A mathematical model of performance measurement of defined contribution pension funds

KELEKELE LILOO Didier Joel

A mini-thesis submitted in partial fulfilment of the requirements

for the degree of Master of Science (Computational Finance),

Supervisor: Prof P.J. Witbooi

University of the Western Cape,

Department of Mathematics and Applied Mathematics,

South Africa.

February 18, 2015

Declaration

I declare that this is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Kelekele Liloo Didier Joel.   February, 2015

Signed: ....

Acknowledgements

I would like to thank my Saviour and Lord Jesus for rendering it possible to complete this programme; albeit with many difficulties we have endured in the process. A special thanks goes to the Professor Peter Witbooi who have accepted to supervise this mini-thesis. I would also like to thank all my family, parents and siblings, my fiancee Ruth Lumu Nsase, my pastor and fellow brethren in Christ and my fellow students who have been supportive. May you all find my heartfelt gratitude. A special thought goes to Felly Ilunga Bitokuela, a soldier who has fallen on the field of battle so early.
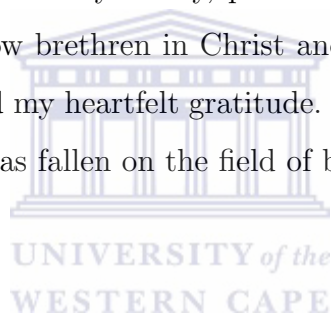
Key words

Performance measurement, asset allocations, defined contribution, defined benefit, minimum guarantee, sharing rule, pension products, stochastic control, martingale, Brownian motion, power utility, growth optimal portfolio, benchmark function, sensitivity analysis.

Abstract

The industry of pension funds has become one of the drivers of today's economic activity by its important volume of contribution in the financial market and by creating wealth. The increasing importance that pension funds have acquired in today's economy and financial market, raises special attention from investors, financial actors and pundits in the sector. Regarding this economic weight of pension funds, a thorough analysis of the performance of different pension funds plans in order to optimise benefits need to be undertaken. The research explores criteria and invariants that make it possible to compare the performance of different pension fund products. Pension fund companies currently do measure their performances with those of others. Likewise, the individual investing in a pension plan compares different products available in the market. There exist different ways of measuring the performance of a pension fund according to their different schemes. Generally, there exist two main pension funds plans. *The defined benefit* (DB) pension funds plan which is mostly preferred by pension members due to his ability to hold the risk to the pension fund manager. *The defined contributions* (DC) pension fund plan on the other hand, is more popularly preferred by the pension fund managers due to its ability to transfer the risk to the pension fund members. One of the reasons that motivate pension fund members' choices of entering into a certain programme is that their expectations of maintaining their living lifestyle after retirement are met by the pension fund strategies. This dissertation investigates the various properties and characteristics of the defined contribution pension fund plan with a minimum guarantee and benchmark in order to mitigate the risk that pension fund members are subject to. For the pension fund manager the aim is to find the optimal asset allocation strategy which optimises its

retribution which is in fact a part of the surplus (the difference between the the pension fund value and the guarantee) (2004) [19] and to analyse the effect of sharing between the contributor and the pension fund. From the pension fund members' perspective it is to define a optimal guarantee as a solution to the contributor's optimisation programme. In particular, we consider a case of a pension fund company which invests in a bond, stocks and a money market account. The uncertainty in the financial market is driven by Brownian motions. Numerical simulations were performed to compare the different models.

Kelekele Liloo Didier Joel

MSc Thesis, Department of Mathematics and Applied Mathematics, University of the Western Cape.

February 2015.

List of Acronyms

AL: actuarial liability;

CARA: constant absolute risk aversion;

CRRA: constant relative risk aversion;

CIR model: Cox-Ingersoll-Ross model;

CLI: cost of living index;

DB: defined benefit;

DC: defined contribution;

HJB: Hamiltion-Jacobi-Bellman;

NC: normal contribution;

PR: pension ratio;

RCR: recommended contribution rate;

TSR: total salary roll;

UK: United Kingdom;

US: United States;

List of Tables

List of Figures

Contents

Chapter 1

General introduction

1.1 Introduction

The need for assessing the performance of pension fund managers or investment managers
has become imperative, in view of the large amounts of funds that pension managers are
handling, and the scale of competition this market is facing. From the investor's point of
view the need is even more serious when one considers the increasing volatility that the
financial market is subjected to. Consider the global financial crisis of 2008, the scandal
of the banking system of 2009, the Eurozone crisis of 2010 and all the economic turmoil
that have been daily reported to the media and news feeds. These events underline
the importance, so paramount, of improving the management of financial institutions,
even more so of mutual funds, when taking into account their social and economic value
and impact. The management of mutual funds such as in pension schemes is so complex,
requiring some high level of skill and sophistication. Pension funds managers and investors
are aware of the above problems and they are concerned to know the performance of
different pension fund schemes that are available in the financial market. To analyse
and assess the performance of financial institutions in general and specifically those of
mutual funds, some practitioners and pundits on the financial field has laid down some
methods and practices. The literature engaging the management of investment strategies

of pension funds is prolific and present methodologically two approaches:

- The stochastic control method.
  This method was pioneered by Merton (1969,1971) [48] [49]. The stochastic methodology is mainly based on solving the Hamilton-Jacobi-Bellman (HJB) equation coming from the dynamic programming under the real world probability measure. Several authors have laid down theories related to the stochastic control approach. For instance, Vigna and Haberman (2001) [56] used stochastic dynamic programming to analyse the financial risk in a defined contribution (DC) pension scheme under Gaussian interest rate models. They attempted to find an optimal investment strategy, given a final target linked to the net replacement ratio and a set of interim targets. Haberman and Vigna (2002) [32] extended their paper of earlier to study the investment allocation in a DC scheme whose fund is invested in several assets, and considered three different risk measures to analyse the final net replacement ratios. Devolder *et al* (2003) [20] studied the management of an annuity contract under an interest rate model described by a geometric Brownian motion. Menoncin and Scaillet (2006) [51] provided the same exercise for a life annuity contract. Xiao *et al* (2007) [58] applied the constant elasticity of variance model to an annuity contract and derived the dual solution for the logarithm utility via the Legendre transform. However, these studies generally suppose the short rate to be a constant. As the contribution period in a pension plan is very long, generally from 20 to 40 years, the assumption of constant rates is then difficult to accept in the field of DC plans (see Gao 2008) [27].

- The second approach, is the martingale method.
  It aims generally to calculate an expectation under a risk-neutral measure. This method was first developed by Cox and Huang (1989) [16] in the setting of complete markets and relies on the theory of Lagrange multipliers. In the last few years, it was successfully applied by other authors such as Boulier *et al* (2001) [7] and Deelstra *et al* (2003, 2004) [18] [19] to study the optimal design and asset allocation of a

pension plan. Recently, Hainaut and Devolder (2006) [33] analysed the dividend policy and asset allocation of a pension fund under the Vasicek interest model.

While a pension fund manager are called to use these different methods to produce high returns in the management of assets, one can question the true degree of performance these funds managers achieve. When a pension member enters a pension programme, his or her expectation is that the mutual fund will be able to keep safe his wealth and to increase it in some extent. Because guided by the axiom of non-satiation, which claims that *"plus is preferred to minus"* the pension fund member is expected to steer the investment to yield results that can allow him to continue living the lifestyle he is used to. One of the instruments of measuring the level of lifestyle is the individual income. The investment manager is called to make optimal asset allocation which can yield outcomes that are close to the level of the pension member's income just prior to retirement.

1.2 Research Problem

The defined contribution pension fund plan has gained important consideration in the investment portfolio management in the last two decades. Some practitioners of the world of finance have been paying attention to the maintenance of lifestyle of the pension fund member as the main motivation guiding their choice of different investment products available in the financial market and thus the presence of some guarantee in order to mitigate the risk exposure of pension fund members. Authors such as Boulier *et al* [7] considers the Vasicek model for the interest rate and have presented a minimum guarantee in the form of deterministic process. Deelstra *et al* (2003) [18] considers stochastic dynamics for the interest rate, covering as a special case of the Vasicek model and of the Cox-Ingersoll-Ross model. Their method considers a stochastic minimum guarantee to the worker at retirement plus a fraction of the surplus (the difference between the final wealth and the minimum guarantee). The same authors, in Deelstra *et al* (2004) [19], present a model with various choices of guarantees that are available to the pension members. The main objective of the present study is to analyse the defined contribution pension fund model

3

with a minimum stochastic guarantee serving as a benchmark and a sharing rule between the fund manager and the pension fund member.

1.3 Research Objectives
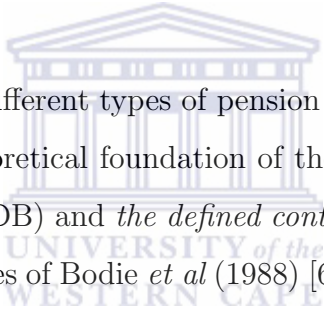
The present research attempts to find an optimal guarantee for a DC pension fund plan subject to a sharing rule between the fund members and the fund manager. Thus the objective is then to maximise a certain expected utility of the portfolio manager from his part of the surplus; and the determination of the optimal sharing rule between the fund manager and the pension fund member. The structure of the objective function in the present research is widely used in the investment management literature; the model being used in the present research assumes that the parameters are time varying. The objective function depends on a risky asset, riskless asset, the zero coupon bond and the guarantee.

1.4 Structure

The present dissertation proceeds as follows: Chapter two discusses the different types of pension fund plan. The third chapter present the mathematical model of the defined contribution (DC) pension fund. The fourth chapter presents the pension fund model with benchmark under the framework of Browne (1999) [10] with the introduction of the sharing rule and *the growth optimal portfolio* (GOP). The fifth chapter gives a conclusion.

Chapter 2

Different types of pension fund schemes

The present chapter reviews different types of pension fund schemes that are available in the market. It revisit the theoretical foundation of the two main pension fund schemes, which are *the defined benefit* (DB) and *the defined contribution* (DC). The main reference of the chapter are the researches of Bodie *et al* (1988) [6], Boulier *et al* (1995) [8] , Andrew Cairns (2000) [11], Andrew Cairns (2003) [12] and Cairns *et al* (2000) [13] . The chapter discusses how the DB and the DC pension fund schemes have evolved in years and how they have been adapted.

2.1   The defined benefit plan

The defined benefit (DB) schemes can be viewed as a plan where the employees' pension benefit entitlement is determined by a formula which takes into account years of service for the employer and, in most cases, wages or salary. Many defined benefit formulas also take into account the Social Security benefits to which an employee is entitled. These are the so-called integrated plans (1988) [6]. The defined benefit pension sheme can be analysed in many ways. In what follows we briefly review the different mathematical approaches to DB plans.

2.1.1 Deterministic methods

A process is called deterministic if its value as a function of time can be pre-determined at a given time. We are presenting in this paragraph a simple discrete time deterministic model of a pension fund, which aims to build a pension that will be in step with the final salary at retirement. The model features in the paper of Cairns (2003) [12] and it has the following assumptions:

- Salaries are increased each year in line with a *cost-of-living index* CLI(*t*).

- At the start of the year $[t, t+1)$, one new member joins the scheme at age 25, with a salary of $10000 \times$ CLI(*t*)/CLI(*0*).

- All members stay with the scheme until age 65 and mortality before age 65 is assumed to be zero.

- At age 65 the member retires and receives a pension equal to a fraction of the final salary for each year as a member of the scheme, payable annually in advance for life. This pension is secured by the pension fund member purchasing an annuity from a life insurer in order to secure his pension and the pension fund has no further responsibility such as making payments to the member. The final salary is defined as the salary rate at age 65 including the cost-of-living salary increase at age 65.

- Salaries are increased at the start of each year and include a combination of age-related and cost-of-living increases.

Let $S(t, x)$ represent the salary at time t for a member aged x at that time. The scheme structure described above indicates that

$$S(t+1, x) = S(t, x) \times \frac{w(x+1)}{w(x)} \times \frac{\text{CLI}(t+1)}{\text{CLI}(t)} \qquad (2.1)$$

where the function $w(x)$ is called the wage profile and determines age-related increases. Initially it is assumed that $S(0, x) = 10000w(x)/w(25)$ for $x = 25, \ldots, 65$, and it follows that we have the identity:

6

$$S(t, x) = S(0, x)\frac{\text{CLI}(t)}{\text{CLI}(0)} \tag{2.2}$$

In this equation, we find that the actuary has a twofold traditional role : Firstly, the actuary must determine the actuarial liability at time t which is then compared with the value of the assets. Secondly, the actuary must recommend a contribution rate. There are different approaches for responding to these two questions. We present two of the most popular methods in this regard.

- The first one is the Entry Age Method approach. This approach has emerged in the UK. The method regards the contract as a life insurance policy and poses the question of : *what contribution rate (as a percentage of salary) should a new member pay throughout his career in order to have the right amount of cash available at age 65 to buy the promised pension?* The answer to this question is what paves the way to calculate the fund overall liability [12].

- The second approach is the Projected Unit Method. It is firstly based on the calculation of the actuarial liability. In this method only the contribution of fund accrued to date is valued and the method does not make any allowance for benefits arising from continued service in the future. The valuation also takes into account projected future salary increases (age-related and cost-of-living). In this case the actuarial liability at time t is

$$\text{AL}(t) = \sum_{x=25}^{64} \frac{x-25}{60} \left(S(t, x)\frac{w(65)}{w(x)}(1+\varepsilon)^{65-x} \right) v^{65-x}\kappa_{65} + \frac{40}{60}S(t, 65)\kappa_{65} \tag{2.3}$$

In this equation ε represents the assumed rate of growth of $\text{CLI}(t), (v = 1/(1+i))$ where i is the valuation rate of interest and κ_{65} is the assumed price for a unit of pension from age 65. It is generally known that pensions are bought out at age 65. The component of $\text{AL}(t)$ of the equation (2.3) represents the liability for the member who has just attained age 65 but for whom a pension has not yet been

purchased. The incorporation of some decrements such as mortality, ill-health retirement, resignation from the company before the age of 65 and benefits coming along with such as early exits are often seen as not appropriate. Since salaries at each constant age x increase each year in line with CLI(t), we note that:

AL$(t+1)$ = AL(t)CLI$(t+1)$/CLI(t).

Having assets equal to liabilities at time t and at the time $t+1$, the normal contribution rate of the pension fund member, NC(t), payable at time t encompasses the following elements:

– the sponsor pays in NC(t) times the total salary roll at t,

$$\text{TSR}(t) = \sum_{i=25}^{65} S(t, x); \tag{2.4}$$

– the pension-fund trustees immediately secure the purchase of a pension (for a price $B(t)$) for the member who has just attained age 65;

With experience suggesting that anticipation in the valuation basis of investment return, the increase of salary at time $t+1$ and no deaths occurring before time $t+1$, this gives the following:

$$(\text{AL}(t) + \text{NC}(t)\text{TSR}(t) - B(t))(1+i) = \text{AL}(t)(1+\varepsilon). \tag{2.5}$$

This means that

$$\text{NC}(t) = (B(t) - \text{AL}(t)(1-\rho))/\text{TSR}(t) \tag{2.6}$$

where $\rho = (1+\varepsilon)/(1+i)$ is the assumed real discount factor. The remaining element of a funding valuation is the recommendation of a contribution rate and NC(t) is only appropriate if assets equal liabilities. This leads us to the concept of amortisation of surplus or deficit. Let $F(t)$ represent the fund size at time t, so that the deficit on the funding basis is AL(t)-$F(t)$ The recommended contribution rate is

$$\text{RCR}(t) = \text{NC}(t) + \left(\frac{\text{AL}(t) - F(t)}{\text{TSR}(t)\kappa \Sigma_\eta} \right) \tag{2.7}$$

where

$$\kappa\Sigma_\eta = \sum_{k=0}^{\eta-1} \rho^k = \frac{1-\rho^\eta}{1-\rho} \tag{2.8}$$

The constant η is the amortisation period. This could be set according to how rapidly the fund sponsor wants to get rid of surplus or deficit. Often, though, it is set equal to the expected future working lifetime of the active membership, in line with certain accounting guidelines. A related approach to amortisation is mostly used in North America. The adjustment is divided into η components which are each relating to the amortisation of the surplus or deficit arising in each of the last η years. Additionally, there may be a corridor around the actuarial liability within which surplus or deficit is not amortised. Both of these differences, in comparison to the UK approach, lead to a greater volatility in contribution rates [12].

2.1.2 Dufresne method

Early work on the stochastic methods was conducted by Dufresne in a series of papers (Dufresne, 1988, 1989, 1990) [22] [23] [24]. Dufresne took the series of actuarial liabilities, AL(n) as given. The valuation method and basis were assumed to be given too; these assumptions were later relaxed by the work of other authors such as Haberman [30] and it was found that the original conclusions remained broadly intact. In particular, all elements of the basis were best estimates of the relevant quantities. The method developed by Dufresne is seen as a precursor of the stochastic method which became widely used in finance literature and research. In Dufresne's method the focus is more on the dynamics of the fund size and the contribution rate,

$$F(t+1) = (1 + i(t+1))[F(t) + \mathrm{RCR}(t)\mathrm{TSR}(t) - B(t)], \tag{2.9}$$

Where $i(t+1)$ is the achieved return on the fund from time t to time $t+1$, and

$$\mathrm{RCR}(t) = \mathrm{NC}(t) + \left(\frac{\mathrm{AL}(t) - \mathrm{F}(t)}{\mathrm{TSR}(t)\kappa\Sigma_\eta}\right). \tag{2.10}$$

Simple models for $i(t)$ allow us to derive analytical (or semi-analytical) formulas for the unconditional mean and variance of both $F(t)$ and $\text{RCR}(t)$. The key feature of these investigations is the assessment of how these values depend upon the amortisation period η. It is found that if η is too large (typically greater than 10 years) then the amortisation strategy will be inefficient: a lower value for η would reduce the variance of both the fund size and the contribution rate. Below a certain threshold for η, however, there would be a trade-off between continued reductions in $\text{Var}[F(t)]$ and an increasing $\text{Var}[\text{RCR}(t)]$.

2.1.3 Other stochastic methods

Cairns and Parker (1997) [15] and Huang(2000) [36] are among those who have made some of further advances in the Dufresne method to make it more sophisticated. In the earlier works the only control variable was the amortisation period [11]. Cairns and Parker extended this to include the valuation rate of interest, while Huang extended it furthermore to include the asset strategy as a control variable. They all found that having a valuation rate of interest different from $\mathbb{E}[i(t)]$ enriched the analysis. This meant that the decision process now had to take into account the mean contribution rate as well as the variances. Cairns and Parker also conducted a comprehensive sensitivity analysis with respect to various model parameters and took a close look at conditional, in addition to unconditional means and variances with finite time horizons.

Up to this point the decision-making process was still relatively subjective. There was no formal objective which would result in the emergence of one specific strategy out of the range of efficient strategies. Then came the introduction of stochastic control theory as a means of assisting in the decision making process. This approach has been taken using both continuous time modeling [8] and [11] and discrete time, Haberman and Sung (1994) [31]. The former approach yields some stronger results. Once the objective function has been specified, dynamic stochastic control theory identifies the dynamic control strategy which is optimal at all times in the future and in all possible future states of the world. This is in contrast to the previous approach where a limited range of controls might

be considered and which might only result in the identification of a strategy which is optimal for only one state of the world. In stochastic control there is no automatic requirement to conduct actuarial valuations or set contribution rates with respect to a normal contribution rate augmented by rigid amortisation guidelines. The use of a clean sheet and the formulation of an objective function which takes into account the interests of the various stakeholders in the pension fund has been used. As Cairns [11] proposed, let the fund size $F(t)$ at time t and the corresponding contribution rate $C(t)$ be governed by the stochastic differential equation,

$$dF(t) = F(t)\Big[(d\delta(t) + C(t) - B)dt + \sigma dZ(t)\Big]. \tag{2.11}$$

Where: $F(t)$ is the fund size, $d\delta(t)$ is the return on asset between t and $t + dt$, $C(t)$ is the contribution rate, B is the expected rate of benefit, σ is the volatility.

This equation assumes that the outgo benefit has a constant mean with fluctuations around the mean to account for demographic and other uncertainty in the benefit payments. The first element in (2.11) gives us the instantaneous investment gain on the fund from t to $t + \delta dt$. The $d\delta(t)$ term represents the instantaneous gain per unit invested and contains the usual mixture of drift (dt) and Brownian motion $dZ(t)$ terms. The second element represents the contributions paid in by the fund sponsor. The third term in brackets represents the outgo benefit [8] and [11] assumes that both the contribution rate, $C(t)$, and the possibly dynamic asset-allocation strategy, $p(t)$, are both control variables which could be used to optimise the fund's objective function. The objective function is similar to that introduced by Merton (1969, 1971,1990) [48] [49] [50] and can be described as the discounted expected loss function:

$$\Lambda(t, f)(C, p) = \mathbb{E}\Big[\int_t^\infty e^{-\beta s} L(s, (C), F(s))d_s | F(t)\Big]. \tag{2.12}$$

Following Boulier *et al* (1995) [8], $L(s, C, F(s))$ is the loss function, $C(t)$ is the contribution rate and $P(t)$ is the pension payement.

From the current fund size $F(t) = f$ the expected discounted loss depends on the choice of control strategies C^* and p^*. These strategies depend upon the time of application,

11

s, in the future as well as the state of the market at that time (that is, $F(s)$). The function $L(t, c, f)$ is a loss function which measures how unhappy (in a collective sense) the stakeholders are about what is happening at time s given that $F(s) = f$ and that the contribution rate will be $C(s) = C^*$. The discount function $e^{-\beta s}$ determines the relative weight attached to outcomes at various points in the future: for example, a large value of β will place more emphasis on the short term. The aim of the exercise is then to determine what strategies C^* and p^* will minimise $\Lambda_{(t,f)}(C, p)$. Thus the programme is

$$V(t, f) = \inf_{C,s} \Lambda_{(t,f)}(C, p). \tag{2.13}$$

It is well known that problems of this type $V(t, f)$ can be solved using the Hamilton-Jacobi-Bellman equation (HJB) as some authors have stipulated Fleming and Rishel, (1975) [26], Korn, (1997) [42], or Björk, (1998) [4] and Cairns [11]. The HJB is used for determing the optimal contribution and asset strategies. Cairns has analysed examples where the loss function is quadratic in C and f as well as power and exponential loss in C only. These led to some interesting conclusions about the optimal C and p. Some of these were intuitively sensible but others were less so and this could be connected to aspects of the original loss function. Cairns concluded that alternative loss functions needed to be developed to address these problems [11].

2.2 The Defined Contribution plan

A *defined contribution* (DC) scheme is a plan whereby each employee has an account into which the employer and, if it is a contributory plan, the employee make regular contributions. Benefit levels depend on the total contributions and investment earnings of the accumulation in the account. Often, the employee has some choice regarding the type of assets in which the accumulation is invested and can easily monitor its value at any time. Defined contribution plans are, in effect, tax deferred savings accounts in trust for the employees, and they are by definition fully funded, Bodie *et al* [6].

Mathematically, the defined contribution plan is characterized by the way it operates.

The defined contribution pensions fund plan operates quite differently from the defined benefit pensions fund plan. In the latter case the company sponsoring the fund usually takes on the majority of risk, especially investment risk. In a DC pension fund the individual members take on all of the risk. In a typical occupational DC pension fund the contribution rate payable by both the member and the employer is a fixed percentage of the salary. This is invested in a variety of managed funds with some control over the choice of funds in the hands of the member. The result is that there is considerable uncertainty over the amount of pension that might be achieved at the time of retirement. Again this is in contrast to a DB pension which delivers a well-defined level of pension. A DC pension plan is a personal pension which offers additional flexibility over occupational schemes through variation of the contribution rate. This means that a pension fund member might choose to pay more if their pension fund investments have not been performing very well.

2.2.1   Deterministic projection method

This method has emerged since 2003 in the UK, the approach requires that at the point of sale of a personal pension policy, the insurer is compelled to provide to the potential policyholder some deterministic projections to help their decision making with regard to their contribution rate. This is used as means to present some guarantees to pensions fund members. There is not any formal requirement for actuarial or other advice to help DC fund members to choose how to invest or what level of contributions they should pay, but existing DC pension fund members (personal pension policyholders and occupational fund members) may be provided with deterministic projections. This lamentable situation has left DC fund members largely ignorant. about the risk that they are exposed to. So the present situation is that they just have to accept the risks that face them. However, the growing use of stochastic methods in DB pension funds has spawned similar work in DC pensions funds.

For a DC pensions funds the aim of a stochastic modeling is to:

- Inform existing members of the potential risks that they face if they continue with their present strategy;

- Inform potential new DC fund members or new personal pension policy holders of the risks that they face, to allow them to choose between the DC pension funds and some alternatives;

- Allow existing members to manage the risks that they face by choosing an investment and a contribution strategy which is consistent with their appetite for risk and with the current status of their personal DC-pension account;

- Allow members to adopt a strategy which will, with high probability, permit them to retire by a certain age with a comfortable level of pension.

Let us look at occupational DC pension funds, where the contribution rate is a fixed percentage of salary. This type of scheme has been analysed extensively by Cairns *et al* (2001) [13], Haberman and Vigna (2002) [32] and Cairns *et al* (2003) [14] and (2001) [13], and the authors have looked at the problem in discrete time from an empirical point of view. The fund dynamics are

$$F(t+1) = (1 + i(t+1))(F(t) + C(t)), \tag{2.14}$$

Where the annual contribution $C(t) = k \times S(t)$ is a fixed proportion k of the member's current salary $S(t)$, and $i(t+1)$ is the investment return for the period from t to $t+1$. The pension ratio (PR) represented by the equation (2.15) encompasses the following elements:

- The salary process which is stochastic and is correlated with investment returns,

- The time of retirement T ,

- The fund size $F(T)$,

- The Market price or rate $a_{(T)}$, which depends primarily on long-term interest rates at time T but could also incorporate changes in mortality expectations,

14

- The DB benchmark ratio which is the two-thirds of final salary.

When the time of retirement is for instance 65 the pension ratio (PR) is then given by the equation (2.15) here below with $\mathrm{PEN}(T)$ being the pension payement at time T.

$$\mathrm{PR}(T) = \frac{\mathrm{PEN}(T)}{\frac{2}{3}S(T)} = \frac{3F(T)}{2a_{65}(T)S(T)} \tag{2.15}$$

A variety of investment strategies which are commonly used in practice and offered by pension funds and pension providers have been investigated by Cairns *et al* [14] They have considered a variety of different models for investment returns on 6 asset classes in order to assess the extent of model risk. They conclude that if the models are all calibrated to the same historical data set then differences between models are relatively small when compared with the differences which arise as a result of adopting different investment strategies. They have used the model to generate by simulation the distribution of the pension ratio at retirement for each investment strategy and their empirical analysis was simple to avoid the relative complexity of the individual asset models. This then allows users to compare strategies using a variety of different measures of risk although Cairns *et al* [13] have concentrated their analysis on Value at Risk measures. They conclude that some of the more sophisticated strategies such as the lifestyle strategy, which is popular with insurers, do not deliver superior performance over the long period of the contract. Instead they found that a static strategy with a high equity content is likely to be best for all but the most risk-averse policyholders (2001, 2003) [13] [14]. Haberman and Vigna (2002) [32] also consider a discrete-time model as in equation (2.14) but restrict themselves to a fixed contribution rate in monetary terms, in a framework which is consistent with constant salary over the working lifetime of the policyholder. They use a simpler model for investment returns and this allows them to conduct a more rigorous analysis using stochastic control with quadratic or mean-shortfall loss functions.

2.2.2 Stochastic projection method

Cairns *et al* (2001, 2003) [13] [14] formulate the DC pension problem in continuous time. Their aim was to tackle the same problem as in their paper (2001) [13] with the require-

ment to optimise the expected value of a terminal utility function. This requires a simpler model (multivariate geometric Brownian motion) than before for asset values and salary growth but also includes a full, arbitrage-free model for the term structure of interest rates which allowed them to calculate accurately the price of the annuity at time T. Terminal utility is assumed to be of the power-utility form

$$\frac{1}{\gamma}\text{PR}(T)^{\gamma} \tag{2.16}$$

for $\gamma < 1$ and $\gamma \neq 0$. Cairns *et al* [14] et al apply the same HJB technology as used by Cairns [11] in tackling the DB pension problem to determine the optimal dynamic asset-allocation strategy for the policy holder with a given level of risk aversion. A crucial feature of the DC pension policy is that the policyholder needs to take into account future contributions. The result of this feature is that the optimal strategy should vary significantly over time, starting with a high proportion in high-return-high-risk assets, gradually reducing to a mixed portfolio consistent with their degree of risk aversion. They found that the optimal strategy depends significantly on the current fund size. This is then compared with a variety of static and deterministic but dynamic asset allocation strategies, and they concluded that the optimal stochastic strategy delivers a significantly higher expected terminal utility.

Chapter 3

Mathematical model of a DC pension fund with minimum guarantee and sharing rule

3.1 Introduction

In this chapter, we lay down the mathematical model of the defined contribution pension fund with minimum guarantee. The main references for this chapter are the papers Federico (2008) [25], Gozzi *et al* (2011) [21], Deelstra *et al* (2003) [18] and Deelstra *et al* (2004) [19]. The model under analysis assumes a complete market and in a continuous time. Essentially we assume the following for our model:

- Competitiveness: assume that the investor's behaviour is to optimise its utility function and the time horizon.

- Frictionless: assume that all the assets are perfectly divisible and there exist no transaction costs or taxes.

- Arbitrage free: any gain opportunity is assumed with positive risk probability.

- Default free: financial institutions issuing assets are no-defaulting.

17

- Continuously open: investors can continuously trade in the market.

The research supposes that the pension fund wealth must be above a suitable positive function which is the solvency level. Also assumed is a demographic stationary hypothesis, which states that the flow of people who enter into the fund starts at time $t = 0$ and is constant over time and that there is an exogenous constant $T \geq 0$ which is the time during which the members adhere to the pension fund. Therefore the exit flow of people is null in the interval $[0; T]$ and is constant after time T, balancing exactly the entrance flow. Gozzi *et al* (2011) [21].

3.2 The model

The model under analysis encompasses four different elements, which are the wealth, the contributions, the benefits and the solvency level. As per Gozzi*et al* (2011) [21] the analysis is made to see how the combination of the different elements drives the model behaviour.
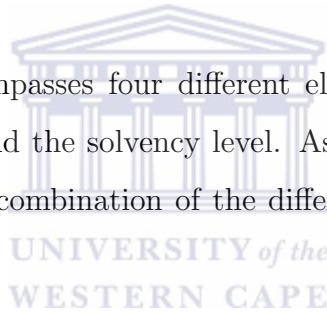
3.2.1 The dynamic of wealth

The mathematical model is set up by considering a complete probability space $(\Omega; F; \mathbb{P})$ with a filtration $\{F_t\}_{t \geq 0}$, where $t \geq 0$ is the time variable. The filtration $\{F_t\}_{t \geq 0}$ describes the information structure of the model. It is generated by the trajectories of a one-dimensional standard Brownian motion $B(t)$, $t \geq 0$, defined on the same probability space and completed with the addition of the null measure sets of F. The financial market is composed of two kinds of assets: a riskless asset and risky assets. The price of the riskless asset $S^0(t)$, $t \geq 0$, evolves according to the equation:

$$
\begin{cases}
S^0(t) = rS^0(t)dt, \\
\quad S^0(0) = 1.
\end{cases}
\tag{3.1}
$$

where $r \geq 0$ is the instantaneous spot rate of return. The price $S(t)$, $t \geq 0$, of the risky assets follows an Itô process and satisfies the stochastic differential equation

$$\begin{cases} dS^i(t) = \mu_i S^i(t)dt + \sigma_i S^i(t)dB(t), \\ \qquad S^i(0) = s_0^i. \end{cases} \tag{3.2}$$

Where $i = 1, 2, 3, ..., n$, μ is the instantaneous rate of expected return and $\sigma > 0$ is the instantaneous rate of volatility. We assume that the market assigns a premium for the risky investment, i.e. $\mu > r$. The drift μ can also be expressed by the relation $\mu = r + \sigma\lambda$, where $\lambda > 0$ is the instantaneous risk premium of the market, i.e., the price that the market assigns to the randomness expressed by the standard Brownian motion $B(.)$. The value of λ is assumed to be positive and $\mu \geq r$. The interest rate is stochastic in line with the works of Boulier *et al* (2001) [7], Deelstra *et al* (2003) [18], Cairns *et al* (2000) [13], but with a solvency constraint introduced this time.

The state variable, represented by $X(t)$, $t \geq 0$, is the $\{F_t\}_{t \geq 0}$ -adapted process that gives the amount of the pension fund wealth at any time. It is supposed that the pension fund starts its activity at the date $t = 0$ and at this time it owns a starting amount of wealth $X_0 \geq 0$. There is a control variable, denoted by $\theta(t)$, $t \geq 0$, which is the $\{F_t\}_{t \geq 0}$ adapted process that represents the proportion of fund wealth invested in the risky assets. The positivity of the wealth is due to the solvency constraints and the borrowing and short selling constraints determines the choice of $\theta(t) \in [0, 1]$ for every t. Thus the dynamics of wealth is formally expressed by the following state equation:

$$\begin{cases} dX(t) = \frac{\theta(t)X(t)}{\sum_{i=1}^{n} S^i(t)}dS^i(t) + \frac{[1-\theta(t)]X(t)}{S^0(t)}dS^0(t) + c(t)dt - b(t)dt, \qquad t \geq 0 \\ \qquad\qquad X(0) = x_0 \geq 0. \end{cases} \tag{3.3}$$

Where $\frac{\theta(t)X(t)}{S^i(t)}$ and $\frac{[1-\theta(t)]X(t)}{S^0(t)}$ are the quantities in the portfolio of risky and riskless assets, respectively, while the non-negative integrable function $c(t)$, indicates the flow of contributions and the non-negative function $b(t)$, represents the flow of benefits.

Following the Itô representation, the equation (3.3) can be rewritten as:

$$\begin{cases} dX(t) = [r + \sigma\lambda\theta(t)X(t) + c(t) - b(t)]dt + \sigma\theta(t)X(t)dB(t), & t \geq 0 \\ X(0) = x \geq 0. \end{cases} \qquad (3.4)$$

3.2.2 Contributions

In the context of this work we assume that the population is stationary, the flow of contributions $c(.)$ is exogenous. The present work assumes also that the workers who enter into the pension fund are a homogeneous class, that is to say a class of people having the same characteristics. We suppose that their entrance flow is constant over time and that each participant adheres for a length of time represented by an exogenous constant $T > 0$. We assume that there is a fixed number N of fund members after time T. The flow of aggregate contributions of pension members can be written such as:

$$c(t) = \frac{t}{T}\alpha N\omega. \qquad \text{with} \qquad 0 \leq t \leq T \qquad (3.5)$$

Where $\alpha \in (0,1)$ represents the average contribution rate (per time and per salary) and $\overline{W}(t) \geq 0$, $t \geq 0$, the average per capita wage rate of the fund members. For simplicity we take $\overline{W}(.)$ equal to a constant $\omega > 0$.

3.2.3 The benefit

The benefit of the pension fund member is composed of the minimum guarantee a part of the surplus of the contributions. By assuming the hypothesis of the demographic stationarity, the function describing the minimum guarantee is as follows:

$$g(t) = \int_{t_0}^{\tau} \bar{c}(u)e^{\delta(\tau - u)}du, \qquad \tau \geq t_0 \qquad (3.6)$$

Where $\delta > 0$ is the guaranteed rate of return, and with $r \geq \delta$. This inequality is actually justified by the fact that pension fund members are forced by law to delegate the management of their funds to mutual fund institutions. Thus they cannot invest directly into the financial market. The process of investment delegation involves management

costs which are paid by accepting a guaranteed rate of return δ lower than the risk free rate r. Due to the solvency level, as developed in the subsequent subsection, the pension fund manager will always pay the benefit which is the following:

$$b(t, X(.) \mid [t_0, \tau]) = g(t) + Y(t, X(.) \mid [t_0, \tau]), \qquad \text{with} \qquad \tau \geq t_0. \tag{3.7}$$

Here $Y(.,.)$ the surplus, is a function depending on the time τ and on the fund level within the interval of time $[t_0, \tau]$.

3.2.4 The solvency level

Generally the solvency level is imposed by law or by a supervisory authority to avoid improper behavior of the fund manager and to guarantee that the mutual fund is able to pay at least part of the due benefits at any time $t \geq 0$. Without the imposition of this constraint the fund manager may be tempted to use strategies that may bring the mutual fund to be mismatched with the social target of the pension fund. The solvency level $l(.)$ is a non-decreasing continuous function of time T and has the following equation:

$$l(t) = l_0 + \beta \int_0^t \bar{c}(u) e^{\delta(t-u)} du \quad \text{with} \quad t \geq 0. \tag{3.8}$$

This equation shows the following:

- At the beginning, the pension fund should hold a given minimum start-up level which is $l_0 \geq 0$.

- At any time $t > 0$ the solvency level is l_0 plus a fraction $\beta > 0$ of the annual contribution at time t capitalized at rate δ.

This implies that the value of the pension fund members benefit would be almost surely equal or superior to the current minimum guarantee.

From the equation (3.8) and (3.5) it gives the equation:

$$l(t) = l_0 + \beta \alpha N \omega \frac{e^{\delta(t)} - 1}{\delta T}, \qquad t \geq 0. \tag{3.9}$$

3.3   The objective function

The objective function of a pension fund is usually different from that of other forms of firms intervening directly into the financial market. This is due mostly to the fact that pension funds are subject to a certain number of restrictions from regulatory authorities in order to enforce a guarantee for subscribers to obtain the promised benefits. While firms intervening directly into the financial market are willing to optimise their welfare by taking direct advantage from stock market opportunities, a pension fund subscription is usually a process of investment delegation forced by the social security laws. It is well known that the process of investment delegation involves costs for the contributors and a potential divergence between the interests of the collectivity of subscribers and the manager of the fund. In order to bring some incentive to the manager, it is common practice to introduce a variable component in the management fee proportional to the absolute level of funds wealth as shown by Goetzmann *et al* (2003) [29]. This is basically to say that the optimization criterion for the management of a pension fund does take into account two different points of view:

- The point of view of the pension fund members: the fund manager is directly delegated by the members to invest in the risky market in order to generate their benefits and the function programme is proportional to contributions of pension fund members.

- The point of view of the pension fund manager: the manager is led to invest in risky assets in order to improve his fee, which is an increasing function of the absolute level of the funds wealth.

  In the following subsection the research looks at the aforementioned views and translate them into a suitable optimisation programme.

3.3.1 The optimisation programme of the contributor

The optimisation programme of the contributor is set such that the contributor pays a flow to the pension fund which consists of a lump sum at date 0, denoted by X_0 and a continuously paid premium, at a rate denoted by $c(t), t \in [0, T]$. The flow of contributions is assumed to be a non-negative, progressively measurable process such that:

$$\int_0^T c^2(t)dt < \infty, \tag{3.10}$$

Let H_t be the deflator price process and it is defined as:

$$H_t = \exp(-\int_0^t (r_s d_s - \int_0^t \lambda_s' dW_s - \frac{1}{2} \int_0^t ||\lambda_s'||^2 d_s). \tag{3.11}$$

The value of the cash given by the contributor to the pension fund at the date 0 is equal to:

$$X_0' = X_0 + \mathbb{E}\Big[\int_0^T H(s)c(s)ds\Big]. \tag{3.12}$$

At time T the pension fund member will receive from the fund a certain benefit which encompasses two parts, the guarantee G_t and a certain fraction of the surplus Y_t which is equal to $X_t - G_t$. Following prior studies on the incentives of agents under a constant absolute risk aversion (CARA) utility function, the model of surplus is linear (Holstrom and Milgrom (1987) [35]). Thus the benefit process will be

$$B_t = G_t + (1 - \beta)Y_t, \tag{3.13}$$

With :

- $Y_t = X_t - G_t$ is the surplus process,

- G_t is the guarantee.

In the hypothesis of a stochastic guarantee and with the other variables being constant, one can ask where lies the optimal guarantee that gives the pension fund member a desirable benefit. The cases of $\beta = 0$ or $\beta = 1$ are trivial, and so will not be pursued in

the present dissertation. Thus we will consider only $0 < \beta < 1$.

The programme of the contributor amounts to finding

$$\max \left[\mathbb{E}[U(1-\beta)Y_t + G_t]\mathbb{E}_t \int_t^T \frac{H_s}{H_t}C_s ds \right], \tag{3.14}$$

$$\text{subject to} \quad \begin{cases} (1-\beta)Y_t H_t + G_t = k, \\ \quad G_t > 0. \end{cases} \tag{3.15}$$

The solution of this programme will require to first find the characteristics of the surplus process. Let $Y_t = X_t + D_t - G_t$ be the surplus process with $t \geq 0$ and

$$D_t = \mathbb{E}_t \int_t^T \frac{H_s}{H_t}C_s ds; \quad G_t = \mathbb{E}_t \int_t^T \frac{H_s}{H_t}C_s ds, \tag{3.16}$$

Which gives,

$$Y_t = X_t + \mathbb{E}_t \int_t^T \frac{H_s}{H_t}C_s ds - \mathbb{E}_t \int_t^T \frac{H_s}{H_t}C_s ds. \tag{3.17}$$

The surplus process is in fact the value of the portfolio X_t plus the discounted value of the future engagements coming from the pension fund member D_t and minus the discounted value of the pension fund future engagement which is the guarantee G_t. The surplus process is self financing as defined by Karatzas and Shreve (1998) [39]. This means that there exists a progressively measurable random process $Y_t = (y_{1t}, ..., y_{nt})'$, $t \in [0, T]$ such that

$$dY_t = Y_t r_t d_t + y_t'(b_t - r_t)d_t + y_t'\sigma_t dW_t, \tag{3.18}$$

and the final surplus process is such:

$$Y_T = Y_0 \exp(\int_0^T (r_t + y_t'(b_t - r_t))dt + \int_0^T y_t'\sigma_t dW_t - \frac{1}{2}\int_0^T ||y_t'\sigma_t||^2 d_t), \tag{3.19}$$

with

$$Y_0 = X_0 + \mathbb{E}[\int_0^T H_s c_s ds] - \mathbb{E}[H_T G_T] \geq 0. \tag{3.20}$$

The value of Y_T has two parts; the deterministic part which is represented by Y_0 and link Y_T to G_T and the random part which is represented by

$$\Psi = \exp(\int_0^T (r_t + y_t'(b_t - r_t))dt + \int_0^T y_t'\sigma_t dW_t - \frac{1}{2}\int_0^T ||y_t'\sigma_t||^2 d_t). \tag{3.21}$$

24

3.3.2 The optimisation programme of the pension fund manager

The present section describes the problem faced by the pension fund manager. He maximises the expected utility of his terminal wealth which is his part of the surplus. The form of the pension fund manager function is of the type of power utility, $U(Y) = \frac{Y^{1-\gamma}}{1-\gamma}$. The choice of the power utility function is guided by the fact that the power utility presents the characteristics that better capture the features of pension fund companies due to the large scale of money they are managing. Pensions funds companies are regulated in such a way as to not reach negatives values. The power utility function endorses this through the infinite marginal utility at zero [18].

Let X_t being the wealth of the fund at date $t \in [0,T]$, and π_t the proportion of wealth invested into the risky assets so that $1 - \sum_{i=1}^{n} \pi_t$ is the proportion of wealth invested into the riskless asset $p_0(t)$. The optimisation programme of the pension fund will be

$$\max \frac{1}{1-\gamma} \mathbb{E}(\beta X_T - G_T)^{1-\gamma}, \tag{3.22}$$

subject to :

$$dX_t = [X_t r_t + X_t \pi_t'(b_t - r_t) + c_t]dt + X_t \pi_t' \sigma_t dW_t, \tag{3.23}$$

with $X(0) = X_0 > 0$ and lets $\Lambda^x = \pi_t = (\pi_{1t}, ..., \pi_{nt})', t \in [0,T]$, being an $F_t -$ adapted process such that $\int_0^T ||X_t \pi_t' \sigma_t||^2 dt < \infty$ and $X_t - G_t \geq 0$.

The condition:

$$\mathbb{E}[H_T G_T] < X_0 + \mathbb{E}[\int_0^T H_t c_t d_t] = X_0'. \tag{3.24}$$

shows that the market value of the contributions is larger than the market value of the guarantee, and this condition guarantees the non emptiness of the set of admissible strategies.

Ideas from the framework developed in this chapter will be used in the Browne's framework, which constitute the main results of this dissertation and are developed in the following chapter.

Chapter 4

A defined contribution pension fund scheme with benchmark

In this chapter, the research presents the portfolio proposed in the paper of Browne (1999) [10] and it is attached to it a certain sharing rule, similarly as in the paper of Deelstra *et al* (2004) [19]. The research analyses the impact of the different parameters on the stochastic evolution of the model. Then it observes the performance of the investment portfolio with respect to a certain benchmark which is also stochastic. Benchmarking has been widely used in the pension fund literature and in practice. In this regard, see for example the works of Lioui and Poncet (2013) [45], Gaivonronski *et al* (2005) [28], Platen and Heath (2006) [34]. In the present case there is a guarantee that serves as a benchmark. The portfolio manager has as goal to maximise his surplus, which is $X_t - G_t$, where X_t is the wealth process and G_t is the guarantee process. When the surplus is positive, the fund manager is performing well and vice versa. As mentioned above, the analysis of a benchmark function in the financial literature has been prominent in recent years. The notion has its roots in portfolio management theory. In portfolio management there are two alternative approaches that managers of different assets management units can use to run their activities. Those types are the passive portfolio management and the active portfolio management.

- The passive portfolio management is a financial strategy in which a fund manager invest in a predetermined manner which mostly consist of trying to replicate a return of a certain benchmark which are mostly market indices. Once the portfolio has been set, the manager does not alter it until maturity.

- The active portfolio management is a financial strategy where the fund manager makes a specific investment and continually changes it with the goal of outperforming the market represented by an investment benchmark index, see Bodie *et al* (2004) [5] for instance.

Active portfolio management has gained considerable importance in financial investment practice, due to its ability to take advantage of the market inefficiencies by using arbitrage. As argued in Browne (1999) [10], active portfolio management may have many possible objectives related to outperforming a benchmark, such as:

- the maximisation of the probability of beating the benchmark,

- the maximisation or (respectively) the minimisation of the expected time of beating or (respectively) being beaten by the benchmark,

- the maximisation or the minimisation of the expected discounted rewards upon beating or being beaten by the benchmark,

- the maximisation of terminal utility.

The above objectives are important for financial institutions, especially those who are managing portfolio funds, in the sense that their performances are judged with respect to the return they realise against a certain benchmark.

4.1 Analysis of the wealth process with respect to a benchmark

In the present section, we are analysing the model that follows the framework of Browne [10]. The wealth process given in the equation (5.4) below is composed of the following processes :

- The risk free asset B_t evolves as:

$$dB_t = r_t B_t dt, \qquad (4.1)$$

  with $r \geq 0$.

- The risky asset S_t under the filtered probability space $(\Omega, \mathbb{F}, P)$ satisfies the stochastic differential equation:

$$dS_t^i = \sum_{i=1}^{K} S_t^i [\mu_i dt + \sigma_i dW_t^i], \qquad (4.2)$$

  with $i = 1, 2, 3, ..., K$. In order to avoid any kind of triviality, we impose the condition that $\mu_i > r_t$ for all $i = 1, 2, 3, ..., K$.

- The contribution from the pension fund member is C_t and from the equation (3.5) it takes the following form:

$$dC_t = \alpha N \omega_t dt, \qquad (4.3)$$

  Where, for some constant α, N denotes the number of members in the pension fund, and ω the flow of the salary.

The wealth process as in Browne takes the form:

$$dX_t = X_t \big[(r_t + \pi_t'(\mu_t - r_t) + c_t) \big] dt + X_t \pi_t' \sigma_t dW_t. \qquad (4.4)$$

It entails the risky asset which is composed of stocks, the risk free asset which is a bond and the contribution of pension fund members and where π' is the vector of the investment

28

policy of the portfolio.

The guarantee is assumed to be a stochastic process of the form :

$$dG_t = G_t r'_t dt + G_t \gamma_t (b_t - r'_t) dt + G_t \gamma_t \sigma_t dW_t. \tag{4.5}$$

The research is interested in determining an investment strategy that is optimal with respect to the performance against a certain benchmark. The benchmark that is presented in this work is another stochastic process. Generally the benchmark can take many forms. It might be an inflation rate, or an exchange rate, it might also represent the value process of a non-traded asset or even a wealth process from a different portfolio strategy. In the present research a non-traded asset such as the guarantee serves as the benchmark. Then the fund manager, in order for him to observe the performance of his fund against it, will be interested in the ratio of the wealth process as compared to the guarantee process. The ratio is $Y_t = X_t/G_t$ and it is as per Browne's notation [10], and its equation is of the form:

$$dY_t = Y_t \big[(\hat{r}_t + (\theta_t \hat{f}_t + c_t)) dt + \theta_t \sigma_t dW_t. \big] \tag{4.6}$$

Where : $\hat{r}_t = r_t - r'_t$, $\theta_t = \pi'_t - \gamma_t$, $\hat{f}_t = b_t - r_t$.

After presenting the benchmarking problem, there is the task analysing its optimality. Thus the following section will treat issues relative to the portfolio optimisation.

4.2   The portfolio optimisation problem.

To find the maximal expected terminal utility of the ratio process at some fixed terminal time $T < \infty$, we can either use the Hamilton - Jacobi - Bellman method or the martingale method. The martingale method has been applied in the present research to find the expected values as per Rogers and Williams (1987)[53] or Davis and Norman (1990)[17]. It entails finding an appropriate function that is either uniformly integrable martingale under a certain chosen optimal policy or a supermartingale under any other admissible policy, with respect to the filtration $\mathbb{F}_t$.

Let $\Phi(t) = \int_s^t \lambda(Y_t) d_t$ be a stochastic integral and let us introduce the following functional process, which is a martingale with respect to the filtration $\mathbb{F}_t$.

$$M(t, Y_t) = \exp(\Phi(t))\Pi(Y_t) + \int_0^t \exp(\Phi(s))g(Y_s)ds \quad \text{for} \quad t \geq 0 \tag{4.7}$$

Similarly to the study in Browne [10] we consider a reward function $v(Y_t)$ which, for a portfolio f, is defined by the equation below.

$$v^f(y) = \mathbb{E}_y\left[\int_0^{\tau^f} g(Y_t) \exp\{-\int_0^t \lambda(Y_t)ds\}dt + h(Y_{\tau^f})\exp\{-\int_0^{\tau^f} \lambda(Y_t)ds\} \right] \tag{4.8}$$

where τ^f is the first escaping time of $Y^f(t)$ from a predetermined interval (l, u) with $l < Y(0) < u$. There is also a non-negative function $\lambda(Y_t)$, and a bounded continuous function $g(Y_t)$. There exist a function $h(Y_t)$ which takes l and u values such $y = l$ or $y = u$, with $h(u) < \infty$. The value function $v(Y_t)$ and the optimal portfolio $f_v^*(Y_t)$ are as follows:

$v(Y_t) = \sup\{v^f(Y_t): f \in \mathbb{A}\}$,

$f_v^*(Y_t) = \arg_f \sup\{v(Y_t): f \in \mathbb{A}\}$,

Where $\mathbb{A}$ is the set of admissible policies.

As demonstrated in Browne [10, Theorem 1] an optimal control vector $f_v(Y_t)$ does exist, and is of the form:

$$f_v(Y_t) = -\delta^{-1}\widehat{\mu}\frac{(\Pi_y(Y_t))}{Y_t\Pi_{yy}(Y_t)} + (\sigma^{-1})'b. \tag{4.9}$$

In the subsequent section we introduce the sharing rule parameter in order to analyse the risk mitigation that might ensue from its presence in the model.

4.3 Analysis of the portfolio with respect to the sharing rule

In this section, we analyse the impact of the parameter β which represents the sharing rule between the fund manager and the contributor. With the introduction of the sharing rule [54], the ratio function became: $\beta X_t / G_t$ and this determines the ratio process Y_t which is

a geometric Brownian motion because X_t and G_t are geometric Brownian motions. The ratio is of the form:

$$\widehat{Y}_t = \beta Y_t \big[(\hat{r}_t + (\theta_t \hat{f}_t + c_t)) d_t + \theta_t \sigma_t dW_t \big] \tag{4.10}$$

Finding the optimal control value of the terminal utility of the ratio process with the presence the parameter β, is analogous to the process without β with the only difference being the introduction of the β parameter. Thus we introduce β into the functional martingale process given by the equation (4.7) giving the following:

$$M(t, \widehat{Y}_t) = \exp(\Phi(t))\Pi(\beta Y_t) + \int_0^t \exp(\Phi(s)) g(\beta Y_s) ds \quad \text{for} \quad t \geq 0 \tag{4.11}$$

Following the same reasoning as in the section (4.2) we have a reward function defined in equation (4.8) and a stopping time given by τ^f which has an optimal control vector given by the equation (4.9). With the introduction of the sharing rule the optimal control solution vector of the equation (4.9) given in the equation (4.8) becomes:

$$f_v(\widehat{Y}_t) = \beta \left[-\delta^{-1} \widehat{\mu} \frac{(\Pi_y(Y_t))}{Y_t \Pi_{yy}(Y_t)} + (\sigma^{-1})'b \right]. \tag{4.12}$$

Now as we have the grasp of the impact of the introduction of the sharing rule parameter β, let us take a look at the notion of *growth optimal portfolio* (GOP) which is popularly used in the world of finance.

4.4 Growth optimal portfolio

The notion of GOP has been widely used in the financial industry and in the financial literature since the work of Kelly (1956) [40]. Many authors have studied the growth optimal portfolio such as Latane (1959) [43], Breiman (1960) [9], Thorp (1961) [55], Markowitz (1976) [47] and Long (1990) [46], in gambling problems, portfolio optimization and the pricing of derivatives. Authors such as Browne [10] and Heath and Platen (2006) [34] have studied the GOP as the benchmarking model of portfolio optimisation. The GOP has the characteristic of being the portfolio that maximises the expected log-utility from terminal

31

wealth T, that is the quantity $\mathbb{E}[\ln(V_T)]$. The GOP possesses also some other remarkable properties. For instance, it maximises the expected growth rate over any time horizon. It is also a strictly positive portfolio which almost surely outperforms any other strictly positive portfolio over a sufficiently long time horizon [34]. Following the definition of the GOP in the works of Heath and Platen [34], the GOP satisfies the following stochastic differential equation :

$$dV_t = V_t\Big[r_t d_t + \sum_{k=1}^{d} \theta_t^k (\theta_t^k d_t + dW_t^k)\Big]. \tag{4.13}$$

where θ_t^k denote the market price of the risk and it is represented by the equation.

$$\theta_t^k = \sum_{j=1}^{d} b_t^{-1}(\alpha_t - r_t). \tag{4.14}$$

with b_t^{-1} being the inverse of the volatility matrix process, α_t is the appreciation rate process and r_t is the vector of the interest rate process. As shown in Browne [10] the proportion of investment for the GOP is given by the equation 4.15 below:

$$\pi = \Sigma^{-1}\widehat{\mu} + (\sigma^{-1})'b. \tag{4.15}$$

The following section will focus on the numerical and graphical analysis of the different portfolios treated in the previous sections namely: the wealth process X_t, the guarantee process G_t, the ratio process Y_t and the GOP V_t.

4.5 Numerical and graphical illustration of the portfolio.

In this section, we present a numerical and graphical comparative analysis of different portfolios. We consider the wealth process, the guarantee, the growth optimal portfolio and the ratio process. Computations have been made by calculating the values of these portfolios. On the page 35 we have the graph Fig 4.1, illustrating the evolution of the wealth process which is following the Browne's portfolio comparatively to the guarantee

and the ratio. It can be seen on the graph how the ratio in green just evolve below the wealth in blue with the same pattern but where the difference is the guarantee in red as it is the denominator of the ratio with respect to the wealth portfolio. The Table 4.2 on the same page 35 correspond to the Fig 4.1 and it illustrates the numerical evolution of wealth process, the guarantee, and the ratio process. On the page 36 we have the Fig 4.2 illustrating the stochastic evolution of wealth process, the guarantee, and the growth optimal portfolio (GOP); and the Table 4.3 which contains the corresponding computed values of the wealth process, the guarantee and the GOP. On the page 37 the Fig 4.3 illustrates the evolution of the guarantee, the wealth process and the ratio process where we have introduced the parameter β which is the sharing rule [54]. It can be seen on the graph that β does not change the pattern of the ratio but it diminishes its value, by doing so it mitigates the risk by sharing it between the pension fund manager and the pension fund member [19].

The graphs in Fig 4.1 and 4.2 on pages 35 and 36 show the stochastic evolution of the wealth process, the guarantee, the ratio process and the growth optimal portfolio. As it can be seen in these figures, the evolution of the guarantee is less volatile than the wealth process and the growth optimal portfolio. The guarantee reach hardly negatives values [19]. This is because of its state imposed positive constraint structure. The reading of the Tables 4.2, 4.3 and 4.4 on pages 35 to 37 is quite interesting, especially observing how the parameter β impacts on the values of the ratio process. This can be observed on the computed values of the Table 4.4 on page 37 by looking at the value on the column Y_t and $Z_t = \beta Y_t$. It can be spotted that the introduction of the sharing rule parameter is mitigating the risk in the process; as the value of the surplus process with β comparatively to the value of the surplus process without β is dwindling. It shows how the value of β does impact the ratio process as it dwarfs its value comparing to the other portfolios. The fact that the value of the ratio process is reduced shows how the risk is shared as the return does diminish, going along with the theoretical reasoning which link returns to risks [6]. The ratio process is positive when the values of the wealth process are over the

33

guarantee's. The fund manager will always try to find a combination of asset portfolio that allows him to have a ratio that is positive.

The following Table contains num. value used to analyse the model under this dissertation with the Matlab Software.

Table 4.1: Num. value relative to Fig 4.1 to 4.12 and Tables 4.1 to 4.9

Symbols	$\sigma_{11}=\sigma_{22}$	$\sigma_{12}=\sigma_{21}$	α	β	μ_1	μ_2
Num. value	1	0	0.01	0.9	0.52	0.58
Symbols	dt	b_1	b_2	b_3	alf	c_1
Num. value	1	0.15	0.25	0.5	0.4	0.41

Here σ is the volatility, μ is the drift, β is the sharing parameter, $b_1 b_2$ b_3 alf c_1 are coefficients, dt is the differential with respect to time.
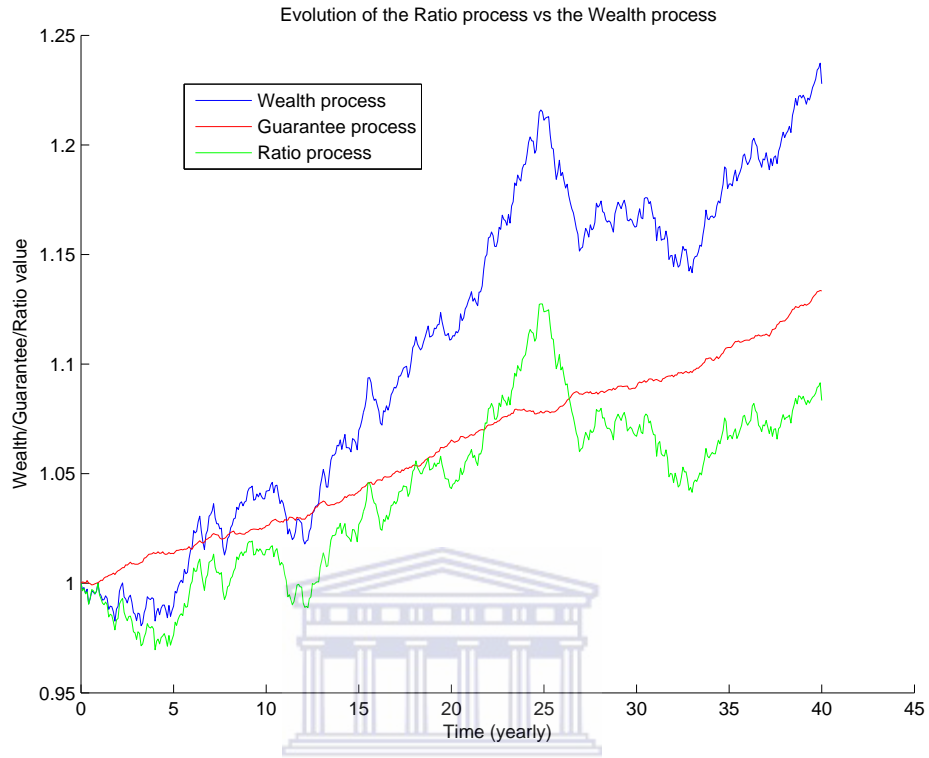
34

Figure 4.1: Evolution of the Ratio process vs the Wealth process.

Table 4.2: The wealth X_t, guarantee G_t, and ratio process Y_t.

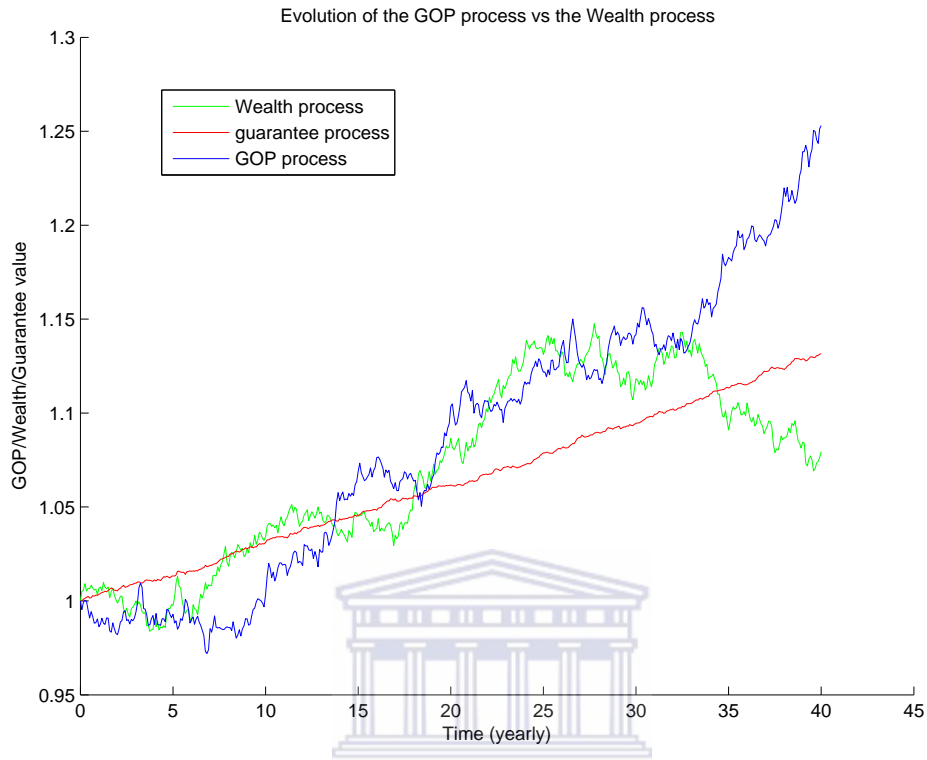| Time	X_t	G_t	Y_t
0	1.0000	1.0000	1.0000
10	1.0385	1.0254	1.0128
20	1.1109	1.0642	1.0439
30	1.1625	1.0889	1.0676
40	1.2373	1.1336	1.0915

Figure 4.2: Evolution of the GOP process vs the Wealth process.

Table 4.3: The wealth X_t, guarantee G_t, and The GOP process V_t.

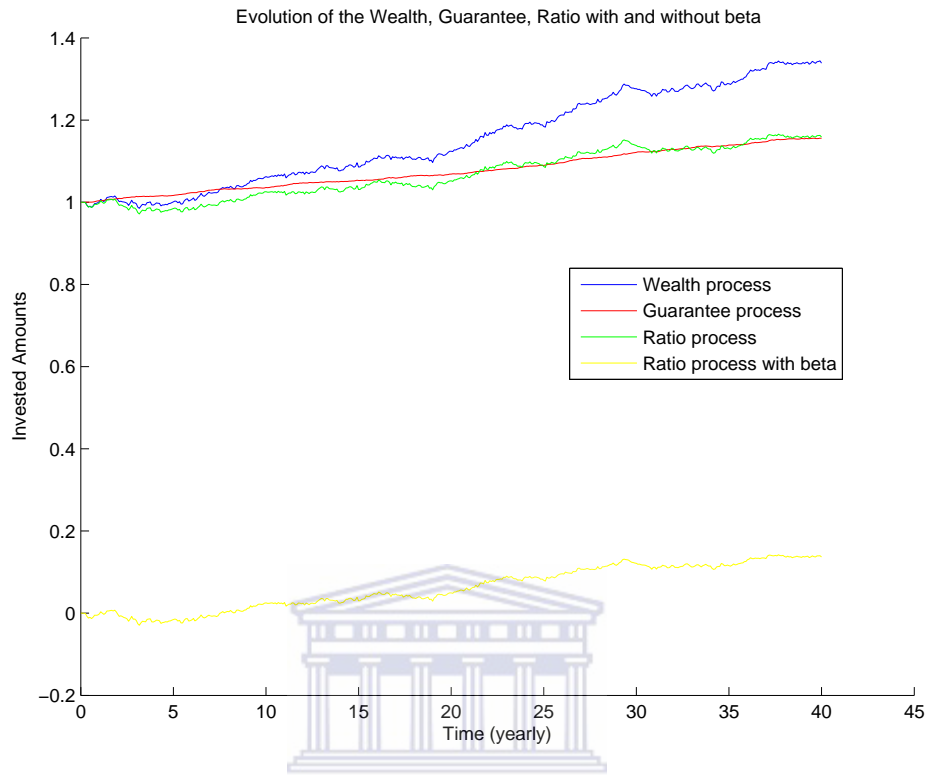| Time	X_t	G_t	V_t
0	1.0000	1.000	1.0000
10	1.0322	1.0303	0.9968
20	1.0819	1.0615	1.0634
30	1.1128	1.0940	1.1409
40	1.0762	1.1313	1.2435

Figure 4.3: Evolution of the Wealth, Guarantee, Ratio with and without β.

Table 4.4: The wealth X_t, guarantee G_t, ratio process Y_t before sharing and Z_t the ratio after sharing with $\beta = 0.9$.

Time	X_t	G_t	Y_t	$Z_t = (\beta Y_t)$
0	1.0000	1.000	1.0000	0.0000
10	1.0313	1.0408	0.9998	0.0008
20	1.0284	1.0103	1.0018	0.0016
30	1.0339	1.0136	1.0245	0.0041
40	1.0408	1.0157	1.0159	0.0053

37

4.6 Sensitivity analysis of the portfolio

The sensitivity analysis is a technique used to determine how a value of particular asset is impacted by variations occurring to some factors in the market such interest rate, time, volatility,... [37]. The sensitivity analysis is one of the means used by financial institutions to manage risk. Generally the measures used in sensitivity analysis are restricted with the black-scholes parameters set and called as the Greeks [2]. The Greeks are:

- Delta (δ): is the rate of change of the value of the portfolio with respect to it market value ($\delta = \frac{\partial \Pi}{\partial S}$).

- Theta (θ): is the rate of change of the value of the portfolio with respect to time, it often referred as the time decay ($\theta = \frac{\partial \Pi}{\partial T}$).

- Gamma (γ): is the rate of change of the value of the portfolio's delta with respect to market value of the portfolio ($\gamma = \frac{\partial^2 \Pi}{\partial S^2}$).

- Vega [1] (ν): is the rate of change of the value of the portfolio with respect to the volatility of the portfolio ($\nu = \frac{\partial \Pi}{\partial \sigma}$).

- Rho (ρ): is the rate of change of the value of the portfolio with respect to the interest rate ($\rho = \frac{\partial \Pi}{\partial r}$).

In the present research, the focused is put on the level of the volatility of the Browne's wealth portfolio (X_t), the ratio process (Y_t), the guarantee (G_t) and the GOP (V_t). The financial theory states that the value of the portfolio is very sensitive to small changes in volatility [37]. The graph Fig 4.4 on page 40 illustrates the rate of the variation of the Wealth with respect the value of Vega (ν), it shows how volatile is the wealth portfolio (X_t). It can be seen in the graph that any small variation of vega (ν) impact the value of the wealth portfolio; this is captured by the degree of the envelop of the curve. The graph Fig 4.5 on page 40 illustrates how a combined analysis of the volatility of the Wealth

---

[1]it is worthwhile to notice that Vega is not a Greek letter, but the symbol ν has been used as its representation in the finance literature.

process (X_t) following Browne portfolio, the Guarantee process (G_t), the Ratio process (Y_t) and the GOP (V_t) (growth optimal portfolio) vary with respect of Vega. It can be spotted that a little variation of vega (ν) does significantly affect the level of the portfolios values. The degree of acutancy of the curve in graph as it goes along with vega's and portfolios' values.
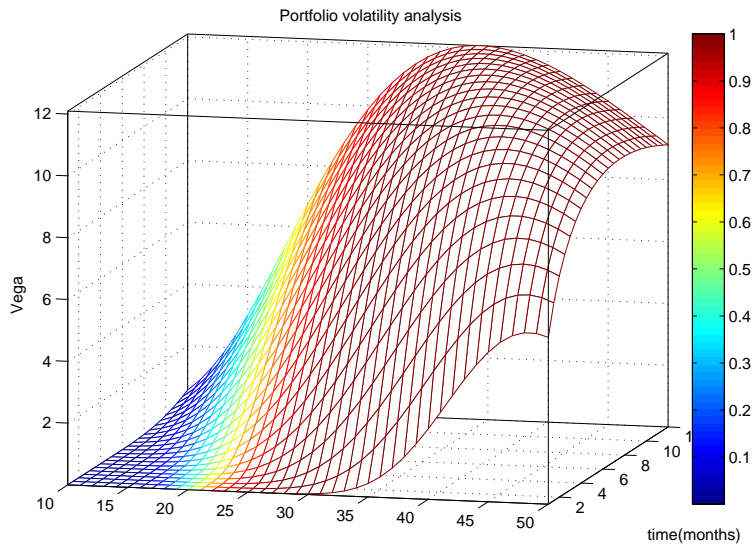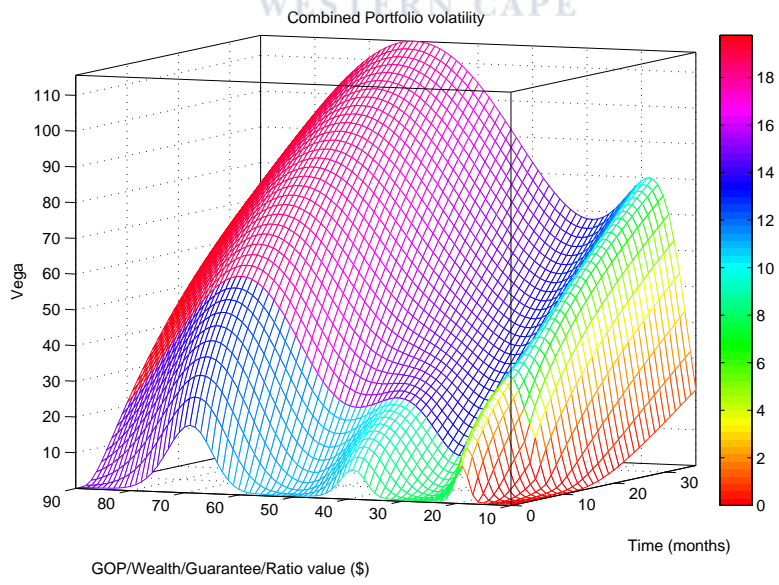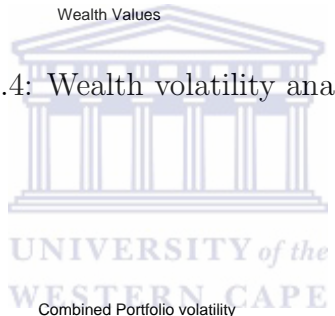
Figure 4.4: Wealth volatility analysis.,

Figure 4.5: Portfolios volatility analysis.,

40

4.7   Utility function of the portfolio

The present paragraph illustrates graphically and numerically the estimation of the power utility function. Computations have been made, comparing the value of the power utility function under constant relative risk aversion (CRRA) of the pension fund manager $U(Z_t)$ and of the pension fund members $U(\alpha Z_t)$, with each having its own pattern of risk aversion. The literature on the utility function has shown that fund managers and fund members have conflicting patterns of risk aversion with the former tending to be less risk averse than the latter [38]. The power utility function is of the form:

$$U(Y_t) = \frac{\beta(Y_t)^{1-\zeta}}{1-\zeta} \quad \text{with} \quad \zeta \neq 1 \tag{4.16}$$

In Tables 4.5 to 4.8 we computed the utility values of the ratio process, of the share of the pension fund manager as well as of the share of the pension fund member. In Table 4.5 on page 42 we assigned a high value of coefficient of risk aversion to both the pension fund manager and the pension fund member. The finding is that when both are risk averse their respective utility values are low. This goes along with the theoretical reasoning which stipulate that risk aversion is inversely related to return [52] [57].
In Table 4.6 on page 42 a lower value of the coefficient of risk aversion was assigned for both the pension fund manager and the pension fund members. The outcome is that the utility value for both are higher as the coefficient of risk aversion is low [37]. In the subsequent tables on page 43 which follow, the computation has been made by allocating either high or low the value of risk aversion for both the pension fund manager and the pension fund members. In those tables the theoretical reasoning of risk aversion and return does stand as well. The coefficients of risk aversion on these tables are namely ζ for the pension fund manager and η for the pension fund members. It is worthwhile to note that the ratio process appearing in the tables and represented by Y_t, has been assigned the coefficient ζ for the pension fund manager, only for the sake of comparative analysis. Graphs in Fig 4.6 to 4.9 on pages 44 and 45 are related to Tables 4.5 to 4.8 . The results illustrates even better how utility values are rising and dwindling inversely with the coefficient of risk aversion.

Table 4.5: Utility value of the PF Manager ($\zeta = 0.85$) and of PF Members ($\eta = 0.85$).

Time	$U(Y_t)$	$U(Z_t)$	$U(\alpha Z_t)$
0	0.000	0.0000	0.0000
10	2.9499	2.6978	0.4168
20	2.9562	2.7028	0.4176
30	2.9484	2.6962	0.4166
40	2.9447	2.6939	0.4162

Table 4.6: Utility value of the PF Manager ($\zeta = 0.3$) and of PF Members ($\eta = 0.3$).

Time	$U(Y_t)$	$U(Z_t)$	$U(\alpha Z_t)$
0	0.0000	0.0000	0.0000
10	5.5835	5.4095	2.7982
20	5.5821	5.4103	2.7987
30	5.5825	5.4109	2.7990
40	5.5823	5.4100	2.7985

Table 4.7: Utility value of the PF Manager($\zeta = 0.85$) and of PF Members ($\eta = 0.3$).

Time	$U(Y_t)$	$U(Z_t)$	$U(\alpha Z_t)$
0	0.0000	0.0000	0.0000
10	2.9509	2.6987	2.3116
20	2.9524	2.6986	2.3113
30	2.9494	2.6975	2.3110
40	2.9517	2.6993	2.3115

Table 4.8: Utility value of the PF Manager($\zeta = 0.3$) and of PF Members ($\eta = 0.85$).

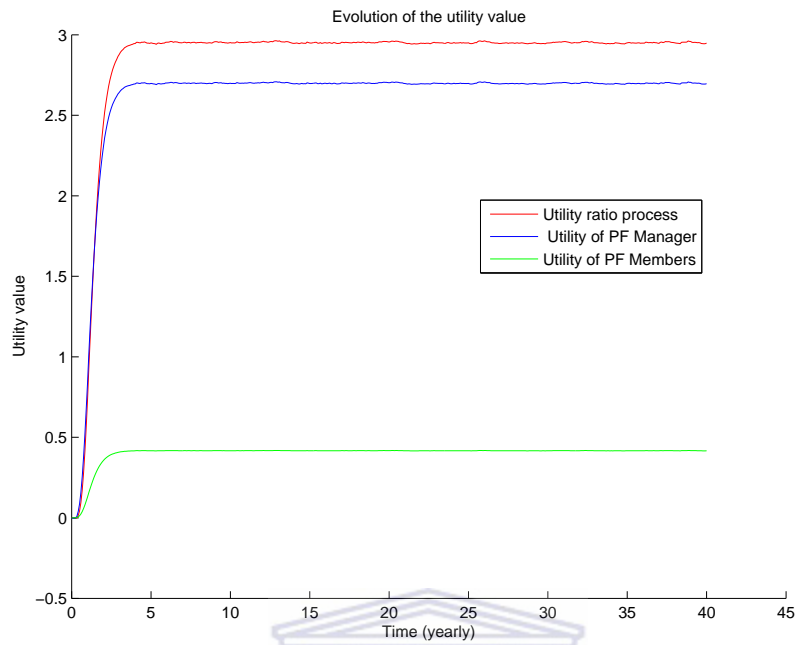| Time	$U(Y_t)$	$U(Z_t)$	$U(\alpha Z_t)$
0	0.0000	0.0000	0.0000
10	5.5838	2.6978	0.4168
20	5.5849	2.7028	0.4176
30	5.5830	2.6962	0.4166
40	5.5850	2.6939	0.4162

Figure 4.6: Evolution of the utility function of the fund manager with $\zeta, \eta = 0.85$
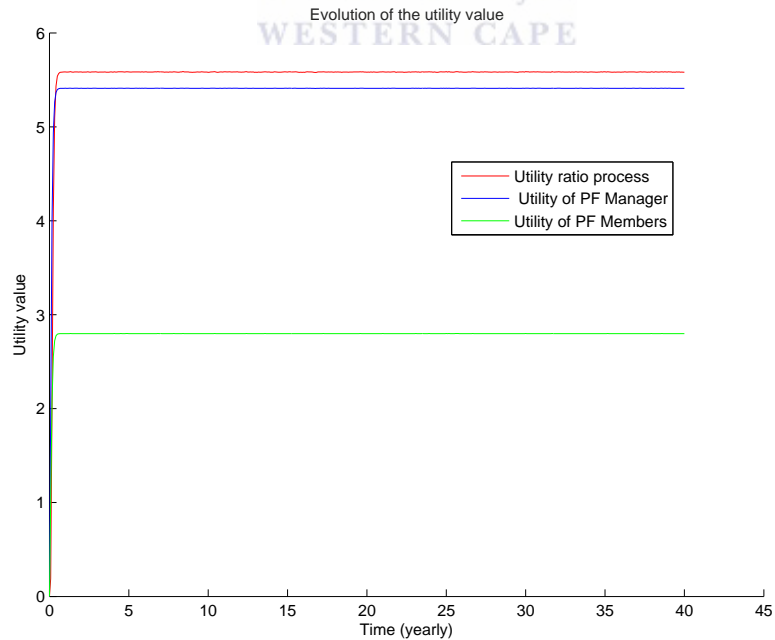

Figure 4.7: Evolution of the utility function of the fund manager with $\zeta = 0.3, \eta = 0.3$
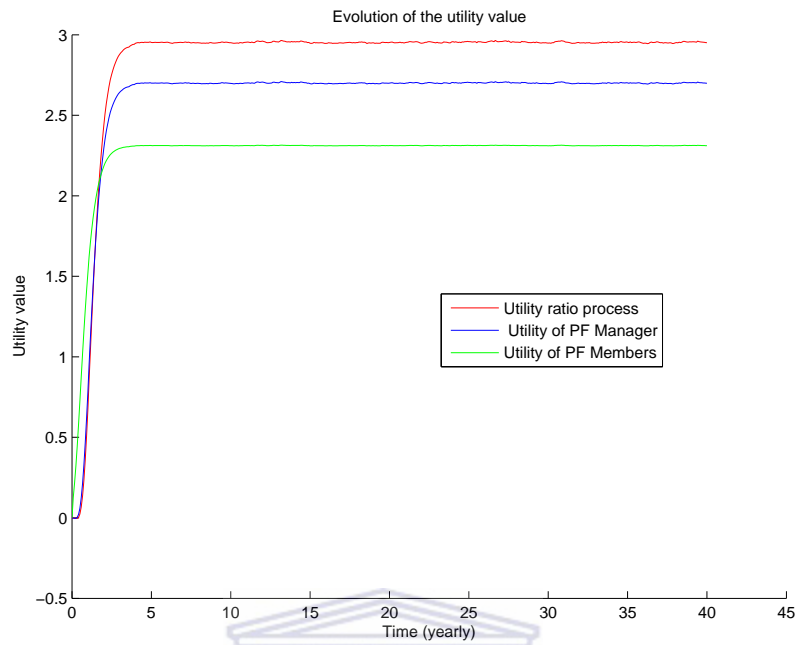
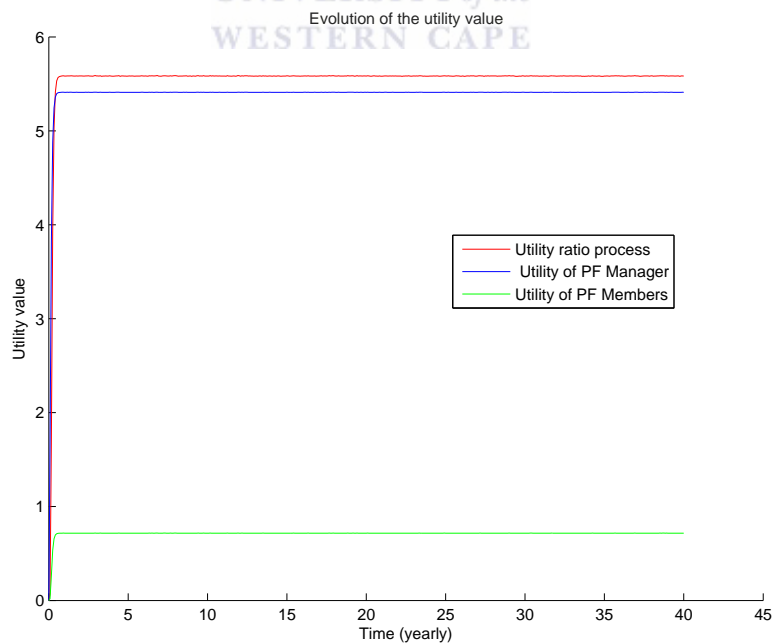Figure 4.8: Evolution of the utility function of the fund manager with $\zeta = 0.85, \eta = 0.3$

Figure 4.9: Evolution of the utility function of the fund manager with $\zeta = 0.3, \eta = 0.85$

4.8   The share of the pension fund manager

The present paragraph illustrates graphically and analyses numerically the share C of the pension fund manager and the benefit B of the pension fund member. As demonstrated by Deelstra *et al* [19] the share of the pension fund manager is defined as $C = \beta K_t$, where $K_t = X_t - G_t$ is the surplus process; and the benefit of the pension fund member is defined as $B = G_t + (1 - \beta)K_t$ with $(1 - \beta)K_t$ being the share of the pension fund member. Computations have been made by calculating the values of the share C of the pension fund manager, the surplus K_t and the benefit B of the pension fund member. Those values are provided by the Table 4.9 on page 47. They correspond to the graph Fig 4.12 on page 49. The Table 4.10 on the page 47 provide the values of the surplus process with different levels of β (1, 0.985, 0.95, 0.85) and we observed that the values of the surplus process are dwindling with β. The reduction of the value of the surplus with the reduction of β is of the financial importance as it suggest that the risk is shared between the pension fund stakeholders namely the pension fund manager and the pension fund member [19][54].

In the graph Fig 4.10 on the page 48 is the illustration of the evolution of the surplus process with different levels of β. The impact of this parameter is seen on the graph by showing how the surplus process is lowering with the level of β. The presence of the parameter β in the surplus process illustrates how the risk is shared by pension funds and their clients. This provides some guarantee to the pension fund member investment, thus rendering the programme more attractive in the client investment point of view. The graph in Fig 4.11 on page 48 illustrates the evolution of the share of the surplus process of the pension fund manager $C = \beta K_t$ and of the pension fund member $(1 - \beta)K_t$. In the graph we can see how the gap between pension fund managers and pension fund members widen as years goes, it shows how the effect of β is felt by the two stakeholders of the pension fund during the investment life. The graph in Fig 4.12 on page 49 illustrates the evolution of the surplus process (K_t), the share of the pension fund manager $(C = \beta K_t)$ and the benefit of the pension fund member $B = G_t + (1 - \beta)K_t$.

Table 4.9: Share of the PF Manager(C) and Benefit of PF Members (B)

Time	Surplus($X_t - G_t$)	C, Share of PF Manager	B, benefit of PF Member)
0	0.0000	0.0000	0.0000
10	0.0764	0.0688	1.0482
20	0.1221	0.1099	1.0768
30	0.1261	0.1135	1.1138
40	0.1846	0.1661	1.1440

Table 4.10: Evolution of the surplus with different levels of β

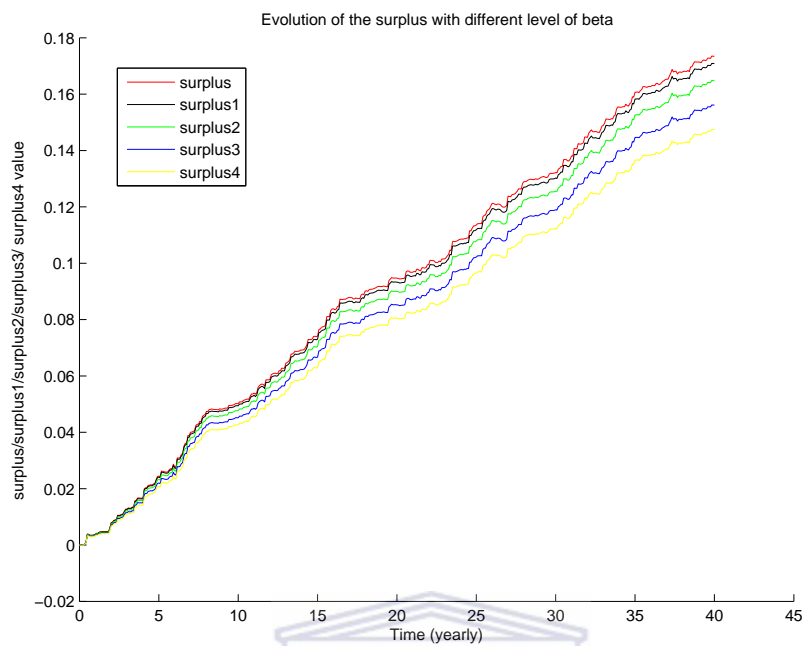| Time	Surp,$\beta = 1$	Surp1,$\beta = 0.985$	Surp2,$\beta = 0.950$	Surp3,$\beta = 0.90$	Surp4,$\beta = 0.85$
0	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.0484	0.0477	0.0460	0.0436	0.0411
20	0.0966	0.0952	0.0918	0.0868	0.0821
30	0.1415	0.1394	0.1344	0.1274	0.1203
40	0.1706	0.1680	0.1620	0.1535	0.1450

Figure 4.10: Evolution of the surplus with different levels of β.
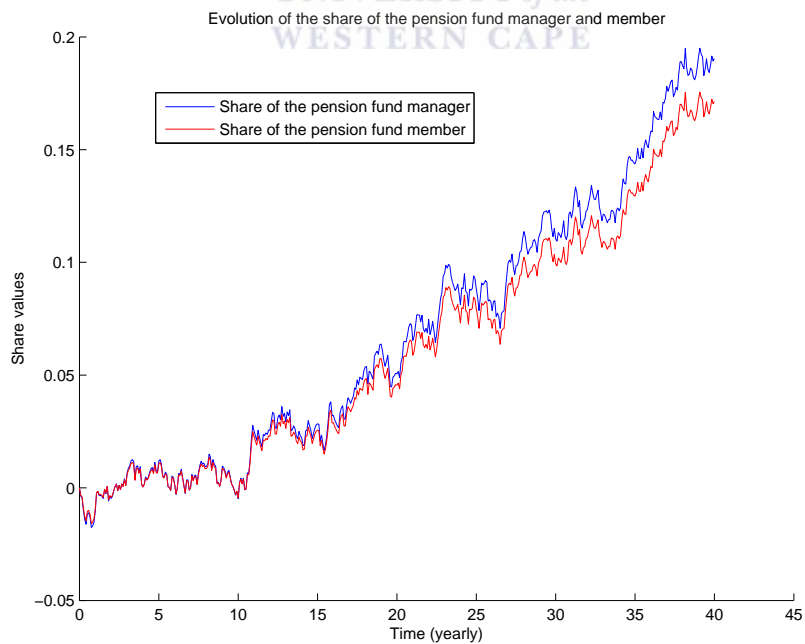

Figure 4.11: Evolution of the share of the pension fund manager and member.
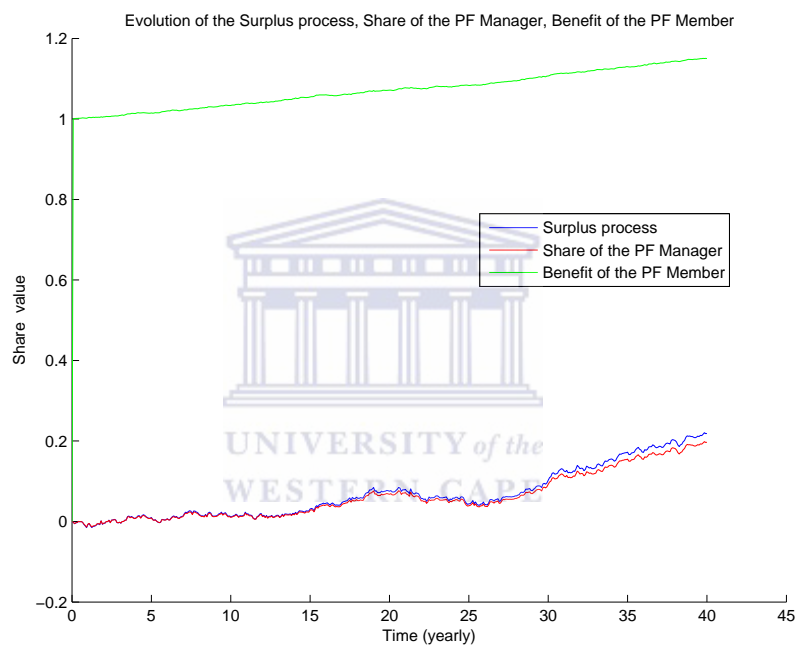
48

Figure 4.12: Evolution of the Surplus process, Share of the PF Manager, Benefit of the PF Member.

Chapter 5

Conclusion

The present dissertation presents an analysis of the problem of the optimal performance measurement of a defined contribution pension fund scheme where we have introduced two parameters: the guarantee and the sharing rule. The defined contribution pension fund scheme has seen its importance increased in the last two decades and this is due to its capacity of transferring risk to pension fund members. This has pushed most fund managers to adopt the aforementioned pension fund scheme. The defined contribution pension fund programme present a considerable advantage such as providing more benefits than the ever present programme of defined benefit pension fund. This is due to the fact that the DC pension fund scheme goes into the market and thus present the possibility of returns which are deemed higher than of the DB pension fund plan. By the same token, because it presents a possibility of a higher return, it bears with a higher risk as well. This possibility of higher risk which is mostly borne by pension fund members, is its downside. The question raised here is to find means to solve this problem encountered in the DC pension fund programme in order to mitigate the risk borne by the pension fund contributor.

Two elements have been introduced as part of the solution to try to improve the defined contribution pension fund plan; those element are the guarantee and the sharing rule represented by the parameter β. The research has aimed to analyse the impact of these elements into the defined contribution programme, in the context benchmarking

model with minimum guarantee.

The numerical and graphical analysis have been conducted in order to measure the impact of the two parameter on the evolution of the portfolio. It has for instance shown that the introduction of the parameter β which represent the sharing rule, has contributed to reduce the volatility of the portfolio, thus its risk; although this reduce the expected return at the same time. The ideal here is that as the defined contribution is a marketable pension fund programme, the introduction of the parameter β serves as a trade-off between the pension fund member and the pension fund manager in terms of risk.

- The guarantee, unlike to the sharing rule's parameter β is a state imposed parameter, which makes it quite difficult to the pension fund manager to have full control of the variable. Nonetheless, finding a guarantee that gives an optimal benefit to the programme is paramount against the risk of running the pension fund plan in loss. In Chapter 4, we have analysed the benchmarking model of pension fund with minimum guarantee in the Browne's framework [10]. The finding is that the parameter guarantee by its status of being state imposed, presents an incentive element for pension fund members to embrace the defined contribution programme. Thus pension fund members are almost sure of getting their investment back. This limits the reckless behavior of pension fund managers and make them more cautious when entering in the financial market.

- The sharing rule on the other hand is a variable mostly controlled by pension fund managers. A comparative numerical and graphical analysis of the pension fund programme with different level of the parameter β representing the sharing rule have been conducted. The numerical analysis has shown in Chapter 4 that the volatility of the portfolio does decrease with the sharing parameter β. For the graphical analysis the finding is that, although the portfolio without the sharing parameter β present some return higher than the portfolio with the sharing parameter β; its does not present any guarantee to the pension fund member due to the fact that it does not mitigate the risk mostly borne by the the pension fund member. The

present research has tried to confront the problem of the optimal sharing rule found in the existing literature and it is shown that the parameter β is a trade-off between pension fund managers and pension fund members in terms of risk mitigation. [19] [54].

We have conducted a comparative analysis between the growth optimal portfolio (GOP) with the guarantee process, the Browne's wealth process and the ratio process. The sensitivity analysis of the portfolios have been also conducted. It has been found that our portfolios are so sensitive to the market volatility as illustrated in graphs Fig 4.4 and Fig 4.5 on the page 40. We have also conducted a numerical analysis on the level of the utility and its degree of risk aversion between the pension fund manager and the pension fund member. The finding is that the level of the utility for both stakeholder evolve inversely with the level of risk aversion [52]. Finally we have computed the share of the pension fund manager and the benefit of the pension fund member.

Concerning these issues of the guarantee and of the sharing rule, there are different ways for taking it forward. So for instance one can use the tracking method to compare the guarantee with other benchmark instruments such as the GOP.

Bibliography

[1] Alan Bain (2008). *Stochastic calculus, class notes.* Chiark.greenend.org.uk.

[2] Devis Bayezit (2010). *Sensitivity analysis of options under Levy processes via Malliavin calculus.* Doctorate thesis, The Florida State University, College of Arts and Sciences.

[3] D. Blake, A. J. G. Cairns and K. Dowd (2001). Pensionmetrics: Stochastic pension plan design and value at risk during the accumulation phase. *Insurance: Mathematics and Economics* vol 29, p 187-215.

[4] T. Björk (1998). *Arbitrage Theory in Continuous Time.* OUP, Oxford.

[5] Zvi Bodie, Alex Kane, Alan J. Marcus (2004). *Essential of investments, fifth edition.* The McGraw-Hill/Irwin series in Finance, Insurance, and Real estate. International edition.

[6] Zvi Bodie, Alan J. Marcus, and Robert C. Merton (1988). Defined Benefit versus Defined Contribution Pension Plans: What are the Real Trade-offs? *National Bureau of Economic Research: Pensions in the U.S. Economy* vol 0-226-06285-6, p 139-162.

[7] J. F. Boulier, S. Huang and G. Taillard (2001). Optimal management under stochastic interest rates: The case of a protected defined contribution pension fund. *Insurance: Mathematics and Economics* vol 28, p 173-189.

[8] J. F. Boulier, E. Trussant and D. Florens (1995). A dynamic model for pension funds management. *Proceedings of the 5th AFIR International Colloquium* vol 1, p 361-384.

[9] L. Breiman (1960). Investment policies for expanding business optimal in a long run sense. *Naval Research Logistics Quarterly* vol 7(4): p 647-651

[10] Sid Browne (1999). Beating a moving target: Optimal portfolio strategies for outperforming a stochastic benchmark. *Finance and Stochastics*. vol 3, p 275-294.

[11] Andrew J. G. Cairns (2001). Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time. *ASTIN Bulletin* vol 30, p 19-55.

[12] Andrew J. G. Cairns (2003). Pension-Fund Mathematics: Discussion paper PI-0315. *The Pensions Institute Cass Business School*

[13] A. J. G. Cairns, D. Blake, and K. Dowd (2000). Optimal dynamic asset allocation for defined-contribution pension plans. *Proceedings of the 10th AFIR Colloquium*, Tromsoe, June, 2000, p 131-154

[14] A. J. G. Cairns, D. Blake, and K. Dowd (2006). Stochastic Lifestyling: Optimal Dynamic Asset Allocation for Defined Contribution Pension Plans.*Journal of Economic Dynamic and Control* Vol 30, p 843-877.

[15] A. J. G. Cairns, and G. Parker (1997). Stochastic pension fund modelling. *Insurance: Mathematics and Economics* vol 21, p 43-79.

[16] J. C. Cox and C. F. Huang (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory* vol 49, p 33-83.

[17] M. H. A. Davis, A. Norman (1990). Portfolio selection with transactions costs. *Mathematical operational research.* Vol 15, p 676 - 713.

[18] Griselda Deelstra, Martino Grasselli and Pierre-Francois Koehl (2003). Optimal investment strategies in the presence of a minimum guarantee. *Insurance: Mathematics and Economics* vol 33, p 189-207.

[19] Griselda Deelstra, Martino Grasselli and Pierre-Francois Koehl (2004). Optimal design of the guarantee for defined contribution funds. *Journal of Economic Dynamics and Control.* vol 28, p 2239-2260.

[20] P. Devolder, P. M. Bosch and F. I. Dominguez (2003). Stochastic optimal control of annuity contracts. *Insurance: Mathematics and Economics* Vol 33 (2), p 227-238

[21] Marina Di Giacinto, Fausto Gozzi and Salvatore Federico (2011). Pension funds with a minimum guarantee: stochastic control approach. *Finance Stochastic.* Vol 15, p 297-342.

[22] D. Dufresne (1988). Moments of pension contributions and fund levels when rates of return are random. *Journal of the Institute of Actuaries* vol. 115, p 535-544.

[23] D. Dufresne (1989). Stability of pension systems when rates of return are random. Insurance: *Mathematics and Economics* vol 8, p 71-76.

[24] D. Dufresne (1990). The distribution of a perpetuity, with applications to risk theory and pension funding. *Scandinavian Actuarial Journal* 1990, p 39-79.

[25] S. Federico (2008). A pension fund in the accumulation phase: a stochastic control approach. *Banach Center Publications, 83: Advances in Mathematics of Finance,* 2008.

[26] W. H. Fleming and R. W. Rishel (1975).*Deterministic and stochastic optimal control.* Springer-Verlag, New York

[27] Jianwei Gao (2008). Stochastic optimal control of DC pension funds. *Insurance: Mathematics and Economics* vol 42 p 1159-1164

[28] A. Gaivoronski, S. Krylov and Nico van der Wijst (2005). Optimal portfolio selection and dynamic benchmark tracking. *European journal of Operational Research.* vol 163, p 115- 131.

[29] W. N. Goetzmann, J.E. Ingersoll and S.A. Ross (2003). High-Water Marks and Hedge Fund Management Contracts. *The Journal of Finance* vol 58, 4, p 1685-1718.

[30] S. Haberman (1993). Pension funding with time delays and autoregressive rates of investment return. *Insurance: Mathematics and Economics* vol 13, p 45-56.

[31] S. Haberman and J. H Sung (1994). Dynamic approaches to pension funding. *Insurance: Mathematics and Economics* vol 15,p 151-162.

[32] S. Haberman, and E. Vigna (2002). Optimal investment strategies and risk measures in defined contribution pension schemes. *Insurance: Mathematics and Economics* vol 31, p 35-69.

[33] D. Hainaut and P. Devolder (2006). A martingale approach applied to the management of life insurances. *In: Working Paper. Institute of Actuarial Sciences* UCL.

[34] E. Platen and D. Heath (2006). *A Benchmark Approach to Quantitative Finance.* Springer-Verlag Berlin Heidelberg.

[35] B. Holstrom and P. Milgrom (1987). Aggregation and Linearity in the Provision of Intertemporal Incentives. *Econometrica* vol 55, P 303-328.

[36] H. C. Huang (2000). *Stochastic modelling and control of pension plans. Ph.D.Thesis,* Heriot-Watt University

[37] John. C. Hull (2012).*Options, Futures, And Others Derivatives. 8th ed.* Pearson Education Limited, p 299- 328.

[38] Ireland Peter (2014).*EC 379.01. Financial Economics,* Boston College, Spring 2014.

[39] Ioannis Karatzas and Steven E. Shreve (1998). *Methods of Mathematical Finance.* Springer Verlag, New York, LLC.

[40] J. R. Kelly (1956). A new interpretation of information rate, *Bell Syst. Techn. J.* 35: 917- 926.

[41] Kyriaki Kosmidou and Constantin Zopounidis (2004). *Goal Programming Techniques for Bank Asset Liability Management.* Kluwer Academic Publishers.

[42] R. Korn (1997). Optimal Portfolios. *World Scientific, Singapore*

[43] H. Latane (1959). Criteria for choice among risky ventures. *Journal of Political Economy* vol 38: p145-155.

[44] Andrew E. B. Lim and Bernard Wong (2010). A benchmarking approach to optimal asset allocation for insurers and pension funds. *Insurance: Mathematics and Economics.* vol 46, p 317-327.

[45] A. Lioui and P. Poncet (2013). Optimal benchmarking for active portfolio managers. *European Journal of Operational Research.* vol 226, p 268- 276.

[46] J. B. Long (1990). The numeraire portfolio. *Journal of Financial Economics* vol 26: p 29- 69.

[47] H. Markowitz (1976). Investment for the long run: New evidence for an old rule. *Journal of Finance* vol XXXI(5): p 1273-1286.

[48] R. Merton (1969). Lifetime portfolio selection under uncertainty: The continuous time case. *Review of Economics and Statistics* vol 51, p 247-257.

[49] R. Merton (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* vol 3, p 373-413

[50] R. Merton (1990). *Continuous-Time Finance.* Blackwell, Cambridge, Mass.

[51] F. Menoncin and O. Scaillet (2006). Optimal asset management for pension funds. *Managerial Finance* vol 32 (4), p 347-374.

[52] MIT (2010) *Microeconomy Theory III, Chap.4 Attitude towards risk.* MIT Open CourseWare/http://OCW.mit.edu. spring 2010.

[53] L. C. G. Rogers and D. Williams (1987). *Diffusions, Markow process and Martingales, Vol. 2.* New York: Wiley.

[54] Johanna Scheller and Jacques Pezier (2009). Optimal Investment Strategies and Performance Sharing Rules for Pension Schemes with Minimum Guarantee. *ICMA Centre Discussion Papers in Finance, The business school for financial market, University of Reading* DP 2009-10.

[55] E. O. Thorp (1961). A favourable strategy for twenty-one. *Proceedings of the National Academy of Sciences* Vol. 47, p 110-112.

[56] E. Vigna and S. Haberman (2001). Optimal investment strategy for defined contribution pension schemes. *Insurance: Mathematics and Economics* vol 28, p 233-262.

[57] William T. Ziemba and Raymond G. Vickson (2006) *Stochastic Optimization Models in Finance, 2nd edition.* World Scientific Handbook in Financial Economics Series: Volume 1.

[58] J. Xiao, H. Zhai, and C. Qin (2007). The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts. *Insurance: Mathematics and Economics* vol 40, p 302-310.