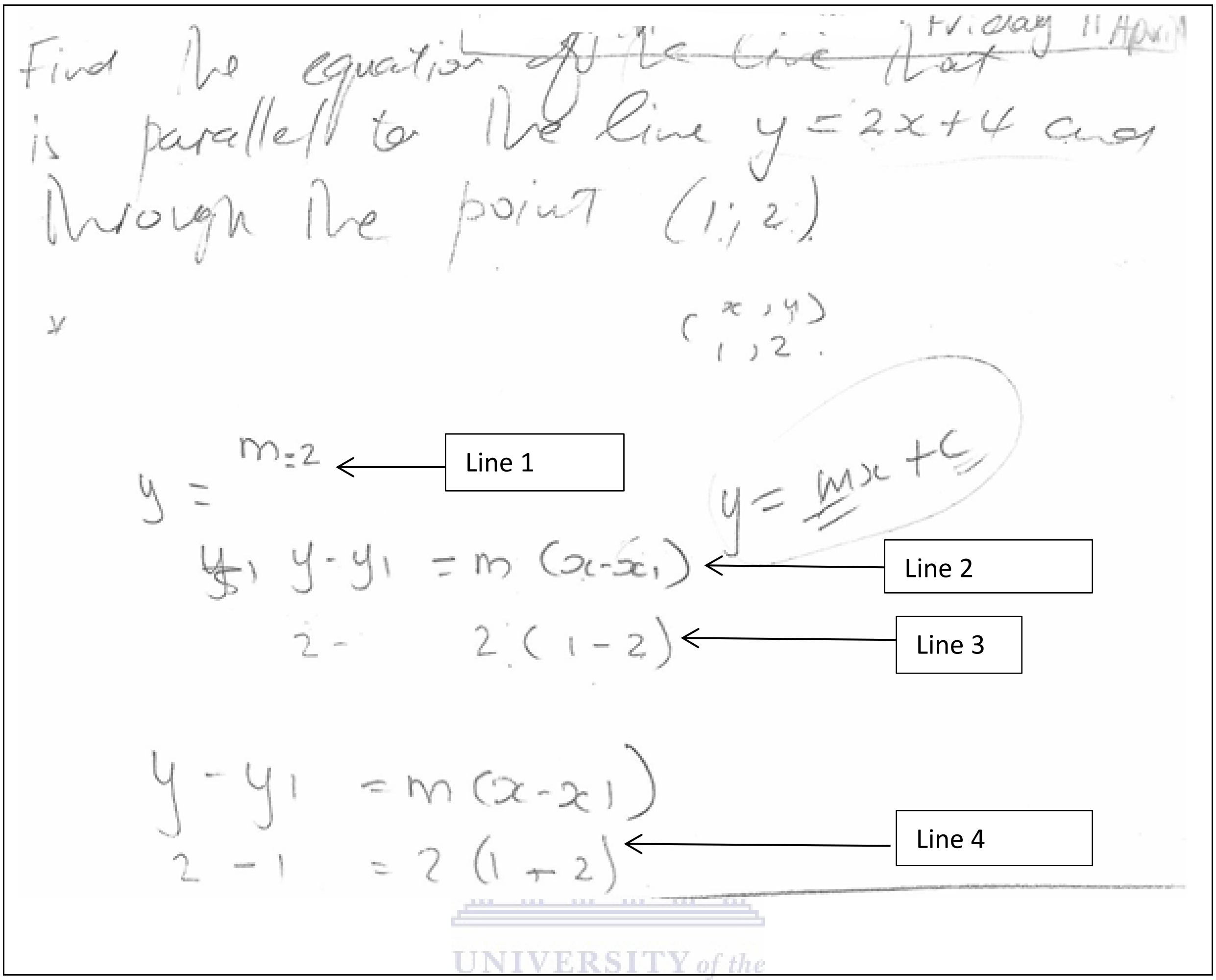


Figure 6.49: Discussion with Student B

What was very puzzling however was that when I checked Tests 1 and 2 which had been written in February and March respectively and which covered linear functions, I was surprised to discover that the student had produced correct solutions to similar and even more challenging problems to the one I presented her with in our discussion (see Figures 6.2 & 6.8).

In order to probe her retention and transfer abilities further I posed a question based on remainder and factor theorem. This content concerning remainder and factor theorem had been covered more recently.

The following is the problem that was presented to her:
$Given\ f(x) = x^3 - 2x + 5.\ If\ x - 2\ is\ divided\ into\ f, find\ the\ remainder.$

Student B provided a correct response and when prompted produced an alternative method which was also correct. The fact that student B performed better on content that had been covered more recently (remainder and factor theorem) than on content which had been covered much earlier (linear functions) may have been the result of memory decay.

Rittle-Johnson (2006) contends that self-explanation is an essential and effective way of improving learning and transfer of knowledge. Furthermore she proposes that direct instruction on a correct procedure, combined with a conceptual explanation for the procedure would lead to the greatest gains in learning if students were also prompted to self-explain. Self-explanation is defined as the ability to generate explanations of correct material by oneself.

In spiral revision one cannot revise all content covered all the time. An instructional challenge therefore is to design revision exercises in such a way so as to revise the majority of previously covered content appropriately. Furthermore it is not easy to design revision in a way that does not encourage rote learning, but rather to ensure that each revision session results in a deeper understanding and an awareness of the interconnectedness of concepts. Since the task that I presented to Student B in our discussion was based on work that had not been recently revised in class it meant that if the student was able to self-explain correctly while studying on her own, her chances of providing a correct response would increase. Since she could not at that point produce a correct solution, but could produce a correct solution in an earlier test (this test had been written very shortly after the content had been covered), I suspected that the student's individual ability to self-explain when studying previously covered content was not well-developed. It was also possible that the student had not developed the ability to read and interpret mathematical texts to the appropriate level. When I investigated test responses of all participating students I discovered that many of the lower ability students exhibited this phenomenon (better performance on recent content and a not-so-good performance on content covered at an earlier point). The majority of the more able students however did not exhibit this phenomenon.

In order to test my hypotheses I presented the whole class with a copied mathematical text from a discrete mathematics textbook (Epp, 2011). The text was based on the concepts of floor and ceiling which forms part of elementary number theory. The concepts did not form part of the content covered by our course and therefore were completely new to the students. The text contained definitions and illustrative examples based on the definitions. The text also contained an exposition of direct proofs of properties of floor and ceiling. Students were required to study the theory and examples, and then to attempt exercises provided in the text. No teaching was offered for the entire duration of this exercise. This intervention exercise was done as a once-off three hour session on a Saturday morning. It should be noted that since it was a voluntary exercise not all students participated.

Some of the exercise problems based on the presented text required solutions that were premised on imitative reasoning based on procedural knowledge. These included problems requiring students to compute the floor and ceiling of numbers similar to the completed examples. The exercises also contained questions of a higher cognitive level and, for the greater part of the solution required creative reasoning based on conceptual knowledge. Questions that required the construction of a direct proof fell into this category. Students were allowed to discuss with one another while working through the text, but were required to attempt the exercises individually.

The following is the definition of floor as provided in the text:

> Given any real number x, the floor of x, denoted $\lfloor x \rfloor$, is defined as follows:
> $\lfloor x \rfloor = \textit{that unique integer } n \textit{ such that } \ n \leq x < n+1.$
> Symbolically, if x is a real number and n is an integer, then
> $\lfloor x \rfloor = n \iff n \leq x < n+1.$

The following is the definition of ceiling as provided in the text:

> Given any real number x, the ceiling of x, denoted $\lceil x \rceil$, is defined as follows:
> $\lceil x \rceil = \textit{that unique integer } n \textit{ such that } \ n-1 < x \leq n.$
> Symbolically, if x is a real number and n is an integer, then
> $\lceil x \rceil = n \iff n-1 < x \leq n.$

The text provided completed examples for the following values of x: $\frac{25}{4}$; 0.999; -2.01

Since I wanted to determine whether students could apply the definition of floor and ceiling to cases that were not covered by the examples, I presented them with the following additional question (referred to as Question 1), which was slightly different and not contained in the textbook exercises. This was done in order to determine how well the text was read and understood:

Compute the floor and ceiling for each of the following: (a) 3 and (b) -7

The definition of floor implies that if given any real number (that is not an integer) then the floor of the real number is the integer immediately to the left of the given number. Implicit in the definition of floor is the fact that if the number provided is an integer, then the floor is that integer. One should therefore be cognizant of the fact that the set of integers is contained in the set of real numbers and therefore that all integers are real numbers. Thus if an integer is provided then the floor is that integer. A similar argument holds for problems based on the concept of ceiling.

When I assessed their solutions I discovered that none of the participating students – with the exception of one – made the connection that if an integer is provided, then the floor is that integer. As I have stated previously not all students participated in the exercise since participation was voluntary and hence only the responses of those case study students that participated (Students A, B, E and F) will be provided and discussed (see Figures 6.50 to 6.53).

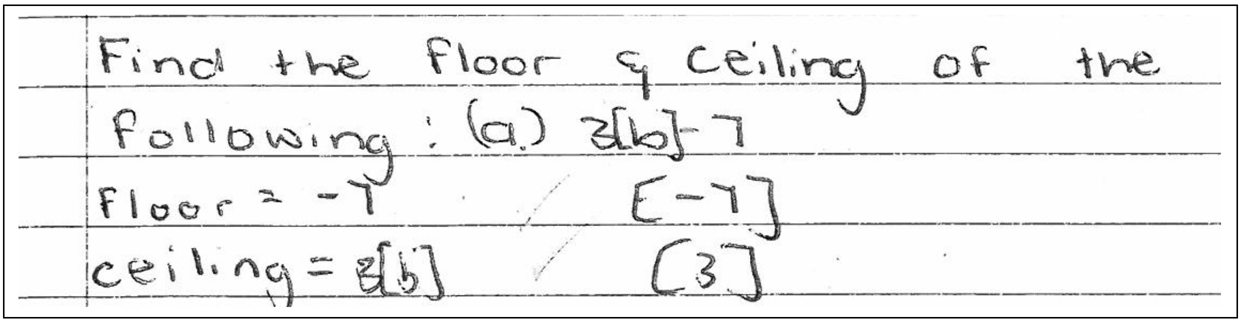


Figure 6.50: Student A's response to Question 1

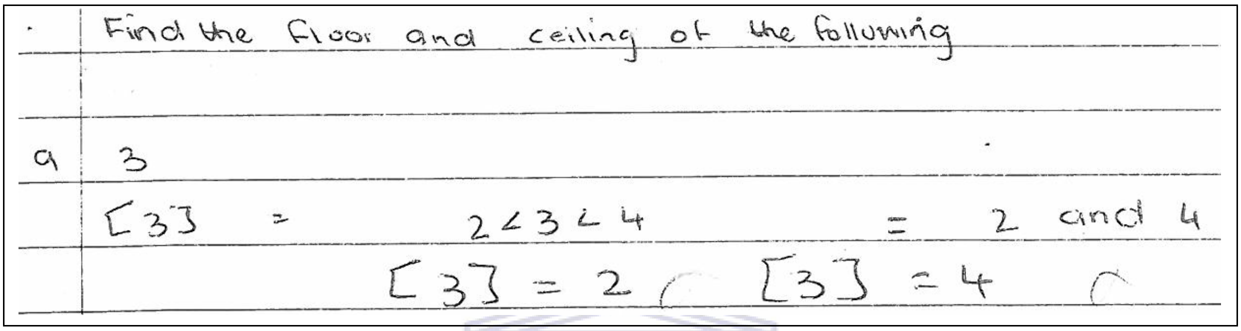


Figure 6.51: Student B's response to Question 1

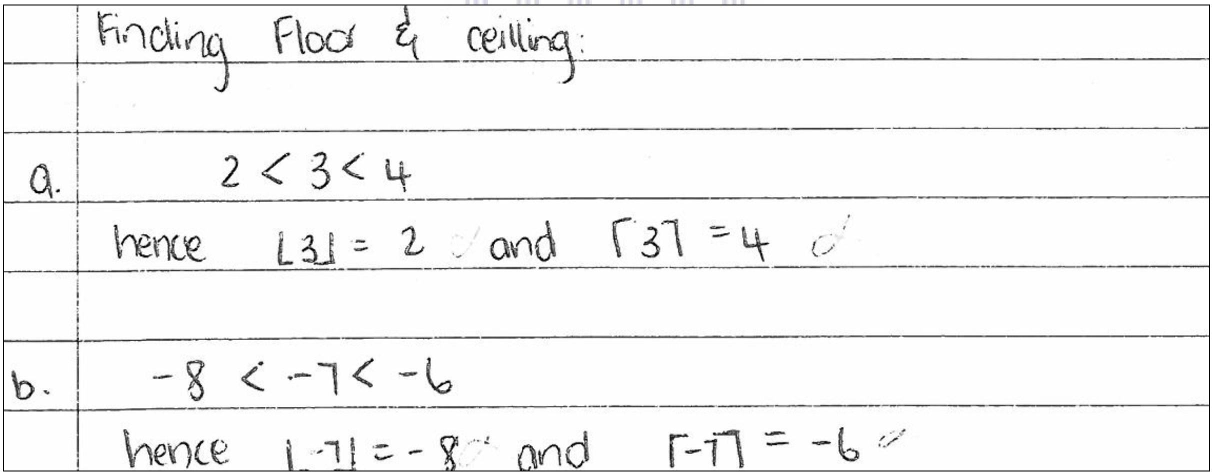
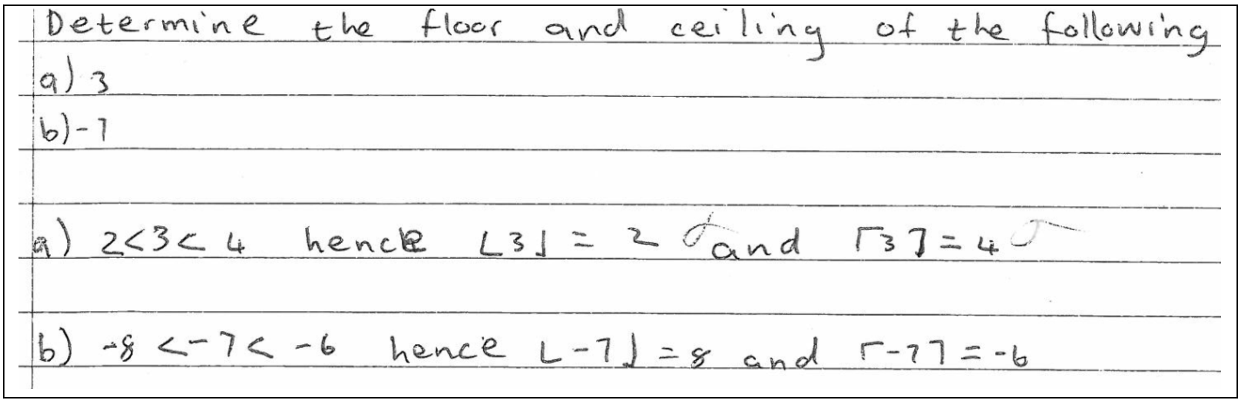


Figure 6.52: Student E's response to Question 1


Determine the floor and ceiling of the following
a) 3
b) -7

a) $2 < 3 < 4$ hence $\lfloor 3 \rfloor = 2$ and $\lceil 3 \rceil = 4$

b) $-8 < -7 < -6$ hence $\lfloor -7 \rfloor = 8$ and $\lceil -7 \rceil = -6$

Figure 6.53: Student F's response to Question 1

Analyses of student responses indicate that only one student (Student A) provided a response that shows the student was aware of the meaning implicit in the definition. That is, only one student made the connection that if the provided number is an integer then both floor and ceiling are that number. If one analyses responses to other similar questions (referred to as Question 2 - see Figure 6.54) it seems most students were imitating the completed examples without making a connection with the implicit information in the definition. In other words they applied procedures correctly by imitating the completed examples provided, without a full conceptual understanding. This is illustrated by their responses to these problems which were similar to the completed examples (see Figures 6.55 to 6.58). Since most of these responses are correct this hides the fact that they did not make the connection with the implicit information in the definitions provided. This implies that they could not self-explain correctly. I also found it strange that Student A, who had been identified as one of the weaker students, was the only student who made the implicit connections.

Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ $for\ each\ of\ the\ following\ values\ of\ x\ in$ $1-4$:

1. 37.999	2. $\frac{17}{4}$
3. -14.00001	4. $-\frac{32}{5}$

Figure 6.54: Question 2

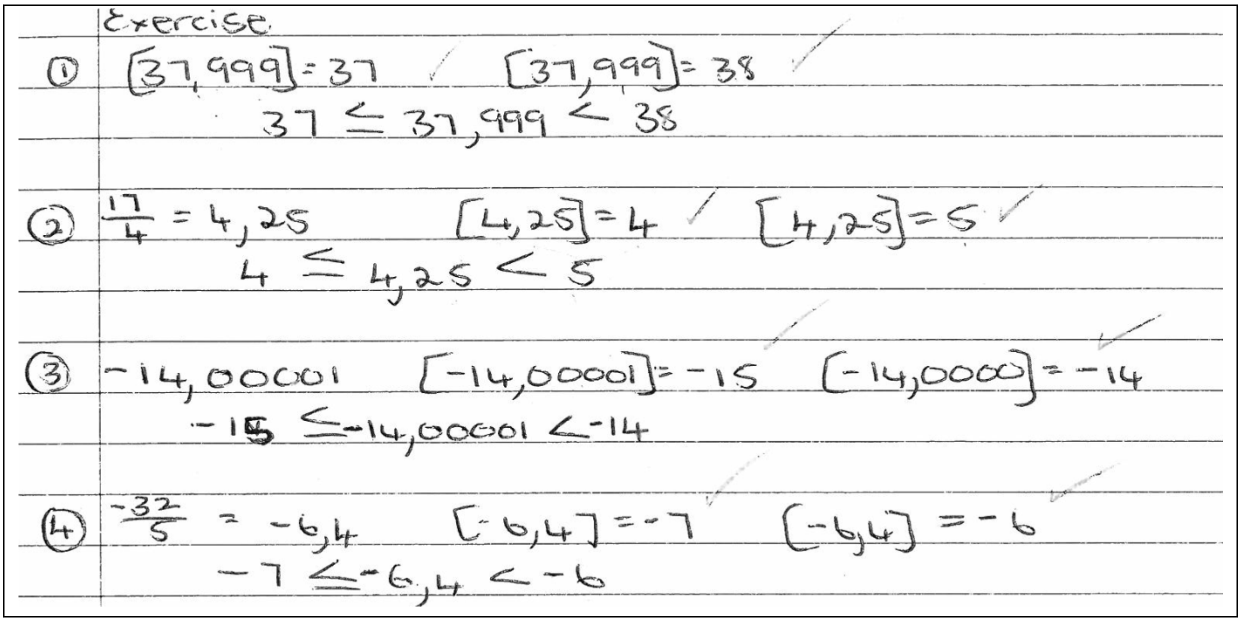


Figure 6.55: Student A's response to Question 2

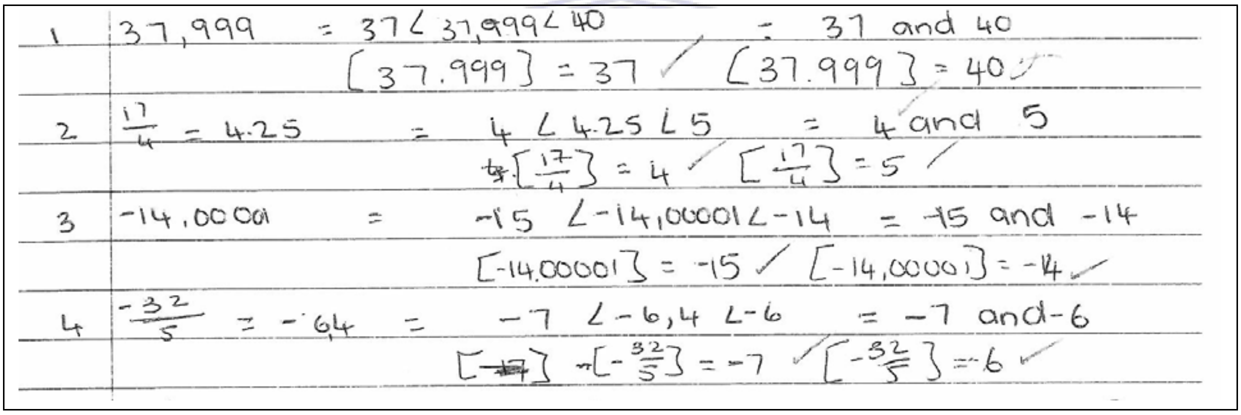
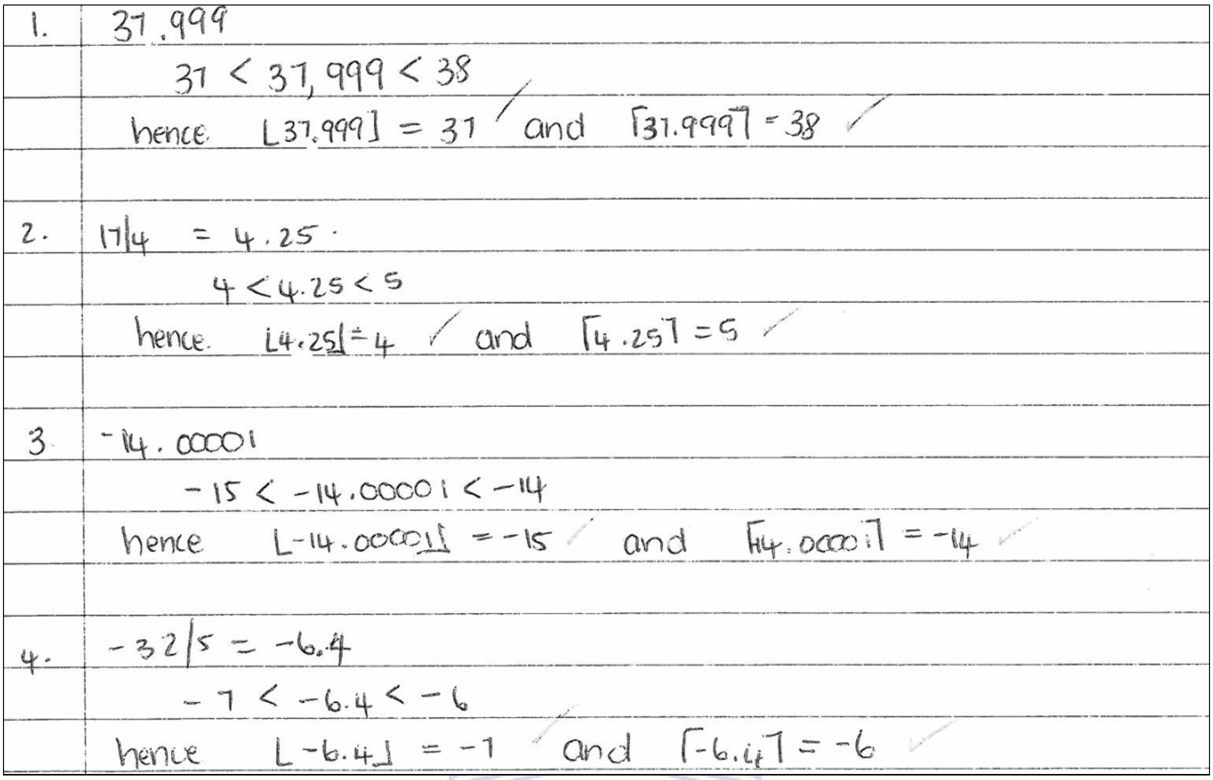


Figure 6.56: Student B's response to Question 2


1. 37.999

37 < 37.999 < 38

hence ⌊37.999⌋ = 37 and ⌈37.999⌉ = 38

2. 17/4 = 4.25

4 < 4.25 < 5

hence ⌊4.25⌋ = 4 and ⌈4.25⌉ = 5

3. −14.00001

−15 < −14.00001 < −14

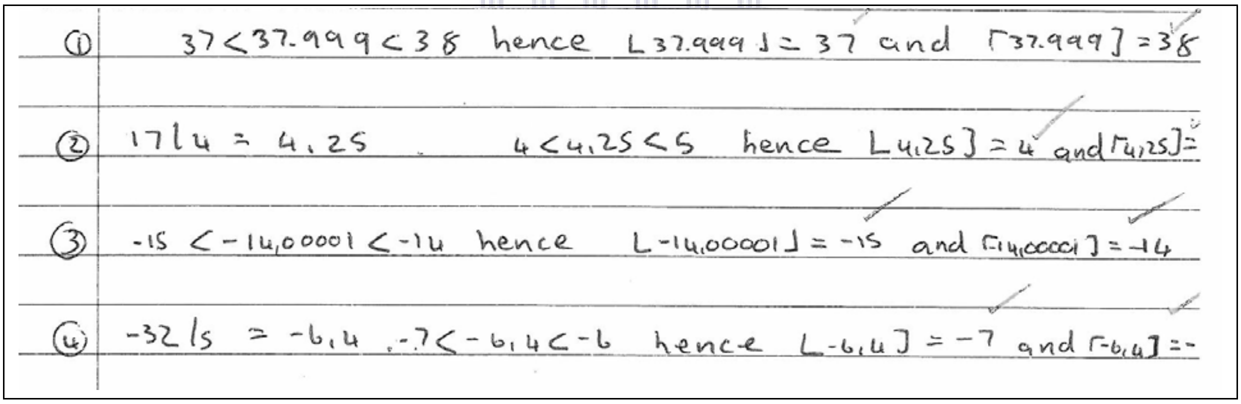
hence ⌊−14.00001⌋ = −15 and ⌈14.00001⌉ = −14

4. −32/5 = −6.4

−7 < −6.4 < −6

hence ⌊−6.4⌋ = −7 and ⌈−6.4⌉ = −6

Figure 6.57: Student E's response to Question 2


① 37<37.999<38 hence ⌊37.999⌋ = 37 and ⌈37.999⌉ = 38

② 17/4 = 4.25 4<4.25<5 hence ⌊4.25⌋ = 4 and ⌈4.25⌉=

③ -15 <-14.00001<-14 hence ⌊-14.00001⌋ = -15 and ⌈14.00001⌉ = -14

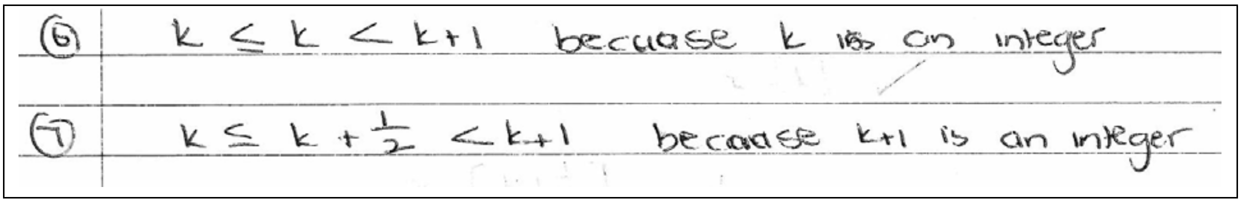
④ -32/5 = -6.4 , -7<-6.4<-6 hence ⌊-6.4⌋ = -7 and ⌈-6.4⌉ =-

Figure 6.58: Student F's response to Question 2

The exercises based on the text also contained problems which were explicitly conceptual in nature. Some of these problems (referred to as Questions 6 and 7) are shown in Figure 6.59. Student responses to these problems are presented in Figures 6.60 to 6.63.

Question 6: $If\ k\ is\ an\ integer, what\ is\ \lceil k \rceil?\ Why?$
Question 7: $If\ k\ is\ an\ integer, what\ is\ \left\lceil k + \frac{1}{2} \right\rceil ? Why?$

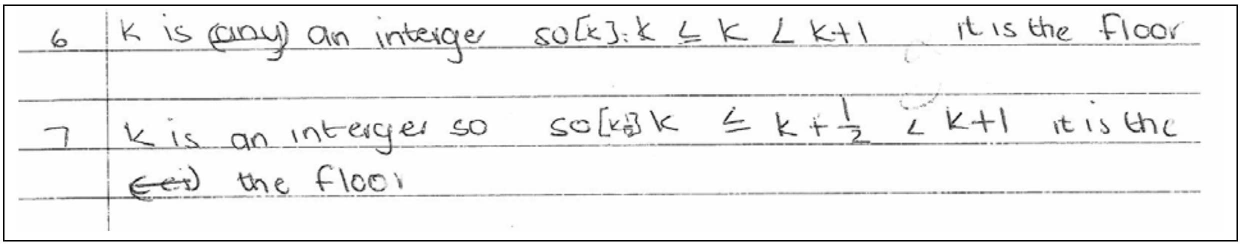
Figure 6.59: Questions 6 and 7



(6) k ≤ k < k+1 because k is an integer

(7) k ≤ k + ½ < k+1 because k+1 is an integer

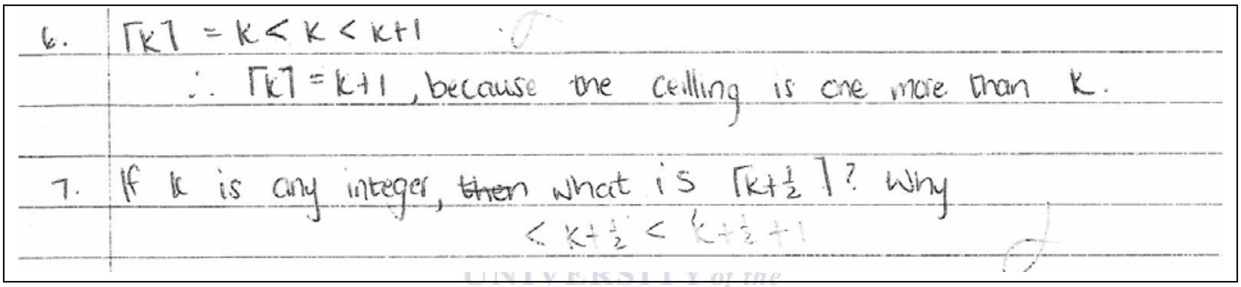
Figure 6.60: Student A's responses to Questions 6 and 7



6 k is ~~(any)~~ an integer so [k]: k ≤ k < k+1 it is the floor

7 k is an integer so so [k+½] k ≤ k + ½ < k+1 it is the ~~ceil~~ the floor

Figure 6.61: Student B's response to Questions 6 and 7



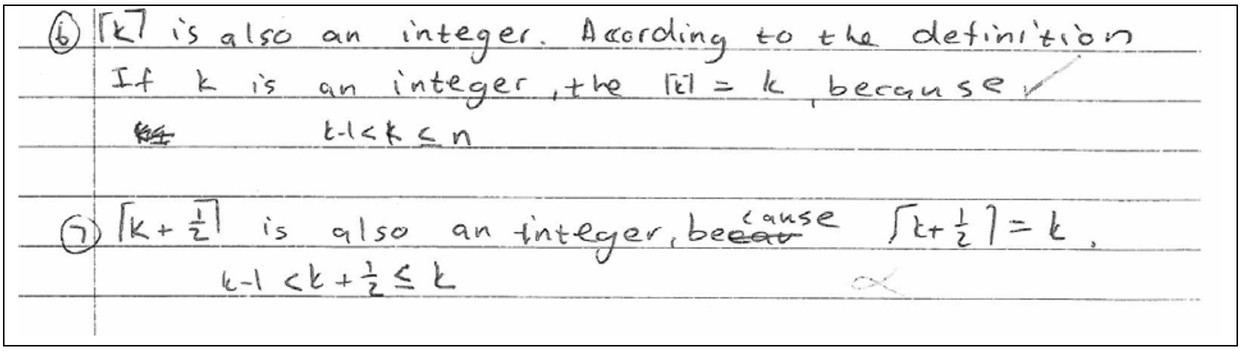
6. ⌈k⌉ = k < k < k+1

∴ ⌈k⌉ = k+1, because the ceiling is one more than k.

7. If k is any integer, ~~then~~ what is ⌈k+½⌉? Why

< k+½ < k+½+1

Figure 6.62: Student E's response to Questions 6 and 7



(6) ⌈k⌉ is also an integer. According to the definition

If k is an integer, the ⌈k⌉ = k, because

k-1 < k ≤ n

(7) ⌈k + ½⌉ is also an integer, because ⌈k + ½⌉ = k,

k-1 < k + ½ ≤ k

Figure 6.63: Student F's response to Questions 6 and 7

Only Student F provided a correct reason and solution to Question 6. Unfortunately he did not reconcile his response to this question with his response to Question 1. All other students did not supply a correct solution and reason.

Students A and B provided a partially correct reason in Question 7. All the other students did not provide a correct solution or reason. Based on all these responses I concluded that most of the students could not make all the requisite cognitive connections with the implicit information in the definitions provided.

The research-related question arising was, ‘What does this indicate?’ Does one conclude that the students have not developed the ability to self-explain correct material to the appropriate level? Since I suspected that this was the case I revised the teaching strategy by including exercises that required the students to self-explain. In classroom interactions students were prompted by the researcher to make visible their self-explanation of mathematical texts presented and problem and solution statements in order to correct and enhance their ability to self-explain. Peer discussions in tutorial classes were also utilized to make reasoning visible in order to enhance self-explanation abilities.

6.5 Quality of response

Instead of analysing student responses only for correctness one could also analyse responses in terms of quality of response. An aspect of quality in this regard is efficiency of response. Efficiency of response can in many instances provide insight into the level of proficiency. If analysis of a written response indicates that some intermediate steps were skipped and/ or some steps were performed simultaneously (and were effectively performed as a single step) then such a response would be deemed to be more efficient than a response that would show the converse.

I presented the argument in chapter 3 that expertise level of students is a function of their ability to either skip some (or all) intermediate steps or to perform some steps simultaneously. I am of the opinion that if a student in a particular class of tasks shows a consistent ability to skip intermediate steps and/or to perform some steps simultaneously then this would be assumed to be an indicator of well-developed procedural and conceptual knowledge in that domain. In what follows I will analyse a few solution procedures of the case study students to determine if they skipped intermediate steps or performed some steps simultaneously. It should be noted that in all the analysis that follows solution procedures will be evaluated in terms of how steps are performed and correctness of steps will not necessarily be evaluated.

Question 1.1 of class test 1 was as follows: “*Determine the equation of the straight line through* $(2;1)$ *and perpendicular to* $3y + 2x = 6$”

(For the ensuing discussion **figures 6.1 to 6.6 of section 6.3.1** will be utilized.)

The solution required students to transform the provided two-variable linear equation to the form $y = mx + c$ in order to determine m the gradient of the line. The fact that the product of the gradients of perpendicular lines is equal to -1 is then utilized to determine the gradient of the required line. This new gradient together with the coordinate $(2;1)$ is then substituted in the point-slope form of the line i.e. $y - y_1 = m(x - x_1)$ to obtain the required equation.

There are thus three sub goals in the solution procedure namely:

(i) To transform the equation $3y + 2x = 6$ to the form $y = mx + c$

(ii) Subsequently to use the gradient of this line to determine the gradient of the other line

(iii) This new gradient is used together with the provided coordinate pair to determine the equation of the perpendicular line

In order to achieve the first sub goal the following steps need to be performed:

1. The additive inverse of $2x$ must be added on both sides: $3y + 2x - 2x = -2x + 6$
2. The subsequent equation must be simplified: $3y = -2x + 6$
3. Next each term must be divided by 3 (stated differently each term must be multiplied by the reciprocal of 3 namely $\frac{1}{3}$): $\frac{3y}{3} = -\frac{2x}{3} + \frac{6}{3}$
4. The resulting equation must then be simplified: $y = -\frac{2}{3}x + 2$

If one compares the solutions of the six students the following is observed:

- Students A, B, D and F show steps 2, 3 and 4.
- Student C shows steps 2, 3 and 4 but steps 2 and 3 are performed together
- Student E skipped steps 1 to 3 and only shows the final form namely step 4.

It has been my experience that many teachers at the school level often teach that the $2x$ of the equation $3y + 2x = 6$ should be 'taken over and the sign changed'. Hence learners that were exposed to this way of instruction would only show the $-2x$ on the right side of the equation. But this is not an indication that they have skipped the step of adding the additive inverse on both sides in order to preserve equality. Rather it shows that they have 'taken over' the $2x$. It is my contention therefore that students A, B, D and F performed all four steps as indicated above.

In order to achieve the third sub goal the following steps need to be performed:

5. Either the point-slope form of the two variable linear equation $[y - y_1 = m_{\perp}(x - x_1)]$ or the slope-y-intercept form $(y = mx + c)$ is written down. (I will only use point-slope)
6. The perpendicular gradient and the $x - and\ y -$coordinates are substituted into the equation: $y - 1 = \frac{3}{2}(x - 2)$
7. The brackets is removed by applying the distributive law: $y - 1 = \frac{3}{2}x - 3$
8. The additive inverse of 1 is added on both sides: $y - 1 + 1 = \frac{3}{2}x - 3 + 1$
9. The resulting equation is simplified: $y = \frac{3}{2}x - 2$

If the solutions of the students are compared again the following is observed:

- Students B,C and D show all five steps
- Students A, E and F show only four steps since steps 7 and 8 are done simultaneously and therefore shown as one line

Question 1.5 of the first semester examination is similar in certain ways to question 1.1 of class test 1. Question 1.5 was as follows:

Determine the equation of the line parallel to PQ and through the point R.

For the discussion that follows **figures 6.28 to 6.33** will be utilized.

The solution procedure for question 1.5 required that one is aware that parallel lines have equal gradients. This gradient is then utilized together with the provided coordinates to determine the required two-variable linear equation. The following are possible steps for the solution procedure:

1. Either the point-slope form of the two variable linear equation [$y - y_1 = m_{\perp}(x - x_1)$] or the slope-y-intercept form $(y = mx + c)$ is written down. I will only use the former.
2. The parallel gradient and the x- and y-coordinates are substituted into the equation: $y - 6 = -(x - 6)$
3. The distributive law is applied to remove the brackets: $\boldsymbol{y - 6 = -x + 6}$
4. Six is added on both sides of the equation: $\boldsymbol{y - 6 + 6 = -x + 6 + 6}$
5. The resulting equation is simplified: $\boldsymbol{y = -x + 12}$

If one compares the solution procedures of the six students the following is observed (student E was not considered since she utilized a different method):

- Students A,B and D showed all the possible steps
- Students C and F performed steps 3 and 4 simultaneously

Question 4.1 of the end of module examination of the second semester of 2014 was as follows: *Determine the centre and radius of the following circle*: $\boldsymbol{x^2 - 2x + y^2 - 7 = 0}$.

Student responses are shown in **figures 6.70 to 6.75**

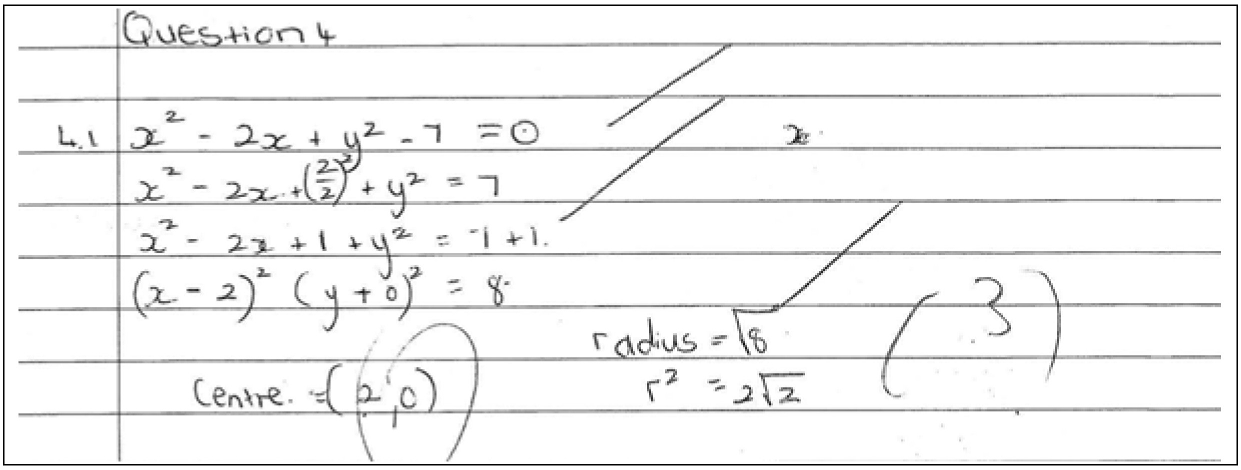


Figure 6.70: Student A's response to question 4.1 of the end of module examination of the second semester

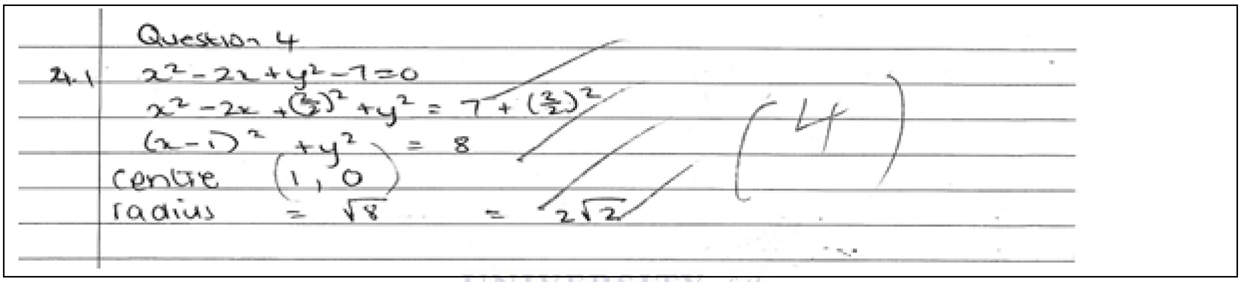


Figure 6.71: Student B's response to question 4.1 of the end of module examination of the second semester

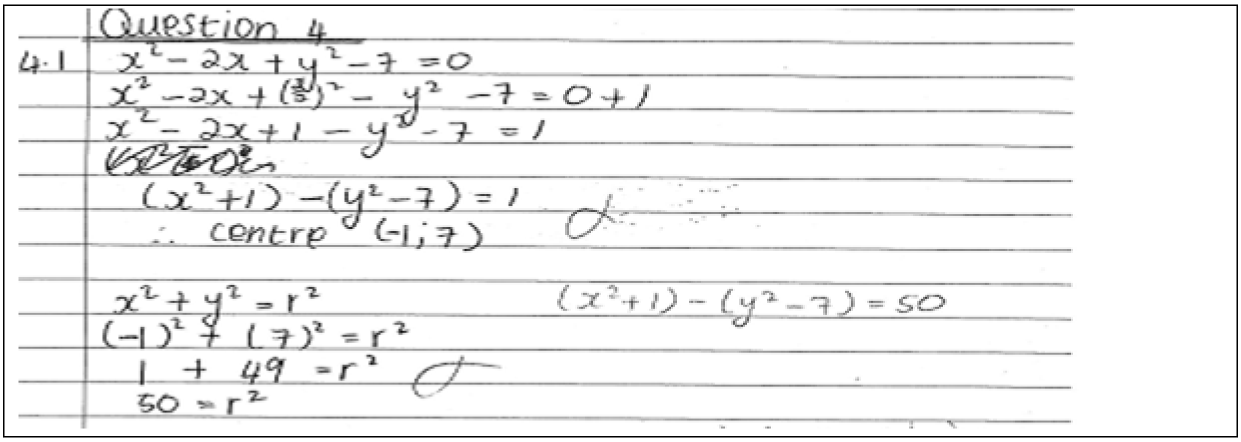


Figure 6.72: Student C's response to question 4.1 of the end of module examination of the second semester

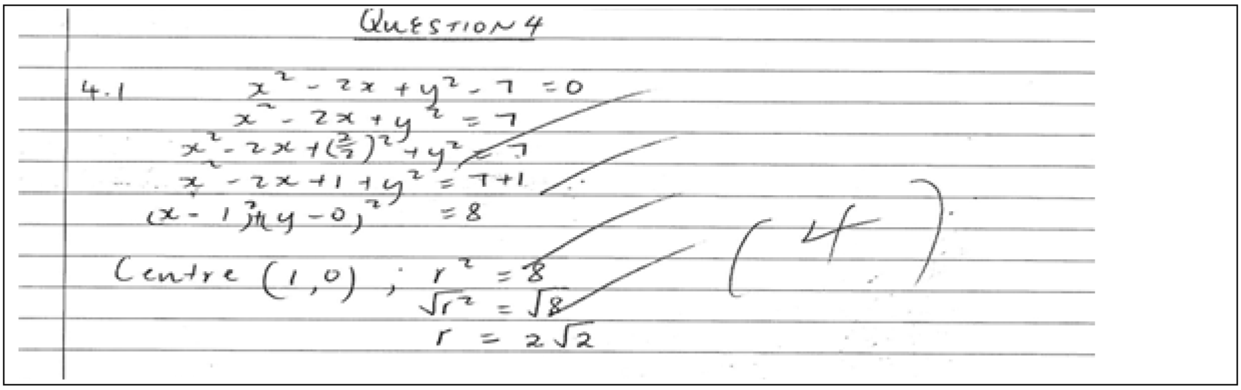


Figure 6.73: Student D's response to question 4.1 of the end of module examination of the second semester

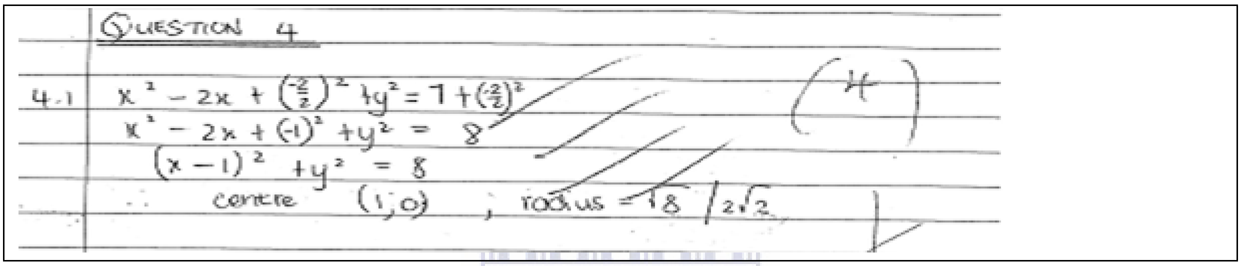


Figure 6.74: Student E's response to question 4.1 of the end of module examination of the second semester

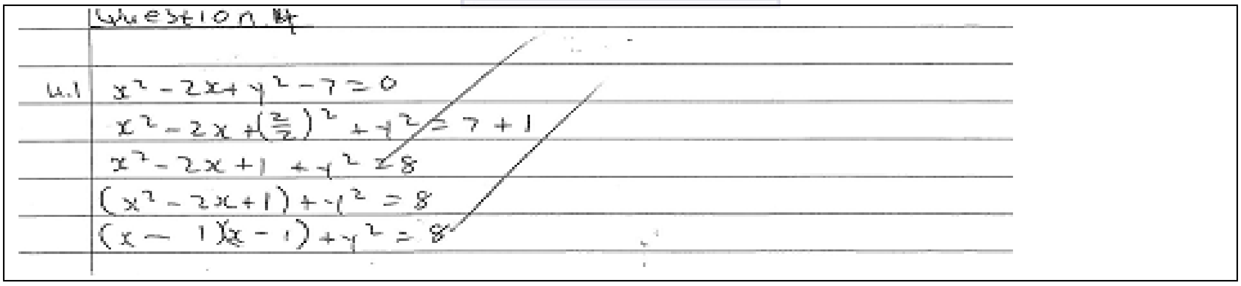


Figure 6.75: Student F's response to question 4.1 of the end of module examination of the second semester

The solution procedure for question 4.1 requires that the equation be transformed to the form: $(\boldsymbol{x}-\boldsymbol{a})^2 + (\boldsymbol{y}-\boldsymbol{b})^2 = \boldsymbol{r}^2$

In order to achieve this form with the above equation the following steps should be performed:

1. Add 7 on both sides : $\boldsymbol{x^2 - 2x + y^2 - 7 + 7 = 7}$
2. Simplify the resulting equation: $\boldsymbol{x^2 - 2x + y^2 = 7}$
3. The coefficient of the $\boldsymbol{x - term}$ is halved and then squared. This number is then added on both sides: $\boldsymbol{x^2 - 2x + \left(\frac{-2}{2}\right)^2 + y^2 = 7 + \left(\frac{-2}{2}\right)^2}$
4. The equation is simplified: $\boldsymbol{x^2 - 2x + 1 + y^2 = 7 + 1}$
5. The equation is written in the required form: $\boldsymbol{(x-1)^2 + y^2 = 8}$
6. The centre and radius is written down: $\boldsymbol{centre\ (1;0)\ and\ radius = 2\sqrt{2}}$

The following is observed if one compares student responses:

- If one assumes that all of these students performed ‘take over’ as explained earlier, then students B, C, D, E and F performed steps 1,2,3 together, skipped step 4 and then performed steps 5 and 6.
- Student A performed all the required steps.

It is not practically possible to show this kind of analysis for all test and examination items. The above findings however were evident in most of the participating students’ written productions. In other words if a student laboriously showed all possible steps in the analyzed items then the same way of working would be evident in their other responses to test or examination items. The question is why do some students have the ability to perform steps more efficiently than others despite the fact that all participating students were exposed to the same teaching under uniform conditions? A possible answer to this question will be provided in chapter 7.

6.6 Summary of student performance

A summary of the six students’ performance in the selected test and examination items (the items discussed in the previous sections of this chapter) is shown in Table 6.2. The table shows each of the six students’ performance in terms of retention of indispensable knowledge, errors committed, ability to transfer requisite knowledge, ability to utilize flexible procedural knowledge and ability to use prior knowledge creatively. Not all of these performance criteria were represented in the different test items and hence the summary does not include all knowledge and reasoning competencies that were measured for each test and examination item, but only salient competencies for the particular item.

A distinction was drawn between procedural errors and conceptual errors. Procedural errors are perceived to be errors where manipulation or calculation is the cause of the error. Conceptual errors are errors that are the result of a misconception. ‘Y’ indicates that an error was committed whilst ‘N’ indicates that no error was committed. Similarly, ‘Y’ indicates that knowledge was retained or transferred and ‘N’ indicates negation. The symbol ‘P’ indicates that retention or transfer was only partial. ‘Y’ indicates use of flexible procedural knowledge or creative thinking and ‘N’ signifies that it was not utilized.

Since I was focusing on only a narrow part of linear functions in this part of the study, the algebraic component consisted mostly of utilization of the two-variable linear equation. In the majority of cases therefore the requisite procedural knowledge consisted of knowledge of manipulation rules of these types of linear equations. I therefore found it convenient to draw a distinction between procedural and conceptual errors. Procedural errors in this instance refer to errors that occur as a result of a violation of the manipulation rules, application of an inappropriate procedure, or a calculation error. For example if the task requires manipulation of the equation $6x + 3y = 12$ in order to determine the gradient, and if the presented solution included an error such as $y = 2x + 4$, then it was be classified as a procedural error.

The concepts which enjoyed focus in this part of the study include gradient, intercepts, parallelism of lines, perpendicularity of lines, coordinates, midpoint of a line, perpendicular bisector, intersection of lines and tangent. Conceptual errors are more difficult to define. For the purpose of the study I distinguish between two categories of conceptual errors. A conceptual error could be inferred if a participant failed to make appropriate connections with provided information and prior knowledge, or between seemingly disparate pieces of information. Conceptual errors could also be inferred if analysis of a written response exposed a misunderstanding of a concept or definition.

It is generally assumed that experts in a domain tend to make fewer errors than novices. One criterion that can be used to determine level of expertise therefore is number of errors committed. Hence for this study the fewer the number of errors a student committed the higher the level of expertise was perceived to be and conversely. It should be noted that this was not the only criterion used to determine level of expertise, but forms one of a range of criteria.

	Semester 1																				Semester 2												
	Test 1			Test 2				Test 3					Test 4					Exam				Test 3						Exam					
	B1bi			D1bi				C1bi					B1BI					D1BI				D1BI						D2A					
	Errors			Errors				Errors					Errors					Errors				Errors						Errors					
Students	Procedural	Conceptual	Retention	Procedural	Conceptual	Retention	Transfer	Procedural	Conceptual	Retention	Transfer	Flexible	Procedural	Conceptual	Retention	Transfer	Flexible	Procedural	Conceptual	Retention	Transfer	Procedural	Conceptual	Retention	Transfer	Flexible	Creative	Procedural	Conceptual	Retention	Transfer	Flexible	Creative
A	Y		Y		Y	Y	Y			N	N	N			P	N	N			Y	Y	Y	Y	P	N	N	N	Y	Y	P	N	N	N
B			Y			Y	Y	Y		N	N	N		Y	P	N	N			Y	Y	Y	Y	P	N	N	N	Y	Y	P	P	N	N
C			Y		Y	Y	Y			Y	N	N			Y	Y	Y	Y		Y	Y		Y	Y	Y	Y	N	Y	Y	P	P	N	N
D			Y			Y	Y		Y	Y	N	N			Y	Y	N			Y	Y	Y	Y	P	Y	N	N	Y	Y	P	P	N	N
E			Y			Y	Y			Y	Y	Y			Y	Y	N			Y	Y			Y	Y	Y	Y			Y	Y	Y	Y
F			Y			Y	Y			Y	Y	Y			Y	Y	N			Y	Y		P	Y	Y	Y	N			Y	Y	Y	Y

Table 6.2: Summary of case study student performance on linear functions

6.7 Level of expertise

The six students selected for the case study had varied mathematical backgrounds. Students A and B had taken mathematics up to grade 9 whereas the other four had taken mathematics up to grade 12. Students A and B did not therefore have the same level of exposure to mathematical concepts and procedures at school level as the other four students. It could be argued therefore that the prior knowledge (concerning mathematical topics discussed in the course) of these two students was not developed to the same level as that of the other four. It was the intention of this study to determine how mathematical competency of participating students (with their different levels of prior knowledge in terms of course topics) was affected by the teaching strategy.

It should be noted that I use the phrases 'level of competence' and 'level of expertise' synonymously. As indicated earlier an objective of the study was to develop task-specific expertise in participating students. Task specific expertise is the ability to perform fluently in a specific class of tasks. I agree with the argument that task- specific expertise is a prerequisite for a student to become an expert in a larger mathematical domain. The class of tasks used to determine competency in this case was the two-variable linear equation and its attendant concepts. As has been indicated, learning is defined as a change in long-term memory. If knowledge can be retrieved from long-term memory it implies that it has been retained. The retained knowledge may or may not be appropriately connected to other relevant knowledge.

But even if the knowledge is in some ways isolated, the fact that it is retained is in my view a start to some learning process. In this study students were subjected to repeated testing on the same content and hence were required to retrieve some knowledge repeatedly. Repeated retrievals cause changes in the way knowledge is subsequently retained. Retention and transfer abilities can therefore be utilized as a measure of reasoning and knowledge development (and current level of expertise) in participating students.

For this study the main categories of knowledge retention considered were procedural and conceptual knowledge whilst the main reasoning types were imitative and creative. It is very rare however that mathematical tasks require solution procedures which exclusively involve either purely procedural or purely conceptual knowledge. It is argued that knowledge that is initially conceptual can be converted to knowledge that is procedural (Hiebert & Lefevre, 1986). Therefore knowledge type is also dependent on the student involved. A student's level of mathematical development will determine whether knowledge is conceptual or procedural. Because of this it is a very difficult exercise to categorize test or task items accurately in terms of procedural and conceptual knowledge requirements.

So categorization was based on perceived level of mathematical development in the majority of participating students in order to be applicable to most of the students. Retention abilities of the different knowledge types were inferred from written productions.

Expertise level was also thought to be a function of flexible procedural knowledge. I have argued previously that skilled problem-solvers in mathematics are also flexible in their use of known procedures. A student who does not possess flexible procedural knowledge will not always be able to solve unfamiliar problems where the solution requires the student to use assimilated procedural knowledge. Also, the student will not be able to produce a maximally efficient solution in the absence of such flexible procedural knowledge. Thus such flexibility endows students who possess relevant knowledge with the ability to generate maximally efficient solutions for known and even sometimes unknown problem situations. Star (2005) contends that flexible procedural knowledge is deep procedural knowledge that would allow a student to use appropriate mathematical procedures in a known or novel problem situation.

The two knowledge types (procedural and conceptual) are usually intricately entwined in the reasoning requirements of most mathematical tasks. In this study the level of development of mathematical competence was determined by ability to deal effectively with tasks of varying degrees of difficulty. Thus some mathematical tasks were perceived to be of a greater degree of difficulty than others. A novice-to-expert competency scale was created (see Table 6.3) based on the perceived knowledge and reasoning requirements (which determine the difficulty level), and the correctness of written solutions (which provide the inferred knowledge levels and correctness of reasoning) to assigned tasks.

The levels of the scale are based on the Dreyfus (1980) model of skill acquisition. The Dreyfus (1980) model is normally used to provide a means of assessing and supporting progress in the development of skills or competencies. The descriptors delineate levels of knowledge and reasoning. Perceived ability of participants therefore is described in terms of these two. A distinction is drawn between two types of knowledge and two types of reasoning. All the test and examination items were utilized to determine an expertise level for the six students.

As already argued it is expected that students with low expertise in a mathematical domain will have fragmented knowledge and will usually lack the ability to ‘see’ how procedures and concepts relate to each other in the domain. Yet as expertise in a mathematical domain increases so does the ability to integrate pieces of conceptual and procedural knowledge into a coherent knowledge structure (Baroody & Dowker, 2003; Linn, 2006; Schneider & Stern, 2009; Schneider et al, 2011).

Student ability to generate advance steps of a solution procedure and to skip intermediate steps or alternatively to perform some advance steps in a single step, was also used to gauge level of expertise for a specified class of tasks (see 6.5). If in test or examination items a student skipped intermediate steps of a solution procedure or performed intermediate steps in a single step, it was deemed to be an indicator of an instance of expert performance.

Error rate was also used as a measure by which to determine level of expertise. As it is generally assumed that experts in a domain tend to make fewer errors than novices, level of expertise can therefore be inferred from the number of errors committed. In this study the fewer the number of errors a student committed the higher the level of expertise was assumed to be and the converse also applied. It should be noted that error classification has been discussed in the previous section.

The category in which a student is placed is not viewed as the final competency level of the student. It is an indication of their developmental level at that point and should be seen as the point from which the student will progress should further learning occur. The competency level is therefore not regarded as fixed, but is considered to be in a constant state of flux. The scale reflects the expertise level after exposure to the interventions of the study. The scale is neither utilized to indicate initial levels of competency nor subsequent final levels of competency, but rather to indicate the level the student is currently at, from which they might progress.

There is general consensus in the mathematics education community that a well-connected deep understanding of fundamental mathematics is an absolute necessity for primary and high school teachers. In terms of scale categories therefore, at the very least teachers should operate at the competent expertise level since it is inevitable that teachers learn much in the process of teaching. Hence if prospective teachers enter the profession at the competent level, the expectation is that they are likely progress to a higher level of expertise as a consequence of the different teaching interactions. The preferred categories that teachers entering the teaching profession would fall into should thus be proficient or expert.

To explore the school grounding that might ultimately produce such readiness I shall use the scale in order to categorize the six students under investigation in this study.

LEVEL	DESCRIPTOR (of Knowledge and Reasoning)
1. Novice	(i) Rigid adherence to taught procedures or rules is in evidence and therefore student relies to a very large extent on imitative reasoning. No signs of imitative reasoning (based on) flexible procedural knowledge are present, and/or student exhibits algorithmic fixation. (ii) Student exhibits procedural errors in more than 80% of imitative reasoning based on procedural knowledge tasks. (iii) Student exhibits misconceptions in more than 80% of tasks. (iv) Creative thinking is non-existent. (v) There is no instance where intermediate steps are skipped or intermediate steps are performed in a single step.
2. Advanced Beginner	(i) Slight flexibility in the use of taught procedures and rules is in evidence; relies to a large extent on imitative reasoning, but also shows hints of ability to use flexible procedural knowledge. (ii) Student exhibits procedural errors in 50% to 80% of imitative reasoning based on procedural knowledge tasks. (iii) Student exhibits misconceptions in 50% to 80% of presented tasks. (iv) Creative thinking is exhibited in fewer than half the required cases. (v) There is one case in which intermediate steps are skipped or intermediate steps are performed in a single step.
3. Competent	(i) Flexible procedural knowledge is used correctly in 50% – 60% of required cases; imitative reasoning is used correctly in the majority of required cases. (ii) Student exhibits procedural errors in 20% to 50% of imitative reasoning based on procedural knowledge tasks. (iii) Student exhibits misconceptions in 20% to 50% of presented tasks. (iv) Creative thinking is shown in slightly more than half required instances. (v) There are two to four cases where intermediate steps are skipped or intermediate steps are performed in a single step.
4. Proficient	(i) Flexible procedural knowledge is used correctly in 60% to 80% of required cases; imitative reasoning is used correctly in nearly all cases. (ii) Student exhibits procedural errors in 0% to 20% of imitative reasoning based on procedural knowledge tasks. (iii) Student exhibits misconceptions in 0% to 20% of tasks. (iv) Creative thinking is shown in 60% - 80% of required instances. (v) In more than half the cases where applicable, intermediate steps are skipped or intermediate steps are performed in a single step.
5. Expert (highly skilled)	(i) Flexible procedural knowledge is used correctly in all cases; imitative reasoning is used correctly in all cases. (ii) Student exhibits no procedural errors in imitative reasoning based on procedural knowledge tasks. (iii) Student exhibits no misconceptions in any presented tasks. (iv) Creative thinking is shown in 80% – 100% of required cases (v) In all applicable cases intermediate steps are skipped or intermediate steps are performed in a single step.

Table 6.3: Novice-to-expert competency scale

Students A and B are very similar in terms of level of mathematical knowledge and reasoning. An analysis of their test and examination responses indicates that they relied mostly on imitative reasoning and they showed a moderate degree of flexibility in procedural knowledge, but also committed a number of errors. An analysis of these errors shows that some were the result of misconceptions. Both these students struggled with the more complex problems. Neither of these students could produce a coherent solution to either of the more complex test items. Student B however performed slightly better with the more complex problems. Based on these arguments Student B was thought to be midway in the competent expertise level whereas Student A was thought to be near the start of the competent level.

Students C and D also exhibited a high level of similarity in their performance. Yet while they also relied largely on imitative reasoning, they showed more advanced abilities in terms of procedural skills and were more accurate in the execution of procedures. Student C utilized the more efficient solution strategy in two instances, which is an indication of flexibility in the application of procedural knowledge. Although Student D did not identify those instances where a more efficient strategy might be employed, she was much more accurate in the execution of procedures and hence is grouped with Student C. Although these students did not supply a completely correct solution to the more complex problems, they performed much better than Students A and B in these items. Their responses to these items were more coherent and showed that their conceptual understanding was slightly more developed than that of Students A and B. Both these students were thought to be midway in the proficient expertise level with Student C slightly ahead of Student D.

Students E and F were identified as the highest performing students. Neither made any procedural errors across all the different test and examination items based on linear functions. Student F made one conceptual error in one of the more complex test items. Surprisingly, an analysis of this error showed that Student F did not have a well-developed understanding of parallelism of linear functions. Although these students had not used more efficient strategies where applicable, it became evident that if one were to prompt them, they would have been able use their procedural knowledge flexibly. These students produced near flawless responses to the more complex tasks. Student E was the only student who could use prior knowledge creatively. Based on the foregoing arguments, Student E appeared to be near the top of the proficient level for these types of linear function problems. Because Student F exhibited a fundamental misconception, he was considered to be slightly above the middle of the proficient level.

6.8 Conclusion

The main aim of the analysis in this chapter was to determine whether there was any evidence of knowledge retention and transfer in the written production of participating students with regard to the two-variable linear equation and its attendant concepts. A related objective was to determine level of expertise after exposure to the teaching strategy. The six students selected for the case studies showed varying degrees of retention and transfer of knowledge over time.

In the next paragraphs I summarize the main findings (of this chapter) based on student responses to the tests and examinations. I then use these as a basis for synthesizing my concluding findings (for this part of the study).

Students A and B seemed to have followed a similar pattern of retention and transfer of requisite indispensable knowledge in the given context of linear functions. Analyses of the responses of Students A and B (in terms of the selected test and examination items) indicate that initially the students retained and transferred a major part of the indispensable linear function knowledge in two problems of differing difficulty levels. Responses to the last two tests in the latter part of the first semester however indicated that the students later struggled to retain and transfer the relevant indispensable knowledge. In the end of module examination of the first semester these problems seem to have been overcome since the students produced completely correct responses. The responses of the students to the more complex problems presented in the second semester however produced evidence that the retention and transfer problems had not been eradicated completely. Student B however, seemed to cope better with the more demanding questions than Student A.

Students C and D also seem to have followed similar knowledge retention and transfer patterns. Responses to the first four tests of the first semester as well as to the end of module examination seem to indicate that the indispensable linear function knowledge had been retained and the ability to transfer this knowledge in relevant problem situations had been developed to a high degree. Their responses to the more complex problem presented in the second semester however painted a slightly different picture. From these responses it became clear that their transfer abilities in terms of linear functions was lacking in certain areas. For Student C these problems persisted in the case of a similar complex problem presented in the examination of the second semester. Student D seemed to have resolved these issues by the time of this examination and produced a near perfect response.

The responses of Students E and F to the test and examination items seem to indicate that the indispensable linear function knowledge and the concomitant ability to transfer this knowledge to relevant problem situations had been very well-developed.

It is not normally the intention of qualitative researchers to generalize findings beyond the original research participants. However, as I have argued in Chapter 3, if a finding in a qualitative study is replicated with different people and in different circumstances then the finding can be generalized beyond the people in the study even if there was no random sampling. In this study six students with different abilities comprised the case studies. All their written work over two semesters was incorporated. These written productions were produced in testing situations that were not all the same since some testing situations were high stakes (such as the examinations) whereas other tests (such as class tests) were not at the same high stakes level. One can therefore argue that the circumstances for the different sets of data collected were not the same. Moreover, the test and examination items were of different degrees of difficulty. If a research finding therefore is replicated in the six case studies one can argue that because of replication logic the findings can be generalized beyond the current population.

The analysis of the student responses shows that indispensable knowledge relating to linear functions was retained across the greater portion of the selected test and examination items by a majority of the six students. This was the case despite the fact that four of the six students struggled with the more complex linear function problems. A similar argument can be advanced for transfer of knowledge in that the majority of students were able to transfer relevant linear function knowledge, but struggled to transfer in the domain of the more complex items. Following replication logic one can therefore argue there is a high probability that this finding holds for many of the other participating students.

While it may not be possible to extend this finding to other content domains that formed part of the study, I nonetheless have high expectations that similar results will be obtained for the other content areas.

In the next chapter I discuss findings of both the qualitative and quantitative parts of the study, synthesize conclusions and make some suggestions for future research.

CHAPTER 7: DISCUSSION, SUGGESTIONS FOR FUTURE RESEARCH, LIMITATIONS AND CONCLUSION

7.1 Introduction

An examination of the literature regarding mathematics education reveals that not a lot is known about the efficacy of mathematics teacher preparation in terms of the organization of opportunities to learn mathematics and how the form of instructional delivery affects prospective teachers' knowledge (Tatto, Lerman, & Novotna, 2010). Furthermore very little research was done to determine the effectiveness of mathematics teacher education programs and very little information is shared about teacher education programs across tertiary institutions hence teacher educators mostly start anew when presenting teacher education courses (Hiebert et al, 2003). This research was an attempt to address some of these issues in the South African context.

In an attempt to deal with some of the knowledge and reasoning issues of pre-service mathematics teachers this study was designed. The hypothesis of this study was as follows: *If* South African mathematics education pre-service students are exposed to a teaching strategy that is premised on spiral revision and productive practice *then* their procedural fluency, conceptual understanding, knowledge retention and knowledge transfer abilities will be enhanced. A mixed methods approach was employed to investigate this hypothesis.

In this chapter I will provide a summary of results and conclusions; provide some explanations for the findings; offer recommendations for teaching mathematics to pre-service teachers (including the issue of learning resources); discuss contributions of the study to theory; discuss limitations of the study; discuss issues for further research and provide overall concluding comments. Some of these will not be discussed separately, but will be discussed together since it provides for a logical flow. For example some sections might include summary and discussion of results, explanations, conclusions and recommendations.

7.2. Summary of findings: Procedural fluency

How would exposure to a teaching strategy that is based on spiral revision and productive practice, in requisite content areas of the specified curriculum, influence the mathematical competencies of procedural fluency and conceptual understanding of pre-service mathematics teachers was the main research question for this study. The secondary question was concerned with how retention and transfer abilities of participants would be influenced if they experience mathematics through a teaching strategy underpinned by spiral revision and productive practice. In this section findings related to procedural fluency are discussed. I will then proceed to discuss findings related to conceptual understanding and finally I will discuss findings related to retention and transfer.

The categories memorized reasoning based on factual knowledge, familiar algorithmic reasoning based on procedural knowledge, familiar algorithmic reasoning based on flexible procedural knowledge and delimiting algorithmic reasoning based on flexible procedural knowledge was deemed to be a measure of the competency procedural fluency. These categories therefore formed the measuring instrument that was utilized to measure knowledge and reasoning proficiency of research participants. As indicated in chapter 4, the instrument used to pursue the hypothesis was based on these categories. Student written responses to test and examination items were analysed to determine their proficiency levels.

The statistical analysis (see chapter 5) indicated that students scored high on the majority of the variables for procedural fluency. Furthermore the stratified analysis showed that the majority of students improved from pre-test to post-test in terms of procedural fluency. The stratified analysis however also indicates that some students struggled with the category familiar algorithmic reasoning based on flexible procedural knowledge. Approximately a quarter of students fell in the fail grades for this category of question. The teaching strategy appears to be effective in improving some aspects of procedural fluency, but lack in others. It would seem that a possible weakness of the teaching strategy was that it did not effectually assist all students to acquire the ability to flexibly use known procedural knowledge in novel contexts.

The findings in chapter 6 indicate that participating students had varying levels of competencies after exposure to the teaching strategy. Moreover the analysis done in chapter 6.5 indicates that some students performed steps of solution procedures more efficiently than others.

Possible explanations for the above phenomena will be presented next.

7.2.1 Explanations, conclusions and recommendations: Procedural fluency

As argued previously schema acquisition and the automatization of learned procedures are two essential mechanisms in the learning process (Sweller,1994). In cognitive load theory mechanisms of learning is utilized to determine which features of material make it hard to learn. The theory is based on the assumption that gaining of knowledge and cognition *based* on this knowledge is heavily dependent on schema acquisition. The notion of schemas is thought to be able to explain a major part of learning-mediated cognitive performance. Sweller (1988) argues that experts in a domain have more domain specific knowledge (in the form of schemas) than novices in the domain.

It is generally accepted that newly presented information is not internalized in the exact form that it is presented; rather new knowledge is altered so that it fits in with current knowledge. It is argued that knowledge of subject matter is organized into schemas and it is these schemas that determine how new information is dealt with (see chapter 3.4). Sweller (1994) contends that people utilize schemas to deal with mathematical problems. These schemas allow for problems to be categorized based on the solution procedure. For example students that have been exposed to algebra will not only know how to solve a specific linear equation such as $2x + 5 = 7$, but will know the solution procedure for this category of problem and hence would be able to solve all problems of the form $ax \pm b = c, \quad \forall a, b, c \in \mathbb{R}$. These schemas therefore reduces the amount of mental effort required to solve such problems and allow people to potentially solve an infinite variety of such problems.

Schemas are not acquired in an all or nothing manner, but rather it is assimilated gradually over a period of time. Consequently when a student is exposed to new knowledge, the ability to use this knowledge is initially severely constrained since the schema has not been fully developed.

Cognitive processing of information can either be controlled or automatic (Schneider & Shiffrin, 1977). Controlled cognition is said to occur when information is consciously attended to. In other words the information is either not in the long-term memory or is not well-established in the long-term memory and therefore have to be processed in the working memory.

When a complex mental skill is first acquired it can only be utilized with considerable cognitive effort; however over time and with enough practice the skill may become automated (Sweller, 1994). Consequently if mundane procedural elements of a task have been practiced to the extent that it became automated it would free cognitive capacity for more creative reasoning and applying prior knowledge in unfamiliar situations. Moreover if skills operate under automatic processing then cognitive load will be reduced. This I think is what teachers of mathematics would ideally want to achieve with instruction in mathematics.

It is an accepted fact that a function of learning is to store information in long-term memory. A function of learning according to cognitive load theory is to store automated schemas in long-term memory. As mentioned previously working memory has a limited capacity and duration and hence the amount of information that can be processed in the working memory is limited. This limitation can affect learning negatively and hence a function of instruction and learning should be to find ways to reduce working memory load. Schema acquisition and automatization have precisely this effect namely to reduce working memory load.

Generally when students are first introduced to a mathematical topic their initial knowledge of the topic tends to be very limited. So it is often the case that students know a little about a topic, but do not fully understand the topic. Conceptual knowledge is indispensable for the construction, selection and correct application of procedures (Hiebert & Lefevre, 1986). On the other hand practicing known procedures is thought to help students develop and deepen understanding of concepts. The main argument therefore is that both kinds of knowledge are required for effective mathematical learning and that each type of knowledge is required to strengthen the other over time. Instruction in mathematics should therefore strive to develop both procedural and conceptual knowledge. Development of procedural and conceptual knowledge in a mathematical domain should however also include development of the ability to categorize problems based on required solution procedures. It is my contention that the spiral testing component of the teaching strategy played a major role in developing the ability to discriminate between different kinds of problems and to associate each kind of problem with an appropriate strategy.

As argued previously students with low expertise in a mathematical domain usually has fragmented knowledge and do not have the ability to 'see' how procedures and concepts relate to each other in the domain. As expertise in a domain increase the ability to integrate the conceptual and procedural pieces of knowledge into a coherent knowledge structure increases (Baroody & Dowker, 2003; Linn, 2006; Schneider & Stern, 2009; Schneider et al, 2011). Schemas of low expertise students therefore will not contain well-developed procedural and conceptual knowledge and ways of reasoning for a specific class of problem but will lack in some areas. High expertise individuals on the other hand have schemas that have more mature conceptual and procedural development and the concomitant reasoning abilities.

My argument is that although problems that require flexible use of procedural knowledge was practiced in class what happened was that since student schemas (the students that struggled with this kind of problem) regarding manipulation was not fully developed for that class of task because their focus was on how the different parts of the steps fit together. Moreover they were focusing on the reasoning that connects the individual steps. That is their understanding of the procedure as a whole was not fully developed. This limits their ability to compare features of the procedure to procedures of other similar problems. That is they did not compare it in terms of comparing problem features, but were focusing on the reasoning behind the individual steps and the ideas that connected the steps. They have not developed to the point where they can compare problems based on problem features such as for example if it is a reversal problem of a kind that has been done previously. This is evidenced by the fact that the struggling students still wrote all the steps for this class of problem. In turn the writing of all the steps indicates what is happening cognitively. What is happening in terms of reasoning is that the ideas that connect each step are not fully automated and hence these have to be brought into conscious focus (in the working memory) in order to be processed. These students therefore do not have mature schemas for a particular class of tasks and have to consciously attend to tasks in working memory. They have to laboriously reason through all of the steps that constitute the solution procedure for the particular task. What I am arguing therefore is that it is practically impossible to become flexible in the use of procedures if one still needs to focus on each individual step in a routine procedure.

The central executive is one of the components of the working memory. As I have mentioned in chapter 3.2 the central executive is an attentional control system of limited processing capacity that has the role of controlling action. As I have mentioned previously the fact that all the steps are written is an indication that the reasoning that connect each step in the procedure has to be brought into consciousness (in the working memory). Since the reasoning that connects steps are not fully automated it implies that the central executive has to direct the majority of attention to these thoughts which leaves less capacity to shift attention to other features of the solution procedure such as comparing it with other similar known cases. I believe this is one of the main reasons why these students struggle to develop their ability to use known procedures flexibly and in novel situations. In other words since the procedure for a particular type of task is not fully automated and the schema for the type of task is not at a mature level it compromises ability to use known procedures creatively. Hence what the data is indicating (struggles with the category of question familiar algorithmic reasoning based on flexible procedural knowledge) is that these students are not at the same developmental level as their peers. The question is what inferences can be drawn from these arguments regarding the teaching strategy and what can be done to improve the situation?

A premise of my version of spiral revision is that development and increase of procedural knowledge will aid in procedural flexibility and conceptual knowledge in a given mathematical domain. The statistical analysis and qualitative analysis however has shown that this is not true for the development of procedural flexibility of some students (in particular students identified as low ability students). A primary goal of the teaching strategy was to make explicit conceptual underpinnings of a procedure and conversely to show how concepts are utilized in the development of a procedure. The expectation was that this way of doing would negate the need to explicitly compare solution procedures of different practice examples as a function of problem features. The findings however indicate that the expectation was not met in all cases and hence there is a need to compare solution procedures of presented problems.

Schema development for a specific class of tasks allow for problems to be categorized based on solution procedure. As argued earlier schema acquisition is one of the essential mechanisms in the learning process. If problems are categorized based on solution procedures then deeper insight would be gained if problem features can be linked to procedure type. What I am proposing therefore is that the teaching strategy be adapted by including exercises where solution procedures of different problems (of the same class of tasks) are compared for similarities and differences and links are made between problem features and procedure type. Such exercises would require more complex cognitive processes since it requires one to consider individual steps that constitute a solution procedure, the ideas that link the steps and then to compare these to another solution procedure while simultaneously attempting to link problem features to the procedures. It is my contention that although it would be a more demanding exercise it would allow one to gain deeper insight as to how and why procedures are applied for a specific problem type, the limitations in terms of when and under what circumstances it can be applied and what are the circumstances that allow for the procedure to be adapted. This deeper insight is in my opinion a prerequisite for the development of the ability to identify those instances where known procedural knowledge can be applied in novel contexts or to use procedures more flexibly and more efficiently. The question is which practice strategies could be exploited to achieve such learning objectives?

It is theorized that mass practice assists mathematics learning by providing immediate opportunity to practice what one has been exposed to. It allows students to focus on the execution of a strategy for a particular type of mathematical problem (Rohrer, et al, 2015). This type of practice however does not provide for students to compare what they have learnt to other similar problems. Furthermore there is no guarantee that mastery of the particular type of problem is achieved since it is possible that students can imitate solution procedures without having an understanding of the underlying ideas.

A distributed practice format on the other hand provides many opportunities for students to master a particular skill or concept. Furthermore it can be utilized to uncover the different facets of concepts. Once students has been exposed to many different types of problems of a particular class of tasks over an extended period of time they can be presented with tasks that require comparison of solution procedures. It is my opinion that distributed practice is best suited for this type of learning activity.

The quantitative analysis indicates that the categories memorized reasoning based on factual knowledge, memorized reasoning based on procedural knowledge and familiar algorithmic reasoning based on procedural knowledge had high pre- and post-test means. All three categories also showed significant increases in means between pre- and post-test. These are some of the categories that constitute the competency procedural fluency. The cognitive process dimension for all of these categories is imitative reasoning. These categories therefore require less complex reasoning and hence can be classified as lower order. An analysis of the categories familiar algorithmic reasoning based on flexible procedural knowledge and delimiting algorithmic reasoning based on flexible procedural knowledge on the other hand indicates that there was very little improvement between pre- and post-test for these categories. These category of question required more complex reasoning since its solution procedure require one to use known procedures in a new way or use known procedures in tasks that have not been encountered before. These categories can therefore be classified as higher order. The question is what inference can be made from these findings regarding how the teaching strategy affected competency levels of participants? In other words what is the answer to the main research question in terms of procedural fluency?

The teaching strategy enhanced some aspects of procedural fluency (memorized reasoning based on factual knowledge, memorized reasoning based on procedural knowledge and familiar algorithmic reasoning based on procedural knowledge) but did not improve other aspects (familiar algorithmic reasoning based on flexible procedural knowledge and delimiting algorithmic reasoning based on flexible procedural knowledge) of procedural fluency (see chapter 5). This finding was more pronounced for lower ability students than for higher ability students. The categories where improvements were noted fall in those categories where highly imitative reasoning is required. The categories where the imitative reasoning requirements were less and flexibility in reasoning was a requirement were the categories where improvement were either negligible or a regression was registered. This implies that the teaching strategy was effective in improving ability to reason imitatively with known procedural knowledge, but was not as effective to improve ability to reason flexibly with known procedural knowledge or to use known procedural knowledge in novel situations. The teaching strategy affected higher and lower ability students differently. The higher ability students show an improvement in both the higher order categories and the lower order categories. The lower ability students on the other hand show an improvement in the lower order categories, but do not show the same level of improvement in the higher order categories.

A premise of the teaching strategy was that development and increase of procedural knowledge and concomitant reasoning abilities will aid in improving procedural flexibility and conceptual knowledge in a given mathematical domain. This seems to hold true for the higher ability students that participated in the study, but in most cases does not hold true for the lower ability students. The majority of higher ability students had some prior exposure at the school level with some of the presented content whereas some of the lower ability students did not have the same level of exposure to the content. An inference one can therefore make is that level of prior knowledge in a content area played a major role in how individual participants were affected by the teaching strategy.

7.3 Summary of findings: Conceptual understanding

In this section I will discuss findings related to effectiveness of the teaching strategy with regards to conceptual understanding of participants. As mentioned previously conceptual understanding refers to the comprehension of mathematical concepts, operations and relations. The categories of Memorised Reasoning based on Conceptual knowledge (D1a), Familiar Algorithmic Reasoning based on Conceptual knowledge (D1bi), Delimiting

Algorithmic Reasoning based on Conceptual knowledge (D1bii) and Local Creative reasoning based on Conceptual knowledge (D2a) were considered to be measures of conceptual understanding.

Analysis of the data indicates that the majority of students entered the course with weak conceptual understanding. The mean for the pre-test was 48.99% which is below the pass cut-off score. The post-test mean was 59.94% which represents an improvement of 10.95%. The stratified analysis indicates that the lowest ability students were affected most by the implemented teaching strategy. These students showed the biggest improvement (27% moved to higher ranks post-test). This represents the biggest movement of students in terms of improvement of scores for conceptual understanding. The higher ability students also showed a significant improvement. There was thus an overall improvement in terms of conceptual understanding.

Closer inspection of the data however reveals that this positive result does not hold for all the categories that constitute conceptual understanding. The categories familiar algorithmic reasoning based on conceptual knowledge and delimiting algorithmic reasoning based on conceptual knowledge showed significant improvement from pre- to post-test. The categories memorized reasoning based on conceptual knowledge and local creative reasoning based on conceptual knowledge on the other hand showed a near zero difference between pre- and post-test. This implies that after intervention there was no improvement in terms of performance in these categories. Furthermore the data reveals that the majority of students entered with very weak abilities in terms of Local Creative Reasoning based on Conceptual Knowledge. I am of the opinion that this is the most difficult category to improve on. This is also the category that I believe teachers should excel in in terms of school level mathematics. I believe that in order for mathematics teachers to explain concepts adequately they must be able to view and utilize concepts from different angles which in turn require the ability to reason creatively with known concepts.

Possible explanations for the above phenomena will be presented next.

7.3.1 Explanations, conclusions and recommendations: Conceptual understanding

As I have mentioned previously conceptual understanding is defined to be the comprehension of mathematical concepts, operations and relations. Conceptual understanding is therefore premised on knowledge of concepts. Conceptual knowledge is defined as knowledge that is rich in relationships (Hiebert & Lefevre, 1986). Conceptual knowledge is developed by establishing cognitive links between different pieces of information. The linking process is possible between pieces of information that is already stored in memory or between an existing piece of knowledge and one that is newly learned. The result of the linking process is that the new knowledge is assimilated into appropriate knowledge structures and hence becomes part of the existing network. In other words it becomes part of the schemas in the long term-memory.

The term abstract is used to determine the level of connection between pieces of information. The term abstract refers to the degree to which a unit of knowledge is tied to a specific context. Abstractness is said to increase as knowledge is freed from a specific context. If the relationship is established at the same level of abstractness or at a less abstract level than the level at which the original information was presented then the relationship is said to be at the primary level. On the other hand if the relationship between pieces of information is established at a higher abstract level than the pieces of information they connect then the relationship is said to be at the reflective level. Such relationships normally are less tied to specific contexts. Building the cognitive connections in this case requires that one reflects on the information being connected and consequently more of the mathematical terrain will become 'visible'. The reflective level is therefore perceived to be at a higher level than the primary level.

The description and definition of procedural and conceptual knowledge of Hiebert and Lefevre (1986) however has drawn some criticism. Star (2005) maintains that the definitions of procedural and conceptual knowledge of Hiebert and Lefevre (1986) do not fully account for knowledge type and knowledge quality in their definitions. An assumption that can be drawn from their definition of conceptual knowledge is that conceptual knowledge is always deep and rich in relationships. On the contrary cognitive connections of conceptual knowledge might be limited and superficial or it might be extensive and deep (Star, 2005). He maintains that a mathematics students' initial conceptual knowledge generally is limited and superficial, but over time might develop and become deeper and have more connections.

As I have mentioned before a primary goal of the teaching strategy was to make explicit conceptual underpinnings of a procedure and conversely to show how concepts are utilized in the development of a procedure. The main teaching strategy that was used to develop conceptual knowledge was productive practice (see chapter 1.5). Productive practice is a strategy where deepening thinking-like problems is employed to enrich the conceptual knowledge of students in requisite content areas of the specified mathematics curriculum.

As I have indicated before I support the iterative model as proposed by Rittle-Johnson, et al (2001). A major premise of this theory is that either procedural or conceptual knowledge might develop first, but one type of knowledge does not as a rule develop before the other. The contention is that it is often the case that a particular type of knowledge is incomplete. More specifically one type of knowledge might be better developed at a particular point in time, but this does not imply that the other type of knowledge is totally absent. Furthermore initial knowledge in a domain generally is very limited. The contention is that levels of prior knowledge in a domain determine which type of knowledge will emerge first and set the learning process in motion. Whichever knowledge type is developed first then in turn forms a basis from which the other type of knowledge develops. According to the iterative model therefore improved conceptual knowledge results in improved procedural knowledge. Improved procedural knowledge in turn leads to improved conceptual knowledge, which then leads to improved procedural knowledge and so on.

Analysis of the data revealed that students entered the study with very low levels of conceptual knowledge in the content areas that were covered. The data further revealed that the low ability students benefitted most from the teaching strategy with regards to conceptual understanding. These students however did not move from being advanced beginners to expert but rather in my estimate moved to the competent level. Students that entered the study with a more or less competent level generally moved to a more proficient level. Most participants however did not show an improvement in the category local creative knowledge based on conceptual knowledge as mentioned before. Next I will advance theories as to the underlying reasons for these different levels and change in proficiency level of the participating students.

Mathematics educators in the majority of cases use students' solution procedures to presented tasks to determine proficiency levels of students in terms of type and quality of knowledge and concomitant reasoning abilities. Broadly speaking problem solving in mathematics can be divided into three phases. The first phase comprises understanding of the problem statement in terms of provided information and what is to be determined or proved. The second phase consists of developing a strategy to provide a solution procedure or proof. The third phase involves producing a written or verbal solution procedure based on the strategy developed in the second phase. The first and second phase is mostly invisible since it is done mentally. I am aware that according to the literature there are other phases involved, but for my purposes these three phases suffice. Each of the phases necessitates adequate conceptual understanding in order to produce a correct solution procedure. Conceptual understanding mediates interpretation of the problem statement. Thus if conceptual understanding for a particular class of tasks is not well developed then the problem statements for the tasks will not be completely understood which in turn will lead to the production of an incorrect solution procedure. The second and third phases are also mediated by conceptual understanding. It is important to keep in mind that procedures that are produced normally are the result of conceptual knowledge that is highly abstracted. Procedures are said to access and act on conceptual knowledge translating it into something observable. Procedures therefore informs on the state of conceptual knowledge (Hiebert & Lefevre, 1986). Next I will use an example to explicate the foregoing arguments:

Prove that $\forall\, x \in \mathbb{R}\ and\ \forall m \in \mathbb{Z},\ \lfloor x + m \rfloor = \lfloor x \rfloor + m.$

The first phase requires reading and interpretation of the statement. The very first word in this case states what must be done namely that a proof must be provided of a mathematical statement. Next the symbolism has to be translated into English which in this case is: the *for all* symbol; an *element of a set*; the *set of real numbers* and the *set of integers*. Conceptual understanding is required to understand what $\forall\, x \in \mathbb{R}\ and\ \forall m \in \mathbb{Z}$ mean. That is one needs to make the connection that the statement must be proved in such a way that it holds for all real numbers and for all integers with no exceptions. Next one has to understand what one has to prove. That is an understanding of $\lfloor x + m \rfloor = \lfloor x \rfloor + m$ is required. That is one needs to understand that what is required is that one must show that the sum of the floor of a real number and an integer is equal to the sum of the integer and the floor of the real number.

This is premised on an understanding of the concept floor. Correct interpretation of the problem statement therefore is mediated by conceptual understanding.

The second phase requires that a strategy be developed to tackle the problem. This requires that one realize that one should start with some prior knowledge and what is provided in the problem statement. Conceptual knowledge mediates the reasoning that is required to produce a strategy.

In the last phase it is required that a coherent solution procedure is produced based on the strategy developed in phase two. In this instance the definition of floor is applied directly as the first step. That is:

let $n = \lfloor x \rfloor, for\ n \in \mathbb{Z}\ and\ \ x \in \mathbb{R}$, then by definition of floor we have: $n \leq x < n + 1$

Subsequently we add m to all three parts of the inequality: $n + m \leq x + m < n + 1 + m$

This step is based on understanding of properties of inequalities. Finally the definition of floor is now used in the opposite direction. That is we interpret this statement to mean that $\lfloor x + m \rfloor = n + m,$ but since $n = \lfloor x \rfloor$ we have $\lfloor x + m \rfloor = \lfloor x \rfloor + m$. What we have shown therefore is that the statement is true for an arbitrarily chosen integer and real number. The statement is therefore true for all real numbers and all integers. It is clear that this last phase is also mediated by conceptual understanding.

It is highly unlikely that the above problem can be correctly understood and a correct solution procedure produced without having the requisite conceptual knowledge that is appropriately connected to other knowledge. The first point I am making is that procedural knowledge is the visible part of our conceptual knowledge and as argued by Hiebert and Lefevre (1986) it informs on how well developed conceptual knowledge in a mathematical domain is and to what level reasoning abilities with the conceptual knowledge is developed. In other words it informs on our conceptual understanding. The second point I am making is that conceptual understanding mediates almost all acts involved in problem solving. Therefore mathematical understanding in general is premised on conceptual understanding.

The question remains why students that are exposed to the same teaching under uniform conditions show different proficiency levels in terms of level of development of procedural fluency and conceptual understanding. Students entered the course with differing levels of proficiency is the first part of the answer. That is the starting point for participating students were not all the same in terms of prior knowledge. More particularly their initial types and quality of knowledge were not all the same. In some cases procedural knowledge were more developed than conceptual knowledge and in other cases it was vice versa. For argument's sake let us presume that a participating student entered the course with a well-developed ability to manipulate linear equations, but the student is not as developed in the underlying conceptual knowledge, whilst another student entered with well-developed procedural and conceptual knowledge. According to the iterative model improved procedural knowledge leads to improved conceptual knowledge. The first student needs to develop in terms of conceptual knowledge first before further progress can be made, while the other student is already at this stage and can therefore improve on procedural knowledge which in turn will lead to improvement of conceptual knowledge that is already ahead of the other student. The second student therefore will tend to learn more under the same conditions as a consequence of better developed procedural and conceptual knowledge. It will therefore be difficult for the first student to first catch up and then overtake the other student if they receive the same kind of instruction under uniform conditions.

The second part of the answer is based on how the knowledge is held by the different students. In other words the quality of their knowledge is not all the same. To determine the quality of knowledge one has to determine if the knowledge is surface level or deep level knowledge. Surface level knowledge is normally associated with a low level of abstractness whereas deep level knowledge is associated with a high level of abstractness.

Although some participating students have been exposed to some of the content before entering the course their level of abstractness for the content was at the primary level. That is the relationship between relevant pieces of information is tied to a specific context and hence they find it difficult to transfer their knowledge to a different context where the knowledge might be required. The more these students are exposed to different problems in different contexts the more their knowledge will be freed from a specific context and hence the abstractness of knowledge will improve. That is they will develop a better and deeper conceptual understanding. On the other hand students that were already at the reflective level in terms of some content could deal more effectively with more advanced problems. Evidence for this can be seen in responses to the more complex problems that were discussed in chapter 6. Students that were at the reflective level could progress much further with problems in a different context (circles) than students that were at the primary level. In the majority of cases the primary level students could not progress beyond the first sub procedure.

The last part of the answer has to do with the to be learned material itself. In cognitive load theory an element is defined as any material that is to be learned. If in order to learn material it is required that mental connections needs to be made between many other elements then the material is said to have a high element interactivity and is perceived to be harder to learn. Conversely if elements of a task can be learned without making other mental connections then it is said to have low element interactivity and is perceived to be easier to learn. Mathematical tasks rarely have low element interactivity. In particular mathematical tasks that require the utilization of concepts and definitions at both primary and high school level usually necessitate that many mental connections be made. It can therefore be argued that such tasks have high element interactivity.

Examples of material that has high element interactivity that were part of the presented course can be found in the complex examples that were discussed in chapter 6, such as the theorem about tangents to circles.

This theorem is as follows: *If a tangent to a circle is drawn, then it is perpendicular to the radius at the point of contact.*

A number of concepts appear in this theorem. These concepts are tangent, circle, perpendicular and radius. A failure to fully comprehend any of these will result in not understanding the theorem. Furthermore connections need to be made with tangent and radius and straight line. That is one needs to be cognizant of the fact that a tangent and a radius are in fact straight lines and therefore can be represented by the two-variable linear equation and all the other attendant concepts hold such as gradient, intercepts, etc.

Furthermore one has to realize that the theorem is written in the *if-then* format. This implies that it is a conditional statement which has a hypothesis and a conclusion. This in turn implies that if the hypothesis is true then it follows that the conclusion is true. In some problem situations the converse might be presented. That is a radius is presented perpendicular to a straight line. In such a situation one need to be aware that this is the converse of the stated theorem and that the conclusion that can be drawn is that the straight line is a tangent. Problems that rely for their solution on an understanding of such theorems therefore have high element interactivity.

Total cognitive load is the sum of intrinsic and extraneous cognitive load (Sweller, 1994). Extraneous cognitive load is imposed by instructional methods whereas intrinsic cognitive load is determined by element interactivity. If a content area has a high number of interacting elements it is associated with a high intrinsic cognitive load. Conversely if material has a low element interactivity it is thought to have a low cognitive load. Instructors have no control over intrinsic cognitive load since it is only dependent on the element interactivity of to be learned material. It is argued that if people have acquired schemas of a content area with high element interactivity, then the content is understood. If these schemas are automated then the material is understood very well (Sweller, 1994).

One can deduce from the responses of the low ability students to the complex problems that although they know the concepts involved, the schemas for these are not fully automated. Their laborious reasoning is evident in the fact that they show each step. This in turn provides evidence that the schemas for these content areas are not fully automated. I think it is to be expected that material that have high element interactivity will be more difficult to learn and hence will be more difficult to apply in problem situations. A possible solution is that students are exposed to such problems more often and over an extended period. I think it takes time and effort to really understand some concepts in all of its different nuances. Exposure more often will allow for engagement repeatedly with the same concepts increasing the probability that it will become abstracted. I have attempted to expose students often and over an extended period of time to the different classes of tasks in this study. I could not however do this consistently for all the tasks because of time constraints. The more complex tasks are an example of such tasks.

Exposure more often and over an extended period will allow for the creation of appropriate cognitive connections between conceptual and procedural knowledge which is thought to contribute to efficient memory storage and successful retrieval of procedures in applicable circumstances. Hiebert and Lefevre (1986) advance several reasons why procedures are stored and retrieved more successfully when it is connected to conceptual knowledge. If procedures are connected with conceptual knowledge it becomes part of a network of information that is held together by semantic relationships that do not deteriorate easily since memory endures longer for meaningful relationships. Since the procedures are part of a knowledge network many alternate cognitive links can be used to access and retrieve it. It is thought that conceptual knowledge also has an executive control function in that it is utilized to monitor not only the selection and use of a procedure, but also to determine the reasonableness of the procedural outcome.

Conceptual knowledge only becomes useful for solving mathematical tasks when it is converted into an appropriate form. When a student is introduced to a new mathematical topic, the student initially do not know procedures to solve mathematical problems based on the topic. Since no procedures are known problems is solved by applying mathematical facts and concepts in an arduous way. However as the student are exposed to similar problems and these are solved repeatedly conceptual knowledge is gradually transformed into procedural knowledge. In this way knowledge that was initially conceptual can become procedural. Hiebert and Lefevre (1986) argues that this is a very important process in the learning of mathematics since well-known procedures can reduce cognitive effort required and hence very complex problems can be solved. They maintain that automated procedures release cognitive resources that can be utilized to for example look for relationships between novel aspects of problems or where relevant conceptual knowledge can be applied.

All of the above arguments however still do not adequately address the issue of why the teaching strategy did not make inroads into enhancing creative reasoning based on conceptual knowledge abilities of participants. The majority of participating students struggled to deal effectively with this category of question. The teaching strategy therefore affected most of the participating students in the same way for this category of problem. I think the debate about nature or nurture is relevant in this instance. I am of the opinion that teaching strategies can be designed and utilized to enhance creative mathematical reasoning of students. Creativity is however also a function of innate abilities and therefore teaching strategies can have limited success in this regard.

7.4 Retention and Transfer as a function of Temporality

There is evidence that participants of the study have retained the majority of indispensable knowledge for the different classes of tasks. Transfer of knowledge to problems that were not structurally the same but that required a known solution procedure were not dealt with in the same way by the different ability groups. The data suggest that in most cases the lower ability group struggled to transfer knowledge to such problems. The higher ability group on the other hand performed slightly better in terms of transfer of knowledge in such cases. The teaching strategy therefore was effective in terms of retention of indispensable knowledge, but was not as effective for all categories of students in terms of transfer of knowledge.

The argument was presented in chapter 6 that humans forget approximately half of newly learned knowledge in a matter of days or weeks (Ebbinghaus, 1964; Rubin & Wenzel, 1996; Averell & Heathcote, 2010; Murre & Dros, 2015) unless they consciously review the learned material. In chapter 1.5 retention interval was defined as time elapsed between the most recent learning session and the test (Rohrer & Taylor, 2006). The first assessment of the study was done on 21 February 2014. In order to determine how effective the teaching strategy was in terms of retention and transfer of indispensable knowledge over an extended period I presented student B with a two-variable linear equation problem on 17 August 2015.

The retention interval therefore was 18 months. It should be noted that student B would have had no exposure to content that covered two- variable linear equations from the start of 2015 onwards. The student also was not informed that she would be presented with this type of problem beforehand and therefore would not have reviewed the content. This implies that the student would not have reviewed similar content for approximately eight months. The expectation therefore was that student B would have forgotten how to deal with such problems. One should also keep in mind that student B struggled with this type of problem in the second and third test of the first semester of the study. The idea therefore was to determine the effectiveness of the teaching strategy as a function of temporality. The presented problem and student B's response is presented in **figure 7.7**.

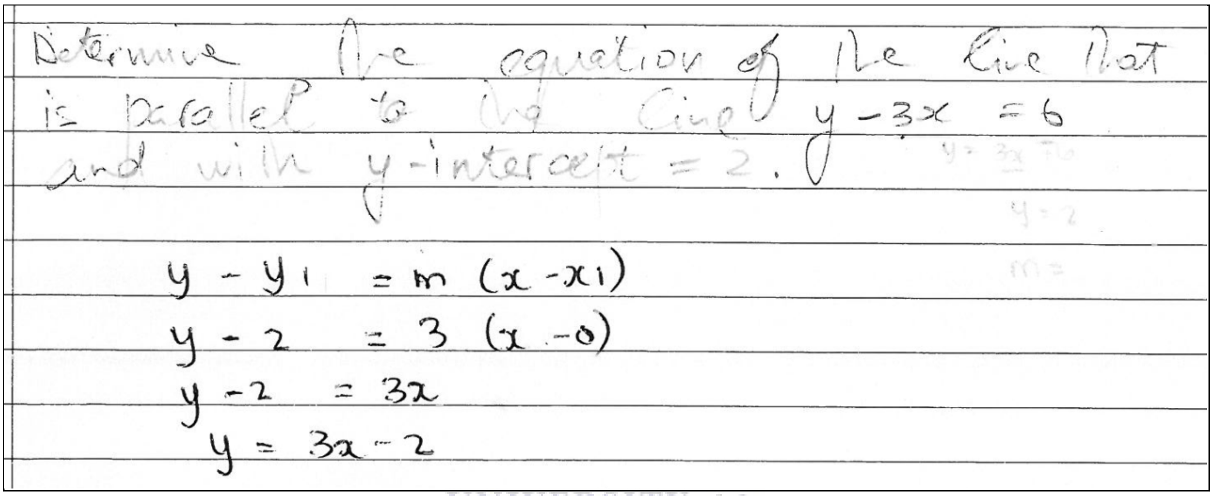


Figure 7.1: Student B and temporality

The solution procedure requires that one is cognisant of the fact that parallel lines have equal gradients and that one therefore has to transform the provided equation to the form $\boldsymbol{y = mx + c}$**.** One should then be aware that when the equation is in this form the coefficient of the $\boldsymbol{x - term}$ is the gradient. One could then either substitute into the point-slope form or the slope-intercept form of the two-variable linear equation. In this case utilization of the slope intercept form would be the more efficient way to deal with the problem.

Although student B did not use the more efficient way to solve the problem she demonstrated with her solution procedure that she had retained all the indispensable knowledge for this class of problem and could transfer her knowledge successfully. One can therefore argue that the thought processes involved in the production of such procedures (schema) is part of the long-term memory structures of student B. This could therefore be construed as evidence that the teaching strategy achieved its goal of retention of indispensable knowledge for student B. Although one cannot extrapolate this finding to the entire population my sense is that it would be similar for other students since student B was identified to be in the low ability group.

7.5 Contributions to theory

Usually taxonomies of educational objectives are utilized as a framework to classify learning and instructional activities. Many of these taxonomies however are not always very clear and consistent as to the criteria that it utilizes for classification purposes. The South African Further Education and Training (FET) mathematics CAPS document for example utilizes four categories (referred to as cognitive levels) to categorize mathematical tasks and assessment items. The four cognitive levels are: knowledge, routine procedures, complex procedures and problem solving. Criteria utilized for classification in the CAPS taxonomy includes type of problem (e.g. non-routine problem), cognitive processes (e.g. straight recall) and skill required (e.g. perform well known procedures). The criteria for each of the different cognitive levels sometimes include all of these criteria and in other cases only some. For some cognitive levels the knowledge requirements are stated whereas in others it is not. Similarly reasoning requirements are included in some instances and in others it is not clear what the reasoning requirements are.

The taxonomy developed in this study (see chapter 4.7) was designed to classify learning and instructional activities in terms of mathematical reasoning and mathematical knowledge requirements. The reasoning and knowledge categories utilized are those that are predominant in mathematics learning. It is different from other similar taxonomies in that classification is based on cognitive processes that operate on knowledge structures. It can therefore be argued that its categories are more closely aligned to mathematical cognition. It is also different in terms of the categories that are usually employed in the knowledge dimension (e.g. flexible procedural knowledge) and cognitive process dimension (e.g. imitative reasoning). The scheme of the taxonomy contributes to theories concerning design of taxonomies that are specific to mathematics. In particular it strives to employ discrimination criteria that align with human mathematical cognitive processes. This in my opinion provides a tool that is more fine grained and is more specific (in terms of mathematical knowledge and reasoning demands) in its classification of learning and instructional activities. The taxonomy table also provides a ready for use instrument that can be implemented in other similar studies or to determine how well learning and instructional objectives are achieved in the ordinary South African mathematics classroom.

An essential strategy employed in sport to improve individual skills and overall proficiency is practice. Without practice no athlete can improve performance. Even in sport like running, (where movements are elementary) practice is required to improve performance. Similarly there is general agreement in the mathematics education community that practice forms an important part of learning. However there is not always consensus as to the type of practice that should be employed or the frequency and timing of practice in the various learning contexts. The majority of studies that investigated effectiveness of the different types of practice focussed on the effectiveness of one type of practice (e.g. meta-analysis of benefits of overlearning done by Driskell, Willis & Cooper, 1992; benefit of distributed presentation done by Seabrook et al, 2005) or compared the different types of practice for effectiveness (e.g. Rohrer & Taylor, 2006; Seabrook et al, 2005). The foci of some of these studies are: retention, test performance, ability to connect a problem with an appropriate procedure, ability to discriminate between problem types and ability to distinguish among similar concepts. Furthermore there is a paucity of studies that investigated effects of a combination of types of practice.

Investigations on how the different types of practice affects retention of mathematical knowledge do not specify the type of mathematical knowledge affected (e.g. Pashler et al, 2007; Rohrer & Pashler, 2007; Seabrook et al, 2005). In other words it is not specified which type of mathematical knowledge is retained or alternatively not retained. None of the studies that I have encountered focused on how reasoning based on knowledge is affected by the different kinds of practice. I have also not encountered studies that investigated how the various practice forms affects transfer of mathematical knowledge.

Theories on mathematical practice are silent on how a combination of practice strategies affects mathematical proficiency in terms of enhancement of the different knowledge types and reasoning abilities. This study makes a contribution in this regard. It provides evidence on how a combination of practice strategies affects the different knowledge types and reasoning abilities. This study provides evidence that a revision strategy premised on spiral revision and productive practice enhanced the mathematical competencies of procedural fluency and conceptual understanding. One can therefore argue that if mathematics education pre-service students are exposed to a teaching strategy that is premised on spiral revision and productive practice then their procedural fluency, conceptual understanding, knowledge retention and knowledge transfer abilities will be enhanced.

Theories regarding the practice strategy of overlearning (repetitive practice) indicate that overlearning increases performance for a short while, but that the benefit diminishes sharply over time (Rohrer & Pashler, 2007). Most of these overlearning studies however were done only with word-definition pairs. Notable exceptions are studies done by Rohrer and Taylor (2006) and Rohrer and Taylor (2007) which were done with mathematics problems. These studies however were done with only one type of task (permutations) and therefore is of only limited value. These studies do however extend the theory to include more abstract types of learning.

Findings from the present study indicate that the strategy of overlearning increased test performance a short while after exposure to content, but that performance declined in later tests (see chapter 6.4). These findings therefore can be viewed as an instance of confirmation of the theory that overlearning increases performance for a short while, but that the benefit decreases over time. Since this study employed a range of mathematical tasks that require more than rote learning it can be argued that the findings extend the theory to include more abstract cognitive tasks. Also since many of the previous studies were done in the domain of language this study assists in generalizing results of such studies to more abstract cognitive tasks.

Rohrer and Pashler (2007) proposes a theory that final test performance is a function of the spacing gaps between practice or study sessions. According to this proposition very small gaps results in poorer performance when compared to excessively long gaps. Additionally they argue that the spacing effect gets bigger for longer-term retention. The present study provides evidence of both of these propositions (see chapters 5 and 6) and therefore can be perceived to be confirmation of the propositions. Evidence is presented in the study that performance improved as a consequence of spacing of exposures to learning over long gaps (see discussion of student B in chapter 7). Evidence is presented in chapter 5 of the proposition that the spacing effect gets bigger for longer term retention.

Rohrer et al (2015) argues that the solution of the majority of mathematical problems requires two distinct steps:

- Discrimination between different kinds of problems. That is the student has to decide which type of problem they are presented with.
- Subsequently the type of problem has to be associated with an appropriate strategy which has to be executed.

Mass practice is important to achieve mastery of recently learned procedures or concepts. This is especially so in the case of more complex procedures or algorithms. Repetitive mass practice where one type of procedure or concept is practiced however suffers from some weaknesses. The following weaknesses of repetitive mass practice are identified by Rohrer et al (2015):

- No discrimination between different types of problems is required. The fact that students do not need to discriminate between different types of problems reduce the difficulty of the presented problems.
- Students can solve this kind of problem without reading instructions since each consecutive problem are of the same type. They can therefore solve these kinds of problems without being aware of the kind of problem they are solving.
- This kind of practice therefore impedes the learning of the association between a problem and an appropriate strategy. Consequently it is theorized that this kind of practice in most cases do not support long-term retention.

The results of the present study indicate that discrimination abilities of the majority of participants were enhanced (see chapter 5). Based on the foregoing arguments it is highly unlikely that the repetitive mass practice was responsible for this enhanced discrimination ability. It is my contention that the spiral testing component of this version of spiral revision was responsible for this finding.

My contention is that the spiral testing component of spiral revision has a number of things in common with interleaving practice. In spiral tests different types of problems are presented consecutively. It thus requires that students discriminate between the different types of problems presented. Once a decision is made as to the type of problem, the student subsequently has to decide on an appropriate strategy. Spiral testing therefore enhances *ability to discriminate between problem types* and *strengthens the association between a problem and an appropriate strategy*. Spiral testing therefore also enhances long-term retention.

Spiral testing is premised on retrieval practice. As I have stated earlier retrieval practice has been found to be more effective than restudying in terms of enhancement of learning. Retrieval practice has also been found to be more effective at reducing the forget rate in comparison with restudying. Most studies concerning retrieval practice however has been done with verbal tasks (e.g. Carrier & Pashler, 1992; McDaniel, 2007). Although this study is premised on more than one practice type, spiral testing formed a foundational part of the intervention strategy. One can therefore argue that this study provides evidence of enhancement of learning and reduction of the forget rate in the domain of more abstract cognitive mathematics tasks and hence provides an instance of generalization of these theories (see chapters 5 and 6).

Rohrer and Taylor (2006) maintain that theories of distributed practice and overlearning do not declare how transfer abilities are affected by these practice types. The present study provides evidence that near transfer abilities was significantly enhanced by the implemented strategy. Far transfer abilities however were not significantly changed (see chapter 6).

The present study is premised on the notion that many factors contribute to learning during teaching and learning interactions. It is my opinion that no single factor on its own can dramatically enhance learning of mathematics. A teaching approach that is premised on different types of practice, restudying and an expository instructional approach is better suited to:

1. Retard forgetting;

2. Improve procedural skills and procedural flexibility;

3. Enhance conceptual understanding.

This study therefore contributes to theory in the sense that it provides evidence that more than one factor needs to be utilized to enhance learning. Each of the factors (continuous restudying; distributed practice; retrieval practice; mass practice; repetitive practice and an expository instructional method) contributed to more effective learning. In a minority of instances each of these factors contributed uniquely to learning while in the majority of cases all the factors (in an overlapping manner) together contributed to enhancement of learning.

Hiebert et al (2003) maintains that there is a paucity of research that set out to determine the effectiveness of mathematics teacher education programs. In particular there is a scarcity of research that investigated how the form of instructional delivery affects pre-service mathematics teachers' mathematical knowledge and reasoning. Moreover it is very rare that tertiary institutions share knowledge about teacher education programs.

The present research contributes to current understandings of how and in what ways an instructional strategy premised on a continuous revision strategy affects pre-service mathematics teachers' mathematical competency in terms of procedural fluency and conceptual understanding. This is especially the case for low knowledge participants since many pre-service teachers in South Africa enter mathematics courses with substantial gaps in their knowledge base.

From a pedagogical point of view students also were exposed to the implementation of a teaching strategy and could learn from this and can when they are teaching themselves implement such a strategy. This implies that they also learnt how to implement practice strategies to enhance learning. Hence learning was on the one hand of the content itself, but on the other hand they were also learning about teaching strategies to enhance learning.

7.6 Limitations of the study

In the quantitative part of this study a non-experimental research method was followed. This implies that independent variables were not manipulated and no random assignment to groups is effected. Since there is no random assignment to groups and no manipulation of the independent variable in the non-experimental method, it is perceived to provide weaker evidence of a causal relationship between variables.

In order to provide stronger evidence of cause-and-effect relationship between variables the teaching strategy employed (the independent variable) in the study need to be manipulated. This implies that on the one hand the type of teaching employed should be varied and on the other hand the continuous practice strategy should be varied.

Since no control group was utilized there is no evidence that the chosen sample of students performed better or worse than another group that were not subjected to the implemented teaching strategy. One way of dealing with this weakness (in order to provide stronger evidence or corroborating evidence) is to utilize the implemented teaching strategy with an experimental group and to teach the control group in an ordinary way. Ordinary here might mean that no continuous practice is employed and only repetitive mass practice is used at the conclusion of a lesson and revision is done as a once-off exercise just before a test or exam. The type of teaching employed for the control group might be unguided or minimally guided instruction only.

Higher order problems or tasks appear less frequently in practice tasks and hence they are practised less. In the present study higher order problems (problems of the category Familiar Algorithmic Reasoning based on Flexible Procedural Knowledge and Local Creative reasoning based on Conceptual knowledge) also appeared less frequently since a major portion of the intervention time was spent on the routine problems. This meant that higher order problems was spiralled less frequently and this might be a contributing factor to the finding that students tend to perform worse on these type of problems.

The fact that only one type of question was analysed in the qualitative part of the study is another possible weakness of the study. In possible successive studies this weakness can be addressed by utilizing two or three question types from different content areas.

7.7 Issues for further research

It is not clear from findings of the study how each of the different components of the teaching strategy contributed to participant competency in terms of retention of knowledge, transfer abilities and knowledge and reasoning development. Issues that might be pursued in further research are how each of the components of the teaching strategy uniquely contributes to participant competency development. For example an investigation could be done on how a teaching strategy premised on spiral revision and direct teaching influences low knowledge participants in terms of transfer abilities (near or far transfer or both). Alternatively a study could be done on the effectiveness of a teaching strategy premised on productive practice and minimally guided instruction.

It is also not clear from findings of the study how each of the constituent parts of spiral revision (review-as-you-go; repetitive practice, mass practice; distributed practice and spiral testing) contributed to the overall competency development of participants. Moreover it is also not entirely clear what if any influence the different practice forms had on each other. For example one could investigate how repetitive practice and spiral testing would influence competency as compared to distributed practice and spiral testing.

Can one for example do distributed practice without previously doing mass or repetitive or is the one necessary before the other is done. This implies that one will be investigating if mass and repetitive practice is utilized for mastery of procedures, or if distributed practice on its own will produce mastery and retention.

7.8 Overall concluding comments

This research was concerned about how continuous revision influences mathematical ability. In particular it was concerned about how the different categories of knowledge were retained and how the reasoning ability with this knowledge was influenced. Also the research was concerned about how knowledge was transferred to the different mathematical contexts. The findings indicate that the continuous revision constantly updated the memory trace and hence memory was retained even up to five months after initial exposure to the content. It was found however that transfer to similar contexts (near transfer) was easier than far transfer where the context was considerably different from the context in which the original learning occurred. This phenomenon was more pronounced for the lower ability students than for the higher ability students. That is the lower ability students tended to transfer knowledge to similar contexts well, but not so well to different contexts.

I believe that it is essential that teachers of mathematics at all levels include in their teaching strategies well thought out plans to practice using previously learnt procedures and concepts during problem solving activities. Things to consider in such plans includes aims of the practice session, frequency of practice, temporality and what format the practice sessions should take, should students be allowed to struggle during practice sessions (productive struggle), should retrieval practice be utilized, etc. The kinds of practice that are currently available in the literature are mass practice, distributed practice, interleaved practice and repetitive practice. The objectives of these kinds of practice include the following:

- to practice until error free productions of a type of procedure or procedures is achieved (or alternatively mastering is achieved)
- retention of a procedure and its attendant concepts
- deepening understanding of a concept or concepts and the associated procedure(s)

In this research I have used a spiral revision teaching strategy to improve procedural knowledge. This improved procedural knowledge would subsequently allow for the improvement of conceptual knowledge. But this does not imply that participating students would as a result of being exposed to this teaching strategy now have finished and complete procedural and conceptual knowledge in the topics they received instruction in. Rather I see this as a good start in the right direction in terms of knowledge and reasoning development.

I utilized productive practice as a teaching strategy to ameliorate conceptual understanding of participating students. The improved conceptual understanding allowed for the improvement and development of procedural knowledge. Again my argument is that participating students would as a rule not have a complete conceptual understanding of topics that were discussed. I believe that a major part of their learning would occur when they are teaching. What normally would happen is that the new teacher would be forced to examine his/her understanding of concepts as they prepare to teach a topic. This in itself would cause a change in information stored in long-term memory. That is as the information is being retrieved it will be examined for relevance and level of detail required by the teaching process. In some instances some of the knowledge that the new teacher possess would have to be 'dumbed down' for the level the teacher is teaching at. This would require the teacher to identify breadth and depth of required knowledge. This in turn requires the teacher to step back and look over the entire mathematical terrain that the topic covers, since decisions regarding levels and depth of knowledge cannot be done in isolation. For the teacher this requires that new cognitive connections are made between schemas. This is in my opinion a powerful learning experience. The teaching strategy employed in this research however developed crucial fundamental knowledge that is essential in the learning trajectory of these prospective teachers.

A deduction one can make based on the data of the study is that the assessment component of this spiral revision teaching strategy has revealed the quality and type of knowledge and reasoning ability of students after exposure to the teaching strategy. If one produces a correct solution in two tests (on the same topic) it does not necessarily imply that task specific expertise and conceptual understanding has been achieved as can be seen in the case of student B (discussed in chapter 6). More specifically it does not imply that retention of requisite knowledge is permanent and that transfer would occur automatically to structurally similar problems. It was only when student B was assessed the third and fourth time on the same topic that it became apparent that the student perhaps did not understand the content as well as the results of the first two tests implied. It is my contention therefore that the assessment component (spiral testing) of my version of spiral revision can be utilized to uncover the quality and type of knowledge a student would possess after exposure to instruction. By utilizing different kinds of questions with varying levels of difficulty in each successive test one can determine if the underlying concepts of a topic are connected appropriately, if the necessary procedural skills has been developed and if the student is able to identify the relevance of prior knowledge when attempting to provide solutions to given mathematical tasks. Furthermore since the same content is assessed over and over one can determine how well established the knowledge is in the long term memory, the level of abstractness of the knowledge and if appropriate connections were made with other relevant knowledge.

It has been my experience that both at school and at tertiary level in South Africa that the majority of mathematics textbooks are written in a format where the type of practice that is mooted is mass and repetitive practice. In this format nearly all of the problems concerning a given topic appear in the exercises immediately after the lesson and examples of the topic (Rohrer & Taylor, 2006). Based on the textbook format a highly probable inference one can make is that many teachers follow this way of doing and hence the type of practice in such classes would overwhelmingly be mass practice and repetitive practice. Furthermore in many cases in the South African context revision in mathematics is done mostly as preparation for tests or exams, is done in the same way as the original teaching was done and normally is done as a once off exercise. It has been my experience that this kind of revision does not influence in a major way procedural skill or levels of understanding. Distributed practice where practice of the same skills is practiced across multiple class sessions is very rare. Research has shown that retention of original well learnt mathematical procedures are enhanced by distributed practice and are unaffected by repetitive practice (Rohrer & Taylor, 2006). I therefore contend that one possible way of making inroads into the dismal mathematical performance of South African students at all levels is to change the format of textbooks exercises to include mass practice, repetitive practice and distributed practice.

I am of the opinion that not enough attention is given in South African research to how regular revision of previously covered mathematical content influences retention, transfer, proficiency and level of understanding. I am not claiming however that regular practice of previously covered mathematical content is a panacea for all our teaching and learning ills. I do believe however that spiral revision together with productive practice and self-explanation exercises hold promise for improving student mathematical ability.

As mentioned previously an analysis of the literature regarding mathematics education of pre-service mathematics teachers reveal that many pre-service teachers in South Africa enter mathematics teacher education programmes with substantial gaps in their knowledge bases. There is thus a need for research that investigates which teaching methods are best suited for such pre-service students. This study is such an investigation. That is it was the intention of the study to determine if a teaching strategy based on spiral revision and productive practice will enhance procedural fluency and conceptual understanding of participants and consequently attend to knowledge gaps. The findings of the study indicate that both procedural and conceptual knowledge of participants were improved.

It is the intention of many of the mathematics courses for prospective secondary school teachers in South Africa to teach these students more advanced mathematical content. Research has however shown that taking more advanced courses in mathematics do not necessarily translate into deeper understanding of fundamental mathematics (Ma, 1999). An important question therefore is what kind of teaching would allow pre-service mathematics education students to develop requisite procedural skills and a well-connected conceptual understanding. An analysis of the literature done by Hiebert and Grouws (2007) indicates that students can acquire conceptual understanding if teaching attends explicitly to conceptual underpinnings and to connections among mathematical facts, procedures and ideas. The teaching strategy employed in this research attempted to do exactly this.

Furthermore I am of the opinion that prospective teachers would be better prepared for their teaching if courses designed for them would deal explicitly with the content they would teach in schools. The course content of this study was school level content where the emphasis was on developing depth of knowledge in school level content. I am of the opinion that well developed knowledge and a deeper understanding of school level mathematics will result in better teaching at the school level.

Shulman (1986) distinguishes between three categories of teacher content knowledge namely subject matter content knowledge, pedagogical content knowledge and curricular content knowledge. Mathematics teachers need to be well developed in all three categories in order to deliver effective and quality teaching. All three categories are used together in most teaching encounters in mathematics. Therefore a lack in any category would negatively affect the teaching. Hence although this research only attended to content knowledge it contributes to the overall development of the prospective teachers.

An advantage of the revision strategy implemented in the study is that it strengthens the association of the different kinds of problems that students were presented with and the appropriate strategy. Consequently students tended to choose the correct strategy more often than before the intervention. Since the revision strategy required students to execute these appropriate chosen strategies many times during the study they became proficient in the execution of correct procedures. Another advantage therefore was development of proficiency in executing appropriate procedures. The revision strategy of this study aided in developing ability of participants to choose an appropriate strategy for a presented problem and to execute the chosen strategy based on the features of the presented problem. One could therefore argue that the revision strategy utilized in this study improved problem solving abilities of participants in terms of the topics that were dealt with in the study.

Mathematics teachers are required to explain their understanding of mathematical concepts and procedures to students during instruction. This understanding is always under construction, i.e. it is continually updated and corrected. I believe that it is especially during the teaching process that teachers uncover gaps in their knowledge base. This discovery prompts them to attempt to address the gap which requires them to make more appropriate cognitive connections and hence move their understanding to a new level.

In conclusion I think that the teaching strategy achieved the majority of its objectives in terms of enhancing procedural fluency and conceptual understanding of participants. The categories where the teaching strategy was not effective requires further research in order to reveal which strategies would be effective to deal with these shortcomings.

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APPENDICES

APPENDIX A

Ethics Statement

Before undertaking this study various sources have been consulted in order to ensure that this study meets acceptable ethical guidelines. The researcher also undertakes to abide by the Code of Research Ethics of the Human Sciences Research Council and of the American Psychologist Association.

The researcher undertakes to treat as confidential all information obtained from the University of the Western Cape and elsewhere and that all such information will be kept in a locked cabinet.

The ethical principles that this research will strive to adhere to include the following: Autonomy, beneficence, nonmaleficence, justice and confidentiality. Autonomy refers to the right of every individual to self-determination or the freedom of the individual to decide and/or act for him or herself. The principle of beneficence refers to the idea of doing good in the process of conducting the research; Nonmaleficence is the idea that research should not harm participants; Justice is the requirement that all research participants be treated with impartiality and conscientiousness and that there is an equitable allocation of resources, opportunities, benefits, and burdens for all participants (Adu-Gyamfi and Okech, 2010).

It is important that research participants are informed of the nature, duration and purpose of the research and also the methodology that will be followed. The consent of participants should be voluntary and they should have the right to withdraw their participation at any stage. Participants should also be aware that all information obtained in the course of the research will be treated as confidential and that the right to privacy of participants will be respected.

Operationalization of Ethical Principles

The right to full disclosure about the research

In order to give research participants a clear understanding of the study the researcher will explain the general purpose and process of the study as well as a description of what each participant will be expected to do and the conditions that they will be exposed to. All participants will be fully briefed about the research procedure and how findings of the research will be disseminated.

The right to privacy

Research participants have the right to refuse to be interviewed. They also have the right to refuse any mention of them in the study. Research participants will also be informed that they may terminate their participation in the study at any time and that any data obtained will be held confidential. They will also be informed that all data collected will be accessible to the study supervisor as well.

The right to anonymity

The participants have the right to remain anonymous. The researcher will ensure that all participants remain anonymous in the recording of data as well as dissemination of findings of the study. The researcher will make certain that each participant remain anonymous during the transcribing of interviews.

The right to confidentiality

The information gathered from participants will remain confidential. Furthermore all data such as recordings, transcriptions and written work will be kept secure in a locked cabinet.

All participants will complete a Participant Consent Form. A completed Consent form will be viewed as the respondents' consent to participate in the study.

APPENDIX B

PARTICIPANT CONSENT FORM

VERIFICATION OF ADULT INFORMED CONSENT FOR OWN PARTICIPATION

I,……………………………………………………………………………………………….

(Please print full name and surname)

voluntarily give my consent to serve as a participant in the study entitled:

A Teaching Strategy to Enhance Mathematical Competency of Pre-Service Teachers at UWC

I have received a satisfactory explanation of the general purpose and process of this study, as well as a description of what I will be asked to do and the conditions that I will be exposed to.

It is my understanding that my participation in this study is voluntary and I will receive no remuneration for my participation.

It is further my understanding that I may terminate my participation in this study at any time and that any data obtained will be held confidential. I am aware that the researcher has to report to his supervisor and that all data collected will be accessible to the supervisor as well.

Signature of participant:……………………………………………………………………….

Date:………………………………………………………………………………………..

APPENDIX C

CLASS TEST 1 OF SEMESTER 1

MAE 211

CLASS TEST 1 **21 February 2014** **TOTAL: 50**

Question 1

1.1 Determine the equation of the straight line through $(2;1)$ and perpendicular to $3y+2x=6$ (5)

1.2 The point $A(2a;3a)$ lies on the straight line joining the points $P(1;2)\ and\ Q(2;6)$. Calculate the value of a. (5)

1.3 Determine whether the following points are collinear: $A(-2;-6),\ B(2;-4)\ and\ C(4;-3)$ (5)

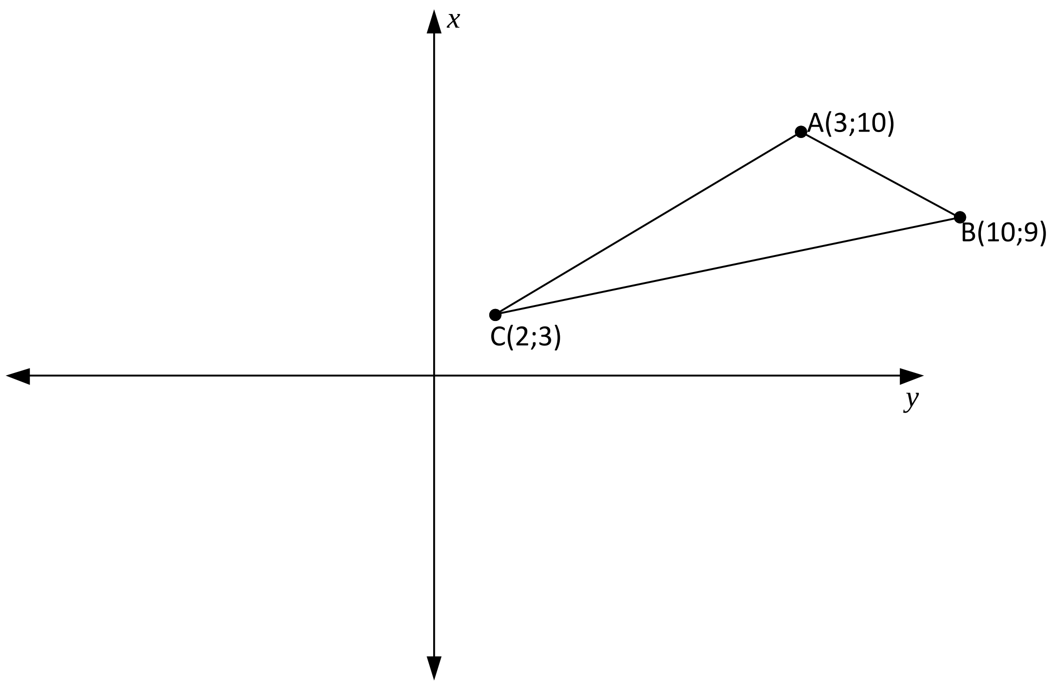
1.4 Determine the value of y if the distance of $A(-4;y)$ from the origin is 5. (4)

1.5 Determine the value of $x\ and\ y$ if $(\frac{3}{2};1)$ is the midpoint of the line segment joining $A(4;-1)\ and\ B(x;y)$ (5)

/24/

Question 2

ABC is a triangle with vertices $A(3;10),\ B(10;9)\ and\ C(2;3)$



2.1 Find the gradient of *AC*. (2)

2.2 Determine the gradient of *AB*. (2)

2.3 Show that triangle *ABC* is right-angled at *A*. (2)

2.4 Find the equation of the line parallel to *AC* and through the point *B*. (3)

2.5 Determine the equation of line *AC*. (2)

2.6 Show that the point $D(1;-4)$ lie on the line *AC*. (2)

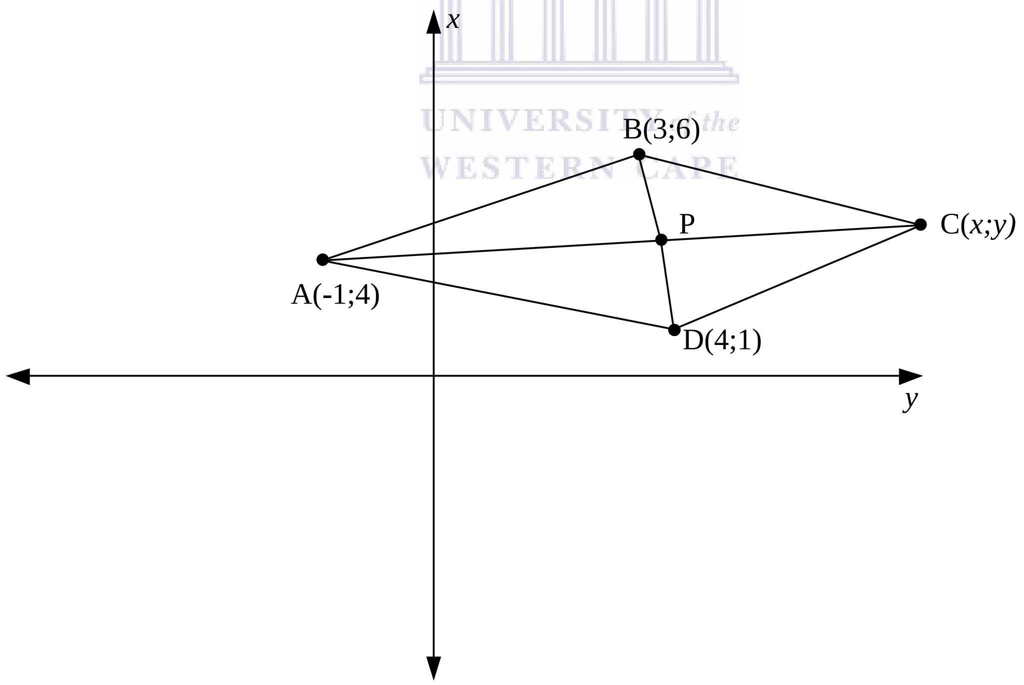
2.7 Determine the coordinates of the midpoint *M* of *CB*. (2)

2.8 Determine the equation of the perpendicular bisector of *CB*. (4)

/19/

QUESTION 3

In the sketch below $ABCD$ is a parallelogram with vertices $A(-1;4)$, $B(3;6)$ and $D(4;1)$and diagonals *AC* and *BD*:



3.1 Determine the length of *AB*. (2)

3.2 Determine the midpoint *P* of *BD*. (2)

3.3 Determine the coordinates of *C*. (5)

/9/

APPENDIX D

CLASS TEST 2 OF SEMESTER 1

MAE 211

CLASS TEST 2 **14 March 2014** **TOTAL: 50**

QUESTION 1

1.1 Determine the equation of the line with y-intercept -2 and passing through the point (2;1). (3)

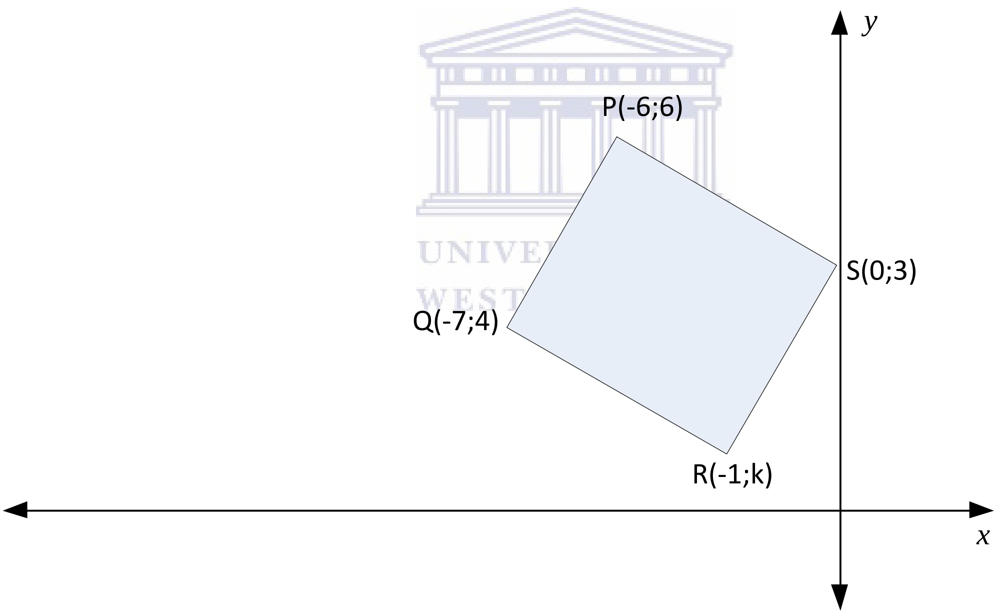
1.2 In the sketch below PQRS is a rectangle. Determine the following:

(a) The length of PS. (2)

(b) The length of QS (2)

(c) The gradient of PQ (2)

(d) The value of k (3)



/12/

QUESTION 2

2.1 Given $f(x) = -x^2 + 2x + 3$

2.1.1 Write $f(x)$ in the form $f(x) = a(x-p)^2 + q$ (3)

2.1.2 Determine the equation of the axis of symmetry (1)

2.1.3 Does the function have a maximum or minimum value? Write an equation for this value. (2)

2.1.4 Write the coordinates of the turning point. (2)

2.2 Determine the equation of a quadratic function with turning point (2;3) and through the point (1;2). (4)

/12/

QUESTION 3

3.1 The figure below shows the graphs of $y = ax^2 + bx + c$ *and* $y = dx + e$

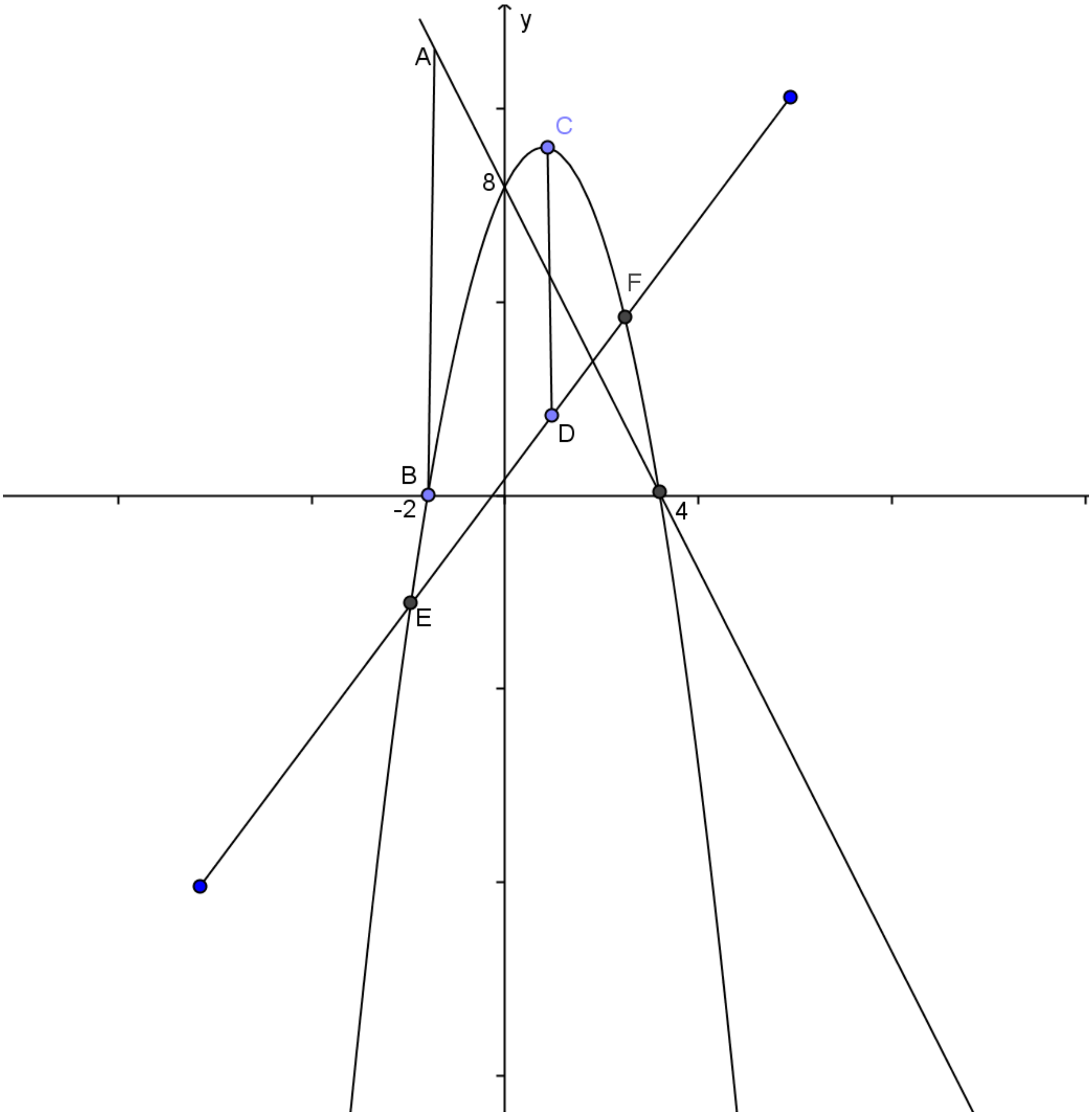
3.1.1 Determine the numerical values of d *and* e (4)

3.1.2 Determine the numerical values of a, b *and* c (6)

3.1.3 Determine the length of AB if AB is parallel to the $y - axis$ (2)

3.1.4 If the equation of line EF is $y = 3x + 2$. Determine the coordinates of the points of intersection of the parabola and line EF. (6)

3.1.5 Determine the length of CD if CD is parallel to the $y - axis$ and C is the turning point (4)



3.2 Draw a rough sketch of $y = ax^2 + bx + c$:
if $a < 0$; $b < 0$; $c > 0$ (4)

/26/

APPENDIX E

CLASS TEST 3 OF SEMEMSTER 1

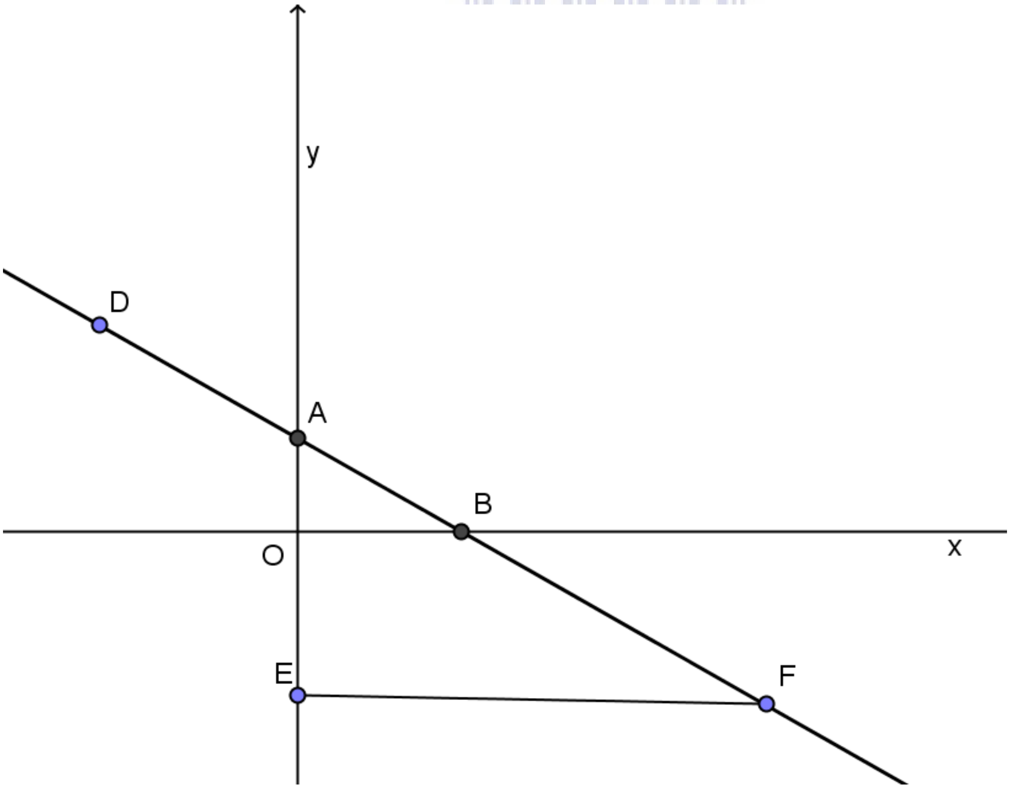
MAE 211

CLASS TEST 3 **11 April 2014** **TOTAL: 50**

QUESTION 1

1. Given line DF with equation: $x + 2y - 2 = 0$ (as shown in the sketch)

1.1 Find the slope of line DF. (2)
1.2 Find the length of OA. (2)
1.3 Determine the length of OB. (2)
1.4 The $y - coordinate$ of E is -1. Find the length of EF if EF is parallel to the $x - axis$. (3)
1.5 Write the coordinates of A and F. (2)
1.6 Determine the midpoint of line AF. (2)
1.7 Determine the length of AF. (2)



/15/

QUESTION 2

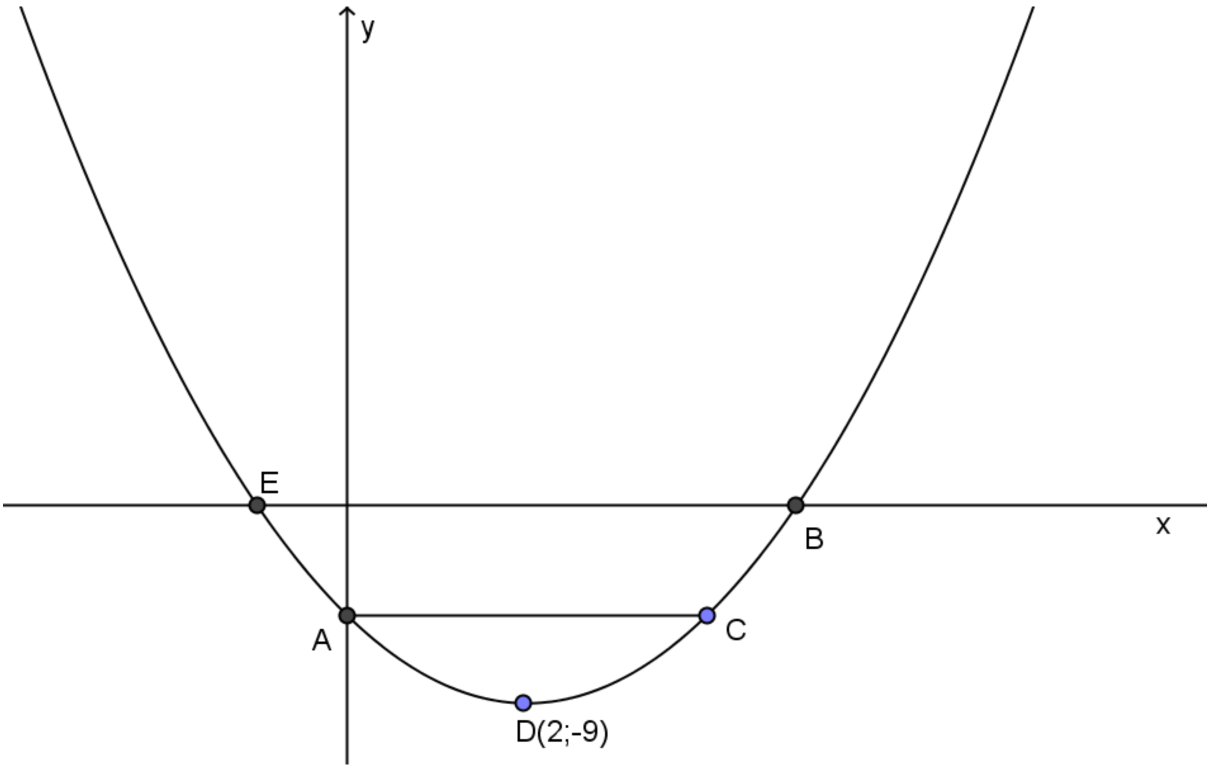
2 In the sketch below $D(2;-9)$ is the turning point and $A(0;-5)$ is the $y-intercept$.

2.1 Show that $f(x) = x^2 - 4x - 5$. (4)

2.2 Determine the coordinates of E. (3)

2.3 Write the equation of the axis of symmetry. (1)

2.4 Determine the coordinates of C if C and A are symmetrical about the axis of symmetry and then determine the distance from A to C. (4)



/12/

QUESTION 3

3.1 Find the value of a if $f(x) = x^3 + ax^2 - x + 5$ is divided by $x - 2$ and gives a remainder of 23. (3)

3.2 If $x - 1$ is a factor of $f(x) = ax^3 - 7x + 3$, determine a. (3)

3.3 If $2x^2 + kx + 4 = (x-2) \cdot Q(x) + 6$. Find $k\ and\ Q(x)$. (6)

3.4 Given $f(x) = x^3 - 2x^2 - 4x + 8$:

(a) Find the $y - intercept$. (1)

(b) Find the $x - intercept(s)$. (3)

(c) Find the turning point(s). (7)

/23/

APPENDIX F

CLASS TEST 4 OF SEMESTER 1

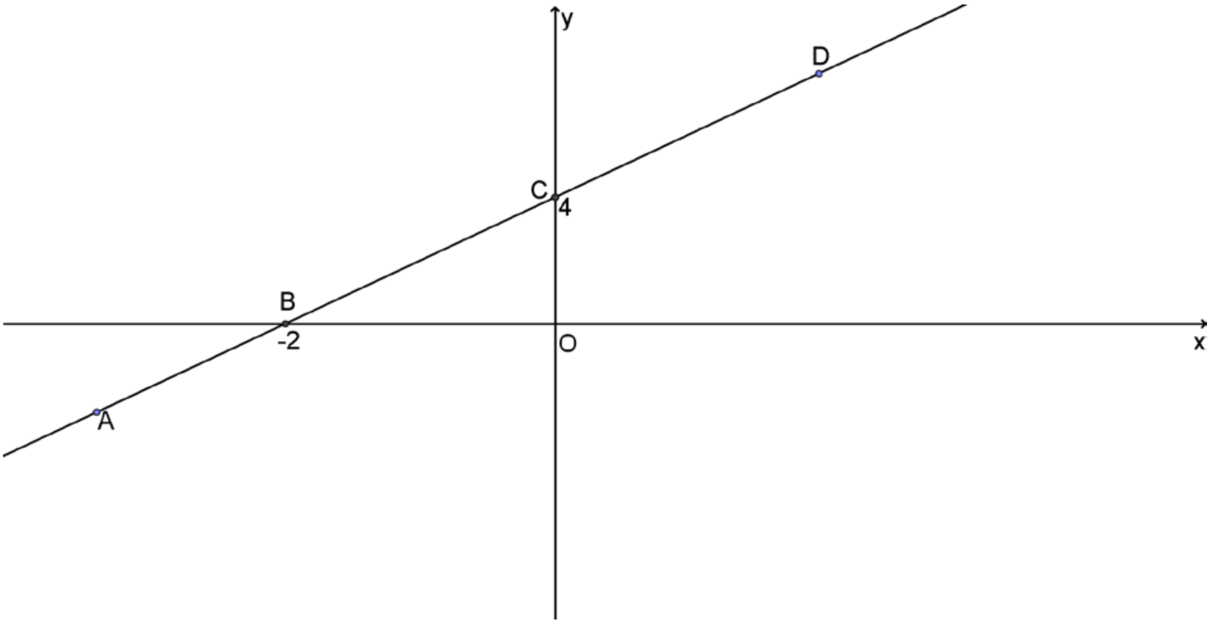
MAE 211

CLASS TEST 4 **2 MAY 2014** **TOTAL: 50**

QUESTION 1

A sketch of line AD is shown below. Use the sketch to answer the following questions:

1.1 Find the equation of line AD. (3)
1.2 Determine the length of BC. (3)
1.3 Determine the equation of the line perpendicular to line AD and through the point (4;-2). (4)

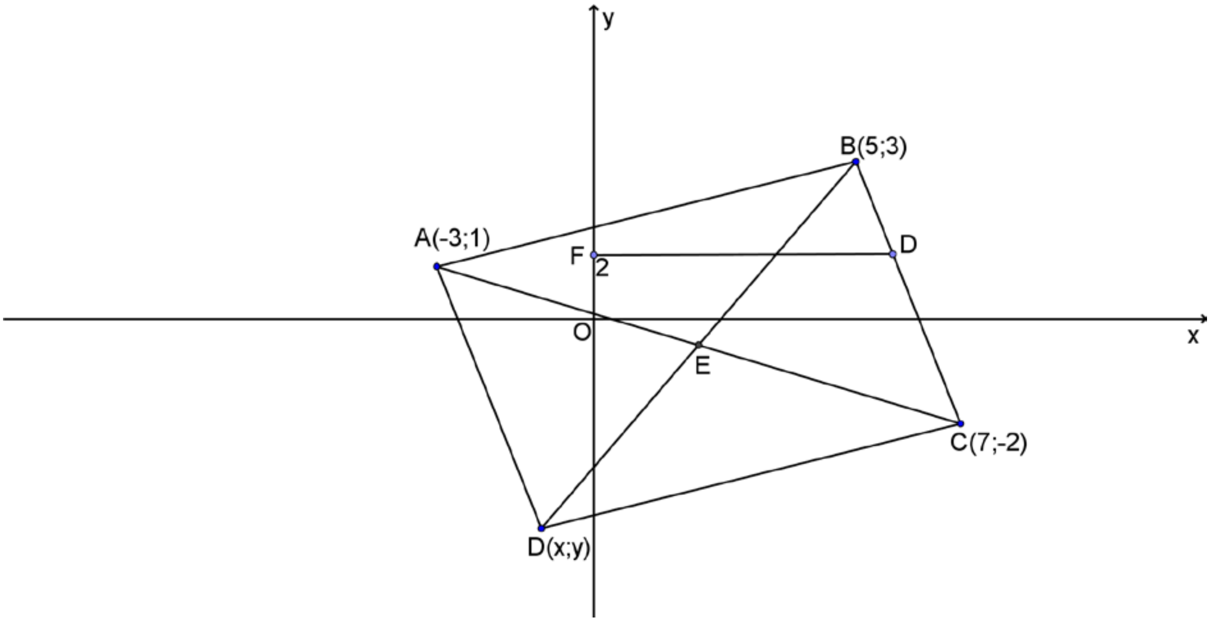


/10/

QUESTION 2

ABCD is a parallelogram with vertices $A(-3;1)$, $B(5;3)$, $C(7;-2)$ and $D(x;y)$. The point E is the midpoint of AC and DB. See sketch below.

2.1 Determine the length of *BC*. (3)
2.2 Determine the midpoint *E* of *AC*. (3)
2.3 Determine the midpoint of line *BD*. (2)
2.4 Determine the coordinates of *D*. (4)
2.5 Determine the length of line *DF* if the $y-coordinate\ of\ F\ is\ 2$ and *DF* is parallel to the $x-axis$. (7)



/19/

QUESTION 3

Given the following rational function: $f(x) = \frac{x^2-9}{x^2-2x-3}$

3.1 Determine the domain of f. (3)

3.2 Find the $y - intercept$ if it exists. (2)

3.3 Find the $x - intercept(s)$ if it exists. (4)

3.4 Find the vertical asymptote(s) if it exists. (4)

3.5 Find the horizontal asymptote(s) if it exists. (2)

3.6 Sketch the graph of f. (6)

/21/

APPENDIX G

CLASS TEST 5 OF SEMESTER 1

MAE 211

CLASS TEST 5 **16 MAY 2014** **TOTAL: 50**

QUESTION 1

1.1 Determine the value of k if $f(x) = x^3 + 6x^2 - kx - 8$ has a remainder of 15 when divided by $x - 2$. (3)

1.2 Determine the value of a if $x - 1$ is a factor of $g(x) = 2x^3 + ax^2 - 8x + 3$. (3)

1.3 Find p if $k + 2$ is a factor of $k^{50} - p^{25}$. (4)

/10/

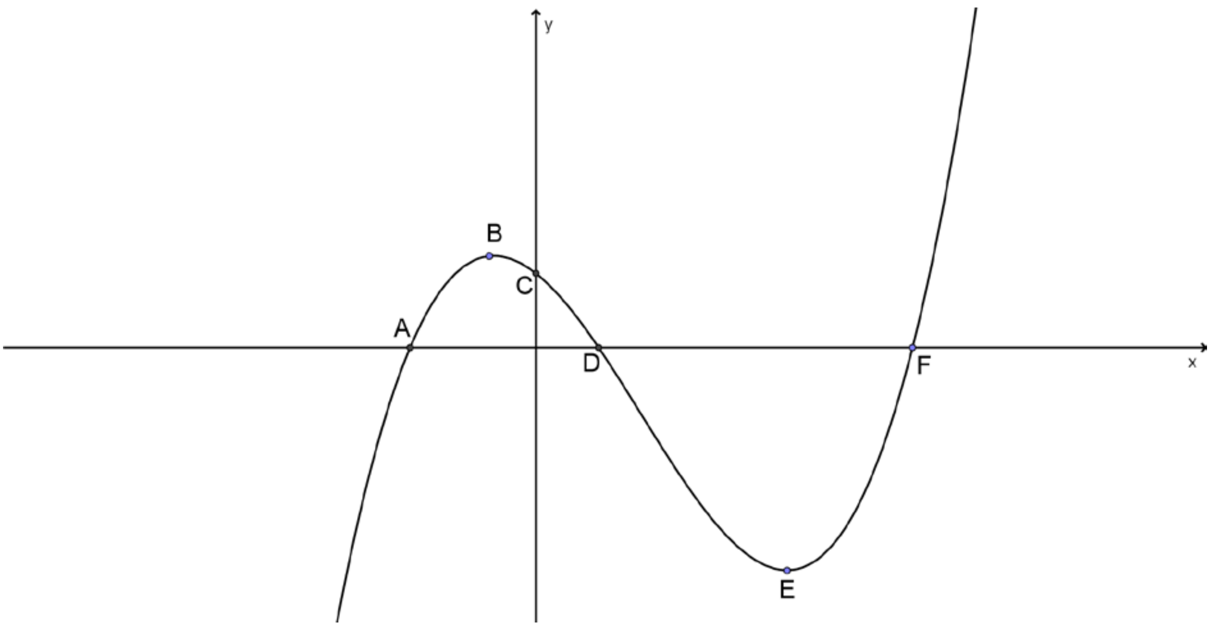
QUESTION 2

The sketch below shows the graph of $f(x) = x^3 - 5x^2 - 8x + 12$.

2.1 Determine the $y - intercept$. (1)

2.2 Determine the $x - intercepts$. (5)

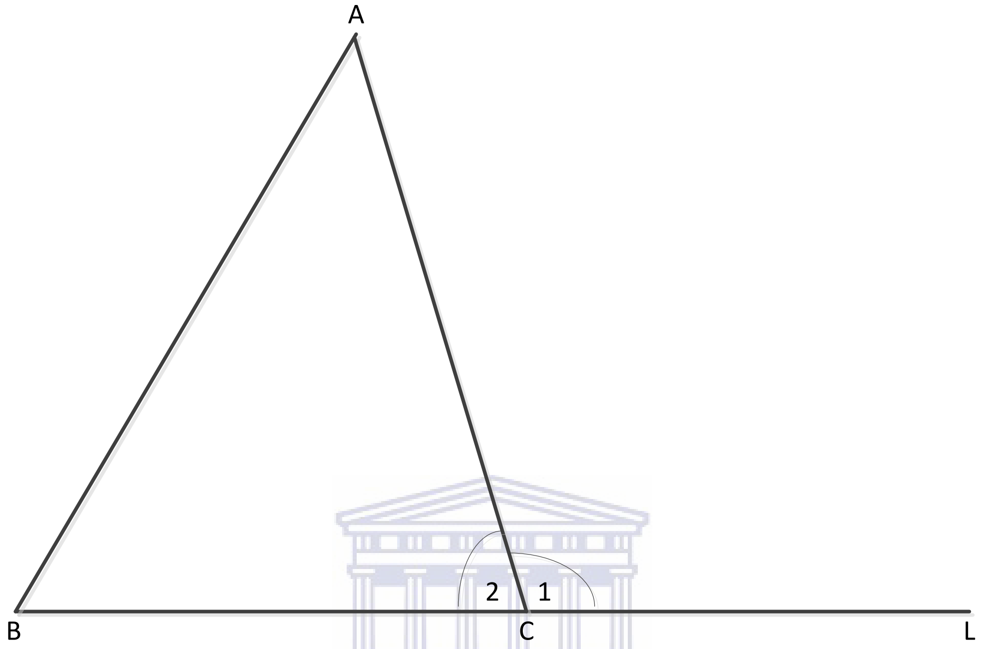
2.3 Determine the coordinates of the turning points B and E. (8)



/14/

QUESTION 3

3.1 Use the provided sketch to prove the theorem which states: *If any side of a triangle is produced, then the exterior angle so formed is equal to the sum of the opposite two interior angles*. You are given ΔABC with side *BC* produced to *L* and exterior angle *ACL* and opposite interior angles $\hat{A}$ and $\hat{B}$. Note angle *ACL* is also known as $\hat{C}_1$.



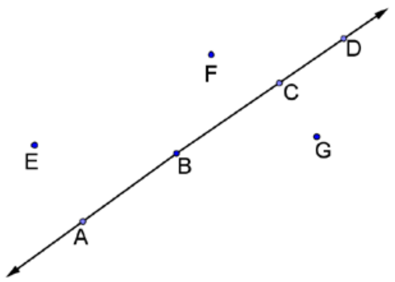
(4)

3.2 What can you deduce from the following?

(a) Two lines l_1 *and* l_2 are coplanar and $l_1 \cap l_2 = \emptyset$ (1)

(b) M and N are two planes such that $M \cap N = \emptyset$ (1)

3.3 Refer to the diagram and complete the following statements based on it:



(a) $\overline{AB} + \overline{BC} =$……….. (1)

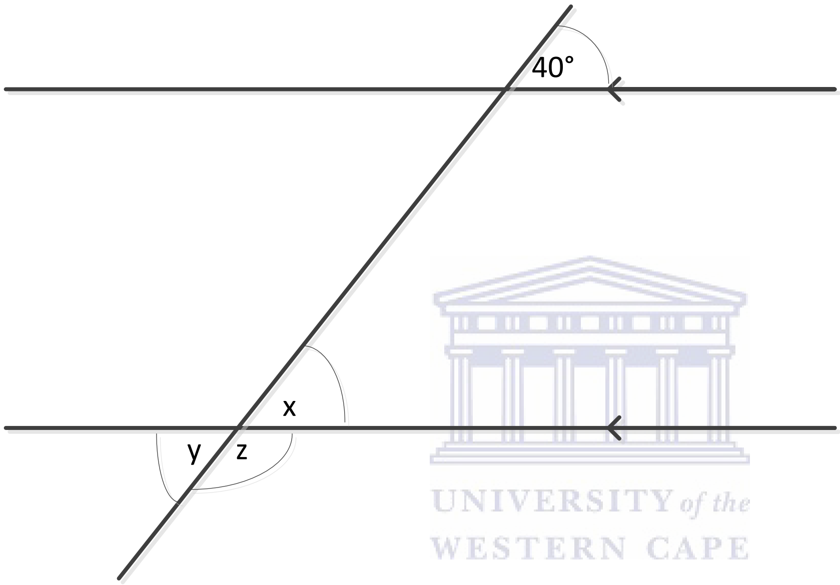
(b) $\overline{AB} \cup \overline{BC} =$………. (1)

(c) $\overline{AB} \cap \overline{BA} =$……….. (1)

(d) $\overrightarrow{BA} \cup \overrightarrow{BD} =$………. (1)

(e) $\overrightarrow{BA} \cap \overrightarrow{BD} =$……… (1)

3.4 Use the provided sketch to determine the following (provide reasons for your answers):

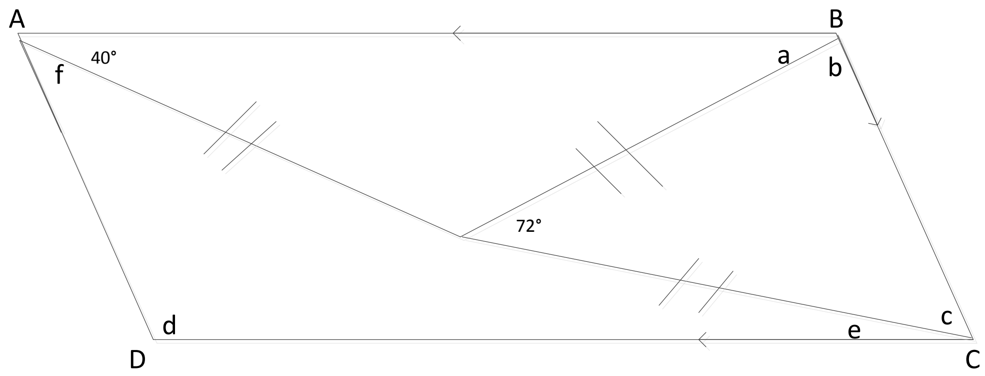


(a) x (1)

(b) y (1)

(c) z (1)

3.5 ABCD is a parallelogram. Use the sketch to answer the questions that follow:



(a) Determine the value of a. (1)

(b) Determine the value of b (3)

(c) Determine the value of c. (1)

(d) Determine the value of d. (2)

(e) Determine the value of e. (3)

(f) Determine the value of f. (2)

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APPENDIX H

END OF MODULE EXAMINATION SEMESTER 1

UNIVERSITY OF THE WESTERN CAPE

FINAL ASSESMENT

MAY / JUNE 2014

MODULE NAME : MATHEMATICS (B.ED) 211
MODULE CODE : MAE 211

DURATION : 3 HOURS **MARKS : 100**

STUDENTS WILL BE NOTIFIED VIA THEIR G – MAIL ACCOUNT WHETHER THEY QUALIFY FOR A SPECIAL OR SUPPLEMENTARY ASSESSMENT IN A PARTICULAR MODULE

STUDENTS CAN ALSO CONSULT THE DEPARTMENTAL NOTICE BOARD TO ASCERTAIN WHETHER THEY QUALIFY FOR A SPECIAL OR SUPPLEMENTARY ASSESSMENT IN A PARTICULAR MODULE.

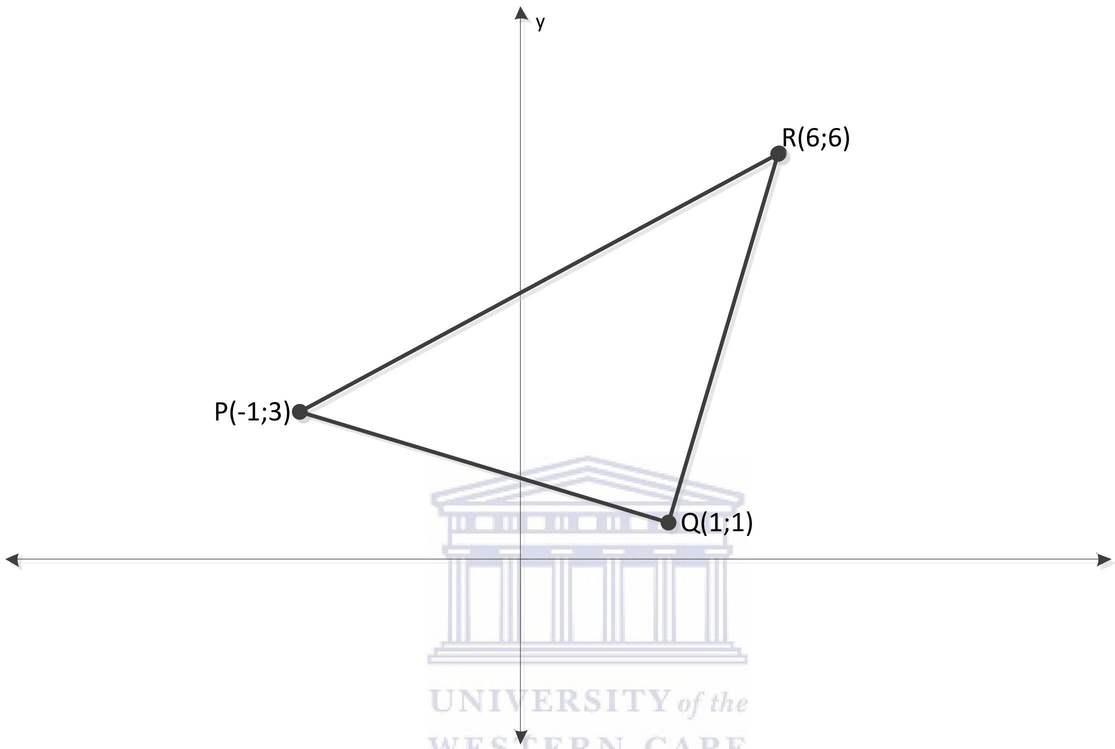
ACADEMIC DEPARTMENTS MUST PUBLISH A LIST OF STUDENTS WHO QUALIFY TO WRITE THE ABOVE MENTIONED CATEGORIES OF ASSESSMENTS WITHIN 48 HOURS BEFORE THE WRITING TIME OF A MODULE.

LECTURER(S) INSTRUCTIONS TO STUDENTS (OPTIONAL):

1. Answer **ALL** questions.
2. Make sure that you write your *name*, *surname* and *student number* in the appropriate spaces on your answer book.

QUESTION 1

PQR is a triangle with vertices $P(-1;3)$, $Q(1;1)$ and $R(6;6)$.



1.1	Determine the gradient of PQ.	(2)
1.2	Determine the gradient of QR.	(2)
1.3	Show that triangle PQR is right-angled at Q.	(2)
1.4	Determine the equation of line PQ.	(2)
1.5	Determine the equation of the line parallel to PQ and through the point R.	(3)
1.6	Show that the point (-3;5) lie on the line PQ.	(2)
1.7	Determine the coordinates of the midpoint M of PR.	(2)
1.8	Determine the equation of the perpendicular bisector of PR.	(4)

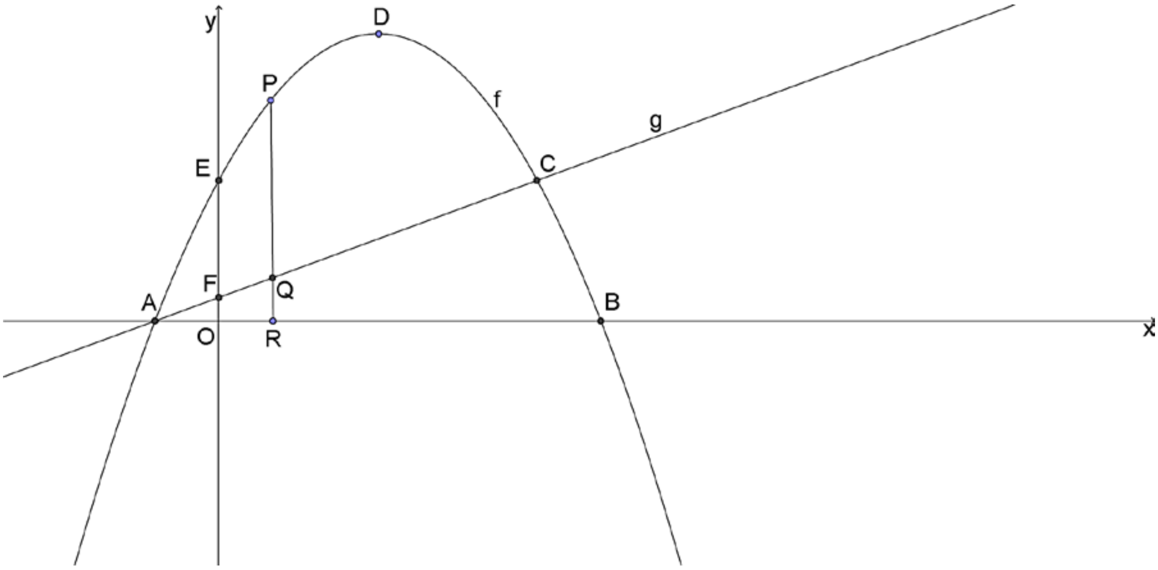
/19/

QUESTION 2

2.1 The sketch shows the graphs of two functions:

$$\boldsymbol{f(x) = -x^2 + 10x + 24 \ \ and \ \ g(x) = 2x + 4}$$

2.1.1	Determine the coordinates of the turning point D.	(4)
2.1.2	Determine the length of AB.	(4)
2.1.3	Determine the coordinates of C.	(5)
2.1.4	Determine the length of PQ if $\boldsymbol{OR = 3}$. [PQ is parallel to the $\boldsymbol{y-axis}$]	(3)



2.2 Use the following information to draw a rough sketch of $\boldsymbol{y = ax^2 + bx + c}$ if:
$\boldsymbol{a < 0,\ b < 0,\ c < 0,\ and\ b^2 - 4ac < 0}$ (4)

/20/

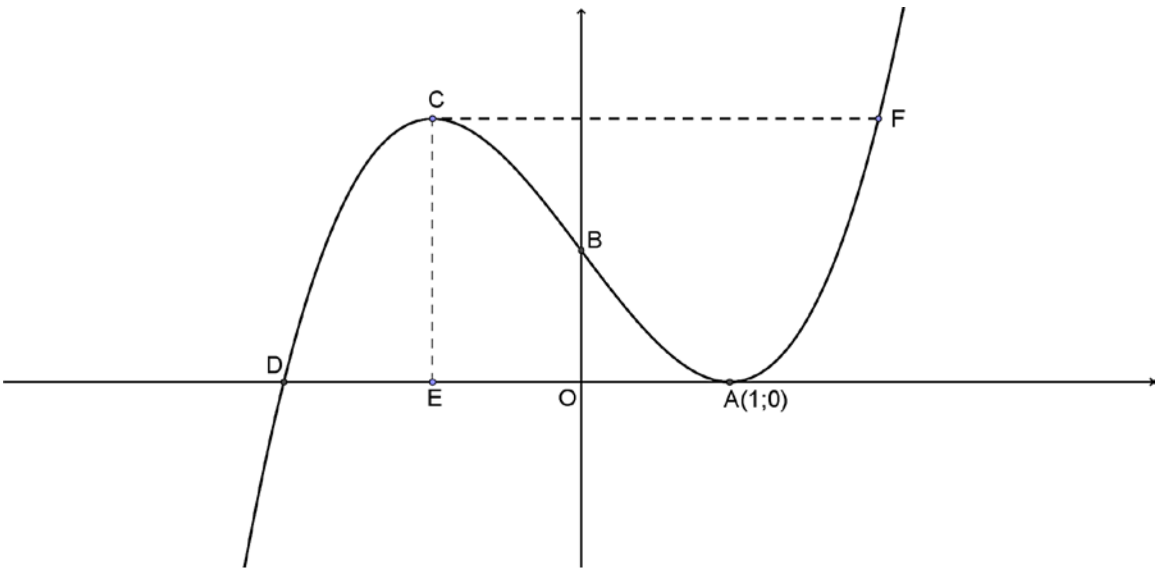
QUESTION 3

3.1 If $\boldsymbol{x - 1}$ is a factor of $\boldsymbol{f(x) = x^4 - 15x^2 + px + 24}$ determine $\boldsymbol{p}$. (3)

3.2 If $\boldsymbol{2x^3 + x^2 - kx + 1 = (x - 1) \cdot Q(x) + 1}$. Find $\boldsymbol{k\ and\ Q(x)}$. (6)

3.3 The sketch below shows the graph of $\boldsymbol{g(x) = x^3 - 3x + 2}$. $\boldsymbol{A\ and\ C}$ are turning points.

3.3.1 Determine the length of OB. (1)
3.3.2 Determine the length of OD. (3)
3.3.3 Determine the coordinates of C. (4)
3.3.4 Determine the length of EA. (2)
3.3.5 Determine the length of CF if CF is parallel to the $\boldsymbol{x-axis}$. (4)



/23/

QUESTION 4

Given the rational function : $\boldsymbol{f(x)} = \frac{x^2+3x}{x^2+x-6}$

4.1 Determine the domain of the function. (3)

4.2 Determine the $\boldsymbol{x-}$ $\boldsymbol{and}$ $\boldsymbol{y-intercepts}$ if it exists. (4)

4.3 Determine the vertical asymptote(s) if it exists. (4)

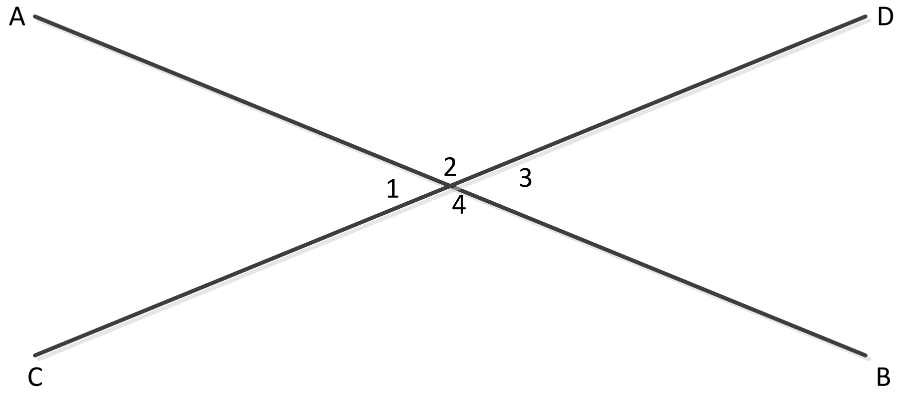
4.4 Determine the horizontal asymptotes if it exists. (2)

4.5 Sketch the graph of $\boldsymbol{f}$. (5)

/18/

QUESTION 5

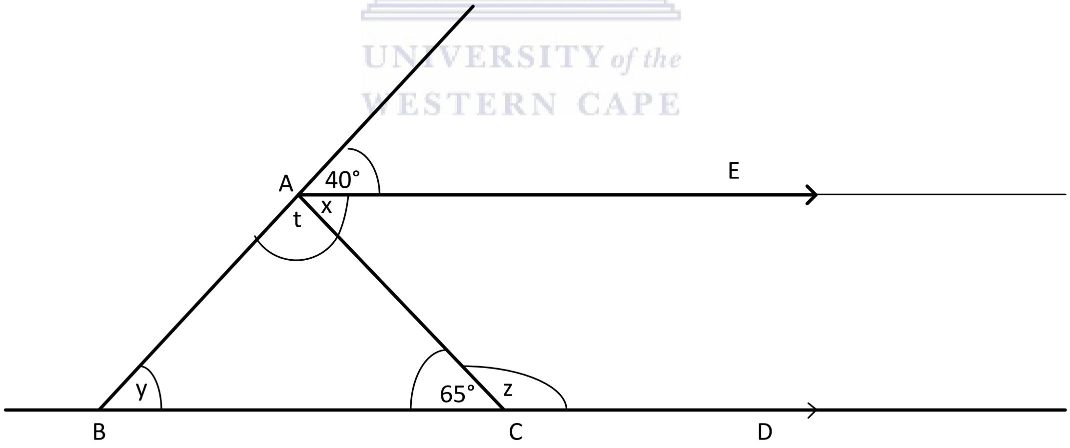
5.1 Use the sketch below to prove the theorem that states: If two straight lines intersect then the vertically opposite angles are equal.



(4)

5.2 Use the sketch below to determine the following: [Provide reasons for your answers]

5.2.1 x (2)
5.2.2 y (2)
5.2.3 z (2)
5.2.4 t (2)



5.3 ABCD is a parallelogram, AE = FC. Prove that EBFD is a parallelogram.

(8)

/20/

APPENDIX I
CLASS TEST 1 OF SEMESTER 2

MAE 221

CLASS TEST 1 12 AUGUST 2014 TOTAL: 40

1. Use the graph of function f to describe the transformation that yields the graphs of g:
   (a) $f(x) = 2^x \quad and \quad g(x) = 2^{x+2} + 3$ (2)
   (b) $f(x) = (\frac{3}{4})^x \; and \;\; g(x) = -(\frac{9}{4})^{-x+1}$ (2)

2. Use the one-to-one property to solve the equation for x:
   (a) $e^{x^2+6} = e^{5x}$ (4)
   (b) $0.25^x = 4$ (4)

3. On the day of a child's birth her father deposits an amount of R50 000 into a banking account, that pays 5% interest compounded continuously. Determine the amount in the account on the child's 21st birthday. (3)

4. Use the properties of logarithms to simplify the expression:

   (a) $\log_y y^3$ (2)
   (b) $5^{\log_5 10}$ (1)

5. Given the following logarithmic function: $f(x) = -\log_2(x+3)$
   Determine the following:
   (a) The domain of the function (2)
   (b) The $x - intercept$ (2)
   (c) The $y - intercept \;\; if \; it \; exists$ (2)
   (d) asymptote (1)
   (e) sketch the graph (4)

6. Find the exact value of the logarithmic expression without using a calculator:

   (a) $\log_3 81^{-3}$ (3)
   (b) $6 \ln e^2 - \ln e^4$ (3)

7. Condense the expression to the logarithm of a single quantity:
   (a) $\ln x - [\ln(x+1) + \ln(x-1)]$ (2)

   (b) $\frac{1}{2}[\log_4 x + 2\log_4(x+4) - \log_4(x-1)]$ (3)

APPENDIX J
CLASS TEST 2 OF SEMESTER 2

MAE 221

CLASS TEST 2 [MATRICES] **TOTAL: 40**

1. Given the matrix: $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 5 & 5 \\ 2 & 6 & 3 \end{bmatrix}$

   (a) Perform the following sequence of row operations on $matrix\ A$:

   (i) $R_2 - R_1$

   (ii) $R_3 - 2R_1$

   (iii) $\frac{1}{2} R_2$

   (iv) $R_2 - 2R_3$

   (v) $R_1 - 3R_2$

   (vi) $R_1 - R_3$ (6)

   (b) What did the operations accomplish? (1)

2. Given the following system of linear equations: $\begin{Bmatrix} x + y - z = 2 \\ 2x + 3y - z = 7 \\ 3x - 2y + z = 9 \end{Bmatrix}$

   (a) Write an augmented matrix for the system. (2)

   (b) Use Gauss-Jordan elimination to solve the system of equations. (10)

3. Find the determinant of the following matrix: $M = \begin{bmatrix} 3 & 3 \\ -1 & 0 \end{bmatrix}$ (1)

4. Given the matrix: B = $\begin{bmatrix} -2 & 4 & 1 \\ -6 & 0 & 2 \\ 5 & 3 & 4 \end{bmatrix}$. Find the determinant of B by expanding by cofactors on the row or column that would make the computations easiest. (8)

5. Find the determinants to verify the equations:

   (a) $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & b + ka \\ c & d + kc \end{vmatrix}$ (4)

   (b) $\begin{vmatrix} x & y \\ z & w \end{vmatrix} = -\begin{vmatrix} z & w \\ x & y \end{vmatrix}$ (4)

6. Solve for x: $\begin{vmatrix} x-1 & 2 \\ 3 & x-2 \end{vmatrix} = 6$ (5)

APPENDIX K

CLASS TEST 3 OF SEMESTER 2

MAE 221

CLASS TEST 3 **3 OCTOBER 2014** **TOTAL: 40**

8. Use the graph of function f to describe the transformation that yields the graphs of g:

(c) $f(x) = (\frac{3}{2})^x \quad and \quad g(x) = -(\frac{3}{2})^{x+2} + 3$ (2)

(d) $f(x) = logx \; and \quad g(x) = 3 - lox(x-2)$ (2)

2. Draw sketch graphs on the same system of axes of the functions given by $f(x) = 2^x \; and \; h(x) = 3^x$ and use the graphs to solve the inequalities:

(a) $3^x < 2^x$

(b) $3^x > 2^x$ (4)

3. Is the following an equation of a circle? (Answer yes or no).

(a) $x^2 - y^2 - 16 = 0$ (1)

(b) $3x^2 + 2y^2 = 12$ (1)

4. Determine the equation of the circle with centre at the origin and which passes through (2;3). (3)

5. Determine the equation of the circle with centre (1;-2) and passing through the point (-2;-1). (4)

6. If $(a; 3)$ is a point on the circle $x^2 + y^2 = 25$. Determine all possible values of a. (3)

7. Determine the coordinates of the centre and the radius of the following circle:

$x^2 - 4x + y^2 + 6y = -4$ (4)

8. Find the equations of the tangents to $x^2 + (y+1)^2 = 20$ which are parallel to $y + 2x = 0$. (16)

APPENDIX L
CLASS TEST 4 OF SEMESTER 2

MAE 221

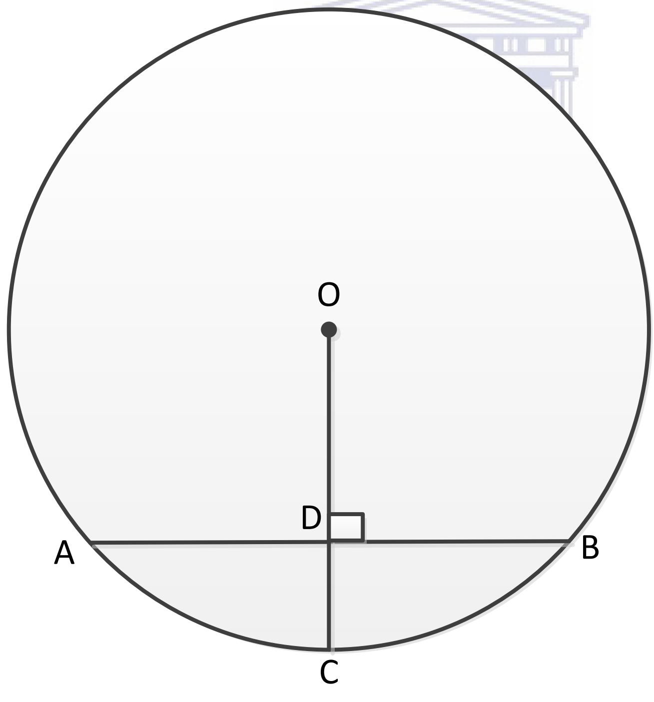
CLASS TEST 4 17 OCTOBER 2014 TOTAL: 40

1. Use Gauss-Jordan elimination to solve the following system of equations:

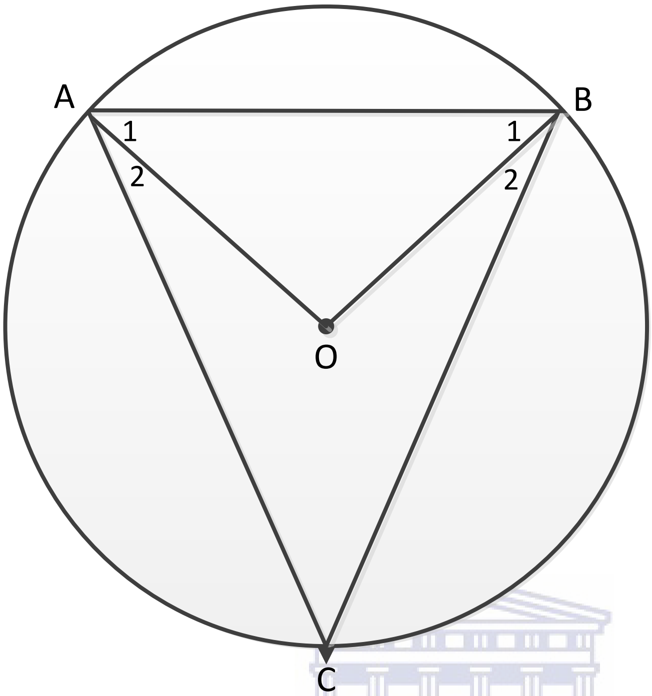
$x + 3y + z = 3$
$x + 5y + 5z = 1$
$2x + 6y + 3z = 8$ (6)

2. Solve for x:
$\begin{vmatrix} x-2 & 1 \\ -3 & x \end{vmatrix} = 6$ (4)

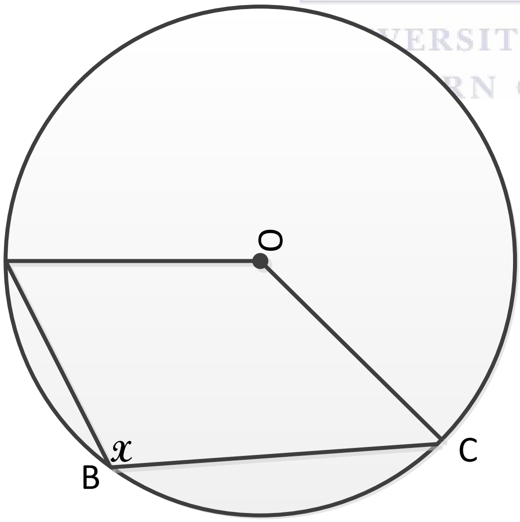
3. If $AB = 8mm$ and $DC = 2mm$ and $OD \perp AB$. Find the length of OD. (See sketch below)



4. In the sketch below O is the centre of the circle. Use the sketch to prove that: $\angle B_1 + \angle C = 90°$ **(5)**

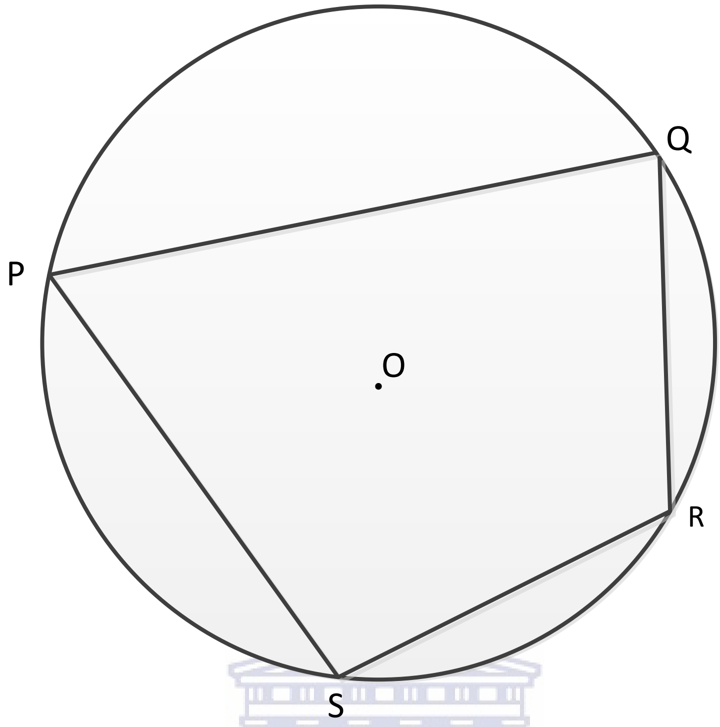


5. Determine the value of x: O is the centre of the circle.



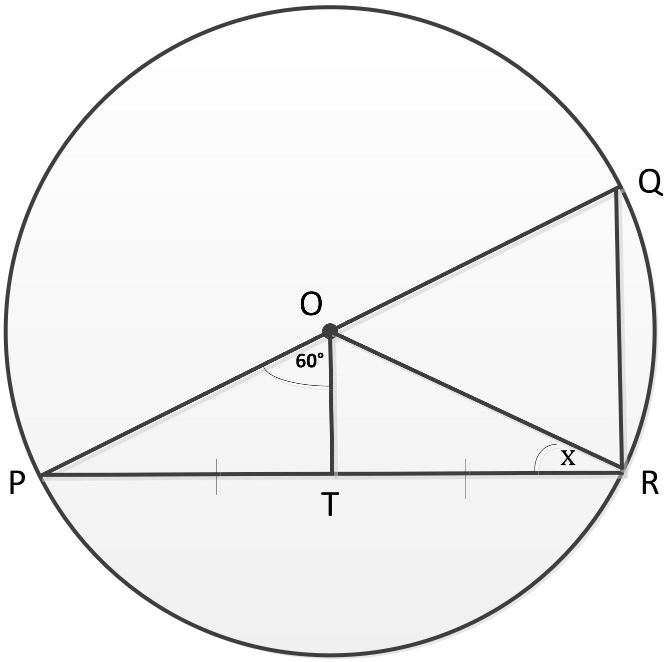
(4)

6. Use the provided sketch to prove the theorem which states: If a quadrilateral is cyclic then the opposite angles are supplementary. O is the centre of the circle.



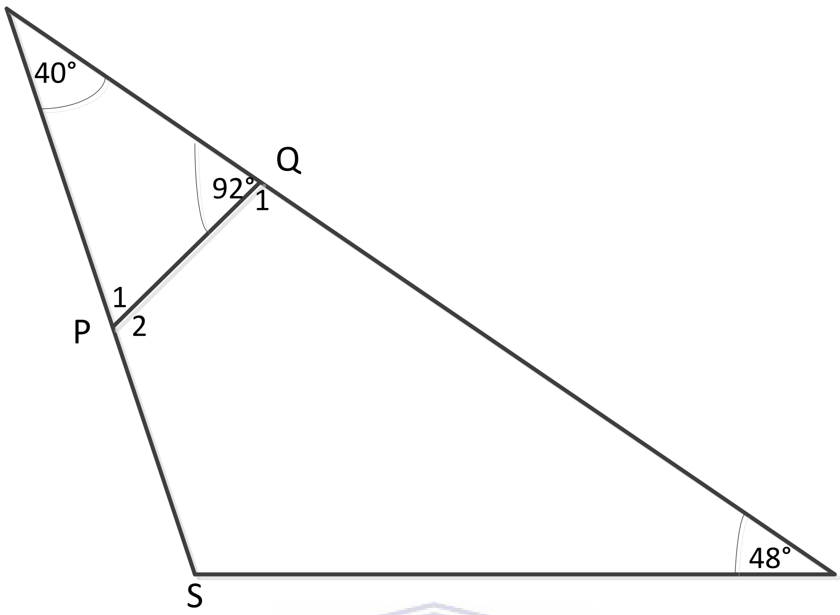
(6)

7. PQ is a diameter ,PR is a chord and $PT = TR$ in the sketch below. Determine the value of x.



(4)

8. Use the sketch below to prove that PQRS is a cyclic quadrilateral.



APPENDIX M

END OF MODULE EXAMINATION SEMESTER 2

UNIVERSITY OF THE WESTERN CAPE

FINAL ASSESMENT

OCTOBER / NOVEMBER 2014

MODULE NAME : MATHEMATICS (B.ED) 221

MODULE CODE : MAE 221

DURATION : 3 hours **MARKS : 100**

EXAMINER: B. MAY

DEPARMENTAL CHAIRPERSON: M. MBEKWA

INTERNAL MODERATOR: M. MBEKWA

STUDENTS WILL BE NOTIFIED VIA THEIR G – MAIL ACCOUNT WHETHER THEY QUALIFY FOR A SPECIAL OR SUPPLEMENTARY ASSESSMENT IN A PARTICULAR MODULE

STUDENTS CAN ALSO CONSULT THE DEPARTMENTAL NOTICE BOARD TO ASCERTAIN WHETHER THEY QUALIFY FOR A SPECIAL OR SUPPLEMENTARY ASSESSMENT IN A PARTICULAR MODULE.

ACADEMIC DEPARTMENTS MUST PUBLISH A LIST OF STUDENTS WHO QUALIFY TO WRITE THE ABOVE MENTIONED CATEGORIES OF ASSESSMENTS WITHIN 48 HOURS BEFORE THE WRITING TIME OF A MODULE.

LECTURER(S) INSTRUCTIONS TO STUDENTS (OPTIONAL):

1. Answer **ALL** questions
2. Make sure that you write your *name*, *surname* and *student number* in the appropriate spaces on your answer book.

QUESTION 1

1.1 Use the graph of f to describe the transformation that yields the graph of g.

(a) $f(x) = 3^x$; $g(x) = 1 - 3^{x+1}$ (2)
(b) $f(x) = 10^x$; $g(x) = 10^{-x-1} + 2$ (2)

1.2 Use the one-to-one property of functions to solve the following equations for x:

(a) $e^{x^2-3} = e^{2x}$ (4)
(b) $(0.5)^x = 32$ (4)

1.3 Given the logarithmic function: $\log(-x-5)$. Determine the following:

(a) The domain of the function. (2)
(b) The $x - intercept$. (2)
(c) The equation of the asymptote. (1)

1.4 Condense the following expressions to the logarithm of a single quantity:

(a) $5\log(x^2+10) - 6\log x$ (3)
(b) $[\ln(x-1) - \ln(x+1)] - \ln(x+1)$ (2)

1.5 If R75 000 is invested at in interest rate of 12% compounded continuously, determine the amount of the investment at the end of 5 years. (3)

/25/

QUESTION 2

2.1 Write the augmented matrix for the following system of equations and then use Gauss-Jordan elimination to solve the system of equations:

$$-x + y + 2z = 1$$
$$2x + 3y + z = -2$$
$$5x + 4y + 2z = 4$$ (8)

2.2 Solve for x:

$$\begin{vmatrix} x-3 & 5 \\ 2 & x+2 \end{vmatrix} = 4x - 10$$ (5)

2.3 Find the determinant of the matrix, by expanding by cofactors on the row or column that makes the computations the easiest.

$$A = \begin{bmatrix} -2 & 4 & 1 \\ -6 & 0 & 2 \\ 5 & 3 & 4 \end{bmatrix}$$ (8)

/21/

QUESTION 3

ABCD is a square with vertices $A(-2;4)$, $B(6;3)$, $C(7;11)$ and $D(x;y)$. The point G is the midpoint of AC and DB. (See sketch below).

3.1 Determine the length of A*B*. (3)

3.2 Determine the midpoint *G* of *AC*. (2)

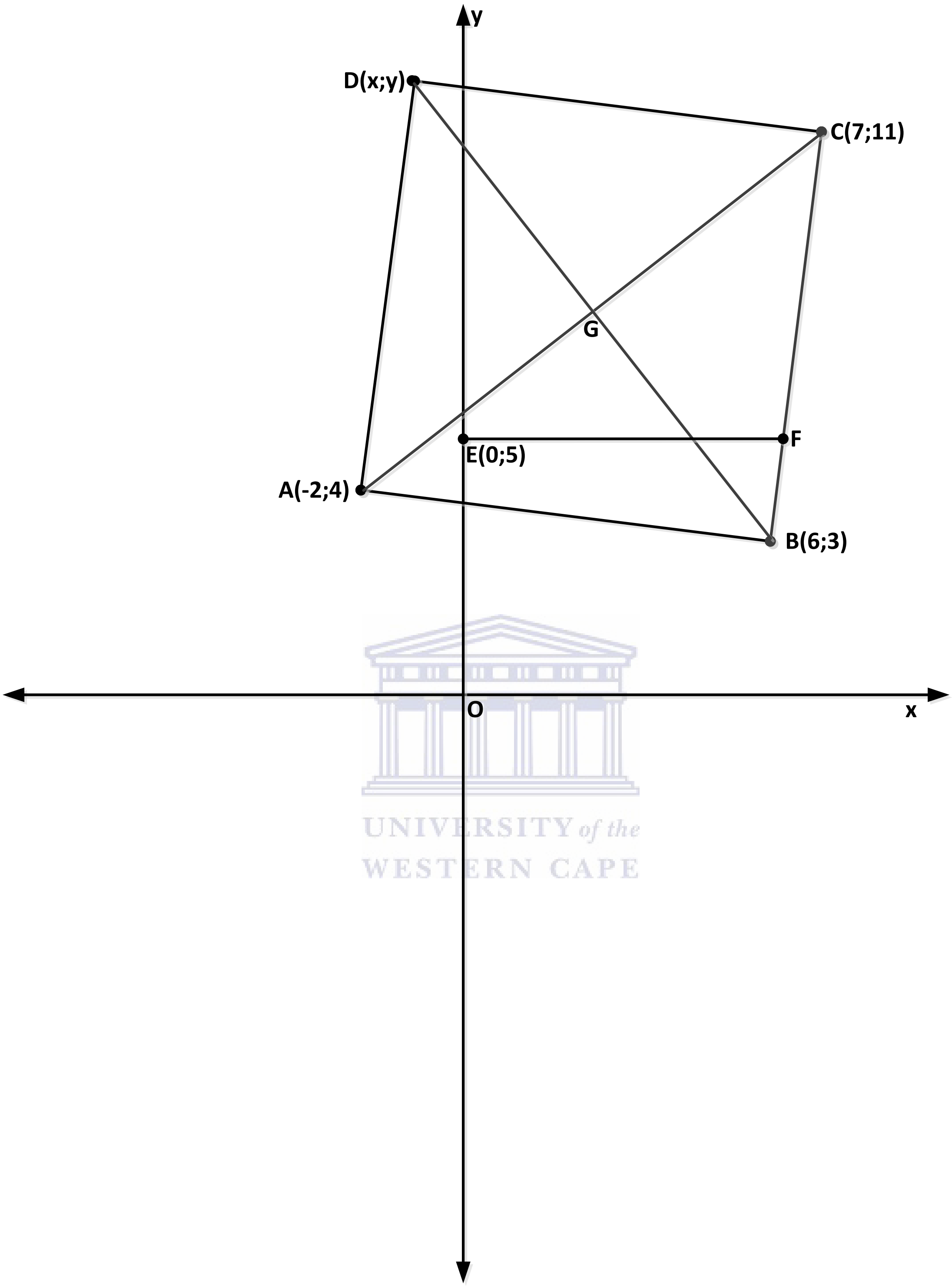
3.3 Determine the midpoint of line *BD*. (2)

3.4 Determine the coordinates of *D*. (4)

3.5 Calculate the gradient of *BC* (3)

3.6 Determine the equation of line *BC* (3)

3.7 Determine the length of line *EF* if the $y - coordinate\ of\ E\ is\ 5$ and *EF* is parallel to the $x - axis$. (3)



/20/

QUESTION 4

4.1 Determine the centre and radius of the following circle:
$x^2 - 2x + y^2 - 7 = 0$ (4)

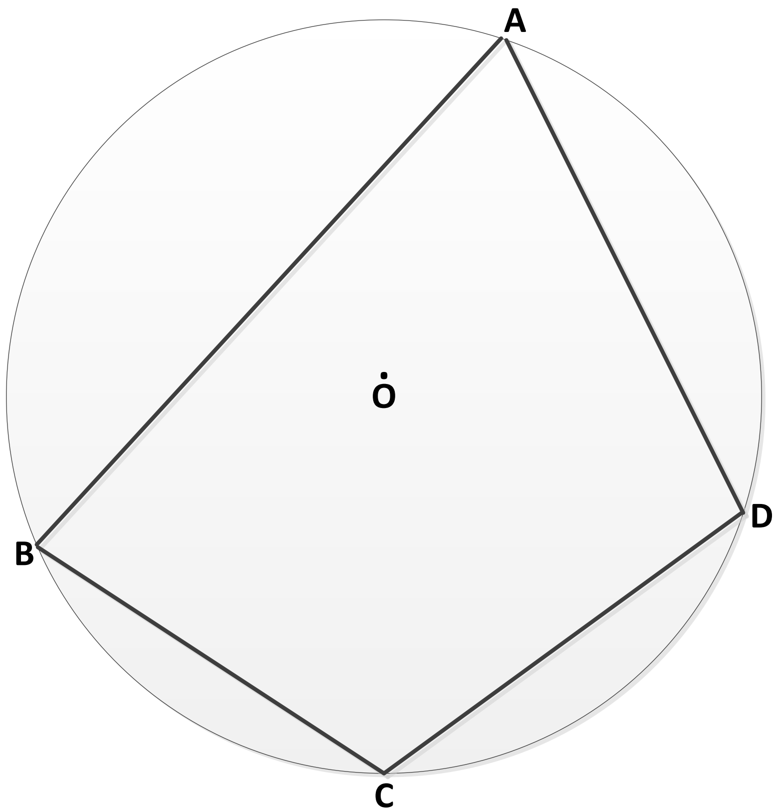
4.2 Find the equation of the tangents to the circle in 4.1 which are parallel to $y - x + 3 = 0$ (14)

4.3 Do the tangents that you determined in 4.2 intersect at some point (yes or no)? Explain your reasoning. (3)

/21/

QUESTION 5

5.1 Use the sketch below (circle with centre O) to prove the theorem which states: *If a quadrilateral is cyclic, then the opposite angles are supplementary.*

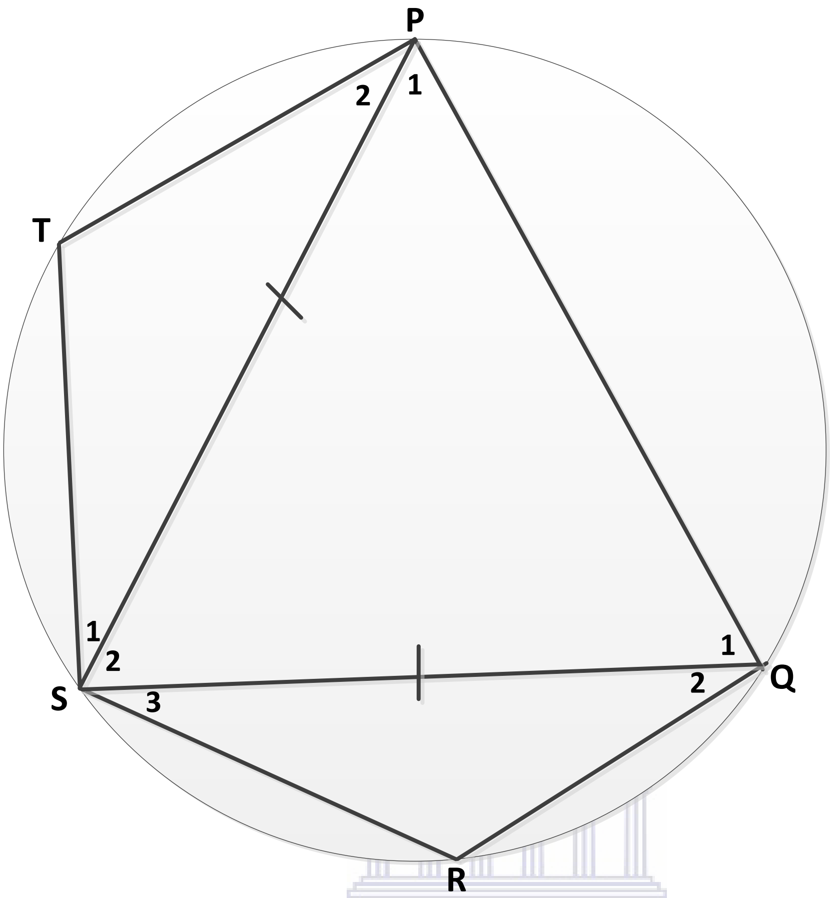


(5)

5.2 In the sketch below: $PS = QS$ and $\angle S_1 = \angle S_3$

a) Prove that $\angle T = \angle R$ (4)

b) Prove that $TS = RS$ (4)



/13/

[TOTAL:100]

APPENDIX N
MATHEMATICAL TEXT TO TEST FOR SELF-EXPLANATION ABILITIES

hat any integer n can be written in one of the three

$n = 3q$ or $n = 3q + 1$ or $n = 3q + 2$

me integer q.

e the quotient-remainder theorem with $d = 3$ to prove
it the product of any three consecutive integers is
visible by 3.

se the *mod* notation to rewrite the result of part (a).

se the quotient-remainder theorem with $d = 3$ to prove
iat the square of any integer has the form $3k$ or $3k + 1$
or some integer k.

Jse the *mod* notation to rewrite the result of part (a).

Jse the quotient-remainder theorem with $d = 3$ to prove
that the product of any two consecutive integers has the
form $3k$ or $3k + 2$ for some integer k.

Use the *mod* notation to rewrite the result of part (a).

33, you may use the properties listed in Example 4.2.3.

Prove that for all integers m and n, $m + n$ and $m - n$ are either both odd or both even.

. Find all solutions to the equation $m^2 - n^2 = 56$ for which both m and n are positive integers.

. Find all solutions to the equation $m^2 - n^2 = 88$ for which both m and n are positive integers.

Given any integers a, b, and c, if $a - b$ is even and $b - c$ is even, what can you say about the parity of $2a - (b + c)$? Prove your answer.

Given any integers a, b, and c, if $a - b$ is odd and $b - c$ is even, what can you say about the parity of $a - c$? Prove your answer.

Given any integer n, if $n > 3$, could n, $n + 2$, and $n + 4$ all be prime? Prove or give a counterexample.

ove each of the statements in 35–46.

. The square of any integer has the form $4k$ or $4k + 1$ for some integer k.

5. The product of any four consecutive integers is divisible by 8.

7. The fourth power of any integer has the form $8m$ or $8m + 1$ for some integer m.

38. For any integer n, $n^2 + 5$ is not divisible by 4.

39. The sum of any four consecutive integers has the form $4k + 2$ for some integer k.

40. For all integers m, $m^2 = 5k$, or $m^2 = 5k + 1$, or $m^2 = 5k + 4$ for some integer k.

41. For any integer n, $n(n^2 - 1)(n + 2)$ is divisible by 4.

H 42. Every prime number except 2 and 3 has the form $6q + 1$ or $6q + 5$ for some integer q.

43. If n is an odd integer, then n^4 *mod* $16 = 1$.

H 44. For all real numbers x and y, $\lfloor x \rfloor \cdot \lfloor y \rfloor = \lfloor xy \rfloor$.

45. For all real numbers r and c with $c \geq 0$, if $-c \leq r \leq c$, then $|r| \leq c$.

46. For all real numbers r and c with $c \geq 0$, if $|r| \leq c$, then $-c \leq r \leq c$.

47. A matrix **M** has 3 rows and 4 columns.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

The 12 entries in the matrix are to be stored in *row major* form in locations 7,609 to 7,620 in a computer's memory. This means that the entries in the first row (reading left to right) are stored first, then the entries in the second row, and finally the entries in the third row.

a. Which location will a_{22} be stored in?

b. Write a formula (in i and j) that gives the integer n so that a_{ij} is stored in location $7{,}609 + n$.

c. Find formulas (in n) for r and s so that a_{rs} is stored in location $7{,}609 + n$.

48. Let **M** be a matrix with m rows and n columns, and suppose that the entries of **M** are stored in a computer's memory in row major form (see exercise 47) in locations $N, N + 1, N + 2, \ldots, N + mn - 1$. Find formulas in k for r and s so that a_{rs} is stored in location $N + k$.

∗49. If m, n, and d are integers, $d > 0$, and m *mod* $d = n$ *mod* d, does it necessarily follow that $m = n$? That $m - n$ is divisible by d? Prove your answers.

∗50. If m, n, and d are integers, $d > 0$, and $d \mid (m - n)$, what is the relation between m *mod* d and n *mod* d? Prove your answer.

∗51. If m, n, a, b, and d are integers, $d > 0$, and m *mod* $d = a$ and n *mod* $d = b$, is $(m + n)$ *mod* $d = a + b$? Is $(m + n)$ *mod* $d = (a + b)$ *mod* d? Prove your answers.

∗52. If m, n, a, b, and d are integers, $d > 0$, and m *mod* $d = a$ and n *mod* $d = b$, is (mn) *mod* $d = ab$? Is (mn) *mod* $d = ab$ *mod* d? Prove your answers.

53. Prove that if m, d, and k are integers and $d > 0$, then $(m + dk)$ *mod* $d = m$ *mod* d.

Answers for Test Yourself

1. integers; $n = dq + r$; $0 \leq r < d$ 2. the quotient obtained when n is divided by d; the nonnegative remainder obtained when n ... 4. $0, 1, 2, \ldots, (d - 1)$; $dq, dq + 1, dq + 2, \ldots, dq + (d - 1)$ 5. If A_1, then C. If A_2, then ... odd or even

4.5 Direct Proof and Counterexample V: Floor and Ceiling

Proof serves many purposes simultaneously. In being exposed to the scrutiny and judgment of a new audience, [a] proof is subject to a constant process of criticism and revalidation. Errors, ambiguities, and misunderstandings are cleared up by constant exposure. Proof is respectability. Proof is the seal of authority.

Proof, in its best instances, increases understanding by revealing the heart of the matter. Proof suggests new mathematics. The novice who studies proofs gets closer to the creation of new mathematics. Proof is mathematical power, the electric voltage of the subject which vitalizes the static assertions of the theorems.

Finally, proof is ritual, and a celebration of the power of pure reason.

— Philip J. Davis and Reuben Hersh, *The Mathematical Experience*, 1981

Imagine a real number sitting on a number line. The *floor* and *ceiling* of the number are the integers to the immediate left and to the immediate right of the number (unless the number is, itself, an integer, in which case its floor and ceiling both equal the number itself). Many computer languages have built-in functions that compute floor and ceiling automatically. These functions are very convenient to use when writing certain kinds of computer programs. In addition, the concepts of floor and ceiling are important in analyzing the efficiency of many computer algorithms.

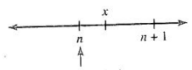
• Definition

Given any real number x, the **floor of x**, denoted $\lfloor x \rfloor$, is defined as follows:

$$\lfloor x \rfloor = \text{that unique integer } n \text{ such that } n \leq x < n + 1.$$

Symbolically, if x is a real number and n is an integer, then

$$\lfloor x \rfloor = n \quad \Leftrightarrow \quad n \leq x < n + 1.$$



floor of $x = \lfloor x \rfloor$

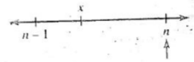
• Definition

Given any real number x, the **ceiling of x**, denoted $\lceil x \rceil$, is defined as follows:

$$\lceil x \rceil = \text{that unique integer } n \text{ such that } n - 1 < x \leq n.$$

Symbolically, if x is a real number and n is an integer, then

$$\lceil x \rceil = n \quad \Leftrightarrow \quad n - 1 < x \leq n.$$



ceiling of $x = \lceil x \rceil$

Example 4.5.1 Computing Floors and Ceilings

Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of x:

a. 25/4 b. 0.999 c. −2.01

Solution

a. $25/4 = 6.25$ and $6 < 6.25 < 7$; hence $\lfloor 25/4 \rfloor = 6$ and $\lceil 25/4 \rceil = 7$.

b. $0 < 0.999 < 1$; hence $\lfloor 0.999 \rfloor = 0$ and $\lceil 0.999 \rceil = 1$.

c. $-3 < -2.01 < -2$; hence $\lfloor -2.01 \rfloor = -3$ and $\lceil -2.01 \rceil = -2$.

Note that on some calculators $\lfloor x \rfloor$ is denoted INT (x). ■

Example 4.5.2 An Application

The 1,370 students at a college are given the opportunity to take buses to an out-of-town game. Each bus holds a maximum of 40 passengers.

a. For reasons of economy, the athletic director will send only full buses. What is the maximum number of buses the athletic director will send?

b. If the athletic director is willing to send one partially filled bus, how many buses will be needed to allow all the students to take the trip?

Solution

a. $\lfloor 1370/40 \rfloor = \lfloor 34.25 \rfloor = 34$ b. $\lceil 1370/40 \rceil = \lceil 34.25 \rceil = 35$ ■

Example 4.5.3 Some General Values of Floor

If k is an integer, what are $\lfloor k \rfloor$ and $\lfloor k + 1/2 \rfloor$? Why?

Solution Suppose k is an integer. Then

$$\lfloor k \rfloor = k \text{ because } k \text{ is an integer and } k \le k < k+1,$$

and

$$\left\lfloor k + \frac{1}{2} \right\rfloor = k \text{ because } k \text{ is an integer and } k \le k + \frac{1}{2} < k+1.$$ ■

Example 4.5.4 Disproving an Alleged Property of Floor

Is the following statement true or false?

For all real numbers x and y, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.

Solution The statement is false. As a counterexample, take $x = y = \frac{1}{2}$. Then

$$\lfloor x \rfloor + \lfloor y \rfloor = \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor = 0 + 0 = 0,$$

whereas

$$\lfloor x + y \rfloor = \left\lfloor \frac{1}{2} + \frac{1}{2} \right\rfloor = \lfloor 1 \rfloor = 1.$$

Hence $\lfloor x + y \rfloor \ne \lfloor x \rfloor + \lfloor y \rfloor$.

To arrive at this counterexample, you could have reasoned as follows: Suppose x and y are real numbers. Must it necessarily be the case that $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$, or could x and y be such that $\lfloor x + y \rfloor \ne \lfloor x \rfloor + \lfloor y \rfloor$? Imagine values that the various quantities could take. For instance, if both x and y are positive, then $\lfloor x \rfloor$ and $\lfloor y \rfloor$ are the integer parts of $\lfloor x \rfloor$ and $\lfloor y \rfloor$ respectively; just as

$$2\frac{3}{5} = 2 + \frac{3}{5}$$

integer part fractional part

so is

$$x = \lfloor x \rfloor + \text{fractional part of } x$$

and

$$y = \lfloor y \rfloor + \text{fractional part of } y.$$

where the term *fractional part* is understood here to mean the part of the number to the right of the decimal point when the number is written in decimal notation. Thus if x and y are positive,

$$x + y = \lfloor x \rfloor + \lfloor y \rfloor + \textit{the sum of the fractional parts of } x \textit{ and } y.$$

But also

$$x + y = \lfloor x + y \rfloor + \textit{the fractional part of } (x + y).$$

These equations show that if there exist numbers x and y such that the sum of the fractional parts of x and y is at least 1, then a counterexample can be found. But there do exist such x and y; for instance, $x = \frac{1}{2}$ and $y = \frac{1}{2}$ as before. ■

The analysis of Example 4.5.4 indicates that if x and y are positive and the sum of their fractional parts is less than 1, then $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. In particular, if x is positive and m is a positive integer, then $\lfloor x + m \rfloor = \lfloor x \rfloor + \lfloor m \rfloor = \lfloor x \rfloor + m$. (The fractional part of m is 0; hence the sum of the fractional parts of x and m equals the fractional part of x, which is less than 1.) It turns out that you can use the definition of floor to show that this equation holds for all real numbers x and for all integers m.

Example 4.5.5 Proving a Property of Floor

Prove that for all real numbers x and for all integers m, $\lfloor x + m \rfloor = \lfloor x \rfloor + m$.

Solution Begin by supposing that x is a particular but arbitrarily chosen real number and that m is a particular but arbitrarily chosen integer. You must show that $\lfloor x + m \rfloor = \lfloor x \rfloor + m$. Since this is an equation involving $\lfloor x \rfloor$ and $\lfloor x + m \rfloor$, it is reasonable to give one of these quantities a name: Let $n = \lfloor x \rfloor$. By definition of floor,

$$n \text{ is an integer} \quad \text{and} \quad n \le x < n + 1.$$

This double inequality enables you to compute the value of $\lfloor x+m \rfloor$ in terms of n by adding m to all sides:

$$n+m \le x+m < n+m+1.$$

Thus the left-hand side of the equation to be shown is

$$\lfloor x+m \rfloor = n+m.$$

On the other hand, since $n = \lfloor x \rfloor$, the right-hand side of the equation to be shown is

$$\lfloor x \rfloor + m = n+m$$

also. Thus $\lfloor x+m \rfloor = \lfloor x \rfloor + m$. This discussion is summarized as follows:

Theorem 4.5.1

For all real numbers x and all integers m, $\lfloor x+m \rfloor = \lfloor x \rfloor + m$.

Proof:

Suppose a real number x and an integer m are given. *[We must show that $\lfloor x+m \rfloor = \lfloor x \rfloor + m$.]* Let $n = \lfloor x \rfloor$. By definition of floor, n is an integer and

$$n \le x < n+1.$$

Add m to all three parts to obtain

$$n+m \le x+m < n+m+1$$

[since adding a number to both sides of an inequality does not change the direction of the inequality].

Now $n+m$ is an integer *[since n and m are integers and a sum of integers is an integer]*, and so, by definition of floor, the left-hand side of the equation to be shown is

$$\lfloor x+m \rfloor = n+m.$$

But $n = \lfloor x \rfloor$. Hence, by substitution,

$$n+m = \lfloor x \rfloor + m,$$

which is the right-hand side of the equation to be shown. Thus $\lfloor x+m \rfloor = \lfloor x \rfloor + m$ *[as was to be shown].* ■

The analysis of a number of computer algorithms, such as the binary search and merge sort algorithms, requires that you know the value of $\lfloor n/2 \rfloor$, where n is an integer. The formula for computing this value depends on whether n is even or odd.

Theorem 4.5.2 The Floor of $n/2$

For any integer n,

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof:

Suppose n is a *[particular but arbitrarily chosen]* integer. By the quotient-remainder theorem, either n is odd or n is even.

Case 1 (n is odd): In this case, $n = 2k+1$ for some integer k. *[We must show that $\lfloor n/2 \rfloor = (n-1)/2$.]* But the left-hand side of the equation to be shown is

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k+1}{2} \right\rfloor = \left\lfloor \frac{2k}{2} + \frac{1}{2} \right\rfloor = \left\lfloor k + \frac{1}{2} \right\rfloor = k$$

because k is an integer and $k \le k + 1/2 < k+1$. And the right-hand side of the equation to be shown is

$$\frac{n-1}{2} = \frac{(2k+1)-1}{2} = \frac{2k}{2} = k$$

also. So since both the left-hand and right-hand sides equal k, they are equal to each other. That is, $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$ *[as was to be shown].*

Case 2 (n is even): In this case, $n = 2k$ for some integer k. *[We must show that $\lfloor n/2 \rfloor = n/2$.]* The rest of the proof of this case is left as an exercise.

Given any integer n and a positive integer d, the quotient-remainder theorem guarantees the existence of unique integers q and r such that

$$n = dq + r \quad \text{and} \quad 0 \le r < d.$$

The following theorem states that the floor notation can be used to describe q and r as follows:

$$q = \left\lfloor \frac{n}{d} \right\rfloor \quad \text{and} \quad r = n - d\left\lfloor \frac{n}{d} \right\rfloor.$$

Thus if, on a calculator or in a computer language, floor is built in but *div* and *mod* are not, *div* and *mod* can be defined as follows: For a nonnegative integer n and a positive integer d,

$$n \text{ div } d = \left\lfloor \frac{n}{d} \right\rfloor \quad \text{and} \quad n \text{ mod } d = n - d\left\lfloor \frac{n}{d} \right\rfloor. \qquad 4.5.1$$

Note that d divides n if, and only if, $n \text{ mod } d = 0$, or, in other words, $n = d\lfloor n/d \rfloor$. You are asked to prove this in exercise 13.

Theorem 4.5.3

If n is any integer and d is a positive integer, and if $q = \lfloor n/d \rfloor$ and $r = n - d\lfloor n/d \rfloor$, then

$$n = dq + r \quad \text{and} \quad 0 \le r < d.$$

Proof:

Suppose n is any integer, d is a positive integer, $q = \lfloor n/d \rfloor$, and $r = n - d\lfloor n/d \rfloor$. *[We must show that $n = dq + r$ and $0 \le r < d$.]* By substitution,

$$dq + r = d\left\lfloor \frac{n}{d} \right\rfloor + \left(n - d\left\lfloor \frac{n}{d} \right\rfloor\right) = n.$$

So it remains only to show that $0 \le r < d$. But $q = \lfloor n/d \rfloor$. Thus, by definition of floor,

$$q \le \frac{n}{d} < q + 1.$$

Then

$$dq \le n < dq + d \qquad \text{by multiplying all parts by } d$$

and so

$$0 \le n - dq < d \qquad \text{by subtracting } dq \text{ from all parts}$$

But

$$r = n - d\left\lfloor \frac{n}{d} \right\rfloor = n - dq.$$

Hence

$$0 \le r < d \qquad \text{by substitution.}$$

[This is what was to be shown.]

Example 4.5.6 Computing *div* and *mod*

Use the floor notation to compute 3850 *div* 17 and 3850 *mod* 17.

Solution By formula (4.5.1),

$$3850 \; div \; 17 = \lfloor 3850/17 \rfloor = \lfloor 226.4705882\ldots \rfloor = 226$$
$$3850 \; mod \; 17 = 3850 - 17 \cdot \lfloor 3850/17 \rfloor = 3850 - 17 \cdot 226 = 3850 - 3842 = 8.$$

Test Yourself

1. Given any real number x, the floor of x is the unique integer n such that ____.
2. Given any real number x, the ceiling of x is the unique integer n such that ____.

Exercise Set 4.5

Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the values of x in 1–4.

1. 37.999
2. 17/4
3. −14.00001
4. −32/5
5. Use the floor notation to express 259 *div* 11 and 259 *mod* 11.
6. If k is an integer, what is $\lceil k \rceil$? Why?
7. If k is an integer, what is $\lceil k + \frac{1}{2} \rceil$? Why?
8. Seven pounds of raw material are needed to manufacture each unit of a certain product. Express the number of units that can be produced from n pounds of raw material using either the floor or the ceiling notation. Which notation is more appropriate?
9. Boxes, each capable of holding 36 units, are used to ship a product from the manufacturer to a wholesaler. Express the number of boxes that would be required to ship n units of the product using either the floor or the ceiling notation. Which notation is more appropriate?
10. If 0 = Sunday, 1 = Monday, 2 = Tuesday, …, 6 = Saturday, then January 1 of year n occurs on the day of the week given by the following formula:
$$\left(n + \left\lfloor \frac{n-1}{4} \right\rfloor - \left\lfloor \frac{n-1}{100} \right\rfloor + \left\lfloor \frac{n-1}{400} \right\rfloor\right) mod\ 7.$$
   a. Use this formula to find January 1 of
      i. 2050 ii. 2100 iii. the year of your birth.
   H b. Interpret the different components of this formula.
11. Suppose n and d are integers and $d \ne 0$. Prove each of the following.
   a. If $d \mid n$, then $n = \lfloor n/d \rfloor \cdot d$.
   b. If $n = \lfloor n/d \rfloor \cdot d$, then $d \mid n$.
   c. Use the floor notation to state a necessary and sufficient condition for an integer n to be divisible by an integer d.
12. Prove that if n is any even integer, then $\lfloor n/2 \rfloor = n/2$.
13. State a necessary and sufficient condition for the floor of a real number to equal that number.

Some of the statements in 14–22 are true and some are false. Prove each true statement and find a counterexample for each false statement, but do not use Theorem 4.5.1 in your proofs.

14. For all real numbers x and y, $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$.
15. For all real numbers x, $\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1$.
16. For all real numbers x, $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$.

H 17. For all integers n,
$$\lfloor n/3 \rfloor = \begin{cases} n/3 & \text{if } n \; mod \; 3 = 0 \\ (n-1)/3 & \text{if } n \; mod \; 3 = 1 \\ (n-2)/3 & \text{if } n \; mod \; 3 = 2 \end{cases}.$$

H 18. For all real numbers x and y, $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

H 19. For all real numbers x, $\lceil x - 1 \rceil = \lceil x \rceil - 1$.

20. For all real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \cdot \lceil y \rceil$.
21. For all odd integers n, $\lceil n/2 \rceil = (n+1)/2$.
22. For all real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \cdot \lfloor y \rfloor$.

Prove each of the statements in 23–29.

23. For any real number x, if x is not an integer, then $\lfloor x \rfloor + \lfloor -x \rfloor = -1$.
24. For any integer m and any real number x, if x is not an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1$.

H 25. For all real numbers x, $\lfloor \lfloor x/2 \rfloor / 2 \rfloor = \lfloor x/4 \rfloor$.

26. For all real numbers x, if $x - \lfloor x \rfloor < 1/2$ then $\lfloor 2x \rfloor = 2\lfloor x \rfloor$.
27. For all real numbers x, if $x - \lfloor x \rfloor \ge 1/2$ then $\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$.
28. For any odd integer n,
$$\left\lceil \frac{n^2}{4} \right\rceil = \frac{n^2 + 3}{4}.$$
29. For any odd integer n,
$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right).$$
30. Find the mistake in the following "proof" that $\lfloor n/2 \rfloor = (n-1)/2$ if n is an odd integer.
   "**Proof:** Suppose n is any odd integer. Then $n = 2k + 1$ for some integer k. Consequently,
$$\left\lfloor \frac{2k+1}{2} \right\rfloor = \frac{(2k+1) - 1}{2} = \frac{2k}{2} = k.$$
   But $n = 2k + 1$. Solving for k gives $k = (n-1)/2$. Hence, by substitution, $\lfloor n/2 \rfloor = (n-1)/2$."

Answers for Test Yourself

1. $n \le x < n + 1$ 2. $n - 1 < x \le n$