Higher Order Numerical Methods
for
Singular Perturbation Problems

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KEYWORDS

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Higher order numerical methods

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Self-adjoint problems

Turning point problems

Parabolic problems

Elliptic problems

Maximum and minimum principles

Convergence analysis

Abstract

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PhD Thesis, Department of Mathematics and Applied Mathematics, University of the Western Cape.

In recent years, there has been a great interest towards the higher order numerical methods for singularly perturbed problems. As compared to their lower order counterparts, they provide better accuracy with fewer mesh points. Construction and/or implementation of direct higher order methods is usually very complicated. Thus a natural choice is to use some convergence acceleration techniques, e.g., Richardson extrapolation, defect correction, etc. In this thesis, we will consider various classes of problems described by singularly perturbed ordinary and partial differential equations. For these problems, we design some novel numerical methods and attempt to increase their accuracy as well as the order of convergence. We also do the same for existing numerical methods in some instances. We find that, even though the Richardson extrapolation technique always improves the accuracy, it does not perform equally well when applied to different methods for certain classes of problems. Moreover, while in some cases it improves the order of convergence, in other cases it does not. These issues are discussed in this thesis for linear and non-linear singularly perturbed ODEs as well as PDEs. Extrapolation techniques are analyzed thoroughly in all the cases, whereas the limitations of the defect correction approach for certain problems is indicated at the end of the thesis.

May 2009.

Declaration

I declare that **Higher Order Numerical Methods for Singular Perturbation Problems** is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Justin Bazimaziki Munyakazi May 2009

Signed:..

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A Adeline,

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A tous ceux qui,

de près ou de loin,

souffrent les affres des guerres

récurrentes et injustes

imposées a mon pays,

la R.D.Congo.

"There's no honorable way to kill, no gentle way to destroy.
There is nothing good in war except its ending."

-Abraham Lincoln

"Never think that war, no matter how necessary,
nor how justified, is not a crime."

-Ernest Hemingway

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Part of this thesis has already been published/submitted in form of the following research papers:

1. Justin B. Munyakazi and Kailash C. Patidar, On Richardson extrapolation for fitted operator finite difference methods, *Applied Mathematics and Computation* **201** (2008) 465-480.

2. Justin B. Munyakazi and Kailash C. Patidar, Limitations of Richardson's Extrapolation for a High Order Fitted Mesh Method for Self-adjoint Singularly Perturbed Problems, *Journal of Applied Mathematics and Computing*, in press.

3. Justin B. Munyakazi and Kailash C. Patidar, A fitted operator finite difference method for a singularly perturbed turning point problem whose solution has boundary layers, submitted for publication.

4. Justin B. Munyakazi and Kailash C. Patidar, Performance of convergence acceleration techniques on various numerical methods for a singularly perturbed turning point problem whose solution has boundary layers, submitted for publication.

5. Justin B. Munyakazi and Kailash C. Patidar, A high accuracy fitted operator finite difference method for a nonlinear singularly perturbed two-point boundary value problem, submitted for publication.

6. Justin B. Munyakazi and Kailash C. Patidar, Higher order numerical method for singularly perturbed parabolic problems in one dimension, submitted for publication.

7. Justin B. Munyakazi and Kailash C. Patidar, Higher order numerical methods for singularly perturbed elliptic problems, submitted for publication.

Chapter 1

General Introduction

In this chapter, we provide a state-of-the-art on some works on higher order methods developed in recent years for singular perturbation problems (SPPs). To motivate the works, firstly we present some singularly perturbed models and briefly review the methods of solving them with a particular attention to the fitted methods. Two popular meshes (Bakhvalov mesh and Shishkin mesh) for resolving the difficulties associated with the layer(s) in the solutions of SPPs are also discussed. Finally, we present the summary of this thesis at the end of this chapter.

1.1 Introduction

In real life we often encounter many problems which are described by parameter dependent differential equations. The behaviour of the solutions of these differential equations depend on the magnitude of the parameters. If the parameter is small and multiplies the highest derivative term in such an equation, then the problem is said to be singularly perturbed and the small parameter is referred to as the singular perturbation parameter. More precisely, consider a problem depending on a small parameter ε (the singular perturbation parameter) which we denote by P_ε where ε is multiplied to the highest derivative term(s). Setting $\varepsilon = 0$ in P_ε, we obtain a reduced problem which we denote by P_0. Let us assume further that $u(x,\varepsilon)$ is a solution of P_ε, and $u(x,0)$ is the solution of the reduced problem.

Now, if

$$\lim_{\varepsilon \to 0} u(x,\varepsilon) = u(x,0)$$

then P_ε is a regular perturbation problem (RPP); otherwise P_ε is a singular perturbation problem (SPP). Notice that the solutions of this type of differential equations typically contain layers [124]. We explain the layer behaviour of the solutions through the following examples.

Example 1.1.1. Consider the following initial value problem [102]

$$\begin{aligned} \varepsilon u'(x,\varepsilon) + u(x,\varepsilon) &= 0 \\ u(0,\varepsilon) &= u_0. \end{aligned}$$

The exact solution of the above problem is $u(x,\varepsilon) = u(0,\varepsilon)e^{-x/\varepsilon}$. The reduced problem has the trivial solution $v(x,0) = 0$, which does not agree with the initial condition unless $u_0 = 0$. This explains that there is a boundary layer in the neighbourhood of $x = 0$.

Example 1.1.2. Consider the reaction-diffusion problem [102]

$$\begin{aligned} -\varepsilon u''(x,\varepsilon) + u(x,\varepsilon) &= 0, x \in [0,1], \\ u(0,\varepsilon) = u_0, u(1,\varepsilon) &= u_1. \end{aligned}$$

When $u_0 = u_1 = 1$, the exact solution of the above problem will be

$$u(x,\varepsilon) = \frac{e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}}.$$

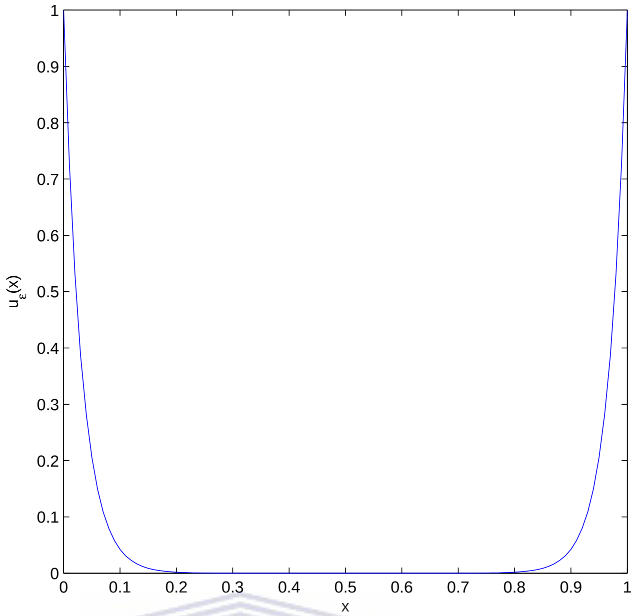

Figure 1.1: Exact solution of Example 1.1.2 for $\varepsilon = 10^{-3}$

The solution to the reduced problem of this reaction-diffusion problem is again the trivial function $v(x, 0) = 0$. It does not agree with the boundary values u_0 and u_1, unless these values vanish. Thus, the solution possesses two boundary layers: one in the neighourhood of $x = 0$ and the other in the neighbourhood of $x = 1$.

Example 1.1.3. Consider the linear convection-diffusion problem [102]

$$-\varepsilon u''(x, \varepsilon) + u'(x, \varepsilon) = 0$$
$$u(0, \varepsilon) = u_0, \quad u(1, \varepsilon) = u_1.$$

The exact solution is of the form

$$u(x, \varepsilon) = A + Be^{-(1-x)/\varepsilon}$$

The solution of the reduced problem solves the first order ordinary differential equation

$v_0'(x) = 0$ in which only one integration parameter is allowed. Therefore only one boundary condition can be used to determine the solution of the reduced problem. Since the problem does not agree with the other boundary condition, a layer will occur. It is clear from the form of $u(x, \varepsilon)$ that, unless $u_0 = u_1$, a boundary layer arises in the neighbourhood of $x = 1$.

Example 1.1.4. Consider the following two-point boundary value problem for the Burger's equation on the interval $\Omega = (-1, 1)$ [102]

$$-\varepsilon u''(x,\varepsilon) + u(x,\varepsilon)u'(x,\varepsilon) + u(x,\varepsilon) = 0$$
$$u(-1,\varepsilon) = u_{-1}, \quad u(1,\varepsilon) = u_1.$$

The reduced equation $v(x,0)v'(x,0) + v(x,0) = 0$ has two families of solutions, namely $v(x,0) = 0$ and the solutions of $v'(x,0) = -1$ which are $v^+(x,0) = -(x+1) + u_{-1}$ and $v^-(x,0) = -(x-1) + u_1$.

The layer occurs at:

$$x_s = \frac{u_{-1} + u_1}{2}$$

The terminology *boundary layers* was introduced by Ludwig Prandtl at the Third International Congress of Mathematicians in Heidelberg [124]. In his paper, Prandtl explained the boundary layer phenomenon which occurs in fluid and gas dynamics. It is however believed that the idea of boundary layer has its roots in the early nineteenth century [36]. The great natural philosophers of that era such as Laplace and Lorenz applied this idea first to the static liquid drop of meniscus, and then to elasticity, creeping viscous flow, electrostatics and acoustics.

Singular perturbation problems arise in many other areas of applied mathematics. Fluid mechanics, quantum mechanics, plasticity, chemical-reaction theory, aerodynamics, rarefied-gas dynamics, oceanography, meteorology, modelling of semiconductor devices, diffraction theory and reaction-diffusion processes are some of these areas. The singularly

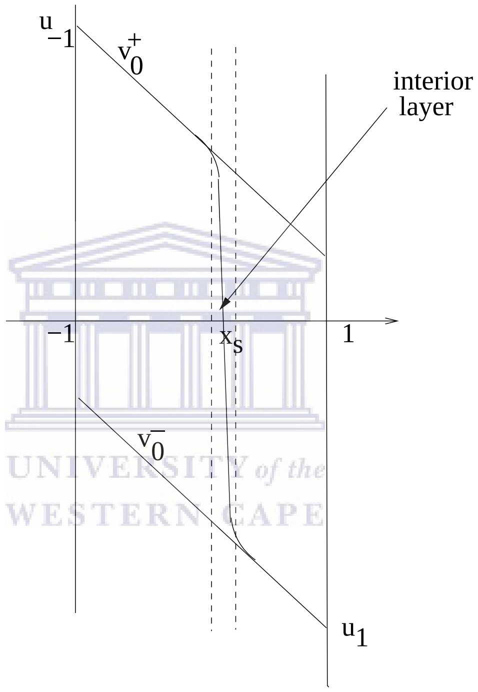

Figure 1.2: The exact solution of the Burger's problem and the two reduced solutions v_0^+ and v_0^-

perturbed differential equations have a variety of features depending on the situations that they describe. These features may be taken into account in the selection of the methods for solutions.

Asymptotic methods can be used to give qualitative information about the solutions, for instance the width and the location of layers. When analytical solutions are not available, SPPs can be solved by means of numerical methods (finite difference methods, finite elements methods, spline approximation methods, etc). However, these standard methods fail to resolve the layer(s) for all values of the parameter ε, unless a very fine grid is considered, which unfortunately raises up the computational complexities. Therefore, methods providing reliable numerical results on a mesh with a reasonable number of grid points are to be sought.

The rest of this chapter is organized as follows. Section 1.2 presents some models describing singularly perturbed problems. Methods for solution of SPPs are discussed in Section 1.3. Two mesh selection strategies for resolving the layer difficulties occurring in the solution of SPPs, namely the Bakhvalov-type and the Shiskhin-type meshes are also dealt within this section. The focus of Section 1.4 is to provide a brief account of works on higher order methods which are applied so far to solve SPPs, and finally in Section 1.5, we give a short discussion about different issues presented in this chapter.

1.2 Some models of singular perturbation problems (SPPs)

Several real life situations are described by singularly perturbed differential equations. Below, we give some models describing these situations.

1. Fluid and gas dynamics are described by Navier-Stokes equations [102]. In two dimensions, these are made of the following system of four nonlinear partial differential

equations for the conservation of mass, momentum, and energy.

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} &= 0, \\
\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \frac{\partial (\rho v u)}{\partial y} - \mu \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) &= 0, \\
\frac{\partial \rho v}{\partial t} + \frac{\partial (\rho u v)}{\partial x} + \frac{\partial (\rho v^2 + p)}{\partial y} - \mu \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \right) &= 0, \\
\frac{\partial \rho e}{\partial t} + \frac{\partial}{\partial x} \left(\rho u \left(e + \frac{p}{\rho} \right) \right) + \frac{\partial}{\partial y} \left(\rho v \left(e + \frac{p}{\rho} \right) \right) & \\
-\mu \left(\frac{\partial}{\partial x} (u \tau_{xx} + v \tau_{xy}) + \frac{\partial}{\partial y} (u \tau_{yx} + v \tau_{yy}) \right) - k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) &= 0,
\end{aligned}
$$

where ρ, u, v and e are the dependent variables; ρ is density of the material (fluid), u and v, the components of the velocity of the fluid, and e the internal energy. The coefficient μ and k are respectively the inverse of the Reynolds number Re and that of the Prandtl number Pr. The component τ_{xx}, τ_{xy}, τ_{yx} and τ_{yy} of the viscous stress tensor τ are expressed in terms of the rate of change in space of the velocities by the relations:

$$
\tau_{xx} = \frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \frac{\partial v}{\partial y}; \ \tau_{yy} = -\frac{2}{3} \frac{\partial u}{\partial x} + \frac{4}{3} \frac{\partial v}{\partial y}; \ \tau_{xy} = \tau_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.
$$

Notice that last three of the above mentioned Navier-Stokes equations are of second order. When $\mu = 0$ and $k = 0$ in these equations, their orders drops to first order. The equations thus obtained are the Euler equations. The solutions of the Navier-Stokes equations contain more integration parameters than those of the Euler equations and, consequently, more boundary conditions are required to specify the solution of the Navier-Stokes equations. For instance, the imposition of a condition of zero velocity (the 'no-slip' condition) at the surface of the plate, in the case of steady incompressible laminar flow over an infinite flat plate is allowed for the Navier-Stokes equations and not for the Euler equations. In this case the 'no-slip' condition creates a layer near the surface of the infinite flat plate.

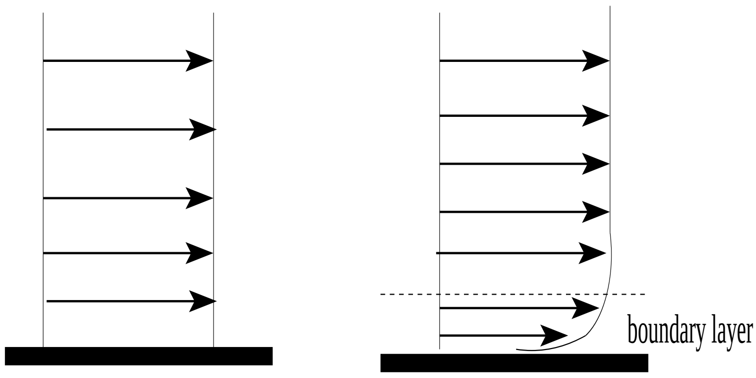

Figure 1.3: The profile of a viscous flow for Euler and Navier-Stokes models

2. Consider the free motion of the undamped linear spring mass system with a very resistant spring [114]. Let the prescribed specific displacement be at times $t = 0$ and 1. Then one can obtain the two-point problem

$$\varepsilon^2\ddot{x} + x = 0, \quad 0 \le t \le 1, \quad x(0) = 0, \quad x(1) = 1$$

where ε^2 (the ratio of the mass to the spring constant) is small. For non-exceptional small positive values of ε the exact solution oscillates rapidly, so no pointwise limit exists as $\varepsilon \to 0$.

3. Consider the Dirichlet problem [113, 152]:

$$\varepsilon\ddot{x} + x\dot{x} = 0 \quad \text{on} \quad 0 \le t \le 1,$$

where $x(0)$ and $x(1)$ are prescribed. It could describe the motion of a mass moving in a medium with damping proportional to the displacement, where either the mass is small or the damping is large. Depending on the particular end values $x(0)$ and $x(1)$, the solution may have initial/shock/boundary layers.

4. The example:

$$\varepsilon \ddot{x} - \left(t - \frac{1}{2}\right)\dot{x} = 0, \quad 0 \le t \le 1, \;\; x(0) \;\text{ and }\; x(1) \quad \text{are prescribed}$$

   relates to an exit time problem for randomly perturbed dynamical systems [127].

5. Consider the swirling flow between two rotating, coaxial disks, located at $x = 0$ and at $x = 1$ [13]. The BVP is

$$\begin{aligned} \varepsilon f'''' + f''' + g' &= 0, \\ \varepsilon g'' + f g' - f' g &= 0, \\ f(0) = f(1) = f'(0) = f'(1) &= 0, \\ g(0) = \Omega_0, g(1) &= \Omega_1, \end{aligned}$$

   where Ω_0 and Ω_1 are the angular velocities of the infinite disks, $|\Omega_0| + |\Omega_1| \neq 0$, and ε is a velocity parameter, $0 < \varepsilon \ll 1$. For this BVP, multiple solutions are possible. Taking, e.g., $\Omega_1 = 1$, one can obtain different cases for different values of Ω_0. If $\Omega_0 < 0$ (with a special symmetry when $\Omega_0 = -1$), then the disks are counter-rotating; if $\Omega_0 = 0$ then one disk is at rest, while if $\Omega_0 > 0$ then the disks are co-rotating.

6. The mathematical model describing the motion of the sunflower is [120]

$$\varepsilon x''(t) + a x'(t) + b \sin x(t - \varepsilon) = 0, \quad \varepsilon > 0, \;\; t \in [-\varepsilon, 0],$$

   with $x'(0)$ prescribed. Here the function $x'(t)$ is the angle of the plant with the vertical, the time lag say ε is geotropic reaction, and a and b are positive parameters which can be obtained experimentally.

7. In the modelling of a semiconductor device, the model equations [101] governing the

static one-dimensional case are

$$
\begin{aligned}
\psi'' &= \frac{q}{\varepsilon}(n - p - C(z)) \;\; \text{Poisson's equation,} \\
n' &= \frac{\mu_n}{D_n} n\psi' + \frac{I}{qD_n} J_n \;\; \text{electron current relation,} \\
p' &= -\frac{\mu_p}{D_p} p\psi' - \frac{I}{qD_p} J_p \;\; \text{hole current relation,} \\
J_n' &= qR(n,p) \;\; \text{continuity equation for electron,} \\
J_p' &= -qR(n,p) \;\; \text{continuity equation for holes, for} - l \leq z \leq l
\end{aligned}
$$

subject to the boundary conditions

$$
\begin{aligned}
\psi(-l) &= U_T \ln \frac{n_i}{p(-l)} + U_A \;\; \text{(anode),} \\
\psi(l) &= U_T \ln \frac{n(l)}{n(i)} + U_C \;\; \text{(cathode),} \\
n(\pm l)p(\pm l) &= n_i^2, \\
n(\pm l) - p(\pm l) - C(\pm l) &= 0,
\end{aligned}
$$

where ψ is potential, J_n is electron current density, J_p is hole current density, n is electron density, p is hole density, q is electron charge, ε is permittivity constant, μ_n is electron mobility, μ_p is hole mobility, D_n is electron diffusion constant, D_p is hole diffusion constant, n_i is intrinsic number, $U_T \equiv D_n/\mu_n \equiv D_p/\mu_p$ is thermal voltage, $C(z) = N_D^+(z) - N_A^-(z)$ is impurity distribution, N_D^+ is the donor density, N_A^- is the accepter density and $R(n,p)$ is the recombination rate.

8. A model of an armature controlled DC-motor [79] is

$$
\begin{aligned}
\dot{x} &= az, \\
L\dot{z} &= bx - Rz + u
\end{aligned}
$$

where x, z and u are, respectively, speed, current, and voltage, R and L are armature resistance and inductance, and a and b are some motor constants. In most DC-motors L is small parameter which we consider as the singular perturbation parameter ε.

9. The point mass equations of motion for two-dimensional flight using the sum of kinetic and potential energy

$$E = h + \frac{v^2}{2g} \tag{1.2.1}$$

as a state variable, can be written as [79]

$$\begin{aligned} \dot{x} &= v\cos\gamma,\ \ v = \sqrt{(E-h)/2g}, \\ \varepsilon\dot{E} &= \frac{(T-D)v}{W}, \\ \varepsilon^2\dot{h} &= v\sin\gamma, \\ \varepsilon^3\dot{h} &= g\frac{L - W\cos\gamma}{Wv}, \end{aligned}$$

where T is thrust, D is drag, L is lift, W is weight, γ is the flight path angle, x is down range position, h is altitude, g is the gravitation constant and v is velocity, in this case not a state variable.

More models can be found in the standard texts on singular perturbation problems. We refer the readers to Kadalbajoo and Patidar ([66]) for an exhaustive list of related works on some of these models.

1.3 A brief survey of some numerical techniques for solving SPPs

The main difficulty lies in resolving the boundary and/or interior layers. The use of Standard Finite Difference like methods fail to resolve the layers when $\varepsilon \to 0$. The truncation error is reduced in refining the mesh more and more. A better level of accuracy may be achieved with a large number of mesh points and this makes the methods expensive. A very fine mesh may resolve the layers but if considered on the whole interval, then it may increase the round off errors and therefore such a solution is not really appreciable.

Asymptotic methods (Matched Asymptotic Expansion (MAE), Method of Multiple Scales (MMS), etc.) are used to analyze the qualitative behavior of solutions to singular perturbation problems. Finite Difference Methods (FDM), Finite Element Methods (FEM), Spline Approximation Methods are some of the numerical methods that can be modified in order to capture the difficulties arising in the layers. Two families of FDM are commonly used in this respect: the Fitted Mesh Finite Difference Methods (FMFDM) and the Fitted Operator Finite Difference Methods (FOFDM).

The use of FMFDM requires the knowledge of the location of the layer(s). The method aims at designing a mesh which is more refined in the layers. However, it is not always easy to detect the location of the layers, even for some simple singularly perturbed ordinary differential equations, e.g., turning point problems. In this case, FOFDM is a possible approach. In these methods, a fitting factor is sought. The fitting factor is then utilized to construct the finite difference operator for approximating the differential operator of the concerned problem.

The FMFDMs are easily extendable to higher dimensional and nonlinear problems (provided a suitable mesh selection strategy is chosen). However, they require some *a priori* knowledge of the location and the width of the layer(s). On the other hand, the FOFDMs give reliable results on a uniform mesh. The only major disadvantage of this later class of methods is that they are sometimes difficult to extend to higher dimensional problems.

Fitted (also called "layer adapted") meshes lie under two classes: graded and piecewise uniform meshes. The most successful and popular ones are those of Bakhvalov-type and Shishkin-type [72]. A Bakhvalov mesh is designed in such a way that in the layer region the mesh is fine at one end and gradually becomes coarse and outside the layer region the mesh is uniform. A Shishkin mesh is a union of two or more uniform meshes with different discretization parameters. Below we explain these two meshes briefly.

Bakhvalov-type meshes

The basic tool for the construction of a layer adapted mesh is the mesh generating function. It is a strictly monotone function $\varphi : [0,1] \to [0,1]$ that maps a uniform mesh in ξ onto a layer-adapted mesh in x by $x = \varphi(\xi)$. We now discuss how this tool is used to generate meshes of Bakhvalov-type [87].

Bakhvalov's idea is to use an equidistant ξ-grid near $x = 0$, then to map this grid back onto the x-axis by means of the (scaled) boundary layer function. That is, grid points x_i near $x = 0$ are defined by

$$q\left(1 - e^{-\frac{\beta x_i}{\sigma\varepsilon}}\right) = \xi_i = \frac{i}{N} \text{ for } i = 0, 1, \dots, \tag{1.3.2}$$

where the scaling parameters $q \in (0,1]$ and $\sigma > 0$ are user chosen: q is the ratio of mesh points used to resolve the layer, while σ determines the grading of the mesh inside the layer. Away from the layer a uniform mesh in x is used with the transition point τ such that the resulting mesh generating function is $C^1[0,1]$, i.e.,

$$\varphi(\xi) = \begin{cases} \chi(\xi) := -\frac{\sigma\varepsilon}{\beta} \ln\left(1 - \frac{\xi}{q}\right) & \text{for } \xi \in [0,\tau], \\ \pi(\xi) := \chi(\tau) + \chi'(\tau)(\xi - \tau) & \text{for } \xi \in [\tau, 1], \end{cases}$$

where the point τ satisfies

$$\chi(\tau) + \chi'(\tau)(1 - \tau) = 1. \tag{1.3.3}$$

Geometrically this means that $(\tau, \chi(\tau))$ is the contact point of the tangent π to $x = \chi(\xi)$ that passes through the point (1,1).

Equation (1.3.2) gives

$$x_i = \chi(\xi_i) = -\frac{\sigma\varepsilon}{\beta} \ln\left(1 - \frac{\xi_i}{q}\right). \tag{1.3.4}$$

The transition point τ is chosen such that

$$\chi(\tau) = \gamma\frac{\varepsilon}{\beta}|\ln\varepsilon|. \tag{1.3.5}$$

Using (1.3.5) in (1.3.3), we obtain

$$\chi'(\tau) = \frac{1 - \gamma\frac{\varepsilon}{\beta}|\ln\varepsilon|}{1-\tau}.$$

Therefore

$$x_i = \pi(\xi_i) = \gamma\frac{\varepsilon}{\beta}|\ln\varepsilon| + \left(1 - \gamma\frac{\varepsilon}{\beta}|\ln\varepsilon|\right)\frac{\xi_i - \tau}{1-\tau}. \tag{1.3.6}$$

Equations (1.3.4) and (1.3.6) serve to determine the mesh points inside and outside the layer region, respectively.

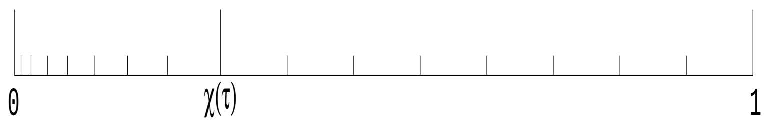

Figure 1.4: A presentation of a Bakhvalov mesh

Shishkin-type meshes

Another frequently studied mesh is the so-called Shishkin mesh. We describe this mesh for the problem

$$-\varepsilon u'' - bu' + cu = f \text{ in } (0,1), u(0) = u(1) = 0,$$

where ε is a small positive parameter, $b(x) \geq \beta > 0$ and $c(x) \geq 0$ for $x \in [0, 1]$. Let $q \in (0, 1)$ and $\sigma > 0$ be two mesh parameters.

We define a mesh transition point τ by

$$\tau = \min \left\{ q, \frac{\sigma\varepsilon}{\beta} \ln N \right\}.$$

Then the intervals $[0, \tau]$ and $[\tau, 1]$ are divided into qN and $(1-q)N$ equidistant subintervals (assuming that qN is an integer). This mesh may be regarded as generated by the mesh generating function

$$\varphi(\xi) = \begin{cases} \frac{\sigma\varepsilon}{\beta} \ln N \frac{\xi}{q} & \text{for } \xi \in [0, q], \\ 1 - \left(1 - \frac{\sigma\varepsilon}{\beta} \ln N\right) \frac{1-\xi}{1-q} & \text{for } \xi \in [q, 1], \end{cases}$$

if $q \geq \tau$. The mesh points are therefore the x_i's such that $x_i = \varphi(\xi_i)$, $\xi_i = i/N$, $i = 0, 1, \ldots$

Again the parameter q is the amount of mesh points used to resolve the layer. The mesh transition point τ has been chosen such that the layer term $e^{\beta x/\varepsilon}$ in

$$|u^{(k)}(x)| \leq C\{1 + \varepsilon^{-k} e^{-\beta x/\varepsilon}\} \text{ for } k = 0, 1, \ldots, q \text{ and } x \in [0, 1],$$

is smaller than $N^{-\sigma}$ on $[\tau, 1]$. Typically σ will be chosen equal to the formal order of the method or sufficiently large to accommodate the error analysis.

Note that unlike the Bakhvalov mesh (and Vulanović modification of it) the underlying mesh generating function is only piecewise $C^1[0, 1]$ and depends on N, the number of mesh elements. For simplicity, it is assumed that $q \geq \tau$ as otherwise N is exponentially large compared to $1/\varepsilon$ and a uniform mesh is sufficient to cope with the problem.

A Shishkin-type mesh can be constructed on $(0, 1)$ for the initial value problem of Example 1.1.1 as follows: Choose τ such that $0 < \tau \leq 1/2$ and assume $N = 2^r, r \geq 2$. The transition point τ divides $(0, 1)$ into $(0, \tau)$ and $(\tau, 1)$. Divide each of these subintervals into $N/2$ equal subintervals. The transition point is located at $\tau = \min\{1/2, \varepsilon \ln N\}$. For N sufficiently large, $\varepsilon \ln N \geq 1/2$, therefore the mesh is uniform.

A typical presentation of a Shishkin mesh is given in Figure 1.5.

Figure 1.5: A presentation of a Shishkin mesh

Using variable mesh schemes on one of these meshes, reliable results can be obtained for a class of SPPs.

Fitted operators and fitted meshes are well discussed in many research works, some example of these being [72], [87] and [102].

The use of fitted meshes is immensely documented. The work by Bakhvalov in [15] pioneered the use of an *a priori* mesh to solve a singular perturbation problem. Vulanović [142] later performed a generalization of this mesh. Numerical methods based on Bhakhvalov meshes have successfully solved a wide range of SPPs (see, e.g., [46, 95, 94, 96, 143, 144, 146]).

The idea that ε-uniform results can be obtained by using a simple piecewise equidistant mesh was put forward by Shishkin [130]. More researchers then adhered to the use of the piecewise uniform meshes (of Shishkin-type) even though they appear to be inferior to the graded ones (of Bakhvalov-type), as far as convergence and accuracy are concerned. The superiority of methods based on Bakhavalov meshes is due to the fact that these meshes are better adapted to the layer structure [148]. Comparative results to support this fact can be found in [92, 125, 148]

Research has been conducted also in the line of improving on performance of Shishkin meshes while retaining some of their simplicity. The use of a piecewise uniform mesh with several transition points is suggested in [150]. Strategies of combining ideas of Bakhvalov and Shishkin are exposed in [90, 91]. An idea of equidistribution [34] combined with

Shishkin type transition point is presented in [17].

The extendability of the methods using meshes of Shishkin type to higher dimensional problem explains why people are interested in using them. Another advantage of Shishkin meshes over Bakhvalov ones, pointed out in [150], is the convenience to handle complicated higher order methods. Since, in this thesis, we aim at constructing higher order methods, we will rather use Shishkin type meshes.

The fitted operator methods were introduced by Allen and Southwell [10] to solve the problem of viscous fluid pass a cylinder. Subsequently, Doolan et al. [33] studied one type of exponentially fitted methods considered by Liniger and Willoughby [89] which is in fact a special class of the θ-method of Lambert [83]. The discussion about the construction of a suitable fitting factor in the above methods is provided in [33].

The research is ongoing in this field and hence there is no end to the literature accountable to this topic.

1.4 Literature review on higher order numerical methods for SPPs

In this section, we survey some of the works done so far on higher order methods for singular perturbation problems in recent years, some of which are found in [66]. The works are presented in the chronological order.

Fitted methods have been shown to be superior to standard methods in solving singular perturbation problems because they attempt to capture the singular behaviour of the solution in the layers. However, higher order methods can be used to obtain an expected degree of accuracy with fewer mesh points as compared to lower order methods. Table 1.1 shows that the maximum error is reduced by a factor of 1/16 if the number of subintervals of a mesh is multiplied by 16 for a first order method. The same degree of accuracy is attained when the number of subintervals of the mesh is only doubled for a fourth order method. Another comparison can be made as follows: if one multiplies the number of subintervals of a mesh by 16, the maximum error is only divided by 16 for a first order method whereas this error is divided by 65536 for a fourth order method. This explains our interest in designing higher order methods.

Vulanović [145] solved the singularly perturbed problem

$$-\varepsilon u'' - b(x)u' + c(x)u = f(x),$$

subject to one of the following boundary conditions

$$u(0) = \gamma_0, \;\; u(1) = \gamma_1,$$

or

$$-\varepsilon u'(0) = \gamma_0, \;\; u(1) = \gamma_1.$$

The functions b, c, f are sufficiently smooth and $b(x) > \beta > 0$, $c(x) \geq 0$, while $0 < \varepsilon \ll 1$. He obtained the second-order convergence uniform in ε due to the treatment of the boundary layer function, to a special non-equidistant mesh (dense in the layer), and to the use of a combination of central and mid-point finite difference schemes.

Stynes and O'Riordan [137] examined the problem

$$\varepsilon u'' + a(x)u' - b(x)u = f(x),$$

Table 1.1: The reduction of the maximum error by higher order methods.

Order	10	20	40	80	160
1		1/2	1/4	1/8	1/16
2		1/4	1/16	1/64	1/256
3		1/8	1/64	1/512	1/4096
4		1/16	1/256	1/4096	1/65536

for $0 < x < 1$, $a(x) \geq \alpha > 0$, $b(x) \geq \beta$, $\alpha^2 + 4\alpha\beta > 0$; a, b and f in $C^2[0,1]$, ε in $(0,1]$, $u(0)$ and $u(1)$ given. Using finite elements and a discretized Green's function, they showed that the El-Mistikawy and Werle difference scheme on an equidistant mesh of width h is uniformly second order accurate for this problem. With a natural choice of trial functions, they obtained uniform first order accuracy in $L^\infty(0,1)$ norm. Choosing piecewise linear trial functions ("hat" functions) they obtained the same accuracy in the $L^1(0,1)$ norm.

O'Riordan and Stynes [115] considered the numerical solutions of the differential equation

$$\varepsilon(p(x)u')' + (q(x)u)' - r(x)u = f(x),$$

$$0 < x < 1;\ u(0) = u_0;\ u(1) = u_1,$$

where $p > 0$, $q > 0$, $r \geq 0$, $0 < \varepsilon \leq 1$, and p, q, r and $f \in C^2[0,1]$. Using finite elements with uniform mesh h, they generated a tridiagonal difference scheme which has uniform $O(h^2)$ nodal accuracy. Using piecewise linear trial functions, they obtained uniform $O(h)$ accuracy in the $L^1(0,1)$ norm. Using certain other trial functions ($\overline{L}$-splines), they obtained uniform $O(h)$ accuracy in the $L^\infty(0,1)$ norm.

Farell [38] gave some results which characterize the behavior of a linear nonselfadjoint singular perturbation problem. He also gave criteria for uniform convergence of a nonturning, simple turning point and one multiple turning point case and indicated the uniform methods for higher-order cases. Then he discussed the consequences for quasilinear problems.

Using a finite difference framework of Doedel [32] and Lynch and Rice [99], Gartland [45] constructed a family of uniformly accurate finite difference schemes for the problem

$$-\varepsilon u''(x) + a(x)u' + b(x)u = f(x),$$
$$0 < x < 1\ ;\ \ u(0) = g_0\ ,\ u(1) = g_1.$$

with the assumptions that a, b and f are bounded continuous functions and $a(x) \geq \underline{a} > 0$

on [0,1]. A scheme of order h^p (uniform in ε) is constructed to be exact on a collocation of functions of the type

$$1, x, \cdots\cdots, x^p, \exp\left(\int_x^1 a\right), x\ \exp\left(\int_x^1 a\right), \cdots\cdots,$$
$$x^{p-1}\ \exp\left(\int_x^1 a\right).$$

The high order is achieved through extra evaluations of f. He also presented some numerical experiments which exhibit uniform orders h^p, $p = 1, 2, 3$ and 4.

Sklyar [134] constructed a conservative difference scheme for singularly perturbed differential problems. In the construction a suitable decomposition of a symmetric bilinear form is applied. The method is presented for the model problem

$$\varepsilon u'' + au' = f, \;\; x \in (0, 1); \;\; u(0) = \alpha_0, \;\; u(1) = \alpha_1.$$

The coefficients of the scheme are obtained by recursion; the number of iterations depends on ε. The order of convergence is proved to be $O(h^2)$ and is independent of ε.

Herceg et al. [56] considered singularly perturbed semilinear selfadjoint two-point boundary value problems, with Dirichlet boundary conditions. Using a Bakhvalov-type mesh, they gave a difference scheme for numerically solving such problems. It is shown that the solution of this difference scheme is amenable to Richardson extrapolation, and that one can thereby obtain sixth-order convergence at each node, uniformly in the singular perturbation parameter.

Herceg [57] used the Hermitian approximation of the second order derivative for a linear singularly perturbed nonlocal problem

$$\varepsilon^2 u'' + b(x)u = f(x)\ , \qquad 0 \le x \le 1,$$

$$u(0) = 0, \;\; u(1) = \sum_{i=1}^{m} c_i u(s_i) + d,$$

$$d, c_i \in R,$$

$$s_i \in (0,1), \; i = 1, 2, \cdots, m,$$

$$0 < \varepsilon << 1, \; b \in C^k[0,1],$$

$$k \in N, \; b(x) \geq \beta^2 > 0,$$

for some positive constant β. He proved that the technique is fourth order uniformly convergent.

Stojanovic [136] considered a linear, self-adjoint, singularly perturbed, two-point boundary value problem. She generated a difference scheme for this problem by approximating the forcing term with a piecewise cubic polynomial, and approximating the coefficient of the zero-order term with a piecewise constant function. This scheme is shown to be second-order accurate, uniformly in the singular perturbation parameter.

Schmitt [126] constructed a symmetric difference scheme for linear, stiff, or singularly perturbed boundary value problems of first-order with constant coefficients. His scheme is based on a stability function containing a matrix square root. Its essential feature is the unconditional stability function in the absence of purely imaginary eigenvalues of the coefficient matrix. He proved local damping of errors, uniform stability, and uniform second-order convergence. He also discussed the computation of the specific matrix square root by a well-known stable variant of Newton's method.

Based on the coupling of the central difference scheme with the Abrahamsson-Keller-Kreiss box scheme on a special nonuniform mesh, Sun and Wu [138] proposed a scheme for the numerical solution of the singular boundary value problem

$$\varepsilon u'' + b(x)u' - c(x)u = f(x), \;\; u(0) = \alpha, \;\; u(1) = \beta.$$

They proved that this scheme is uniformly second-order convergent.

Kadalbajoo and Bawa [65] presented a variable-mesh method based on cubic spline approximation for nonlinear singularly perturbed boundary-value problems of the form

$$\varepsilon y'' = f(x,y) \ , \ \ y(a) = \alpha \ , \ \ y(b) = \beta.$$

They gave convergence analysis and the method is shown to have third-order convergence.

In [151], Wang solved a nonlinear singular perturbation problem numerically on non-equidistant meshes which are dense in the boundary layers. The method is based on the numerical solution of integral equations. He proved the fourth-order uniform accuracy of the scheme.

Grekov and Krasnikov [51] examined a linear singularly perturbed reaction-diffusion problem in one dimension. Assuming that its coefficients are piecewise smooth, they considered any mesh whose nodes include the points of discontinuity of these coefficients. The solution u is expressed as a series, each term of which can be computed by numerically solving a singularly perturbed reaction-diffusion problem with piecewise constant coefficients. They proved that by truncating this series, u can be approximated in the L^∞-norm, uniformly in the singular perturbation parameter, up to $O(h^m)$, where h is the mesh diameter and m is an arbitrary positive integer.

Hu et al. [62] developed a discretization method for one-dimensional singular perturbation problems based on Petrov-Galerkin finite element, or an equivalent finite volume, scheme. The model one-dimensional problem which they considered was

$$-\varepsilon u'' + \beta u' + \sigma u = f \ \text{ in } \ (a,b),$$

$$u(a) = u_a, \ \ u(b) = u_b.$$

This problem has its origin in the physical conservation law

$$q' + (\sigma - \beta')u = f,$$

and Fick's diffusion law

$$q = -\varepsilon u' + \beta u,$$

where q is the flux, ε the diffusivity, β the velocity, and σ the absorbing coefficient (or reactivity). The scheme that they developed is not only $O(h^2)$ accurate uniformly in ε, but also satisfies certain discrete versions of both the conservative law and maximum principle.

Beckett and Mackenzie [18] studied the numerical approximation of a singularly perturbed reaction-diffusion equation using a p-th order Galerkin finite element method on a non-uniform grid. The grid was constructed by equidistributing a strictly positive monitor function which is a linear combination of a constant floor and a power of the second derivative of a representation of the boundary layers-obtained using a suitable decomposition of the analytical solution. By the appropriate selection of the monitor function parameters they proved that the numerical solution is insensitive to the size of the singular perturbation parameter and achieves the optimal rate of convergence with respect to the mesh density.

In [43], a defect correction method based on finite difference schemes is considered for a singularly perturbed boundary value problem on a Shishkin mesh. The method combines the stability of the upwind difference scheme and the higher-order convergence of the central difference scheme. The almost second-order convergence of the scheme with respect to the discrete maximum norm, uniformly in the perturbation parameter, is proved.

A boundary value problem for a singularly perturbed parabolic equation of convection diffusion type on an interval is studied. For the approximation of the boundary value problem, Hemker et al. [53] use earlier developed finite difference schemes, epsilon-uniformly of a high order of accuracy with respect to time, based on defect correction.

Vulanović [150] solved numerically a class of singularly perturbed quasilinear boundary value problems with two small parameters by finite differences on a Shishkin-type mesh. The discretization combined a four-point third-order scheme inside the boundary layers with the standard central scheme outside the layers. This results in an almost third-order accuracy which is uniform with respect to the perturbation parameters. The paper also showed that the Shishkin meshes are more suitable for higher-order schemes than the Bakhvalov meshes, since complicated non-equidistant schemes can be avoided.

Hemker et al. [54] used a defect correction technique to construct ε-uniformly convergent schemes of high-order time-accuracy. The efficiency of the new defect-correction schemes is confirmed with numerical experiments. An original technique for an experimental study of convergence orders is developed for cases when the orders of convergence in the x-direction and in the t-direction can be essentially different.

Until an approach by Roos (in one of his technical reports in 2005: complete citation details are not available), the best way to construct high order uniformly convergent schemes for singular perturbation problems was to apply exponentially fitted compact difference schemes. His approach consists of the following steps: firstly solve an auxiliary problem with piecewise or nearly piecewise constant coefficients, secondly improve the approximation iteratively using the defect correction idea and piecewise polynomial approximations of higher order. An important advantage of this approach lies in the fact that it is possible to start from a classical or weak formulation of a boundary value problem. Therefore, the approach is useful for singular perturbations related to ordinary as well as partial differential equations.

Patidar [118] considered the self-adjoint singularly perturbed two-point boundary value problems

$$-\varepsilon(a(x)y')' + b(x)y = f(x),\ x \in [0,1],$$

$$y(0) = \eta_0, \; y(1) = \eta_1.$$

Highest possible order of uniform convergence for such problems achieved so far via fitted operator methods, was one. Reducing the original problem into the normal form and then using the theory of inverse monotone matrices, he derived a FOFDM via the standard Numerov's method. His scheme is fourth order accurate for moderate values of ε and ε-uniformly convergent with order two for very small values of ε.

A one-dimensional singularly perturbed problem of mixed type is considered by Brayanov [24]. The domain under consideration is partitioned into two subdomains. In the first subdomain a parabolic reaction-diffusion problem is given and in the second one an elliptic convection-diffusion-reaction problem. The solution is decomposed into regular and singular components. The problem is discretized using an inverse-monotone finite volume method on condensed Shishkin meshes. He establishes an almost second-order global pointwise convergence in the space variable.

Gracia et al. [49] constructed a second order monotone numerical method for a singularly perturbed ordinary differential equation with two small parameters affecting the convection and diffusion terms. The monotone operator is combined with a piecewise-uniform Shishkin mesh. An asymptotic error bound in the maximum norm is established theoretically whose error constants are shown to be independent of both singular perturbation parameters.

A numerical study is made in [71] to examine a singularly perturbed parabolic initial-boundary value problem in one space dimension on a rectangular domain. The solution of this problem exhibits the boundary layer on the right side of the domain. They constructed a Crank-Nicolson finite difference method consisting of an upwind finite difference operator on a fitted piecewise uniform mesh. The resulting method has been shown to be almost first order accurate in space and second order in time. Numerical experiments have been carried out, which validate the theoretical results. It is also shown that a numerical method consisting of same finite difference operator on uniform mesh does not converge uniformly with respect to the singular perturbation parameter.

Mohanthy and Singh [108] derived a difference method of $O(h^4)$, so called, arithmetic average discretization for the solution of two dimensional non-linear singularly perturbed elliptic partial differential equation of the form

$$\varepsilon(u_{xx} + u_{yy}) = f(x, y, u, u_x, u_y),$$

$$0 < x,\ y < 1,$$

subject to appropriate Dirichlet boundary conditions where $\varepsilon > 0$ is a small parameter. They also derived new methods of higher order for the estimates of $\partial u/\partial n$, which are quite often of interest in many physical problems. In all cases, only 9-grid points and a single computational cell were required. The main advantage of the proposed methods is that the methods are directly applicable to singular problems.

In [12], a high-order (second and fourth of convergence, but with first and third-order local truncation error, respectively) compact finite difference schemes for elliptic equations with intersecting interfaces is derived. The approach uses the differential equation and the jump (interface) relations as additional identities which can be differentiated to eliminate higher order local truncation errors. Numerical experiments are carried out to demonstrate the high-order accuracy and to show that our method is effective to sharp contrast in the diffusion coefficients of the problems.

Rao and Kumar [122] present a B-spline collocation method of higher order for a class of self-adjoint singularly perturbed boundary value problems. The essential idea in this method is to divide the domain of the differential equation into three non-overlapping subdomains and solve the regular problems obtained by transforming the differential equation with respective boundary conditions on these subdomains using the present higher order B-spline collocation method. The boundary conditions at the transition points are obtained by the asymptotic approximation of order zero to the solution of the problem. The convergence analysis is given and the method is shown to have optimal order convergence; by collocating the perturbed differential equation, which is satisfied by a special cubic spline interpolate of the true solution.

Franz [42] analyzed a continuous interior penalty (CIP) method for elliptic convection-diffusion problems with characteristic layers on a Shishkin mesh. The method penalizes jumps of the normal derivative across interior edges. He shows that it is of the same order of convergence as the streamline diffusion finite-element method and is superclose in the CIP norm induced by its bilinear form for the difference between the FEM solution and the bilinear nodal interpolant of the exact solution. Furthermore, he studies numerically the behaviour of the method for different choices of the stabilization parameter.

A fourth-order finite-difference method for singularly perturbed one-dimensional reaction-diffusion problem is presented by Herceg and Herceg in [58]. The problem is discretized using a Bakhvalov-type mesh. They gave a uniform convergence with respect to the perturbation parameter.

Kadalbajoo and Kumar [73] develop a method which deals with the singularly perturbed boundary value problem for a linear second order differential-difference equation of the convection-diffusion type with small delay parameter τ of $O(\varepsilon)$ whose solution has a boundary layer. The fitted mesh technique is employed to generate a piecewise-uniform mesh condensed in the neighborhood of the boundary layers. B-spline collocation method is used with fitted mesh. Parameter-uniform convergence analysis of the method is discussed. The method is shown to have almost second order parameter-uniform convergence. The effect of small delay τ on boundary layer has also been discussed.

Kadalbajoo and Yadaw [74] presented a B-spline collocation method for solving a class of two-parameter singularly perturbed boundary value problems. They used B-spline collocation method on piecewise-uniform Shishkin mesh, which leads to a tridiagonal linear system. They analyzed the method for convergence and showed that it is uniformly convergent of second order.

Lin et al. [88] developed a new method by detecting the boundary layer of the solution of a singular perturbation problem. On the non-boundary layer domain, the singular perturbation problem is dominated by the reduced equation which is solved with standard techniques for initial value problems. While on the boundary layer domain, it is controlled by the singular perturbation. Its numerical solution is obtained using finite difference methods. The numerical error is maintained at the same level with a constant number of mesh points for a family of singular perturbation problems.

Shahraki and Hosseini [128] presented a new scheme for discretization of singularly perturbed boundary value problems based on finite difference methods. This method is a combination of simple upwind scheme and central difference method on a special non-uniform mesh (Shishkin mesh) for the space discretization. Numerical results show that the convergence of method is uniform with respect to singular perturbation parameter and has a higher order of convergence.

In the paper, Solin and Avila [135] present a new piecewise-linear finite element mesh suitable for the discretization of the one-dimensional convection-diffusion equation

$$-\varepsilon u'' - bu' = 0, \quad u(0) = 0, \quad u(1) = 1.$$

The solution to this equation exhibits an exponential boundary layer which occurs also in more complicated convection-diffusion problems of the form

$$-\varepsilon \Delta u - b\frac{\partial u}{\partial x} + cu = f.$$

Their new mesh is based on the equidistribution of the interpolation error and it takes into account finite computer arithmetic. It is demonstrated numerically that for the above problem, the new previous mesh has remarkably better convergence properties than the well-known previous shishkin and Bakhvalov meshes.

Xie et al. [154] presented a novel approach for solving parameterized singularly perturbed two-point boundary value problems with a boundary layer. By the boundary layer correction technique, the original problem is converted into two non-singularly perturbed problems which can be solved using traditional numerical methods, such as Runge-Kutta methods. Several non-linear problems are solved to demonstrate the applicability of the method.

The bilinear finite element methods on appropriately graded meshes are considered in Zhu and Chen [155] both for solving singular and semisingular perturbation problems. In each case, the quasi-optimal order error estimates are proved in the ε-weighted H1-norm uniformly in singular perturbation parameter ε, up to a logarithmic factor. By using the interpolation postprocessing technique, the global superconvergent error estimates in ε-weighted H1-norm are obtained.

Kadalbajoo and Gupta [75] designed a numerical scheme to solve a singularly perturbed convection-diffusion problem. The scheme involves B-spline collocation method and appropriate piecewise-uniform Shishkin mesh. Bounds were established for the derivative of the analytical solution. Moreover, the method is boundary layer resolving as well as second-order uniformly convergent in the maximum norm. They give a comprehensive analysis to prove the uniform convergence with respect to singular perturbation parameter.

Surla et al. [139] considered finite difference approximation of a singularly perturbed one-dimensional convection-diffusion two-point boundary value problem. The problem is numerically treated by a quadratic spline collocation method on a piecewise uniform slightly modified Shishkin mesh. The position of collocation points is chosen so that the obtained scheme satisfies the discrete minimum principle. They prove pointwise convergence of order $O(N^{-2}\ln^2 N)$ inside the boundary layer and second order convergence elsewhere. Further, they approximate normalized flux and give estimates of the error at the mesh points and between them.

They determine the conditions under which the difference schemes, applied independently on subdomains may accelerate (epsilon-uniformly) the solution of the boundary value problem without losing the accuracy of the original schemes. Hence, the simultaneous solution on subdomains can in principle be used for parallelization of the computational method.

Ilicasu and Schultz [63] developed a high-order finite-difference technique for the second-order, singularly perturbed linear BVP in one dimension. Taylor series expansions and error conversions are used for the development of the techniques. Convergence and stability conditions of these techniques are proved.

Liu and Shen [97] proposed a new spectral Galerkin method for the convection-dominated convection-diffusion equation. This method employs a new class of trial function spaces. The available error bounds provide a clear theoretical interpretation for the higher accuracy of the new method compared to the conventional spectral methods when applied to problems with this boundary layers.

Some of the works that are more specific for the problems considered in the individual chapters are described further in the introduction sections of those chapters. There might be little repetitions but we do so in order for the chapters to be self contained.

1.5 Summary of the thesis

The order of the various numerical methods that we mentioned in previous section vary from less that one to three or four. Quite often a numerical analyst prefers a higher order method due to the fact that it offers the opportunity to attain a better degree of accuracy with fewer mesh points as compared to lower order methods. Since in most cases, techniques of constructing directly higher order methods are tedious, we will rather focus on Richardson's extrapolation which is one of the convergence acceleration techniques. It consists of taking a linear combination of k solutions ($k \geq 2$) corresponding to different but nested meshes on the intersection of these meshes which is in fact the coarsest mesh [41]. Due to time limitation the other convergence technique, the defect correction is not considered in this thesis. While we will investigate the effect of Richardson extrapolation on some existing fitted methods in some instances, we will use this technique on some novel fitted methods in other instances.

In Chapter 2, we investigate the effect of Richardson extrapolation on the fitted mesh finite difference method (FMFDM) of [119] for a self-adjoint problem. We note that even though the accuracy is improved, the order of convergence remains unchanged. This unexpected fact contradicts the assertion met in the literature about Richardson extapolation that "A numerical solution of required accuracy can be obtained by using Richardson extrapolation method [11, 133] and it can be used to improve the ε-uniform rates of convergence of computed solutions [133].

We go on investigating what impact the extrapolation technique will have on other methods to solve the above mentioned self-adjoint problem in Chapter 3. We consider two fitted operator finite difference methods (FOFDMs) which we denote by FOFDM-I and FOFDM-II, presented in [118] and [98], respectively. In the first case, Richardon extrapolation does not improve the convergence which is of order four and two for some moderate and smaller values of ε. In the latter case, the second order accuracy is improved up to four, irrespective of the value of ε.

Chapter 4 deals with construction and analysis of a FMFDM and a FOFDM to solve a singularly perturbed turning point problem whose solution has boundary layers. We study the performance of Richardson extrapolation on these methods.

In Chapter 5, we consider a singularly perturbed nonlinear two-point boundary value problem. We first apply the quasilinearization process [19] to linearize the problem. Then the resulting sequence of linear problems is solved by a FOFDM.

A time-dependent nonlinear Burgers' equation is considered in Chap 6. We again linearize the problem using the quasilinearization process. The process results in a sequence of linear problems at each time level which we solve using a FOFDM.

The FOFDM-II of Chapter 3 is extended to singularly perturbed elliptic problems in

2-dimensions in Chapter 7. This method is of order 2 in both x- and y-direction. The fourth order convergence is achieved after applying Richardson extrapolation.

Due to the space limitations, we give only necessary details in the latter chapters.

Finally, some concluding remarks and directions for further research are provided in Chapter 8.

Chapter 2

Higher Order Fitted Mesh Finite Difference Scheme for a Singularly Perturbed Self-adjoint Problem

Numerous methods have been developed for singularly perturbed self-adjoint boundary value problems in past three decades. The order of these methods vary from less than one to three or four. Quite often a numerical analyst prefers a higher order method due to the fact that it offers the opportunity to attain a better degree of accuracy with fewer mesh points as compared to lower order methods. Motivated by this fact, we would like to investigate in this chapter whether we can accelerate the order of convergence of existing high order methods.

We consider the fitted mesh finite difference method of Patidar [119] applied on a Shishkin-type mesh for the solution of self-adjoint problem which is ε-uniform convergent of order four. We attempt to increase the order of convergence by Richardson's extrapolation and notice that this well-known convergence acceleration technique has some limitations. We observe that even though Richardson extrapolation improves the accuracy slightly, this technique does not increase the rate of convergence which is originally four for the underlying method for the problem above. This fact was unexpected and contradicts the assertion met in the literature so far about Richardson extrapolation.

2.1 Introduction

Solutions of singular perturbation problems (SPPs) present large gradients when the perturbation parameter approaches zero. The solution of these SPPs typically contains layers. This behavior lowers the order of convergence of the underlying numerical method and results in low accuracy. Standard methods have failed to resolve the layer(s) for all values of ε (the singular perturbation parameter), unless a very fine mesh is considered, which unfortunately increases the computational complexities. To overcome this difficulty, fitted methods have been considered by various authors since they provide reliable numerical results on a mesh with a reasonable number of grid points and hence make the method practically applicable. However, some fitted methods perform better than others.

Various numerical methods have been developed so far to solve such problems, some of which we will mention below. The order of these methods vary from less than one to three or four. Since quite often a numerical analyst prefers a higher order method due to the fact that it offers the opportunity to attain a better degree of accuracy with fewer mesh points as compared to lower order methods, we would like to investigate in this paper whether we can accelerate the order of convergence of existing high order methods.

Direct techniques to obtain high order methods for singularly perturbed problems are well documented. We provide few examples. For a one-dimensional convection-diffusion problem, a second order ε-uniformly convergent method was designed in [145]. In [137], the same order of convergence was obtained using finite elements and discretized Green's functions and applying the El-Mistikawy and Werle difference scheme on an equidistant mesh. This problem was also examined in [45] where a scheme of order p $(p = 1, 2, 3, 4)$ was constructed using collocation approach.

By using Hermitian approximation of the second order derivative, a fourth order uniformly convergent scheme for a reaction-diffusion problem was presented in [57].

For a self-adjoint problem, a second-order nodal accuracy using finite elements with uniform mesh is obtained in [115], whereas in [118] a fitted operator finite difference method (FOFDM) was derived via Numerov's method that showed to be fourth order accurate for moderate value of ε and second order uniformly convergent for small values of ε. On the other hand in [119] a fitted mesh finite difference method (FMFDM) was shown to be fourth order ε-uniformly convergent. The third order of convergence was found for quasilinear problems in [149] and [150]. The third and fourth order of convergence was obtained for a nonlinear problem [151] by using numerical solution of integral equations. While none of these methods is of order higher than four, there exist methods of arbitrary order (see, e.g., [51]). However, because designing and implementing such methods appears not to be an easy task, no numerical experiment has supported this assertion.

Beside the various techniques of constructing directly higher order methods (which are tedious in most cases), one would rather use a convergence acceleration strategy. Several methods for improvement of solutions have been designed (see, e.g., [11, 47, 64, 123, 132, 133] and the references therein). One of these methods, which was subsequently termed as the Richardson extrapolation, is a post-processing procedure where a linear combination of two computed solutions approximating a particular quantity gives a third and better approximation. It was implemented in [77] for a system of first order linear ordinary differential equation, in [93] and [111] for a one-dimensional linear convection-diffusion problem and in [133] for a quasilinear parabolic singularly perturbed convection-diffusion equations.

Our aim in this Chapter is to investigate the limitations of the Richardson's extrapolation when it is applied for a method which is already of high order. To this end, we consider the following problem (2.1.1) for which Patidar [119] constructed a fourth order

ε-uniformly convergent FMFDM (on a mesh of Shishkin-type):

$$Ly \equiv -\varepsilon(a(x)y')' + b(x)y = f(x),\ x \in (0,1),\ y(0) = \eta_0,\ y(1) = \eta_1, \qquad (2.1.1)$$

where η_0 and η_1 are given constants and $\varepsilon \in (0,1]$. The functions $f(x), a(x)$ and $b(x)$ are assumed to be sufficiently smooth and to satisfy the conditions

$$a(x) \geq a > 0,\ b(x) \geq b > 0.$$

The rest of this chapter is organized as follows. Some theoretical estimates are provided in Section 2.2. For the sake of completeness, we present the FMFDM of [119] in Section 2.3. We establish the extrapolation formula in Section 2.5. This section deals also with the error analysis of the FMFDM after extrapolation where analysis before extrapolation is reviewed. Two numerical examples are considered in section 2.5 to confirm our theoretical results. Section 2.7 is devoted to the conclusions and further research plans.

2.2 Reduction to normal form and some theoretical estimates

The following lemmas [119] are necessary for the existence and uniqueness of the solution and for the problem to be well-posed.

Lemma 2.2.1. (Maximum Principle) *Let L be the operator as in (2.1.1) such that $B_0y(0) \equiv y(0) = \eta_0$, $B_1y(1) \equiv y(1) = \eta_1$. Suppose $\phi(x)$ is any smooth function satisfying $B_0y(0) \geq 0, B_1y(1) \geq 0$ and let $L\phi(x) \geq 0, \forall\ 0 < x < 1$ then $\phi(x) \geq 0, \forall\ 0 \leq x \leq 1$.*

Proof. The proof is by contradiction. Let x^* be such that $\phi(x^*) = \min_{x\in[0,1]} \phi(x)$ and assume that $\phi(x^*) < 0$. Clearly, $x^* \notin \{0,1\}$ and therefore $\phi'(x^*) = 0$ and $\phi''(x^*) \geq 0$.

Further,

$$L\phi(x^*) = -\varepsilon(a(x^*)\phi'(x^*))' + b(x^*)\phi(x^*) < 0,$$

which is a contradiction. It follows that $\phi(x^*) \geq 0$ and thus $\phi(x) \geq 0 \forall\ x \in [0,1]$.

The uniqueness of the solution is implied by this maximum principle. Its existence follows trivially (as for linear problems, the uniqueness of the solution implies its existence). This principle is now applied to prove that the solution of (2.1.1) is bounded.

Lemma 2.2.2. *Let $y(x)$ be the solution of the problem (2.1.1), then we have*

$$||y|| \leq b^{-1}||f|| + \max(\eta_0, \eta_1).$$

Proof. We construct two barrier functions $\Pi^{\pm}$ defined by

$$\Pi^{\pm}(x) = b^{-1}||f|| + \max(\eta_0, \eta_1) \pm y(x).$$

Then we have

$$\begin{aligned}
\Pi^{\pm}(0) &= b^{-1}||f|| + \max(\eta_0, \eta_1) \pm y(0) \\
&= b^{-1}||f|| + \max(\eta_0, \eta_1) \pm \eta_0 \\
&\geq 0, \\
\Pi^{\pm}(1) &= b^{-1}||f|| + \max(\eta_0, \eta_1) \pm y(1) \\
&= b^{-1}||f|| + \max(\eta_0, \eta_1) \pm \eta_1 \\
&\geq 0,
\end{aligned}$$

and we have

$$\begin{aligned}
L\Pi^{\pm}(x) &= -\varepsilon(a(x)(\Pi^{\pm}(x))')' + b(x)\Pi^{\pm}(x) \\
&= b(x)(b^{-1}||f|| + \max(\eta_0, \eta_1)) \pm Ly(x)
\end{aligned}$$

$$
\begin{aligned}
&= b(x)[b^{-1}||f|| + \max(\eta_0, \eta_1)] \pm f(x) \\
&\geq 0, \quad \text{since} \quad ||f|| \geq f(x).
\end{aligned}
$$

Therefore, by the maximum principle (Lemma 2.2.1), we obtain $\Pi^{\pm}(x) \geq 0$, for all $x \in [0, 1]$, which gives the required estimate.

Now, let

$$
P(x) = \frac{a'(x)}{a(x)}, \; Q(x) = -\frac{b(x)}{\varepsilon a(x)} \text{ and } R(x) = -\frac{f(x)}{\varepsilon a(x)}.
$$

Equation (2.1.1) therefore becomes

$$
y'' + P(x)y' + Q(x)y = R(x). \tag{2.2.2}
$$

Via the substitutions

$$
U(x) = \exp\Big(-\frac{1}{2}\int_0^x P(\zeta)d\zeta\Big)
$$

and

$$
y(x) = U(x)V(x), \tag{2.2.3}
$$

into equation (2.2.2), the problem (2.1.1) is transformed into the normal form

$$
\widetilde{L}V \equiv -\varepsilon V'' + W(x)V = Z(x), \tag{2.2.4}
$$

$$
V(0) = \alpha_0 \left(\equiv \frac{y(0)}{U(0)}\right), \; V(1) = \alpha_1 \left(\equiv \frac{y(1)}{U(1)}\right), \;\; \alpha_0, \; \alpha_1 \in \mathbb{R}
$$

where

$$
W(x) = -\varepsilon\left(Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}(P(x))^2\right),
$$

$$
Z(x) = -\varepsilon\left(R(x)\exp\Big(\frac{1}{2}\int_0^x P(\zeta)d\zeta\Big)\right).
$$

It is worthwhile noting that the operator $\widetilde{L}$ also satisfies the maximum principle:

Lemma 2.2.3. *Let $\psi(x)$ be any sufficiently smooth function such that $\psi(0) \geq 0$ and $\psi(1) \geq 0$. Then $\widetilde{L}\psi(x) \geq 0, \forall\, x \in (0,1)$ implies that $\psi(x) \geq 0, \forall x \in [0,1]$.*

In the error analysis of problem (2.2.4), it is convenient to decompose the solution V_ε into a smooth (regular) component $V_{\varepsilon,r}$ and a singular component $V_{\varepsilon,s}$. Bounds on these components and on their derivatives are provided in the following Lemma [105]:

Lemma 2.2.4. *The solution V_ε of the problem (2.2.4) can be decomposed into the form*

$$V_\varepsilon := V_{\varepsilon,r} + V_{\varepsilon,s}$$

where for all $k \in 0, 1, 2, \cdots, 6$ and $x \in [0,1]$, the regular component $V_{\varepsilon,r}$ satisfies

$$|V_{\varepsilon,r}^{(k)}| \leq M[1 + \varepsilon^{-(k-2)/2} E(x,\beta)],$$

and the singular component $V_{\varepsilon,s}$ satisfies

$$|V_{\varepsilon,s}^{(k)}| \leq M\varepsilon^{-k/2} E(x,\beta),$$

where

$$0 < \beta \leq W(x)$$

and

$$E(x,\beta) = \Big\{ \exp\Big(-x/\sqrt{\beta/\varepsilon}\Big) + \exp\Big(-(1-x)\sqrt{\beta/\varepsilon}\Big)\Big\}$$

.

Lemma 2.2.5. *For a fixed mesh and for all integers k, we have*

$$\lim_{\varepsilon \to 0} \max_{1 \leq j \leq n-1} \frac{\exp(-Mx_j/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0,$$

and

$$\lim_{\varepsilon\to 0}\max_{1\le j\le n-1}\frac{\exp(-M(1-x_j)/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0,$$

where $x_j = jh, h = 1/n, \forall j = 1(1)n-1$.

Proof. Consider the partition

$$[0,1] := \{0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1\}.$$

It is clear that, for the interior grid points, we have

$$\max_{1\le j\le n-1}\frac{\exp(-Mx_j/\sqrt{\varepsilon})}{\varepsilon^{k/2}} \le \frac{\exp(-Mx_1/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = \frac{\exp(-Mh/\sqrt{\varepsilon})}{\varepsilon^{k/2}}$$

and

$$\max_{1\le j\le n-1}\frac{\exp(-M(1-x_j)/\sqrt{\varepsilon})}{\varepsilon^{k/2}} \le \frac{\exp(-M(1-x_n)/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = M\frac{\exp(-Mh/\sqrt{\varepsilon})}{\varepsilon^{k/2}}$$

(as $x_1 = h, 1 - x_{n-1} = 1-(n-1)h = h$). An application of L'Hospital's rule the gives

$$\lim_{\varepsilon\to 0}\frac{\exp(-Mh/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = \lim_{p(=1/\sqrt{\varepsilon})\to\infty}\frac{p^k}{\exp(Mhp)} \equiv \lim_{p\to\infty}\frac{k!}{(Mh)^k\exp(Mhp)} = 0,$$

which completes the proof.

2.3 The numerical method

The FMFDM that we use in Chapter has been derived in Patidar [119] on the Shishkin mesh described below.

The interval $[0,1]$ is divided into three sub-intervals:

$$[0,1] := [0,\delta]\cup[\delta,1-\delta]\cup[1-\delta,1],$$

where δ is the width of the boundary layer. Let n be a positive integer such that $n = 2^m$ with $m \geq 5$. The intervals $(0, \delta)$ and $(1-\delta, 1)$ are each divided into $n/4$ equal mesh elements, while the interval $(\delta, 1-\delta)$ is divided into $n/2$ equal mesh elements. Therefore, we have $n/4+1$ equidistant grid points in the intervals $[0, \delta]$ and $[1-\delta, 1]$ and $n/2-1$ equidistant grid points in $(\delta, 1-\delta)$. The parameter δ is defined by

$$\delta = \min\left\{1/4, 4\left(\sqrt{\varepsilon/\beta}\right)\ln(n/16)\right\} \tag{2.3.5}$$

where $0 < \beta \leq W(x),\ \forall x \in [0,1]$. Assuming that $j_0 = n/4,\ \ x_{j_0} = \delta,\ \ x_{n-j_0} = 1-\delta$ and

$$[0,1] := 0 = x_0 < x_1 < \cdots < x_{j_0} < \cdots < x_{n-j_0} < \cdots < x_n = 1,$$

we have $h_j = x_j - x_{j-1}$ where the mesh spacing is given by

$$h_j = \begin{cases} 4\delta n^{-1}, & j = 1, \ldots, j_0, n-j_0+1, \ldots, n, \\ \\ 2(1-2\delta)n^{-1}, & j = j_0+1, \ldots, n-j_0. \end{cases} \tag{2.3.6}$$

We denote this mesh by $\mu_{n,\delta}$ and assume that

$$\delta = 4\left(\sqrt{\varepsilon/\beta}\right)\ln(n/16), \tag{2.3.7}$$

since if $\delta = 1/4$, i.e, $1/4 < 4\left(\sqrt{\varepsilon/\beta}\right)\ln(n/16)$, then n^{-1} is very small relative to ε. (This is very unlikely in practice and in such a case the method can be analyzed using the standard techniques).

We use the notation $V_j = V(x_j), W_j = W(x_j),$ and $Z_j = Z(x_j)$, the approximations of V_j at the grid points are denoted by the unknowns ν_j. The scheme of [119] is given by the tridiagonal system

$$A\nu = F, \tag{2.3.8}$$

where A is the matrix of the system and ν and F are corresponding vectors. The various entries of this matrix and the components of the right-hand-side vector are given by

$$\left.\begin{aligned}
&(\mathrm{diag}(A))_j = r_j^c, && j = 1, 2, \ldots, n-1,\\
&(\mathrm{subdiag}(A))_j = r_j^-, && j = 2, 3, \ldots, n-1,\\
&(\mathrm{supdiag}(A))_j = r_j^+, && j = 1, 2, \ldots, n-2,\\
&F_1 = q_1^- Z_0 + q_1^c Z_1 + q_1^+ Z_2 - r_1^- \nu_0, &&\\
&F_j = q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ Z_{j+1}, && j = 2,\ 3, \ldots,\ n-2,\\
&F_{n-1} = q_{n-1}^- Z_{n-2} + q_{n-1}^c Z_{n-1} + q_{n-1}^+ Z_n - r_{n-1}^+ \nu_n, &&\\
&\nu_0 = \alpha_0,\ \ \nu_n = \alpha_1, &&
\end{aligned}\right\} \tag{2.3.9}$$

where

$$\left.\begin{aligned}
q_j^- &= \frac{h_{j+1}}{h_j(h_j+h_{j+1})}\Big(\frac{h_j^3-h_{j+1}^3}{6}+\frac{h_j^4+h_{j+1}^4}{12h_{j+1}}\Big),\\
q_j^+ &= -\frac{h_j}{h_{j+1}(h_j+h_{j+1})}\Big(\frac{h_j^3-h_{j+1}^3}{6}-\frac{h_j^4+h_{j+1}^4}{12h_j}\Big),\\
q_j^c &= \frac{h_j^2+h_{j+1}^2}{2}+\frac{2(h_j-h_{j+1})(h_j^3-h_{j+1}^3)-(h_j^4+h_{j+1}^4)}{12h_jh_{j+1}},\\
r_j^- &= -\varepsilon\Big\{1-\frac{h_{j+1}(h_j-h_{j+1})}{h_j(h_j+h_{j+1})}\Big\}+q_j^-W_{j-1},\\
r_j^+ &= -\varepsilon\Big\{1+\frac{h_j(h_j-h_{j+1})}{h_{j+1}(h_j+h_{j+1})}\Big\}+q_j^+W_{j+1},\\
r_j^c &= \varepsilon\Big\{2+\frac{(h_j-h_{j+1})^2}{h_jh_{j+1}}\Big\}+q_j^cW_j.
\end{aligned}\right\}\tag{2.3.10}$$

If $h_j = h_{j+1} = h$ (i.e, uniform mesh throughout the region), then (2.3.10) reduces to

$$\left.\begin{aligned}
r_j^- = -\varepsilon+h^2W_{j-1}/12,\;\; r_j^+ = -\varepsilon+h^2W_{j+1}/12,\;\; r_j^c = 2\varepsilon+5h^2W_j/6,\\
q_j^\pm = \frac{h^2}{12},\;\; q_j^c = \frac{5h^2}{6}.
\end{aligned}\right\}\tag{2.3.11}$$

Using equations (2.3.8)-(2.3.10) or (2.3.11), we get the approximate solution of $V(x)$ at the grid points x_j. The solution of the original problem (2.1.1) at these grid points is obtained using (2.2.3) since $U(x)$ is known.

The method consisting of (2.3.8)-(2.3.10) is referred to as the Fitted Mesh Finite Difference Method (FMFDM) whereas the method consisting of (2.3.8), (2.3.9), and (2.3.11) is the Standard Numerov's Finite Difference Method (SNFDM).

In the rest of the chapter, M denotes various positive constants independent of the mesh spacing h_j and of ε and may take different values in different equations and inequalities.

The discrete operator in the FMFDM, which we denote by $\widetilde{L}^h$, satisfies the following

Lemmas (see [105] for proofs).

Lemma 2.3.1. (Discrete maximum principle) *For any mesh function ϕ_i satisfying $\phi_0 \geq 0$, $\phi_n \geq 0$ and $\widetilde{L}^h \phi_i \geq 0$, $\forall \;\; 0 < i < n$, we have $\phi_i \geq 0, \forall \;\; 0 \leq i \leq n$.*

Lemma 2.3.2. (Uniformly stability estimate) *If ζ_i is any mesh function such that $\zeta_0 = \zeta_n = 0$, then*

$$|\zeta_i| \leq \frac{1}{\beta} \max_{1 \leq j \leq n-1} |\widetilde{L}^h \zeta_j| \;\; \textit{for} \;\; 0 \leq i \leq n.$$

2.4 Convergence analysis of the method

In this section, we succinctly present the relevant results of [119] and then we provide convergence analysis.

Error estimates before extrapolation

The main result of [119] which is mentioned in (2.5.12) is stated in the following theorem.

Theorem 2.4.1. *Let $W(x), Z(x)$ be sufficiently smooth so that $V(x) \in C^6[0,1]$ and $W(x) \geq \beta > 0$. Let $\nu_j, j = 0(1)n$, be the approximate solution of (2.2.4), obtained using (2.3.8)-(2.3.10) with $\nu_0 = V(0), \nu_n = V(1)$. Then, there is a constant M independent of ε and h such that*

$$\sup_{0<\varepsilon\leq 1} \max_{0<j\leq n} |V_j - \nu_j| \leq M n^{-6} \ln^6(n/16) \leq M n^{-4},$$

since $\ln^3(n/16) \leq Mn, \; \forall \; n.$

Next, we derive the extrapolation formula that will be used in the extrapolation technique.

2.5 Extrapolation

Richardson extrapolation is a convergence acceleration technique where a linear combination of two computed solutions approximating a particular quantity gives a third and better approximation. These solutions are calculated on two different but nested meshes. This method is used to increase the accuracy of computed approximations of the solutions of classical boundary value problems and to improve the ε-uniform rates of convergence of computed solutions for linear singularly perturbed problems ([133] and some of the references therein).

2.5.1 Extrapolation formula

We outline below how we implement this procedure to the solution of FMFDM (2.3.8)-(2.3.10).

Consider the mesh $\mu_{2n,\delta}$ where δ is given by (2.3.7), and $\mu_{2n,\delta}$ is obtained from $\mu_{n,\delta}$ by bisecting each mesh sub-interval. Thus,

$$\mu_{n,\delta} = \{x_j\} \subset \mu_{2n,\delta} = \{\widetilde{x}_j\}$$

and $\widetilde{x}_j - \widetilde{x}_{j-1} = \widetilde{h}_j = h_j/2$.

Solving the discrete analogue of (2.2.4) on $\mu_{n,\delta}$, the following estimate was established in [119]:

$$\sup_{0<\varepsilon\leq 1} \max_{0<j\leq n} |V_j - \nu_j| \leq Mn^{-6}\ln^6(n/16) \leq Mn^{-4}, \text{ since } \ln^3(n/16) \leq Mn, \ \forall\ n. \quad (2.5.12)$$

Denoting by $\widetilde{\nu}$ the numerical solution computed on the mesh $\mu_{2n,\delta}$, (2.5.12) reads:

$$\sup_{0<\varepsilon\leq 1} \max_{0<j\leq n} |V_j - \widetilde{\nu}_j| \leq M(2n)^{-6}\ln^6(n/16) \leq M(2n)^{-4}. \quad (2.5.13)$$

It is to be noted that the factor $\ln(n/16)$ in both (2.5.12) and (2.5.13) comes from equation

(2.3.7) and one need not substitute n by $2n$ in this factor on the mesh $\mu_{2n,\delta}$, since the two meshes use the same mesh transition parameter δ. It follows that

$$V(x_j) - \nu(x_j) \;\; = \;\; Mn^{-6}\ln^6(n/16) + R_n(x_j), \;\; \forall \;\; x_j \in \mu_{n,\delta}$$

and

$$V(\widetilde{x}_j) - \widetilde{\nu}(\widetilde{x})_j \;\; = \;\; M(2n)^{-6}\ln^6(n/16) + R_{2n}(\widetilde{x}_j), \;\; \forall \;\; \widetilde{x}_j \in \mu_{2n,\delta},$$

where the remainders $R_n(x_j)$ and $R_{2n}(\widetilde{x}_j)$ are of $o(n^{-6}\ln^6(n/16))$.

A combination of the two equations above gives

$$\Big(V(x_j) - \nu(x_j)\Big) - 64\Big(V(x_j) - \widetilde{\nu}(x_j)\Big) = R_n(x_j) - 64R_{2n}(x_j) = o(n^{-6}\ln^6(n/16)),$$

$$\forall \;\; x_j \in \mu_{n,\delta}.$$

Hence,

$$V(x_j) - \frac{64\widetilde{\nu}_j(x_j) - \nu(x_j)}{63} = o(n^{-6}\ln^6(n/16)), \;\; \forall \;\; x_j \in \mu_{n,\delta}$$

and therefore we set

$$\nu_j^{ext} := \frac{64\widetilde{\nu}_j(x_j) - \nu(x_j)}{63}, \;\; \forall \;\; x_j \in \mu_{n,\delta}. \tag{2.5.14}$$

as the extrapolation formula which we shall use in next section.

2.5.2 Error estimates after extrapolation

For any $j \in \{1, 2, \ldots, n-1\}$, the local truncation error of the scheme (2.3.8)-(2.3.10) after extrapolation is

$$\Big[\widetilde{L}^h\Big(V - \nu^{ext}\Big)\Big]_j = \frac{64}{63}\Big(\widetilde{L}^h(V - \widetilde{\nu})\Big)_j - \frac{1}{63}\Big(\widetilde{L}^h(V - \nu)\Big)_j. \tag{2.5.15}$$

Now

$$\begin{aligned}\left(\widetilde{L}^h(V-\nu)\right)_j &= (r_j^- - q_j^- W_{j-1})V_{j-1} + (r_j^c - q_j^c W_j)V_j + (r_j^+ - q_j^+ W_{j+1})V_{j+1} \\ &\quad +\varepsilon(q_j^- V''_{j-1} + q_j^c V''_j + q_j^+ V''_{j+1}) \qquad (2.5.16)\end{aligned}$$

and

$$\begin{aligned}\left(\widetilde{L}^h(V-\widetilde{\nu})\right)_j &= (\widetilde{r}_j^- - \widetilde{q}_j^- W_{j-1})V_{j-1} + (\widetilde{r}_j^c - \widetilde{q}_j^c W_j)V_j + (\widetilde{r}_j^+ - \widetilde{q}_j^+ W_{j+1})V_{j+1} \\ &\quad +\varepsilon(\widetilde{q}_j^- V''_{j-1} + \widetilde{q}_j^c V''_j + \widetilde{q}_j^+ V''_{j+1}). \qquad (2.5.17)\end{aligned}$$

The quantities $r_j^-, r_j^c, r_j^+, q_j^-, q_j^c$ and q_j^+ are given in (2.3.10) while the quantities $\widetilde{r}_j^-, \widetilde{r}_j^c, \widetilde{r}_j^+, \widetilde{q}_j^-, \widetilde{q}_j^c$ and $\widetilde{q}_j^+$, are obtained by substituting h_j by $\widetilde{h}_j$ in the expressions for $r_j^-, r_j^c, r_j^+, q_j^-, q_j^c,$ and q_j^+, respectively.

We will use two versions of the expansions of V_{j-1}, V_{j+1} and their derivatives depending on whether we want to apply them in (2.5.16) or in (2.5.17).

Expansions to be used in (2.5.16):

$$V_{j-1} = V_j - h_j V'_j + \frac{h_j^2}{2}V''_j - \frac{h_j^3}{6}V'''_j + \frac{h_j^4}{24}V_j^{(4)} - \frac{h_j^5}{120}V_j^{(5)} + \frac{h_j^6}{720}V^{(6)}(\xi_{1,j}),$$

$$V_{j+1} = V_j + h_{j+1}V'_j + \frac{h_{j+1}^2}{2}V''_j + \frac{h_{j+1}^3}{6}V'''_j + \frac{h_{j+1}^4}{24}V_j^{(4)} + \frac{h_{j+1}^5}{120}V_j^{(5)} + \frac{h_{j+1}^6}{720}V^{(6)}(\xi_{2,j}),$$

$$\begin{aligned}V''_{j-1} &= V''_j - h_j V'''_j + \frac{h_j^2}{2}V_j^{(4)} - \frac{h_j^3}{6}V_j^{(5)} + \frac{h_j^4}{24}V^{(6)}(\xi_{3,j}),\\ V''_{j+1} &= V''_j + h_{j+1}V'''_j + \frac{h_{j+1}^2}{2}V(4)_j + \frac{h_{j+1}^3}{6}V_j^{(5)} + \frac{h_{j+1}^4}{24}V^{(6)}(\xi_{4,j}),\end{aligned}$$

where

$$\xi_{1,j} \in (x_{j-1}, x_j), \quad \xi_{3,j} \in (x_{j-1}, x_j)$$

and

$$\xi_{2,j} \in (x_j, x_{j+1}), \quad \xi_{4,j} \in (x_j, x_{j+1}).$$

Expansions to be used in (2.5.17):

$$V_{j-1} = V_j - \widetilde{h}_j V'_j + \frac{\widetilde{h}_j^2}{2} V''_j - \frac{\widetilde{h}_j^3}{6} V'''_j + \frac{\widetilde{h}_j^4}{24} V_j^{(4)} - \frac{\widetilde{h}_j^5}{120} V_j^{(5)} + \frac{\widetilde{h}_j^6}{720} V^{(6)}(\widetilde{\xi}_{1,j}),$$

$$V_{j+1} = V_j + \widetilde{h}_{j+1} V'_j + \frac{\widetilde{h}_{j+1}^2}{2} V''_j + \frac{\widetilde{h}_{j+1}^3}{6} V'''_j + \frac{\widetilde{h}_{j+1}^4}{24} V_j^{(4)} + \frac{\widetilde{h}_{j+1}^5}{120} V_j^{(5)} + \frac{\widetilde{h}_{j+1}^6}{720} V^{(6)}(\widetilde{\xi}_{2,j}),$$

$$V''_{j-1} = V''_j - \widetilde{h}_j V'''_j + \frac{\widetilde{h}_j^2}{2} V_j^{(4)} - \frac{\widetilde{h}_j^3}{6} V_j^{(5)} + \frac{\widetilde{h}_j^4}{24} V^{(6)}(\widetilde{\xi}_{3,j}),$$

$$V''_{j+1} = V''_j + \widetilde{h}_{j+1} V'''_j + \frac{\widetilde{h}_{j+1}^2}{2} V(4)_j + \frac{\widetilde{h}_{j+1}^3}{6} V_j^{(5)} + \frac{\widetilde{h}_{j+1}^4}{24} V^{(6)}(\widetilde{\xi}_{4,j}),$$

where

$$\widetilde{\xi}_{1,j} \in \left(\frac{x_{j-1} + x_j}{2}, x_j\right), \quad \widetilde{\xi}_{3,j} \in \left(\frac{x_{j-1} + x_j}{2}, x_j\right)$$

and

$$\widetilde{\xi}_{2,j} \in \left(x_j, \frac{x_j + x_{j+1}}{2}\right), \quad \widetilde{\xi}_{4,j} \in \left(x_j, \frac{x_j + x_{j+1}}{2}\right).$$

Equations (2.5.16) and (2.5.17), respectively, become

$$\left(\widetilde{L}^h(V - \nu)\right)_j = T_0 V_j + T_1 V'_j + T_2 V''_j + T_3 V'''_j + T_4 V(4)_j + T_5 V_j^{(5)} + T_{6,1} V^{(6)}(\xi_{1,j})$$

$$+ T_{6,2} V^{(6)}(\xi_{2,j}) + T_{6,3} V^{(6)}(\xi_{3,j}) + T_{6,4} V^{(6)}(\xi_{4,j}) \qquad (2.5.18)$$

and

$$\left(\widetilde{L}^h(V-\widetilde{\nu})\right)_j = \widetilde{T}_0 V_j + \widetilde{T}_1 V_j' + \widetilde{T}_2 V_j'' + \widetilde{T}_3 V_j''' + \widetilde{T}_4 V(4)_j + \widetilde{T}_5 V_j^{(5)} + \widetilde{T}_{6,1} V^{(6)}(\widetilde{\xi}_{1,j})$$

$$+\widetilde{T}_{6,2} V^{(6)}(\widetilde{\xi}_{2,j}) + \widetilde{T}_{6,3} V^{(6)}(\widetilde{\xi}_{3,j}) + \widetilde{T}_{6,4} V^{(6)}(\widetilde{\xi}_{4,j}) \tag{2.5.19}$$

where

$$\xi_{1,j},\ \xi_{3,j} \in (x_{j-1}, x_j),\ \ \xi_{2,j},\ \xi_{4,j} \in (x_j, x_{j+1})$$

and

$$\widetilde{\xi}_{1,j},\ \widetilde{\xi}_{3,j} \in \left(\frac{x_{j-1}+x_j}{2}, x_j\right),\ \widetilde{\xi}_{2,j},\ \widetilde{\xi}_{4,j} \in \left(x_j, \frac{x_j+x_{j+1}}{2}\right).$$

In the above

$$\begin{aligned}
T_0 &= (r_j^- + r_j^c + r_j^+) - (q_j^- W_{j-1} + q_j^c W_j + q_j^+ W_{j+1}),\\
T_1 &= h_{j+1}(r_j^+ - q_j^+ W_{j+1}) - h_j(r_j^- - q_j^- W_{j-1}),\\
T_2 &= \frac{h_j^2}{2}(r_j^- - q_j^- W_{j-1}) + \frac{h_{j+1}^2}{2}(r_j^+ - q_j^+ W_{j+1}) + \varepsilon(q_j^- + q_j^c + q_j^+),\\
T_3 &= -\frac{h_j^3}{6}(r_j^- - q_j^- W_{j-1}) + \frac{h_{j+1}^3}{6}(r_j^+ - q_j^+ W_{j+1}) + \varepsilon(h_{j+1}q_j^+ - h_j q_j^-),\\
T_4 &= \frac{h_j^4}{24}(r_j^- - q_j^- W_{j-1}) + \frac{h_{j+1}^4}{24}(r_j^+ - q_j^+ W_{j+1}) + \frac{\varepsilon}{2}(h_{j+1}^2 q_j^+ + h_j^2 q_j^-),\\
T_5 &= -\frac{h_j^5}{120}(r_j^- - q_j^- W_{j-1}) + \frac{h_{j+1}^5}{120}(r_j^+ - q_j^+ W_{j+1}) + \frac{\varepsilon}{6}(h_{j+1}^3 q_j^+ - h_j^3 q_j^-),\\
T_{6,1} &= \frac{h_j^6}{720}(r_j^- - q_j^- W_{j-1}),\ T_{6,2} = \frac{h_{j+1}^6}{720}(r_j^+ - q_j^+ W_{j+1})
\end{aligned} \tag{2.5.20}$$

$$T_{6,3} = \frac{\varepsilon h_j^4 q_j^-}{24}, \; T_{6,4} = \frac{\varepsilon h_{j+1}^4 q_j^+}{24}.$$

The expressions for $\widetilde{T}_0, \widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3, \widetilde{T}_4, \widetilde{T}_5, \widetilde{T}_{6,1}, \widetilde{T}_{6,2}, \widetilde{T}_{6,3},$ and $\widetilde{T}_{6,4}$ can similarly be found as in (3.3.19) by replacing h_j by $\widetilde{h}_j$.

Now two cases are to be considered:

Either

$$j \in \{1, \ldots, j_0 - 1\} \cup \{n - j_0 + 1, \ldots, n - 1\} \tag{2.5.21}$$

(i.e, the grid point x_j lies in the fine mesh),

or

$$j \in \{j_0, \ldots, n - j_0\} \tag{2.5.22}$$

(i.e, the point lies in the coarse mesh).

Using (2.3.6), (2.3.7) and (2.3.10), we see that in both cases, all T's and $\widetilde{T}$'s vanish except $T_{6,1}, \ldots, T_{6,4}$, each of which being equal to $M\varepsilon h_j^6$ and $\widetilde{T}_{6,1}, \ldots, \widetilde{T}_{6,4}$ each of which being equal to $M\varepsilon \widetilde{h}_j^6$.

It follows from (2.5.18) and (2.5.19) that

$$\left| \Big(L^h (V - \nu) \Big)_j \right| \leq M\varepsilon h_j^6 \left| V^{(6)}(\xi) \right| \;\text{ and }\; \left| \Big(\widetilde{L}^h (V - \nu) \Big)_j \right| \leq M\varepsilon \widetilde{h}_j^6 \left| V^{(6)}(\widetilde{\xi}) \right|,$$

where

$$\xi \in (x_{j-1}, x_{j+1}) \;\text{ and }\; \widetilde{\xi} \in \Big(\frac{x_{j-1} + x_j}{2}, \frac{x_j + x_{j+1}}{2} \Big).$$

Therefore, (2.5.15) leads to

$$\left| \Big[\widetilde{L}^h \Big(V - \nu_j^{ext} \Big) \Big]_j \right| \leq M\varepsilon h_j^6 \left| V^{(6)}(\xi) \right|, \tag{2.5.23}$$

where

$$\xi \in (x_{j-1}, x_{j+1}).$$

We denote $\nu_{\varepsilon,r}^{ext}$ and $\nu_{\varepsilon,s}^{ext}$, respectively, the regular and the singular components of ν^{ext}. By virtue of Lemma (2.2.4), we have,

$$\left|\left[\widetilde{L}^h\Big(V_{\varepsilon,r} - \nu_{\varepsilon,r}^{ext}\Big)\right]_j\right| \leq M\varepsilon h_j^6 \left|V_{\varepsilon,r}^{(6)}(\xi)\right| \leq M\varepsilon h_j^6\Big(1 + \varepsilon^{-2}E(x_j,\beta)\Big) \tag{2.5.24}$$

which finally leads to the estimate

$$\left|\left[\widetilde{L}^h\Big(V_{\varepsilon,r} - \nu_{\varepsilon,r}^{ext}\Big)\right]_j\right| \leq M \begin{cases} n^{-6}\ln^6(n/16), & \text{in case of (2.5.21)}, \\ \\ n^{-6}\Big(\frac{\varepsilon^{-x_{j0}\sqrt{\beta/\varepsilon}} + e^{-(1-x_{j0})\sqrt{\beta/\varepsilon}}}{\varepsilon}\Big), & \text{in case of (2.5.22)}. \end{cases} \tag{2.5.25}$$

Likewise,

$$\left|\left[\widetilde{L}^h\Big(V_{\varepsilon,s} - \nu_{\varepsilon,s}^{ext}\Big)\right]_j\right| \leq M\varepsilon h_j^6 \left|V_{\varepsilon,s}^{(6)}(\xi)\right| \leq M\varepsilon h_j^6 \varepsilon^{-3}E(x_j,\beta) \tag{2.5.26}$$

leads to

$$\left|\left[\widetilde{L}^h\Big(V_{\varepsilon,s} - \nu_{\varepsilon,s}^{ext}\Big)\right]_j\right| \leq M \begin{cases} n^{-6}\ln^6(n/16), & \text{in case of (2.5.21)}, \\ \\ n^{-6}\Big(\frac{e^{-x_{j0}\sqrt{\beta/\varepsilon}} + \varepsilon^{-(1-x_{j0})\sqrt{\beta/\varepsilon}}}{\varepsilon}\Big), & \text{in case of (2.5.22)}. \end{cases} \tag{2.5.27}$$

Combining (2.5.25) and (2.5.27), and using Lemma (2.2.5) along with Lemma (2.3.2) (uniform stability estimate), we obtain our main result stated in the following theorem.

Theorem 2.5.1. *Let $W(x)$, $Z(x)$ be sufficiently smooth so that $V(x) \in C^6[0,1]$ and $W(x) \geq \beta > 0$. Let ν_j^{ext}, $j = 0(1)n$ be the approximate solution of (2.2.4) after using the*

Richardson extrapolation. Then, there is a constant M independent of ε and h such that

$$\sup_{0<\varepsilon\leq 1}\max_{0<j\leq n}\left|V_j - \nu_j^{ext}\right| \leq Mn^{-6}\ln^6(n/16) \leq Mn^{-4}.$$

2.6 Numerical results

To demonstrate the theoretical outcomes, we consider the following examples and present the results before and after extrapolation.

Example 2.6.1. [119] Consider problem (2.1.1) with

$$\begin{aligned} a(x) &= 1+x^2,\ b(x) = 1+x(1-x), \\ f(x) &= 1+x(1-x)-\exp(-x/\sqrt{\varepsilon})[x(2x^2-3x+1)-2\sqrt{\varepsilon}(2x^2-x(1+\sqrt{\varepsilon})+1)] \\ &\quad +\exp(-(1-x)/\sqrt{\varepsilon})[x^2(2x-1)+2\sqrt{\varepsilon}(2x^2+x\sqrt{\varepsilon}+1)]. \end{aligned}$$

Its exact solution is given by

$$y(x) = 1+(x-1)\exp[-x/\sqrt{\varepsilon}] - x\exp[-(1-x)/\sqrt{\varepsilon}].$$

Example 2.6.2. [119] Consider problem (2.1.1) with

$$a(x) = 1,\ b(x) = 1,\ f(x) = -(\cos^2\pi x + 2\varepsilon\pi^2\cos 2\pi x).$$

Its exact solution is given by

$$y(x) = \left(\exp[-(1-x)/\sqrt{\varepsilon}] + \exp[-x/\sqrt{\varepsilon}]\right) / \left(1+\exp[-1/\sqrt{\varepsilon}]\right) - \cos^2\pi x.$$

Maximum errors at all the mesh points are evaluated using the formulae:

$$E_{n,\varepsilon} := \max_{0\leq j\leq n}|y(x_j) - v_j|, \quad \text{before extrapolation,}$$

and

$$E_{n,\varepsilon}^{ext} := \max_{0\le j\le n} |y(x_j) - v_j^{ext}|, \quad \text{after extrapolation,}$$

where ν_j is the solutions of (2.1.1) obtained using (2.2.3) and (2.2.4) and ν_j^{ext} is the solution after extrapolation of ν_j.

The numerical rates of convergence are computed using the formula [33]:

$$r_k \equiv r_{\varepsilon,k} := \log_2(\widetilde{E}_{n_k}/\widetilde{E}_{2n_k}),\ k = 1, 2, \ldots$$

where $\widetilde{E}$ stands for $E_{n,\varepsilon}$ and $E_{n,\varepsilon}^{ext}$, respectively. Further, we compute

$$E_n = \max_{0<\varepsilon\le 1} E_{n,\varepsilon}$$

and

$$E_n^{ext} = \max_{0<\varepsilon\le 1} E_{n,\varepsilon}^{ext}$$

whereas the numerical rate of uniform convergence is computed as

$$R_n := \log_2(E_n/E_{2n})$$

and

$$R_n^{ext} := \log_2(E_n^{ext}/E_{2n}^{ext}).$$

2.7 Discussion

In this chapter we have investigated whether Richardson extrapolation improves the accuracy of the numerical solution obtained through a high order method applied to a self-adjoint singular perturbation problem.

We observe that even though Richardson extrapolation improves the accuracy slightly,

Table 2.1: Results for Example 2.6.1 **before** extrapolation (Maximum errors)

ε	n=64	n=128	n=256	n=512	n=1024
1.00E-02	1.10E-06	6.87E-08	4.30E-09	2.69E-10	1.68E-11
1.00E-03	4.57E-05	4.75E-06	2.99E-07	1.87E-08	1.17E-09
1.00E-04	6.26E-04	8.41E-06	1.01E-06	1.55E-07	2.01E-08
1.00E-05	4.46E-03	9.06E-05	1.13E-06	1.48E-07	1.92E-08
1.00E-06	1.00E-02	3.98E-04	1.13E-05	1.87E-07	1.90E-08
1.00E-07	1.31E-02	6.73E-04	3.07E-05	1.14E-06	2.80E-08
1.00E-08	1.43E-02	7.98E-04	4.29E-05	2.18E-06	9.70E-08
1.00E-09	1.47E-02	8.42E-04	4.77E-05	2.68E-06	1.47E-07
1.00E-10	1.48E-02	8.56E-04	4.93E-05	2.87E-06	1.68E-07
1.00E-11	1.49E-02	8.61E-04	4.98E-05	2.93E-06	1.75E-07
1.00E-12	1.49E-02	8.62E-04	5.00E-05	2.95E-06	1.77E-07
1.00E-13	1.49E-02	8.63E-04	5.00E-05	2.96E-06	1.78E-07
E_n	1.49E-02	8.63E-04	5.00E-05	2.96E-06	1.78E-07

Table 2.2: Results for Example 2.6.1 **after** extrapolation (Maximum errors)

ε	n=64	n=128	n=256	n=512	n=1024
1.00E-02	5.24E-08	3.27E-09	2.05E-10	1.28E-11	7.54E-13
1.00E-03	1.08E-05	2.27E-07	1.42E-08	8.89E-10	5.56E-11
1.00E-04	1.06E-04	1.82E-06	4.83E-08	7.38E-09	9.56E-10
1.00E-05	1.30E-03	1.55E-05	1.78E-07	7.07E-09	9.16E-10
1.00E-06	5.90E-03	1.71E-04	2.78E-06	2.58E-08	9.02E-10
1.00E-07	9.98E-03	4.72E-04	1.77E-05	4.33E-07	5.52E-09
1.00E-08	1.18E-02	6.59E-04	3.41E-05	1.53E-06	5.17E-08
1.00E-09	1.25E-02	7.33E-04	4.21E-05	2.32E-06	1.18E-07
1.00E-10	1.27E-02	7.58E-04	4.50E-05	2.65E-06	1.53E-07
1.00E-11	1.27E-02	7.67E-04	4.60E-05	2.77E-06	1.67E-07
1.00E-12	1.28E-02	7.69E-04	4.63E-05	2.81E-06	1.71E-07
1.00E-13	1.28E-02	7.70E-04	4.64E-05	2.82E-06	1.73E-07
E_n	1.28E-02	7.70E-04	4.64E-05	2.82E-06	1.73E-07

Table 2.3: Results for Example 2.6.2 **before** extrapolation (Maximum errors)

ε	n=64	n=128	n=256	n=512	n=1024
1.00E-02	4.81E-07	3.01E-08	1.88E-09	1.18E-10	7.42E-12
1.00E-03	7.41E-05	2.85E-06	1.78E-07	1.12E-08	6.97E-10
1.00E-04	4.10E-04	7.94E-06	6.93E-07	1.81E-07	4.56E-08
1.00E-05	2.10E-03	5.05E-05	7.78E-07	1.05E-07	1.37E-08
1.00E-06	4.03E-03	1.70E-04	5.37E-06	1.07E-07	1.37E-08
1.00E-07	5.00E-03	2.60E-04	1.22E-05	4.85E-07	1.37E-08
1.00E-08	5.35E-03	2.97E-04	1.60E-05	8.27E-07	3.86E-08
1.00E-09	5.47E-03	3.10E-04	1.74E-05	9.82E-07	5.43E-08
1.00E-10	5.51E-03	3.15E-04	1.79E-05	1.04E-06	6.05E-08
1.00E-11	5.52E-03	3.16E-04	1.81E-05	1.06E-06	6.27E-08
1.00E-12	5.52E-03	3.16E-04	1.81E-05	1.07E-06	6.45E-08
1.00E-13	5.52E-03	3.17E-04	1.82E-05	1.10E-06	7.10E-08
E_n	5.52E-03	3.17E-04	1.82E-05	1.10E-06	7.10E-08

Table 2.4: Results for Example 2.6.2 **after** extrapolation (Maximum errors)

ε	n=64	n=128	n=256	n=512	n=1024
1.00E-02	2.29E-08	1.43E-09	8.97E-11	5.67E-12	3.62E-13
1.00E-03	1.77E-05	1.36E-07	8.50E-09	5.31E-10	3.32E-11
1.00E-04	7.76E-05	1.81E-06	1.67E-07	4.32E-08	1.09E-08
1.00E-05	7.05E-04	1.03E-05	1.38E-07	5.02E-09	9.21E-10
1.00E-06	2.47E-03	8.06E-05	1.59E-06	1.79E-08	6.50E-10
1.00E-07	3.79E-03	1.86E-04	7.52E-06	2.12E-07	3.34E-09
1.00E-08	4.35E-03	2.44E-04	1.29E-05	6.07E-07	2.26E-08
1.00E-09	4.54E-03	2.67E-04	1.53E-05	8.57E-07	4.47E-08
1.00E-10	4.60E-03	2.74E-04	1.62E-05	9.56E-07	5.56E-08
1.00E-11	4.62E-03	2.76E-04	1.65E-05	9.90E-07	5.99E-08
1.00E-12	4.63E-03	2.77E-04	1.66E-05	1.00E-06	6.09E-08
1.00E-13	4.63E-03	2.77E-04	1.66E-05	1.00E-06	6.13E-08
E_n	4.63E-03	2.77E-04	1.66E-05	1.00E-06	6.13E-08

Table 2.5: Results for Example 2.6.1 **before** extrapolation (Rates of convergence) $n_k = 64, 128, 256, 512, 1024$

ε	r_1	r_2	r_3	r_4	r_5
1.0E-02	4.00	4.00	4.00	4.00	4.07
1.0E-03	3.27	3.99	4.00	4.00	4.00
1.0E-04	6.22	3.05	2.71	2.95	3.11
1.0E-05	5.62	6.33	2.93	2.95	3.11
1.0E-06	4.65	5.15	5.91	3.30	3.11
1.0E-07	4.28	4.45	4.76	5.34	3.67
1.0E-08	4.16	4.22	4.30	4.49	4.88
1.0E-09	4.12	4.14	4.15	4.19	4.31
1.0E-10	4.11	4.12	4.10	4.10	4.12
1.0E-11	4.11	4.11	4.09	4.07	4.06
1.0E-12	4.11	4.11	4.08	4.06	4.04
1.0E-13	4.11	4.11	4.08	4.05	3.34
R_n	4.11	4.11	4.08	4.05	3.34

Table 2.6: Results for Example 2.6.1 **after** extrapolation (Rates of convergence) $n_k = 64, 128, 256, 512, 1024$

ε	r_1	r_2	r_3	r_4	r_5
1.0E-02	4.00	4.00	4.00	4.08	-0.29
1.0E-03	5.57	4.00	4.00	4.00	3.99
1.0E-04	5.86	5.23	2.71	2.95	3.11
1.0E-05	6.39	6.44	4.66	2.95	3.11
1.0E-06	5.11	5.94	6.75	4.84	3.09
1.0E-07	4.40	4.74	5.35	6.29	5.62
1.0E-08	4.16	4.27	4.48	4.89	5.63
1.0E-09	4.09	4.12	4.18	4.30	4.57
1.0E-10	4.06	4.07	4.08	4.11	4.19
1.0E-11	4.06	4.06	4.05	4.05	4.07
1.0E-12	4.05	4.05	4.04	4.04	4.03
1.0E-13	4.05	4.05	4.04	4.03	3.30
R_n	4.05	4.05	4.04	4.03	3.30

Table 2.7: Results for Example 2.6.2 **before** extrapolation (Rates of convergence) $n_k = 64, 128, 256, 512, 1024$

ε	r_1	r_2	r_3	r_4	r_5
1.0E-02	4.00	4.00	4.00	3.99	4.01
1.0E-03	4.70	4.00	4.00	4.00	4.00
1.0E-04	5.69	3.52	1.94	1.99	2.28
1.0E-05	5.38	6.02	2.88	2.95	3.11
1.0E-06	4.57	4.98	5.65	2.97	3.11
1.0E-07	4.27	4.41	4.65	5.14	3.11
1.0E-08	4.17	4.22	4.27	4.42	4.61
1.0E-09	4.14	4.15	4.15	4.18	4.27
1.0E-10	4.13	4.13	4.11	4.10	4.11
1.0E-11	4.13	4.13	4.10	4.07	4.06
1.0E-12	4.13	4.12	4.08	4.05	3.72
1.0E-13	4.12	4.12	4.05	3.95	2.02
R_n	4.12	4.12	4.05	3.95	2.02

Table 2.8: Results for Example 2.6.2 **after** extrapolation (Rates of convergence) $n_k = 64, 128, 256, 512, 1024$

ε	r_1	r_2	r_3	r_4	r_5
1.0E-02	4.00	4.00	3.98	3.97	0.23
1.0E-03	7.02	4.00	4.00	4.00	3.16
1.0E-04	5.42	3.44	1.95	1.99	2.28
1.0E-05	6.10	6.23	4.78	2.45	1.67
1.0E-06	4.94	5.66	6.47	4.78	3.09
1.0E-07	4.35	4.63	5.15	5.99	5.35
1.0E-08	4.15	4.24	4.41	4.75	5.39
1.0E-09	4.09	4.12	4.16	4.26	4.48
1.0E-10	4.07	4.08	4.08	4.10	4.16
1.0E-11	4.06	4.07	4.06	4.05	4.07
1.0E-12	4.06	4.06	4.05	4.04	3.64
1.0E-13	4.06	4.06	4.05	4.03	1.81
R_n	4.06	4.06	4.05	4.03	1.81

this technique does not increase the rate of convergence which is originally four. This fact was unexpected and contradicts the assertion met in the literature so far about Richardson extrapolation that "A numerical solution of required accuracy is obtained by using Richardson extrapolation method to increase the accuracy of the difference solution [11, 133] and to improve the ε-uniform rates of convergence of computed solutions [133]."

Since the ε-uniform rate of convergence of the FMFDM remains unimproved after applying Richardson extrapolation to a method of order four for a self-adjoint SPP, it is natural to check up to which extent this technique improves the order of convergence for a particular class of SPPs. We are also interested in investigating the limitations of the technique as applied to methods for high-dimensional SPPs.

Chapter 3

Higher Order Fitted Operator Finite Difference Scheme for a Singularly Perturbed Self-adjoint Problem

Recently, there has been a great interest towards the higher order methods for singularly perturbed problems. As compared to their lower order counterparts, they provide better accuracy with fewer mesh points. Construction and/or implementation of direct higher order methods is usually very complicated. Thus a natural choice is to use some convergence acceleration techniques, e.g., Richardson extrapolation, etc. However, as we see in this chapter, such techniques do not perform equally well on all type of methods. To investigate this, we consider two fitted operator finite difference methods (FOFDMs) developed by Patidar [118] and Lubuma and Patidar [98], referred to as FOFDM-I and FOFDM-II, respectively. The FOFDM-I is fourth and second order accurate for moderate and smaller values of ε, respectively. Unfortunately, the Richardson extrapolation does not improve the order of this method. The FOFDM-II is second order uniformly convergent and we show that its order can be improved up to four by using Richardson extrapolation. Both the methods are analyzed for convergence and comparative numerical results supporting theoretical estimates are provided.

3.1 Introduction

The main aim of this chapter is to investigate the performance of Richardson extrapolation when applied to various FOFDMs for Singular Perturbation Problems (SPPs).

It is known that the solutions of SPPs have large gradients when the singular perturbation parameter ε approaches zero. In such limiting cases, boundary/interior layers are developed. The layer behavior of the solution lowers the order of convergence of the underlying numerical method. Standard methods have failed to resolve these problems unless a very fine mesh is considered, which unfortunately raises the computational complexities. To overcome this difficulty, fitted mesh methods have been considered by various authors (see, e.g., [68, 111, 105, 119]) since they provide reliable numerical results on a mesh with a reasonable number of grid points and hence make the method practically applicable. However, there are certain limitations of these fitted mesh methods (when one intend to design a direct higher order method) and therefore we consider in this paper the fitted operator type of methods.

Direct techniques to obtain high order methods for singularly perturbed problems are well documented. We provide here some of those works.

Gartland [45] examined a one-dimensional convection-diffusion problem where he constructed a scheme of order p $(p = 1, 2, 3, 4)$ using collocation approach whereas a fourth order uniformly convergent scheme for a reaction-diffusion problem was presented in [57] where the Hermitian approximation of the second order derivative was used.

For a self-adjoint problem, O'Riordan and Stynes [115] gave a method using finite elements with uniform mesh. This method is second order accurate in L^∞-norm. In [118] a fitted operator finite difference method (FOFDM) was derived via Numerov's method and shown to be fourth order accurate for moderate value of ε and second order accurate for very small values of ε. On the other hand, in [119] a fitted mesh finite difference method (FMFDM) was shown to be fourth order ε-uniformly convergent.

On the other hand, Vulanovic presented a second order ε-uniformly convergent method in [145] for a nonlinear problem. The same author gave a third order method for quasi-linear problems in [149] and [150] whereas Wang [151] achieved third and fourth order convergence for a nonlinear problem.

While none of the methods above is of order higher than four, there exist methods of arbitrary order (see, e.g., [51]) for certain class of problems.

Since the aim is to achieve a better accuracy, one would rather use a convergence acceleration strategy than any of the direct methods (which are tedious in most cases). Several methods for improving the accuracy have been designed in the past (see for example [11, 47, 64, 123, 132, 133] and the references therein). One of these convergence acceleration techniques (presented in [64]) was subsequently termed as the Richardson extrapolation. It is a postprocessing procedure where a linear combination of two computed solutions approximating a particular quantity gives a third and better approximation ([111]). It was implemented in [77] for a system of first order linear ordinary differential equation, in [93] and [111] for a one-dimensional linear convection-diffusion problem, and in [133] for a quasilinear parabolic singularly perturbed convection-diffusion equations.

In this paper, we consider two FOFDMs for the solution of the self-adjoint problem

$$Ly \equiv -\varepsilon(a(x)y')' + b(x)y = f(x),\ x \in [0,1],\ y(0) = \eta_0,\ y(1) = \eta_1, \tag{3.1.1}$$

where η_0 and η_1 are given constants and $\varepsilon \in (0,1]$. The functions $f(x)$, $a(x)$ and $b(x)$ are assumed to be sufficiently smooth that satisfy the conditions

$$a(x) \geq a > 0,\ b(x) \geq b > 0.$$

The existence and uniqueness of a solution of the above problem can be obtained by using the following two results (both of which are proved in Patidar [118]):

Lemma 3.1.1. *Let $\Psi(x)$ be any sufficiently smooth function that satisfies $\Psi(0) \geq 0$ and $\Psi(1) \geq 0$. Then $L\Psi(x) \geq 0$ for all $x \in (0,1)$ implies that $\Psi(x) \geq 0$ for all $x \in [0,1]$.*

Lemma 3.1.2. *Let $y(x)$ be the solution of the problem (3.1.1), then we have*

$$||y|| \leq b^{-1}||f|| + \max(\eta_0, \eta_1),$$

where $\|.\|$ is the usual maximum norm.

The rest of this chapter is organized as follows. We present two FOFDMs in Section 3.2 which are analyzed in Section 3.3. Comparative numerical results (before and after extrapolation) for these two methods are presented in Section 3.4. Finally, we conclude the chapter in Section 3.5.

3.2 Two fitted operator finite difference methods

Now, let n be a positive integer. Consider the following partition of the interval $[0,1]$:

$$x_0 = 0, \;\; x_j = x_0 + jh, \;\; j = 1(1)n, \;\; h = x_j - x_{j-1}, \;\; x_n = 1.$$

We denote the above mesh by μ_n whereas the mesh μ_{2n} is obtained by bisecting each mesh interval in μ_n , i.e.,

$$\mu_{2n} = \{\widetilde{x}_j\} \text{ with } \widetilde{x}_0 = 0, \; \widetilde{x}_n = 1 \text{ and } \widetilde{x}_j - \widetilde{x}_{j-1} = \widetilde{h} = h/2, \; j = 1(1)2n.$$

These two meshes will be used to derive the extrapolation formulae in the next section. Furthermore, we use the notations $V_j = V(x_j)$, $W_j = W(x_j)$ and $Z_j = Z(x_j)$ and we denote the approximations of V_j at the grid points x_j by the unknowns ν_j.

3.2.1 FOFDM-I

Using the theory of inverse monotone matrices, Patidar [118] designed a high order FOFDM to solve (3.1.1) via (2.2.4) and (2.2.3) as follows:

He defined the fitting comparison problem associated with (2.2.4) by

$$-\sigma(x,\varepsilon)V'' + W(x)V = Z(x),\ V(0)=\alpha_0,\ V(1)=\alpha_1, \tag{3.2.2}$$

where $\sigma(x,\varepsilon)$ is a fitting factor. Then the approximate solution of the problem (3.2.2) is sought by the Numerov's method:

$$\begin{aligned} -\left[\sigma_j^- - \frac{h^2}{12}W_{j-1}\right]\nu_{j-1} + \left[2\sigma_j^c + \frac{5h^2}{6}W_j\right]\nu_j - \left[\sigma_j^+ - \frac{h^2}{12}W_{j+1}\right]\nu_{j+1} \\ = \frac{h^2}{12}[Z_{j-1} + 10Z_j + Z_{j+1}], \end{aligned} \tag{3.2.3}$$

where $\sigma_j^{\pm}$ and σ_j^c are given by

$$\sigma_j^{\pm} = \frac{h^2 W_{j\pm1}}{12}\left(1 + \frac{3}{\sinh^2(\frac{\rho_j h}{2})}\right) \quad \text{and} \quad \sigma_j^c = \frac{h^2 W_j}{12}\left(1 + \frac{3}{\sinh^2(\frac{\rho_j h}{2})}\right). \tag{3.2.4}$$

In matrix notation, the scheme (3.2.3) can be written as the following tridiagonal system

$$A\nu = F. \tag{3.2.5}$$

The entries corresponding to A and F in this case are

$$A_{ij} = r_j^-,\ i = j+1;\ \ j = 1,2,\cdots,n-2;$$

$$A_{ij} = r_j^c,\ i = j;\ \ j = 1,2,\cdots,n-1;$$

$$A_{ij} = r_j^+,\ i = j-1;\ \ j = 2,3,\cdots,n-1;$$

$$F_1 = q_1^- Z_0 + q_1^c Z_1 + q_1^+ Z_2 - r_1^- \alpha_0,$$

$$F_j = q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ + Z_{j+1}, \quad j = 2, 3, \ldots, n-2,$$

$$F_{n-1} = q_{n-1}^- Z_{n-2} + q_{n-1}^c Z_j + q_{n-1}^+ Z_n - r_{n-1}^+ \alpha_1,$$

where

$$\left.\begin{aligned} r_j^- &= -\left[\sigma_j^- - \tfrac{h^2}{12} W_{j-1}\right], \\ r_j^c &= \left[2\sigma_j^c + \tfrac{5h^2}{6} W_j\right], \\ r_j^+ &= -\left[\sigma_j^+ - \tfrac{h^2}{12} W_{j+1}\right], \\ q_j^- = q_j^+ &= \tfrac{h^2}{12}, \quad q_j^c = \tfrac{5h^2}{6}; j = 1, 2, \ldots, n-1. \end{aligned}\right\} \tag{3.2.6}$$

3.2.2 FOFDM-II

Subsequent to Patidar [118], Lubuma and Patidar [98] developed the following FOFDM (using the nonstandard finite difference modeling rules of Mickens [103]) to solve (2.2.4):

$$-\varepsilon \frac{\nu_{j-1} - 2\nu_j + \nu_{j+1}}{\widetilde{\phi}_j^2} + \widetilde{W}_j \nu_j = Z_j, \tag{3.2.7}$$

where

$$\widetilde{W}_j = \frac{W_{j-1} + W_j + W_{j+1}}{3}, \quad \widetilde{\rho}_j = \sqrt{\frac{\widetilde{W}_j}{\varepsilon}}, \quad \text{and} \quad \widetilde{\phi}_j \equiv \frac{2}{\widetilde{\rho}_j} \sinh\left(\frac{\widetilde{\rho}_j h}{2}\right).$$

This leads to a tridiagonal system of linear equations

$$A\nu = F. \tag{3.2.8}$$

Corresponding entries of A and F in this case are

$$A_{ij} = r_j^-,\ i = j+1;\ \ j = 1, 2, \cdots, n-2,$$

$$A_{ij} = r_j^c,\ i = j;\ \ j = 1, 2, \cdots, n-1,$$

$$A_{ij} = r_j^+,\ i = j-1;\ \ j = 2, 3, \cdots, n-1,$$

$$F_1 = Z_1 - r_1^- \alpha_0,\ F_{n-1} = Z_{n-1} - r_{n-1}^+ \alpha_1,$$

$$F_j = Z_j;\ \ j = 2, 3, \cdots, n-2,$$

where

$$r_j^- = -\frac{\varepsilon}{\widetilde{\phi}_j^2},\ \ r_j^+ = -\frac{\varepsilon}{\widetilde{\phi}_j^2},\ \ \text{and}\ \ r_j^c = \frac{2\varepsilon}{\widetilde{\phi}_j^2} + \widetilde{W}_j. \tag{3.2.9}$$

We analyze the above FOFDMs in next section whereas the comparative numerical results obtained via these methods are presented in Section 4.

3.3 Analysis of the numerical methods

FOFDM-I was analyzed for convergence (before extrapolation) in [118]. Here we provide additional analysis, that is, the one after the extrapolation. Regarding FOFDM-II, we revisit the analysis (before extrapolation) presented in [98] and then present the analysis after extrapolation.

The analysis for each of the methods is divided into three parts. Firstly, we provide the error estimates where the approximate solution is the one obtained before extrapolation. These estimates are then used to derive the extrapolation formula. Finally, we provide the error estimates in which the approximate solution obtained after extrapolation is used.

3.3.1 Analysis of FOFDM-I

Error estimates before extrapolation

The following estimates are obtained in [118]:

$$\max_{1\le j\le n-1}|V(x_j)-\nu_j| \le \begin{cases} \frac{Mh^4}{\varepsilon}\left[1+\max_{1\le j\le n-1}\frac{E(x_j,\beta)}{\varepsilon^2}\right], & \text{when } Ch\le\varepsilon,\\ \\ Mh^2\left[1+h^2\max_{1\le j\le n-1}\frac{E(x_j,\beta)}{\varepsilon}\right], & \text{when } Ch\ge\varepsilon. \end{cases} \tag{3.3.10}$$

where

$$E(x,\beta)=\Big\{\exp\Big(-x/\sqrt{\beta/\varepsilon}\Big)+\exp\Big(-(1-x)\sqrt{\beta/\varepsilon}\Big)\Big\},\ \text{ and }\ 0<\beta\le W(x).$$

Here and after, M and C denote positive constants which may take different values in different equations and inequalities but are always independent of h and ε.

Extrapolation formula

The FOFDM-I on the mesh μ_n satisfies (3.3.10). Denoting by $\widetilde{\nu}$ the numerical solution computed on the mesh μ_{2n}, the estimate (3.3.10) reads

$$\max_{1\le\ j\le 2n-1}|V(\widetilde{x}_j)-\widetilde{\nu}_j| \le \begin{cases} \frac{M}{\varepsilon}(\frac{h}{2})^4\left[1+\max_{1\le\ j\le 2n-1}\frac{E(\widetilde{x}_j,\beta)}{\varepsilon^2}\right], & \text{when } Ch\le\varepsilon,\\ \\ M(\frac{h}{2})^2\left[1+(\frac{h}{2})^2\max_{1\le\ j\le 2n-1}\frac{E(\widetilde{x}_j,\beta)}{\varepsilon}\right], & \text{when } Ch\ge\varepsilon. \end{cases} \tag{3.3.11}$$

To establish the suitable extrapolation formula, it is important to consider the two cases separately.

We start with the case in which $Ch\le\varepsilon$.

It follows, from (3.3.10) and (3.3.11), that

$$V(x_j) - \nu_j = \frac{Mh^4}{\varepsilon}\left[1 + \frac{E(x_j, \beta)}{\varepsilon^2}\right] + R_n(x_j), \;\; 1 \leq j \leq n-1,$$

and

$$V(\widetilde{x}_j) - \widetilde{\nu}_j = \frac{M}{\varepsilon}\left(\frac{h}{2}\right)^4 \left[1 + \frac{E(\widetilde{x}_j, \beta)}{\varepsilon^2}\right] + R_{2n}(\widetilde{x}_j), \;\; 1 \leq j \leq 2n-1,$$

where both the remainders, $R_n(x_j)$ and $R_{2n}(\widetilde{x}_j)$, are of $O(h^4)$.

Therefore,

$$(V(x_j) - \nu_j) - 16\,(V(x_j) - \widetilde{\nu}_j) = R_n(x_j) - 16R_{2n}(x_j) = o(h^4), \;\; \forall \;\; x_j \in \mu_n.$$

Hence,

$$V(x_j) - \frac{16\widetilde{\nu}_j - \nu_j}{15} = O(h^4), \;\; \forall \;\; x_j \in \mu_n. \tag{3.3.12}$$

In the case when $Ch \geq \varepsilon$, estimates (3.3.10) and (3.3.11), respectively, give

$$V(x_j) - \nu_j = Mh^2\left[1 + h^2\frac{E(x_j, \beta)}{\varepsilon}\right] + R_n^*(x_j), \; 1 \leq j \leq n-1,$$

and

$$V(\widetilde{x}_j) - \widetilde{\nu}_j = M\left(\frac{h}{2}\right)^2 \left[1 + \left(\frac{h}{2}\right)^2 \frac{E(\widetilde{x}_j, \beta)}{\varepsilon}\right] + R_{2n}^*(\widetilde{x}_j), \; 1 \leq j \leq 2n-1,$$

where both the remainders, $R_n^*(x_j)$ and $R_{2n}^*(\widetilde{x}_j)$ are of $O(h^2)$.

Thus,

$$(V(x_j) - \nu_j) - 16\,(V(x_j) - \widetilde{\nu}_j) = O(h^2), \;\; \forall \;\; x_j \in \mu_n$$

and consequently,

$$V(x_j) - \frac{16\widetilde{\nu}_j - \nu_j}{15} = O(h^2), \quad \forall \ \ x_j \in \mu_n. \tag{3.3.13}$$

In view of equations (3.3.12) and (3.3.13), it is natural to use the formula

$$\nu_j^{ext} := \frac{16\widetilde{\nu}_j - \nu_j}{15}, \ j = 1(1)n-1 \tag{3.3.14}$$

in the extrapolation process, irrespective of the cases $Ch \leq \varepsilon$ or $Ch \geq \varepsilon$.

Error estimates after extrapolation

Unless indicated otherwise, in what follows, the functions with a symbol '$\sim$' means that they are evaluated at the mesh μ_{2n}. The only exceptions to this notation are with the denominator functions used in FOFDM-II and the functions W used in (3.3.47) where we use ' $-$ ' with the denominator function evaluated at the mesh μ_{2n} whereas the one at the mesh μ_n has a '$\sim$' on top of it.

The local truncation error of the scheme (3.2.5) and (3.2.6) after extrapolation is

$$\left[\widetilde{L}^h\left(V - \frac{16\widetilde{\nu} - \nu}{15}\right)\right]_j = \frac{16}{15}\left[\widetilde{A}(V - \widetilde{\nu})\right]_j - \frac{1}{15}\left[A(V-\nu)\right]_j, \quad j = 1(1)n-1. \tag{3.3.15}$$

Here

$$\begin{aligned}(A(V-\nu))_j &= (r_j^- - q_j^- W_{j-1})V_{j-1} + (r_j^c - q_j^c W_j)V_j + (r_j^+ - q_j^+ W_{j+1})V_{j+1} \\ &\quad + \varepsilon(q_j^- V_{j-1}'' + q_j^c V_j'' + q_j^+ V_{j+1}''),\end{aligned} \tag{3.3.16}$$

and

$$(\widetilde{A}(V-\widetilde{\nu}))_j = (\widetilde{r}_j^- - \widetilde{q}_j^- W_{j-1})V_{j-1} + (\widetilde{r}_j^c - \widetilde{q}_j^c W_j)V_j + (\widetilde{r}_j^+ - \widetilde{q}_j^+ W_{j+1})V_{j+1}$$

$$+\varepsilon(\widetilde{q}_j^- V''_{j-1} + \widetilde{q}_j^c V''_j + \widetilde{q}_j^+ V''_{j+1}). \tag{3.3.17}$$

Using the Taylor series expansions, we obtain (when $Ch \leq \varepsilon$)

$$\begin{aligned}(A(V-\nu))_j &= T_0V_j + T_1V'_j + T_2V''_j + T_3V'''_j \\ &\quad + T_4V^{(4)}(\xi_{1,j}) + T_4V^{(4)}(\xi_{2,j}),\end{aligned} \tag{3.3.18}$$

and

$$\begin{aligned}\left(\widetilde{A}(V-\widetilde{\nu})\right)_j &= \widetilde{T}_0V_j + \widetilde{T}_1V'_j + \widetilde{T}_2V''_j + \widetilde{T}_3V'''_j \\ &\quad + \widetilde{T}_4V^{(4)}(\widetilde{\xi}_{1,j}) + \widetilde{T}_4V^{(4)}(\widetilde{\xi}_{2,j}),\end{aligned} \tag{3.3.19}$$

where

$$\xi_{1,j} \in (x_{j-1}, x_j),\ \xi_{2,j} \in (x_j, x_{j+1}),$$

and

$$\widetilde{\xi}_{1,j} \in \left(\frac{x_{j-1}+x_j}{2}, x_j\right),\ \widetilde{\xi}_{2,j} \in \left(x_j, \frac{x_j+x_{j+1}}{2}\right).$$

Also

$$\left.\begin{aligned} T_0 &= -\sigma_j^- + 2\sigma_j^c - \sigma_j^+, \\ T_1 &= h(\sigma_j^- - \sigma_j^+), \\ T_2 &= -h^2\left[\tfrac{1}{2}(\sigma_j^- + \sigma_j^+) - \varepsilon\right], \\ T_3 &= \tfrac{h_j^3}{6}(\sigma_j^- - \sigma_j^+), \\ T_4 &= -\tfrac{h_j^4}{24}[\sigma_j^- + \sigma_j^+ - 2\varepsilon]. \end{aligned}\right\} \tag{3.3.20}$$

The different expressions for $\widetilde{T}_0, \widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3,$ and $\widetilde{T}_4$, are equivalently found as in (3.3.20) by replacing h by $\widetilde{h}$ and σ by $\widetilde{\sigma}$.

Some algebraic manipulations yield

$$\left.\begin{array}{c} |T_0| \le \frac{Mh^6}{\varepsilon}, \ \ |\widetilde{T}_0| \le \frac{M\widetilde{h}^6}{\varepsilon}; |T_1| \le \frac{Mh^6}{\varepsilon}, \ \ |\widetilde{T}_1| \le \frac{M\widetilde{h}^6}{\varepsilon}; \\ \\ |T_2| \le \frac{Mh^6}{\varepsilon}, \ \ |\widetilde{T}_2| \le \frac{M\widetilde{h}^6}{\varepsilon}; \\ \\ |T_3| \le \frac{Mh^8}{\varepsilon}, \ \ |\widetilde{T}_3| \le \frac{M\widetilde{h}^8}{\varepsilon}; |T_4| \le \frac{Mh^8}{\varepsilon}, \ \ |\widetilde{T}_4| \le \frac{M\widetilde{h}^8}{\varepsilon}. \end{array}\right\} \quad (3.3.21)$$

On the other hand, the following lemma (proved in [105]) provides bounds on the derivatives of solution:

Lemma 3.3.1. *For all* $k \in \{0, 1, 2, 3, 4\}$ *and* $x \in [0, 1]$, *the solution* $V(x)$ *of (2.2.4) satisfies*

$$|V^{(k)}(x)| \le M[1 + \varepsilon^{-k/2} E(x, \beta)],$$

where

$$0 < \beta \le W(x) \quad \text{and} \quad E(x, \beta) = \left\{\exp\left(-x/\sqrt{\beta/\varepsilon}\right) + \exp\left(-(1-x)\sqrt{\beta/\varepsilon}\right)\right\}.$$

Using the above lemma, relations (3.3.18)-(3.3.19), and the fact that $\widetilde{h} < h$, we obtain

$$\max_{1\le j\le n-1} \left|(A(V-\nu))_j\right| \le \frac{Mh^6}{\varepsilon}\left[1 + \max_{1\le j\le n-1} \frac{E(x_j, \beta)}{\varepsilon^2}\right], \quad (3.3.22)$$

and

$$\max_{1\le j\le n-1} \left|\left(\widetilde{A}(V-\widetilde{\nu})\right)_j\right| \le \frac{Mh^6}{\varepsilon}\left[1 + \max_{1\le j\le n-1} \frac{E(x_j, \beta)}{\varepsilon^2}\right]. \quad (3.3.23)$$

Furthermore, the matrices A and $\widetilde{A}$ are diagonally dominant by rows, therefore, we have the following result (due to Varah [140]) to estimate the norm of the associated matrix:

$$||A^{-1}|| \leq \max_j \{|r_j^c| - (|r_j^-| + |r_j^+|)\}^{-1}. \tag{3.3.24}$$

Using (3.2.6), we get

$$\{|r_j^c| - (|r_j^-| + |r_j^+|)\} \geq Mh^2.$$

Hence from the inequality (3.3.24), we have

$$||A^{-1}|| \leq \frac{M}{h^2}, \tag{3.3.25}$$

and similarly

$$||\widetilde{A}^{-1}|| \leq \frac{M}{\widetilde{h}^2}. \tag{3.3.26}$$

Now, using the inequality

$$\max_j \left| V(x_j) - \frac{16\widetilde{\nu}_j - \nu_j}{15} \right| \leq \frac{16}{15} \max_j |V(x_j) - \widetilde{\nu}_j| + \frac{1}{15} \max_j |V(x_j) - \nu_j|,$$

along with (3.3.22)-(3.3.23) and (3.3.25)-(3.3.26) into

$$\max_j |V_j - \nu_j| \leq ||A^{-1}|| \max_j \left| (A(V - \nu))_j \right|, \tag{3.3.27}$$

and

$$\max_j |V_j - \widetilde{\nu}_j| \leq ||\widetilde{A}^{-1}|| \max_j \left| \left(\widetilde{A}(V - \widetilde{\nu})\right)_j \right|, \tag{3.3.28}$$

we obtain

$$\max_j \left| V(x_j) - \nu_j^{ext} \right| \leq \frac{Mh^4}{\varepsilon} \left(1 + \max_j \frac{E(x_j, \beta)}{\varepsilon^2} \right). \tag{3.3.29}$$

On the other hand, when $Ch \geq \varepsilon$, we introduce some new notations

$$\begin{aligned} r_j^- &= r_j^-(W_{j-1}), \quad r_j^+ = r_j^+(W_{j+1}), \quad r_j^c = r_j^c(W_j), \\ R_j^- &= r_j^-(W_0), \quad R_j^+ = r_j^+(W_0), \quad R_j^c = r_j^c(W_0), \end{aligned}$$

$$
\begin{aligned}
\widetilde{r}_j^- &= \widetilde{r}_j^-(W_{j-1}), \quad \widetilde{r}_j^+ = \widetilde{r}_j^+(W_{j+1}), \quad \widetilde{r}_j^c = \widetilde{r}_j^c(W_j),\\
\widetilde{R}_j^- &= \widetilde{r}_j^-(W_0), \quad \widetilde{R}_j^+ = \widetilde{r}_j^+(W_0), \quad \widetilde{R}_j^c = \widetilde{r}_j^c(W_0),
\end{aligned}
$$

and since q_j's and $\widetilde{q}_j$'s are independent of W_j's, we will have $Q_j = q_j$, $\widetilde{Q}_j = \widetilde{q}_j$, etc.

In this case, then we have

$$
\begin{aligned}
(A(V-\nu))_j &= \{[(r_j^- - q_j^- W_{j-1})V_{j-1} + (r_j^c - q_j^c W_j)V_j + (r_j^+ - q_j^+ W_{j+1})V_{j+1}\\
&\quad +\varepsilon(q_j^- V''_{j-1} + q_j^c V''_j + q_j^+ V''_{j+1})]\\
&\quad -[(R_j^- - Q_j^- W_0)V_{j-1} + (R_j^c - Q_j^c W_0)V_j + (R_j^+ - Q_j^+ W_0)V_{j+1}\\
&\quad +\varepsilon(Q_j^- V''_{j-1} + Q_j^c V''_j + Q_j^+ V''_{j+1})]\}
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\widetilde{A}(V-\widetilde{\nu})\right)_j &= \{[(\widetilde{r}_j^- - \widetilde{q}_j^- W_{j-1})V_{j-1} + (\widetilde{r}_j^c - \widetilde{q}_j^c W_j)V_j + (\widetilde{r}_j^+ - \widetilde{q}_j^+ W_{j+1})V_{j+1}\\
&\quad +\varepsilon(\widetilde{q}_j^- V''_{j-1} + \widetilde{q}_j^c V''_j + \widetilde{q}_j^+ V''_{j+1})]\\
&\quad -[(\widetilde{R}_j^- - \widetilde{Q}_j^- W_0)V_{j-1} + (\widetilde{R}_j^c - \widetilde{Q}_j^c W_0)V_j + (\widetilde{R}_j^+ - \widetilde{Q}_j^+ W_0)V_{j+1}\\
&\quad +\varepsilon(\widetilde{Q}_j^- V''_{j-1} + \widetilde{Q}_j^c V''_j + \widetilde{Q}_j^+ V''_{j+1})]\},
\end{aligned}
$$

which when simplified, reduce to

$$
\begin{aligned}
(A(V-\nu))_j &= [(r_j^- - R_j^-) - q_j^-(W_{j-1} - W_0)]V_{j-1} + [(r_j^c - R_j^c) - q_j^c(W_j - W_0)]V_j\\
&\quad +[(r_j^+ - R_j^+) - q_j^+(W_{j+1} - W_0)]V_{j+1} \qquad (3.3.30)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\widetilde{A}(V-\widetilde{\nu})\right)_j &= [(\widetilde{r}_j^- - \widetilde{R}_j^-) - \widetilde{q}_j^-(W_{j-1} - W_0)]V_{j-1} + [(\widetilde{r}_j^c - \widetilde{R}_j^c) - \widetilde{q}_j^c(W_j - W_0)]V_j\\
&\quad +[(\widetilde{r}_j^+ - \widetilde{R}_j^+) - \widetilde{q}_j^+(W_{j+1} - W_0)]V_{j+1}, \; j = 1(1)n-1. \qquad (3.3.31)
\end{aligned}
$$

Simplifying (3.3.30), we obtain

$$\left|[A(V-\nu)]_j\right| \leq Mh^4\left[3V_j + h^2V_j'' + \frac{h^4}{24}\left(V^{(4)}(\xi_{1,j}) + V^{(4)}(\xi_{2,j})\right)\right]. \qquad (3.3.32)$$

Applying Lemma 3.3.1, we get

$$\max_{1\leq j\leq n-1}\left|[A(V-\nu)]_j\right| \leq Mh^4\left[1 + h^2\max_{1\leq j\leq n-1}\frac{E(x_j,\beta)}{\varepsilon}\right]. \qquad (3.3.33)$$

Similarly, Eq. (3.3.31) yields

$$\max_{1\leq j\leq n-1}\left|[\widetilde{A}(V-\widetilde{\nu})]_j\right| \leq Mh^4\left[1 + h^2\max_{1\leq j\leq n-1}\frac{E(x_j,\beta)}{\varepsilon}\right]. \qquad (3.3.34)$$

Hence, from (3.3.25)-(3.3.28) and (3.3.33)-(3.3.34), we obtain

$$\max_{0<j\leq n}\left|V_j - \nu_j^{ext}\right| \leq Mh^2\left[1 + h^2\max_{0<j\leq n}\frac{E(x_j,\beta)}{\varepsilon}\right]. \qquad (3.3.35)$$

We have therefore established that

$$\max_{1<j\leq n-1}\left|V_j - \nu_j^{ext}\right| \leq \begin{cases} \frac{Mh^4}{\varepsilon}\left[1 + \max_{1\leq j\leq n-1}\frac{E(x_j,\beta)}{\varepsilon^2}\right], & \text{when } Ch \leq \varepsilon, \\ \\ Mh^2\left[1 + h^2\max_{1\leq j\leq n-1}\frac{E(x_j,\beta)}{\varepsilon}\right], & \text{when } Ch \geq \varepsilon. \end{cases} \qquad (3.3.36)$$

which means that the Richardson extrapolation does not improve the order of convergence of FOFDM-I.

3.3.2 Analysis of FOFDM-II

Error estimates before extrapolation

The local truncation error of the scheme (3.2.8) and (3.2.9) is given by

$$\tau_j(V) = T_0 V_j + T_1 V_j' + T_2 V_j'' + T_3 V_j''' + T_4 V^{(4)}(\xi_j); \quad \xi_j \in (x_{j-1}, x_{j+1}), \tag{3.3.37}$$

where

$$\left.\begin{aligned} T_0 &= r_j^- + r_j^c + r_j^+ - \widetilde{W}_j, \\ T_1 &= h(r_j^+ - r_j^-), \\ T_2 &= \tfrac{h^2}{2}(r_j^+ + r_j^-) + \varepsilon, \\ T_3 &= \tfrac{h^3}{6}(r_j^+ - r_j^-), \\ \text{and } T_4 &= \tfrac{h^4}{24}(r_j^+ + r_j^-). \end{aligned}\right\} \tag{3.3.38}$$

Further simplifications yield

$$T_0 = T_1 = T_3 = 0, \quad |T_2| \le Mh^2, \text{ and } |T_4| \le Mh^2. \tag{3.3.39}$$

Finally using Lemma 3.3.1 we obtain

$$\max_{1\le j\le n-1} |\tau_j(V)| \le Mh^2 \left[1 + \max_{1\le j\le n-1} \frac{E(x_j,\beta)}{\varepsilon^2}\right]. \tag{3.3.40}$$

Now since A is diagonally dominant by rows, we can estimate $||A^{-1}||$ by the relation (3.3.24). Since, $\{|r_j^c| - (|r_j^-| + |r_j^+|)\} \geq M$, we conclude that,

$$\|A^{-1}\| \leq M. \tag{3.3.41}$$

But the relation

$$\tau_j(V) = (AV)_j - \left(\widetilde{L}V\right)_j = (A(V-\nu))_j \tag{3.3.42}$$

implies that

$$\max_j |V_j - \nu_j| \leq \|A\|^{-1} \max_j \left|(A(V-\nu))_j\right|. \tag{3.3.43}$$

Hence, using (3.3.40) and (3.3.41), we obtain

$$\max_{1\leq j\leq n-1} |V_j - \nu_j| \leq Mh^2 \left[1 + \max_{1\leq j\leq n-1} \frac{E(x_j,\beta)}{\varepsilon^2}\right]. \tag{3.3.44}$$

Now using the lemma (see, [118] for details) on exponential behavior of the solution, we find that

$$\sup_{0<\varepsilon\leq 1} \max_{1\leq j\leq n-1} |V_j - \nu_j| \leq Mh^2. \tag{3.3.45}$$

Extrapolation formula

In this case, ν and $\widetilde{\nu}$ denote the computed solutions of problem (2.2.4) by the scheme (3.2.8) and (3.2.9) on the meshes μ_n and μ_{2n}, respectively. This implies that

$$|V_j - \nu_j| \leq Mh^2 \left[1 + \frac{E(x_j,\beta)}{\varepsilon^2}\right],\ j = 1(1)n-1$$

and

$$|V_j - \widetilde{\nu}_j| \leq M(h/2)^2 \left[1 + \frac{E(\widetilde{x}_j,\beta)}{\varepsilon^2}\right],\ j = 1(1)2n-1.$$

Therefore,

$$V_j - \nu_j = Mh^2\left[1 + \frac{E(x_j,\beta)}{\varepsilon^2}\right] + R_n(x_j), \forall\ \ x_j \in \mu_n$$

and

$$V_j - \widetilde{\nu}_j = M(h/2)^2\left[1 + \frac{E(\widetilde{x}_j,\beta)}{\varepsilon^2}\right] + R_{2n}(\widetilde{x}_j), \forall\ \ \widetilde{x}_j \in \mu_{2n}.$$

where both the remainders, $R_n(x_j)$ and $R_{2n}(\widetilde{x}_j)$ are $o(h^2)$.

Hence,

$$(V_j - \nu_j) - 4(V_j - \widetilde{\nu}_j) = R_n(x_j) - 4R_{2n}(x_j) = o(h^2), \forall\ \ x_j \in \mu_n$$

indicates that in the extrapolation process, we should use the formula

$$\nu_j^{ext} := \frac{4\widetilde{\nu}_j - \nu_j}{3},\ j = 1(1)n-1. \tag{3.3.46}$$

Error estimates after extrapolation

An analogue of (3.3.15) implies that the local truncation error of the scheme (3.2.8) and (3.2.9) after extrapolation should be given by

$$\begin{aligned}(L_*^h(V-\nu^{ext}))_j &= \frac{4}{3}(L_*^{\tilde{h}}(V-\widetilde{\nu}))_j - \frac{1}{3}(L_*^h(V-\nu))_j\\ &= \frac{4}{3}\left[\left(-\varepsilon\widetilde{V}_j'' + \bar{W}_j\widetilde{V}_j\right)\right.\\ &\quad\left. - \left(-\varepsilon\frac{\widetilde{V}_{j+1} - 2\widetilde{V}_j + \widetilde{V}_{j-1}}{\bar{\phi}_j^2} + \bar{W}_j\widetilde{V}_j\right)\right]\\ &\quad -\frac{1}{3}\left[\left(-\varepsilon V_j'' + \widetilde{W}_jV_j\right)\right.\\ &\quad\left. - \left(-\varepsilon\frac{V_{j+1} - 2V_j + V_{j-1}}{\widetilde{\phi}_j^2} + \widetilde{W}_jV_j\right)\right],\end{aligned} \tag{3.3.47}$$

where L_*^h and $L_*^{\widetilde{h}}$ denote the discrete operators associated with FOFDM-II (i.e., relations (3.2.8) and (3.2.9)) when considered on meshes μ_n and μ_{2n}, respectively. (Note that $\bar{\phi}_j$ is obtained from $\widetilde{\phi}_j$ by replacing h by $\widetilde{h}$).

Some algebraic manipulations yield

$$(L_*^h(V - \nu^{ext}))_j \leq Mh^4 V^{(vi)}(\xi_j), \quad \xi_j \in (x_{j-1}, x_{j+1}). \tag{3.3.48}$$

Using Lemma 3.3.1, we obtain

$$\max_{1\leq j\leq n-1} |L_*^h(V - \nu^{ext})|_j \leq Mh^4 \left[1 + \max_{1\leq j\leq n-1} \frac{E(x_j, \beta)}{\varepsilon^3}\right]. \tag{3.3.49}$$

Finally, using the lemma (see, [118] for details) on exponential behavior of the solution, we find that

$$\sup_{0<\varepsilon\leq 1} \max_{0<j\leq n} \left|V_j - \nu_j^{ext}\right| \leq Mh^4.$$

In summary, we have the following main result:

Theorem 3.3.1. *Let $W(x)$, $Z(x)$ be sufficiently smooth so that $V(x) \in C^4[0,1]$. Let ν_j^{ext}, $j = 0(1)n$ be the approximate solutions of (2.2.4) obtained after extrapolation, with $\nu_0 = \nu_0^{ext} = V(0)$, and $\nu_n = \nu_n^{ext} = V(1)$. Then, there is a constant M independent of ε and h such that*

$$\sup_{0<\varepsilon\leq 1} \max_{0<j\leq n} \left|V_j - \nu_j^{ext}\right| \leq Mh^2 \text{ for FOFDM-I.}$$

$$\sup_{0<\varepsilon\leq 1} \max_{0<j\leq n} \left|V_j - \nu_j^{ext}\right| \leq Mh^4 \text{ for FOFDM-II.}$$

3.4 Numerical results

In this section we present some comparative numerical results for two test problems considered in [118].

Example 3.4.1. Consider problem (3.1.1) with

$$a(x) = 1 + x^2, \ \ b(x) = (\cos x)/(3 - x)^3, \ f(x) = 4(3x^2 - 3x + 1)\left[(x - 1/2)^2 + 2\right];$$

$$y(0) = -1, \ \ y(1) = 0.$$

The exact solution for this problem is not available.

Example 3.4.2. Consider problem (3.1.1) with

$$a(x) = 1, \ \ b(x) = 1 + x(1 - x),$$

$$\begin{aligned} f(x) &= 1 + x(1 - x) + \left[2\sqrt{\varepsilon} - x^2(1 - x)\right] \exp\left[-(1 - x)/\sqrt{\varepsilon})\right] \\ &\quad + \left[2\sqrt{\varepsilon} - x(1 - x)^2\right] \exp\left[-x/\sqrt{\varepsilon}\right]. \end{aligned}$$

Its exact solution is given by

$$y(x) = 1 + (x - 1)\exp\left[-x/\sqrt{\varepsilon}\right] - x\exp\left[-(1 - x)/\sqrt{\varepsilon}\right].$$

Since the exact solution is available for Example 3.4.2, the maximum errors at all the mesh points are calculated using the formula

$$e_{\varepsilon,n} := \max_{0\le j\le n} |y(x_j) - \nu_j|, \quad \text{before extrapolation,}$$

and

$$e^{ext}_{\varepsilon,n} := \max_{0\le j\le n} |y(x_j) - \nu^{ext}_j|, \quad \text{after extrapolation,}$$

where ν_j is the solution of (3.1.1) obtained by using (2.2.4) and (2.2.3), and ν^{ext}_j is the solution after extrapolation.

For Example 3.4.1, the exact solution is not available and therefore we use the double mesh principle [33] to evaluate the maximum errors at all the mesh points:

$$e_{\varepsilon,n} := \max_{0\leq j\leq n} |\nu_j^n - \nu_{2j}^{2n}|, \quad \text{before extrapolation,}$$

and

$$e_{\varepsilon,n}^{ext} := \max_{0\leq j\leq n} |\nu_j^{ext} - \nu_{2j}^{ext}|, \quad \text{after extrapolation,}$$

where ν_{2j}^{2n} is the numerical solution of (3.1.1) obtained by using (2.2.4) and (2.2.3) on the mesh μ_{2n}. The numerical rates of convergence are computed using the formula [33]: $r_k \equiv r_{\varepsilon,k} := \log_2(\widetilde{e}_{n_k}/\widetilde{e}_{2n_k}),\ k = 1, 2, \ldots$ where $\widetilde{e}$ stands for $e_{\varepsilon,n}$ and $e_{\varepsilon,n}^{ext}$, respectively.

Furthermore, we compute $e_n := \max_{0<\varepsilon\leq 1} e_{\varepsilon,n}$ and $e_n^{ext} = \max_{0<\varepsilon\leq 1} e_{\varepsilon,n}^{ext}$ whereas the numerical rate of uniform convergence is computed as $r_n := \log_2(e_n/e_{2n})$ and $r_n^{ext} := \log_2(e_n^{ext}/e_{2n}^{ext})$. (Note that the negative entries in some of the tables for rates of convergence are due to the fact that the round-off errors propagate which can be seen from the corresponding entries in the error tables).

Table 3.1: Results for example 3.4.1 **before** extrapolation (maximum errors using FOFDM-I)

ε	n=20	n=40	n=80	n=160	n=320	n=640	n=1280
1.0e-01	3.62e-06	2.26e-07	1.41e-08	8.84e-10	5.59e-11	1.55e-12	8.33e-12
1.0e-02	3.79e-05	2.37e-06	1.48e-07	9.23e-09	5.84e-10	2.24e-11	9.42e-11
1.0e-04	1.29e-02	8.94e-04	5.78e-05	3.67e-06	2.30e-07	1.44e-08	8.86e-10
1.0e-06	2.67e-01	3.56e-02	1.89e-02	3.17e-03	2.25e-04	1.48e-05	9.35e-07
1.0e-08	2.99e-01	7.79e-02	1.99e-02	4.77e-03	7.15e-04	2.56e-03	6.54e-04
1.0e-10	2.99e-01	7.79e-02	1.99e-02	5.02e-03	1.26e-03	3.16e-04	7.80e-05
1.0e-11	2.99e-01	7.79e-02	1.99e-02	5.02e-03	1.26e-03	3.16e-04	7.92e-05
1.0e-12	2.99e-01	7.79e-02	1.99e-02	5.02e-03	1.26e-03	3.16e-04	7.92e-05
e_n	2.99e-01	7.79e-02	1.99e-02	5.02e-03	1.26e-03	3.16e-04	7.92e-05

Table 3.2: Results for example 3.4.1 **after** extrapolation (maximum errors using FOFDM-I)

ε	n=20	n=40	n=80	n=160	n=320	n=640	n=1280
1.0e-01	3.64e-10	5.86e-12	3.89e-13	7.31e-13	2.68e-12	8.81e-12	4.15e-11
1.0e-02	6.99e-09	1.10e-10	4.07e-12	7.99e-12	1.62e-11	9.91e-11	3.26e-10
1.0e-04	9.68e-05	1.64e-06	2.61e-08	4.10e-10	1.03e-11	3.68e-11	9.51e-11
1.0e-06	2.34e-02	2.91e-03	7.28e-04	2.91e-05	5.19e-07	8.49e-09	1.36e-10
1.0e-08	5.97e-02	1.56e-02	3.73e-03	4.97e-04	2.06e-04	1.38e-04	9.11e-06
1.0e-10	5.97e-02	1.56e-02	3.98e-03	1.00e-03	2.52e-04	6.21e-05	1.01e-05
1.0e-11	5.97e-02	1.56e-02	3.98e-03	1.00e-03	2.52e-04	6.33e-05	1.58e-05
1.0e-12	5.97e-02	1.56e-02	3.98e-03	1.00e-03	2.52e-04	6.33e-05	1.58e-05
e_n^{ext}	5.97e-02	1.56e-02	3.98e-03	1.00e-03	2.52e-04	6.33e-05	1.58e-05

Table 3.3: Results for example 3.4.1 **before** extrapolation (rates of convergence using FOFDM-I) $n_k = 20 \times 2^{k-1}$, $k = 1(1)5$

ε	r_1	r_2	r_3	r_4	r_5
1.0e-01	4.00e+00	4.00e+00	4.00e+00	3.98e+00	5.17e+00
1.0e-02	4.00e+00	4.00e+00	4.00e+00	3.98e+00	4.70e+00
1.0e-04	3.85e+00	3.95e+00	3.98e+00	4.00e+00	4.00e+00
1.0e-06	2.91e+00	9.18e-01	2.57e+00	3.81e+00	3.93e+00
1.0e-08	1.94e+00	1.97e+00	2.06e+00	2.74e+00	-1.84e+00
1.0e-10	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
1.0e-11	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
1.0e-12	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
r_n	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00

Table 3.4: Results for example 3.4.1 **after** extrapolation (rates of convergence using FOFDM-I) $n_k = 20 \times 2^{k-1}$, $k = 1(1)5$

ε	r_1	r_2	r_3	r_4	r_5
1.0e-01	5.96e+00	3.91e+00	-9.08e-01	-1.87e+00	-1.72e+00
1.0e-02	5.99e+00	4.75e+00	-9.71e-01	-1.02e+00	-2.62e+00
1.0e-04	5.89e+00	5.97e+00	5.99e+00	5.32e+00	-1.84e+00
1.0e-06	3.01e+00	2.00e+00	4.64e+00	5.81e+00	5.93e+00
1.0e-08	1.94e+00	2.06e+00	2.91e+00	1.27e+00	5.81e-01
1.0e-10	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.02e+00
1.0e-11	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
1.0e-12	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
r_n^{ext}	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00

Table 3.5: Results for example 3.4.2 **before** extrapolation (maximum errors using FOFDM-I)

ε	n=20	n=40	n=80	n=160	n=320	n=640	n=1280
1.0e-01	5.19e-07	3.27e-08	2.05e-09	1.28e-10	8.00e-12	5.38e-13	1.14e-12
1.0e-02	2.14e-05	1.43e-06	8.99e-08	5.63e-09	3.52e-10	2.21e-11	1.32e-12
1.0e-04	1.08e-03	6.00e-04	9.44e-05	6.53e-06	4.31e-07	2.71e-08	1.70e-09
1.0e-06	3.82e-04	9.94e-05	1.39e-04	1.52e-04	9.33e-05	1.98e-05	1.67e-06
1.0e-08	3.82e-04	9.94e-05	2.54e-05	1.01e-05	1.50e-05	1.62e-05	1.63e-05
1.0e-10	3.96e-04	1.00e-04	2.54e-05	6.43e-06	1.62e-06	1.26e-06	1.56e-06
1.0e-11	3.97e-04	1.01e-04	2.54e-05	6.43e-06	1.62e-06	4.06e-07	4.25e-07
1.0e-12	3.98e-04	1.02e-04	2.56e-05	6.43e-06	1.62e-06	4.06e-07	1.02e-07
e_n	3.98e-04	1.02e-04	2.57e-05	6.43e-06	1.62e-06	4.06e-07	1.02e-07

Table 3.6: Results for example 3.4.2 **after** extrapolation (maximum errors using FOFDM-I)

ε	n=20	n=40	n=80	n=160	n=320	n=640	n=1280
1.0e-01	3.86e-11	6.13e-13	1.61e-14	2.50e-14	1.12e-13	1.25e-12	7.47e-13
1.0e-02	1.61e-08	2.79e-10	4.41e-12	7.47e-14	6.35e-14	4.34e-13	6.75e-13
1.0e-04	5.56e-05	3.11e-06	3.51e-07	7.25e-09	1.25e-10	2.43e-12	2.37e-12
1.0e-06	9.02e-05	3.12e-05	1.58e-05	8.13e-06	8.54e-08	9.81e-08	2.78e-09
1.0e-08	8.06e-05	2.14e-05	6.24e-06	2.40e-06	1.43e-06	1.20e-06	1.06e-06
1.0e-10	7.97e-05	2.04e-05	5.25e-06	1.40e-06	4.35e-07	1.92e-07	1.31e-07
1.0e-11	7.96e-05	2.04e-05	5.18e-06	1.33e-06	3.60e-07	1.16e-07	5.54e-08
1.0e-12	7.96e-05	2.03e-05	5.16e-06	1.31e-06	3.36e-07	9.23e-08	3.14e-08
e_n^{ext}	7.96e-05	2.03e-05	5.15e-06	1.30e-06	3.28e-07	8.48e-08	7.41e-08

Table 3.7: Results for example 3.4.2 **before** extrapolation (rates of convergence using FOFDM-I) $n_k = 20 \times 2^{k-1}$, $k = 1(1)5$

ε	r_1	r_2	r_3	r_4	r_5
1.0e-01	3.99e+00	4.00e+00	4.00e+00	4.00e+00	3.89e+00
1.0e-02	3.90e+00	4.00e+00	4.00e+00	4.00e+00	4.00e+00
1.0e-04	8.49e-01	2.67e+00	3.85e+00	3.92e+00	3.99e+00
1.0e-06	1.94e+00	-4.82e-01	-1.33e-01	7.06e-01	2.23e+00
1.0e-08	1.94e+00	1.97e+00	1.33e+00	-5.71e-01	-1.15e-01
1.0e-10	1.99e+00	1.98e+00	1.98e+00	1.99e+00	3.63e-01
1.0e-11	1.97e+00	1.99e+00	1.98e+00	1.99e+00	2.00e+00
1.0e-12	1.97e+00	1.99e+00	1.99e+00	1.99e+00	2.00e+00
r_n	1.97e+00	1.99e+00	2.00e+00	1.99e+00	2.00e+00

Table 3.8: Results for example 3.4.2 **after** extrapolation (rates of convergence using FOFDM-I) $n_k = 20 \times 2^{k-1}$, $k = 1(1)5$

ε	r_1	r_2	r_3	r_4	r_5
1.0e-01	5.98e+00	5.25e+00	-6.34e-01	-2.16e+00	-3.48e+00
1.0e-02	5.85e+00	5.98e+00	5.88e+00	2.35e-01	-2.77e+00
1.0e-04	4.16e+00	3.15e+00	5.60e+00	5.86e+00	5.69e+00
1.0e-06	1.53e+00	9.84e-01	9.55e-01	6.57e+00	-1.99e-01
1.0e-08	1.91e+00	1.78e+00	1.38e+00	7.44e-01	2.55e-01
1.0e-10	1.96e+00	1.96e+00	1.90e+00	1.69e+00	1.18e+00
1.0e-11	1.97e+00	1.98e+00	1.96e+00	1.89e+00	1.63e+00
1.0e-12	1.97e+00	1.98e+00	1.98e+00	1.96e+00	1.86e+00
r_n^{ext}	1.97e+00	1.98e+00	1.99e+00	1.98e+00	1.95e+00

Table 3.9: Results for example 3.4.2 **before** extrapolation (maximum errors using FOFDM-II)

ε	n=16	n=32	n=64	n=128	n=256	n=512	
1.0e-01	2.66E-03	6.55E-04	1.63E-04	4.07E-05	1.02E-05	2.54E-06	6.36E-07
1.0e-02	8.15E-03	2.02E-03	5.04E-04	1.26E-04	3.15E-05	7.87E-06	1.97E-06
1.0e-04	9.48E-03	2.58E-03	8.41E-04	1.86E-04	7.21E-05	2.06E-05	5.49E-06
1.0e-06	9.48E-03	2.47E-03	6.32E-04	1.60E-04	4.69E-05	4.70E-05	1.10E-05
1.0e-08	9.48E-03	2.47E-03	6.32E-04	1.60E-04	4.04E-05	1.01E-05	2.54E-06
1.0e-10	9.48E-03	2.47E-03	6.32E-04	1.60E-04	4.04E-05	1.01E-05	2.54E-06
1.0e-11	9.48E-03	2.47E-03	6.32E-04	1.60E-04	4.04E-05	1.01E-05	2.54E-06
1.0e-12	9.48E-03	2.47E-03	6.32E-04	1.60E-04	4.04E-05	1.01E-05	2.54E-06
e_n	9.48E-03	2.47E-03	6.32E-04	1.60E-04	4.04E-05	1.01E-05	2.54E-06

Table 3.10: Results for example 3.4.2 **after** extrapolation (maximum errors using FOFDM-II)

ε	n=16	n=32	n=64	n=128	n=256	n=512	
1.0e-01	1.31E-05	8.39E-07	5.28E-08	3.30E-09	2.07E-10	1.33E-11	3.17E-12
1.0e-02	1.83E-04	1.53E-05	1.03E-06	6.58E-08	4.13E-09	2.58E-10	1.63E-11
1.0e-04	2.17E-05	2.47E-05	4.34E-05	3.21E-05	3.48E-06	2.35E-07	1.50E-08
1.0e-06	2.23E-05	1.52E-06	9.99E-08	6.24E-09	1.81E-06	9.90E-07	4.93E-06
1.0e-08	2.23E-05	1.52E-06	9.99E-08	6.42E-09	4.08E-10	2.57E-11	2.44E-11
1.0e-10	2.23E-05	1.52E-06	9.99E-08	6.42E-09	4.08E-10	2.57E-11	1.61E-12
1.0e-11	2.23E-05	1.52E-06	9.99E-08	6.42E-09	4.08E-10	2.57E-11	1.61E-12
1.0e-12	2.23E-05	1.52E-06	9.99E-08	6.42E-09	4.08E-10	2.57E-11	1.61E-12
e_n^{ext}	2.23E-05	1.52E-06	9.99E-08	6.42E-09	4.08E-10	2.57E-11	1.61E-12

Table 3.11: Results for example 3.4.2 **before** extrapolation (rate of convergence using FOFDM-II), $n_k = 8 \times 2^{k-1}, k = 1(1)6$

ε	r_2	r_3	r_4	r_5	r_6	
1.0e-01	2.02E+00	2.01E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
1.0e-02	2.01E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
1.0e-04	1.88E+00	1.62E+00	2.17E+00	1.37E+00	1.80E+00	1.91E+00
1.0e-06	1.94E+00	1.96E+00	1.98E+00	1.77E+00	-3.53E-03	2.09E+00
1.0e-08	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00
1.0e-10	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00
1.0e-11	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00
1.0e-12	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00
r_n	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00

Table 3.12: Results for example 3.4.2 **after** extrapolation (rate of convergence using FOFDM-II), $n_k = 8 \times 2^{k-1}, k = 1(1)6$

ε	r_2	r_3	r_4	r_5	r_6	
1.0e-01	3.96E+00	3.99E+00	4.00E+00	4.00E+00	3.96E+00	2.06E+00
1.0e-02	3.58E+00	3.89E+00	3.96E+00	3.99E+00	4.00E+00	3.98E+00
1.0e-04	-1.86E-01	-8.14E-01	4.38E-01	3.20E+00	3.89E+00	3.97E+00
1.0e-06	3.88E+00	3.93E+00	4.00E+00	-8.18E+00	8.74E-01	-2.32E+00
1.0e-08	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	7.26E-02
1.0e-10	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	3.99E+00
1.0e-11	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	3.99E+00
1.0e-12	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	3.99E+00
r_n^{ext}	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	3.99E+00

3.5 Discussion

In this Chapter, we have investigated the performance of the Richardson extrapolation on some fitted operator finite difference methods. We considered two FOFDMs referred to as FOFDM-I and FOFDM-II which were designed to solve a class of self-adjoint problems in [119] and [98], respectively. These methods are analyzed for convergence (where the solution before and after extrapolation is used to derive the error estimates).

Richardson extrapolation does not improve the convergence of FOFDM-I which is of order four and two for some moderate and smaller values of ε respectively. In the case of FOFDM-II, its second order accuracy is improved up to four, irrespective of the value of ε. The observations (made through tables 3.1-3.12) and the associated analysis show that the performance of Richardson extrapolation is scheme dependent.

Chapter 4

Performance of Richardson Extrapolation on Various Numerical Methods for a Singularly Perturbed Turning Point Problem whose Solution has Boundary Layers

In this chapter, we consider singularly perturbed turning point problems. There exist two classes of such type of problems: the one whose solution possesses boundary layer(s) and the one whose solution possesses interior layer(s). After we design some fitted methods, the performance of the Richardson extrapolation is studied here for the problems of the former class. The same for the later class of problems is being considered elsewhere.

4.1 Introduction

In this Chapter we develop a fitted operator finite difference method (FOFDM) and a fitted mesh finite difference method (FMFDM) to solve a singularly perturbed turning point problem (TPP) whose solution displays boundary layers. Since we aim at achieving high order of convergence, we investigate the performance of Richardson extrapolation on these methods.

Several authors have attempted to solve singularly perturbed TPPs, but up to the best of our knowledge, the acceleration techniques such as the one above have not yet been explored.

Abrahamsson [4] and Berger et al. [21] derived a number of *a priori* estimates for solutions of singularly perturbed TPPs. Adzic in [5], [6] and [8] developed modified standard spectral methods for singularly perturbed problems without turning points, with turning point with boundary layers and with turning point with interior layer, respectively. The same author used a domain decomposition method (in [7]) to solve some turning point problems via the asymptotic behavior of the exact solution.

We consider the problem

$$Lu := \varepsilon u'' + a(x)u' - b(x)u = f(x), x \in \Omega = (-1, 1), \tag{4.1.1}$$

$$u(-1) = A, \;\; u(1) = B. \tag{4.1.2}$$

where A and B are given constants and $\varepsilon \in (0, 1]$, and the coefficients $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions in $\bar{\Omega}$.

The distinct zeros $\alpha_i, \;\; i = 1, 2, \cdots, r$ of $a(x)$ in the interval $\bar{\Omega}$, if they exist, are called the turning points of (4.1.1)-(4.1.2), provided that $a(-1)a(1) \neq 0$.

Berger et al. [21] showed that the bounds of the solution $u(x)$ near a given turning point α_i depend on ε and the constant $\beta_i = b(\alpha_i)/a'(\alpha_i)$. For $\beta_i < 0$, $u(x)$ is "smooth" near $x = \alpha_i$ whereas $\beta_i > 0$ indicates that $u(x)$ has a large gradient at $x = \alpha_i$ resulting in an "interior layer". Moreover, $u(x)$ has a boundary layer at $x = -1$ and $x = 1$ if and only if $a(-1) > 0$ and $a(1) < 0$, respectively.

In the rest of this chapter, we assume that

$$a(0) = 0, \;\; a'(0) \leq 0, \;\; a(-1) > 0, \;\; a(1) < 0, \;\; \text{and} \;\; |a(x)| \geq a_0 > 0, \;\; \text{for} \;\; 0 < |x| \leq 1,$$

thus ensuring that the solution to (4.1.1)-(4.1.2) has two boundary layers. Also it is required that $b(x) \geq b_0 > 0$ so as to ensure that the solution of (4.1.1)-(4.1.2) satisfies a minimum principle. The condition $|a'(x)| \geq |a'(0)/2|, \;\; -1 \leq x \leq 1$ guarantees the uniqueness of the turning point in the interval [-1,1].

Under the requirements mentioned above, the operator L admits the following continuous minimum principle

Lemma 4.1.1. *Let ξ be a smooth function satisfying $\xi(-1) \geq 0$, $\xi(1) \geq 0$ and $L\xi(x) \leq 0, \;\; \forall x \in (-1, 1)$. Then $\xi(x) \geq 0, \;\; \forall x \in [-1, 1]$.*

Proof Let $x^* \in [-1, 1]$ such that $\xi(x^*) = \min_{x \in [-1,1]} \xi(x)$ and assume $\xi(x^*) < 0$. Then, obviously, $x^* \notin \{-1, 1\}, \;\; \xi'(x^*) = 0$ and $\xi''(x^*) \geq 0$. We have

$$L\xi(x^*) = \varepsilon\xi''(x^*) + a(x^*)\xi'(x^*) - b(x^*)\xi(x^*) > 0,$$

which is a contradiction. It follows that, $\xi(x^*) \geq 0$ and thus, $\xi(x) \geq 0, \;\; \forall x \in [-1, 1]$.

The minimum principle implies the existence and unicity of the solution. We use this principle to prove the following results which states that the solution depends continuously on the data.

Lemma 4.1.2. *Let $u(x)$ be the solution of (4.1.1)-(4.1.2). Then, we have*

$$||u|| \leq C \left(b_0^{-1} ||f|| + \max\{|A|, |B|\} \right), \forall x \in [-1, 1].$$

Proof Consider the comparison function

$$\Pi^{\pm}(x) = b_0^{-1}||f|| + \max\{|A|, |B|\} \pm u(x).$$

Then we have

$$L\Pi^{\pm}(x) = \pm f(x) - \frac{b(x)}{b_0}||f|| - b(x)\max\{|A|, |B|\} \leq 0.$$

implying that $\Pi^{\pm}(x) \geq 0, \ \ \forall x \in [-1, 1]$, which completes the proof.

The rest of this chapter is organized as follows. In section 4.2, we state some *a priori* estimates of the bounds of the solution and its derivatives, the use of which will be apparent in the analysis of the numerical methods. The construction and analysis of FOFDM and FMFDM are presented in sections 4.3 and 4.4. In these sections, the performance of extrapolation on the underlying methods is studied. Numerical results to support our theoretical findings are displayed in section 4.5. A short discussion on these results is provided in section 4.6.

4.2 Some *a priori* estimates for the bounds of the solution and its derivatives

In this section, we present the bounds on the solution of the problem (4.1.1)-(4.1.2) and its derivatives.

We shall denote by $\Omega_l = [-1, -\delta], \ \ \Omega_c = [-\delta, \delta], \ \ \Omega_r = [\delta, 1], \ \$ where $\ 0 < \delta \leq \frac{1}{2}$; the left, central and right part of the domain, respectively. Note that $\beta = b(0)/a'(0) < 0$.

Let k be a positive integer. We define

$$S_1(k) = \{||a||_k, ||b||_k, ||f||_k, a_0, 1 - \delta, |B|, u(\delta), k\},$$

$$S_2(k) = \{||a||_k, ||b||_k, ||f||_k, a_0, 1-\delta, |A|, u(-\delta), k\}$$

and

$$S_3(k) = \{||a||_k, ||b||_k, ||f||_k, \beta_s, b_0, |A|, |B|, k\}.$$

Depending on whether x belongs to Ω_l, Ω_c or Ω_r, the appropriate bounds are provided in the following lemmas.

Lemma 4.2.1. [21] *If $u(x)$ is the solution of the TPP (4.1.1)-(4.1.2) and a, b and $f \in C^k(\bar{\Omega}), k > 0$, then there exist positive constants η and C depending only on $S_1(k)$ such that*

$$|u^{(j)}(x)| \leq C[1 + \varepsilon^{-j} \exp(-a_0(1-x)/\varepsilon)], j = 1(1)k+1, \; x \in \Omega_r.$$

Proof. See [21].

Lemma 4.2.2. [21] *If $u(x)$ is the solution of the TPP (4.1.1)-(4.1.2) and a, b and $f \in C^k(\bar{\Omega}), k > 0$, then there exist positive constants η and C depending only on $S_2(k)$ such that*

$$|u^{(j)}(x)| \leq C[1 + \varepsilon^{-j} \exp(-a_0(1+x)/\varepsilon)], j = 1(1)k+1, \; x \in \Omega_l.$$

Proof. See [21].

Lemma 4.2.3. [21] *If $u(x)$ is the solution of the TPP (4.1.1)-(4.1.2) and a, b and $f \in C^k(\bar{\Omega}), k > 0$, then there exists a positive constant C depending only on $S_3(k)$ such that*

$$|u^{(j)}| \leq C, \;\; \forall x \in \Omega_c, \;\; j = 0(1)k.$$

Proof. See [21].

Lemma 4.2.4. [105] *The solution u of the TPP (4.1.1)-(4.1.2) can be decomposed as*

$$u = v + w,$$

where, for all j, $0 \le j \le k$, and all $x \in [-1, 1]$, the smooth component v satisfies

$$\left|v^{(j)}(x)\right| \le C(1 + \varepsilon^{-(k-2)} \exp(-a_0(1 + x)/\varepsilon), \quad x \in [-1, 0],$$

$$\left|v^{(j)}(x)\right| \le C(1 + \varepsilon^{-(k-2)} \exp(-a_0(1 - x)/\varepsilon), \quad x \in [0, 1],$$

and the singular component w satisfies

$$\left|w^{(j)}(x)\right| \le C\varepsilon^{-k} \exp(-a_0(1 + x)/\varepsilon), \quad x \in [-1, 0],$$

$$\left|w^{(j)}(x)\right| \le C\varepsilon^{-k} \exp(-a_0(1 - x)/\varepsilon), \quad x \in [0, 1],$$

for some constant C independent of ε.

Proof. See [105].

4.3 Richardson extrapolation on fitted operator finite difference method

Here we first present the FOFDM which is developed in [110] and the associated error estimates. Then we analyze the effect of Richardson extrapolation on this scheme.

4.3.1 The fitted operator finite difference method (FOFDM)

Let n be any positive integer. Consider the following partition of the interval $[-1, 1]$: $\mu_n = \{x_j = x_0 + jh, \;\; x_0 = -1, \;\; x_n = 1, \;\; j = 1(1)n, \;\; h = x_j - x_{j-1}\}$.

The denominator function ϕ_j^2 appearing in the discrete form of the approximation of the second derivative term of the differential equation (4.1.1) is considered as

$$\phi_j^2 = \begin{cases} \frac{h\varepsilon}{a_j}\left(\exp\left(\frac{a_j h}{\varepsilon}\right) - 1\right), & j = 0(1)\frac{n}{2} - 1, \\ \frac{h\varepsilon}{a_j}\left(1 - \exp\left(-\frac{a_j h}{\varepsilon}\right)\right), & j = \frac{n}{2} + 1(1)n, \\ h^2, & j = \frac{n}{2}. \end{cases} \tag{4.3.3}$$

The above is normally obtained by using the theory of difference equations (see, e.g., [103]).

Hence, the problem (4.1.1)-(4.1.2) is discretized as follows

$$L^h U_j \equiv \begin{cases} \varepsilon\frac{U_{j+1} - 2U_j + U_{j-1}}{\widetilde{\phi}_j^2} + \widetilde{a}_j\frac{U_{j+1} - U_j}{h} - \widetilde{b}_j U_j = \widetilde{f}_j, & j = 1(1)\frac{n}{2} - 1, \\ \varepsilon\frac{U_{j+1} - 2U_j + U_{j-1}}{\widetilde{\phi}_j^2} + \widetilde{a}_j\frac{U_j - U_{j-1}}{h} - \widetilde{b}_j U_j = \widetilde{f}_j, & j = \frac{n}{2}(1)n - 1, \end{cases} \tag{4.3.4}$$

$$U_0 = A, \quad U_n = B. \tag{4.3.5}$$

where

$$\widetilde{a}_j = \frac{a_j + a_{j+1}}{2},$$

$$\widetilde{b}_j = \frac{b_{j-1} + b_j + b_{j+1}}{3},$$

$$\widetilde{f}_j = \frac{f_{j-1} + f_j + f_{j+1}}{3},$$

and $\widetilde{\phi}_j$ is obtained as in (4.3.3) by substituting a_j by $\widetilde{a}_j$.

Equations (4.3.4) can be written in the form

$$r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = \widetilde{f}_j, \quad j = 1(1)n - 1. \tag{4.3.6}$$

where

$$r_j^+ = \frac{\varepsilon}{\widetilde{\phi}_j^2} + \frac{\widetilde{a}_j}{h}, \;\; r_j^c = -\frac{2\varepsilon}{\widetilde{\phi}_j^2} - \frac{\widetilde{a}_j}{h} - \widetilde{b}, \;\; r_j^- = \frac{\varepsilon}{\widetilde{\phi}_j^2}, \text{ for } j = 1, 2, \cdots, \frac{n}{2} - 1,$$

and

$$r_j^+ = \frac{\varepsilon}{\widetilde{\phi}_j^2}, \;\; r_j^c = -\frac{2\varepsilon}{\widetilde{\phi}_j^2} + \frac{\widetilde{a}_j}{h} - \widetilde{b}, \;\; r_j^- = \frac{\varepsilon}{\widetilde{\phi}_j^2} - \frac{\widetilde{a}_j}{h}, \text{ for } j = \frac{n}{2}, \frac{n}{2} + 1, \cdots, n - 1.$$

In view of the scheme above, we now prove the following Lemma which states that the discrete problem $L^h U_j = f_j, \;\; 1 \leq j \leq n-1, \;\; U_0 = A, \;\; U_n = B$, satisfies the discrete minimum principle.

Lemma 4.3.1. *For any mesh function ξ_j such that $L^h \xi_j \leq 0, \; \forall j = 1(1)n-1, \; \xi_0 \geq 0$ and $\xi_n \geq 0$, we have $\xi_j \geq 0, \;\; \forall j = 0(1)n$.*

Proof

Let k be such that $\xi_k = \min_{0 \leq j \leq n} \xi_j$ and suppose that $\xi_k < 0$. It's clear that $k \notin \{0, n\}$. Also $\xi_{k+1} - \xi_k \geq 0, \;\; \xi_k - \xi_{k-1} \leq 0$.

On one hand we have

$$L^h \xi_k = \varepsilon \frac{\xi_{k+1} - 2\xi_k + \xi_{k-1}}{\phi_k^2} + a_k \frac{\xi_{k+1} - \xi_k}{h} - b_k \xi_k > 0, \;\; \text{for } \; 1 \leq k \leq n/2 - 1.$$

On the other hand

$$L^h \xi_k = \varepsilon \frac{\xi_{k+1} - 2\xi_k + \xi_{k-1}}{\phi_k^2} + a_k \frac{\xi_k - \xi_{k-1}}{h} - b_k \xi_k > 0, \;\; \text{for } \; n/2 \leq k \leq n - 1.$$

Thus $L^h \xi_k > 0, \;\; 1 \leq k \leq n-1$, which is a contradiction. It follows that $\xi_k \geq 0$ and thus $\xi_j \geq 0, \; 0 \leq j \leq n$.

This minimum principle is used to prove the following lemma.

Lemma 4.3.2. *If Z_i is any mesh function such that $Z_0 = Z_n = 0$, then*

$$|Z_i| \leq \frac{1}{a^*} \max_{1\leq j\leq n-1} |L^h Z_j| \;\; \textit{for} \;\; 0 \leq i \leq n.$$

where

$$a^* = \begin{cases} -a_0 & \textit{if} \;\; 0 \leq i \leq n/2-1, \\ a_0 & \textit{if} \;\; n/2 \leq i \leq n. \end{cases}$$

Proof Let us define two comparison functions $Y_i^{\pm}$ by

$$Y_i^{\pm} = \frac{x_i}{a^*} \max_{1\leq j\leq n-1} |L^h Z_j| \pm Z_i, \;\; 0 \leq i \leq n.$$

It is clear that $Y_0^{\pm} \geq 0$ and $Y_n^{\pm} \geq 0$. Also, observe that

$$L^h Y_i^{\pm} = \frac{a_i - b_i x_i}{a^*} \max_{1\leq j\leq n-1} |L^h Z_j| + L^h Z_i, \;\; 0 \leq i \leq n.$$

If $0 \leq i \leq n/2-1$, then $a_i > 0, a_i > a_0$ and since $b_i > 0$ and $x_i < 0$, we have $(a_i - b_i x_i)/(-a_0)) < -1$. Likewise, if $n/2 \leq i \leq n$, then $a_i < 0, |a_i| > a_0$ and since $b_i > 0$ and $x_i > 0$, we have $(a_i - b_i x_i)/a_0 < -1$. In either case, $L^h Y_i^{\pm} \leq 0$. By the discrete minimum principle (Lemma 4.3.1), we conclude that $Y_i \geq 0, \;\; \forall 0 \leq i \leq n$ and this completes the proof.

We will be requiring the following lemma in the analysis below.

Lemma 4.3.3. *For a fixed mesh and for all integers k, we have*

$$\lim_{\varepsilon\to 0} \max_{1\leq j\leq n/2-1} \frac{\exp(-M(1+x_j)/\varepsilon)}{\varepsilon^k} = 0$$

and

$$\lim_{\varepsilon\to 0} \max_{n/2\leq j\leq n-1} \frac{\exp(-M(1-x_j)/\varepsilon)}{\varepsilon^k} = 0,$$

where $x_j = jh - 1, \;\; h = 2/n, \;\; \forall j = 1(1)n - 1.$

Proof

We extend the proof provided in [118] to cater for the turning point problems here. In fact, to prove the second limit above we consider the partition $[0, 1] := \{0 = x_{\frac{n}{2}} < x_{\frac{n}{2}+1} < \cdots < x_{n-1} < x_n = 1\}$. The first limit is established by replacing x_j by $-x_j$ in the second limit. In this case, we use the partition $[-1, 0] := \{-1 = x_0 < x_1 < x_2 < \cdots < x_{\frac{n}{2}-1} < x_{\frac{n}{2}} = 0\}$.

Now, the truncation error of our method is calculated as follows.
For $j = 1(1)n/2 - 1$ we have

$$\begin{aligned}\widetilde{L}^h(u_j - U_j) &= \left(r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}\right) - \widetilde{f}_j \\ &= T_0 u_j + T_1 u_j' + T_2 u_j'' + T_3 u_j''' + T_4 u^{(iv)}(\xi_j),\end{aligned} \tag{4.3.7}$$

where $\xi_j \in (x_{j-1}, x_{j+1})$ and

$$\begin{aligned}T_0 &= r_j^- + r_j^c + r_j^+ + b_j + \frac{1}{3}h^2 b_j'', \\ T_1 &= h\left(r_j^+ - r_j^-\right) - a_j - \frac{1}{3}h^2\left(a_j'' - 2b_j'\right), \\ T_2 &= \frac{h^2}{2}\left(r_j^+ + r_j^-\right) - \varepsilon - \frac{1}{3}h^2\left(2a_j' - b_j - \frac{h^2}{2}b_j''\right), \\ T_3 &= \frac{h^3}{6}\left(r_j^+ - r_j^-\right) - \frac{1}{3}h^2\left(a_j + \frac{h^2}{2}a_j''\right), \\ T_4 &= \frac{h^4}{24}\left(r_j^+ + r_j^-\right) - \frac{1}{3}h^2\varepsilon.\end{aligned}$$

We note that $T_0 = 0, \;\; |T_1| \le Mh, \;\; |T_2| \le Mh, \;\; |T_3| \le Mh^2$, and $|T_4| \le Mh^2$. (Hereinafter, M denotes a positive constant which may take different values in different equations and inequalities but is always independent of h and ε.) Therefore, (4.3.7) leads to

$$|\widetilde{L}^h(u_j - U_j)| \le Mh\left[1 + \frac{\exp\left(-a_0(1 + x_j)/\varepsilon\right)}{\varepsilon}\right], \tag{4.3.8}$$

where we have used Lemma 4.2.2 and considered only dominating terms.

Following the same procedure, and using both lemmas 4.2.2 and 4.2.3, we establish that for $j = n/2(1)n-1$,

$$|\widetilde{L}^h(u_j - U_j)| \leq Mh\left[1 + \frac{\exp(-a_0(1-x_j)/\varepsilon)}{\varepsilon}\right]. \tag{4.3.9}$$

Finally, using lemmas 4.3.2 and 4.3.3, we have the following result.

Theorem 4.3.1. *Let $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth functions in the problem (4.1.1)-(4.1.2) and so that $u(x) \in C^4([-1,1])$. The numerical solution U obtained via the FOFDM (4.3.6)-(4.3.5) satisfy the following estimate:*

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq j\leq n} |u_j - U_j| \leq Mh. \tag{4.3.10}$$

4.3.2 Richardson extrapolation for FOFDM

Let us denote by μ_{2n} the mesh obtained by bisecting each mesh interval in μ_n, i.e,

$$\mu_{2n} = \{\bar{x}_j\} \text{ with } \bar{x}_0 = -1, \ \bar{x}_n = 1 \text{ and } \bar{x}_j - \bar{x}_{j-1} = \bar{h} = h/2, \ \ j = 1(1)2n.$$

We denote the analytical and numerical solutions on the mesh μ_{2n} by $\bar{u}_j$ and $\bar{U}_j$, respectively.

From estimate (4.3.10), we have on one hand

$$u_j - U_j = Mh + R_n(x_j), \ \ 1 \leq j \leq n-1.$$

On the other hand, we have

$$\bar{u}_j - \bar{U}_j = M\bar{h} + R_{2n}(\bar{x}_j), \ \ 1 \leq j \leq 2n-1.$$

Therefore,

$$u_j - (2\bar{U}_j - U_j) = O(h), \ \ \forall 1 \leq j \leq n-1.$$

Let

$$U_j^{ext} := 2\bar{U}_j - U_j.$$

Thus U^{ext} is another numerical approximation of u_j.

Using some algebraic manipulations, we established that

$$\left|L^h(u_j - U_j)\right| \leq Mh.$$

Therefore,

$$\left|L^{\bar{h}}(u_j - \bar{U}_j)\right| \leq Mh$$

and consequently

$$\max_{1\leq j\leq n-1} |u_j - U_j| \leq Mh$$

and

$$\max_{1\leq j\leq n-1} \left|u_j - \bar{U}_j\right| \leq Mh.$$

Finally, the inequality

$$\max_{1\leq j\leq n-1} \left|u_j - U_j^{ext}\right| \leq 2 \max_{1\leq j\leq n-1} \left|u_j - \bar{U}_j\right| + \max_{1\leq j\leq n-1} |u_j - U_j| \tag{4.3.11}$$

leads to

Theorem 4.3.2. *Let $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth functions in the problem (4.1.1)-(4.1.2) and so that $u(x) \in C^4([-1,1])$. Then the numerical solution U^{ext} obtained via Richardson extrapolation based on FOFDM (4.3.6) along with (4.3.5) satisfies the following estimate:*

$$\sup_{0<\varepsilon\leq 1} \max_{1\leq j\leq n-1} |u_j - U_j^{ext}| \leq Mh. \tag{4.3.12}$$

4.4 Richardson extrapolation on fitted mesh finite difference method

The idea from Chapter 8 of Miller et al. [105] is used in this section to develop a fitted mesh finite difference scheme. The convergence of the scheme is analyzed before embarking on the study of the effect of Richardson extrapolation on its accuracy and rate of convergence.

4.4.1 The fitted mesh finite difference method (FMFDM)

It is assumed that there are two boundary layers, one at each end, and let the interval $[-1, 1]$ be partitioned as

$$[-1,1] := [-1,-1+\tau] \cup [-1+\tau, 1-\tau] \cup [1-\tau, 1],$$

where τ is a parameter denoting the width of the boundary layer.

Let n be a positive integer such that $n = 2^m$ with $m \geq 3$.

To construct the piece-wise mesh (of Shishkin type), we subdivide both the intervals $[-1, -1+\tau]$ and $[1-\tau, 1]$ into $n/4$ equal mesh elements while we subdivide the interval $[-1+\tau, 1-\tau]$ into $n/2$ equal mesh elements. This gives

$$[-1,1] := -1 = x_0 < x_1 < \cdots < x_{n/4} < \cdots < x_{n/2} = 0 < \cdots < x_{3n/4} < \cdots < x_n = 1.$$

The parameter τ is defined by

$$\tau = \min\left\{\frac{1}{4}, \frac{\varepsilon}{a_0}\ln\left(\frac{n}{4}\right)\right\}. \tag{4.4.13}$$

The mesh spacing $h_j = x_j - x_{j-1}$ is given by

$$h_j = \begin{cases} 4\tau n^{-1} & j = 1, 2, \cdots, n/2, 3n/4 + 1, 3n/4 + 2, \cdots, n-1, n \\ 4(1-\tau)n^{-1} & j = n/2 + 1, n/2 + 2, \cdots, 3n/4. \end{cases} \quad (4.4.14)$$

We denote this mesh by $\mu_{n,\tau}$.

Using the above conventions, we discretize the problem (4.1.1)-(4.1.2) on $\mu_{n,\tau}$ as

$$\begin{cases} \varepsilon \widetilde{D} U_j + a_j D^+ U_j - b_j U_j = f_j, & a_j > 0; \\ \varepsilon \widetilde{D} U_j + a_j D^- U_j - b_j U_j = f_j, & a_j \leq 0; \end{cases} \quad (4.4.15)$$

$$U_0 = A, \;\; U_n = B, \quad (4.4.16)$$

where

$$D^+ U_j = \frac{U_{j+1} - U_j}{h_{j+1}},$$

$$D^- U_j = \frac{U_j - U_{j-1}}{h_j},$$

and

$$\widetilde{D} U_j = \frac{2}{h_j + h_{j+1}} \left(D^+ U_j - D^- U_j \right).$$

Equations (4.4.15) can be written in the form

$$r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = f_j, \;\; j = 1(1)n-1, \quad (4.4.17)$$

where, for $j = 1, 2, \cdots, \frac{n}{2} - 1$, we have

$$r_j^+ = \frac{2\varepsilon}{h_{j+1}(h_{j+1} + h_j)} + \frac{a_j}{h_{j+1}},$$

$$r_j^c = -\frac{2\varepsilon}{h_j h_{j+1}} - \frac{a_j}{h_{j+1}} - b_j,$$

$$r_j^- = \frac{2\varepsilon}{h_j(h_j + h_{j+1})},$$

and for $j = \frac{n}{2}, \frac{n}{2} + 1, \cdots, n-1$, we have

$$r_j^+ = \frac{2\varepsilon}{h_{j+1}(h_{j+1} + h_j)},$$

$$r_j^c = -\frac{2\varepsilon}{h_j h_{j+1}} + \frac{a_j}{h_j} - b_j,$$

$$r_j^- = \frac{2\varepsilon}{h_j(h_j + h_{j+1})} - \frac{a_j}{h_j}.$$

Convergence analysis of FMFDM

The restrictions of problem (4.1.1)-(4.1.2) to the intervals $[0, 1]$ and $[-1, 0]$ feature like the convection-diffusion problem of Chapter 8 in [105]. In our analysis, we will implement the ideas provided in this work, for the interval $[0, 1]$. The analysis on $[-1, 0]$ follows similar steps.

We decompose the solution U of the discrete problem (4.4.15)-(4.4.16) in its regular part V and singular part W. The components V and W of U are solutions of the problems

$$L^n V = f, \;\; V(-1) = v(-1), V(1) = v(1)$$

and

$$L^n W = 0, \;\; W(-1) = w(-1), W(1) = w(1),$$

respectively, where L^n denotes the discrete operator associated with (4.4.17).

We can write the error in the form

$$U - u = (V - v) + (W - w) \tag{4.4.18}$$

and estimate the components of the error separately.

We start with the regular component.

The local truncation error is given by

$$L^n(V - v) = \varepsilon \left(\frac{d^2}{dx^2} - \widetilde{D} \right) v + a \left(\frac{d}{dx} - D^- \right) v. \tag{4.4.19}$$

Using Lemma 4.1 ([105]), we obtain

$$|L^n(V_j - v_j)| \leq \frac{\varepsilon}{3} (x_{j+1} - x_{j-1}) |v_j'''| + \frac{a_j}{2} (x_j - x_{j-1}) |v_j''|, \text{ for } \frac{n}{2} \leq j \leq n - 1. \tag{4.4.20}$$

Since $h_j = x_j - x_{j-1} \leq 4n^{-1}$ for any j, therefore using lemma 4.2.4,we obtain

$$|L^n(V_j - v_j)| \leq Mn^{-1}. \tag{4.4.21}$$

Hence, by Lemma 4.3.2,

$$|V_j - v_j| \leq Mn^{-1}. \tag{4.4.22}$$

The estimate on $L^n(W - w)$ depends on whether $\tau = 1/4$ or $\tau = (\varepsilon/a_0) \ln(n/4)$.

If $\tau = 1/4$, the mesh is uniform and $1/4 \leq (\varepsilon/a_0) \ln(n/4)$. The local truncation error $L^n(W - w)$ is given by

$$|L^n(W_j - w_j)| \leq \frac{\varepsilon}{3}(x_{j+1} - x_{j-1})|w_j'''| + \frac{a_j}{2}(x_j - x_{j-1})|w_j''|, \text{ for } \frac{n}{2} \leq j \leq n - 1. \tag{4.4.23}$$

By Lemma 4.2.4 and the fact that $h_j = x_j - x_{j-1} = 4n^{-1}$, the above inequality gives

$$|L^n(W_j - w_j)| \leq M\varepsilon^{-2}n^{-1}. \tag{4.4.24}$$

Now since ε^{-1} is less than $(4/a_0)\ln(n/4)$, we have

$$|L^n(W_j - w_j)| \leq Mn^{-1}(\ln(n/4))^2 \tag{4.4.25}$$

Using Lemma 4.3.2 then we obtain

$$|W_j - w_j| \leq Mn^{-1}(\ln(n/4))^2. \tag{4.4.26}$$

If $\tau = (\varepsilon/a_0)\ln(n/4)$, then the mesh is piecewise uniform. In each of the subintervals $[0, 1-\tau]$ and $[1-\tau, 1]$, a different argument is used to bound $W - w$.

Both W and w are small on the subinterval with no boundary layer, namely $[0, 1-\tau]$. Therefore, since $|W - w| \leq |W| + |w|$, we will bound W and w separately. Before we move any further, let us note that w can also be decomposed as $w = w_0 + \varepsilon w_1$ (see [105], p.59). Introducing the function φ by

$$\varphi(x) = \frac{\int_x^1 \exp(-A(t)/\varepsilon)dt}{\int_0^1 \exp(-A(t)/\varepsilon)dt}, \quad A(t) = \int_x^1 a(s)ds.$$

It can be shown that w_0 can be written in the form

$$w_0(x) = w_0(0)\varphi(x) + w_0(1)(1 - \varphi(x))$$

and therefore

$$w_0'(x) = (w_0(0) - w_0(1))\varphi'(x).$$

But $w_0(0) = w_0(1)\exp(-a_0/\varepsilon)$ and hence

$$\frac{w_0'(x)}{w_0(1)} = -(1 - \exp(-a_0/\varepsilon)\varphi'(x) > 0.$$

It follows that $w_0(x)/w_0(1)$ is positive and increasing in the interval $[0, 1]$.

Thus

$$0 \le \frac{w_0(x)}{w_0(1)} \le \frac{w_0(1-\tau)}{w_0(1)}$$

and hence

$$|w_0(x)| \le |w_0(1-\tau)|, \ \ \forall x \in [0, 1-\tau].$$

The same is true for $w_1(x)$ and since $w = w_0 + \varepsilon w_1$, it follows that

$$|w(x)| \le |w(1-\tau)|, \ \ \forall x \in [0, 1-\tau].$$

Using the estimate for $|w|$ and the fact that $\tau = (\varepsilon/a_0)\ln(n/4)$, we obtain

$$|w(x)| \le M \exp(-a_0\tau/\varepsilon) = Mn^{-1}. \tag{4.4.27}$$

Now we define an auxiliary mesh function $\widetilde{W}$ analogous to W except that the coefficient a in the difference operator L^n is replaced by a_0. Then Lemma 7.5 on page 53 of [105] suggests that

$$|W_j| \le |\widetilde{W}_j|, \ \ \forall 0 \le j \le n.$$

Thus by Lemma 7.3(p.51 of [105]) , we conclude that

$$|W_j| \le Mn^{-1}, \ \text{ for } \ n/2 \le j \le 3n/4. \tag{4.4.28}$$

Hence, from inequalities (4.4.27) and (4.4.28) , we have

$$|W_j - w_j| \le Mn^{-1}, \ \text{ for } \ n/2 \le j \le 3n/4. \tag{4.4.29}$$

In the subinterval $[1-\tau, 1]$, the classical argument leads to

$$|L^n(W_j - w_j)| \le M\varepsilon^{-2}|x_{j+1} - x_{j-1}| = 8M\varepsilon^{-2}\tau n^{-1}.$$

But also from the (4.4.29), we have

$$|W_n - w_n| = 0$$

and

$$|W_{3n/4} - w_{3n/4}| \leq |W_{3n/4}| + |w_{3n/4}| \leq Mn^{-1}.$$

By introducing the barrier function

$$\Phi_j = (x_j - (1-\tau))M_1\varepsilon^{-2}\tau n^{-1} + M_2 n^{-1},$$

we see that the mesh functions

$$\Psi_j^{\pm} = \Phi_j \pm (W_j - w_j)$$

satisfy

$$\Psi_{3n/4}^{\pm} \geq 0, \quad \Psi_n^{\pm} = 0,$$

provided that the constants M_1 and M_2 are chosen suitably.

Note that

$$L^n\Psi_j^{\pm} \leq 0, \quad 3n/4 + 1 \leq j \leq n-1.$$

By the discrete minimum principle (Lemma 4.3.1), on $[1-\tau, 1]$ we get

$$\Psi_j^{\pm} \geq 0, \quad 3n/4 \leq j \leq n.$$

Consequently,

$$|W_j - w_j| \leq \Phi_j \leq M_1\varepsilon^{-2}\tau^2 n^{-1} + M_2 n^{-1},$$

and making use of the inequality $\tau \leq (\varepsilon/a_0)\ln(n/4)$, we obtain

$$|W_j - w_j| \leq Mn^{-1}(\ln(n/4))^2. \tag{4.4.30}$$

Combining (4.4.29) and (4.4.30), we obtain the following estimate on the singular component of the error over the interval [0,1]:

$$|W_j - w_j| \leq Mn^{-1}(\ln(n/4))^2, \quad n/2 \leq j \leq n. \tag{4.4.31}$$

Estimates (4.4.22) and (4.4.31) along with the inequality (4.4.18) immediately gives

$$|U_j - u_j| \leq Mn^{-1}(\ln(n/4))^2, \quad n/2 \leq j \leq n. \tag{4.4.32}$$

Similarly,

$$|U_j - u_j| \leq Mn^{-1}(\ln(n/4))^2, \quad 0 \leq j \leq n/2 - 1. \tag{4.4.33}$$

We therefore have the following result.

Theorem 4.4.1. *Let $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth functions in the problem (4.1.1)-(4.1.2) and so that $u(x) \in C^4([-1,1])$. The numerical solution U obtained via FMFDM (4.4.17) along with (4.4.16) satisfies*

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq j\leq n} |u_j - U_j| \leq Mn^{-1}(\ln(n/4))^2. \tag{4.4.34}$$

4.4.2 Richardson extrapolation for FMFDM

We bisect each mesh sub-interval of $\mu_{n,\tau}$ and obtain a new mesh which we denote by $\mu_{2n,\tau}$.

$$\mu_{2n,\tau} = \{\bar{x}_j, 0 \leq j \leq 2n+1\} \supset \mu_{n,\tau} \text{ and } \bar{x}_j - \bar{x}_{j-1} = \bar{h}_j = h_j/2.$$

We denote the numerical solution computed on the mesh $\mu_{2n,\tau}$ by $\bar{U}$.

From the estimate (4.4.34), we have

$$u_j - U_j = Mn^{-1}(\ln(n/4))^2 + R_n(x_j),\ \forall x_j \in \mu_{n,\tau} \tag{4.4.35}$$

and

$$u_j - \bar{U}_j = M(2n)^{-1}(\ln(n/4))^2 + R_{2n}(\bar{x}_j),\ \forall \bar{x}_j \in \mu_{2n,\tau}. \tag{4.4.36}$$

The remainders $R_n(x_j)$ and $R_{2n}(\bar{x}_j)$ are of $O(n^{-1}(\ln(n/4))^2)$. It is to be noted that in practice, we assume

$$\tau = \frac{\varepsilon}{a_0}\ln\left(\frac{n}{4}\right), \tag{4.4.37}$$

because the possibility $\tau = 1/4$ suggested in equation (4.4.13) means that $1/4 < (\varepsilon/a_0)\ln(n/4)$, and so n^{-1} is very small relative to ε. This unlikely situation can be dealt with using the standard techniques.

The presence of the factor $\ln(n/4)$ in both (4.4.35) and (4.4.36) explains the fact that the two meshes $\mu_{n,\tau}$ and $\mu_{2n,\tau}$ use the same parameter τ given by (4.4.37).

A combination of equations (4.4.35) and (4.4.36) suggests that

$$u_j - (2\bar{U}_j - U_j) = O(n^{-1}(\ln(n/4))^2),\ \ \forall j = 1, \cdots, n-1$$

and therefore we set

$$U_j^{ext} := 2\bar{U}_j - U_j,\ \ \forall j = 1, \cdots, n-1 \tag{4.4.38}$$

as the numerical approximation of u at the grid point $x_j \in \mu_{n,\tau}$ resulting from the extrapolation process.

The decomposition of the error after extrapolation in a similar manner as in (4.4.18) gives

$$U^{ext} - u = \left(V^{ext} - v\right) + \left(W^{ext} - w\right), \tag{4.4.39}$$

where V^{ext} and W^{ext} are the regular and singular components of U^{ext}, respectively. We will estimate the components of the error separately.

For similar reasons as mentioned in the previous subsection, we will provide the analysis only on the interval $[0,1]$.

The local truncation error of the scheme (4.4.17) along with (4.4.16) at the grid point x_j after extrapolation is given by

$$\begin{aligned} L^n(u - U^{ext}) &= \left[2L^n_*(v_j - \bar{V}_j) - L^n(v_j - V_j)\right] \\ &\quad + \left[2L^n_*(w_j - \bar{W}_j) - L^n(w_j - W_j)\right], \end{aligned} \tag{4.4.40}$$

where, like L^n, L^n_* is a discrete operator associated with (4.4.17) along with (4.4.16) but on the mesh $\mu_{2n,\tau}$.

For the regular part of the local truncation after extrapolation, we use Lemma 4.1 ([105]). An analogous result as in (4.4.20) is

$$\begin{aligned} \left|2L^n_*(v_j - \bar{V}_j) - L^n(v_j - Vj)\right| &\leq \frac{2\varepsilon}{3}(x_{j+1/2} - x_{j-1/2})|v''_j| + a_j(x_j - x_{j-1/2})|v'_j| \\ &\quad + \frac{\varepsilon}{3}(x_{j+1} - x_{j-1})|v'''_j| + \frac{a_j}{2}(x_j - x_{j-1})|v''_j|, \\ &\quad \text{for } \frac{n}{2} \leq j \leq n-1. \end{aligned}$$

Using Lemma 4.3.2, we therefore have

$$\left|v_j - V^{ext}_j\right| \leq Mn^{-1}, \text{ for } \frac{n}{2} \leq j \leq n-1. \tag{4.4.41}$$

For the estimates on $w_j - W^{ext}_j$, we discuss two different cases.

If $\tau = 1/4$, the mesh is uniform and we have $\varepsilon^{-1} \leq (4/a_0)\ln(n/4)$. Therefore, by Lemma (4.4.17), we have

$$\left|2L^n_*(w_j - \bar{W}_j) - L^n(w_j - W_j)\right| \leq Mn^{-1}\varepsilon^{-2} \leq Mn^{-1}(\ln(n/4))^2.$$

An application of Lemma 4.3.2 then gives

$$\left|w_j - W_j^{ext}\right| \leq Mn^{-1}(\ln(n/4))^2, \text{ for } \frac{n}{2} \leq j \leq n-1. \tag{4.4.42}$$

If $\tau = (\varepsilon/a_0)\ln(n/4)$, the mesh is piecewise uniform with mesh spacing of $4(1-\tau)/n$ in the interval $[0, 1-\tau]$ and $4\tau/n$ in the interval $[1-\tau, 1]$.

In the subinterval $[0, 1-\tau]$, the functions w, W and $\bar{W}$ are small and therefore we have

$$\left|w_j - W_j^{ext}\right| \leq |w| + 2|\bar{W}| + |W|.$$

The bounds on $|w|$ and $|W|$ are obtained in the previous subsection. Also, bounds of $|W|$ are those of $|\bar{W}|$. Hence,

$$\left|w_j - W_j^{ext}\right| \leq Mn^{-1}, \text{ for } \frac{n}{2} \leq j \leq \frac{3n}{4}. \tag{4.4.43}$$

In the subinterval $[1-\tau, 1]$, we use the discrete minimum principle (Lemma 4.3.1) to bound $\left|w_j - W_j^{ext}\right|$. For $3n/4 + 1 \leq j \leq n-1$, we have

$$L^n(w_j - W_j^{ext}) \leq M\varepsilon^{-2}|x_{j+1} - x_{j-1}| = M\varepsilon^{-2}\tau n^{-1}.$$

Furthermore, $\left|w_{3n/4} - W_{3n/4}^{ext}\right| \leq Mn^{-1}$ and $|w_n - W_n^{ext}| = 0$.

Defining the barrier function

$$\bar{\Phi}_j = (x_j - (1-\tau))M_1\varepsilon^{-2}\tau n^{-1} + M_2 n^{-1},$$

we notice that, for a suitable choice of M_1 and M_2, the mesh function

$$\bar{\Psi}_j^{\pm} = \bar{\Phi}_j \pm (w_j - W_j^{ext})$$

satisfies

$$\bar{\Psi}^{\pm}_{3n/4} \geq 0, \quad \bar{\Psi}^{\pm}_{n} = 0$$

and

$$L^n \bar{\Psi}^{\pm}_j \leq 0, \quad \text{for } \frac{3n}{4} + 1 \leq j \leq n - 1.$$

It follows, by the discrete minimum principle (Lemma 4.3.1) that on the interval $[1-\tau, 1]$

$$\bar{\Psi}^{\pm}_j \geq 0, \quad \text{for } \frac{3n}{4} + 1 \leq j \leq n - 1.$$

Therefore

$$|w_j - W^{ext}_j| \leq \bar{\Phi}_j \leq M_1 \varepsilon^{-2} \tau^2 n^{-1} + M_2 n^{-1}.$$

Hence

$$|w_j - W^{ext}_j| \leq M n^{-1} (\ln(n/4))^2, \quad \text{for } \frac{3n}{4} + 1 \leq j \leq n - 1. \tag{4.4.44}$$

Combining estimates (4.4.43) and (4.4.44), we obtain

$$|w_j - W^{ext}_j| \leq M n^{-1} (\ln(n/4))^2, \quad \text{for } \frac{n}{2} \leq j \leq n. \tag{4.4.45}$$

By virtue of (4.4.39), estimates (4.4.41) and (4.4.45) lead to

$$|u_j - U^{ext}_j| \leq M n^{-1} (\ln(n/4))^2, \quad \text{for } \frac{n}{2} \leq j \leq n. \tag{4.4.46}$$

Following the similar lines on the interval $[-1, 0]$, i.e, when $a > 0$, we obtain the same estimate.

Combining the two, we have

Theorem 4.4.2. *Let $a(x)$, $b(x)$ and $f(x)$ be sufficiently smooth functions in the problem (4.1.1)-(4.1.2) and so that $u(x) \in C^4([-1, 1])$. The numerical solution U^{ext} obtained via*

FMFDM (4.4.17) along with (4.4.16) after extrapolation satisfies

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq j\leq n} |u_j - U_j^{ext}| \leq Mn^{-1}(\ln(n/4))^2. \tag{4.4.47}$$

In next section, we provide test examples to support these theoretical estimates.

4.5 Numerical results

For the following two test examples we provide comparative numerical results before and after extrapolation using the two fitted methods.

Example 4.5.1. [82] Consider problem (4.1.1)-(4.1.2) with

$$a(x) = 2(1-2x), \quad b(x) = 4, \quad f(x) = 0$$

for $0 < x < 1$.

The exact solution is

$$u(x) = \exp\left(-2x\frac{1-x}{\varepsilon}\right).$$

The solution to this problem has a turning point at $x = 0.5$.

Example 4.5.2. Consider problem (4.1.1)-(4.1.2) with

$$a(x) = -2x^3, \quad b(x) = \exp(x^2),$$

$$f(x) = \left[2\left(1+\frac{2x^2}{\varepsilon}-\frac{2x^4}{\varepsilon}\right)-\exp(x^2)\right]\exp\left[\frac{x^2-1}{\varepsilon}\right].$$

Its exact solution is given by

$$u(x) = \exp\left[-\frac{(1-x)(1+x)}{\varepsilon}\right].$$

The solution has a turning point at $x = 0$.

The maximum errors before extrapolation at all mesh points are evaluated using the fomulae

$$e_{\varepsilon,n} := \max_{0\le j\le n} |u_j - U_j|, \tag{4.5.48}$$

for both FOFDM (4.3.6) along with (4.3.5) and FMFDM (4.4.17) along with (4.4.16). After extrapolation, the maximum errors are calculated as

$$e^{ext}_{\varepsilon,n} := \max_{0\le j\le n} |u_j - U^{ext}_j|. \tag{4.5.49}$$

The numerical rates of convergence are computed by using the formula [33]:

$$r_k \equiv r_{\varepsilon,k} := \log_2(\widetilde{e}_{n_k}/\widetilde{e}_{2n_k}), k = 1, 2, ...$$

where $\widetilde{e}$ stands for $e_{\varepsilon,n}$, and $e^{ext}_{\varepsilon,n}$, respectively.

Table 4.1: Results for Example 4.5.2: Maximum errors via FOFDM (4.3.6) along with (4.3.5) **before** extrapolation.

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
1.0E-02	3.62E-02	1.46E-02	8.06E-03	9.07E-03	7.21E-03	3.17E-03	1.34E-03
1.0E-04	3.60E-02	1.45E-02	6.37E-03	2.99E-03	1.45E-03	7.12E-04	3.53E-04
1.0E-05	3.60E-02	1.45E-02	6.37E-03	2.99E-03	1.45E-03	7.12E-04	3.53E-04
1.0E-06	3.60E-02	1.45E-02	6.37E-03	2.99E-03	1.45E-03	7.12E-04	3.53E-04
1.0E-07	3.60E-02	1.45E-02	6.37E-03	2.99E-03	1.45E-03	7.12E-04	3.53E-04
1.0E-08	3.60E-02	1.45E-02	6.37E-03	2.99E-03	1.45E-03	7.12E-04	3.53E-04
1.0E-09	3.60E-02	1.45E-02	6.37E-03	2.99E-03	1.45E-03	7.12E-04	3.53E-04
1.0E-10	3.60E-02	1.45E-02	6.37E-03	2.99E-03	1.45E-03	7.12E-04	3.53E-04
1.0E-11	3.60E-02	1.45E-02	6.37E-03	2.99E-03	1.45E-03	7.12E-04	3.53E-04
e_n	3.60E-02	1.45E-02	6.37E-03	2.99E-03	1.45E-03	7.12E-04	3.53E-04

Table 4.2: Results for Example 4.5.2: Maximum errors via FOFDM (4.3.6) along with (4.3.5) **after** extrapolation.

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
1.0E-02	2.43E-02	1.01E-02	6.24E-03	3.86E-03	1.54E-03	5.05E-04	1.49E-04
1.0E-04	2.42E-02	9.78E-03	4.28E-03	2.00E-03	9.67E-04	4.75E-04	2.36E-04
1.0E-05	2.42E-02	9.78E-03	4.28E-03	2.00E-03	9.67E-04	4.75E-04	2.36E-04
1.0E-06	2.42E-02	9.78E-03	4.28E-03	2.00E-03	9.67E-04	4.75E-04	2.36E-04
1.0E-07	2.42E-02	9.78E-03	4.28E-03	2.00E-03	9.67E-04	4.75E-04	2.36E-04
1.0E-08	2.42E-02	9.78E-03	4.28E-03	2.00E-03	9.67E-04	4.75E-04	2.36E-04
1.0E-09	2.42E-02	9.78E-03	4.28E-03	2.00E-03	9.67E-04	4.75E-04	2.36E-04
1.0E-10	2.42E-02	9.78E-03	4.28E-03	2.00E-03	9.67E-04	4.75E-04	2.36E-04
1.0E-11	2.42E-02	9.78E-03	4.28E-03	2.00E-03	9.67E-04	4.75E-04	2.36E-04
e_n^{ext}	2.42E-02	9.78E-03	4.28E-03	2.00E-03	9.67E-04	4.75E-04	2.36E-04

Table 4.3: Results for Example 4.5.2: Rates of convergence via FOFDM (4.3.6) along with (4.3.5) **before** extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
1.0E-02	1.31	0.85	-0.17	0.33	1.19	1.24
1.0E-04	1.32	1.18	1.09	1.05	1.02	1.01
1.0E-05	1.32	1.18	1.09	1.05	1.02	1.01
1.0E-06	1.32	1.18	1.09	1.05	1.02	1.01
1.0E-07	1.32	1.18	1.09	1.05	1.02	1.01
1.0E-08	1.32	1.18	1.09	1.05	1.02	1.01
1.0E-09	1.32	1.18	1.09	1.05	1.02	1.01
1.0E-10	1.32	1.18	1.09	1.05	1.02	1.01
1.0E-11	1.32	1.18	1.09	1.05	1.02	1.01
R_n	1.32	1.18	1.09	1.05	1.02	1.01

Table 4.4: Results for Example 4.5.2: Rates of convergence via FOFDM (4.3.6) along with (4.3.5) **after** extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
1.0E-02	1.27	0.69	0.69	1.33	1.61	1.76
1.0E-04	1.31	1.19	1.10	1.05	1.02	1.01
1.0E-05	1.31	1.19	1.10	1.05	1.02	1.01
1.0E-06	1.31	1.19	1.10	1.05	1.02	1.01
1.0E-07	1.31	1.19	1.10	1.05	1.02	1.01
1.0E-08	1.31	1.19	1.10	1.05	1.02	1.01
1.0E-09	1.31	1.19	1.10	1.05	1.02	1.01
1.0E-10	1.31	1.19	1.10	1.05	1.02	1.01
1.0E-11	1.31	1.19	1.10	1.05	1.02	1.01
R_n^{ext}	1.31	1.19	1.10	1.05	1.02	1.01

Table 4.5: Results for Example 4.5.1: Maximum errors via FMFDM (4.4.17) along with (4.4.16) **before** extrapolation.

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
1.0E-02	1.36E-01	9.02E-02	5.83E-02	3.67E-02	2.25E-02	1.33E-02	7.71E-03
1.0E-04	1.43E-01	9.44E-02	6.04E-02	3.77E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-05	1.43E-01	9.44E-02	6.04E-02	3.77E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-06	1.43E-01	9.44E-02	6.04E-02	3.77E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-07	1.43E-01	9.44E-02	6.04E-02	3.77E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-08	1.43E-01	9.44E-02	6.04E-02	3.77E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-09	1.43E-01	9.44E-02	6.04E-02	3.77E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-10	1.43E-01	9.44E-02	6.04E-02	3.77E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-11	1.43E-01	9.44E-02	6.04E-02	3.77E-02	2.29E-02	1.36E-02	7.84E-03
e_n	1.43E-01	9.44E-02	6.04E-02	3.77E-02	2.29E-02	1.36E-02	7.84E-03

Table 4.6: Results for Example 4.5.1: Maximum errors via FMFDM (4.4.17) along with (4.4.16) **after** extrapolation.

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
1.0E-02	6.61E-02	4.23E-02	2.46E-02	1.56E-02	1.11E-02	8.37E-03	6.60E-03
1.0E-04	6.46E-02	4.35E-02	2.56E-02	1.61E-02	1.13E-02	8.51E-03	6.69E-03
1.0E-05	6.45E-02	4.35E-02	2.56E-02	1.61E-02	1.13E-02	8.51E-03	6.69E-03
1.0E-06	6.45E-02	4.35E-02	2.56E-02	1.61E-02	1.13E-02	8.51E-03	6.69E-03
1.0E-07	6.45E-02	4.35E-02	2.56E-02	1.61E-02	1.13E-02	8.51E-03	6.69E-03
1.0E-08	6.45E-02	4.35E-02	2.56E-02	1.61E-02	1.13E-02	8.51E-03	6.69E-03
1.0E-09	6.45E-02	4.35E-02	2.56E-02	1.61E-02	1.13E-02	8.51E-03	6.69E-03
1.0E-10	6.45E-02	4.35E-02	2.56E-02	1.61E-02	1.13E-02	8.51E-03	6.69E-03
1.0E-11	6.46E-02	4.35E-02	2.56E-02	1.61E-02	1.14E-02	8.51E-03	6.75E-03
e_n^{ext}	6.45E-02	4.35E-02	2.56E-02	1.61E-02	1.14E-02	8.51E-03	6.69E-03

Table 4.7: Results for Example 4.5.1: Rates of convergence via FMFDM (4.4.17) along with (4.4.16) **before** extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
1.0E-02	0.59	0.63	0.67	0.71	0.75	0.79
1.0E-04	0.60	0.64	0.68	0.72	0.76	0.79
1.0E-05	0.60	0.64	0.68	0.72	0.76	0.79
1.0E-06	0.60	0.64	0.68	0.72	0.76	0.79
1.0E-07	0.60	0.64	0.68	0.72	0.76	0.79
1.0E-08	0.60	0.64	0.68	0.72	0.76	0.79
1.0E-09	0.60	0.64	0.68	0.72	0.76	0.79
1.0E-10	0.60	0.64	0.68	0.72	0.76	0.79
1.0E-11	0.60	0.64	0.68	0.72	0.76	0.79
R_n	0.60	0.64	0.67	0.72	0.76	0.79

Table 4.8: Results for Example 4.5.1: Rates of convergence via FMFDM (4.4.17) along with (4.4.16) **after** extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
1.0E-02	0.64	0.78	0.66	0.50	0.40	0.34
1.0E-04	0.57	0.77	0.67	0.51	0.41	0.35
1.0E-05	0.57	0.77	0.67	0.51	0.41	0.35
1.0E-06	0.57	0.77	0.67	0.51	0.41	0.35
1.0E-07	0.57	0.77	0.67	0.51	0.41	0.35
1.0E-08	0.57	0.77	0.67	0.51	0.41	0.35
1.0E-09	0.57	0.77	0.67	0.51	0.41	0.35
1.0E-10	0.57	0.77	0.67	0.51	0.41	0.35
1.0E-11	0.57	0.77	0.67	0.50	0.42	0.33
R_n^{ext}	0.57	0.77	0.67	0.50	0.42	0.33

Table 4.9: Results for Example 4.5.2: Maximum errors via FMFDM (4.4.17) along with (4.4.16) **before** extrapolation.

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
1.0E-02	9.82E-02	8.49E-02	5.82E-02	3.71E-02	2.27E-02	1.35E-02	7.78E-03
1.0E-04	9.85E-02	8.58E-02	5.88E-02	3.75E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-05	9.85E-02	8.58E-02	5.88E-02	3.75E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-06	9.85E-02	8.58E-02	5.88E-02	3.75E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-07	9.85E-02	8.58E-02	5.88E-02	3.75E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-08	9.85E-02	8.58E-02	5.88E-02	3.75E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-09	9.85E-02	8.58E-02	5.88E-02	3.75E-02	2.29E-02	1.36E-02	7.84E-03
1.0E-10	9.85E-02	8.58E-02	5.88E-02	3.75E-02	2.29E-02	1.36E-02	7.83E-03
1.0E-11	9.85E-02	8.58E-02	5.88E-02	3.74E-02	2.29E-02	1.35E-02	7.83E-03
e_n	9.85E-02	8.58E-02	5.88E-02	3.74E-02	2.29E-02	1.35E-02	7.83E-03

Table 4.10: Results for Example 4.5.2: Maximum errors via FMFDM (4.4.17) along with (4.4.16) **after** extrapolation.

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
1.0E-02	3.78E-02	3.47E-02	2.35E-02	1.55E-02	1.11E-02	8.40E-03	6.62E-03
1.0E-04	3.83E-02	3.53E-02	2.40E-02	1.58E-02	1.13E-02	8.50E-03	6.69E-03
1.0E-05	3.83E-02	3.53E-02	2.40E-02	1.58E-02	1.13E-02	8.50E-03	6.69E-03
1.0E-06	3.83E-02	3.53E-02	2.40E-02	1.58E-02	1.13E-02	8.50E-03	6.69E-03
1.0E-07	3.83E-02	3.53E-02	2.40E-02	1.58E-02	1.13E-02	8.50E-03	6.69E-03
1.0E-08	3.83E-02	3.53E-02	2.40E-02	1.58E-02	1.13E-02	8.50E-03	6.69E-03
1.0E-09	3.83E-02	3.53E-02	2.40E-02	1.58E-02	1.13E-02	8.50E-03	6.69E-03
1.0E-10	3.83E-02	3.53E-02	2.40E-02	1.59E-02	1.13E-02	8.54E-03	6.69E-03
1.0E-11	3.83E-02	3.53E-02	2.40E-02	1.58E-02	1.15E-02	8.50E-03	1.18E-02
e_n^{ext}	3.83E-02	3.53E-02	2.40E-02	1.59E-02	1.13E-02	8.54E-03	6.69E-03

Table 4.11: Results for Example 4.5.1: Rates of convergence via FMFDM (4.4.17) along with (4.4.16) **before** extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
1.0E-02	0.75	0.86	0.92	0.96	0.98	0.99
1.0E-04	0.27	0.54	0.63	0.75	0.97	0.98
1.0E-05	0.24	0.55	0.64	0.70	0.75	0.79
1.0E-06	0.22	0.54	0.65	0.71	0.75	0.79
1.0E-07	0.21	0.54	0.65	0.71	0.75	0.79
1.0E-08	0.20	0.54	0.65	0.71	0.76	0.79
1.0E-09	0.20	0.54	0.65	0.71	0.76	0.79
1.0E-10	0.20	0.54	0.65	0.71	0.76	0.79
1.0E-11	0.20	0.54	0.65	0.71	0.76	0.79
R_n	0.20	0.54	0.65	0.71	0.76	0.79

Table 4.12: Results for Example 4.5.1: Rates of convergence via FMFDM (4.4.17) along with (4.4.16) **after** extrapolation

ε	r_1	r_2	r_3	r_4	r_5	r_6
1.0E-02	0.12	0.56	0.60	0.49	0.40	0.34
1.0E-04	0.12	0.56	0.60	0.49	0.41	0.35
1.0E-05	0.12	0.56	0.60	0.49	0.41	0.35
1.0E-06	0.12	0.56	0.60	0.49	0.41	0.35
1.0E-07	0.12	0.56	0.60	0.49	0.41	0.35
1.0E-08	0.12	0.56	0.60	0.49	0.41	0.35
1.0E-09	0.12	0.56	0.60	0.49	0.41	0.35
1.0E-10	0.12	0.56	0.60	0.49	0.40	0.35
1.0E-11	0.12	0.56	0.60	0.47	0.43	-0.48
R_n^{ext}	0.12	0.56	0.60	0.49	0.40	0.35

4.6 Discussion

In this chapter, we have constructed two fitted finite difference methods to solve the singularly perturbed turning point problem (4.1.1)-(4.1.2): a FOFDM and a FMFDM. The former is first order convergent and the latter is almost first order. These theoretical results are confirmed by our numerics presented in Table 4.3 in the case of FOFDM and tables 4.7 and 4.11 in the case of FMFDM. We have also investigated the performance of the Richardson extrapolation when applied on these methods and noticed that this convergence acceleration technique does not improve the order of convergence of either of the methods above as seen in tables 4.4, 4.8 and 4.12. However, the Richardson extrapolation improves the accuracy of the methods as supported by the comparative results before and after extrapolation in tables 4.1 and 4.2, 4.5 and 4.6 and 4.9 and 4.10.

Chapter 5

A High Accuracy Fitted Operator Finite Difference Method for a Nonlinear Singularly Perturbed Two-point Boundary Value Problem

In this chapter, we extend the ideas developed for the singularly perturbed linear two-point boundary value problems to solve a class of singularly perturbed nonlinear two-point boundary value problems. The original nonlinear problem is linearized and each of the linear problems is then solved using an appropriate FOFDM. The Richardson extrapolation is then carried out to find whether we can achieve higher accuracy. Error estimates before and after extrapolation are also provided.

5.1 Introduction

We consider the following class of singularly perturbed nonlinear two-point boundary value problems ([33])

$$Ly \equiv \varepsilon y'' = F(x, y, y'), \tag{5.1.1}$$

$$y(0) = \eta_0, \quad y(1) = \eta_1, \tag{5.1.2}$$

where $y(0) = \eta_0, y(1) = \eta_1 \in \mathbb{R}$ and $x \in (0, 1)$ and ε is a small positive parameter. We assume that F is a smooth function satisfying the following conditions:

- $(\partial/\partial z)F(x, y, z) \le 0$,
- $(\partial/\partial y)F(x, y, z) \ge 0$,
- $(\partial/\partial y - \partial/\partial z)F(x, y, z) \ge \alpha > 0$,
- the growth condition $F(x, y, z) = O(|z|^2)$, as $z \to \infty$, for all $x \in [0, 1]$ and all real y and z.

Under the above assumptions, a unique solution exists for the problem (5.1.1)-(5.1.2). The details can be found in [29, 61].

The solution to the above problem is sought in two steps: The first step is the quasi-linearization and the second one is to solve the sequence of linear problems.

In general, one linearizes the equation (5.1.1) around a nominal solution, which satisfies the specified boundary conditions [19]. This process leads to a sequence of linear two-point boundary-value problems in which the solution of the k-th linear problem satisfies the specified boundary conditions and is taken as the nominal profile for the $(k + 1)$-th linear problem. Each of the linear problems in this sequence is then solved by using an appropriate method for a linear problem. The iterative procedure is continued until the desired convergence is achieved.

The readers are referred to chapter 1 for some of the works found in the literature regarding the numerical solutions of nonlinear two-point boundary value problems. However, we are not aware of the use of the convergence acceleration techniques for such problems and hence, the present work is an attempt to fill this gap.

The rest of this chapter is organized as follows. We linearize the nonlinear problem above via quasilinearization and prove the convergence of this process in Section 5.2. The sequence of linear boundary value problems obtained via quasilinearization is solved using a fitted operator finite difference method (FOFDM) which we introduce in Section 5.3. In Section 5.4, the analysis of this FOFDM is presented. Section 5.5 provides the error estimate of the extrapolation of this FOFDM. Numerical results which validate our findings are displayed in Section 5.7. Finally, a brief discussion of our results is given in Section 5.8.

5.2 Quasilinearization process and its convergence

Following some of the works, for example, [19, 67], in this section we discuss the quasilinearization process and its convergence.

5.2.1 Quasilinearization

Let $\left(y^{(k)}(x), (y')^{(k)}(x)\right)$ be the kth nominal solution to problem (5.1.1)-(5.1.2) over the interval $[0, 1]$. This means that the profiles $y^{(k)}(x)$ and $(y')^{(k)}(x)$ satisfy the boundary conditions exactly and the differential equation (5.1.1) approximately.

Taking the Taylor expansion of the right-hand side of (5.1.1) up to first-order terms around the above nominal solution, we get

$$\varepsilon(y'')^{(k+1)} \approx F\left(y^{(k)}, (y')^{(k)}\right) + F_y\left[y^{(k+1)} - y^{(k)}\right] + F_{y'}\left[(y')^{(k+1)} - (y')^{(k)}\right], \qquad (5.2.3)$$

and therefore

$$\varepsilon(y'')^{(k+1)} - F_{y'}(y')^{(k+1)} - F_y(y)^{(k+1)} = F\left(y^{(k)}, (y')^{(k)}\right) - F_y y^{(k)} - F_{y'}(y')^{(k)}, \qquad (5.2.4)$$

which is linear in $y^{k+1}(x)$.

Now, instead of solving the nonlinear problem (5.1.1)-(5.1.2), we will solve a sequence of linear problems (5.2.4) for $k = 0, 1, 2, \cdots$ along with the boundary conditions

$$y^{(k)}(0) = y(0) = \eta_0, \ \ y^{(k)}(1) = y(1) = \eta_1. \qquad (5.2.5)$$

Theoretically, for a solution to the nonlinear problem, we require that

$$\lim_{k\to\infty} y^{(k)}(x) = y^*(x), \ \ 0 \le x \le 1,$$

where $y^*(x)$ is the solution of the nonlinear problem. Numerically, we require that

$$\left|y^{(k+1)}(x) - y^{(k)}(x)\right| \le \text{Tol}, \ \ 0 \le x \le 1.$$

where Tol is a small tolerance prescribed by us.

5.2.2 Convergence of the quasilinearization process

For the sake of simplicity, in this subsection we will denote $F(x, y, y')$ by $F(y)$. Consider the problem

$$\varepsilon y'' = F(y), \qquad (5.2.6)$$

$$y(x_0) = 0, \ \ y(x_n) = 0. \qquad (5.2.7)$$

We recall that x_0 and x_n are respectively 0 and 1 in this chapter. However, to keep the exposition wider, we present the analysis in this section by considering the general values.

Let $y_0(x)$ be some initial approximation and consider the sequence $\{y_k\}$ determined by the recurrence relation

$$\varepsilon y_k'' = F(y_{k-1}) + (y_k - y_{k-1})F'(y_{k-1}), \tag{5.2.8}$$

$$y_k(x_0) = 0, \;\; y_k(x_n) = 0. \tag{5.2.9}$$

Equation (5.2.8) implies that

$$\varepsilon(y_{k+1} - y_k)'' = F(y_k) - F(y_{k-1}) + (y_{k+1} - y_k)F'(y_k) - (y_k - y_{k-1})F'(y_{k-1}). \tag{5.2.10}$$

This equation can be regarded as a differential equation in $y_{k+1} - y_k$. We convert it into the following integral equation

$$\begin{aligned}\varepsilon\,(y_{k+1} - y_k) \;&= \int_{x_0}^{x_n} G(x,s)\,[F(y_k) - F(y_{k-1}) + (y_{k+1} - y_k)F'(y_k) \\ &\quad -(y_k - y_{k-1})F'(y_{k-1})]\,ds,\end{aligned} \tag{5.2.11}$$

where the Green function $G(x,s)$ is given by

$$G(x,s) = \begin{cases} \frac{(x_n - x)(s - x_0)}{x_n - x_0}, & x_0 \le s \le x, \\ \frac{(x - x_0)(x_n - s)}{x_n - x_0}, & x \le s \le x_n. \end{cases} \tag{5.2.12}$$

It is to be noted that the function $G(x,s)$ reaches its maximum value $(x_n - x_0)/4$ at $s = (x_0 + x_n)/2$. Also, we note from the mean value theorem that

$$F(y_k) - F(y_{k-1}) - (y_k - y_{k-1})F'(y_{k-1}) = (y_k - y_{k-1})F''(\theta), \;\; y_{k-1} < \theta < y_k.$$

Define

$$K = \max_{|y|\le 1} |F''(y)|, \;\; \text{and} \;\; \widetilde{M} = \max_{|y|\le 1} |F'(y)|.$$

It follows from equation (5.2.11) that

$$|y_{k+1} - y_k| \leq \frac{x_n - x_0}{4\varepsilon} \int_{x_0}^{x_n} \left[\frac{K}{2}(y_k - y_{k-1})^2 + \widetilde{M}|y_{k+1} - y_k| \right] ds.$$

Taking maximum over x after rearranging the terms, we get

$$\max_x (y_{k+1} - y_k) \leq \left[\frac{K(x_n - x_0)^2}{8\varepsilon} \Big/ \left(1 - \frac{\widetilde{M}(x_n - x_0)^2}{4\varepsilon} \right) \right] \max_x (y_k - y_{k-1})^2. \quad (5.2.13)$$

This shows a quadratic convergence, provided that

$$\left[\frac{K(x_n - x_0)^2}{8\varepsilon} \Big/ \left(1 - \frac{\widetilde{M}(x_n - x_0)^2}{4\varepsilon} \right) \right] < 1.$$

If $x_n - x_0$ is small enough, the above inequality holds. If $x_n - x_0$ is large, an adequate choice of the initial approximation $y_0(x)$ will keep $|y_1(x) - y_0(x)|$ sufficiently small. It follows that $\max |y_{k+1} - y_k|$ is small enough for all $x \in (x_0, x_n)$, which is sufficient for convergence.

5.3 Fitted operator finite difference method (FOFDM) for the sequence of linear problems

At each iteration, equation (5.2.4) can be written as

$$Lu \equiv -\varepsilon u'' + a(x)u' + b(x)u = f, \quad \text{for } x \in (0,1), \quad (5.3.14)$$

where

$$a(x) = F_{y'}, \quad b(x) = F_y, \quad f(x) = F_y y^{(k)} + F_{y'}(y^{(k)})' - F(y^{(k)}, (y^{(k)})'),$$

and

$$u = y^{(k+1)}.$$

The boundary conditions at each iteration are given by

$$u(0) = \eta_0, \quad \text{and} \quad u(1) = \eta_1. \tag{5.3.15}$$

It is to be noted that the solution profile of problem (5.3.14)-(5.3.15) depends on the sign patterns of the coefficient functions $a(x)$ and $b(x)$.

- If $a(x) \neq 0$ for $0 \leq x \leq 1$, then the solution has a boundary layer at $x = 0$ for $a(x) < 0$ and at $x = 1$ for $a(x) > 0$.
- If $a(x) \equiv 0$ for $0 \leq x \leq 1$, then the solution may have boundary layers at $x = 0$ and $x = 1$ for $b(x) > 0$ and may oscillate rapidly for $b(x) < 0$.

We will develop a FOFDM for one of the above cases, namely, the case where $a(x) \geq \alpha > 0$.

We discretize problem (5.3.14)-(5.3.15) as in Chapter 4 and obtain the following FOFDM (note that the notation U used below denotes the approximations for u):

$$L^h U_j \equiv -\varepsilon \frac{U_{j+1} - 2U_j + U_{j-1}}{\phi_j^2} + \widetilde{a}_j \frac{U_j - U_{j-1}}{h} + \widetilde{b}_j U_j = \widetilde{f}_j,$$

where

$$\widetilde{a}_j = \frac{a_j + a_{j+1}}{2},$$

$$\widetilde{b}_j = \frac{b_{j-1} + b_j + b_{j+1}}{3},$$

$$\widetilde{f}_j = \frac{f_{j-1} + f_j + f_{j+1}}{3},$$

and

$$\phi_j^2 = \frac{h\varepsilon}{\widetilde{a}_j} \left(\exp\left(\frac{\widetilde{a}_j h}{\varepsilon} \right) - 1 \right).$$

In the above, L^h is the discrete operator associated with the linear operator L. This discretization results in the following tridiagonal system

$$AU = G. \tag{5.3.16}$$

The corresponding entries of A and G are

$$\begin{aligned}
A_{ij} &= r_j^-, \ \ i = j+1; \ \ j = 1, 2, \ldots, n-2, \\
A_{ij} &= r_j^c, \ \ i = j; \ \ j = 1, 2, \ldots, n-1, \\
A_{ij} &= r_j^+, \ \ i = j-1; \ \ j = 2, 3, \ldots, n-1, \\
G_1 &= \widetilde{f}_1 - r_1^- \eta_0, \ \ F_{n-1} = \widetilde{f}_{n-1} - r_{n-1}^+ \eta_1, \\
G_j &= \widetilde{f}_j, \ \ j = 2, 3, \ldots, n-1,
\end{aligned}$$

where

$$r_j^- = -\frac{\varepsilon}{\phi_j^2} - \frac{\widetilde{a}_j}{h}, \ r_j^c = \frac{2\varepsilon}{\phi_j^2} + \frac{\widetilde{a}_j}{h} + \widetilde{b}_j, \ \text{ and } \ r_j^+ = -\frac{\varepsilon}{\phi_j^2}. \tag{5.3.17}$$

If the form of the linear equation is different from the one considered in this chapter, then the above process can be suitably adjusted.

In next section, we analyze this method for convergence.

5.4 Convergence analysis of FOFDM

The local truncation error of the scheme (5.3.16) and (5.3.17) is given by

$$L^h(u_j - U_j) = T_0 u_j + T_1 u_j' + T_2 u_j'' + T_3 u_j''' + T_4 u^{(iv)}(\xi_j), \tag{5.4.18}$$

where $\xi_j \in (x_{j-1}, x_{j+1})$ and

$$T_0 = r_j^- + r_j^c + r_j^+ - b_j - \frac{h^2}{3} b_j'',$$

$$
\begin{aligned}
T_1 &= h(r_j^+ - r_j^-) - a_j - \frac{1}{3}h^2\left(a_j'' + 2b_j'\right), \\
T_2 &= \frac{h^2}{2}\left(r_j^+ + r_j^-\right) + \varepsilon - \frac{h^2}{3}\left(2a_j' + b_j + \frac{1}{2}b_j''h^2\right), \\
T_3 &= \frac{h^3}{6}\left(r_j^+ - r_j^-\right) - \frac{h^2}{3}\left(a_j + \frac{1}{2}a_j''h^2\right), \\
T_4 &= \frac{h^3}{24}\left(r_j^+ + r_j^-\right) + \frac{1}{3}\varepsilon h^2.
\end{aligned} \tag{5.4.19}
$$

After some algebraic manipulations, we obtain

$$T_0 = 0,\ \ |T_1| \leq Mh,\ \ |T_2| \leq Mh + Mh^2/\varepsilon,\ \ |T_3| \leq Mh^2,\ \ |T_4| \leq Mh^2. \tag{5.4.20}$$

Considering the dominating terms and using Lemma 4.2.2 we obtain

$$|L^h(u_j - U_j)| \leq Mh\left(1 + \frac{h}{\varepsilon}\right)\left[1 + \frac{\exp(-\alpha(1-x_j)/\varepsilon)}{\varepsilon^2}\right],\ \text{ for } j = 1, 2, \cdots, n-1. \tag{5.4.21}$$

where α is such that $a(x) \geq \alpha > 0,\ \ \forall x \in [0,1]$.

Finally, using lemmas (4.3.2) and (2.2.5), we have the following result.

Theorem 5.4.1. *Let U be the numerical approximation to u of (5.3.14)-(5.3.15) obtained by using (5.3.16)-(5.3.17). Then there is a positive constant M, independent of h and ε, such that*

$$\max_{0\leq j\leq n} |u_j - U_j| \leq Mh\left(1 + \frac{h}{\varepsilon}\right), \tag{5.4.22}$$

where M is a constant independent of h and ε.

5.5 Richardson extrapolation

We have used the FOFDM presented in an earlier section to solve the sequence of linear problems. The extrapolation formula as well as the error estimates after extrapolation are derived in this section.

5.5.1 Extrapolation formula for linear problems

Let μ_{2n} be the mesh obtained by bisecting each mesh interval in μ_n. We have

$$\mu_{2n} = \{\widetilde{x}_j\}, \text{ with } \widetilde{x}_0 = 0, \ \ \widetilde{x}_n = 1, \text{ and } \widetilde{x}_j - \widetilde{x}_{j-1} = \widetilde{h} = h/2, \ \ j = 1(1)2n.$$

Denoting by $\bar{U}$ the numerical approximation of u computed using (5.3.16)-(5.3.17) on the mesh μ_{2n}, the estimate in Theorem 5.4.1 suggests that

$$u_j - U_j = Mh + O(h^2/\varepsilon), \ \ j = 0(1)n.$$

Similarly,

$$u_j - \bar{U}_j = M\widetilde{h} + O(\widetilde{h}^2/\varepsilon), \ \ j = 0(1)2n.$$

A straightforward calculation therefore shows that

$$u_j - (2\widetilde{U}_j - U_j) = O(h^2/\varepsilon), \ \ j = 0(1)n.$$

Thus, we will use

$$U_j^{ext} := 2\bar{U}_j - U_j, \ \ j = 0(1)n. \tag{5.5.23}$$

as the approximation of u after extrapolation.

5.5.2 Error estimates for the linear problems after extrapolation

The local truncation error of the scheme (5.3.16)-(5.3.17) after extrapolation is

$$\widetilde{L}^h \left(u_j - U_j^{ext}\right) = 2L^{\widetilde{h}}(u_j - \bar{U}_j) - L^h(u_j - U_j), \ \ j = 1(1)n-1. \tag{5.5.24}$$

The quantity $L^h(u_j - U_j)$ is given by equation (5.4.18) and

$$L^{\widetilde{h}}(u_j - \bar{U}_j) = \widetilde{T}_0 u_j + \widetilde{T}_1 u_j' + \widetilde{T}_2 u_j'' + \widetilde{T}_3 u_j''' + \widetilde{T}_4 u^{(iv)}(\widetilde{\xi}_j), \qquad (5.5.25)$$

where $\widetilde{\xi}_j \in (x_j - \widetilde{h}, x_j + \widetilde{h})$ and

$$\begin{aligned}
\widetilde{T}_0 &= \widetilde{r}_j^- + \widetilde{r}_j^c + \widetilde{r}_j^+ - b_j - \frac{\widetilde{h}^2}{3} b_j'', \\
\widetilde{T}_1 &= \widetilde{h}(\widetilde{r}_j^+ - \widetilde{r}_j^-) - a_j - \frac{1}{3}\widetilde{h}^2 \left(a_j'' + 2b_j'\right), \\
\widetilde{T}_2 &= \frac{\widetilde{h}^2}{2}\left(\widetilde{r}_j^+ + \widetilde{r}_j^-\right) + \varepsilon - \frac{\widetilde{h}^2}{3}\left(2a_j' + b_j + \frac{1}{2} b_j'' \widetilde{h}^2\right), \\
\widetilde{T}_3 &= \frac{\widetilde{h}^3}{6}\left(\widetilde{r}_j^+ - \widetilde{r}_j^-\right) - \frac{\widetilde{h}^2}{3}\left(a_j + \frac{1}{2} a_j'' \widetilde{h}^2\right), \\
\widetilde{T}_4 &= \frac{\widetilde{h}^3}{24}\left(\widetilde{r}_j^+ + \widetilde{r}_j^-\right) + \frac{1}{3}\varepsilon \widetilde{h}^2,
\end{aligned} \qquad (5.5.26)$$

and the $\widetilde{r}$'s are obtained from the r's by substituting h by $\widetilde{h}$. It follows that

$$\widetilde{L}^h\left(u_j - U_j^{ext}\right) = (2\widetilde{T}_0 - T_0)u_j + (2\widetilde{T}_1 - T_1)u_j' + (2\widetilde{T}_2 - T_2)u_j'' + (2\widetilde{T}_3 - T_3)u_j''' + (2\widetilde{T}_4 - T_4)u^{(iv)}(\bar{\xi}_j), \qquad (5.5.27)$$

where $\bar{\xi}_j \in (x_j - \widetilde{h}, x_j + \widetilde{h})$. We note that

$$\begin{aligned}
2\widetilde{T}_0 - T_0 = 0,\ |2\widetilde{T}_1 - T_1| \leq Mh^2,\ |2\widetilde{T}_2 - T_2| \leq Mh^2/\varepsilon, \\
|2\widetilde{T}_3 - T_3| \leq Mh^2,\ |2\widetilde{T}_4 - T_4| \leq Mh^2.
\end{aligned} \qquad (5.5.28)$$

Thus, from (5.5.27), we obtain

$$\left|\widetilde{L}^h\left(u_j - U_j^{ext}\right)\right| \leq Mh^2|u_j'| + M\frac{h^2}{\varepsilon}|u_j''| + Mh^2|u_j'''| + Mh^2|u_j^{iv}|.$$

Using Lemma 4.2.2, we get

$$\left|\widetilde{L}^h\left(u_j - U_j^{ext}\right)\right| \leq Mh^2\left(1+\frac{1}{\varepsilon}\right)\left[1+\frac{\exp(-\alpha(1-x_j)/\varepsilon)}{\varepsilon^2}\right] \tag{5.5.29}$$

Finally, using lemmas (4.3.2) and (2.2.5) we obtain the following result

Theorem 5.5.1. *Let U^{ext} be the numerical approximation of the solution u of problem (5.3.14)-(5.3.15) obtained after extrapolation of (5.3.16)-(5.3.17). Then there exists a positive constant M independent of h and ε, such that*

$$\max_{0\leq j\leq n}|u_j - U_j^{ext}| \leq Mh^2\left(1+\frac{1}{\varepsilon}\right), \tag{5.5.30}$$

where M is a constant independent of h and ε.

5.6 The case $a(x)\equiv 0$, $b(x)>0$, for all $x\in(0,1)$

For this case, we briefly describe the method, give the basic steps of its analysis then we embark in Richardson extrapolation.

5.6.1 The method

The continuous problem (5.3.14)-(5.3.15) is discretized on the mesh μ_n as follows.

$$L^hU_j \equiv -\varepsilon\frac{U_{j+1}-2U_j+U_{j-1}}{\widetilde{\psi}_j^2} + \widetilde{b}_jU_j = \widetilde{f}_j,$$

where

$$\widetilde{b}_j = \frac{b_{j-1}+b_j+b_{j+1}}{3},$$

$$\widetilde{f}_j = \frac{f_{j-1}+f_j+f_{j+1}}{3},$$

and

$$\widetilde{\psi}_j \equiv \frac{2}{\widetilde{\rho}_j}\sinh\left(\frac{\widetilde{\rho}_j h}{2}\right), \quad \widetilde{\rho}_j = \sqrt{\frac{\widetilde{b}_j}{\varepsilon}}.$$

The tridiagonal system resulting from this discretization is

$$AU = G, \tag{5.6.31}$$

where the corresponding entries of A and G are

$$\begin{aligned}
A_{ij} &= r_j^-, \quad i = j+1; \quad j = 1, 2, \ldots, n-2,\\
A_{ij} &= r_j^c, \quad i = j; \quad j = 1, 2, \ldots, n-1,\\
A_{ij} &= r_j^+, \quad i = j-1; \quad j = 2, 3, \ldots, n-1,\\
G_1 &= \widetilde{f}_1 - r_1^- \eta_0, \quad F_{n-1} = \widetilde{f}_{n-1} - r_{n-1}^+ \eta_1,\\
G_j &= \widetilde{f}_j, \quad j = 2, 3, \ldots, n-1,
\end{aligned}$$

and

$$r_j^- = -\frac{\varepsilon}{\phi_j^2}, \quad r_j^c = \frac{2\varepsilon}{\phi_j^2} + \widetilde{b}_j, \quad \text{and} \quad r_j^+ = -\frac{\varepsilon}{\phi_j^2}. \tag{5.6.32}$$

5.6.2 Convergence analysis of the method

The local truncation error of the scheme above is calculated as in (5.4.18). Note that in this case, in the expressions for the T_js, all the a_js vanish and the ϕ_js are substituted by ψ_j as necessary. We obtain

$$T_0 = T_3 = 0; \quad |T_1| \le Mh^2; \quad |T_2| \le Mh^2(1 + h^2/\varepsilon); \quad |T_4| \le Mh^2.$$

The following lemmas will be used below.

Lemma 5.6.1. *If $u(x)$ is the solution of the problem (5.3.14)-(5.3.15), and $b,\ f \in$*

$C^k(\bar{\Omega}),\ k > 0$, *with* $a(x) \equiv 0$, *then there exists a constant* C *such that*

$$|u^{(k)}| \leq C\left[1 + \varepsilon^{-k/2}\left(\exp(-x\sqrt{\beta/\varepsilon}) + \exp(-(1-x)\sqrt{\beta/\varepsilon})\right)\right],$$

where $0 < \beta \leq b(x),\ x \in [0,1]$.

Proof. See [105].

Lemma 5.6.2. *If* Z_i *is any mesh function such that* $Z_0 = Z_n = 0$, *then*

$$|Z_i| \leq \frac{1}{\beta}\max_{1\leq j\leq n-1}|L^h Z_j| \ \text{ for } \ 0 \leq i \leq n.$$

Proof. See [105].

Using lemmas (4.3.2), (2.2.5) and (5.6.1), we obtain the following result.

Theorem 5.6.1. *Assume that* $a(x) \equiv 0$, *and* $b(x)$ *and* $f(x)$ *are sufficiently smooth functions in equation (5.3.14) for* $x \in [0,1]$. *Let* $U_j,\ j = 0(1)n$, *be the approximate solution of (5.3.14)-(5.3.15) obtained using the method (5.6.31)-(5.6.32). Then we have*

$$\max_{0\leq j\leq n}|u_j - U_j| \leq Mh^2\left(1 + \frac{h^2}{\varepsilon}\right). \tag{5.6.33}$$

5.6.3 Richardson extrapolation

Extrapolation formula

In this section, U and $\widetilde{U}$ denote the computed solutions of problem (5.3.14) by the scheme (5.6.31)-(5.6.32) on the meshes μ_n and μ_{2n}, respectively. This implies that

$$u_j - U_j = Mh^2\left(1 + \frac{h^2}{\varepsilon}\right) + R_n(x_j),\quad x_j \in \mu_n$$

and

$$u_j - \widetilde{U}_j = M\left(\frac{h}{2}\right)^2\left(1 + \frac{1}{\varepsilon}\left(\frac{h}{2}\right)^2\right) + R_{2n}(\widetilde{x}_j),\quad \widetilde{x}_j \in \mu_{2n}$$

where both the remainders $R_n(x_j)$ and $R_{2n}(\widetilde{x}_j)$ are $0(h^4)$. The linear combination

$$4(u_j - \widetilde{U}_j) - (u_j - U_j) = O(h^4/\varepsilon)$$

suggests that we should use

$$U_j^{ext} := \frac{4\widetilde{U}_j - U_j}{3}, \quad j = 1(1)n-1,$$

as the approximation of u_j after extrapolation.

Error estimates after extrapolation

The local truncation error of the scheme (5.6.31)-(5.6.32) after extrapolation is

$$\widetilde{L}^h\left(u_j - U_j^{ext}\right) = \frac{4}{3}L^{\tilde{h}}(u_j - \bar{U}_j) - \frac{1}{3}L^h(u_j - U_j), \quad j = 1(1)n-1. \tag{5.6.34}$$

An analogous of (5.5.27) is then obtained in the form

$$\begin{aligned}\widetilde{L}^h\left(u_j - U_j^{ext}\right) &= \left(\frac{4}{3}\widetilde{T}_0 - \frac{1}{3}T_0\right)u_j + \left(\frac{4}{3}\widetilde{T}_1 - \frac{1}{3}T_1\right)u_j' + \left(\frac{4}{3}\widetilde{T}_2 - \frac{1}{3}T_2\right)u_j'' \\ &\quad + \left(\frac{4}{3}\widetilde{T}_3 - \frac{1}{3}T_3\right)u_j''' + \left(\frac{4}{3}\widetilde{T}_4 - \frac{1}{3}T_4\right)u^{(iv)}(\bar{\xi}_j),\end{aligned} \tag{5.6.35}$$

where $\bar{\xi}_j \in (x_j - \widetilde{h}, x_j + \widetilde{h})$.

The $\widetilde{T}$s are obtained from the Ts by substituting h by $\widetilde{h}$. Straightforward calculations show that

$$\frac{4}{3}\widetilde{T}_0 - \frac{1}{3}T_0 = \frac{4}{3}\widetilde{T}_1 - \frac{1}{3}T_1 = \frac{4}{3}\widetilde{T}_3 - \frac{1}{3}T_3 = 0$$

and

$$\left|\frac{4}{3}\widetilde{T}_2 - \frac{1}{3}T_2\right| \le Mh^4\left(1 + \frac{1}{\varepsilon}\right), \quad \left|\frac{4}{3}\widetilde{T}_4 - \frac{1}{3}T_4\right| \le Mh^4.$$

With these bounds, applying lemmas (4.3.2), (2.2.5) and (5.6.1) to equation (5.6.35), we obtain the following result.

Theorem 5.6.2. *Assume that $a(x) \equiv 0$, and $b(x)$ and $f(x)$ are sufficiently smooth functions in equation (5.3.14) for $x \in [0,1]$. Let U_j^{ext}, $j = 0(1)n$, be the approximate solution of (5.3.14)-(5.3.15) obtained using the method (5.6.31)-(5.6.32) after extrapolation. Then we have*

$$\max_{0\le j\le n} \left|u_j - U_j^{ext}\right| \le Mh^4\left(1+\frac{1}{\varepsilon}\right). \qquad (5.6.36)$$

5.7 Numerical results

In this section we solve the following singularly perturbed nonlinear problems in order to illustrate our theoretical results.

Example 5.7.1. ([112]) Consider the problem

$$\varepsilon y'' - yy' - y = 0, \quad y(0) = 1, \quad y(1) = 1.$$

Its exact solution is not available.

The quasilinear process equations are

$$-\varepsilon(y'')^{(k+1)}(x) + y^{(k)}(x)(y')^{(k+1)}(x) + \left(1 + (y')^{(k)}(x)\right) y^{(k+1)}(x) = y^{(k)}(x)(y')^{(k)}(x),$$

$$y^{(k)}(0) = 1, \quad y^{(k)}(1) = 1.$$

Example 5.7.2. ([28]) Consider the problem

$$\varepsilon y'' - y - y^2 = -\exp(-2x/\sqrt{\varepsilon}), \quad y(0) = 1, \quad y(1) = \exp(-1/\sqrt{\varepsilon}).$$

The exact solution of this problem is

$$y(x) = \exp(-x/\sqrt{\varepsilon}).$$

The quasilinear process equations are

$$\varepsilon(y'')^{(k+1)}(x) - \left[1 + 2y^{(k)}(x)\right] y^{(k+1)} = -\left[y^{(k)}(x)\right]^2 - \exp(-2x/\sqrt(\varepsilon)),$$

$$y^{(k)}(0) = 1, \;\; y^{(k)}(1) = \exp(-1/\sqrt{\varepsilon}).$$

Example 5.7.3. ([20]) Consider the problem

$$\varepsilon y'' - xy - y^2 = 0; \;\; y(0) = 1, \;\; y(1) = 0. \tag{5.7.37}$$

Its exact solution is not available.

The quasilinear process equations are

$$-\varepsilon(y'')^{(k+1)}(x) + \left[x + 2y^{(k)}(x)\right] y^{(k+1)}(x) = \left(y^{(k)}(x)\right)^2,$$

$$y^{(k)}(0) = 1, \;\; y^{(k)}(1) = 0.$$

The maximum errors as tabulated in tables 5.1 and 5.2 at all mesh points are calculated using the formula

$$E_{\varepsilon,n} := \max|u_j - U_j| \;\text{ and }\; E^{ext}_{\varepsilon,n} := \max|u_j - U^{ext}_j|$$

before and after extrapolation, respectively for Example 5.7.2 since its exact solution is available. For examples 5.7.1 and 5.7.3, the exact solutions are not available. Therefore, we use the formula

$$E_{\varepsilon,n} := \max|U^n_j - U^{2n}_{2j}| \;\text{ and }\; E^{ext}_{\varepsilon,n} := \max|U^{ext}_j - U^{ext}_{2j}|$$

before and after extrapolation, respectively, where U^{2n}_{2j} and U^{ext}_{2j} are the computed solutions before and after extrapolation on the mesh μ_{2n}, respectively. The maximum errors for these examples are presented in tables 5.5, 5.6, 5.9 and 5.10.

The numerical rates of convergence are calculated using the formula [33]:

$$r_k \equiv r_{\varepsilon,k} := \log_2(\widetilde{E}_{n_k}/\widetilde{E}_{2n_k}),\ k = 1, 2, \ldots$$

where $\widetilde{E}$ stands for E and E^{ext}, respectively. These rates are given in tables 5.3, 5.4, 5.7, 5.8, 5.11 and 5.12.

Table 5.1: Results for Example 5.7.1: Maximum errors via FOFDM **before** extrapolation

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-1}	5.83E-04	2.36E-04	1.02E-04	4.66E-05	2.22E-05	1.08E-05	5.35E-06
2^{-2}	2.64E-03	1.10E-03	4.73E-04	2.14E-04	1.01E-04	4.90E-05	2.41E-05
2^{-3}	9.38E-03	3.95E-03	1.69E-03	7.48E-04	3.45E-04	1.65E-04	8.04E-05
2^{-4}	2.65E-02	1.15E-02	4.87E-03	2.10E-03	9.40E-04	4.38E-04	2.10E-04
2^{-5}	6.07E-02	2.83E-02	1.24E-02	5.31E-03	2.31E-03	1.04E-03	4.86E-04
2^{-6}	1.08E-01	6.38E-02	2.89E-02	1.28E-02	5.52E-03	2.41E-03	1.09E-03
2^{-7}	1.59E-01	1.13E-01	6.49E-02	2.91E-02	1.30E-02	5.59E-03	2.44E-03
2^{-8}	2.03E-01	1.65E-01	1.15E-01	6.50E-02	2.93E-02	1.30E-02	5.59E-03
2^{-9}	2.29E-01	2.08E-01	1.67E-01	1.15E-01	6.49E-02	2.92E-02	1.29E-02

Table 5.2: Results for Example 5.7.1: Maximum errors via FOFDM **after** extrapolation

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-1}	1.15E-04	3.23E-05	8.76E-06	2.30E-06	5.91E-07	1.50E-07	3.78E-08
2^{-2}	4.73E-04	1.52E-04	4.52E-05	1.26E-05	3.32E-06	8.55E-07	2.17E-07
2^{-3}	1.57E-03	5.82E-04	1.94E-04	5.81E-05	1.61E-05	4.24E-06	1.09E-06
2^{-4}	3.97E-03	1.75E-03	6.68E-04	2.21E-04	6.54E-05	1.80E-05	4.73E-06
2^{-5}	8.38E-03	4.55E-03	1.90E-03	7.04E-04	2.30E-04	6.77E-05	1.85E-05
2^{-6}	1.56E-02	1.01E-02	4.75E-03	1.95E-03	7.13E-04	2.31E-04	6.75E-05
2^{-7}	2.68E-02	1.98E-02	1.07E-02	4.78E-03	1.95E-03	7.08E-04	2.28E-04
2^{-8}	4.17E-02	3.56E-02	2.10E-02	1.08E-02	4.74E-03	1.93E-03	6.99E-04
2^{-9}	5.19E-02	5.89E-02	3.75E-02	2.13E-02	1.08E-02	4.71E-03	1.91E-03

Table 5.3: Results for Example 5.7.1: Rates of convergence via FOFDM **before** extrapolation $n_k = 16, 32, 64, 128, 256, 512$.

ε	r_1	r_2	r_3	r_4	r_5	r_6
2^{-1}	1.31	1.21	1.13	1.07	1.04	1.02
2^{-2}	1.27	1.21	1.14	1.08	1.05	1.02
2^{-3}	1.25	1.23	1.17	1.11	1.07	1.04
2^{-4}	1.21	1.24	1.21	1.16	1.10	1.06
2^{-5}	1.10	1.19	1.23	1.20	1.15	1.10
2^{-6}	0.76	1.14	1.18	1.21	1.20	1.15
2^{-7}	0.49	0.80	1.16	1.17	1.21	1.20
2^{-8}	0.30	0.52	0.82	1.15	1.17	1.22

Table 5.4: Results for Example 5.7.1: Rates of convergence via FOFDM **after** extrapolation $n_k = 16, 32, 64, 128, 256, 512$.

ε	r_1	r_2	r_3	r_4	r_5	r_6
2^{-1}	1.83	1.88	1.93	1.96	1.98	1.99
2^{-2}	1.64	1.74	1.85	1.92	1.96	1.98
2^{-3}	1.43	1.58	1.74	1.85	1.92	1.96
2^{-4}	1.18	1.39	1.60	1.75	1.86	1.93
2^{-5}	0.88	1.26	1.43	1.61	1.77	1.87
2^{-6}	0.63	1.09	1.28	1.45	1.63	1.77
2^{-7}	0.44	0.89	1.16	1.29	1.46	1.63
2^{-8}	0.23	0.76	0.96	1.19	1.30	1.46

Table 5.5: Results for Example 5.7.2: Maximum errors via FOFDM **before** extrapolation

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-1}	2.21E-03	5.10E-04	1.23E-04	3.02E-05	7.48E-06	1.86E-06	4.65E-07
2^{-2}	3.39E-03	8.03E-04	1.94E-04	4.77E-05	1.18E-05	2.94E-06	7.33E-07
2^{-3}	6.50E-03	1.57E-03	3.79E-04	9.25E-05	2.28E-05	5.67E-06	1.41E-06
2^{-4}	1.28E-02	3.17E-03	7.65E-04	1.86E-04	4.57E-05	1.13E-05	2.81E-06
2^{-5}	2.42E-02	6.44E-03	1.56E-03	3.76E-04	9.19E-05	2.27E-05	5.64E-06
2^{-6}	4.54E-02	1.28E-02	3.17E-03	7.64E-04	1.86E-04	4.56E-05	1.13E-05
2^{-7}	7.64E-02	2.42E-02	6.44E-03	1.56E-03	3.76E-04	9.19E-05	2.27E-05
2^{-8}	1.18E-01	4.54E-02	1.28E-02	3.17E-03	7.64E-04	1.86E-04	4.56E-05
2^{-9}	1.82E-01	7.64E-02	2.42E-02	6.44E-03	1.56E-03	3.76E-04	9.19E-05

Table 5.6: Results for Example 5.7.2: Maximum errors via FOFDM **after** extrapolation

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-1}	9.36E-05	2.80E-05	7.49E-06	1.93E-06	4.89E-07	1.23E-07	3.09E-08
2^{-2}	1.04E-04	3.00E-05	8.22E-06	2.13E-06	5.42E-07	1.37E-07	3.44E-08
2^{-3}	1.52E-04	2.45E-05	7.02E-06	1.85E-06	4.72E-07	1.19E-07	3.00E-08
2^{-4}	3.67E-04	5.47E-05	7.88E-06	1.13E-06	2.91E-07	7.37E-08	1.86E-08
2^{-5}	1.18E-03	1.38E-04	2.10E-05	2.91E-06	3.84E-07	4.94E-08	7.13E-09
2^{-6}	3.49E-03	3.68E-04	5.45E-05	7.87E-06	1.06E-06	1.38E-07	1.76E-08
2^{-7}	9.22E-03	1.18E-03	1.38E-04	2.09E-05	2.91E-06	3.84E-07	4.94E-08
2^{-8}	2.11E-02	3.49E-03	3.68E-04	5.45E-05	7.87E-06	1.06E-06	1.38E-07
2^{-9}	4.10E-02	9.22E-03	1.18E-03	1.38E-04	2.09E-05	2.91E-06	3.84E-07

Table 5.7: Results for Example 5.7.2: Rates of convergence via FOFDM **before** extrapolation, $n_k = 16, 32, 64, 128, 256, 512$.

ε	r_1	r_2	r_3	r_4	r_5	r_6
2^{-1}	2.11	2.05	2.03	2.01	2.01	2.00
2^{-2}	2.08	2.05	2.03	2.01	2.01	2.00
2^{-3}	2.05	2.05	2.03	2.02	2.01	2.00
2^{-4}	2.01	2.05	2.04	2.02	2.01	2.01
2^{-5}	1.91	2.05	2.05	2.03	2.02	2.01
2^{-6}	1.83	2.01	2.05	2.04	2.02	2.01
2^{-7}	1.66	1.91	2.05	2.05	2.03	2.02
2^{-8}	1.38	1.83	2.01	2.05	2.04	2.02
2^{-9}	1.26	1.66	1.91	2.05	2.05	2.03

Table 5.8: Results for Example 5.7.2: Rates of convergence via FOFDM **after** extrapolation, $n_k = 16, 32, 64, 128, 256, 512$.

ε	r_1	r_2	r_3	r_4	r_5	r_6
2^{-1}	1.74	1.90	1.96	1.98	1.99	1.99
2^{-2}	1.79	1.87	1.95	1.98	1.99	1.99
2^{-3}	2.63	1.80	1.93	1.97	1.98	1.99
2^{-4}	2.75	2.79	2.81	1.96	1.98	1.98
2^{-5}	3.10	2.72	2.85	2.92	2.96	2.79
2^{-6}	3.25	2.76	2.79	2.89	2.94	2.97
2^{-7}	2.96	3.10	2.72	2.85	2.92	2.96
2^{-8}	2.60	3.25	2.76	2.79	2.89	2.94
2^{-9}	2.15	2.96	3.10	2.72	2.85	2.92

Table 5.9: Results for Example 5.7.3: Maximum errors vi FOFDM **before** extrapolation

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-1}	5.96E-04	1.45E-04	3.56E-05	8.80E-06	2.19E-06	5.46E-07	1.36E-07
2^{-2}	1.11E-03	2.64E-04	6.39E-05	1.57E-05	3.88E-06	9.65E-07	2.41E-07
2^{-3}	2.11E-03	4.99E-04	1.19E-04	2.89E-05	7.11E-06	1.76E-06	4.39E-07
2^{-4}	4.14E-03	9.87E-04	2.33E-04	5.59E-05	1.37E-05	3.38E-06	8.41E-07
2^{-5}	8.13E-03	2.00E-03	4.68E-04	1.11E-04	2.70E-05	6.64E-06	1.65E-06
2^{-6}	1.52E-02	4.02E-03	9.47E-04	2.23E-04	5.35E-05	1.31E-05	3.24E-06
2^{-7}	2.70E-02	7.91E-03	1.92E-03	4.48E-04	1.07E-04	2.59E-05	6.38E-06
2^{-8}	4.52E-02	1.50E-02	3.89E-03	9.11E-04	2.14E-04	5.14E-05	1.26E-05
2^{-9}	6.80E-02	2.65E-02	7.67E-03	1.86E-03	4.32E-04	1.03E-04	2.49E-05

Table 5.10: Results for Example 5.7.3: Maximum errors via FOFDM **after** extrapolation

ε	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-1}	5.84E-06	9.28E-07	1.28E-07	1.67E-08	2.14E-09	2.70E-10	3.23E-11
2^{-2}	1.97E-05	2.96E-06	3.98E-07	5.14E-08	6.52E-09	8.21E-10	1.05E-10
2^{-3}	5.60E-05	8.39E-06	1.13E-06	1.45E-07	1.84E-08	2.31E-09	2.90E-10
2^{-4}	1.41E-04	2.22E-05	3.03E-06	3.94E-07	5.00E-08	6.29E-09	7.90E-10
2^{-5}	3.25E-04	5.62E-05	8.04E-06	1.06E-06	1.35E-07	1.71E-08	2.14E-09
2^{-6}	9.98E-04	1.38E-04	2.11E-05	2.85E-06	3.68E-07	4.66E-08	5.87E-09
2^{-7}	2.92E-03	3.26E-04	5.42E-05	7.65E-06	1.00E-06	1.28E-07	1.61E-08
2^{-8}	7.69E-03	9.33E-04	1.34E-04	2.03E-05	2.73E-06	3.52E-07	4.45E-08
2^{-9}	1.78E-02	2.78E-03	3.21E-04	5.26E-05	7.40E-06	9.69E-07	1.23E-07

Table 5.11: Results for Example 5.7.3: Rates of convergence via FOFDM **before** extrapolation, $n_k = 16, 32, 64, 128, 256, 512$.

ε	r_1	r_2	r_3	r_4	r_5	r_6
2^{-1}	2.04	2.03	2.02	2.01	2.00	2.00
2^{-2}	2.06	2.05	2.03	2.01	2.01	2.00
2^{-3}	2.08	2.07	2.04	2.02	2.01	2.01
2^{-4}	2.07	2.09	2.06	2.03	2.02	2.01
2^{-5}	2.02	2.10	2.07	2.04	2.02	2.01
2^{-6}	1.92	2.09	2.09	2.06	2.03	2.02
2^{-7}	1.77	2.04	2.10	2.07	2.04	2.02
2^{-8}	1.59	1.95	2.09	2.09	2.06	2.03
2^{-9}	1.36	1.79	2.05	2.10	2.07	2.04

Table 5.12: Results for Example 5.7.3: Rates of convergence via FOFDM **after** extrapolation, $n_k = 16, 32, 64, 128, 256, 512$.

ε	r_1	r_2	r_3	r_4	r_5	r_6
2^{-1}	2.65	2.86	2.94	2.97	2.98	3.06
2^{-2}	2.74	2.89	2.95	2.98	2.99	2.96
2^{-3}	2.74	2.90	2.96	2.98	2.99	2.99
2^{-4}	2.67	2.87	2.95	2.98	2.99	2.99
2^{-5}	2.53	2.81	2.92	2.97	2.99	2.99
2^{-6}	2.85	2.71	2.89	2.95	2.98	2.99
2^{-7}	3.17	2.59	2.82	2.93	2.97	2.99
2^{-8}	3.04	2.79	2.73	2.89	2.96	2.98
2^{-9}	2.68	3.11	2.61	2.83	2.93	2.97

5.8 Discussion

In this chapter, we considered a singularly perturbed nonlinear two point boundary value problem. We first linearized the problem via the quasilinearization method. This process led to a sequence of linear problems. For the case where the functional F of equation (5.1.1) contains the argument y', we developed and analyzed a fitted operator method designed for the resulting sequence of linear problems. We noted that the method is first order convergent. Richardson extrapolation was then carried out and both the accuracy and order of convergence were improved.

Similar steps are also taken for the case where the functional F does not contain the argument y'. However this is done with less details. The second order convergence of the underlying fitted operator finite difference method is improved to four.

Some numerical results are not of higher order as expected but this is due to the convergence properties of the sequence of linear problems. This is issue is being investigated further.

Chapter 6

Higher Order Numerical Method for Singularly Perturbed Parabolic Problems in One Dimension

This chapter deals with singularly perturbed parabolic problems. Our basic aim is to extend the ideas generated in chapters 4 and 5 to solve this class of problems. After we linearize the problem, each of the linear problems is discretized as follows: the time derivative is approximated by a forward Euler approximation and then the stationary problem is solved using a fitted operator finited difference method (FOFDM). The overall method is analyzed for convergence. We also discuss why the extrapolation process can not improve the order of convergence of the proposed FOFDM.

6.1 Introduction

We consider the problem

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}, \quad (x,t) \in \Omega \times (0,T], \tag{6.1.1}$$

where $\Omega = (0, 1)$.

The initial and boundary conditions are respectively given by

$$u(x, 0) = f(x), \quad \text{for} \quad 0 \leq x \leq 1, \tag{6.1.2}$$

$$u(0, t) = 0 \quad u(1, t) = 0 \quad \text{for} \quad 0 \leq t \leq T. \tag{6.1.3}$$

Equation (6.1.1) is a one-dimensional quasi-linear parabolic partial differential equation, which is referred to in the literature as Burgers' equation (see [25, 26, 40]).

The parameter $\varepsilon \in (0, 1]$ is the coefficient of kinematic viscosity and the function $f(x)$ is sufficiently smooth. In order for the data to match at the two corners $(0, 0)$ and $(1, 0)$ of the domain $\bar{\Omega} \times [0, T]$, we impose the compatibility conditions

$$f(0) = 0, \quad f(1) = 0. \tag{6.1.4}$$

In [22], it was proved that there exists a constant C such that

$$|u(x, t) - f(x)| \leq Ct, \quad t \in (0, T), \tag{6.1.5}$$

$$|u(x, t) - 0| \leq Cx, \quad x \in (0, 1), \tag{6.1.6}$$

with $(x, t) \in \overline{\Omega} \times [0, T]$.

Equation (6.1.1) was introduced by Bateman [16], presenting its steady state solutions. It was after Burgers who studied this model for turbulent flows in [26], that it is referred to as Burgers' equation. Several researchers have studied this important fluid dynamic model whose use was subsequently extended to other fields such as gas dynamics, heat conduction, elasticity, turbulence and shock wave theory [26, 31, 60]. These references, amongst others, provide the exact solutions (in the form of infinite series) to Burgers' equations for given initial and boundary conditions.

One important fact presented by Miller [104] is that these exact solutions have no practical meaning when ε is very small due to the occurrence of slow convergence. Numerous numerical schemes are available in the literature to circumvent this difficulty.

Abbasbandy and Darvishi [1] used the modified Adomian's decomposition method for calculating a numerical solution of problem (6.1.1)-(6.1.3) without recourse to any transformation in the above equation such as Hopf-Cole transformation. Kutluay et al. [80] presented the exact-explicit finite difference scheme to achieve a reliable accuracy. A variational method built on the method of time discretization was suggested by Aksan and Ödzes [9]. Subsequently, Kadalbajoo et al. [70] semidiscretized the equation in time by backward Euler scheme with uniform time step and then used the quasilinerization process [19] to linearize the stationary Burgers' equation. There are other numerical methods to solve the Burgers' equation that are based on finite differences [37, 80, 117, 141], on finite element approaches [27, 81, 76] and on splines [3]. Other notable works include [2, 59, 106, 107].

In this chapter, we intend to extend the FOFDM developed in Chapter 5 to solve the Burgers' equation. The quasilinearization of the original problem gives us a sequence of linear parabolic problems. After the time semi-discretization for these parabolic problems, the stationary problems are solved using this FOFDM.

The rest of this chapter is organized as follows. Section 6.2 deals with quasilinearization and time semi-discretization. Section 6.3 is concerned with the FOFDM and its error analysis. A result of Richardson extrapolation on this FOFDM is presented in Section 6.4. Some numerical illustrations are given in Section 6.5 and these results are discussed in Section 6.6.

6.2 Quasilinearization and time semidiscretization

6.2.1 Quasilinearization

Writing Eq. (6.1.1) in the form

$$u_t = \varepsilon u_{xx} - uu_x \text{ on } \Omega \times (0, T], \tag{6.2.7}$$

and setting

$$g(u, u_x) = -uu_x,$$

we have $g_u = -u_x$ and $g_{u_x} = -u$.

Assuming that (u^n, u_x^n) is the n-th nominal solution to problem (6.1.1) along with the initial and boundary conditions (6.1.2)-(6.1.3) and taking the Taylor expansion of g up to first-order terms around this nominal function, we get

$$g(u^{n+1}, u_x^{n+1}) = g(u^n, u_x^n) + g_u(u^{n+1} - u^n) + g_{u_x}(u_x^{n+1} - u_x^n).$$

Hence the quasilinearization process for equation (6.2.7) (see [19]) gives

$$u_t^{n+1} = \varepsilon u_{xx}^{n+1} - u_x^{n+1}u^n - u_x^n u^{n+1} + u_x^n u^n, \tag{6.2.8}$$

along with

$$u^{n+1}(0, t) = 0, \quad u^{n+1}(1, t) = 0 \text{ for } 0 \le t \le T \tag{6.2.9}$$

and

$$u^{n+1}(x, 0) = f(x) \text{ for } 0 \le x \le 1. \tag{6.2.10}$$

Letting $u^{n+1} = U$, we get the quasilinear process equations

$$U_t = \varepsilon U_{xx} - u^n U_x - u_x^n U + u_x^n u^n. \tag{6.2.11}$$

With this new notation, the boundary and initial conditions take the form

$$U(0,t) = 0, \quad U(1,t) = 0 \ \text{ for } \ 0 \le t \le T \tag{6.2.12}$$

and

$$U(x,0) = f(x) \ \text{ for } \ 0 \le x \le 1. \tag{6.2.13}$$

6.2.2 Time semidiscretization

We discretize the time variable by means of the implicit Euler method (IEM). To do so, firstly we partition the time interval $[0,T]$ into M subintervals such that the time step Δt is given by $\Delta t = T/M$.

Then the IEM reads:

$$\frac{U_{j+1} - U_j}{\Delta t} = \varepsilon (U_{j+1})_{xx} - u^n_{j+1}(U_{j+1})_x - (u^n_{j+1})_x U_{j+1} + (u^n_{j+1})_x u^n_{j+1}. \tag{6.2.14}$$

Rearranging this equation and using the notation

$$\begin{aligned} u^n_{j+1} &= a^n(x), \\ \frac{1}{\Delta t} + (u^n_{j+1})_x &= b^n(x), \\ \frac{U_j}{\Delta t} + (u^n_{j+1})_x u^n_{j+1} &= F^n(x), \end{aligned}$$

we obtain from equation (6.2.14)

$$-\varepsilon \left(U_{j+1}(x)\right)_{xx} + a^n(x) \left(U_{j+1}(x)\right)_x + b^n(x) U_{j+1} = F^n(x), \tag{6.2.15}$$

with $j = 0(1)M-1$.

Finally, equations (6.2.10) and (6.2.15) can be written in the form

$$\begin{aligned} U_0 &= f(x), \\ LU_{j+1}(x) &= F^n(x), \quad 0 \le j \le M-1, \end{aligned} \tag{6.2.16}$$

where

$$LU_{j+1}(x) \equiv -\varepsilon \left(U_{j+1}(x)\right)_{xx} + a^n(x)\left(U_{j+1}(x)\right)_x + b^n(x)U_{j+1}(x),$$

and the boundary conditions are

$$U_{j+1}(0) = 0, \quad U_{j+1}(1) = 0.$$

Letting

$$k = \max_{|U_{j+1}|\le 1} |H_{U_{j+1}U_{j+1}}(U_{j+1})| \text{ and } m = \max_{|U_{j+1}|\le 1} |H_{U_{j+1}}(U_{j+1})|,$$

the following result was proved in [70].

$$\max_x \left|\left(U_{j+1}^{(n+2)} - U_{j+1}^{(n+1)}\right)\right| \le \left(\frac{\frac{k}{8}}{1-\frac{m}{4}}\right) \max_x \left(U_{j+1}^{(n+1)} - U_{j+1}^{(n)}\right)^2. \tag{6.2.17}$$

This inequality shows that the quasilinearization process enjoys a quadratic convergence.

The linear operator L satisfies the following maximum principle.

Lemma 6.2.1. *Let $\psi \in C^2(\bar{\Omega})$ be any function satisfying $\psi(0) \ge 0$, $\psi(1) \ge 0$ and $L\psi(x) \ge 0$ for all $x \in \Omega$. Then $\psi(x) \ge 0$ for all $x \in \bar{\Omega}$.*

Proof. See [105].

6.3 A fitted operator finite difference method for the solution of Burgers' equation

6.3.1 The method

Let N be a positive integer. The interval $[0,1]$ is partitioned as follows.

$$x_0 = 0,\ h = 1/N,\ \ x_i = x_0 + ih,\ i = 1(1)N.$$

For the sake of simplicity, we assume that $a^n(x) \geq \alpha > 0$, for all $x \in (0,1)$.

At each iteration n, and each time level $j = 1, 2, \dots, M-1$, we discretize (6.2.16) as

$$L^h U(i,j) := -\varepsilon \frac{U_{i+1,j} - 2U_{i,j} + U_{i+1,j}}{\phi_i^2} + \widetilde{a}_i^n \frac{U_{i,j} - U_{i-1,j}}{h} + \widetilde{b}_i^n U_{i,j} = \widetilde{F}_{i,j}^n \tag{6.3.18}$$

where the function ϕ is given by

$$\phi_i^2 = \frac{h\varepsilon}{\widetilde{a}_i^n}\left(\exp\left(\frac{\widetilde{a}_i^n h}{\varepsilon}\right) - 1\right).$$

where

$$\widetilde{a}_i^n = \frac{a_i^n + a_{i+1}^n}{2},$$

$$\widetilde{b}_i^n = \frac{b_{i-1}^n + b_i^n + b_{i+1}^n}{3},$$

and

$$\widetilde{F}_i^n = \frac{F_{i-1}^n + F_i^n + F_{i+1}^n}{3}.$$

Equation (6.3.18) leads to a tridiagonal system of linear equation

$$AU = G. \tag{6.3.19}$$

Corresponding entries of A and G in this case are

$$\begin{aligned}
A_{ik} &= r_k^-, i = k+1; k = 1, 2, \ldots, N-2, \\
A_{ik} &= r_k^c, i = k; k = 1, 2, \ldots, N-1, \\
A_{ik} &= r_k^+, i = k-1; k = 2, 3, \ldots, N-1, \\
G_1 &= F_1^n - r_1^- U(0), \\
G_{N-1} &= F_{N-1}^n - r_{N-1}^+ U(1), \\
G_k &= F_k^n, k = 2, 3, \ldots, N-1,
\end{aligned}$$

where

$$r_k^- = -\frac{\varepsilon}{\phi_k^2} - \frac{\widetilde{a}_k^n}{h}, \; r_k^+ = -\frac{\varepsilon}{\phi_k^2}, \; \text{and} \; r_k^c = 2\frac{\varepsilon}{\phi_k^2} + \frac{\widetilde{a}_k^n}{h} + \widetilde{b}_k^n. \tag{6.3.20}$$

6.3.2 Convergence analysis

Below we present the bounds on the solution of Burgers' equation. Then we provide the error analysis of both the time discretization (see [70]) and the FOFDM introduced above. Finally we will summarize the two results at the end of this subsection.

Lemma 6.3.1. *The solution $u(x,t)$ of (6.1.1) enjoys the following bound*

$$|u(x,t)| \le C, \;\; \textit{for all} \;\; (x,t) \in \Omega \times [0,T].$$

Proof. Inequality (6.1.5) implies that

$$|u| \le Ct + |f|.$$

The proof follows using the fact that $f(x)$ is sufficiently smooth and x and t lie in bounded intervals.

Lemma 6.3.2. *By keeping x fixed along the line $\{(x,t) : 0 \le t \le T\}$, the bound of u_t is* *given by*

$$\left|\frac{\partial^i u(x,t)}{\partial t^i}\right| \leq C, \quad \textit{for} \;\; i = 0, 1, 2, 3. \tag{6.3.21}$$

Proof. Assuming that the solution $u(x,t)$ is sufficiently smooth in the domain $\Omega \times [0,T]$, the mean value theorem suggests that there exists $t^* \in (t, t+k)$ along the line $\{(x,t) : 0 \leq t \leq T\}$ such that

$$u_t(x,t^*) = \frac{u(x,t+k) - u(x,t)}{k}$$

thus implying that

$$|u_t(x,t^*)| \leq \frac{2|u(x,t)|}{k}.$$

By Lemma 6.3.1, we get

$$u_t(x,t) \leq C.$$

We get the bounds on $u_{tt}(x,t)$ and $u_{ttt}(x,t)$ along the line $\{(x,t) : 0 \leq t \leq T\}$ in a similar manner.

It follows from this lemma that, by keeping x fixed along the line $\{(x,t), 0 \leq t \leq T\}$, the solution U of equation (6.2.11) satisfies

$$U(t_j) = U(t_{j+1}) - \Delta t \frac{\partial U(t_{j+1})}{\partial t} + \int_{t_{j+1}}^{t_j} (t_j - s) \frac{\partial^2 U}{\partial t^2}(s) ds, \tag{6.3.22}$$

i.e.,

$$\begin{aligned} U(t_j) &= U(t_{j+1}) - \Delta t \left(\varepsilon U_{xx} - u^n U_x - u_x^n U + u_x^n u^n \right)(t_{j+1}) \\ &\quad + \int_{t_{j+1}}^{t_j} (t_j - s) \frac{\partial^2 U}{\partial t^2}(s) ds. \end{aligned} \tag{6.3.23}$$

Subtracting (6.2.14) from (6.3.23) and denoting the local truncation error $U(t_{j+1}) - U_{j+1}$ by e_{j+1}, we obtain

$$\max_j |\Delta t L e_{j+1}| \leq C(\Delta t)^2 \tag{6.3.24}$$

and since the operator $\Delta t L$ satisfies the maximum principle (Lemma 6.2.1), we deduce that

$$\max_j |e_{j+1}| \leq C(\Delta t)^2. \tag{6.3.25}$$

We now derive the estimate of the global error at the $(j+1)$th time step, E_{j+1} as follows

$$\begin{aligned} \max_j |E_{j+1}| &= \max_j \left| \sum_{l=1}^{j} e_l \right|, \quad j \leq T/\Delta t, \\ &\leq \sum_{l=1}^{j} \max_j |e_l|, \\ &\leq jC(\Delta t)^2, \\ &\leq CT\Delta t, \\ &\leq C\Delta t. \end{aligned} \tag{6.3.26}$$

We have thus proved that

Theorem 6.3.1. *If*

$$\left| \frac{\partial^k u(x,t)}{\partial^k t} \right| \leq C, \quad (x,t) \in \overline{\Omega} \times [0,T], \quad 0 \leq k \leq 2$$

then the local and global error estimates of the time discretization satisfy the following estimates

$$\max_j |e_{j+1}| \leq C(\Delta t)^2,$$

$$\max_j |E_{j+1}| \leq C\Delta t, \quad \textit{for all } \; j \leq T/\Delta t.$$

In other words, the time discretization process is uniformly convergent of first order.

For the solution U_{j+1} of (6.2.16) and its derivatives, the estimates contained in the following lemma hold (see [70, 105]).

Lemma 6.3.3. *If U_{j+1} is the solution of (6.2.16) then there exists a constant C such that*

$$|U_{j+1}(x)| \leq C, \quad \textit{for all} \ \ x \in \overline{\Omega}.$$

Lemma 6.3.4. *If U_{j+1} is the solution of (6.2.16), then the bounds on its derivatives are given by*

$$\left|U_{j+1}^{(k)}\right| \leq C(1+\varepsilon^{-k}\exp\left(-\alpha(1-x)/\varepsilon\right); \quad \textit{for all} \ \ x \in \overline{\Omega}, \quad k = 1,2,3. \qquad (6.3.27)$$

For the sake of notational simplicity, we drop the index $j+1$. Therefore, at each grid point x_i, $i = 0, 1, \ldots, N$, $U(x_i)$ and U_i represent the solution of (6.2.16) and (6.3.19) respectively.

The local truncation error of FOFDM (6.3.19)-(6.3.20) is therefore given by

$$\begin{aligned} L^h\left(U_i - U(x_i)\right) = &- \left[r_i^+ U(x_{i+1}) + r_i^c U(x_i) + r_i^- U(x_{i-1})\right] \\ &- \left[-\varepsilon U''(x_i) + a^n(x_i)U'(x_i) + b^n(x_i)U(x_i)\right]. \end{aligned}$$

This equation implies that

$$Ł^h\left(U(x_i) - U_i\right) = T_0 U(x_i) + T_1 U'(x_i) + T_2 U''(x_i) + T_3 U'''(x_i) + T_4 U^{iv}(\xi_i), \quad (6.3.28)$$

where $\xi_i \in (x_{i-1}, x_{i+1})$ and

$$\begin{aligned} T_0 &= r_i^- + r_i^c + r_i^+ - b_i^n - \frac{h^2}{3} b_i'', \\ T_1 &= h(r_i^+ - r_i^-) - a_i^n - \frac{1}{3}h^2\left((a_i^n)'' + 2(b_i^n)'\right), \\ T_2 &= \frac{h^2}{2}\left(r_i^+ + r_i^-\right) + \varepsilon - \frac{h^2}{3}\left(2(a_i^n)' + b_i^n + \frac{1}{2}(b_i^n)''h^2\right), \\ T_3 &= \frac{h^3}{6}\left(r_i^+ - r_i^-\right) - \frac{h^2}{3}\left(a_i^n + \frac{1}{2}(a_i^n)''h^2\right), \end{aligned} \qquad (6.3.29)$$

$$T_4 = \frac{h^3}{24}\left(r_i^+ + r_i^-\right) + \frac{1}{3}\varepsilon h^2.$$

Further simplifications yield

$$T_0 = 0, \;\; |T_1| \le Mh, \;\; |T_2| \le Mh + Mh^2/\varepsilon, \;\; |T_3| \le Mh^2, \;\; |T_4| \le Mh^2. \tag{6.3.30}$$

Considering the dominating terms and using lemmas 4.2.2 and 6.3.3 we obtain

$$|L^h(U_i - U(x_i))| \le Mh\left(1 + \frac{h}{\varepsilon}\right)\left[1 + \frac{\exp(-\alpha(1-x_i)/\varepsilon)}{\varepsilon^2}\right], \text{ for } i = 1, 2, \cdots, n-1. \tag{6.3.31}$$

where α is such that $a^n(x) \ge \alpha > 0, \;\; \forall x \in [0, 1]$.

Finally, using Lemma 4.3.2 and re-instating the dropped time indice $j+1$, we obtain

$$|U_{i,j+1} - U_{j+1}(x_i)| \le Mh\left(1 + \frac{h}{\varepsilon}\right)\left[1 + \frac{\exp(-\alpha(1-x_i)/\varepsilon)}{\varepsilon^2}\right], \text{ for } i = 1, 2, \cdots, n-1.$$

and therefore applying Lemma 2.2.5, we get

Theorem 6.3.2. *If $U_{j+1}(x_i)$ is the solution of problem (6.2.16) and $U_{i,j+1}$ the solution of the discrete problem (6.3.18) at the point x_i and the $(j+1)$-th time level, there is a constant C such that*

$$\max_i |U_{i,j+1} - U_{j+1}(x_i)| \le Mh\left(1 + \frac{h}{\varepsilon}\right).$$

The results of theorems 6.3.1 and 6.3.2 are now combined to give an estimate of the fully discrete scheme.

Theorem 6.3.3. *Let $U(x_i, t_{j+1})$ be the solution of the linearized problem (6.2.11) of equation (6.1.1), and $U_{i,j+1}$ be the solution of the totally discrete equation (6.3.18). Then, there* *exists a constant M such that*

$$\max_{i,j} |U_{i,j+1} - U_{j+1}(x_i, t_{j+1})| \leq M\left(\Delta t + h\left(1 + \frac{h}{\varepsilon}\right)\right),$$

where $i = 0, 1, \ldots, N$ and $j = 0, 1, \ldots, M-1$.

6.4 Richardson extrapolation

In this section, we adapt the Richardson extrapolation of Section 5.5 to FOFDM 6.3.18.

Denoting by $\bar{U}_{i,j+1}$ and $U^{ext}_{i,j+1}$ the solutions of equation (6.3.19) on the mesh μ_{2N} and after extrapolation, respectively, we have

$$U^{ext}_{i,j+1} := 2\bar{U}_{i,j+1} - U_{i,j},\ i = 1(1)N-1.$$

Following the same lines as in Section 5.5, we obtain the following result.

Theorem 6.4.1. *Let $U(x_i, t_{j+1})$ be the solution of the linearized equation (6.2.11) of equation (6.1.1) and $U^{ext}_{i,j+1}$ the solution of the totally discrete equation (6.3.18). Then, there exists a constant M such that:*

$$\max_{i,j} \left|U^{ext}_{i,j+1} - U(x_i, t_{j+1})\right| \leq M\left(\Delta t + h^2\left(1 + \frac{1}{\varepsilon}\right)\right). \tag{6.4.32}$$

6.5 Numerical results

Example 6.5.1. We consider the equation

$$u_t + uu_x = \varepsilon u_{xx},\quad (x,t) \in (0,1) \times (0,T)$$

with initial condition

$$u(x,0) = \sin(\pi x),\quad 0 \leq t \leq T,$$

and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leq x \leq 1.$$

The exact (Fourier) solution is given by ([31])

$$u(x,t) = 2\pi\varepsilon \frac{\sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\varepsilon t) n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\varepsilon t)\cos(n\pi x)},$$

where a_0 and $a_n (n = 1, 2, \ldots)$ are the following Fourier coefficients

$$\begin{aligned} a_0 &= \int_0^1 \exp\{-(2\pi\varepsilon)^{-1}[1-\cos(\pi x)]\} dx, \\ a_n &= 2\int_0^1 \exp\{-(2\pi\varepsilon)^{-1}[1-\cos(\pi x)]\}\cos(n\pi x) dx, \quad n = 1, 2, 3, \ldots. \end{aligned}$$

Many researchers have used the above solution to evaluate the errors in their approximations.

6.6 Discussion

We have considered viscous Burgers' equation which is a nonlinear parabolic PDE and shown the quasilinearization for this equation. The set of quasilinear process equations are then solved at each time level by a novel FOFDM. Each of the quasilinear process equation is time dependent linear SPP for which it is known that the standard methods do not perform well.

The profile of the numerical solution is shown in Figure 6.1.

One remarkable issue here is the use of extrapolation with respect to the spatial variable x. We noted that the order of convergence (in x-direction) after extrapolation has improved. Due to the convergence properties of the quasilinearization, the FOFDM appears not to be ε-uniform. This aspect is under investigation.

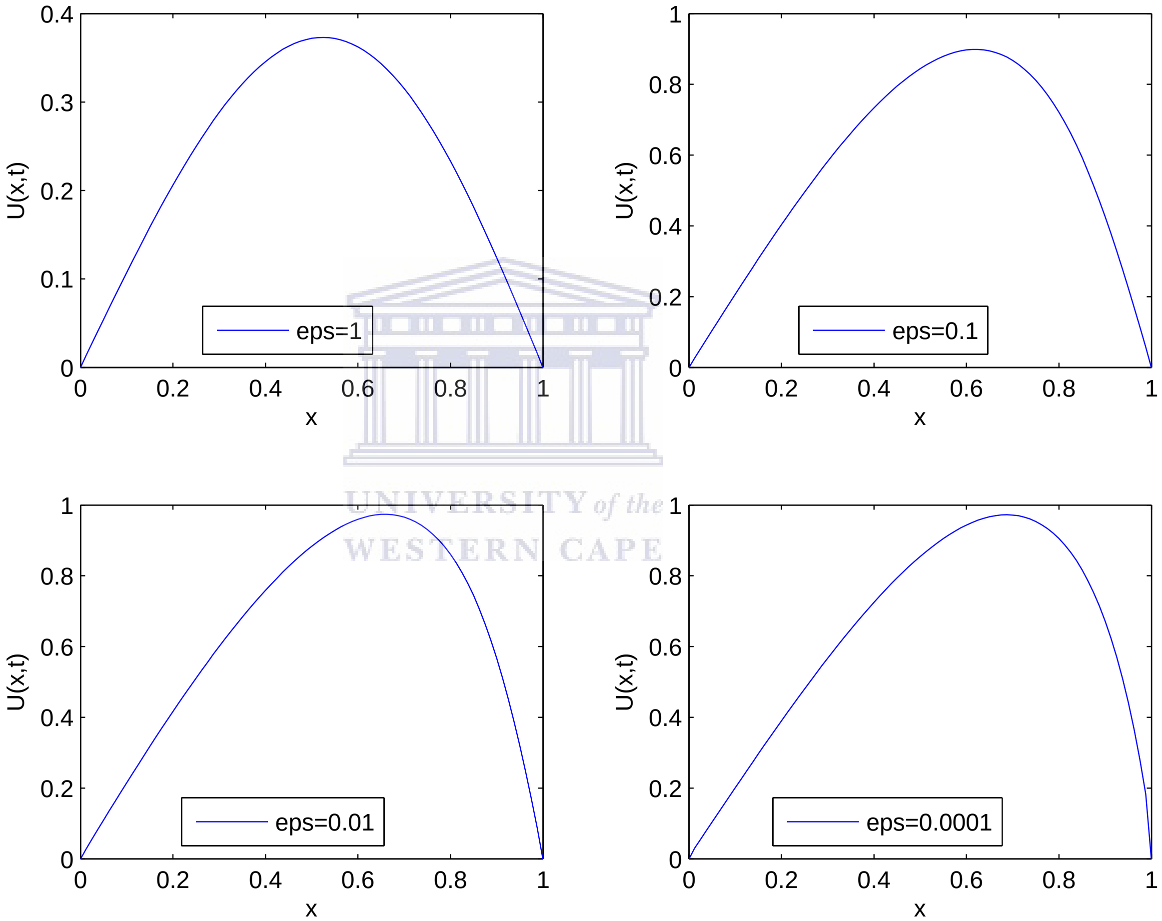

Figure 6.1: Profile of the numerical solution of the problem in Example 6.5.1 for various values of ε.

Chapter 7

Higher Order Numerical Methods for Singularly Perturbed Elliptic Problems

This chapter is devoted to a family of singularly perturbed elliptic problems in two dimensions. We extend FOFDM-II (p.64) to solve such problems. Through a rigorous convergence analysis, we show that the method is second order in both variables. This order of convergence is improved to four through extrapolation.

7.1 Introduction

We consider the problem

$$Lu := -\varepsilon \Delta u + b(x,y)u = f(x,y), \text{ in } \Omega = (0,1)^2, \tag{7.1.1}$$

$$u = 0, \text{ on } \partial\Omega. \tag{7.1.2}$$

where $\varepsilon \in (0,1]$ and b and f are sufficiently smooth functions in Ω. It is assumed that $b(x,y) \geq \alpha^2 > 0, \quad \text{in} \quad \Omega$. Also, we impose the following compatibility conditions [116, 124] which guarantee that the solution $u(x,y)$ to problem (7.1.1)-(7.1.2) is a member of $C^4(\Omega) \cap C^2(\overline{\Omega})$, where $\overline{\Omega} = \partial\Omega \cup \Omega$:

$$f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0.$$

While singularly perturbed two-point boundary value problems are well studied from different angles, their higher dimensional counterparts are not tackled sufficiently, as far as FOFDMs are concerned. There were some attempts made to extend the approaches developed for singularly perturbed ordinary differential equation but the success was very limited.

On the other hand, some researchers tried to solve these higher dimensional problems directly. Some notable works include [39, 52, 86, 124, 129, 131, 147].

A careful reading of the work of Kadalbajoo and Patidar [69] indicates that there are no extensions of any fitted operator methods developed for singularly perturbed ODEs that can solve the singularly perturbed PDEs, in particular the elliptic ones. To fill this gap, the first aim of this chapter is to extend a FOFDM (which is developed for singularly perturbed ODEs) to solve the elliptic singular perturbation problem. Then, in order to achieve a higher order convergence, we perform the Richardson extrapolation.

The rest of this chapter is organized as follows. In Section 7.2, we presents some qualitative features of the solution and its derivatives. Section 7.3 is concerned with the construction and analysis of the numerical method Section 7.4 deals with the extrapolation of the method developed in Section 7.3. Numerical results to support the theory are provided in Section 7.5. We end the chapter with a discussion of the results and related issues in Section 7.6.

7.2 Bounds on the solution and its derivatives

Lemma 7.2.1. [48] (Continuous maximum principle)
Let $\xi(x,y)$ be any sufficiently smooth function such that $\xi(x,y) \geq 0$ on $\partial\Omega$. Then $L\xi(x,y) \geq 0$ on Ω implies that $\xi(x,y) \geq 0, \ \forall (x,y) \in \overline{\Omega} = \partial\Omega \cup \Omega$.

Proof. Let (x^*, y^*) be such that

$$\xi(x^*, y^*) = \min_{(x,y)\in\overline{\Omega}} \xi(x,y)$$

and assume that $\xi(x^*, y^*) < 0$. Clearly, $(x^*, y^*) \notin \partial\Omega$. We have

$$\frac{\partial}{\partial x}\xi(x,y)\Big|_{(x^*,y^*)} = 0,$$

$$\frac{\partial}{\partial y}\xi(x,y)\Big|_{(x^*,y^*)} = 0,$$

$$\frac{\partial^2}{\partial x^2}\xi(x,y)\Big|_{(x^*,y^*)} \geq 0,$$

and

$$\frac{\partial^2}{\partial y^2}\xi(x,y)\Big|_{(x^*,y^*)} \geq 0.$$

Therefore

$$L\xi(x^*, y^*) = -\varepsilon\Delta\xi(x^*, y^*) + b(x^*, y^*)\xi(x^*, y^*) < 0,$$

which is a contradiction.

The following lemmas provide bounds on the solution of the problem (7.1.1)-(7.1.1) as well as those of its derivatives [85]. A suitable choice of barrier functions [84, 153] may be made in the proofs.

Lemma 7.2.2. *Let $u(x,y)$ be the solution of problem (7.1.1)-(7.1.2). Then we have*

(a). $|u(x,y)| \leq C\left(1 - e^{-\alpha x/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$, (b). $|u(x,y)| \leq C\left(1 - e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$,

(c). $|u(x,y)| \leq C\left(1 - e^{-\alpha y/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$,

(d). $|u(x,y)| \leq C\left(1 - e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$.

Proof. (a). Using the barrier function

$$\phi(x,y) = C(1 - e^{(-\alpha x/\sqrt{\varepsilon})}),$$

we see that

$$\begin{aligned} L(\phi \pm u) &= -\varepsilon\Delta(\phi \pm u) + b(\phi \pm u), \\ &= C\alpha^2 e^{(-\alpha x/\sqrt{\varepsilon})} + bC(1 - e^{(-\alpha x/\sqrt{\varepsilon})}) \pm f, \\ &= C(\alpha^2 - b)\left(e^{(-\alpha x/\sqrt{\varepsilon})} - 1\right) + C\alpha^2 \pm f. \end{aligned}$$

Since

$$(\alpha^2 - b)\left(e^{(-\alpha x/\sqrt{\varepsilon})} - 1\right) \geq 0,$$

we have

$$L(\phi \pm u) \geq C\alpha^2 \pm f \geq 0.$$

Using the maximum principle and the fact that $(\phi \pm u)|_{\partial\Omega} \geq 0$, we get

$$|u| \leq \phi.$$

The proof of part (b), (c) and (d) is done in a similar way by choosing the barrier functions

$$\phi(x,y) = \left(1 - e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right),$$

$$\phi(x,y) = \left(1 - e^{-\alpha y/\sqrt{\varepsilon}}\right),$$

and

$$\phi(x,y) = \left(1 - e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right),$$

respectively.

Now we have

Lemma 7.2.3. *Let $u(x,y)$ be the solution of problem (7.1.1)-(7.1.2). Then we have*

(a). $|u_x(x,y)| \leq C\varepsilon^{-1/2}$ *on* $\partial\Omega$,

(b). $|u_y(x,y)| \leq C\varepsilon^{-1/2}$ *on* $\partial\Omega$.

Proof

By Lemma 7.2.2, we have

$$\begin{aligned} |u_x(0,y)| &= \left|\lim_{x\to 0^+} \frac{u(x,y)-u(0,y)}{x}\right| \\ &\leq \lim_{x\to 0^+} \frac{C(1-e^{(-\alpha x/\sqrt{\varepsilon})})}{x} = C\frac{\alpha}{\varepsilon^{1/2}} \\ &\leq C\varepsilon^{-1/2}. \end{aligned}$$

Similarly, applying the estimate in part (b) of Lemma 7.2.2, we get the estimate for $u_x(1,y)$.

Differentiating the given boundary conditions $u(x,y) = 0$ at $y = 0$ and $y = 1$ with respect to x gives us $u_x(x,0) = u_x(x,1) = 0$ and this finishes the proof.

Similarly,

$$\begin{aligned} |u_y(x,0)| &= \left|\lim_{y\to 0^+} \frac{u(x,y)-u(x,0)}{y}\right| \\ &\leq \lim_{y\to 0^+} \frac{C(1-e^{(-\alpha y/\sqrt{\varepsilon})})}{y} \\ &\leq C\varepsilon^{-1/2}. \end{aligned}$$

We get the estimate of $u_y(x,1)$ by applying the estimate in part (d) of Lemma 7.2.2. Differentiating the given boundary conditions $u(x,y)=0$ at $x=0$ and $x=1$ with respect to y we get $u_y(0,y)=u_y(1,y)=0$. This completes the proof.

Lemma 7.2.4. *Let $u(x,y)$ be the solution of problem (7.1.1)-(7.1.2). Then we have*

(a). $|u_x(x,y)| \le C\left(1-\varepsilon^{-1/2}e^{-\alpha x/\sqrt{\varepsilon}}+\varepsilon^{-1/2}e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$,

(b). $|u_y(x,y)| \le C\left(1-\varepsilon^{-1/2}e^{-\alpha y/\sqrt{\varepsilon}}+\varepsilon^{-1/2}e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$.

Proof. By choosing the barrier function

$$\phi(x,y) = C\left(1-\varepsilon^{-1/2}e^{-\alpha x/\sqrt{\varepsilon}}+\varepsilon^{-1/2}e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right),$$

we obtain

$$L(\phi \pm u_x) \ge bC \pm (f_x - b_x u) \ge 0,$$

and since $(\phi \pm u_x)|_{\partial\Omega} \ge 0$, the proof is completed by making use of the maximum principle (Lemma [48]).

For the proof of the estimate in part (b), we can use the barrier function

$$\phi(x,y) = C\left(1-\varepsilon^{-1/2}e^{-\alpha y/\sqrt{\varepsilon}}+\varepsilon^{-1/2}e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right).$$

Now, the following results for the bounds on the second derivatives hold.

Lemma 7.2.5. *Let $u(x,y)$ be the solution of problem (7.1.1)-(7.1.2). Then we have*

(a). $|u_{xx}(x,y)| \le C\varepsilon^{-1}$ *on* $\partial\Omega$,

(b). $|u_{yy}(x,y)| \le C\varepsilon^{-1}$ *on* $\partial\Omega$. *%endLemma*

(c). $|u_{xx}(x,y)| \le C\left(1+\varepsilon^{-1}e^{-\alpha x/\sqrt{\varepsilon}}+\varepsilon^{-1}e^{-\alpha(1-x)/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$, (d). $|u_{yy}(x,y)| \leq C\left(1+\varepsilon^{-1}e^{-\alpha y/\sqrt{\varepsilon}}+\varepsilon^{-1}e^{-\alpha(1-y)/\sqrt{\varepsilon}}\right)$ *on* $\bar{\Omega}$.

Proof. See [85].

7.3 Construction and analysis of the fitted operator finite difference method

Let n and m be positive integers.

We consider the following partitions of the interval [0,1]:

$$x_0 = 0, \;\; x_i = x_0 + ih, \;\; i = 1(1)n, \;\; h = x_i - x_{i-1}, \;\; x_n = 1.$$

$$y_0 = 0, \;\; y_j = y_0 + jk, \;\; j = 1(1)m, \;\; k = y_j - y_{j-1}, \;\; y_m = 1.$$

The tensor product of these two partitions gives the mesh grid

$$\mu_{(n,m)} = \{(x_i, y_j), \;\; i = 0(1)n, \;\; j = 0(1)m\}.$$

In the rest of this chapter, we adopt the notation $W_i^j = W(x_i, y_j)$ and denote the approximations of u_i^j at the grid point (x_i, y_j) by the unknown v_i^j.

Using the theory of difference equations for problems in one dimension, we construct the following FOFDM (looking at the one dimension at a time):

$$-\varepsilon\left[\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\phi_i^j)_h^2} + \frac{v_i^{j+1} - 2v_i^j + v_i^{j-1}}{(\phi_i^j)_k^2}\right] + b_i^j v_i^j = f_i^j, \tag{7.3.3}$$

with the discrete boundary conditions

$$v_i^0 = v_0^j = v_i^m = v_n^j = 0, \; i = 0(1)n, \; j = 0(1)m, \tag{7.3.4}$$

where

$$(\phi_i^j)_h \equiv \phi_i^j(h,\varepsilon) := \frac{2}{\rho_i^j}\sinh\left(\frac{\rho_i^j h}{2}\right) \qquad (7.3.5)$$

and

$$(\phi_i^j)_k \equiv \phi_i^j(k,\varepsilon) := \frac{2}{\rho_i^j}\sinh\left(\frac{\rho_i^j k}{2}\right), \qquad (7.3.6)$$

with $\rho_i^j = \sqrt{b_i^j/\varepsilon}$.

Note that

$$\phi_i^j(h,\varepsilon) = h + O\left(\frac{h^3}{\varepsilon}\right),$$

and

$$\phi_i^j(k,\varepsilon) = k + O\left(\frac{k^3}{\varepsilon}\right).$$

For the sake of simplicity, we assume that $h = k$, and hence the common denominator will then be $\phi_i^j (= (\phi_i^j)_h = (\phi_i^j)_k)$. Thus equation (7.3.3) becomes

$$-\varepsilon\left[\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\phi_i^j)^2} + \frac{v_i^{j+1} - 2v_i^j + v_i^{j-1}}{(\phi_i^j)^2}\right] + b_i^j v_i^j = f_i^j, \qquad (7.3.7)$$

which we rewrite as

$$-\frac{\varepsilon}{(\phi_i^j)^2}\left[v_{i+1}^j + v_{i-1}^j + v_i^{j+1} + v_i^{j-1} - 4v_i^j\right] + b_i^j v_i^j = f_i^j. \qquad (7.3.8)$$

One should note that, in the above we have considered $h = k$ merely for the sake of simplicity. However, in the analysis below, we keep the general set up.

We start with stating the following two lemmas whose roles are primordial in the analysis of the method developed in previous section.

Lemma 7.3.1. *(Discrete maximum principle) Let* $\{\xi_i^j\}$ *be any mesh function satisfying*
$\xi_i^0 \geq 0, \;\; i = 1(1)n-1,$
$\xi_i^m \geq 0, \;\; i = 1(1)n-1,$
$\xi_0^j \geq 0, \;\; i = 1(1)m-1,$

$\xi_n^j \geq 0, \;\; i = 1(1)m-1,$

$\xi_0^0 \geq 0, \, \xi_n^0 \geq 0, \, \xi_0^m \geq 0, \, \xi_m^n \geq 0,$

and $L_h^k \xi_i^j \geq 0, \; i = 1(1)n-1; j = 1(1)m-1.$

Then $\xi_i^j \geq 0, \; \forall i = 0(1)n, j = 0(1)m.$

Proof Let (s,t) be indices such that

$$\xi_s^t = \min_{(i,j)} \xi_i^j, \;\; \forall \;\; (i,j) \in \{0,1,\ldots,n\} \times \{0,1,\ldots,m\}.$$

Assume that $\xi_s^t < 0$. It is clear that

$$(s,t) \in \{1,2,\ldots,n-1\} \times \{1,2,\ldots,m-1\}$$

or else, $\xi_s^t \geq 0$.

We observe that

$$\xi_{s+1}^t - \xi_s^t > 0,$$

$$\xi_{s-1}^t - \xi_s^t > 0,$$

$$\xi_s^{t+1} - \xi_s^t > 0,$$

$$\xi_s^t - \xi_s^{t-1} > 0.$$

Therefore

$$L_h^k \xi_s^t < 0,$$

which is a contradiction.

Lemma 7.3.2. *If Z_i^j is any mesh function such that $Z_i^j = 0$ on $(\partial\Omega)_i^j$, then there exists a constant C such that*

$$|Z_l^s| \leq C \max_{1 \leq i \leq n-1; 1 \leq j \leq m-1} |L_h^k Z_i^j|, \;\; \textit{for} \;\; 0 \leq l \leq n; 0 \leq s \leq m.$$

Proof. The proof follows similar lines as those for the FOFDMs developed for singularly perturbed linear two-point boundary value problems.

7.3.1 Error estimate before extrapolation

The local truncation error of the FOFDM (7.3.3)-(7.3.4) is

$$\begin{aligned}
L_h^k(u_i^j - v_i^j) &= \left\{-\varepsilon(\Delta u)_i^j + b_i^j u_i^j\right\} \\
&\quad - \left\{-\varepsilon\left[\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\phi_i^j)_h^2} + \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\phi_i^j)_k^2}\right] + b_i^j u_i^j\right\} \\
&= -\varepsilon(u_{xx})_i^j - \varepsilon(u_{yy})_i^j \\
&\quad + \frac{\varepsilon}{(\phi_i^j)_h^2}\left[h^2(u_{xx})_i^j + \frac{h^4}{12}(u_{xxxx})_i^j + \cdots\right] \\
&\quad + \frac{\varepsilon}{(\phi_i^j)_k^2}\left[k^2(u_{yy})_i^j + \frac{k^4}{12}(u_{yyyy})_i^j + \cdots\right] \\
&= -\varepsilon(u_{xx})_i^j - \varepsilon(u_{yy})_i^j \\
&\quad + \left(\frac{\varepsilon}{h^2} - \frac{b_i^j}{12} + \frac{h^2(b_i^j)^2}{240\varepsilon} + \cdots\right)\left[h^2(u_{xx})_i^j + \frac{h^4}{12}(u_{xxxx})_i^j + \cdots\right] \\
&\quad + \left(\frac{\varepsilon}{k^2} - \frac{b_i^j}{12} + \frac{k^2(b_i^j)^2}{240\varepsilon} + \cdots\right)\left[h^2(u_{yy})_i^j + \frac{k^4}{12}(u_{yyyy})_i^j \cdots\right]
\end{aligned}$$

This implies that

$$\begin{aligned}
L_h^k(u_i^j - v_i^j) &= \frac{\varepsilon h^2}{12}(u_{xxxx})_i^j - \frac{h^2(b_i^j)^2}{12}(u_{xx})_i^j - \frac{h^4(b_i^j)}{144}(u_{xxxx})_i^j + \frac{h^4(b_i^j)^2}{240\varepsilon}(u_{xx})_i^j \\
&\quad + \frac{\varepsilon k^2}{12}(u_{yyyy})_i^j - \frac{k^2(b_i^j)^2}{12}(u_{yy})_i^j - \frac{k^4(b_i^j)}{144}(u_{yyyy})_i^j + \frac{k^4(b_i^j)^2}{240\varepsilon}(u_{yy})_i^j + \cdots
\end{aligned}$$

Using Lemma 7.2.5 we obtain

$$|L_h^k(u_i^j - v_i^j)| \leq M\left[h^2\left(1 + \frac{h^2}{\varepsilon}\right) + k^2\left(1 + \frac{k^2}{\varepsilon}\right)\right].$$

Then by Lemma 7.3.2, we have

$$\max_{0\le i\le n}\max_{0\le j\le m}|u_i^j - v_i^j| \le M\left[h^2\left(1+\frac{h^2}{\varepsilon}\right)+k^2\left(1+\frac{k^2}{\varepsilon}\right)\right]. \quad (7.3.9)$$

Note that, if $h = k$, then we have the estimate

$$\max_{0\le i\le n}\max_{0\le j\le n}|u_i^j - v_i^j| \le Mh^2\left(1+\frac{h^2}{\varepsilon}\right). \quad (7.3.10)$$

7.4 Extrapolation on the fitted operator finite difference method

7.4.1 Extrapolation formula

Let $\mu_{(2n,2m)} = \{(\bar{x}_i, \bar{y}_j)\}$ be the mesh with $\bar{x}_0 = 0$, $\bar{x}_n = 1$, $\bar{y}_0 = 0$, $\bar{y}_m = 1$, and $\bar{x}_i - \bar{x}_{i-1} = \bar{h} = h/2$, $j = 1(1)2n$, and $\bar{y}_j - \bar{y}_{j-1} = \bar{k} = k/2$, $j = 1(1)2m$, and $\bar{v}_i^j$ denote the numerical solution computed on the mesh $\mu_{(2n,2m)}$.

On one hand, we have from (7.3.9),

$$u_i^j - v_i^j = M(h^2+k^2) + R_n(x_i) + R_m(y_j),$$

$$1 \le i \le n-1, \;\; 1 \le j \le m-1.$$

On the other hand, we have

$$\bar{u}_i^j - \bar{v}_i^j = M(\bar{h}^2+\bar{k}^2) + R_{2n}(\bar{x}_i) + R_{2m}(\bar{y}_j),$$

$$1 \le i \le 2n-1, \;\; 1 \le j \le 2m-1.$$

Therefore,

$$u_i^j - \frac{4\bar{v}_i^j - v_i^j}{3} = O(h^2+k^2), \;\; \forall (x_i, y_j) \in \mu_{(2n,2m)}.$$

We therefore set

$$(v_i^j)^{ext} := \frac{4\bar{v}_i^j - v_i^j}{3}$$

as the extrapolation formula.

7.4.2 Analysis of the extrapolation process

The local truncation error after extrapolation is

$$\bar{L}_h^k\left(u_i^j - (v_i^j)^{ext}\right) = \frac{4}{3}L_{\bar{h}}^{\bar{k}}(u_i^j - \bar{v}_i^j) - \frac{1}{3}L_h^k(u_i^j - v_i^j). \tag{7.4.11}$$

While $L_h^k(u_i^j - v_i^j)$ is given by equation (7.3.9), $L_{\bar{h}}^{\bar{k}}(u_i^j - \bar{v}_i^j)$ is obtained from $L_h^k(u_i^j - v_i^j)$ by substituting h and k by $\bar{h}$ and $\bar{k}$, respectively. It follows that

$$\begin{aligned}\bar{L}_h^k\left(u_i^j - (v_i^j)^{ext}\right) \;=\; & \frac{4}{3}\left[\frac{\varepsilon\bar{h}^2}{12}(u_{xxxx})_i^j - \frac{\bar{h}^2(b_i^j)^2}{12}(u_{xx})_i^j - \frac{\bar{h}^4(b_i^j)}{144}(u_{xxxx})_i^j + \frac{\bar{h}^4(b_i^j)^2}{240\varepsilon}(u_{xx})_i^j\right.\\ & \left.+\frac{\varepsilon\bar{k}^2}{12}(u_{yyyy})_i^j - \frac{\bar{k}^2(b_i^j)^2}{12}(u_{yy})_i^j - \frac{\bar{k}^4(b_i^j)}{144}(u_{yyyy})_i^j + \frac{\bar{k}^4(b_i^j)^2}{240\varepsilon}(u_{yy})_i^j + \cdots\right]\\ & -\frac{1}{3}\left[\frac{\varepsilon h^2}{12}(u_{xxxx})_i^j - \frac{h^2(b_i^j)^2}{12}(u_{xx})_i^j - \frac{h^4(b_i^j)}{144}(u_{xxxx})_i^j + \frac{h^4(b_i^j)^2}{240\varepsilon}(u_{xx})_i^j\right.\\ & \left.+\frac{\varepsilon k^2}{12}(u_{yyyy})_i^j - \frac{k^2(b_i^j)^2}{12}(u_{yy})_i^j - \frac{k^4(b_i^j)}{144}(u_{yyyy})_i^j + \frac{k^4(b_i^j)^2}{240\varepsilon}(u_{yy})_i^j + \cdots\right].\end{aligned}$$

A simplification leads to

$$\bar{L}_h^k\left(u_i^j - (v_i^j)^{ext}\right) := \frac{b_i^j h^4}{576}(u_{xxxxx})_i^j - \frac{(b_i^j)^2 h^4}{960\varepsilon}(u_{xx})_i^j + \frac{b_i^j k^4}{576}(u_{xxxxx})_i^j - \frac{(b_i^j)^2 k^4}{960\varepsilon}(u_{xx})_i^j + \cdots.$$

Using Lemma 7.2.5 and its analogues for fourth order derivative terms we obtain

$$\left|\bar{L}_h^k\left(u_i^j - (v_i^j)^{ext}\right)\right| \le M(h^4 + k^4)\left(1 + \frac{1}{\varepsilon}\right). \tag{7.4.12}$$

By Lemma 7.3.2, we obtain

$$\left|u_i^j - (v_i^j)^{ext}\right| \leq M(h^4 + k^4)\left(1 + \frac{1}{\varepsilon}\right). \tag{7.4.13}$$

We summarize the results in the following theorem

Theorem 7.4.1. *Let $b(x,y)$ and $f(x,y)$ be sufficiently smooth functions in the problem (7.1.1)-(7.1.2) so that $u(x,y) \in C^4([0,1]^2)$. Then the numerical solutions v and v^{ext} obtained via the FOFDM (7.3.3)-(7.3.4) before and after extrapolation, respectively, satisfy the following estimates*

$$\max_{0\leq i\leq n}\max_{0\leq j\leq m}|u_i^j - v_i^j| \leq M\left[h^2\left(1 + \frac{h^2}{\varepsilon}\right) + k^2\left(1 + \frac{k^2}{\varepsilon}\right)\right]. \tag{7.4.14}$$

$$\max_{0\leq i\leq n}\max_{0\leq j\leq m}|u_i^j - (v_i^j)^{ext}| \leq M(h^4 + k^4)\left(1 + \frac{1}{\varepsilon}\right). \tag{7.4.15}$$

7.5 Numerical results

In this section, we give some numerical results for a test example corresponding to the problem (7.1.1)-(7.1.2). In the implementation of the FOFDM (7.3.3)-(7.3.4) before and after extrapolation, we assume that the step-sizes h and k in x- and y-directions, respectively, are equal.

Example 7.5.1. Consider problem (7.1.1)-(7.1.2) with $b = 2$,

$$\begin{aligned} f(x,y) &= -\frac{e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} - \frac{e^{-y/\sqrt{\varepsilon}} + e^{-(1-y)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} \\ &\quad +2\left[1 + \varepsilon\left(x(1-x) + y(1-y) + xy(1-x)(1-y)\right)\right]. \end{aligned}$$

The exact solution is

$$u(x,y) = \left(1 - \frac{e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}}\right)\left(1 - \frac{e^{-y/\sqrt{\varepsilon}} + e^{-(1-y)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}}\right) + xy(1-x)(1-y).$$

The maximum errors at all mesh points are calculated using the formulas

$$E_{\varepsilon,n} := \max_{0\le i,j\le m} |u_i^j - v_i^j|, \text{ before extrapolation}$$

and

$$E_{\varepsilon,n}^{ext} := \max_{0\le i,j\le m} |u_i^j - (v_i^j)^{ext}|, \text{ after extrapolation.}$$

The numerical rates of convergence are computed using the formula [33]

$$r_{\varepsilon,s} := \log_2(\widetilde{E}_{n_s}/\widetilde{E}_{2n_s}),\ s = 1, 2, \ldots$$

where $\widetilde{E}$ stands for $E_{\varepsilon,n}$ and $E_{\varepsilon,n}^{ext}$, respectively.

Table 7.1: Maximum errors **before** extrapolation

ε	n=8	n=16	n=32	n=64
1	1.62E-04	4.05E-05	1.01E-05	2.53E-06
2^{-1}	3.59E-04	8.98E-05	2.25E-05	5.62E-06
2^{-2}	8.98E-04	2.25E-04	5.64E-05	1.41E-05
2^{-3}	2.26E-03	5.68E-04	1.42E-04	3.56E-05
2^{-4}	4.52E-03	1.15E-03	2.89E-04	7.24E-05
2^{-5}	6.71E-03	1.76E-03	4.46E-04	1.12E-04
2^{-6}	1.10E-02	3.07E-03	7.88E-04	1.99E-04
2^{-7}	1.95E-02	5.76E-03	1.51E-03	3.83E-04
2^{-8}	2.65E-02	1.04E-02	2.91E-03	7.53E-04
2^{-9}	2.30E-02	1.91E-02	5.67E-03	1.49E-03

Table 7.2: Maximum errors **after** extrapolation

ε	n=8	n=16	n=32
1	1.12E-07	7.06E-09	4.43E-10
2^{-1}	8.71E-07	5.49E-08	3.44E-09
2^{-2}	6.01E-06	3.80E-07	2.38E-08
2^{-3}	2.96E-05	1.90E-06	1.19E-07
2^{-4}	1.02E-04	6.78E-06	4.30E-07
2^{-5}	3.51E-04	2.48E-05	1.60E-06
2^{-6}	1.19E-03	9.31E-05	6.19E-06
2^{-7}	3.18E-03	3.40E-04	2.40E-05
2^{-8}	5.14E-03	1.18E-03	9.19E-05
2^{-9}	4.31E-03	3.15E-03	3.38E-04

Table 7.3: Rates of convergence **before** extrapolation, $n_s = 8, 16, 32$.

ε	r_1	r_2	r_3
1	2.00	2.00	2.00
2^{-1}	2.00	2.00	2.00
2^{-2}	2.00	2.00	2.00
2^{-3}	1.99	2.00	2.00
2^{-4}	1.97	1.99	2.00
2^{-5}	1.93	1.98	2.00
2^{-6}	1.84	1.96	1.99
2^{-7}	1.76	1.93	1.98
2^{-8}	1.35	1.84	1.95

Table 7.4: Rates of convergence **after** extrapolation, $n_s = 8, 16$

ε	r_1	r_2
1	3.98	3.99
2^{-1}	3.99	4.00
2^{-2}	3.99	4.00
2^{-3}	3.98	4.00
2^{-4}	3.96	3.99
2^{-5}	3.91	3.98
2^{-6}	3.82	3.95
2^{-7}	3.68	3.91
2^{-8}	3.23	3.82

7.6 Concluding remarks

This chapter was concerned with singularly perturbed elliptic problems in two dimensions. Our aim was to design a fitted operator finite difference method for these problems and to investigate the effect of extrapolation on the convergence of this novel method. The method showed to be second order convergent. The extrapolation improves this convergence up to fourth order. Numerical results presented in tables 7.1-7.4 confirm the theoretical estimates given in (7.4.14)-(7.4.15).

Chapter 8

Concluding remarks and scope for future research

In this thesis, we have considered many classes of singularly perturbed problems. This include, linear and non-linear two-point boundary value problems, turning point problems, time dependent parabolic problems and elliptic problems. Our aim was to construct some higher order methods for these problems. This could be done either by designing direct methods or by making use of the convergence acceleration techniques (for example, Richardson extrapolation, defect corrections, etc.). Due to the fact that the convergence acceleration techniques can cater for the large class of problems, we have decided to choose this later option.

The main observation that we have made through the work in this thesis is that if a singular perturbation model involves the first derivative term(s) of the solution, then the extrapolation technique improves the accuracy of the underlying fitted method, while the rate of convergence remains intact in many cases. However, if the model does not involve the first derivative term(s), the rate of convergence can also be improved. This depends on the fitted method utilized.

In Chapter 2, we investigated the effect of Richardson extrapolation on the fitted mesh finite difference method (FMFDM) for a self-adjoint problem. We noted that even though the accuracy is improved, the order of convergence remains unchanged. This unexpected fact contradicts the assertion met in the literature about Richardson extrapolation. This motivated us to investigate, for the same class of problems, which impact the extrapolation technique will have on other methods to solve the above mentioned self-adjoint problem in Chapter 3. We considered two fitted operator finite difference methods (FOFDMs) which we denoted by FOFDM-I and FOFDM-II. In the first case, the extrapolation does not improve the convergence which is of order four and two for some moderate and smaller values of ε. In the latter case, the second order accuracy is improved up to four, irrespective of the value of ε. We are investigating this issue in more details.

Chapter 4 dealt with the construction and analysis of a FMFDM and a FOFDM to solve a singularly perturbed turning point problem whose solution has boundary layers. We studied the performance of Richardson extrapolation on these methods. The conclusions drawn after analysis are in line with the observations made earlier: The turning point problem involves a first order derivative term and therefore, the rate of convergence is not increased even though the accuracy is improved in both cases. As a scope of future work, we intend to explore the proposed method in this chapter to solve multiple turning point problems.

In Chapter 5, we considered a singularly perturbed nonlinear two-point boundary value problem. We first applied the quasilinearization process to linearize the problem. Then the resulting sequence of linear problems was solved by a FOFDM. The ideas developed here are extended in Chapter 6 to solve a time-dependent nonlinear Burgers' equation. Currently we are investigating whether we can use a direct approach to solve these types of nonlinear problems.

The FOFDM-II of Chapter 3 is extended for solving singularly perturbed elliptic problems of reaction-diffusion type in 2-dimensions in Chapter 7. This method is of order 2 in both x- and y-directions. A remarkable fact is that the fourth order convergence is achieved after applying Richardson extrapolation.

Due to the space limitations, we did not include the impact of defect correction technique on the various classes of problems mentioned in this thesis. Some work has been done in this regard (see, e.g., [43]) on a singularly perturbed problem of the convection-diffusion type in one dimension. We would like to deepen this study to various classes of singular perturbation problems and come up with general conclusions. Currently, we are also studying the singularly perturbed turning point problems whose solution has interior layers.

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