Proposition 6.1.4. *Let $\sqsubseteq$ be a topogenous order. The class of $\sqsubseteq$-closed morphisms in $\mathcal{C}$*

(1) *is closed under composition,*

(2) *is left cancellable with respect to $\mathcal{M}$ that is, $g \circ f$ $\sqsubseteq$-closed and $g \in \mathcal{M} \Rightarrow f$ is $\sqsubseteq$-closed,*

(3) *is right cancellable with respect to $\mathcal{E}$ that is, $g \circ f$ $\sqsubseteq$-closed $\Rightarrow g$ is $\sqsubseteq$-closed provided $\mathcal{E}$ is stable under pullback.*

Proof. (1) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are $\sqsubseteq$-closed then,

$$\begin{aligned} f^{-1}(g^{-1}(m)) \sqsubseteq n &\Rightarrow \exists\, p \mid g^{-1}(m) \sqsubseteq p \text{ and } f^{-1}(p) \leq n \\ &\Rightarrow \exists\, q \mid m \sqsubseteq q \text{ and } g^{-1}(q) \leq p \\ &\Rightarrow m \sqsubseteq q \text{ and } f^{-1}(g^{-1}(q)) \leq f^{-1}(p) \leq n \end{aligned}$$

(2) If $g \circ f$ $\sqsubseteq$-closed and $g \in \mathcal{M}$

$$f^{-1}(m) \sqsubseteq n$$

$$\ldots^{-1}(g^{-1}(p)) \leq n$$

$$\Rightarrow m \sqsubseteq g^{-1}(p) \text{ and } f^{-1}(g^{-1}(p)) \leq n$$

(3) If $g \circ f$ $\sqsubseteq$-closed and $f \in \mathcal{E}$ with $\mathcal{E}$ stable under pullback, then

$$\begin{aligned} g^{-1}(m) \sqsubseteq n &\Leftrightarrow f(f^{-1}(g^{-1}(m))) \sqsubseteq n \\ &\Rightarrow f^{-1}(g^{-1}(m)) \sqsubseteq f^{-1}(n) && \text{by } (T3) \\ &\Rightarrow \exists\, p \mid m \sqsubseteq p \text{ and } f^{-1}(g^{-1}(p)) \leq f^{-1}(n) \\ &\Rightarrow m \sqsubseteq p \text{ and } g^{-1}(p) \leq n \end{aligned}$$

□

It is important to observe from the above two propositions that $\sqsubseteq$-closed and $\sqsubseteq$-initial morphisms behave in a similar way as their conterpart in neighbourhood notations.

We next provide an immediate connection between the notions of $\sqsubseteq$-initiality and $\sqsubseteq$-closedness.

Proposition 6.1.5. *For a topogenous order $\sqsubseteq$;*

(1) *Every $\sqsubseteq$-closed morphism in $\mathcal{M}$ is $\sqsubseteq$-initial.*

(2) *Every $\sqsubseteq$-initial morphism in $\mathcal{E}$ is $\sqsubseteq$-closed provided $\mathcal{E}$ is pullback stable.*

Proof. (1) If $f : X \longrightarrow Y$ is $\sqsubseteq$-closed in $\mathcal{M}$, then for any $m; n \in \text{sub}X$, $m \sqsubseteq n \Leftrightarrow f^{-1}(f(m)) \sqsubseteq n \Rightarrow \exists\, p \mid f(m) \sqsubseteq p$ and $f^{-1}(p) \leq n$.

(2) If $f : X \longrightarrow Y$ is $\sqsubseteq$-initial in $\mathcal{E}$ with $\mathcal{E}$ stable under pullback, then

$$f^{-1}(m) \sqsubseteq n \Rightarrow \exists\, p \mid f(f^{-1}(m) \sqsubseteq p \text{ and } f^{-1}(p) \leq n$$
$$\Rightarrow m \sqsubseteq p \text{ and } f^{-1}(p) \leq n$$

□

Our next proposition links $\sqsubseteq$-initiality and $\sqsubseteq$-strictness.

Proposition 6.1.6. (1) *Any $\sqsubseteq$-initial morphism in $\mathcal{E}$ maps $\sqsubseteq$-strict subobjects to $\sqsubseteq$-strict subobjects.*

(2) *Any $\sqsubseteq$-strict morphism in $\mathcal{M}$ is $\sqsubseteq$-initial.*

Proof. (1) If $f : X \longrightarrow Y$ is $\sqsubseteq$-initial in $\mathcal{E}$ with $\mathcal{E}$ stable under pullback, then

$$m \sqsubseteq m \Rightarrow \exists\, p \mid f(m) \sqsubseteq p \text{ and } f^{-1}(p) \leq m$$
$$\Rightarrow f(m) \sqsubseteq p \text{ and } p = f(f^{-1}(p)) \leq f(m)$$
$$\Rightarrow f(m) \sqsubseteq f(m)$$

(2) If $f : X \longrightarrow Y$ is $\sqsubseteq$-strict in $\mathcal{M}$, then $m \sqsubseteq n \Rightarrow f(m) \sqsubseteq f(n)$. Put $p = f(n)$ to get $f^{-1}(p) = f^{-1}(f(n)) = n$

□

6.2 Finality and openness

Final and open morphisms with respect to a topogenous order are also obtained by the same process used to get the $\sqsubseteq$-closed and $\sqsubseteq$-initial.

Definition 6.2.1. A $\mathcal{C}$-morphism $f : X \longrightarrow Y$ is said to be $\sqsubseteq$-final if for any $n \in \text{sub}Y$ and $k \geq n$,

$$f^{-1}(n) \sqsubseteq f^{-1}(k) \Rightarrow n \sqsubseteq k$$

Definition 6.2.2. A morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is said to be $\sqsubseteq$-open if

$$(\exists\, p \mid m \sqsubseteq p \text{ and } f(p) \leq n) \Rightarrow f(m) \sqsubseteq n$$

for all $m \in \text{sub}X$ and $n \in \text{sub}Y$.

The notion of $\sqsubseteq$-open morphism that we have obtained is just the $\sqsubseteq$-strict morphism as observed in the previous chapter. In fact if f is $\sqsubseteq$-strict and $(\exists\, p \mid m \sqsubseteq p$ and $f(p) \leq n)$, then $p \leq f^{-1}(n)$ by adjointness. So $m \sqsubseteq f^{-1}(n)$ by $(T2)$ and $f(m) \sqsubseteq n$. Conversely f is $\sqsubseteq$-open, one puts $p = f^{-1}(n)$ in Definition 2.2.2 to see that f is $\sqsubseteq$-strict.

We shall now provide some basic properties of the $\sqsubseteq$-final and $\sqsubseteq$-strict morphisms

Proposition 6.2.3. *Let $\sqsubseteq$ be a topogenous order. The following statements hold true*

(1) *$\sqsubseteq$-final morphisms are closed under composition.*

(2) *If $g \circ f$ is $\sqsubseteq$-final then then g is $\sqsubseteq$-final.*

(3) *If $g \circ f$ is $\sqsubseteq$-final and g is mono, then f is $\sqsubseteq$-final.*

Proof. (1) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are $\sqsubseteq$-final then,

$$\begin{aligned} f^{-1}(g^{-1}(n)) \sqsubseteq f^{-1}(g^{-1}(k)) &\Rightarrow g^{-1}(n) \sqsubseteq g^{-1}(k) \\ &\Rightarrow n \sqsubseteq k \end{aligned}$$

(2) If $g \circ f$ is $\sqsubseteq$-final and $k \geq n$ for any $n \in \text{sub}Z$, then

$$\begin{aligned} g^{-1}(n) \sqsubseteq (g^{-1}(k)) &\Rightarrow f^{-1}(g^{-1}(n)) \sqsubseteq f^{-1}(g^{-1}(k)) \\ &\Rightarrow n \sqsubseteq k \end{aligned}$$

(3) If g is mono and $k \geq n$, then $k = g^{-1}(g(k))$ and $n = g^{-1}(g(n))$ for any $n \in \text{sub}Y$. Hence,

$$\begin{aligned} f^{-1}(n) \sqsubseteq (f^{-1}(k)) &\Leftrightarrow f^{-1}(g^{-1}(g(n))) \sqsubseteq f^{-1}(g^{-1}(g(k))) \\ &\Rightarrow g(n) \sqsubseteq g(k) \\ &\Rightarrow n \sqsubseteq k \end{aligned}$$

□

The following proposition is the same as Proposition 5.2.2. We shall leave the proof as it was already given in the previous chapter.

Proposition 6.2.4. *Let $\sqsubseteq$ be a topogenous order. The class of $\sqsubseteq$-strict morphisms in $\mathcal{C}$*

(1) *is closed under composition,*

(2) *is left cancellable with respect to $\mathcal{M}$ that is, $g \circ f$ $\sqsubseteq$-strict and $g \in \mathcal{M} \Rightarrow f$ is $\sqsubseteq$-strict,*

(3) *is right cancellable with respect to $\mathcal{E}$ that is, $g \circ f$ $\sqsubseteq$-strict $\Rightarrow g$ is $\sqsubseteq$-strict provided $\mathcal{E}$ is stable under pullback.*

The following is a further relationship between the types of morphisms with respect to a topogenous order.

Proposition 6.2.5. *Let $\sqsubseteq$ a topogenous order;*

(1) *Every $\sqsubseteq$-strict morphism in $\mathcal{E}$ is $\sqsubseteq$-final provided $\mathcal{E}$ is pullback stable.*

(2) *If $g \circ f = 1$ in $\mathcal{C}$ then f is a $\sqsubseteq$-initial morphism and g is a $\sqsubseteq$-final morphism in $\mathcal{E}$.*

(3) *Every $\sqsubseteq$-closed morphism in $\mathcal{E}$ is $\sqsubseteq$-final.*

Proof. (1) If $f : X \longrightarrow Y$ is $\sqsubseteq$-strict and $k \geq n$ for any $n \in \text{sub}Y$, then

$$\begin{aligned} f^{-1}(n) \sqsubseteq f^{-1}(k) &\Rightarrow f(f^{-1}(n)) \sqsubseteq f(f^{-1}(k)) \\ &\Rightarrow n \sqsubseteq k \end{aligned}$$

(2) Follows from 6:1:3(2) and 6:2:2(2) respectively.

(3) If $k \leq n$ and $f : X \longrightarrow Y$ in $\mathcal{E}$ with $\mathcal{E}$ pullback stable, then

$$
\begin{aligned}
f^{-1}(n) \sqsubseteq f^{-1}(k) &\Rightarrow \exists\, p \mid n \sqsubseteq p \text{ and } f^{-1}(p) = f^{-1}(k) \\
&\Rightarrow n \sqsubseteq p \text{ and } p = f(f^{-1}(p)) = f(f^{-1}(k)) = k
\end{aligned}
$$

□

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