Numerical Methods
for
Mathematical Models on Warrant Pricing

by

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Abstract

Numerical methods for mathematical models on warrant pricing

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Warrant pricing has become very crucial in the present market scenario. See, for example, M. Hanke and K. Potzelberger, Consistent pricing of warrants and traded options, *Review Financial Economics* **11**(1) (2002) 63-77 where the authors indicate that warrants issuance affects the stock price process of the issuing company. This change in the stock price process leads to subsequent changes in the prices of options written on the issuing company's stocks. Another notable work is W.G. Zhang, W.L. Xiao and C.X. He, Equity warrant pricing model under Fractional Brownian motion and an empirical study, *Expert System with Applications* **36**(2) (2009) 3056-3065 where the authors construct equity warrants pricing model under Fractional Brownian motion and deduce the European options pricing formula with a simple method. We study this paper in details in this mini-thesis. We also study some of the mathematical models on warrant pricing using the Black-Scholes framework. The relationship between the price of the warrants and the price of the call accounts for the dilution effect is also studied mathematically. Finally we do some numerical simulations to derive the value of warrants.

Declaration

I declare that *Numerical methods for mathematical models on warrant pricing* is my work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

M Londani May 2010

Signed.....

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General introduction

Warrant is a kind of stock option which gives the holder the right but not the obligation to buy (if it is a call warrant) or to sell (if it is a put warrant) the stock or underlying asset by a certain date (for a European style warrant) or up until the expiry date (for an American style warrant) at a specified price (or strike price).

Warrants are classified as special options and can be divided into covered warrants and equity warrants according to the way they are issued. Covered warrants operate like options, only with a longer time frame and they are of American type. Covered warrants are typically issued by the traders and financial sectors and are for those who do not raise the company's stock after the day of expiration. Equity warrants are different from covered warrants because only the listed companies are recommended to issue them and the underlying assets are the issued stock of their company.

The warrant pricing can be affected by the supply and demand for its underlying asset such as stock price, volatility of the stock price, remaining time to expiry, interest rate and the expected dividend payments on stock.

Noreen and Wolfson (1981) used stock prices in companies with warrants to approximate the standard deviation of the return rate, since the volatility of the warrant pricing is higher than the assets of the company, the volatility

of the equity in such company is lower than that of its assets [25].

Different statistical and mathematical models have been developed to take the price effects and probabilities into consideration to decide the fair value of warrants. These models decide the fair value of a warrant based on certain assumptions. Black and Scholes (1973) state that their model can be used in many cases as an approximation to estimate the warrant pricing value and they used warrant pricing commonly as it was an extension of their call option model.

There are many complications in warrant pricing model. Black and Scholes (1973) mentioned that not only warrant pricing models have complications but also there are limitations inherent in the option pricing models. They investigated the error occurring when warrants are mistakenly priced as standard options ignoring the dilution effects. Therefore, it was very crucial to modify Black-Scholes call option model, because warrants are not written by other traders, they are provided by the company. Merton (1973) showed and proved that the Black-Scholes model can be modified to incorporate stochastic interest rates.

The volatility of warrant is described by the warrant pricing models, but under the framework of the existing pricing warrants analysis, the model based on stochastic volatility does not have an analytical solution. To this end, numerical methods such as Monte Carlo simulation or those based on Brownian motions (Fractional Brownian motion and Geometric Brownian motion) can be used to calculate the warrant pricing.

1.1 History of warrants

1.1.1 Warrant pricing: 1960s and before

The long history of warrant pricing began very early. Warrant pricing was not usually the financial theory property. Lot of researchers were focusing on the option pricing because warrant pricing was complicated than option pricing.

Louis Bachelier's (1900) work on option pricing included warrants and it was based on the assumption that stock prices follow an arithmetic Brownian motion, saying that prices can be negative. Bachelier's work was unknown for a period of time in the financial literature. Many of his models were derived by Osborne (1959) independently after fifty years of Bachelier's time. He developed the theory of random walks in stock prices and say that the random walks have two basic hypotheses in stock prices: (i) the changes in successive price are independent; and (ii) the changes in price follow some probability distributions.

The model that Osborne derived from Bachelier (1900) proposes that the price changes from transaction to transaction in an independent individual security is identically distributed random variables. However, this normality was not satisfied by the majority of stock.

Samuelson (1965) credited Bachelier's work related to warrants by considering the geometric Brownian Motion. He came with the assumption that warrants will only be exercised on its expiration date.

Mandelbrot and Taylor (1967) observed that there are fractal behaviour in stock prices. Sidney (1949) released a warrant survey book "The Speculative Merits of Common Stock Warrants". It was regarded as the first book to reveal the common stock warrants which turn in the most spectacular performance of any group of securities and this common stock warrants are very huge.

McKean (1965) and Samuelson (1965b) showed the warrant valuation which consider the non-negative value to the warrants holder who has the right to exercise a warrants at any time (being an American warrant) before its maturity.

1.1.2 Warrant pricing: 1970s and 1980s

A crucial influence in 1970s research on warrant pricing was the work of Chen (1970). He gave the equation of warrants expected value on its exercising date. Chen (1969) derived an equation to value warrants by making use of dynamic programming technique. Chen (1970) defined warrant pricing differ-

ently. His formulas for computing the future expected value of the warrants were preferred by other researchers. Chen's work aligned with Sidney's work by comparing the perpetual warrants (warrants with indefinite length of life) with common stocks. Chen affirmed that the perpetual warrants cannot be worth more than the common stocks because the company which owns the perpetual warrants are exercisable at zero exercise price which is the same as owning common stocks.

The market price of the stock is always below the exercise price at the time of issue. The mathematical analysis of warrant evaluation to analyze the relationship between the prices of a warrants and common stocks are used in the literature. The most popular method for valuing options are based on the Black and Scholes (1973) and Merton (1973) models (see Bernstein (1992) for the full story of how they developed their model). Their models for pricing options have been taken into consideration to many different commodities and payoff structures and they have become the most popular method for valuing options and warrants.

Option and warrant pricing are defined using Black-Scholes framework. Black and Scholes derived their formulas and assumed that the option price is the function of the stock price. It is noted that the changes in the option price are completely correlated with the changes in the stock price. Black and Scholes (1973) showed how their formulas can be modified to value European warrants. Merton's model is the same as Black-Sholes model despite that the maturity for default free bond which matures at the same time as the options' expiration date is used for the interest rate.

Merton's model of the option pricing was not appropriate for warrant pricing because he assumed that the variance of the default free bond is constant and the variance of bond prices may change due to long life of warrants. European call option was the easiest one in the stock options. Black and Scholes (1973) achieved their formulas for European call option considering the call option. They also developed the warrant pricing formulas from the call option formulas.

Schwartz (1977) used a numerical technique to value the AT & T (Amer-

ican Telephone and Telegraph) warrants. Galai and Schneller (1978) derived the value of the warrants and the value of the company that issues warrants by discussing the equality between the value of the warrants and the value of the call options on a share of the company which warrants hold for any other financial or investment decisions of the company. Several studies on warrants have ignored the dilution effects and equated the warrants to the call options.

1.1.3 Warrant pricing: 1990s and beyond

Lot of researchers measured warrant's life comparing with option's life and found that warrants have a long life. Kremer and Roenfeldt (1993) used jump-diffusion more often to price warrants. There is a high possibility that the stock price might jump during the life of warrants. These diffusions of the stock returns are more relevant for warrant pricing than for option pricing. Jump diffusion is listed as a bias model for pricing options, but it is more efficient for pricing warrants.

Schulz and Trautmann (1994) compared the warrants value resulting from their valuation model with the value obtained by using the Black-Scholes formula and affirm that when warrants are exercised there is dilution of equity and dividend.

Hanke and Potzelberger (2002) investigated the effects of warrants issuance on the prices of traded options bought and sold by third parties which are already outstanding at the time of warrants issuance. They said that if one use any dilution effect pricing model for pricing warrants and use the very same model (but without dilution effect) for pricing options expiring after warrants issuance then they are inconsistent.

Zhang et al. (2009) used the data of Changdian warrants collected from 25 May 2006 to 29 January 2007 (the expiration date) and considered the probability distribution. The yield series distribution of Changdian warrants are greater than zero, which implies that the yield series distribution of Changdian warrants are not normally distributed. Figures 1.1 and 1.2 show the Changdian warrants which are the first warrants in China. Figure 1.1 shows a bell-shaped

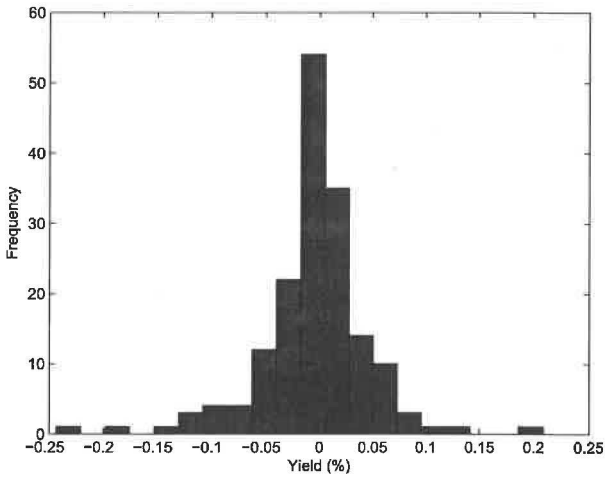

Figure 1.1: Histogram of Changdian warrant pricing returns from 25 May 2006 to 29 January 2007 and its probability distribution. [Programmed in MATLAB and reproduced by using the data from Zhang et al. (2009)]

and symmetrical histogram with data points equally distributed around the middle. The graph is skewed to the right and kurtosis is greater than three which implies that the yield series of Changdian warrants is leptokurtic. Figure 1.2 shows the independent variable of observation times, which has high volatility to the percentage of yield no matter what happens. It is hard to see the consistent pattern in this figure. This gives the insight of using fractional Brownian motion. Zhang et al. (2009) developed the fractional Browian motion considering the mathematical models of strong correlated stochastic processes.

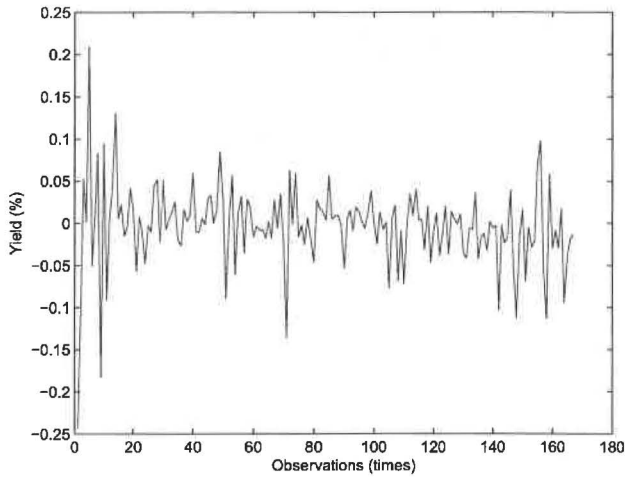

Figure 1.2: Plot of Changdian warrant pricing returns from 25 May 2006 to 29 January 2007 and its probability distribution. [Programmed in MATLAB and reproduced by using the data from Zhang et al. (2009)]

1.2 Warrant pricing vs. Option pricing

Warrant pricing and option pricing carry the right to buy the shares of an underlying asset at a certain price and can be exercised anytime during their life (if they are of American style) or on expiration date (if they are of European style). While the call options are issued by an individual, the warrants are issued by a company. Warrants proceeds increase the company's equity and when it is time to exercise them, new shares are always issued and the payment of cash increases the assets of the issuing company because of the dilution of equity and dividend. When options are exercised, the shares can come from another investor or public exchange.

In warrant pricing, many researchers ignored the dilution effects and valued warrants as the call options on common stocks of the company. The valuation of warrants and call options involves making assumptions about the capital structure of the company and future dividend policy.

The call options can uniquely be priced and the price can be independent in the amount of written call options, given the fact that all call options can

be exercised simultaneously, each call option is a separate stake. Nevertheless, when warrants are exceptional, they can be exercised and new shares can be formed and the changes in the capital structure of the company and dividend policy can occur. Warrant and option pricing are based on the underlying asset such as stocks and bonds. Researchers have used the following formulas for pricing options and warrants and to study the dilution effects.

1.2.1 Formula for pricing options

The Black and Scholes (1973) option pricing model specifies the following price for a call and put option on a nondividend-paying stock

$$C = SN(d_1) - Xe^{-r(T-t)}N(d_2), \tag{1.1}$$

$$P = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{1.2}$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

C is the value of the call option,
P is the value of the put option,
S is the price of the underlying stock,
X is the exercise price of the call and put option,
r is the annualized risk-free interest rate,
$T - t$ is the time until expiration,
σ is the annualized standard deviation of the logarithmic stock return, and
$N(\cdot)$ is the probability from the cumulative standard normal distribution.

1.2.2 Formula for pricing warrants

Galai and Schneller (1978) presented the first solution for the warrant pricing problem in which they incorporated the dilution effect by deriving the following equation:

$$W = \frac{N}{N+n} C_w, \tag{1.3}$$

where

W is the value of the warrant,
N is the number of shares outstanding,
n is the number of new shares to be issued if warrants are exercised,
C_w is the value of a call option written on the stock of a firm without warrants.
The equation (1.3) is based on the assumption that the company with capital structure consists only equity warrants and it is defined as

$$V = NS + nW, \tag{1.4}$$

where

V is the value of the company's equity,
S is the stock price.
It is assumed in Equation (1.4) that not only the value of the company's stock follows the diffusion process, but also the value of the company's equity (V).

Schulz and Trautmann (1994) have compared the outcomes of the original Black-Sholes model with the outcomes of the correct warrant valuation model and they concluded that although the high dilution effects are assumed, the Black-Sholes models produce small biases. Crouhy and Galai (1991) note that warrant prices are always calculated by multiplying the outcome from the option pricing model (such as the Black-Scholes model) by the dilution effects $(\frac{N}{N+n})$.

If the standard deviation of the return in the company's equity is constant,

it leads to the following equation [25]:

$$C_w = \widehat{N}[\widehat{S}N(d_1) - Xe^{r(T-t)}N(d_2)], \tag{1.5}$$

where

$d_1 = \frac{\left[\ln\left(\frac{\widehat{S}}{X}\right)+(T-t)(r+\frac{1}{2}\sigma^2)\right]}{\sigma\sqrt{T-t}}$,

$d_2 = d_1 - \sigma\sqrt{T-t}$,

$\widehat{N} = \frac{N}{[(\frac{N}{k})+M]}$,

$\widehat{S} = S + (\frac{M}{N})W - PV_D$,

W is the value warrant price,
N is the number of outstanding shares,
k is the number of shares that can be purchased with each warrant,
M is the number of outstanding warrants,
S is the stock price,
PV_D is the present value of dividends expected over the life of the warrant,
X is the warrant exercise price,
r is the risk-free interest rate,
σ is the firm-value process volatility,
T is the time to expiration on the option, and
$N(d)$ is the probability that a standard normal variable will take on a value less than equal to d.

1.3 Outline of the thesis

The rest of the thesis is organized as follows. We discuss some mathematical models on warrant pricing in Chapter 2. In Chapter 3 we present some numerical methods which are used to price warrants in the past, in particular, Section 3.1 introduces background information regarding the use of Brownian

motion to price warrants, Section 3.2 describes the use of geometric Brownian motion on warrant pricing, in Section 3.3 we used fractional Brownian motion to value warrant pricing by showing the equations of warrant pricing, in Section 3.4 we used risk-neutral valuation to price warrants, in Section 3.5 we used lattice methods to value warrants and Section 3.6 covers Monte Carlo simulation and discusses memory requirements of the least-squares algorithms. Results obtained by using some of these numerical methods are presented in Chapter 4. Finally in Chapter 5 we give some concluding remarks where we also discuss the scope for future research.

Chapter 2

Mathematics models on warrant pricing

In this chapter we discuss some mathematical models on warrant pricing used by other researchers in the past.

2.1 Model in the work of Chen (1970)

Chen (1970) formulated the model of warrant pricing using the theory of expectations. If a warrant is expected to be exercised t periods from now, the expected value of the warrant at time t can be expressed by the following equation

$$EV[W_t] = \int_{y_t}^{\infty} (x_t - y_t) f(x_t) dx_t, \tag{2.1}$$

where x_t is the market price of the associated stock, t periods from now, and is a random variable,
y_t is the exercise price at time t,
$f(x_t)$ is the density function of X_t,

T is the expiration date of the warrant $0 \leq T < \infty$.

Warrants can not be exercised if the exercise price is above the market price of the associated stock.

The present discounted value of the expected value of the warrant can be expressed as

$$PV[W(t)] = e^{-\beta t} EV[W_t] = e^{-\beta t} \int_{y_t}^{\infty} (x_t - y_t f(x_t)) dx_t, \tag{2.2}$$

where

$PV[W(t)]$ is the present value of the expected value of a warrant to be exercised t periods from now,
β is the discount rate.

2.2 Model in the work of Merton (1976a & b)

Merton (1976a) introduced the occurrence of price jumps in the stock markets. He compares warrant prices computed by his model with those obtained by Black-Scholes model to study the jump-diffusion. He affirms that more accurate warrant prices can be described by jump-diffusions, since the jumps in warrant prices can not be hedged using traded securities. Navas (2003) used mathematical models to elaborate Merton's jump-diffusion model. The continuous trading economy with trading interval $[0, \tau]$, where $\{Z_t : t \in [0, \tau]\}$ represents the standard Brownian motion and $\{N_t : t \in [0, \tau]\}$ represents a Poisson distribution with mean λ. Y_t is a sequence of independent identically distributed random numbers that follows a standard normal distribution.

Navas defines $F_t = F_t^Z \vee F_t^N$ and $F = F_\tau$, where F_t^W and F_t^N are the smallest right-continuous complete σ-algebras generated by $\{Z_s : s \leq t\}$ and $\{N_s : s \leq t\}$ respectively.

Merton assumed that the warrant prices are described by following stochas-

tic differential equation.

$$\frac{dS(t)}{S(t)} = (\alpha - \lambda k)dt + \sigma dZ_t + (Y_t - 1)dN_t, \tag{2.3}$$

where $k = E[Y_t - 1]$ is the expected relative jump of S_t.

The jump sizes follows a normal distribution with parameters μ and σ with a normal density

$$F(S(t), \tau, K, \sigma^2, r; \mu, \sigma^2, \lambda) = \sum_{n=0}^{\infty} \frac{e^{-\hat{\lambda}\tau}(\hat{\lambda}\tau)}{n!} W(S(t), \tau, K, \sigma_n^2, r_n), \tag{2.4}$$

where $W(S(t), \tau, K, \sigma_n^2, r_n)$ is the Black-Scholes warrant price for a European warrant with exercise price K and maturity τ on a non-dividend stock, σ is the volatility of the stock returns, r is the risk-free interest rate, and

$$r_n = r + \frac{n}{\tau}\left(\mu + \frac{\delta^2}{2}\right) - \lambda k,$$

$$\sigma_n^2 = \sigma^2 + \frac{n}{\tau}\sigma^2,$$

$$\hat{\lambda} = \lambda(1 + k) = \lambda e^{\left(\mu + \frac{\delta^2}{2}\right)}.$$

Unlike in the jump-diffusion case, when the warrant price changes are given by Equation (2.3) the stock returns are not normally distributed, because the distribution will have non-zero skewness and when comparing to Gaussian distribution it will be leptokurtic which is consistent with the empirical evidence.

2.3 Model in the work of Lauterbach and Schultz (1990)

Lauterbach and Schultz (1990) used the Black-Scholes model to price warrants after adjusting it for the dilution that occurs when warrants are exercised.

Their warrant pricing model is given by

$$W = \left(\frac{N}{\frac{N}{\gamma} + M}\right)\left[\left(S - \sum_i e^{-rt_i} D_i + \frac{M}{N} W\right) N(d_1) - e^{-rT} X N(d_2)\right], \quad (2.5)$$

where

$$d_1 = \frac{\ln\left(\frac{S - \sum_i e^{-rt_i} D_i + (\frac{M}{N})W}{X}\right) + rT}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2},$$

$d_2 = d_1 - \sigma\sqrt{T}$,
W is the warrant price,
S is the stock price,
X is the exercise price,
N is the number of outstanding shares of stock,
M is the number of warrants,
γ is the number of shares that can be purchased with each warrant,
r is the risk-free interest rate,
T is the time until expiration,
$N(d)$ is the cumulative normal distribution function,
t_i is the time until the i-th dividend is paid, and
D_i is the dollar amount (per share) of the i-th dividend.

Leemakdej et al. (1998) expanded the above model of Lauterbach and Schultz (1990) by their warrant pricing model in the approximated closed-form reads

$$\widehat{W} = \frac{n\gamma}{n + m\gamma}\left(\frac{V}{n} N(q(4)) - X e^{-r\tau} N(q(0))\right), \quad (2.6)$$

where, for ν=0 or 4,

$$q(\nu) = \frac{1 + h(h-1)p - \frac{1}{2}h(h-1)(2-h)(1-3h)p^2 - \left(\frac{z}{\nu+y}\right)^h}{2h^2 p(1-(1-h)(1-3h)p)},$$

$$h(\nu) = 1 - \frac{2(\nu+y)(\nu+3y)}{3(\nu+2y)^2},$$

$p = \frac{\nu+2y}{(\nu+y)^2}$,

$y = \frac{4r\frac{\nu}{n}}{\sigma^2(1-e^{-r\tau})}$,

$z = \frac{4rX}{\sigma^2(e^{r\tau}-1)}$.

The model in equation (2.6) is often referred in the literature as the Cox square root of constant elasticity of variance (CEV) model as it assumes return standard deviations which are related to the square root of the equity value. Leemakdej et al. (1998) mention that the option priced by the Cox Square Root model at-the-money will be higher than the counterpart priced by the Black-Scholes model. They compared the Cox Root and Black-Scholes warrant pricing models with the equity standard deviations by estimating for each warrant on a daily basis for both models.

2.4 Model in the work of Hanke and Potzelberger (2002)

Hanke and Potzelberger (2002) derived the aggregate value MW_t of the warrant pricing at maturity. They mention that if the warrants holders exercise they get a share of $(m/(M+N))$ in the equity of the company for the aggregate strike price MX. The equity of the company consists of the equity just before exercise plus the strike paid by the warrant holders:

$$MW_T = \left(\left(\left(1 + \frac{MW_{t_0}}{V_{t_0}}\right) V_T + Mx\right) \frac{M}{M+N} - Mx\right)_\xi . \tag{2.7}$$

This can be simplified to

$$W_T = \frac{1}{M+N}\left(\left(1 + \frac{MW_{t_0}}{V_{t_0}}\right) V_T - Nx\right)_\xi \tag{2.8}$$

and therefore the value of the warrant at time t ($t_0 \leq t \leq T$) is given by

$$W_t = E^{\theta}\left[D_{t,T}\frac{1}{M+N}\left(\left(1+\frac{MW_{t_0}}{V_{t_0}}\right)V_T - Nx\right)_+ |F_t\right], \tag{2.9}$$

which can further be simplified to

$$\begin{aligned} W_t &= \frac{N}{M+N}E^{\theta}\left[D_{t,T}\left(\left(1+\frac{MW_{t_0}}{V_{t_0}}\right)\frac{V_T}{N} - x\right)_+ |F_t\right], \\ &= \frac{N}{M+N}C_t\left(\left(1+\frac{MW_{t_0}}{V_{t_0}}\right)\frac{V_T}{N}, x, T\right). \end{aligned} \tag{2.10}$$

Where θ is equivalent martingale.

2.5 Model in the work of Lim and Terry (2002)

Another crucial model is of Lim and Terry (2002). They used parameter A for series-A warrants and parameter B for series-B warrants. They computed the warrants value at T_A and the expiry date of the series A warrants. When series-A warrants expire, the company will be left with only one series of warrants outstanding after T_A. They used warrant pricing formula of Galai and Schneller (1978) to value the series-B warrants. They assumed that the series-A warrants are not exercised at T_A. In such case, the value of each series-B warrants at T_A will be

$$W_{B,T_A} = \frac{1}{n+n_B}[V_{T_A}N(d_1) - nK_Be^{-r(T_B-T_A)}N(d_2)], \tag{2.11}$$

where

$$d_1 = \frac{\ln\left(\frac{V_{T_A}}{nK_B}\right) + \left[r + \frac{\sigma^2}{2}\right](T_B - T_A)}{\sigma\sqrt{T_B - T_A}},$$

$$d_2 = d_1 - \sigma\sqrt{T_B - T_A}.$$

n is the common shares,
n_B is the series-B warrants of the company which are outstanding,
K_B is the exercise price of series-B warrants which mature at T_B,
r is the continuously compounded riskless interest rate,
σ is the instantaneous return variance on the firm, and
$N(d)$ represents the standard cumulative normal distribution.

Assuming that the series-A warrants are exercised at T_A, the value of the company will increase by $n_A K_A$ and number of outstanding shares will rise to $n + n_A$ at that time. The value of each series B warrant will then be

$$W^e_{B,T_A} = \frac{1}{n + n_A + n_B} \left\{ V_{T_A} N(d^e_1) + [n_A K_A - (n + n_A) K_B e^{-r(T_B - T_A)}] N(d^e_2) \right\},$$

where

$$d^e_1 = \frac{\ln(V_{T_A}/[(n+n_A)K_B - n_A K_A e^{r(T_B - T_A)}]) + [r + \sigma^2/2](T_B - T_A)}{\sigma\sqrt{T_B - T_A}},$$

$d^e_2 = d^e_1 - \sigma\sqrt{T_B - T_A}$, and
K_A is the exercise price of series-A warrants which matures at T_A,

The value of each series-A warrant at time T_A is given by the boundary condition

$$\begin{aligned} W_{A,T_A} &= \max\{0, S_{T_A} - K_A\}, \\ &= \max\left\{0, \frac{1}{n + n_A}(V_{T_A} + n_A K_A - n_B W^e_{B,T_A}) - K_A\right\}, \end{aligned}$$

where S_t is the company's share price at time t. This is simplified to

$$W_{A,T_A} = \max\left\{0, \frac{1}{n + n_A}(V_{T_A} - nK_A - n_B W^e_{B,T_A})\right\}. \tag{2.12}$$

In equation (2.12) the series-A warrants will be exercised at T_A whenever the

value of the company exceeds V^*, and it is given by

$$V^* = nK_A + n_B W^e_{B,T_A}(V^*). \tag{2.13}$$

In equation (2.13) W_{B,T_A} is shown as an explicit function of V^*. The current value can be determined using the risk-neutral pricing method of Cox and Ross (1976), if the prices of the two warrants series at T_A are given.

2.6 Model in the work of Ukhov (2003)

Ukhov (2003) considered the valuation of conventional warrants issued by a company for its own risk. In his model, $W(V,\tau)$ denotes the value of each warrant in a company of value V. When n warrants are exercised the company receives the amount nX and issues kn new shares of stock. According to his model, the value of the warrant under the Black-Scholes assumptions is

$$\begin{aligned} W(V,\tau;X,\sigma,r,k,N,n) &= \frac{1}{N+kn}C(kV,\tau;NX,\sigma,r), \\ &= \frac{1}{N+kn}[kV\cdot\Phi(\eta_W) - e^{-r\tau}\cdot NX\cdot\Phi(\eta_W - \sigma\sqrt{\tau})], \end{aligned} \tag{2.14}$$

where

$$\eta_W = \frac{\ln(kV/NX) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}},$$

$C(\cdot)$ is the Black-Scholes call option price,
τ is the remaining time,
V is the value of the company's assets,
r is the continuously compounded interest rate,
e^r is the annual interest rate,
σ is the annual standard deviation in (logarithmic) returns on the value process, and
$\Phi(\cdot)$ is the cumulative normal distribution function.

2.7 Model in the work of Bajo and Barbi (2008)

Bajo and Bologna (2008) used constant elasticity of variance (CEV) with the feasible elasticity parameter (gamma parameter) $\gamma^* = 1 + \epsilon^*_{\sigma_s,S}$ to compute the value of warrants. The feasible elasticity parameter can allow the risk-neutral dynamics of the stock return to be written as

$$\frac{ds_t}{s_t} = rdt + \bar{\sigma} s_t^{\gamma-1} d\tilde{z}_t, \tag{2.15}$$

where
S_t is the value of common stocks,
r is the risk-free rate,
$\tilde{z}_t$ is a risk-neutral standard Brownian motion,
γ is straightforwardly related to the elasticity of the return volatility, and
$\bar{\sigma}$ is positive and constant.

Without taking dilution and risk-shifting effect into consideration the value warrant can be computed. Having

$$\hat{d}_2 = \frac{\log[\frac{s_t}{ke^{-r(T-t)}}] - \frac{1}{2}\sigma_s^2(T-t)}{\sigma_s\sqrt{T-t}},$$

the warrant's price is calculated as

$$w_t = e^{-r(T-t)} \int_k^{\infty} (s_T - k) f(s_T) ds_T,$$

where $f(\cdot)$ is the density function of the stock price at maturity. In [66], the CEV warrant pricing formula is provided in terms of the non-central Chi-square distribution.

$$w_t = s_t[1 - \chi^2_{2+a_2,a_3}(a_1)] - ke^{-r(T-t)}\chi^2_{a_2,a_1}(a_3) \qquad \text{if } 0 < \hat{\gamma} < 1,$$

and

$$w_t = s_t[1 - \chi^2_{-a_2,a_1}(a_3)] - ke^{-r(T-t)}\chi^2_{2-a_2,a_3}(a_1) \qquad \text{if } \hat{\gamma} > 1,$$

where

$a_1 = hk^{2(1-\hat{\gamma})}$,

$a_2 = \frac{1}{1-\hat{\gamma}}$,

$a_3 = hs_t^{2(1-\hat{\gamma})}e^{2r(1-\hat{\gamma})(T-t)}$,

$h = \frac{2r}{\sigma^2(1-\hat{\gamma})[e^{2r(1-\hat{\gamma})(T-t)}-1]}$,

and $\chi^2_{\nu,\theta}(\cdot)$ denotes the cumulative non-central Chi-square distribution with ν degrees of freedom and non-centrality parameter θ.

2.8 Model in the work of Zhang et al. (2009)

Zhang et al. (2009) priced equity warrants using fractional Brownian motion. They denoted company's equity by V_T at time T, saying that the company will receive a cash inflow from the payment of the exercise price of MlX. If warrant holders exercise the warrants, the value of the company's equity will increase to $V_T + MlX$. This value is distributed among $N + Ml$ shares so that the price of share after exercise becomes

$$\frac{V_T + MlX}{N + Ml},$$

where
N is the number of shares of outstanding stocks,
M is the number of warrants issued,
l is the number of shares of stock that can be bought with each warrant, and
X is the strike price of option.

The warrants can be exercised only if the payoff is greater than minimum

guarantee provision, i.e.,

$$l\left(\frac{V_T + MlX}{N + Ml} - X\right) > B, \tag{2.16}$$

where B is the minimum guarantee provision. This shows that the warrants value at expiration time satisfies

$$\begin{aligned} W_T &= l\max\left[\frac{V_T + MlX}{N + Ml} - \left(X + \frac{B}{l}\right), 0\right] + B, \\ &= \frac{Nl}{N + Ml}\max\left[\frac{V_T}{N} - X - \frac{N + Ml}{Nl}B, 0\right] + B. \end{aligned}$$

Letting $\alpha = \frac{M}{N}$ and $\hat{X} = X + \frac{N+Ml}{Nl}B$, above implies

$$W_T = \frac{l}{1 + \alpha l}\max\left(\frac{V_T}{N} - \hat{X}, 0\right) + B. \tag{2.17}$$

Since V_T denotes the company's equity (including the warrants) at time T. We have, $V_T = NS_T + MW_T = NS_T + \alpha NW_T$. Setting $\hat{S}_T = S_T + \alpha W_T$, equation (2.17) implies

$$W_T = \frac{l}{1 + \alpha l}\max(\hat{S}_T - \hat{X}, 0) + B. \tag{2.18}$$

In the fractional Brownian motion and risk-neutral world, the price at every $t(t \in [0, T])$ of an equity warrant with strike price X and maturity T is given by

$$W_t = \frac{l}{1 + \alpha l}[\hat{S}_t N(d_1) - \hat{X}e^{-r(T-t)}N(d_2)] + Be^{-r(T-t)}, \tag{2.19}$$

where

$$d_1 = \frac{\ln\frac{S_t}{X} + r(T - t) + \frac{\sigma_V^2}{2}(T^{2H} - t^{2H})}{\sigma_V\sqrt{T^{2H} - t^{2H}}},$$

and

$$d_2 = \frac{\ln \frac{S_t}{X} + r(T-t) - \frac{\sigma_V^2}{2}(T^{2H} - t^{2H})}{\sigma_V \sqrt{T^{2H} - t^{2H}}},$$

where

r is the risk-free interest rate,

$T - t$ is the time to expiration of warrant,

σ_V is the firm-value process volatility,

H is the Hurst parameter,

α denotes the percentage of warrants issued in shares of stock outstanding, and

$N(\cdot)$ is the cumulative probability distribution function of a standard normal distribution.

Chapter 3

Numerical Methods for warrant pricing

3.1 Warrant pricing using Brownian motion

The Brownian motion theory has been applied in many fields including physics, astronomy, medicine (medical imaging), robotics, and stock markets. Louis Bachelier (1900) was the first one to propose that the theory of Brownian motion can be used for option pricing. He used the same reasoning as Robert Brown and state that the latter follows a random walk. The previous change in the value of variable is unrelated to the future or past changes. Before we proceed, let us first discuss what do we mean by Brownian motion.

Definition 3.1.1. *A Brownian motion in [78] is a stochastic process K_t, for $t \geq 0$, with the following properties.*

- *Every increment $K_t - K_s$ over an interval of length $t - s$ is normally distributed with mean 0 and variance $t - s$.*
- *For every pair of disjoint time intervals $[t_1, t_2]$ and $[t_3, t_4]$, with $t_1 < t_2 < t_3 < t_4$, the increments $K_{t_4} - K_{t_3}$ and $K_{t_2} - K_{t_1}$ are independent random*

variables with distributions given as in the first bullet above same applies for n disjoint time intervals, with n being an arbitrary integer.

- $K_0 = 0$.
- *For all t,* K_t *must be continuous.*

3.1.1 The assumptions of Brownian motion

The three critical assumptions that underline the Brownian motion model for stock price as discussed by Martinelli and Neil (2006).

(i) Statistical independence of price changes (price changes or increments are uncorrelated or follow a random walk). This means that the current change of a price is not influenced by the past changes and does not have any influence on the future changes. This assumption seems to be relevant, at least on a long enough term. From time to time, price changes are probably independent. It has been documented by several studies and constitutes the essence of the Efficient Market Hypothesis, or the Random Walk Hypothesis, which states that if the price changes are random and therefore unpredictable, it is because investors are properly doing their jobs. In this case, all arbitrage opportunities are exploited as much as possible.

(ii) Normality of price changes (meaning that changes follow a bell shaped curve). This assumption provides a distribution function characterized by only the mean and the volatility, and implies a certain behaviour of the changes. It also seem realistic for stock price fluctuations but does not take into consideration the fact that negative stock prices could result from large negative changes. This problem is solved by using the log normal distribution from the geometric Brownian motion.

(iii) The price-change indexes or statistics do not vary with time. In other words, the mean and the standard deviation of price changes do not

change with time.

3.1.2 Application of the Brownian motion for warrant pricing

In the past years, the application of the Brownian motion process to analyze financial time series and stock prices has been under the scrutiny of empirical research. Some of the works (applications) are listed below.

- Osborne (1972) applied the Brownian motion in stock markets and showed that the logarithms of common stock prices and the value of the money can be regarded as a collection of decisions in a statistical equilibrium. He found that this ensemble of price changing with time is similar to that of the coordinates of a large number of molecules in the Brownian motion theory.
- The model of Black and Scholes (1973) for pricing options and warrants is also based on the statistical properties of the Brownian motion.
- The model by Smith (1994) consists of the application of the Brownian motion theory in the investigation of price controls. In this model, he analyzed the effects of price stabilization schemes on investment when the demand is vague, by using the method of regulated Brownian motion. He came up with the methods and conclusions which are applicable to any economic situation involving smooth costs of adjustment of stocks when there is uncertainty of prices.

3.2 Warrant pricing using Geometric Brownian motion

The geometric Brownian motion, is also called the exponential Brownian motion, is a continuous time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion. It is used to model financial markets data, especially in option and warrant pricing (see Black and Scholes (1973)) because it accommodates positive values, and only fractional changes in the random variates are significant. Prices that follow a random walk, a Brownian motion, and a geometric Brownian motion meet the independence condition, and their volatilities increase with $\sqrt{t}$ the square root of time.

Samuelson (1965a) solved the option and warrant pricing problems assuming that the stock price follows a geometric Brownian motion. This overcomes the shortcoming of the Bachelier findings of allowing the price to be negative. The resulting implicit valuation equation is based on modeling the equity of the firm (i.e., the total value of stocks and warrants together) as a geometric Brownian motion.

As an illustration, let S_t be a stochastic process. Then S_t is said to follow a geometric Brownian motion if the following stochastic differential equation is satisfied.

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{3.1}$$

where W_t is a Brownian motion and μ and σ are percentage drift and volatility parameters, respectively. The analytic solution of the equation is as follows

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \tag{3.2}$$

where S_0 is an initial value of the asset under consideration. The random variable $\log(\frac{S_t}{S_0})$ is normally distributed with mean $(\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$.

Merton's (1976a) model modified Black and Scholes (1973) model to accom-

modate the jumps of the warrants. N_t is the Poisson distribution with mean λ, $Z_t \approx (\mu_s, {\sigma_s}^2)$ is the sequence of independent identically distributed random numbers that follows a standard normal distribution. In Black-Scholes model, μ is the drift parameter, σ is the stock volatility and W_t is the warrant which follows the geometric Brownian motion.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{3.3}$$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t} Z_i - 1\right), \tag{3.4}$$

Z_i follows a normal distribution with density

$$f_Z(Z) \approx \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(Z-\mu)^2}{2\sigma}\right), \tag{3.5}$$

where μ and σ is the mean and standard deviation of Z.

The stochastic differential equations in equation (3.3) and (3.4) can be used for the following equation which calculates the stock price at a given time t.

$$S_t = S_0 e^{([\mu - \frac{1}{2}\sigma^2]t + \sigma W_t)}, \tag{3.6}$$

$$S_t = S_0 e^{([\mu - \frac{1}{2}\sigma^2]t + \sigma W_t) * \prod_{i=1}^{N_t} Z_i}. \tag{3.7}$$

The Equations (3.6) and (3.7) show the solutions for stochastic differential Equations (3.3) and (3.4) respectively. S_0 is the first stock price and S_t is the stock price in the time period t.

The disadvantage of Merton's model is when analyzing the size of stock jumps Z_i. He assumes that the size of stock jumps Z_i follows a standard normal distribution which is symmetric with a bell-shape. However, Kou (2008) disproved Merton's assumption by showing that the standard normal distribution does not always represent the size of stock jumps seen in the stock

markets.

3.3 Warrant pricing using fractional Brownian motion

Many authors used fractional Brownian motion to avoid independence on warrant pricing. Kolmogorov (1940) introduced the fractional Brownian motion for the first time within the Hilbert Space framework. However, the name fractional Brownian motion is firstly seen in the work of Mandelbrot and Van Ness (1968). Before we proceed, below we give the formal definition of fractional Brownian motion and its properties.

Definition 3.3.1 (Biagini et al. (2008)). *Let $H \in (0,1)$ be a constant. A fractional Brownian motion $(B^{(H)}(t))_{t\geq 0}$ of Hurst index H is a continuous and centered Gaussian process with covariance function*

$$E[B^{(H)}(t)B^{(H)}(s)] = \tfrac{1}{2}t^{2H} + s^{2H} - |t-s|^{2H}, \forall s,t \in R^{+}.$$

For $H = \frac{1}{2}$, the fractional Brownian motion is a standard Brownian motion. By definition, a standard fractional Brownian motion $B^{(H)}$ has the following properties:

- $B^{(H)}(0) = E[B^{(H)}(t)] = 0$, for all $t \geq 0$;
- $B^{(H)}$ has homogeneous increments, i.e., $B^{(H)}(t+s) - B^{(H)}(s)$ has the same law of $B^{(H)}(t)$ for $s,t \geq 0$;
- $B^{(H)}$ is a Gaussian process and $E[B^{(H)}(t)^2] = t^{2H}, t \geq 0$, for all $H \in (0,1)$;
- $B^{(H)}$ has continuous trajectories .

Parameter H of $B^{(H)}$ was named by Mandelbrot after the name of the hydrologist Hurst, who made a statistical study of yearly water run-offs of the Nile river. Mandelbrot (1983) used this process to model some economic time series. Most recently these processes have been used to model telecommunication traffic [45].

The values of the Hurst exponent range from zero to one. In [79] it is mentioned that

- $H = \frac{1}{2}$ or close to that value indicate a random walk or a Brownian motion. In this case no correlation is present between any past, current, and future elements. In other words, there is no independence behaviour in the series. Such series is not easy to predict.
- $H < \frac{1}{2}$ indicates the presence of anti-persistence, meaning that if there is an increase, the decrease will automatically follow and vice versa. This behaviour is also called the mean reversion in the sense that the future values will always tend to return to a longer term mean value.
- $H > \frac{1}{2}$ indicates the presence of the persistence behaviour, meaning that the time series is trending. It may be a decreasing or increasing trend.

Fractional Brownian motion has two crucial properties: self-similarity and long-range dependence. The self-similarity is $a > 0$ then $(B^{(H)}(at), t \geq 0)$ if $(a^{(H)}B^{(H)}, t \geq 0)$. The long range means that if $r(n) = E[B^{(H)}(t)(B^{(H)}(n+1) - B^{(H)}(n))]$ then $\sum_{n=1}^{\infty} r(n) = \infty$. These two properties make the fractional Brownian motion a suitable instrument in different applications such as mathematical finance:

The assumptions which are used to derive the warrant pricing formula in fractional Brownian motion are as follows (see [80] for further details):

(i) The warrant price is the function of the time and underlying stock's price,

(ii) The shorting of assets with all use of proceeds is allowed,

(iii) There are no transactions costs or taxes and all securities are perfectly divisible,

(iv) Risk less arbitrage opportunities are controlled,

(v) The trading of the asset is continuous,

(vi) The risk-free rate of interest and all the maturities is constant,

(vii) The price of the stock follows fractional Brownian motion process and the dynamics of the risk adjusted process $(S_t, t \geq 0)$ are defined as

$$dS_t = S_t(\mu dt) + \sigma_v dB^{(H)}(t), 0 \leq t \leq T, \tag{3.8}$$

where

$B^{(H)} = B^{(H)}(t, x), t > 0$ is the Fractional Brownian motion,
μ is the expectation of the yield rate,
σ_v is the firm-value process volatility,
T is the option expiration time,
S_t is the stock price at time t.

Hu and Øksendal (2000) made use of Itô integrals with respect to $B^{(H)}$ and showed that the fractional Black-Scholes market presents no arbitrage opportunity. In that case, the following lemma [31] holds:

Lemma 3.3.2. *(Geometric Fractional Brownian motion) The solution of fractional differential equation*

$$dS(t) = \mu S(t) d_t + \sigma S(t) dB^{(H)}(t) \text{ where } S(0) = s_0$$

is given by

$$S(t) = s_0 e^{(\sigma B^{(H)}(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H})}. \tag{3.9}$$

Using the results of Hu and Øksendal (2000), Necula (2002) proved the following theorem which deals with European call option price.

Theorem 3.3.3. *(Fractional Black-Scholes formula) The price at every $t \in [0, T]$ of an European call option with strike price X and maturity T is given by*

$$C(t, S_t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2), \tag{3.10}$$

where

$$d_1 = \frac{\ln(\frac{S_t}{K}) + r(T-t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}},$$

and

$$d_2 = \frac{\ln(\frac{S_t}{K}) + r(T-t) - \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}},$$

$N(\cdot)$ is the cumulative probability of the standard normal distribution.

Proof. See Necula (2002). □

3.4 Warrant pricing using risk-neutral valuation

The assumption of Risk-neutral valuation is a crucial concept in the warrant pricing theory. This risk-neutral valuation approach was first introduced by Cox and Ross (1976). Harrison & Kreps (1979) and Harrison & Pliska (1981) applied it and stated that the theoretical price of a European-style claim is the discounted expected value of its future cash flows under risk-neutral valuation. The theory of risk-neutral valuation is crucial for warrant pricing because all the return assets should be equal to the risk-free interest rate and it can be

concluded that the expected return of the investors does not play a role in warrant pricing. Risk-neutral valuation theory assumed that the price of the stock follows the geometric Brownian motion. The resulting equation is

$$E_t\left[\frac{dS_t}{S_t} + \sigma d_t\right] = rdt. \tag{3.11}$$

This shows that the expected total return on the stock equals to the risk-free rate. Where E_t is the expected value at time t with respect to the geometric Brownian motion W_t.

Necula (2002) used fractional Black-Scholes formula and a fractional risk-neutral valuation theorem to price options.

Theorem 3.4.1 (Necula (2002)). *(fractional risk-neutral valuation) The price at every $t \in [0, T]$ of a bounded $F_T{}^H$ - measurable claim $F \in L^2(\mu)$ is given by*

$$F(t) = e^{-r(T-t)}\tilde{E}_t[F]. \tag{3.12}$$

Proof. We provide some basic steps for the proof of above theorem from Necula (2002). There is a replicating portfolio of the claim $(m(t), s(t))$ as the market is complete, and therefore

$$F(t) = m(t)M(t) + s(t)S(t).$$

This gives

$$\begin{aligned} dF(t) &= m(t)dM(t) + s(t)dS(t) \\ &= rF(t)dt + \sigma s(t)S(t)dB_H(t). \end{aligned}$$

Multiplying both sides by e^{-rt} and integrating we obtain

$$e^{-rt}F(t) = F(0) + \int_0^t e^{-r\tau}\sigma s(\tau)S(\tau)dB_H(\tau), 0 \leq t \leq T. \tag{3.13}$$

Now from the fractional Clark-Ocone theorem [3], we have

$$F(w) = E[F] + \int_0^T \tilde{E}_t[D_t F] dB_H(t).$$

Therefore

$$e^{-rT}F = E[F] + e^{-rT}\int_0^T \tilde{E}_\tau[D_\tau F] dB_H(\tau).$$

When the market is complete, we have

$$\tilde{E}_\tau[D_\tau F] = e^{T-\tau}\sigma s(\tau)S(\tau), 0 \leq t \leq T. \tag{3.14}$$

This implies

$$e^{-rT}F = E[F] + \int_0^T e^{-rT}\sigma s(\tau)S(\tau)dB_H(\tau),$$

it follows that

$$\tilde{E}_t[e^{-rT}F] = E[F] + \tilde{E}_t[\int_0^T e^{-rt}\sigma s(\tau)S(\tau)dB_H(\tau)].$$

Using quasi-martingale we get

$$\tilde{E}_t[e^{-rT}F] = E[F] + \int_0^t e^{-rt}\sigma s(\tau)S(\tau)dB_H(\tau). \tag{3.15}$$

Finally using equations (3.13) and (3.15) we, obtain

$$F(t) = e^{-r(T-t)}\tilde{E}_t[F].$$

□

Risk-neutral valuation is not just used to price options and warrants but it can hold for valuing other securities, such as forward contracts.

3.5 Warrant pricing using lattice methods

The lattice methods are widely used as the valuation technique for pricing American warrants and options. These methods are usually categorized in three categories which are binomial tree method $m = 2$ (which means a total of two possible outcomes for each successive step), trinomial tree method $m = 3$ (has three possible outcomes) and multinomial method for general m. The idea of lattice methods is to discretize the risk-neutral process and use dynamic programming to solve for the warrant and option prices. Below we discuss binomial and trinomial methods.

3.5.1 Binomial method

The binomial method was introduced by Cox, Ross and Rubinstein in 1979. It had a very profound impact on option pricing and warrant pricing ever since and it is widely used in option pricing model until today. The following are some of the features that the binomial method has:

- It can be easily implemented and produces fairly accurate results and it is usually preferred for pricing easy options and warrants.
- It is very straightforward to calculate while maintaining the clear insight behind it.
- It uses discrete-time and discrete-state approximations of differential equations to price American and European options.
- It is absolutely well-established in the economic theory of option pricing during reproduction under no-arbitrage conditions.

Rubinstein (1994) has extended standard binomial method into implied binomial method. Some of the advantages of the binomial method are even applied in the implied binomial method while expanding applicability beyond the

Black-Scholes economy as they permit for arbitrary ending risk-neutral probability distributions (binomial methods are only consistent with log-normal distributions).

Cox et al. (1979) define the binomial tree (see Figure 3.1) as a discretized description of geometric Brownian motion which is used often to describe asset behaviour. The up and down factors in the prices are given by

$$u = e^{\sigma\sqrt{st}}, d = \frac{1}{u} = e^{-\sigma\sqrt{st}}. \tag{3.16}$$

The probabilities with which these factors move up or down are given by

$$p_u = \frac{e^{rt} - e^{-\sigma\sqrt{t}}}{e^{\sigma\sqrt{t}} - e^{-\sigma\sqrt{t}}}, \tag{3.17}$$

and

$$p_d = 1 - p_u, \tag{3.18}$$

where

u is up-factor,
d is down-factor,
σ is a volatility,
t is a time step,
r is a yield of the underlying asset,
p_u is the probability of an up movement, and
p_d is the probability of a down movement.

The binomial tree method has been used very widely.

3.5.2 Trinomial method

The trinomial tree is more advanced than the binomial tree by allowing a stock price to stay the same apart from moving up or down with a certain probability

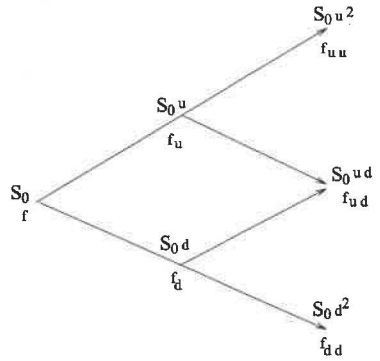

Figure 3.1: A typical example of a binomial tree

as in binomial method.

The trinomial tree is applied to the following problems:

- Pricing different European and American options;
- Pricing barrier options;
- Calculating the greeks (various hedging parameters that can be computed from the underlying option price).

Building a trinomial tree can be similar to building the binomial tree. In the trinomial model the price at the next time level is given by

$$S(t) = \begin{cases} S(t)u & \text{with probability } p_u \\ S(t) & \text{with probability } 1 - p_u - p_d \\ S(t)d & \text{with probability } p_d \end{cases}$$

and matching the first two moments of the distribution according to the no-arbitrage condition, we obtain

$$E[S(t_{i+1})|S(t_i)] = e^{rt}S(t_i), \tag{3.19}$$

$$Var[S(t_{i+1})|S(t_i)] = tS(t_i)^2\sigma^2 + O(t), \tag{3.20}$$

where the volatility of the underlying asset σ is assumed to be constant and the asset price follows a geometric Brownian motion; r is the risk-free rate of interest. The value of u, d, p_u and p_d, for the trinomial models are

$$u = e^{\sigma\sqrt{st}},\ d = e^{-\sigma\sqrt{st}}, \tag{3.21}$$

$$p_u = \left(\frac{e^{\frac{rt}{2}} - e^{-\sigma\sqrt{\frac{t}{2}}}}{e^{\sigma\sqrt{\frac{t}{2}}} - e^{-\sigma\sqrt{\frac{t}{2}}}}\right)^2, \tag{3.22}$$

and

$$p_d = \left(\frac{e^{\sigma\sqrt{\frac{t}{2}}} - e^{\frac{rt}{2}}}{e^{\sigma\sqrt{\frac{t}{2}}} - e^{-\sigma\sqrt{\frac{t}{2}}}}\right)^2, \tag{3.23}$$

also

$$p_m = 1 - p_u - p_d, \tag{3.24}$$

where
u is up-factor,
d is down-factor,
m is middle-factor
σ is volatility,
t is a time-step,
r is the yield of underlying asset,
p_u is the probability of an up movement,
p_m is the probability of a middle movement,
p_d is the probability of a down movement.

3.6 Warrant pricing using Monte Carlo simulation

Monte Carlo simulation was first introduced by Boyle (1976) to value options and warrants, in the context of claims contingent to a single underlying asset. The main purpose of Monte Carlo simulation was to provide a method of obtaining numerical solutions to option valuation problems. It has been used widely to price European-style claims. Only recently have there been endeavors to extend the method to price American-style claims. Bossaerts (1989) and Tilley (1993) were the first people to attempt to price American-style claims using Monte Carlo simulation. Now there is a benefit of using Monte Carlo simulation, because it allows a continuous pricing region, which in turn can price American-style claims with optimal accuracy.

Monte Carlo simulation is also used when there is a lack of continuous pricing region which can be a deficiency of many numerical approximation algorithms. DeHaven (2007) used the discrete event simulation program namely Rockwell Softwares Arena 10.0 to present a Monte Carlo simulation approach. This simulation is a process for valuing options by making use of numerical probability to generate a series of prices for the underlying instrument.

Monte Carlo simulation is different from other methods because its pricing region remains continuous which is the advantage of this simulation over the other pricing methods to produce very accurate results. Fouse (2009) compared Monte Carlo simulation with the binomial method saying that there is an advantage of using Monte Carlo simulation when computational costs effort because accuracy is considered. He emphasized it by the following example. If generated a simulation of 200 paths and compared to a binomial method, the simulation has an advantage as it has 200 possible pricing nodes in the first period (when comparing this to two nodes in the first period of the binomial method there is an enormous difference). It is straightforward that a Monte Carlo simulation has an advantage looking at the computational costs effort when comparing the accuracy.

Longstaff and Schwartz (2001) introduced the least-squares method which

unravel the backward-looking simulation to value warrants and options of American type.

Computations in the method of Tilley (1993) demand a lot of memory and grows in order of $O(MN)$ for stock prices at all simulation times and paths where M represent the number of paths and N represent the number of time periods, and it was limiting the accuracy of this simulation because of storage requirement. Chan et al. (2003) and Longstaff and Schwartz (2001) attempted to reduce the large amount of storage required in Tilley's model by replacing the forward path simulation of a given method with the backward one. Their solution reduced the memory storage to $O(M)$ by not storing all the intermediate asset prices and by generating each random number twice instead of once. The method had biases like other methods for pricing American options in terms of achieving high accuracy, because of not using large M and N.

In general, Monte Carlo simulation generates M pricing paths of an underlying asset, using the traditional valuing system to calculate the increase of that path (depending on the warrant you have, whether a call or a put), it then finds the anticipated warrant value discounted to the initial time steps. The discounted present value is therefore the estimated price associated with the warrant. When generating M paths and finding the mean warrant value of these paths, Monte Carlo simulation uses a stochastic sampling technique to create the expected value.

Rasmussen (2002) defined the Monte Carlo estimate using a stopping time $\tau \in (t, T)$, where

$$\frac{L_t}{\beta_t} = E_t^{\theta}\left[\frac{X_\tau}{\beta_\tau}\right],$$

is determined as the conditional expectation when information of time t is given. He used the underlying model to generate N independent paths of the variables determining the payoffs process $\{X_t\}_{0 \leq t \leq T}$ and the discount process

$\{\beta_t\}_{0\leq t\leq T}$, and gave the crude Monte Carlo estimate by

$$\frac{L_t^{(N)}}{\beta_t} = \frac{1}{N}\sum_{i=1}^{N}\frac{X_\tau^i}{\beta_\tau^i}, \tag{3.25}$$

where $\frac{X_\tau^i}{\beta_\tau^i}$ is the discounted payoff from the i'th path using the stopping time τ.

To see that the Monte Carlo estimate is unbiased, we note that the expectation of the estimate is given by

$$E_t^\theta\left[\frac{L_t^{(N)}}{\beta_t}\right] = \frac{1}{N}\sum_{i=1}^{N} E_t^\theta\left[\frac{X_\tau^i}{\beta_\tau^i}\right] = E_t^\theta\left[\frac{X_\tau^i}{\beta_\tau^i}\right] = \frac{L_t}{\beta_t},$$

and its variance is given by

$$Var_t^\theta\left[\frac{L_t^{(N)}}{\beta_t}\right] = \frac{1}{N^2}\sum_{i=1}^{N} Var_t^\theta\left[\frac{X_\tau^i}{\beta_\tau^i}\right] = \frac{1}{N}Var_t^\theta\left[\frac{X_\tau}{\beta_\tau}\right],$$

and then from the Central Limit Theorem we can see that $\frac{L_t^{(N)}}{\beta_t}$ follows normal distribution as $N \to \infty$.

Chan et al. (2003) used an algorithm which was very similar to that of Longstaff and Schwartz (2001). They generated paths in the time decreasing direction which follow the geometric Brownian motion. Their algorithm reads:

$S_1 = S_0 e^{([r-\frac{1}{2}\sigma^2]\delta t+\sigma\sqrt{\delta t}\epsilon_N)}$,

$S_i = S_0 e^{(i[r-\frac{1}{2}\sigma^2]\delta t+\sigma\sqrt{\delta t}(\epsilon_N+\epsilon_{N-1}+\ldots+\epsilon_{N-i+1}))}$,

$S_N = S_0 e^{(N[r-\frac{1}{2}\sigma^2]\delta t+\sigma\sqrt{\delta t}(\epsilon_N+\epsilon_{N-1}+\ldots+\epsilon_1))}$

where $\epsilon \approx N(0,1)$ and
S_0 is the initial stock price,
r is the risk-free interest rate,
σ is the volatility of the stock,
N is the number of time periods,
M is the number of paths,

δt is the length of each time period, and finally
ϵ_i are independent identically distributed from $N(0,1)$ for $i = 1, 2, ..., N$.

In this algorithm the benefit is that the initial starting seed value can provide each random number required during the simulation, and this allows the initial seed value to generate the random number set. In such cases each random number is generated twice, but it does not affect memory storage needed to carry out the simulation because these random numbers are not required to be stored.

3.6.1 The standard Monte Carlo method

This section provides the algorithm given in Chan et al. (2003) and later on modified by DeHaven (2007). The algorithm is as follows:

Step 1. Inputs:

- Strike price (K);
- Length of time horizon in years (T);
- Initial price of the call or put option.

Step 2. Initialization:

- Set the seed of the path to any given positive integer;
- Randomly generate $\epsilon_i \approx N(0,1)$ for each path $i = 1, 2, ..., N$ and compute their sum ω_N;
  $\omega = 0$; for $i = 1 : N$,
  $\omega = \omega$+randn;
  end;

- Compute S_N (asset price) at the expiration date T using:

$$S_i = S_0 \exp\left(i\left[r - \tfrac{1}{2}\sigma^2\right]\delta t + \sigma\sqrt{\delta t}\omega_i\right).$$

Step 3. Generate S_{i-1} from S_i for $i = N, ..., 1$ in the step of 1:

- Set $\delta t = \frac{T}{N}$.
- Extract ϵ_{N-i+1} and compute $\omega_{i-1} = \omega_i - \epsilon_{N-i+1}$ using the same seed sequence.
- Extract the new seed value.

Step 4. Compute if the warrant is in the money for each path k. For each path:

- Let Y be the vector containing the corresponding cash flows received at $i+1$ time period and X be the vector containing asset prices S_i, which have the been discounted backward to the ith time period.
- Determine whether to exercise the warrant immediately or hold the warrant until the next time period, based on which gives the higher value. Establish the current cash flows conditional on not exercising prior to time period i using:

$$C_i(k) = \begin{cases} cashflow & \text{if cash flow} \geq E[Y|X] \\ 0 & \text{otherwise} \end{cases}$$

- Compute the present value of the cash flows $P_i(k)$ given by:

$$P_i(k) = C_i(k) + e^{-r\delta t}P_i(k) \tag{3.26}$$

3.6.2 The Least-Squares Monte Carlo method

The Least-Squares approach was first introduced by Longstaff and Schwartz (2001). This approach is very useful in the use of Monte Carlo simulation for pricing options. This approach became more famous due to its ease to price options with complex payoff functions. The method of Longstaff and Schwartz (2001) uses backward analysis to decide if an option would be exercised in the given time, by comparing the immediate profit.

Rasmussen (2002) improved the Monte Carlo valuation by applying control variates to the sampled payoffs which are discounted and used in the Least-Squares approach and scattering the variables of initial state from the paths used in the Least-Squares approach.

Rodrigues and Armada (2006) described the Least-Square method for the contingent claims on underlying assets whose prices follow a geometric Brownian motion:

$$dX_t = (\mu - \delta)X_t dt + \sigma X_t dW, \tag{3.27}$$

where X_t is greater than 0, μ and σ are drift volatility and parameters, δ is the dividend and dW is the increment of a Wiener process. Assuming market completeness, there is a unique risk-neutral probability measure under which the asset price stochastic process is

$$dX_t = rX_t dt + \sigma X_t dW, \tag{3.28}$$

where r is the risk free interest rate. The American option value that can be exercised from time interval $[0, T]$, or $[t, T]$ with payoff function is $\prod(t, X_t)$. It can be expressed as

$$F(t, X_t) = \max_{\tau} \left\{ E_t^*[e^{-r(\tau - t)} \prod(\tau, X_\tau)] \right\}, \tag{3.29}$$

where τ is the optional stopping time ($\tau \in [t, T]$) and E_t^* is the risk neutral expectation, conditional on the information available at t.

Longstaff and Schwartz (2001) suggested a Monte Carlo simulation algorithm to value American options (NB. After necessary modifications, this algorithm can be used to price warrants). They stated that the optimal stopping time can be obtained by using the following Bellman equation

$$F(t_n, X_{t_n}) = \max\left\{\prod(t_n, X_{t_n}), e^{-r(t_{n+1}-t)}E^*_{t_n}[F(t_{n+1}, X_{t_{n+1}})]\right\}. \tag{3.30}$$

The continuation value is denoted by

$$\Phi(t_n, X_{t_n}) = e^{-r(t_{n+1}-t_n)}E^*_{t_n}[F(t_{n+1}, X_{t_{n+1}})]. \tag{3.31}$$

The continuation value equals to zero at the expiration date, because the option is no longer available. It is written as

$$\Phi(T, X_T) = 0.$$

Starting from T and moving backwards, the optimal stopping time for each path ($\tau(\omega)$) is computed. i.e.,

$$\text{if } \Phi(t_n, X_{t_n}(\omega)) \leq \prod(t_n, X_{t_n}) \text{ then } \tau(\omega) = t_n,$$

where $\tau(\omega)$ is optimal stopping time. At time t_n, prior to T, the option holder must compare the payoff with the immediate exercise ($\Phi(t_n, X_{t_n})$), which has an unknown continuation value and it is the expected conditional value of future cash flows. When this condition holds, $\tau(\omega)$ is updated. The value of the option is calculated by averaging the values of all (K) paths.

The computation of the continuation value (Φ) is the main contribution of the Least-Square approach and it is the expected value of the future cash flows from optimal exercise. Let $\prod(t, s, \tau, \omega)$ to be the cash flows from the ω-path. If option is exercised in an optimal manner at $s(t < s \leq T)$ with the assumption that it has not been exercised at or before time t, the expected

value at t_n would be

$$\Phi(t_n, X_{t_n}) = E^*_{t_n}\left[\sum_{i=n+1}^{N} e^{-r(t_i-t_n)} \prod(t_n, t_i, \tau, \cdot)\right], \tag{3.32}$$

with

$$\prod(t, s, \tau, \omega) = \begin{cases} \prod(s, X_s(\omega)) & \text{if } \tau(\omega) = s \\ 0 & \text{otherwise} \end{cases}$$

In [42] Φ belongs to a Hilbert space L^2, it can be represented by a countable orthonormal basis. The conditional expectation can be expressed by a linear combination of the elements of the basis, $\Phi(t, X_t) = \sum_{j=i}^{\infty} \phi(t) L_j(t, X_t)$. The continuation value can be calculated using the first $J < \infty$ basis: $\Phi^J(t, X_t) = \sum_{j=1}^{J} \phi(t) L_j(t, X_t)$ with $\phi(t)$ estimated by a least squares regression. The continuation value estimated by the regression is then used to compute the optimal stopping time

$$\hat{\Phi}^J(t_n, X_{t_n}) = \sum_{j=1}^{J} \hat{\phi}(t) L_j(t_n, X_{t_n}). \tag{3.33}$$

The use of in-the-money paths in the Least-Square regression produces faster algorithm estimate of the option value with lower standard errors. Stentoft (2004) states that the standard error of the Least-Square method algorithm can be decomposed in to two kinds of biases:

- An approximation error of the continuation value is estimated and it leads to a low bias, as a result of using finite number of basis functions, this bias is written as

$$\Phi(t, X_t) \approx \hat{\Phi}^J(t, X_t),$$

- The stochastic error as a result of the Monte Carlo simulation is written:

$$F(0, x) \approx \frac{1}{k} \sum_{\omega=1}^{k} e^{r\tau(\omega)} \prod(\tau(\omega), X_{\tau(\omega)}(\omega)).$$

This chapter presented the numerical methods that are used for option and warrant pricing. In the next chapter we present some numerical results obtained by some of these methods.

Chapter 4

Numerical results and discussion

In this chapter we illustrate some results that are calculated for warrant pricing.

Table 4.1 shows the Changdian stock price which was left out when simulating warrant pricing in [80]. This is to compare the descriptive statistics of stock price and warrant price using the same Changdian market. The logarithmic returns is defined as

$$x_t = \ln \frac{y_{t+1}}{y_t} = \ln y_{t+1} - \ln y_t,$$

where y_t is the closing quotation of Changdian stocks and warrants at time t. The average of Changdian warrants is 0.0042 and that of the Changdian stocks is 0.003. This implies that in Changdian stocks the gaps between the data are closer to each other than in Changdian warrants. The standard deviation for Changdian warrants is 0.0490 and for Changdian stocks is 0.0197. It is known that the smaller the standard deviation the closer they are. For skewness, kurtosis and Jarque-Bera, Table 4.1 is compared with Table 4.2 (from Zhang et al. 2009). Table 4.1 is the yield series of Changdian stocks' descriptive statistics and Table 4.2 is the yield series of Changdian warrants' descriptive statistics. The formulas used to calculate the mean, standard deviation, skewness, kurtosis and Jarque-Bera are as follows:

Table 4.1: The yield series of Changdian stocks' descriptive statistics

Observations	Mean	Standard deviation	Skewness	Kurtosis	Jarque-Bera
168	0.0030	0.0197	0.4393	2.2126	9.7436

Mean=$\frac{\sum_{i=1}^{n} x_i}{n}$,
Standard deviation=$\sqrt{\frac{\sum_{i=1}^{n}(x-\bar{x})^2}{n-1}}$,
Skewness=$\frac{\mu_3}{\sigma^3}$,
Kurtosis=$\frac{\mu_4}{\sigma^4}$,
Jarque-Bera=$(\frac{n}{6})(S^2 + \frac{1}{4}(K-3)^2)$.

where n is the number of observations, μ_3 is the third moment about the mean, μ_4 is the fourth moment about the mean, σ is the standard deviation, $\bar{x}$ is the mean, S is coefficient of skewness and K is coefficient of kurtosis.

Table 4.2: The yield series of Changdian warrants' descriptive statistics

Observations	Mean	Standard deviation	Skewness	Kurtosis	Jarque-Bera
168	0.0042	0.0490	0.5773	8.6698	234.3679

The yield distribution of Changdian stocks is greater than zero, which implies that the yield distribution of Changdian stocks is not a normal distribution like in Changdian warrants. When kutosis is greater than three, it implies that the yield of Changdian stocks is leptokurtic. The value of Jarque-Bera in Changdian stocks implies that the yield distribution is less probability group near the starting point and in the tails. While the value of Jarque-Bera in Changdian warrants implies that the yield distribution of Changdian warrants have more probability group near the starting point and in the tails.

In Table 4.3 the results are obtained from the Changdian stocks and warrants data. We calculate the probabilities in the binomial tree and trinomial tree methods using the formulas from Sections 3.5.1 and 3.5.2. The results calculated using Changdian warrants data are: r=0.0225, t=0.75 and σ=0.3016 and are comparable with those seen in [80]. The second results are calculated from Changdian stock data: r=0.014, t=0.75 and σ=0.2. In binomial method

we calculated up-movement and down-movement. In the trinomial method we calculated up-movement, down-movement and middle-movement.

Table 4.3: Results of warrant pricing via the Binomial and Trinomial methods (Probabilities)

	Binomial Tree		Trinomial Tree		
	u	d	u	d	middle movement
1	0.46727	0.53273	0.22736	0.27372	0.49892
2	0.48713	0.51287	0.24092	0.25925	0.49983

We expanded the descriptive statistics of Changdian warrants and Changdian stocks and calculated a regression analysis and Anova. Letting Changdian warrants to be the dependent variable (Y) and Changdian stocks to be independent variable (X) we compute the results as shown in Table 4.4. It shows the multiple regression of 0.8762 which is a strong (+) correlation coefficient. R^2 is equal to 0.7677 which means that 77% of the variance is shared between Changdian warrants and stocks. We only have Changdian stocks as an independent variable in our regression analysis. Changdian stocks are the same as the square of the correlation between the Changdian warrants and stocks. The

Table 4.4: Results of regression analysis and statistical testing for the Changdian warrants and stocks

Observations	Multiple R	R^2	Adjusted R^2	Standard error
168	0.8762	0.7677	0.7663	0.4335

formulas used to calculate Multiple R, R^2, Adjusted R^2 and Standard error.

Multiple regression $(r) = \frac{SS_{xy}}{\sqrt{SS_x SS_y}}$,

where: $SS_{xy} = \sum xy - \frac{(\sum x)(\sum y)}{n}$, $SS_x = \sum x^2 - \frac{(\sum x)^2}{n}$ and $SS_y = \sum y^2 - \frac{(\sum y)^2}{n}$,

$R^2 = \sqrt{\frac{SS_{xy}}{\sqrt{SS_x SS_y}}}$

Adjusted $R^2 = 1 - (R^2)\frac{n-1}{n-k-1}$

where R^2 is the coefficient of determination, n is the number of observations and k is the number of independent variables.

Standard error (SE) $= \frac{\sigma}{\sqrt{n}}$

where σ is the standard deviation. Table 4.5 is the analysis of variance (ANOVA) obtained from the data of Changdian warrants and Changdian stocks. The test statistic in ANOVA is the F of 548.59. A large value of F indicates relatively more difference between groups than within groups. Since the test statistic is much larger than the critical value, and P-value is less than 0.05 and 0.01 we reject the null hypothesis of two means (Changdian warrants and Changdian stocks) when it is true and conclude that there is a statistically significant difference among these means, where df is the degree of freedom,

Table 4.5: ANOVA results for the Changdian warrants and stocks

	df	SS	MS	F	P-value
Regression	1	103.09	103.09	548.59	1.7E-54
Residual	166	31.20	0.19	–	–
Total	167	134.29	–	–	–

SS is the sum of squares, MS is the mean square and F is the F-value

Chapter 5

Concluding remarks and scope for future research

In this thesis different methods are shown for pricing warrants. The warrant pricing models are based on the variables used in the Black-Scholes option pricing formula. Mathematical models show that the better predictions of warrant pricing are given by a special case of the Constant Elasticity of Variance (CEV) model, over the Black-Scholes model.

The warrant pricing has been compared with option pricing theoretically and practically, showing the similarities and how they differ. The results have shown that the Black-Scholes model associated for dilution as the stock price (S) are replaced by the value of the company (V). The standard deviation of the stock's return (σ) is replaced by the standard deviation of the value (σ_v) and the outcome model is multiplied by the dilution factor ($1/(1+q)$).

In order to generate the initial approximation to the warrant pricing, certain numerical methods were also used. For example, we have mentioned *fractional Brownian motion*, which is used by many authors to avoid independency. To derive warrant pricing formulas in fractional Brownian motion the assumptions and fractional Black-Scholes formuli were taken into consideration. The warrant pricing in fractional Brownian motion is similar to the

European call option.

It is shown that if warrant prices are calculated twice using the Black-Scholes model, they give the same results, but if using fractional Brownian motion they give different results because of the long memory property.

Another method used to price warrants is *Monte Carlo simulation.* The purpose of this simulation is to provide a method obtaining numerical solutions to warrant valuation problems. Monte Carlo simulation was compared with the binomial methods. Figure 3.1 shows that the binomial tree has only two nodes in the first period, while Monte Carlo simulation has 200 nodes which means that the Monte Carlo simulation is more accurate than the binomial tree and the other methods. Monte Carlo simulation calculates much higher prices for the American option.

The Least-Squares approach is crucial in the use of Monte Carlo simulation for pricing warrants and to estimate the expected payoff to the holder of American warrants. Longstaff and Schwartz (2001) suggested the use of in-the-money paths which improves the accuracy of warrants' valuation.

Lattice methods are also preferred to price warrants. In this thesis it has been shown that the trinomial method is more advanced than the binomial method by allowing the stock to stay the same or to move up or down with certain probability.

This thesis has covered a wide range of methods and models for warrant pricing. Out of these methods, some of them are easy to implement and produce accurate results, such as, lattices (binomial tree and trinomial tree), risk-neutral valuation, Monte Carlo simulation and Least-Squares approach. The Monte-Carlo algorithm of reducing the storage capacity to price American options is shown.

The geometric Brownian motion does not allow us to verify all the applicability of the warrant pricing. Therefore, a certain question arises: Why are geometric Brownian motion and random walk preferred to price warrants while they can not capture the stock price behaviour? Such question can be researched further.

In fractional Brownian motion, we utilised the Hurst exponent, which is a

tool used to test the memory in time series, and therefore helps to determine the behaviour and efficiency of the markets. A Hurst exponent which is equal to $\frac{1}{2}$ indicates the independence behaviour of the series, whereas the Hurst exponent values different from $\frac{1}{2}$ show the presence of long memory or long range dependence which is characterised by the fractional Brownian motion model.

Interesting questions that can be researched further are: 1. Why some pricing methods do not consider the individuality of the warrants and make use of the Black-Scholes formula to price warrants? 2. Why some researchers replaced firm-value process volatility by stock return volatility when pricing warrants which can lead to inaccuracy of warrant pricing results?

In Monte Carlo simulation, Chan (2003) presented a reduced memory algorithm for pricing American options that does not store all of the intermediate prices and values to calculate the prices; (e.g. Z_i and V_i). The methods that store each of the prices are far more expensive than the memory requirement method. The cost of this method is only the computational requirements, because each seed value must be calculated twice, once for the backward pricing and once for the forward pricing. When pricing multiple warrants in a multidimensional domain this memory reduction technique can be used. Chan's (2003) reduced memory method is not utilised and as a result requires a large amount of memory to perform the simulation.

As indicated earlier, some of the methods discussed above are purely for options but after necessary modifications, they can be used to price warrants. Such a modification is being done but due to time limitations, we would explore them further in the near future.

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