High Accuracy Fitted Operator Methods for Solving Interior Layer Problems

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Abstract

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Fitted operator finite difference methods (FOFDMs) for singularly perturbed problems have been explored for the last three decades. The construction of these numerical schemes is based on introducing a fitting factor along with the diffusion coefficient or by using principles of the non-standard finite difference methods. The FOFDMs based on the latter idea, are easy to construct and they are extendible to solve partial differential equations (PDEs) and their systems. Noting this flexible feature of the FOFDMs, this thesis deals with extension of these methods to solve interior layer problems, something that was still outstanding. The idea is then extended to solve singularly perturbed time-dependent PDEs whose solutions possess interior layers. The second aspect of this work is to improve accuracy of these approximation methods via methods like Richardson extrapolation. Having met these three objectives, we then extended our approach to solve singularly perturbed two-point boundary value problems with variable diffusion coefficients and analogous time-dependent PDEs. Careful analyses followed by extensive numerical simulations supporting theoretical findings are presented where necessary.

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Key words

High accuracy fitted operator methods for solving interior layer problems

Singular perturbation problems

Turning point problems

Interior layers

Two-point boundary value problems

Evolutionary partial differential equations

Fitted operator finite difference methods

Higher order numerical methods

Extrapolation methods

Convergence analysis

Stability analysis.

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Declaration

I declare that *High accuracy fitted operator methods for solving interior layer problems* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Mbani T. Sayi August 2, 2020

Signed: ....

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List of Publications

Part of this thesis has already been published/submitted in the form of the following research papers:

1. J.B. Munyakazi, K.C. Patidar, M.T. Sayi, A robust fitted operator finite difference method for singularly perturbed problems whose solution has an interior layer,*Mathematics and Computer in Simulation* **160**(2019) 155-167.

2. J.B. Munyakazi, K.C. Patidar, M.T. Sayi, A fitted numerical method for parabolic turning point singularly perturbed problems with an interior layer, *Numerical Methods Partial Differential Equations* **370**(2019) 2407-2422.

3. J.B. Munyakazi, K.C. Patidar, M.T. Sayi, A fitted numerical method for interior layer convection-diffusion problems with a quadratic factor affecting the second derivative, ready for submission.

4. J.B. Munyakazi, K.C. Patidar, M.T. Sayi, A discretization of turning point parabolic problems with a quadratic diffusion coefficient, ready for submission.

5. J.B. Munyakazi, K.C. Patidar, M.T. Sayi, A robust fitted operator finite difference method for singularly perturbed turning point problems with a linear diffusion factor and an interior layer, in preparation.

6. J.B. Munyakazi, K.C. Patidar, M.T. Sayi, A uniformly convergent fitted numerical method for turning point singularly perturbed parabolic problems with a linear diffusion coefficient and an interior layer, in preparation.

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Chapter 1

General Introduction

In this chapter we present a general overview of the work accomplished in this thesis. After circumscribing the scope and providing some background information about this thesis, we provide a literature review.

1.1 Introduction

Singular perturbation problems (SPPs) started attracting the attention of researchers since the beginning of last century. They quickly became popular due to the wide range of applications in many domains of science and engineering including but not limited to fluid dynamics, fluid mechanics, solid mechanics, quantum mechanics, chemical reactor theory, aerodynamics, optimal control, reaction-diffusion process, geophysical, oceanic and atmospheric circulation, plasticity, chemical reaction theory, meteorology, modelling of semi-conductor devices, diffraction theory, plasma dynamics, magneto-hydrodynamics process, etc [5, 59, 70].

The solution to SPPs undergoes abrupt changes in narrow regions known as the boundary or interior layer regions depending on the location in the domain of the problem being considered. The parameter responsible for these abrupt changes, known as the perturbation parameter, is the coefficient of the highest derivative term of the differential equation that underlies the problem concerned. When the perturbation parameter, also known as the diffusion parameter, approaches zero, the problems become harder to handle qualitatively and quantitatively as analytical methods fail to capture some important dynamics of the solutions. Thus, researchers have resolved to resort to numerical approaches such as the finite difference methods, finite elements methods, finite volume methods and spectral methods.

Finite difference methods used to solve singular perturbation problems are grouped in two categories: the fitted mesh finite difference methods (FMFDMs) and the fitted operator

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finite difference methods (FOFDMs). In this thesis, we will focus on the latter.

The construction of FOFDMs remained a challenge for quite some time. Such methods were initially named exponentially fitted methods. The construction of these numerical schemes was based on introducing a fitting factor along with the diffusion coefficient. The derivation of this fitting factor required that the analogous schemes satisfy the sufficient conditions for uniform convergence. However, such an approach was limited to only two-point boundary value problems. Its extension to partial differential equations was a difficult task due to too much technical details. To fill this gap, the literature in the last decade witnessed the so-called non-standard finite difference methods as fitted operator methods for the singularly perturbed problems.

The FOFDMs based on this idea are easy to construct and to extend to solve partial differential equations (PDEs) and their systems. Keeping in mind this flexible feature of the FOFDMs, this thesis deals with extension of these methods to solve two-point boundary value singularly perturbed interior layer problems. The next milestone of this thesis is the extension of this idea to singularly perturbed time-dependent PDEs whose solutions possess interior layers. The second aspect of this work is to improve accuracy of these approximation methods via methods like Richardson extrapolation. After attaining these three objectives, we then extend our approach to solve singularly perturbed two-point boundary value problems (TPBVPs) with variable diffusion coefficients and analogous time-dependent PDEs. Careful analyses followed by extensive numerical simulations supporting theoretical findings are presented where necessary.

In the next session we speak broadly about the fitted methods that have been used in the literature in the framework of finite difference techniques.

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1.2 Fitted methods

The fitted finite difference methods are best suited to solve SPPs as compared to their standard finite difference counterpart. Despite their convergence property, standard finite difference fail to produce reliable approximations when the perturbation parameter becomes very small, unless a very fine mesh is considered. However, consideration of such a fine mesh will increase the round-off error. The fitted finite difference methods do not suffer from this drawback as they produce good approximations with a reasonable number of mesh points which allow to find the right balance between the truncation and the round-off errors.

In this section we describe both the FMFDMs and the FOFDMs and explain how Richardson extrapolation is used.

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Fitted mesh finite difference methods (FMFDMs)

These methods are constructed on layer-adapted meshes (see e.g. [35]). They are easy to extend to problems in higher dimension and work well on nonlinear problems provided that a good mesh construction strategy is put in place. However, they necessitate prior knowledge of the size and location of the layers. This is a serious drawback of these methods since, in many cases, it is challenging to determine the size and the location of the layers, especially for problems in higher dimensions.

The FMFDMs may be constructed on two types of layer-adapted meshes namely the piecewise uniform meshes and the graded meshes. A piecewise-uniform mesh often referred to as a Shishkin mesh is a concatenation of two or more uniform meshes having different discretisation parameters. In order to resolve the layer issues, the Shishkin meshes are fine in the layer region and coarse elsewhere. The graded meshes, also known as the Bakhvalov meshes are fine in the layer region and become gradually uniform away from the region. For more information, readers are referred to [37, 47, 58].

Fitted operator finite difference methods (FOFDMs)

The FOFDMs are classified into the exponentially fitted methods and the nonstandard finite difference methods.

The construction of exponentially fitted methods require that a fitting factor be introduced along with the diffusion coefficient of the problem being investigated. The derivation of this fitting factors is done in such a way that the resultant scheme is uniformly convergent. Research informs that theses schemes are suitable only for two-point boundary value problems and are not extendible to partial differential equations.

The advent of nonstandard finite difference methodology [57] more than two decades ago brought an alternative framework for solving SPPs. This alternative framework constitutes the FOFDMs that consist of the substitution of the denominator functions of the standard or classical derivatives by the positive functions designed to capture major properties of the governing differential equations [8]. The construction of these methods does not require any knowledge of the width and location of the layers. Moreover, these methods are easy to extend to problems in higher dimensions and their systems.

Higher order approximations

There are two ways to achieve higher order accuracies: (i) through the construction of direct higher order methods, or (ii) via some extrapolation techniques. The work presented in this

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thesis makes use of the latter approach. The specific extrapolation technique that we use is the Richardson extrapolation. Some relevant work where this technique was used for SPPs are [65, 66, 67, 68, 91].

In what follows, we provide a brief review on literature pertaining to the work presented in this thesis.

1.3 Literature review

The solution of singularly perturbation ordinary differential equations is quite difficult to find due to the computational problem [71] and more so, when it comes to singularly perturbed partial differential equations [50]. Theoretically, in differential equations, there are methods that one can use to foretell the existence and the uniqueness of the solution, although it is almost impossible to get the analytical solution. More often, researchers use numerical methods to provide approximations to the solution; and the great challenge has always been that of constructing numerical methods that lead to better approximations.

Patidar *et al.* [73] reviewed existing literature on asymptotic and numerical methods for solving singularly perturbed turning point and interior layer problems. The aim of this survey article was to identify the problems which have been treated; the numerical and asymptotic methods utilised for their solutions. It covers the period from 1970 to 2011.

Below, we mention some of the works surveyed in the above article as well as some other works published in the liturature.

Geng *et al.*[30] considered the following singularly perturbed turning point problems with an interior layer near $x = 0$,

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), -1 < x < 1,$$

with

$$u(-1) = \alpha, \; u(1) = \gamma,$$

where $0 < \varepsilon << 1$, $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently smooth, such that the current problem has a unique solution. The problem was split into an interior layer problem and a regular domain problem. Asymptotic expansion method was used to tackle the regular domain problem while the method of stretching variable and reproducing kernel method (RKM) was utilised to treat the interior layer problem. Though reproducing kernel theory has important application in numerical analysis, differential equations, probability and statistics, its applications to singularly perturbed differential equations are more often complicated. In particular, RKM failed to solve turning point singularly perturbed problems.

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A singularly perturbed parabolic periodic boundary value problem for a reaction-advection-diffusion equation was studied by Nefedov *et al.* [69]. They constructed an asymptotic method for the interior layer type and proposed a modified procedure to get asymptotic lower and upper solutions. And by using sufficiently precise lower and upper solutions they proved the existence of the parabolic solution with an interior layer and estimated the accuracy of its asymptotic.

In [89], Shih and Tung studied a transient convection-diffusion problem with moving sharp fronts by using perturbation methods. They obtained a uniformly valid approximate solution for two cases: shock layer and angular layer.They showed that the shock layer function can be described by the complementary error function, while the angular layer function can be described by the first iterated of the complemantary. Note that the solution to this problem has an interior layer.

Shishkin [90] in a composed domain on an axis $\mathbb{R}$ with the moving interface boundary between two subdomains, considered an initial value problem for a singularly perturbed parabolic reaction-diffusion equation in the presence of a concentrated source on the interface boundary. He stated that, monotone classical difference schemes for problems from this class converge only when $\varepsilon >> N^{-1} + N_0^{-1}$, where ε is the perturbation parameter, N and N_0 define the number of mesh points with respect to x (on segments of unit length) and t. He carried on and said that in the case of such problems with moving interior layers, it is necessary to develop special numerical methods whose errors depend rather weakly on the parameter ε and, in particular, are independent of ε (i.e., ε-uniformly convergent methods).

In this work, he studied schemes on adaptive meshes which are locally condensing in a neibhbourhood of the set γ^*, that is, the trajectory of the moving source. He also said, in the class of difference schemes consisting of standard finite difference operator on rectangular meshes which are (a priori or a posteriori) locally condensing in x and t, there are no schemes that converge $\varepsilon-$uniformly, and in particular, even under the condition $\varepsilon \approx N^{-2} + N_0^{-2}$, if the total number of mesh points between the cross-sections x_0 and $x_0 + 1$ for any $x_0 \in \mathbb{R}$ has order of NN_0. Thus, the adaptive mesh refinement techniques used directly did not allow him to widen essentially the convergence range of classical numerical methods. On the other hand, he said, the use of condensing meshes in local coordinate system fitted to the set γ^* made it possible to construct schemes which converged $\varepsilon-$uniformly for $N, N_0 \longrightarrow \infty$. Such scheme converges at the rate $\mathcal{O}\left(N^{-1}\ln N + N_0^{-1}\right)$.

A streamline-diffusion finite element method to solve a two-dimensional singular perturbation convection-diffusion problem whose solution has boundary and interior layers was proposed by Kopteva [43].

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Carey *et al.* [16] considered the use of adaptive mesh strategies for solution of problems exhibiting boundary and interior layer solutions. As the presence of these layer structures suggests, reliable and accurate solution of this class of problems using finite difference, finite volume or finite element schemes requires grading the mesh into the layers and due attention to the associated algorithms. They said, when the nature and structure of the layer is known, mesh grading can be achieved during the grid generation by specifying an appropriate grading function. However, in many applications the location and nature of the layer behavior is not known in advance. Consequently, adaptive mesh techniques that employ feedback from intermediate grid solutions are an appealing approach.

O'Riordan and Quinn [70] examined a linear time dependent singularly perturbed convection-diffusion problems, where the convective coefficient got interior layer. They design and analyse a monotone finite difference operator on a piecewise-uniform Shiskin mesh.

Gracia and O'Riordan [31] constructed and analysed a numerical method consisting on a monotone finite difference operator and piecewise uniform mesh. This method was used to solve a linear singularly perturbed time dependent convection-diffusion problem, in which initial condition was designed to have steep gradient in the vicinity of the inflow, transported in time to create a moving interior chock layer.

A fitted operator finite difference methods for boundary value problems where the solution of a singularly perturbed delay differential equations with turning point and mixed shifts is introduced by Rai and Sharma [80].

Martin [55] considered the rotational gravity water flows having two jumps in the vorticity distribution. The dispersion relation and a fourth order algebraic equation with intricate coefficients being naturally complicated the author of this paper derives the appropriate dispersion relation for periodic travelling waves propagating at the surface of the water. The waves are assumed to have a layer of constant non-zero vorticity situated between two layers of the irrotational flow. He suggested an estimate of a very simple form involving only the levels at which the vorticity has jumps. The formula derived generalizes a corresponding one from [80].

Bennett *et al.* [6] studied interior layer problems. They considered the film blowing which is a highly complex industrial process used to manufacture thin sheets of polymer. The models describing this process are nonlinear. It is well known that numerical instabilities often occur when solving the highly nonlinear differential equations. Both shear-thinning and shear-thickening polymers are considered in this paper and they used a balance of orders argument to identify the structure of a region of rapid expansion in the radial profile of the film. To obtain an accurate form, a mixture of heuristic and singular perturbation

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techniques is applied to obtain a closed form approximate expression for the radial profile of the film which displays the interior layer phenomenon. They demonstrate how approximate solutions to the highly nonlinear two point boundary value problem describing this process may be constructed using this expression as an initial estimate in an iterative scheme. Numerical solutions for the radial temperature, velocity and thickness profiles of the film are subsequently obtained by iteration.

In [2], Aifeng and Mingkang constructed an asymptotic expansion formula using the methods of boundary and fractional steps for a nonlinear singularly perturbed second order differential-difference equation with interior layer. They proved the existence of the smooth solution and the uniform validity of asymptotic expansion using differential inequality technique.

Singularly perturbed BVPs for delay differential equations with a turning point were studied by Rai and Sharma [79]. They used fitted mesh technique to generate a piecewise uniform mesh, condensed in the neighbourhood of the boundary layers. The finite difference method derived is uniformly convergent with respect to the perturbation parameter. Numerical experiments are used to illustrate, in practice, the result of convergence proved theoretically and demonstrate the effect of the delay argument and the coefficient of the delay term on the layer behaviour of the solution.

Munyakazi *et al.* [64] constructed and studied a fitted operator finite difference method (FOFDM) for the class of singularly perturbed problems whose solution exhibits an interior layer due to the presence of a turning point

$$\varepsilon u'' + a(x)u' - b(x)u = f(x), x \in (-1, 1),$$

with the boundary conditions

$$u(-1) = A, u(1) = B,$$

where A and B are given real numbers, $0 < \varepsilon < 1$, and $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions such that this problem has a unique solution. They showed that the Scheme designed is uniformly convergent of order one. They also applied Richardson extrapolation as the acceleration technique to improve the accuracy and the order of convergence of the Scheme up to two.

Du and Gui [22] considered the problem

$$\varepsilon^2 \Delta U + V(y)U(1 - V^2) = 0, y \in \Omega,$$

and

$$\frac{\partial U}{\partial n} = 0, n \in \Omega,$$

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where Ω is a small and bounded domain in $\mathbb{R}^2$ and V is a positive smooth function in $\bar{\Omega}$. Let Γ be a closed, non-degenerate geodesic with respect to the metric $ds^2 = V(y)(dy_1^2 + dy_2^2)$ in Ω. They proved that there exist two interior transition layer solutions $U_\varepsilon^{(1)}, U_\varepsilon^{(1)}$ when ε is sufficiently small. One of the layer solutions $U_\varepsilon^{(1)}$ approaches -1 and $+1$ as ε approaches 0. The other solution $U_\varepsilon^{(2)}$ exhibits a transition layer in the opposite direction of the previous solution.

Kopteva [43] considered two model two-dimensional singularly perturbed convection-diffusion problems whose solutions may have characteristic boundary and interior layers. She solved them numerically by the stream-line-diffusion finite element method using piecewise linear or bilinear elements. She investigated how accurate the computed solution is in characteristic-layer regions if anisotropic layer adapted meshes are used. She showed theoretically and practically that the stream line-diffusion formulation may, in maximum norm, imply only first-order accuracy in characteristic-layer region.

Geng *et al.* [30] presented a numerical method based on the asymptotic expansion technique and the reproducing kernel method (RKM) for solving the following singularly perturbed turning point problems exhibiting an interior layer

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), -1 < x < 1,$$

with

$$u(-1) = \alpha, u(1) = \gamma,$$

where $0 < \varepsilon << 1, a(x), b(x)$ and $f(x)$ assumed to be sufficiently smooth, such that this problem has a unique solution. They reduced the original problem to interior layer and regular domain problems. While the regular domain problems were solved by using the asymptotic expansion method; the interior layer problems are treated by the method of stretching variable and the RKM. They proved the method to provide very accurate approximate solutions.

Asher [5], considered the following singularly perturbed boundary value ordinary differential problem where the problem defining the reduced solution is singular

$$\varepsilon y' = A(t, \varepsilon)y + f(t, \varepsilon), 0 \leq t \leq 1, 0 < \varepsilon < 1,$$

under the boundary conditions

$$B_0(\varepsilon)y(0, \varepsilon) + B_1(\varepsilon)y(1, \varepsilon) = \beta(\varepsilon).$$

Here A, B_0 and B_1 are $n \times n$ real valued matrices of size n.
For the numerical approximations, he used families of symmetric difference schemes; which

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are equivalent to certain collocation schemes based on Gauss and Lobatto points. He extended the convergence results, previously obtained for the " regular" singularly perturbed case. He also found out that; Gauss Schemes are extended with no change but Lobatto Schemes require a small modification in the mesh selection procedure.

Kadalbajoo and Patidar [37] considered general singular perturbation problems of the form

$$C_\varepsilon y''(x) + a(x)y' + b(x)y(x) = f(x), x \in [0,1],$$

with the boundary conditions

$$y(0) = \eta_0, y(1) = \eta_1,$$

where C_ε equals to $+\varepsilon$ and $-\varepsilon$, $a(x), b(x), f(x)$ positive throughout the interval and $\eta_0, \eta_1 \in \mathbb{R}$. They indicated that, the indirect methods (those which do not use any acceleration of convergence techniques, e.g., Richardson's extrapolation or defect correction, etc.) for such problems on a mesh of Shishkin type lead the error as $\mathcal{O}(n^{-1}\ln n)$ where n denotes the total number of sub-intervals of $[0,1]$. In this work, they described a very simple and direct method which reduces the error to $\mathcal{O}(n^{-2}\ln^2 n)$. They proved theoretically and numerically that the method was $\varepsilon-$uniformly convergent with the above error bounds, on a piece-wise uniform mesh of Shishkin type.

Kadalbajoo *et al.* [38] considered some problems arising from the following singularly perturbed general differential equations

$$\varepsilon y''(x) + a(x)y'(x) + \alpha(x)y(x-\delta) + \zeta(x)y(x) + \beta(x)y(x+\eta) = f(x), x \in \Omega = (0,1),$$

with boundary conditions

$$y(x) = \phi(x), -\eta \le x \le 0,$$

and

$$y(x) = \gamma, 1 \le x \le 1+\eta,$$

where $a(x),\ \alpha(x),\ \zeta(x),\ \beta(x),\ f(x),\ \phi(x)$, and $\gamma(x)$ are sufficiently smooth functions, $0 < \varepsilon << 1$, is the singular perturbation parameter and $0 < \eta = \mathcal{O}(\varepsilon), 0 < \eta = \mathcal{O}(\varepsilon)$ are the delay and the advance parameters. They first constructed and analyzed a "fitted operator finite difference method (FOFDM)" which they showed to be first order $\varepsilon-$uniformly convergent. Using one function evaluation at each step, they derived a higher order method via Shishkin mesh to which they referred as the " fitted mesh finite difference method (FMFDM)", a direct method and $\varepsilon-$uniformly convergent with the nodal error as $\mathcal{O}(\bar{n}^2\ln^2 n)$ which is an improvement over the existing direct methods for such problems on a mesh of Shishkin type that lead to the error as $\mathcal{O}(n^{-1}\ln n)$ where n denotes the total

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number of sub-intervals of $(0, 1)$. They presented comparative numerical results to support their theory.

Lubuma and Patidar [53] constructed and analyzed a non-Standard finite difference method for a class of singularly perturbed differential equations with two types of problems: (i) those equations having solutions with layer behaviour and (ii) those having solutions with oscillatory behaviour. One of their aim was to design a special method to resolve the latter type of problems. Which was effectively proved and supported by several numerical examples.

O'Riordan and Quinn [70] examined the following linear time dependent singularly perturbed convection-diffusion problem where the convective coefficient contains an interior layer, which in turn induces an interior layer in the solution.

$$-\varepsilon u_{xx} + au_x + bu + Cu_t = f, (x,t) \in (0,1) \times (0,T],$$

with $b \geq 0,\ C \geq 0,\ 0 < \varepsilon << 1$. Functions $u(0,t), u(1,t), u(x,0)$ are specified and the convective coefficient $a(x)$ is assumed to be discontinuous across a curve $\Gamma_1 := \{(d(t),t), t \in [0,T],\ 0 < d(t) < 1\}$ and to have the particular sign pattern $a(x) > 0,\ x < d(t);\ a(x) < 0,\ x > d(t)$. They constructed and analyzed a numerical method consisting of a monotone finite difference operator and a piece-wise-uniform Shishkin mesh. And they showed theoretically and numerically that the scheme is first order parameter uniform convergent.

Aifeng and Mingkang [2] considered the Interior layer for a second order nonlinear singularly perturbed differential equation. They constructed the formula of asymptotic expansion, using the method of boundary function and fractional steps. They pointed out that the boundary layer at $t = 0$ has a great influence upon the interior layer at $t = \sigma$. And they proved the existence of the smooth solution and the uniform validity of the asymptotic expansion using differential inequality techniques. An example was also used to demonstrate the effectiveness of their results.

Nefedov *et al.* [69] considered the following singularly perturbed parabolic boundary value problem for a reaction-convection-diffusion equation:

$$N_\varepsilon := \varepsilon \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} \right) - A(u,x,t)\frac{\partial u}{\partial x} - B(u,x,t) = 0,$$

for

$$(x,t) \in D := (x,t) \in \mathbb{R}^2 := -1 < x < 1, t \in \mathbb{R},$$

$$u(-1,t,\varepsilon) = u^{(-)}(t), u(1,t,\varepsilon) = u^{(+)}(t), \forall\ x \in \mathbb{R},$$

$$u(x,t,\varepsilon) = u(x,t+T,\varepsilon) \forall\ t \in \mathbb{R}, -1 \leq x \leq 1,$$

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and

$$\varepsilon \in I_{\varepsilon_0} := \{0 < \varepsilon \leq \varepsilon_0\}.$$

The functions $A, B, u^{(-)}$ and $u^{(+)}$ are sufficiently smooth and $T-$ periodic in t.
After constructing the interior layer type formal asymptotic, they proposed a modified procedure to get asymptotic lower and upper solution to prove the existence of a periodic solution with an interior layer and estimate the accuracy of its asymptotic. Nevertheless, they manage to establish the asymptotic stability of this solution.

Bennett and Shepherd [7] investigated the structure of typical solutions that arise when the polymer is assumed to be described by a power-law fluid operating under non isothermal conditions. They considered both a shear-thinning and shear-thickening polymer and used a balance of orders argument to identify the structure of a region of rapid expansion in the radial profile of film. They then applied a mixture of heuristic and singular perturbation techniques to obtain a closed form approximate expression for the radial profile of the film which displays the interior layer phenomenon. They finally demonstrated theoretically and practically how approximate solutions to the highly non linear two-point boundary value problem describing this process may be constructed using this expression as an initial estimate in an iterative scheme.

Kopteva [43] used a streamline-diffusion finite element method to solve a two-dimensional singularly perturbed convection-diffusion problem whose solution has boundary and interior layers. She showed that the streamline-diffusion formulation is only first-order accurate.

Herceg *et al.* [34], considered the following singularly perturbed boundary value problem:

$$-\varepsilon^2 u'' + C(x,u) = 0, x \in I = [0,1], u(0) = u(1) = 0,$$

with a small perturbation parameter $\varepsilon, \varepsilon \in (0,1)$ and $C \in C^8(I \times R)$.
They used finite difference schemes on non-equidistant meshes, dense in the layers to solve the above semilinear singular perturbation problem. The fourth order uniform accuracy of the Hermitian approximation was improved to sixth order by Richardson extrapolation.

In [66], Munyakazi and Patidar considered the singularly perturbed two-point boundary value problem (BVP)

$$-\varepsilon(a(x)y')' + b(x)y = f(x), x \in (0,1),$$

with the boundary conditions

$$y(0) = \eta_0, y(1) = \eta_1,$$

where η_0 and η_1 are given constants and $\varepsilon \in (0,1]$. The functions $f(x), a(x)$ and $b(x)$ are assumed to be sufficiently smooth satisfying the conditions $a(x) \geq a > 0, b(x) \geq b > 0$.

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This problem was resolved by Patidar [74], who used mesh of Shishkin type and showed the method to be fourth order $\varepsilon-$uniformly convergent. After attempting to increase the order of convergence by Richardson extrapolation, Munyakazi and Patidar above, discovered that this well-known convergence acceleration technique has some limitations. Though the extrapolation technique improves the accuracy slightly, but it does not increase the rate of convergence which is originally four order for the underlying method of the current problem.

In [66], Munyakazi and Patidar investigated whether they could increase the order of convergence of existing high order methods to solve some singularly perturbed two-point BVPs. They proceeded by considering a fitted mesh finite difference method of Patidar [74] applied on a mesh of Shishkin type for the solution of self-adjoint problem which is ε-uniformly convergent of order four. After attempting to improve the order of convergence by Richardson's extrapolation, they discovered that though it improves slightly the accuracy; this acceleration technique is limited. It could not increase the rate of convergence which is originally four for the underlying method for the given problem.

Richards [81] extended the Richardson extrapolation of the time-independent problems to time-dependent problems. The technique he presented is completed in the sense that the extrapolated solution is calculated at all special grid nodes which coincide with nodes of the finest grid considered. Richards compared the results obtained after applying Richardson extrapolation, and he concluded that, the extrapolation can be an easy and efficient way in which one can produce accurate numerical solutions to time-dependent problems.

Bujanda *et al.* [14] designed and analyzed an efficient numerical method to solve two dimensional initial-boundary value reaction–diffusion problems, for which the diffusion parameter can be very small with respect to the reaction term. They defined the method by combining the Peaceman and Rachford alternating direction method to discretize in time, together with a HODIE finite difference scheme constructed on a tailored mesh. They proved that the resulting scheme is ε-uniformly convergent of second order in time and of third order in spatial variables. They provided some numerical examples to illustrate the efficiency of the method and the orders of uniform convergence proved theoretically. They also showed that it is easy to avoid the well-known order reduction phenomenon, which is usually produced in the time integration process when the boundary conditions are time dependent.

Clavero, *et al.* [18] designed and analyzed a finite difference scheme used to solve a class of 2D time-dependent convection–diffusion problems, for which they supposed that the convection term is positive in both spatial directions. They used the Peaceman and Rachford method to discretize in time such problems and higher-order differences via an identity expansion finite difference scheme, defined on a piecewise uniform Shishkin mesh, to

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discretize in space. They proved that the method is uniformly convergent with respect to the diffusion parameter, reaching almost order two in space. A brief discussion concerning the theoretical and practical orders of convergence in time was included, pointing out possible theoretical advances in the future. They presented some numerical examples illustrating such a behaviour; they indicated that the numerical method is also suitable in a wider set of singularly perturbed problems than the ones defined by the theoretical restrictions.

Allen and Southwell [3] were among the first researchers to consider the Fitted Finite Difference methods while solving the problem of viscous fluid pass a cylinder. Thereafter Doolan *et al.* [21] took it further and came up with the exponentially fitted methods which were afterwards developed by Liniger and Wlloughby [48] as a special class of teta-mathods of Lambert [46]. Lubuma and Patidar [53, 54] were also amongst the earlier researchers to develop the FOFDMs when they constructed the Non-standard finite difference methods (NSFDMs) using the modelling rules of Mickens [57]. Collection of papers and books related to the discussions on the construction and analysis of FOFDMs can be found in [9, 21, 59] and the references therein.

In [45] Kumar and Rao introduced a high order parameter-robust numerical method to solve a Dirichlet problem for one-dimensional time dependent singularly perturbed reaction-diffusion equation. A small parameter ε is multiplied with the second order spatial derivative in the equation. The parabolic boundary layers appear in the solution of the problem as the perturbation parameter ε tends to zero. To obtain the approximate solution of the problem they constructed a numerical method by combining the Crank–Nicolson method on an uniform mesh in time direction, together with a hybrid scheme which is a suitable combination of a fourth order compact difference scheme and the standard central difference scheme on a generalized Shishkin mesh in spatial direction. They proved that the resulting method was parameter-robust or ε-uniform in the sense that its numerical solution converges to the exact solution uniformly well with respect to the singular perturbation parameter ε. More specifically, they proved that the numerical method is uniformly convergent of second order in time and almost fourth order in spatial variable, if the discretization parameters satisfy a non-restrictive relation. They presented numerical experiments to validate their theoretical results and also they indicated that the relation between the discretization parameters is not necessary in practice.

Munyakazi and Patidar [67] considered two fitted operator finite difference methods (FOFDMs) for the solution of the self-adjoint problem

$$-\varepsilon(a(x)y')' + b(x)y = f(x), x \in (0, 1],$$

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with the boundary conditions

$$y(0) = \eta_0, y(1) = \eta_1,$$

where η_0 and η_1 are given constants and $\varepsilon \in (0, 1]$. The functions $f(x), a(x)$ and $b(x)$ are assumed to be sufficiently smooth functions to ensure the smoothness of the solution, and in addition $a(x) \geq a > 0,\ b(x) \geq b > 0$. One method FOFDM-I was constructed by Patidar [75] and the second one FOFDM-II designed by Lubuma and Patidar [54]. The FOFDM-I and FOFDM-II were respectively showed to be fourth and second order for small and moderate values of ε. At the end of their study, it came out that Richardson extrapolation did not improve the order of FOFDM-I. However, the FOFDM-II was improved up to the fourth order with respect to the perturbation parameter ε.

1.4 Outline of the thesis

In this thesis we aim at designing and analysing fitted operator finite difference methods (FOFDMs) to solve various classes of singularly perturbed problems (SPPs) whose solution possesses an interior layer due to the presence of a turning point. Furthermore, we seek to increase the accuracy of the constructed methods via Richardson extrapolation. The work is outlined as follows.

In Chapter 2, we construct and analyse an FOFDM to solve two-point boundary value singularly perturbed problems with a constant diffusion coefficient. Chapter 3 is devoted to the extension of this FOFDM to a time-dependent convection-diffusion problems. We use the classical Euler method to discretize the time variable.

Chapters 4 and 5 deal with the construction and analysis of FOFDMs to solve singularly perturbed problems with a variable diffusion coefficient of the form $\varepsilon + x^2$. While Chapter 4 considers the stationary case, the time-dependent counterpart is studied in Chapter 5.

Construction of FOFDMs for singularly perturbed problems with a variable coefficient of the form $\varepsilon + x$ is dedicated to Chapter 6 for the stationary case and Chapter 7 for the time dependent case.

Convergence analyses of the methods above show to be first order uniformly convergent for the stationary cases and first order in both time and space in the case of evolutionary ones. Application of Richardson extrapolation improves the order of convergence in space from first to second order.

Finally, in Chapter 8, we discuss some concluding remarks and provide directions for the future works.

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Chapter 2

A robust fitted operator finite difference method for singularly perturbed problems whose solution has an interior layer

In this chapter, we propose a fitted operator finite difference method to solve the class of singularly perturbed problems whose solution exhibits an interior layer due to the presence of a turning point. This method is then analyzed by making use of the bounds on the solutions that we derive. We show that the scheme is uniformly convergent of order one. We also apply Richardson extrapolation as the acceleration technique to improve the accuracy and the order of convergence of the scheme up to two. Numerical investigations are carried out to demonstrate the efficacy and robustness of the scheme.

2.1 Introduction

Numerical methods have become essential tools in science and engineering due to the fact that most differential equations that model real life situations have no closed form analytical solutions. The challenge would be that of designing better methods *i.e.,* prone to produce the best approximations.

Singular perturbation problems (SPPs) have attracted many researchers' attention since the beginning of last century. These are problems in which the underlying (ordinary or partial) differential equation involves a small parameter as the coefficient of the highest derivative. The presence of this small parameter renders classical methods unfit for SPPs as their

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solutions present large gradients in narrow parts of the domain termed as boundary or interior layers. In the context of finite difference methods, the two widely adopted strategies to circumvent the difficulty inherent to handling large gradients are that of chosing a fine mesh in the layer region(s) or that of designing a difference formula that captures the behaviour of the solution. These strategies are known as the fitted mesh and the fitted operator finite difference methods, respectively. Several works have been done by researchers where such methods are presented, analysed and/or implemented. See for instance [25, 39, 40, 59, 62, 83]. One of the objective herein is to investigate the action of Richardson extrapolation, as the acceleration technique to improve the accuracy and the order of convergence of the fitted operator finite difference method(s) having interior layer(s) [65, 66].

In this chapter we consider the following singularly perturbed internal layer problem

$$Lu := \varepsilon u'' + a(x)u' - b(x)u = f(x),\ x \in \Omega = (-1, 1), \tag{2.1.1}$$

$$u(-1) = \gamma,\ u(1) = \beta, \tag{2.1.2}$$

where γ and β are given real numbers, $0 < \varepsilon \ll 1$ and $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth functions such that problem (2.1.1)-(2.1.2) has a unique solution. The problem above is said to be a turning point problem if there exists α_i with $-1 < \alpha_i < 1$ such that $a(\alpha_i) = 0$, and $a(-1)a(1) \neq 0$. The r zeros α_i, $i = 1, 2, \ldots, r$ of $a(x)$ are called turning points of problem (2.1.1)-(2.1.2). The assumptions

$$\left.\begin{array}{lll} \text{(i)} & a(0) = 0, & a'(0) > 0, \\ \text{(ii)} & b(x) \geq b(0) > 0, & x \in [-1, 1], \\ \text{(iii)} & |a'(x)| \geq \frac{|a'(0)|}{2}, & x \in [-1, 1], \end{array}\right\} \tag{2.1.3}$$

guarantee an interior layer at $x = 0$. In fact, (i) guarantees the existence of the turning point, (ii) ensures that the problem satisfies a minimum principle and (iii) implies that zero is the only turning point in $[-1, 1]$. In addition, stability of the solution requires that $|a(x) - xb(x)| \geq b_0 > 0$. Note that interior layers may occur as a result of discontinuous data (see for instance [4, 11, 24]).

These problems captured the interest of researchers starting in the late 60s [88] whereby asymptotic and numerical approaches were adopted (see, [2, 21, 28, 41, 61, 79, 92]).

It is no doubt that turning point problems are more challenging than the non-turning point ones due to the change in the sign of the convective coefficient, and more so with the case where solutions exhibit interior layers (see for instance [6, 16, 17, 22, 30, 64] and more in the literature review for works where interior layer problems were studied).

As in to the best of our knowledge, this is the first time the work like this is done. As for as we know, the discretisations of turning point problems are done on some layer adapted

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meshes except for the works presented in [65] where turning point problems whose solutions has boundary layers were discretised on a uniform mesh in the framework of fitted operator methods.

The rest of this chapter is structured as follows. Section 2.2 presents some a priori estimates on the bounds of the solution and its derivatives. Using techniques presented in [1] and [9] we provide sharp bounds on the solution and its derivatives. In section 2.3, we propose a numerical scheme whose convergence is analysed in section 2.4. Section 2.5 is devoted to Richardson extrapolation. To show the effectiveness of the proposed scheme, we carry out and discuss some numerical experiments in section 2.6. In section 2.7, we provide some concluding remarks.

2.2 Some qualitative results

In this section, a number of results related to the continuous problem are presented. These results will be used in the error analysis in Section 2.4.

The operator L satisfies the following continuous minimum principle

Lemma 2.2.1. Let ψ be a smooth function satisfying $\psi(-1) \geqslant 0$, $\psi(1) \geqslant 0$ and $L\psi(x) \leq 0$, $\forall x \in (-1, 1)$. Then $\psi(x) \geqslant 0$, $\forall x \in [-1, 1]$.

Proof. Let $x^* \in [-1, 1]$ such that $\psi(x^*) = \min_{x\in[-1,1]} \psi(x)$ and assume that $\psi(x^*) < 0$. Then, obviously, $x^* \notin \{-1, 1\}$, $\psi'(x^*) = 0$ and $\psi''(x^*) \geqslant 0$. We have

$$L\psi(x^*) := \varepsilon\psi''(x^*) + a(x^*)\psi'(x^*) - b(x^*)\psi(x^*) > 0,$$

which is a contradiction. It follows that, $\psi(x^*) \geqslant 0$ and thus $\psi(x) \geqslant 0$, $\forall x \in [-1, 1]$.

Lemma 2.2.2. Let $u(x)$ be the solution of (2.1.1)-(2.1.2). Then, we have:

$$||u(x)|| \leq C\left(b_0^{-1}||f|| + \max\left(|\gamma|, |\beta|\right)\right), \forall x \in [-1, 1],$$

where $||.||$ denotes the maximum norm and b_0 is such that $0 < b_0 < b(x)$ for all $x \in \Omega$.

Proof. Consider the comparison function

$$\Pi^{\pm}(x) = b_0^{-1}||f|| + \max\left(|\gamma|, |\beta|\right) \pm u(x).$$

We have

$$L\Pi^{\pm}(x) = \pm f(x) - \frac{b(x)}{b_0}||f|| - b(x)\max\left(|\gamma|, |\beta|\right) \leq 0,$$

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implying that

$$\Pi^{\pm}(x) \geq 0, \forall x \in [-1, 1],$$

which completes the proof.

We denote by $\Omega_L = [-1, -\delta),\ \Omega_C = [-\delta, \delta],\ \Omega_R = (\delta, 1]$, where $0 < \delta \leq 1/2$; respectively the left, central and the right part of the domain. Further, $\Omega_C = \Omega_C^- \cup \Omega_C^+$, where $\Omega_C^- = [-\delta, 0)$ and $\Omega_C^+ = [0, \delta]$.

The bounds on the solution and its derivatives are provided in the following Lemmas.

Lemma 2.2.3. *If $u(x)$ is the solution to (2.1.1)-(2.1.2) and a, b, and f sufficiently smooth functions in $\bar{\Omega}$; there exists a constant C such that*

$$|u^{(j)}(x)| \leq C,\ \forall x \in \Omega_L U \Omega_R,\quad j = 0, 1, 2, 3, 4. \tag{2.2.1}$$

Proof. See [9]

The next two lemmas provide bounds on the solution u and its derivatives in the layer region Ω_C. Bounds in the immediate left side of the turning point (that is in Ω_C^-) are given in Lemma 2.2.1 while those in the immediate right side of the turning point (Ω_C^+) are presented in Lemma 2.2.5.

Lemma 2.2.4. *Let $u(x)$ be the solution of (2.1.1)-(2.1.2) and f a smooth function. Then there exist positive constants α and C, such that*

$$|u^{(j)}(x)| \leq C\left[1 + \varepsilon^{-j} \exp\left(\frac{\alpha x}{\varepsilon}\right)\right],\ \forall x \in \Omega_C^-,\ j = 1, 2, 3, 4. \tag{2.2.2}$$

Proof. The proof is by induction. We follow the works done by Berger *et al.* [9] and Munyakazi and Patidar [65]. Let u be the solution of (2.1.1)-(2.1.2). From Lemma 2.2.2, we have

$$|u(x)| \leq C,\ \forall x \in \Omega_C.$$

Let us assume that $\forall j,\ 0 \leq j \leq k$, the following estimates holds

$$|u^{(j)}(x)| \leq C\left[1 + \varepsilon^{-j} \exp\left(\frac{\alpha x}{\varepsilon}\right)\right], \forall x \in \Omega_C^-. \tag{2.2.3}$$

Notice that differentiating successively (2.1.1) - (2.1.2) leads to

$$Lu^{(k)} = f_k, \tag{2.2.4}$$

with

$$f_0 = f \text{ and } f_k = f^{(k)} - \sum_{l=0}^{k-1} \binom{k}{l} a^{(k-l)} u^{(l+1)} + \sum_{l=0}^{k-1} \binom{k}{l} b^{(k-l)} u^{(l)}. \tag{2.2.5}$$

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From (2.2.5), we see that

$$|f_k(x)| \leq C\left[1 + \varepsilon^{-k}\exp\left(\frac{\alpha x}{\varepsilon}\right)\right]. \tag{2.2.6}$$

Also, from (2.2.3) we have

$$|u^{(k)}(0)| \leq C\left(1 + \varepsilon^{-k}\right),$$

and

$$|u^{(k)}(-1)| \leq C\left[1 + \varepsilon^{-k}\exp\left(-\frac{\alpha}{\varepsilon}\right)\right].$$

Since $\varepsilon^{-1}\exp(\alpha x/\varepsilon) \leq C$, we have $|u^{(k)}(-1)| \leq C\left[1 + \varepsilon^{-(k-1)}\right]$.

Let

$$\theta_k(x) = \frac{1}{\varepsilon}\int_x^0 f_k(t)\exp\left[-\frac{A(x)-A(t)}{\varepsilon}\right]dt,$$

where

$$A(x) = -\int_x^0 a(s)d,s$$

and

$$u_p^{(k)}(x) = -\int_x^0 \theta_k(t)dt,$$

which is a particular solution of

$$Lu^{(k)} = f_k.$$

Its general solution can therefore be written as

$$u^{(k)} = u_p^{(k)} + u_h^{(k)},$$

where the solution of the homogeneous equation

$$Lu_h^{(k)} = 0,$$

satisfies

$$u_h^{(k)}(-1) = u^{(k)}(-1) - u_p^{(k)}(-1,)$$

and

$$u_h^{(k)}(0) = u^{(k)}(0).$$

Using the function

$$\varphi(x) = \frac{\int_x^0 \exp\left[-\frac{A(t)}{\varepsilon}\right]dt}{\int_{-1}^0 \exp\left[-\frac{A(t)}{\varepsilon}\right]dt},$$

it follows that

$$L\varphi(x) = 0;\ \varphi(-1) = 1; \varphi(0) = 0;\ \text{and } 0 \leq \varphi(x) \leq 1.$$

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The derivative $u_h^{(k)}$ can be given as

$$u_h^{(k)}(x) = \left[u^{(k)}(-1) - u_p^{(k)}(-1)\right][1+\varphi(x)] + u^{(k)}(0)\,[-\varphi(x)]\,.$$

The above leads to the following expression for $u_\varepsilon^{(k+1)}$:

$$u^{(k+1)} = u_p^{(k+1)} + u_h^{(k+1)} = \theta_k + \left[u^{(k)}(-1) - u_p^{(k)}(-1) - u^{(k)}(0)\right]\varphi'(x).$$

Since

$$\varphi'(x) = \frac{-\exp\left[-\frac{A(x)}{\varepsilon}\right]}{\int_{-1}^{0}\exp\left[-\frac{A(t)}{\varepsilon}\right]dt},$$

the upper and the lower bounds of $a(x)$ lead to the estimates

$$|\varphi'(x)| \le C\,\varepsilon^{-1}\exp\left(\frac{\alpha x}{\varepsilon}\right).$$

Furthermore, the lower bound on the coefficient $a(x)$ and the estimate for f_k lead to

$$|\theta_k(x)| = \left|\frac{1}{\varepsilon}\int_x^0 f_k(t)\exp\left[-\frac{A(x)-A(t)}{\varepsilon}\right]dt\right|.$$

Using (2.2.6) we obtain

$$|\theta_k(x)| \le C\varepsilon^{-1}\int_x^0\left[1+\varepsilon^{-k}\exp\left(\frac{\alpha t}{\varepsilon}\right)\right]\exp\left[-\frac{\alpha(t-x)}{\varepsilon}\right]dt.$$

After integration, we can easily see that

$$|\theta_k(x)| \le \frac{C}{\alpha}\left[1-\exp\left(\frac{\alpha x}{\varepsilon}\right) - \varepsilon^{-(k+1)}x\exp\left(\frac{\alpha x}{\varepsilon}\right)\right],$$

or

$$|\theta_k(x)| \le C\left[1+\varepsilon^{-(k+1)}\exp\left(\frac{\alpha x}{\varepsilon}\right)\right],$$

since

$$u_p^{(k)}(0) = -\int_0^0\theta_k(t)dt = 0,$$

and

$$u_p^{(k)}(-1) = -\int_{-1}^0\theta_k(t)dt,$$

it follows that

$$u_p^{(k)}(-1) \le C\exp(-k).$$

But

$$|u^{(k+1)}(x)| \le |\theta_k| + \left[|u^{(k)}(-1)| + |u_p^{(k)}(-1)| + |u^{(k)}(0)|\right]|\varphi'|.$$

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Therefore, the above estimate gives

$$|u^{(k+1)}(x)| \leq C\left[1+\varepsilon^{-(k+1)}\exp\left(\frac{2\alpha x}{\varepsilon}\right)\right]+\left[|u^{(k)}(-1)|+|u_p^{(k)}(-1)|+|u^{(k)}(0)|\right]C\varepsilon^{-1}\exp\left(\frac{\alpha x}{\varepsilon}\right),$$

implying that

$$|u^{(k+1)}(x)| \leq C\left[1+\varepsilon^{-(k+1)}\exp\left(\frac{\alpha x}{\varepsilon}\right)\right],$$

which completes the proof.

Lemma 2.2.5. *Let $u(x)$ be the solution of (2.1.1)-(2.1.2) and $f \in C^k(\bar{\Omega}), k > 0$, then there exist positive constants α and C such that*

$$|u^{(j)}(x)| \leq C\left[1+\varepsilon^{-j}\exp\left(\frac{-\alpha x}{\varepsilon}\right)\right], \forall x \in \Omega_C^+, j = 1, 2, ...$$

Proof. The proof of Lemma 2.2.5 is similar to that of Lemma 2.2.4 above.

2.3 Construction of the FOFDM

The numerical methods developed in this paper is based on the modelling rules of Mickens [57]. Lubuma and Patidar [53, 54] were the first to consider these rules and derive non-standard finite difference methods for singularly perturbed ordinary differential equations. Subsequently, these methods were explored by Munyakazi and Patidar [65] to solve singularly perturbed turning point problem whose solution has boundary layers. Towards the usage of the name "fitted operator finite difference methods" as compared to the "non-standard finite difference methods", it was indicated in Patidar [72] that the term FOFDM classifies NSFDMs to a very specialized research domain.

Now to proceed with, let n be a positive and even integer and let denote by Ω_n the following partition of the interval $[-1, 1]$: $x_0 = -1; x_j = x_0 + jh; j = 1, ..., n-1, h = x_j - x_{j-1}, x_n = 1$. Following the similar idea as in [38], on Ω_n, our discretization of problem (2.1.1)-(2.1.2) reads

$$\left.\begin{array}{l} L_1^h U_j := \varepsilon\delta^2 U_j + \tilde{a}_j D^- U_j - \tilde{b}_j U_j = \tilde{f}_j,\ j = 0, 1, 2, \cdots, \frac{n}{2}-1, \\ L_2^h U_j := \varepsilon\delta^2 U_j + \tilde{a}_j D^+ U_j - \tilde{b}_j U_j = \tilde{f}_j,\ j = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1, \end{array}\right\} \tag{2.3.1}$$

$$U_0 = \gamma,\ U_n = \beta, \tag{2.3.2}$$

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where

$$D^-U_j = \frac{U_j - U_{j-1}}{h}, \quad D^+U_j = \frac{U_{j+1} - U_j}{h}, \quad \delta^2 U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{\tilde{\phi}_j^2}.$$

and

$$\left.\begin{array}{l} \tilde{\phi}_j^2 = \frac{h\varepsilon}{\tilde{a}_j}\left[1 - \exp\left(\frac{-\tilde{a}_j h}{\varepsilon}\right)\right], \; j = 0, 1, 2, ..., \frac{n}{2} - 1, \\ \tilde{\phi}_j^2 = \frac{h\varepsilon}{\tilde{a}_j}\left[\exp\left(\frac{\tilde{a}_j h}{\varepsilon} - 1\right)\right], \; j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1. \end{array}\right\} \tag{2.3.3}$$

Also, we have adopted the following convention.

$$\left.\begin{array}{l} \tilde{a}_j = \dfrac{a_j + a_{j-1}}{2} \text{ for } j = 0, 1, 2, ..., \dfrac{n}{2} - 1, \\ \\ \tilde{a}_j = \dfrac{a_j + a_{j+1}}{2} \text{ for } j = \dfrac{n}{2}, \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, ..., n - 1, \\ \\ \tilde{b}_j = \dfrac{b_{j-1} + b_j + b_{j+1}}{3}; \tilde{f}_j = \dfrac{f_{j-1} + f_j + f_{j+1}}{3} \text{ for } j = 0, 1, 2, ..., n - 1. \end{array}\right\} \tag{2.3.4}$$

We rewrite (2.3.1) as

$$L^h U_j := r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = \tilde{f}_j, \tag{2.3.5}$$

where

$$\left.\begin{array}{l} r_j^- = \dfrac{\varepsilon}{\tilde{\phi}_j^2} - \dfrac{\tilde{a}_j}{h}; \; r_j^c = \dfrac{-2\varepsilon}{\tilde{\phi}_j^2} + \dfrac{\tilde{a}_j}{h} - \tilde{b}_j; \; r_j^+ = \dfrac{\varepsilon}{\tilde{\phi}_j^2}, \text{ if } \; j = 0, 1, 2, ..., \dfrac{n}{2} - 1, \\ \text{and} \\ r_j^- = \dfrac{\varepsilon}{\tilde{\phi}_j^2}; \; r_j^c = \dfrac{-2\varepsilon}{\tilde{\phi}_j^2} - \dfrac{\tilde{a}_j}{h} - \tilde{b}_j; \; r_j^+ = \dfrac{\varepsilon}{\tilde{\phi}_j^2} + \dfrac{\tilde{a}_j}{h}, \text{ if } \; j = \dfrac{n}{2}, \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, ..., n - 1. \end{array}\right\} \tag{2.3.6}$$

This proposed FOFDM satisfies the following two lemmas.

Lemma 2.3.1. (Discrete minimum principle) . *For any mesh function ξ_j such that $\xi_0 \geq 0$, $\xi_n \geq 0$ and $L^h \xi_j \leq 0$, $\forall j = 1(1)n - 1$, we have $\xi_j \geq 0$, $\forall j = 0(1)n$.*

Proof. Let k be such that $\xi_k = \min_{0 \leq j \leq n} \xi_j$, suppose that $\xi_k < 0$. It is clear that $k \neq 0, n$. Also $\xi_{k+1} - \xi_k \geqslant 0$, and $\xi_k - \xi_{k-1} \leq 0$. Remembering that $a_k < 0$, for $1 \leq k \leq n/2 - 1$ and $a_k \geq 0$, for $n/2 \leq k \leq n - 1$, on one hand we have

$$L_1^h \xi_k = \varepsilon \delta^2 \xi_k + a_k D^- \xi_k - b_k \xi_k > 0, \; 1 \leq k \leq n/2 - 1,$$

and

$$L_2^h \xi_k = \varepsilon \delta^2 \xi_k + a_k D^+ \xi_k - b_k \xi_k > 0, \; n/2 \leq k \leq n - 1,$$

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on the other. Thus $L^h\xi_k > 0,\ 1 \le k \le n-1$, which is a contradiction. It follows that $\xi_j \geqslant 0,\ j = 1, 2, ..., n$.

The above minimum principle is used to prove the following Lemma.

Lemma 2.3.2. *Let Z_i be any mesh function such that $Z_0 = Z_n = 0$. Then*

$$|Z_i| \le \frac{1}{a^*} \max_{1 \le j \le n-1} |L^h Z_j|,\ for\ \ 0 \le i \le n,$$

where

$$\left.\begin{aligned} a^* &= -a_0\ if\, 0 \le i \le n/2-1,\\ a^* &= a_0\ if\, n/2 \le i \le n. \end{aligned}\right\} \tag{2.3.7}$$

Proof. Let us define two comparison functions $Y_i^{\pm}$ by

$$Y_i^{\pm} = \frac{x_i}{a^*} \max_{1 \le j \le n-1} |L^h Z_j| \pm L^h Z_j,\ \text{for}\ \ 0 \le i \le n,$$

where

$$\left.\begin{aligned} a^* &= -a_0\ if\ 0 \le i \le n/2-1,\\ a^* &= a_0\ if\ n/2 \le i \le n. \end{aligned}\right\} \tag{2.3.8}$$

It is clear that $Y_0^{\pm} \geqslant 0$ and $Y_n^{\pm} \geqslant 0$. Also, observe that

$$L^n Y_0^{\pm} = \frac{a_i - b_i x_i}{a^*} \max_{1 \le i \le n-1} |L^n Z_i| \pm z_i,\ \text{for}\ \ 0 \le i \le n.$$

If $0 \le i \le n/2 - 1$, then $(a_i - b_i x_i)/(-a_0) \le -1$.
Likewise, if $n/2 \le i \le n$, then $(a_i - b_i x_i)/a_0 \le -1$. In either case, $L^n Y_i^{\pm} \le 0$. By the discrete minimum principle Lemma 2.3.1, we conclude that $Y_i \le 0, \forall 0 \le i \le n$ and this completes the proof.

2.4 Convergence analysis of the FOFDM

In this section, we analyse the FOFDM described in the previous section. We present the full analysis on the interval$[0, 1]$. The analysis on $[-1, 0)$ can be done in a similar manner.

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The truncation error on the interval [0, 1] is given by

$$\begin{aligned}
L_2^h(U_j-u_j) &= L_2^hU_j-L_2^hu_j,\\
&= \tilde{f}_j-\left[\frac{\varepsilon}{\tilde{\phi}_j^2}(u_{j+1}-2u_j+u_{j-1})+\frac{\tilde{a}_j}{h}(u_{j+1}-u_j)-\tilde{b}_ju_j\right]\\
&= \frac{1}{3}\left[\varepsilon u''_{j+1}+a_{j+1}u'_{j+1}-b_{j+1}u_{j+1}\right]+\frac{1}{3}[\varepsilon u''_j+a_ju'_j-b_ju_j]\\
&\quad +\frac{1}{3}[\varepsilon u''_{j-1}+a_{j-1}u'_{j-1}-b_{j-1}u_{j-1}]\\
&\quad -\left[\frac{\varepsilon}{\tilde{\phi}_j^2}(u_{j+1}-2u_j+u_{j-1})+\frac{\tilde{a}_j}{h}(u_{j+1}-u_j)-\tilde{b}_ju_j\right].
\end{aligned}$$

Note that, in the above, we have used the fact that $\tilde{f}_j=(f_{j+1}+f_j+f_{j-1})/3$ as suggested in (2.3.4). Using the expression for $\tilde{a}_j$ and $\tilde{b}_j$ in reference to (2.3.4) and the Taylor expansions of u_{j+1}, u_{j-1} a_{j+1}, a_{j-1}, b_{j+1}, b_{j-1}, u'_{j+1}, u'_{j-1}, u''_{j+1}, and u''_{j-1} and the truncated Taylor expansion of $\frac{1}{\tilde{\phi}_j^2}$ of order four, we have

$$\begin{aligned}
L_2^h\left(U_j-u_j\right) &= \left[\frac{hb_j}{3}-\frac{2h^2b'_j}{3}+\frac{h^2b''_j}{6}+\frac{h^3b'''_j}{9}+\frac{5h^4b^{(iv)}\left(\xi_{1_j}\right)}{72}\right]u_j\\
&\quad +\left[ha'_j-\frac{h^2a''_j}{6}-\frac{h^3a'''_j}{6}-\frac{h^4a^{(iv)}(\xi_{2_j})}{36}+\frac{h^4a^{(iv)}\left(\xi_{3_j}\right)}{72}-\frac{hb_j}{3}\right.\\
&\quad \left.-\frac{h^2b'_j}{3}-\frac{h^3b''_j}{3}-\frac{h^4b'''_j}{9}-\frac{h^5b^{(iv)}\left(\xi_{4_j}\right)}{72}\right]u'_j\\
&\quad +\left[-\frac{ha_j}{2\varepsilon}-\frac{ha_j}{2}+\frac{h^2a'_j}{6}-\frac{h^2a'_j}{2\varepsilon}-\frac{h^3a''_j}{4\varepsilon}-\frac{h^3a''_j}{4}+\frac{h^4a'''_j}{36}-\frac{h^4a'''_j}{12\varepsilon}\right.\\
&\quad +\frac{h^5a^{(iv)}(\xi_{5_j})}{3}-\frac{h^5a^{(iv)}(\xi_{6_j})}{72}-\frac{h^5a^{(iv)}(\xi_{5_j})}{48\varepsilon}-\frac{h^5a^{(iv)}(\xi_{7_j})}{48}-\frac{h^2b_j}{3}\\
&\quad \left.-\frac{h^4b''_j}{4}-\frac{h^6b^{(iv)}(\xi_{1_j})}{144}-\frac{h^6b^{(iv)}(\xi_{1_j})}{144}\right]u''_j\\
&\quad +\left[-\frac{h^3a_j}{6}+\frac{h^2a'_j}{6}-\frac{h^3a'_j}{6}-\frac{h^4a'_j}{12}+\frac{h^4a''_j}{12}-\frac{h^5a''_j}{24}-\frac{h^5a'''_j}{36}-\frac{h^6a'''_j}{72}\right.\\
&\quad \left.+\frac{h^6a^{(iv)}(\xi_{3_j})}{144}-\frac{h^4b'_j}{9}+\frac{h^5b''_j}{36}-\frac{45h^6b'''_j}{1944}-\frac{h^7b^{(iv)}(\xi_{1_j})}{432}+\frac{h^7b^{(iv)}(\xi_{8_j})}{432}\right]u'''_j\\
&\quad +\kappa\left(\varepsilon,h^2,h^3,\cdots,h^7,a_j,a'_j,\cdots,a_j^{(iv)},b_j,b'_j,\cdots,b_j^{(iv)}\right)u^{(iv)}(\xi_{*_j}),
\end{aligned} \tag{2.4.9}$$

where κ is a function of its arguments and the ξ_{*_j}'s lie in the interval (x_{j-1},x_{j+1}). Note that the coefficients of $u_j,u'_j,\cdots,u^{(iv)}(\xi_{*_j})$ can be bounded by a constant.

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Now, applying the triangular inequality, Lemma 2.2.5 and using Lemma 7 in [66], it follows that

$$|L^h(U_j - u_j)| \leq Ch, \forall j = 1(1)(n-1).$$

The use of Lemma 2.3, leads to the following main result.

Theorem 2.4.1. *Let $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth functions in (2.1.1)-(2.1.2). Then the numerical solution approximation U of u obtained via the FOFDM (2.3.1)-(2.3.2) satisfies*

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq j\leq n} |u_j - U_j| \leq Ch, \tag{2.4.10}$$

where C is a constant independent of ε and h.

In the next section we use Richardson extrapolation to improve the accuracy and convergence rate of the proposed scheme.

2.5 Richardson extrapolation on the FOFDM

The purpose of this section is to improve the accuracy and the order of convergence (2.5.7). To begin, we look back to (2.4.9) that can also be written as:

$$L_2^h(u_j - U_j) = M_1 h + M_2 h^2 + R_n(x_j), \tag{2.5.1}$$

where

$$M_1 = \frac{b_j}{3}u_j + \left(a_j' - \frac{b_j}{3}\right)u_j' - \frac{a_j u_j''}{2},$$

$$M_2 = \left(\frac{-2b_j'}{3} + \frac{b_j''}{6}\right)u_j - \left(\frac{a_j''}{6} + \frac{b_j'}{3}\right)u_j' + \left(\frac{a_j'}{6} - \frac{b_j}{3}\right)u_j'' + \frac{a_j'}{6},$$

$$R_n(x_j) = h^3\left[\frac{b_j'''}{9}u_j - \left(\frac{a_j'''}{6} + \frac{b_j''}{3}\right)u_j' - \frac{a_j''}{4} - \left(\frac{a_j}{6} + \frac{a_j'}{6}\right)u_j'''\right]$$

$$+h^4\left[\frac{5b_j^{(iv)}\left(\xi_{1_j}\right)}{72}u_j + \left(-\frac{a^{(iv)}\left(\xi_{2_j}\right)}{36} + \frac{a^{(iv)}\left(\xi_{3_j}\right)}{72} - \frac{b_j'''}{9}\right)u_j' + \left(\frac{a_j'''}{36} - \frac{b_j''}{4} - \frac{a_j'}{12} + \frac{a_j'''}{12} - \frac{b_j'}{9}\right)u_j'''\right]$$

$$+h^5\left[-\frac{b^{(iv)}\left(\xi_{4_j}\right)}{72}u_j'' + \left(\frac{a^{(iv)}\left(\xi_{5_j}\right)}{3} - \frac{a^{(iv)}\left(\xi_{6_j}\right)}{72} - \frac{a^{(iv)}(\xi_{7_j})}{48}\right)u_j'' + \left(-\frac{a_j''}{24} - \frac{a_j'''}{36} + \frac{b_j''}{36}\right)u_j'''\right]$$

$$+h^6\left[-\frac{b_j^{(iv)}\left(\xi_{1_j}\right)}{72}u_j'' + \left(-\frac{a_j'''}{72} + \frac{a^{(iv)}\left(\xi_{1_j}\right)}{144} - \frac{45b_j'''}{1944}\right)u_j'''\right]$$

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$$+h^7\left[\left(-\frac{b_j^{(iv)}\left(\xi_{1_j}\right)}{432}+\frac{b_j^{(iv)}\left(\xi_{8_j}\right)}{432}\right)u_j''\right]$$
$$+\kappa\left(\varepsilon,h^2,h^3,\cdots,h^7,a_j,a_j',\cdots,a_j^{(iv)},b_j,b_j',\cdots,b_j^{(iv)},u_j'',u^{(iv)}(\xi_{*_j})\right).$$

The descriptions of κ, ξ's and $u_j,u_j',\cdots,u^{(iv)}(\xi_{*_j})$, remain the same as the ones specified in (2.4.9). Notice that L_1^h and L_2^h also stand for the operator L^h respectively for $j=0(1)n-1$ and $j=n(1)2n-1$.

Let Ω_{2n} be the mesh obtained by bisecting each mesh interval in Ω_n, i.e.,

$$\Omega_{2n}=\{\bar{x}_i\}\text{ with }\ \bar{x}_0=-1,\ \ \bar{x}_n=1\ \text{ and }\ \bar{x}_j-\bar{x}_{j-1}=\bar{h}=h/2,\ \ j=1,2,...,2n.$$

Let $\bar{U}_j$ be the numerical solution on Ω_{2n}. M and p positive constants. Equation (2.5.1), can be generalized on Ω_{2n} in terms of U_j and on Ω_{2n} in terms of $\bar{U}_j$ as follows:

$$L^h\left(u_j-U_j\right)=Mh+ph^2+R_n(x_j),1\le j\le n. \tag{2.5.2}$$

and

$$L^h\left(\bar{u}_j-\bar{U}_j\right)=M\bar{h}+p\bar{h}^2+R_{2n}(\bar{x}_j),1\le j\le 2n-1. \tag{2.5.3}$$

Multiplying (2.5.3) by 2, we get

$$2L^n\left(\bar{u}_j-\bar{U}_j\right)=2M\bar{h}+2p\bar{h}^2+2R_{2n}(\bar{x}_j),1\le j\le 2n-1, \tag{2.5.4}$$

or

$$L^n\left(2\bar{u}_j-2\bar{U}_j\right)=2M\bar{h}+2p\bar{h}^2+2R_{2n}(\bar{x}_j),1\le j\le 2n-1. \tag{2.5.5}$$

Note that $\bar{u}_j=u_j,\forall\ j\in\Omega_n$. Substracting (2.5.2) from (2.5.5), we obtain

$$L^n\left(u_j-(2\bar{U}_j-U_j)\right)=\frac{ph^2}{2}+2R_{2n}(\bar{x}_j)-R_n(x_j),1\le j\le 2n-1 \tag{2.5.6}$$

or

$$L^n\left(u_j-(2\bar{U}_j-U_j)\right)=\mathcal{O}(h^2),1\le j\le 2n-1,$$

Let

$$U_j^{ext}:=2\bar{U}_j-U_j.$$

U_j^{ext} is also another numerical approximation of u_j. Using Lemma 2.3.2, we obtain the following result.

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Theorem 2.5.1. *Let $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth functions in (2.1.1)-(2.1.2) and also $u(x) \in C^4([-1,1])$. Then the numerical solution approximation U_j^{ext} obtained via the Richardson extrapolation based on FOFDM (2.3.1)-(2.3.2) satisfies:*

$$\sup_{0<\varepsilon\leq 1} \max_{1\leq j\leq 2n} \left| u_j - U_j^{ext} \right| \leq Mh^2, \tag{2.5.7}$$

where M is a constant independent of ε and h.

In the next section, we use the proposed schemes on two numerical examples to confirm its accuracy and robustness.

2.6 Numerical examples

In this section we present the numerical results obtained in the integration of some problems of type (2.1.1)-(2.1.2).

Example 2.6.1. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{aligned} \varepsilon u'' + xu' - u = -(1+\varepsilon\pi^2)\cos\pi x - \pi x \sin\pi x \\ u(-1) = -1, \quad u(1) = 1. \end{aligned}\right\}$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$. The exact solution is

$$u(x) = \cos(\pi x) + x + \frac{x\, erf(x/\sqrt{2\varepsilon}) + \sqrt{2\varepsilon/\pi}\exp(-x^2/2\varepsilon)}{erf(1/\sqrt{2\varepsilon}) + \sqrt{2\varepsilon/\pi}\exp(-1/2\varepsilon)},$$

where erf is the error function (special function of sigmoid shape that describes diffusion in mathematical modelling).

The Exponentially Fitted Weighted-Residual (EFWR) and the classical Galerkin (GAL) methods in [33] were used to solve the example above. The author computed the error outside the layer region (away from the turning point). The essence of studying singularly perturbed problems is to design numerical schemes suitable to provide reasonable approximations in the layer region. We will compute the maximum nodal error of the method we propose in the whole domain. Our numerical results confirm the ε-uniform convergence established theoretically in the previous section.

Example 2.6.2. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{aligned} \varepsilon u'' + 2(x-0.5)u' - 2u = f(x) \\ u(0) = \tfrac{\varepsilon}{4}\exp(\tfrac{-1}{4\varepsilon}), u(1) = -\tfrac{\varepsilon}{4}\exp(\tfrac{-1}{4\varepsilon}). \end{aligned}\right\}$$

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This problem has an interior layer of width $\mathcal{O}(\varepsilon)$. The exact solution is

$$u(x) = \cos[\pi(x-0.5)] - \frac{\varepsilon}{2}(x-0.5)\exp[-(x-0.5)^2/\varepsilon].$$

and

$$f(x) = -\Big[(\varepsilon\pi^2+2)\cos[\pi(x-0.5)] + 2(x-0.5)\pi\sin[\pi(x-0.5)]\Big] + \\ +3\varepsilon(x-0.5)\exp[-(x-0.5)^2/\varepsilon].$$

The maximum errors at all mesh points and the numerical rates of convergence before extrapolation are evaluated using the formulas

$$E_{\varepsilon,n} := \max_{0\le j\le n}|u_j - U_j| \text{ and } r_k \equiv r_{\varepsilon,k} := \log_2\left(\tilde{e}_{n_k}/\tilde{e}_{2n_k}\right), k=1,2,...$$

respectively, where $\tilde{e}_n$ stands for $E_{\varepsilon,n}$. Furthermore, we compute $E_n = \max_{0<\varepsilon\le 1} E_{\varepsilon,n}$.

For a fixed mesh, we see that the maximum nodal errors remain constant for small values of ε (see tables 2.1 and 2.5). Moreover, results in tables 2.3 and 2.7 show that the proposed method is essentially first order convergent.

After extrapolation the maximum errors at all mesh points and the numerical rates of convergence are evaluated using the formulas

$$E^{ext}_{\varepsilon,n} := \max_{0\le j\le 2n}|u_j - U^{ext}_j| \text{ and } R_k \equiv R_{\varepsilon,k} := \log_2\left(E^{ext}_{n_k}/E^{ext}_{2n_k}\right), k=1,2,...$$

respectively, where $E^{ext}_{n_k}$ stands for $E_{\varepsilon,2n}$.

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Table 2.1: Maximum errors for Example 2.6.1 (before extrapolation)

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
10^{-4}	1.29E-01	6.81E-02	3.50E-02	1.77E-02	8.69E-03	4.16E-03	1.94E-03
10^{-6}	1.29E-01	6.85E-02	3.55E-02	1.81E-02	9.12E-03	4.58E-03	2.29E-03
10^{-7}	1.29E-01	6.85E-02	3.55E-02	1.81E-02	9.13E-03	4.59E-03	2.30E-03
10^{-14}	1.29E-01	6.85E-02	3.55E-02	1.81E-02	9.13E-03	4.59E-03	2.30E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-18}	1.29E-01	6.85E-02	3.55E-02	1.81E-02	9.13E-03	4.59E-03	2.30E-03

Table 2.2: Maximum errors for Example 2.6.1 (after extrapolation)

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$	$n = 1024$
10^{-4}	3.03E-02	8.23E-03	4.23E-03	2.13E-03	5.32E-04	3.68E-04	2.64E-04
10^{-6}	3.06E-02	8.49E-03	2.22E-03	5.66E-04	4.85E-04	5.21E-04	2.55E-04
10^{-7}	3.06E-02	8.49E-03	2.22E-03	5.92E-04	1.43E-04	1.51E-04	2.15E-04
10^{-14}	3.06E-02	8.49E-03	2.22E-03	8.43E-04	3.02E-04	9.98E-05	3.12E-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-18}	3.06E-02	8.49E-03	2.22E-03	8.43E-04	3.02E-04	9.98E-05	3.12E-05

Table 2.3: Rates of convergence for Example 2.6.1 (before extrapolation, $n_k = 16, 32, 64, 128, 256, 512, 1026$)

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-4}	0.92	0.96	0.99	1.02	1.06	1.10
10^{-6}	0.91	0.95	0.97	0.99	0.99	1.00
10^{-7}	0.91	0.95	0.97	0.99	0.99	1.00
10^{-14}	0.91	0.95	0.97	0.99	0.99	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-18}	0.91	0.95	0.97	0.99	0.99	1.00

Table 2.4: Rates of convergence for Example 2.6.1 (after extrapolation, $n_k = 16, 32, 64, 128, 256, 512, 1026$)

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-4}	2.38	1.46	1.49	2.50	1.03	0.98
10^{-6}	2.35	2.43	2.47	0.72	0.40	1.53
10^{-7}	2.35	2.43	2.41	2.54	0.42	-0.01
10^{-14}	2.35	2.43	1.90	1.98	2.10	2.18
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-18}	2.35	2.43	1.90	1.98	2.10	2.18

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Table 2.5: Maximum errors for Example 2.6.2 (before extrapolation)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$	$n=1024$
10^{-4}	7.08E-02	3.79E-02	1.94E-02	9.79E-03	4.87E-03	2.38E-03	1.14E-03
10^{-5}	7.09E-02	3.80E-02	1.95E-02	9.89E-03	4.97E-03	2.49E-03	1.24E-03
10^{-7}	7.09E-02	3.80E-02	1.96E-02	9.91E-03	4.99E-03	2.50E-03	1.25E-03
10^{-9}	7.09E-02	3.80E-02	1.96E-02	9.91E-03	4.99E-03	2.50E-03	1.25E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-18}	7.09E-02	3.80E-02	1.96E-02	9.91E-03	4.99E-03	2.50E-03	1.25E-03

Table 2.6: Maximum errors for Example 2.6.2 (after extrapolation)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$	$n=1024$
10^{-4}	4.31E-03	2.12E-03	3.12E-04	2.58E-04	1.75E-04	1.50E-04	1.06E-04
10^{-5}	4.43E-03	1.96E-03	8.42E-04	2.84E-04	7.04E-05	3.08E-05	2.18E-05
10^{-7}	4.45E-03	1.93E-03	8.18E-04	2.97E-04	9.91E-05	3.13E-05	9.45E-06
10^{-9}	4.45E-03	1.93E-03	8.18E-04	2.97E-04	9.88E-05	3.10E-05	9.36E-06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-18}	4.45E-03	1.93E-03	8.18E-04	2.97E-04	9.88E-05	3.10E-05	9.36E-06

Table 2.7: Rates of convergence for Example 2.6.2 (before extrapolation, $n = 16, 32, 64, 128, 256, 512, 1026$)

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-4}	0.90	0.96	0.99	1.01	1.03	1.07
10^{-5}	0.90	0.96	0.98	0.99	1.00	1.00
10^{-7}	0.90	0.96	0.98	0.99	1.00	1.00
10^{-9}	0.90	0.96	0.98	0.99	1.00	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-18}	0.90	0.96	0.98	0.99	1.00	1.00

Table 2.8: Rates of convergence for Example 2.6.2 (after extrapolation, $n_k = 16, 32, 64, 128, 256, 512, 1026$)

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-4}	1.03	2.76	0.27	0.56	0.23	0.50
10^{-5}	1.18	1.22	1.57	2.01	1.20	0.50
10^{-7}	1.20	1.24	1.46	1.59	1.66	1.73
10^{-9}	1.20	1.24	1.46	1.59	1.67	1.73
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-18}	1.20	1.24	1.46	1.59	1.67	1.73

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2.7 Summary

Singularly perturbed turning point problems are difficult to solve due to the presence of boundary or interior layers in their solutions. Usually, when seeking numerical solutions of layers problems, layer adapted meshes are used. These meshes are fine in the layer region and coarse away from the layer region. Due to the nature of these meshes, convergence analysis is complex. The main aim of this chapter was to design a fitted operator finite difference method to solve a class of singularly perturbed turning point problems whose solution has interior layer. This approach utilizes uniform meshes to obtain a discrete problem. We first established sharp bounds on the solution and its derivatives. These bounds were then used to prove uniform convergence of the proposed numerical method. The first order uniform convergence shown theoretically was confirmed numerically through two test examples. We also investigated the effects of Richardson extrapolation to improve the accuracy and the convergence of the numerical solution of the fitted operator finite difference method with interior layer obtained. It came out that Richardson extrapolation improves slightly both the accuracy of the errors and the rates of convergence.

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Chapter 3

A fitted numerical method for parabolic turning point singularly perturbed problems with an interior layer

The objective of this chapter is to construct and analyse a fitted operator finite difference method for the family of time-dependent singularly perturbed parabolic convection-diffusion problems. The solution to the problems exhibits an interior layer due to the presence of a turning point. We first establish sharp bounds on the solution and its derivatives. Then, we discretize the time variable using the classical Euler method. Afterwards, we propose a fitted operator finite difference method to solve the problem. Through a rigorous error analysis, we show that the scheme is uniformly convergent of order one with respect to both time and space variables. Moreover, we apply Richardson extrapolation to enhance the accuracy and the order of convergence of the proposed scheme. Numerical investigations are carried out to demonstrate the efficacy and robustness of the scheme.

3.1 Introduction

In this chapter, we consider the turning point parabolic singularly perturbed problems with interior layer

$$Lu := -d(x,t)u_t + \varepsilon u_{xx} + a(x,t)u_x - \ b(x,t)u = \ f(x,t), -1 \leq x \leq 1; \ t \in [0,T]; \quad (3.1.1)$$

$$u(-1,t) = \alpha, \ \ u(1,t) = \gamma, \ u_0(x) = u(x,0), \quad (3.1.2)$$

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where α and γ are given real numbers and the perturbation parameter ε satisfies $0 < \varepsilon \ll 1$. The coefficients functions $a(x,t)$, $b(x,t)$, $d(x,t)$, $f(x,t)$ and $u_0(x)$ are assumed to be sufficiently smooth to ensure the smoothness of the solution. Also $d(x,t) > 0 \ \forall (x,t) \in [-1,1] \times [0,T]$. The condition of the reaction factor $b(x,t) \geq b_0 > 0, \forall\, t \in [0,T]$ ensures the uniqueness of the solution [59].

The problem (3.1.1)-(3.1.2) is said to be a turning point problem, if there exists α_i with $-1 < \alpha_i < 1$ such that $a(\alpha_i,t) = 0$ and $a(-1,t)a(1,t) \neq 0, \forall\ t \in [0,T]$. The r zeros α_i, $i = 1,2,\ldots,r$ of $a(x,t)$ are called turning points. This definition can also be found in Berger et al.[9] where they showed the bounds of the solution to the problem (3.1.1)-(3.1.2) near the given turning point α_i to depend on ε and the constants $\beta_i = b(\alpha_i,t)/a_x(\alpha_i,t)$. When $\beta_i < 0$, the solution to $u(x,t)$ is "smooth" near $(x,t) = (\alpha_i,t)$, and if $\beta_i > 0$, the solution $u(x,t)$ presents a rapid change at $(x,t) = (\alpha_i,t)\ \forall\, t \in [0,T]$ termed "interior layer" which is often demonstrated by the change in signs of the convection coefficient $a(x,t)$ near $(\alpha_i,t)\ \forall (x,t) \in [-1,1] \times [0,T]$. In the case where the convection coefficient $a(x,t)$ does not change the sign throughout the spatial domain, the boundary layer may occur near -1 or/and 1. In addition, the existence of $\alpha_0 \in [-1,1]$, such that $|a_x(x,t)| \geq |a_x(\alpha_0,t)|/2, \forall t \in [0,T]$, ensures the uniqueness of the turning point in $[-1,1]$.

In this paper, we consider the assumptions below to guarantee the interior layer of the solution to problem (3.1.1)-(3.1.2) at $x = 0, \forall\, t \in [0,T]$,

$$\left.\begin{array}{ll} a(0,t) = 0, & a_x(0,t) > 0, t \in [0,T], \\ |a_x(x,t)| \geq \frac{|a_x(0,t)|}{2}, & x \in \ [-1,1], t \in [0,T], \\ \frac{b(0,t)}{a_x(0,t)} > 0, & x \in t \in [0,T], \\ b(x,t) \geq b(0,t) > 0, & x \in \ [-1,1], t \in [0,T]. \end{array}\right\} \tag{3.1.3}$$

The interior layers may also originate from discontinuous data (see [4, 11, 24]).

Parameter-sensitive problems such as (3.1.1)-(3.1.2) in which the perturbation parameter multiplies the highest derivative of the underlying differential equation are termed singularly perturbed problems. They have attracted researchers' attention over the last few decades because of the existence of oscillations or spurious solutions when trying to solve them numerically. These challenges are more pronounced as the parameter approaches zero and classical numerical methods fail to generate suitable approximations to the solution.

In the context of finite difference discretisations, two families of methods are widely used namely the fitted mesh finite difference methods(FMFDM) (see for example [66, 68, 74]) and the fitted operator finite difference methods (FOFDM)(see [54, 67, 75]).

Recently, a very large number of special methods have been developed by various authors to solve non-turning and turning points time dependent singularly perturbed parabolic

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problems using implicit Euler method for time discretisation. Some authors developed appropriate spatial discretisations adapted to the conditions of their problems.

Beside the references provided in the literature of chapter 1; readers who need more information related to non-turning points time dependent singularly perturbed parabolic problems may refer to [13, 26, 27, 42], and those who are interested in time dependent singularly perturbed parabolic problems when the turning points lead to boundary and/or interior layer(s) are referred to [10, 30, 44, 65, 80].

Discussions on fitted finite difference methods to solve time dependent singularly perturbed convection-diffusion problems whose solution exhibits an interior layer are rare. Few references are given in the literature of chapter 1.

In several works on time dependent problems, as we can notice from the references in the literature of chapter 1, in the the discretization of interior layer problems based on difference equation theory (see [57]), there has never been singularly perturbed problem with smooth coefficients depending on both space and time variables.

The main aim of this chapter is to construct and analyse a fitted operator finite difference method. We use difference equation theory and implicit Euler method to obtain piecewise uniform meshes respectively on space and time. This strategy approximates the solution of time dependent singularly perturbed problems (3.1.1)-(3.1.2), where the coefficients are functions of space and time variables and the solution to the problem exhibits an interior layer due to the presence of a turning point. The method obtained is first order uniformly converges uniformly in both space and time variables. We also use Richardson extrapolation (see [65, 66]), to improve the accuracy and the order of convergence of the fitted operator finite difference method designed up to order two in space only.

The chapter has been organized as follows: in section 3.2 we provide qualitative results on the bounds of the solution and its derivatives at every time level t in $[0, T]$. Using techniques presented in [1, 9, 19], we then provide sharp error estimates specific to the class of problems (3.1.1)-(3.1.2). In Section 3.3, we introduce the proposed scheme which is analysed in section 3.4. Section 3.5 deals with Richardson extrapolation. To show the effectiveness of the proposed scheme, we carry out and discuss some numerical experiments in Section 3.6. Section 3.7 is devoted to some concluding remarks.

3.2 Qualitative results

This section, considers some results of the continuous problem. We use these results in Section 3.4 which deals with the error analysis. Functions $f(x, t)$ and $u(x, 0)$ are assumed to

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be smooth, to secure the continuity and ε-uniform bound of the solution with its derivatives to the problem (3.1.1)-(3.1.2). These conditions are required for appropriate space and time accuracy when using the maximum norm on the domain $\bar{D} = \bar{\Omega} \times [0, T]$, with $\Omega = (-1, 1)$ and $D = \Omega \times (0, T]$.

Lemma 3.2.1. *(Minimum principle)* Consider ψ a smooth function such that $\psi(-1, t) \geq 0$, $\psi(1, t) \geq 0$, $\forall t \in [0, T]$ and $L\psi(x, t) \leq 0, \forall (x, t) \in D$. It follows that, $\psi(x, t) \geq 0, \forall (x, t) \in \bar{D}$.

Proof. The proof is by contradiction.
Assume that there exists a point $(x^*, t^*) \in \bar{D}$ and $\psi(x^*, t^*) = \min \psi(x, t) < 0$. It follows from the given boundary values that (x^*, t^*) cannot be $(-1, 0); (-1, 1); (1, 0)$ or $(1, 1)$. From the definition; $\psi_x(x^*, t^*) = 0, \psi_t(x^*, t^*) = 0$ and $\psi_{xx}(x^*, t^*) \geq 0$. But then

$$L\psi(x^*, t^*) = \varepsilon\psi_{xx}(x^*, t^*) + a(x^*, t^*)\psi_x(x^*, t^*) - b(x^*, t^*)\psi(x^*, t^*) + \psi_t(x^*, t^*) > 0,$$

leading to a contradiction. Consequently $\psi(x, t) \;\geq 0 \;\forall\; (x, t) \;\in\; \bar{D}$.

We use this minimum principle to proof Lemma 3.2.2 below.

Lemma 3.2.2. *(Uniform stability estimate)* Given $u(x, t)$ the solution of (3.1.1)-(3.1.2). Then,

$$||u(x, t)|| \leq b_0^{-1}||f(x, t)|| + \max\left(|\alpha|, |\gamma|\right), \forall (x, t) \in \bar{D},$$

the notation $||.||$ refers to the maximum norm on the domain $\bar{D}$, and b_0 a positive constant as specified above in the introduction.

Proof. Let $\Pi^{\pm}$ be the comparison function such that

$$\Pi^{\pm}(x, t) = b_0^{-1}||f(x, t)|| + \max\left(|\alpha|, |\gamma|\right) \pm u(x, t), x \in \bar{D},$$

and C and positive constant given by

$$C = b_0^{-1}||f(x, t)|| + \max\left(|\alpha|, |\gamma|\right).$$

It follows that

$$L\Pi^{\pm}(x, t) = -Cb(x, t) \pm Lu(x, t) \leq 0.$$

Using the minimum principle above we get

$$\Pi^{\pm}(x, t) \geq 0, \forall (x, t) \in \bar{D}.$$

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Consequently

$$||u(x,t)|| \leq b_0^{-1}||f(x,t)|| + \max\left((|\alpha|,|\gamma|)\right), \forall (x,t) \in \bar{D},$$

which ends the proof.

For the rest of this work we consider the following partition of $\bar{\Omega} = [-1,1]$: $\Omega_L = [-1,-\delta)$, $\Omega_C = [-\delta,\delta]$, $\Omega_R = (\delta,1]$, where $0 < \delta \leq 1/2$. Further, $\Omega_C = \Omega_C^- \cup \Omega_C^+$, with $\Omega_C^- = [-\delta, 0)$, $\Omega_C^+ = [0,\delta]$ and $\bar{D} = \bar{\Omega} \times [0,T]$.

The Lemmas below provide the appropriate bounds on the solution to the problem (3.1.1)-(3.1.2) and its derivatives, depending on whether x belongs to Ω_L, Ω_C or Ω_R.

It is well known that if $u(x,t)$ is the solution to the problem (3.1.1)-(3.1.2), then there exists a positive constant C such that $|u(x,t)| \leq C,\ \forall (x,t) \in \bar{D}$.

Lemma 3.2.3. *Consider $u(x,t)$ the solution to (3.1.1)-(3.1.2), then,*

$$\left|\frac{\partial^i u(x,t)}{\partial x^i}\right| \leq C,\ \forall (x,t) \in \bar{D} \setminus \Omega_C,$$

where C is a positive constant.

Proof. See [19].

Lemma 3.2.4. *Let $u(x,t)$ be the solution to (3.1.1)-(3.1.2). Then, there exist positive constants η and C such that*

$$|\frac{\partial^i u(x,t)}{\partial x^i}| \leq C\left[1 + \varepsilon^{-i}\exp\left(\frac{\eta x}{\varepsilon}\right)\right], \forall x \in \Omega_C^-, t \in [0,T],\ i = 0,1,2,$$

and

$$|\frac{\partial^i u(x,t)}{\partial x^i}| \leq C\left[1 + \varepsilon^{-i}\exp\left(\frac{-\eta x}{\varepsilon}\right)\right], \forall x \in \Omega_C^+, t \in [0,T],\ i = 0,1,2.$$

Proof. We prove this Lemma on Ω_C^-. The proof on Ω_C^+ can be done in similar manner. To start let us rewrite the equation (3.1.1) as follows

$$L_{x,\varepsilon}u = d(x,t)\frac{\partial u}{\partial t} + f(x,t) = g(x,t), \forall x \in \Omega_C^-, t \in [0,T], \tag{3.2.1}$$

where

$$L_{x,\varepsilon}u = \varepsilon\frac{\partial^2 u}{\partial x^2} + a(x,t)\frac{\partial u}{\partial x} - b(x,t)u,$$

Assuming $u_0 = u(x,0)$, d and f smooth functions, then $g(x,t)$ is continuous and ε-uniformly bounded. We use the technique of [42] and equation (3.2.1), to get

$$\left|\frac{\partial^i u(x,t)}{\partial x^i}\right| \leq C\left[1 + \varepsilon^{-i}\exp\left(\frac{\eta x}{\varepsilon}\right)\right], \forall x \in \Omega_C^-, t \in [0,T], i = 0,1. \tag{3.2.2}$$

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To deduce the similar bounds for higher values of i, we consider $v(x,t) = \partial u(x,t)/\partial x$, and after differentiating (3.2.1) with respect to x , it follows that $\forall\, x \in \Omega_C^-,\ t \in [0,T]$;

$$-d(x,t)\frac{\partial v(x,t)}{\partial t} + L_{x,\varepsilon}v = m(x,t) = \frac{\partial f(x,t)}{\partial x} + \frac{\partial d(x,t)}{\partial x}\frac{\partial u}{\partial t} - \frac{\partial a(x,t)}{\partial x}\frac{\partial u}{\partial t} + \frac{\partial b(x,t)}{\partial x}u,$$

$$v(-1,t) = \frac{\partial u(-1,t)}{\partial x} = \alpha_1, v(1,t) = \frac{\partial u(1,t)}{\partial x} = \gamma_1, v_0(x) = \frac{\partial u(x,0)}{\partial x}.$$

Assuming $m(x,t)$ smooth function and applying the above technique for the second time, yields

$$\left|\frac{\partial v}{\partial x}\right| \le C\left[1 + \varepsilon^{-1}\exp(\frac{\eta x}{\varepsilon})\right], \forall x \in \Omega_C^-, t \in [0,T],$$

which is a bound for $\partial^2 u/\partial x^2$,

In the next section we introduce the scheme which we analyse in a subsequent section.

3.3 Construction of the FOFDM

Time dicretization

In this section, we discretize the problem (3.1.1)-(3.1.2) with respect to time, with uniform step-size τ, using Euler implicit method. The partition of the time interval [0, T] is given by:

$$\bar{\omega}^k = \{t_k = k\tau,\ 0 \le k \le K,\ \tau = T/K\}. \tag{3.3.1}$$

The discretization of the problem (3.1.1)-(3.1.2) on $\bar{\omega}^k$ is given by

$$-\,d(x,t_k)\frac{u(x,t_k) - u(x,t_{k-1})}{\tau} + L_{x,\varepsilon}(u(x,t_k)) = f(x,t_k), 1 \le k \le K, \tag{3.3.2}$$

$$u(x,t_0) = u_0(x), \forall x \in (-1,1),\ u(-1,t_k) = \alpha,\ u(1,t_k) = \gamma. \tag{3.3.3}$$

Equation (3.3.2) can also be written as:

$$(-d(x,t_k)I + \tau L_{x,\varepsilon})(u(x,t_k)) = \tau f(x,t_k) - d(x,t_k)u(x,t_{k-1}). \tag{3.3.4}$$

The discretization above is the result of the turning point singularly perturbed problems, at each time level $t_k = k\tau$ which will be examined later. The global error E_k at the time level t_k is the sum of local errors e_k at each time level t_k. This local truncation error e_k is given as: $e_k = u(x,t_k) - \tilde{u}(x,t_k)$,

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where $\tilde{u}(x,t_k)$ is the solution of

$$(-d(x,t)I+\tau L_{x,\varepsilon})(u(x,t_k))=\tau f(x,t_k)-d(x,t)u(x,t_{k-1}),\, u(-1,t_k)=\alpha,\ u(1,t_k)=\gamma. \quad (3.3.5)$$

We find out that the operator $(-d(x,t)I+\tau L_{x,\varepsilon})$ verifies the maximum principle leading to:

$$||(-d(x,t_k)I+\tau L_{x,\varepsilon})^{-1}|| \leq \frac{1}{\max_{0\leq k\leq K,\, x\in[-1,1]}(|d(x,t_k)|^{order(I)})+\tau\beta}, \quad (3.3.6)$$

where $order(I)$ is the order of the identity matrix I. This proves the stability of the discretization with respect to time. It is also known that the local error and the global error are respectively bounded as follows: $||e_k||_\infty \leq c\tau^2, 1\leq k\leq K$ and $||E_k||_\infty \leq c\tau, 1\leq k\leq K$.

Lemma 3.3.1. *Let $u(x,t_k)$ be the solution of (3.3.2) - (3.3.3) at time level t_k, then there exists a positive constant C such that*

$$|u^{(m)}(x,t_k)| \leq C\left[1+\varepsilon^{-m}\exp\left(\frac{\eta x}{\varepsilon}\right)\right],\ m=0,1,2,3, \forall\, x\in\Omega_C^-,$$

and

$$|u^{(m)}(x,t_k)| \leq C\left[1+\varepsilon^{-m}\exp\left(\frac{-\eta x}{\varepsilon}\right)\right],\ m=0,1,2,3, \forall\, x\in\Omega_C^+.$$

Proof. See [19].

Let n be a positive and even integer and let us denote by $\bar{\Omega}^n$ the following partition of the interval $[-1,1]$:

$$x_0=-1;\ x_j=x_0+jh;\ j=1,...,n-1,\ h=x_j-x_{j-1},\ x_n=1.$$

Let $\bar{Q}^{n,K}=\bar{\Omega}^n\times\bar{\omega}^K$ be the grid of (x_j,t_k). To simplify, we adopt the following;
$\forall\,(x_j,t_k)\ \in\bar{Q}^{n,K},\ \Xi(x_j,t_k):=\Xi_j^k$. And U_j^k the approximation of u_j^k.
Using difference equation theory on $\bar{Q}^{n,K}$ (see [57]), we discretize the problem (3.1.1)-(3.1.2) as:

$$L^{n,K}U_j^k := \begin{cases} \varepsilon\delta^2 U_j^k+\tilde{a_j^k}D^-U_j^k-\left(\tilde{b_j^k}+\frac{\tilde{d_j^k}}{\tau}\right)U_j^k=\tilde{f_j^k}-\tilde{d_j^k}\frac{U_j^{k-1}}{\tau}, \\ j=1,2,\cdots,\frac{n}{2}-1,\ k=1,...,K, \\ \\ \varepsilon\delta^2 U_j^k+\tilde{a_j^k}D^+U_j^k-\left(\tilde{b_j^k}+\frac{\tilde{d_j^k}}{\tau}\right)U_j^k=\tilde{f_j^k}-\tilde{d_j^k}\frac{U_j^{k-1}}{\tau}, \\ j=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots,n-1,\ k=1,...,K, \end{cases} \quad (3.3.7)$$

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$$U_0^k = \alpha, \; U_n^k = \gamma, \tag{3.3.8}$$

with

$$D^- U_j^k = \frac{U_j^k - U_{j-1}^k}{h}, \quad D^+ U_j^k = \frac{U_{j+1}^k - U_j^k}{h}, \quad \delta^2 U_j^k = \frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{\tilde{\phi}_j^{k\,2}},$$

and

$$\tilde{\phi}_j^{k\,2} = \begin{cases} \frac{h\varepsilon}{\tilde{a_j}^k}\left[\exp\left(\frac{\tilde{a_j}^k h}{\varepsilon}\right) - 1\right], \; j = 1, 2, ..., \frac{n}{2} - 1, \\ \\ \frac{h\varepsilon}{\tilde{a_j}^k}\left[1 - \exp\left(\frac{-\tilde{a_j}^k h}{\varepsilon}\right)\right], \; j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1. \end{cases} \tag{3.3.9}$$

We also utilize the following convention for $k = 1, ..., K$.

$$\left.\begin{array}{l} \tilde{a}_j^k = \dfrac{a_j^k + a_{j-1}^k}{2} \text{ for } j = 0, 1, 2, ..., \dfrac{n}{2} - 1, \\ \\ \tilde{a}_j^k = \dfrac{a_j^k + a_{j+1}^k}{2} \text{ for } j = \dfrac{n}{2}, \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, ..., n - 1, \\ \\ \tilde{b}_j^k = \dfrac{b_{j-1}^k + b_j^k + b_{j+1}^k}{3}, \tilde{f}_j^k = \dfrac{f_{j-1}^k + f_j^k + f_{j+1}^k}{3} \text{ for } j = 1, 2, ..., n - 1, \\ \tilde{F}_j^k = \tilde{f}_j^k - \tilde{d}_j^k \dfrac{U_j^{k-1}}{\tau}, \text{for } j = 1, 2, ..., n - 1, \\ \tilde{d}_j^k = \dfrac{d_{j-1}^k + d_j^k + d_{j+1}^k}{3}; \text{ for } j = 0, 1, 2, ..., n - 1. \end{array}\right\} \tag{3.3.10}$$

The system of equations (3.3.7) can be rewritten as

$$\left.\begin{array}{l} r_{j,k}^- U_{j-1}^k + r_{j,k}^c U_j^k + r_{j,k}^+ U_{j+1}^k = \tilde{f}_j^{\,k}, \; j = 0, 1, 2, ..., \frac{n}{2} - 1; \\ k = 0, 1, ..., K, \\ r_{j,k}^- U_{j-1}^k + r_{j,k}^c U_j^k + r_{j,k}^+ U_{j+1}^k = \tilde{f}_j^{\,k}, \; j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1; \\ k = 0, 1, ..., K. \end{array}\right\} \tag{3.3.11}$$

Where

$$\left.\begin{array}{l} r_{j,k}^- = \dfrac{\varepsilon}{\tilde{\phi}_j^{k\,2}} - \dfrac{\tilde{a_j}^k}{h}; \; r_{j,k}^c = \dfrac{-2\varepsilon}{\tilde{\phi}_j^{k\,2}} + \dfrac{\tilde{a_j}^k}{h} - \left(\tilde{b_j}^k + \dfrac{\tilde{d}_j^k}{\tau}\right); \; r_{j,k}^+ = \dfrac{\varepsilon}{\tilde{\phi}_j^{k\,2}}, \; j = 0, 1, 2, ..., \dfrac{n}{2} - 1, \\ r_{j,k}^- = \dfrac{\varepsilon}{\tilde{\phi}_j^{k\,2}}; \; r_{j,k}^c = \dfrac{-2\varepsilon}{\tilde{\phi}_j^{k\,2}} - \dfrac{\tilde{a_j}^k}{h} - \left(\tilde{b_j}^k + \dfrac{\tilde{d}_j^k}{\tau}\right); \; r_{j,k}^+ = \dfrac{\varepsilon}{\tilde{\phi}_j^{k\,2}} + \dfrac{\tilde{a_j}^k}{h}, \; j = \dfrac{n}{2}, \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, ..., n - 1. \end{array}\right\} \tag{3.3.12}$$

The fitted operator finite difference method (FOFDM) (3.3.11) along with the boundary conditions (3.3.8) verifies the following lemmas:

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Lemma 3.3.2. (Discrete minimum principle) . *Consider a mesh function* ξ_j^k *such that,* $L^{n,k}\xi_j^k \leq 0 \; \forall (j,k) \in Q^{n,K}$, $\xi_j^0 \geq 0, \; 0 \leq j \leq n$, $\xi_0^k \geq 0$, *and* $\xi_n^k \geq 0, \; 1 \leq k \leq K$. *Then* $\xi_j^k \geq 0$, $\forall (j,k) \in \bar{Q}^{n,K}$.

Proof. Given (s,l) such that $\xi_s^l = \min_{(j,k)} \xi_j^k < 0$, $\xi_j^k \in \bar{Q}^{n,K}$. Obviously $s \neq 1,2,...,n-1$ and $l \neq 1,2,...,K$. In addition $\xi_{s+1}^l - \xi_s^l \geq 0$, $\xi_s^l - \xi_{s-1}^l \leq 0$, and $\xi_s^l - \xi_s^{l-1} \leq 0$.
We have

$$L^{n,K}\xi_s^l = \begin{cases} \varepsilon\bar{\delta}^2\xi_s^l + a_s^l D^-\xi_s^l - \left(b_s^l + \frac{d_s^l}{\tau}\right)\xi_s^l > 0, s = 1,2,...,\frac{n}{2}-1, \; l = 1,2,...,K, \\ -\left(b_s^l + \frac{d_s^l}{\tau}\right)\xi_s^l > 0, s = \frac{n}{2}, \; l = 1,2,...,K, \\ \varepsilon\bar{\delta}^2\xi_s^l + a_s^l D^+\xi_s^l - \left(b_s^l + \frac{d_s^l}{\tau}\right)\xi_s^l > 0, s = \frac{n}{2}+1,..,n-1 \; , \; l = 1,2,...,K. \end{cases} \tag{3.3.13}$$

Giving $L^{n,K}\xi_k^l > 0, s = 1,2,...,n-1$ and $l = 1,2,...,K$, which leads to a contradiction. It follows that $\xi_s^l \geq 0$, and thus $\xi_j^k \geq 0, \; \forall \, (j,k) \in \bar{Q}^{n,K}$.
The above minimum principle is used to prove the following Lemma.

Lemma 3.3.3. *(Uniform stability estimate) Let* Z_j^k *be a mesh function at a time level such that* $Z_0^k = Z_n^k = 0$. *Then*

$$|Z_j^k| \leq \frac{1}{b_0} \max_{1 \leq i \leq n-1} |L^{n,K} Z_i^k|, \textit{ for } 1 \leq j \leq n, \textit{and } 1 \leq k \leq K.$$

Proof. Consider the mesh function

$$(\xi^{\pm})_j^k = \frac{1}{b_0} \max_{1 \leq i \leq n-1} \left| L_\varepsilon^{n,K} Z_i^k \right| \pm Z_j^k, 1 \leq j \leq n, \text{and } 1 \leq k \leq K,$$

with $b_j^k \geq b_0 > 0$ to ensure the uniqueness of the solution to the problem (3.3.7) - (3.3.8). It is clear that $(\xi^{\pm})_0^k \geq 0$ and $(\xi^{\pm})_n^k \geq 0$. Also, for $0 \leq j \leq n$, and $1 \leq k \leq K$,

$$L^{n,K}(\xi^{\pm})_j^k = \frac{-b_j^k}{b_0} \max_{1 \leq i \leq n-1} \left| L^{n,K} Z_i^k \right| \pm L^{n,K} z_j^k, 1 \leq j \leq n, \text{and } 1 \leq k \leq K.$$

For $0 \leq j \leq n, (-b_j^k)/(b_0) \leq -1$. This leads to $L^{n,K}(\xi^{\pm})_j^k \leq 0$. By the discrete minimum principle Lemma 3.3.2, we conclude that $(\xi^{\pm})_j^k \geq 0, \; \forall \; 0 \leq j \leq n, 1 \leq k \leq K$ and this ends the proof.

Lemma 3.3.4. *For a fixed mesh and for all integers m, we have*

$$\lim_{\varepsilon -> 0} \max_{1 \leq j \leq \frac{n}{2}-1} \frac{\exp(Mx_j/\sqrt{\varepsilon})}{\varepsilon^{m/2}} = 0, \textit{and} \lim_{\varepsilon -> 0} \max_{\frac{n}{2} \leq j \leq n-1} \frac{\exp(-Mx_j/\sqrt{\varepsilon})}{\varepsilon^{m/2}} = 0.$$

Proof. *See [74]*

In the next section we concentrate on convergence analysis of the FOFDM derived.

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3.4 Convergence analysis of the FOFDM

This section is devoted to the analysis of the FOFDM proposed in section 3.3 above. The analysis will be conducted on $x \in [-1,0]$, since the case when $x \in (0,1]$ can be done similarly.

Let us define the operator L^K from (3.3) as:

$$\begin{aligned} L^K z(x,t_k) &= \varepsilon \frac{d^2 z(x,t_k)}{dx^2} + a(x,t_k)\frac{dz(x,t_k)}{dx} - (b(x,t_k) + \frac{d(x,t_k)}{\tau})z(x,t_k), \\ &= f(x,t_k) - d(x,t_k)\frac{z(x,t_{k-1})}{\tau}. \end{aligned} \tag{3.4.14}$$

The local truncation error of the space discretization on $[-1,0]\times[0,T]$ (e.g. $j = 1,2,...,n/2-1, k = 1,2,...,K$) can be given as:

$$\begin{aligned} L^{n,K}(U_j^k - z_j^k) &= \left(L^K - L^{n,K}\right) z_j^k \\ &= \varepsilon z''_{j,k} + \tilde{a}_j^k z_j^k - \left[\frac{\varepsilon}{\tilde{\phi}_j^{2^k}}(z_{j+1}^k - 2z_j^k + z_{j-1}^k) + \frac{\tilde{a}_j^k}{h}(z_j^k - z_{j-1}^k)\right] \\ &= \varepsilon u''_{j,k} - \frac{\varepsilon}{\tilde{\phi}_j^{2^k}}\left[h^2 u''_{j,k} + \frac{h^4}{24}(z^{(iv)})^k(\xi_1) + \frac{h^4}{24}(z^{(iv)})^k(\xi_2)\right] \\ &+ \frac{\tilde{a}_j^k h}{2} z''_{j,k} - \frac{\tilde{a_j}^k h^2}{6} z'''_{j,k} + \frac{\tilde{a_j}^k h^3}{24}(z^{(iv)})^k(\xi_3), \end{aligned} \tag{3.4.15}$$

with $\xi_1, \in (x_j, x_{j+1})$, $\xi_2, \xi_3 \in (x_{j-1}, x_j)$. Using the expression for $\tilde{a}_j^k$ as specified in to (3.3.10), the Taylor expansions of a_{j-1}^k up to order four, and the truncated Taylor expansion $1/\tilde{\phi}_j^{2^k} = 1/h^2 - \tilde{a_j}^k/\varepsilon h$, we get

$$\begin{aligned} L_1^{n,K}\Big(U_j^k - z_j^k\Big) = \frac{3}{2}a_j^k u''_{j,k} h! &+ \left[-\frac{3a'_{j,k}}{2} z''_{j,k} - \frac{\varepsilon}{24}\Big((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\Big) - \frac{a_j^k}{6} z'''_{j,k}\right] h^2 \\ &+ \left[\frac{3a''_{j,k}}{4} z''_{j,k} - \frac{a_j^k}{24}\Big((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\Big) + \frac{a'_{j,k}}{12} z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3)\right] h^3 \\ &+ \left[-\frac{13a'''_{j,k}}{24} z''_{j,k} - \frac{a'_{j,k}}{48}\Big((u^{(iv)})^k(\xi_1)) + (z^{(iv)})^k(\xi_2)\Big) - \frac{a''_{j,k}}{24} z'''_{j,k}\right] h^4 \\ &+ \left[-\frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3)\right] h^4. \end{aligned} \tag{3.4.16}$$

With ξ's in the interval (x_{j-1}, x_{j+1}). The coefficients of $u_j^k, z'_{j,k}, \cdots, (z^{(iv)})^k(\xi_{*_j})$ can be bounded by a constant. Applying both Lemma 3.3.1 and Lemma 3.3.4, we obtain the following

$$|L_1^{n,K}(U_j^k - z_j^k)| \le Mh, \forall j = 1(1)\frac{n}{2} - 1.$$

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In a similar way, we can prove that

$$|L_2^{n,K}(U_j^k - z_j^k)| \le Mh, \forall j = \frac{n}{2}(1)n+1.$$

From Lemma 3.3.3, we come to the following results

Theorem 3.4.1. *Let U_j^k be the numerical solution of (3.3.7) along with (3.3.10) and z_j^k the solution to (3.3.2) - (3.3.3) at time level t_k. Then, there exists a constant M independent of ε, τ, h and k such that*

$$\max_{1\le j\le n+1} |U_j^k - z_j^k| \le Mh \quad k = 1(1)K+1. \tag{3.4.17}$$

The triangular inequality $|U_j^k - u_j^k| \le |U_j^k - z_j^k| + |z_j^k - u_j^k|$ along with Lemma 3.3.3, Theorem 3.4.1 and the global error; lead to the following main result.

Theorem 3.4.2. *Let U_j^k be the numerical solution of (3.3.7)-(3.3.10) and u_j^k the solution to (3.1.1)-(3.1.2) at the grid point (x_j, t_k). Then, there exists a constant M independent of ε, τ, h and k such that*

$$\max_{0\le j\le n} |U_j^k - u_j^k| \le M(h+\tau). \tag{3.4.18}$$

Section 3.5 below deals with Richardson extrapolation which is an acceleration technique. We use this technique to improves the estimate (3.4.18).

3.5 Richardson extrapolation on the FOFDM

Richardson extrapolation is the extrapolation technique based on linear combination of p solutions, $p \ge 0$ corresponding to different, nested meshes.

In this section we improve the accuracy and the order of convergence of (3.4.18). To start, rewrite the equation (3.4.16) as follows:

$$L^{n,K}\left(U_j^k - z_j^k\right) = M_1 h + M_2 h^2 + R_n(x_j), \tag{3.5.1}$$

with

$$M_1 = \frac{3a_j}{2} z''_{j,k},$$

$$M_2 = \frac{3a'_{j,k}}{3} - \frac{\varepsilon}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a_j^k}{6} z'''_{j,k},$$

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$$R_n^k(x_j) = h^3\left[\frac{3a''_{j,k}}{4}z''_{j,k} - \frac{a_j^k}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) + \frac{a'_{j,k}}{12}z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3)\right]$$
$$+h^4\left[\frac{13a'''_{j,k}}{24}z''_{j,k} - \frac{a'_{j,k}}{48}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a''_{j,k}}{24}z'''_{j,k} - \frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3)\right].$$

The ξ's and $z_j^k, z'_{j,k}, \cdots, (z^{(iv)})^k(\xi_{*_j})$ remain the same as described in (3.4.15). Now, let μ_{2n} be the mesh obtained by bisecting each mesh interval in μ_n, i.e.,

$$\mu_{2n} = \{\bar{x}_i\} \text{ with } \bar{x}_0 = -1, \;\; \bar{x}_n = 1 \;\text{ and }\; \bar{x}_j - \bar{x}_{j-1} = \bar{h} = h/2, \;\; j = 1, 2, ..., 2n.$$

Consider $\bar{U}_j^k$ the numerical solution on μ_{2n}, M and p positive real numbers. We rewrite the equation (6.5.1) in terms of $\bar{U}_j^k$ as follows:

$$L^{n,K}\left(\bar{U}_j^k - \bar{z}_j^k\right) = M\bar{h} + p\bar{h}^2 + R_{2n}^k(\bar{x}_j), 1 \le j \le 2n-1. \tag{3.5.2}$$

We note that $\bar{z}_j^k \equiv z_j^k$.

After multiplying (3.5.2) by 2, we get

$$2L^{n,K}\left(\bar{U}_j^k - \bar{z}_j^k\right) = 2M\bar{h} + 2p\bar{h}^2 + 2R_{2n}^k(\bar{x}_j), 1 \le j \le 2n-1, \tag{3.5.3}$$

meaning

$$L^{n,K}\left(2\bar{U}_j^k - 2\bar{z}_j^k\right) = 2M\bar{h} + 2p\bar{h}^2 + 2R_{2n}^k(\bar{x}_j), 1 \le j \le 2n-1. \tag{3.5.4}$$

Let (3.5.1) be in terms of M and p. After subtracting (3.5.1) from (3.5.4), we obtain

$$L^{n,K}\left((2\bar{U}_j^k - U_j^k) - z_j^k\right) = p\bar{h}^2 + 2R_{2n}^k(\bar{x}_j), 1 \le j \le 2n-1, \tag{3.5.5}$$

meaning

$$L^{n,K}\left((2\bar{U}_j^k - U_j^k) - z_j^k\right) = 0(h^2), 1 \le j \le 2n-1,$$

Consider

$$U_j^{ext,k} := 2\bar{U}_j^k - U_j^k.$$

The numerical solution $U_j^{ext,k}$ is another numerical approximation of z_j^k.

From Lemma 3.3.3 we come to the following result:

Theorem 3.5.1. *Let $U_j^{ext,k}$ be the numerical solution approximation, obtained via the Richardson extrapolation based on FOFDM (3.3.11) along with the boundary conditions (3.3.8) and*

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z_j^k the solution to (3.3.2)-(3.3.3) at time level t_k. Then, there exists a constant M independent of ε, τ, h and k such that

$$\max_{0\leq j\leq n} |U_j^{ext,k} - z_j^k| \leq Mh^2. \tag{3.5.6}$$

Applying triangular inequality leads to

$$|U_j^{ext,k} - u_j^k| \leq |U_j^{ext,k} - z_j^k| + |z_j^k - u_j^k|. \tag{3.5.7}$$

From Lemma 3.3.1, Theorem 3.5.1 and the global error, we get the following result.

Theorem 3.5.2. *Let $U_j^{ext,k}$ be the numerical solution of (3.3.11) along with the boundary conditions (3.3.8) and z_j^k the solution to (3.1.1)-(3.1.2) at the grid point (x_j, t_k). Then, there exists a constant M independent of ε, τ, h and k such that*

$$\max_{0\leq j\leq n} |U_j^{ext,k} - u_j^k| \leq M(h^2 + \tau). \tag{3.5.8}$$

In section 3.6 below we implement the proposed scheme on two examples and present numerical results which confirm the accuracy and robustness of the solution of some problems of type (3.1.1)-(3.1.2).

3.6 Numerical examples

In this section we present the numerical results of some problems of type (3.1.1)-(3.1.2).

Example 3.6.1. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{array}{r} \varepsilon u_{xx} + a(x,t)u_x - b(x,t)u - du_t = f(x,t), -1 \leq x \leq 1;\ \varepsilon, t \in [0,1], \\ u(-1,0) = u(1,0) = 0. \end{array}\right\}$$

Where $d = 1$, $a(x,t) = 2x[1 + \sqrt{\varepsilon}t^2)]$ and $b(x,t) = 2(2 + xt)$.

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$. The exact solution is

$$u(x,t) = \varepsilon\left(1 - x^2\right)\exp\left(-\frac{t}{\varepsilon}\right) erf\left(\frac{x}{\sqrt{\varepsilon}}\right).$$

To get the expression of $f(x,t)$ we substitute $a(x,t)$; $b(x,t)$ and $u(x,t)$ into equation (3.6.1).

Example 3.6.2. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{array}{r} \varepsilon u_{xx} + a(x,t)u_x - b(x,t)u - du_t = f(x,t),\ 0 \leq x \leq 1,\ \varepsilon, t \in [0,1], \\ u(0,0) = \varepsilon\tanh(\frac{1}{2\varepsilon}) - \varepsilon^{\frac{3}{2}}; u(1,0) = \varepsilon\tanh(-\frac{1}{2\varepsilon}) - \varepsilon^{\frac{3}{2}}, \end{array}\right\}$$

where $d = (1 + x^2)\exp(-xt)$, $a(x,t) = 2(2x-1)(1+t^2)$ and $b(x,t) = 2(1 + xt)$.

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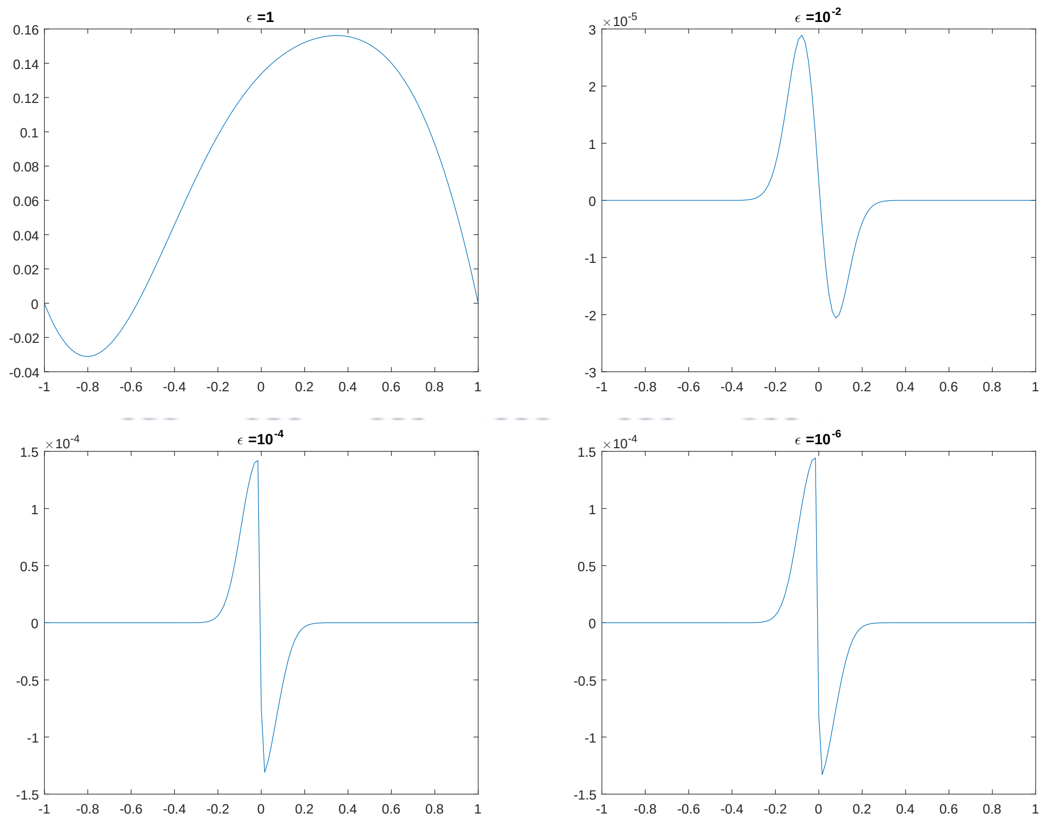

Figure 3.1: Plots of the numerical solution for Example 3.6.1 for $\varepsilon = 1, 10^{-2}, 10^{-4}$ and 10^{-6} with $n = 128$ and $K = 128$. These plots are the solution profile at the final time T ($u(x, T)$ is plotted against the space variable x)

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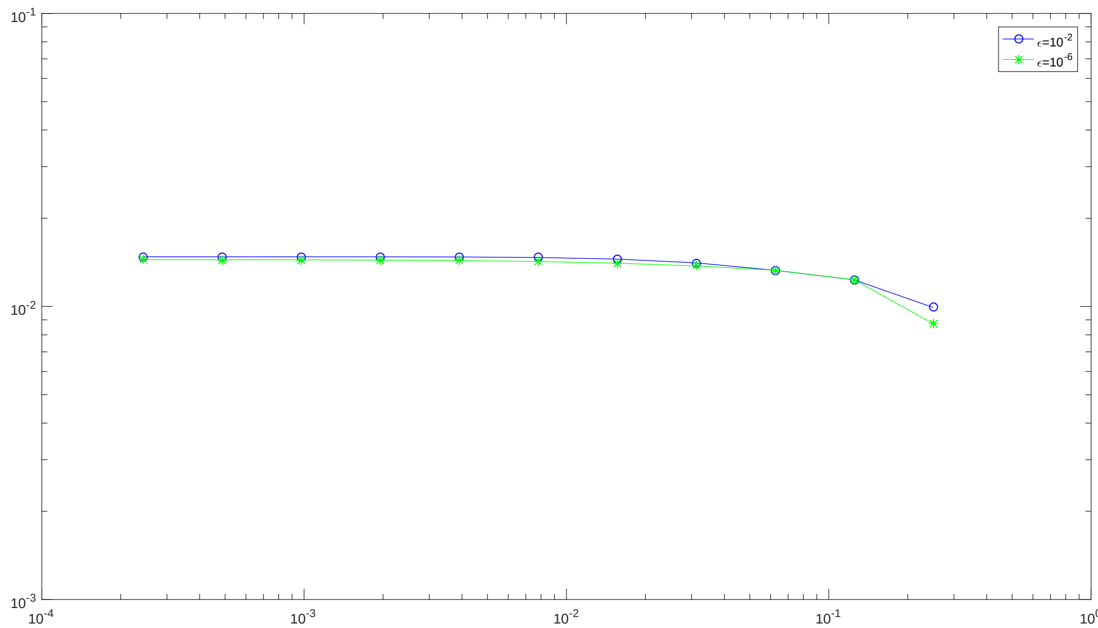

Figure 3.2: Loglog plot for Example 3.6.2: the logarithm of pointwise maximum errors is plotted against the logarithm of step size h for at time $t = 1$ with values of n from 4 to 4096 and for $\varepsilon = 10^{-2}$ and 10^{-6}

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$. The exact solution is

$$u(x,t) = \varepsilon \exp\left[-\frac{t}{\varepsilon}\right] \tanh\left(\frac{0.5-x}{\varepsilon}\right) - \varepsilon^{\frac{3}{2}} \exp\left[-(1-2x)\,t\right],$$

and $f(x,t)$ is obtained after substituting $u(x,t)$ into equation (3.6.2).

The maximum errors at all mesh points and the numerical rates of convergence before extrapolation are evaluated using the formulas

$$E^{\varepsilon,n,K} := \max_{0\le j\le n; 0\le k\le K} \left| U_{j,k}^{\varepsilon,n,K} - u_{j,k}^{\varepsilon,n,K} \right|.$$

In case the exact solution is unknown, we use a variant of the double mesh principle

$$E^{\varepsilon,n,K} := \max_{0\le j\le n; 0\le k\le K} \left| U_{j,k}^{\varepsilon,n,K} - U_{j,k}^{\varepsilon,2n,2K} \right|.$$

Where $u_{j,k}^{\varepsilon,n,K}$ and $U_{j,k}^{\varepsilon,n,K}$ in the above represent respectively the exact and the approximate solutions obtained using a constant time stepτ and space step h. Similarly, $U_{j,k}^{\varepsilon,2n,2K}$ is found

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using the constant time step $\frac{\tau}{2}$ and space step $\frac{h}{2}$. Nevertheless, the computation of numerical rates of convergence is given by:

$$r_l = r_k \equiv r_{\varepsilon,k} := \log_2\left(E^{\varepsilon,n,K}/E^{\varepsilon,2n_l,2K_l}\right), l = 1, 2, ...$$

Also, we compute $E_{n,K} = \max\limits_{0<\varepsilon\leq 1} E_{\varepsilon,n,K}$.
And the numerical rate of uniform convergence are:

$$R_{n,k} := \log_2\left(E_{n,K}/E_{2n,2K}\right).$$

For a fixed mesh, we see that the maximum nodal errors remain constant for small values of ε (see tables 3.1 and 3.5). Moreover, results in tables 3.3 and 3.7 show that the proposed method is essentially first order convergent.
After extrapolation the maximum errors at all mesh points and the numerical rates of convergence are evaluated using the formulas

$$E^{ext}_{\varepsilon,n,K} := \max_{0\leq j\leq 2n; 0\leq k\leq 2K} |U^{ext}_j - u^{\varepsilon,n,K}_{j,k}| \text{ and } R_k \equiv R_{\varepsilon,k} := \log_2\left(E^{ext}_{n_k}/E^{ext}_{2n_k}\right), k = 1, 2, ...$$

respectively, where $E^{ext}_{n_k}$ stands for $E^{\varepsilon,2n,2K}$.

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Table 3.1: Maximum errors for Example 3.6.1 (before extrapolation)

ε	$N=16$ $K=10$	$N=32$ $K=20$	$N=64$ $K=40$	$N=128$ $K=80$	$N=256$ $K=160$
10^{-3}	6.34E-02	4.06E-02	2.25E-02	1.18E-02	6.05E-03
10^{-5}	6.34E-02	4.07E-02	2.26E-02	1.19E-02	6.09E-03
10^{-7}	6.34E-02	4.07E-02	2.26E-02	1.19E-02	6.10E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	6.34E-02	4.07E-02	2.26E-02	1.19E-02	6.10E-03

Table 3.2: Maximum errors for Example 3.6.1 (after extrapolation)

ε	$N=16$ $K=10$	$N=32$ $K=40$	$N=64$ $K=160$	$N=128$ $K=640$	$N=256$ $K=2560$
10^{-3}	8.97E-02	2.75E-02	6.24E-03	1.55E-03	3.88E-04
10^{-5}	8.99E-02	2.98E-02	8.09E-03	2.07E-03	4.99E-04
10^{-7}	8.99E-02	2.98E-02	8.09E-03	2.07E-03	5.20E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	8.99E-02	2.98E-02	8.09E-03	2.07E-03	5.20E-04

Table 3.3: Rates of convergence for Example 3.6.1 (before extrapolation)

ε	r_1	r_2	r_3	r_4
10^{-3}	0.64	0.86	0.93	0.96
10^{-5}	0.64	0.85	0.93	0.96
10^{-7}	0.64	0.85	0.93	0.96
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	0.64	0.85	0.93	0.96

Table 3.4: Rates of convergence for Example 3.6.1 (after extrapolation)

ε	r_1	r_2	r_3	r_4
10^{-3}	1.70	2.14	2.01	2.00
10^{-5}	1.60	1.88	1.97	2.05
10^{-7}	1.60	1.88	1.97	1.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	1.60	1.88	1.97	1.99

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Table 3.5: Maximum errors for Example 3.6.2 (before extrapolation)

ε	$N = 16$ $K = 10$	$N = 32$ $K = 20$	$N = 64$ $K = 40$	$N = 128$ $K = 80$	$N = 256$ $K = 160$
10^{-3}	1.17E-01	7.10E-02	4.05E-02	2.20E-02	1.16E-02
10^{-4}	1.17E-01	7.10E-02	4.05E-02	2.20E-02	1.16E-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	1.17E-01	7.10E-02	4.05E-02	2.20E-02	1.16E-02

Table 3.6: Maximum errors for Example 3.6.2 (after extrapolation)

ε	$N = 16$ $K = 10$	$N = 32$ $K = 40$	$N = 64$ $K = 160$	$N = 128$ $K = 640$	$N = 256$ $K = 2560$
10^{-3}	1.36E-01	4.14E-02	1.17E-02	3.09E-03	1.40E-03
10^{-4}	1.36E-01	4.14E-02	1.17E-02	3.09E-03	7.86E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	1.36E-01	4.14E-02	1.17E-02	3.09E-03	7.86E-04

Table 3.7: Rates of convergence for Example 3.6.2 (before extrapolation)

ε	r_1	r_2	r_3	r_4
10^{-3}	0.72	0.81	0.88	0.92
10^{-4}	0.72	0.81	0.88	0.92
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	0.72	0.81	0.88	0.92

Table 3.8: Rates of convergence for Example 3.6.2 (after extrapolation)

ε	r_1	r_2	r_3	r_4
10^{-3}	1.72	1.82	1.92	1.14
10^{-4}	1.72	1.82	1.92	1.97
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-14}	1.72	1.82	1.92	1.97

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3.7 Summary

Singularly perturbed turning point problems are difficult to solve using standard/classical methods due to the presence of boundary or interior layers in their solutions. Usually, when seeking for numerical solutions of layer problems, layer adapted meshes are used. These meshes are fine in the layer region and coarse away from the layer region. Due to the nature of these meshes, and especially when time is involved, the computation with regards to the convergence analysis becomes more complex.

The main aim of this chapter was to design and analyse a fitted operator finite difference method to solve a class of time dependent singularly perturbed turning point problems whose solution exhibits an interior layer. We first established bounds on the solution and its derivatives. Then, we discretized the time variable before proceeding to space discretization. Bounds were used to prove uniform convergence of the proposed numerical method. The first order uniform convergence shown theoretically, with respect to space and time variables was confirmed numerically through two test examples.

We provided plots of the numerical solution for Example 3.6.1 for various values of the perturbation parameter ε to see the layer behavior. In addition, we presented a loglog plot for Example 3.6.2.

We also applied Richardson extrapolation to improve the accuracy and the convergence of the numerical scheme in the space variable. Indeed, convergence order improved from one to two.

The problem investigated in this chapter depends on the perturbation parameter ε which multiplies the highest order derivative that appears in the problem. One would like to understand how replacing ε by some function of ε and x affects the design of numerical methods. We are currently working in that direction.

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Chapter 4

A fitted discretization for interior-layer problems with a quadratic factor affecting the second derivative

We consider a family of singularly perturbed turning point problems in which the quadratic function $\varepsilon + x^2$ multiplies the second derivative of the unknown function. The study is focussed on the case where the turning point gives rise to an interior layer. Despite their importance as far as applications are concerned, these problems have attracted little attention from the research community. The aim of this chapter is two-fold. On one hand, we establish sharp bounds on the solution and its derivatives. On the other hand, we construct a numerical method and analyse its convergence properties. We use Richardson extrapolation to increase the accuracy of the proposed method.

4.1 Introduction

Consider the singularly perturbed two-point boundary values problems

$$Lu \;:=\; \varepsilon u'' + a(x)u' - \; b(x)u = \; f(x), x \in \Omega = (-1, 1), \tag{4.1.1}$$

$$u(-1) = \gamma_1,\; u(1) = \gamma_2, \tag{4.1.2}$$

where γ_1 and γ_2 are given real numbers, the perturbation parameter ε satisfies $0 < \varepsilon \ll 1$, and the functions $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently smooth in $\bar{\Omega}$, to ensure

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the smoothness of the solution. The condition $b(x) \geq b_0 > 0$ guarantees the uniqueness of the solution [59].

Problems such as (4.1.1)-(4.1.2) occur in many domains of science and engineering, including fluid mechanics, solid mechanics, quantum mechanics, chemical reactor theory, aerodynamics, optimal control, reaction-diffusion process and geophysics to name but a few. The solution possesses large gradients in narrow region(s) of the domain called boundary or interior layer(s) when ε is very small.

The location and the number of layers depend on the coefficient functions of the equation (4.1.1). If $a(x) > 0$ or $a(x) < 0$ for all $x \in [-1, 1]$, then the solution has a boundary layer on either the left or the right end of the interval. If, instead, $a(x) \equiv 0$ for all $x \in [-1, 1]$, then the solution has twin boundary layers or an oscillatory behavior. These cases, referred to as non-turning point problems, have extensively been studied in the literature (see e.g [12, 17, 29, 49, 54, 59]).

The turning point problems are those where $a(x)$ possesses at least a zero in the domain. Turning point problems give rise to interior layer or to twin boundary layers. For more information on turning point problems, interested reader may wish to consult [1, 9, 16, 30, 43, 65]. It is worth noting that interior layers may also be caused by non-smooth coefficient functions or discontinuous data (see e.g [4, 11, 24]).

The turning and non-turning point problems referred to above are widely studied in the literature. However, equally important for their applications in fluid dynamics and biology are problems in which the coefficient of the highest derivatives are functions of the form $g(x, \varepsilon)$. These problems have received little attention from the research community. Liseikin [52] considered the case $g(x, \varepsilon) = -(\varepsilon + px)^\beta$ for $\beta \geq 1$ and studied the problem for $p = 0$ and $p = 1$. In ([51], pp. 106-111) he derived bounds on the solution and its derivatives for the case $g(x, \varepsilon) = -(\varepsilon + x)^\beta$ for some prescribed values of β. Additionally, for $\beta = 1$ (see pp. 256-262), he designed a numerical method and analysed its convergence. In real life, the case where $p = 1$ and $\beta = 1$ describes filtration of a liquid through a neighbourhood about a circular orifice or radius $r = \varepsilon$ [52, 77]. And when $p = 1$ and $\beta = 2$, the model describes a steady diffusive-drift motion [52, 93]. For the best of our knowledge, no other works is recorded in the literature.

In this chapter, we consider the case $g(x, \varepsilon) = \varepsilon + x^2$. Thus, the aim of this chapter is to study the problem

$$Lu := (\varepsilon + x^2)u'' + a(x)u' - b(x)u = f(x), x \in [-1, 1]; \tag{4.1.3}$$

$$u(-1) = \gamma_1,\ u(1) = \gamma_2, \tag{4.1.4}$$

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with the assumptions

$$\left.\begin{array}{ll} (i)\ a(0) = 0, & a'(0) > 0, \\ (ii)\ b(x) \geq b_0 > 0, & x \in\ [-1, 1], \\ (iii)\ |a'(x)| \geq \frac{|a'(0)|}{2}, & x \in\ [-1, 1], \end{array}\right\} \tag{4.1.5}$$

(i) guarantees the existence of the turning point, (ii) indicates that the problem has only one solution and satisfies the minimum principle, and (iii) implies that zero is the unique turning point in $[-1, 1]$. These assumptions guarantee an interior layer at $x = 0$.

One interesting aspect about problem (4.1.3)-(4.1.4) is the fact that when one sets $\varepsilon = 0$, the order of the underlying equation does not reduce as it is the case in classical singularly perturbed problems and in particular for problem (4.1.1)-(4.1.2).

We will start by deriving bounds on the solution and its derivatives in section 4.2. Then in section 4.3, we will construct a fitted operator finite difference method (FOFDM). Section 4.4 will be concerned with convergence analysis of the proposed numerical method. We will show that the method is first order accurate, uniformly with respect to ε. This accuracy will be improved upon to second order by postprocessing the FOFDM through Richardson extrapolation in section 4.5. Results on numerical experiments to confirm the theoretical findings shall be presented in section 4.6. Section 4.7 will be devoted to some concluding remarks .

4.2 Qualitative results

This section deals with continuous bounds which are later on used in section 4.4 for the convergence analysis of the FOFDM.

We note that the operator L verifies continuous minimum principle below.

Lemma 4.2.1. Given ψ a smooth function where $\psi(-1) \geqslant 0$, $\psi(1) \geqslant 0$ and $L\psi(x) \leq 0$, $\forall\, x \in (-1, 1)$. Then $\psi(x) \geqslant 0$, $\forall\, x \in [-1, 1]$.

Proof. The proof of this Lemma is by contradiction. Consider $x^* \in [-1, 1]$ and $\psi(x^*) = \min\limits_{x\in[-1,1]} \psi(x) < 0$. We have $x^* \notin \{-1, 1\}$, $\psi'(x^*) = 0$ and $\psi''(x^*) \geqslant 0$. Then

$$L\psi(x^*) := (\varepsilon + x^2)\psi''(x^*) + \ a(x^*)\psi'(x^*) - \ b(x^*)\psi(x^*)\ > 0,$$

leading to a contradiction. Meaning that $\psi(x^*) \geqslant 0$ consequently $\psi(x) \geqslant 0$, $\forall x \in [-1, 1]$.

Lemma 4.2.2. Consider $u(x)$ the solution of (1.1)-(1.2). It follows that

$$||u(x)|| \leq C \left(b_0^{-1}||f|| + \max\left(|\gamma_1|, |\gamma_2|\right)\right), \forall x \in [-1, 1],$$

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the notation $||.||$ stands for the maximum norm, and b_0 a positive real number, as specified in the introduction above.

Proof. Given the comparison function

$$\Pi^{\pm}(x) = b_0^{-1}||f|| + \max\left(|\gamma_1|, |\gamma_2|\right) \pm u(x),$$

where b_0, is a positive number such that $b(x) \geq b_0 > 0$ to ensure the uniqueness of the solution to the problem (4.1.3)-(4.1.4), $\gamma_1 = u(-1)$ and $\gamma_2 = u(1)$.

We have $\Pi^{\pm}(-1) \geq 0$, $\Pi^{\pm}(1) \geq 0$, and

$$L\Pi^{\pm}(x) = -\frac{b(x)}{b_0}||f|| - b(x)\max\left(|\gamma_1|, |\gamma_2|\right) \pm Lu(x) \leq 0,$$

which in virtue of Lemma 4.2.1 above, implies that

$$\Pi^{\pm}(x) \geq 0, \forall x \in [-1, 1].$$

This ends the proof.

The following Lemma focuses on the Inverse Monotonicity.

Lemma 4.2.3. *[51] Let $F(x, u, u') = a(x)u' - b(x)u - f(x)$ be a smooth function in $[-1, 1] \times \mathbb{R}^2$, where $a(x), b(x), f(x)$ are functions as described in (4.1.3)-(4.1.4). The problem (4.1.3)-(4.1.4) is said to be inverse monotone for $F(x, u, u') \in C^2\left((-1, 1)\right) \cap C\left([-1, 1]\right)$ if one of the following conditions imposed on F is satisfied:*

(1) $F(x, u, u')$ is strictly increasing in u, i.e., $F(x, u_1, z) < F(x, u_2, z)$ if $u_1 < u_2$,

(2) $F(x, u, u')$ is weakly increasing in u and there exists a positive constant $C > 0$, such that $|F(x, u, z_1) - F(x, u, z_1)| \leq C\,|z_1 - z_2|$.

Proof. See [51] with $d(x) = x^2,\ l = 1,\ \forall x \in [-1, 1]$.

Throughout this chapter we consider the following partition of $\bar{\Omega} = [-1, 1]$:

$\Omega_L = [-1, -\delta),\ \ \Omega_C = [-\delta, \delta], \Omega_R = (\delta, 1]$,where $0 < \delta \leq 1/2$, respectively the left side of the layer region, the layer region and the right side of the layer region. Also, $\Omega_C = \Omega_C^- \cup \Omega_C^+$, with $\Omega_C^- = [-\delta, 0)$ and $\Omega_C^+ = [0, \delta]$.

The following Lemmas deal with the appropriates bounds on the derivatives of the solution to the problem (4.1.3)-(4.1.4), where x is either in Ω_L, in Ω_C or in Ω_R.

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Lemma 4.2.4. *Let $u(x)$ be the solution to the problem (4.1.3)-(4.1.4), we have*
$|u^{(j)}| \leq C, \forall x \in \Omega_L U \Omega_R,$
where C is a positive real number, independent of the singular perturbation ε but depending on δ.

Proof. This Lemma is the immediate consequence of Theorem 4.2.3 for the inverse monotonicity with $C = M$ as specified in [51]:
$\forall x \in \Omega_R$, $F[x, -M, u'] \leq F[x, u, u'] \leq F[x, M, u']$ leading to $-M \leq u(x) \leq M$.
This completes the proof. Similarly, we can prove this Lemma for $x \in \Omega_L$.

In the Lemma 4.2.5 below, we discuss the bounds on the solution of the problem (4.1.3)-(4.1.4) and its derivatives in the layer region. Thereupon, we follow Liseikin [51] work to adapt it to our problem. In this Lemma, the convection coefficient at a specific point x_0 is given by $a(x_0) = a$ depending on whether $x_0 \in \Omega_C^+$ or $x_0 \in \Omega_C^-$.

Lemma 4.2.5. *[51] Consider $u(x)$ the solution to (4.1.3)-(4.1.4). Then, it follows that*

1) for $x \in \Omega_C^+$ and $x_0 \in \Omega_C^+$ such that $a(x_0) = a > 0$, $j = 0, 1, 2, 3, 4$; we have

$$\left|u^{(j)}(x)\right| \leq M \begin{cases} 1 + (\varepsilon + x^2)^{1-a-j}, \; if \; 0 < a < 1, \\ \\ 1 + (\varepsilon + x^2)^{-j}, \; if \; a = 1, \\ \\ 1 + \varepsilon^{a-1} (\varepsilon + x^2)^{1-a-j}, \; if \; a > 1. \end{cases} \tag{4.2.1}$$

2) For $x \in \Omega_C^-$, and $x_0 \in \Omega_C^-$, $a(x_0) = a \leq 0, j = 0, 1, 2, 3, 4$; and p a whole number such that $a + p \geq 0, a + p - 1 < 0$; we have

$$\left|u^{(j)}(x)\right| \leq M \begin{cases} 1, \; if \; a < 0, j \leq p, \\ \\ 1 + (\varepsilon + x^2)^{1-j-p} \left|\arctan\left(\frac{x}{\sqrt{(\varepsilon)}}\right)\right|, \; if \; a + p = 0, \; j > p, \\ \\ 1 + (\varepsilon + x^2)^{-a-j}, \; if \; a + p > 1, \; j > p, \end{cases} \tag{4.2.2}$$

where M is a positive constant independent of ε.

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Proof.

1) We first prove Lemma 4.2.5 for $x \in \Omega_C^+$, also we consider $x_0 \in \Omega_C^+$ such that $a(x_0) = a > 0$. Let u be the solution to (4.1.3)-(4.1.4). From the inverse monotonicity Lemma 4.2.3 above, we have

$$|u(x)| \leq M. \tag{4.2.3}$$

Also, according to Liseikin [51], there exists a positive constant m such that (4.1.3)-(4.1.4) and 4.2.3 lead to

$$\left|u^{(j)}(x)\right| \leq M \begin{cases} 1, \; 0 < m \leq x \leq \delta, \\ \\ \varepsilon^{-j}, \; 0 \leq x \leq \delta, \end{cases} \tag{4.2.4}$$

$$j = 1, 2, 3, 4.$$

Supposed that $a > 0$. We can rewrite (4.1.3) as follows

$$u''(x) = -\frac{a(x)u'(x)}{\varepsilon + x^2} + \frac{b(x)u(x) + f(x)}{\varepsilon + x^2},$$

or

$$u'(x) = -\int_0^x \frac{a(\eta)u'(\eta)}{\varepsilon + \eta^2} d\eta + \int_0^x \frac{b(\eta)u(\eta) + f(\eta)}{\varepsilon + x^2} d\eta.$$

This derivative can be expressed by the following formula:

$$u'(x) = u'(0)\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^a \exp\left[-g_1(x)\right] + g_2(x), 0 \leq x \leq \delta, \tag{4.2.5}$$

where

$$g_1(x) = \int_0^x \frac{a(\eta)}{\varepsilon + \eta^2} d\eta,$$

and the integration by parts leads to

$$g_1(x) = \frac{a(x)}{\sqrt{\varepsilon}} \arctan\left(\frac{x}{\sqrt{\varepsilon}}\right) + \frac{1}{\sqrt{\varepsilon}} \int_0^x a'(\eta) \arctan\left(\frac{\eta}{\sqrt{\varepsilon}}\right) d\eta,$$

with $g_1(0) = 0$, since $a(0) = 0$. We also have

$$g_2(x) = (\varepsilon + x^2)^{-a} \int_0^x [b(\eta)u(\eta) + f(\eta)] \left(\varepsilon + \eta^2\right)^{a-1} \exp[g_1(\eta) - g_1(x)] d\eta.$$

From (4.2.4) with $a > 0$, we have

$$|g_j(x)| \leq M, \; j = 1, 2; \; 0 < x \leq \delta.$$

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Applying triangular inequalities, (4.2.5) leads to

$$\begin{aligned}
|u'(x)| &\leq \left| u'(0) \left(\frac{\varepsilon}{\varepsilon + x^2} \right)^a \exp\left[-g_1(x)\right] \right| + |g_2(x)|, \\
|u'(x)| &\leq M \left|u'(0)\right| \left(\frac{\varepsilon}{\varepsilon + x^2} \right)^a + M, \\
|u'(x)| &\leq M \left[1 + |u'(0)| \left(\frac{\varepsilon}{\varepsilon + x^2} \right)^a \right], 0 < x \leq \delta. \qquad (4.2.6)
\end{aligned}$$

Considering $0 < a < 1$, $0 < \varepsilon << 1$, and a positive constant m, with $x = m$ such that (4.2.4) and (4.2.5) lead to

$$|u'(0)| \left(\frac{\varepsilon}{\varepsilon + m^2} \right)^a \leq M,$$

or

$$|u'(0)| \leq M \left(\frac{\varepsilon + m^2}{\varepsilon} \right)^a \leq M\varepsilon^{-a}.$$

Thus (4.2.6) leads to

$$|u'(x)| \leq M \left[1 + \varepsilon^{-a}\varepsilon^a \left(\varepsilon + x^2 \right)^{-a} \right],$$

giving

$$|u'(x)| \leq M \left[1 + \left(\varepsilon + x^2 \right)^{-a} \right], \; 0 < a < 1, \; 0 < x \leq \delta.$$

Also, from (4.1.3) we have the following

$$u'''(x) = -\frac{[2x + a(x)]u''(x)}{\varepsilon + x^2} + \frac{[-a'(x)u'(x) - b(x)]u'(x) + b'(x)u(x) + f'(x)}{\varepsilon + x^2}, \qquad (4.2.7)$$

or

$$u''(x) = -\int_0^x \frac{[2\eta + a(\eta)]u''(\eta)}{\varepsilon + \eta^2} d\eta + \int_0^x \frac{[-a'(\eta)u'(\eta) + b(\eta)u'(\eta) + b'(\eta)u(\eta) + f'(\eta)}{\varepsilon + \eta^2} d\eta.$$

This derivative can also be expressed by the following formula:

$$u''(x) = u''(0) \left(\frac{\varepsilon}{\varepsilon + x^2} \right)^{a+1} \exp\left[-g_3(x)\right] + g_4(x), 0 \leq x \leq \delta, \qquad (4.2.8)$$

where

$$g_3(x) = \int_0^x \frac{[2\eta + a(\eta)]}{\varepsilon + \eta^2} d\eta,$$

and the integration by parts leads to

$$g_3(x) = \frac{(2x + a(x))}{\sqrt{\varepsilon}} \arctan\left(\frac{x}{\sqrt{\varepsilon}} \right) + \int_0^x \frac{(2 + a'(\eta))}{\sqrt{\varepsilon}} \arctan\left(\frac{\eta}{\sqrt{\varepsilon}} \right) d\eta,$$

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with $g_3(0) = 0$, since $a(0) = 0$. We also have

$$g_4(x) = (\varepsilon + x^2)^{-a-1} \int_0^x [-a'(\eta)u'(\eta) + b(\eta)u'(\eta) + b'(\eta)u(\eta) \\ + f'(\eta)] \left(\varepsilon + \eta^2\right)^a \exp[g_3(\eta) - g_3(x)]d\eta.$$

From (4.2.4) with $a > 0$, we have

$$|g_3(x)| \leq M, \ |g_4(x)| \leq M \ \ 0 < x \leq \delta.$$

The triangular inequality applied to (4.2.8) leads to

$$\begin{aligned} |u''(x)| &\leq \left| u''(0) \left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1} \exp\left[-g_3(x)\right] \right| + |g_4(x)|, \\ |u''(x)| &\leq M \, |u''(0)| \left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1} + M, \\ |u''(x)| &\leq M \left[1 + |u''(0)| \left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1}\right], 0 < x \leq \delta. \end{aligned} \tag{4.2.9}$$

Also, let us consider $0 < a < 1$, $0 < \varepsilon << 1$, and m' a positive constant, with $x = m'$ such that (4.2.4) and (4.2.8) lead to

$$|u''(0)| \left(\frac{\varepsilon}{\varepsilon + m'^2}\right)^{a+1} \leq M,$$

$$i.e., |u''(0)| \leq M \left(\frac{\varepsilon + m'^2}{\varepsilon}\right)^{a+1} \leq M\varepsilon^{-a-1}.$$

Thus from (4.2.11) it follows that

$$|u''(x)| \leq M \left[1 + \varepsilon^{-a-1}\varepsilon^{a+1} \left(\varepsilon + x^2\right)^{-a-1}\right],$$

leading to

$$|u''(x)| \leq M \left[1 + \left(\varepsilon + x^2\right)^{-a-1}\right], \ 0 < a < 1, \ 0 < x \leq \delta.$$

Thereafter, from (4.1.3) and (4.2.3), we come to the following result for $0 < a < 1$, $0 < x \leq \delta$;

$$\left|u^{(j)}(x)\right| \leq M \left[1 + \left(\varepsilon + x^2\right)^{-a+1-j}\right], \ 0 < \varepsilon << 1, j = 1, 2, 3, 4. \tag{4.2.10}$$

Lemma 4.2.5 is fulfilled for $0 < a < 1$.

If $a = 1$, the integration of (4.2.5) from 0 to δ leads to

$$\int_0^\delta u'(\eta)d\eta = \int_0^\delta u'(0) \left[\frac{\varepsilon}{\varepsilon + \eta^2}\right] \exp\left[-g_1(\eta)\right] d\eta + \int_0^\delta g_2(\eta)d\eta, 0 \leq x \leq \delta.$$

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Integrating by parts leads to

$$\begin{aligned} A_{\delta_1} - A_0 &= \frac{u'(0)}{\sqrt{\varepsilon}}[\arctan\left(\frac{\delta}{\sqrt{\varepsilon}}\right)\exp[-g_1(\delta)] \\ &\quad + \int_0^\delta \arctan\left(\frac{\eta}{\sqrt{\varepsilon}}\right) g_1'(\eta)\exp[-g_1(\eta)]d\eta] + \int_0^\delta g_2(\eta)d\eta, \\ or\, A_{\delta_1} - A_0 &= \frac{u'(0)}{\sqrt{\varepsilon}}[\arctan\left(\frac{\delta}{\sqrt{\varepsilon}}\right)\exp[-g_1(\delta)] \\ &\quad - \int_0^\delta \arctan\left(\frac{\eta}{\sqrt{\varepsilon}}\right)[2\eta + a(\eta)](\varepsilon+\eta^2)^{-1}\exp[-g_1(\eta)]d\eta] + \int_0^\delta g_2(\eta)d\eta. \end{aligned}$$

We know that

$$\left|\arctan\left(\frac{\delta}{\sqrt{\varepsilon}}\right)\exp[-g_1(\delta)] - \int_0^\delta \arctan\left(\frac{\eta}{\sqrt{\varepsilon}}\right)[2\eta+1](\varepsilon+\eta^2)^{-1}\exp[-g_1(\eta)]d\eta\right| \leq M,$$

then,

$$\sqrt{\varepsilon}\,|u'(0)| \leq M,$$

meaning

$$|u'(0)| \leq M\varepsilon^{-\frac{1}{2}},$$

and (4.2.6) leads to

$$|u'(x)| \leq M\left[1 + (\varepsilon + x^2)^{-1}\right], 0 \leq x \leq \delta,\ 0 < \varepsilon << 1,\ a = 1.$$

Thereafter, after differentiating (4.1.3) and using (4.2.3), we come to the following result with $a = 1$ and $0 < x \leq \delta$

$$\left|u^{(j)}(x)\right| \leq M\left[1 + \left(\varepsilon + x^2\right)^{-j}\right],\ 0 < \varepsilon << 1, j = 1,2,3,4. \tag{4.2.11}$$

For $a > 1$, (4.2.4) into (4.2.5) and using triangular inequality; we get the following

$$|u'(x)| \leq M\varepsilon^{-1}\left[\frac{\varepsilon}{\varepsilon + x^2}\right]^a + M,$$

meaning

$$|u'(x)| \leq M\left[1 + \varepsilon^{a-1}(\varepsilon + x^2)^{-a}\right],$$

and using (4.1.3), we come to the same derivative as specified in (4.2.8)

$$u''(x) = u''(0)\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1}\exp\left[-g_1(x)\right] + g_2(x), 0 \leq x \leq \delta, \tag{4.2.12}$$

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and the triangular inequality of (4.2.12) in connection with (4.1.3) and (4.2.3) leads to

$$|u''(x)| \leq M\,|u''(0)|\,.\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1} + M,$$

or

$$|u''(x)| \leq M\varepsilon^{-2}\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1} + M,$$

meaning that

$$|u''(x)| \leq M\left[1 + \varepsilon^{-2}\varepsilon^{a+1}\left(\varepsilon + x^2\right)^{-a-1}\right],$$

or

$$|u''(x)| \leq M\left[1 + \varepsilon^{a-1}\left(\varepsilon + x^2\right)^{-a-1}\right].$$

Thereafter from (4.1.3) and (4.2.3) we conclude that

$$\left|u^{(j)}(x)\right| \leq M\left[1 + \varepsilon^{a-1}(\varepsilon + x^2)^{1-a-j}\right],\ 0 < x \leq \delta,\ 0 < \varepsilon << 1, a > 1,\ j = 1,2,3,4, \tag{4.2.13}$$

which ends the proof for $x \in \Omega_C^+$ and $a > 0$.

2) Consider $x \in\ \Omega_C^- = [-\delta, 0]$, and suppose there exists a constant $x_0 \in \Omega_C^-$ such that $a(x_0) = a \leq 0$.
Then, solving (4.1.3) - (4.1.4) with respect to $u'(x)$ leads to

$$u'(x) = u'(x_0)\exp(\psi(x)) + \int_{x_0}^{x} \frac{[b(\eta)u(\eta) + f(\eta)]}{\varepsilon + \eta^2}\exp\left[\psi(\eta)\right]d\eta, \tag{4.2.14}$$

or

$$u'(x) = u'(x_0)\exp(\psi(x)) + (\varepsilon + x^2)^{-p}\int_{x_0}^{x}[b(\eta)u(\eta) + f(\eta)](\varepsilon + \eta^2)^{p-1}\exp\left[\psi(\eta)\right]d\eta, \tag{4.2.15}$$

with $\psi(x)$ given by

$$\psi(x) = \int_{x_0}^{x} \frac{a(\eta)}{\varepsilon + \eta^2}d\eta.$$

It is clear that

$$|\psi(x)| \leq M, -\delta \leq x \leq 0;\ 0 < \varepsilon << 1.$$

Given $x_0 \in\ [-\delta, 0]$, using (4.2.4), $|u'(x_0)| \leq M$, and applying triangular inequality, we come to the following

$$|u'(x)| \leq M\left[1 + \arctan\left(\frac{x}{\sqrt{\varepsilon}}\right)\right].$$

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This proves Lemma 4.2.5 for $j = 1, a(0) = a = 0, p = 0$.
From (4.1.3) - (4.1.4), (4.2.7), (4.2.4) and for $j = 2, a(0) = a = 0, p = 0$; we can easily show that

$$|u''(x)| \leq M\left[1 + (\varepsilon + x^2)^{-1}\arctan\left(\frac{x}{\sqrt{\varepsilon}}\right)\right].$$

From (4.1.3) and (4.2.4), we come to the following result, with $a + p = 0, j > p$

$$\left|u^{(j)}(x)\right| \leq M\left[1 + \left(\varepsilon + x^2\right)^{1-j-p}\arctan\left(\frac{x}{\sqrt{\varepsilon}}\right)\right],\ 0 < \varepsilon << 1, j = 1, 2, 3, 4. \tag{4.2.16}$$

Now, let $x \in [-\delta, 0],\ 0 < \varepsilon << 1,\ p \geq 1, a(x) < 0,\ m_3$ be a positive constant given by $-\delta \leq m_3 \leq x \leq 0$, such that

$$\psi(x) \leq -m_3 \ln\left(\frac{\varepsilon + \eta^2}{\varepsilon + x^2}\right), -\delta \leq m_3 \leq \eta \leq x \leq 0.$$

It follows that

$$\exp\left[\psi(x)\right] \leq M\left(\frac{\varepsilon + x^2}{\varepsilon + \eta^2}\right)^{m_3}, -\delta \leq m_3 \leq \eta \leq x \leq 0.$$

Using (4.2.4) and letting $x_0 = m_3$; (4.2.14) leads to

$$|u'(x)| \leq M, -\delta \leq m_3 \leq x \leq 0;$$

which gives the proof for $j = 1 \leq p, a(x) = a < 0$, also, using (4.2.4) and letting $x_0 = m_3$, we can easily show that

$$|u'(x)| \leq M, -\delta \leq m_3 \leq x \leq 0, j = 2 \leq p, a < 0.$$

Form (4.1.3) - (4.1.4); we conclude that

$$\left|u^{(j)}(x)\right| \leq M, -\delta \leq x \leq 0,\ j \leq p, a(x) = a < 0,\ j = 1, 2, 3, 4. \tag{4.2.17}$$

Finally, let $x \in [-\delta, 0], 0 < \varepsilon << 1, a(x) = a < 0$. We can defined a formula for the first derivative of the problem (4.1.3) - (4.1.4) similar to (4.2.5) as follows

$$u'(x) = u'(0)\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1}\exp\left[-g_1(x)\right] + g_2(x), -\delta \leq x \leq 0, \tag{4.2.18}$$

with g_1 and g_2 as specified in (4.2.5).

Applying triangular inequality and following the same process as (4.2.5), we get

$$|u'(x)| \leq M\left[1 + (\varepsilon + x^2)^{-a-1}\right], -\delta \leq x \leq 0, j = 1, a(x) = a < 0.$$

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We also defined the formula of the second derivative in connection with (4.2.8) as

$$u''(x) = u''(0)\left(\frac{\varepsilon}{\varepsilon+x^2}\right)^{a+2}\exp\left[-g_5(x)\right] + g_6(x), -\delta \le x \le 0, \tag{4.2.19}$$

where g_5 and g_6 are obtained after solving the differentiation of the equation (4.1.3) with respect to x. In the same manner, applying triangular inequality, we come to the following:

$$|u''(x)| \le M\left[1 + (\varepsilon + x^2)^{-a-2}\right], -\delta \le x \le 0, j = 2, a(x) = a < 0.$$

Thereafter, from (4.1.3) - (4.1.4); we get

$$\left|u^{(j)}(x)\right| \le M\left(\varepsilon + x^2\right)^{-a-j}, -\delta \le x \le 0, a(x) = a < 0,\ j = 1, 2, 3, 4. \tag{4.2.20}$$

This complete the proof of Lemma 4.2.5 for $x \in \Omega_C^-$ and $a(x) \le 0$.

The next section describes the method used to solve the problem (4.1.3)-(4.1.4).

4.3 Construction of the FOFDM

Let n be a positive and even integer and Ω_n the partition on $[-1, 1]$ given by: $x_0 = -1; x_j = x_0 + jh; j = 1, ..., n-1, h = x_j - x_{j-1}, x_n = 1$.

The dicritization of (4.1.3)-(4.1.4) on Ω_n can be given by

$$L^hU_j := \begin{cases} (\varepsilon + x_j^2)\delta^2 U_j + \tilde{a}_j D^- U_j - \tilde{b}_j U_j = \tilde{f}_j, \\ j = 0, 1, 2, \cdots, \frac{n}{2} - 1, \\ \\ (\varepsilon + x_j^2)\delta^2 U_j + \tilde{a}_j D^+ U_j - \tilde{b}_j U_j = \tilde{f}_j, \\ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n-1, \end{cases} \tag{4.3.1}$$

$$U_0 = \gamma_1,\ U_n = \gamma_2, \tag{4.3.2}$$

with

$$D^-U_j = \frac{U_j - U_{j-1}}{h},\ \ D^+U_j = \frac{U_{j+1} - U_j}{h},\ \ \delta^2 U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{\tilde{\phi}^2}.$$

and

$$\tilde{\phi}_j^2 = \begin{cases} \frac{h(\varepsilon + x_j^2)}{\tilde{a_j}}\left[\exp\left(\frac{\tilde{a_j} h}{\varepsilon + x_j^2}\right) - 1\right],\ j = 0, 1, 2, ..., \frac{n}{2} - 1, \\ \\ \frac{h[\varepsilon + x_j^2)}{\tilde{a_j}}\left(1 - \exp\left(-\frac{\tilde{a_j} h}{\varepsilon + x_j^2}\right)\right],\ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n-1. \end{cases} \tag{4.3.3}$$

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We also use the following convention.

$$\left.\begin{aligned}
\tilde{a}_j &= \frac{a_j + a_{j-1}}{2} \text{ for } j = 0, 1, 2, ..., \frac{n}{2} - 1, \\
\tilde{a}_j &= \frac{a_j + a_{j+1}}{2} \text{ for } j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1, \\
\tilde{b}_j &= \frac{b_{j-1} + b_j + b_{j+1}}{3}; \tilde{f}_j = \frac{f_{j-1} + f_j + f_{j+1}}{3} \text{ for } j = 0, 1, 2, ..., n - 1.
\end{aligned}\right\} \tag{4.3.4}$$

The equation (4.3.1) can be rewritten as

$$\left.\begin{aligned}
r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = \tilde{f}_j,\ j = 0, 1, 2, ..., \tfrac{n}{2} - 1, \\
r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = \tilde{f}_j,\ j = \tfrac{n}{2}, \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, ..., n - 1
\end{aligned}\right\} \tag{4.3.5}$$

with

$$\left.\begin{aligned}
r_j^- &= \frac{\varepsilon + x_j^2}{\tilde{\phi}_j^2} - \frac{\tilde{a}_j}{h}; r_j^c = \frac{-2(\varepsilon + x_j^2)}{\tilde{\phi}_j^2} + \frac{\tilde{a}_j}{h} - \tilde{b}_j; r_j^+ = \frac{\varepsilon + x_j^2}{\tilde{\phi}_j^2}, j = 0, 1, ..., \frac{n}{2} - 1, \\
r_j^- &= \frac{\varepsilon + x_j^2}{\tilde{\phi}_j^2};\ r_j^c = \frac{-2(\varepsilon + x_j^2)}{\tilde{\phi}_j^2} - \frac{\tilde{a}_j}{h} - \tilde{b}_j;\ r_j^+ = \frac{\varepsilon + x_j^2}{\tilde{\phi}_j^2} + \frac{\tilde{a}_j}{h}, j = \frac{n}{2}, \frac{n}{2} + 1, ..., n - 1.
\end{aligned}\right\} \tag{4.3.6}$$

The fitted operator finite difference method (FOFDM) (4.3.5) - (4.3.2) satisfies the following lemmas:

Lemma 4.3.1. (Discrete minimum principle) . *Given a mesh function ξ_j with $\xi_0 \geq 0$, $\xi_n \geq 0$ and $L^n \xi_j \leq 0,\ \forall j = 1(1)n - 1$, it follows that $\xi_j \geq 0,\ \forall j = 0(1)n$.*

Proof. The proof of this Lemma is by contradiction.
Consider k such that $\xi_k = \min_{0 \leq j \leq n} \xi_j$ and $\xi_k < 0$. It is clear that $k \neq 0, n.$, $\xi_{k+1} - \xi_k \geqslant 0$, and $\xi_k - \xi_{k-1} \leq 0$. Then

$$L^n \xi_k = \begin{cases} (\varepsilon + x_k^2)\delta^2 \xi_k + a_k D^- \xi_k - b_k \xi_k > 0,\ a_k < 0,\ 1 \leq k \leq n/2 - 1, \\ -b_k \xi_k > 0, k = n/2,\ a_{n/2} = 0, \\ (\varepsilon + x_k^2)\delta^2 \xi_k + a_k D^+ \xi_k - b_k \xi_k > 0,\ a_k > 0,\ n/2 + 1 \leq k \leq n - 1. \end{cases} \tag{4.3.7}$$

We have $L^n \xi_k > 0,\ 1 \leq k \leq n - 1$, leading to a contradiction. Thereupon $\xi_j \geqslant 0, 1 \leq j \leq n$.

We use this minimum principle to prove the following Lemma

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Lemma 4.3.2. *(Uniform stability estimate) Consider Z_i a mesh function where $Z_0 = Z_n = 0$. We have*

$$|Z_i| \leq \frac{1}{b_0} \max_{1 \leq j \leq n-1} |L^n Z_j|, \text{ for } 0 \leq i \leq n,$$

with $b_i \geq b_0 > 0$, to ensure the uniqueness of the solution to the problem (4.3.1) - (4.3.2).

Proof. Consider two comparison functions $Y_i^{\pm}$

$$Y_i^{\pm} = \frac{1}{b_0} \max_{1 \leq j \leq n-1} |L^n Z_j| \pm Z_j, \text{ for } 0 \leq i \leq n,$$

with $b_i \geq b_0 > 0$, to guarantee the uniqueness of the solution to (4.3.1) - (4.3.2). We have $Y_0^{\pm} \geqslant 0$, $Y_n^{\pm} \geqslant 0$. And

$$L^n Y_i^{\pm} = \frac{-b_i}{b_0} \max_{1 \leq j \leq n-1} |L^n Z_j| \pm L^n Z_i, \text{ with } 0 \leq i \leq n.$$

For $0 \leq i \leq n, -b_i/(b_0) \leq -1$.
We also have $L^n Y_i^{\pm} \leq 0$. Lemma 4.3.1 leads to $Y_i \leq 0, \forall\ 0 \leq i \leq n$ which completes the proof.

Section 4.4 below is devoted to convergence analysis of the FOFDM.

4.4 Convergence analysis of the FOFDM

In this section, we focus on the convergence analysis of the FOFDM developed in section 3. The analysis focuses on the left part of the interval, viz $[-1, 0)$, and on the right part $[0, 1]$ we can do it similarly. Consider the truncation error on the interval [-1, 0), given by:

$$\begin{aligned}
L^n(U_j - u_j) &= L^n U_j - L^n u_j, \\
&= \tilde{f}_j - \left[\frac{\varepsilon + x_j^2}{\tilde{\phi}_j^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{\tilde{a}_j}{h}(u_{j+1} - u_j) - \tilde{b}_j u_j\right], \\
&= \frac{1}{3}\left[(\varepsilon + x_j^2)u''_{j+1} + a_{j+1}u'_{j+1} - b_{j+1}u_{j+1}\right] + \frac{1}{3}[(\varepsilon + x^2)u''_j + a_j u'_j - b_j u_j] \\
&\quad + \frac{1}{3}[(\varepsilon + x_j^2)u''_{j-1} + a_{j-1}u'_{j-1} - b_{j-1}u_{j-1}] \\
&\quad - \left[\frac{\varepsilon + x_j^2}{\tilde{\phi}_j^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{\tilde{a}_j}{h}(u_{j+1} - u_j) - \tilde{b}_j u_j\right].
\end{aligned}$$

To develop the truncation error above, we consider $\tilde{f}_j = (f_{j+1} + f_j + f_{j-1})/3$ as given in (4.3.4); the expression of $\tilde{a}_j$, $\tilde{b}_j$ as suggested in (4.3.4), the Taylor expansions of u_{j+1}, u_{j-1}

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a_{j+1}, a_{j-1}, b_{j+1}, b_{j-1}, u'_{j+1}, u'_{j-1}, u''_{j+1}, u''_{j-1} and the truncated Taylor expansion of $\frac{1}{\tilde{\phi}_j^2}$ up to order four, gives

$$\begin{aligned}
L^n\left(U_j - u_j\right) \;=\;& \left[-\frac{h^4 b^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{h^4 b^{(iv)}\left(\xi_{9_j}\right)}{72} + \frac{h^4 b^{(iv)}\left(\xi_{15_j}\right)}{72} + \frac{h^4 b^{(iv)}\left(\xi_{16_j}\right)}{72}\right] u_j \\
&+ \left[-h a'_j - \frac{h^2 a''_j}{6} - \frac{h^3 a'''_j}{6} + \frac{h^4 a^{(iv)}(\xi_{2_j})}{72} + \frac{h^4 a^{(iv)}\left(\xi_{7_j}\right)}{72} - \frac{h^4 a^{(iv)}\left(\xi_{13_j}\right)}{24}\right. \\
&\left. - \frac{2h^2 b'_j}{3} - \frac{h^4 b'''_j}{9} - \frac{h^5 b^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{h^5 b^{(iv)}\left(\xi_{9_j}\right)}{72}\right] u'_j \\
&+ \left[\frac{h a_j}{2} + \frac{7h^2 a'_j}{6} + \frac{h^3 a''_j}{4} + \frac{7h^2 a'''_j}{36} + \frac{h^5 a^{(iv)}(\xi_{5_j})}{3} - \frac{h^5 a^{(iv)}(\xi_{6_j})}{72}\right. \\
&- \frac{h^5 a^{(iv)}(\xi_{2_j})}{72} - \frac{h^5 a^{(iv)}(\xi_{7_j})}{72} + \frac{h^5 a^{(iv)}(\xi_{13_j})}{48} - \frac{h^2 b_j}{3} - \frac{h^4 b''_j}{6} \\
&\left. - \frac{h^6 b^{(iv)}(\xi_{4_j})}{144} - \frac{h^6 b^{(iv)}(\xi_{9_j})}{144}\right] u''_j \\
&+ \left[\frac{h^2 a_j}{6} - \frac{h^3 a'_j}{6} - \frac{h^4 a''_j}{12} - \frac{h^5 a'''_j}{36} + \frac{h^6 a^{(iv)}(\xi_{2_j})}{144} + \frac{h^6 a^{(iv)}(\xi_{7_j})}{144}\right. \\
&\left. - \frac{h^6 a^{(iv)}(\xi_{13_j})}{144} + \frac{h^3 b_j}{18} - \frac{h^3 b'_j}{18} + \frac{h^5 b''_j}{36} - \frac{h^6 b'''_j}{108} + \frac{h^7 b^{(iv)}(\xi_{9_j})}{432}\right] u'''_j \\
&+ \kappa\left(\varepsilon, h^2, h^3, \cdots, h^7, a_j, a'_j, \cdots, a_j^{(iv)}, b_j, b'_j, \cdots, b_j^{(iv)}\right) u^{(iv)}(\xi_{*_j}),
\end{aligned} \tag{4.4.8}$$

with κ a function of its arguments and the ξ's lie in (x_{j-1}, x_{j+1}). The coefficients of $u_j, u'_j, \cdots, u^{(iv)}(\xi_{*_j})$ can be bounded by a constant.

Equation (4.4.8) above can be rewritten as follows

$$L^n\left(U_j - u_j\right) = M_1 h + R_n(x_j),\ \forall j = 1(1)\frac{n}{2} - 1, \tag{4.4.9}$$

where

$$\begin{aligned}
M_1 &= -a'_j u'_j + \frac{a_j}{2} u''_j \\
R_n(x_j) &= h^2\left[\left(\frac{-a''_j}{6} - \frac{2b'_j}{3}\right) u'_j + \left(\frac{7a'_j}{6} + \frac{7a'''_j}{36} - \frac{b_j}{3}\right) u''_j + \frac{a_j}{6} u'''_j\right] \\
&+ h^3\left[\frac{-a'''_j}{6} u'_j + \frac{a''_j}{4} u''_j + \left(\frac{-a'_j}{6} + \frac{b_j}{18} - \frac{b'}{18}\right) u'''_j\right] \\
&+ h^4\left[\left(\frac{-b_j^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{b^{(iv)}\left(\xi_{9_j}\right)}{72} + \frac{b^{(iv)}\left(\xi_{15_j}\right)}{72} + \frac{b^{(iv)}\left(\xi_{15_j}\right)}{72}\right) u_j\right]
\end{aligned}$$

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$$+h^4\left[\left(\frac{a^{(iv)}\left(\xi_{2_j}\right)}{72}+\frac{a^{(iv)}\left(\xi_{7_j}\right)}{72}-\frac{a^{(iv)}\left(\xi_{13_j}\right)}{24}-\frac{b_j''}{9}\right)u_j'-\frac{b_j'''u_j''}{6}-\frac{a_j''}{12}u_j'''\right]$$

$$+h^5\left[\left(-\frac{b^{(iv)}\left(\xi_{4_j}\right)}{72}-\frac{b^{(iv)}\left(\xi_{9_j}\right)}{72}\right)u_j'\right]$$

$$+h^5\left[\left(\frac{a^{(iv)}\left(\xi_{5_j}\right)}{3}-\frac{a^{(iv)}\left(\xi_{6_j}\right)}{72}-\frac{a^{(iv)}\left(\xi_{2_j}\right)}{72}-\frac{a^{(iv)}\left(\xi_{7_j}\right)}{72}+\frac{a^{(iv)}\left(\xi_{13_j}\right)}{48}\right)u_j''+\left(\frac{-a'''}{36}+\frac{b_j''}{36}\right)u_j'''\right]$$

$$+h^6\left[\left(-\frac{b_j^{(iv)}\left(\xi_{4_j}\right)}{144}-\frac{b_j^{(iv)}\left(\xi_{9_j}\right)}{144}\right)u_j''+\left(\frac{a_j^{(iv)}\left(\xi_{2_j}\right)}{144}+\frac{a_j^{(iv)}\left(\xi_{7_j}\right)}{144}-\frac{a_j^{(iv)}\left(\xi_{13_j}\right)}{144}-\frac{b_j'''}{108}\right)u_j'''\right]$$

$$+h^7\left[\frac{b_j^{(iv)}\left(\xi_{9_j}\right)}{432}u_j'''\right]$$

$$+\kappa\left(\varepsilon,h^3,h^4,\cdots,h^7,a_j,a_j',\cdots,a_j^{(iv)},b_j,b_j',\cdots,b_j^{(iv)},u_j'',u^{(iv)}(\xi_{*_j})\right),$$

or

$$L^n\left(U_j-u_j\right)=\mathcal{O}(h),\ \forall j=1(1)\frac{n}{2}-1,$$

leading to

$$|L^n(U_j-u_j)|\le Ch,\forall j=1(1)\frac{n}{2}-1.$$

Similarly, we have

$$|L^n(U_j-u_j)|\le Ch,\forall j=\frac{n}{2}(1)n-1.$$

From Lemma 4.3.2 above, we come to the following main result of this work:

Theorem 4.4.1. *Consider u the solution of (4.1.3)-(4.1.4) and U the numerical solution approximation of u obtained via the FOFDM (4.3.1)-(4.3.2) then there exist a positive constant C independent of ε and h such that*

$$\sup_{0<\varepsilon\le 1}\max_{0\le j\le n}|u_j-U_j|\le Ch. \tag{4.4.10}$$

<u>**Remark 4.1:**</u> (This remark concerns chaters 2-7).

The theorem above provides the main result of the problem. We notice that, the first few steps in the proof of this result differ from one chapter to another, depending on the case studied. Afterwards, they line to the previous chapter(s) for both time in-dependent and time dependent problems.

Section 4.5 below deals with Richardson extrapolation as a technique used to improve the accuracy and the order of convergence of the estimates (4.4.10) above.

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4.5 Richardson extrapolation on the FOFDM

Richardson extrapolation is the acceleration technique used to improve the accuracy and the order of convergence of the fitted operator finite difference method designed. It is based on linear combination of k solutions, $k \geq 0$; corresponding to different and nested meshes.

Let us rewrite (4.4.9) as follows

$$L^n\left(U_j - u_j\right) = M_1 h + M_2 h^2 + R_n(x_j), \tag{4.5.1}$$

where

$$M_1 = -a'_j u'_j + \frac{a_j}{2} u''_j,$$

$$M_2 = \left(\frac{-a''_j}{6} - \frac{2b'_j}{3}\right) u'_j + \left(\frac{7a'_j}{6} + \frac{7a'''_j}{36} - \frac{b_j}{3}\right) u''_j + \frac{a_j}{6} u'''_j,$$

$$\begin{aligned}
R_n(x_j) = h^3 &\left[\frac{-a'''_j}{6} u'_j + \frac{a''_j}{4} u''_j + \left(\frac{-a'_j}{6} + \frac{b_j}{18} - \frac{b'}{18}\right) u'''_j\right] \\
+h^4 &\left[\left(\frac{-b_j^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{b^{(iv)}\left(\xi_{9_j}\right)}{72} + \frac{b^{(iv)}\left(\xi_{15_j}\right)}{72} + \frac{b^{(iv)}\left(\xi_{15_j}\right)}{72}\right) u_j\right] \\
+h^4 &\left[\left(\frac{a^{(iv)}\left(\xi_{2_j}\right)}{72} + \frac{a^{(iv)}\left(\xi_{7_j}\right)}{72} - \frac{a^{(iv)}\left(\xi_{13_j}\right)}{24} - \frac{b''_j}{9}\right) u'_j - \frac{b'''_j u''_j}{6} - \frac{a''_j}{12} u'''_j\right] \\
+h^5 &\left[\left(-\frac{b^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{b^{(iv)}\left(\xi_{9_j}\right)}{72}\right) u'_j\right] \\
+h^5 &\left[\left(\frac{a^{(iv)}\left(\xi_{5_j}\right)}{3} - \frac{a^{(iv)}\left(\xi_{6_j}\right)}{72} - \frac{a^{(iv)}\left(\xi_{2_j}\right)}{72} - \frac{a^{(iv)}\left(\xi_{7_j}\right)}{72} + \frac{a^{(iv)}\left(\xi_{13_j}\right)}{48}\right) u''_j + \left(\frac{-a'''}{36} + \frac{b''_j}{36}\right) u'''_j\right] \\
+h^6 &\left[\left(-\frac{b_j^{(iv)}\left(\xi_{4_j}\right)}{144} - \frac{b_j^{(iv)}\left(\xi_{9_j}\right)}{144}\right) u''_j + \left(\frac{a_j^{(iv)}\left(\xi_{2_j}\right)}{144} + \frac{a_j^{(iv)}\left(\xi_{7_j}\right)}{144} - \frac{a_j^{(iv)}\left(\xi_{13_j}\right)}{144} - \frac{b'''_j}{108}\right) u'''_j\right] \\
+h^7 &\left[\frac{b_j^{(iv)}\left(\xi_{9_j}\right)}{432} u'''_j\right] \\
&+\kappa\left(\varepsilon, h^3, h^4, \cdots, h^7, a_j, a'_j, \cdots, a_j^{(iv)}, b_j, b'_j, \cdots, b_j^{(iv)}, u''_j, u^{(iv)}(\xi_{*_j})\right).
\end{aligned}$$

We keep κ, ξ's and $u_j, u'_j, \cdots, u^{(iv)}(\xi_{*_j})$ the same as the ones described in (4.4.8). To start, consider μ_{2n} the mesh obtained by bisecting each mesh interval in μ_n, i.e.,

$$\mu_{2n} = \{\bar{x}_i\} \text{ with } \bar{x}_0 = -1, \;\; \bar{x}_n = 1 \;\text{ and }\; \bar{x}_j - \bar{x}_{j-1} = \bar{h} = h/2, \;\; j = 1, 2, ..., 2n.$$

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Consider $\bar{U}_j$ the numerical solution of (4.1.3)-(4.1.4) on μ_{2n}. Using $\bar{U}_j$ into the the equation (4.5.1) leads to

$$L^n\left(\bar{U}_j - \bar{u}_j\right) = M_1\bar{h} + M_2\bar{h}^2 + R_{2n}(\bar{x}_j), 1 \le j \le 2n-1. \tag{4.5.2}$$

The numerical solution $\bar{u}_j$, remains the same as u.
After multiplying (4.5.2) by 2, we get

$$2L^n\left(\bar{U}_j - \bar{u}_j\right) = 2M_1\bar{h} + 2M_2\bar{h}^2 + 2R_{2n}(\bar{x}_j), 1 \le j \le 2n-1, \tag{4.5.3}$$

meaning

$$L^n\left(2\bar{U}_j - 2\bar{u}_j\right) = 2M_1\bar{h} + 2M_2\bar{h}^2 + 2R_{2n}(\bar{x}_j), 1 \le j \le 2n-1. \tag{4.5.4}$$

Subtracting (4.5.4) from (4.5.1) yields

$$L^n\left(u_j - (2\bar{U}_j - U_j)\right) = \frac{M_2h^2}{2} + R_n - 2R_{2n}(\bar{x}_j), 1 \le j \le 2n-1, \tag{4.5.5}$$

or

$$L^n\left(u_j - (2\bar{U}_j - U_j)\right) = \mathcal{O}(h^2), 1 \le j \le 2n-1,$$

Let

$$U_j^{ext} := 2\bar{U}_j - U_j.$$

The numerical solution U_j^{ext} above is another numerical approximation of u_j.
Using Lemma 4.3.2, we come to the following result.

Theorem 4.5.1. *Consider U_j^{ext} the numerical solution of (4.1.3)-(4.1.4) obtained via the Richardson extrapolation based on FOFDM (4.3.1)-(4.3.2). Then there exists a positive constant M independent of ε and h such that*

$$\sup_{0<\varepsilon\le 1} \max_{1\le j\le 2n} \left|u_j - U_j^{ext}\right| \le Mh^2. \tag{4.5.6}$$

Section 4.6 below deals with two numerical examples to confirm its accuracy and robustness of the scheme.

4.6 Numerical examples

In this section we deal with numerical results obtained in the integration of some problems of type (4.1.3)- (4.1.4).

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Example 4.6.1. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{aligned}(\varepsilon + x^2)u'' + 2xu' - 2u = x^2\\ u(-1) = u(1) = 1\end{aligned}\right\}$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$ near $x = 0$. The exact solution is

$$u(x) = -\frac{1}{4}\frac{\left(x\sqrt{\varepsilon}\arctan\left(\frac{x}{\sqrt{\varepsilon}}\right) + \varepsilon\right)(-3+\varepsilon)}{\sqrt{\varepsilon}\arctan\left(\frac{1}{\sqrt{\varepsilon}}\right) + \varepsilon} + \frac{x^2}{4} + \frac{\varepsilon}{4},$$

Example 4.6.2. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{aligned}(\varepsilon + x^2)u'' + xu' - u = 1 + x^2\\ u(-1) = u(1) = 1\end{aligned}\right\}$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$ near $x = 0$. And the exact solution is given by

$$u(x) = -\frac{1}{3}\frac{\sqrt{x^2+\varepsilon}(2\varepsilon - 5)}{\sqrt{1+\varepsilon}} + \frac{1}{3}x^2 - 1 + \frac{2}{3}\varepsilon$$

The formula of the maximum errors at all mesh points and the numerical rates of convergence before extrapolation are given by

$$E_{\varepsilon,n} := \max_{0\le j\le n} |u_j - U_j| \;\text{ and }\; r_k \equiv r_{\varepsilon,k} := \log_2\left(\tilde{e}_{n_k}/\tilde{e}_{2n_k}\right), k = 1, 2, ...$$

where $\tilde{e}_n$ stands for $E_{\varepsilon,n}$. The calculation of E_n is as follows $E_n = \max_{0<\varepsilon\le 1} E_{\varepsilon,n}$.

Also for a fixed mesh, the maximum nodal errors remain constant for small values of ε (see tables 4.1 and 4.5). Notice, the results in tables 4.3 and 4.7 show that the method we derived is first order convergent.

Lastly, the computation of the maximum errors at all mesh points and the numerical rates of convergence after extrapolation also requires the following formulas

$$E^{ext}_{\varepsilon,n} := \max_{0\le j\le 2n} |u_j - U^{ext}_j| \;\text{ and }\; R_k \equiv R_{\varepsilon,k} := \log_2\left(E^{ext}_{n_k}/E^{ext}_{2n_k}\right), k = 1, 2, ...$$

Where $E^{ext}_{n_k}$ stands for $E_{\varepsilon,2n}$.

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Table 4.1: Maximum errors for Example 4.6.1 (before extrapolation)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$
10^{-2}	3.57E-02	1.91E-02	9.76E-03	4.94E-03	2.48E-03	1.24E-03
10^{-4}	4.20E-02	2.27E-02	1.16E-02	5.84E-03	2.93E-03	1.47E-03
10^{-10}	4.31E-02	2.27E-02	1.17E-02	5.90E-03	2.96E-03	1.49E-03
10^{-11}	4.31E-02	2.27E-02	1.17E-02	5.90E-03	2.96E-03	1.49E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	4.31E-02	2.27E-02	1.17E-02	5.90E-03	2.96E-03	1.49E-03

Table 4.2: Maximum errors for Example 4.6.1 (after extrapolation)

ε	$n=16$	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$
10^{-2}	9.95E-03	2.86E-03	5.09E-04	9.75E-05	2.07E-05	4.73E-06
10^{-4}	2.57E-03	4.30E-03	3.23E-03	5.78E-04	3.89E-04	6.48E-05
10^{-10}	2.22E-03	5.63E-04	1.48E-04	3.79E-05	9.62E-06	4.45E-06
10^{-11}	2.22E-03	5.63E-04	1.48E-04	3.79E-05	9.62E-06	2.42E-06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	2.22E-03	5.63E-04	1.48E-04	3.79E-05	9.62E-06	2.42E-06

Table 4.3: Rates of convergence for Example 4.6.1 (before extrapolation, $n_k = 16, 32, 64, 128, 256, 512$)

ε	r_1	r_2	r_3	r_4	r_5
10^{-2}	0.91	0.97	0.98	0.99	1.00
10^{-4}	0.89	0.97	0.99	0.99	1.00
10^{-10}	0.92	0.96	0.99	0.99	1.00
10^{-11}	0.92	0.96	0.99	0.99	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	0.92	0.96	0.99	0.99	1.00

Table 4.4: Rates of convergence for Example 4.6.1 (after extrapolation, $n_k = 16, 32, 64, 128, 256, 512$)

ε	r_1	r_2	r_3	r_4	r_5
10^{-2}	1.80	2.49	2.38	2.24	2.13
10^{-4}	-0.74	0.41	2.48	0.57	2.59
10^{-10}	1.98	1.93	1.96	1.98	1.11
10^{-11}	1.98	1.93	1.96	1.98	1.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.98	1.93	1.96	1.98	1.99

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Remark 4.2: We notice a notable deviation of computed rates of convergence for Richardson extrapolation from the theoretical one for certains values of ε and n. This is due to the fact that the denominator function computed as part of Richardson extrapolation algorithms for a specific value of n is not similar to the one obtained by the algorithm before extrapolation for $2n$. This is one of the issues researchers have encountered as far as Richardson extrapolation (see e.g. [15, 82, 91]).

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Table 4.5: Maximum errors for Example 4.6.2 (before extrapolation)

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
10^{-2}	5.80E-02	3.16E-02	1.64E-02	8.33E-03	4.20E-03	2.10E-03
10^{-4}	8.42E-02	4.62E-02	2.32E-02	1.16E-02	5.86E-03	2.93E-03
10^{-6}	9.09E-02	4.76E-02	2.50E-02	1.28E-02	6.39E-03	3.19E-03
10^{-11}	9.25E-02	4.93E-02	2.54E-02	1.29E-02	6.47E-03	3.25E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	9.25E-02	4.93E-02	2.54E-02	1.29E-02	6.47E-03	3.25E-03

Table 4.6: Maximum errors for Example 4.6.2 (after extrapolation)

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
10^{-2}	2.41E-02	4.89E-03	9.21E-04	1.93E-04	4.36E-05	1.03E-05
10^{-4}	1.07E-02	1.47E-02	3.54E-03	2.68E-03	6.69E-04	1.07E-04
10^{-6}	5.67E-03	1.50E-03	1.31E-03	1.58E-03	1.64E-03	9.13E-04
10^{-11}	6.02E-03	1.52E-03	3.86E-04	9.77E-05	2.46E-05	6.17E-06
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	6.03E-03	1.52E-03	3.86E-04	9.77E-05	2.46E-05	6.17E-06

Table 4.7: Rates of convergence for Example 4.6.2 (before extrapolation, $n_k = 16, 32, 64, 128, 256, 512$)

ε	r_1	r_2	r_3	r_4	r_5
10^{-2}	0.88	0.95	0.98	0.99	1.00
10^{-4}	0.86	0.99	1.00	0.99	1.00
10^{-6}	0.93	0.93	0.97	1.00	1.00
10^{-11}	0.91	0.96	0.98	0.99	0.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	0.91	0.96	0.98	0.99	1.00

Table 4.8: Rates of convergence for Example 4.6.2 (after extrapolation, $n_k = 16, 32, 64, 128, 256, 512$)

ε	r_1	r_2	r_3	r_4	r_5
10^{-2}	2.30	2.41	2.26	2.14	2.08
10^{-4}	-0.46	2.05	0.40	2.00	2.64
10^{-6}	1.92	0.20	-0.27	-0.05	0.85
10^{-11}	1.99	1.97	1.98	1.99	1.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.99	1.97	1.98	1.99	1.99

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4.7 Summary

In this chapter we considered a class of singularly perturbed turning point problems in which the underlying differential equation involves a quadratic coefficient function which multiplies the highest derivative of the unknown function. The study was focussed on the case where the turning point gives rise to an interior layer.

We first established bounds on the solution and its derivatives. Next, we proposed a fitted operator finite difference method to solve this class of problems. We proved that the method is uniformly convergent of order one with respect to the perturbation parameter. We used Richardson extrapolation to improve the accuracy of the proposed method and achieved uniform convergence of order two. The theoretical results were supported by simulations conducted on two test examples.

Future work in this direction involve designing similar methods to two dimensional problems perhaps with time dependency.

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Chapter 5

A discretization of turning-point parabolic problems with a quadratic diffusion coefficient

We study a family of time-dependent convection-diffusion problems whose solution displays an interior layer. The coefficient of the second derivative term in the differential equation is the quadratic function $\varepsilon + x^2$, where ε is a perturbation parameter. We establish bounds on the solution and its derivatives. Then, we construct a numerical scheme which consists of an Euler time-discretization followed by a fitted operator finite difference method for the space variable. Through a rigorous error analysis, we show that the scheme is uniformly convergent with respect to the perturbation parameter ε. Furthermore, we use Richardson extrapolation to improve the accuracy of the proposed method. To illustrate the theoretical results, we implement the proposed method on some numerical examples.

5.1 Introduction

Consider the family of time dependent singularly perturbed problems

$$Lu := \varepsilon u_{xx} + a(x,t)u_x - b(x,t)u - d(x,t)u_t = f(x,t),\ (x,t) \in D, \tag{5.1.1}$$

with

$$D = \Omega \times (0,T],\ \Omega = (-1,1),$$

and

$$u(-1,t) = \gamma_1,\ u(1,t) = \gamma_2, u(x,0) = u_0(x), \tag{5.1.2}$$

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where γ_1 and γ_2 are given real numbers, $0 < \varepsilon \ll 1$, $t \in [0, T]$, $a(x,t)$, $b(x,t)$, $d(x,t)$, $f(x,t)$, and $u_0(x)$ are sufficiently smooth functions to ensure the smoothness of the solution. We impose the condition $b(x,t) \geq b_0 > 0, \forall\ (x,t) \in \bar{D}$ to ensure that the problem (5.1.1)-(5.1.2) satifies a minimum principle and guarantees the uniqueness of the solution [59].

Problems such as (5.1.1)-(5.1.2) occur in many domain of science and engineering, including control system analysis and design, fluid dynamics, non linear mechanic, jump phenomena in electrical circuits, electrical networks, power systems, reactor systems, chemical kinetics, diffusion processes, population biology models, flight dynamic low thrust (aircraft) and high thrust (missile), jet engine control, missile guidance and energy management (see e.g [84] and the references therein).

When the perturbation parameter ε becomes very small, the solution to the problem (5.1.1)-(5.1.2) presents a rapid change in narrow regions known as boundary or interior layer(s). The behavior of the convection and reaction coefficients throughout the domain determine the location and the number of these layers. Indeed, if $a(x,t) > 0$ or $a(x,t) < 0$, $\forall\ (x,t) \in \bar{D}$, the solution of (5.1.1)-(5.1.2) presents a boundary layer respectively near $x = -1$ or near $x = 1 \quad \forall\ t \in [0, T]$. If $a(x,t) \equiv 0$, $\forall\ (x,t) \in \bar{D}$, then the solution of (5.1.1) presents two-boundary layers (see e.g [24, 29, 49, 54, 60]).

However, if there exist $\alpha_i \in \bar{\Omega}$ such that $a(\alpha_i, t) = 0$ and $a(-1,t)a(1,t) \neq 0, \forall\ t \in [0,T]$, then α_i, $i = 1, 2, \ldots, r$ are called turning point(s) of the problems (5.1.1). These turning points give rise to either interior layer(s) or twin boundary layers. For more information on turning point problems leading to interior layer(s) or twin boundary laters, interested readers may consult for instance [10, 17, 19, 23, 30, 31, 44, 64, 65, 80, 70]. Interior layer(s) may also originate from the non-smooth coefficient functions or discontinuous data. (see e.g. [4, 11, 13, 26, 27, 42]).

Turning-point and non turning-point time dependent singularly perturbed problems are widely studied in the literature. However, their applications in fluid dynamics and biology are problems in which the highest derivatives are affected by functions depending on the variable x and the perturbation parameter ε. Nevertheless, little attention has been given to these problems. Liseikin [51] (pp. 106-111 and pp. 256-262) and [52] considered the case where the coefficient of the highest derivative has the form $g(x,\varepsilon) = -(\varepsilon + px)^{\beta}$ for $\beta \geq 1$ and studied the problem for $p = 0$ and $p = 1$ and established bounds on the solution and its derivatives in each case. Additionally, in [51] he presented a numerical method and analysed its convergence properties, however he did not validate his theoretical findings via numerical experiments.

The application of these problems for $p = 1$ and $\beta = 1$ appears in the description

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of filtration of a liquid through a neighbourhood about a circular orifice or radius $r = \varepsilon$ [52, 77]. For $p = 1$ and $\beta = 2$, they describe a steady diffusive-drift motion [52, 93].

To the best of our knowledge, all the works above have considered problems with a constant coefficient (viz ε) multiplying the highest derivative terms, except the works of Liseikin. In the present work, we consider the coefficient functions of the form $g(x, \varepsilon) = \varepsilon + x^2$ and whose solution exhibits an interior layer. It is worth noting that in (5.1.1)-(5.1.2) the order of the reduced equation (when $\varepsilon = 0$) is lowered to one, unlike what happens when the diffusion coefficient is $\varepsilon + x^2$ for which case the order remains unchanged.

To be more precise, we consider the singularly perturbed parabolic problems (SPP)

$$Lu := (\varepsilon + x^2)u_{xx} + a(x,t)u_x - b(x,t)u - d(x,t)u_t = f(x,t),\ (x,t) \in D. \tag{5.1.3}$$

$$u(-1,t) = \gamma_1,\ u(1,t) = \gamma_2, u(x,0) = u_0(x). \tag{5.1.4}$$

We assume that:

$$\left.\begin{array}{ll} (i)\ a(0,t) = 0, & a_x(0,t) > 0, t \in [0,T], \\ (ii)\ |a_x(x,t)| \geq \frac{|a_x(0,t)|}{2}, & (x,t) \in \bar{D}, \\ (iii)\ \frac{b(0,t)}{a_x(0,t)} > 0, & \in t \in [0,T], \\ (iv)\ b(x,t) \geq b_0 > 0, & (x,t) \in \bar{D}, \end{array}\right\} \tag{5.1.5}$$

where (i) ensures the existence of the turning point, (ii) implies that zero is the only turning point in $\bar{\Omega}$, $\forall\ t \in [0,T]$, (iii) confirms that the interior layer of the solution $u(x,t)$ occurs around the point $(0,t)$, $\forall t \in [0,T]$, (iv) guarantees the uniqueness of the solution and also confirms that the problem (5.1.3)-(5.1.4) satisfies a minimum principle.

To address the interior layer problem, we aim to construct and analyse a fitted operator finite difference method based on modeling rules of Mickens [57] in conjunction with the implicit Euler method. We then show that the scheme is first order uniformly convergent in both space and time variables with respect to the perturbation parameter ε. We also apply Richardson extrapolation (see [32, 65, 66, 67, 68]), to improve the accuracy and the order of convergence of the scheme.

The rest of the chapter is structured as follows. In section 5.2 we present some qualitative results on the bounds of the solution and its derivatives at every time level t in $[0,T]$ following [1, 9, 20]. Section 5.3 is devoted to the construction of the method. In section 5.4, we conduct an error analysis of the proposed scheme. Section 5.5 deals with Richardson extrapolation. To confirm the theoretical findings of the proposed scheme, we carry out some numerical experiments in section 5.6. Section 5.7 is devoted to some concluding remarks.

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5.2 Qualitative results

This section deals with a number of qualitative properties of the continuous problem. We rely on these results in section 5 for the analysis of the maximum error and the rate of convergence of the problem. Functions $f(x,t)$ and $u_0(x)$ are assumed to be smooth and compatible to secure the continuity and ε-uniform bound of the solution of the problem (5.1.3)-(5.1.4) and its derivatives. We use these conditions to get the appropriate space and time accuracy when using the maximum norm on $\bar{D} = \bar{\Omega} \times [0,T]$, where $\Omega = (-1,1)$ and $D = \Omega \times (0,T]$.

Lemma 5.2.1. *(Minimum principle)* Consider ψ a smooth function with $\psi(-1,t) \geq 0$, $\psi(1,t) \geq 0, \forall\, t \ \in [0,T]$ and $L\psi(x,t) \leq 0, \forall\ (x,t) \ \in D$. It follows that $\psi(x,t) \geq 0, \forall\ (x,t) \ \in \bar{D}$.

Proof. We prove this Lemma by contradiction.
Given $(x^*,t^*) \in \bar{D}$ and $\psi(x^*,t^*) = \min \psi(x,t) < 0$. It follows that $(x^*,t^*) \neq (-1,0)$; $(-1,1)$; $(1,0)$ or $(1,1)$. From the definition of the minimum principle we have $\psi_x(x^*,t^*) = 0,\ \psi_t(x^*,t^*) = 0$ and $\psi_{xx}(x^*,t^*) \geq 0$. But

$$L\psi(x^*,t^*) = (\varepsilon + x^{*2})\psi_{xx}(x^*,t^*) + a(x^*,t^*)\psi_x(x^*,t^*) - b(x^*,t^*)\psi(x^*,t^*) + \psi_t(x^*,t^*) \geq 0.$$

Leading to a contradiction. Thus $\psi(x,t) \ \geq 0\ \forall\ (x,t) \ \in \ \bar{D}$.
The above minimum principle is used to proof Lemma 5.2.2 below.

Lemma 5.2.2. *(Uniform stability estimate)* Given $u(x,t)$ the solution of (5.1.3)-(5.1.4). We get

$$||u(x,t)|| \leq b_0^{-1}||f(x,t)|| + \max\left(|\gamma_1|,|\gamma_2|\right), \forall (x,t) \in \bar{D},$$

the notation $||.||$ stands for the maximum norm on $\bar{D}$, and the condition $b(x,t) \geq b_0 > 0, \forall\ (x,t) \in \bar{D}$ unsures the uniqueness of the solution to the problem (5.1.3)-(5.1.4), γ_1 and γ_2 are boundary conditions.

Proof. Given the comparison function

$$\Pi^{\pm}(x,t) = b_0^{-1}||f(x,t)|| + \max\left(|\gamma_1|,|\gamma_2|\right) \pm u(x,t), x \in \bar{D},$$

we have

$$L\Pi^{\pm}(x,t) = -\frac{b(x,t)}{b_0}||f(x,t)|| - b(x,t)\max\left(|\gamma_1|,|\gamma_2|\right) \pm Lu(x,t) \leq 0, (x,t) \in \bar{D}.$$

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Applying the minimum principle, it comes out that

$$\Pi^{\pm}(x,t) \geq 0, \forall\ (x,t) \in \bar{D}.$$

Finally

$$||u(x,t)|| \leq b_0^{-1}||f(x,t)|| + \max\left(|\gamma_1|, |\gamma_2|\right), \forall (x,t) \in \bar{D},$$

which ends the proof.

Let us consider the partition on $\bar{\Omega} = [-1,1]$ given as by:
$\Omega_L = [-1,-\delta)$, $\Omega_C = [-\delta,\delta]$, $\Omega_R = (\delta,1]$, with $0 < \delta \leq 1/2$; respectively the left side of the layer region, central part (or the layer region) and the right side of the layer region. We also have $\Omega_C = \Omega_C^- \cup \Omega_C^+$, where $\Omega_C^- = [-\delta, 0)$, $\Omega_C^+ = [0,\delta]$. and $\bar{D} = \bar{\Omega}$ x $[0,T]$.

From the literature, it is known that if $u(x,t)$ is the solution of the problem (5.1.3)-(5.1.4), then there exists a positive constant C such that $|u(x,t)| \leq C$, $\forall (x,t) \in \bar{D}$.

Lemma 5.2.3. *Under the above assumption and that of Lemma 5.2.1, the bound on the derivative of u with respect to t is given by.*

$$|u_t| \leq C,\ \forall\ (x,t) \in \bar{D}.$$

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Proof. See [36]

The following Lemma focuses on the Inverse Monotonicity.

Lemma 5.2.4. *[51] Let $F(x,u,u_x) = a(x,t)u_x(x,t) - b(x,t)u_t + d(x,t)u_t(x,t) - f(x,t)$ be a smooth function in $([-1,1] \times [0,T]) \times \mathbb{R}^2$, where $a(x,t), b(x,t), d(x,t)$ and $f(x)$ are functions described in (5.1.3)-(5.1.4). The problem (5.1.3)-(5.1.4) is said to be inverse monotone for $F(x,u,u_x) \in C^2\left((-1,1) \times [0,T]\right) \cap C\left([-1,1] \times [0,T]\right)$ if one of the following conditions imposed on F is satisfied:*

(1) $F(x,u,u_x)$ is strictly increasing in u, i.e., $F(x,u_1,z) < F(x,u_2,z)$ if $u_1 < u_2$,

(2) $F(x,u,u_x)$ is weakly increasing in u and there exists a positive constant $C > 0$, such that $|F(x,u,z_1) - F(x,u,z_1)| \leq C\,|z_1 - z_2|$.

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Proof. See [51] with $d(x) = x^2,\ l = 1,\ \forall x \in [-1, 1]$.

The following lemmas deals with the appropriates bounds on the derivatives of the solution to the problem (5.1.3) - (5.1.4) where $t \in [0, T]$ and x is either in Ω_L, in Ω_C or in Ω_R.

Lemma 5.2.5. *Let u(x, t) be the solution to (5.1.3)-(5.1.4), we have*

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq C,\ \forall x \in \Omega_L U \Omega_R,\ t \in [0, T].$$

Where C is a positive real number, free from the singular perturbation ε but depending on δ .

Proof. This Lemma is the immediate consequence of Theorem 5.2.4 for the inverse monotonicity with $C = M$ as specified in [51],
$\forall (x, t) \in \Omega_R \times [0, T]$, $F[x, -M, u_x] \leq F[x, u, u_x] \leq F[x, M, u_x]$ leading to $-M \leq u(x, t) \leq M$. This completes the proof. In the similar way, we can proof this Lemma for $(x, t) \in \Omega_L \times [0, T]$.

In Lemma 5.2.6 below, we discuss the bounds of the solution and its derivatives in the layer region. We herein follow Liseikin [51] work to adapt it to our problem. We also consider the convection coefficient at a specific point (x_0, t) to be given by $a(x_0, t) = a$ where $(x_0, t) \in \Omega_C^+ \times [0, T]$ or $(x_0, t) \in \Omega_C^- \times [0, T]$.

Lemma 5.2.6. *[51] (Continuous results) Consider u(x,t) the solution to the problem (5.1.3)-(5.1.4). Then, we have:*

1) for $x \in \Omega_C^+$ and $x_0 \in \Omega_C^+, t \in [0, T]$, such that $a(x_0, t) = a > 0$ and $j = 0, 1, 2, 3, 4$;

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M \begin{cases} 1 + (\varepsilon + x^2)^{1-a-j},\ if\ 0 < a < 1, \\ \\ 1 + (\varepsilon + x^2)^{-j},\ if\ a = 1, \\ \\ 1 + \varepsilon^{a-1} (\varepsilon + x^2)^{1-a-j},\ if\ a > 1. \end{cases} \tag{5.2.1}$$

2) for $x \in \Omega_C^-$ and $x_0 \in \Omega_C^-, t \in [0, T]$, such that , $a(x_0, t) = a \leq 0, j = 0, 1, 2, 3, 4$ and p a whole number such that $a + p \geq 0, a + p - 1 < 0$

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$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M \begin{cases} 1,\ if\ a<0, j\leq p, \\ \\ 1+(\varepsilon+x^2)^{1-j-p}\left|\arctan\left(\frac{x}{\sqrt{(\varepsilon)}}\right)\right|,\ if\ a+p=0,\ j>p, \\ \\ 1+(\varepsilon+x^2)^{-a-j},\ if\ a+p>1,\ j>p, \end{cases} \tag{5.2.2}$$

where M is a positive constant independent of ε.

Proof.

1) Let us first prove Lemma 5.2.6 for $x\in\Omega_C^+=(0,\delta,m],\ t\in[0,T]$, also with $x_0\in\Omega_C^+$ such that $a(x_0,t)=a>0$.
Consider u the solution to the problem (5.1.3)-(5.1.4). From the inverse monotonicity Lemma 5.2.4, we have

$$|u(x,t)|\leq M. \tag{5.2.3}$$

Also, according to Liseikin [51], there exists a positive constant m such that (5.1.3)-(5.1.4) and (5.2.3) lead to

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M \begin{cases} 1,\ 0<m\leq x\leq\delta,\ t\in[0,T], \\ \\ \varepsilon^{-j},\ 0\leq x\leq\delta,\ t\in[0,T] \end{cases} \tag{5.2.4}$$

$$j=1,2,3,4.$$

Supposed that $a>0$. We can rewrite (5.1.3) as follows

$$\frac{\partial^2 u(x,t)}{\partial x^2} = -\frac{a(x,t)\frac{\partial u(x,t)}{\partial x}}{\varepsilon+x^2}+\frac{b(x,t)u(x,t)+f(x,t)}{\varepsilon+x^2}+\frac{d(x,t)\frac{\partial u(x,t)}{\partial t}}{\varepsilon+x^2},$$

or

$$\partial u(x,t) = -\int_0^x \frac{a(\eta,t)\frac{\partial u(\eta,t)}{\partial\eta}}{\varepsilon+\eta^2}d\eta+\int_0^x\frac{b(\eta,t)u(\eta,t)+f(\eta,t)}{\varepsilon+\eta^2}d\eta+\int_0^x\frac{d(\eta,t)\frac{\partial u(\eta,t)}{\partial t}}{\varepsilon+\eta^2}d\eta,$$

which can be expressed by the formula:

$$\frac{\partial u(x,t)}{\partial x}=\frac{\partial u(0,t)}{\partial x}\left(\frac{\varepsilon}{\varepsilon+x^2}\right)^a\exp\left[-g_1(x,t)\right]+g_2(x,t), 0\leq x\leq\delta,\ t\in[0,T], \tag{5.2.5}$$

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where

$$g_1(x,t) = \int_0^x \frac{a(\eta,t))}{\varepsilon+\eta^2} d\eta,\ t \in [0,T],$$

and the integration by parts leads to

$$g_1(x,t) = \frac{a(x,t)}{\sqrt{\varepsilon}} \arctan\left(\frac{x}{\sqrt{\varepsilon}}\right) - \frac{1}{\sqrt{\varepsilon}} \int_0^x \left[\frac{\partial a(\eta,t)}{\partial \eta}\right] \arctan\left(\frac{\eta}{\sqrt{\varepsilon}}\right) d\eta,$$

with $g_1(0,t) = 0$, since $a(0,t) = 0$ and $\arctan(0) = 0$, $\forall\, t \in [0,T]$. We also have

$$\begin{aligned} g_2(x,t) \quad &= \quad (\varepsilon+x^2)^{-a} \int_0^x [b(\eta,t)u(\eta,t) + d(\eta,t)\frac{\partial u(\eta,t)}{\partial t}. \\ &+ f(\eta,t)] \left(\varepsilon+\eta^2\right)^{a-1} \exp[g_1(\eta,t) - g_1(x,t)] d\eta,\ \forall t \in [0,T]. \end{aligned}$$

From (5.2.4) with $a > 0$, we have

$$|g_j(x,t)| \le M,\ j = 1,2;\ 0 < x \le \delta,\ t \in [0,T].$$

Applying triangular inequalities, (5.2.5) leads to

$$\begin{aligned} \left|\frac{\partial u(x,t)}{\partial x}\right| &\le \left|\partial u(0,t)\left(\frac{\varepsilon}{\varepsilon+x^2}\right)^a \exp\left[-g_1(x,t)\right]\right| + |g_2(x,t)|\,, \\ \left|\frac{\partial u(x,t)}{\partial x}\right| &\le M \left|\frac{\partial u(0,t)}{\partial x}\right| \left(\frac{\varepsilon}{\varepsilon+x^2}\right)^a + M, \\ \left|\frac{\partial u(x,t)}{\partial x}\right| &\le M \left[1 + \left|\frac{\partial u(0,t)}{\partial x}\right| \left(\frac{\varepsilon}{\varepsilon+x^2}\right)^a\right], 0 < x \le \delta,\ t \in [0,T]. \end{aligned} \tag{5.2.6}$$

Considering $0 < a < 1$, $0 < \varepsilon << 1$, $t \in [0,T]$, and a positive constant m, with $x = m$ such that (5.2.4) and (5.2.5) lead to

$$\left|\frac{\partial u(0,t)}{\partial x}\right| \left(\frac{\varepsilon}{\varepsilon+m^2}\right)^a \le M,\ t \in [0,T],$$

$$i.e., \left|\frac{\partial u(0,t)}{\partial x}\right| \le M \left(\frac{\varepsilon+m^2}{\varepsilon}\right)^a \le M\varepsilon^{-a},\ t \in [0,T].$$

Thus (5.2.6) leads to

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \le M \left[1 + \varepsilon^{-a}\varepsilon^a \left(\varepsilon+x^2\right)^{-a}\right],$$

giving

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \le M \left[1 + \left(\varepsilon+x^2\right)^{-a}\right],\ 0 < a < 1,\ 0 < x \le \delta,\ t \in [0,T].$$

Also, from (5.1.3) we have the following

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$$\frac{\partial^3 u(x,t)}{\partial x^3} = -\frac{[2x+a(x,t)]\frac{\partial^2 u(x,t)}{\partial x^2}}{\varepsilon + x^2} + \frac{[-\frac{\partial a(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial x} - b(x)]\frac{\partial u(x,t)}{\partial x}}{\varepsilon + x^2}$$
$$+\frac{\frac{\partial b(x,t)}{\partial x}u(x,t) + \frac{\partial f(x,t)}{\partial x} + \frac{\partial d(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial t} + d(x,t)\frac{\partial u_t(x,t)}{\partial x}}{\varepsilon + x^2}, \tag{5.2.7}$$

or

$$\frac{\partial^2 u(x,t)}{\partial x^2} = -\int_0^x \frac{[2\eta + a(\eta,t)]\frac{\partial^2 u(\eta,t)}{\partial \eta^2}}{\varepsilon + \eta^2}d\eta + \int_0^x \frac{[-\frac{\partial a(\eta,t)}{\partial \eta}\frac{\partial u(\eta,t)}{\partial \eta} - b(\eta,t)]\frac{\partial u(\eta,t)}{\partial \eta}}{\varepsilon + \eta^2}d\eta$$
$$+\int_0^x \frac{\frac{\partial b(\eta,t)}{\partial \eta}u(\eta,t) + \frac{\partial f(\eta,t)}{\partial \eta} + \frac{\partial d(\eta,t)}{\partial \eta}\frac{\partial u(\eta,t)}{\partial t} + d(\eta,t)\frac{\partial u_t(\eta,t)}{\partial \eta}}{\varepsilon + \eta^2}d\eta.$$

We can express this derivative by the formula:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(0,t)}{\partial x^2}\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1}\exp\left[-g_3(x)\right] + g_4(x,t), 0 \le x \le \delta, \tag{5.2.8}$$

where

$$g_3(x,t) = \int_0^x \frac{[2\eta + a(\eta,t)]}{\varepsilon + \eta^2}d\eta,$$

and the integration by parts leads to

$$g_3(x,t) = \frac{(2x + a(x,t))}{\sqrt{\varepsilon}}\arctan\left(\frac{x}{\sqrt{\varepsilon}}\right) - \frac{1}{\sqrt{\varepsilon}}\int_0^x \left(2 + \frac{\partial a(\eta)}{\partial \eta}\right)\arctan\left(\frac{\eta}{\sqrt{\varepsilon}}\right)d\eta,$$

with $g_3(0,t) = 0$, since $a(0,t) = 0$, and $\arctan(0) = 0, \forall\ t\ \in [0,T]$. On the other hand, we have

$$\begin{aligned} g_4(x,t) &= (\varepsilon + x^2)^{-a-1}\int_0^x [-\frac{\partial a(\eta,t)}{\partial \eta}\frac{\partial u(\eta,t)}{\partial \eta} + b(\eta,t)\frac{\partial u(\eta,t)}{\partial \eta} \\ &\quad +\frac{\partial b(\eta,t)}{\partial \eta}u(\eta,t) + \frac{\partial f(\eta,t)}{\partial \eta} + \frac{\partial d(\eta,t)}{\partial \eta}u_t(\eta,t) \\ &\quad +d(\eta,t)\frac{\partial u_t(\eta,t)}{\partial \eta}u_t(\eta,t)]\left(\varepsilon + \eta^2\right)^a \exp[g_3(\eta,t) - g_3(x,t)]d\eta. \end{aligned}$$

From (5.2.4) with $a > 0$, we have

$|g_3(x,t)| \le M,\ |g_4(x,t)| \le M\ \ 0 < x \le \delta$, and $t \in [0,T]$.

The triangular inequality applied to (5.2.8) leads to

$$\left|\frac{\partial^2 u(x,t}{\partial x^2}\right| \le \left|\frac{\partial^2 u(0,t)}{\partial x^2}\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1}\exp\left[-g_3(x,t)\right]\right| + |g_4(x,t)|\,,$$
$$\left|\frac{\partial^2 u(x,t}{\partial x^2}\right| \le M\left|\frac{\partial^2 u(0,t}{\partial x^2}\right|\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1} + M,$$

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$$\left|\frac{\partial^2 u(x,t}{\partial x^2}\right| \leq M\left[1+\left|\frac{\partial^2 u(0,t}{\partial x^2}\right|\left(\frac{\varepsilon}{\varepsilon+x^2}\right)^{a+1}\right], 0<x\leq \delta, t\in[0,T]. \tag{5.2.9}$$

Considering $0<a<1,\ 0<\varepsilon<<1, t\in[0,T]$, and m' a positive constant, with $x=m'$ such that (5.2.4) and (5.2.8) lead to

$$\left|\frac{\partial^2 u(0,t}{\partial x^2}\right|\left(\frac{\varepsilon}{\varepsilon+m'^2}\right)^{a+1} \leq M,$$

$$i.e., \left|\frac{\partial^2 u(0,t}{\partial x^2}\right| \leq M\left(\frac{\varepsilon+m'^2}{\varepsilon}\right)^{a+1} \leq M\varepsilon^{-a-1}.$$

Thus (5.2.9) gives

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left[1+\varepsilon^{-a-1}\varepsilon^{a+1}\left(\varepsilon+x^2\right)^{-a-1}\right],$$

leading to

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left[1+\left(\varepsilon+x^2\right)^{-a-1}\right], 0<a<1, 0<x\leq\delta, t\in[0,T].$$

Thereafter, from (5.1.3) and (5.2.3), we come to the following result for $0<a<1, 0<x\leq \delta; t\in[0,T]:$

$$\left|\frac{\partial^j u(x,t)}{\partial x^2}\right| \leq M\left[1+\left(\varepsilon+x^2\right)^{-a+1-j}\right],\ 0<\varepsilon<<1, j=1,2,3,4. \tag{5.2.10}$$

The Lemma 5.2.6 is fulfilled for $0<a<1$.
If $a=1$, the partial integration with respect to x of (5.2.5) from 0 to δ leads to

$$\int_0^\delta \frac{\partial u(\eta,t)}{\partial \eta}d\eta = \int_0^\delta \frac{\partial u(0,t)}{\partial \eta}\left[\frac{\varepsilon}{\varepsilon+\eta^2}\right]\exp\left[-g_1(\eta,t)\right]d\eta + \int_0^\delta g_2(\eta,t)d\eta, 0\leq x\leq\delta.$$

Integrating by parts leads to

$$\begin{aligned}
A_\delta - A_0 &= \frac{1}{\sqrt{\varepsilon}}\frac{\partial u(0,t)}{\partial \eta}\arctan\left(\frac{\delta}{\sqrt{\varepsilon}}\right)\exp[-g_1(\delta,t)] \\
&\quad -\frac{1}{\sqrt{\varepsilon}}\int_0^\delta \arctan\left(\frac{\eta}{\sqrt{\varepsilon}}\right)\frac{\partial g_1(\eta,t)}{\partial\eta}\exp[-g_1(\eta,t)]d\eta] + \int_0^\delta g_2(\eta,t)d\eta, \\
A_\delta - A_0 &= \frac{1}{\sqrt{\varepsilon}}\frac{\partial u(0,t)}{\partial \eta}[\arctan\left(\frac{\delta_1}{\sqrt{\varepsilon}}\right)\exp[-g_1(\delta,t)], \\
&\quad -\int_0^\delta \arctan\left(\frac{\eta}{\sqrt{\varepsilon}}\right)[2\eta+a(\eta)](\varepsilon+\eta^2)^{-1}\exp[-g_1(\eta,t)]d\eta] + \int_0^\delta g_2(\eta,t)d\eta,
\end{aligned}$$

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we know that

$$\left|\arctan\left(\frac{\delta}{\sqrt{\varepsilon}}\right)\exp[-g_1(\delta,t)] - \int_0^\delta \arctan\left(\frac{\eta}{\sqrt{\varepsilon}}\right)[2\eta+1](\varepsilon+\eta^2)^{-1}\exp[-g_1(\eta,t)]d\eta\right| \leq M,$$

then,

$$\frac{1}{\sqrt{\varepsilon}}\left|\frac{\partial u(0,t)}{\partial x}\right| \leq M,$$

meaning

$$\left|\frac{\partial u(0,t)}{\partial x}\right| \leq M\varepsilon^{\frac{1}{2}}.$$

From (5.2.6) we have

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M\left[1+(\varepsilon+x^2)^{-1}\right], 0 \leq x \leq \delta_1,\ 0 < \varepsilon << 1, t \in [0,T]\ a = 1.$$

Thereafter, after differentiating (5.1.3) and using (5.2.3), we come to the following result with $a = 1$ and $0 < x \leq \delta, t \in [0,T]$;

$$\left|\frac{\partial^j u(x,t)}{\partial x}\right| \leq M\left[1+\left(\varepsilon+x^2\right)^{-j}\right],\ 0 < \varepsilon << 1, t \in [0,T], j = 1,2,3,4. \tag{5.2.11}$$

For $a > 1$, (5.2.4) into (5.2.5) and using triangular inequality; we get the following

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M\varepsilon^{-1}\left[\frac{\varepsilon}{\varepsilon+x^2}\right]^a + M,$$

or

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M\left[1+\varepsilon^{a-1}(\varepsilon+x^2)^{-a}\right],$$

and using (5.1.3) we come to the same derivative as specified in (5.2.8)

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(0,t)}{\partial x^2}\left(\frac{\varepsilon}{\varepsilon+x^2}\right)^{a+1}\exp\left[-g_1(x)\right] + g_2(x), 0 \leq x \leq \delta, \tag{5.2.12}$$

and the triangular inequality of (5.2.12) in connection with (5.1.3) and (5.2.4) leads to

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left|\frac{\partial^2 u(0,t)}{\partial x^2}\right| \cdot \left(\frac{\varepsilon}{\varepsilon+x^2}\right)^{a+1} + M,$$

meaning

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\varepsilon^{-2}\left(\frac{\varepsilon}{\varepsilon+x^2}\right)^{a+1} + M,$$

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which leads to

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left[1+\varepsilon^{-2}\varepsilon^{a+1}\left(\varepsilon+x^2\right)^{-a-1}\right],$$

or

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left[1+\varepsilon^{a-1}\left(\varepsilon+x^2\right)^{-a-1}\right].$$

Thereafter from (5.1.3) and (5.2.4) we conclude that

$$\begin{aligned}\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \quad &\leq \quad M\left[1+\varepsilon^{a-1}(\varepsilon+x^2)^{1-a-j}\right],\\ &\qquad 0<x\leq\delta,\ 0<\varepsilon<<1, a>1, t\in[0,T]\ j=1,2,3,4,\end{aligned} \tag{5.2.13}$$

which ends the proof for $x\in\Omega_C^+$ and $a>0$.

2) Consider $x\in\ \Omega_C^-=[-\delta,0]$, and let $x_0\in\Omega_C^-=[-\delta,0]$ such that $a(x_0,t)=a\leq 0, t\in[0,T]$. Solving (5.1.3) - (5.1.4) with respect to $u'(x)$ leads to

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} \quad &= \quad \frac{\partial u(x_0,t)}{\partial x}\exp(\psi(x,x_0,t))\\ &\quad +\int_{x_0}^{x}\frac{[b(\eta,t)u(\eta,t)+f(\eta,t)+d(\eta,t)u_t(\eta,t)]}{\varepsilon+\eta^2}\exp\left[\psi(\eta,x_0,t)\right]d\eta,\end{aligned} \tag{5.2.14}$$

or

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} \quad &= \quad \frac{\partial u(0,t)}{\partial x}\exp(\psi(x,x_0,t))+(\varepsilon+x^2)^{-p}\int_{x_0}^{x}[b(\eta,t)u(\eta,t)\\ &\quad +f(\eta,t)+d(\eta,t)u_t(\eta,t)](\varepsilon+\eta^2)^{p-1}\exp\left[\psi(\eta,x_0,t)\right]d\eta,\end{aligned} \tag{5.2.15}$$

with $\psi(x,x_0,t)$ given by

$$\psi(x,x_0,t)=\int_{x_0}^{x}\frac{a(\eta,t)}{\varepsilon+\eta^2}d\eta.$$

It is clear that

$$|\psi(x,x_0,t)|\leq M, -\delta\leq x\leq 0;\ 0<\varepsilon<<1,\ t\in[0,T].$$

Given $x_0\in\ [-\delta,0]$, using (5.2.4), $\left|\frac{\partial u(x_0,t)}{\partial x}\right|\leq M$, and applying triangular inequality, we come to the following

$$\left|\frac{\partial u(x,t)}{\partial x}\right|\leq M\left[1+\arctan\left(\frac{x}{\sqrt{\varepsilon}}\right)\right].$$

This proves Lemma 5.2.6 for $j=1, a(0)=a=0, p=0,\ t\in[0,T]$.

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From (5.1.3) - (5.1.4), (5.2.7), (5.2.4) and for $j = 2, a(0,t) = a = 0, p = 0, t \in [0,T]$; we can easily show that

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left[1 + (\varepsilon + x^2)^{-1}\arctan\left(\frac{x}{\sqrt{\varepsilon}}\right)\right].$$

Thereafter, from (5.1.3) and (5.2.4), we come to the following result, with $a+p = 0, j > p$

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M\left[1 + \left(\varepsilon + x^2\right)^{1-j-p}\arctan\left(\frac{x}{\sqrt{\varepsilon}}\right)\right],\ 0 < \varepsilon << 1, t \in [0,T], j = 1,2,3,4. \tag{5.2.16}$$

Now, consider $x \in\ [-\delta, 0],\ 0 < \varepsilon << 1,\ p \geq 1, t \in [0,T], a(x,t) < 0$ and m_3 a positive constant given by $-\delta \leq m_3 \leq x \leq 0$, then we have

$$\psi(x, x_0, t) \leq -m_3 \ln\left(\frac{\varepsilon + \eta^2}{\varepsilon + x^2}\right), -\delta \leq m_3 \leq \eta \leq x \leq 0, t \in [0,T],$$

which follows that

$$\exp\left[\psi(x, x_0, t)\right] \leq M\left(\frac{\varepsilon + x^2}{\varepsilon + \eta^2}\right)^{m_3}, -\delta \leq m_3 \leq \eta \leq x \leq 0, t \in [0,T].$$

Using (5.2.4) and letting $x_0 = m_3$; (5.2.14) leads to

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M, -\delta \leq m_3 \leq x \leq 0, t \in [0,T];$$

which gives the proof for $j = 1 \leq p, a(x,t) = a < 0$.

Also, using (5.2.4) and letting $x_0 = m_3$; we can easily show that

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M, -\delta \leq m_3 \leq x \leq 0, j = 2 \leq p, a < 0, t \in [0,T].$$

Thus, form (5.1.3) - (5.1.4); we conclude that

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M, -\delta \leq x \leq 0,\ j \leq p, a(x,t) = a < 0, t \in [0,T]\ j = 1,2,3,4. \tag{5.2.17}$$

Finally, let $x \in\ [-\delta, 0], 0 < \varepsilon << 1, t \in [0,T], a(x_0,t) = a < 0$, we can defined a formula for the first derivative of the problem (5.1.3) - (5.1.4) similar to (5.2.5) as follows

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x_0,t)}{\partial x}\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1}\exp\left[-g_1(x,t)\right] + g_2(x,t), \delta \leq x \leq 0, \tag{5.2.18}$$

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with g_1 and g_2 as specified in (5.2.5),
Applying triangular inequality and following the same process as (5.2.5) we get

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M\left[1+(\varepsilon+x^2)^{-a-1}\right], -\delta \leq x \leq 0, t \in [0,T], j=1, a(x_0,t)=a<0.$$

We also defined the formula of the second derivative in connection with (5.2.8) as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(0,t)}{\partial x^2}\left(\frac{\varepsilon}{\varepsilon+x^2}\right)^{a+2} \exp\left[-g_5(x,t)\right] + g_6(x,t), -\delta \leq x \leq 0, t \in [0,T], \quad (5.2.19)$$

where g_5 and g_6 are obtained after integrating the derivative of (5.1.3) with respect to x. After applying triangular inequality, we come to the following:

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left[1+(\varepsilon+x^2)^{-a-2}\right], -\delta \leq x \leq 0, t \in [0,T], j=2, a(x_0,t)=a<0.$$

Thereafter, from (5.1.3) - (5.1.4); we get

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M\left(\varepsilon+x^2\right)^{-a-j}, \delta \leq x \leq 0, t \in [0,T], a(x_0,t)=a<0, \; j=1,2,3,4. \quad (5.2.20)$$

This complete the proof of Lemma 5.2.6 for $x \in \Omega_C^-$ and $a(x,t) \leq 0$.

In the next section, we derive the method to solve the interior layer time dependent singularly perturbed problem (5.1.3)-(5.1.4).

5.3 Construction of the FOFDM

In this part we concentrate on the discretization of the problem (5.1.3)-(5.1.4) in time, with uniform step-size τ. We herein use Euler implicit method. Consider the partition of the time interval [0, T] as follow:

$$\bar{\omega}^k = \left\{t_k = k\tau, \; 0 \leq k \leq K, \; \tau = T/K\right\}. \quad (5.3.1)$$

We discretize (5.1.3)-(5.1.4) on $\bar{\omega}^k$ as follows:

$$-d(x,t_k)\frac{u(x,t_k)-u(x,t_{k-1})}{\tau} + L_{x,\varepsilon}(u(x,t_k)) = f(x,t_k), 1 \leq k \leq K, \quad (5.3.2)$$

$$u(x,0) = u_0(x), \forall x \in (-1,1), \; u(-1,t_k) = \gamma_1, \; u(1,t_k) = \gamma_2. \quad (5.3.3)$$

The equation (5.3.2) becomes:

$$(-d(x,t)I + \tau L_{x,\varepsilon})(u(x,t_k)) = \tau f(x,t_k) - d(x,t)u(x,t_{k-1}). \quad (5.3.4)$$

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We consider the discretization (5.3.4) above as the result of the turning point singularly perturbed problems at each time level $t_k = k\tau$. This result is the used in section 5.4 for the error analysis. The global error E_k at the time level t_k is the sum of local errors e_k at each time level t_k. The local truncation error e_k is given by $e_k = u(x,t_k) - \tilde{u}(x,t_k)$, with $\tilde{u}(x,t_k)$ the solution of

$$(-d(x,t)I+\tau L_{x,\varepsilon})(u(x,t_k))=\tau f(x,t_k) - d(x,t)u(x,t_{k-1}), u(-1,t_k)=\alpha,\ u(1,t_k)=\gamma. \tag{5.3.5}$$

The operator $(-d(x,t)I + \tau L_{x,\varepsilon})$ satisfies the maximum principle, leading to:

$$\left|\left|(-d(x,t_k)I + \tau L_{x,\varepsilon})^{-1}\right|\right| \leq \frac{1}{\max_{0\leq k\leq K,\ x\in[-1,1]}(|d(x,t_k)|^{order(I)}) + \tau\beta}. \tag{5.3.6}$$

with order (I) in (5.3.6) is the order of the identity matrix I. This proves the stability of the discretization in time.

In the other hand, we know from the literature that the local error and the global error are respectively bounded as: $||e_k||_\infty \leq c\tau^2, 1 \leq k \leq K$ and $||E_k||_\infty \leq c\tau, 1 \leq k \leq K$.

Lemma 5.3.1. *Consider $u(x,t_k)$ the solution of (5.3.2) - (5.3.3) at time level t_k, we have*
$\left|u^{(m)}(x,t_k)\right| \leq C\left[1 + (\varepsilon + x^2)^{-m}\exp\left(\frac{\eta x}{\varepsilon}\right)\right], m = 0,1,2,3,$
and
$\left|u^{(m)}(x,t_k)\right| \leq C\left[1 + (\varepsilon + x^2)^{-m}\exp\left(\frac{-\eta x}{\varepsilon}\right)\right], m = m = 0,1,2,3$
where C is a positive constant independent of ε.

Proof. See [20].

Given n a positive and even integer and $\bar{\Omega}^n$ the following partition on $\bar{\Omega} = [-1,1]$:

$$x_0 = -1; x_j = x_0 + jh; j = 1, ..., n-1, h = x_j - x_{j-1}, x_n = 1.$$

Consider $\bar{Q}^{n,K} = \bar{\Omega}^n \times \bar{\omega}^K$ the grid of (x,t).
We also adopt the following: $\forall\ (x_j,t_k)\ \in \bar{Q}^{n,K},\ \Xi(x_j,t_k) := \Xi_j^k$, Where U_j^k represents the numerical approximation of u_j^k.
Applying difference equation theory on $\bar{Q}^{n,K}$ (see [57]), we get the discretization of the problem (5.1.3)-(5.1.4) as follows

$$L^{n,K}U_j^k := \begin{cases} -\tilde{d}_j^k\frac{U_j^k - U_j^{k-1}}{\tau} + (\varepsilon + x_j^2)\delta^2 U_j^k + \tilde{a}_j^k D^- U_j^k - \tilde{b}_j^k U_j^k = \tilde{f}_j^k, \\ j = 0,1,2,\cdots,\frac{n}{2}-1, k = 0,1,...,K, \\ \\ -\tilde{d}_j^k\frac{U_j^k - U_j^{k-1}}{\tau} + (\varepsilon + x_j^2)\delta^2 U_j^k + \tilde{a}_j^k D^+ U_j^k - \tilde{b}_j^k U_j^k = \tilde{f}_j^k, \\ j = \frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n-1, k = 0,1,...,K, \end{cases} \tag{5.3.7}$$

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$$U_0 = \gamma_1,\ U_n = \gamma_2, \tag{5.3.8}$$

we again use the following

$$D^- U_j^k = \frac{U_j^k - U_{j-1}^k}{h},\ \ D^+ U_j^k = \frac{U_{j+1}^k - U_j^k}{h},\ \ \delta^2 U_j^k = \frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{\tilde{\phi}^2},$$

with

$$\tilde{\phi_j^k}^2 = \begin{cases} \frac{h(\varepsilon + x_j^2)}{\tilde{a_j}^k}\left[\exp\left(\frac{\tilde{a_j}^k h}{\varepsilon + x_j^2}\right) - 1\right],\ j = 0, 1, 2, ..., \frac{n}{2} - 1, \\ \\ \frac{h(\varepsilon + x_j^2)}{\tilde{a_j}^k}\left[1 - \exp\left(\frac{-\tilde{a_j}^k h}{\varepsilon + x_j^2}\right)\right],\ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1. \end{cases} \tag{5.3.9}$$

In addition, we adopt the following conventions.

$$\left.\begin{aligned} \tilde{a}_j^k &= \frac{a_j^k + a_{j-1}^k}{2} \text{ for } j = 0, 1, 2, ..., \frac{n}{2} - 1, \\ \tilde{a}_j^k &= \frac{a_j^k + a_{j+1}^k}{2} \text{ for } j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1, \\ \tilde{b}_j^k &= \frac{b_{j-1}^k + b_j^k + b_{j+1}^k}{3}; \tilde{f}_j = \frac{f_{j-1}^k + f_j^k + f_{j+1}^k}{3} \text{ for } j = 0, 1, 2, ..., n - 1, \\ \tilde{d_j}^k &= \frac{d_{j-1}^k + d_j^k + d_{j+1}^k}{3} \text{ for } j = 0, 1, 2, ..., n - 1. \end{aligned}\right\} \tag{5.3.10}$$

Using (5.3.10) above, we rewrite (5.3.7) as follows

$$\left.\begin{aligned} & r_{j,k}^- U_{j-1}^k + r_{j,k}^c U_j^k + r_{j,k}^+ U_{j+1}^k = \tilde{f_j}^k,\ j = 0, 1, 2, ..., \tfrac{n}{2} - 1 \\ & k = 0, 1, ..., K, \\ & r_{j,k}^- U_{j-1}^k + r_{j,k}^c U_j^k + r_{j,k}^+ U_{j+1}^k = \tilde{f_j}^k,\ j = \tfrac{n}{2}, \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, ..., n - 1, \\ & k = 0, 1, ..., K, \end{aligned}\right\} \tag{5.3.11}$$

the coefficients of this system of equations are given by

$$\left.\begin{aligned} & r_{j,k}^- = \frac{\varepsilon + x_j^2}{\tilde{\phi_j^k}^2} - \frac{\tilde{a_j}^k}{h}; r_{j,k}^c = \frac{-2(\varepsilon + x_j^2)}{\tilde{\phi_j^k}^2} + \frac{\tilde{a_j}^k}{h} - \tilde{b_j}^k - \frac{\tilde{d_j^k}}{\tau}; r_{j,k}^+ = \frac{\varepsilon + x_j^2}{\tilde{\phi_j^k}^2}, j = 0, 1, 2, ... \frac{n}{2} - 1, \\ & r_{j,k}^- = \frac{\varepsilon + x_j^2}{\tilde{\phi_j^k}^2}; r_{j,k}^c = \frac{-2(\varepsilon + x_j^2)}{\tilde{\phi_j^k}^2} - \frac{\tilde{a_j}^k}{h} - \tilde{b_j}^k - \frac{\tilde{d_j^k}}{\tau}; r_{j,k}^+ = \frac{\varepsilon + x_j^2}{\tilde{\phi_j^k}^2} + \frac{\tilde{a_j}^k}{h} j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1. \end{aligned}\right\} \tag{5.3.12}$$

$$\tilde{F_j^k} = \tilde{f_j^k} - \frac{\tilde{d_j^k}}{\tau} U_j^{k-1}. \tag{5.3.13}$$

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The numerical method obtained called fitted operator finite difference method (FOFDM) (5.3.11)-(5.3.8) satisfies the Lemmas below.

Lemma 5.3.2. (Discrete minimum principle) . *Given* ξ_j^k *with,* $L^{n,k}\xi_j^k \leq 0 \ \forall \ (j,k) \in Q^{n,K}$, $\xi_j^0 \geq 0, \ 0 \leq j \leq n, \ \xi_0^k \geq 0,$ *and* $\xi_n^k \geq 0, \ 1 \leq k \leq K$. *Then* $\xi_j^k \geq 0, \ \forall \ (j,k) \in \bar{Q}^{n,K}$.

Proof. Given (s,l) with $\xi_s^l = \min_{(j,k)} \xi_j^k < 0, \ \xi_j^k \in \bar{Q}^{n,K}$. We have $s \neq 1,2,...,n-1$ and $l \neq 1,2,...,K$; otherwise $\xi_s^l \geq 0$. In the other hand $\xi_{s+1}^l - \xi_s^l \geq 0, \ \xi_s^l - \xi_{s-1}^l \leq 0$, and $\xi_s^l - \xi_s^{l-1} \leq 0$.
We get

$$L^{n,K}\xi_s^l = \begin{cases} (\varepsilon + x_s^2)\bar{\delta}^2\xi_s^l + a_s^l D^- \xi_s^l - \left(b_s^l + \frac{d_s^l}{\tau}\right)\xi_s^l > 0, \ a_s^l < 0, s = 1,2,...,\frac{n}{2}-1, \\ -\left(b_s^l + \frac{d_s^l}{\tau}\right)\xi_s^l > 0, s = \frac{n}{2}, \\ (\varepsilon + x_s^2)\bar{\delta}^2\xi_s^l + a_s^l D^+ \xi_s^l - \left(b_s^l + \frac{d_s^l}{\tau}\right)\xi_s^l > 0, \ a_s^l > 0, \ s = \frac{n}{2}+1,..,n-1, \end{cases} \tag{5.3.14}$$

with l = 1, 2, ..., K. Leading to $L^{n,K}\xi_k^l > 0, s = 1,2,...,n-1$ and $l = 1,2,...,K$, which is a contradiction. We conclude that $\xi_j^k \ \geq \ 0, \ \forall \ (j,k) \ \in \ \bar{Q}^{n,K}$.
This minimum principle is used as a tool to prove the following Lemma.

Lemma 5.3.3. *(Uniform stability estimate) Given* Z_j^k *a mesh function at a certain time level* t_k *with* $Z_0^k = Z_n^k = 0$. *It follows that*

$$\left|Z_j^k\right| \leq \frac{1}{b_0} \max_{1 \leq i \leq n-1} \left|L^{n,K} Z_i^k\right|, \ for \ 1 \leq j \leq n, and\, 1 \leq k \leq K.$$

Where b_0 *remain the same as specified in section 5.1 above.*

Proof. Given the mesh function

$$(\xi^{\pm})_j^k = \frac{1}{b_0} \max_{1 \leq i \leq n-1} \left|L_\varepsilon^{n,K} Z_i^k\right| \pm Z_j^k, 1 \leq j \leq n, \text{and } 1 \leq k \leq K,$$

with $b_j^k \geq b_0 > 0$ to ensure the uniqueness of the solution to (5.3.7) - (5.3.8). It follows that $(\xi^{\pm})_0^k \geq 0$ and $(\xi^{\pm})_n^k \geq 0$. In addition, for $0 \leq j \leq n$, and $1 \leq k \leq K$,

$$L^{n,K}(\xi^{\pm})_j^k = \frac{-b_j^k}{b_0} \max_{1 \leq i \leq n-1} \left|L^{n,K} Z_i^k\right| \pm L^{n,K} z_j^k, 1 \leq j \leq n, \text{and } 1 \leq k \leq K,$$

with $0 \leq j \leq n, (-b_j^k)/(b_0) \leq -1$; and $L^{n,K}(\xi^{\pm})_j^k \leq 0$. Using the discrete minimum principle Lemma 5.3.2, we obtain $(\xi^{\pm})_j^k \geq 0, \ \forall \ 0 \leq j \leq n, 1 \leq k \leq K$,
which completes the proof.

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5.4 Convergence analysis of the FOFDM

This section analyse the FOFDM which we described in section 5.3. We focus on the interval $[-1, 0]$ for the analysis of the model, since the analysis on $(0, 1]$ can also be done similarly.

To start, let consider the operator L^K from (5.3.3) as:

$$
\begin{aligned}
L^K z(x,t_k) \; &:= \; (\varepsilon + x^2)\frac{d^2 z(x,t_k)}{dx^2} + a(x,t_k)\frac{dz(x,t_k)}{dx} - \left(b(x,t_k) + \frac{d(x,t_k)}{\tau}\right) z(x,t_k) \\
&= \; f(x,t_k) - d(x,t_k)\frac{z(x,t_{k-1})}{\tau}. \qquad (5.4.15)
\end{aligned}
$$

The local truncation error of the space discretization on $[-1, 0] \times [0, T]$
(e.g. $j = 1, 2, ..., n/2 - 1, k = 1, 2, ..., K$), is given by:

$$
\begin{aligned}
L^{n,K}(U_j^k - z_j^k) \; &= \; \left(L^K - L^{n,K}\right) z_j^k, \\
&= \; (\varepsilon + x_j^2) z''_{j,k} + \tilde{a}_j^k z_j^k - \left[\frac{(\varepsilon + x_j^2)}{\tilde{\phi}_j^{2^k}}(z_{j+1}^k - 2z_j^k + z_{j-1}^k) + \frac{\tilde{a}_j^k}{h}(z_j^k - z_{j-1}^k)\right] \\
&= \; (\varepsilon + x_j^2) u''_{j,k} - \frac{(\varepsilon + x_j^2)}{\tilde{\phi}_j^{2^k}}\left[h^2 u''_{j,k} + \frac{h^4}{24}(z^{(iv)})^k(\xi_1) + \frac{h^4}{24}(z^{(iv)})^k(\xi_2)\right] \\
&+ \; \frac{\tilde{a}_j^k h}{2} z''_{j,k} - \frac{\tilde{a}_j{}^k h^2}{6} z'''_{j,k} + \frac{\tilde{a}_j{}^k h^3}{24}(z^{(iv)})^k(\xi_3), \qquad (5.4.16)
\end{aligned}
$$

with $\xi_1, \in (x_j, x_{j+1})$, $\xi_2, \xi_3 \in (x_{j-1}, x_j)$.
Using the expression of $\tilde{a}_j^k$ from (5.3.10) and the Taylor expansions of a_{j-1}^k up to order four, and the truncated Taylor expansion $1/\tilde{\phi}_j^{2^k} = 1/h^2 - \tilde{a}_j{}^k/\varepsilon h$, give

$$
\begin{aligned}
L^{n,K}\left(U_j^k - z_j^k\right) \; &= \; \frac{3}{2} a_j^k u''_{j,k} h + \left[-\frac{3a'_{j,k}}{2} z''_{j,k} - \frac{\varepsilon}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a_j^k}{6} z'''_{j,k}\right] h^2 \\
&+ \left[\frac{3a''_{j,k}}{4} z''_{j,k} - \frac{a_j^k}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) + \frac{a'_{j,k}}{12} z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3)\right] h^3 \\
&+ \left[-\frac{13a'''_{j,k}}{24} z''_{j,k} - \frac{a'_{j,k}}{48}\left((u^{(iv)})^k(\xi_1)) + (z^{(iv)})^k(\xi_2)\right) - \frac{a''_{j,k}}{24} z'''_{j,k}\right] h^4 \\
&+ \left[-\frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3)\right] h^4, \qquad (5.4.17)
\end{aligned}
$$

with ξ's in (x_{j-1}, x_{j+1}). We can also bound the coefficients of $u_j^k, z'_{j,k}, \cdots, (z^{(iv)})^k(\xi_{*_j})$ by a constant.

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The equation (5.4.17) may be written as follows:

$$L^{n,K}\left(U_j^k - z_j^k\right) = M_1 h + R_n(x_j), \tag{5.4.18}$$

where the coefficients M_1 and R_n are given by:

$$M_1 = \frac{3a_j}{2} z''_{j,k},$$

$$\begin{aligned}
R_n^k(x_j) &= h^2 \left[\frac{3a'_{j,k}}{3} - \frac{\varepsilon}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a_j^k}{6} z'''_{j,k} \right] \\
&+ h^3 \left[\frac{3a''_{j,k}}{4} z''_{j,k} - \frac{a_j^k}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) + \frac{a'_{j,k}}{12} z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3) \right] \\
&+ h^4 \left[\frac{13a'''_{j,k}}{24} z''_{j,k} - \frac{a'_{j,k}}{48}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a''_{j,k}}{24} z'''_{j,k} - \frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3) \right],
\end{aligned}$$

or

$$\left| L_1^{n,K}(U_j^k - z_j^k) \right| = \mathcal{O}(h),\ \forall j = 1(1)\frac{n}{2} - 1.$$

Meaning

$$\left| L_1^{n,K}(U_j^k - z_j^k) \right| \le Mh, \forall j = 1(1)\frac{n}{2} - 1.$$

Similarly, we can easily come to the following result

$$\left| L_2^{n,K}(U_j^k - z_j^k) \right| \le Mh, \forall j = \frac{n}{2}(1)n + 1.$$

From Lemma 5.3.3, we obtain

Theorem 5.4.1. *Consider U_j^k the numerical solution of* (5.3.7)-(5.3.10) *and z_j^k the solution to* (5.3.2) - (5.3.3) *at time level t_k. Then,*

$$\max_{0\le j\le n} \left| U_j^k - z_j^k \right| \le Mh,\ k = 1(1)K + 1. \tag{5.4.19}$$

Where M is a positive constant independent of ε, τ and h.

Applying the triangular inequality $\left|U_j^k - u_j^k\right| \le \left|U_j^k - z_j^k\right| + |z_j^k - u_j^k|$, and using Lemma 5.3.3, Theorem 5.4.1 and the global error, we come to the following result:

Theorem 5.4.2. *Let U_j^k be the numerical solution of* (5.3.7)-(5.3.10) *and u_j^k the solution to* (5.1.3)-(5.1.4) *at the grid point (x_j, t_k). Then, there exists a positive constant M independent of ε, τ and h such that*

$$\max_{0\le j\le n} \left| U_j^k - u_j^k \right| \le M(h + \tau),\ k = 1(1)K + 1. \tag{5.4.20}$$

In section 5.5 below, we apply Richardson extrapolation to improve the accuracy and the order of convergence of the proposed scheme.

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5.5 Richardson extrapolation on the FOFDM

Let us rewrite equation (5.4.18) as follows:

$$L^{n,K}\left(U_j^k - z_j^k\right) = M_1 h + M_2 h^2 + R_n(x_j), \tag{5.5.1}$$

the coefficients M_1, M_2 and R_n are given by:

$$M_1 = \frac{3a_j}{2} z''_{j,k}.$$

$$M_2 = \frac{3a'_{j,k}}{3} - \frac{\varepsilon}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a_j^k}{6} z'''_{j,k}.$$

$$R_n^k(x_j) = h^3\left[\frac{3a''_{j,k}}{4} z''_{j,k} - \frac{a_j^k}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) + \frac{a'_{j,k}}{12} z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3)\right]$$
$$+h^4\left[\frac{13a'''_{j,k}}{24} z''_{j,k} - \frac{a'_{j,k}}{48}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a''_{j,k}}{24} z'''_{j,k} - \frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3)\right].$$

The symbols ξ's and $z_j^k, z'_{j,k}, \cdots, (z^{(iv)})^k(\xi_{*_j})$ are defined in the same way as the ones used in (5.4.16). Consider μ_{2n} the mesh obtained after bisecting each mesh interval in μ_n, i.e.,

$$\mu_{2n} = \{\bar{x}_i\} \text{ with } \bar{x}_0 = -1, \;\; \bar{x}_n = 1 \;\text{ and }\; \bar{x}_j - \bar{x}_{j-1} = \bar{h} = h/2, \;\; j = 1, 2, ..., 2n.$$

Consider $\bar{U}_j^k$ the numerical solution on μ_{2n}. M and p positive real number. After rewriting the equation (6.5.1) in terms of $\bar{U}_j^k$ we come to the following

$$L^{n,K}\left(\bar{U}_j^k - \bar{z}_j^k\right) = M\bar{h} + p\bar{h}^2 + R_{2n}^k(\bar{x}_j), 1 \le j \le 2n-1. \tag{5.5.2}$$

We also note that $\bar{z}_j^k \equiv z_j^k$.
After multiplying (5.5.2) by 2, it follows that

$$2L^{n,K}\left(\bar{U}_j^k - \bar{z}_j^k\right) = 2M\bar{h} + 2p\bar{h}^2 + 2R_{2n}^k(\bar{x}_j), 1 \le j \le 2n-1, \tag{5.5.3}$$

meaning

$$L^{n,K}\left(2\bar{U}_j^k - 2\bar{z}_j^k\right) = 2M\bar{h} + 2p\bar{h}^2 + 2R_{2n}^k(\bar{x}_j), 1 \le j \le 2n-1. \tag{5.5.4}$$

Let (6.5.1) be in terms of M and p. After subtracting (5.5.1) from (5.5.4), we get

$$L^{n,K}\left((2\bar{U}_j^k - U_j^k) - z_j^k\right) = p\bar{h}^2 + 2R_{2n}^k(\bar{x}_j), 1 \le j \le 2n-1, \tag{5.5.5}$$

or

$$L^{n,K}\left((2\bar{U}_j^k - U_j^k) - z_j^k\right) = 0(h^2), 1 \le j \le 2n-1,$$

The numerical solution $U_j^{ext,k} := 2\bar{U}_j^k - U_j^k$ is another numerical approximation of z_j^k.

From Lemma 5.3.3 we come to the following result

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Theorem 5.5.1. *Given $U_j^{ext,k}$ the numerical solution approximation, obtained via the Richardson extrapolation based on FOFDM (5.3.7)-(5.3.10) and z_j^k the solution to (5.3.2) - (5.3.3) at time level t_k. Then, there exists a constant M independent of ε, τ and h such that*

$$\max_{0\le j\le n} |U_j^{ext,k} - z_j^k| \le Mh^2,\ k = 1(1)K+1. \tag{5.5.6}$$

After applying triangular inequality; the local error leads to

$$\left|U_j^{ext,k} - u_j^k\right| \le \left|U_j^{ext,k} - z_j^k\right| + |z_j^k - u_j^k|. \tag{5.5.7}$$

Lemma 5.2.1 together with the theorem 5.5.1,lead to the result below:

Theorem 5.5.2. *Consider $U_j^{ext,k}$ the numerical solution of (5.3.7)-(5.3.10) and z_j^k the solution to (5.1.3)-(5.1.4) at the grid point (x_j, t_k). Then, there exists a constant M independent of ε, τ and h such that*

$$\max_{0\le j\le n} \left|U_j^{ext,k} - u_j^k\right| \le M(h^2+\tau),\ k = 1(1)K+1. \tag{5.5.8}$$

The next section deals with two numerical examples. The results of these examples are presented in tables to confirm the theoretical results on the accuracy and the order of convergence of the scheme. The discussion on these results are in included in the last section 8 of some concluding remarks.

5.6 Numerical examples

Example 5.6.1. *Consider the following time dependent singularly perturbed turning point problem:*

$$\left.\begin{aligned}(\varepsilon + x^2)\, u_{xx} + 2x\,(1+t^2)\, u_x - [x^2+1+\cos(\pi x t)]\, u - 2u_t = f(x,t)\\ u(-1,t) = u(1,t) = 1; \forall t\in[0,1]; 0<\varepsilon\le 1.\end{aligned}\right\}$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$ near $(x,t) = (0,t), \forall t \in [0,1]$. The exact solution is

$$u(x,t) = \varepsilon \exp\left(-\frac{t}{\varepsilon}\right) \arctan\left(\frac{x}{\sqrt{\varepsilon}}\right) - \varepsilon^{2/3}\exp(-xt),$$

at $t = 0$

$$u(x,0) = \varepsilon \arctan\left(\frac{x}{\sqrt{\varepsilon}}\right) - \varepsilon^{2/3}.$$

We obtain the expression of $f(x,t)$ after substituting $u(x,t)$ and its derivatives into the equation (5.6.1).

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Example 5.6.2. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{array}{r}(\varepsilon+x^2)\,u_{xx}+2x\,(1+t^2)\,u_x-\left[x^2+1+\cos(\pi xt)\right]u-(3+xt)\,u_t=f(x,t)\\ u(-1,t)=u(1,t)=\varepsilon\exp\left(-\frac{t}{\varepsilon}\right)\exp\left[\arctan\left(\frac{1}{\sqrt{\varepsilon}}\right)\right];\forall t\in[0,1];0<\varepsilon\le 1.\end{array}\right\}$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$near $(x,t)=(0,t),\forall t\in[0,1]$. The exact solution is

$$u(x,t)=\varepsilon\exp\left(-\frac{t}{\varepsilon}\right)\exp\left[\arctan\left(\frac{x^2}{\sqrt{\varepsilon}}\right)\right].$$

The expression of $f(x,t)$ is obtained after substituting $u(x,t)$ into (5.6.2).

The formula to calculate the maximum errors at all mesh points and the numerical rates of convergence before extrapolation are given by

$$E^{\varepsilon,n,K}:=\max_{0\le j\le n;0\le k\le K}\left|U_{j,k}^{\varepsilon,n,K}-u_{j,k}^{\varepsilon,n,K}\right|$$

In case the exact solution is unknown, we use a variant of the double mesh principle

$$E^{\varepsilon,n,K}:=\max_{0\le j\le n;0\le k\le K}\left|U_{j,k}^{\varepsilon,n,K}-U_{j,k}^{\varepsilon,2n,2K}\right|,$$

where $u_{j,k}^{\varepsilon,n,K} and U_{j,k}^{\varepsilon,n,K}$ above represent respectively the exact and the approximate solutions obtained using a constant time step τ and space step h. Similarly, $U_{j,k}^{\varepsilon,2n,2K}$ is found using the constant time step $\frac{\tau}{2}$ and space step $\frac{h}{2}$:

$$r_l=r_k\equiv r_{\varepsilon,k}:=\log_2\left(E^{\varepsilon,n,K}/E^{\varepsilon,2n_l,2K_l}\right),l=1,2,...$$

In addition, we compute $E_{n,K}=\max\limits_{0<\varepsilon\le 1}E_{\varepsilon,n,K}$.
Finally, the numerical rate of uniform convergence are given by

$$R_{n,k}:=\log_2\left(E_{n,K}/E_{2n,2K}\right).$$

For a fixed mesh, we see that the maximum nodal errors remain constant for small values of ε (see tables 5.1 and 5.5). Moreover, results in tables 5.3 and 5.7 show that the proposed method is essentially first order convergent.

The calculation of both the maximum errors and numerical rates of convergence after extrapolation are given by the following formulae

$$E_{\varepsilon,n,K}^{ext}:=\max_{0\le j\le 2n;0\le k\le 2K}|U_j^{ext}-u_{j,k}^{\varepsilon,n,K}|\ \text{ and }\ R_k\equiv R_{\varepsilon,k}:=\log_2\left(E_{n_k}^{ext}/E_{2n_k}^{ext}\right),k=1,2,...$$

respectively, where $E_{n_k}^{ext}$ stands for $E^{\varepsilon,2n,2K}$.

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Table 5.1: Maximum errors for Example 5.6.1 before extrapolation

ε	$n = 16$, $K = 16$	$n = 32$, $K = 32$	$n = 64$, $K = 64$	$n = 128$, $K = 128$	$n = 256$, $K = 256$
10^{-4}	1.64E-01	8.99E-02	4.66E-02	2.38E-02	1.20E-02
10^{-5}	1.67E-01	9.13E-02	4.72E-02	2.40E-02	1.21E-02
10^{-8}	1.68E-01	9.21E-02	4.76E-02	2.42E-02	1.22E-02
10^{-11}	1.68E-01	9.21E-02	4.76E-02	2.42E-02	1.22E-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.68E-01	9.21E-02	4.76E-02	2.42E-02	1.22E-02

Table 5.2: Maximum errors for Example 5.6.1 after extrapolation

ε	$n = 16$, $K = 16$	$n = 32$, $K = 32$	$n = 64$, $K = 64$	$n = 128$, $K = 128$	$n = 256$, $K = 256$
10^{-4}	2.03E-01	5.33E-02	1.20E-02	3.55E-03	3.55E-03
10^{-5}	2.13E-01	5.92E-02	1.46E-02	3.34E-03	7.82E-04
10^{-8}	2.17E-01	6.19E-02	1.61E-02	4.05E-03	1.01E-03
10^{-11}	2.17E-01	6.20E-02	1.61E-02	4.08E-03	1.02E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	2.17E-01	6.20E-02	1.61E-02	4.08E-03	1.02E-03

Table 5.3: Rates of convergence for the Example 5.6.1 before extrapolation, $n_k = 16,\ 32, 64, 128, 256$

ε	r_1	r_2	r_3	r_4
10^{-4}	0.87	0.95	0.97	0.98
10^{-5}	0.87	0.95	0.98	0.99
10^{-8}	0.87	0.95	0.98	0.99
10^{-11}	0.87	0.95	0.98	0.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	0.87	0.95	0.98	0.98

Table 5.4: Rates of convergence for the Example 5.6.1 after extrapolation, $n_k = 16, 32, 64, 128, 256$

ε	r_1	r_2	r_3	r_4
10^{-4}	1.93	2.15	1.76	0.00
10^{-5}	1.85	2.02	2.13	2.09
10^{-8}	1.81	1.94	1.99	2.00
10^{-11}	1.81	1.94	1.98	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.81	1.94	1.98	2.00

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Table 5.5: Maximum errors for Example 5.6.2 before extrapolation

ε	$n=16$ $K=16$	$n=32$ $K=32$	$n=64$ $K=64$	$n=128$ $K=128$	$n=256$ $K=256$
10^{-4}	6.99E-01	3.93E-01	2.08E-01	1.07E-01	5.45E-02
10^{-5}	7.44E-01	4.09E-01	2.14E-01	1.09E-01	5.54E-02
10^{-9}	7.77E-01	4.23E-01	2.18E-01	1.11E-01	5.59E-02
10^{-13}	7.77E-01	4.23E-01	2.19E-01	1.11E-01	5.59E-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	7.77E-01	4.23E-01	2.19E-01	1.11E-01	5.59E-02

Table 5.6: Maximum errors for Example 5.6.2 after extrapolation

ε	$n=16$ $K=16$	$n=32$ $K=32$	$n=64$ $K=64$	$n=128$ $K=128$	$n=256$ $K=256$
10^{-4}	7.26E-01	2.08E-01	5.45E-02	1.38E-02	3.47E-03
10^{-5}	7.66E-01	2.14E-01	5.54E-02	1.40E-02	3.51E-03
10^{-9}	9.51E-01	2.66E-01	6.49E-02	1.41E-02	3.52E-03
10^{-13}	9.55E-01	2.71E-01	7.03E-02	1.77E-02	4.33E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	9.55E-01	2.71E-01	7.03E-02	1.77E-02	4.43E-03

Table 5.7: Rates for convergence of the Example 5.6.2 before extrapolation, $n_k = 16, 32, 64, 128,$ 256

ε	r_1	r_2	r_3	r_4
10^{-4}	0.83	0.92	0.96	0.98
10^{-5}	0.86	0.94	0.97	0.98
10^{-9}	0.88	0.95	0.98	0.99
10^{-13}	0.88	0.95	0.98	0.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	0.88	0.95	0.98	0.99

Table 5.8: Rates for convergence of the Example 5.6.2 after extrapolation, $n_k = 16\ 32, 64, 128,$ 256

ε	r_1	r_2	r_3	r_4
10^{-4}	1.80	1.93	1.98	1.99
10^{-5}	1.84	1.95	1.99	2.00
10^{-9}	1.84	2.03	2.21	2.00
10^{-13}	1.82	1.95	1.99	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.82	1.95	1.99	2.00

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5.7 Summary

The main aim of this chapter was to design a fitted operator finite difference method to solve a class of time dependent singularly perturbed problems when the highest spatial derivative is affected by a quadratic perturbation coefficient $(\varepsilon + x^2)$ with the solution exhibiting an interior layer due to the presence of a turning point. This approach utilizes uniform meshes to obtain a discrete problem in time and space with respect to the perturbation parameter ε.

We first established sharp bounds on the solution and its derivatives and then we discretized the problem in time and space. These bounds were used to prove uniform convergence of the proposed numerical method in both time and space. The first order uniform convergence shown theoretically, in time and space variables; was confirmed numerically through two test examples. We also applied Richardson extrapolation to improve the accuracy and the order of convergence of the numerical solution of the proposed fitted operator finite difference method. We concluded our study with numerical simulations to confirm the theoretical results.

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Chapter 6

A robust fitted operator finite difference method for singularly perturbed turning point problems with a linear diffusion factor and an interior layer

We onsider a family of singularly perturbed problems with a linear diffusion factor $\varepsilon+x$, where ε is a perturbation parameter. The convection coefficient of these equations possesses a root called turning point, which induces an interior layer in the solution. In the process of solving the problem, we first start with the analysis, then we derive the bounds of the solution and its derivatives. Afterwards, we construct the method and analyse its convergence properties. The scheme obtained is first order uniformly convergent with respect to the perturbation parameter ε. We also use Richardson extrapolation to improve the accuracy and the order of convergence of the proposed scheme up to two. Numerical results are presented to support the theoretical findings.

6.1 Introduction

We consider the class of singularly perturbed differential equations

$$Lu \ := \ \varepsilon u'' + a(x)u' - \ b(x)u = \ f(x), \ x \in \Omega = (0,1), \tag{6.1.1}$$

with the actual boundary conditions

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$$u(0) = \gamma_1, \; u(1) = \gamma_2, \tag{6.1.2}$$

where γ_1 and γ_2 are given constants, and ε is a small positive parameter ($0 < \varepsilon \ll 1$). Moerover, the functions $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently smooth in $\bar{\Omega}$ to ensure the smoothness of the solution. The condition $b(x) \geq b_0 > 0, \forall\, x \in \bar{\Omega}$ guarantees the uniqueness of the solution [59, 64].

Problems such as (6.1.1)-(6.1.2) arise in various fields of science and engineering. For instance fluid mechanics, solid mechanics, elasticity, quantum mechanics, chemical reactor theory, aerodynamics, optimal control, reaction-diffusion process, hydrodynamics, geophysics, etc (see for instance [64, 84] and the references therein).

When the perturbation parameter ε becomes very small, the solution to the problems (6.1.1)-(6.1.2) presents sudden change(s) (non-uniformly) or large gradient(s) in narrow region(s) of the domain termed layer(s). These layer(s) may be situated either at the end point(s) of the domain called boundary layer(s) or in the domain near the root(s) x_i of $a(x)$, which are called turning point(s) leading to interior layer(s).

The position(s) and the number of layer(s) within the domain depend on the properties of the convection coefficient $a(x)$ and that of the reaction coefficient $b(x)$ of the differential equation (6.1.1).

If $a(x) \neq 0$, for $0 \leq x \leq 1$, then we have one boundary layer at $x = 0$ or $x = 1$ respectively when $a(x) > 0$ or $a(x) < 0$. When $a(x) \equiv 0$ throughout the domain, and $b(x) < 0$; then the solution of the problem presents twin boundary layers at $x = 0$ and $x = 1$. But if $b(x) > 0$, then we have a rapidly oscillatory solution. These cases are called non-turning point problems, and they have extensively been studied in the literature (see e.g [29, 49, 54, 59]). Nevertheless, when $b(x)$ changes the sign we have a classic turning point.

In case we have the existence of $x_i, i = 1, 2, 3, ..., n$ such that $a(x_i) = 0$ and $a'(x_i) \neq 0$, then the x_i are called turning points, leading to interior layers or possible boundary layers respectively when $a'(x_i) > 0$ or $a'(x_i) < 0$.

Turning point problems give rise to interior layer or to twin boundary layers. For more information on turning point problems, interested reader may wish to consult [1, 9, 16, 43, 56, 64, 65, 76, 85, 86, 87, 90]. We also note that interior layers may be caused by non-smooth coefficient functions or discontinuous data (see e.g [4, 11, 24]).

The turning and non-turning point problems mentioned above are widely studied in the literature. Nevertheless, their applications in fluid dynamics and biology are problems in which the coefficient of the highest derivatives are functions of x and ε.

These problems have received little attention from the research community. Liseikin [52]

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considered the case $g(x,\varepsilon) = -(\varepsilon + px)^{\beta}$ for $\beta \geq 1$ and studied the problem for $p = 0$ and $p = 1$. In [51] (pp. 106-111), he derived bounds on the solution and its derivatives for the case $g(x,\varepsilon) = -(\varepsilon + x)^{\beta}$ for some prescribed values of β. In addition, for $\beta = 1$ (see pp. 256-262), he designed a numerical method and analysed its convergence. Nevertheless, in particular, when $p = 1$ and $\beta = 1$, the problem describes filtration of a liquid through a neighbourhood about a circular orifice of radius $r = \varepsilon$ [52, 77]. From the best of our knowledge, there is no other work recorded in the literature. To this end, we consider the case $g(x,\varepsilon) = \varepsilon + x$.

The main objectives of this chapter are (1) to construct and analyse the fitted operator finite difference method and (2) to improve the accuracy and the order of convergence of the scheme designed using Richardson extrapolation [66].

In this chapter, we aim to study the following singularly perturbed problems

$$Lu := (\varepsilon + x)u'' + a(x)u' - b(x)u = f(x), x \in \Omega; \tag{6.1.3}$$

$$u(0) = \gamma_1, \ u(1) = \gamma_2, \tag{6.1.4}$$

with the following assumptions to guarantee an interior layer near $x = 0.5$: (i) $a(0.5) = 0$, and $a'(0.5) > 0$, guaranteeing the existence of the turning point, (ii) $b(x) \geq b_0 > 0, \forall\, x \in \bar{\Omega}$, indicating that the problem (6.1.3) has only one solution and satisfies the minimum principle, and (iii) $|a'(x)| \geq |a'(0.5)|/2, \forall\, x \in \bar{\Omega}$, implying that 0.5 is the unique turning point in $\bar{\Omega}$.

The interesting aspect of the problem (6.1.3)-(6.1.4) is that, the order of the underlying reduced equation ($\varepsilon = 0$), remain the same with that of the original equation; contrary to the cases in classical singularly perturbed problems and in particular for problems like (6.1.1)-(6.1.2), whose order of the reduced equation is lowered to one.

In addition, in order to keep the diffusion coefficient positive; we restrict our discussion throughout the paper to positive values of $x \in \bar{\Omega}$. However, translation or assumptions can be made to deal with both positive and negative diffusion coefficient. Thereupon numerical examples are provided to confirm the uniform convergence of these problems for all x in $[-1, 1]$.

Briefly, the outline is as follows. In section 6.2, we derive bounds on the solution and its derivatives. The construction of a fitted operator finite difference method (FOFDM) is given in section 6.3. In section 6.4, we discuss the convergence analysis of the proposed numerical method and we show that the scheme is first order uniformly convergent with respect to the perturbation parameter ε. In section 6.5, Richardson extrapolation is used as an acceleration technique to improve the accuracy and the order of convergence of the method up to two. The results on numerical experiments to confirm the theoretical findings are given in section

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6.6 and finally some concluding remarks are presented in section 6.7.

6.2 Qualitative results

The operator L satisfies the following continuous minimum principle

Lemma 6.2.1. Consider ψ a smooth function such that $\psi(0) \geqslant 0$, $\psi(1) \geqslant 0$ and $L\psi(x) \leq 0$, $\forall\, x \in \Omega$. We have $\psi(x) \geqslant 0$, $\forall\, x \in \bar{\Omega}$.

Proof. We proceed by contradiction to prove this Lemma. To start, we consider $x^* \in \bar{\Omega}$ and $\psi(x^*) = \min_{x\in\bar{\Omega}} \psi(x) < 0$. It is evident that, $x^* \notin \{0, 1\}$, $\psi'(x^*) = 0$ also $\psi''(x^*) \geqslant 0$. Then

$$L\psi(x^*) := (\varepsilon + x)\psi''(x^*) + \; a(x^*)\psi'(x^*) - \; b(x^*)\psi(x^*) \; > 0,$$

this leads to a contradiction. Thus, $\psi(x^*) \geqslant 0$ and consequently $\psi(x) \geqslant 0$, $\forall x \in \bar{\Omega}$.

Lemma 6.2.2. Consider $u(x)$ the solution of (6.1.3)-(6.1.4). It follows that

$$||u(x)|| \leq b_0^{-1}||f|| + \max\left(|\gamma_1|, |\gamma_2|\right), \forall x \in \bar{\Omega}.$$

The notation $||.||$ stands for the maximum norm, and b_0 is a positive constant as specified in the introduction.

Proof. Let us consider the following comparison function

$$\Pi^{\pm}(x) = b_0^{-1}||f|| + \max\left(|\gamma_1|, |\gamma_2|\right) \pm u(x), \forall x \in \bar{\Omega}.$$

The positive constant b_0, is chosen such that $b(x) \geq b_0 > 0, \forall x \in \bar{\Omega}$ to guarantee the uniqueness of the solution to (6.1.3)-(6.1.4), $\gamma_1 = u(0)$ and $\gamma_2 = u(1)$.

Thus $\Pi^{\pm}(0) \geq 0$, $\Pi^{\pm}(1) \geq 0$ and

$$L\Pi^{\pm}(x) = -\frac{b(x)}{b_0}||f|| - b(x)\max\left(|\gamma_1|, |\gamma_2|\right) \pm Lu(x) \leq 0, \forall x \in \bar{\Omega}.$$

From Lemma 6.2.1, we get

$$\Pi^{\pm}(x) \geq 0, \forall x \in \bar{\Omega},$$

which ends the proof.

The lemma below describes the Inverse Monotonicity.

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Theorem 6.2.1. *[51] Let $F(x,u,u') = a(x)u' - b(x)u - f(x)$ be a smooth function in $[0,1]\times\mathbb{R}^2$, where $a(x), b(x), f(x)$ are functions as described in (6.1.3)-(6.1.4). The problem (6.1.3)-(6.1.4) is said to be inverse monotone for $F(x,u,u') \in C^2((0,1)) \cap C([0,1])$ if one of the following conditions imposed on F is satisfied:*

(1) $F(x,u,u')$ is strictly increasing in u, i.e., $F(x,u_1,z) < F(x,u_2,z)$ if $u_1 < u_2$,

(2) $F(x,u,u')$ is weakly increasing in u and there exists a positive constant $C > 0$, such that $|F(x,u,z_1) - F(x,u,z_1)| \le C|z_1 - z_2|$.

Proof. See [51] with $d(x) = x,\ l = 1,\ \forall x \in [0,1]$.

Throughout this paper we consider the partition on $\bar{\Omega} = [0,1]$ given by $\Omega_L = [0,\delta)$, $\Omega_C = [\delta,\delta_1], \Omega_R = (\delta_1, 1]$,with $0 < \delta, \delta_1 \le 1/4$. where L stands for the left side of the layer region, C for the central part (or the layer region) and R for the right side of the layer region. In addition, $\Omega_C = \Omega_C^- \cup \Omega_C^+$, with $\Omega_C^- = [\delta, 0.5)$ and $\Omega_C^+ = [0.5, \delta_1]$.

Lemma 6.2.3. *Let $u(x)$ be the solution to (6.1.3)-(6.1.4); then*
$|u^{(j)}| \le C, \forall x \in \Omega_L U \Omega_R$. Where C is a positive real number.

Proof. This Lemma is the immediate consequence of Theorem 6.2.1 for inverse monotonicity with $C = M$ as specified in [51], $\forall x \in \Omega_R$,
$F[x,-M,u'] \le F[x,u,u'] \le F[x,M,u']$ leading to $-M \le u(x) \le M$. This completes the proof. We can in the similar way show the proof for $x \in \Omega_L$.

The lemma 6.2.4 below concentrates on the bounds of the solution and its derivatives in the layer region. In this Lemma, we follow Liseikin [51] work to adapt it to our problem. We note that in this Lemma, the convection coefficient at a point x_0 is given by $a(x_0) = a$ where $x_0 \in \Omega_C^+$ or $x_0 \in \Omega_C^-$.

Lemma 6.2.4. *[51] Consider $u(x)$ the solution to (6.1.3)-(6.1.4). Then, it follows that*

1) for $x \in \Omega_C^+ = (0.5, \delta_1]$ and $x_0 \in \Omega_C^+$ such that $a(x_0) = a > 0$, $j = 0,1,2,3,4$; we have

$$\left|u^{(j)}(x)\right| \le M \begin{cases} 1 + (\varepsilon + x)^{1-a-j}, \ if\ 0 < a < 1, \\ \\ 1 + (\varepsilon + x)^{-j}\left|\ln^{-1}\left(\varepsilon + \frac{1}{2}\right)^{-1}\right|, \ if\ a = 1, \\ \\ 1 + \varepsilon^{a-1}(\varepsilon + x)^{1-a-j}, \ if\ a > 1. \end{cases} \tag{6.2.1}$$

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2) for $x \in \Omega_C^- = [\delta, 0.5], x_0 \in \Omega_C^-$, $a(x_0) = a \leq 0, j = 0, 1, 2, 3, 4$; and p a whole number such that $a + p \geq 0, a + p - 1 < 0$; we have

$$\left|u^{(j)}(x)\right| \leq M \begin{cases} 1, \; if \; a < 0, j \leq p, \\ \\ 1 + (\varepsilon + x)^{1-j-p} \left|\ln(\varepsilon + x)\right|, \; if \; a + p = 0, \; j > p, \\ \\ 1 + (\varepsilon + x)^{-a-j}, \; if \; a + p > 1, \; j > p, \end{cases} \tag{6.2.2}$$

where M is a positive constant independent of ε.

Proof. 1) We first prove Lemma 6.2.4 for $x \in \Omega_C^+ = (0.5, \delta_1], a > 0$.
Consider u the solution to (6.1.3)-(6.1.4), then from the Lemma on the inverse monotonicity (6.2.1), we have

$$|u(x)| \leq M. \tag{6.2.3}$$

According to Liseikin [51], there exists a positive constant m such that (6.1.3)-(6.1.4) and (6.2.3) lead to

$$\left|u^{(j)}(x)\right| \leq M \begin{cases} 1, \; 0.5 < m \leq x \leq \delta_1, \\ \\ \varepsilon^{-j}, \; 0.5 \leq x \leq \delta_1, \end{cases} \tag{6.2.4}$$

$$j = 1, 2, 3, 4.$$

Consider $a > 0$. The equation (6.1.3) can be rewritten as

$$u''(x) = -\frac{a(x)u'(x)}{\varepsilon + x} + \frac{b(x)u(x) + f(x)}{\varepsilon + x},$$

and integrating both sides we have

$$u'(x) = -\int_{1/2}^{x} \frac{a(\eta)u'(\eta)}{\varepsilon + \eta} d\eta + \int_{1/2}^{x} \frac{b(\eta)u(\eta) + f(\eta)}{\varepsilon + x} d\eta,$$

also we can express this derivative by the formula:

$$u'(x) = u'(0.5) \left(\frac{\varepsilon}{\varepsilon + x}\right)^a \exp\left[-g_1(x)\right] + g_2(x), 0.5 \leq x \leq \delta_1, \tag{6.2.5}$$

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where

$$g_1(x) = \int_{1/2}^{x} \frac{a(\eta)}{\varepsilon + \eta} d\eta.$$

Integrating by parts leads to

$$g_1(x) = a(x) \ln(\varepsilon + x) + \int_{1/2}^{x} a'(\eta) \ln(\varepsilon + \eta)\, d\eta,$$

with $g_1(0.5) = 0$, since $a(0.5) = 0$. We also have

$$g_2(x) = (\varepsilon + x)^{-a} \int_{1/2}^{x} [b(\eta)u(\eta) + f(\eta)] (\varepsilon + \eta)^{a-1} \exp[g_1(\eta) - g_1(x)] d\eta.$$

From (6.2.4) with $a > 0$, we have

$$|g_j(x)| \leq M,\ j = 1, 2;\ 0.5 < x \leq \delta_1.$$

Using triangular inequalities, (6.2.5) leads to

$$|u'(x)| \leq \left| u'(0.5) \left(\frac{\varepsilon}{\varepsilon + x} \right)^{a} \exp\left[-g_1(x)\right] \right| + |g_2(x)|,$$

$$|u'(x)| \leq M\, |u'(0.5)| \left(\frac{\varepsilon}{\varepsilon + x} \right)^{a} + M,$$

$$|u'(x)| \leq M \left[1 + |u'(0.5)| \left(\frac{\varepsilon}{\varepsilon + x} \right)^{a} \right], 0.5 < x \leq \delta_1. \tag{6.2.6}$$

Let $0 < a < 1,\ 0 < \varepsilon << 1$, and m a positive constant, with $x = m \in \Omega_C^+$ such that (6.2.4) and (6.2.5) lead to

$$|u'(0.5)| \left(\frac{\varepsilon}{\varepsilon + m} \right)^{a} \leq M,$$

$$i.e., |u'(0.5)| \leq M \left(\frac{\varepsilon + m}{\varepsilon} \right)^{a} \leq M\varepsilon^{-a}.$$

From (6.2.6) we have

$$|u'(x)| \leq M \left[1 + \varepsilon^{-a} \varepsilon^{a} (\varepsilon + x)^{-a} \right],$$

leading to

$$|u'(x)| \leq M \left[1 + (\varepsilon + x)^{-a} \right],\ 0 < a < 1,\ 0.5 < x \leq \delta_1.$$

On the other hand, (6.1.3) leads to

$$u'''(x) = -\frac{[1 + a(x)]u''(x)}{\varepsilon + x} - \frac{[a'(x) - b(x)]u'(x) + b'(x)u(x) + f'(x)}{\varepsilon + x},$$

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integrating both sides lead to

$$u''(x) = -\int_{1/2}^{x} \frac{[1+a(\eta)]u''(\eta)}{\varepsilon+\eta} d\eta + \int_{1/2}^{x} \frac{[-a'(\eta)u'(\eta) - b(\eta)u'(\eta) + b'(\eta)u(\eta)f'(\eta)}{\varepsilon+\eta} d\eta.$$

Also, we can express this derivative by the following formula:

$$u''(x) = u''(0.5)\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1} \exp\left[-g_3(x)\right] + g_4(x), 0.5 \le x \le \delta_1, \tag{6.2.7}$$

where

$$g_3(x) = \int_{1/2}^{x} \frac{[1+a(\eta)]}{\varepsilon+\eta} d\eta,$$

and the integration by parts leads to

$$g_3(x) = -[1+a(x)]\ln(\varepsilon+x) + \ln(\varepsilon+\frac{1}{2}) + \int_{1/2}^{x} a'(\eta)\ln(\varepsilon+\eta)\, d\eta,$$

with $g_3(0.5) = 0$, since $a(0.5) = 0$. We also have

$$g_4(x) = (\varepsilon+x)^{-a-1}\int_{1/2}^{x}[-a'(\eta)u'(\eta) - b(\eta)u'(\eta) + b'(\eta)u(\eta) + f'(\eta)](\varepsilon+\eta)^{a}\exp[g_3(\eta) - g_3(x)]d\eta.$$

From (6.2.4) with $a > 0$, we have

$$|g_3(x)| \le M,\ |g_4(x)| \le M,\ 0.5 < x \le \delta_1.$$

The triangular inequality applied to (6.2.11) leads to

$$|u''(x)| \le \left|u''(0.5)\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1}\exp\left[-g_3(x)\right]\right| + |g_4(x)|,$$

$$|u''(x)| \le M\,|u''(0.5)|\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1} + M,$$

$$|u''(x)| \le M\left[1 + |u''(0.5)|\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1}\right], 0.5 < x \le \delta_1. \tag{6.2.8}$$

Consider $0 < a < 1,\ 0 < \varepsilon << 1$, and a positive constant m', with $x = m'$ such that (6.2.4) and (6.2.11), then

$$|u''(0.5)|\left(\frac{\varepsilon}{\varepsilon+m}\right)^{a+1} \le M,$$

$$i.e., |u''(0.5)| \le M\left(\frac{\varepsilon+m}{\varepsilon}\right)^{a+1} \le M\varepsilon^{-a-1}.$$

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Thus (6.2.8) gives

$$|u''(x)| \le M\left[1+\varepsilon^{-a-1}\varepsilon^{a+1}(\varepsilon+x)^{-a-1}\right],$$

or

$$|u''(x)| \le M\left[1+(\varepsilon+x)^{-a-1}\right],\ 0<a<1,\ 0.5<x\le\delta_1.$$

From (6.1.3) and (6.2.3), we conclude that $0<a<1$,
$0.5<x\le\delta_1$;

$$\left|u^{(j)}(x)\right| \le M\left[1+(\varepsilon+x)^{-a+1-j}\right],\ 0<\varepsilon<<1, j=1,2,3,4. \tag{6.2.9}$$

The Lemma 6.2.4 is proved for $0<a<1$.

If $a=1$, the integration of (6.2.5) from 0.5 to δ_1 gives

$$\int_{1/2}^{\delta_1} u'(\eta)d\eta = \int_{1/2}^{\delta_1} u'(0.5)\left[\frac{\varepsilon}{\varepsilon+\eta}\right]\exp\left[-g_1(\eta)\right]d\eta + \int_{1/2}^{\delta_1} g_2(\eta)d\eta, 0.5\le x\le\delta_1.$$

Using integration by parts, we get

$$A_{\delta_1}-A_{\frac{1}{2}} = u'(0.5)\left[\varepsilon ln(\varepsilon+\eta)\exp[-g_1(\eta)]|_{1/2}^{\delta_1} + \varepsilon\int_{1/2}^{\delta_1}\ln(\varepsilon+\eta)g_1'(\eta)\exp(-g_1(\eta))d\eta\right]$$

$$+\int_{1/2}^{\delta_1} g_2(\eta)d\eta.$$

$$A_{\delta_1}-A_{\frac{1}{2}} = u'(0.5)\varepsilon\ln(\varepsilon+\delta_1)\exp[-g_1(\delta_1)] - \varepsilon ln(\varepsilon+\frac{1}{2})$$

$$+\varepsilon u'(0.5)\int_{1/2}^{\delta_1}\ln(\varepsilon+\eta)\,g_1'(\eta)\exp(-g_1(\eta))d\eta + \int_{1/2}^{\delta_1} g_2(\eta)d\eta,$$

with

$$g_1'(x) = -a(x)(\varepsilon+x)^{-1},$$

$$A_{\delta_1}-A_{\frac{1}{2}} = u'(0.5)\varepsilon ln(\varepsilon+\delta_1)\exp[-g_1(\delta_1)] - \varepsilon ln\left(\varepsilon+\frac{1}{2}\right)$$

$$+\varepsilon u'(0.5)\int_{1/2}^{\delta_1}[-a(\eta)](\varepsilon+\eta)^{-1}\ln(\varepsilon+\eta)\exp(-g_1(\eta))d\eta + \int_{1/2}^{\delta_1} g_2(\eta)d\eta.$$

Knowing that

$$\left|\ln(\varepsilon+\delta_1)\exp[-g_1(\delta_1)] - \int_{1/2}^{\delta_1}[a(\eta)](\varepsilon+\eta)^{-1}\ln(\varepsilon+\eta)\exp(-g_1(\eta))d\eta\right| \le M,$$

we have,

$$|u'(0.5)|\left(\varepsilon-\varepsilon\ln\left(\varepsilon+\frac{1}{2}\right)\right) \le C\,|u'(0.5)|\,\varepsilon\left|\ln\left(\varepsilon+\frac{1}{2}\right)^{-1}\right| \le M,$$

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meaning

$$|u'(0.5)| \leq M\varepsilon^{-1}.\left|\ln^{-1}\left(\varepsilon+\frac{1}{2}\right)\right|.$$

From (6.2.6) we have

$$|u'(x)| \leq M\left[1+\varepsilon^{-1}\left|\ln^{-1}\left(\varepsilon+\frac{1}{2}\right)^{-1}\right|\varepsilon.(\varepsilon+x)^{-1}\right].$$

$$|u'(x)| \leq M\left[1+(\varepsilon+x)^{-1}\left|\ln^{-1}\left(\varepsilon+\frac{1}{2}\right)^{-1}\right|\right], 0.5 \leq x \leq \delta_1,\ 0<\varepsilon<<1,\ a=1.$$

The differentiation of (6.1.3) along with (6.2.4), lead to

$$\left|u^{(j)}(x)\right| \leq M\left[1+(\varepsilon+x)^{-j}\left|\ln^{-1}\left(\varepsilon+\frac{1}{2}\right)^{-1}\right|\right],\ 0<\varepsilon<<1, a=1,\ j=1,2,3,4. \quad (6.2.10)$$

For $a>1$, (6.2.4) in (6.2.5) and using triangular inequality; we come to the following

$$|u'(x)| \leq M\varepsilon^{-1}\left[\frac{\varepsilon}{\varepsilon+x}\right]^{a}+M,$$

leading to

$$|u'(x)| \leq M\left[1+\varepsilon^{a-1}(\varepsilon+x)^{a}\right].$$

Also, using the equation (6.1.3), we get the following formula for the second derivative

$$u''(x) = u''(0.5)\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1}\exp\left[-g_5(x)\right]+g_6(x), 0.5 \leq x \leq \delta_1, \quad (6.2.11)$$

where $g_5(x)$ and $g_6(x)$ are given by

$$g_5(x) = \int_{0.5}^{x}\frac{2}{\varepsilon+\eta}d\eta,$$

and

$$g_6(x) = (\varepsilon+x)^{-2}\int_{0.5}^{x}\frac{b'(\eta)u(\eta)+b(\eta)u'(\eta)+f'(\eta)}{\varepsilon+\eta}d\eta,$$

with $|g_5(x)| \leq M$, and $|g_6(x)| \leq M$.

Applying triangular inequality of (6.2.11) in connection with (6.1.3) and (6.2.4) we get

$$|u''(x)| \leq M\left|u''(0.5)\right|.\left(\frac{\varepsilon}{\varepsilon+x^2}\right)^{a+1}+M,$$

or

$$|u''(x)| \leq M\varepsilon^{-2}\left(\frac{\varepsilon}{\varepsilon+x^2}\right)^{a+1}+M,$$

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meaning that

$$|u''(x)| \leq M\left[1+\varepsilon^{-2}\varepsilon^{a+1}\left(\varepsilon+x^2\right)^{-a-1}\right],$$

or

$$|u''(x)| \leq M\left[1+\varepsilon^{a-1}\left(\varepsilon+x^2\right)^{-a-1}\right].$$

Thereafter from (6.1.3) and (6.2.1) we conclude that

$$\left|u^{(j)}(x)\right| \leq M\left[1+\varepsilon^{a-1}(\varepsilon+x^2)^{1-a-j}\right],\ 0.5 < x \leq \delta_1,\ 0<\varepsilon<<1, a>1,\ j=1,2,3,4, \tag{6.2.12}$$

which ends the proof for $x \in \Omega_C^+$ and $a > 0$.

2) Consider $x \in \ \Omega_C^- = [\delta, 0.5]$, and suppose the there exists a constant $x_0 \in \Omega_C^-$ such that $a(x_0) = a \leq 0$.

Then, solving (6.1.3) - (6.1.4) with respect to $u'(x)$ leads to

$$u'(x) = u'(x_0)\exp(\psi(x)) + \int_{x_0}^{x} \frac{[b(\eta)u(\eta)+f(\eta)]}{\varepsilon+\eta}\exp\left[\psi(\eta)\right]d\eta, \tag{6.2.13}$$

with $\psi(x)$ given by

$$\psi(x) = \int_{x_0}^{x}\frac{a(\eta)}{\varepsilon+\eta}d\eta.$$

It follows that

$$|\psi(x)| \leq M, \delta \leq x \leq 0.5;\ 0<\varepsilon<<1.$$

Let $x_0 \in \ [\delta, 0.5]$, using (6.2.4), we get

$$|u'(x_0)| \leq M,$$

and using triangular inequality we get

$$|u'(x)| \leq M + M\left|ln(\varepsilon+x)\right|,$$

or

$$|u'(x)| \leq M\left[1+|ln(\varepsilon+x)|\right], \delta \leq x \leq 0.5;\ 0<\varepsilon<<1, a(0.5)=0,$$

which proves Lemma 6.2.4 for $j=1, a(0.5)=a=0, p=0$.

After differentiating (6.1.3), we come to the following

$$\left|u^{(j)}(x)\right| \leq M\left[1+(\varepsilon+x)^{1-j-p}\left|ln(\varepsilon+x)\right|\right], a+p=0, j>p, j=1,2,3,4. \tag{6.2.14}$$

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Now, let $x \in [\delta, 0.5]$, $0 < \varepsilon << 1$, $p \geq 1, a < 0$ and m_3 a positive constant given by $\delta \leq m_3 \leq x \leq 0.5$, then we have

$$\psi(x) \leq -m_3 ln\left(\frac{\varepsilon+\eta}{\varepsilon+x}\right), \delta \leq m_3 \leq \eta \leq x \leq 0.5.$$

Taking exponential both sides leads to

$$\exp\left[\psi(x)\right] \leq M\left(\frac{\varepsilon+x}{\varepsilon+\eta}\right)^{m_3}, \delta \leq m_3 \leq \eta \leq x \leq 0.5,$$

Using (6.2.4) and letting $x_0 = m_3$, we can easily show that

$$|u'(x)| \leq M,\ \delta \leq m_3 \leq x \leq 0.5,$$

which gives the proof for $j = 1 \leq p, a < 0$.
Form (6.1.3) - (6.1.4); we conclude that

$$\left|u^{(j)}(x)\right| \leq M, \delta \leq x \leq 0.5, j \leq p, a < 0,\ j = 1, 2, 3, 4. \tag{6.2.15}$$

Finally, let $x \in [\delta, 0.5], 0 < \varepsilon << 1, a < 0$, and m_1 a constant given by $\delta \leq m_1 \leq x \leq 0.5$ such that

$$\psi(x) \leq -m_1 ln\left[(\varepsilon+x)^{\frac{a+1}{m_1}}\right], \delta \leq m_1 \leq x \leq 0.5,$$

it follows that

$$\exp\left[\psi(x)\right] \leq M_1\left[(\varepsilon+x)^{-a-1}\right].$$

Using (6.2.13) and letting $x_0 = m_1$, we come to the following

$$|u'(x)| \leq M\left(\varepsilon+x\right)^{-a-1},_1 \delta \leq m_1 \leq x \leq 0.5, a < 0, 0 < \varepsilon << 1;$$

This also proves Lemma (6.2.4) for $j = 1, a < 0$.
Form (6.1.3) - (6.1.4); we get

$$\left|u^{(j)}(x)\right| \leq M\left(\varepsilon+x\right)^{-a-j}, \delta \leq x \leq 0.5, a < 0,\ j = 1, 2, 3, 4, \tag{6.2.16}$$

which complete the proof of Lemma (6.2.4) for $x \in \Omega_C^-$ and $a \leq 0$.

Section 6.3 below constructs the fitted operator finite difference method (FOFDM) useful to solve the problem (6.1.3)-(6.1.4).

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6.3 Construction of the FOFDM

Consider n and $\bar{\Omega}_n$ respectively an even positive integer and the partition of the interval $\Omega = [0,1]$ such that:$x_0 = 0; x_j = x_0 + jh; j = 1, ..., n-1, h = x_j - x_{j-1}, x_n = 1$.

We dicritize (6.1.3)-(6.1.4) on $\bar{\Omega}_n$ as follows

$$L^h U_j := \begin{cases} (\varepsilon + x_j)\delta^2 U_j + \tilde{a}_j D^- U_j - \tilde{b}_j U_j = \tilde{f}_j, \\ j = 0, 1, 2, \cdots, \frac{n}{2} - 1, \\ \\ (\varepsilon + x_j)\delta^2 U_j + \tilde{a}_j D^+ U_j - \tilde{b}_j U_j = \tilde{f}_j, \\ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n-1 \end{cases} \tag{6.3.1}$$

$$U_0 = \gamma_1, \ U_n = \gamma_2, \tag{6.3.2}$$

with

$$D^- U_j = \frac{U_j - U_{j-1}}{h}, \quad D^+ U_j = \frac{U_{j+1} - U_j}{h}, \quad \delta^2 U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{\tilde{\phi}_j^2}.$$

The denominator functions $\tilde{\phi}_j^2$ are given by

$$\tilde{\phi}_j^2 = \begin{cases} \frac{h(\varepsilon + x_j)}{\tilde{a}_j}\left[\exp\left(\frac{\tilde{a}_j h}{\varepsilon + x_j}\right) - 1\right], \ j = 0, 1, 2, ..., \frac{n}{2} - 1, \\ \\ \frac{h[\varepsilon + x_j)}{\tilde{a}_j}\left(1 - \exp\left(-\frac{\tilde{a}_j h}{\varepsilon + x_j}\right)\right], \ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n-1. \end{cases} \tag{6.3.3}$$

In addition,

$$\left.\begin{aligned} \tilde{a}_j &= \frac{a_j + a_{j-1}}{2} \text{ for } j = 0, 1, 2, ..., \frac{n}{2} - 1, \\ \tilde{a}_j &= \frac{a_j + a_{j+1}}{2} \text{ for } j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n-1, \\ \tilde{b}_j &= \frac{b_{j-1} + b_j + b_{j+1}}{3}; \tilde{f}_j = \frac{f_{j-1} + f_j + f_{j+1}}{3} \text{ for } j = 0, 1, 2, ..., n-1. \end{aligned}\right\} \tag{6.3.4}$$

The equation (6.3.1) becomes

$$\left.\begin{aligned} r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = \tilde{f}_j, \ j = 0, 1, 2, ..., \frac{n}{2} - 1, \\ r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = \tilde{f}_j, \ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n-1, \end{aligned}\right\} \tag{6.3.5}$$

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with

$$\left.\begin{array}{l} r_j^- = \dfrac{\varepsilon+x_j}{\tilde{\phi}_j^2} - \dfrac{\tilde{a}_j}{h}; r_j^c = \dfrac{-2(\varepsilon+x_j)}{\tilde{\phi}_j^2} + \dfrac{\tilde{a}_j}{h} - \tilde{b}_j; r_j^+ = \dfrac{\varepsilon+x_j}{\tilde{\phi}_j^2}, j=0,1,\ldots,\dfrac{n}{2}-1, \\ \\ r_j^- = \dfrac{\varepsilon+x_j}{\tilde{\phi}_j^2};\; r_j^c = \dfrac{-2(\varepsilon+x_j)}{\tilde{\phi}_j^2} - \dfrac{\tilde{a}_j}{h} - \tilde{b}_j;\; r_j^+ = \dfrac{\varepsilon+x_j}{\tilde{\phi}_j^2} + \dfrac{\tilde{a}_j}{h}, j=\dfrac{n}{2}, \dfrac{n}{2}+1,\ldots,n-1. \end{array}\right\} \tag{6.3.6}$$

The system of equations (6.3.5) - (6.3.2) is called the fitted operator finite difference method (FOFDM). It satisfies the following Lemmas:

Lemma 6.3.1. (Discrete minimum principle) . *Consider a mesh function ξ_j where $\xi_0 \geq 0$, $\xi_n \geq 0$ and $L^n\xi_j \leq 0$, $\forall j = 1(1)n-1$, then $\xi_j \geq 0$, $\forall j = 0(1)n$.*

Proof. We proceed by contradiction to prove this Lemma.

Consider k such that $\xi_k = \min_{0\leq j\leq n} \xi_j$ and $\xi_k < 0$. It is clear that $k \neq 0, n$. In addition $\xi_{k+1} - \xi_k \geqslant 0$, also $\xi_k - \xi_{k-1} \leq 0$. Then

$$L^n\xi_k = \begin{cases} (\varepsilon + x_k)\delta^2\xi_k + a_k D^-\xi_k - b_k\xi_k > 0,\ a_k < 0,\ 1 \leq k \leq n/2-1, \\ -b_k\xi_k > 0, k = n/2,\ a_{n/2} = 0, \\ (\varepsilon + x_k)\delta^2\xi_k + a_k D^+\xi_k - b_k\xi_k > 0,\ a_k > 0,\ n/2+1 \leq k \leq n-1. \end{cases} \tag{6.3.7}$$

It follows that $L^n\xi_k > 0,\ 1 \leq k \leq n-1$, Leading to a contradiction. Consequently $\xi_j \geqslant 0, 1 \leq j \leq n$.

Lemma 6.3.2. *(Uniform stability estimate) Let Z_i be any mesh function such that $Z_0 = Z_n = 0$. Then*

$$|Z_i| \leq \frac{1}{b_0} \max_{1\leq j\leq n-1} |L^n Z_j|, \text{ for } 0 \leq i \leq n,$$

with $b_i \geq b_0 > 0$, to ensure the uniqueness of the solution to the problem (6.3.1) - (6.3.2).

Proof. Given two comparison functions $Y_i^{\pm}$

$$Y_i^{\pm} = \frac{1}{b_0} \max_{1\leq j\leq n-1} |L^n Z_j| \pm Z_j, \text{ for } 0 \leq i \leq n,$$

with $b_i \geq b_0 > 0$. We have $Y_0^{\pm} \geqslant 0$ and $Y_n^{\pm} \geqslant 0$. Then

$$L^n Y_i^{\pm} = \frac{-b_i}{b_0} \max_{1\leq j\leq n-1} |L^n Z_j| \pm L^n Z_i, \text{ for } 0 \leq i \leq n.$$

With $0 \leq i \leq n, -b_i/(b_0) \leq -1$.
This leads to $L^n Y_i^{\pm} \leq 0$. Lemma 6.3.1 leads to $Y_i \leq 0, \forall\, 0 \leq i \leq n$, which ends the proof.

Now, in the next section, we concentrate on the analysis of the scheme we derived in section 6.3.

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6.4 Convergence analysis of the FOFDM

This section concentrates on the convergence analysis of the FOFDM designed in the section 6.3 above. We only focus $[0, 0.5)$, since, the study on $[0.5, 1]$ can be done similarly. The truncation error on [0, 0.5) is given by

$$\begin{aligned}
L^n(U_j - u_j) &= L^nU_j - L^nu_j,\\
&= \tilde{f}_j - \left[\frac{\varepsilon + x_j}{\tilde{\phi}_j^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{\tilde{a}_j}{h}(u_{j+1} - u_j) - \tilde{b}_ju_j\right]\\
&= \frac{1}{3}\left[(\varepsilon + x_j)u''_{j+1} + a_{j+1}u'_{j+1} - b_{j+1}u_{j+1}\right] + \frac{1}{3}[(\varepsilon + x_j)u''_j + a_ju'_j - b_ju_j]\\
&\quad +\frac{1}{3}[(\varepsilon + x_j)u''_{j-1} + a_{j-1}u'_{j-1} - b_{j-1}u_{j-1}]\\
&\quad -\left[\frac{\varepsilon + x_j}{\tilde{\phi}_j^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{\tilde{a}_j}{h}(u_{j+1} - u_j) - \tilde{b}_ju_j\right].
\end{aligned}$$

Note that, in the truncation error above, we have used the fact that $\tilde{f}_j = (f_{j+1}+f_j+f_{j-1})/3$ as suggested in (6.3.4). Using the expression of $\tilde{a}_j$, $\tilde{b}_j$ as given in (6.3.4), the Taylor expansions of u_{j+1}, u_{j-1} a_{j+1}, a_{j-1}, b_{j+1}, b_{j-1}, u'_{j+1}, u'_{j-1}, u''_{j+1}, u''_{j-1} and the truncated Taylor expansion of $\frac{1}{\tilde{\phi}_j^2}$ up to order four, we get

$$\begin{aligned}
L^n\left(U_j - u_j\right) &= \left[-\frac{h^4b^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{h^4b^{(iv)}\left(\xi_{9_j}\right)}{72} + \frac{h^4b^{(iv)}\left(\xi_{15_j}\right)}{72} + \frac{h^4b^{(iv)}\left(\xi_{16_j}\right)}{72}\right]u_j\\
&\quad + \left[-ha'_j - \frac{h^2a''_j}{6} - \frac{h^3a'''_j}{6} + \frac{h^4a^{(iv)}(\xi_{2_j})}{72} + \frac{h^4a^{(iv)}\left(\xi_{7_j}\right)}{72} - \frac{h^4a^{(iv)}\left(\xi_{13_j}\right)}{24}\right.\\
&\quad \left.-\frac{2h^2b'_j}{3} - \frac{h^4b'''_j}{9} - \frac{h^5b^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{h^5b^{(iv)}\left(\xi_{9_j}\right)}{72}\right]u'_j\\
&\quad + \left[\frac{ha_j}{2} + \frac{7h^2a'_j}{6} + \frac{h^3a''_j}{4} + \frac{7h^2a'''_j}{36}\right.\\
&\quad +\frac{h^5a^{(iv)}(\xi_{5_j})}{3} - \frac{h^5a^{(iv)}(\xi_{6_j})}{72} - \frac{h^5a^{(iv)}(\xi_{2_j})}{72} - \frac{h^5a^{(iv)}(\xi_{7_j})}{72} + \frac{h^5a^{(iv)}(\xi_{13_j})}{48}\\
&\quad \left.-\frac{h^2b_j}{3} - \frac{h^4b''_j}{6} - \frac{h^6b^{(iv)}(\xi_{4_j})}{144} - \frac{h^6b^{(iv)}(\xi_{9_j})}{144}\right]u''_j\\
&\quad + \left[\frac{h^2a_j}{6} - \frac{h^3a'_j}{6} - \frac{h^4a''_j}{12} - \frac{h^5a'''_j}{36} + \frac{h^6a^{(iv)}(\xi_{2_j})}{144}\right.\\
&\quad \left.+\frac{h^6a^{(iv)}(\xi_{7_j})}{144} - \frac{h^6a^{(iv)}(\xi_{13_j})}{144} + \frac{h^3b_j}{18} - \frac{h^3b'_j}{18} + \frac{h^5b''_j}{36} - \frac{h^6b'''_j}{108} + \frac{h^7b^{(iv)}(\xi_{9_j})}{432}\right]u'''_j\\
&\quad +\kappa\left(\varepsilon, h^2, h^3, \cdots, h^7, a_j, a'_j, \cdots, a_j^{(iv)}, b_j, b'_j, \cdots, b_j^{(iv)}\right)u^{(iv)}(\xi_{*_j}),
\end{aligned} \tag{6.4.8}$$

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where κ is a function of its arguments and the ξ's lie in the interval (x_{j-1}, x_{j+1}). Note that the coefficients of $u_j, u'_j, \cdots, u^{(iv)}(\xi_{*_j})$ can be bounded by a constant.

The equation (6.4.8) can be rewritten as follows

$$L^n\left(U_j - u_j\right) = M_1 h + R_n(x_j), \tag{6.4.9}$$

with

$$M_1 = -a'_j u'_j + \frac{a_j}{2} u''_j.$$

$$
\begin{aligned}
R_n(x_j) = {} & h^2\left[\left(\frac{-a''_j}{6} - \frac{2b'_j}{3}\right)u'_j + \left(\frac{7a'_j}{6} + \frac{7a'''_j}{36} - \frac{b_j}{3}\right)u''_j + \frac{a_j}{6}u'''_j\right] \\
& + h^3\left[\frac{-a'''_j}{6}u'_j + \frac{a''_j}{4}u''_j + \left(\frac{-a'_j}{6} + \frac{b_j}{18} - \frac{b'}{18}\right)u'''_j\right] \\
& + h^4\left[\left(\frac{-b_j^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{b^{(iv)}\left(\xi_{9_j}\right)}{72} + \frac{b^{(iv)}\left(\xi_{15_j}\right)}{72} + \frac{b^{(iv)}\left(\xi_{15_j}\right)}{72}\right)u_j\right] \\
& + h^4\left[\left(\frac{a^{(iv)}\left(\xi_{2_j}\right)}{72} + \frac{a_j^{(iv)}\left(\xi_{7_j}\right)}{72} - \frac{a^{(iv)}\left(\xi_{13_j}\right)}{24} - \frac{b''_j}{9}\right)u'_j - \frac{b'''_j u''_j}{6} - \frac{a''_j}{12}u'''_j\right] \\
& + h^5\left[\left(\frac{b^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{b^{(iv)}\left(\xi_{9_j}\right)}{72}\right)u'_j\right] \\
& + h^5\left[\left(\frac{a^{(iv)}\left(\xi_{5_j}\right)}{3} - \frac{a^{(iv)}\left(\xi_{6_j}\right)}{72} - \frac{a^{(iv)}\left(\xi_{2_j}\right)}{72} - \frac{a^{(iv)}\left(\xi_{7_j}\right)}{72} + \frac{a^{(iv)}\left(\xi_{13_j}\right)}{48}\right)u''_j + \left(\frac{-a'''_j}{36} + \frac{b''_j}{36}\right)u'''_j\right] \\
& + h^6\left[\left(\frac{b_j^{(iv)}\left(\xi_{4_j}\right)}{144} - \frac{b_j^{(iv)}\left(\xi_{9_j}\right)}{144}\right)u''_j + \left(\frac{a_j^{(iv)}\left(\xi_{2_j}\right)}{144} + \frac{a^{(iv)}\left(\xi_{7_j}\right)}{144} - \frac{a_j^{(iv)}\left(\xi_{13_j}\right)}{144} - \frac{b'''_j}{108}\right)u'''_j\right] \\
& + h^7\left[\frac{b_j^{(iv)}\left(\xi_{9_j}\right)}{432}u'''_j\right] \\
& + \kappa\left(\varepsilon, h^3, h^4, \cdots, h^7, a_j, a'_j, \cdots, a_j^{(iv)}, b_j, b'_j, \cdots, b_j^{(iv)}, u''_j, u^{(iv)}(\xi_{*_j})\right).
\end{aligned}
$$

or

$$L^n\left(U_j - u_j\right) = \mathcal{O}(h),\ \forall j = 1(1)\frac{n}{2} - 1,$$

leading to

$$|L^n(U_j - u_j)| \leq Ch, \forall j = 1(1)\frac{n}{2} - 1.$$

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Also
$$|L^n(U_j - u_j)| \le Ch, \forall j = \frac{n}{2}(1)n-1.$$
From Lemma 6.3.2, we come to the following main result of this work:

Theorem 6.4.1. *Consider u the solution of (6.1.3)-(6.1.4) and U the numerical solution, approximation of u obtained via the FOFDM (6.3.1)-(6.3.2) then*
$$\sup_{0<\varepsilon\le 1}\max_{0\le j\le n} |u_j - U_j| \le Ch. \tag{6.4.10}$$
Where C is a positive real number, free from ε and h.

Section 6.5 below deals with Richardson extrapolation as a technique used to improve the accuracy and the order of convergence of the estimates 6.4.10 above.

6.5 Richardson extrapolation on the FOFDM

The equation (6.4.9) can be rewritten as follows
$$L^n\left(U_j - u_j\right) = M_1 h + M_2 h^2 + R_n(x_j), \tag{6.5.1}$$
with
$$M_1 = -a_j' u_j' + \frac{a_j}{2} u_j''.$$
$$M_2 = \left(\frac{-a_j''}{6} - \frac{2b_j'}{3}\right) u_j' + \left(\frac{7a_j'}{6} + \frac{7a_j'''}{36} - \frac{b_j}{3}\right) u_j'' + \frac{a_j}{6} u_j'''.$$
$$R_n(x_j) = h^3\left[\frac{-a_j'''}{6} u_j' + \frac{a_j''}{4} u_j'' + \left(\frac{-a_j'}{6} + \frac{b_j}{18} - \frac{b'}{18}\right) u_j'''\right]$$
$$+h^4\left[\left(\frac{-b_j^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{b^{(iv)}\left(\xi_{9_j}\right)}{72} + \frac{b^{(iv)}\left(\xi_{15_j}\right)}{72} + \frac{b^{(iv)}\left(\xi_{15_j}\right)}{72}\right) u_j\right]$$
$$+h^4\left[\left(\frac{a^{(iv)}\left(\xi_{2_j}\right)}{72} + \frac{a^{(iv)}\left(\xi_{7_j}\right)}{72} - \frac{a^{(iv)}\left(\xi_{13_j}\right)}{24} - \frac{b_j''}{9}\right) u_j' - \frac{b_j''' u_j''}{6} - \frac{a_j''}{12} u_j'''\right]$$
$$+h^5\left[\left(-\frac{b^{(iv)}\left(\xi_{4_j}\right)}{72} - \frac{b^{(iv)}\left(\xi_{9_j}\right)}{72}\right) u_j'\right]$$
$$+h^5\left[\left(\frac{a^{(iv)}\left(\xi_{5_j}\right)}{3} - \frac{a^{(iv)}\left(\xi_{6_j}\right)}{72} - \frac{a^{(iv)}\left(\xi_{2_j}\right)}{72} - \frac{a^{(iv)}\left(\xi_{7_j}\right)}{72} + \frac{a^{(iv)}\left(\xi_{13_j}\right)}{48}\right) u_j'' + \left(\frac{-a'''}{36} + \frac{b_j''}{36}\right) u_j'''\right]$$

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$$+h^6\left[\left(-\frac{b_j^{(iv)}\left(\xi_{4_j}\right)}{144}-\frac{b_j^{(iv)}\left(\xi_{9_j}\right)}{144}\right)u_j''+\left(\frac{a_j^{(iv)}\left(\xi_{2_j}\right)}{144}+\frac{a_j^{(iv)}\left(\xi_{7_j}\right)}{144}-\frac{a_j^{(iv)}\left(\xi_{13_j}\right)}{144}-\frac{b_j'''}{108}\right)u_j'''\right]$$

$$+h^7\left[\frac{b_j^{(iv)}\left(\xi_{9_j}\right)}{432}u_j'''\right]$$

$$+\kappa\left(\varepsilon,h^3,h^4,\cdots,h^7,a_j,a_j',\cdots,a_j^{(iv)},b_j,b_j',\cdots,b_j^{(iv)},u_j'',u^{(iv)}(\xi_{*_j})\right).$$

The descriptions of κ, ξ's and $u_j, u_j', \cdots, u^{(iv)}(\xi_{*_j})$ remain the same as the ones specified in (6.4.8).

We consider μ_{2n} the mesh obtained by bisecting each mesh interval in μ_n, i.e.,

$$\mu_{2n}=\{\bar{x}_i\} \text{ with } \bar{x}_0=0,\ \ \bar{x}_n=1 \text{ and } \bar{x}_j-\bar{x}_{j-1}=\bar{h}=h/2,\ \ j=1,2,...,2n.$$

Let $\bar{U}_j$ be the numerical solution of (6.1.3)-(6.1.4) based on μ_{2n}. After substituting $\bar{U}_j$ and the mesh size into the equation (6.5.1) we come to the following

$$L^n\left(\bar{U}_j-\bar{u}_j\right)=M_1\bar{h}+M_2\bar{h}^2+R_{2n}(\bar{x}_j), 1\le j\le 2n-1. \tag{6.5.2}$$

Note that $\bar{u}_j$ is the same as u.

After multiplying (6.5.2) by 2, we get

$$2L^n\left(\bar{U}_j-\bar{u}_j\right)=2M_1\bar{h}+2M_2\bar{h}^2+2R_{2n}(\bar{x}_j), 1\le j\le 2n-1, \tag{6.5.3}$$

meaning

$$L^n\left(2\bar{U}_j-2\bar{u}_j\right)=2M_1\bar{h}+2M_2\bar{h}^2+2R_{2n}(\bar{x}_j), 1\le j\le 2n-1. \tag{6.5.4}$$

Subtracting (6.5.4) from (6.5.2) we get

$$L^n\left(u_j-(2\bar{U}_j-U_j)\right)=\frac{M_2h^2}{2}+R_n-2R_{2n}(\bar{x}_j), 1\le j\le 2n-1, \tag{6.5.5}$$

or

$$L^n\left(u_j-(2\bar{U}_j-U_j)\right)=\mathcal{O}(h^2), 1\le j\le 2n-1,$$

Let

$$U_j^{ext}:=2\bar{U}_j-U_j.$$

The numerical solution U_j^{ext} above is another numerical approximation of u_j.
Using lemma 6.3.2, we come to the following

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Theorem 6.5.1. *Consider U_j^{ext} the numerical solution of (6.1.3)-(6.1.4) derived from the Richardson extrapolation based on FOFDM (6.3.1)-(6.3.2). Then there exists a positive constant M independent of ε and h such that*

$$\sup_{0<\varepsilon\leq 1}\max_{1\leq j\leq 2n}\left|u_j - U_j^{ext}\right| \leq Mh^2 \tag{6.5.6}$$

Section 6.6 treats two numerical examples to confirm the accuracy and robustness of the FOFDM designed.

6.6 Numerical examples

In this section we present the numerical results obtained in the integration of some problems of type (6.1.3)-(6.1.4).

The test examples 6.6.1 below relies on a wider domain $[0, 1]$. But the example 6.6.2 relies on $[-1, 1$. The values of $x \in [-1, 1]$ leads to both positive and negative coefficients of the highest derivative of the differential equations.

Example 6.6.1. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{aligned} (\varepsilon + x)u'' + \tfrac{x-0.5}{100}u' - 80u = f(x) \\ u(0) = \varepsilon\exp\left[\arctan\left(\frac{1}{2\sqrt{(\varepsilon)}}\right)\right] + \varepsilon^{\frac{2}{3}}\arctan\left(\tfrac{1}{2\sqrt{\varepsilon}}\right) - 2.25\pi^2; \\ u(1) = \varepsilon\exp\left[-\arctan\left(\frac{1}{2\sqrt{(\varepsilon)}}\right)\right] - \varepsilon^{\frac{2}{3}}\arctan\left(\tfrac{1}{2\sqrt{\varepsilon}}\right) + 2.25\pi^2; \end{aligned}\right\}$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$ near $x = 0.5$. The exact solution is given by:

$$u(x) = \varepsilon\exp\left[-\arctan\left(\frac{x-0.5}{\sqrt{(\varepsilon)}}\right)\right] - \varepsilon^{\frac{2}{3}}\arctan\left(\frac{x-0.5}{\sqrt{\varepsilon}}\right) + 4.5\pi^2\sin[\pi(x-0.5)].$$

The function $f(x)$ is obtained after substituting $u(x), u'(x)$ and $u''(x)$ into the above equation.

Example 6.6.2. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{aligned} (\varepsilon + x)u'' + 5xu' - u = f(x) \\ u(-1) = -1; u(1) = 1 \end{aligned}\right\}$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$ near $x = 0$. The exact solution is given by

$$u(x) = \cos(\pi x) + x + \frac{x\, erf\left(\frac{x}{\sqrt{2\varepsilon}}\right) + \sqrt{\frac{2\varepsilon}{\pi}}\exp(\frac{-x^2}{2\varepsilon})}{erf\left(\frac{1}{\sqrt{2\varepsilon}}\right) + \sqrt{\frac{2\varepsilon}{\pi}}\exp\left(\frac{-1}{2\varepsilon}\right)},$$

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Similarly to the previous examples; $f(x)$ is obtained after substituting $u(x), u'(x)$ and $u''(x)$ into the above equation.

The maximum errors at all mesh points and the numerical rates of convergence before extrapolation are calculated using the formulas

$$E_{\varepsilon,n} := \max_{0\leq j\leq n} |u_j - U_j| \text{ and } r_k \equiv r_{\varepsilon,k} := \log_2\left(\tilde{e}_{n_k}/\tilde{e}_{2n_k}\right), k = 1, 2, ...$$

respectively, where $\tilde{e}_n$ stands for $E_{\varepsilon,n}$. Furthermore, we compute $E_n = \max_{0<\varepsilon\leq 1} E_{\varepsilon,n}$.

For a fixed mesh, we see that the maximum nodal errors remain constant for small values of ε (see tables 6.1 and 6.5). Moreover, results in tables 6.3 and 6.7 show that the proposed method is essentially first order convergent.

After extrapolation the maximum errors at all mesh points and the numerical rates of convergence are evaluated using the formulas

$$E^{ext}_{\varepsilon,n} := \max_{0\leq j\leq 2n} |u_j - U^{ext}_j| \text{ and } R_k \equiv R_{\varepsilon,k} := \log_2\left(E^{ext}_{n_k}/E^{ext}_{2n_k}\right), k = 1, 2, ...$$

respectively, where $E^{ext}_{n_k}$ stands for $E_{\varepsilon,2n}$.

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Table 6.1: Maximum errors for Example 6.6.1 (before extrapolation)

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
10^{-4}	3.08E-03	2.09E-03	9.19E-04	2.35E-04	5.39E-05	1.43E-05
10^{-5}	1.72E-03	1.29E-03	8.84E-04	4.68E-04	1.38E-04	2.96E-05
10^{-11}	1.47E-03	9.48E-04	5.40E-04	2.88E-04	1.49E-04	7.58E-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.47E-03	9.48E-04	5.40E-04	2.88E-04	1.49E-04	7.58E-05

Table 6.2: Maximum errors for Example 6.6.1 (after extrapolation)

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
10^{-4}	6.75E-04	1.09E-03	6.79E-04	1.27E-04	2.69E-05	7.16E-06
10^{-5}	3.64E-04	2.49E-04	1.63E-04	3.09E-04	9.99E-05	1.50E-05
10^{-11}	4.22E-04	1.32E-04	3.68E-05	9.68E-06	2.48E-06	6.28E-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	4.22E-04	1.32E-04	3.68E-05	9.68E-06	2.48E-06	6.28E-07

Table 6.3: Rates of convergence for Example 6.6.1 (before extrapolation, $n_k = 16, 32, 64, 128, 256, 512$)

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	0.56	1.18	1.97	2.12	1.91
10^{-5}	0.42	0.54	0.92	1.76	2.22
10^{-11}	0.64	0.81	0.90	0.95	0.98
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	0.64	0.81	0.90	0.95	0.98

Table 6.4: Rates of convergence for Example 6.6.1 (after extrapolation, $n_k = 16, 32, 64, 128, 256, 512$)

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	-0.70	0.69	2.42	2.24	1.91
10^{-5}	0.55	0.61	-0.92	1.63	2.74
10^{-11}	1.67	1.85	1.93	1.97	1.98
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.67	1.85	1.93	1.96	1.98

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Table 6.5: Maximum errors for Example 6.6.2 (before extrapolation)

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
10^{-4}	3.20E-01	2.22E-01	1.32E-01	6.63E-02	3.65E-02	2.23E-02
10^{-5}	3.19E-01	2.21E-01	1.32E-01	7.23E-02	3.64E-02	1.71E-02
10^{-6}	3.19E-01	2.21E-01	1.32E-01	7.22E-02	3.76E-02	1.91E-02
10^{-8}	3.19E-01	2.21E-01	1.32E-01	7.22E-02	3.76E-02	1.91E-02
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	3.19E-01	2.21E-01	1.32E-01	7.22E-02	3.76E-02	1.91E-02

Table 6.6: Maximum errors for Example 6.6.2 (after extrapolation)

ε	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
10^{-4}	1.35E-01	4.46E-02	5.75E-03	6.82E-03	8.27E-03	3.84E-03
10^{-5}	1.34E-01	4.54E-02	1.29E-02	2.62E-03	2.37E-03	2.62E-03
10^{-6}	1.34E-01	4.53E-02	1.28E-02	3.18E-03	6.87E-04	1.39E-03
10^{-8}	1.34E-01	4.53E-02	1.28E-02	3.17E-03	6.99E-04	1.38E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.34E-01	4.53E-02	1.28E-02	3.17E-03	6.99E-04	1.38E-04

Table 6.7: Rates of convergence for Example 6.6.2 (before extrapolation, n_k = 16, 32, 64, 128, 256, 512)

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	0.53	0.75	0.99	0.86	0.71
10^{-5}	0.53	0.74	0.87	0.99	1.10
10^{-6}	0.53	0.74	0.87	0.94	0.97
10^{-8}	0.53	0.74	0.87	0.94	0.97
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	0.53	0.74	0.87	0.94	0.97

Table 6.8: Rates of convergence for Example 6.6.2 (after extrapolation, n_k = 16, 32, 64, 128, 256, 512)

ε	r_1	r_2	r_3	r_4	r_5
10^{-4}	1.60	2.96	-0.25	-0.28	1.11
10^{-5}	1.56	1.81	2.30	0.15	-0.14
10^{-6}	1.57	1.82	2.01	2.21	-1.02
10^{-8}	1.57	1.82	2.01	2.18	2.34
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.57	1.82	2.01	2.18	2.34

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6.7 Summary

In this chapter we dealt with singularly perturbed turning point problems whose solution exhibits an interior layer due to the presence of the turning point. The highest derivative of these equations is affected by a linear diffusion coefficient.

We first analysed the problem qualitatively, then we established sharp bounds on the solution and its derivatives.

The next step was to propose a numerical approach to solve this class of problems. To this end, we designed a fitted operator finite difference method and proved it to be uniformly convergent of order one with respect to the perturbation parameter ε. To improve the accuracy, we post-processed the numerical solution via Richardson extrapolation and achieved second order uniform convergence.

We conducted numerical simulations on two test examples to confirm the theoretical results.

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Chapter 7

A fitted numerical method for turning point singularly perturbed parabolic problems with a linear diffusion coefficient and an interior layer

This chapter deals with time-dependent singularly perturbed convection-diffusion turning point problems. The highest spatial derivative is multipied by a linear diffusion coefficient $\varepsilon + x$, where ε is a singular perturbation parameter. The solution of these problems possesses an interior layer induced by the turning point. In the process of solving the problem, we first start with time discritization using the classical backward Euler method, and afterwards we follow nonstandard methodology of Mickens to discritize the problem in space on a uniform mesh. A fitted operator finite difference method is then constructed and its convergence properties analysed. The scheme we design is first order uniformly convergent in both time and space variables with respect to the singular perturbation parameter. Thereafter, we apply Richardson extrapolation as a convergence acceleration technique to improve the accuracy and the order of convergence of the scheme up to two in space only. To support theoretical results, we implement the proposed method on some numerical examples.

7.1 Introduction

In this chapter, we consider the following singularly perturbed parabolic convection-diffusion problems

$$Lu := \varepsilon u_{xx} + a(x,t)u_x - b(x,t)u - d(x,t)u_t = f(x,t), \ (x,t) \in D, \tag{7.1.1}$$

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where

$$D = \Omega \times (0, T],\ \Omega = (0, 1),$$

The boundary conditions are given by

$$u(0,t) = \gamma_1,\ u(1,t) = \gamma_2, u(x,0) = u_0(x), \tag{7.1.2}$$

with γ_1 and γ_2 given constants, $0 < \varepsilon \ll 1$, $t \in [0, T]$. The functions $a(x,t)$, $b(x,t)$, $d(x,t)$, $f(x,t)$, and $u_0(x)$ sufficiently smooth in $\bar{D}$ to ensure the smoothness of the solution of (7.1.1)-(7.1.2). The inequality $b(x,t) \geq b_0 > 0, \forall\ (x,t) \in \bar{D}$ confirms that the problem (7.1.1)-(7.1.2) satisfies the minimum principle and it also ensures the uniqueness of the solution [59].

This type of problems where the variable coefficient and small parameter ε multiplies the highest spacial derivative arise in various fields of science and engineering, including fluid mechanics, solid mechanics, quantum mechanics, chemical reactor theory, aerodynamics, optimal control, reation-diffusion process and geophysics etc (see e.g [64, 84] and the references therein).

Problems (7.1.1) become singularly perturbed when the perturbation parameter ε approaches zero. This is revealed by the non-uniform or the rapid change(s) in behaviour(s) of the solution in the narrow part(s) of $\bar{\Omega}$ termed layer(s). They may occur at the extreme points of $\bar{\Omega}$ termed boundary layer(s) or near the roots x_i of $a(x,t)$ in $\bar{\Omega}$, $\forall\ t \in [0,T]$ for $i = 1, 2, ...$ called turning point(s) which lead to interior layer(s) or twin boundary layer(s). The number and location(s) of the layer(s) depend on the proprieties of both the convection and reaction coefficients $a(x,t$ and $b(x,t)$ respectively.

When $a(x,t) \neq 0$; $\forall (x,t) \in \bar{D}$, we have a boundary layer at $(-1,t)$, if $a(x,t) > 0$, or a boundary layer at $(1,t)$ if $a(x,t) < 0$. In case where $a(x,t) \equiv 0$; $\forall (x,t) \in \bar{D}$, then the problem leads to a boundary layer at $(-1,t)$ and $(1,t)$ if $b(x,t) < 0$. But, if $b(x,t) > 0$, the solution is said to be a rapidly oscillatory solution. If $b(x,t)$ changes signs on the domain, then we have turning points.

Finally, when there exist $\forall (x_i, t) \in \bar{D}, i = 1, 2, ...$ such that $a(x_i,t) = 0$ and $a_x(x_i,t) \neq 0$, then, if $a_x(x_i,t) < 0$, we have a signal of no boundary layers, but there is a turning point leading to interior layer at (x_i,t) and if $a_x(x_i,t) > 0$, we have possible boundary layers, no interior layer.

For more information on turning point problems leading to interior layer(s) or twin boundary laters, readers who are interested may consult for instance [9, 10, 17, 23, 30, 31, 44, 64, 65, 80, 70]. Interior layer(s) may also occur from the non-smooth coefficient functions or discontinuous data (See e.g. [4, 11, 13, 26, 27, 42, 54, 59]). We notice that non-turning point problems have extensively been studied in the literature (see e.g [29, 49, 54, 59]).

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Though the turning points and non-turning points time dependent singularly perturbed problems are widely studied in the literature. But, their applications in fluid dynamics and biology are problems in which the coefficient of the highest derivatives are functions of x and ε). These problems have received little attention from the research community. Liseikin [52] considered the case $g(x,\varepsilon) = -(\varepsilon + px)^{\beta}$ for $\beta \geq 1$ and studied the problem for $p = 0$ and $p = 1$. In ([51], pp. 106-111), Liseikin derived bounds on the solution and its derivatives for the case $g(x,\varepsilon) = -(\varepsilon + x)^{\beta}$ for some particular values of β. Moreover, for $\beta = 1$ (see pp. 256-262), he did not provide any numerical example but only designed a numerical method and analysed its convergence. In the application, the case where $p = 1$ and $\beta = 1$ describes filtration of a liquid through a neighbourhood about a circular orifice or radius $r = \varepsilon$ [52, 77]. When $p = 1$ and $\beta = 2$, the model describes a steady diffusive-drift motion [52, 93].

Up to the best of our knowledge and considering various works of time dependent, as we can notice from the references above; the discretization of interior layer problems based on difference equation theory[57] and implicit Euler method has never dealt with singularly perturbed problems with smooth coefficients, where the diffusion coefficient is a linear perturbation function $(\varepsilon + x)$, and whose solution exhibits an interior layer due to the presence of the turning point.

The aim of this work is to study the following time-dependent problem

$$Lu := (\varepsilon + x)u_{xx} + a(x,t)u_x - b(x,t)u - d(x,t)u_t = f(x,t), \ x \in D. \tag{7.1.3}$$

with the boundary conditions

$$u(0,t) = \gamma_1, \ u(1,t) = \gamma_2, u(x,0) = u_0(x). \tag{7.1.4}$$

To guarantee an interior layer at $(0.5,t), \forall t \in [0,T]$; we consider the problem (7.1.3)-(7.1.4) along with the following assumptions

$$\left.\begin{array}{ll} (i)\ a(0.5,t) = 0, & a_x(0.5,t) > 0, t \in [0,T], \\ (ii)\ |a_x(x,t)| \geq \frac{|a_x(0.5,t)|}{2}, & (x,t) \in \bar{D}, \\ (iii)\ \frac{b(0.5,t)}{a_x(0.5,t)} > 0, & t \in [0,T], \\ (iv)\ b(x,t) \geq b_0 > 0, & (x,t) \in \bar{D}, \end{array}\right\} \tag{7.1.5}$$

with, (i) to ensure the existence of the turning point, (ii) confirms that $(0.5,t)$ is the only turning point in $\bar{\Omega}$, $\forall\, t \in [0,T]$, (iii) specifies that $(0.5,t)$ is an interior layer of the solution $u(x,t)$ in $\bar{\Omega}, \forall\, t \in [0,T]$, (iv) guarantees the uniqueness of the solution and also confirms that the problem (7.1.3)-(7.1.4) satisfies the minimum principle.

The problem (7.1.3)-(7.1.4) differs from the classical time-dependent singularly perturbed problems (SPPs) due to the fact that; the order of the reduced equation $(\varepsilon = 0)$ remains the

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same order of the highest spacial derivative, contrary to the classical problems in particular (7.1.1)-(7.1.2) whose order is lowered to one when $\varepsilon = 0$.

The main objectives of this chapter are to design and analyse a fitted operator finite difference method based on difference equation theory and implicit Euler method, to obtain piecewise uniform meshes respectively on time and space variables. This strategy approximates the solution of time dependent singularly perturbed problems (7.1.3)-(7.1.4), having the linear diffusion coefficient $\varepsilon + x$, and whose solution exhibits an interior layer induced by the turning point. The coefficients of these problems are smooth functions depending on space and time variables. We show that the method converges uniformly of order one in both space and time variables. We use Richardson extrapolation (see [65, 66]), as the convergence acceleration technique to improve the accuracy and the order of convergence of the fitted operator finite difference method we design.

This chapter is organized in the following manner: In section 7.2, we studies qualitative properties of the solution and its derivatives at every time level t in [0, T]. We use techniques presented in [1, 9, 20], to provide sharp error estimates specific to the class of problems (7.1.3)-(7.1.4). Section 7.3 is devoted to the design of the scheme which is analysed in section 7.4. Section 7.5 deals with Richardson extrapolation as an acceleration technique to improve the accuracy and the order of convergence of the method up to two in space. To show the effectiveness of the proposed scheme, we carry out and discuss some numerical experiments in section 7.6, and in section 7.7 we end the chapter with some concluding remarks.

7.2 Qualitative results

We consider the continuous problem whose results are later on used in section 7.4 for the error analysis. And let $f(x,t)$ and $u_0(x)$ be smooth and compatible functions to guarantee the continuity and ε-uniform bound of the solution of (7.1.3)-(7.1.4) and its derivatives. We use these mentioned conditions to obtain relevant space and time reliability while applying the maximum norm on $\bar{D} = \bar{\Omega} \times [0, T]$, where $\Omega = (0, 1)$ and $D = \Omega \times (0, 1]$.

Lemma 7.2.1. *(Minimum principle)* Let ψ be a smooth function, with $\psi(0,t) \geq 0$, $\psi(1,t) \geq 0$, $\forall\ t\ \in [0,1]$ and $L\psi(x,t) \leq 0, \forall\ (x,t)\ \in D$. Then, $\psi(x,t) \geq 0, \forall\ (x,t)\ \in \bar{D}$.

Proof. Let us proceed by contradiction to prove this Lemma. Consider $(x^*, t^*) \in \bar{D}$ and $\psi(x^*, t^*) = \min \psi(x,t) < 0$. It is evident that
$(x^*, t^*) \notin \{(0, 0.5); (0, 1); (1, 0); (1, 1)\}$, and applying the minimum principle, leads to $\psi_x(x^*, t^*) =$

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0, $\psi_t(x^*,t^*) = 0$ and $\psi_{xx}(x^*,t^*) \geq 0$. However

$$L\psi(x^*,t^*) = (\varepsilon + x^*)\psi_{xx}(x^*,t^*) + a(x^*,t^*)\psi_x(x^*,t^*) - b(x^*,t^*)\psi(x^*,t^*) + \psi_t(x^*,t^*) > 0,$$

which is a contradiction. Consequently $\psi(x,t) \geq 0 \; \forall \; (x,t) \in \bar{D}$.
The next Lemma refers to the stability of the estimate. We use the minimum principle above to prove

Lemma 7.2.2. *(Uniform stability estimate)* Consider $u(x,t)$ the solution of (1.1)-(1.2). We have

$$||u(x,t)|| \leq b_0^{-1}||f(x,t)|| + \max(|\gamma_1|, |\gamma_2|), \forall (x,t) \in \bar{D},$$

with $||.||$ the maximum norm on $\bar{D}$, and $b(x,t) \geq b_0 > 0$ to secure the uniqueness of the solution (7.1.3)-(7.1.4), γ_1 and γ_2 the boundary conditions of the problem.

Proof. Consider the following comparison function

$$\Pi^{\pm}(x,t) = b_0^{-1}||f(x,t)|| + \max(|\gamma_1|, |\gamma_2|) \pm u(x,t), x \in \bar{D},$$

applying the operator on both sides of the equality, we come to the following

$$L\Pi^{\pm}(x,t) = -\frac{b(x,t)}{b_0}||f(x,t)|| - b(x,t)\max(|\gamma_1|, |\gamma_2|) \pm Lu(x,t) \leq 0.$$

Using the minimum principle, it follows that

$$\Pi^{\pm}(x,t) \geq 0, \forall (x,t) \in \bar{D}.$$

consequently

$$||u(x,t)|| \leq b_0^{-1}||f(x,t)|| + \max(|\gamma_1|, |\gamma_2|), \forall (x,t) \in \bar{D},$$

which completes the proof.

Let us define the partition on $\bar{\Omega} = [0,1]$as follows:
$\Omega_L = [0,\delta)$, $\Omega_C = [\delta,\delta_1]$, $\Omega_R = (\delta_1, 1]$, with $0 < \delta, \delta_1 \leq 1/4$; respectively the left, central and the right parts of the domain. Moreover, $\Omega_C = \Omega_C^- \cup \Omega_C^+$, with $\Omega_C^- = [\delta, 0.5)$, $\Omega_C^+ = [0.5, \delta_1]$, and $\bar{D} = \bar{\Omega}$ x $[0,T]$.

We know from the literature that if $u(x,t)$ is the solution to (7.1.3)-(7.1.4), then

$$|u(x,t)| \leq C, \;\; \forall (x,t) \in \bar{D},$$

where C is a positive real number.

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Lemma 7.2.3. *Let us consider the assumption above and Lemma 7.2.1. The partial derivative of u with respect to t can be bounded as follows*

$$|u_t| \leq C, \ \forall (x,t) \in \bar{D},$$

where C is a positive constant.

Proof. See [36].
The following Lemma focuses on the Inverse Monotonicity.

Lemma 7.2.4. *[51] Let $F(x,u,u_x) = a(x,t)u_x(x,t) - b(x,t)u_t + d(x,t)u_t(x,t) - f(x,t)$ be a smooth function in $([0,1] \times [0,T]) \times \mathbb{R}^2$, where $a(x,t), b(x,t), d(x,t)$ and $f(x)$ are functions described in (7.1.3)-(7.1.4). The problem (7.1.3)-(7.1.4) is said to be inverse monotone for $F(x,u,u_x) \in C^2\left((0,1) \times [0,T]\right) \cap C\left([0,1] \times [0,T]\right)$ if one of the following conditions imposed on F is satisfied:*

(1) $F(x,u,u_x)$ is strictly increasing in u, i.e., $F(x,u_1,z) < F(x,u_2,z)$ if $u_1 < u_2$,

(2) $F(x,u,u_x)$ is weakly increasing in u and there exists a positive constant $C > 0$, such that $|F(x,u,z_1) - F(x,u,z_1)| \leq C\,|z_1 - z_2|$.

Proof. See [51] with $d(x) = x, \ l = 1, \ \forall x \in [0,1]$.

The following lemmas deal with the appropriates bounds on the derivatives of the solution to the problem (7.1.3) - (7.1.4) where $t \in [0,T]$ and x is either in Ω_L, in Ω_C or in Ω_R.

Lemma 7.2.5. *Let u(x, t) be the solution to (7.1.3)-(7.1.4), we have*

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq C, \ \forall x \in \Omega_L U \Omega_R, \ t \in [0,T],$$

where C is a positive real number, free from the singular perturbation ε but depending on δ.

Proof. This Lemma is the immediate consequence of Theorem 7.2.4 for the inverse monotonicity with $C = M$ as specified in [51],
$\forall (x,t) \in \Omega_R \times [0,T]$, $F[x,-M,u_x] \leq F[x,u,u_x] \leq F[x,M,u_x]$ leading to $-M \leq u(x,t) \leq M$, which completes the proof. Similarly, we can proof this Lemma for $(x,t) \in \Omega_L \times [0,T]$.

In the following Lemma 7.2.6, we focus on the bounds of the solution and its derivatives in the layer region. We rely on Liseikin [51] work to adapt it to our problem. We also assume that the convection coefficient at a specific point (x_0,t) is given by $a(x_0,t) = a$, where $(x_0,t) \in \Omega_C^+ \times [0,T]$ or $(x_0,t) \in \Omega_C^- \times [0,T]$.

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Lemma 7.2.6. *[51] (Continuous results) Consider u(x,t) the solution to the problem (7.1.3)-(7.1.4). Then, we have*

1) for $x \in \Omega_C^+$ and $x_0 \in \Omega_C^+, t \in [0,T]$, such that $a(x_0,t) = a > 0$ and $j = 0,1,2,3,4$;

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq MM \begin{cases} 1+(\varepsilon+x)^{1-a-j}, \; if \; 0<a<1, \\ \\ 1+(\varepsilon+x)^{-j}\left|\ln^{-1}\left(\varepsilon+\frac{1}{2}\right)^{-1}\right|, \; if \; a=1, \\ \\ 1+\varepsilon^{a-1}(\varepsilon+x)^{1-a-j}, \; if \; a>1. \end{cases} \tag{7.2.1}$$

2) for $x \in \Omega_C^-$ and $x_0 \in \Omega_C^-, t \in [0,T]$, such that , $a(x_0,t) = a \leq 0, j = 0,1,2,3,4$ and p a whole number such that $a+p \geq 0, a+p-1 < 0$

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M \begin{cases} 1, \; if \; a<0, j \leq p, \\ \\ 1+(\varepsilon+x)^{1-j-p}\left|\ln(\varepsilon+x)\right|, \; if \; a+p=0, \; j>p, \\ \\ 1+(\varepsilon+x)^{-a-j}, \; if \; a+p>1, \; j>p, \end{cases} \tag{7.2.2}$$

where M is a positive constant independent of ε.

Proof.

1) Let us first prove Lemma 7.2.6 for $x \in \Omega_C^+$, $t \in [0,T]$, also with $x_0 \in \Omega_C^+$ such that $a(x_0,t) = a > 0$.
Consider u the solution to the problem (7.1.3)-(7.1.4). From the inverse monotonicity (7.2.4), we have

$$|u(x,t)| \leq M. \tag{7.2.3}$$

In the other hand, according to Liseikin [51], there exists a positive constant m such that (7.1.3)-(7.1.4) and (7.2.3) lead to

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M \begin{cases} 1, \; 0.5<m \leq x \leq \delta, \; t \in [0,T], \\ \\ \varepsilon^{-j}, \; 0.5 \leq x \leq \delta, \; t \in [0,T], \end{cases} \tag{7.2.4}$$

$$j = 1,2,3,4.$$

Supposed that $a > 0$. We can rewrite (7.1.3) as follows

$$\frac{\partial^2 u(x,t)}{\partial x} = -\frac{a(x,t)\frac{\partial u(x,t)}{\partial x}}{\varepsilon+x} + \frac{b(x,t)u(x,t)+f(x,t)}{\varepsilon+x} + \frac{d(x,t)\frac{\partial u(x,t)}{\partial t}}{\varepsilon+x},$$

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or

$$\frac{\partial u(x,t)}{\partial x} = -\int_{0.5}^{x} \frac{a(\eta,t)\frac{\partial u(\eta,t)}{\partial \eta}}{\varepsilon+\eta} d\eta + \int_{0.5}^{x} \frac{b(\eta,t)u(\eta,t)+f(\eta,t)}{\varepsilon+\eta} d\eta + \int_{0.5}^{x} \frac{d(\eta,t)\frac{\partial u(\eta,t)}{\partial t}}{\varepsilon+\eta} d\eta,$$

which can be expressed by the formula

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(0.5,t)}{\partial x}\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a} \exp\left[-g_1(x,t)\right] + g_2(x,t), 0.5 \le x \le \delta_1,\ t \in [0,T], \quad (7.2.5)$$

where

$$g_1(x,t) = \int_{0.5}^{x} \frac{a(\eta,t))}{\varepsilon+\eta} d\eta,\ t \in [0,T],$$

and integrating by parts leads to

$$g_1(x,t) = a(x,t)\ln(\varepsilon+x) - \int_{0.5}^{x} \left[\frac{\partial a(\eta,t)}{\partial \eta}\right] \ln(\varepsilon+x) d\eta,$$

with $g_1(0.5,t) = 0$, since $a(0.5,t) = 0$, $\forall\, t \in [0,T]$. We also have

$$\begin{aligned} g_2(x,t) \quad &= \quad (\varepsilon+x)^{-a} \int_{0.5}^{x} [b(\eta,t)u(\eta,t) + d(\eta,t)\frac{\partial u(\eta,t)}{\partial t}. \\ &\quad +f(\eta,t)]\,(\varepsilon+\eta)^{a-1} \exp[g_1(\eta,t) - g_1(x,t)]d\eta,\ \forall t \in [0,T]. \end{aligned}$$

From (7.2.4) with $a > 0$, we have

$$|g_j(x,t)| \le M,\ j = 1,2;\ 0.5 < x \le \delta_1,\ t \in [0,T].$$

Applying triangular inequalities, (7.2.5) gives

$$\begin{aligned} \left|\frac{\partial u(x,t)}{\partial x}\right| &\le \left|\partial u(0.5,t)\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a} \exp\left[-g_1(x,t)\right]\right| + |g_2(x,t)|\,, \\ \left|\frac{\partial u(x,t)}{\partial x}\right| &\le M\left|\frac{\partial u(0.5,t)}{\partial x}\right|\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a} + M, \\ \left|\frac{\partial u(x,t)}{\partial x}\right| &\le M\left[1 + \left|\frac{\partial u(0.5,t)}{\partial x}\right|\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a}\right], 0.5 < x \le \delta_1,\ t \in [0,T]. \quad (7.2.6) \end{aligned}$$

Considering $0 < a < 1,\ 0 < \varepsilon << 1,\ t \in [0,T]$, and a positive constant m, with $x = m$ such that (7.2.4) and (7.2.5) lead to

$$\left|\frac{\partial u(0.5,t)}{\partial x}\right|\left(\frac{\varepsilon}{\varepsilon+m}\right)^{a} \le M,\ t \in [0,T],$$

$$i.e., \left|\frac{\partial u(0.5,t)}{\partial x}\right| \le M\left(\frac{\varepsilon+m}{\varepsilon}\right)^{a} \le M\varepsilon^{-a},\ t \in [0,T].$$

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Thus (7.2.6) leads to

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M\left[1+\varepsilon^{-a}\varepsilon^{a}(\varepsilon+x)^{-a}\right],$$

giving

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M\left[1+(\varepsilon+x)^{-a}\right],\ 0<a<1,\ 0.5<x\leq\delta_1,\ t\in[0,T].$$

Also, from (7.1.3) we have the following

$$\frac{\partial^3 u(x,t)}{\partial x^3} = -\frac{[1+a(x,t)]\frac{\partial^2 u(x,t)}{\partial x^2}}{\varepsilon+x} + \frac{[-\frac{\partial a(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial x} - b(x)]\frac{\partial u(x,t)}{\partial x}}{\varepsilon+x} + \frac{\frac{\partial b(x,t)}{\partial x}u(x,t) + \frac{\partial f(x,t)}{\partial x} + \frac{\partial d(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial t} + d(x,t)\frac{\partial u_t(x,t)}{\partial x}}{\varepsilon+x}, \tag{7.2.7}$$

or

$$\frac{\partial^2 u(x,t)}{\partial x^2} = -\int_{0.5}^{x}\frac{[1+a(\eta,t)]\frac{\partial^2 u(\eta,t)}{\partial \eta^2}}{\varepsilon+\eta}d\eta + \int_{0.5}^{x}\frac{[-\frac{\partial a(\eta,t)}{\partial \eta}\frac{\partial u(\eta,t)}{\partial \eta} - b(\eta,t)]\frac{\partial u(\eta,t)}{\partial \eta}}{\varepsilon+\eta}d\eta + \int_{0.5}^{x}\frac{\frac{\partial b(\eta,t)}{\partial \eta}u(\eta,t) + \frac{\partial f(\eta,t)}{\partial \eta} + \frac{\partial d(\eta,t)}{\partial \eta}\frac{\partial u(\eta,t)}{\partial t} + d(\eta,t)\frac{\partial u_t(\eta,t)}{\partial \eta}}{\varepsilon+\eta}d\eta.$$

We can express this derivative by the formula:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(0.5,t)}{\partial x^2}\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1}\exp\left[-g_3(x)\right] + g_4(x,t), 0.5\leq x\leq\delta_1, \tag{7.2.8}$$

where

$$g_3(x,t) = \int_{0.5}^{x}\frac{[1+a(\eta,t)]}{\varepsilon+\eta}d\eta,$$

and integration by parts leads to

$$g_3(x,t) = -\left[1+a(x,t)\right]\ln(\varepsilon+x) + \ln\left(\varepsilon+\frac{1}{2}\right) + \int_{0.5}^{x}\left(\frac{\partial a(\eta)}{\partial \eta}\right)\ln(\varepsilon+x)\,d\eta,$$

with $g_3(0.5,t)=0$, since $a(0.5,t)=0, \forall\ t\ \in[0,T]$. On the other hand, we have

$$\begin{aligned} g_4(x,t) &= (\varepsilon+x)^{-a-1}\int_{0.5}^{x}[-\frac{\partial a(\eta,t)}{\partial \eta}\frac{\partial u(\eta,t)}{\partial \eta} + b(\eta,t)\frac{\partial u(\eta,t)}{\partial \eta} \\ &\quad + \frac{\partial b(\eta,t)}{\partial \eta}u(\eta,t) + \frac{\partial f(\eta,t)}{\partial \eta} + \frac{\partial d(\eta,t)}{\partial \eta}u_t(\eta,t) \\ &\quad + d(\eta,t)\frac{\partial u_t(\eta,t)}{\partial \eta}u_t(\eta,t)]\,(\varepsilon+\eta)^{a}\exp[g_3(\eta,t)-g_3(x,t)]d\eta. \end{aligned}$$

From (7.2.4) with $a>0$, we have

$|g_3(x,t)|\leq M,\ |g_4(x,t)|\leq M\ \ 0.5<x\leq\delta_1$, and $t\in[0,T]$.

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Triangular inequality applied to (7.2.8) leads to

$$\left|\frac{\partial^2 u(x,t}{\partial x}\right| \leq \left|\frac{\partial^2 u(0.5,t)}{\partial x^2}\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1}\exp\left[-g_3(x,t)\right]\right| + |g_4(x,t)|,$$

$$\left|\frac{\partial^2 u(x,t}{\partial x^2}\right| \leq M\left|\frac{\partial^2 u(0.5,t}{\partial x^2}\right|\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1} + M,$$

$$\left|\frac{\partial^2 u(x,t}{\partial x^2}\right| \leq M\left[1+\left|\frac{\partial^2 u(0,t}{\partial x^2}\right|\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1}\right], 0.5 < x \leq \delta_1, t \in [0,T]. \tag{7.2.9}$$

Considering $0 < a < 1$, $0 < \varepsilon << 1, t \in [0,T]$, and m' a positive constant, with $x = m'$ such that (7.2.4) and (7.2.8) lead to

$$\left|\frac{\partial^2 u(0,t}{\partial x^2}\right|\left(\frac{\varepsilon}{\varepsilon+m'}\right)^{a+1} \leq M,$$

$$i.e., \left|\frac{\partial^2 u(0.5,t}{\partial x^2}\right| \leq M\left(\frac{\varepsilon+m'}{\varepsilon}\right)^{a+1} \leq M\varepsilon^{-a-1}.$$

Thus (7.2.9) gives

$$\left|\frac{\partial^2 u(x,t}{\partial x^2}\right| \leq M\left[1+\varepsilon^{-a-1}\varepsilon^{a+1}(\varepsilon+x)^{-a-1}\right],$$

leading to

$$\left|\frac{\partial^2 u(x,t}{\partial x^2})\right| \leq M\left[1+(\varepsilon+x)^{-a-1}\right], 0 < a < 1, 0.5 < x \leq \delta_1, t \in [0,T].$$

Thereafter, from (7.1.3) and (7.2.3), we come to the following result for $0 < a < 1, 0.5 < x \leq \delta_1; t \in [0,T]$:

$$\left|\frac{\partial^j u(x,t}{\partial x^2}\right| \leq M\left[1+(\varepsilon+x)^{-a+1-j}\right],\ 0 < \varepsilon << 1, j = 1,2,3,4. \tag{7.2.10}$$

The Lemma 7.2.6 is fulfilled for $0 < a < 1$.

If $a = 1$, the partial integration with respect to x of (7.2.5) from 0 to δ leads to

$$\int_{0.5}^{\delta_1}\frac{\partial u(\eta,t)}{\partial \eta}d\eta = \int_{0.5}^{\delta_1}\frac{\partial u(0.5,t)}{\partial \eta}\left[\frac{\varepsilon}{\varepsilon+\eta}\right]\exp\left[-g_1(\eta,t)\right]d\eta + \int_{0.5}^{\delta_1}g_2(\eta,t)d\eta, 0.5 \leq x \leq \delta_1, \forall t \in [0,T].$$

Integrating by parts leads to

$$A_{\delta_1} - A_{0.5} = \frac{\partial u(0.5,t)}{\partial x}\left[\varepsilon ln(\varepsilon+\eta)\exp[-g_1(\eta,t)]|_{0.5}^{\delta_1} + \varepsilon\int_{0.5}^{\delta_1}\ln(\varepsilon+\eta)\frac{\partial g_1(\eta,t)}{\partial \eta}\exp(-g_1(\eta,t))d\eta\right]$$

$$+\int_{0.5}^{\delta_1}g_2(\eta,t)d\eta, \forall t \in [0,T],$$

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or

$$A_{\delta_1} - A_{0.5} = \frac{\partial u(0.5,t)}{\partial x}\left[\varepsilon \ln(\varepsilon+\delta_1)\exp[-g_1(\delta_1,t)] - \varepsilon\ln(\varepsilon+0.5)\right]$$
$$+\frac{\partial u(0.5,t)}{\partial x}\left[\varepsilon\int_{0.5}^{\delta_1}\ln\left(\varepsilon+\eta\right)\frac{\partial g_1(\eta,t)}{\partial \eta}\exp(-g_1(\eta,t))d\eta\right]+$$
$$+\int_{0.5}^{\delta_1} g_2(\eta,t)d\eta, \forall t\in[0,T],$$

with

$$\frac{\partial g_1(x,t)}{\partial x} = -a(x,t)(\varepsilon+x)^{-1},$$
$$A_{\delta_1}-A_{0.5}=\frac{\partial u(0.5,t)}{\partial x}\left[\varepsilon\ln(\varepsilon+\delta_1)\exp[-g_1(\delta_1,t)]-\varepsilon\ln\left(\varepsilon+0.5\right)\right]$$
$$\frac{\partial u(0.5,t)}{\partial x}\left[+\varepsilon\int_{0.5}^{\delta_1}[-a(\eta,t)](\varepsilon+\eta)^{-1}\ln(\varepsilon+\eta)\exp(-g_1(\eta,t))d\eta\right]$$
$$+\int_{0.5}^{\delta_1} g_2(\eta,t)d\eta, \forall t\in[0,T].$$

Knowing that

$$\left|\ln(\varepsilon+\delta_1)\exp[-g_1(\delta_1)]-\int_{0.5}^{\delta_1}[a(\eta)](\varepsilon+\eta)^{-1}\ln(\varepsilon+\eta)\exp(-g_1(\eta,t))d\eta\right|\leq M, \forall t\in[0,T],$$

we have,

$$\left|\frac{\partial u(0.5,t)}{\partial x}\right|(\varepsilon-\varepsilon\ln\left(\varepsilon+0.5\right))\leq C\left|\frac{\partial u(0.5,t)}{\partial x}\right|\varepsilon\left|\ln\left(\varepsilon+0.5\right)^{-1}\right|\leq M, \forall t\in[0,T],$$

meaning

$$\left|\frac{\partial u(0.5,t)}{\partial x}\right|\leq M\varepsilon^{-1}.\left|\ln^{-1}\left(\varepsilon+0.5\right)\right|, \forall t\in[0,T].$$

From (7.2.6), we have

$$\left|\frac{\partial u(x,t)}{\partial x}\right|\leq M\left[1+\varepsilon^{-1}\left|\ln^{-1}\left(\varepsilon+\frac{1}{2}\right)^{-1}\right|\varepsilon.(\varepsilon+x)^{-1}\right], \forall t\in[0,T],$$

or

$$\left|\frac{\partial u(x,t)}{\partial x}\right|\leq M\left[1+(\varepsilon+x)^{-1}\left|\ln^{-1}\left(\varepsilon+\frac{1}{2}\right)^{-1}\right|\right], 0.5\leq x\leq\delta_1, \forall t\in[0,T]\ 0<\varepsilon<<1,\ a=1.$$

The differentiation of (7.1.3) along with (7.2.4), lead to

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right|\leq M\left[1+(\varepsilon+x)^{-j}\left|\ln^{-1}\left(\varepsilon+0.5\right)^{-1}\right|\right], t\in[0,T]\ 0<\varepsilon<<1, a=1,\ j=1,2,3,4. \quad (7.2.11)$$

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For $a > 1$, (7.2.4) into (7.2.5) and using triangular inequality; we get the following

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M\varepsilon^{-1}\left[\frac{\varepsilon}{\varepsilon+x}\right]^{a} + M,$$

or

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M\left[1+\varepsilon^{a-1}(\varepsilon+x)^{-a}\right].$$

Using (7.1.3) we come to the same derivative as specified in (7.2.8)

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(0.5,t)}{\partial x^2}\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1}\exp\left[-g_1(x)\right] + g_2(x), 0 \leq x \leq \delta_1, \tag{7.2.12}$$

and the triangular inequality of (7.2.12) in connection with (7.1.3) and (7.2.4) leads to

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left|\frac{\partial^2 u(0,t)}{\partial x^2}\right|.\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1} + M,$$

meaning

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\varepsilon^{-2}\left(\frac{\varepsilon}{\varepsilon+x}\right)^{a+1} + M,$$

which leads to

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left[1+\varepsilon^{-2}\varepsilon^{a+1}\left(\varepsilon+x\right)^{-a-1}\right],$$

or

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left[1+\varepsilon^{a-1}\left(\varepsilon+x\right)^{-a-1}\right].$$

Thereafter, from (7.1.3) and (7.2.4) we conclude that

$$\begin{aligned}\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| &\leq M\left[1+\varepsilon^{a-1}(\varepsilon+x)^{1-a-j}\right],\\ &\qquad 0 < x \leq \delta_1,\ 0 < \varepsilon << 1, a > 1, t \in [0,T],\ j = 1,2,3,4.\end{aligned} \tag{7.2.13}$$

This ends the proof for $x \in \Omega_C^+$ and $a > 0$.

2) Consider $x \in \Omega_C^- = [-\delta, 0.5]$, and let $x_0 \in \Omega_C^- = [-\delta, 0.5]$ such that $a(x_0,t) = a \leq 0, t \in [0,T]$. Solving (7.1.3) - (7.1.4) with respect to $u'(x,t)$ leads to

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} &= \frac{\partial u(x_0,t)}{\partial x}\exp(\psi(x,x_0,t))\\ &+ \int_{x_0}^{x}\frac{[b(\eta,t)u(\eta,t)+f(\eta,t)+d(\eta,t)u_t(\eta,t)]}{\varepsilon+\eta}\exp\left[\psi(\eta,x_0,t)\right]d\eta,\end{aligned} \tag{7.2.14}$$

or

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} &= \frac{\partial u(0,t)}{\partial x}\exp(\psi(x,x_0,t)) + (\varepsilon+x)^{-p}\int_{x_0}^{x}[b(\eta,t)u(\eta,t)\\ &\quad + f(\eta,t) + d(\eta,t)u_t(\eta,t)](\varepsilon+\eta)^{p-1}\exp\left[\psi(\eta,x_0,t)\right]d\eta,\end{aligned} \tag{7.2.15}$$

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with $\psi(x, x_0, t)$ given by

$$\psi(x, x_0, t) = \int_{x_0}^{x} \frac{a(\eta, t)}{\varepsilon + \eta} d\eta.$$

It is clear that

$$|\psi(x, x_0, t)| \leq M, -\delta \leq x \leq 0;\ 0 < \varepsilon << 1,\ t \in [0, T].$$

Given $x_0 \in\ [-\delta, 0]$, using (7.2.4), $\left|\frac{\partial u(x_0,t)}{\partial x}\right| \leq M$, and applying triangular inequality, we come to the following

$$\left|\frac{\partial u(x, t)}{\partial x}\right| \leq M\left[1 + |\ln(\varepsilon + x)|\right].$$

This proves Lemma 7.2.6 for $j = 1, a(0, t) = a = 0, p = 0,\ t \in [0, T]$.

From (7.1.3) - (7.1.4), (7.2.7), (7.2.4) and for $j = 2, a(0, t) = a = 0, p = 0, t \in [0, T]$; we can easily show that

$$\left|\frac{\partial^2 u(x, t)}{\partial x^2}\right| \leq M\left[1 + (\varepsilon + x)^{-1}|\ln(\varepsilon + x)|\right].$$

Thereafter, from (7.1.3) and (7.2.4), we come to the following result, with $a + p = 0, j > p$

$$\left|\frac{\partial^j u(x, t)}{\partial x^j}\right| \leq M\left[1 + (\varepsilon + x)^{1-j-p}|\ln(\varepsilon + x)|\right],\ 0 < \varepsilon << 1, t \in [0, T], j = 1, 2, 3, 4. \tag{7.2.16}$$

Now, consider $x \in\ [\delta, 0.5],\ 0 < \varepsilon << 1,\ p \geq 1, t \in [0, T], a(x, t) < 0$ and m_3 a positive constant given by $-\delta \leq m_3 \leq x \leq 0$, then we have

$$\psi(x, x_0, t) \leq -m_3 \ln\left(\frac{\varepsilon + \eta}{\varepsilon + x}\right), -\delta \leq m_3 \leq \eta \leq x \leq 0.5, t \in [0, T],$$

which follows that

$$\exp\left[\psi(x, x_0, t)\right] \leq M\left(\frac{\varepsilon + x}{\varepsilon + \eta}\right)^{m_3}, -\delta \leq m_3 \leq \eta \leq x \leq 0, t \in [0, T].$$

Using (7.2.4) and letting $x_0 = m_3$; (7.2.14) leads to

$$\left|\frac{\partial u(x, t)}{\partial x}\right| \leq M, \delta \leq m_3 \leq x \leq 0.5, t \in [0, T],$$

which proves the Lemma for $j = 1 \leq p, a(x, t) = a < 0$.

Also, using (7.2.4) and letting $x_0 = m_3$; we can easily show that

$$\left|\frac{\partial u(x, t)}{\partial x}\right| \leq M, \delta \leq m_3 \leq x \leq 0.5, j = 2 \leq p, a < 0, t \in [0, T].$$

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Thus, from (7.1.3) - (7.1.4); we conclude that

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M, \delta \leq x \leq 0.5,\ j \leq p, a(x,t) = a < 0, t \in [0,T]\ j = 1,2,3,4. \tag{7.2.17}$$

Finally, let $x \in\ [\delta, 0], 0 < \varepsilon << 1, t \in [0,T], a(x_0,t) = a < 0$, we can defined a formula for the first derivative of the problem (7.1.3) - (7.1.4) similar to (7.2.5) as follows

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x_0,t)}{\partial x}\left(\frac{\varepsilon}{\varepsilon + x^2}\right)^{a+1} \exp\left[-g_1(x,t)\right] + g_2(x,t), \delta \leq x \leq 0.5, \tag{7.2.18}$$

with g_1 and g_2 as specified in (7.2.5). Applying triangular inequality and following the same process as (7.2.5) we get

$$\left|\frac{\partial u(x,t)}{\partial x}\right| \leq M\left[1 + (\varepsilon + x)^{-a-1}\right], \delta \leq x \leq 0.5, t \in [0,T], j = 1, a(x_0,t) = a < 0.$$

We also defined the formula of the second derivative in connection with (7.2.8) as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(0,t)}{\partial x^2}\left(\frac{\varepsilon}{\varepsilon + x}\right)^{a+2} \exp\left[-g_5(x,t)\right] + g_6(x,t), \delta \leq x \leq 0.5, t \in [0,T], \tag{7.2.19}$$

where g_5 and g_6 are obtained after integrating the derivative of (7.1.3) with respect to x. After applying triangular inequality, we come to the following:

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leq M\left[1 + (\varepsilon + x)^{-a-2}\right], -\delta \leq x \leq 0, t \in [0,T], j = 2, a(x_0,t) = a < 0.$$

Thereafter, from (7.1.3) - (7.1.4); we get

$$\left|\frac{\partial^j u(x,t)}{\partial x^j}\right| \leq M\left(\varepsilon + x\right)^{-a-j}, \delta \leq x \leq 0.5, t \in [0,T], a(x_0,t) = a < 0,\ j = 1,2,3,4. \tag{7.2.20}$$

This complete the proof of Lemma 7.2.6 for $x \in \Omega_C^-$ and $a(x,t) \leq 0$.

In Section 7.3 below, we construct the fitted operator finite difference method used to solve the problem (7.1.3)-(7.1.4).

7.3 Construction of the FOFDM

Consider the partition of time interval [0, T] given by

$$\bar{\omega}^k = \left\{t_k = k\tau,\ 0 \leq k \leq K,\ \tau = T/K\right\}. \tag{7.3.1}$$

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Time discretization of (7.1.3)-(7.1.4) on $\bar{\omega}^k$ is given by

$$-d(x,t_k)\frac{u(x,t_k)-u(x,t_{k-1})}{\tau}+L_{x,\varepsilon}(u(x,t_k))=f(x,t_k), 1\leq k\leq K, \tag{7.3.2}$$

$$u(x,0)=u_0(x), \forall x\in(0,1),\ u(-1,t_k)=\gamma_1,\ u(1,t_k)=\gamma_2. \tag{7.3.3}$$

From the equation (7.3.2) we get

$$(-d(x,t)I+\tau L_{x,\varepsilon})(u(x,t_k))=\tau f(x,t_k)-d(x,t)u(x,t_{k-1}). \tag{7.3.4}$$

The discretization (7.3.4) above is the result of the turning point singularly perturbed problems at each time level $t_k=k\tau$ which is examined in section 7.4 for the error analysis. The global error E_k at the time level t_k is the sum of local errors e_k at each time level t_k. The local truncation error e_k is given by $e_k=u(x,t_k)-\tilde{u}(x,t_k)$, where $\tilde{u}(x,t_k)$ is the solution of

$$(-d(x,t)I+\tau L_{x,\varepsilon})(u(x,t_k))=\tau f(x,t_k)-d(x,t)u(x,t_{k-1}), u(0,t_k)=\alpha,\ u(1,t_k)=\gamma. \tag{7.3.5}$$

The operator $(-d(x,t)I+\tau L_{x,\varepsilon})$ satisfies the maximum principle and we have:

$$||(-d(x,t_k)I+\tau L_{x,\varepsilon})^{-1}||\leq\frac{1}{\max_{0\leq k\leq K,\, x\in[0,1]}(|d(x,t_k)|^{order(I)})+\tau\beta}, \tag{7.3.6}$$

where $order(I)$ stands for the order of the identity matrix I, which proves the stability of the discretization with respect to time.

It is also known that the local error and the global error are respectively bounded as $||e_k||_\infty\leq c\tau^2, 1\leq k\leq K$ and $||E_k||_\infty\leq c\tau, 1\leq k\leq K$.

Lemma 7.3.1. *Given $u(x,t_k)$ the solution of (7.3.2) - (7.3.3) at time level t_k, Then there exist positive constants C such that*
$|u^{(m)}(x,t_k)|\leq C\left[1+(\varepsilon+x)^{-m}\exp\left(\frac{\eta x}{\varepsilon}\right)\right], m=0,1,2,3,$
and
$|u^{(m)}(x,t_k)|\leq C\left[1+(\varepsilon+x)^{-m}\exp\left(\frac{-\eta x}{\varepsilon}\right)\right], m=0,1,2,3.$

Proof. (See [20]).

Consider n a positive and even integer and $\bar{\Omega}^n$ the partition of the interval $[0,1]$ given by

$$x_0=0; x_j=x_0+jh; j=1,...,n-1, h=x_j-x_{j-1}, x_n=1.$$

With $\bar{Q}^{n,K}=\bar{\Omega}^n\times\bar{\omega}^K$ the grid of (x,t).
We also adopt the following: $\forall\ (x_j,t_k)\ \in\bar{Q}^{n,K},\ \Xi(x_j,t_k):=\Xi_j^k$, where U_j^k is the approximation of u_j^k.

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After applying difference equation theory on $\bar{Q}^{n,K}$ (see [57]); the discretization of the problem (7.1.3)-(7.1.4) is given as follows:

$$L^{n,K}U_j^k := \begin{cases} -\tilde{d}_j^k \frac{U_j^k - U_j^{k-1}}{\tau} + (\varepsilon + x_j)\delta^2 U_j^k + \tilde{a}_j^k D^- U_j^k - \tilde{b}_j^k U_j^k = \tilde{f}_j^k, \\ j = 0, 1, 2, \cdots, \frac{n}{2} - 1, k = 0, 1, ..., K, \\ \\ -\tilde{d}_j^k \frac{U_j^k - U_j^{k-1}}{\tau} + (\varepsilon + x_j)\delta^2 U_j^k + \tilde{a}_j^k D^+ U_j^k - \tilde{b}_j^k U_j^k = \tilde{f}_j^k, \\ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n - 1, k = 0, 1, ..., K, \end{cases} \tag{7.3.7}$$

$$U_0 = \gamma_1,\ U_n = \gamma_2, \tag{7.3.8}$$

with

$$D^- U_j^k = \frac{U_j^k - U_{j-1}^k}{h},\quad D^+ U_j^k = \frac{U_{j+1}^k - U_j^k}{h},\quad \delta^2 U_j^k = \frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{\tilde{\phi}^2},$$

where

$$\tilde{\phi_j^k}^2 = \begin{cases} \frac{h(\varepsilon + x_j)}{\tilde{a_j}^k} \left[\exp\left(\frac{\tilde{a_j}^k h}{\varepsilon + x_j}\right) - 1\right],\ j = 0, 1, 2, ..., \frac{n}{2} - 1, \\ \\ \frac{h(\varepsilon + x_j)}{\tilde{a_j}^k} \left[1 - \exp\left(\frac{-\tilde{a_j}^k h}{\varepsilon + x_j}\right)\right],\ j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1. \end{cases} \tag{7.3.9}$$

We also consider the following

$$\left.\begin{aligned} &\tilde{a}_j^k = \frac{a_j^k + a_{j-1}^k}{2} \text{ for } j = 0, 1, 2, ..., \frac{n}{2} - 1, \\ &\tilde{a}_j^k = \frac{a_j^k + a_{j+1}^k}{2} \text{ for } j = \frac{n}{2}, \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n - 1, \\ &\tilde{b}_j^k = \frac{b_{j-1}^k + b_j^k + b_{j+1}^k}{3}; \tilde{f}_j = \frac{f_{j-1}^k + f_j^k + f_{j+1}^k}{3} \text{ for } j = 0, 1, 2, ..., n - 1, \\ &\tilde{d_j}^k = \frac{d_{j-1}^k + d_j^k + d_{j+1}^k}{3} \text{ for } j = 0, 1, 2, ..., n - 1. \end{aligned}\right\} \tag{7.3.10}$$

Using the conventions above, (7.3.7) can be rewritten as

$$\left.\begin{aligned} &r_{j,k}^- U_{j-1}^k + r_{j,k}^c U_j^k + r_{j,k}^+ U_{j+1}^k = \tilde{f_j}^k,\ j = 0, 1, 2, ..., \tfrac{n}{2} - 1 \\ &k = 0, 1, ..., K, \\ &r_{j,k}^- U_{j-1}^k + r_{j,k}^c U_j^k + r_{j,k}^+ U_{j+1}^k = \tilde{f_j}^k,\ j = \tfrac{n}{2}, \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, ..., n - 1, \\ &k = 0, 1, ..., K, \end{aligned}\right\} \tag{7.3.11}$$

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where

$$\left.\begin{array}{l} r^-_{j,k}=\dfrac{\varepsilon+x_j}{\tilde{\phi}^{k\,2}_j}-\dfrac{\tilde{a_j}^k}{h};r^c_{j,k}=\dfrac{-2(\varepsilon+x_j)}{\tilde{\phi}^{k\,2}_j}+\dfrac{\tilde{a_j}^k}{h}-\tilde{b_j}^k-\dfrac{\tilde{d}^k_j}{\tau};r^+_{j,k}=\dfrac{\varepsilon+x_j}{\tilde{\phi}^{k\,2}_j},j=0,1,2,...,\dfrac{n}{2}-1,\\ r^-_{j,k}=\dfrac{\varepsilon+x_j}{\tilde{\phi}^{k\,2}_j};r^c_{j,k}=\dfrac{-2(\varepsilon+x_j)}{\tilde{\phi}^{k\,2}_j}-\dfrac{\tilde{a_j}^k}{h}-\tilde{b_j}^k-\dfrac{\tilde{d}^k_j}{\tau};r^+_{j,k}=\dfrac{\varepsilon+x_j}{\tilde{\phi}^{k\,2}_j}+\dfrac{\tilde{a_j}^k}{h}j=\dfrac{n}{2},\dfrac{n}{2}+1,\dfrac{n}{2}+2,...,n-1. \end{array}\right\} \tag{7.3.12}$$

$$\tilde{F}^k_j=\tilde{f}^k_j-\frac{\tilde{d}^k_j}{\tau}U^{k-1}_j. \tag{7.3.13}$$

The system of equations (7.3.11)-(7.3.8) is called the fitted operator finite difference method (FOFDM). This scheme satisfies the following Lemmas:

Lemma 7.3.2. (Discrete minimum principle) . *For any mesh function ξ^k_j such that, $L^{n,k}\xi^k_j \le 0$ $\forall (j,k)\in Q^{n,K}$, $\xi^0_j \ge 0$, $0\le j\le n$, $\xi^k_0\ge 0$, and $\xi^k_n \ge 0$, $1\le k\le K$. Then $\xi^k_j\ge 0$, $\forall (j,k)\in \bar{Q}^{n,K}$.*

Proof. Let (s,l) be such that $\xi^l_s=\min_{(j,k)}\xi^k_j<0$, $\xi^k_j\in\bar{Q}^{n,K}$. It follows that $s\neq 1,2,...,n-1$ and $l\neq 1,2,...,K$; otherwise $\xi^l_s\ge 0$. In the other hand $\xi^l_{s+1}-\xi^l_s\ge 0$, $\xi^l_s-\xi^l_{s-1}\le 0$, and $\xi^l_s-\xi^{l-1}_s\le 0$.
Leading to

$$L^{n,K}\xi^l_s=\begin{cases}(\varepsilon+x_s)\bar{\delta}^2\xi^l_s+a^l_sD^-\xi^l_s-\left(b^l_s+\frac{d^l_s}{\tau}\right)\xi^l_s>0,\ a^l_s<0, s=1,2,...,\frac{n}{2}-1,\\ -\left(b^l_s+\frac{d^l_s}{\tau}\right)\xi^l_s>0, s=\frac{n}{2},\\ (\varepsilon+x_s)\bar{\delta}^2\xi^l_s+a^l_sD^+\xi^l_s-\left(b^l_s+\frac{d^l_s}{\tau}\right)\xi^l_s>0,\ a^l_s>0,\ s=\frac{n}{2}+1,..,n-1,\end{cases} \tag{7.3.14}$$

where $l=1,2,...,K$. This implies that $L^{n,K}\xi^l_k>0, s=1,2,...,n-1$ and $l=1,2,...,K$, which is a contradiction. Thus $\xi^k_j\ \ge\ 0,\ \forall\,(j,k)\ \in\ \bar{Q}^{n,K}$.
The minimum principle above is used to prove the Lemma below for the uniform stability of the estimate.

Lemma 7.3.3. *(Uniform stability estimate) Consider Z^k_j a mesh function at a time level such that $Z^k_0=Z^k_n=0$. Then*

$$|Z^k_j|\le\frac{1}{b_0}\max_{1\le i\le n-1}|L^{n,K}Z^k_i|,\ \textit{for}\ 1\le j\le n, \textit{and}\, 1\le k\le K,$$

where b_0 is defined as specified in the introduction.

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Proof. Given the mesh function

$$(\xi^{\pm})_j^k = \frac{1}{b_0} \max_{1\le i\le n-1} \left| L_{\varepsilon}^{n,K} Z_i^k \right| \pm Z_j^k, 1 \le j \le n, \text{and } 1 \le k \le K,$$

with $b_j^k \ge b_0 > 0$ to ensure the uniqueness of the solution to the problem (7.3.7) - (7.3.8). We have $(\xi^{\pm})_0^k \ge 0$ and $(\xi^{\pm})_n^k \ge 0$. In addition, for $0 \le j \le n$, and $1 \le k \le K$;

$$L^{n,K}(\xi^{\pm})_j^k = \frac{-b_j^k}{b_0} \max_{1\le i\le n-1} \left| L^{n,K} Z_i^k \right| \pm L^{n,K} z_j^k, 1 \le j \le n, \text{and } 1 \le k \le K.$$

With $0 \le j \le n, (-b_j^k)/(b_0) \le -1$. It follows that $L^{n,K}(\xi^{\pm})_j^k \le 0$. After applying the discrete minimum principle Lemma 7.3.2, we get $(\xi^{\pm})_j^k \ge 0, \ \forall\ 0 \le j \le n, 1 \le k \le K$.

Section 7.4 below focuses on convergence analysis of FOFDM constructed in this chapter.

7.4 Convergence analysis of the FOFDM

Let us first concentrate on the interval $[0, 0.5]$ to analyse the scheme, the analysis on $(0.5, 1]$ can be done similarly.

The operator L^K from (7.3.3) can be rewritten as follows:

$$\begin{aligned} L^K z(x,t_k) \ &:= \ (\varepsilon + x)\frac{d^2 z(x,t_k)}{dx^2} + a(x,t_k)\frac{dz(x,t_k)}{dx} - \left(b(x,t_k) + \frac{d(x,t_k)}{\tau} \right) z(x,t_k) \\ &= \ f(x,t_k) - d(x,t_k)\frac{z(x,t_{k-1})}{\tau}. \end{aligned} \tag{7.4.15}$$

The local truncation error of the space discretization on $[0, 0.5] \times [0, T]$ (e.g. $j = 1, 2, ..., n/2 - 1, k = 1, 2, ..., K$) is defined by

$$\begin{aligned} L^{n,K}(U_j^k - z_j^k) \ &= \ \left(L^K - L^{n,K} \right) z_j^k, \\ &= \ (\varepsilon + x) z''_{j,k} + \tilde{a}_j^k z_j^k - \left[\frac{(\varepsilon + x)}{\tilde{\phi}_j^{2^k}} (z_{j+1}^k - 2z_j^k + z_{j-1}^k) + \frac{\tilde{a}_j^k}{h}(z_j^k - z_{j-1}^k) \right] \\ &= \ (\varepsilon + x) u''_{j,k} - \frac{(\varepsilon + x)}{\tilde{\phi}_j^{2^k}} \left[h^2 u''_{j,k} + \frac{h^4}{24}(z^{(iv)})^k(\xi_1) + \frac{h^4}{24}(z^{(iv)})^k(\xi_2) \right] \\ &\quad + \frac{\tilde{a}_j^k h}{2} z''_{j,k} - \frac{\tilde{a_j}^k h^2}{6} z'''_{j,k} + \frac{\tilde{a_j}^k h^3}{24}(z^{(iv)})^k(\xi_3), \end{aligned} \tag{7.4.16}$$

with $\xi_1, \in (x_j, x_{j+1})$, $\xi_2, \xi_3 \in (x_{j-1}, x_j)$.

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Considering the expression of $\tilde{a}_j^k$ from (7.3.10), the Taylor expansions of a_{j-1}^k up to order four, and the truncated Taylor expansion $1/\tilde{\phi}_j^{2^k} = 1/h^2 - \tilde{a_j}^k/\varepsilon h$, we get

$$\begin{aligned}
L^{n,K}\left(U_j^k - z_j^k\right) &= \frac{3}{2}a_j^k u''_{j,k}h + \left[-\frac{3a'_{j,k}}{2}z''_{j,k} - \frac{\varepsilon}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a_j^k}{6}z'''_{j,k}\right]h^2 \\
&+\left[\frac{3a''_{j,k}}{4}z''_{j,k} - \frac{a_j^k}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) + \frac{a'_{j,k}}{12}z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3)\right]h^3 \\
&+\left[-\frac{13a'''_{j,k}}{24}z''_{j,k} - \frac{a'_{j,k}}{48}\left((u^{(iv)})^k(\xi_1)) + (z^{(iv)})^k(\xi_2)\right)\right. \\
&\left. -\frac{a''_{j,k}}{24}z'''_{j,k} - \frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3)\right]h^4,
\end{aligned} \tag{7.4.17}$$

where ξ's lie in the interval (x_{j-1}, x_{j+1}). We note that the coefficients of $u_j^k, z'_{j,k}, \cdots, (z^{(iv)})^k(\xi_{*_j})$ can be bounded by a constant.

Let us reformulate the equation (7.4.17) as follows

$$L^{n,K}\left(U_j^k - z_j^k\right) = M_1 h + R_n(x_j), \tag{7.4.18}$$

with

$$M_1 = \frac{3a_j}{2}z''_{j,k},$$

$$\begin{aligned}
R_n^k(x_j) = h^2&\left[\frac{3a'_{j,k}}{3} - \frac{\varepsilon}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a_j^k}{6}z'''_{j,k}\right] \\
+h^3&\left[\frac{3a''_{j,k}}{4}z''_{j,k} - \frac{a_j^k}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) + \frac{a'_{j,k}}{12}z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3)\right] \\
+h^4&\left[\frac{13a'''_{j,k}}{24}z''_{j,k} - \frac{a'_{j,k}}{48}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a''_{j,k}}{24}z'''_{j,k} - \frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3)\right].
\end{aligned}$$

The above implies that

$$\left|L^{n,K}(U_j^k - z_j^k)\right| = \mathcal{O}(h),\ \forall j = 1(1)\frac{n}{2} - 1,$$

or

$$|L_1^{n,K}(U_j^k - z_j^k)| \le Mh, \forall j = 1(1)\frac{n}{2} - 1.$$

In the same way, we can easily show that

$$|L_2^{n,K}(U_j^k - z_j^k)| \le Mh, \forall j = \frac{n}{2}(1)n + 1.$$

Applying Lemma 7.3.3, we get

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Theorem 7.4.1. *Given U_j^k the numerical solution of (7.3.7)-(7.3.10) and z_j^k the solution to (7.3.2) - (7.3.3) at time level t_k. Then, there exists a positive constant M independent of ε, τ and h such that*

$$\max_{0\le j\le n} |U_j^k - z_j^k| \le Mh,\ k = 1(1)K+1. \tag{7.4.19}$$

Using the triangular inequality $|U_j^k - u_j^k| \le |U_j^k - z_j^k| + |z_j^k - u_j^k|$, Lemma 7.3.3, Theorem 7.4.1 and the global error; we come to the following main result:

Theorem 7.4.2. *Let U_j^k be the numerical solution of (7.3.7)-(7.3.10) and u_j^k the solution to (7.1.3)-(7.1.4) at the grid point (x_j, t_k). Then, there exists a positive constant M independent of ε, τ and h such that*

$$\max_{0\le j\le n} |U_j^k - u_j^k| \le M(h+\tau),\ k = 1(1)K+1. \tag{7.4.20}$$

The next section deals with Richardson extrapolation as an acceleration technique to investigate the improvement of the result of the scheme constructed from theorem (7.4.1) above.

7.5 Richardson extrapolation on the FOFDM

This section concentrates on the improvement of the accuracy and the order of convergence of (7.4.20).

To start, let us rewrite the equation (7.4.18) as follows:

$$L^{n,K}\left(U_j^k - z_j^k\right) = M_1 h + M_2 h^2 + R_n(x_j), \tag{7.5.1}$$

with

$$M_1 = \frac{3a_j}{2} z''_{j,k},$$

$$M_2 = \frac{3a'_{j,k}}{3} - \frac{\varepsilon}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a_j^k}{6} z'''_{j,k}.$$

$$R_n^k(x_j) = h^3\left[\frac{3a''_{j,k}}{4} z''_{j,k} - \frac{a_j^k}{24}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) + \frac{a'_{j,k}}{12} z'''_{j,k} + \frac{a_j^k}{24}(z^{(iv)})^k(\xi_3)\right]$$
$$+h^4\left[\frac{13a'''_{j,k}}{24} z''_{j,k} - \frac{a'_{j,k}}{48}\left((z^{(iv)})^k(\xi_1) + (z^{(iv)})^k(\xi_2)\right) - \frac{a''_{j,k}}{24} z'''_{j,k} - \frac{a'_{j,k}}{48}(z^{(iv)})^k(\xi_3)\right],$$

where ξ's and $z_j^k, z'_{j,k}, \cdots, (z^{(iv)})^k(\xi_{*_j})$ remain the same as the ones specified in (7.4.16).

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Given μ_{2n} the mesh obtained by bisecting each mesh interval in μ_n, i.e.,

$$\mu_{2n} = \{\bar{x}_i\} \text{ with } \bar{x}_0 = -1, \ \ \bar{x}_n = 1 \ \text{ and } \ \bar{x}_j - \bar{x}_{j-1} = \bar{h} = h/2, \ \ j = 1, 2, ..., 2n.$$

Let $\bar{U}_j^k$ be the numerical solution on μ_{2n}. We can rewrite the equation (6.5.1) in terms of $\bar{U}_j^k$ as follows:

$$L^{n,K}\left(\bar{U}_j^k - \bar{z}_j^k\right) = M\bar{h} + p\bar{h}^2 + R_{2n}^k(\bar{x}_j), 1 \leq j \leq 2n-1, \tag{7.5.2}$$

where M and p are positive real numbers.

We also note that $\bar{z}_j^k \equiv z_j^k$.
After multiplying (7.5.2) by 2, we get

$$2L^{n,K}\left(\bar{U}_j^k - \bar{z}_j^k\right) = 2M\bar{h} + 2p\bar{h}^2 + 2R_{2n}^k(\bar{x}_j), 1 \leq j \leq 2n-1, \tag{7.5.3}$$

meaning

$$L^{n,K}\left(2\bar{U}_j^k - 2\bar{z}_j^k\right) = 2M\bar{h} + 2p\bar{h}^2 + 2R_{2n}^k(\bar{x}_j), 1 \leq j \leq 2n-1. \tag{7.5.4}$$

Let (7.5.1) be in terms of M and p and after subtracting (7.5.1) from (7.5.4) we come to the following

$$L^{n,K}\left((2\bar{U}_j^k - U_j^k) - z_j^k\right) = p\bar{h}^2 + 2R_{2n}^k(\bar{x}_j), 1 \leq j \leq 2n-1, \tag{7.5.5}$$

or

$$L^{n,K}\left((2\bar{U}_j^k - U_j^k) - z_j^k\right) = 0(h^2), 1 \leq j \leq 2n-1,$$

The numerical solution $U_j^{ext,k} := 2\bar{U}_j^k - U_j^k$ is another numerical approximation of z_j^k.

After applying Lemma 7.3.3 we get the following result:

Theorem 7.5.1. *Let $U_j^{ext,k}$ be the numerical solution approximation, obtained via the Richardson extrapolation based on FOFDM (7.3.7)-(7.3.10) and z_j^k the solution to (7.3.2) - (7.3.3) at time level t_k. Then, there exists a positive constant M independent of ε, τ and h such that*

$$\max_{0 \leq j \leq n} |U_j^{ext,k} - z_j^k| \leq Mh^2, \ k = 1(1)K+1. \tag{7.5.6}$$

Applying triangular inequality; the local error leads to

$$|U_j^{ext,k} - u_j^k| \leq |U_j^{ext,k} - z_j^k| + |z_j^k - u_j^k|. \tag{7.5.7}$$

Lemma 7.2.1 along with the theorem 7.5.1 lead to the following result.

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Theorem 7.5.2. *Let $U_j^{ext,k}$ be the numerical solution of (7.3.7)-(7.3.10) and z_j^k the solution to (7.1.3)-(7.1.4) at the grid point (x_j, t_k). Then, there exists a constant M independent of ε, τ and h such that*

$$\max_{0\le j\le n} |U_j^{ext,k} - u_j^k| \le M(h^2+\tau),\ k = 1(1)K+1. \tag{7.5.8}$$

In the next section, we use the proposed schemes on two numerical examples to confirm the accuracy and robustness of the solution. After presenting the examples, we display the results in tables and end our discussion with some concluding remarks.

7.6 Numerical examples

Example 7.6.1. *Consider the following time dependent singularly perturbed turning point problem:*

$$\left.\begin{array}{r}(\varepsilon+x)\,u_{xx}+2\,(x-0.5)\,[1+t]\,u_x-(1+xt)\,u-(1+xt)\exp\left(-xt\right)u_t=f(x,t)\\ \forall t\in[0,1];0\le\varepsilon\le 1.\end{array}\right\}$$

$$u(0,t) = -1+\varepsilon\exp\left(-\frac{t}{\varepsilon}\right)\tanh\left(\frac{1}{2\varepsilon}\right); u(1,t) = 1+\varepsilon\exp\left(-\frac{t}{\varepsilon}\right)\tanh\left(-\frac{1}{2\varepsilon}\right).$$

This problem has an interior layer of width $\mathcal{O}(\varepsilon)$ near $(0.5,t), \forall t\in[0,1]$. The exact solution is

$$u(x,t) = \sin\left(\pi(x-0.5)\right)+\varepsilon\exp\left(-\frac{t}{\varepsilon}\right)\tanh\left(\frac{0.5-x}{\varepsilon}\right),$$

at $t=0$

$$u(x,0) = \sin\left(\pi(x-0.5)\right)+\varepsilon\tanh\left(\frac{0.5-x}{\varepsilon}\right).$$

The expression of $f(x,t)$ is obtained after substituting $u(x,t)$ and its derivatives into the equation (7.6.1).

Example 7.6.2. *Consider the following singularly perturbed turning point problem*

$$\left.\begin{array}{r}(\varepsilon+x)\,u_{xx}+2\,(2x-1)\,[1+t^2]\,u_x-2\,(1+xt)\,u-(1+x^2)\exp\left(-xt\right)u_t=f(x,t)\\ \forall t\in[0,1];0\le\varepsilon\le 1.\end{array}\right\}$$

$$u(0,t) = \varepsilon\exp\left(-\frac{t}{\varepsilon}\right)\tanh\left(\frac{1}{2\varepsilon}\right)-\varepsilon^{-3/2}\exp(-t); u(1,t) = \varepsilon\exp\left(-\frac{t}{\varepsilon}\right)\tanh\left(-\frac{1}{2\varepsilon}\right)-\varepsilon^{-3/2}\exp(t).$$

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This problem has an interior layer of width $\mathcal{O}(\varepsilon)$ near $(0,t)\forall t \in [0,1]$. The exact solution is

$$u(x,t) = \varepsilon \exp\left(-\frac{t}{\varepsilon}\right) \tanh\left(\frac{0.5-x}{\varepsilon}\right) - \varepsilon^{-3/2}\exp(-(1-2x)t),$$

at $t=0$

$$u(x,0) = \varepsilon \tanh\left(\frac{0.5-x}{\varepsilon}\right) - \varepsilon^{-3/2}.$$

The expression of $f(x,t)$ is obtained after substituting $u(x,t)$ and its derivatives into the equation (7.6.2).

The maximum errors at all mesh points and the numerical rates of convergence before extrapolation are evaluated using the formulas

$$E^{\varepsilon,n,K} := \max_{0\le j\le n; 0\le k\le K}\left|U_{j,k}^{\varepsilon,n,K} - u_{j,k}^{\varepsilon,n,K}\right|.$$

In case the exact solution is unknown, we use a variant of the double mesh principle

$$E^{\varepsilon,n,K} := \max_{0\le j\le n; 0\le k\le K}\left|U_{j,k}^{\varepsilon,n,K} - U_{j,k}^{\varepsilon,2n,2K}\right|,$$

where $u_{j,k}^{\varepsilon,n,K} and U_{j,k}^{\varepsilon,n,K}$in the above represent respectively the exact and the approximate solutions obtained using a constant time stepτ and space step h. Similarly,$U_{j,k}^{\varepsilon,2n,2K}$ is found using the constant time step $\frac{\tau}{2}$ and space step $\frac{h}{2}$. Nevertheless, the computation of numerical rates of convergence can be given as

$$r_l = r_k \equiv r_{\varepsilon,k} := \log_2\left(E^{\varepsilon,n,K}/E^{\varepsilon,2n_l,2K_l}\right), l = 1,2,...$$

Also, we compute $E_{n,K} = \max_{0<\varepsilon\le 1} E_{\varepsilon,n,K}$.
And the numerical rate of uniform convergence are:

$$R_{n,k} := \log_2\left(E_{n,K}/E_{2n,2K}\right).$$

For a fixed mesh, we see that the maximum nodal errors remain constant for small values of ε (see tables 7.1 and 7.5). Moreover, results in tables 7.3 and 7.7 show that the proposed method is essentially first order convergent.
After extrapolation the maximum errors at all mesh points and the numerical rates of convergence are evaluated using the formulas

$$E_{\varepsilon,n,K}^{ext} := \max_{0\le j\le 2n; 0\le k\le 2K}|U_j^{ext} - u_{j,k}^{\varepsilon,n,K}| \text{ and } R_k \equiv R_{\varepsilon,k} := \log_2\left(E_{n_k}^{ext}/E_{2n_k}^{ext}\right), k = 1,2,...$$

respectively, where $E_{n_k}^{ext}$ stands for $E^{\varepsilon,2n,2K}$.

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Table 7.1: Maximum errors for Example 7.6.1 (before extrapolation)

ε	$n=16$ $K=16$	$n=32$ $K=32$	$n=64$ $K=64$	$n=128$ $K=128$	$n=256$ $K=256$
10^{-4}	4.79E-02	2.89E-02	1.54E-02	7.84E-03	3.92E-03
10^{-5}	4.79E-02	2.89E-02	1.54E-02	7.84E-03	3.92E-03
10^{-6}	4.79E-02	2.89E-02	1.54E-02	7.84E-03	3.93E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	4.79E-02	2.89E-02	1.54E-02	7.84E-03	3.93E-03

Table 7.2: Maximum errors for Example 7.6.1 (after extrapolation)

ε	$n=16$ $K=16$	$n=32$ $K=32$	$n=64$ $K=64$	$n=128$ $K=128$	$n=256$ $K=256$
10^{-4}	4.81E-02	1.52E-02	3.89E-03	9.80E-04	2.49E-04
10^{-5}	4.81E-02	1.52E-02	3.89E-03	9.80E-04	2.49E-04
10^{-6}	4.81E-02	1.52E-02	3.89E-03	9.80E-04	2.49E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	4.81E-02	1.52E-02	3.89E-03	9.80E-04	2.49E-04

Table 7.3: Rates of convergence for Example 7.6.1 (before extrapolation, $n_k = 16,\ 32,\ 64,\ 128,\ 256$)

ε	r_1	r_2	r_3	r_4
10^{-4}	0.73	0.90	0.98	1.00
10^{-5}	0.73	0.90	0.98	1.00
10^{-6}	0.73	0.90	0.98	1.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	0.73	0.90	0.98	1.00

Table 7.4: Rates of convergence for Example 7.6.1 (after extrapolation, $n_k = 16,\ 32,\ 64,\ 128,\ 256$)

ε	r_1	r_2	r_3	r_4
10^{-4}	1.66	1.96	1.99	1.98
10^{-5}	1.66	1.96	1.99	1.98
10^{-6}	1.66	1.96	1.99	1.98
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.66	1.96	1.99	1.98

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Table 7.5: Maximum errors for Example 7.6.2 (before extrapolation)

ε	$n = 16$ $K = 16$	$n = 32$ $K = 32$	$n = 64$ $K = 64$	$n = 128$ $K = 128$	$n = 256$ $K = 256$
10^{-4}	4.25E-02	2.89E-02	1.91E-02	1.12E-02	6.12E-03
10^{-5}	4.24E-02	2.89E-02	1.91E-02	1.12E-02	6.12E-03
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	4.24E-02	2.89E-02	1.91E-02	1.12E-02	6.12E-03

Table 7.6: Maximum errors for Example 7.6.2 (after extrapolation)

ε	$n = 16$ $K = 16$	$n = 32$ $K = 32$	$n = 64$ $K = 64$	$n = 128$ $K = 128$	$n = 256$ $K = 256$
10^{-4}	4.43E-02	1.91E-02	6.12E-03	1.70E-03	4.50E-04
10^{-5}	4.43E-02	1.91E-02	6.12E-03	1.70E-03	4.50E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	4.43E-02	1.91E-02	6.12E-03	1.70E-03	4.50E-04

Table 7.7: Rates of convergence for Example 7.6.2 (before extrapolation, $n_k = 16, 32, 64, 128, 256$)

ε	r_1	r_2	r_3	r_4
10^{-4}	0.55	0.60	0.77	0.87
10^{-5}	0.55	0.60	0.77	0.87
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	0.55	0.60	0.77	0.87

Table 7.8: Rates of convergence for Example 7.6.2 (after extrapolation, $n_k = 16, 32, 64, 128, 256$)

ε	r_1	r_2	r_3	r_4
10^{-4}	1.21	1.64	1.85	1.91
10^{-5}	1.21	1.64	1.85	1.91
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-16}	1.21	1.64	1.85	1.91

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7.7 Summary

This chapter dealt with a family of time dependent singularly perturbed convection-diffusion turning point problems, with a linear diffusion coefficient function multiplying the highest spatial partial derivative of the differential equation. Our attention in this chapter was based on the turning point which induces an interior layer in the solution. To solve the interior layer problem, we first discretized the main problem in time and space with respect to the singular perturbation parameter, and afterwards, we established sharp bounds on the solution and its derivatives. We used these bounds to prove that the proposed scheme is first order uniformly convergent in both time and space variables. Two numerical test examples were thereupon used to confirm the theoretical results. We also applied Richardson extrapolation as a convergence acceleration technique to increase the accuracy and the rate of convergence of the scheme to order two in space only.

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Chapter 8

Concluding remarks and scope for the future research

The aim of this thesis was to design and analyse fitted operator finite difference methods (FOFDMs) to solve various classes of two-point boundary value singularly perturbed problems and time-dependent problems whose solution possesses an interior layer. Problems investigated had either constant diffusion parameter or a variable diffusion factor. Furthermore, we sought to increase the accuracy of the constructed methods via Richardson extrapolation.

In tackling each of the problems, we first established bounds on the solution and its derivatives before constructing the numerical method. Then the bounds established were utilised in the convergence analysis of each method.

For the two-point boundary value problems, analysis showed that the FOFDMs were first order convergent. Moreover, for the time-dependent problems, the FOFDMs were first order convergent in time and space. For all the methods presented, the order of convergence was increased to two in space upon application of Richardson extrapolation.

To confirm the theoretical findings, numerical simulations were performed on several test examples and numerical results were tabulated in each relevant chapter.

As far as the **scope for further research** is concerned, we intend to

- Explore the possibility of extending the proposed approach for elliptic singular perturbation problems having variable diffusion coefficients.
- Construct higher order FOFDMs to solve the problems considered in this thesis.
- Explore problems in mathematical biology where solutions change rapidly in the interior of the domain.

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