Testing gravity with redshift-space distortions, using MeerKAT and the SKA

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Declaration

I declare that Testing gravity with redshift-space distortions, using MeerKAT and the SKA is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Signature

Jan-Albert Viljoen
March 2019
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Abstract

The growth rate of large-scale structure is a key probe of gravity in the accelerating Universe. Standard models of Dark Energy within General Relativity predict essentially the same growth rate, whereas Modified Gravity theories without Dark Energy predict a different growth rate. Redshift-space distortions lead to anisotropy in the power spectrum, and extracting the monopole and quadrupole allows us to determine the growth rate and thus test theories of gravity. We investigate redshift-space distortions in the intensity maps of the 21cm emission line of neutral hydrogen (HI) in galaxies after the Epoch of Reionization. HI intensity mapping delivers very accurate redshifts. We first use the standard approach based on the Fourier power spectrum. Then we explored an alternative approach, based on the spherical-harmonic angular power spectrum. Fisher forecasting was used to make predictions of the accuracy with which MeerKAT will measure the growth rate parameter, via the proposed MeerKAT Large Area Synoptic Survey (MeerKLASS). Then we extend the forecasts to consider the planned HI intensity mapping survey in Phase 1 of the Square Kilometre Array. These forecasts enable us to predict at what level of accuracy General relativity and various alternative theories could be ruled out.
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Chapter 1

Overview

At the start of the 20th century a young physicist named Albert Einstein developed a theory of gravity called General Relativity (GR), Section 2.1.4. The theory greatly improved our previous Newtonian understanding of the mechanics of the universe by describing the mechanism responsible for gravitational attraction. Even though Newton’s law of gravitation accurately predicted the motions in the solar system, the theory lacked an explanation as to why there is this force in the first place. Einstein described gravitational attraction as the curvature of a four-dimensional manifold called space-time. The most fascinating implication of this interpretation is that not only is space malleable, but so is time.

The success of the theory was celebrated when GR correctly predicted an observed deviation in position of a known star during a solar eclipse. Photons from the source was deflected from its original course due to the curvature imposed by the sun - an effect called gravitational lensing. Newton’s gravitational theory excluded interactions with light since photons are massless, and so began the age modern of cosmology.

Despite the fact that GR withstood the tests of every experiment over the past century, almost all the contents of the universal model is either unknown or undetected. Many different theories of gravity have been proposed that could possibly explain the late time expansion, not through an unknown ‘dark’ energy field, but by redefining our theory of gravitational attraction. These models are referred to as modified theories of gravity (MG), Section 2.1.6. A popular suggestion is that at low-redshift gravity can drive the accelerated expansion of the universe, thus it is possible to explain current observations without dark energy.
The growth rate of large scale structure formation \( f \), is essentially how quickly matter clump together as result of their gravitational interaction. If we can measure the velocities of galaxies falling into to large structures, like galaxy clusters or groups, we can determine \( f \). Thus the rate at which large scale structures grow is an essential tool to determine which gravitational theory more closely resembles reality. The growth rate can be parametrized by a power law, in terms of matter density \( \Omega_m \), and the growth index \( \gamma \). Different theories predict distinctly different \( \gamma \), making this a key discriminant to experimentally test the aforementioned theories.

In order to extract information on the growth rate of large scale structure formation we need to take into account the effects of Redshift Space Distortions (RSD), Section 2.4. RSD arise from the peculiar velocities of galaxies, since the peculiar motion induces a Doppler shift in the frequency of the emitter and hence in redshift position. This begs the question, exactly how well can we measure this parameter or how confident can we be in our assumed gravitational model?

Our confidence in a specific theory is quantified by means of Bayesian statistics, which describes the conditional probability of an event based on data, prior information and conditions related to the event. We can use a Bayesian statistical method called Fisher forecasting, Chapter 5, with which determine the accuracy by which we can measure an individual cosmological parameter by also taking into account uncertainties in other parameters of the cosmological model.

The next generation of galaxy surveys promises to have unprecedented observational sensitivity, like the Square Kilometer Array (SKA), and its precursor MeerKAT. The survey observes in the radio spectrum and the main cosmological observable used for the forecasts is the intensity mapping emission from Neutral hydrogen (HI), Chapter 4.

The tool used to interpret the data gathered from these large surveys is called a two point correlation function, Chapter 3. By looking at the galaxy number counts per volume in a galaxy surveys, we can determine the distribution of matter. The correlation function can be analysed in either the Fourier domain or the spherical harmonic space, and in this thesis both are considered. Taking the Fourier transform of the two-point correlation function is called the Fourier power spectrum \( P(k) \), and allows us look at the different scales in wave-number space.
The analysis of the correlation function in spherical harmonic space is called the angular power spectrum $C_\ell$. This method has the ability to cross-correlate different redshift bins, and account for wide-angle correlations, thus we expect that more information will be available to constrain cosmological parameters.

These methods of analysis will be used to investigate our ability to constrain cosmological parameters using Fisher forecasting on the MeerKLASS and SKA1 survey. Specifically we focus on the rate of large scale structure formation, which is extracted by taking into account RSD. But also other parameters are marginalized over to perform a more realistic forecast.

First we discuss what redshift space distortion is and how it is quantified in the perturbation theories. Great care is taken to understand the differences of a Fisher forecast, depending on which power spectrum is analyzed. Importantly the difference between conditional and marginal is discussed, and how they are generated using Fisher forecasts. Also understanding exactly how the Fisher matrix marginalize over different parameters in a model, and why the Alcock-Paczynski effect (AP) should necessarily be included when working with $P(k)$. The technicalities of cross-correlating redshift bins when using $C_\ell$ is also an important aspect of this thesis.

Another important aspect of the thesis is applying the theory to an actual future survey. We simulate the expected noise and telescope beam for Neutral Hydrogen Intensity Maps, and show how the power spectrum is adjusted to reproduce the expected HI signal given the assumed survey specifications.
Chapter 2

Standard model of cosmology

Everything we know about the evolution of the universe is derived from our cosmological theory and observational verifications thereof. This enables us to explain and model the expansion of the universe for different eras in history.

Fig. 2.1 The evolution of the expanding universe given the standard cosmological model, $\Lambda CDM$. Image credit - NASA/WMAP Science Team.

http://etd.uwc.ac.za/
In the first few fractions of a second after the big bang, the universe expanded at super luminous speeds, known as inflation, Fig 2.1. The inflation theory is not assumed to be within the standard cosmological model ($\Lambda$CDM), but it is probably the best candidate among the theories for the evolution of the early universe. Inflation proposes to solve some of the issues and fine tuning within the hot big bang model. It best explains why the temperature fluctuations observed on the surface of last scattering is uniform to within a hundred-thousandths of a degree Kelvin, despite the great physical separation between points. The surface of last scattering is represented by the “Afterglow Light Pattern” in Fig 2.1, and is an observable known as the Cosmic Microwave Background (CMB).

Before the epoch of last scattering the primordial universe only consisted of sub-atomic particles interacting in a super hot plasma, and the radiative energy was the main source of the universal expansion. After around 375 000 years of expansion the density and temperature of the universe decreased enough for Hydrogen atoms (H) to form, known as recombination, which in turn allowed photons to free stream.

For a significant time Hydrogen gas accumulated around the primordial matter over-densities, which is observed as temperature fluctuations in the CMB. Since this is a time that precedes the first stars, this era is known as the Dark Ages, Fig 2.1.

Around $400 \times 10^6$ years after the big bang, the mass of the over-densities was sufficient to overcome the outward pressure exerted by the Hydrogen gas, causing the gravitational collapse of the clouds and formation of the first stars. Through gravitational interaction these stars would form part of the first galaxies, and in turn form galaxy clusters in a hierarchical process to create the structures observed today.

Given the gravitational attraction between matter, we expected the expansion of the universe would eventually slow down, but it has been observed as accelerating. The standard cosmological model explains the late time expansion of the universe using a dark energy field (DE), which started dominating the expansion roughly $5 \times 10^9$ years ago, Fig 2.1. $\Lambda$CDM enables us to measure cosmological parameters needed to accurately model the distribution of mass we observe today.
2.1 Background cosmology

Before we can forecast constraints on cosmological parameters, we first need to define the assumptions made to describe the expansion of the universe, given its constituents. In order to explain these concepts we first define some notation.

2.1.1 Metric tensor and covariant derivative

On a curved manifold the basis vectors are functions of the time and space coordinates, which means we need to quantify the relation between basis vectors using a metric. The metric tensor is a function relating different basis vectors $e_\mu$ with each other and is defined in terms of the dot product,

$$g_{\mu\nu} = g(e_\mu, e_\nu) = e_\mu \cdot e_\nu$$

where the Greek letters indicate that we are working with a four-dimensional manifold. The Christoffel symbols are an array of numbers describing a metric connection, which is a specialisation of the affine connection to a manifold endowed with a metric. The Christoffel symbols are defined in terms of the metric and the differential operator $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \gamma_\mu$,

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^\mu\nu \left[ g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\beta\alpha,\nu} \right]$$

and it enables us to define a covariant derivative on a manifold. A covariant transformation is a rule that specifies how entities like tensors and vectors transform with a change of basis, and a covariant derivative in a general coordinate system is a derivative that transforms covariantly. The covariant derivative is defined in terms of the Christoffel symbols,

$$\nabla_\nu T^\mu = \partial_\nu T^\mu + \Gamma^\mu_{\nu\alpha} T^\alpha$$
$$\nabla_\nu T_\mu = \partial_\nu T_\mu - \Gamma^\alpha_{\nu\mu} T_\alpha$$

where $T^\mu$ is an arbitrary tensor, and the metric is used as a lowering operator $T_\mu = g_{\mu\nu} T^\nu$. 
2.1.2 Curvature tensor

By parallel transporting a vector along a closed loop on a curved manifold, we see the resulting vector has changed along its path, Fig 2.2. In differential geometry the Riemann curvature tensor described this difference.

![Fig. 2.2](https://en.wikipedia.org/wiki/Parallel_transport)

The curvature tensor can be expressed in terms of Christoffel symbols

\[ R^{\mu}_{\nu\alpha\beta} = \partial_{\alpha} \Gamma^{\mu}_{\nu\beta} - \partial_{\beta} \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\sigma\alpha} \Gamma_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta} \Gamma_{\nu\alpha} \]  

(2.4)

and assigns a tensor to each point on the Riemannian manifold. Rewriting this expression in terms of the metric

\[ R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R_{\beta\lambda\mu\nu} = \frac{1}{2} (g_{\alpha\nu,\beta\nu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}) \]  

(2.5)

it is easy to verify the following identities:

\[ R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}, \]

\[ R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\beta\nu} = 0. \]  

(2.6)

Thus \( R_{\alpha\beta\mu\nu} \) is anti-symmetric in the first and second pair of indices. The anti-symmetry of the Riemann tensor allows us to contract two indices and construct the Ricci tensor

\[ R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = -R^{\alpha}_{\nu\alpha\mu}. \]  

(2.7)
2.1 Background cosmology

Then we can define the Ricci scalar as a contraction of the Ricci tensor with the metric

\[ R = R_{\mu\nu}g^{\mu\nu} \]  \hspace{1cm} (2.8)

which is the simplest curvature invariant of a Riemannian manifold. A special case of transport is such that the tangent vector to a curve is transported parallel to itself, called auto parallel transport. It is defined by the vanishing of the covariant derivative of the tangent vector \( A^\mu = dx^\mu/dv \), in the direction of motion. Given that \( v \) is an affine parameter

\[ A^\nu \nabla_\nu A^\mu = \frac{dA^\mu}{dv} + \Gamma^\mu_{\nu\alpha} A^\nu A^\alpha = 0 \]  \hspace{1cm} (2.9)

which is the geodesic equation. Free-falling particles and photons follow geodesics.

2.1.3 Energy-momentum tensor

Now that the mathematical description of the geometry of space have been recapped, it is necessary to describe the interaction of matter there in. We can model the distribution of matter as a fluid, and in any space-time the definition of the energy-momentum tensor of a perfect fluid is given by

\[ T_{\mu\nu} = \rho u_\mu u_\nu + p (u_\mu u_\nu + g_{\mu\nu}) \]  \hspace{1cm} (2.10)

which depends on the pressure and density of the cosmological fluid. The four-velocity of a particle is defined as the rate of change of four-position with respect to \( \tau \), the proper time (measured by a clock comoving with the particle):

\[ u^\mu = \frac{dx^\mu}{d\tau} \]  \hspace{1cm} (2.11)

The four-velocity in the comoving rest frame is given by \( u^\mu(t) = \delta^\mu_0 \), such that \( g_{\mu\nu} u^\mu u^\nu = -1 \), with speed of light set to one.

The law of conservation of energy and momentum is used to constrain the flow of matter by

\[ 0 = \nabla_\mu T^{\mu}_\nu \]

\[ = u_\nu u^\mu \nabla_\mu (\rho + p) + (\rho + p) \left[ u^\mu \nabla_\mu u_\nu + u_\nu \nabla_\mu u^\mu \right] + \nabla_\nu p \]  \hspace{1cm} (2.12)
2.1.4 General relativity

The curvature of time and space not only influence the matter and energy contained there in, but the matter itself influences the geometry of space and time. An object with mass will influence this 'fabric' of time and space by creating a potential field around it, subsequently other objects in the field experience a potential difference that can be interpreted as a force. The curvature at a position in space-time is determined by the distribution of mass around it, such that we recover the classical Newtonian theory on local scales.

![Image](https://oneminuteastronomer.com/9237/gravitational-lens/)

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**Fig. 2.3** The deflection of apparent position of a star due to the space-time curvature imposed by a gravitational field, called gravitational lensing. Image credit - https://oneminuteastronomer.com/9237/gravitational-lens/.

The Einstein tensor describes the geometric part of GR, and is written in terms of the Ricci tensor (2.7) and scalar (2.8),

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \]  

(2.13)

The Einstein field equations are

\[ G_{\mu\nu} = \kappa T_{\mu\nu} \]  

(2.14)

where \( \kappa = 8\pi G/c^4 \). The fundamental constants considered are \( G \), Newton’s gravitational constant, and \( c \) is the speed of light.

http://etd.uwc.ac.za/
The consequence of this relation is that the space containing matter would distort due to the stress induced by the mass distribution there in, and therefore massless photons would adjust their trajectories according to the geodesic, Fig 2.3.

### 2.1.5 $\Lambda$CDM

Over the years Einstein’s field equations have been used to develop a model that describes the evolution universe, called $\Lambda$CDM. It assumes the Cosmological principle, which states that the spatial distribution of matter in the universe is both isotropic and homogeneous, when viewed on large enough scales. That is to say from any point in space (homogeneity), the universe should look the same in every direction (isotropy), Fig 2.4.

![Fig. 2.4 An indication of a homogeneous distribution (left). Isotropic but not homogeneous distribution (right). Image credit - [33].](http://etd.uwc.ac.za/)

The spherical symmetry that this implies enable us to express the metric as

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$  \hspace{1cm} (2.15)

where we assume that the spatial hypersurfaces at $t = const$ are flat. This is called the flat FLRW metric, named after Friedmann, Lemaître, Robertson and Walker, [22].

The rate at which the universe expands is quantified using the expansion parameter, or Hubble rate $H$, and it is dependent on the contents of the universe, like matter and radiation. Considering that the expansion parameter is the rate of change of the scale of the universe, we can write

$$H = \frac{\dot{a}}{a} = \frac{d \ln a}{dt} = \frac{1}{a^2} \frac{da}{d\eta} = \frac{a'}{a^2}$$  \hspace{1cm} (2.16)
hence the conformal Hubble rate is then given by

\[ H = \frac{a'}{a} = aH. \] (2.17)

In order to model the accelerated expansion of the late universe, we require an extra component in the theory. Either extra component in the Einstein equations, called modified theories of gravity, Section 2.1.6. Or traditionally it is explained as an extra species i.e dark energy, \( \Lambda \). The vacuum energy \( \Lambda \) has constant pressure and energy density that obey

\[ p_\Lambda = -\frac{\Lambda}{\kappa} = -\rho_\Lambda \] (2.18)

which means DE has a negative equation of state.

Another non-baryonic component is needed to reproduce observations. We also have to assume a component of Cold Dark Matter (CDM), which does not interact with photons. The influence of CDM have been observed in galactic rotation curves, and gravitational lensing experiments, but still no confirmation from the side of high-energy physics.

In this thesis we consider the universe to be flat, thus from (2.2) and (2.15) the non-vanishing Christoffel symbols are

\[ \Gamma^0_{ij} = Ha^2 \delta_{ij}; \quad \Gamma^i_{0j} = H \delta^i_j; \quad \Gamma^i_{jk} = 0 \] (2.19)

The Riemann tensor can then be computed by using definition (2.4), which leads to the Ricci tensor using (2.7). Thus the Ricci tensor for a flat isotropic and homogeneous universe is given by

\[ R_{00} = -3\frac{\ddot{a}}{a}; \quad R_{ij} = \left(2H^2 + \frac{\ddot{a}}{a}\right)a^2 \delta_{ij} \] (2.20)

after splitting up the tensor into temporal and spatial components. The Ricci scalar is then computed by contracting the above tensor with the metric

\[ R = 6\left(H^2 + \frac{\ddot{a}}{a}\right), \] (2.21)

and now we can compute the Einstein tensor.
A useful relation needed to simplify the Ricci tensor and scalar is given by

\[ \frac{dH}{dt} = \frac{dt}{a} \frac{\dot{a}}{a} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \Rightarrow \frac{\ddot{a}}{a} = H + H^2 \tag{2.22} \]

such that we can simplify \( G_{00} \) and \( G_{ij} \) as follows

\[ R_{00} - \frac{1}{2} R g_{00} = 3H^2 \]

\[ R_{ij} - \frac{1}{2} R g_{ij} = -a^2 \left[ H^2 + 2\frac{\dot{a}}{a} \right] \delta_{ij} \tag{2.23} \]

and hence we described the geometrical part of FLRW field equations.

In a comoving rest frame the stress tensor can easily be separated into its temporal and spatial parts, but first we need to consider the Dark Energy contribution, \( \Lambda \). By simply adding the DE contribution to the density and pressure of matter, it is trivial to show from (2.10) the stress energy tensor can be written as

\[ T_{00} = \rho + \rho_\Lambda \quad \text{and} \quad T_{ij} = a^2 \delta_{ij} (p + p_\Lambda) \tag{2.24} \]

such that we can now model the expansion of the Universe with GR using Einstein’s field equations (2.14). From the temporal component of the Einstein equations, the Friedmann equation is determined

\[ H^2 = \frac{k \rho}{3} + \frac{\Lambda}{3} \tag{2.25} \]

using the DE relation (2.18). This equation describes the expansion of the universe as a function of it’s constituents, matter and DE. Since we are considering a time sufficiently later than equality, we neglect the effect of radiation pressure.

The Friedmann equation is then substituted into \( G_{ij} \) and equated to the stress energy tensor, which gives

\[ \frac{\ddot{a}}{a} = -\kappa \left( \rho + 3p \right) + \frac{\Lambda}{3} \tag{2.26} \]

which is the Raychaudhuri equation. This equation shows that accelerated expansion \( \ddot{a} > 0 \) occurs when

\[ \rho + 3p < 2\rho_\Lambda. \tag{2.27} \]
From (2.25, 2.26) we can derive the energy conservation equation, by taking the derivative of the first and equating to the second,

\[ \dot{\rho} + H(\rho + 3P) = \left( \frac{\Lambda}{3} - H^2 \right) \frac{6H}{\kappa} \]  

(2.28)

then substituting back in (2.25), the DE terms cancel

\[ \dot{\rho} + 3H(\rho + p) = 0 \]  

(2.29)

resulting in the conservation equation.

The expansion can be rewritten in a dimensionless form by dividing by a critical density \( \rho_c = 3H^2/\kappa \), thus the density parameters are defined \( \Omega_X = \rho_X/\rho_c \). In the Friedmann equation we divide on both sides by \( H^2 \) giving

\[ \frac{1}{\kappa} \frac{\dot{a}^2}{a^2} + \frac{\kappa \rho_X}{3H^2} = \Omega_m + \Omega_\Lambda \]  

(2.30)

given that the cosmological constant is written in terms of DE density. At present day the values are conventionally denoted by a subscript 0, and by taking the ratio of the matter density with present day values, we can determine the evolution in terms of values measured today,

\[ \Omega_m = \frac{\Omega_{m0}}{H_0^2} \text{ and } \Omega_\Lambda = \frac{\Omega_{\Lambda0}}{H_0^2} \]  

(2.31)

where \( \rho_m(a) = \rho_{m0} a^{-3} \) by (2.29). The total density of the universe can then be written in terms of the different constituents relating to present day values,

\[ 1 = \left( \Omega_{m0} a^{-3} + \Omega_{\Lambda0} \right) \frac{H_0^2}{H^2} \]  

(2.32)

and since redshift is related to the scale factor \( (1 + z) = a^{-1} \), it easily follows that

\[ H^2(z) = H_0^2 \left[ \Omega_{m0}(1 + z)^3 + \Omega_{\Lambda0} \right] \]  

(2.33)

giving the expansion rate of the universe in terms of unitless matter and DE density.
The Friedmann equations can be converted to conformal time

\[ H^2 = \frac{\kappa}{3} \rho a^2 + \frac{\Lambda}{3} a^2 \]

\[ H' = -\frac{\kappa}{6} a^2 (\rho + 3p) + \frac{\Lambda}{3} a^2 \]  

(2.34)

\[ \rho' + 3H(\rho + p) = 0 \]

where the primes denote the derivative with respect to conformal time \( \eta \). This set of equations govern how the density of matter and energy contained in the universe expands space, and how the expansion decreases the density and pressure there in. The model is what will be used subsequently to describe the evolution of the background cosmology, and deviations from this universal average are called perturbations, which will be discussed in the next section.

2.1.6 Dark energy and Modified theories of gravity

Despite the many observational successes of \( \Lambda CDM \) model, there are a few theoretical shortcomings. One such example is the fine tuning problem, where the measured and theorised value for the vacuum energy differ by many orders of magnitude. In order to avoid this problem, alternative dark energy models make use of a dynamical field to describe the dark energy. A useful parameterisation of dynamical DE is

\[ w(z) = w_0 + w_a(z) \]  

(2.35)

which is defined in terms of pressure and density, \( P = \rho p \). Here \( w_0 = -1 \), using the \( \Lambda CDM \) equation of state as baseline, such that \( \omega_a = 0 \) recovers \( \Lambda CDM \).

Scalar field models with canonical kinetic energy, and given Einstein’s gravity, are called quintessence models. These models are prototypical DE models and are studied are physically well defined, and can drive the accelerated expansion. Recently many proposals have been put forward on how gravity could be modified to produce acceleration without DE, called Modified gravity theories (MG) [12].

Results from solar system and pulsar experiments have constrained possible contenders to reduce almost exactly to GR on local scales. Thus the most viable MG theories
Standard model of cosmology

use a 'screening mechanism' to switch off deviations from GR at small scales, leaving them the freedom to significantly differ on cosmological scales [21]. But GR+$\Lambda CDM$ expansion histories very often fall within the parameter space of MG models, making it difficult to distinguish them on background evolution alone [10]. Often in MG theories the effective strength of gravity is changed on non-linear scales, and can have significant implications even if the linear regime is left relatively unmodified [47].

2.2 Distances and volume

An observer comoving with a galaxy receives light from distant galaxy at redshift $z$.

The comoving radial distance to the galaxy is

$$\chi(z) = \int_0^z \frac{c \, dz'}{H(z')}$$

which follows from considering infinitesimal radial displacements corresponding to displacements along the lightray.

This enables us to determine the distances to objects at a time the light was emitted, as opposed to the separation observed today. The past light cone determines the furthest
possible point observable, given that the light takes time to reach an observer, Fig 2.5. So assuming the ΛCDM model, the size of the observable universe is \( \sim 45 \) Gpc.

The Hubble radius indicates at which separation distance the expansion of space between two points exceed the speed of light, and therefore no interaction would be possible. Hence the event horizon determines the furthest distance that will ever be observable, assuming the accelerated expansion of the universe.

The particle horizon is the maximum distance a photon emitted could travel, thus intersects present day surface at the same distance as the past light cone at recombination.

The angular diameter distance \( D_A \) is defined as follows. If a distant object of proper area \( A \) subtends a solid angle \( \Delta \Omega \) at the observer (Fig 2.6), then

\[
D_A^2 = \frac{A}{\Delta \Omega} \quad (2.37)
\]

If the radius \( R \) of the object subtends an angle \( \Delta \theta \), where \( \Delta \Omega = \Delta \theta^2 \) then

\[
D_A = \frac{R}{\Delta \theta}. \quad (2.38)
\]

In a flat FLRW model, (2.38) becomes

\[
D_A(z) = \frac{\chi(z)}{1 + z}, \quad (2.39)
\]

\[\text{Fig. 2.6} \text{ The angular diameter distance, } D_A.\]
and in order to determine density distributions in the sky, it is also necessary to find the volume of a sky survey. This is dependent on sky fraction considered

$$f_{\text{sky}} = \frac{\Omega_s}{4\pi}$$ (2.40)

given solid angle of survey area $\Omega_s$. The comoving volume of a redshift bin width $\Delta z$ at redshift $z$ is

$$\Delta V(z) = 4\pi f_{\text{sky}} \chi^2(z) \Delta \chi$$ (2.41)

which can be integrated from $\chi_{\text{min}}$ to $\chi_{\text{max}}$ to get

$$V = \frac{4\pi f_{\text{sky}}}{3} \left[ \chi_{\text{max}}^3 - \chi_{\text{min}}^3 \right]$$ (2.42)
2.3 Perturbation theory

The assumption that the universe is perfectly isotropic and homogeneous breaks down on smaller scales, as evident from the observations of galaxy clusters and voids. In order to model this we make use of perturbation theory.

A background quantity is denoted by $\bar{\theta}$, and is defined as the average value throughout the universe. The perturbed parameter $\theta$ gives the value at a specific position, then deviations from the average is denoted by $\delta \theta$, and defined by

$$\delta \theta(\eta, \mathbf{x}) = \frac{\theta(\eta, \mathbf{x}) - \bar{\theta}(\eta)}{\bar{\theta}(\eta)} \Rightarrow \theta(\eta, \mathbf{x}) = \bar{\theta}(\eta) \left( 1 + \delta \theta(\eta, \mathbf{x}) \right) \quad (2.43)$$

and in this thesis we consider the background parameters to evolve on a $\Lambda$CDM background.

### 2.3.1 Perturbed Metric tensor and Christoffel symbols

The above definition is applied to the background metric in (2.1), such that the perturbed metric is given by

$$g_{\mu \nu} = \bar{g}_{\mu \nu} \left( 1 + \delta g_{\mu \nu} \right) \quad (2.44)$$

and then the FLRW metric (2.15) has the form

$$ds^2 = g_{\mu \nu} \, dx^\mu \, dx^\nu = a^2(\eta) \left[ - \left( 1 + 2 \Phi \right) d\eta^2 + \left( 1 - 2 \Psi \right) \delta_{ij} \, dx^i \, dx^j \right] \quad (2.45)$$

considering the temporal and spatial potentials, $\Phi$ and $\Psi$ respectively. If there is no anisotropic stress these potentials are the same $\Phi = \Psi$, as in GR, and thus perturbed Christoffel symbols can be determined from (2.2). It is found that

$$\Gamma^0_{00} = \mathcal{H} + \Phi'$$
$$\Gamma^i_{00} = \partial^i \Phi$$
$$\Gamma^0_{0i} = \partial_i \Phi$$
$$\Gamma^i_{0j} = \left( \mathcal{H} + \Phi' \right) \delta^i_j$$
$$\Gamma^0_{ij} = \left( \mathcal{H} - [\Phi' + 4 \mathcal{H} \Phi] \right) \delta_{ij}$$
$$\Gamma^i_{jk} = \delta_{jk} \partial^i \Phi - \delta^i_j \partial^k \Phi - \delta^i_k \partial^j \Phi$$

(2.46)
2.3.2 Perturbed Energy-momentum tensor

Proper time $\tau$, is what an observer would experience and is influenced by the perturbed space-time. It is related to the proper distance in (2.45) by

$$-\Delta \tau^2 \approx -a^2 [1 + 2\Phi] \Delta \eta^2 + a^2 [1 - 2\Psi] \Delta x^2$$  \hspace{1cm} (2.47)

where $\Delta \eta, \Delta x$ are increments along the observer world line. The perturbed four-velocity is defined as the rate of change of four-position with respect to proper time

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{(1 - \Phi)}{a} \frac{dx^\mu}{d\eta}$$  \hspace{1cm} (2.48)

to lowest order perturbations. Here $v^i = dx^i/d\eta$, is the velocity of the fluid relative to the background. Assuming that the velocity vector field is irrotational, it can be written in terms of velocity potential field $V$,

$$v^i = \frac{\partial V}{\partial x^i} = \partial^i V$$

which means that the perturbed four-velocity can be written as

$$\bar{u}^\mu + \delta u^\mu = a^{-1} \left( 1 - \Phi, \partial_i V \right); \hspace{0.5cm} u_{\mu,} + \delta u_{\mu,} = a \left( 1 - \Phi, \partial_i V \right)$$  \hspace{1cm} (2.49)

by using the condition $u^\mu u_\nu = -1$.

Consider a function which consists of the product of two functions $AB$, then the perturbed form of the function is computed analogous to the product rule, since we neglect $2^{nd}$ order perturbations

$$\delta (AB) = B \delta A + A \delta B.$$  \hspace{1cm} (2.50)

Hence the perturbed form of (2.10) is is given by

$$\delta T^\mu_\nu = (\delta \rho + \delta p) \bar{u}^\mu \bar{u}_\nu + (\bar{\rho} + \bar{p}) (\bar{u}^\mu \delta u_\nu + \bar{u}_\nu \delta u^\mu) + \delta p \delta^\mu_\nu$$  \hspace{1cm} (2.51)

and computed by splitting up the components.
The time component of the perturbed stress tensor is considered first, and

\[ \bar{u}^0 \bar{u}_0 = -1 \quad \text{and} \quad \bar{u}^0 \delta u_0 = -\Phi = -\bar{u}_0 \delta u^0 \]
\[ \Rightarrow \delta T^0_0 = -\delta \rho. \] (2.52)

Looking at the 0i components, only the middle term doesn’t cancel, so

\[ \bar{u}^0 \delta u_i = \partial^i V \Rightarrow \delta T^i_0 = (\bar{\rho} + \bar{p}) \partial^i V \] (2.53)

and finally since \( \bar{u}^\mu = a^{-1} \delta^\mu_0 \) the spatial part is simply given by

\[ \delta T^i_j = \delta \rho \delta^i_j. \] (2.54)

The perturbed conservation equations is calculated by combining (2.3) and (2.50)

\[ 0 = \delta \left( \nabla_{\mu} T^{\mu}_{\nu} \right) = \delta \left( \partial_{\mu} T^{\mu}_{\nu} + \Gamma^{\mu}_{\mu\rho} T^{\rho}_{\nu} - \Gamma^{\rho}_{\mu\nu} T^{\mu}_{\alpha} \right) \]
\[ = \delta_{\mu} T^{\mu}_{\nu} + \delta \Gamma^{\mu}_{\mu\rho} T^{\rho}_{\nu} - \delta \Gamma^{\rho}_{\mu\nu} T^{\mu}_{\alpha} - \Gamma^{\rho}_{\mu\nu} \delta T^{\mu}_{\alpha} \] (2.55)

Then setting \( \nu = 0 \) we can determine the energy conservation equation

\[ \delta \rho' + 3 \mathcal{H} \left( \delta \rho + \delta p \right) = \delta \left( \bar{\rho} + \bar{p} \right) \left( 3 \Phi' - \nabla^2 V \right) \] where \( \nabla^2 = \partial^i \partial_i \) (2.56)

and \( \nu = i \), for momentum conservation

\[ \left[ \left( \bar{\rho} + \bar{p} \right) V \right]' = - \left( \bar{\rho} + \bar{p} \right) \left( \Phi + 4 \mathcal{H} V \right). \] (2.57)

For a perfect fluid the speed of sound, defined as

\[ c_s^2 = \frac{\delta p}{\delta \rho} = \frac{\bar{p}'}{\bar{\rho}'} \] (2.58)

where primes denote derivatives with respect to conformal time. The energy and momentum conservation equations can then be rewritten in terms of the matter perturbation \( \delta = \delta \rho / \bar{\rho} \),

\[ \delta' + 3 \left( c_s^2 - w \right) \mathcal{H} \delta = \left( 1 + w \right) \left( 3 \Phi' - \nabla^2 V \right) \] (2.59)
\[
V' + \left(1 - 3c_s^2\right)\mathcal{H}V = -\Phi - \frac{\epsilon_s^2}{1 + w} \delta
\]  
(2.60)

respectively, and equation of state \( w = p/\rho \).

### 2.3.3 Perturbed Einstein tensor

The Einstein tensor in geometrical form (2.13) is then perturbed using (2.50),

\[
\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} \left( \bar{g}_{\mu\nu} \delta R + \bar{R} \delta g_{\mu\nu} \right)
\]  
(2.61)

which means the perturbed Ricci tensor and scalar needs to be calculated. Substituting the perturbed Christoffel symbols (2.46), into the Riemann tensor (2.4), then using the definition of Ricci tensor (2.7),

\[
\begin{align*}
\delta R_{00} &= 3\left( \Phi'' + 2\mathcal{H}\Phi' + \nabla^2 \Phi \right) \\
\delta R_{0i} &= 2\partial_i \left( \Phi' + \mathcal{H}\Phi \right) \\
\delta R_{ij} &= \left[ \nabla^2 \Phi - 3\Phi'' - 12\mathcal{H}\Phi' - 4\left( \mathcal{H}' + 2\mathcal{H}^2 \right) \Phi \right] \delta_{ij}.
\end{align*}
\]  
(2.62)

The perturbed Ricci scalar is found by contracting the Ricci tensor with the metric (2.8),

\[
\bar{R} + \delta R = \left( \bar{g}^{\mu\nu} + \delta g^{\mu\nu} \right) \left( \bar{R}_{\mu\nu} + \delta R_{\mu\nu} \right)
\]  
(2.63)

which yields

\[
\delta R = 2a^{-2} \left[ \nabla^2 \Phi - 3\Phi'' - 12\mathcal{H}\Phi' - 6 \left( \mathcal{H}' + \mathcal{H}^2 \right) \Phi \right].
\]  
(2.64)

It is now possible to write (2.61) in terms of perturbed potential and expansion parameter,

\[
\begin{align*}
\delta G_{00} &= 2\nabla^2 \Phi - 6\mathcal{H}\Phi' \\
\delta G_{0i} &= 2\partial_i \left( \Phi' + \mathcal{H}\Phi \right) \\
\delta G_{ij} &= \left[ 2\Phi'' + 6\mathcal{H}\Phi' + 4 \left( 2\mathcal{H}' + \mathcal{H}^2 \right) \Phi \right] \delta_{ij}.
\end{align*}
\]  
(2.65)

Before the geometrical term can be equated to the stress-energy tensor, we first account for the contribution of dark energy. \( \Lambda \) enters the energy-momentum tensor, \( T_{\mu\nu} \),

\[
T_{\mu\nu} = 8\pi G \left[ \left( \rho + p \right) u_\mu u_\nu + pg_{\mu\nu} + \bar{p} \Lambda g_{\mu\nu} \right]
\]  
(2.66)

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2.3 Perturbation theory

with a relation as in (2.18). This enables us to include $\Lambda$ in the perturbation by simply adding the contribution from dark energy to the background. Again equating the temporal and spatial components of the perturbed Einstein (2.65) and energy-momentum tensor (2.51),

$$\delta G_{\mu\nu} = \kappa \delta T_{\mu\nu}$$  \hspace{1cm} (2.67)

the perturbed field equations can be written as

\begin{align*}
\text{Poisson (00)} : \nabla^2 \Phi &= 4\pi G a^2 \delta \rho + 3 \mathcal{H} (\Phi' + \mathcal{H} \Phi) \\
\text{Momentum Constraint (0i)} : \Phi' + \mathcal{H} \Phi &= -4\pi G a^2 \bar{\rho} (1 + w) V \\
\text{Bardeen (ij)} : \Phi'' + 3 \mathcal{H} \Phi' + \left[ 4 \mathcal{H}' + (2 + 3w) \mathcal{H}^2 \right] \Phi &= 4\pi G a^2 \delta p
\end{align*}  \hspace{1cm} (2.68)

and can be used to determine the evolution of a perturbed universe, assuming the existence dark energy.

2.3.4 Bardeen equation

We write $\delta \rho$ in terms of $\Phi$ and $\mathcal{H}$ using the Poisson in (2.68). By relating density to pressure $\delta p = c_s^2 \delta \rho$, and considering the evolution of the expansion parameter

$$\mathcal{H}' = -\frac{1}{2} (1 + 3w) \mathcal{H}^2$$

the Bardeen equation in (2.68) is rewritten as

$$\Phi'' + 3 \left( 1 + c_s^2 \right) \mathcal{H} \Phi' + 3 \left( c_s^2 - w \right) \mathcal{H}^2 \Phi = c_s^2 \nabla^2 \Phi$$  \hspace{1cm} (2.69)

and is used to determine the evolution of $\Phi$. The field equations are then written in the form

\begin{align*}
\text{Poisson (00)} : \nabla^2 \Phi &= 4\pi G a^2 \bar{\rho} \delta + 3 \mathcal{H} (\Phi' + \mathcal{H} \Phi) \\
\text{Momentum Constraint (0i)} : \Phi' + \mathcal{H} \Phi &= -4\pi G a^2 \bar{\rho} (1 + w) V \\
\text{Bardeen (ij)} : \Phi'' + 3 \left( 1 + c_s^2 \right) \mathcal{H} \Phi' + 3 \left( c_s^2 - w \right) \mathcal{H}^2 \Phi &= c_s^2 \nabla^2 \Phi
\end{align*}  \hspace{1cm} (2.70)

where the matter density contrast is

$$\delta = \frac{\delta \rho}{\bar{\rho}}$$  \hspace{1cm} (2.71)
the Poisson equation is then used to determine the $\delta$, which is used in momentum conservation to determine velocity potential $V$.

### 2.3.5 Co moving density contrast

It is useful to define comoving matter density perturbation

\[ \Delta_m = \delta_m + \frac{\rho_m'}{\rho_m} V_m = \delta_m - 3H V_m \]

which is the density contrast that is measured by observers comoving with the matter. On super-Hubble scales the rate of change of velocity potential is proportional to the potential

\[ |V_m'| \sim H|V_m| \Rightarrow 2H|V_m| \sim |\Phi| \]

by the momentum conservation equation (2.59). On sub-Hubble scales where $|\Phi| \ll |\delta_m| \Rightarrow \Delta_m \approx \delta_m$.

The perturbed conservation and field equations can be written in terms of the comoving matter perturbation. The continuity (2.59) is then expressed as

\[ \Delta_m'' - \nabla^2 V_m = 0 \]  

and the Euler equation (2.60) is written as

\[ V_m' + H V_m = -\Phi. \]

The perturbed field equations (2.70) then become

- Poisson: $\nabla^2 \Phi = 4\pi G a^2 \rho_m \Delta_m$
- momentum constraint: $\Phi' + H\Phi = -4\pi G a^2 \rho_m V_m$
- Bardeen: $\Phi'' + 3H\Phi' + \Lambda^2 \Phi = 0$

By taking the derivative of (2.74) with respect to time and substituting in (2.75),

\[ \Delta_m'' + H\Delta_m' - \frac{3}{2} \Omega_m H^2 = 0 \]

which gives evolution of the density contrast.
2.3.6 Initial conditions

In order to determine the evolution of the universe via the differential equations in (2.76), we need to assume initial conditions from an inflationary model. Cosmic inflation or alternative theories are supposed to generate the seed of the anisotropies imprinted on the cosmic microwave background radiation and the inhomogeneity of the matter distribution. Thus primordial overdensities can be statistically measured using the CMB observations from surveys like WMAP and Planck. Even though this surface is almost perfectly smooth (up to the order of $10^{-5}$K), we believe the inhomogeneity in the CMB make up the seeds of the earliest galaxies.

![Fig. 2.7 Primordial perturbations are determined by looking at temperature fluctuations on the surface of last scattering. These irregularities have grown over time and is considered the seeds of the large scale structures we observe today. Image credit - http://www.bingotelescope.org/wp-content/uploads/2017/08/cartoon-bao.jpg](http://etd.uwc.ac.za/)

At the largest of scales these primordial fluctuations are frozen into the large scale structure of the universe, since these scales exceed that of the horizon, and hence do not experience gravitational interactions due to causality.
The Poisson equation in (2.76) can be rewritten in the Fourier domain

\[-k^2 \Phi(k, z) = \frac{3}{2} \frac{\Omega_{m0} H_0^2}{a} \delta(k, z) \quad (2.78)\]

since the Fourier transform of \( \nabla \) is given by \( ik \). Before we write the potential in terms of the primordial curvature fluctuations, first consider separating out large scale modes \( k_{LS} \), and times sufficiently close to decoupling \( z_+ \), where these remnants could still be observed. Then \( \Phi \) can be expressed in terms of ratios

\[\Phi(k, z) = \Phi(k_{LS}, z_+) \frac{\Phi(k, z_+)}{\Phi(k_{LS}, z_+)} \frac{\Phi(k, z)}{\Phi(k, z_+)} \quad (2.79)\]

such that we define the transfer and growth function as

\[T(k) = \frac{\Phi(k, z_+)}{\Phi(k_{LS}, z_+)} \quad \text{and} \quad \frac{D(z)}{a} = \frac{\Phi(k, z)}{\Phi(k, z_+)} \quad (2.80)\]

respectively. On large scales during matter domination the primordial curvature perturbations \( \zeta \) is given by

\[\Phi_m(k_{LS}, z_+) = -\frac{3}{5} \zeta(k_{LS}) \quad (2.81)\]

where subscript \( m \) indicates the matter domination era. This means we have separated the potential into evolution of scale and time, and can be expressed in terms of \( T(k) \) and \( D(z) \) as

\[\Phi(k, z) = -\frac{3}{5} T(k) \frac{D(z)}{a} \zeta(k). \quad (2.82)\]

Substituting this into the Poisson equation (2.78), and after some simplification

\[\delta(k, z) = \frac{2}{5} \frac{T(k)}{\Omega_{m0} H_0^2} \frac{D(z)}{k^2} \zeta(k) \quad (2.83)\]

is the expression for matter perturbation in terms of primordial curvature perturbation in the Fourier domain. For simplicity let

\[Q(k, z) = k^2 T(k) D(z) \quad \text{and} \quad R_o = \frac{2}{5} \Omega_{m0} H_0^2 \quad (2.84)\]

such that the perturbation is given by

\[\delta(k, z) = R_o Q(k, z) \zeta(k). \quad (2.85)\]
The primordial curvature perturbations from inflation on super-Hubble scales is

$$|\zeta(k)| = C_k^{(n_s-4)/2}$$

(2.86)

with spectral index $n_s \approx 1$.

## 2.4 Growth rate and RSD

A decisive way of discerning between different models of gravity is the linear growth rate of large scale structure formation $f$, which is increasingly better constrained via Redshift Space Distortions and other peculiar velocity measurements [38]. One can model RSD on large scales using linear cosmological perturbation theory [16].

![Fig. 2.8](http://etd.uwc.ac.za/) The evolution of the large scale structure of the universe with redshift. Image credit - http://english.icosmo.ir/research/large-scale-structure-formation/
2.4.1 Evolution of $f$

The rate at which structure grows from small perturbations offers a key observational discriminant between different cosmological models [46]. The evolution equation for the matter density contrast (2.77) can be written in terms of the growth rate of large scale structure formation.

$$f = \frac{1}{H} \frac{D'}{D} = \frac{d \ln D}{d \ln a} \Rightarrow \Delta'_m = f \mathcal{H} \Delta_m$$

(2.87)

since $\Delta_m(k, z) = D(z) \Delta_m(k, 0)$. Thus $f$ is defined as the logarithmic rate of change of comoving matter perturbation $\Delta_m$. The evolution equation of growth rate can be found by rewriting (2.77) in terms of $f$,

$$f' + \frac{1}{2} (4 - 3 \Omega_m) f + f^2 = \frac{3}{2} \Omega_m.$$  

(2.88)

The growth rate can be parameterized in a number of different ways, but the simplest is using a power-law of the matter density of the universe [19],

$$f(z) = \Omega_m(z)^\gamma.$$  

(2.89)

The exponent $\gamma$ is called the growth index, and is assumed to be constant as a function of time. Different theories of gravity predict distinctly different $\gamma$, and is therefore useful to determine constraints on this parameter as well. For example the standard $\Lambda CDM$ predicts $\gamma \approx 0.55$, whereas self-accelerating MG theories expect $\gamma$ significantly different, eg $\gamma \approx 0.65$ [27] and $\gamma \approx 0.68$ [23].

2.4.2 Redshift Space Distortions

The cause of redshift space distortions is due to how we infer distances using the redshift of an emitter. The redshift of a galaxy is influenced not only by the background expansion, but the peculiar velocity as well, thus we observe the position of the galaxy as a combination of the two, called redshift space. Therefore in order to make precision measurements, this distortion needs to be taken into account.
2.4 Growth rate and RSD

The Hubble redshift is modified by a Doppler correction due to the peculiar velocities of galaxies, so to first order we can use the background light ray with energy $E$,

$$\tilde{k}^\mu = a^{-1}E(1, -n), \quad u_\mu = a(-1 - \Phi, v)$$

$$\Rightarrow u_\mu \tilde{k}^\mu = -E(1 + v \cdot n + \Phi) \tag{2.90}$$

to determine the positional shift in redshift space. By neglecting the gravitational potential relative to the peculiar velocity, we can write down the observed redshift as

$$1 + z = \frac{\left(\frac{u_\mu \tilde{k}^\mu}{u_\mu \tilde{k}^\mu}_s\right)}{E_s(1 + v_s \cdot n)} \approx \frac{E_s(1 + v_s \cdot n)}{E_o(1 + v_o \cdot n)}. \tag{2.91}$$

Considering the observer as stationary $v_o = 0$, and the change in energy as due to the expanding background,

$$1 + \tilde{z} = \frac{E_s}{E_o} \tag{2.92}$$

the redshift can be expressed as

$$1 + z = (1 + \tilde{z})(1 + v_s \cdot n)$$

$$\Rightarrow \delta z = (1 + \tilde{z})(v \cdot n) \tag{2.93}$$

when dropping the subscript. This is the change in redshift space due to the peculiar velocity of the source. The real and observed comoving positions are

$$x = \chi(z)n, \quad x_{\text{obs}} = \chi_{\text{obs}}(z)n \tag{2.94}$$

respectively.

By Taylor expanding the observed comoving distance to the first order,

$$\chi_{\text{obs}}(z) = \chi(\tilde{z} + \delta z) = \chi(\tilde{z}) + \frac{1}{(1 + \tilde{z})H} \delta z \tag{2.95}$$

means the observed position is given by

$$x_{\text{obs}} = x + \frac{v \cdot n}{H}. \tag{2.96}$$
The conservation of number counts can relate the two quantities via number density of galaxies observed $n_g$, thus

$$dN = n_{g\,\text{obs}} \, d^3x_{\text{obs}} = n_g \, d^3x$$

$$\Rightarrow \tilde{n}_g(1 + \delta_g) \, d^3x = \tilde{n}_g(1 + \delta_{g\,\text{obs}}) \, d^3x_{\text{obs}}.$$  

Assuming the line of sight is along the $z$-direction and using cylindrical coordinates,

$$d^3x_{\text{obs}} = \frac{\partial \chi_{\text{obs}}}{\partial \chi} \, d^3x.$$  

From (2.95) it can be shown that

$$\frac{\partial \chi_{\text{obs}}}{\partial \chi} = 1 + \frac{1}{(1 + \tilde{z})H} \frac{\partial \delta z}{\partial \chi} + \frac{\delta z}{(1 + \tilde{z})} \frac{\partial H^{-1}}{\partial \chi}$$  

but since Hubble rate changes very little, the term

$$\frac{1}{H^2} \frac{\partial H}{\partial \chi} \ll \frac{1}{(1 + \tilde{z})H}$$  

which means we can exclude it. Thus we approximate (2.97) as

$$(1 + \delta_g) \, d^3x = (1 + \delta_{g\,\text{obs}}) \left(1 + \frac{1}{H} \frac{\partial v \cdot n}{\partial \chi}\right) \, d^3x$$  

after substituting in $\delta z$ from (2.93). Simplifying this equation and ignoring terms that is second order in perturbation, leads to the final form

$$\delta_{g\,\text{obs}} = \delta_g - \frac{1}{H} \frac{\partial}{\partial \chi} v \cdot n$$  

called the Kaiser formula [20], demonstrated in Fig 2.9. In order to relate the observed galaxies to the underlying dark matter distribution, we assume galaxy velocity is equal to the DM velocity. Also the galaxy overdensities in the number counts are related to the DM overdensities,

$$\delta_g = b \delta_m$$  

given a bias model, $b$. 

http://etd.uwc.ac.za/
In Fourier space the continuity equation (2.74) becomes
\[ \delta_m' = k^2 V_m, \]  \hspace{1cm} (2.104)

thus by (2.87)
\[ k^2 V_m = f^2 H \delta_m, \]  \hspace{1cm} (2.105)

Now considering the radial velocity term in the form of velocity potential,
\[ \partial_\chi \mathbf{n} \cdot \mathbf{v} = n^i \partial_i \left( n^j \partial_j V_m \right) \rightarrow (\mathbf{n} \cdot \mathbf{k})^2 V_m - (\mathbf{n} \cdot \hat{k})^2 k^2 V_m \]  \hspace{1cm} (2.106)

after the Fourier transform. If we consider the angle between the wave vector and pointing direction,
\[ \mu = \mathbf{n} \cdot \hat{k} \]  \hspace{1cm} (2.107)

the radial velocity terms can be written in terms of growth rate,
\[ \partial_\chi \mathbf{v} \cdot \mathbf{n} = -\mu^2 f^2 H \delta_m \]  \hspace{1cm} (2.108)

and hence the Kaiser becomes
\[ \delta_{\text{obs}} = (b + f \mu^2) \delta_m. \]  \hspace{1cm} (2.109)
Chapter 3

Two-point correlation function

Redshift surveys allow us to simultaneously measure the cosmic expansion history and growth rate of structure formation, by statistically analysing the three-dimensional clustering of galaxies [32]. In order to understand the distribution of matter in the universe given observations from a sky survey, we need to determine how galaxies and larger structures are distributed in space and how frequently they are found.

![Image](http://www.astro.ucla.edu/wright/BAO-cartoon.jpg)

The method used to correlate the average matter density perturbations at different separations is called the 2-point correlation function. Let \( \mathbf{x}(z) \) and \( \mathbf{r}(z) \) denote the position and separation vectors respectively, such that \( \mathbf{x'} = \mathbf{x} + \mathbf{r} \).

![Fig. 3.1 Determining \( \xi(r) \) from sky surveys have shown that there is a strong correlation corresponding \( \sim 150(1 + z)^{-1} \) Mpc, called the Baryonic Acoustic Oscillations. Image credit - http://www.astro.ucla.edu/wright/BAO-cartoon.jpg](http://etd.uwc.ac.za/)

http://etd.uwc.ac.za/
Two-point correlation function

Then the correlation of density perturbations $\delta$, are given by

$$\xi(r) = \langle \delta(x) \delta(x') \rangle,$$

(3.1)

where $\xi$ is independent of $x$, reflecting statistical homogeneity, and depends on $r$ not $r'$, reflecting statistical isotropy. This analysis have been applied to the recent sky surveys and it has been shown that there is a strong correlation at $\sim 150(1+z)^{-1}$ Mpc due to the Baryonic Acoustic Oscillations. Before the era of recombination there was primordial over densities that trapped light oscillating around the center of mass. The slight excess of over densities at $r = 150(1+z)^{-1}$ Mpc can be used as a standard measure of the expansion of the universe.

### 3.1 Fourier power spectrum

Recall the Fourier and inverse Fourier transform is defined as

$$\delta(k) = \int dx \delta(x)e^{-ikx} \quad \text{and} \quad \delta(x) = \int \frac{dk}{(2\pi)^3} \delta(k)e^{ikx}$$

(3.2)

respectively. Then the matter correlation function in $k$-space can be found by correlating the Fourier transform of $\delta(x)$

$$\langle \delta(k)\delta^*(k') \rangle = \int \int dx dx' \langle \delta(x)\delta^*(x') \rangle e^{-ikx}e^{ik'x'}$$

$$= \int \int dx dr \xi(r)e^{-ikx}e^{ik'(x+r)}$$

(3.3)

and since by definition the Dirac delta has the form

$$\delta^D(k - k') = \frac{1}{(2\pi)^3} \int dx e^{ik'x},$$

the correlation reduces to

$$\langle \delta(k)\delta^*(k') \rangle = (2\pi)^3\delta^D(k' - k) \int dr \xi(r)e^{ik'r}.$$

(3.4)

The Fourier power spectrum $P(k)$, is then defined as the Fourier transform of the 2-point correlation function,

$$P(k') = \int dr \xi(r)e^{ik'r}$$

(3.5)
such that
\[(2\pi)^3\delta^D(\mathbf{k}' - \mathbf{k})P(k') = \langle \delta(\mathbf{k})\delta^*(\mathbf{k}') \rangle\] (3.6)

To include the effect of RSD we substitute (2.109) in (3.6),
\[\langle \delta_{\text{obs}}(\mathbf{k})\delta^*_{\text{obs}}(\mathbf{k}') \rangle = (b + f\mu^2)^2 \langle \delta(\mathbf{k})\delta^*(\mathbf{k}') \rangle \]
\[= (2\pi)^3\delta^D(\mathbf{k}' - \mathbf{k})(b + f\mu^2)^2 P_m(k).\] (3.7)

This method assumes a flat-sky approximation, which means we do not account for wide-angle correlations.

For a complete model of RSD on the power spectrum, one needs to consider a “dispersion model”, which includes a damping effect along with the Kaiser formula [31]. The small-scale damping takes into account a non-linear effects. This effect would have to be considered when the forecasts include the non-linear scales, but in this thesis we consider only linear scales.

The bias and growth rate parameters can separately be extracted by measuring the monopole and quadrupole, and is calculated by expanding the power spectrum in Legendre polynomials \(P_\ell\),
\[P_{g,\ell}(\eta, k) = \frac{2\ell + 1}{2} \int_{-1}^{1} d\mu P_\ell P_g(\eta, k, \mu).\] (3.8)

For the monopole \(\ell = 0\), the Legendre polynomial is given by \(P_0 = 1\), thus
\[P_{g,0}(\eta, k) = \frac{1}{2} \int_{-1}^{1} d\mu (b + f\mu^2)^2 P_g(\eta, k) \]
\[= \left[ b^2 + \frac{2}{3}f + \frac{1}{5}f^2 \right] P_g(\eta, k).\] (3.9)

The quadropole \(\ell = 2\), has the Legendre polynomial \(P_2 = \frac{1}{2}(3\mu^2 - 1)\),
\[P_{g,2}(\eta, k) = \frac{5}{2} \int_{-1}^{1} d\mu \frac{1}{2} (3\mu^2 - 1) (b + f\mu^2)^2 P_g(\eta, k) \]
\[= \left[ \frac{4}{3}bf + \frac{4}{7}f^2 \right] P_g(\eta, k)\] (3.10)

which means we can simultaneously solve the above equations to extract \(f\) and \(b\) (assuming we know the amplitude of \(P_g\)). If we know \(b\) we can take the ratio and solve for \(f\).
3.2 Angular power spectrum

Another transform of the 2-point correlation is called spherical harmonic transform, which produces the angular power spectrum $C_\ell$. Let us first consider the convention used to describe the positional coordinate of a astrophysical source. It is convenient to define the position in the sky in terms of comoving distance and pointing direction $\mathbf{n}$,

$$x(z) = \chi(z) \mathbf{n} \quad \text{with} \quad \mathbf{n} \cdot \mathbf{n} = 1$$

such that the exponential term in the inverse Fourier transform can be expressed as

$$k \cdot x = k \mu \chi.$$

3.2.1 Spherical harmonic coefficient

The matter perturbations can be expanded in spherical harmonics on the sky at each $z$:

$$\delta(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\chi(z)) Y_{\ell m}(\mathbf{n})$$

where the multipole $\ell$ corresponds to the angular size considered. By multiplying both sides in (3.12) with the Hermitian conjugate of the spherical harmonic $Y_{\ell m}^*$, and
3.2 Angular power spectrum

integrating over the entire sky, using the property of orthogonality

\[ \int d\Omega_n Y_{\ell m}(\mathbf{n}) Y_{\ell' m'}^*(\mathbf{n'}) = \delta_{\ell \ell'} \delta_{m m'} \]  \hspace{1cm} (3.13)

the harmonic coefficients of the density contrast are

\[ a_{\ell m}(z) = \int d\Omega_n \delta(x) Y_{\ell m}^*(\mathbf{n}). \]  \hspace{1cm} (3.14)

The matter perturbation is expanded to the Fourier domain, and rewriting the exponential term by using the identity

\[ e^{i k \mu \chi} = \sum_{\ell=0}^{\infty} (2\ell + 1) (i)^\ell P_\ell(\mu) j_\ell(k \chi) \]  \hspace{1cm} (3.15)

(3.14) then becomes

\[ a_{\ell m}(z) = \frac{1}{(2\pi)^3} \int \int d\Omega_n \int d\mathbf{k} \delta(k) Y_{\ell m}^*(\mathbf{n}) \sum_{\ell'=0}^{\infty} (2\ell' + 1) (i)^{\ell'} P_{\ell'}(\mu) j_{\ell'}(k \chi). \]  \hspace{1cm} (3.16)

The Legendre polynomial has the property

\[ \int d\Omega_n P_{\ell'}(\mu) Y_{\ell m}^*(\mathbf{n}) = \frac{4\pi}{2\ell' + 1} Y_{\ell m}^*(\mathbf{n}) \delta_{\ell \ell'} \]  \hspace{1cm} (3.17)

hence we can simplify \( a_{\ell m}(z) \) to

\[ a_{\ell m}(z) = \frac{(i)^\ell}{2\pi^2} \int d\mathbf{k} \delta(k) j_\ell(k \chi) Y_{\ell m}^*(\mathbf{n}) \]  \hspace{1cm} (3.18)

and thus the complex conjugate is given by

\[ a_{\ell m}^*(z) = \frac{(-i)^\ell}{2\pi^2} \int d\mathbf{k} \delta(k) j_\ell(k \chi) Y_{\ell m}(\mathbf{n}). \]  \hspace{1cm} (3.19)
3.2.2 Angular correlation

The angular power spectrum is the correlation of the spherical harmonic coefficients, and we can define $C_\ell$ similar to (3.1)

$$ \langle a_\ell m(z_i) a_{\ell' m'}^*(z_j) \rangle = C_\ell(z_i, z_j) \delta_{\ell \ell'} \delta_{m m'} \quad (3.20) $$

thus to determine $C_\ell$, (3.18, 3.19) are substituted into (3.20)

$$ \langle a_\ell m(z_i) a_{\ell' m'}^*(z_j) \rangle = \left( -i \right)^{\ell\ell'} \int \frac{dk \, dk'}{2(2\pi)^2} \int \frac{\delta(k')}{k'} \delta^*(k) \, Y_{\ell m}(n) \, Y_{\ell' m'}^*(n') \quad (3.21) $$

Conventionally the angular correlations are written in terms of the primordial curvature perturbations $\zeta$, where

$$ P_\zeta(k) = \frac{A_s}{k^3} \left( \frac{k}{k_s} \right)^{n_s-1} \quad (3.22) $$

with scalar amplitude $A_s = 2.14 \times 10^{-9}$, and pivot scale $k_s = 0.05 \text{ Mpc}^{-1}$, from Planck 2015 [2]. We can go from $\delta$ to $\zeta$ using relation (2.83) as follows

$$ \langle \delta(k, z_i) \delta^*(k', z_j) \rangle = \left( 2\pi \right)^3 \delta_D(k' - k) R^2 \Omega Q(k, z_i) Q(k, z_j) P_\zeta(k) \quad (3.23) $$

which yields the result

$$ \langle a_\ell m(z_i) a_{\ell' m'}^*(z_j) \rangle = \frac{2}{\pi} \left( -i \right)^{\ell\ell'} \int k^2 dk \, j_\ell(k\chi(z_i)) Q(k, z_i) j_{\ell'}(k\chi(z_j)) Q(k, z_j) \times R^2 \Omega P_\zeta(k) \int d\Omega_n Y_{\ell m}(n) Y_{\ell' m'}^*(n') \quad (3.24) $$

after the integral is split into angular and radial parts by converting to spherical coordinates $\int dk = \int k^2 dk \int d\Omega_n$.

Using the orthogonal property of spherical harmonics, it then easily follows from definition (3.20) that

$$ C_\ell(z_i, z_j) = 4\pi R^2 \int d\ln k \, P_\zeta(k) \, j_\ell(k\chi(z_i)) Q(k, z_i) \, j_{\ell'}(k\chi(z_j)) Q(k, z_j) \quad (3.25) $$

where the dimensionless primordial power spectrum is defined by

$$ P_\zeta(k) = \frac{k^3 P_\zeta}{2\pi^2} \quad (3.26) $$
The output given by the CAMB_sources software is in the form of a \( n \times n \) matrix,

\[
C(\ell) = \begin{bmatrix}
C_{11} & \cdots & C_{1n} \\
\vdots & \ddots & \vdots \\
C_{n1} & \cdots & C_{nn}
\end{bmatrix}
\]  \quad (3.27)

with components \( C_{ij}(\ell) = C_\ell(z_i, z_j) \).

The galaxy angular power spectrum (without RSD) follows from \( \delta_g = b \delta \), so that (3.25) becomes

\[
C_g^\ell(z_i, z_j) = 4\pi R_0^2 b(z_i) b(z_j) \int d\ln k \, \mathcal{P}_\zeta(k) \, j_\ell(k \chi(z_i)) Q(k, z_i) \, j_\ell(k \chi(z_j)) Q(k, z_j)
\]  \quad (3.28)

### 3.2.3 Influence of RSD

The angular power spectrum will also be influenced by the effect of redshift space distortions. It was shown in (2.109) that the contribution of RSD to the observed galaxy density contrast is

\[
\delta_{\text{RSD}}^\ell = \mu^2 \delta_m(k)
\]  \quad (3.29)

whose angular transform is

\[
a_{\ell m}^{\text{RSD}}(z) = \frac{1}{(2\pi)^3} \int dk \int d\Omega_n \, f(z) \mu^2 \delta(k) e^{ik \mu X^*_\ell m(n)}.
\]  \quad (3.30)

Considering that

\[
\frac{\partial^2}{\partial(k \chi)^2} e^{ik \mu X} = -\mu^2 e^{ik \mu X}
\]

we can rewrite the angular perturbation as

\[
a_{\ell m}^{\text{RSD}}(z) = -\frac{1}{(2\pi)^3} \int dk \ f(z) \delta(k) \ \frac{\partial^2}{\partial(k \chi)^2} \int d\Omega_n \ e^{ik \mu X^*_\ell m(n)}.
\]  \quad (3.31)

The exponential is expanded using (3.15), and the property of the Legendre polynomial (3.17) used as before, then let primes denote derivatives with respect to \( k \chi \)

\[
a_{\ell m}^{\text{RSD}}(z) = \frac{-i\ell}{2\pi^2} \int dk \ f(z) \delta(k) \ j''_\ell(k \chi) Y^*_\ell m(n).
\]  \quad (3.32)
The influence of RSD on the angular power spectrum can now be determined with (3.20),

\[ C_{\ell}^{RSD}(z_i, z_j) = 4\pi R_0^2 \int d\ln k \, \mathcal{P}_\zeta(k) \, f(z_i) \, Q(k, z_i) \, j''_\ell(k \chi(z_i)) \]
\[ \times f(z_j) \, Q(k, z_j) \, j''_\ell(k \chi(z_j)). \]  

(3.33)

### 3.2.4 Total correlation

The spherical harmonic coefficient of the total correlation including RSD and matter perturbation can be determined by combining (3.28) and (3.33) such that

\[ C_{\ell}^{\text{obs}}(z_i, z_j) = 4\pi R_0^2 \int d\ln k \, \mathcal{P}_\zeta(k) \, Q_i \, Q_j \left[ b_{ij} \, j'_\ell(k \chi_i) \, b_{ij} \, j'_\ell(k \chi_j) + f_{ij} \, j''_\ell(k \chi_i) \, f_{ij} \, j''_\ell(k \chi_j) - b_{ij} \, j'_\ell(k \chi_i) \, f_{ij} \, j''_\ell(k \chi_j) - f_{ij} \, j'_\ell(k \chi_i) \, b_{ij} \, j'_\ell(k \chi_j) \right] \]

(3.34)

where the subscript \( i \) indicates the \( i \)th redshift bin is considered. The cross-correlation between RSD and galaxy density contrast

\[ C_{\ell}^{gRSD}(z_i, z_j) = -4\pi R_0^2 \int d\ln k \, \mathcal{P}_\zeta(k) \, b_{ij} \, Q_i \, j''_\ell(k \chi_i) \, f_j \, Q_j \, j''_\ell(k \chi_j) \]

(3.35)

with galaxy and RSD perturbations taken at redshift bin \( i \) and \( j \) respectively. Thus the total correlation can be written in terms of

\[ C_{\ell}^{\text{obs}}(z_i, z_j) = C_{\ell}^g(z_i, z_j) + C_{\ell}^{RSD}(z_i, z_j) + C_{\ell}^{gRSD}(z_i, z_j) + C_{\ell}^{RSDg}(z_i, z_j). \]

(3.36)
Chapter 4

Cosmology with Intensity mapping

In order to observe the growth rate of large scale structure formation, one will need to perform an experiment. This thesis will focus on data expected from future HI intensity mapping survey called MeerKAT Large Area Synoptic Survey (MeerKLASS) produced by MeerKAT telescope. The MeerKAT telescope array will eventually be expanded into what is known as the Square Kilometer Array (SKA). Forecasts were done to estimate the expected constraints from the first phase called SKA1. The future surveys will use large telescope arrays, with low-noise wideband receivers to map the matter distribution over an unprecedented volume and redshift range [28].

HI IM observes the distribution of neutral hydrogen via the intensity of the 21cm line, without resolving individual galaxies [25]. The emission is a result of a spin-flip transition in HI, as the proton and electron transition from parallel spin to anti-parallel spin. The main advantages of this technique, as compared to galaxy surveys, is that larger volumes can be surveyed in relatively short period of time. There is a one-to-one relation between observed frequency and redshift, and is therefore very accurate. All the signal, including the inter-galactic gas is recorded, and therefore is expected to be a good tracer of matter distribution [35].

The Green Bank Telescope (GBT) has made the first detection of the HI emission at $z \approx 0.8$, [11]. This detection showed the HI IM is indeed feasible, and a tool to study the large-scale structure of the Universe. It is also difficult to do, where the main challenges was the astrophysical contamination and the systematics that are present in the observed HI signal.
Fig. 4.1 A simulated full-sky map of the temperature fluctuations expected from HI intensity mapping. The colour bar is given in mK. Image generated using HEALPix

The cross-correlation of intensity maps with galaxy surveys is a robust measure of the power spectrum which diminishes systematics caused by instrumental effects and foreground removal, [50]. Results from 21-cm intensity maps acquired from the Parkes radio telescope was cross-correlated with galaxy maps from the 2dF galaxy survey, with cross correlation detected at a significance of 5.18 σ, [3]. This technique is called multi-tracing, and will be considered in future work.

4.1 Neutral hydrogen bias

The nature of dark matter (DM) is such that it does not interact with the electromagnetic spectrum, it only interacts gravitationally, and therefore can not be directly detected in observation. In order to measure the DM perturbations, we use a tracer of the underlying distribution via visible baryonic matter. Generally one looks at number counts in a galaxy survey, but there is a high bias due to the point-like distribution of galaxy numbers. Neutral Hydrogen being the most abundant element in the universe, not only trace galaxies but also the distribution of the filaments in the large scale structure. The expected HI density as a function of dark matter mass $M$ can be determined as in [42],

$$\rho_{\text{HI}}(z) = \int_{M_{\min}}^{M_{\max}} dM \frac{dn}{dM} M_{\text{HI}}(M, z)$$

(4.1)

where the interval of the halo masses $[M_{\min}, M_{\max}]$ is dependent on the ability to detect HI in different sized perturbations i.e. galaxy clusters.
4.1 Neutral hydrogen bias

In order to model the power spectrum of the HI content, we need to know the relation between the dark matter and HI distributions,

\[ b_{\text{HI}}(z) = \rho_{\text{HI}}^{-1} \int_{M_{\text{min}}}^{M_{\text{max}}} dM \frac{dn}{dM} M_{\text{HI}}(M, z) b(z, M) \] (4.2)

called the HI bias \( b_{\text{HI}} \).

The following function describes the dimensionless HI matter density of the universe, written in terms of the average HI density \( \rho_{\text{HI}} \),

\[ \Omega_{\text{HI}}(z) \equiv \frac{\rho_{\text{HI}}(z)}{\rho_{c,0} [\Omega_m(1 + z)^3 + \Omega_\Lambda]} \] (4.3)

assuming a ΛCDM flat model and \( \rho_{c,0} \) being the critical density at present day \( (z = 0) \).

A 2nd order polynomial fit describes the HI content as in [43],

\[ \Omega_{\text{HI}}(z) = 4.83 \times 10^{-1} + 3.88 \times 10^{-1} z - 6.51 \times 10^{-4} z^2 \] (4.4)

and from the same paper the HI bias was modeled by the fit

\[ b_{\text{HI}}(z) = 6.67 \times 10^{-1} + 1.78 \times 10^{-1} z + 5.02 \times 10^{-2} z^2. \] (4.5)

The current analytical efforts to model HI bias is based on modeling the halo-matter bias, by relying on a perturbation theory framework [6]. A particular interest that is being investigated is the scale dependence of bias, which can be a powerful probe to test initial conditions of the universe as well as the nature of gravity. It has recently been shown that the standard cosmological model also shows the halo-bias scale dependence due to GR effects, and not only due to non-gaussianities [4]. According to another recent study based on N-body simulations [13], scale independent bias was found at a level of \( 2 - 5\% \) for scales larger than 20Mpc/h, and thus scale dependence was ignored since we only consider linear scales in this thesis.
4.2 Brightness temperature

Since this is not a galaxy survey, the power spectrum is not determined by number counts. Instead we observe the integrated photons from HI, which can be translated to temperature fluctuations. To determine the perturbations of HI, we first need to model the average expected signal.

The mean background temperature brightness $\bar{T}_b$, is modeled by a fit from [43],

$$\bar{T}_b(z) = 5.59 \times 10^{-2} + 2.32 \times 10^{-1}z - 2.41 \times 10^{-2}z^2 \text{ mK}$$  \hspace{1cm} (4.6)

in order to describe the temperature fluctuations, $T_b = \bar{T}_b + \delta T_b$. The neutral hydrogen power spectrum $P_{HI}$ is determined by the multiplication of the matter power spectrum with background HI temperature. Considering a model of HI bias $b_{HI}$, from (3.7) it can be shown that

$$P_{HI}(k) = \bar{T}_b^2 (b_{HI} + f\sigma_8^2) \sigma_8^2 P_m(k)$$  \hspace{1cm} (4.7)

is the final form of the HI Fourier power spectrum including effects of RSD. However a problem arises when taking the physical observation of the cosmological model, since there is a degeneracy with the normalisation of the power spectrum, $\sigma_8$. This means we can only measure the combination of $b\sigma_8$ and $f\sigma_8$ directly [7]. Thus the power spectrum was adapted as follows

$$P_{HI}(k) = \bar{T}_b^2 (b_{HI}\sigma_8 + f\sigma_8\mu^2) \sigma_8^2 P_m(k)$$  \hspace{1cm} (4.8)

The total angular correlation (3.36) expected from HI intensity mapping is found similarly to $P(k)$ by using the average background temperature

$$C_{\ell}^{HI}(z_i, z_j) = \bar{T}_b(z_i) \bar{T}_b(z_j) C_{\ell}^{\mu obs}(z_i, z_j)$$  \hspace{1cm} (4.9)

and assuming a model for HI bias $b_{HI}$. 

http://etd.uwc.ac.za/
4.3 Modeling noise for HI intensity mapping

It was shown in [29] that the shot noise contribution to the angular power spectrum is not significant, and is therefore ignored.

Noise can be reduced by decreasing the temperature of the receivers or increasing the number of measurements taken. The latter can be done by increasing the number of dishes or time spent observing the sky. The system temperature is a combination of receiver and sky temperature,

\[ T_{sys} = T_{rec} + T_{sky} \]  

(4.10)

where \( T_{sky} = 66 \left( \frac{\nu_o}{300 \text{MHz}} \right)^{-2.55} \) [43]. The relation between observed and emitted frequency for HI observation is assumed \( \nu_o(z) = 1420 \text{MHz}/(1 + z) \).

4.3.1 Fourier power spectrum

It is well known by the Rayleigh criteria that the angular resolution of an observation is limited by the diffraction of light. The effect of diffraction is dependent on the size of the aperture of the instrument that collects the light and wavelength of observation. Considering a circular aperture, the beam resolution of a dish with diameter \( D \) is given by

\[ \theta_b = 1.22 \frac{\lambda}{D} \]  

(4.11)

and in this thesis it was assumed the observed wave length for HI is \( \lambda_o(z) = 0.21(1 + z) \text{m} \). The solid angle in the sky covered by the beam is considered a pixel, and can be written in terms of beam resolution as

\[ \Omega_{pix}(z) \approx 1.13 \times \theta_b(z)^2. \]  

(4.12)

Pixel variance is determined by the physical properties of the telescope array, and survey specifications. Taking into account dual polarization receivers, we can model this variance as

\[ \sigma_{pix}(z) = \frac{T_{sys}(z)}{\sqrt{\delta \nu t_{tot} (\Omega_{pix}(z)/\Omega_s)^2 2N_dN_b}} \]  

(4.13)

given the number of dishes in the array \( N_d \), along with the total time observed \( t_{tot} \) over the survey area \( \Omega_s \). The frequency resolution of the receiver is given by \( \delta \nu \), for HI IM we observed in single dish mode \( N_b = 1 \).
The certainty of measurement on an object is also dependent on \( \mu \). Using the traversal component of the wave vector, \( k_\perp = k \sqrt{1 - \mu^2} \), instrumental response is given by the beam in \( k \)-space,

\[
W^2(k, \mu, z) = \exp \left[ -k_\perp^2 \chi(z)^2 \left( \frac{\theta_b(z)}{\sqrt{8 \ln 2}} \right)^2 \right]
\] (4.14)

which corresponds to the resolution of the telescope. The latest telescopes would provide a very high frequency resolution, which means we can confidently use

\[
\delta z(\delta \nu, \bar{z}) = \left( \frac{\delta \nu}{\nu_s} \right) (1 + \bar{z})^2
\] (4.15)

as approximation for the redshift resolution \( \delta z \) in the radial direction. This is used to determine the redshift interval when computing pixel volume of the survey using (2.42). Taking the above mentioned factors into account the instrumental noise for HI IM is given by

\[
P^N(k, \mu, z) = \sigma_{\text{pix}}^2(z) V_{\text{pix}}(z, \Delta z) W^{-2}(k, \mu, z)
\] (4.16)

and for a more detailed description on modeling noise for \( P(k) \) when using HI IM refer to [5].

### 4.3.2 Angular power spectrum

Similarly to the \( P(k) \) analysis, the angular power spectrum will also be influenced by instrumental noise. The \( C_\ell \) is written in terms of a covariant matrix which means the influence of noise can be quantified using a diagonal matrix, since instrumental noise is independent in different redshift bins. The noise is modeled as in [15],

\[
N_{ij}(z_i) = \frac{4 \pi f_{\text{sky}} T_{\text{sys}}^2(z_i)}{2 N_d \Delta \nu(z_i)} \delta_{ij}
\] (4.17)

and the same survey specifications is used as for \( P(k) \). Note that \( \Delta \nu \) is the frequency band of the redshift bin-width assuming the 21cm line of HI, and the relation between the two can be approximated by (4.15).
4.4 Survey specifications

The telescope beam in spherical harmonic space is given by

\[ B(\ell, z) = \exp \left[ -\ell(\ell + 1) \frac{\theta^2_{\text{FWHM}}(z)}{16 \ln 2} \right] \]  \hspace{1cm} (4.18)

which is multiplied by the signal to take into account the resolution of the telescope,

\[ C^B_{\ell}(z_i, z_j) = C_{\ell}(z_i, z_j) B(\ell, z_i) B(\ell, z_j). \]  \hspace{1cm} (4.19)

4.3.3 Foreground cleaning

The above description only addresses the expected instrumental noise from the future telescope array, but another major contribution to contamination and loss of signal is called Foreground interference, and should be considered for accurate forecasts.

At 1 GHz the galactic synchrotron emission dominates (around 5-6 orders of magnitude brighter than the HI signal), but there is also background emission of extragalactic point sources [5]. Most foregrounds should be spectrally smooth, making it possible to remove using Principle Component Analysis (PCA). Recent simulation work has shown that the existing foreground removal methods can recover the true HI power spectrum to within 5% [44], hence the foreground cleaning was ignored.

However it should be noted when considering foreground cleaning for \( C_{\ell} \), this assumption is probably insufficient. It was found after applying the foreground removal transfer function, about 20 – 40\% of signal is lost, and drops significantly more at the edges of the redshift range [49]. Recent work by [30] has shown that including astrophysical priors can greatly improve the relative uncertainty of foreground cleaning, and hopefully recover the loss of signal sufficiently.

4.4 Survey specifications

The unprecedented sensitivity and field of view of the SKA will allow for dramatically faster survey speeds, making it possible to map the galaxy distribution out to high redshifts over most of the sky [51].

The resulting survey will be a sample variance-limited observation over a truly huge volume, allowing the Neutral Hydrogen surveys to greatly improve on current
cosmological constraints. This also allows us to probe ultra-large scales and novel wide-angle effects [1]. The specifications used to describe the different surveys was mainly sourced from [42] and [8], but other papers was needed to get the full description.

4.4.1 MeerKAT

The area of the sky survey is approximately $\Omega_s = 4 \times 10^3 \text{deg}^2$, and a total time of $t = 4 \times 10^3 \text{h}$ is spent combing the sky. The redshift range of MeerKAT is $0 < z < 1.45$, which is split up in two bands: L-band [$900 < \nu < 1420 \text{ MHz}$] and UHF-band [$580 < \nu < 1000 \text{ MHz}$], [37].

The MeerKat array will consists of $N_d = 64$ dishes with diameter of $D = 13.5\text{m}$. The receivers fitted is expected to yield a frequency resolution of $\delta\nu = 50\text{kHz}$ [36], while functioning at temperatures of around $T_{\text{system}} = 25 \times 10^3 \text{ mK}$ [37].

4.4.2 SKA1

Eventually the MeerKAT array will be upscaled into the first phase of the SKA. The system temperatue is assumed the same as MeerKAT, $T_{\text{system}} = 25 \times 10^3 \text{ mK}$ [37]. The telescope array will consist of $N_d = 194$ dishes with slightly larger diameter of $D = 15\text{m}$. The sky area surveyed is estimated at $\Omega_s = 20 \times 10^3 \text{ deg}^2$, for a total time of $t = 10^4\text{h}$. The frequency range for the new receivers are also increased, which means a larger redshift range observed, L-band [$950 < \nu < 1760 \text{ MHz}$] and UHF-band [$350 < \nu < 1050 \text{ MHz}$], [8]. Approximately $z \in [0, 3]$.

In HI IM all the dishes will work together to form a single beam, $N_b = 1$, which makes it possible to probe the largest of scales.
Chapter 5

Fisher Forecasts

Fisher forecasting is a simple, computationally inexpensive way of predicting the constraints on a set of parameters that should be achieved by a given experimental configuration [7]. This method assumes Gaussianity and neglects systematic biases, which makes it approximate and idealised. Nevertheless the method offers a reliable way of understanding the relative performance of different experimental configurations.

The Fisher analysis takes into account theoretically how much information can be extracted from a specific survey, and how much each parameter would influence the cosmological model we are trying to test. This gives us a confidence interval on how accurately certain parameters could be measured, and hence how confidently we can exclude different models of gravity.

A fully Bayesian statistical approach, called the ‘Internal Robustness’ (iR), was applied to growth rate to find systematics in the latest datasets [40]. This method is not only sensitive to local minimum, like $\chi^2$ comparisons, but by analysing combinations of subsets in the dataset, the Bayesian model in principle can also potentially find group outliers and data affected by systematics. The iR method showed that there is no anomalous behavior in the $f\sigma_8$ dataset, ensuring internal robustness against systematics when using this method.
5.1 Alcock-Paczynski effect

In order to translate a redshift position to coordinates in real space, one needs to assume a model that describes the observed universal expansion, $H(z)$. If the model is slightly inaccurate, the distance measurements will be out, and the error induces a distortion in the correlation function. This phenomenon can be taken into account when forecasting constraints on cosmological parameters, here we considered the Alcock-Paczynski effect (AP).

The distance between a pair of galaxies separated by an angle $d\theta$ is obtained from FLRW metric as in [48]

$$d\ell_\perp = (1 + z) D_A(z) d\theta$$

(5.1)

where $D_A$ represents the angular diameter distance at the redshift of the pair. The separation of galaxies in the radial direction is given in terms of the Hubble rate

$$d\ell_\parallel = c \frac{dz}{H(z)}$$

(5.2)

such that integrating over $z$ will recover the comoving distance (2.36). If a fiducial cosmological model is assumed (denoted by the bars) that is different from reality, we expect to see a transversal distortion

$$d\vec{\ell}_\perp = (1 + z) \bar{D}_A(z) d\theta \left( \frac{D_A}{\bar{D}_A} \right) d\theta \frac{d\ell_\perp}{f_\perp}$$

(5.3)

and radially

$$d\vec{\ell}_\parallel = \frac{c dz}{H(z)} = \left( \frac{H}{\bar{H}} \right) d\ell_\parallel = \frac{d\ell_\parallel}{f_\parallel}$$

(5.4)

where $f_\parallel$ and $f_\perp$ is written as a ratio of $H$ and $D_A$ with their fiducial counterparts respectively. Thus the expected induced anisotropy due to the AP-effect can be expressed as in [9]

$$F = \frac{f_\parallel}{f_\perp} = \frac{D_A}{\bar{D}_A} \frac{\bar{H}}{H}$$

(5.5)

which is how the distance between two points would be influenced.
5.2 Fourier power spectrum

Considering this effect on the Fourier power spectrum $P^{\text{HI}}$, which is a 3D analysis, the final form of the correlation function is given by

$$P^{\text{HI}}_a(k) = \frac{\bar{D}_A^2(z) H(z)}{D_A^2(z) H(z)} P^{\text{HI}}(k)$$  \hspace{1cm} (5.6)$$

since the volume is given by the surface area on the sky $\sim D_A^2$, times depth of the survey, $H$. The factor will cancel to 1 if the assumed cosmology is correct, and distort the power spectrum if it is inaccurate. Equation (5.6) represent the correlation function we will be using for the Fourier space Fisher forecast, and therefore we need to determine the derivatives with respect to the parameters we want to forecast - see A.1.

Given the various different technicalities when it comes to the different mathematical analyses, it should be no surprise that they require different approaches when deriving the constraints on cosmological parameters.

5.2 Fourier power spectrum

In order to know how much information can be extracted from a survey, we first need to determine the $k$-modes/scales we are able to probe and at what resolution we are able to observe them. The largest scale that can practically be measured within a redshift bin is proportional to the volume of that bin. The comoving volume of a redshift bin $V_s$ with central redshift position $z_i$ and bin-width $\Delta z$ can be calculated with (2.42), thus

$$k_{\text{min}}(z_i, \Delta z) = \frac{2\pi}{V_s(z_i, \Delta z)^{1/3}}$$  \hspace{1cm} (5.7)$$

where the cube root projects the volume to one dimensional length, and $2\pi$ is the one dimensional relation to Fourier space. The minimum $k$-mode will be used to determine the smallest mode resolvable, to approximate the discrete nature of sampling modes. Since we use linear perturbation theory to model the expansion of the universe the smallest scales considered in this thesis are determined as in [45],

$$k_{\text{max}}(z) = 0.14 \text{Mpc}^{-1}(1 + z)^{2/3}$$  \hspace{1cm} (5.8)$$

which approximates the boundary of the linear regime. Hence the $k$-modes sampled in each $z$-bin are $k \in [k_{\text{min}}, k_{\text{max}}]$ at intervals of $k_{\text{min}}$. 

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We can add the effect of instrumental noise in the Fisher forecast by adjusting the survey volume in proportion to the S/N level. The effective volume of the survey is given by

\[
V_{\text{eff}}(k, \mu, z, \Delta z) = V_s(z, \Delta z) \left( \frac{P_{\text{HI}}(k, \mu, z)}{P_{\text{HI}}(k, \mu, z) + P^N(k, \mu, z)} \right)^2
\]

(5.9)

where \( P_{\text{HI}} \) (4.7) and \( P^N \) (4.16) are signal and instrumental noise respectively.

The Fisher matrix computes the the influence of a single parameter on the cosmological model by taking the derivative of the correlation with respect to the parameter \( \alpha \), in this case \( \partial_\alpha \ln P_{\text{HI}} \). By numerically integrating over all available \( k \)-scales and \( \mu \), we sum together all the information that is theoretically extractable from \( P_{\text{HI}} \). Thus the Fisher formula for \( P(k) \) analysis is

\[
F_{\alpha\beta}(z_i, \Delta z) = \frac{1}{8\pi^2} \int_{-1}^{1} d\mu \int_{k_{\text{min}}}^{k_{\text{max}}} k^2 dk \left[ \partial_\alpha \ln P_{\text{HI}}(k, \mu, z_i) \partial_\beta \ln P_{\text{HI}}(k, \mu, z_i) \right] V_{\text{eff}}(k, \mu, z_i, \Delta z)
\]

(5.10)

given the survey specifications is considered in \( V_{\text{eff}} \). It is possible to analyse multiple parameters simultaneously and determine the cross-correlation between them, making it possible to make a forecast on a parameter by taking into account the uncertainties on many different parameters.

### 5.3 Angular power spectrum

The angular power spectrum looks at different angular correlations at different scales. In harmonic space this is done by slicing up the redshift range in concentric spherical shells, and projecting the information in the \( z \)-bin on a 2D surface. This means that the redshift resolution (\( \delta z \)) of the analysis is equal to the redshift bin-width, \( \delta z = \Delta z \), as opposed to \( \delta z \propto \delta \nu \) in Fourier space.

Similarly to the \( P(k) \) analysis, the range of angular scales available in the survey had to be determined. The maximum angular scale should also be limited by the size of survey,

\[
\ell_{\text{min}} = \frac{\pi}{\sqrt{4\pi f_{\text{sky}}}}
\]

(5.11)
where $f_{\text{sky}}$ is the survey area in steradians (2.40).

The smallest angular scales considered are again determined by the boundary of the non-linear regime. This is calculated by the product of non-linear cutoff in Fourier space (5.8) and the comoving distance (2.36),

$$\ell_{\text{max}}(z_i) = k_{\text{max}}(z_i) \chi(z_i)$$  \hfill (5.12)

thus the range of multipoles are determined $\ell \in [\ell_{\text{min}}, \ell_{\text{max}}]$, at integer increments.

The survey specifications are simply taken into account by adding the $C_\ell$ covariance matrix (3.27), to the noise determined in (4.17),

$$\Gamma_\ell = C^B(\ell) + N$$

with superscript $B$ indicating the beam has been applied, (4.18). The new covariance relation $\Gamma$ is inverted and multiplied by the derivative-covariance matrices determined in (A.30) for $\gamma$, and (A.35) for $f$. The derivative matrices are also multiplied by the beam. The Fisher forecast from $C_\ell$ analysis also sum together the influence of a parameter on the correlation function over all observed scales. The definition is taken from [15],

$$F_{\alpha\beta} = \sum_\ell \frac{2}{f_{\text{sky}}} \Gamma \left[ \partial_\alpha C_\ell \Gamma^{-1} \partial_\beta C_\ell \Gamma^{-1} \right]$$  \hfill (5.13)

which has the same form as $P(k)$’s Fisher formula, if logarithmic derivatives in (5.10) is expanded, and $P^{\text{HI}}$ canceled in $V_{\text{eff}}$. The main difference is $P(k)$ is a 3D analysis as opposed to projecting the data in a redshift bin to a two dimensional surface, as in $C_\ell$. Thus instead of having a survey volume, we only consider the sky fraction of the survey.

This thesis will only consider growth rate and index of large scale formation in harmonic space.
5.4 Constraints

If we assume that the measurements taken follow a normal distribution centered around the actual parameter value, we can estimate the variance of the distribution by taking the inverse of the Fisher matrix

\[ \sigma^2_{\alpha\beta} = F^-1_{\alpha\beta} \]  

(5.14)

The operation of inverting the Fisher matrix is a marginalisation over the considered parameters, which means that the uncertainties on all cosmological parameters are included in the resulting covariant matrix. Alternatively one can look at an individual parameters independently, which means all other parameters in the model are assumed to be known. This is known as the conditional error, and results in the most optimistic constraints.

By taking the log derivatives in the Fisher formula, \( \partial_{\ln \theta} \), the constraints are given in the form of fractional uncertainty,

\[ \sigma_{\ln \theta} = \frac{\sigma_\theta}{\theta} \]  

(5.15)

given the standard deviation \( \sigma_\theta \) on parameter \( \theta \).

Constraints can be improved by adding prior information from other probes, like CMB temperature anisotropies, weak gravitational lensing and galaxy surveys. This enables us to break degeneracies of certain cosmological quantities, which can greatly improve precision of HI IM measurements [8].

The purpose of this thesis is not to give an extensive forecast of possible cosmological constraints for the coming decade, but merely what we can expect independently for MeerKLASS like surveys, hence prior information is ignored.
Chapter 6

Results

In all forecasts we assumed that the information about the full shape of the power spectrum can reliably be recovered. To this end the $P(k)$ and $C_\ell$ power spectra was simulated by the CAMB and CAMB SOURCES software respectively, which was manipulated with the help of software from [14]. The simulations assumed a $\Lambda$CDM fiducial cosmology and initial conditions from Planck 2015 [2]. The present day matter content of the universe is assumed to be $\Omega_{m0} = 0.3067$, and DE contribution $\Omega_\Lambda = 1 - \Omega_{m0}$ assuming a flat universe. The concordance model assumes that the growth index of large scale structure formation is constant over time and given by $\gamma = 0.545$, which was assumed in this thesis. Also the present day value of the Hubble rate was assumed to be $H_0 = 67.8 \text{ km/s/Mpc}$. 

6.1 Fourier power spectrum

The software output from CAMB is the power spectrum for cold dark matter at a specific redshift, $P_m(z,k)$. The power spectrum is then converted to $P^{\text{HI}}$ using (4.8). For a complete description of the mathematical definitions, and capability of CAMB, see [17].

Firstly we show the simulated HI Fourier power spectrum, along with the noise expected from MeerKAT survey specifications. In order to illustrate the influence of $\mu$ on $P(k)$ and noise, at an arbitrary redshift position let $\mu \in [1, 0.5, 0]$, Fig 6.1a. The signal from the smaller scales are quickly saturated by the beam. Given $\mu = 1$, the noise stays constant as a function of scale, and keeps below the expected signal for both MeerKAT and SKA1 surveys i.e. $0 < z < 3$. 

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Results

Fig. 6.1 Solid curves represent the $P(k)$ signal from HI IM at $z = 0.5$, along with the noise represented by the dashed curves. Colours represent different pointing directions (left). The derivatives of various parameters on the power spectrum was investigated at $z = 0.5$ and $\mu = 0.5$ (right).

The logarithmic derivatives of $P_{HI}$ were determined with respect to the parameters that are going to be considered in this thesis (A.1) is illustrated in Fig 6.1b. As expected the growth rate and growth index are constant as a function of scale, but the plot shows the influence of these parameters on the model is relatively small compared to the other parameters. This in turn will make $f\sigma_8$ and $\gamma$ more difficult to measure and hence constrain. The wiggles around $k \approx 0.1 \text{ Mpc}^{-1}$ are caused by the Baryonic Acoustic Oscillations. This cosmological feature makes it easier to constrain $H$ and $D_A$, and is often used in other works specifically for this purpose. The derivative with respect to $b_{HI}$ indicates that relatively good constraints can be expected. For this reason and the fact that little is still known about the true bias relation, it has been marginalised over as a nuisance parameter in the more advanced analyses.

Having determined the derivatives, the Fourier space Fisher formula (5.10) was used to calculate the fractional uncertainty on $f\sigma_8$. The influence of survey specifications was considered by varying sky fraction, Fig 6.2a, and survey time Fig 6.2b. The effect on constraints was quantified by taking the difference between respective fractional uncertainties.

Given a fractional uncertainty $\theta$, derived using the baseline specifications $\theta_\beta$, and alternate survey specifications $\theta_\alpha$,

$$
\Delta \theta = \theta_\alpha - \theta_\beta
$$

(6.1)
6.1 Fourier power spectrum

Fig. 6.2 Fractional uncertainty on $f \sigma_8$ from $P(k)$ as a function of redshift. Different survey specifications were considered, by varying sky fraction $f_{\text{sky}}$ (on the left), and survey time in hours (on the right). Dashed curves was determined by (6.1), and colour correspond to the specific survey specification considered.

where the baseline specifications selected are $t = 4 \times 10^3$ hours, survey area $f_{\text{sky}} = 0.1$ (MeerKAT survey specifications). It was found that varying the survey time had less of an influence on the desired constraints, as compared to the survey size. Referring to Fig 6.2b, even though the $\Delta \theta$ increases as a function of $z$, we expect at most $\Delta \sigma_{\ln f \sigma_8} \sim 0.005$ improvement after doubling survey time. It was shown in Fig 6.2a that the influence on constraints are larger for lower $z$ ($\Delta \sigma_{\ln f \sigma_8} \sim 0.025$), but for the majority of the range there is also no significant difference between using the MeerKAT and alternate specifications. Consequently the original survey specifications are preserved for the rest of the $P(k)$ analysis.

The constraints from $P(k)$ was found to be highly dependent on the redshift bin-widths, Fig 6.3. This is mostly due to the fact that larger bin sizes would include larger scale structures in the Fisher analysis, as well as increase the number of smaller scale observations. Thus the larger the redshift bins, the better the constraints. The drawback is that the Fisher analysis assumes constant cosmological parameters across the entire redshift bin, which means even though the constraints have improved by increasing the bin width, we now assume time dependent variables constant across larger ranges of $z$.

In order to compromise between the expected precision and the confidence in the accuracy of the measurement, this thesis assumed $\Delta z = 0.1$ for all the $P(k)$ analyses.
Varying redshift bin-width $\Delta z$, was considered on the conditional constraints of $f\sigma_8$. This means we split the survey into bins less than 7% of the total survey size, making sure we account for the evolution of cosmological parameters with redshift, whilst still having acceptable constraints from HI IM.

After taking into consideration the influence of survey specifications and bin-width on the constraints, we determined the conditional errors on the cosmological parameters, $f\sigma_8$ and $\gamma$. The best expected constraints on the aforementioned parameters are shown in Fig 6.4a, since here we assumed all other parameters are known. It was shown that per redshift bin we expect a less precise measurement on $\gamma$ as compared to $f$, due to the fact that $\gamma$ relates exponentially to $f$. In other words since $\partial_\gamma P(k) < \partial_{f\sigma_8} P(k)$, the influence of $\gamma$ on $P(k)$ is smaller than $f$ and hence the parameter is more difficult to observe.

A more realistic forecast on the growth rate of large scale structure formation was achieved by considering the effect of an inaccurate cosmology. The influence of the AP-effect on $f\sigma_8$ was determined by marginalising over the $H$ and $D_A$, then it was compared to the conditional error on $f\sigma_8$, Fig 6.4b. This shows an expected degradation of constraints by 3-8% when considering the AP-effect. It was determined that the main contributor to the increasing uncertainty is $D_A$, as evident from the small influence on $f\sigma_8$ constraints from including only the uncertainty in $H$. 

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6.1 Fourier power spectrum

Fig. 6.4 Conditional constraints on \( f\sigma_8 \) and \( \gamma \), assuming \( \Delta z = 0.1 \) (left). Marginal constraints on \( f\sigma_8 \): Marginalisations included the AP-effect \([\ln f\sigma_8, \ln H, \ln D_A]\), comparing to \([\ln f\sigma_8, \ln H]\) and conditional error \([\ln f\sigma_8]\) (right).

Fig. 6.5 The fractional uncertainty was determined for the parameters \( H \) and \( D_A \), by marginalising over \([\ln f\sigma_8, \ln H, \ln D_A, \ln b_{HI}\sigma_8]\).

A full analysis was considered when \( H \) and \( D_A \) was constrained, Fig 6.5a and 6.5b respectively. That is to say we have taken into account the uncertainty on bias by marginalising over \( b_{HI} \) as well. From HI intensity mapping we expect better constraints from \( H \) compared to \( D_A \), since it has high spectroscopic precision in the radial direction but uses a low angular resolution configuration. The the precision on the measurement of \( H \) and \( D_A \) is comparable, but the angular diameter distance achieves this for a shorter redshift interval. It was found that the constraints could be as low as 2% for MeerKAT and sub 1% for SKA1, at \( z \sim 0.5 \).

For a comparison to previous measurements of \( f\sigma_8 \) from various other surveys (Vipers, BOSS and SDSS), requires us to also include the uncertainty in the HI bias, Fig 6.6.
Fig. 6.6 Fractional uncertainty on $f \sigma_8$ using marginal errors, parameters considered $[\ln f \sigma_8, \ln H, \ln D_A, \ln b_{HI} \sigma_8]$. The blue points is a composition of measurements from previous surveys like BOSS [39], Vipers [34] and SDSS [41].

The full analysis is also used to compare the results from MeerKAT to that of the completed SKA1. Fig 6.6 clearly indicates that MeerKAT is expected to significantly improve constraints on $f$, as compared to previous measurements. Also we find a constraints on $f \sigma_8$ to be under 10% for most of MeerKAT's redshift range. Upon completion of the first phase of the SKA we forecast $\sigma_{\ln f \sigma_8}$ can decrease by 0.02-0.03 over z-range. This means we can expect competitive measurement on $f$ out to $z \sim 2$, which is much further than current measurements.
6.2 Angular power spectrum

The angular power spectrum simulated by CAMB_sources has bias and RSD included, thus we applied the average temperature to the correlation matrix (4.9) to simulate the HI temperature fluctuations. The angular power spectra used in the spherical harmonic analysis is shown by splitting up the auto- and cross-correlations in Fig 6.7a and Fig 6.7b respectively. The auto-correlations are defined by $C_{HI}^{\ell}(z_i, z_i)$ and cross-correlations $C_{HI}^{\ell}(z_i, z_j)$, given $i \neq j$. Colours correspond to the varied redshift bin $z_i$ for $C_{HI}^{\ell}(z_i, z_i)$, and $z_j$ for $C_{HI}^{\ell}(z_i = 0.05, z_j)$. Expected noise levels are indicated by the dashed lines, which shows a high signal to noise ratio for auto-correlations.

The logarithmic derivatives of the total-correlations $C_{HI}^{\ell}(z_i, z_j)$ with respect to $f$ was calculated in (A.35), which was split into the auto- and cross correlations in Fig 6.7c and Fig 6.7d respectively. This indicates we do expect some extra information by including the cross terms, and the larger scales will have the greatest contribution. The cross-correlation of $\partial C_{HI}^{\ell}/\partial \ln f$ is expected to be smooth, but I was unable to increase the accuracy of the CAMB simulation due to insufficient computational resources.

The derivative with respect to $\gamma$ was also considered (A.30), and following the same pattern split up into Fig 6.7e and Fig 6.7f. This shows there is some information in the cross correlations of $f$ and $\gamma$.

By using the spherical harmonic Fisher formula (5.13), the constraints on growth rate was calculated and used to analyse the influence of survey specifications on MeerKAT. Again we considered the variation of sky fraction Fig 6.8a, and time Fig 6.8b, by looking at bins independently. The difference in survey specifications was quantified using $\Delta \theta$ (6.1), indicated by the dashed lines. Similarly to the $P(k)$ analysis, the constraints from $C_{\ell}$ is more influenced by survey area as opposed to time. It should be noted that the influence of sky area is more prominent for the $C_{\ell}$ analysis compared to $P(k)$. Up to $\Delta \sigma_{\ln f} \sim 0.62$ difference can be expected for low redshifts when using $f_{\text{sky}} = 0.2$ as opposed to the baseline specification. Doubling the survey time has a $\Delta \sigma_{\ln f} < 0.01$ influence.
Fig. 6.7: Top panel: The auto-correlations $C_{\ell}^{\text{HI}}(z_i, z_i)$, was determined (on the left) along with the cross-correlations $C_{\ell}^{\text{HI}}(z_i = 0.05, z_j)$ (on the right). Colours correspond to varied redshift bin, $z_i$ for auto- and $z_j$ for cross-correlations. Dashed lines indicate noise levels. Mid panel: Derivatives taken with respect to $\ln f$ of the auto- (on the left), as well as the cross-correlations (on the right). Bottom panel: Derivatives with respect to $\gamma$ for the auto- (on the left) and cross-correlations (on the right).
6.2 Angular power spectrum

Fig. 6.8 Conditional constraints on $f$ from $C_\ell$, with varying sky fraction $f_{sky}$ (left), and survey time in hours (right). Dashed curves determined from (6.1), and colours corresponds to survey specification considered. Bin-width $\delta z = 0.05$.

The influence of cross correlating different redshift bins was determined in harmonic space, by comparing constraints on $f$ derived from total-correlations and auto-correlations, Fig 6.9a. The difference is taken as in (6.1), using total-correlations as the baseline parameter $\theta_\beta$. The analysis has shown that the cross-correlations would improve constraints on $f$ for lower redshifts $\Delta \sigma_{\ln f} \sim 0.23$, but over most of the $z$-range $\Delta \sigma_{\ln f} < 0.01$.

Considering only the conditional error on $f$ and using a single $z$-bin, we compared expected constraints from the MeerKAT and SKA surveys, Fig 6.9b. It was found that the improvement on constraints for SKA1 was mostly influenced by the increase in sky coverage, and not the increase in dish diameter or number of dishes. The most optimistic results from $C_\ell$ shows we expect constraints on $f$ to be as low as $\sim 9\%$ for MeerKAT, and $\sim 4\%$ for SKA1. It also shows $C_\ell$ is expected to better constrain parameters at larger redshifts $z > 1$, which gives it an advantage over $P(k)$’s relative poor ability to constrain at the depths of the survey.

6.2.1 Optimisation

Similarly to $P(k)$, the influence of the redshift bin widths was also considered, but the relation between $\delta z$ and constraints on parameters proved slightly more complicated. The angular correlation and noise are both dependent on $\delta z$, as illustrated in Fig 6.10a-6.10b.
As $\delta z = 0.1$ is reduced the power spectrum increases and eventually converges to a maximum signal, whereas the noise keeps increasing. Since the non-linear cut-off is determined by the angular scale $\ell_{\text{max}}$ (5.12), the bin-widths are not constrained by this limit. It was shown at higher redshifts the survey prefers wider redshift windows $\delta z \sim 10^{-1}$, as opposed to $\delta z \sim 10^{-3}$ for lower redshifts. Thus by varying the bin width across redshift will deliver a better signal to noise ratio (S/N) for $C_\ell$ across the $z$-range.

In order to extract the optimal constraints from $C_\ell$, the S/N is maximized by looking at the fractional uncertainty on $f$ in Fig 6.11a, and $\gamma$ in Fig 6.11b. The central redshift bin position $z_i$ was fixed (corresponding to the different curves) while the bin width $\delta z$ was varied. A linear fit is applied to the points and the minimum fractional uncertainty interpolated. The minimum fractional uncertainty in each bin is used to determine the coordinate $(z_i, \delta z(z_i))$. This allows us to fit a quadratic curve to these coordinates and approximate the function $\delta z(z)$ for maximum S/N, Fig 6.12a. It was found that the function of optimal bin-width is the same for both $f$ and $\gamma$, which is an indication that our method of optimisation is consistent.

**Fig. 6.9** Fractional uncertainty on $f$ from the auto- and total-correlation was compared (left). Auto-correlations from MeerKAT and SKA surveys compared (right).
6.2 Angular power spectrum

Fig. 6.10 Angular power spectrum (solid) along with the noise (dashed). The central bin position is held constant and colours correspond to different bin-widths $\delta z$. Central redshift positions are $z_i = 0.05$ (left), and $z_i = 1$ (right).

Fig. 6.11 Different curves correspond to the central redshift positions $z_i$. $y$-axis is conditional constraints as a function of $\delta z$. Parameters considered are $f$ (on the left) and $\gamma$ (on the right).
Fig. 6.12 Quadratic fit that approximates maximising S/N for $C_\ell$ (left). Optimal slicing has been determined for SKA1 by using a variable bin size $\delta z$ (right).

By using the fit in Fig 6.12a,

$$\delta z(z) = 0.0103 z^2 - 0.0029 z + 0.0012$$ (6.2)

we can determine the sequence to find appropriate $z_i$ with

$$z_i + \frac{\delta z(z_i)}{2} = z_{i+1} + \frac{\delta z(z_{i+1})}{2}$$ (6.3)

then optimally slicing SKA1 range results in around 362 bins, Fig 6.12b.

The optimal bin slicing was used to calculate the best conditional constraints on $f$ we can expect from using $C_\ell$. The effect of the optimisation was determined by comparing the optimised constraints to constraints with constant bin-width $\delta z = 0.05$, for the MeerKAT (Fig 6.13a), and SKA1 (Fig 6.13b) surveys. Again $\Delta \theta$ was used to quantify the difference in fractional uncertainty (6.1), using constraints from constant bin width as the baseline. It was found for lower redshifts $\sigma_{\ln f}$ decreased by as much as $\Delta \sigma_{\ln f} \sim 1.84$ for MeerKAT and shows a difference of $\Delta \sigma_{\ln f} \sim 0.01$ up to $z \sim 1$. This means $\sigma_{\ln f} < 0.1$ for the majority of redshift range, given a conditional analysis. For SKA1 we can expect a decrease of $\Delta \sigma_{\ln f} \sim 0.74$ for low $z$, with a difference of $\Delta \sigma_{\ln f} \sim 0.01$ at $z \sim 1$. By maximising the S/N the fractional uncertainty from SKA1 can be reduced to $\sigma_{\ln f} \sim 0.025$ for $0.25 \lesssim z \lesssim 0.75$. 

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6.2 Angular power spectrum

Fig. 6.13 Difference in constraints on $f$ from optimisation, blue represent constant bin-width $\delta z = 0.05$, compared to $\delta z(z)$ in red. The difference is given by $\Delta \theta$, using optimised constraints as baseline. For MeerKAT (left) and SKA1 (right).

Fig. 6.14 Difference in constraints on $\gamma$ from optimisation, blue represent constant bin-width $\delta z = 0.05$, compared to $\delta z(z)$ in red. The difference is given by $\Delta \theta$, using optimised constraints as baseline. For MeerKAT (left) and SKA1 (right).
6.3 Growth index

The fractional uncertainty on $\gamma$ was analysed at each redshift bin, using conditional errors and considering a non-linear cut-off $\ell_{\text{max}}(z_i)$, (5.12). The difference from optimising the bin-widths was compared to $\delta z = 0.05$ for MeerKAT in Fig 6.14a and SKA1 in Fig 6.14b. By considering a single $z$-bin at a time, it was found that for the MeerKAT survey the fractional uncertainty decreased by as much as $\Delta \sigma_{\ln \gamma} \sim 3.18$ at lower redshift, with a difference of more than $\Delta \sigma_{\ln \gamma} \sim 0.01$ for the entire redshift range. The constraints on $\gamma$ reach a minimum of $\sigma_{\ln \gamma} \sim 0.10$ at $z \sim 0.25$.

Similar improvement is expected from optimising the SKA1 survey, with $\Delta \sigma_{\ln \gamma} \sim 1.28$ at lower redshifts which decreases to around $\Delta \sigma_{\ln \gamma} \sim 0.01$ at $z \sim 1.3$. Optimising the constraints from SKA1 will bring the fractional uncertainty $\sigma_{\ln \gamma} < 0.1$ for $0.2 < z < 0.6$, with a minimum uncertainty $\sigma_{\ln \gamma} \sim 0.05$ at $z \sim 0.25$.

The growth index is assumed constant as a function of time, which means we can combine information across the entire redshift range into a single constraint. The spherical harmonic Fisher matrix automatically takes into account information from all $z$-bins, including the cross-correlation between bins. The constraint $\sigma_{\ln \gamma}$ can easily be computed from the total-correlation by inverting $F_{\gamma \gamma}$. In order to derive the constraint using the auto-correlation, the cross terms was set to zero in both $C_\ell$ and $\partial C_\ell / \partial \gamma$ covariant matrices.

It was found that the auto-correlations yielded constraints of $\sigma_{\ln \gamma} \sim 0.130$ and 0.054 for MeerKAT and SKA1 respectively. Including the cross-correlations showed slight decrease in fractional uncertainty, with $\sigma_{\ln \gamma} \sim 0.119$ and 0.049 for MeerKAT and SKA1 respectively, Table 6.1.

<table>
<thead>
<tr>
<th>$\sigma_{\ln \gamma}$</th>
<th>Auto</th>
<th>Total</th>
<th>$\Delta \sigma_{\ln \gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MeerKAT</td>
<td>0.130</td>
<td>0.119</td>
<td>0.011</td>
</tr>
<tr>
<td>SKA1</td>
<td>0.054</td>
<td>0.049</td>
<td>0.005</td>
</tr>
</tbody>
</table>

Table 6.1 Using both auto- and total-correlations, a combined constraint on $\gamma$ from $C_\ell$ was determined.

In order to optimise the collective constraint on $\gamma$, the $F_{\gamma \gamma}$ matrix for each bin was summed to extract information over all $z$. The combined Fisher matrices was inverted to determine $\sigma_{\ln \gamma}$. This constraint on $\gamma$ neglect the correlation between different bins.
It was found that conditional constraints on $\gamma$, is again $\sigma_{\ln\gamma} \sim 0.130$ and 0.054 for MeerKAT and SKA1 respectively, given $\delta z = 0.05$. By optimising the S/N ratio the constraints on growth index have been influenced significantly, with $\sigma_{\ln\gamma} \sim 0.013$ and 0.005 for MeerKAT and SKA1 respectively, Table 6.2.

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_{\ln\gamma}$</th>
<th>Constant</th>
<th>Optimal</th>
<th>$\Delta\sigma_{\ln\gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MeerKAT</td>
<td>0.130</td>
<td>0.013</td>
<td>0.117</td>
<td></td>
</tr>
<tr>
<td>SKA1</td>
<td>0.054</td>
<td>0.005</td>
<td>0.049</td>
<td></td>
</tr>
</tbody>
</table>

*Table 6.2* Effect of maximising S/N on the combined $\sigma_{\ln\gamma}$ from $C_\ell(z_i, z_i)$. Optimal constraint is calculated using $\delta z(z_i)$, and compared to $\delta z = 0.05$.

The combined constraint on $\gamma$ can also be calculated from Fourier space, by summing the Fisher matrix across the redshift range. In order to compare this to the optimised results from $C_\ell$, we also included the AP-effect (5.1). It was found that the $P(k)$ analysis produced $\sigma_{\ln\gamma} \sim 0.029$ and 0.009 for MeerKAT and SKA1 respectively. This means that $\sigma_{\ln\gamma}$ from a $C_\ell$ analysis is about half of the $P(k)$ constraints, if we consider the uncertainty on the background model. The results are summarized in Table 6.3.

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_{\ln\gamma}$</th>
<th>$P(k)$</th>
<th>$C_\ell$</th>
<th>$\Delta\sigma_{\ln\gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MeerKAT</td>
<td>0.029</td>
<td>0.013</td>
<td>0.016</td>
<td></td>
</tr>
<tr>
<td>SKA1</td>
<td>0.009</td>
<td>0.005</td>
<td>0.004</td>
<td></td>
</tr>
</tbody>
</table>

*Table 6.3* Summarising the combined conditional constraints on $\gamma$, from $P(k)$ and $C_\ell$. Marginal constraints from $P(k)$ include the AP-effect, and $C_\ell$ constraints have been optimised by maximising S/N.

The results from the above tables have been summarised visually for both MeerKAT (in Fig 6.15a) and SKA1 (in Fig 6.15b).

### 6.4 Summary

The influence of the anisotropy on $P(k)$ and associated noise was investigated in Fig 6.1a, showing $\mu \to 0$ the S/N decreases rapidly. This is because the $P(k)$ signal is decreased and noise from the small-scales quickly increase because of the beam. Derivatives of $P(k)$ with respect to various parameters in Fig 6.1b shows that the growth rate and growth index contribute less to the power spectrum compared to the other parameters. Note the derivatives of $f\sigma_8$ and $\gamma$ decreases to zero as $\mu \to 0$ as expected.
Results

Fig. 6.15 A visualisation of combined constraints on $\gamma$, indicating a 1$\sigma$ and 3$\sigma$ error bar. The first three error bars are constraints for $C_{\ell}$ and the last $P(k)$. Error bars are centered around $\gamma = 0.545$ for GR, and rectangle indicate different growth indices for MG from [23][27]. MeerKAT (left) and SKA1 (right).

Considering the $P(k)$ analysis, it was determined that overall survey area has more of an influence on constraints, as compared to survey time, Fig6.2a & Fig6.2b. By doubling the MeerKAT survey time we at most expect a slight difference of $\Delta\sigma_{\ln f_{\sigma}} \sim 0.005$ at the edge of the redshift range. Similarly doubling MeerKAT’s survey area shows only $\Delta\sigma_{\ln f_{\sigma}} \sim 0.01$ difference is expected, except for $z < 0.1$ where $\Delta\sigma_{\ln f_{\sigma}} \sim 0.025$.

The effect of changing the redshift bin-width was considered on the fractional uncertainty of $f_{\sigma_{8}}$, derived using the $P(k)$ analysis, Fig 6.3. It was found that increasing bin-width rendered more precise constraints, but at a significant loss of accuracy. A binning method was selected that relates bin size to under 10% of the total redshift range, ensuring we get good constraints whilst considering the time evolution of the cosmological parameters.

It was shown that the constraints on $f_{\sigma_{8}}$ are better than that of $\gamma$ per redshift bin, Fig 6.4a. However since $\gamma$ is constant across time, the combination of information across the entire survey range makes it a competitive method of analysis, Table 6.3. The constraints on $f_{\sigma_{8}}$ was calculated and it was found the AP-effect has significantly influenced the fractional uncertainty, Fig 6.4b. The greatest contribution to the degradation of constraints is the angular diameter distance.

The constraints on the Hubble parameter and angular diameter distance was done using a full Fisher analysis, showing better constraints are expected on $H$ as compared
to $D_A$, Fig 6.5a and Fig 6.5b. With similar constraints expected to be as low as 2% and sub 1% for MeerKAT and SKA1 respectively, $D_A$ achieves the same level of constraint for a significantly shorter interval.

After including uncertainties in the bias and fiducial cosmology, it was shown that the MeerKLASS survey is expected to greatly improve on previous measurements of $f\sigma_8$. Constraints from SKA1 could be as good as 1% at $z \sim 0.4$, and have the ability to constrain $f$ to much greater redshifts than before, Fig 6.6.

Survey specifications was found to have a greater influence in harmonic space compared to Fourier space. Doubling MeerKAT’s the survey time would have a sub percentage influence of conditional constraints Fig 6.8b, but doubling the sky area could have a significant influence at $z < 0.1$, Fig 6.8a.

The cross-correlations are expected to have little effect on $\sigma_{ln f}$ for the majority of the $z$-range, Fig 6.9a, and was subsequently ignored. From conditional errors in Fig 6.9b, we expect constraints on $f$ from MeerKAT to be as low as 9%, and the best constraint from SKA1 to be $\sim 4\%$.

In spherical harmonic space both the signal and noise depend on the redshift bin-width, which offered the opportunity to maximize the S/N and improve constraints, Fig 6.10a and 6.10b. It was found that lower $z$ prefers narrow bin-widths as opposed to wider bins at higher redshifts. In order to find the optimum bin-width for each redshift bin, the fractional uncertainty of $f$ and $\gamma$ was minimised by varying $\delta z$, Fig 6.11a and Fig 6.11b respectively.

A 2$^{nd}$ order polynomial fit used to approximate the function $\delta z(z)$ (6.2), that produces maximum S/N, Fig 6.12a. A numerical method was then used to determine the central redshift positions $z_i$ and the corresponding bin-widths $\delta z(z_i)$, for optimally slicing $z$-range, Fig 6.12b.

The optimal bin slicing was used to determine the minimum $\sigma_{ln f}$ from $C_\ell$, and compared to constraints with constant bin-width, Fig 6.13a and 6.13b. At lower $z$ we expect a significant improvement on constraints using the optimised bins, $\Delta \sigma_{ln f}$ of up to 1.84 and 0.74 for MeerKAT and SKA1 respectively. With this technique $\sigma_{ln f} < 0.1$ for the majority of $z$-range, hence $C_\ell$ is competitive for $z > 0.125$. 

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The optimisation was also applied to constraints on $\gamma$, and then compared to that of constant bin-width, Fig 6.14a and Fig 6.14b. At lower $z$ a difference $\Delta \sigma_{\ln \gamma}$ of up to 3.18 and 1.28 can be expected for MeerKAT and SKA1 respectively. Also considering SKA1, $\sigma_{\ln \gamma} < 0.1$ for a significant proportion of the redshift range.

The information from all redshift bins was summed to determine the combined constraints on $\gamma$. First the contribution from cross-correlations in $C_\ell$ was considered, and was found to have relatively little effect on decreasing $\sigma_{\ln \gamma}$, Table 6.1.

The influence of optimisation was also considered when calculating the combined constraint on gamma, Table 6.2. It was found that the optimisation technique could potentially have a significant influence on constraining $\gamma$, with around $\sim 1.3\%$ expected from MeerKAT and sub 1% for SKA1.

Comparison of the constraints from the Fourier and Harmonic space is summarised in Table 6.3. To this end the AP-effect was necessarily included in the forecast from $P(k)$. The marginal constraint on $\gamma$ was combined from $P(k)$, and shows to be a useful method of analysis, albeit not as precise as $C_\ell$.

The combined constraints are visualised and summarised in Fig 6.15a-6.15b, which shows the best case constraints expected from future HI IM surveys produced by the MeerKAT and SKA1.
Chapter 7

Discussion and conclusion

7.1 Fourier analysis

The definition of the HI Fourier power spectrum (4.7) makes it easy to see the dependence of the correlation function on $\mu$. As $\mu \to 0$, the amplitude of $P_{\text{HI}}$ decreases since the contribution of RSD declines, Fig 6.1a. Also as $\mu \to 0$, the smaller $k$-scales becomes saturated with noise given the dependence of $k$-space telescope beam(4.14). Thus significantly less information is contributed from smaller scales as the angle of the wavevector and its projection in the line of sight tends to zero.

By varying the survey specifications it was shown that $f_{\text{sky}}$ has a greater influence on constraints, compared to survey time, Fig 6.2a-6.2b. This means that sampling more of the sky leads to better statistics and hence constraints. Overall the influence from varying survey specifications is expected to be minor, considering sub percentage improvement by doubling $f_{\text{sky}}$ and survey time.

Crucially one should consider the effect of $z$-bin width on time dependent cosmological constraints, regarding a $P(k)$ analysis, Fig 6.3. In Fourier space the correlation function assumes constant cosmological parameters across the entire $z$-bin, which means if the bins are too large the extraction of a time dependent parameter becomes highly inaccurate. Larger $z$-bins bins implies better constraints, since larger scales are included as well as an increased number of small scales. Hence for $P(k)$ analysis there is a trade off between accuracy and precision. From $C_\ell$ constraints can more accurately be extracted by considering far smaller bin-widths, and producing around the same precision as $P(k)$.
For independent redshift bins, $\gamma$ is harder to constrain than $f\sigma_8$. Because of the exponential relation between the two parameters (2.89), the growth index has a smaller influence on the correlation function, see Fig 6.4a. However since $\gamma$ is independent of time, we can combine the information from all $z$-bins to determine a single constraint, Table 6.3. Thus the constraints on $\gamma$ derived from HI IM in Fourier space $P^{\text{HI}}$, is a useful test to compare various gravitational theories.

The effect of assuming an inaccurate fiducial cosmology was considered when constraining $f\sigma_8$ in Fourier space. The most significant influence from the AP-effect is the angular diameter distance, which can be attributed to the low angular resolution of IM. The high spectroscopic resolution of the HI line, combined with a great frequency resolution of the MeerKAT telescope, shows only a small influence from $H$ when constraining $f\sigma_8$, Fig 6.4b.

By including the uncertainty in $b_{\text{HI}}$ and $f\sigma_8$, the constraints on $H$ (Fig 6.5a) and $D_A$ (Fig 6.5b) are $\sigma_{\ln \theta} < 0.1$ for the entire $z$-range. Constraints could be as low as 2% for MeerKAT, and less than 1% for SKA1. It should be noted that the angular power spectrum is directly based on observables, therefore in a spherical harmonic analysis we need not assume a background cosmology and hence neglect the AP-effect.

In order to compare the predicted constraints to measurements from previous surveys, the uncertainty on the HI bias has to be considered. Bias relates the HI distribution to the underlying matter perturbations, which means we can compare surveys like HI IM to galaxy surveys. The full analysis was applied when computing the constraints on $f\sigma_8$ for both MeerKAT and SKA1, Fig 6.6. Uncertainty in previous measurements from various surveys are just above $\sigma_{\ln f\sigma_8} \sim 0.1$ uncertainty, see compilation in [24]. For both MeerKAT and SKA the forecast suggest that we could probe well below $\sigma_{\ln f\sigma_8} \sim 0.1$ over a significant proportion of the $z$-range, making this a competitive tool to constrain gravitational theories. We are also able to constrain $f$ accurately at higher redshifts than ever before using SKA1 and $P^{\text{HI}}$.

It should however be noted that generally Fisher forecasts are over optimistic, and we expect constraints in this forecast to degrade since we neglected the effect of foreground cleaning. For example the conditional constraint on $f\sigma_8$ was $\sim 4\%$ for the BOSS survey,
then the dataset yielded constraints $\sim 10\%$ [26]. We hope the decrease in constraints can be compensated for by taking a prior from Planck data and other surveys.

### 7.2 Spherical harmonic analysis

The simulated angular correlation from HI emission was plotted along with the instrumental noise associated with intensity mapping. The signal was split into the auto- (Fig 6.7a), and cross-correlations (Fig 6.7b). This shows that the information from the auto-correlations have large S/N only for the large scales, since the beam in harmonic space exclude the smaller scales (4.18).

The influence of survey specifications was evaluated for the $C_\ell$ analysis, Fig 6.8a, and the sky fraction had a bigger influence compared to time spent observing. The angular correlations turned out to be more sensitive to the changes in survey specifications compared to $P(k)$. A larger sky fraction would have a significant difference at low redshifts ($\Delta \sigma_{\ln f} \sim 0.62$), if MeerKAT’s $f_{\text{sky}}$ could be doubled. However for $z > 0.25$ almost no difference from sky fraction is expected. As with $P(k)$, only $\Delta \sigma_{\ln f} \sim 0.01$ improvement is expected on the $C_\ell$ constraints by doubling the survey time, Fig 6.8b.

One of the main advantages the harmonic space has over Fourier space, is the ability to extract information on the evolution of the universe by cross correlating $z$-bins. The constraint on $f$ was calculated using the total-correlations, and compared to the constraint derived from auto-correlations, thus determining the influence of the cross-correlations, Fig 6.9a. The difference was found to be negligible for most of the $z$-range, with $\Delta \sigma_{\ln f} < 0.01$ for $z > 0.25$.

From $C_\ell^{\text{HI}}(z_i, z_i)$ the constraints on $f$ was forecast for MeerKAT and SKA1 using $\delta z = 0.05$, Fig 6.9b. It was found that $C_\ell$ analysis has relatively little ability to constrain $z < 1$ for the MeerKAT survey. However at higher redshifts $C_\ell$ performs better than $P(k)$, because the AP-effect is only considered in Fourier space. At $z > 1$ and $z > 0.5$ the constraint $\sigma_{\ln f} < 0.1$ for MeerKAT and SKA1 respectively. Minimum constraints are $\sigma_{\ln f_{\text{sk}}} \sim 0.09$ for MeerKAT and $\sigma_{\ln f_{\text{sk}}} \sim 0.05$ for SKA1, at $z \sim 1$.

The influence of $\delta z$ on $C_\ell^{\text{HI}}$ and instrumental noise was investigated at $z_i = 0.05$ (Fig 6.10a), and $z_i = 1$ (Fig 6.10b). For large enough $\delta z$ the effect of RSD vanishes since the peculiar velocity of sources gets averaged out in the bin, hence the signal

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will increase if smaller bins are selected. It was shown by reducing the bin-width the angular correlation increases, and eventually converges to a maximum if all information is extracted. The compromise is that instrumental noise increases with a reduction in bin size as well. The increase in noise is due to reducing the observed frequency band $\Delta \nu$ in (4.17).

A method of maximising the signal to noise ratio was considered by looking at the constraints of the cosmological parameters considered. The minimum fractional uncertainty in each redshift bin was determined for $f$ (Fig 6.11a) and $\gamma$ (Fig 6.11b) by varying $\delta z$ (15 times). The generated uncertainties are linearly fitted and the curves are then interpolated to find the minimum fractional uncertainty. The bin-width corresponding to the minimum constraint is found (called optimised $\delta z$) and recorded along with redshift bin position. The optimum bin-widths as a function of $z$ was the same for both $f$ and $\gamma$, leading us to conclude that the maximisation of S/N is consistent.

The optimised $\delta z$ was computed at four $z_i$ across the $z$-range, and used to model the function that describes optimised bins, Fig 6.12a. Using the optimised redshift bin-width function, the optimal $z_i$ and $\delta z(z_i)$ can easily be determined using (6.3), hence we are able to slice up the $z$-range as demonstrated in Fig 6.12b. A better fit for the optimisation function $\delta z(z)$ can be determined by increasing the number of $\delta z$ considered per $z$-bin, as well as increasing the number of $z_i$ optimised across $z$-range.

The optimal binning was used to determine conditional errors on $f$ and was then compared to constraints calculated using constant bin widths, Fig 6.13a and 6.13b. It should be noted that the constant bin-width was arbitrarily chosen, and serves only as reference for the comparison. But independently of which constant width ($\delta z_c$) is selected the optimised constraints will always be more precise. That is because when $\delta z_c \neq \delta z(z_i)$, optimal constraints will be better, and when $\delta z(z_i) \sim \delta z_c$ the constraints will converge $\sim z_i$. This is true for every $\delta z_c$ and any parameter considered.

The greatest improvements are expected for low $z$, with up to $\Delta \sigma_{\ln f} \sim 1.84$ difference in the fractional uncertainty for MeerKAT and $\Delta \sigma_{\ln f} \sim 0.74$ for SKA1. Crucially using the optimisation means that constraints over the majority of $z$-range is $\sigma_{\ln f} < 0.1$, and could go as low as $\sigma_{\ln f} \sim 0.06$ and $\sigma_{\ln f} \sim 0.03$ for MeerKAT and SKA1 respectively, demonstrating the competitiveness of $C_\ell$ analysis.
Using a $P(k)$ analysis and MeerKAT specifications we can see conditional constraints on $f$ could be as low as $\sigma_{\ln f} \sim 0.01$, Fig 6.4b. But because of the uncertainty in $D_A$ the minimum constraint is expected to increase to $\sigma_{\ln f} \sim 0.03$. Also for the spherical harmonic analysis the fractional uncertainty remain relatively low for higher $z$, since $D_A$ does not affect the analysis.

The $C_\ell$ analysis is favoured in cases when constraints on time dependent cosmological parameters are considered, since the narrow bin widths allows a more accurate measurement without loss in precision.

### Growth index

The optimisation was also demonstrated on conditional constraints on $\gamma$, and proved to be more useful compared to the $f$ case, Fig 6.14a-6.14b. At low redshifts the fractional uncertainty on growth index can be improved by $\Delta \sigma_{\ln \gamma} \sim 3.18$ for MeerKAT and $\Delta \sigma_{\ln \gamma} \sim 1.28$ for SKA1. Fractional uncertainty could go down as low as $\sigma_{\ln \gamma} \sim 0.10$ and $\sigma_{\ln \gamma} \sim 0.05$ for the respective surveys, at $z \sim 0.25$.

The combined constraint on $\gamma$ was computed using information from the entire redshift range and was used to look at the influence of different methods investigated. The improvement can be quantified by taking the difference of the fractional uncertainties $\Delta \sigma_{\ln \gamma}$, and normalising to the worst constraint $\sigma_{\ln \gamma}^*$:

$$\delta \sigma = \frac{\Delta \sigma_{\ln \gamma}}{\sigma_{\ln \gamma}^*}.$$  \hspace{1cm} (7.1)

First the auto-correlations from MeerKAT ($\sigma_{\ln \gamma} \sim 0.130$) and SKA1 ($\sigma_{\ln \gamma} \sim 0.054$) was computed and compared to the total-correlations (considering a $C_\ell$ analysis). It was found that an improvement of around $\delta \sigma = 0.090$ is expected for both MeerKAT and SKA1 by including cross terms, Table 6.1.

The conditional constraints from optimised bin-widths was compared to constant $z$-bin, and showed improvement of around $\delta \sigma = 0.90$ is expected for MeerKAT and SKA1, Table 6.2. The best constraints that we can expect from $C_\ell^{\text{HI}}$ is $\sigma_{\ln \gamma} \sim 0.013$ and $\sigma_{\ln \gamma} \sim 0.005$ for MeetKAT and SKA1 respectively.
The summed constraints was determined for $P(k)$, and compared to that of the optimised constraints of $C_\ell$, Table 6.3. It shows that the angular power spectrum can produce fractional uncertainty on $\gamma$ of around half that expected from the Fourier analysis.

The combined constraints have been used to determine the error bar on $\gamma$ using future HI IM and specifications from MeerKAT (Fig 6.15a) and the SKA1 (Fig 6.15b). The comparison between the different methods of analysis shows that we can expect future constraints to rule out some theories of MG.

In a real survey the foreground cleaning methods would reduce the information we are able to extract from a dataset. Hopefully we can recover some of the precision of the constraints by including prior information.

Constraints could also be improved by taking into account non-linear scales, which is possible if HaloFit is used in CAMB sources. By combining information from different surveys (for instance radio, infrared and optical surveys), we are able to further constrain these cosmological parameters, called multi-tracing. These suggestions will be incorporated in future work to hopefully give more realistic and accurate constraints on the growth rate/index of large scale structure formation.
References


[40] Bryan Sagredo, Savvas Nesseris, and Domenico Sapone. The Internal Robustness of Growth Rate data. 2018.


Appendix A

Derivatives

In order to compute the Fisher matrices, the derivatives of the correlation function with respect to the cosmological parameters needs to be computed first.

A.1 Fourier power spectrum

The temperature correlation function for HI IM in the Fourier domain is given by (4.8) as

\[ P_{\text{HI}}(k) = \bar{T}^2 \left( b_{\text{HI}} \sigma_8 + f_{\sigma_8} \mu^2 \right)^2 P_m(k) \sigma_8^2 \]  

and since only HI is considered, the subscript in the bias will be ignored. Taking the derivative with respect to \( f_{\sigma_8} \) and \( b_{\sigma_8} \) is simply

\[ \frac{\partial P_{\text{HI}}(k)}{\partial f_{\sigma_8}} = 2 \mu^2 \bar{T}^2 \left( b_{\sigma_8} + f_{\sigma_8} \mu^2 \right) \frac{P_m(k)}{\sigma_8^2} \]  

\[ \frac{\partial P_{\text{HI}}(k)}{\partial b_{\sigma_8}} = 2 \bar{T}^2 \left( b_{\sigma_8} + f_{\sigma_8} \mu^2 \right) \frac{P_m(k)}{\sigma_8^2} \]  

after applying the chain rule. In order to immediately recover the fractional uncertainty from the Fisher matrix, the logarithmic derivatives are considered. Hence the derivatives used in the forecast can be related

\[ \frac{\partial \ln C}{\partial \ln \theta} = \frac{\theta \, \partial C}{C \, \partial \theta} \]  

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with $C$ being the correlation function and $\theta$ the parameter considered. It easily follows that the logarithmic derivatives of $P_{HI}$ is then given by

$$\frac{\partial \ln P_{HI}(k)}{\partial \ln f_{\sigma}} = \frac{2f_{\sigma} \mu^2}{(b_{\sigma} + f_{\sigma} \mu^2)}$$

(A.4)

$$\frac{\partial \ln P_{HI}(k)}{\partial \ln b_{\sigma}} = \frac{2b_{\sigma}}{(b_{\sigma} + f_{\sigma} \mu^2)}$$

(A.5)

### A.1.1 AP-effect

The Alcock-Packzynski effect requires us to also take into account the Hubble rate $H(z)$, and angular diameter distance $D_A(z)$. From (5.6) the new correlation function is

$$P_{HI}^a(k) = \frac{D_A^2(z) H(z)}{D_A^2(\bar{z}) \bar{H}(\bar{z})} D^2_b \left( b_{\sigma} + f_{\sigma} \mu^2 \right)^2 \frac{P_m(k)}{\sigma^2}$$

with the barred variables $\bar{H}$ and $\bar{D}_A$, represents the assumed fiducial cosmology. A substitution is performed to simplify the calculation, so to this end let

$$T = \frac{D_A^2(z)}{D_A^2(\bar{z})}; \quad U = \frac{H(z)}{\bar{H}(\bar{z})}; \quad V = T^2_b \left( b_{\sigma} + f_{\sigma} \mu^2 \right)^2; \quad W = \frac{P_m(k)}{\sigma^2};$$

(A.5)

$$\Rightarrow P_{HI}^a(k) = TU VW,$$

thus by the chain rule we find the logarithmic derivative with respect to parameter $\theta$,

$$\frac{\partial \ln P_{HI}^a(k)}{\partial \ln \theta} = \frac{\partial \ln P_{HI}^a(k)}{\partial \ln \theta} \left[ \frac{U V W}{\partial \theta} + \frac{T U}{\partial \theta} + \frac{U V}{\partial \theta} + \frac{T U}{\partial \theta} \right]$$

or

$$\frac{\partial \ln P_{HI}^a(k)}{\partial \ln \theta} = \theta \left[ \frac{1}{T} \frac{\partial T}{\partial \theta} + \frac{1}{U} \frac{\partial U}{\partial \theta} + \frac{1}{V} \frac{\partial V}{\partial \theta} + \frac{1}{W} \frac{\partial W}{\partial \theta} \right].$$

(A.6)

First let us consider the expansion parameter $H$, which technically influences $D_A$ by (2.39) but is considered independent in this case, hence

$$\frac{1}{T} \frac{\partial T}{\partial H} = 0 = \frac{1}{U} \frac{\partial U}{\partial D_A}$$

(A.7)
for the $P(k)$ Fisher analysis. Considering the second term,

$$\frac{H \partial U}{U \partial H} = \frac{\bar{H} \partial U}{\partial H} = \frac{\bar{H}}{H} \frac{\partial H}{\partial H} = 1$$  \hspace{1cm} (A.8)$$

is a rather straight forward calculation. Before we compute the last two terms we need to consider how varying the the expansion parameter will influence the $k$-mode, and hence $\mu$. The Hubble rate will only affect the $k$-mode in the radial direction and so

$$\frac{H}{H} = \frac{k_\parallel}{k_\parallel} \Rightarrow \frac{\partial H}{\partial k_\parallel} = \frac{\bar{H}}{k_\parallel}$$

determines the distortion of $k_\parallel$-mode. We can find the influence of $k$ on $H$ by recalling

$$k^2 = k_\parallel^2 + k_\perp^2 \Rightarrow \frac{\partial k}{\partial k_\parallel} = \frac{k_\parallel}{k}$$

such that

$$\frac{\partial H}{\partial k} = \frac{k_\parallel}{k}$$  \hspace{1cm} (A.9)$$

then we find the rate of change of $\mu$ with respect to $k$-mode

$$\mu, \frac{k_\parallel}{k}, \frac{\partial \mu}{\partial k}, \frac{1 - \mu^2}{k\mu}$$  \hspace{1cm} (A.10)$$

after using the quotient rule and substituting back $k_\parallel = \mu k$. The influence of $H$ on pointing direction is then given by

$$\frac{\partial \mu}{\partial H} = \frac{\partial \mu}{\partial k} \frac{\partial k}{\partial H} = \frac{1 - \mu^2}{k\mu} \frac{k^2}{H} = \frac{\mu(1 - \mu^2)}{H}$$  \hspace{1cm} (A.11)$$

which means we can now determine the derivative of the last two terms $V$ and $W$. 

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Only $\mu$ is influenced by $H$ in the $V$ term is, thus

\[
\begin{align*}
\frac{H}{V} \frac{\partial V(\mu)}{\partial H} &= \frac{H}{V} \frac{\partial V(\mu)}{\partial \mu} \frac{\partial \mu}{\partial H} \\
&= \frac{H}{V} \frac{\partial \mu}{\partial H} 2\bar{T}_b \left( b\sigma_8 + f\sigma_8\mu^2 \right) 2\mu f\sigma_8 \\
&= \frac{H}{V} \frac{\mu(1-\mu^2)}{H} \frac{4\mu f\sigma_8 V}{(b\sigma_8 + f\sigma_8\mu^2)} \\
&= \frac{4f\sigma_8\mu^2(1-\mu^2)}{b\sigma_8 + f\sigma_8\mu^2}
\end{align*}
\]  

(A.12)

Finally we can look at the influence on the matter power spectrum, which is only a function of $k$-scale, hence

\[
\begin{align*}
\frac{H}{W} \frac{\partial W(k)}{\partial H} &= \frac{H}{W} \frac{\partial W(k)}{\partial k} \frac{\partial k}{\partial H} \\
&= \frac{H}{W} \frac{\partial k}{\partial H} 1 \frac{\partial P_m(k)}{\partial k} \\
&= \frac{k\mu^2}{P_m(k)} \frac{\partial P_m(k)}{\partial k} \\
&= \frac{\partial \ln P_m^{HI}(k)}{\partial \ln H} 1 \frac{f\sigma_8\mu^2(1-\mu^2)}{b\sigma_8 + f\sigma_8\mu^2} \frac{k\mu^2}{P(k)} \frac{\partial P(k)}{\partial k}
\end{align*}
\]  

(A.13)

after canceling the expansion parameter and $\sigma_8^2$. It easily follows from (A.6) that

\[
\begin{align*}
\frac{\partial \ln P_m^{HI}(k)}{\partial \ln H} &= 1 \frac{f\sigma_8\mu^2(1-\mu^2)}{b\sigma_8 + f\sigma_8\mu^2} \frac{k\mu^2}{P(k)} \frac{\partial P(k)}{\partial k}
\end{align*}
\]  

(A.14)

Similarly the derivative with respect to $D_A$ can be found, first consider the relation between the transversal direction and angular diameter distance

\[
\frac{D_A}{\bar{D}_A} = \frac{k_\perp}{k} \Rightarrow \frac{\partial D_A}{\partial k_\perp} = -\frac{\bar{D}_A}{k_\perp}
\]

such that the relation to scale is given by

\[
\begin{align*}
\frac{\partial D_A}{\partial k} &= \frac{\partial D_A}{\partial k_\perp} \frac{\partial k_\perp}{\partial k} = -\frac{\bar{D}_A}{k_\perp} \frac{k}{k_\perp} = -\frac{\bar{D}_A}{k(1-\mu^2)}
\end{align*}
\]  

(A.15)

assuming the Pythagorean relation between $k$ and $k_\perp$. 
Since we have already determined the relation between $\mu$ and $k$, it easily follows

$$
\frac{\partial \mu}{\partial D_A} = \frac{\partial \mu}{\partial k} \frac{\partial k}{\partial D_A} = -\frac{1 - \mu^2}{k\mu} \frac{k\(1 - \mu^2\)}{D_A} = -\frac{\(1 - \mu^2\)^2}{\mu D_A} = -\frac{\mu\(1 - \mu^2\)}{D_A}
$$

which means we can now compute the derivative using (A.6). The first term is then easily computed

$$
\frac{D_A}{T} \frac{\partial T}{\partial D_A} = \frac{D_A}{T} \frac{\partial D_A^2}{\partial D_A} = -2 \frac{D_A}{T} \frac{T}{D_A} = -2
$$

and it was already shown the $T$ term is also neglected because of the independence of $H$. The term dependent on $\mu$ is calculated by

$$
\frac{D_A}{V} \frac{\partial V(\mu)}{\partial D_A} = \frac{D_A}{V} \frac{\partial \mu}{\partial \mu} \frac{\partial D_A}{\partial \mu} = \frac{\mu\(1 - \mu^2\)}{(b\sigma_s + f\sigma_s^2)}
$$

The final term is calculated as before, using the relation between $D_A$ and $k$, then

$$
\frac{D_A}{W} \frac{\partial W(k)}{\partial D_A} = \frac{D_A}{W} \frac{\partial W(k)}{\partial k} \frac{\partial k}{\partial D_A} = \frac{D_A}{W} \frac{\partial k}{\partial D_A} \frac{1}{\sigma_s^2} \frac{\partial P_m(k)}{\partial k} = -\frac{k\(1 - \mu^2\)}{P_m(k)} \frac{\partial P_m(k)}{\partial k}
$$

such that the final form of the derivative is given by

$$
\frac{\partial \ln P_{HI}(k)}{\partial \ln D_A} = -2 + \frac{4f\sigma_s \mu^2\(1 - \mu^2\)}{b\sigma_s + f\sigma_s^2} + \frac{k(\mu^2 - 1)P_m(k)}{P_m(k)} \frac{\partial P_m(k)}{\partial k}
$$

(A.20)
A.1.2 Growth index

As mentioned in the thesis it is useful to constrain the growth index directly, since different gravitation theories predict distinctly different values. Thus the derivative of $P^{HI}$ with respect to $\gamma$ needs to be calculated, and can be found by relating growth index to $f\sigma_8$ using (2.89),

$$\frac{\partial \ln f\sigma_8}{\partial \ln \gamma} = \frac{\gamma}{\Omega_m \sigma_8} \frac{\partial \Omega_m \sigma_8}{\partial \gamma} = \frac{\gamma}{\Omega_m} \frac{\partial}{\partial \gamma} e^{\gamma \ln \Omega_m} = \gamma \ln \Omega_m$$

(A.21)

hence it easily follows that after substituting back to $f$, that

$$\frac{\partial \ln P^{HI}(k)}{\partial \ln \gamma} = \frac{\partial \ln P^{HI}(k)}{\partial \ln f\sigma_8} \frac{\partial \ln f\sigma_8}{\partial \ln \gamma} = \frac{2f\sigma_8 \mu^2 \ln f}{(b\sigma_8 + f\sigma_8 \mu^2)}$$

(A.22)
A.2 Angular power spectrum

The Fisher forecast from angular power spectrum requires us to determine derivatives of the total angular correlation $C^g_\ell$ with respect to the cosmological parameters. It is easier to compute the derivatives for individual terms in (3.36)

$$C^{g, \text{obs}}_\ell(z_i, z_j) = C^g_\ell(z_i, z_j) + C^{RSD}_\ell(z_i, z_j) + C^{g, RSD}_\ell(z_i, z_j) + C^{RSD, g}_\ell(z_i, z_j)$$

(A.23)

and add the results together. In the $C_\ell$ analysis we only consider the growth rate and growth index in the Fisher forecast, it should be noted that the angular correlation of the matter power spectrum $C^g_\ell$ is independent of $f$ and hence $\gamma$, so we only need to consider the cross terms and $C^{RSD}_\ell$.

### A.2.1 Growth index

Since $\gamma$ is constant over time the derivative is simpler and will be computed first, now consider the cross terms defined in (3.35),

$$C^{g, RSD}_\ell(z_i, z_j) = -4\pi R_0^2 \int d\ln k P_\zeta(k) b_i Q_i j_i(k\chi_i) f_j Q_j j''_\ell(k\chi_j)$$

(A.24)

which only have $f(z)$ to consider. Using the relation in (A.21) we can determine

$$\frac{\partial C^{g, RSD}_\ell}{\partial \ln \gamma} = -4\pi R_0^2 \int d\ln k P_\zeta(k) b_i Q_i j_i(k\chi_i) Q_j j''_\ell(k\chi_j) f(z_j) \ln f(z_j)$$

(A.25)

given that $\partial_\gamma$ notation implies a correlation with the derivative of the RSD spherical harmonic coefficient, $\partial_\gamma a^{RSD}_{lm}$. The compliment of the cross term is the same as above, except the $i$th and $j$th bins are swapped, $C^{\partial_\gamma, g}_\ell$, thus the transpose of the previous matrix. The correlation function only considering RSD is given in (3.33), and using the subscript notation to indicate the $z$-bin

$$C^{RSD}_\ell(z_i, z_j) = 4\pi R_0^2 \int d\ln k P_\zeta(k) f_i Q_i j''_\ell(k\chi_i) f_j Q_j j''_\ell(k\chi_j)$$

(A.26)
which means we need to employ the product rule to take into account \( f_i f_j \). Then we can write the derivative as

\[
\frac{\partial C^RSD}{\partial \ln \gamma} = 4\pi R_0^2 \int d\ln k \ \mathcal{P}_\ell(k) \ Q_i j_i''(k\chi_i) Q_j j_j''(k\chi_j) f_i f_j \left( \ln f_j + \ln f_i \right)
\]

(A.27)

using the same notation as before. Thus from (3.36) we can determine the derivative for the total correlation by

\[
\frac{\partial C^{g,obs}}{\partial \ln \gamma} = C^{RSD,\partial_r}(z_i, z_j) + C^{\partial_r, RSD}(z_i, z_j)
\]

(A.28)

and by grouping together terms that has \( \partial_r \) in the same position, we can simplify it by

\[
C^{RSD,\partial_r} + C^{g,\partial_r} = 4\pi R_0^2 \int d\ln k \ \mathcal{P}_\ell(k) \ Q_i Q_j j_i''(k\chi_i) j_j''(k\chi_j) f_j \ln f_j [b_i - f_i]
\]

(A.29)

Hence the simplified form of the derivative is given as the sum of two covariant matrices

\[
\frac{\partial C^{g,obs}}{\partial \ln \gamma} = C^{RSD,\partial_r}(z_i, z_j) + C^{g,\partial_r}(z_i, z_j)
\]

(A.30)

using the total and derivative spherical harmonic coefficients in the correlation, \( \langle a_{\ell m}^g a_{\ell m}^{* \partial_r} \rangle \), and its transpose.

---

### A.2.2 Growth rate

Similarly we can determine the derivative with respect to the growth rate, to this end let \( \partial f \) denote the derivative of the spherical harmonic coefficient as as before, \( \partial_f a_{\ell m}^{RSD} \). Thus the derivative of the cross term can be written in terms of \( \delta_f^l \)

\[
\frac{\partial C^RSD}{\partial \ln f_n} = -4\pi R_0^2 \int d\ln k \ \mathcal{P}_\ell(k) \ b_j Q_i j_i''(k\chi_i) Q_j j_j''(k\chi_j) f_n \delta_i^n = C^{\partial_f, g}(z_i, z_j)
\]

(A.31)
which is the correlation between the galaxy number distribution and RSD if the bins correspond, or else zero. The derivative of the auto correlation of RSD is written in terms of the product rule

\[
\frac{\partial C_{RSD}^\ell}{\partial \ln f_n} = 4\pi R_0^2 \int d\ln k \ P_\zeta(k) \ Q_i \ j_i''(k\chi_i) \ Q_j j_j''(k\chi_j) \ f_n \left( f_i \delta_i^n + f_j \delta_j^n \right) \]

(A.32)

and so using (3.36) we can compute the full derivative

\[
\frac{\partial C_{\text{obs}}^\ell(z_i, z_j)}{\partial \ln f_n} = C_{\ell, RSD}^\partial (z_i, z_j) + C_{\ell, f}^\partial (z_i, z_j) + C_{\ell, g}^\partial (z_i, z_j)
\]

(A.33)

and group together similar terms. The simplification is given by

\[
C_{\ell, RSD}^\partial + C_{\ell, f}^\partial = 4\pi R_0^2 \int d\ln k \ P_\zeta(k) \ Q_i \ j_i''(k\chi_i) \ j_i''(k\chi_j) \ f_n \left[ b_i - f_i \right]
\]

(A.34)

The derivative with respect to growth rate in the \( n \)th bin has the same format as with \( \gamma \) and is given by

\[
\frac{\partial C_{\ell, \text{obs}}^\partial(z_i, z_j)}{\partial \ln f_n} = C_{\ell, RSD}^\partial (z_i, z_j) + C_{\ell, f}^\partial (z_i, z_j) + C_{\ell, g}^\partial (z_i, z_j)
\]

(A.35)