Mathematical models for optimal management of bank capital, reserves and liquidity

Garth van Schalkwyk

Supervisor: Prof P.J. Witbooi

A thesis submitted in fulfillment of the requirements
for a degree Philosophiae Doctor in Applied Mathematics,
University of the Western Cape, South Africa.

Contents

Abstract

The aim of this study is to construct and propose continuous-time mathematical models for optimal management of bank capital, reserves and liquidity. This aim emanates from the global financial crisis of $2007-2009$. In this regard and as a first task, our objective is to determine an optimal investment strategy for a commercial bank subject to capital requirements as prescribed by the Basel III Accord. In particular, the objective of the aforementioned problem is to maximize the expected return on the bank capital portfolio and minimize the variance of the terminal wealth. We apply classical tools from stochastic analysis to achieve the optimal strategy of a benchmark portfolio selection problem which minimizes the expected quadratic distance of the terminal risk capital reserves from a predefined benchmark.

Secondly, the Basel Committee on Banking Supervision (BCBS) introduced strategies to protect banks from running out of liquidity. These measures included an increase of the minimum reserves that the bank ought to hold, in response to the global financial crisis. We propose a model to minimize risk for a bank by finding an appropriate mix of diversification, balanced against return on the portfolio. Thirdly and finally, in response to the financial crises, the Basel Committee on Banking Supervision (BCBS) designed a set of precautionary measures (known as Basel III) for liquidity imposed on banks and one of its purposes is to protect the economy from deteriorating. Recently, bank regulators wanted banks to depend on sources such as core deposits and long-term funding from small businesses and less on short-term wholesale funding.

We investigate the money supply process between a central bank and a commercial bank and how it influences the liquidity of a bank. In particular, we formulate a stochastic control problem involving cash which is held as deposits at a central bank.

In this thesis, we propose jump-diffusion models for bank items on a balance sheet in order to simulate some of the main measures in bank management. These measures are the liquidity coverage ratio, capital adequacy ratio, and the reserve requirement ratio. We use data from [89], [30] and [74] to illustrate the importance of our methodology.

Key words: Bank capital management, Capital adequacy ratio, Investment strategies, Stochastic optimal control, Jump-diffusion, Optimal portfolios, Mean-Variance approach, Credit and market risk-weighted assets, Reserve management, Investment strategies, Stochastic optimal control, Deposits, Basel III, Global Financial Crisis.

Acknowledgements

Firstly, I would like to thank the Almighty Father for His grace in enabling me to complete this dissertation.

I dedicate this dissertation to my late mother, Johanna Paulina van Schalkwyk.

I would like to acknowledge the support and encouragement provided by my brother, Frank and sisters Karin, Jocelyn and Lavona.

I am indebted to my supervisor, Prof. Peter J. Witbooi for the guidance, patience and moral support provided during the completion of this dissertation. I thank the members of staff of the Department of Mathematics and Applied Mathematics at the University of the Western Cape and friends for providing moral and a friendly collegial environment during my studies.

Furthermore, I acknowledge financial support from the National Research Foundation (NRF).

Declaration

I declare that *Mathematical models for optimal management of bank capital, reserves and liquidity* is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Garth van Schalkwyk

Signed: G. van Schalkwyk

Key Definitions

The *capital adequacy ratio* (CAR) is a measure of the amount of a bank's capital relative to its risk weighted assets expressed as a percentage, that is,

$$\text{CAR} = \frac{\text{Indicator of Absolute Amount of Bank Capital}}{\text{Indicator of Absolute Level of Bank Risk}}.$$

The *liquidity coverage ratio* (LCR) refers to highly liquid assets held by financial institutions to meet short-term obligations. The ratio is a generic stress test that aims to anticipate market-wide shocks. The liquidity coverage ratio is designed to ensure financial institutions have the necessary assets on hand to ride out short-term liquidity disruptions. It started to be regulated and measured in 2011, but the full 100% minimum won't be enforced until 2015. The LCR is defined as

$$\text{LCR} = \frac{\text{Total Stock of High-Quality Liquid Assets (HQLAs)}}{\text{Total Nett Cash Outflows (TNCOF) over the Next 30 Calendar Days}}.$$

The *reserve requirement ratio* (RRR) is the portion of depositors' balances that banks must have on hand as cash. This is a requirement determined by the country's central bank, which in the United States is the Federal Reserve. The reserve ratio affects the money supply in a country at any given time.

Index of Abbreviations

Basel Committee on Banking Supervision (BCBS)

Capital adequacy ratio (CAR)

Linear-quadratic (LQ)

Non-risk based (NRBCARs)

Risk based (RBCARs)

Federal Deposit Insurance Corporation (FDIC)

Risk-weighted assets (RWAs)

Capital-at-Risk (CaR)

Value-at-Risk (VaR)

Capital Asset Pricing Model (CAPM)

Deposit insurance fund (DIF)

Common Equity Tier 1 capital (CET1)

Hamilton-Jacobi-Bellman equation (HJB)

Federal Reserve (Fed)

Panel Smooth Transition Regression (PSTR)

Automated Teller Machines (ATMs)

Interest on required reserve (IORR)

Interest on excess reserves (IOER)

Peoples Bank of China (PBOC)

Liquidity Coverage Ratio (LCR)

High-Quality Liquid Assets (HQLAs)

Total Nett Cash Outflows (TNCOF)

Level 1 assets (L1As)

Level 2 assets (L2As)

Level 2A assets (L2AAs)

Level 2B assets (L2BAs)

Shareholder Cash Flow Rights (SCFRs)

Net Stable Funding Ratio (NSFR)

Troubled Asset Relief Program (TARP)

Central Bank (CB)

Index of symbols

$R(t)$ - bank reserves at time t

$L(t)$ - bank loans at time t

$S(t)$ - marketable securities at time t

$S_1(t)$ - cash and currency at time t $D(t)$ - bank deposits at time t

$C(t)$ - bank regulatory Capital at time t

$B(t)$ - bank borrowing at time t

$r_S(t)$ - interest rate on marketable securities at time t

σ_S - the security volatility

λ_S - market risk premium

$E(R_m)$ - market expected return

r^A - risk-free rate

β - slope of the security market line

θ_{r_S} - rate of mean reversion of interest rate model

μ_{r_S} - long-run mean of interest rate model

σ_{r_S} - volatility of interest rate model

μ_D - deposit inflow rate

σ_D - volatility in deposits

r^{DD} - rate of demand deposit

r^{TD} - rate of time deposit

fraction of deposits - γ

$C_{T1}(t)$ - Tier 1 Capital at time t

$C_{T2}(t)$ - Tier 2 Capital at time t

$E(t)$ - common equity capital at time t

r_e - return on equity capital

σ_e - volatility of equity

$C_a(t)$ - additional capital at time t

$S_D(t)$ - market value of subordinate debt at time t

r - risk-free interest rate on subordinate debt

α - rate of additional capital

β_{C_a} - perturbation of additional capital

$\mathcal{A}$ - class of admissible control laws

$\mathbb{E}$ - Expectation operator

$\Gamma^{1,2}$ - set containing value function

$\Lambda(t)$ - Basel III capital adequacy ratio at time t

ϕ_1 - Basel III risk-weight for loans

ϕ_2 - Basel III risk-weights for marketable securities

$T_0(t)$ - Treasuries at time t

r^T - interest rate on Treasuries

$r^R(t)$ - rate of return on reserves earned by the bank at time t

$f^R(t)$ - fraction of bank reserves that are available for withdrawal at time t

σ - volatility in the level of reserves

$W(t)$ - provision for deposit withdrawals at time t

$k(t)$ - rate of depository consumption at time t

m_D - expected rate of growth in deposits

σ_D - level of volatility in deposits

$\psi(t)$ - rate at which the Federal Reserve banks pay interest on required reserve balances at time t

$\phi(t)$ - rate at which the Federal Reserve banks pay interest on excess reserves at time t

$D^S(t)$ - stable retail deposits at time t

$D^L(t)$ - less stable retail deposits at time t

$F^U(t)$ - unsecured wholesale funding at time t

$B^I(t)$ - interbank borrowing at time t

m_s - expected rate of growth in stable deposits

m_l - expected rate of growth in less-stable deposits

σ_s - level of volatility in stable deposits

σ_l - level of volatility in less-stable deposits

m_f - rate of brokered deposits

σ_f - level of volatility in brokered deposits

m_B - interbank rate

σ_B - level of volatility in bank borrowing

$A_{L_1}(t)$ - Level 1 assets at time t

$A_{L_2}(t)$ - Level 2 assets at time t

$H_Q(t)$ - Total Stock of High-Quality Liquid Assets at time t

$\Theta(t)$ - Total Nett Cash Outflows at time t

$\Lambda(t)$ - total expected cash outflows at time t

$\Upsilon(t)$ - Banks total expected cash inflows at time t

$l_{M_1}(t)$ - maturing secured lending backed by Level 1 at time t

$l_{M_2}(t)$ - maturing secured lending backed by Level 2 at time t

c^* - optimal rate of currency flow from the vault cash holdings

List of Figures

Chapter 1

Introduction

Commercial banks play a key role in all modern financial systems. One of the most important assurances for banks to perform effectively is for the economic value of its assets to be worth significantly more than its liabilities. The difference between the assets and liabilities represents a cushion of capital that is available to cover any kind of losses.

However, it is important to have a second type of buffer, known as "liquidity", that banks need to have to cover unexpected cash outflows. Even though a bank remains solvent while its holding of assets exceeds its liabilities on an economic and accounting basis, if its depositors and other funders lose confidence in the institution, the bank may still move into a region of bankruptcy. Bank capital and liquidity are concepts that are central to an understanding of bank operations, as the risks they take on and how best to mitigate it can be a tricky situation. Bank capital is a form of funding that can absorb losses which could otherwise threaten a bank's solvency. Meanwhile, liquidity problems arise due to interactions between this funding and the assets of the balance sheet - when a bank does not hold sufficient cash (or assets that can easily be converted into cash) to repay depositors and other creditors. The instability resulting from banks having insufficient financial resources can undermine the vital economic functions they perform.

The financial crisis which occurred between $2007 - 2009$ has put bank capital regulation and liquidity management at the forefront of political and academic discussions (see [3], [18], [53] and [82]). This included a strengthening of bank capital requirements and the introduction of new liquidity requirements, as part of the so-called Basel III package of reforms which was introduced by the Basel Committee on Banking Supervision in 2010 (see [8]). One of the reasons why further restrictions were placed on bank capital requirements was that the crisis highlighted the need for banking systems to be less leveraged, more liquid, more transparent and less prone to taking on excessive risk. Both the private and public sectors have put banks under severe pressure to build larger buffers of high-quality capital and reduce the riskiness of their portfolios. However, if a bank builds up a buffer of capital too rapidly, this would impose considerable short-term macroeconomic costs by forcing the bank to pull back from lending to finance investment. In response to adjust to the higher capital standard, [25] found that banks have a number of options at their disposal. Banks can choose not to pay out dividends to shareholders and instead retained the net earnings to reinvest in its core business or pay the debt. Alternatively, it may try to boost profits themselves or increase the spread between the loan rates and interest rates it pays on its funding. Another way of increasing its net income is to increase profit margins on other business lines and to reduce overall operating expenses. A second approach (but the least attractive option) is to issue new equity, such as through a rights issue to existing shareholders or an equity offering on the open market. Banks may also decide to reduce its risk-weighted assets by replacing riskier (higher-weighted) loans with safer ones, or with government securities. Another strategy for banks to increase its capital adequacy ratio is to either run down its loan portfolio or sell its assets and use the proceeds of loan repayments or asset sales to pay down debt. Also, it can slow down the growth of lending, thereby allowing retained earnings and hence capital to increase. For instance, if banks slow its lending activities or reduce lending to riskier projects then this could reduce investment opportunities. The author of [25] suggested that a gradual decrease in bank lending growth is caused by a reduction in bank loan supply, as opposed to a reduction in demand for loans from borrowers.

Prior to the financial crisis, financial systems had a liquidity surplus which in turn caused risk of liquidity and its management to be monitored far less than other risks. However, the financial crisis spurred on the speed at which a liquidity crisis can appear and at what speed the financial resources can disappear, thereby increasing the assets assessment problem. Banks have the ability to impose major risks, such as insolvency, on the economy. Evasion of such risks and the cost associated with it has been a major concern of prudential regulation. The global financial crises caused the bank to have insufficient liquidity and also spurred on excessive leverage which in turn caused financial instability, and made for banks being unable to withstand large negative shocks. Practitioners in the banking industry has put some proposals forward on the debate of banking regulation and regulatory reform. Some of the main ideas that arise from these proposals are the increase of capital requirements and holding of a buffer of liquid reserves. The reason why capital needs an injection is that if the value of the bank's assets were to decline, this would not automatically lead to distress and the resulting losses would be carried by the bank owners. Banks are expected to hold a buffer of liquid reserves in order to be able to cope with short-term losses as most banks assets are illiquid and raising fresh equity at short notice is costly (see [3]). A primary factor in the subprime mortgage crisis and the global recession was shadow banking. A shadow banking system is a term, coined by Paul McCulley of the Pacific Investment Management Company, referring to a collection of non-bank financial intermediaries (such as hedge funds, private equity firms, special purpose vehicles, insurance companies, crowdfunding organizations, and money market funds) that provide services similar to traditional commercial banks but outside normal banking regulations. These type of unregulated financial institutions acts as a bank but instead of financing activities through deposits, it does so through investors, borrowing, or creating financial products. One of the key differences between a regulated bank and these institutions is that it participates in risky off-balance-sheet activities. Before the crisis, shadow banks originated as an alternative to traditional (regulated) banks to perform similar functions of liquidity, credit, and maturity transformation. Its growth was largely motivated by tightening in the regulatory requirements of banks and financial innovation.

Since shadow banks are not regulated in the same way as banks, it does not have access to either insured deposits or the central bank funding. With regards to the connection of shadow banking and capital regulation, the financial crisis has triggered a broad push toward increased regulation of the financial sector. It was argued that banks should be subject to significantly higher capital requirements in order to mitigate risk-shifting incentives and increase financial stability. However, increased regulation of banks may push intermediation into unregulated entities which in turn may cause increase overall financial fragility and reduce welfare. For a more detailed account of the relationship between capital regulation and shadow banking see the articles [39], [54] and [69].

The academic literature on jointly optimal regulation of bank capital and liquidity seems to increase since the introduction of liquidity measurement standards after 2010 by the Basel Committee on Banking Supervision. The author of [82] addresses the problems of how bank capital and maturity should be regulated to deal with systemic risk (market risk) and how should this interact with regulation against costly bank failures. On the other hand, [53] builds a dynamic model to assess the effects of liquidity and capital requirements on banks' insolvency risk. The article [18] studies the transmission mechanisms of liquidity and capital regulations as well as their effects on the economy and welfare. Their macro-economic model allows a bank regulator to face the following trade-off. On the one hand, banking regulations reduce the aggregate supply of credit. On the other hand, they promote the allocation of credit to its best uses. The article [3] develops a simple structural model of a financial institution that can invest in both liquid and illiquid assets dynamically and whose goal is to maximize the profit of its shareholders while satisfying some regulatory constraints. Furthermore, [3] computes the dynamic optimal portfolio allocations and studies the sensitivity of the shareholders' gain and strategies, as well as the sensitivity of associated bondholders' payoff to the minimal capital requirement and liquidity ratio.

In this thesis, we address some of the issues highlighted in the introduction and literature review, Chapter 1. This dissertation consists of Chapter 1 together with a preliminary Chapter 2, a concluding chapter, Chapter 6, and 3 main chapters. In the preliminary chapter, we cover all relevant mathematical ideas and concepts used in this thesis. The main chapters, i.e., Chapters 3, 4 and 5 focus on three related commercial banking problems. Chapter 6 provides conclusions about the main issues discussed in the thesis, and points out possibilities for future research topics. We will now provide a brief summary of each of these problems that are solved in a jump-diffusion framework. The first problem is addressed in Chapter 3. This problem emanates from Section 3.4 and it involves deriving a capital allocation strategy that optimizes the expected future value of a commercial bank's capital. This entails finding optimal amounts of total capital for investment in different bank assets. In particular, we consider a bank that invests its total capital in a financial market consisting of three assets, viz., a treasury security, a marketable security and a loan. The dynamics of the loan is assumed to be described by a jump-diffusion process and is derived from a balance-sheet constraint. In Section 3.6 we wish to find the capital allocation strategy under the criterion of mean-variance optimization. In Section 3.6 we provide numerical simulations, using data from Barclays Africa Group Limited (see [89]), in order to characterize the behavior of the CAR. Our graphs illustrate that the CAR for Barclays Africa Group Limited remains above the minimum regulatory capital requirements and within its board approved target capital ranges. The second problem is addressed in Chapter 4 and it involves optimal portfolio selection with liability (deposits) under the benchmark criteria. We solve the problem via the stochastic maximum principle method for control problems. We derive an asset allocation strategy for a bank's total reserve portfolio which attempts to minimize the deposit risk associated with it. From the optimal allocation strategy that solves our investment problem in Section 4.2 , we simulate the bank's reserve requirement ratio in Section 4.3 and show that the simulated trajectory of the dynamics for bank reserves exhibit similar characteristics to the realistic graph obtain from [29]. Furthermore, we provide simulations of the reserve requirement ratio for a central bank, Peoples Bank of China listed on [30] and interpret the results. The final problem is tackled in Chapter 5 and it involves constructing a dynamic model for one of the measurements of bank liquidity. These ratios were introduced in an attempt to improve the regulation of the international banking industry in terms of liquidity management. In order to derive the model of one of the liquidity ratios, we require formulae for the Stock of High-Quality Liquid Assets (SHQLAs) and Total Net Cash Outflows (TNCOs). In Section 5.2, we provide numerical simulations in order to characterize the behavior of the Basel III liquidity coverage ratio. Our graphs indicate that the liquidity ratio remains above the recommended threshold of 100% by the Basel Committee on Banking Supervision.

We wish to make explicit the contributions of this project to the literature. The research articles [78] and [79] have already been published. The final research article of the thesis, [80], is currently under review.

In [78], we investigate the money supply process between a central bank and a commercial bank and how it has an effect on the liquidity coverage ratio of a bank. In particular, we formulate a stochastic control problem involving cash which is held as deposits at a central bank. The solution of this problem is readily obtained from existing literature. Using simulations we investigate the manner in which this interplay affects the liquidity coverage ratio of a bank.

In [79], we consider jump-diffusion models of bank reserves in order to address the risk due to deposit withdrawals. We formulate a stochastic optimal control problem related to the minimization of deposit risk and the reserve process, the net cash flows from depository activity, and cumulative cost of the bank's provisioning strategy, respectively. We analyze the main risk management issues arising from the optimization problem, with respect to the reserve requirement ratio, supported by simulations.

In [80], we investigate the continuous-time dynamics of bank items such as loans, reserves, treasuries, marketable securities, deposits and bank regulatory capital. Appropriate stochastic models allow us to formulate an optimal asset allocation problem subject to cash flow, financing and balance sheet constraints. In essence, we study the investment of bank funds in loans, treasuries, and marketable securities with the aim of generating an optimal fund level while minimizing market and credit risk at the same time.

Chapter 2

Preliminaries about Stochastic Processes

In this section we introduce some basic elementary concepts and properties of probability and measure theory that is useful to understand the notion of stochastic differential equations. We provide now definitions and concepts relevant to Brownian motion and Lévy processes. More details about the aforementioned topics can be found in [1], [17], [27], [58], [70] and [71].

2.1 Stochastic Processes

Definition 2.1.1. (see Ash [2] or Cohn [26])
Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a *field* (algebra) if and only if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and finite union, that is,

1. $\Omega \in \mathcal{F}$.

2. For a set A, if $A \in \mathcal{F}$, then also $A^c \in \mathcal{F}$ (A^c is the complement of A).

3. If $A_1, A_2, A_3, \ldots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}$.

Remark: It follows that $\mathcal{F}$ is closed under finite intersection. For if $A_1, A_2, A_3, \ldots, A_n \in \mathcal{F}$, then

$$\bigcap_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n} A_i^c\right)^c \in \mathcal{F}.$$

Definition 2.1.2. (see Ash [2] or Cohn [26])
Let Ω be an arbitrary set and let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a σ-field (σ-algebra) if and only if $\mathcal{F}$ is a field and $\mathcal{F}$ is closed under countable intersection.

For a further discussion on infinite sequences, algebras and σ-algebra we refer the reader to Ash [2] or Cohn [26].

Definition 2.1.3. (see for instance Grimmett and Stirzaker [48])
A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ satisfying

1. $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$;

2. If $A_1, A_2, A_3, \ldots$ is a sequence of mutually disjoint members of $\mathcal{F}$, i.e., so that $A_i \cap A_j = \emptyset$ for all pairs i, j satisfying $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Definition 2.1.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $\{\mathcal{F}_t\}_{t\geq 0}$ of sub $\sigma-$algebras is called a filtration if $\mathcal{F}_t \subset \mathcal{F}_s$, for all $0 \leq s \leq t$. If $\mathcal{F}$ is such that $\mathcal{F}_t = \cap_{t>s}\mathcal{F}_s$, then $\{\mathcal{F}_t\}$ is said to be right-continuous.

Definition 2.1.5. A stochastic process is a family of vector random variables $\{X_t\}_{t\geq 0}$. That is for all $t > 0$, the application

$$X_t : \begin{array}{l} \Omega \to \mathbb{R}^n \\ \omega \mapsto X_t(\omega) \end{array}$$

is measureable. If $\{X_t\}$ is a stochastic process, then for all $t \geq 0$, the application $t \to X_t$ is called a sample path.

Definition 2.1.6. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process $\{M_t\}_{t\geq 0}$ is said to be $\{\mathcal{F}_t\}$-adapted if for all $t \geq 0$, $\{M_t\}$ is $\{\mathcal{F}_t\}$-measurable.

Definition 2.1.7. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process $\{M_t\}_{t\geq 0}$ is called $\{\mathcal{F}_t\}$-martingale if the following properties holds

(i) $\{M_t\}$ is $\{\mathcal{F}_t\}$-adapted.

(ii) $\mathbb{E}||M_t|| < \infty, \ \forall t \geq 0$.

(iii) $\mathbb{E}(M_t|\mathcal{F}_s) = M_s, \forall \ 0 \leq s \leq t$.

Definition 2.1.8. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process $\{X_t\}_{t\geq 0}$ is called an $\{\mathcal{F}_t\}$-predictable process if for all $t > 0$, X_t is measurable with respect to the σ-algebra generated by $\{X_s, s < t\}$.

Definition 2.1.9. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on this space. A $\{\mathcal{F}_t\}$-adapted stochastic process $\{W_t\}_{t\geq 0}$ is called a Wiener process or Brownian motion if:

(i) $W_0 = 0$.

(ii) $t \to W_t$ is almost surely continuous.

(iii) $\{W_t\}_{t\geq 0}$ has independent increments (i.e. $W_t - W_s$ is independent of W_r, $r \leq s$).

(iv) $W_t - W_s \sim \mathcal{N}(0, t-s)$, for $0 \leq s \leq t$. Usually, this property is called stationarity.

For a more detailed description of stochastic processes the reader is referred to Bhattacharya and Waymire [17].

2.2 Preliminaries about Lévy Processes

The following preliminaries about stochastic processes is useful for the dynamic bank models discussed in Chapters 3, 4 and 5.

The Poisson process is a stochastic process type with a discontinuous path, and it is preferred to be used as a building block for constructing more complex jump processes, e.g. Lévy processes.

Definition 2.2.1. (Counting process):

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on this space. Also, let $\{S_k\}_{k\geq 1}$ be an $\{\mathcal{F}_t\}$-adapted stochastic process on the aforementioned space with $S_1(\omega) \leq S_2(\omega) \leq \dots$ for all $k \geq 1$ and $\omega \in \Omega$. The $\{\mathcal{F}_t\}$-adapted process $N = \{N_t\}_{t\geq 0}$ is defined by:

$$N_t := \sum_{k\geq 1} 1_{S_k \leq k}$$

is called a counting process with jump times S_k.

Definition 2.2.2. (Poisson process):

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on this space. An $\{\mathcal{F}_t\}$-adapted counting process $\{N_t\}$ is called a Poisson process with intensity $\lambda > 0$ if:

(i) $N_0 = 0$.

(iii) for all $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $\{N_{t_j} - N_{t_{j-1}}, 1 \leq j \leq n\}$ are independent.

(iii) For $0 \leq s \leq t, N_t - N_s \approx N_{t-s}$, where $\approx$ stands for the equality in probability law.

(iv) For all $t > 0$, N_t follows a Poisson law with parameter λt. That is

$$\mathbb{P}(N_t = k) = e^{\lambda t}\frac{(\lambda t)}{k!}, \quad k \in \mathbb{N}$$

Definition 2.2.3. (Compound Poisson process):

Let $\{Z_n\}$ be a sequence of discrete independent identically distributed random variables with probability law ν_Z. Let $N = \{N_t\}$ be a Poisson process with parameter λ. Let us assume that $\{N_t\}$ and $\{Z_n\}$ are independent. A compound Poisson process with intensity $\lambda > 0$ with a jump law ν_Z is a $\{\mathcal{F}_t\}$-adapted stochastic process $\{Y_t\}$ defined by

$$Y_t := \sum_{k=1}^{N_t} Z_k.$$

Definition 2.2.4. (Compensated Poisson process):

A compensated Poisson process associated to a Poisson process N with intensity λ is a stochastic process $\widetilde{N}$ defined by :

$$\widetilde{N}(t) := N(t) - \lambda t.$$

Definition 2.2.5. (Càdlàg and Càglàd functions):

(i) A function $f : [0,T] \to \mathbb{R}^n$ is said to be right continuous with left limits at $t \in [0,T]$ if

$$f(t^+) := \lim_{s \to t+} f(s) \text{ and } f(t^-) := \lim_{s \to t^-} f(s) \text{ exist and } f(t^+) = f(t).$$

(ii) A function $f : [0,T] \to \mathbb{R}^n$ is said to be left continuous with right limits at $t \in [0,T]$ if

$$f(t^+) := \lim_{s \to t+} f(s) \text{ and } f(t^-) := \lim_{s \to t^-} f(s) \text{ exist and } f(t^-) = f(t).$$

Remark 2.2.6.

- If f is right continuous with left limits at t, then $\Delta f(t) = f(t) - f(t^-)$ called the jump of f at t.
- If f is left continuous with right limits at t, then $\Delta f(t) = f(t^+) - f(t)$ called the jump of f at t.

Definition 2.2.7. A stochastic process $X = \{X_t\}_{t \geq 0}$ is called a jump process if the sample path $s \to X_s$ is right continuous (Càdlàg) or left continuous (Càglàd) for all $s \geq 0$.

Definition 2.2.8. (Lévy process):

A stochastic process $L = \{L_t\}_{t \geq 0}$ is a Lévy process if the following conditions are satisfied:

(i) The increments on disjoint time intervals are independent. That is, $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $\{L_{t_j} - L_{t_{j-1}}, 1 \leq j \leq n\}$ are independent.

(ii) The increments of sample paths are stationary, i.e., $L_t - L_s \approx L_{t-s}$, for $0 \leq s \leq t$.

(iii) The sample paths are right continuous with left limits.

In this thesis we will assume that the Lévy processes we work with are càdlàg . Let B_0 be the family of Borel sets $U \subset R$ whose closure $\bar{U}$ does not contain 0. For $U \in B_0$ we define

$$N(t,U) = N(t,U,\omega) = \sum_{s:0<s\leq t} \chi_U(\Delta\eta(s)).$$

In other words, $N(t,U)$ is the number of jumps of size $\Delta\eta(s) \in U$ which occur before or at time t. Here $N(t,U)$ is called the Poisson random measure (or jump measure) of $\eta(\cdot)$.

Remark 2.2.9. (see [72]) Note that $N(t,U)$ is finite for all $U \in B_0$.

To see why Remark 2.2.9 is true, define

$$T_1(\omega) = \inf\{t > 0; \eta(t) \in U\}.$$

We claim that $T_1(\omega) > 0$ a.s. To prove this, note that by right continuity of paths we have

$$\lim_{t\to 0^+} \eta(t) = \eta(0) = 0 \ a.s.$$

Therefore, for all $\epsilon > 0$ there exists $t(\epsilon) > 0$ such that $|\eta(t)| < \epsilon$ for all $t < t(\epsilon)$. This implies that $\eta(t) \notin U$ for all $t < t(\epsilon)$, if $\epsilon < \text{dist}(0,U)$.

Next we define inductively

$$T_{n+1}(\omega) = \inf\{t > T_n(\omega);\ \Delta\eta(t) \in U\}.$$

Then by the above argument $T_{n+1} > T_n$ a.s. We claim that

$$T_n \to \infty \text{ as } n \to \infty,\ a.s.$$

Assume not, then $T_n \to T < \infty$. But then

$$\lim_{s\to T^-} \eta(s)$$

can not exist, contradicting the existence of left limits of the paths.

Remark 2.2.10. (see [72]) The set function

1. $U \to N(t, U, \omega)$ defines a σ-finite measure on B_0 for each fixed t, ω. The differential form of this measure is written $N(t, dz)$;

2. $[a, b) \times U \to N(b, U, \omega) - N(a, U, \omega); [a, b) \subset [0, \infty),\ U \in B_0$ defines a σ-finite measure for each fixed ω. The differential form of this measure is written $N(dt, dz)$;

3. $\nu(U) = \mathbb{E}[N(1, U)]$, where $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ denotes expectation with respect to $\mathbb{P}$, also defines a σ-finite measure on B_0, called the Lévy measure of $\{\eta(t)\}$;

4. Fix $U \in B_0$. Then the process

$$N_U(t) := N_U(t, \omega) := N(t, U, \omega)$$

is a Poisson process of intensity $\lambda = \nu(U)$.

To find the Lévy measure ν of an Itô process $Y(t)$ note that if $U \in B_0$ then

$$\begin{aligned}
\nu(U) &= \mathbb{E}[N(1, U)] \\
&= \mathbb{E}[\sum_{s:0<s\leq 1} \chi_U(\Delta Y(s))] \\
&= \mathbb{E}[(\text{number of jumps}) \times \chi_U(\text{jump})] \\
&= \mathbb{E}[N(1) \times \chi_U(\text{jump})] \\
&= \lambda \mu_X(U),
\end{aligned}$$

by independence. We conclude that

$$\nu = \lambda \mu_X.$$

This shows that a Lévy process can be represented by a compound Poisson process if and only if its Lévy measure is finite. The following result gives a general description of Lévy processes:

Remark 2.2.11. (see [72]) Let $\{\eta(t)\}$ be a Lévy process. Then $\eta(t)$ has the decomposition

$$\eta(t) = \alpha t + \sigma W(t) + \int_{|z|<R} z\tilde{N}(t, dz) + \int_{|z|\geq R} z\tilde{N}(t, dz), \quad (2.1)$$

for some constants $\alpha, \sigma \in \mathbb{R}$ and $R \in [0, \infty]$. Here

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$$

is the compensated Poisson random measure of $\eta(t)$, and $W(t)$ is a Brownian motion independent of $\tilde{N}(dt, dz)$. For each $A \in \mathcal{B}_0$ the process

$$M(t) := \tilde{N}(t, A)$$

is a martingale. If α and $R = \infty$, we call $\eta(t)$ a Lévy martingale. We can always choose $R = 1$.

Remark 2.2.12. (see [72]) If $\mathbb{E}|\eta(t)| < \infty$ for all $t \geq 0$, then

$$\int_{|z|\geq 1} |z|\nu(dz) < \infty$$

and hence we may choose $R = \infty$ and write

$$\eta(t) = \alpha_1 t + \sigma W(t) + \int_{\mathbb{R}} z\tilde{N}(t, dz),$$

where $\alpha_1 = \alpha + \int_{|z|\geq 1} |z|\nu(dz)$.

Remark 2.2.13. (see [72]) A Lévy process is a strong Markov process.

Remark 2.2.14. (see [72]) A Lévy process is a semimartingale.

Definition 2.2.15. Let D_{ucp} denote the space of càdlàg adapted processes, equipped with the topology of uniform convergence on compacts in probability (ucp):$H_n \to H$ ucp if for all $t > 0$ $\sup_{0\leq s\leq t} |H_n - H(s)| \to 0$ in probability ($A_n - A$ in probability if for all $\epsilon > 0$ there exist $n_\epsilon \in \mathbb{N}$ such that $n \geq n_\epsilon \implies P(|A_n \to A| > \epsilon) < \epsilon$).
Let L_{ucp} denote the space of adapted càglàd processes (left continuous with right limits), equipped with the ucp topology. If $H(t)$ is a step function of the form

$$H(t) = H_0\chi_{\{0\}}(t) + \sum_i H_i\chi_{(T_i, T_{i+1}]}(t),$$

where $H_i \in \mathcal{F}_{T_i}$ and $0 = T_0 \leq T_1 \leq \cdots \leq T_{n+1} < \infty$ are $\mathcal{F}_t$-stopping times and X is càdlàg, we define

$$J_X H(t) := \int_0^t H_s dX(s) := H_0 X(0) + \sum_i H_i(X_{T_{i+1}\wedge t} - X_{T_i\wedge t}), \qquad t \geq 0.$$

Theorem 2.2.1. Let X be a semimartingale. Then the mapping J_X can be extended to a continuous linear map

$$J_X : L_{ucp} \to D_{ucp}.$$

This construction allows us to define stochastic integrals of the form

$$\int_0^t H(s)d\eta(s)$$

for all $H \in L_{ucp}$. In view of the decomposition (2.1) this integral can be split into integrals with respect to $ds, dW(s), \tilde{N}(ds, dz)$ and $N(ds, dz)$. This makes it natural to consider the more general stochastic integrals of the form

$$X(t) = X(0) + \int_0^t \alpha(s,\omega)ds + \int_0^t \beta(s,\omega)dW(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z,\omega)\tilde{N}(ds,dz), \quad (2.2)$$

where the integrands are $\mathcal{F}_t$-predictable and satisfy the growth condition

$$\int_0^t \left\{ |\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}} \gamma^2(s,z)\nu(dz) \right\} ds$$

almost surely for all $t > 0$.

For simplicity we have put

$$\tilde{N}(dt,dz) = \begin{cases} N(ds,dz) - \nu(dz)ds, & \text{if} \quad |z| < R \\ N(ds,dz), & \text{if} \quad |z| \geq R, \end{cases}$$

with R as in Remark 2.2.11.

The following shorthand differential notation for the process X(t) satisfying (2.2) will be used:

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(ds,dz). \quad (2.3)$$

Such processes are known as Itô-Lévy processes.

Recall that a semi-martingale $M(t)$ is called a local martingale up to time T (with respect to $\mathbb{P}$) if there exists an increasing sequence of $\mathcal{F}(t)$-stopping times τ_n such that $\lim_{n\to\infty} \tau_n = T$ almost surely and $M(t \wedge \tau_n)$ is a martingale with respect to $\mathbb{P}$ for all n.

Note that if

1. $$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)dt\Big] < \infty,$$

   then the process

   $$M(t) := \int_0^t \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(ds,dz), \quad 0 \leq t \leq T$$

   is a martingale.

2. If

   $$\int_0^T \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)dt < \infty$$

almost surely then $M(t)$ is a local martingale, $0 \leq t \leq T$.

Chapter 3

A Benchmark Capital Management Problem under the Basel III Framework

Capital adequacy management involves the decision of how a bank manager decides on the amount of capital that the bank should maintain and acquisition of the needed capital. From a shareholder's perspective, retention of capital means less money invested and consequently lower returns. From the regulator's perspective, banks should increase their capital reserves to ensure the safety and soundness in the case where earnings may become negative. Bank regulators are also concerned about the financial risk that could increase the probability of bank failure. In the event where the variability of earnings after taxes increases, the interest, and non-interest expenses may exceed bank earnings and should be absorbed by bank capital. Although requiring a bank to maintain a higher capital level could lower the financial risk, such requirements could disrupt the efficiency and competitiveness of the banking system. Therefore, the aforementioned capital requirement acts as a constraint on the lending activities of a bank. It may also constrain the rate at which bank assets may be expanded. Thus, when determining the amount of capital to hold, the bank owner must decide on how much of the increased benefit that results from the higher capital they are willing to trade off against the lower return on equity that originates from the cost associated with higher capital.

The high cost of capital provides an incentive for bank owners to retain less capital relative to assets than is required by regulatory authorities. In this situation, the amount of bank capital to be held is prescribed by certain capital requirements. The aforementioned requirement was drafted into a document, the 1988 Basel Accord, by the Basel Committee on Banking Supervision (BCBS). The objective of this document was aimed at how banks should manage and regulate their operations. This accord was an attempt to develop regulatory requirements on the banking industry with four objectives in mind:

- to protect depositors and deposit insurance from the ravages of reckless portfolio management by banks;
- to prevent system instabilities arising from bank failures;
- to strengthen the soundness and stability of the international banking system;
- to remove any sources of undesirable competitive behavior among internationally active banks.

The benchmark approach to portfolio management is useful because it monitors investment performances which can bring changes to an investment strategy. In particular, it

- provides a frame of reference;
- provides critical information for assessing a bank's investment strategy, risk and returns;
- is useful in monitoring the progress of a bank's investments and investment performance;

- is helpful towards a better understanding of the right balance of return and risk in a bank's bond investments.

The 1988 Basel Accord consolidated capital requirements as the cornerstone of bank regulation. This Accord required banks to hold a minimum capital-to-risk-weighted assets ratio of at least 8% (see for instance [16]). This ratio is used to protect depositors and deposit insurance schemes from losses due to inadequate or reckless portfolio management and promotes the stability and efficiency of the banking structure. However, the 1988 Basel Accord received widespread criticism for being too crude and oversimplified with the ever-changing standards set for the management and assessment of banking performances. The 1988 Basel Accord [5], also known as the Basel I Accord, was further criticized for treating all credit risk-types alike, which potentially could lead to regulatory arbitrage and it also seemed to neglect contemporary credit risk management techniques. Moreover, the 1988 Basel Accord also failed to take into account the dynamic distortions of capital regulation and complementary regulatory instruments such as supervisory monitoring or prompt corrective regulatory action (see, [55]). Reacting to these criticisms, the BCBS made several adjustments to the 1988 Basel Accord document. This eventually led to a new capital adequacy framework, known as the Basel II Capital Accord (see [6]), which has been implemented by all the major international banks globally from the end of the year 2007. The latter framework was supposed to be more risk sensitive and encouraged banks to use internal models to determine capital levels.

However, the Global Financial Crisis in mid-2007 revealed some loopholes in the Basel II framework which required improved regulation and supervision of financial entities. In particular, financial institutions ignored legislative capital requirements and took on more risk. Unfortunately, the Basel II framework was in the early stage of implementation and was not suitable to deal with some of the characteristics of the crisis. Due to this, the Committee has set out to design a countercyclical framework with higher capital standards that better deals with liquidity risk and adequacy risk. In December 2010 (see [9]) the BCBS released the new global regulatory standards for bank capital adequacy and liquidity. These standards are commonly known as Basel III standards and were endorsed by G20 leaders at their November 2010 summit. The first consultative document of the new framework was released in December 2009, whereby a quantitative impact study was conducted. The Basel III framework also depends on the three pillars of Basel II (that is, capital requirements, market discipline, and supervisory review). However, the most notable changes have occurred in Pillar I (that is capital requirements). The Committee provided more strict rules on both the quantity and quality of eligible capital in the minimum capital requirement ratio. Another contribution to the Basel III framework is the inclusion of two minimum liquidity standards that will be enforced in Pillar I and therefore are regarded as statutory minimum requirements. Under the previous frameworks, the Committee only provided general recommendations for liquidity risk management for financial institutions. The main objective of the Basel III Accord is to further strengthen the resilience of the banking system and to create a competitive level playing field worldwide. Under the Basel III framework, the capital requirements are stricter and the Committee has also supplemented the framework with specific liquidity standards.

A cornerstone of the minimum capital requirement related to this accord is the capital adequacy ratio (CAR). The capital adequacy ratio is a measure of the amount of a bank's reserve capital relative to a combination of its credit exposures. This ratio is normally expressed as a percentage. For example, a capital adequacy ratio of 8% means that a bank's capital is 8% of the size of its credit exposures. In the case where the CAR drops below a certain minimum level due to the exposure to risks (such as credit or market risk), the regulatory body may take certain actions on the bank which could lead, in the worst-case scenario, to the ultimate closure of the bank. This could potentially affect the socio-economic development or financial status of a country. The aim of having minimum capital adequacy ratios is to guarantee that banks are prepared to absorb a reasonable level of losses. Applying minimum capital adequacy ratios helps to promote the stability and effectiveness of the banking system by reducing the likelihood of banks becoming insolvent. When a financial institution, in this case, a bank, becomes insolvent, then this may lead to a loss of confidence in the financial system, causing financial problems for other banks and it might even threaten to distort the smooth functioning of financial markets. Compliance with capital adequacy ratios requires some adjustments to be made to the amount of capital shown on the balance sheet.

Several discussions related to optimal control problems in discrete- and continuous-time settings have recently surfaced in the literature ([73], [72]). Other papers that use dynamic optimization methods to analyze bank regulatory capital policies include [67] for Basel II and [4], [28] for Basel market risk capital requirements. In the paper [73], a discrete-time dynamic banking model of imperfect competition is presented, where the bank can invest in a prudent or a gambling asset. For both these options, a maximization problem that involves the bank value for shareholders is formulated. The study of the dynamics of these risk minimization strategies has always been an important issue in the management of banks. In particular, ([34], [36], [84]) construct continuous-time models which permit optimization problems to be solved in the context of portfolio selection and capital requirements. The paper [15] addresses some of the problems arising from the management of credit risk in the equity and fixed income secondary markets which led to portfolio losses by financial investors. The aforementioned authors consider alternative portfolio optimization approaches in the fixed-income market over the period $2008-2009$. Furthermore, [15] studies dynamic portfolio strategies based on multistage stochastic programming and makes a comparison with policy-rule based methods, in particular, performance against a corporate bond index. In the paper [35] the authors study a benchmarked asset management problem which is relevant to the financial industry: most investment funds have a benchmark, such as a financial index or a customized portfolio, against which their performance is assessed.

Continuous-time portfolio selection problems have been studied by for example [88] and [62]. In the paper of [88], they formulate the mean-variance problem as a stochastic linear-quadratic problem for the very first time. Mean-variance portfolio selection deals with the allocation of wealth amongst a basket of securities with the aim of maximizing the return on an asset portfolio while minimizing the associated risk at the same time. The solution to this problem is obtained using the embedding technique introduced by [62]. The embedding technique was used because dynamic programming could not directly be used to deal with the objective “cost” function. However, in [62], the mean-variance problem was transformed to one where the dynamic programming equation was used to obtain explicit optimal solutions. The authors of [88] used the embedding technique and linear-quadratic (LQ) optimal control theory to solve the continuous-time, mean-variance problem with assets having deterministic diffusion coefficients. Exploiting the stochastic LQ control theory, Zhou, and his colleagues have considerably extended the initial continuous-time, mean-variance results obtained by [88].

The current chapter has close connections with [14], [19], [23], [65], [84] and [66]. The authors of [14] establishes new conditions under which a constrained (no short-selling) time-consistent equilibrium strategy, starting at a certain time, will beat the unconstrained counterpart, as measured by the magnitude of their corresponding equilibrium meanvariance value functions. They further show that the pure strategy of solely investing in a risk-free bond can sometimes simultaneously dominate both constrained and unconstrained equilibrium strategies. The paper [19] sets up an optimal capital management problem which maximizes the expectation of bank capital under a risk constraint on the Capital-at-Risk (CaR), where CaR is defined in terms of Value-at- Risk (VaR). Furthermore, the authors seek an optimal bank capital management strategy in a mean-CaR paradigm. The paper [23] uses existing models to represent bank regulatory capital and bank assets in order to perform simulations of a bank’s Capital Adequacy Ratio and Leverage Ratio in a Brownian motion framework. The paper [65] demonstrates how the CAR can be optimized in terms of bank equity allocation and the rate at which additional debt and equity are raised. The paper [84] derive an optimal equity allocation strategy for the bank and monitor the performance of the Basel II CAR under the allocation strategy. Recently, in the paper [66], the authors’ main objective is to maximize the expected utility of a Basel III compliant bank asset portfolio at a future date. This goal is achieved through the Legendre transform-dual method.

The main novel feature of our study, which is distinct from the aforementioned literature, is to apply classical methods from stochastic control to the investment portfolio problem of a bank in a jump-diffusion setting. We construct a continuous-time portfolio selection problem where it is solved under benchmarking criteria. The objective is to find an optimal investment strategy which minimizes the expected square distance between the final value of a bank capital portfolio and a predefined benchmark for bank capital. This type of analysis has not been reported in the existing banking literature as far as we know. The models considered capture unexpected events such as wars, natural disasters, political uprising. Such processes have an advantage over the more traditional tools of Brownian motion. Brownian motion is continuous and is linked to the normal distribution. However, in the literature, data from equity, fixed income or foreign exchange markets clearly reveal sudden and rare breaks. The empirical distribution of asset returns exhibits fat tails and skewness, behavior that deviates from normality.

The remainder of this chapter is organized as follows. In Section 3.1, we provide a description of the financial market setting which is followed by the modelling of banking items in Section 3.2. In Section 3.3, we construct a dynamic model for bank regulatory capital model under the Basel III framework. In Section 3.4, we solve a benchmark problem for bank capital via the stochastic dynamic programming approach. In Section 3.5 we derive the dynamics of bank loans through balance sheet constraints. In Section 3.6 we illustrate the dynamic behavior of the optimal portfolio strategy as well as the behavior of the CAR. We provide a discussion of the main results in the concluding Section 6.1.

3.1 A Probability Space

We assume $(\Omega, \mathcal{F}, \mathbb{P})$ to be a complete probability space with a filtration $\mathcal{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$ which is right continuous, for some finite T which denotes the investment time horizon. The space Ω represents the different states of the economy in which banks operate and the associated uncertainty. The object $\mathcal{F}_t$ represents the information available to banks up until time t. All random variables and stochastic processes considered in this paper are defined on this space. We introduce some of the important items in this regard.

We assume a 3−dimensional Brownian motion $\{Z(t)\}_{t\geq 0}$, where

$$Z(t) \equiv (Z_0(t,\omega), Z_a(t,\omega), Z_b(t,\omega))', \ \omega \in \Omega,$$

with pairwise independent coordinates, and a 1−dimensional Poisson process $\{N(t)\}_{t\geq 0}$ with intensity λ which is constant in time. For some constant $-1 \leq \rho \leq 1$, we define $Z_1(t)$ as

$$Z_1(t) = \rho Z_0(t) + \sqrt{1-\rho^2} Z_a(t).$$

The processes $\{Z_1(t)\}_{t\geq 0}$ and $\{Z(t)\}_{t\geq 0}$ are correlated, and we note that $dZ_1(t)dZ_0(t) = \rho dt$.

We also assume two sequences of random variables, $(y_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$. Each of these are sequences of independent and identically distributed random variables, and (b_j) is independent of (y_i)

Next we introduce two standard Poisson processes (whose jump sizes are of length 1 and paths are constant between two jumps) with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$ both of which are constant in time. Now we define compound Poisson processes $\{K_1(t)\}_{t\geq 0}$ and $\{K_2(t)\}_{t\geq 0}$ by the formulae:

$$K_1(t) = \sum_{j=1}^{N_1(t)} y_j \quad \text{and} \quad K_2(t) = \sum_{j=1}^{N_2(t)} b_j.$$

The y_i and the b_i can be seen to be the jump sizes of respectively $\{K_1(t)\}_{t\geq 0}$ and $\{K_2(t)\}_{t\geq 0}$. We let $\beta_1 = \mathbb{E}[y_1]$ and $\beta_2 = \mathbb{E}[b_1]$ denote the average jump sizes.

We define the martingales $\{\widetilde{M}_1(t)\}_{t\geq 0}$ and $\{\widetilde{M}_2(t)\}_{t\geq 0}$ as, respectively:

$$\widetilde{M}_1(t) = K_1(t) - \lambda_1 t\beta_1 \text{ and } \widetilde{M}_2(t) = K_2(t) - \lambda_2 t\beta_2 \ .$$

The latter processes are called compensated compound Poisson processes. The jumps arrive randomly according to a Poisson process and the size of the jumps is also random, with a specified probability distribution.

3.2 The Stochastic Banking Model

. Central in the management of a bank, is its balance sheet, which records the bank assets and bank liabilities.
For the purpose of this paper, the bank's balance sheet at time t can be represented as

$$R(t) + L(t) + S(t) = D(t) + C(t). \tag{3.1}$$

Here, the symbols $R(t)$, $L(t)$, $S(t)$, $D(t)$ and $C(t)$ denote the values of bank reserves, loans, marketable securities, deposits and bank capital, respectively. The aformentioned bank items are regarded as stochastic processes.

3.2.1 Bank Reserves

Bank reserves are the deposits held in accounts with the central bank of a country (for instance, the South African Reserve Bank in the case of South Africa) plus money that is physically held by banks (vault cash). Such reserves constitute money that is not lent out but is earmarked to cater for withdrawals by depositors. Since it is uncommon for depositors to withdraw all of their funds simultaneously, only a portion of total deposits may be needed as reserves. Based on the above description, we may represent bank reserves as follows

$$R(t) = \gamma D(t) \tag{3.2}$$

where $0 \leq \gamma \leq 1$ is the fraction of deposits.

The bank uses the remaining deposits to earn a profit, either by issuing loans or by investing in assets such as bonds and equity.

3.2.2 Securities

Marketable securities are debt instruments that can be easily converted to cash such as government bonds, corporate bonds, common stock or certificates of deposit. These instruments are very liquid as they tend to have maturities of less than one year.

The dynamics of the marketable security price (see [23]) is assumed to be given by

$$\frac{dS(t)}{S(t)} = (r_S(t) + \lambda_S)dt + \sigma_S dZ_0(t), \tag{3.3}$$

where σ_S is the security volatility and λ_S denotes the risk premium. Under the Capital Asset Pricing Model (CAPM), λ_S could be quantified by the relation $\lambda_S = \beta[E[R_m] - r^A]$ with $E(R_m)$ representing the market expected return and β the sensitivity of the expected excess asset returns to the expected excess market return. The dynamics of the interest rate $r_S(t)$ is assumed to be given by

$$dr_S(t) = \theta_{r_S}(\mu_{r_S} - r_S(t)))dt + \sigma_{r_S} dZ_b(t), \tag{3.4}$$

where θ_{r_S} is the rate of reversion, μ_{r_S} is the long-run mean and σ_{r_S} is the volatility which are all positive constants. The model (3.4) is known as the Vasicek model. In the case where $r_S(t) < \mu_{r_S}$, the drift term $\theta_{r_S}(\mu_{r_S} - r_S(t)))$ becomes positive for $\theta_{r_S} > 0$, generating a tendency for the interest rate to move upwards (toward the long-run mean). The main disadvantage is that, under Vasicek's model, it is theoretically possible for the interest rate to become negative, which is an undesirable feature under pre-crisis assumptions. Several central banks from European countries such as Sweden and Switzerland as well as Japan have moved to sub-zero rates, with the aim of boosting their economies, growth, and inflation.

3.2.3 Deposits

The majority of a bank's liabilities consists of retail deposits, D, which are fully insured by a deposit insurance fund (DIF). We apply a simple banking principle (see, for instance, Chapter 9 in [64]) that relates the deposits, D, to the reserves, R, by the formula (3.2). This relationship may be understood as follows: In general, funds that are deposited at a bank, are mostly lent out to customers or financial institutions. Commercial banks keep a fraction (known as a reserve-deposit ratio) of those funds as reserves to cover its customer deposit liabilities. Central banks or other banking regulators often mandate the aforementioned reserve requirements to both limits the amount of money creation that occurs in the commercial banking system and ensures that banks have enough ready cash to meet normal demand for withdrawals. The deposit withdrawal sizes are modelled by the jump term.

The dynamics of deposits is modelled as a jump-diffusion process

$$dD(t) \;=\; \mu_D dt + \sigma_D Z_b(t) - dK_1(t), \quad D(0) = D_0, \tag{3.5}$$

with the deposit inflow rate denoted by $\mu_D = r^{DD} + r^{TD}$, $\mu_D(0) = \mu_{D(0)}$ and $\sigma_D > 0$. The rate of demand deposit which is payable on demand is denoted by $r^{DD} : T \to \mathbb{R}_+$ and $r^{TD} : T \to \mathbb{R}_+$ denotes the rate of time deposit which is payable only after a fixed interval of time. The jump term represents the unexpected withdrawals from customers of the particular commercial bank. That is it represents the number of deposits withdrawals occurring in time interval $[0, T]$.

From (3.2) we may conclude that

$$L(t) + S(t) = (1 - \gamma)D(t) + C(t). \tag{3.6}$$

3.3 Total Bank Regulatory Capital

The Basel Committee on Banking Supervision discovered that the severity of the sub-prime mortgage crisis was indirectly due to the insufficiency of the levels of capital that banks were holding. In an attempt to remedy the scenario, the Committee increased the resilience of individual banks by launching new capital requirements that address both the quality and quantity of the eligible capital. In particular, the introduction of capital buffers above the minimum level to capture pro-cyclicality and systemic risk in the financial industry is new extensions to capital requirements. The procyclical effects of bank capital requirements are well-known. In recessions, losses erode banks' capital, while risk-based capital requirements, such as those in Basel II, become higher. If banks cannot quickly raise sufficient new capital, their lending capacity falls and a credit crunch may follow. Fixing the potential contractionary effect on credit supply by relaxing capital requirements in bad times may increase bank failure probabilities precisely when, because of high loan defaults, they are largest. On the procyclicality aspect, Basel III has promoted the build-up of buffers (Countercyclical capital buffer) in good times that can be drawn down in periods of stress. The purpose of the instrument is to protect the banking system against potential losses as these are related to an increase in risks in the system as a result of excessive growth in lending. This directly increases the resilience of the banking system and prevents excessive growth in lending.

3.3.1 Tier 1 Capital

The global banking system entered the crisis with an insufficient level of high quality capital. However, due to the impact of the aforementioned financial crisis, both the quality and quantity of capital requirements needed modifications and adjustments. The Committee specified that Tier 1 capital should mainly consist of common stocks and retained earnings because of their ability to absorb losses (see [9]). In the comprehensive version of Basel II released in June 2006 there is only distinguished between Tier 1 and Tier 2 capital. Nevertheless, financial institutions have been allowed to hold a sub- category of Tier 1 capital called Additional Tier 1 capital. In the Basel II framework Additional Tier 1 mainly consisted of hybrid instruments. The amount of hybrid capital was, however, limited to 50% of total Tier 1 capital which should make up 4% of risk-weighted assets.

This resulted in that banks were able to hold 2% high-quality Tier 1 capital and 2% lower quality hybrid capital to comply with the minimum Tier 1 requirement of 4%. The Basel III framework addresses this issue by categorizing Tier 1 capital into two levels; Common Equity Tier 1 (CET1) and Additional Tier 1. Capital-eligible for inclusion in CET1 should be of the highest quality and for joint stock companies, this capital solely consists of common shares and retained earnings. To be considered properly loss absorbing, common shares should be the most subordinated claim in liquidation and the principal should be perpetual and should never be repaid outside of liquidation. Financial institutions that are not joint stock companies are not eligible to issue common shares and their CET1 can include other forms of capital that meet the criteria. Savings banks can for example issue guarantee capital that is eligible for inclusion in CET1. To strengthen the quality and quantity of capital, the Basel Committee now requires that CET1 should minimum be 4.5% of the risk-weighted asset (see [9]). Thus, the new rules seek to increase the ratio of high-quality capital by 2.5 percentage points.

To be eligible for inclusion in Additional Tier 1 the capital has to comply with 14 requirements. Notably, this form of capital should be subordinated to depositors and general creditors and the bank must be able to convert the capital to common shares or write it down. Moreover, the instruments are required to be perpetual and the bank should be able to cancel dividends or coupons at any time without defaulting (see [9]). The goal of these requirements is to allow banks to manage capital in such a way to absorb losses and avoid insolvency. It should be noted that the complete entry criteria for Additional Tier 1 capital are not yet finalized (see [9]). The Committee does not require a minimum amount of Additional Tier 1 capital. However, total Tier 1 capital equal to the sum of CET1 and Additional Tier 1 capital should be at least 6% of risk-weighted assets. Hence, financial institutions have the discretion to hold 4.5% CET1 and 1.5% Additional Tier 1 capital.

3.3.2 Tier 2 Capital

Tier 2 capital is meant to absorb losses in the case of liquidation. The requirements for eligibility are less strict and this type of capital can have a maturity, although no less than five years. Furthermore, the instruments have to be subordinated to depositors and general creditors of the bank. Moreover, there should be no incentive to redeem Tier 2 capital. The subordinate debt will be the most significant part of Tier 2 capital for the commercial bank under consideration. Subordinate debt is a type of debt whose holders have a claim on the company's assets only after the senior debtholders' claims have been satisfied. For instance, if a company has both subordinated debt and senior debt and has to file for bankruptcy or faces the prospect of liquidation, the senior debt is paid back first before the subordinated debt. Once the senior debt is completely paid back, the company then repays the subordinated debt. Under Basel-III norms, banks must maintain a minimum capital adequacy ratio of 11.5 % by March 2019. Of the total, *Tier 1* capital must be 7 %, with a provision for 2 % in *Tier 2* capital. Banks must also maintain a capital conservation buffer of 2.5 % of risk weighted assets.

In this paper, bank capital under the Basel III paradigm (see [9]) is partitioned into *Tier 1* and *Tier 2* capital,

$$C(t) \;=\; C_{T1}(t) + C_{T2}(t), \tag{3.7}$$

where Tier 1 capital consist of common equity capital, $E(t)$, and additional capital, $C_a(t)$. For *Tier 2* capital, subordinate debt is considered as the most significant part (see for instance, [19], [65]).

$$C_{T1}(t) = E(t) + C_a(t) \tag{3.8}$$

and

$$C_{T2}(t) = S_D(t). \tag{3.9}$$

Therefore we may express the total value of bank capital as

$$C(t) = E(t) + C_a(t) + S_D(t). \tag{3.10}$$

3.3.3 Dynamics of Total Bank Capital

The market value of subordinate debt at time t may be considered to have the form

$$dS_D(t) = rS_D(t)dt, S_D(0) = S_0 > 0,$$

where S_0 is the initial market valueof subordinate debt and r is a positive constant representing the risk-free interest rate. Also, we let the return on bank equity be given by

$$\begin{aligned} dE(t) &= E(t^-)\Big[r_e dt + \sigma_e dZ_0(t) + dK_1(t)\Big], \quad t \in [0,T], \\ E(0) &= e_1 > 0. \end{aligned} \tag{3.11}$$

The constants r_e and σ_e denote the return on equity capital and volatility of equity, respectively. The diffusion term in (3.11) represents shares 'normal' fluctuations, such as temporary imbalance between supply and demand, changes of the economic outlook and so on.The jump term, $K_1(t)$, is the abnormal 'vibration' of shares. These jump amplitudes represent the unanticipated arrival of new information from the equity market that will have an impact on the capital flow from shareholders of the particular bank. Generally, this information is about specific companies and industries and have little effect on the entire market. In our case, *Additional capital*, $C_a(t)$, is described by the following jump-diffusion model

$$\begin{aligned} dC_a(t) &= \alpha dt + \beta_{C_a} dZ_1(t) - dK_2(t), \ t \in [0,T], \\ &= \alpha dt + \beta_{C_a}\rho dZ_0(t) + \beta_{C_a}\sqrt{1-\rho^2} dZ_a(t) - dK_2(t), \\ C_a(0) &= c_a. \end{aligned} \tag{3.12}$$

Here, $\alpha \in \mathbb{R}$ is the rate of additional capital raised in case equity capital was not sufficient, $\beta_{C_a} > 0$ represents the perturbation of the additional capital and $K_2(t)$ represents the rate of deductions from additional capital as required under the Basel III paradigm.

We suppose that the bank starts from an initial capital c_a at time $t = 0$ to invest dynamically in the equity market over a time horizon of length T where $\pi(t)$ denotes the amount invested in equities at time t. The remainder $C(t)-\pi(t)$, is invested in subordinate debt. Taking our lead from the aforementioned arguments, we may express the dynamics of a bank's capital portfolio, $C(t)$, as follows (see for instance [19])

$$
\begin{aligned}
dC(t) &= \pi(t)\frac{dE(t)}{E(t)} + dC_a(t) + \Big(C(t)-\pi(t)\Big)\frac{dS_D(t)}{S_D(t)} \\
&= \pi(t)r_e dt + \pi(t)\sigma_e dZ_0(t) + \pi(t)dK_1(t) \\
&+ \alpha dt + \beta_{C_a}\rho dZ_0(t) + \beta_{C_a}\sqrt{1-\rho^2}dZ_a(t) - dK_2(t) + C^{\pi}(t)rdt - \pi(t)rdt \\
&= \Big(C^{\pi}(t)r + (r_e-r)\pi(t) + \alpha\Big)dt + \Big(\pi(t)\sigma_e + \beta_{C_a}\rho\Big)dZ_0(t) + \beta_{C_a}\sqrt{1-\rho^2}dZ_a(t) \\
&+ \pi(t)dK_1(t) - dK_2(t). \qquad (3.13)
\end{aligned}
$$

We observe that the construction of the model (3.13) is very much similar to many models in pension funds and insurance. The model (3.13) focuses on allocating the capital to a set of securities such that the profit or the risks can be optimized. Due to the uncertainty of the real-world life, the return parameters always take uncertain information in the realistic environments because of the scarcity of the a priori knowledge or uncertain disturbances.

3.4 Benchmark Problem for Bank Capital

3.4.1 Problem Formulation

Since bank capital is a prominent component of the capital adequacy ratio, it would be imperative to formulate a threshold problem for bank capital. A commercial bank may want to find an admissible strategy such that the expected value of bank capital at the final time is close to a predefined threshold $d \geq 0$. Let $\mathcal{A}$ denote the class of admissible control laws given by

$$
\mathcal{A} = \Big\{\pi(\cdot) \mid \text{ the function } \pi : \mathbb{R}_+ \to \mathbb{R} \text{ is bounded and adapted, and renders } C(t) > 0 \text{ almost surely}\Big\} \qquad (3.14)
$$

Towards formulation of the problem we introduce a number c to denote a value assumed by C. We shall sometimes write C as C^{π} when we wish to specify the control π which is in action. Furthermore, we introduce the following functions Y and V associated with (3.13) as

$$Y(t,T,c,d;\pi) = \mathbb{E}\left[\left(C^\pi(T)-d\right)^2 \middle| C^\pi(t)=c\right], \tag{3.15}$$

and

$$V(t,c) = \min_{\pi\in\mathcal{A}} Y(t,T,c,d;\pi). \tag{3.16}$$

The problem is to obtain a solution for the Hamilton-Jacobi-Bellman (HJB) equation for the value function (3.16) (see [72]). Thus we formulate a stochastic optimization problem for a particular bank that will attempt to lower risk and maximize the expected value of bank capital by diversifying its portfolio.

Problem 3.4.1. (Optimal allocation problem): Consider the SDE of the form (3.13). We attempt to find the value function

$$V(t,c) = \min_{\pi\in\mathcal{A}} Y(t,T,c,d;\pi)$$

and the optimal control law, which is given by

$$\pi^*(t) = \arg\inf_{\pi\in\mathcal{A}} Y(t,T,c,d;\pi) \in \mathcal{A}, \quad \text{so that} \quad V(t,c) = Y(t,T,c,d;\pi^*). \tag{3.17}$$

Here, the boundary condition is given by $V(T,c)=(c-d)^2$. Note that the optimal bank capital allocation at time t can then be written $C^{\pi^*}(t)$.

We introduce the infinitesimal generator associated with the process displayed in (3.13). For simpler notation, we write $C(t)$ as x.

$$\begin{aligned}\mathcal{L}M(t,x) &= M_t(t,x) + M_x(t,x)[xr + (r_e - r)\pi(t) + \alpha] \\ &+ \frac{1}{2}M_{xx}(t,x)\left(\pi^2(t)\sigma_e^2 + 2\pi(t)\sigma_e\beta_{C_a}\rho + \beta_{C_a}^2\right) \\ &+ \lambda_1\mathbb{E}\Big[M(t,x+\pi(t)y_1) - M(t,x)\Big] + \lambda_2\mathbb{E}\Big[M(t,x-b_1) - M(t,x)\Big]\end{aligned} \tag{3.18}$$

and we define a set

$$\begin{aligned}\Gamma^{1,2} &= \Big\{\Phi(t,y) \mid \ \Phi \text{ is continuously differentiable in } t \text{ and} \\ &\quad \text{twice continuously differentiable in y and } \Phi(t,y) \\ &\quad \text{satisfies the polynomial growth condition } |\Phi(t,y)| \le \varpi(1+|y|)^{\varepsilon} \\ &\quad \text{for some constants } \varpi \text{ and } \varepsilon\Big\}.\end{aligned}$$

3.4.2 Solution to Problem 3.4.1

We quote the following theorem which is key to solving Problem 3.4.1. Recall that we write x for $C(t)$.

Theorem 3.4.2. [87] *Suppose that $M(t,x) \in \Gamma^{1,2}$ and it satisfies the following equation :*

$$\min_{\pi\in\mathcal{A}} \Big\{\mathcal{L}M(t,C(t))\Big\} = 0, \tag{3.19}$$

$$M(T,C(t)) = (c-d)^2 \tag{3.20}$$

for all $(t,c) \in [0,T]\times\mathbb{R}$, then $M(t,C(t)) \le V(t,C(t))$.

Suppose furthermore there exists a $\pi_1 \in \mathcal{A}$ such that

$$\pi_1(s) = \arg\min_{\pi\in\mathcal{A}} \Big\{\mathcal{L}M(s,C(s))\Big\} \tag{3.21}$$

for $\mathbb{P}-$almost all $(s,\omega)\in[0,T]\times\Omega$. Then $V(t,x) = M(t,x)$ and $\pi_1 = \pi^*$ is optimal.

We can find an expression for the optimal equity allocation strategy π^* as well as for the value function through the following theorem.

Theorem 3.4.3. *Consider the optimization problem for bank regulatory capital formulated in Problem (3.4.1). There exist a function $V(t,x) \in \Gamma^{1,2}$ which is a solution of the partial differential equation (PDE) having the form*

$$\begin{aligned}0 &= \min_{\pi(\cdot)\in\mathcal{A}} \Bigg[V_t(t,x) + V_x(t,x)\Big(xr + (r_e - r)\pi(t) + \alpha\Big) \\ &+ \frac{1}{2}V_{xx}(t,x)\Big(\pi^2(t)\sigma_e^2 + 2\pi(t)\sigma_e\beta_{C_a}\rho + \beta_{C_a}^2\Big) \\ &+ \lambda_1\mathbb{E}\Big[V(t,x+\pi(t)y_1) - V(t,x)\Big] + \lambda_2\mathbb{E}\Big[V(t,x-b_1) - V(t,x)\Big]\Bigg],\end{aligned} \tag{3.22}$$

with boundary condition

$$V(T, C(T)) = (c-d)^2.$$

In this case, a solution to the optimal equity allocation problem stated in Problem (3.4.1) has the form

$$\pi^*(t) = \frac{\sigma_e \beta_{C_a} \rho - \left(r_e - r + \lambda_1 \mathbb{E} y_1\right)\left(C^{\pi^*}(t) + \frac{B}{r} - \left(d + \frac{B}{r}\right) e^{-r(T-t)}\right)}{\sigma_e^2 + \lambda_1 \mathbb{E} y_1^2}, \quad (3.23)$$

and the value function

$$V(t,x) = e^{(2r-A)(T-t)}\left[x + \frac{B}{r} - \left(d + \frac{B}{r}\right)e^{-r(T-t)}\right]^2 + \frac{G}{2r-A}\left[e^{(2r-A)(T-t)} - 1\right] \quad (3.24)$$

where $C^{\pi^*}(t)$ is the optimal value of the bank capital portfolio at time $t \in [0,T]$ corresponding to the optimal investment strategy π^*. The constants A, B, G are defined as

$$A = \frac{(r_e - r + \lambda_1 \mathbb{E} y_1)^2}{\sigma_e^2 + \lambda_1 \mathbb{E} y_1^2},$$

$$B = \frac{(r_e - r + \lambda_1 \mathbb{E} y_1)\sigma_e \beta_{C_a} \rho}{\sigma_e^2 + \lambda_1 \mathbb{E} y_1^2} + \alpha - \lambda_2 \mathbb{E} b_1,$$

and

$$G = \beta_{C_a}^2 + \lambda_2 \mathbb{E} b_1^2 - \frac{\sigma_e^2 \beta_{C_a}^2 \rho^2}{\sigma_e^2 + \lambda_1 \mathbb{E} y_1^2}.$$

The role of Theorem (3.4.3) was to obtain explicit formulae for bank capital allocation. That is, with the influx of capital from shareholders we show how a commercial bank allocates the funds in the equity market through diversification.

In this thesis, we study the Basel III CAR denoted by Λ, and defined as

$$\Lambda(t) = \frac{C(t)}{\bar{\Gamma}(t)}$$

where $C(t)$ is defined as the bank regulatory capital and $\bar{\Gamma}(t)$ represents the total risk-weighted assets. Here, the total risk-weighted assets are assumed to be constituted by only loans since loans are the largest asset on a bank's balance sheet (see [65]). Thus $\bar{\Gamma}(t)$ will be assumed to be equal to $L(t)$. The discussion on capital requirements under the Basel III paradigm in the introduction, motivate us to simulate a continuous time model for the capital adequacy ratio which we illustrate in Section 3.6.

3.5 Dynamics for Bank Loans

From (5.4), we can deduce that the dynamics of loans will be

$$dL(t) + dS(t) = (1-\gamma)dD(t) + dC(t). \tag{3.25}$$

In addition, if we set $\gamma \neq 1$ in (3.25), then

$$dL(t) = (1-\gamma)dD(t) + dC(t) - dS(t). \tag{3.26}$$

Substituting (5.6), (3.5) and (3.13) into (3.26), the dynamics of loans may be represented as

$$\begin{aligned} \phi_1 dL(t) &= (1-\gamma)dD(t) + dC(t) - \phi_2 dS(t) \\ &= (1-\gamma)\Big[\mu_D dt + \sigma_D Z_b(t) - dK_1(t)\Big] + \Big(C^\pi(t)r + (r_e - r)\pi(t) + \alpha\Big)dt \\ &+ \Big(\pi(t)\sigma_e + \beta_{C_a}\rho\Big)dZ_0(t) + \beta_{C_a}\sqrt{1-\rho^2}dZ_a(t) + \pi(t)dK_1(t) - dK_2(t) \\ &- \phi_2 S(t)(r_S(t) + \lambda_S)dt - \phi_2 S(t)\sigma_S dZ_0(t), \end{aligned} \tag{3.27}$$

where $\phi_1 = 0.5$ and $\phi_2 = 0.2$ represent the Basel III risk-weights for loans and marketable securities, respectively (see [66]). Therefore,

$$\begin{aligned} 0.5 \times dL(t) &= (1-\gamma)dD(t) + dC(t) - 0.2dS(t) \\ &= (1-\gamma)\Big[\mu_D dt + \sigma_D dZ_b(t) - dK_1(t)\Big] + \Big(C^\pi(t)r + (r_e - r)\pi(t) + \alpha\Big)dt \\ &+ \Big(\pi(t)\sigma_e + \beta_{C_a}\rho\Big)dZ_0(t) + \beta_{C_a}\sqrt{1-\rho^2}dZ_a(t) + \pi(t)dK_1(t) - dK_2(t) \\ &- 0.2S(t)(r_S(t) + \lambda_S)dt - 0.2S(t)\sigma_S dZ_0(t) \\ &= \Big(C^\pi(t)r + (r_e - r)\pi(t) + \alpha + (1-\gamma)\mu_D - 0.2S(t)(r_S(t) + \lambda_S)\Big)dt \\ &+ (1-\gamma)\sigma_D dZ_b(t) + \Big(\pi(t)\sigma_e + \beta_{C_a}\rho - 0.2S(t)\sigma_S\Big)dZ_0(t) \\ &+ \beta_{C_a}\sqrt{1-\rho^2}dZ_a(t) + \Big(\gamma - 1 + \pi(t)\Big)dK_1(t) - dK_2(t). \end{aligned} \tag{3.28}$$

This implies that the dynamics of bank loans are given by

$$\begin{aligned}
dL(t) &= 2 \times \Bigg[\Bigg(C^{\pi}(t)r + (r_e - r)\pi(t) + \alpha + (1-\gamma)\mu_D - 0.2S(t)(r_S(t) + \lambda_S) \Bigg) dt \\
&+ (1-\gamma)\sigma_D dZ_b(t) + \Bigg(\pi(t)\sigma_e + \beta_{C_a}\rho - 0.2S(t)\sigma_S \Bigg) dZ_0(t) \\
&+ \beta_{C_a}\sqrt{1-\rho^2} dZ_a(t) + \Bigg(\gamma - 1 + \pi(t) \Bigg) dK_1(t) - dK_2(t) \Bigg]. \qquad (3.29)
\end{aligned}$$

We observe that (3.29) depends on the decision variables of other bank items on the balance sheet of a bank. Traditionally, banks rely heavily on portfolio models of bank behavior both by trying, on the one hand, to construct a continuous-time view and, on the other hand, by taking account of the capital regulations effects. This undertaking is motivated by the banks' need to invest in assets with an acceptable risk level and high returns. For example, if the returns on a specific loan turn out to be very high at the end of a loan contract period, the bank might regret not having allocated a fairly large portion of its capital to that particular loan type.

3.6 Numerical Example

Asset allocation for a bank's securities portfolio plays an important role in

- managing the balance sheet's overall interest rate risk,
- managing liquidity (assuring adequate cash is available to meet liabilities),
- producing income, and
- managing credit risk.

The first concern is the most important and dictates an asset-liability management approach to asset allocation. Bank's portfolios of loans and leases are generally not very liquid and may carry substantial credit risk. Therefore, a bank's securities portfolio plays a balancing role in providing a ready source of liquidity and in offsetting loan portfolio credit risk. As with the portfolios of insurers, the public policy usually views bank portfolios as quasi- public trust funds. Thus, banks typically face detailed regulatory restrictions on maximum holdings of asset types, often stated as a percentage of capital. In turn, the risk of assets affects banks' costs through the operation of risk-based capital rules (types of reinsurance).

In this section, we provide numerical simulations in order to characterize the behavior of the CAR. The behavior of the CAR will be influenced by the expression of π^* given by (3.23). In the discussion that follows, the computations are done over $T = 10$ years. The parameter values of some of the bank items are obtained from the table of Barclays Africa Group Limited (see [89]). Barclays Africa Group Limited is owned by Barclays Bank PLC (a bank headquartered in London) which is listed on the Johannesburg Stock Exchange Market. The group is one of Africa's major financial services providers offering personal and business banking, credit cards, corporate and investment banking, wealth and investment management. The executive board of Barclays Africa Group Limited approved target capital ranges of $9.5\% - 11.5\%$ for Common Equity Tier 1 capital, $10.5\% - 12.5\%$ for Tier 1 capital and $13.0\% - 15.0\%$ for total bank capital, respectively.

We note from Table 3.1 that the initial values for $E(t)$, $C_a(t)$ and $S_D(t)$ are given by $E(0) = \mathbf{R}$ 78553 m, $C_a(t) = \mathbf{R}$ 3657 m and $S_D(0) = \mathbf{R}$ 15123 m, respectively. Thus, we have Tier 1 capital and total bank regulatory capital at time $t = 0$ as $C_{T1}(0) = E(0) + C_a(0) = \mathbf{R}$ 82210 m and $C(0) = C_{T1}(0) + C_{T2}(0) = \mathbf{R}$ 97333 m, respectively. From Table 3.1 we observe that loans (Credit risk and counterparty risk) are the largest item on the bank's asset side and thus can act as a proxy for the assets held by the bank. This means that, $\Lambda(0) = \dfrac{C(0)}{L(0)} = \dfrac{97333}{705742} = 13.79\%$. In order to obtain the graphs of the capital adequacy ratio, we consider the following parameters and initial conditions: $S(0) = \mathbf{R}$ 31001 m, $r^A = 0.03$, $r = 0.04$, $r_e = 0.035$, $\sigma_e = 0.07$, $\alpha = 0.01$, $r_S(0) = 0.05$, $\beta = 0.75$, $E(R) = 0.1$, $\lambda_S = 0.0525$, $\sigma_S = 0.04$, $r^{TD} = 0.06$, $r^{DD} = 0.06$, $\mu_D = 0.12$, $\sigma_D = 0.03$, $\theta_{r_s} = 0.03$, $\mu_{r_s} = 0.04$, $\sigma_{r_S} = 0.04$, $\beta_{C_a} = 0.04$, $\mathbb{E}b_1 = 0.05$, $\mathbb{E}y_1 = 0.02$, $\lambda_1 = 0.03$, and $\lambda_2 = 0.04$. Figures (3.1) - (3.4) characterize the behavior of the CAR. Liquid assets (such as marketable securities) are traded on the open market. These assets

Table 3.1: The capital position for Barclays Africa Group Limited

The table below represents the capital position for Barclays Africa Group Limited at 30 September 2016 and the comparatives at 30 June 2016.

	30 Sep 2016[1]		30 Jun 2016[1]	
Regulatory Capital Position (excluding unappropriated profit)	**Rm**	**%**	**Rm**	**%**
Common Equity Tier 1 capital	**78 553**	**11.1%**	**79 249**	**11.3%**
Share capital and premium	6 084		6 106	
Reserves	75 207		75 621	
Non-controlling interest - ordinary shares	2 163		2 219	
Deductions	(4 901)		(4 697)	
Additional Tier 1 capital	**3 657**	**0.5%**	**3 713**	**0.6%**
Tier 1 capital	**82 210**	**11.6%**	**82 962**	**11.9%**
Tier 2 capital	**15 123**	**2.2%**	**13 645**	**1.9%**
Total capital	**97 333**	**13.8%**	**96 607**	**13.8%**
Statutory Capital Position (including unappropriated profit)				
Common Equity Tier 1 capital	82 873	11.7%	84 377	12.1%
Tier 1 capital	86 529	12.3%	88 090	12.6%
Total capital	101 652	14.4%	101 735	14.6%
Board Approved Target Ranges[2]				
Common Equity Tier 1 capital	9.5% - 11.5%		9.5% - 11.5%	
Tier 1 capital	10.5% - 12.5%		10.5% - 12.5%	
Total capital	13.0% - 15.0%		13.0% - 15.0%	

	30 Sep 2016[1]		30 Jun 2016[1]	
Risk Weighted Assets (RWA) and Minimum Required Capital per Risk Type	**RWA Rm**	**Minimum required capital[3] Rm**	**RWA Rm**	**Minimum required capital[3] Rm**
Credit risk	500 900	51 969	506 576	52 558
Counterparty credit risk	33 334	3 459	26 773	2 778
Equity investment risk	9 620	998	10 611	1 101
Market risk	31 001	3 216	25 160	2 610
Operational risk	100 310	10 407	100 310	10 407
Non-customer assets	30 577	3 172	29 255	3 035
Total RWA and minimum required capital	**705 742**	**73 221**	**698 685**	**72 489**

are popular amongst investors due to its high liquidity. Also, the assets are influenced by factors such as the interest rate on the open market.

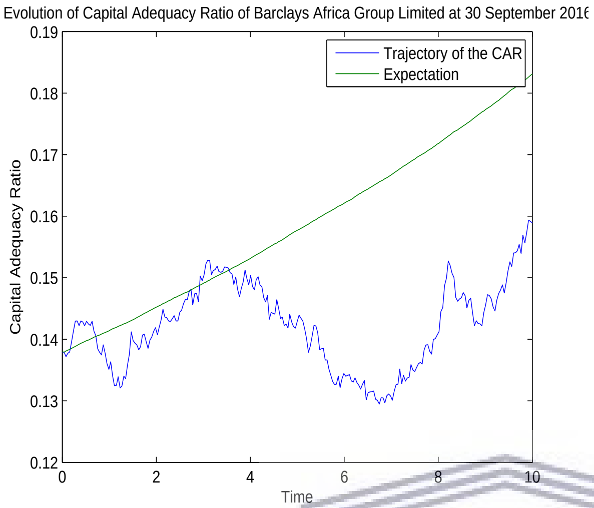

Figure 3.1: Simulation of Capital Adequacy Ratio when $\rho = 0.95$.

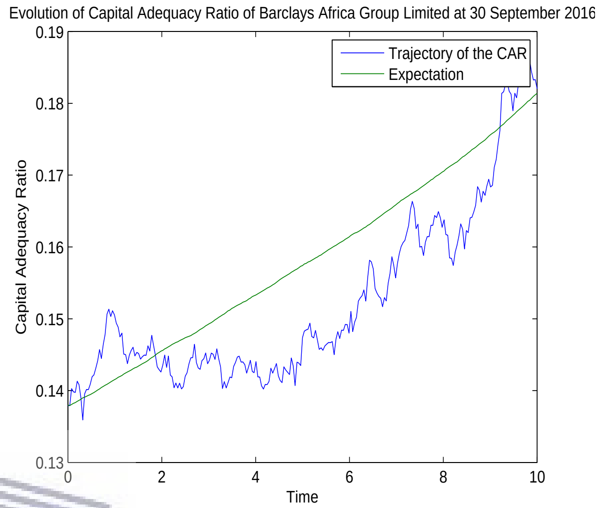

Figure 3.2: Simulation of Capital Adequacy Ratio when $\rho = -0.95$.

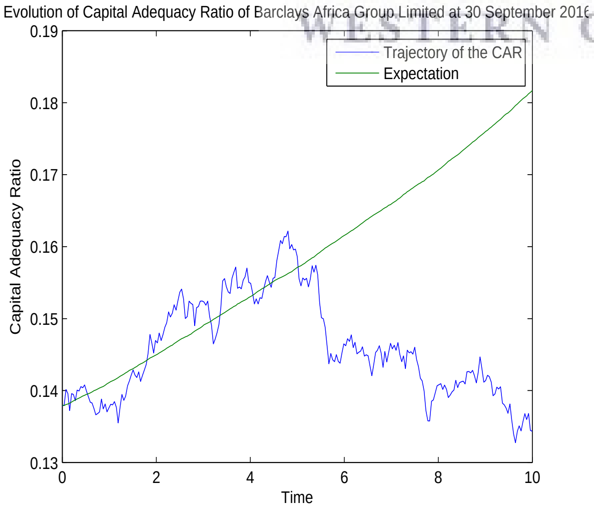

Figure 3.3: Simulation of Capital Adequacy Ratio when $\rho = 0.05$.

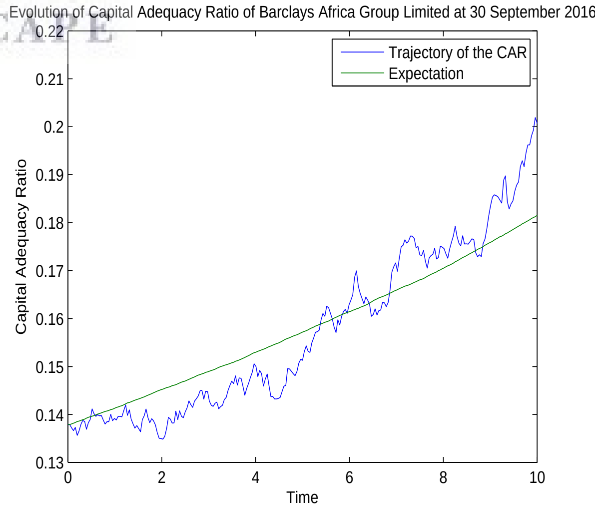

Figure 3.4: Simulation of Capital Adequacy Ratio when $\rho = -0.05$.

From Figures (3.1) - (3.4), we observe how the CAR oscillates between 13% and 15% for the majority of the time over the 10 year period. This is more noticeable when the correlation coefficient is $0 < \rho < 1$. Also, we note that the simulations from Figures (3.1) - (3.4) for Barclays Africa Group Limited remains capitalized above the minimum regulatory capital requirements and within its board approved target capital ranges. Before the global financial crisis of 2008, the minimum capital ratio for banks under these rules was generally around 8%, with higher requirements for institutions with riskier holdings. But the crisis persuaded regulators and policymakers that these standards were insufficient. Thus politicians and regulators around the world are continually crafting new rules to readjust capital-ratio requirements, especially for large banks. In the case where the asset value of Barclays Africa Group Limited falls unexpectedly and it experiences sharp losses, having a sufficient amount of capital allows the bank to continue honoring withdrawals and other obligations, and so to avoid collapse. As we note from Table 3.1 the bank has unprotected sources of funding such as equity, preferred stock and subordinate debt (all of these items falls under Tier 1 and Tier 2 capital). The bank can draw on these financing sources to address shortfalls caused by a sharp decline in asset values, providing security against a default. Thus, the larger the ratio between these financing sources and the bank's overall assets, the better the bank is able to weather losses from loans or other risky activities. Recently, a financial report from Barclays Africa Group Limited was released on 31 of May 2017 (see [90]) which indicates that the Executive Board has decided to increase its total capital requirement range of 13% - 15% to 14 % - 15.5 %. From the aforementioned figures, even though the CAR is well above the minimum requirement of 10.5 % under the Basel III paradigm, a dramatic increase in capital requirements could result in a serious credit crunch. Many banks face the prospect of improving capital standard while trying to avoid jeopardizing the stability of the banking sector and economic growth. This being said, any bank would prefer to operate in a healthy economy and a reliable supply of bank credit. This can be achieved if one understands what deficiencies of Basel II was revealed after the financial crises of 2008, and just what risks are involved in the reforms they are now considering.

Chapter 4

A Model for Bank Reserves versus Treasuries under Basel III

Commercial banks play a crucial role in the financial sector. One of many operations of a bank is to obtain funds through deposits from external sources and to use the said funds to issue loans. In particular, before the subprime mortgage crises, large amounts of deposits flowed into the U.S. banking system. Such funds were utilized on mortgages in a low-interest rate environment. This, in turn, caused the housing and credit markets to flourish since credit was easy to obtain. Of course, credit risk was shifted to investors through diversification. As a consequence, mortgage innovation rose dramatically and the proceeds from these loans were invested in residential mortgage-backed securities and treasuries. The returns from these investments as well as deposit inflows allowed banks to issue new loans.

In this paper we analyze a bank's reserves operations and quantify the extent to which their reserves management policy is optimal. Here, by optimally we mean the minimization of expected reserve costs via developing an appropriate asset allocation strategy. Thus, our paper investigates the dynamics of banking items such as reserves, Treasuries, and deposits which are assumed driven by Lévy processes. These items are constituents of the assets and liabilities held by the bank. It is imperative for banks to measure the volume of Treasuries and reserves that it holds. Treasuries are bonds issued by a national treasury and may be modeled as a risk-free asset (bond) in the usual way. When the Federal Reserve (Fed) buy and sell government securities in the open market, it is subjected to monetary policy. During this process, if the Fed buys (sells) U.S. Treasury securities, it increases (decreases) the volume of bank reserves held by depository institutions. The inclusion (exclusion) of reserves by the Fed can put downward (upward) pressure on the interest rate on federal funds, that is, the market where banks buy and sell reserves, mostly on an overnight basis. In the modern banking industry, it is appropriate to assign a price to reserves and model it by means of a Lévy process because of discontinuity of its evolution and because it provides a good fit to real-life data ([20] and [46]). Banks are interested in establishing the level of Treasuries and reserves on demand deposits that the bank must hold. By setting a bank's individual level of reserves, role players assist in reducing the cost of financial distress. For instance, if the minimum level of required reserves exceeds a bank's optimally determined level of reserves, this may lead to deadweight losses. While the academic literature on the pricing of bank assets is vast and well developed, little attention is given to the minimization of deposit risk. Most bank deposits contain an embedded option which permits the depositor to withdraw funds at will. Demand deposits generally allow costless withdrawal, while time deposits often require payment of an early withdrawal penalty. Managing the risk that depositors will exercise their withdrawal option, is an essential aspect of our contribution. The main thrust of our paper is the hedging of bank deposit withdrawals through an allocation strategy. In this regard, we discuss an optimal risk management problem for a commercial bank, which uses the Treasuries and reserves to cater for such withdrawals. The main risks that can be identified are a reserve, depository, and the intrinsic risk that are associated with the reserve process, the net cash flows from depository activity, and cumulative cost of the bank's provisioning strategy, respectively.

Reserve requirements are one of the three monetary policy tools the Fed uses to implement monetary policy. It is the minimum amount of money (coins and banknotes) that banks must hold in order to provide for deposit withdrawals. It is usually stored in a bank's vault or with a central bank and cannot be loaned out to businesses or individuals. This requirement plays a major role in the economy as it affects the money supply process. In particular, it would also mean that banks earn less interest and could see their share prices fall. Lowering the reserve requirement will have the adverse effect whereby banks will be able to lend more which would increase the money supply and stimulate economic growth. However, this could cause increased risk of being unable to respond to withdrawal requests.

A vast amount of literature exists on the properties of Treasuries and reserves and their connections with deposit withdrawals. Reserves management is dealt with in the contributions [22], [24], [37], [42], [43], [45] and [83]. Castiglionesi [22] investigates the role of a central bank in preventing and avoiding financial distress. Clouse and Dow Jr. [24] applies numerical methods to model the demand for excess reserves by a representative bank in a framework that includes many realistic features of the reserve market structure in the United States. Also, Demiralp and Farley [37] investigates the Domestic Trading Desk (the Desk) at the Federal Reserve's reaction function to test the hypothesis that changes in the pattern of desk operations for supplying liquidity cause funds rate volatility to remain low, even as required balances have declined. Their analysis suggests that interest rate volatility depends essentially on institutional arrangements for providing and absorbing liquidity, rather than the level of reserve requirements. Ennis and Keister [42] investigates how the occurrence of a bank run influences the investment decisions made by a competitive bank. In doing so, they consider the possibility of clients withdrawing all their funds from an account at the same time. The bank will hold an amount of liquid reserves exactly equal to what the withdrawal demand will be if a run does not occur. Precautionary or excess liquidity will not be held. Fernández [43] asserts that the payment of interest on bank reserves by the government assists in the implementation of monetary policy. In particular, it is demonstrated that paying interest on reserves financed by labor tax, reduces welfare. In the paper of Geng and Zhai [45] the authors adopted the Panel

Smooth Transition Regression (PSTR) approach to analyze the effects of the interest rate and the reserve requirement ratio on bank risk empirically. The aforementioned authors analysis indicates that the interest rate and the reserve requirement ratio have a positive impact on bank risk. Finally, Whitesell [83] asserts that reserve requirements allow period-average smoothing of interest rates but are subject to reserve avoidance activities. A system of voluntary, period-average reserve commitments could offer equivalent rate-smoothing advantages. A common theme in the aforementioned contributions about reserves is the fact that they can be viewed as a proxy for general banking assets and that reserve dynamics are closely related to the dynamics of the deposits.

The objective of the paper is to apply a sufficient stochastic maximum principle (see [44]) to a problem related to bank reserves and deposits withdrawal in a jump diffusion market. Stochastic optimization has been applied popularly in finance (see for instance [21]). In particular, such applications include portfolio selection problems, (see, for example, [72], [86]), where problems are commonly formulated as a stochastic linear-quadratic (LQ) problem. Zeng *et al.*, [85], studies an optimal portfolio selection problem for insurers which involves the application of the stochastic maximum principle method to achieve the optimal strategy in a benchmark portfolio selection problem which minimizes the expected quadratic distance of the terminal risk reserve from a predefined benchmark. Moreover, work has been done on extending the stochastic maximum principle to stochastic differential equations involving jumps. In particular, a sufficient maximum principle was developed for stochastic differential equations with jumps (see [56], [59] and [77]).

Our paper has some close connections with Bosch *et al.* [20] and Gideon *et al.* [46] in the sense of establishing an optimal level of bank reserves and the rate at which deposits are consumed. The main difference between our paper and the aforementioned literature is that we derive an explicit solution for the allocation strategy over finite horizon and provide numerical examples to illustrate the importance of portfolio optimization in a realistic setting.

The layout of the rest of the chapter is as follows. In Section 4.1, we extend some of the modeling and optimization issues highlighted in Bosch *et al.* [20] by presenting jump diffusion models for various assets. The stochastic dynamics of bank reserves and the volume of Treasuries and reserves are presented in Section 4.1.2. In Section 4.1.3 we provide a brief discussion on bank deposits and its dynamics. In Section 4.2, we formulate an asset allocation problem and discuss its solution. In Section 4.3 we use the results obtained in Section 4.2 to present numerical simulations of the bank's total reserves and reserve requirement ratio. In addition to solving the problem outlined above, we offer a few concluding remarks in Section 6.2.

4.1 The stochastic banking model

To understand the operation and management of banks, we have to study its balance sheet, which records the bank assets and bank liabilities. The items on the balance sheet behave in an unpredictable manner, arising from the uncertain behavior of the activities related to the evolution of treasuries, loan demand, risky and riskless investments, deposits, loan repayments, borrowings and bank regulatory capital.

Models similar to what we present here, have been used in the papers of [20], [23] and also [46]. In all three of the aforementioned papers, the authors systematically construct continuous time models for bank items. In particular, Chakroun and Abid [23] uses existing models to represent bank regulatory capital and bank assets in order to perform simulations of a bank's Capital Adequacy Ratio and Leverage Ratio in a Brownian motion framework. Gideon *et al.* [46] provides a numerical example involving a simulation of the provisions made for deposit withdrawals via Treasuries and reserves.

4.1.1 Stochastic Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathcal{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$ which is right continuous, for some finite T which denotes the investment time horizon. The space Ω represents the different states of the economy in which banks operate. The sub-algebra $\mathcal{F}_t$ can be regarded as representing the information available to banks up until time t. All random variables considered in this paper are defined on this space. Furthermore, we consider a commercial bank that enters a financial market which is subject to uncertainty, through a $1-$dimensional Brownian motion

$$Z(t) = Z(t, \omega)$$

and a $1-$dimensional Poisson process

$$N(t) = N(t, z), \ t \geq 0, \ z \geq 0.$$

The martingale $\tilde{N}(t) = N(t) - \lambda t$ is called the compensated Poisson process of $N(t)$, where λ is the vector of intensities of $N(t)$. In this paper, we only consider *homogeneous* Poisson processes, that is, with λ being constant over time.

4.1.2 Bank Assets

Securities

Treasury securities or *Treasuries* are government debt obligations. Essentially, a Treasury Bond is a loan to the government. Because there is almost no risk of default by the government, the return on a Treasury bond is relatively low. There are four types of securities issued by a Treasury, namely, treasury bills, treasury notes, treasury bonds, and savings bonds. All of the treasury securities besides savings bonds are very liquid and are heavily traded on the secondary market.

Under such a scenario, one may express the dynamics of the riskless asset (Treasuries) as

$$dT_0(t) \ = \ T_0(t)r^T(t)dt, \ \ T_0(0) = T_0 > 0, \tag{4.1}$$

where $r^T(t) > 0$ represents the riskless interest rate.

Bank Reserves

Bank reserves are the deposits held in accounts with the central bank of a country (for instance, the South African Reserve Bank in the case of South Africa) plus money that is physically held by banks (vault cash). Such reserves constitute money that is not lent out but is earmarked to cater for withdrawals by depositors. Since it is uncommon for depositors to withdraw all of their funds simultaneously, only a portion of total deposits may be needed as reserves. We also note that cash in the Automated Teller Machines (ATMs) network also qualifies as required reserves. Bank reserves may actually have a stochastic nature and banks may earn a positive return on them. For instance, by the end of 2013, China had over \$ 3.82 trillion in foreign exchange reserves (see [75]). Based on the above description, we may represent bank reserves as

$$\begin{aligned} dR(t) &= R(t^-)\left\{[r^R(t) - f^R(t)]dt + \sigma dZ(t)\right\} \\ &+ R(t^-)\left\{\int_{-1}^{\infty} x(t^-, z)\tilde{N}(dt, dz)\right\}, \quad R(0) = R_0 > 0, \end{aligned} \tag{4.2}$$

where r^R is the rate of (positive) return on reserves earned by the bank, f^R is the fraction of bank reserves that are available for withdrawal, and $\sigma > 0$ is the volatility in the level of reserves. Note that $r^R(t) > 0$ and $f^R(t) > 0$ can be deterministic or stochastic. The size of the jumps is denoted by $x(t^-, z)$ and we assume that $x(t^-, z) \geq -1$, ensuring that the jumps are not too large negative. The stochastic model in (4.2) is similar to that as in [72] and is known as a geometric Lévy process.

We assume that the amount of deposits taken from the clients are split into two parts; one of which is invested in Treasuries while the other part is kept as reserves. Also, in order to make provision for deposit withdrawals, it is required of a bank to make rational decisions about the volume of Treasuries and reserves held by the bank. Without loss of generality, in the sequel we suppose that the provision for deposit withdrawals, denoted by $W(t)$, corresponds to the sum of Treasuries and reserves, is $W(t) = T_0(t) + R(t)$. Let $\pi(t)$ be the amount invested in the risky asset, and $(W(t) - \pi(t))$ the amount invested in the risk-free asset. Note that when we want to accentuate the dependence of W on π, then we write W as W^π. This notation is also used when focusing on a particular investment strategy. We introduce a process $k(t)$ as the rate of depository consumption. In other words, it is the rate at which Treasuries and reserves are consumed by anticipated deposit withdrawals from customers. The assumption we make on $k(t)$ is that a bank run must not occur. The dynamics of withdrawals $W(t)$ is given by

$$\begin{aligned} dW(t) &= (W(t)-\pi(t))\frac{dT_0(t)}{T_0(t)} + \pi(t)\frac{dR(t)}{R(t^-)} - k(t)dt \\ &= \Big[W(t)r^T(t) - \pi(t)r^T(t) + \pi(t)[r^R(t) - f^R(t)] - k(t)\Big]dt \\ &+ \pi(t)\sigma dZ(t) + \pi(t)\int_{-1}^{\infty} x(t^-,z)\tilde{N}(dt,dz). \end{aligned} \tag{4.3}$$

It is common in the financial mathematics literature, particularly in portfolio selection problems, to construct a stochastic model for the wealth process in the manner we did for (4.3). Under these conditions, we attempt to solve the stochastic optimal control problem via mean-variance analysis.

4.1.3 Deposits

In general, funds that are deposited at a bank, are mostly lent out to customers or financial institutions. A commercial bank keeps a fraction (known as a reserve-deposit ratio) of those funds as reserves to cover its customer deposit liabilities. Central banks or other banking regulators often mandate the aforementioned reserve requirements in order to limit the amount of money creation that occurs in the commercial banking system and to ensure that banks have enough ready cash to meet normal demand for withdrawals. In our research, the term deposits include both demand and time deposits. Deposits have uncertainty associated with them and thus can be modeled as a stochastic process. Let $r^{DD} : T \to \mathbb{R}_+$ denote the rate of demand deposit which is payable on demand and $r^{TD} : T \to \mathbb{R}_+$ the rate of time deposit which is payable only after a fixed interval of time. The dynamics of deposits can be written as a diffusion process (see, for instance, [36] and

[60]). However, in our article we model deposits as a jump diffusion process

$$dD(t) = m_D dt + \sigma_D dZ(t) + \int_{-1}^{\infty} x(t^-, z)\tilde{N}(dt, dz), \quad D_0 = d \in \mathbb{R}_+, \tag{4.4}$$

where $m_D = r^{DD} + r^{TD}$. The term m_D represents the expected rate of growth in deposits, σ_D the level of volatility in deposits and the jump term captures the unexpected information (such as bank solvency) that will have an influence on deposit taking and deposit rates.

4.2 The Asset Allocation Problem

The optimization problem we present in this section proposes an approach to reserve management. Through simulations, we shall illustrate how the approach can be implemented, and the significance of this approach in comparison with historical data. We do not expect our approach to be able to emulate historic data since the period of interest includes the 2008 financial crisis, when governments had to intervene, and this complicates the modelling around that event.

Towards formulation of the problem we introduce the following functions Q and J associated with a class $\mathcal{A}$ of admissible control laws as below. We shall sometimes write W as W^π when we wish to specify the control π which is in action. The value of w denotes a value assumed by W (or W^π) , and the constant $d = \mathbb{E}[W(T)]$ represents the expected terminal wealth.

$$\begin{aligned}\mathcal{A} &= \Big\{\pi(\cdot): \ \pi \text{ is } \mathcal{F}_t - \text{progressively measurable and satisfies the integrability condition} \\ &\quad \mathbb{E}\int_0^T \pi^2(t)dt < \infty \text{ so that } W(t) > 0 \ \text{ for all } t \in [0, T]\Big\},\end{aligned} \tag{4.5}$$

$$Q(t, T, w; \pi) = \mathbb{E}\Big[-\frac{1}{2}\Big(W(T) - d\Big)^2\Big|W(t) = w\Big], \tag{4.6}$$

and

$$J(w) = \sup_{\pi \in \mathcal{A}} Q(t, T, w; \pi). \tag{4.7}$$

Problem 4.2.1. (Optimal allocation problem): Consider the jump-diffusion model of the form (4.3). We attempt to find the supremum

$$J(w) = \sup_{\pi \in \mathcal{A}} Q(t, T, w; \pi)$$

and the optimal control law, which is given by

$$\pi^*(t) = \arg \sup_{\pi \in \mathcal{A}} Q(t, T, w; \pi) \in \mathcal{A}, \quad \text{so that} \quad J(w) = Q(t, T, w; \pi^*). \tag{4.8}$$

Note that the optimal wealth process at time t can then be written $W^{\pi^*}(t)$. Towards the proof of the following proposition we apply the method used by Framstad *et al.* [44] to solve Problem (4.2.1). In this case the Hamiltonian has the form

$$\begin{aligned} H(t, W^\pi(t), \pi, p, q, r) &= \Big\{ W^\pi(t) r^T(t) - \pi(t) r^T(t) + \pi(t)[r^R(t) - f^R(t)] - k(t) \Big\} p(t) \\ &+ \pi(t)\sigma(t) q(t) + \pi(t) \int_{-1}^{\infty} x(t^-, z) r(t^-, z) \nu(dz). \end{aligned} \tag{4.9}$$

The adjoint equation is

$$\begin{aligned} dp(t) &= -r^T(t) p(t) dt + q(t) dZ(t) + \int_{-1}^{\infty} r(t^-, z) \tilde{N}(dt, dz), \\ p(T) &= -(W^\pi(T) - d). \end{aligned} \tag{4.10}$$

Proposition 4.2.1. *Suppose that W, J, and $\mathcal{A}$ are characterized by (4.3), (4.7) and (4.5), respectively. If we write $K(t) = [r^R(t) - f^R(t)]$ and $\Gamma(t) = \sigma^2 + \int_{-1}^{\infty} x^2(t, z)\nu(dz)$, then the optimal solution $\pi^*(t)$, has the form*

$$\pi^*(t) = -\frac{(r^T(t) + K(t))(\phi(t) W^{\pi^*}(t) + \psi(t))}{\phi(t)\Gamma(t)}, \tag{4.11}$$

with the functions $\psi(t)$ and $\phi(t)$, for $0 \le t \le T$, given by

$$\psi(t) = a \exp\left(\int_t^T \left[r^T(s) - \frac{(r^T(s) + K(s))[K(s) - r^T(s)]}{\Gamma(s)} \right] ds \right), \tag{4.12}$$

and

$$\phi(t) = -\exp\left(\int_t^T \left[\frac{(r^T(s)+K(s))[K(s)-r^T(s)]}{\Gamma(s)} - 2r^T(s) + \frac{k}{W^{\pi^*}(s)}\right]ds\right), \quad (4.13)$$

respectively.

Proof.

In order to find explicit solutions for $\phi(t)$, $\psi(t)$ and $\pi^*(t)$ the proof relies on tools from elementary algebra and differential equations. Let us assume initially that $p(t)$ has the particular form:

$$p(t) = \phi(t)W(t) + \psi(t), \quad (4.14)$$

where $\phi(t)$ and $\psi(t)$ are deterministic differentiable functions. Making use of (4.3), leads to

$$\begin{aligned} dp(t) &= \phi(t)dW(t) + W(t)\phi^{'}(t)dt + \psi^{'}(t)dt \\ &= \phi(t)\Bigg\{\Big[W(t)r^T(t) - \pi(t)r^T(t) + \pi(t)[r^R(t) - f^R(t)] - k(t)\Big]dt \\ &+ \pi(t)\sigma dZ(t) + \pi(t)\int_{-1}^{\infty} x(t^-,z)\tilde{N}(dt,dz)\Bigg\} + W(t)\phi^{'}(t)dt + \psi^{'}(t)dt \\ &= \Big[\phi(t)W(t)r^T(t) - \phi(t)\pi(t)r^T(t) + \phi(t)\pi(t)[r^R(t) - f^R(t)] \\ &- \phi(t)k(t) + W(t)\phi^{'}(t) + \psi^{'}(t)\Big]dt + \phi(t)\pi(t)\sigma dZ(t) \\ &+ \phi(t)\pi(t)\int_{-1}^{\infty} x(t^-,z)\tilde{N}(dt,dz). \end{aligned} \quad (4.15)$$

Comparing (4.10) and (4.15), we get

$$\begin{aligned} &-r^T(t)(\phi(t)W(t) + \psi(t)) \\ &= \phi(t)W(t)r^T(t) - \phi(t)\pi(t)r^T(t) + \phi(t)\pi(t)[r^R(t) - f^R(t)] \\ &- \phi(t)k(t) + W(t)\phi^{'}(t) + \psi^{'}(t). \end{aligned}$$

Also, for $q(t)$ and $r(t^-,z)$ we have

$$q(t) = \phi(t)\pi(t)\sigma, \quad (4.16)$$

$$r(t^-,z) = \phi(t)\pi(t)x(t^-,z). \quad (4.17)$$

Then the Hamiltonian becomes

$$
\begin{aligned}
H(t, W(t), \pi(t), p(t), q(t), r(t,\cdot)) &= \Big\{W(t)r^T(t) - k(t)\Big\}p(t) \qquad (4.18)\\
&+ \pi(t)\Big[r^T(t)p(t) + [r^R(t) - f^R(t)]p(t)\\
&+ \sigma q(t) + \int_{-1}^{\infty} x(t^-, z)r(t^-, z)\nu(dz)\Big]
\end{aligned}
$$

We need to optimize H with respect to π, and to this end we partially differentiate with respect to π and make the derivative to vanish

$$
r^T(t)p(t) + [r^R(t) - f^R(t)]p(t) + \sigma q(t) + \int_{-1}^{\infty} x(t^-, z)r(t^-, z)\nu(dz) = 0. \qquad (4.19)
$$

Now substituting $p(t) = \phi(t)W(t) + \psi(t)$, $q(t) = \phi(t)\pi(t)\sigma$ and $r(t^-, z) = \phi(t)\pi(t)x(t^-, z)$ into (4.19), yields

$$
\pi^*(t) = -\frac{(r^T(t) + [r^R(t) - f^R(t)])(\phi(t)W^{\pi^*}(t) + \psi(t))}{\phi(t)\Big(\sigma^2 + \int_{-1}^{\infty} x^2(t^-, z)\nu(dz)\Big)}. \qquad (4.20)
$$

The parameter $\sigma > 0$ and $\sigma^2 + \int_{-1}^{\infty} x^2(t^-, z)\nu(dz) < \infty$ ensures that the denominator in (4.20) remains nonzero. On the other hand, (4.16) gives

$$
\begin{aligned}
\pi^*(t) &= \frac{-r^T(t)(\phi(t)W^{\pi^*}(t) + \psi(t)) - \phi(t)W^{\pi^*}(t)r^T(t) + \phi(t)k(t) - W^{\pi^*}(t)\phi'(t) + \psi'(t)}{\phi(t)[r^R(t) - f^R(t)] - \phi(t)r^T(t)}\\
&= \frac{-r^T(t)(\phi(t)W^{\pi^*}(t) + \psi(t)) - \phi(t)W^{\pi^*}(t)r^T(t) + \phi(t)k(t) - W^{\pi^*}(t)\phi'(t) + \psi'(t)}{\phi(t)\Big([r^R(t) - f^R(t)] - r^T(t)\Big)}. \qquad (4.21)
\end{aligned}
$$

Note that $r^R(t) > 0$ and $f^R(t) > 0$ can be deterministic or stochastic. We link (4.20) with (4.21) in order to find explicit solutions for $\phi(t)$ and $\psi(t)$. Thus,

$$
\begin{aligned}
&- \frac{(r^T(t) + K(t))p(t)}{\phi(t)\Gamma(t)}\\
&= \frac{-r^T(t)p(t) - \phi(t)W^{\pi^*}(t)r^T(t) + \phi(t)k(t) - W^{\pi^*}(t)\phi'(t) + \psi'(t)}{\phi(t)K(t) - \phi(t)r^T(t)}.
\end{aligned}
$$

Cross multiplication of the latter expression yields:

$$
\begin{aligned}
&(r^T(t) + K(t))[K(t) - r^T(t)](\phi(t)W^{\pi^*}(t) + \psi(t))\\
= \;& 2\Gamma(t)\phi(t)r^T(t)W^{\pi^*}(t) + \Gamma(t)\phi'(t)W^{\pi^*}(t) + \Gamma(t)r^T(t)\psi(t) - \Gamma(t)\phi(t)k(t) - \Gamma(t)\psi'(t) \qquad (4.22)
\end{aligned}
$$

From (4.22), the following holds true (see [44] and [72]):

$$(r^T(t) + K(t))[K(t) - r^T(t)]\psi(t) = \Gamma(t)r^T(t)\psi(t) - \Gamma(t)\psi'(t), \quad \psi(T) = a \quad (4.23)$$

and

$$\begin{aligned} & (r^T(t) + K(t))[K(t) - r^T(t)]\phi(t)W^{\pi^*}(t) \\ = \; & 2\Gamma(t)\phi(t)r^T(t)W^{\pi^*}(t) + \Gamma(t)\phi'(t)W^{\pi^*}(t) - \Gamma(t)\phi(t)k(t), \\ & \phi(T) = -1. \end{aligned} \quad (4.24)$$

Using standard techniques of differential equations to solve for $\phi(t)$ and $\psi(t)$ in (4.23) and (4.24), respectively, yields

$$\psi(t) = a \exp\left(\int_t^T \left[r^T(s) - \frac{(r^T(s) + K(s))[K(s) - r^T(s)]}{\Gamma(s)} \right] ds \right), \quad 0 \le t \le T, \quad (4.25)$$

and

$$\phi(t) = -\exp\left(\int_t^T \left[\frac{(r^T(s) + K(s))[K(s) - r^T(s)]}{\Gamma(s)} - 2r^T(s) + \frac{k}{W^{\pi^*}(s)} \right] ds \right), \; 0 \le t \le T. \; (4.26)$$

With the choices of $\psi(t)$ and $\phi(t)$ in (4.25) and (4.26), respectively, the processes

$$p^*(t) = \phi(t)W^{\pi^*}(t) + \psi(t), \; q^*(t) = \phi(t)\pi^*(t)\sigma, \; r^*(t^-, z) = \phi(t)\pi^*(t)x(t^-, z)$$

solve the adjoint equation (4.10) with $\pi^*(t)$ given by (4.20). □

This proposition solves the Problem 4.2.1 . It will be used in the next section to illustrate the importance of reserve management in banking. In order to interpret the roles of $\psi(t)$ and $\phi(t)$, we consider the following scenario. Bank reserves can be decomposed into required reserves and excess reserves. Excess reserves are capital held by a bank in excess of what is required by regulators, creditors or internal controls. During the global financial crisis, banks were sceptical about the Fed's policies on the stimulation of credit flow. This was one of the reasons why banks allowed funding from the Fed to accumulate instead of lending it out. This can be observed in Figure (4.3). Here, $\psi(t)$ and $\phi(t)$ can be viewed as proxies of the rate at which the Federal Reserve banks pay interest on required reserve (IORR) balances and the interest on excess reserves (IOER), respectively. These rates are determined by the Fed. Central banks of countries such as Sweden, Switzerland, Denmark, European Central Bank and the Bank of Japan have paid negative interest on excess reserves as an expansionary monetary policy tool measure (see expression 4.26).

4.3 Numerical Example

The aim of this section is to provide numerical simulations of the evolution of bank reserves (4.2). Fractional-reserve banking refers to a system in which only a certain fraction of bank deposits are backed by actual cash on hand in case of withdrawal. This is done to stimulate economic growth by freeing up capital that can be loaned out to other parties. Since clients can withdraw funds at any time, we model $f^R(t)$ as a random process. Certain banks are exempt from holding reserves, but all banks are paid a rate of interest on reserves, $r^R(t)$, from the Fed. Increasing the rate earned by banks on reserves, the Fed could encourage banks to put more of their reserves into accounts held at the federal reserve. This means that banks will have less money available to make loans to businesses and individuals. Lowering the rate earned by banks on reserves encourages banks to reduce their balances at the federal reserve and likely increase the amounts which are lent to businesses and consumers. The processes $r^R(t)$ and $f^R(t)$ pertaining to reserves, see equation (4.2), are modelled as mean reversion processes:

$$dr^R(t) = (\kappa - r^R(t))dt + \sigma_r \sqrt{r^R(t)}dZ(t), \tag{4.27}$$

and

$$df^R(t) = \beta(\eta - f^R(t))dt + \sigma_f dZ(t). \tag{4.28}$$

The model for $r^R(t)$ in (4.27) is commonly known as a squareroot process (see [33]) and it ensures that $r^R(t)$ remains positive. The model for $f^R(t)$ in (4.28) is commonly known as the Vasicek model (see [81]). Here $\kappa, \beta, \eta, \sigma_r$ and σ_f are all positive constants. We assume the horizon to be $T = 15$ years (that is, from the period $2000 - 2015$) and the bank is operating for 252 business days. We now produce a simulated trajectory for bank reserves and compare it with a real life example. The values for the parameters used in Table (1) and Table (2) are considered as proxies to real data, except for $R(0) = 42.215$ and $W(0) = 44.229$ which were obtained from [29]. The parameters appearing as Greek letters were chosen after numerous simulations. The parameters and initial values for the dynamics of bank reserves (4.2) are given as follows:

Table 4.1: Parameter values used in the simulation of Figure 4.2.

Parameters	$R(0)$	$r^R(0)$	$f^R(0)$	σ_f	σ	κ	β	η	σ_r	λ
Values	44.229	0.01	0.08	0.02	0.07	0.11	0.9	0.01	0.06	0.02

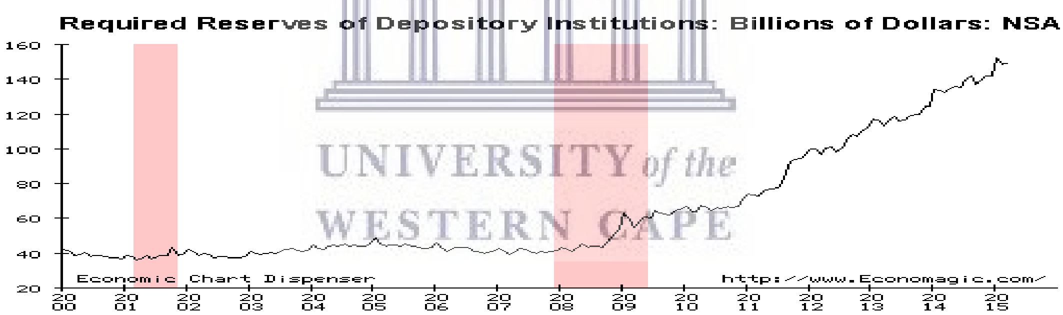

Figure 4.1: Realistic graph of Bank Reserves

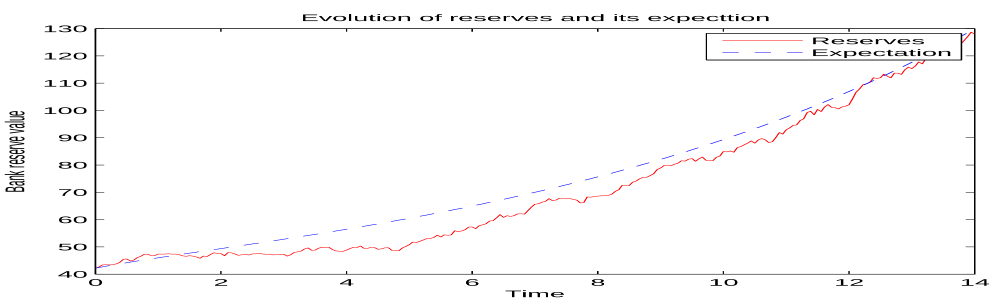

Figure 4.2: Simulation of Bank Reserves

From Figure 4.2 we observe that the simulated trajectory of the dynamics for bank reserves (4.2) exhibit similar characteristics to the realistic graph of Figure 4.1. Here, different values for the banking parameters are collected in Table (1). Figure 4.1 was obtained from the website [29]. Over the first 8 years, from the graphs, we notice that the reserves behave in a mean-reverting fashion. Before the onslaught of the financial crisis, from $2005-2008$, the Federal Reserve required banks to maintain a steady level of reserves as this helps to maintain stability in the banking system and ensure that the banks are able to conduct day-to-day check-clearing and cash-withdrawal transactions. There are three ways in which the Federal Reserve can control the money supply and that is through the discount rate, monetary policy tools and by regulating open market operations. An increase (decrease) in the money supply can be achieved when the Fed lowers (raises) reserve requirements. During the financial crisis of $2007-2009$, the Fed decided that the economy is in or heading toward a recession, consequently, it would be inclined to lower reserve requirements and implement expansionary monetary policy. The reason for such imposition on commercial banks is to help avoid bank failures and related problems when it struggled/failed to cover daily banking activity, including meeting cash withdrawals and processing checks.

The parameters of the simulation for the dynamics of the bank reserves portfolio (4.3) are given in Table (2). The computation for the explicit solutions of $\psi(t)$ and $\phi(t)$ given by (4.25) and (4.26) uses the trapezoidal rule for integration. The values for ψ and ϕ are given by $\psi = 1.2216$ and $\phi = -0.7335$, respectively after substituting the values for each parameter in Table (2) into (4.25) and (4.26).

Table 4.2: Parameter values used in the simulation of Figures 4.4 and 4.5.

Parameters	$W(0)$	$r^R(0)$	$f^R(0)$	σ_f	κ	β	η	σ_r	r^T	k	λ
Values	44.229	6	2	0.03	6	4	5	0.09	5.5	0.4	5

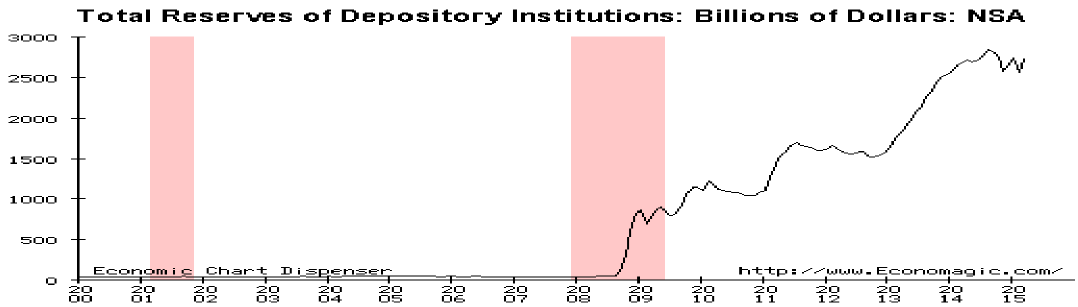

Figure 4.3: Realistic graph of Bank Reserves

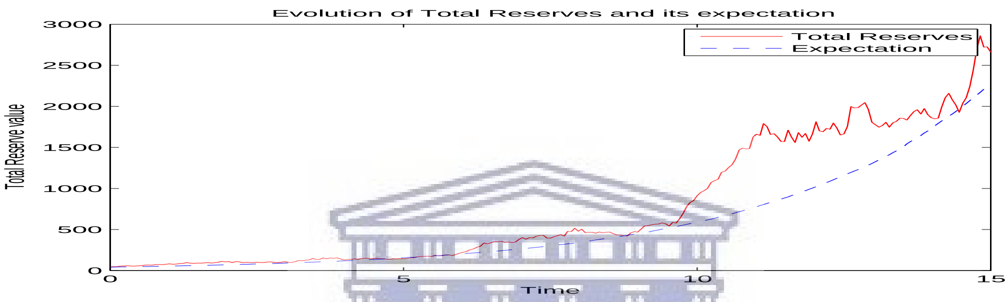

Figure 4.4: Simulation of Total reserves when $\sigma = 0.8$.

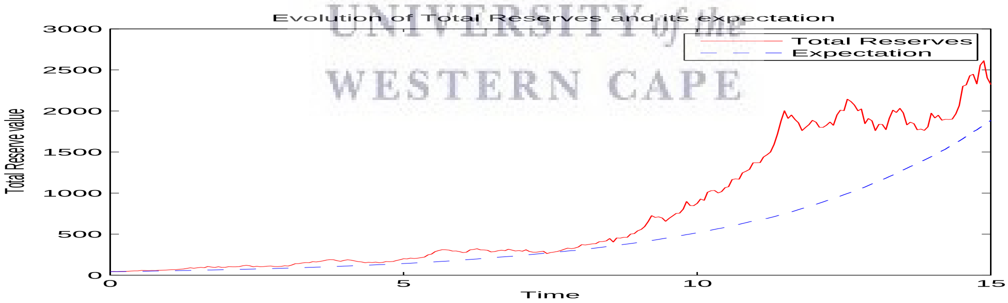

Figure 4.5: Simulation of Total reserves when $\sigma = 0.9$.

From Figures 4.4 and 4.5 we observe that the simulated trajectory of the dynamics for bank reserves starts to exhibit similar characteristics to the realistic graph of Figure 4.3 as the volatility parameter increases. After the 2008 financial crisis, commercial banks have increased their excess reserves over and above the Federal Reserve's requirement. During this period, governments of many countries bailed out banks (such as Citigroup, JPMorgan Chase) in order to stabilize the financial system. Certain measures that were taken include supertax on bankers' bonuses and financial transaction tax. In response to the financial crisis, the United States government announced an emergency plan (known as Emergency Economic Stabilization Act of 2008) to purchase large amounts of illiquid, risky mortgage-backed securities from financial institutions and included a restriction on short-selling of financial stocks. The aftermath of the global financial crisis is still heavily felt by certain countries, especially Greece. Recently, the Greek government submitted an emergency plan meant to avoid the collapse of the Greek banking sector.

Here, different values for the banking parameters are collected in Table (2). The shaded regions in Figure 4.3 indicates periods of recession. During these periods, the Fed has developed certain methods to combat the temporary economic decline and to stimulate its growth. Some of the measures include the raising or lowering of interest rates as dictated by economic circumstances. For instance, if the Fed raises interest rates, banks raise their prime rate, which in turn affects mortgage rates, car loans, business loans, and other consumer loans. If the Fed lower interest rates, then it is usually to spur the economy by making corporate and consumer borrowing easier. During the financial crisis, the level of reserves has risen due to the collapse of Lehman Brothers in mid-September 2008, climbing from roughly $45 billion to more than $900 billion by January 2009. During this period, many banks were aiming to have reserves in excess in case of a deposit run that might occur. However, while providing a cushion for banks against these risks, some economists argue that the high level of excess reserves could be one of the problems behind the credit crunch. Other observers interpreted the large increase in excess reserves as a sign that the Fed's strategies during the crisis have been ineffective. They further criticized the Fed for lending money to banks and other financial intermediaries since September 2008 which is meddling in banks' reserve accounts.

We refer to the liquid assets that a central bank or other body mandates that a bank should keep at all times. It is expressed as a percentage of the bank's total deposits and its objective is to ensure that the bank is able to pay an unusually high number of withdrawals on demand accounts should that event occur. It also helps to ensure that the bank does not over-leverage itself. In certain countries, increasing or decreasing reserve ratios may be used to help control the money supply. That being said, the reserve requirement ratio $\Lambda(t)$ may be expressed as

$$\Lambda(t) = W(t)Y(t), \quad \text{where} \quad Y(t) = \frac{1}{D(t)}. \tag{4.29}$$

The discussion on reserve requirements in the introduction motivates us to simulate a continuous time model for $\Lambda(t)$. In order to illustrate the importance of the reserve requirement, we numerically simulate the dynamics of (4.29). The table on the global reserve requirement ratio, which includes the name of the country, the name of the central bank and the current required reserve ratio for domestic currency deposits can be obtained from [30] . From [30] we observe that many countries use the reserve requirement ratio as a monetary policy tool. We illustrate now through an example how a change in the reserve requirement ratio affects bank credit and the money stock. From [30], the Bank of Albania has a current reserve requirement ratio of 10% on net transaction accounts. If the aforementioned bank experiences a net increase of \$300 million in deposits then it would be required to increase its required reserves by \$30 million. This implies that the bank would be able to lend the remaining \$270 million of deposits, resulting in an increase in bank loans. Deposits will have an influence on the stock in the following manner: Increasing reserve requirement ratio reduces the volume of deposits that can be supported by a given level of reserves and, decreases the money stock and raises the cost of loans. Decreasing the ratios leaves depositories initially with excess reserves, which can induce an expansion of bank credit and deposit levels and a decline in interest rates. Recently, in the month of April 2015, The People's Bank of China (PBOC) lowered the reserve requirement ratio for all Chinese banks by 100 basis points to 18.5%. PBOC have decided to cut the amount of cash that banks must hold as reserves in order to spur on bank lending and combat slow growth. According to the Basel document [10] (see also [63]), the PBOC, heavily relies on the reserve requirement ratio as a monetary policy tool. In fact, it has modified the reserve requirement ratio more than many of its international counterparts since the mid $2000's$. As of 2007, it has implemented the reserve ratio extensively on a regular basis. This means the scale of required reserves maintained by Chinese commercial banks has been among the highest in the world. One of the primary reasons why PBOC prefers this instrument over the other monetary policy tools such as open market operations and the discount rate, was because it is a cheaper substitute for open market operations. Another reason is that it attempted to effectively sterilize foreign exchange interventions at a reduced cost. We list the parameters for simulating the Reserve Requirement Ratio below:

Table 4.3: Parameter values used in the simulation of Figures 4.6 and 4.7.

$\Gamma(0)$	$D(0)$	$r^R(0)$	$f^R(0)$	m_D	σ_D	σ_f	κ	β	η	σ_r	r^T	k	λ
1700	100	6	2	0.12	0.09	0.03	6	4	5	0.09	0.02	0.4	5

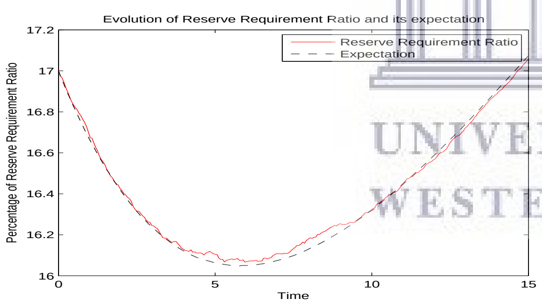

Figure 4.6: Simulation of Reserve Requirement Ratio when $\sigma = 0.01$.

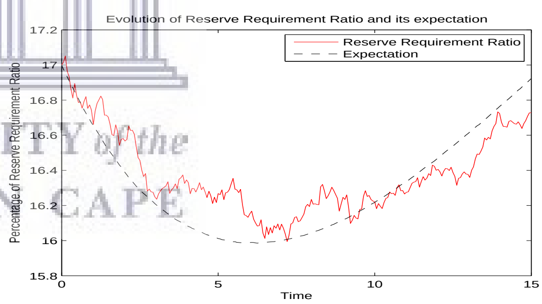

Figure 4.7: Simulation of Reserve Requirement Ratio when $\sigma = 0.08$.

Usually, it is a difficult task to acquire information on real data and graphs of financial institutions. Thus, in the absence of real data and graphs of the reserve requirement ratio, we provide simulations of the current reserve ratio for a central bank listed in [30]. For example, the graphs of Figures 4.6 and 4.7 were generated from the parameters collected in Table (3) and may, for instance, represent a simulated trajectory of China's central bank reserve ratio with its initial value at 17.00%. Last year, the central bank of China's reserve ratio was at 18.50% which indicates a significant drop of 1.50%. This move by the aforementioned bank was an attempt to cushion its economic slowdown amid plunging stock prices and a weakening currency. China's reserves have been declining steadily since 2014. China's central bank, PBOC, has cut its benchmark loan interest rate and reserve requirement ratio a number of times. The most recent reduction in the reserve requirement ratio occurred on 29 February 2016 (see [32]). This was to spur on the sluggish growth of China's economy. According to some observers, PBOC has become extremely anxious about money currently evaporating from Chinese stocks. In an attempt to remedy the situation, PBOC removed funds from the country's financial system to take some heat out of the country's volatile stock market by preventing banks from channeling funds into the stock market. This strategy, in turn, has backfired since the equity market has plunged (see [31]). Some economists believe that this will signal to stock investors that PBOC will tighten its grip on the monetary policy in order to deal with the plummeting market.

Based on the aforementioned observations, Figures 4.6 and 4.7 indicates that lowering the reserve requirement increases the commercial bank's availability of funds to make more loans, thus tending to expand the money stock and to lower interest rates and encourage both consumers and investors to buy more. Once China's economic growth is healthy whereby the central bank does not have to take drastic measures, it can start raising the reserve requirement which will restricts the bankers' ability to make more loans, and those banks that were already operating just barely above the old reserve requirement will probably be forced to call in some of their existing loans to meet the tougher new requirement, thus tending to shrink the money stock, raise interest rates, and thus reduce the volume of purchasing on credit in the economy as a whole.

Chapter 5

An Optimal Strategy for Liquidity Management in Banking

In situations when a financial firm starts raising voluntary savings and using those deposits to finance an asset portfolio, the liquidity and asset-liability management of that institution becomes more complex. For instance, a commercial bank not only has to manage the fluctuating demand and varying interest rates and terms on loans, but it also has to deal with erratic deposit demands, withdrawals, changing interest rates and terms of savings. The management of liquidity should ensure that the commercial bank maintains sufficient cash and liquid assets to satisfy client demand for loans and savings withdrawals, and to pay the bank's expenses. Liquidity refers to the ability of an institution to meet immediate financial obligations. Bank assets should be readily and easily convertible into cash to finance these demands. Daily analysis and detailed estimation of the size and timing of cash inflows and outflows over the coming days and weeks are performed to minimize the risk that savers will be unable to access their deposits in the moments they demand them. In order for a commercial bank to make liquidity projections and realistic growth, certain information is needed such as the history of deposit and loan inflows and outflows and overall daily cash demands to determine the amount of cash that needs to be kept on-site and in demand deposit type accounts. A liquidity shortage can cause great damage to a bank. In particular, a liquidity crisis can result in client confidence to deteriorate over time. The financial crisis caused the banking system in the United States to collapse for the first time, raising fundamental questions about liquidity risk. The global financial system experienced urgent demands for cash from various sources, including counterparties, short-term creditors, and especially, existing borrowers. Central bank emergency lending programs probably mitigated the decline. During the financial crisis, banks that were more exposed to liquidity risk increased their holdings of liquid assets. Liquidity exposure affected behavior along several dimensions. On the asset side, banks holding securities with low liquidity, such as mortgage-backed securities, expanded their cash buffers during the crisis and decreased new lending. Such banks were worried about their ability to finance securitized assets. They protected themselves by hoarding liquidity, to the detriment of borrowers. On the liability side, banks that relied more on wholesale sources of funding, cut new lending significantly more than banks that relied predominantly on traditional deposits and equity capital for funding.

In order to provide some sort of relief for internationally active banks, in December, 2010 the Basel Committee on Banking Supervision (BCBS) issued the Basel III: International framework for liquidity risk measurement, standards, and monitoring. Although Basel II regulation established procedures for assessing credit, market, and operational risk, it did not provide effective protocols for managing liquidity and systemic risks.

Current liquidity risk management procedures can be classified as micro- or macro-prudential. In the case of the former, simple liquidity ratios such as credit-to-deposit ratios (net stable funding ratio), liquidity coverage ratios and the assessment of the gap between short-term liabilities and assets are appropriate to cover the objectives of bank balance sheet analysis. The ratio approach for liquidity risk management is a quantitative internationally accepted standard for alerting banks about any possible adverse economic downturns. In this paper, we mainly focus on the liquidity coverage ratio, which is designed to ensure that commercial banks have the necessary assets on hand to ride out short-term liquidity disruptions. Banks are required to hold an amount of highly-liquid assets, such as cash or Treasury bonds, equal to or greater than their net cash over a 30 day period (having, at least, 100% coverage). The liquidity coverage ratio (LCR) started to be regulated and measured in 2011, but the full 100% minimum would not be enforced until 2015 . The LCR is defined as

$$\text{LCR} = \frac{\text{Total Stock of High-Quality Liquid Assets (HQLAs)}}{\text{Total Nett Cash Outflows (TNCOF) over the next 30 calendar days}}$$

and it is required by BCBS that LCR ≥ 1. The HQLAs in (5.1) refers to the stock of unencumbered (not pledged) high quality liquid assets banks must hold to cover the total net cash outflows over a 30-day period ([13]; [12]; [11]; [8] and [7]). The implementation is intended to favor those assets that are counted as liquid, and at the same time reduce incentives to hold assets that are considered less liquid. The Committee has liberalized the definition of what counts as a liquid asset in their liquidity framework. The high-quality liquid assets can be divided into two categories, namely; Level 1 assets and Level 2 assets. Level 1 assets (L1As) includes cash, central bank reserves, and Government bonds with 0% risk weight under Basel II. Level 2 assets (L2As) mainly comprises government bonds with a 20% risk weight under Basel II and at least AA-rated corporate bonds (issued by a non-bank), and covered bonds which have a proven track record as a reliable source of liquidity (repo or sale) in the capital market. L2As is further categorized into Level 2A assets (L2AAs) and Level 2B assets (L2BAs). L2AAs are subjected to a 15% haircut while L2BAs are subjected to a 50% haircut. The latter assets include corporate debt securities, unencumbered equities, and residential mortgage-backed securities. L2As are limited to a maximum of 40% of the overall liquid asset pool when computing the LCR. In other words, the quantity of L2As included in the calculation of HQLA can be at most 2/3 of the quantity of L1As. In addition, A 15% haircut is applied to the current market value of each Level 2A asset held in the stock of HQLA.(see [8]).

The reason why there are so many haircuts on L2As is to ensure that the majority of a banking organizations' HQLAs consist of Level 1 assets. The amount of Level 1 assets thus acts as a constraint on the recognition of Level 2 assets as HQLAs. Based on the description for HQLAs above, a bank's stock of HQLAs can then be written as

$$\text{HQLAs} = \text{L1A} + \min\left(0.85 \times \text{L2A}, \frac{2}{3} \times \text{L1A}\right). \tag{5.1}$$

In the denominator of the LCR are the TNCOF. TNCOF over a 30-day time period are determined by the total expected cash outflows minus total expected cash inflows and reflect the net amount of funding that may not be realized within the 30 days under a stress scenario. Total expected cash outflows are calculated by multiplying the outstanding balances of various types of liabilities and off-balance sheet commitments by rates at which they are expected to run off or be drawn down (i.e. the amount of funding maturing in the 30-day period that will not be rolled over). Total expected cash inflows are calculated by multiplying the outstanding balances of various categories of contractual receivables by the rates at which they are expected to flow in under the scenario up to an aggregate cap of 75% of total expected cash outflows. The aforementioned haircut prevents bank's from relying solely on these inflows for its liquidity. Thus, it ensures that a bank holds a minimum stock of HQLAs equal to 25% of cash outflows. Symbolically, this means

$$\text{Total Expected Cash Inflows} \leq 0.75 \times \text{Total Expected Cash Outflows}.$$

The NCOF can be calculated as

$$\text{NCOF} = \text{Outflows} - \min\left(\text{Inflows}, 0.75 \times \text{Outflows}\right). \tag{5.2}$$

The formulas for HQLAs and NCOF in (5.1) and (5.2), respectively were obtained from [13].

Our contribution has connections with [38]; [40]; [49]; [51]; [68] and [47].

In the paper [38], the authors explore the relationships between Shareholder Cash Flow Rights (SCFRs), capital stability and liquidity via the Net Stable Funding Ratio (NSFR) and LCR, respectively. In particular, they investigate the effects of shareholder cash flow rights on the aforementioned funding ratio and a non-Basel III liquidity coverage ratio for certain developing countries during the period 2005 Q1 to 2009 Q4. The working paper

[40] examines large capital injections by U.S. financial institutions from 2000 to 2009. These infusions include private as well as government cash injections under the Troubled Asset Relief Program (TARP). The sample period covers both business cycle expansions and contractions, and the recent financial crisis. Elyasiani et. al. [40] show that more financially constrained institutions were more likely to have raised capital through private-market offerings during the period prior to TARP, and firms receiving a TARP injection tended to be riskier and more levered. In the case of TARP recipients, they appeared to finance an increase in lending (as a share of assets) with more stable financing sources such as core deposits, which lowered their liquidity risk. However, in [40] no evidence is found that bank's capital adequacy increased after the capital injections. In the paper [49] the authors discuss liquidity risk management for banks. In particular, their analysis under the Basel III paradigm suggests that overall liquidity risk is best measured using ratio analysis approaches such as the LCR. Their proposition is justified by numerical results which show that bank behavior related to liquidity was highly procyclical during the global financial crisis. The paper [51] provides a comprehensive analysis to calculate the Basel III liquidity coverage ratio (LCR) and the net stable funding ratio (NSFR) of United States commercial banks. Part of their analysis include Call Report data over the period $2001 - 2011$. Their finding suggests that systematic liquidity risk was a major contributor to bank failures in 2009 and 2010.

In [68], actuarial methods are used to solve a nonlinear stochastic optimal liquidity risk management problem for subprime originators with deposit inflow rates and marketable securities allocation as controls. The main objective is to minimize liquidity risk in the form of funding and credit crunch risk in an incomplete market. In order to accomplish this, they construct a stochastic model that incorporates originator mortgage and deposit reference processes. Gideon et al. [47] investigate the Net Stable Funding Ratio, which is one of the liquidity measures, under the Basel III framework. By considering the inverse net stable funding ratio as a measure to quantify the bank's prospects for a stable funding over a one year period, the authors consider an optimal liquidity problem related to the inverse net stable funding ratio whereby optimal choices are made for the inverse net

stable funding targets in order to formulate its cost. The latter was achieved by finding an analytical solution for the value function.
In [41] the authors investigate portfolio optimization problems that consist of maximizing expected terminal wealth under the constraint of an upper bound for the risk. The paper [57] considers maximizing the expected utility from consumption or terminal wealth in a market where logarithmic securities prices follow a Lévy process. The aforementioned authors derive explicit solutions for different utility functions. As a consequence of this approach, the intrinsic risk of the bank arises now not only from the reserve portfolio but also from the deposit withdrawals.

In our contribution issues related to bank liquidity management are addressed in a jump diffusion setting. Our paper has some close connections with the aforementioned literature in the sense of establishing optimal liquidity and a rate of depository consumption that is of importance during a (random) auditing process of the reserve requirements. In particular, we investigate the interplay between a commercial bank and a central bank and how this affects the money supply between the two institutions as well as the LCR. The main motivation for studying the dynamics of LCR is to show that, in principle, banks are able to control their liquidity via an appropriate provisioning strategy. This should ensure that the said ratio does not move below an acceptable level.

The layout of the rest of the section is as follows. In Section 5.1, we extend some of the modeling and optimization issues highlighted in [20] by presenting jump diffusion models for various assets (see also [41] and [57] where asset prices are generally modeled as Lévy processes). The stochastic dynamics of high-quality liquid assets and bank liabilities are presented in Section 5.1.1 and 5.1.2, respectively. In Section 5.1.3 we derive the dynamics of liquid assets and net cash flows. In Section 5.1.4, we formulate a deposit withdrawal problem and solve it via a stochastic control technique which can be found in [72]. In Section 5.2 we use the results obtained in Section 5.1.4 to generate numerical simulations of the Liquidity Coverage Ratio. In addition to the solution of the problem outlined above, we offer a few concluding remarks in Section 6.3.

5.1 The Stochastic Banking Model

To understand the operation and management of banks, we have to study its balance sheet, the items of which are unpredictable and uncertain due to activities related to the evolution of treasuries, loan demand, risky and riskless investments, deposits, loan repayments, borrowings and bank regulatory capital.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathcal{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$ which is right continuous, for some finite T which denotes the investment time horizon. The space Ω represents the different states of the economy in which banks operate. The information available to banks up until time t is represented by $\mathcal{F}_t$. All random variables considered in this paper are defined on this space. Furthermore, we consider a commercial bank that enters a financial market which is subject to uncertainty, through the components of a 4−dimensional Brownian motion

$$Z(t) \equiv (Z_0(t,\omega), Z_a(t,\omega), Z_b(t,\omega), Z_c(t,\omega))', \ t \geq 0, \ \omega \in \Omega,$$

with pairwise independent coordinates, and a 1−dimensional Poisson process

$$N(t) = N(t,z), \ t \geq 0, \ z \geq 0.$$

For some constant $-1 < \rho < 1$, we define $Z_1(t)$ and $Z_2(t)$ as

$$Z_1(t) = \rho Z_0(t) + \sqrt{1-\rho^2} Z_a(t) \text{ and } Z_2(t) = \rho Z_0(t) + \sqrt{1-\rho^2} Z_b(t),$$

respectively. The processes $\{Z_1(t), 0 \leq t \leq T\}$ and $\{Z_2(t), 0 \leq t \leq T\}$ are correlated. Note that $dZ_1(t)dZ_2(t) = \rho^2 dt$.
The martingale $\tilde{N}(t) = N(t) - \lambda t$ is called the compensated Poisson process of $N(t)$, where λ is the vector of intensities of $N(t)$. We point out that λ can be time-dependent. In this paper we only consider *homogeneous* Poisson processes, that is, with λ being constant over time.

Bank capital plays an important role because it balances assets and liabilities by the relation

$$\text{Total Assets} = \text{Total Liabilities} + \text{Bank Capital}. \tag{5.3}$$

A typical commercial bank's asset portfolio at time t can be decomposed into many assets. A useful way of representing the balance sheet of the bank is as follows:

$$T_0(t) + R(t) + S(t) + L(t) = D^S(t) + D^L(t) + F^U(t) + B^I(t) + E(t)$$

where we have T_0, R, S, L, D^S, D^L, F^U, B^I and E represent the market value of cash, reserves, marketable securities, loans, stable retail deposits, less stable retail deposits, unsecured wholesale funding, interbank borrowing and equities, respectively.

5.1.1 Bank Assets

Assets are considered to be high-quality liquid assets (HQLAs) if they can be easily and immediately converted into cash at little or no loss of value (for instance coins, bank notes, reserves, marketable securities and sovereign or central bank debt securities). There are two categories of assets that can be included in the stock. Assets to be included in each category are those that the bank is holding on the first day of a stress period, irrespective of their residual maturity. Level 1 assets (L1As) can be included without limit, while Level 2 assets (L2As) can only comprise up to 40% of the stock.

Level 1 Assets

The first component of stock of high-quality liquid assets is cash that is made up of banknotes and coins. Central Bank (CB) reserves should be able to be drawn down in times of stress ([8]). In this regard, local supervisors should discuss and agree with the relevant CB the extent to which CB reserves should count toward the stock of liquid assets.

The dynamics of the riskless asset (Cash) is given by

$$dS_1(t) \;=\; S_1(t)r^A dt, \quad S_1(0) = S_1 > 0, \tag{5.4}$$

where $r^A > 0$ represents the constant riskless interest rate.

Bank reserves are the deposits held in accounts with the central bank of a country (for instance, the South African Reserve Bank in the case of South Africa) plus money that is physically held by banks (vault cash). Such reserves constitute money that is not lent out but is earmarked to cater for withdrawals by depositors. Since it is uncommon for depositors to withdraw all of their funds simultaneously, only a portion of total deposits may be needed as reserves. It is the amount of money a bank sets aside and does not lend, to meet day-to-day currency withdrawals by its customers. We note that cash in the Automated Teller Machines (ATMs) network also qualifies as required reserves. The investment of bank reserves in the market via bonds and stocks is still possible in many countries. Bank reserves may actually have a stochastic nature and banks may earn a positive return on them. For instance, as reported in [75] by the end of 2013, China had over \$ 3.82 trillion in foreign exchange reserves. A significant proportion of these reserves was invested in U.S. treasury securities since treasuries are seen as a safe investment. Based on the above description, we may represent bank reserves as (see [20])

$$\begin{aligned} dR(t) &= R(t^-)\Big\{[r^{R_1}(t) - f^{R_1}(t)]dt + \sigma dZ_c(t)\Big\} \\ &+ R(t^-)\Big\{\int_{-1}^{\infty} x(t^-, z)\tilde{N}(dt, dz)\Big\} - c(t)dt, \quad R(0) = r > 0, \end{aligned} \tag{5.5}$$

where $r^{R_1}(t) > 0$ is the rate of (positive) return on reserves earned by the bank, $f^{R_1}(t) > 0$ is the fraction of bank reserves that are available for withdrawal, and $\sigma > 0$ is the volatility in the level of reserves. The size of the jumps is denoted by $x(t^-, z)$ and we assume that $x(t^-, z) \geq -1$, ensuring that the jumps are not too large negative. Also, in order to make provisions for deposit withdrawals, $c(t)$, it is required of a bank to make rational decisions about the deposit taking (see [20]).

Level 2 Assets

In this section, the L2As consists mainly of marketable securities. Marketable securities are debt instruments that can be easily converted to cash such as government bonds, corporate bonds, common stock or certificates of deposit. These instruments are very liquid as they tend to have maturities of less than one year.

The dynamics of the marketable security price (see [23]) is assumed to be given by

$$\frac{dS(t)}{S(t)} = (r_S(t) + \lambda_S)dt + \sigma_S dZ_1(t), \tag{5.6}$$

where σ_S is the security volatility and λ_S denotes the risk premium. Under the Capital Asset Pricing Model (CAPM), λ_S could be quantified by the relation $\lambda_S = \beta[E[R_m] - r^A]$ with $E(R_m)$ representing the market expected return and β the sensitivity of the expected excess asset returns to the expected excess market return. The dynamics of the interest rate $r_S(t)$ is given by

$$dr_S(t) \;\; = \;\; \theta_{r_S}(\mu_{r_S} - r_S(t)))dt + \sigma_{r_S} dZ_2(t), \tag{5.7}$$

where θ_{r_S} is the rate of reversion, μ_{r_S} is the long-run mean and σ_{r_S} is the volatility which are all positive constants.

5.1.2 Liabilities

Deposits

A commercial bank creates credit or makes loans, and holds reserves (to satisfy demands for withdrawals) that are less than a number of its customers' deposits. In general, funds that are deposited at a bank, are mostly lent out to customers or financial institutions. Commercial banks keep a fraction (known as a reserve-deposit ratio) of those funds as reserves to cover its customer deposit liabilities. Central banks or other banking regulators often mandate the aforementioned reserve requirements in order to limit the amount of money creation that occurs in the commercial banking system and to ensure that banks have enough ready cash to meet normal demand for withdrawals. In our research, the term deposits include both demand and time deposits. The value of a deposit is ultimately dependent on its stability or likelihood of not being withdrawn. The longer time it will remain in the bank the longer time the bank can lend out that money and, as a long loan is worth more than a short one, charge a higher price for it. Thus, deposits have uncertainty associated with them and thus can be modeled as a stochastic process. Let $r^{DD} : T \to \mathbb{R}_+$ denote the rate of demand deposit which is payable on demand and $r^{TD} : T \to \mathbb{R}_+$ the rate of time deposit which is payable only after a fixed interval of time. Retail deposits are categorized into two types of deposits namely, stable and less stable deposits. Stable deposits are deposits which are fully insured by an effective deposit insurance scheme (i.e. up to the maximum coverage limit the deposit insurance scheme) or by a public guarantee. The aforementioned deposits needs to satisfy certain criteria. These requirements are as follows (see [13] or [8]):

- depositors have one or more established relationships with the banking institution which has existed for at least 12 months; or
- deposits are in transactional accounts, which refer to accounts which are regularly credited or debited (e.g. accounts where salaries are automatically deposited). This refers to a depositor who is contractually bounded to a bank institution for at least the next 12 months.

Retail deposits that does not satisfy the criteria outlined above are referred to as less stable deposits. The dynamics of deposits can be written as a diffusion process (see, for instance, [36]; [50]; [60] and [61]). In our article we model stable deposits as

$$dD^S(t) \;=\; m_s dt + \sigma_s dZ_c(t), \quad D^S(0) = d_s \in \mathbb{R}_+, \tag{5.8}$$

where $m_s = r^{DD} + r^{TD}$ and less stable deposits as

$$dD^L(t) \;=\; m_l dt + \sigma_l dZ_c(t) + \alpha \int_{-1}^{\infty} x(t^-, z)\tilde{N}(dt, dz), \; D^L(0) = d_l \in \mathbb{R}_+, \tag{5.9}$$

where $0 \le \alpha \le 1$ is a constant. The terms m_s and m_l represents the expected rate of growth in deposits, σ_s and σ_l the level of volatility in deposits and the jump term captures the unexpected information (such as bank solvency) that will have an influence in deposit taking and deposit rates.

Unsecured wholesale funding

Banks, as well as commercial finance companies, can both be users of wholesale funding. Since both institutions are regulated differently, their daily operational activities are very distinct. Commercial finance companies solely provide business loans whereas banks provide both business and consumer loans. Wholesale funding refers to a method used by banks in addition to core demand deposits in order to finance operations and managing risk. It includes federal funds, foreign deposits and brokered deposits. Historically, banks used core demand deposits as a source of funding since they are an inexpensive source of financing. Recently, banks have turned to wholesale funding as a way of expanding funding needs. One of the reasons why banks use wholesale funding is to attract new deposits, especially from brokered deposits. In this paper, we model wholesale funding as follows (see [86] where liability is modelled in a similar fashion):

$$dF^U(t) \;=\; m_f dt + \sigma_f dZ_c(t), \quad F^U(0) = d_f \in \mathbb{R}_+, \tag{5.10}$$

where m_f represents the rate at which deposits are received through a broker who takes their wealthy clients' money and finds several different banks in which to deposit it (known as brokered deposits). The parameter σ_f is the level of volatility in these deposits.

Borrowing

Banks usually borrow money from each other in the interbank market. There is an interest rate charged on short-term loans made between banks. It is known as the interbank rate. Banks borrow and lend money in the interbank market in order to manage liquidity and meet the requirements placed on them. Thus, it is required of banks to hold an adequate amount of assets to manage potential withdrawal. If a bank is unable to satisfy this liquidity requirement, it will need to borrow money in the interbank market to cover the shortfall. Banks have to hold a percentage of their deposits with the central bank every night. In the event that a bank is short on cash at a given time, it needs to borrow funds from the central bank at a certain rate. This rate is known as the federal funds rate.

Models of borrowing funds can be found in [68]. Here we model borrowing funds as

$$\frac{dB^I(t)}{B^I(t)} = m_B dt + \sigma_B dZ_c(t), \quad B^I(0) = b_I \in \mathbb{R}_+, \tag{5.11}$$

where $m_B \in \mathbb{R}$ is the interbank rate and $\sigma_B > 0$ is the volatility in the funds being borrowed.

5.1.3 Dynamics of Liquid assets and Net Cash Flows

In this section we mainly focus on deriving the dynamics of L1As, L2As, the bank's total expected cash inflows and total expected cash inflows, respectively. The reason for this is to numerically simulate HQLAs and NCOF given by (5.1) and (5.2), in order to characterize the behavior of LCR.

The evolution of the L1As, A_{L_1}, and L2As, A_{L_2}, are given by

$$\begin{aligned} dA_{L_1}(t) &= \frac{dS_1(t)}{S_1(t)} + dR(t) \\ &= R(t^-)\Big\{[r^{R_1}(t) - f^{R_1}(t)]dt + \sigma dZ_c(t)\Big\} \\ &+ R(t^-)\Big\{\int_{-1}^{\infty} x(t^-, z)\tilde{N}(dt, dz)\Big\} + \Big(r^A - c(t)\Big)dt, \end{aligned} \tag{5.12}$$

and

$$dA_{L_2}(t) = 0.85 \times \frac{dS(t)}{S(t)} = 0.85 \times \Big((r_S(t) + \lambda_S)dt + \sigma_S dZ_1(t)\Big). \tag{5.13}$$

In our case, we assume that the bank's total expected cash inflows, $\Upsilon(t)$, comprises of maturing secured lending backed by Level 1, $l_{M_1}(t)$, and Level 2, $l_{M_2}(t)$, assets as collateral, with the dynamics of the maturing secured lending backed Level 1 and Level 2 assets being described respectively by the equations $(i = 1, 2)$

$$\frac{dl_{M_i}(t)}{dl_{M_i}(t)} = \mu_{M_i} dt + \sigma_{M_i} dZ_i(t). \tag{5.14}$$

In this case the dynamics of the total expected cash inflows is expressed as

$$\begin{aligned} d\Upsilon(t) &= 0 \times \frac{dl_{M_1}(t)}{dl_{M_1}(t)} + 0.15 \times \frac{dl_{M_2}(t)}{dl_{M_2}(t)} \\ &= 0.15\mu_{M_2} dt + 0.15\rho\sigma_{M_2} dZ_0(t) + 0.15\sigma_{M_2}\sqrt{1-\rho^2}dZ_b(t) \end{aligned} \tag{5.15}$$

while the dynamics of the total expected cash outflows, $\Lambda(t)$, is given by

$$\begin{aligned} d\Lambda(t) &= \theta_1 dD^S(t) + \theta_2 dD^L(t) + \theta_3 dF^U(t) + \theta_4 dB^I(t) \\ &= \Big(\theta_1 m_s + \theta_2 m_l + \theta_3 m_f + \theta_4 m_B B^I(t)\Big) dt \\ &+ \Big(\theta_1 \sigma_s + \theta_2 \sigma_l + \theta_3 \sigma_f + \theta_4 \sigma_B B^I(t)\Big) dZ_c(t) \\ &+ \theta_2 \alpha \int_{-1}^{\infty} x(t^-, z) \tilde{N}(dt, dz). \end{aligned} \tag{5.16}$$

In (5.16) θ_1, θ_2, θ_3 and θ_4 are the run-off rate for stable retail deposits, less stable retail deposits, unsecured wholesale funding and overnight interbank borrowing taken to be 7.5%, 15%, 75% and 100%, respectively (see [13]).

5.1.4 The Deposit Withdrawal Problem

In this chapter, we suppose that the bank's behavior towards risk is described by a logarithmic or power utility, respectively. The bank is given full access to withdraw a certain amount of money from the cash vault to satisfy daily demands by depositors. Here, our goal is to minimize deposit risk (a type of liquidity risk) arising from cash outflows of a commercial bank which is caused by changes in depositors' behavior. In order to address the aforementioned problem, we define the performance criterion as follows:

$$J(s, \tau, R; c) = \mathbb{E}\Big[\int_0^{\tau} \exp(-\delta(s+t)) U(c(t))\, dt\Big], \tag{5.17}$$

where $U(c(t))$ is a utility function with $\delta > 0$. The time $\tau = \inf\{t > 0; R(t) \leq 0\}$ represents the time until bankruptcy occurs. The class $\mathcal{A}$ of admissible control laws is defined as follows:

$$\mathcal{A} = \Big\{ c(\cdot) : \ c \text{ is bounded and adapted so that } R(t) > 0 \text{ for all } t \geq 0 \text{ a.s.} \Big\}. \tag{5.18}$$

We define the value function as follows

$$V(s, R) = \sup_{c(\cdot) \in \mathcal{A}} J(s, \tau, R; c).$$

Problem 5.1.1. (Optimal deposit withdrawal problem): Consider the SDE of the form *(5.5)*. We attempt to find the supremum

$$V(s,R) \;=\; \sup_{c(\cdot)\in\mathcal{A}} J(s,\tau,R;c)$$

and the optimal control law, which is given by

$$c^*(t) = \arg\sup_{c\in\mathcal{A}} J(s,\tau,R;c) \in \mathcal{A}, \quad \text{so that} \quad V(s,R) = J(s,\tau,R;c^*). \tag{5.19}$$

The optimal rate of deposit flow, $c^*(t)$, will be determined in Proposition (5.1.1) and Proposition (5.1.3) (see [20] for the infinite horizon case of a bank auditing problem). The role of $R(t)$ should be that it is readily available to satisfy customer withdrawals or transfer to other banks as customers write checks.

It is well understood that this withdrawal should not exceed the cash reserve ratio. The required reserve ratio is sometimes used as a tool in monetary policy, influencing the country's borrowing and interest rates by changing the amount of funds available for banks to make loans with. Commercial banks rarely alter the reserve requirements because it would cause immediate liquidity problems for banks with low excess reserves; they generally prefer to use open market operations (buying and selling government-issued bonds) to implement their monetary policy.

We solve the optimization problem by way of the following two results. We follow an optimization method as discussed in [72]. In fact, the proof of the following Proposition is like ([72], Exercises 3.6)

Proposition 5.1.1. *Suppose that the dynamics of the bank vault cash holdings is described as (5.5) and the value function is characterized by (5.19) where* $U(c(t)) = \ln(c(t))$*. Then the rate of currency outflow from the vault cash holdings is given by*

$$c^*(R(t)) = \hat{c} = \frac{1}{a}R(t). \tag{5.20}$$

Proof.

As a candidate for the value function V, let us test a function of the form

$$\begin{aligned}\varphi(s,R) &= \exp(-\delta s)\chi(R), \text{ with } \chi(R) = a\ln R + b \text{ for some } a>0,\ b>0 \\ &= \exp(-\delta s)(a\ln R + b).\end{aligned}$$

In this case the Hamilton-Jacobi-Bellman equation becomes

$$\begin{aligned}\sup_{c(t)>0}\Bigg\{\ln c - \delta\chi(R) + \Big(R[r^{R_1} - f^{R_1}] - c\Big)\chi^{'}(R) + \frac{1}{2}(\sigma)^2R^2\chi^{''}(R) \\ + \int_{-1}^{\infty}\{\chi(R + Rx(t^-,z)) - \chi(R) - Rx(t^-,z)\chi^{'}(R)\}\nu(dz)\Bigg\} = 0.\end{aligned} \tag{5.21}$$

In order to maximize the relevant entity above with respect to $c(t)$, we must have its partial derivative with respect to $c(t)$ vanishing. This gives us:

$$\begin{aligned}\frac{1}{c} - \chi^{'}(R) = 0, \\ \text{i.e. } c = \frac{1}{\chi^{'}(R)} = \frac{R}{a}.\end{aligned} \tag{5.22}$$

Substituting the expressions for c, $\chi(R)$ and $\chi^{'}(R)$ into (5.21) yields, after some simplification:

$$\begin{aligned}(1-\delta a)\ln R - \ln a - \delta b + [r^{R_1} - f^{R_1}]a - 1 - \frac{1}{2}\sigma^2 a \\ + a\int_{-1}^{\infty}\{\ln(1 + x(t^-,z)) - x(t^-,z)\}\nu(dz) = 0.\end{aligned}$$

The aforementioned expression is possible only if $a = \frac{1}{\delta}$. In that case we obtain:

$$\begin{aligned}-\ln\frac{1}{\delta} - \delta b + [r^{R_1} - f^{R_1}]\frac{1}{\delta} - 1 - \frac{1}{2}\sigma^2\cdot\frac{1}{\delta} \\ + \frac{1}{\delta}\int_{-1}^{\infty}\{\ln(1 + x(t^-,z)) - x(t^-,z)\}\nu(dz) = 0,\end{aligned}$$

and the latter yields a solution for b:

$$b = \frac{1}{\delta^2}\left[\delta\ln\delta + [r^{R_1} - f^{R_1}] - \delta - \frac{\delta^2}{2} + \int_{-1}^{\infty}\{\ln(1 + x(t^-,z)) - x(t^-,z)\}\nu(dz)\right].$$

With these values for a and b, we can conclude that

$$\begin{aligned}\varphi(s,R) &= \exp(-\delta t)(a\ln R + b) \\ &= V(s,R)\end{aligned} \tag{5.23}$$

and that

$$c^*(R(t)) = \frac{R(t)}{a}$$

is the optimal rate of currency flow from the vault cash holdings. $\square$

Remark 5.1.2. Similarly as in [72], we can prove the following proposition, and we omit the proof.

Proposition 5.1.3. *Suppose that the dynamics of the bank vault cash holdings is described as (5.5) and the value function is characterized by (5.19) with* $U(c(t)) = \dfrac{c^\gamma(t)}{\gamma}$ *and* $\gamma \in (0,1)$. *Then the rate of currency outflow from the vault cash holdings is given by*

$$c^*(R(t)) = K^{\frac{1}{\gamma-1}} \gamma^{\frac{1}{\gamma-1}} R(t), \tag{5.24}$$

provided that

$$\begin{aligned} K &= \frac{1}{\gamma}\Bigg[\frac{1}{1-\gamma}\Bigg(\delta - \bigg(\Big[r^{R_1}(t) - f^{R_1}(t)\Big]\bigg)\gamma - \frac{1}{2}(\sigma)^2\gamma(\gamma-1) \\ &- \int_{-1}^{\infty}\Big\{(1+x(t^-,z))^\gamma - 1 - x(t^-,z)\gamma\Big\}\nu(dz)\Bigg)\Bigg]^{\gamma-1}. \end{aligned}$$

In next section, we describe how the optimal rates of currency flow in Proposition 5.20 and Proposition 5.24 affect the LCR.

5.2 Numerical Example Involving Liquidity Coverage Ratio

In this section, we provide numerical simulations in order to characterize the behavior of the LCR. The behavior of the LCR will be influenced by the expressions of c^* given by (5.20) and (5.24), respectively. These two cases can be refered to as log-utility or logarithmic case and power utility, respectively

The instantaneous interest rate dynamics of $r^{R_1}(t)$ and $f^{R_1}(t)$ are modelled as mean reversion processes. That is,

$$dr^{R_1}(t) = (\kappa - r^{R_1}(t))dt + \sigma_r\sqrt{r^{R_1}(t)}dZ_c(t) \quad (5.25)$$

and

$$df^{R_1}(t) = \beta_1(\eta - f^{R_1}(t))dt + \sigma_{f_{R_1}}dZ_c(t), \quad (5.26)$$

respectively.

Here $\kappa, \beta_1, \eta, \sigma_{f_{R_1}}$ and σ_r are all positive constants. In each case, the aforementioned parameters correspond to the degree of mean reversion, long-run mean, and volatility of the interest rate.

In the discussion that follows, the computations are done over T = 30 days. We denote HQLAs and NOCF at time t by $H_Q(t)$ and $\Theta(t)$, respectively. The initial values for HQLAs and TNCOF are given by $H_Q(0) = 316.6667$ and $\Theta(0) = 300.2500$, respectively. Thus, $LCR(0) = H_Q(0)/\Theta(0) = 1.0547$. In order to obtain the aforementioned values, we consider the following parameters and initial conditions: $B^I(0) = 80,\ \Upsilon(0) = 6,\ \Lambda(0) = 306.25,\ A_{L_1}(0) = 190,\ A_{L_2}(0) = 150,\ r^{R_1}(0) = 0.06,\ f^{R_1}(0) = 0.02,\ \sigma_f = 0.011,\ \sigma_r = 0.05,\ \kappa = 5,\ \eta = 6,\ \sigma_r = 0.01,\ r^A = 0.03,\ r_S(0) = 0.05,\ \beta = 0.75,\ E(R) = 0.1,\ \lambda_S = 0.0525,\ \sigma_S = 0.07,\ r^{TD} = 0.06,\ r^{DD} = 0.06,\ m_s = 0.12,\ m_l = 0.04,\ m_f = 0.05,\ m_B = 0.02,\ \sigma_s = 0.03,\ \sigma_f = 0.02,\ \sigma_B = 0.01,\ a = 7,\ \mu_{M_2} = 0.04,\ \theta_{r_s} = 3,\ \mu_{r_s} = 4,\ \sigma_{f_{R_1}} = 0.011,\ \sigma_{r_S} = 0.05,\ \beta_1 = 2, \gamma = 0.4,\ \delta = 1/3,$ and $\sigma_{M_2} = 0.03$. Figures (5.1) - (5.8) characterize the behavior of the LCR when the currency outflow rate is given by (5.20) and (5.24), respectively. Liquid assets (such as marketable securities) are traded on the open market. These assets are very popular amongst investors due to its high liquidity. Also, the assets are influenced by factors such as the interest rate on the open market. From Figures (5.1) - (5.8), we observe how the assets in LCR are influenced by the parameter ρ. For example if the interest rate increases the movement of the liquid asset prices generally increases and vice versa.

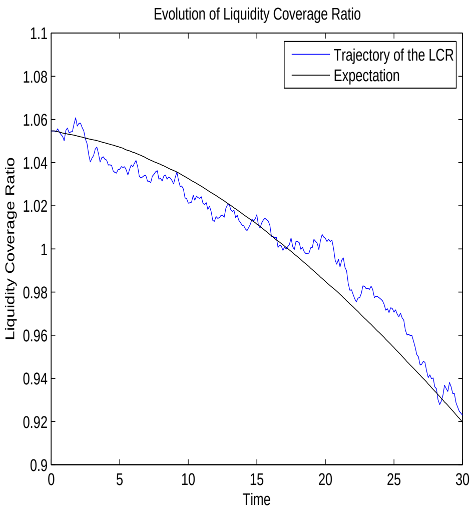

Figure 5.1: Simulation of LCR with c^* given by (5.20) when $\rho = -0.95$.

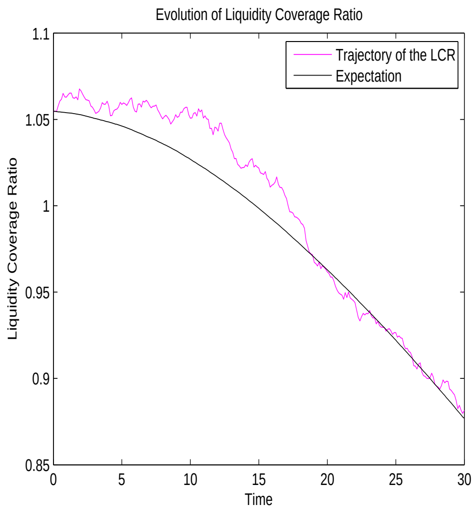

Figure 5.2: Simulation of LCR with c^* given by (5.24) when $\rho = -0.95$.

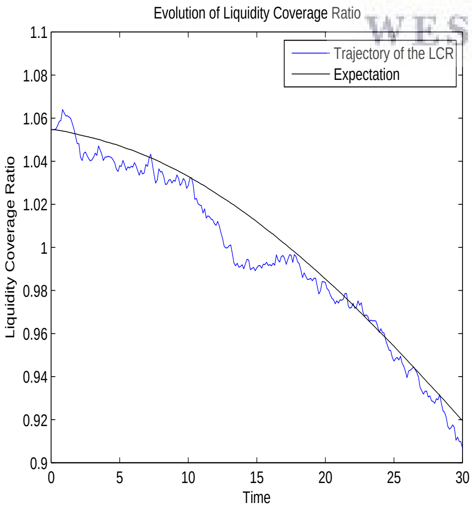

Figure 5.3: Simulation of LCR with c^* given by (5.20) when $\rho = -0.05$.

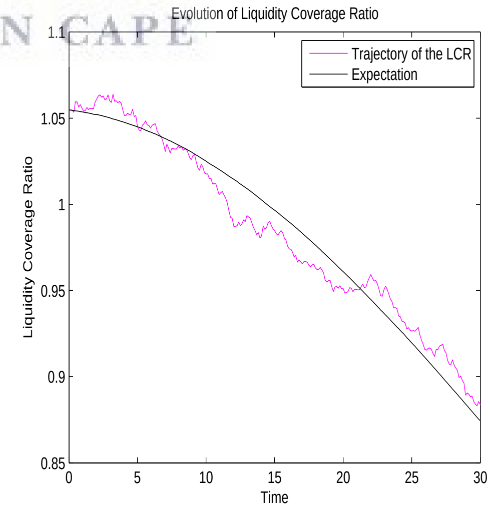

Figure 5.4: Simulation of LCR with c^* given by (5.24) when $\rho = -0.05$.

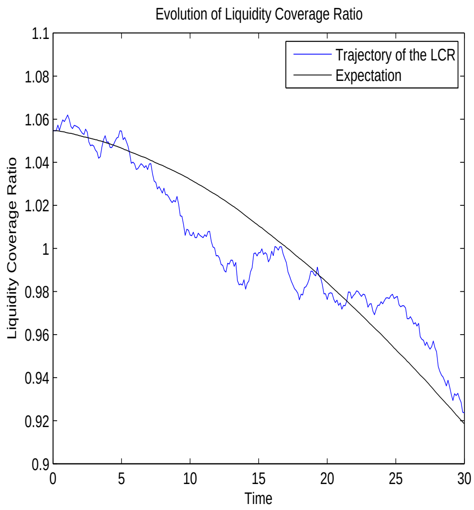

Figure 5.5: Simulation of LCR with c^* given by (5.20) when $\rho = 0.05$.

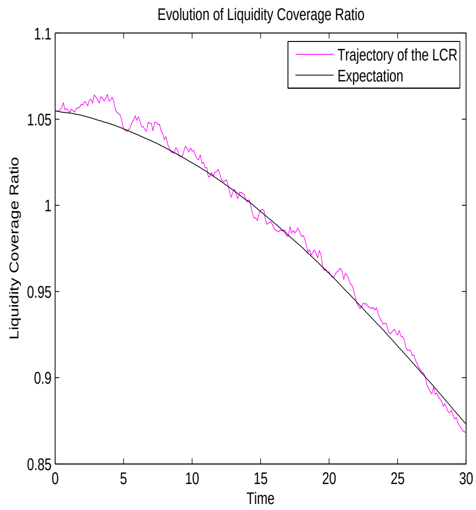

Figure 5.6: Simulation of LCR with c^* given by (5.24) when $\rho = 0.05$.

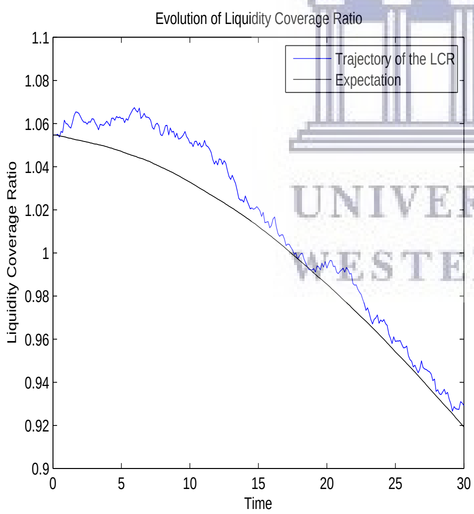

Figure 5.7: Simulation of LCR with c^* given by (5.20) when $\rho = 0.95$.

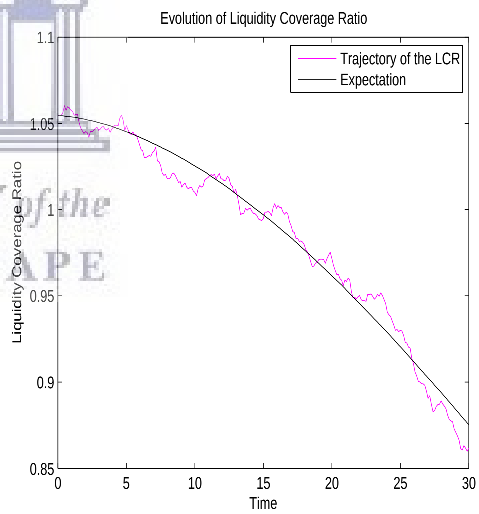

Figure 5.8: Simulation of LCR with c^* given by (5.24) when $\rho = 0.95$.

Comparing the figures above we note that the graphs of LCR exhibit similar characteristics. Over certain periods of time the graphs of Figures (5.1) - (5.8) remains well above the required liquidity threshold of 100%. This is crucial because having enough liquidity to withstand a month of elevated financial stress, gives bank management and regulators additional time to respond if necessary. We also note that the graphs of LCR have a downward trend. This is due to the fact that banks reserves can be drawn down in times of stress.

From Figures (5.1) - (5.8), at some point in time, we observe that the LCR starts to drops below the liquidity threshold. In situations like these, if the shortfall continues for three consecutive business days (as can be seen in the Figures), then it will be required of the bank, who calculates their LCR on a daily basis, to provide a plan for remediation to their primary regulator. If a shortfall occurs when banks are calculating their LCR on a monthly basis, then it must promptly consult with their primary regulator if they need to provide a remediation plan of action. These plans consists of the following:

- a thorough assessment of the bank's liquidity position;
- banks will take action to achieve full compliance with the final rule ;
- they need to provide a time framework for achieving compliance;
- it will be held accountable to report to its regulator no less than weekly on progress to achieve compliance with the plan until full compliance with the final rule is achieved.

In order to further motivate the importance of our numerical example above, we illustrate through a template of the LCR of six Canadian banks as of 31 October 2015 (see [74]) below. Note that the LCR for each bank is well above the liquidity threshold. A LCR shortfall, at a minimum, would result in heightened supervisory monitoring. Also banks facing this LCR shortfall may choose term funding, since it has maturity of greater than 30 days and satisfies both LCR and reserve requirements. This could lower the demand for overnight loans, pushing down the overnight rate and reducing the effectiveness of

Oct 2015	RBC		TD		CIBC		BMO		BNS		NBC	
	UWV[1]	WV	UWV	WV	UWV	WV	UWV	WV	UWV	WV	UWV	WV
High-quality liquid assets												
Total high-quality liquid assets (HQLA)		195		179		98		135		146		32
Cash outflows												
Retail deposits and deposits from small business customers, of which:	181	14	367	26	121	8	142	8	151	10	33	2
Stable deposits	60	2	158	5	58	2	86	3	71	2	16	0
Less stable deposits	120	12	209	21	63	6	57	6	80	8	16	2
Unsecured wholesale funding, of which:	218	97	196	94	111	65	138	78	149	83	43	25
Operational deposits (all counterparties) and deposits in networks of cooperative banks	97	23	85	20	39	10	60	15	35	8	10	2
Non-operational deposits	102	55	77	41	47	31	48	33	90	50	25	13
Unsecured debt	19	19	33	33	25	25	30	30	24	24	9	9
Secured wholesale funding		27		6		3		10		34		3
Additional requirements, of which:	196	51	137	33	64	18	119	24	174	42	34	9
Outflows related to derivative exposures and other collateral requirements	44	18	21	6	10	6	17	6	39	17	8	5
Outflows related to loss of funding on debt products	5	5	7	7	2	2	3	3	4	4	1	1
Credit and liquidity facilities (committed)	147	29	109	20	52	9	99	15	140	21	25	3
Other contractual funding obligations	28	28	12	7	2	2	1	-	4	2	1	0
Other contingent funding obligations (Uncommitted; e.g. credit cards etc.)	433	6	488	7	216	4	253	5	457	8	76	1
Total cash outflows		223		172		99		125		179		39
Cash inflows												
Secured lending (e.g. reverse repos)	119	33	98	15	42	8	93	12	102	27	47	7
Inflows from fully performing exposures	12	8	11	6	13	7	8	6	23	14	7	4
Other cash inflows	29	29	9	9	2	2	4	4	20	20	3	3
Total cash inflows		70		30		17		22		61		14
Total HQLA		195		179		98		135		146		32
Total net cash outflows		153		142		82		104		117		25
Liquidity coverage ratio[2]		127%		126%		119%		130%		124%		131%

[1] UWV= Unweighted Value (average); WV= Weighted Value (average); Average calculated based on month-end values during the quarter
[2] The LCR percentage is published as the simple average of the three month-end LCR percentages

Figure 5.9: LCR of six different Canadian banks (31 October 2015).

traditional monetary policy. Determining the optimal levels for liquidity is a challenging task in itself. For instance, if regulators overestimate the cash outflows in LCR, banks could be forced to hold too much liquidity, introducing inefficiencies into the financial system.

Chapter 6

Summaries, Conclusion and Future Directions

In this section, we interpret the main results encountered in this dissertation. In accordance with the objectives of the Basel III accord, the models of banking items constructed in this study are related to the methods currently being used to assess the riskiness of bank portfolios and their minimum capital requirements (see [9]). The assessment procedure mainly involves a consideration of the capital adequacy, liquidity and reserve requirement. In particular, Chapters 3, 4 and 5 of this dissertation are devoted to the description of the capital adequacy, reserve requirement and liquidity ratio respectively. In each chapter we constructed continuous-time models for the bank items in a jump-diffusion framework. We now provide a summary of what we were able to achieve in each chapter.

6.1 Summary Remarks about Chapter 3

In Chapter 3, we analyzed the issue of bank capital management for banks that raise their capital through equity and subordinate debt and loan loss reserves. The global financial crisis exposed the inadequacy of existing prudential regulatory arrangements, spurring various initiatives for reforming the Basel II capital standard. One of the main lessons from the crisis was that the banking system held insufficient capital. A key question for policymakers is how much more capital the system should have. The high cost of capital provides an incentive for bank owners to retain less capital relative to assets than is required by regulatory authorities. In the case where regulators force banks to build up capital too rapidly, this would impose considerable short-term macroeconomic costs by inducing banks to pull back from lending to finance investment. In this situation, the amount of bank capital to be held is prescribed by certain capital requirements. Therefore, in this study, we address the problem of capital allocation via diversification, which is linked to capital requirements. As we have pointed out in our earlier discussion, Barclays Africa Group Limited for example, has increased their CAR which means that at some point during the financial year it had to reduce its lending operation. One of the main reasons could be that it wanted to avoid the situation of credit default from customers. Also, the lessons learnt from the global financial crises should have caused bank regulators to be more cautious. The political turmoil in South Africa could also have led to the decision of the Board of Barclays Africa Group Limited to increase its capital requirement ratio. A political crises would add higher funding costs to a banking industry already struggling with increasing bad loans, a weak currency and a country's economy expanding at the slowest pace since the financial crisis. We observe that the simulations of the CAR for Barclays Africa Group Limited predicts/indicates an acceptable level of between 13% - 15% for the majority of the time as required by its Executive Board at the time.

6.2 Summary Remarks about Chapter 4

In Chapter 4, we investigated the optimal portfolio selection with liability, in a jump diffusion market under the benchmark criteria. In our setting, the risky assets (bank reserves) and the liabilities (deposits) are governed by different Lévy processes. In particular, we derive an asset allocation strategy for a bank's total reserve portfolio which attempts to minimize the deposit risk associated with it. From the optimal allocation strategy that solves our investment problem, we derive a dynamic formula for the bank's reserve requirement ratio. One of the reasons for developing a model of reserve requirement ratio is the fact that many central banks use it as a monetary policy in order to boost credit lending and spur on economic growth. For example, increasing the reserve ratio causes a reduction in the volume of deposits that can be supported by a given level of reserves. Similarly, decreasing the reserve ratio will leave the bank with excess reserves, which can induce an expansion of bank credit and deposit levels and a decline in interest rates. Thus, the ratio behaves in an unpredictable manner and one can represent it as a stochastic model. We do stress that the computational aspects were challenging especially in the absence of real life data. From the simulation trajectories, the approach and the models developed in this paper appear to be consistent with current practices in banking activity. We point out key differences between our Chapter 4 and [20] (see also, [46]). We derive an explicit solution for the allocation strategy in the finite horizon case by employing the maximum principle method. Another feature of our approach is that we compare the simulations with real graphs of reserves from the Fed's website and found that our trajectories exhibit similar characteristics.

6.3 Summary Remarks about Chapter 5

As far as Chapter 5 is concerned, the financial crisis demonstrated the importance of investments in liquid assets and its dependence on high-risk funding sources. Of course, banks exposed to liquidity risk increased their hold of liquid assets the most. Liquidity risk is part of a core function provided by banks in the sense of maturity transformation. Maturity transformation occurs when there are "mismatches" between liabilities and assets. As an example, a bank who does not have enough liquid assets to meet a sudden increase in demand on its liability side, may be forced to sell assets quickly at reduced prices or to suspend operations. This puts strain on other financial institutions since the bank acts as a lender. Furthermore, it causes disruptions to the financial system.

After providing simulations of the liquidity coverage ratio in Section 5.2, we would like to point out some key similarities of the liquidity coverage ratio and the reserve requirement ratio. In a nutshell, the reserve requirement ratio refers to a regulation set by a central bank and employed by most, but not all, of the world's central banks. It is the minimum fraction of deposits and notes that each commercial bank must hold as reserves. Historically, the reserve requirement could prevent banks from drawing on their liquidity since it was seen as a safety buffer. However, as the financial world became more complex and required more sophisticated tools to manage risk, especially liquidity risk, the popularity of the reserve requirement started to diminish as a means of bank regulation and monetary control. As it became less fashionable, more banks resorted to risk-based capital requirements as implemented through the international Basel Accords. Recently, the widespread practice of paying interest on bank reserves has given central banks an alternative way instead of encouraging banks to hold a certain amount of money. In the process, the liquidity coverage ratio emerged and it mimics many aspects of the reserve requirement ratio.

6.4 Conclusion

This dissertation presents a study of three related commercial banking objectives which is solved in a jump-diffusion setting. In the first objective, we determined an optimal investment strategy for a commercial bank subject to capital requirements as prescribed by the Basel III Accord. In order to address this objective, our analysis depended on the dynamics of the capital adequacy ratio. The risk-weighted assets are solely constituted by loans since its considered to be the largest asset on the balance sheet. We observed from the specific choice of the investment strategy that was derived, the CAR oscillates between 13% and 15% for the majority of the time over the 10 year period and remained capitalized above the minimum regulatory capital requirements and within Barclays Africa Group

Limited approved target capital ranges. For the second objective, we analyzed a banks reserves operations and quantified the extent to which their reserves management policy is optimal. In other words, we minimized the expected costs associated with reserves having developed an appropriate allocation strategy. Our analysis did take into account the reserve requirement ratio which is seen as a monetary policy tool that influences a country's borrowing and interest. Our analysis is supported by simulations of People's Bank of China reserve ratio. From these simulations, we observed that the reserve ratio oscillates steadily between 16% and 17%.

There are many central banks that rarely increase the reserve requirements because it would cause immediate liquidity problems for banks with low excess reserves. Thus, in order to address the third objective, we investigated the money supply process between a central bank and a commercial bank and how it affects the liquidity of a bank. In particular, we formulated a stochastic control problem where the cash of banks is held as deposits at the central bank. We did model the liquidity coverage ratio which is one of the measures to determine the liquidity position of a particular bank. This ratio measures the ability of a company to meet its short-term debt obligations. We provided simulations of this ratio which remains well above the required liquidity threshold of 100%.

6.5 Future Directions

In this dissertation, we discussed on how to minimize liquidity risk and capital adequacy risk management for banks. We investigated the stochastic dynamics of bank items such as loans, reserves, securities, deposits, borrowing, and bank capital. In accordance with Basel III, our study proposes that overall liquidity and capital adequacy is best analyzed using ratio analysis approaches. Here, liquidity risk is measured via the LCR and capital is measured via CAR. We provided numerical results to highlight some of the important issues raised in the introduction of the dissertation. Our numerical quantitative model for liquidity shows that a low LCR stems from a low level of liquid assets or high nett cash outflows. The reliability, transparency and quality of dynamic modelling are crucial to the allocation of resources by key role-players in the banking industry. For example, regulators can benefit greatly from the employment of sound modelling techniques. We were able to add to the debate about the mathematical modelling and simulation of bank capital. We also discussed some of the economic issues arising from the stochastic dynamic models mentioned earlier. Having said that, we do not intend to make any recommendation but rather to take some steps toward clarifying the mechanisms involved in the regulation of financial institutions. In particular, our methodology studies the changes in the liquidity ratio and minimal capital requirement influence the banks management strategies and the payoffs of both shareholders and debt holders.

In future research, we would like to determine the hedging strategies for loan processes that are semi-martingales and are not subject to transformation by an equivalent measure into a martingale. Another risk that becomes important is interest rate risk at the point of loan issuing. In this case, one can formulate an optimization problem that maximizes the rate of interest in order to provide a shareholder with an incentive to invest money. The optimization problems considered in the thesis have analytical solutions. However, we would like to solve optimization problems subjected to complex stochastic models via numerical methods in bank management.

Bibliography

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge University Press, Cambridge, 2nd edition, 2009.

[2] R.B. Ash, *Measure, Integration and Functional Analysis*, Academic Press, inc: New York, 1972.

[3] F. Astic, A. Tourin "Optimal bank management under capital and liquidity constraints", *Journal of Financial Engineering*, vol. 0, no. 03, September 2014. https://doi.org/10.1142/S2345768614500226

[4] S. Basak, A. Shapiro, "Value-at-risk based management: optimal policies and asset prices", *Review of Financial Studies*, vol. 14, no. 2, pp 371-405, 2001.

[5] Basel Committee on Banking Supervision: International convergence of capital measurement and capital standards. Bank for International Settlements (July 1988). http://www.bis.org/publ/bcbs04A.pdf

[6] Basel Committee on Banking Supervision. *International Convergence of Capital Measurement and Capital Standards; A Revised Framework*. Bank for International Settlements. http://www.bis.org/publ/bcbs107.pdf [June 2004].

[7] Basel Committee on Banking Supervision, *Principles for sound liquidity risk management and supervision*, Bank for International Settlements (BIS) Publications, (2008). http://www.bis.org/publ/bcbs144.htm

[8] Basel Committee on Banking Supervision, *Basel III: International framework for liquidity risk measurement, standards and monitoring*, Bank for International Settlements (BIS) Publications, (2010).
http://www.bis.org/publ/bcbs188.pdf

[9] Basel Committee on Banking Supervision, *Basel III: A global regulating framework for more resilient banks and banking systems*. Bank for International Settlements. http://www.bis.org/publ/bcbs189.pdf [June 2011]

[10] Basel Committee on Banking Supervision. *China's evolving reserve requirements*. Bank for International Settlements 2011; http://www.bis.org/publ/work360.pdf.

[11] Basel Committee on Banking Supervision (BCBS), *Basel III: A global regulatory framework for more resilient banks and banking systems- revised version June 2011*, Bank for International Settlements (BIS) Publications, (2011).
http://www.bis.org/publ/bcbs189.htm

[12] Basel Committee on Banking Supervision, *Progress report on Basel III implementation*, Bank for International Settlements (BIS) Publications, (2011). http://www.bis.org/publ/bcbs203.pdf

[13] Basel Committee on Banking Supervision,*Basel III: The liquidity coverage ratio and liquidity risk monitoring tools*, Bank for International Settlements (BIS) Publications, (2013). http://www.bis.org/publ/bcbs238.htm

[14] A. Bensoussan, K. C. Wong, S.C.P. Yam, "A paradox in time-consistency in the meanvariance problem?", *Finance and Stochastics*, vol. 23, no. 1, pp. 173-207, 2019.

[15] P. Beraldi, F. De Simone, A. Viola, G. Consigli and G. Iaquinta , Scenario-based dynamic corporate bond portfolio management, *IMA Journal of Management Mathematics*, vol. 23, no. 4, pp. 341-364, 2012.

[16] A.N. Berger, R.J. Herring and G.P. Szegö, "The Role of Capital in Financial Institutions", *Journal of Banking and Finance*, vol. 19, no. 1-2, pp. 393-430, 1995.

[17] R.N. Bhattacharya and E.C. Waymire, *Stochastic Processes with Applications*, John Wiley and Sons, Inc. ,1990.

[18] F. Boissay, F. Collard, "Macroeconomics of bank capital and liquidity regulations", BIS Working Papers No 596, December 2016. https://www.bis.org/publ/work596.pdf

[19] T. Bosch, J. Mukuddem-Petersen, M.P. Mulaudzi and M.A. Petersen, "Optimal Capital Management in Banking", *Proceedings of the World Congress on Engineering* 2008 Vol II WCE 2008, July 2 - 4, 2008, London, U.K.

[20] T. Bosch, J. Mukuddem-Petersen, M.A. Petersen and I. Schoeman, "Optimal auditing in the banking industry", *Optimal Control Applications and Methods*, vol. 29, no. 2, pp. 127-158, 2008.

[21] A. Cadenillas and U.G. Haussmann, "The Stochastic Maximum Principle for a Singular Control Problem", *Stochastics and Stochastics Reports*, vol. 49, no. 3 - 4, pp. 211-237, 1994.

[22] F. Castiglionesi, "Financial contagion and the role of the central bank", *Journal of Banking and Finance*, vol. 31, no. 1, pp. 81-107, 2007.

[23] F. Chakroun and F. Abid, "Capital adequacy and risk management in banking industry", *Applied Stochastic Models in Business and Industry*, vol. 32, no. 1, 113-132, 2016.

[24] J. Clouse and J.P. Dow Jr., "A computational model of bank's optimal reserve management policy", *Journal of Economic Dynamics and Control*, vol. 26, no. 11, 1787-1814, 2002.

[25] B.H. Cohen, "How Have Banks Adjusted to Higher Capital Requirements?". *BIS Quarterly Review*, September 2013. https://ssrn.com/abstract=2401569

[26] D.L. Cohn, *Measure Theory*, Library of Congress Cataloging in Publication Data, 1990.

[27] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC, 2004. ISBN 1-58488-413-4.

[28] D. Cuoco and H. Liu, "An analysis of VAR-based capital requirements", *Journal of Financial Intermediation*, vol. 15, no. 3, pp. 362-394, 2006.

[29] Chart's Series Name: Required Reserves of Depository Institutions: Billions of Dollars: NSA. `http://www.economagic.com/em-cgi/daychart.exe/form`

[30] Central Bank News - CentralBankNews.info - A trusted and authoritative source on global monetary policy. `http://www.centralbanknews.info/p/reserve-ratios.html`

[31] China cuts interest rates, reserve ratio.
http://www.scmp.com/news/article/1828162/china-cuts-interest-rates-reserve-ratio

[32] China central bank cuts reserve requirement ratio.
http://www.reuters.com/article/us-china-economy-rrr-instantview-idUSKCN0W21A0

[33] J.C. Cox, J.E. Ingersoll, Jr., and S.A. Ross, "A Theory of the Term Structure of Interest Rates", *Econometrica*, vol. 53, no. 2, pp. 385-407, 1985.

[34] T. Dangl and J. Zechner, "Credit Risk and Dynamic Capital Structure Choice", *Journal of Financial Intermediation*, vol. 13, no. 2, pp. 183-204, 2004.

[35] M. Davis and S. Lleo, "Jump-Diffusion Risk-Sensitive Benchmarked Asset Management", *Stochastic Programming*, pp. 97-127, 2013.

[36] J-P. Decamps , J-C. Rochet , B. Roger, "The Three Pillars of Basel II: Optimizing the Mix", *Journal of Financial Intermediation*, vol. 13, no. 2, pp. 132-155, 2004.

[37] S. Demiralp and D. Farley, "Declining required reserves, funds rate volatility, and open market operations", *Journal of Banking and Finance*, vol. 29, no. 5, pp. 1131-1152, 2005.

[38] B. De Waal, M.A. Petersen, L.N.P. Hlatshwayo and J. Mukuddem-Petersen, "A note on Basel III and liquidity", *Applied Economics Letters*, vol. 20, no.8, pp. 777-780, 2013.

[39] J.V. Duca, "How capital regulation and other factors drive the role of shadow banking in funding short-term business credit", *Journal of Banking and Finance*, vol. 69, Supplement 1, pp. S10-S24, 2016.

[40] E. Elyasiani, L.J. Mester, M.S. Pagano, "Large Capital Infusions, Investor Reactions, and the Return and Risk-Performance of Financial Institutions Over the Business Cycle", *Journal of Financial Stability*, vol. 11, pp. 62-81, 2014.

[41] S. Emmer and C. Klüppelberg, "Optimal portfolios when stock prices follow an exponential Lévy process", *Finance and Stochastics*, vol. 8, no. 1, 17-44, 2004.

[42] H.M. Ennis and T. Keister, "Bank runs and investment decisions revisited", *Journal of Monetary Economics*, vol. 53, no. 2, pp. 217-232, 2006.

[43] E. Fernández, "Distorting taxes and interest on reserves", *Economic Modelling*, vol. 22, no. 6, pp. 975-1000, 2005.

[44] N. Framstad, B. Øksendal, A. Sulem, "Sufficient Stochastic Maximum Principle for the Optimal Control of Jump Diffusions and Applications to Finance", *Journal of Optimization Theory and Applications*, vol. 121, no. 1, pp. 77-98, 2004.

[45] Z Geng and X. Zhai, "Effects of the Interest Rate and Reserve Requirement Ratio on Bank Risk in China: A Panel Smooth Transition Regression Approach", *Discrete Dynamics in Nature and Society*, vol. 2015 (2015), Article ID 571384, 8 pages.

[46] F. Gideon, J. Mukuddem-Petersen and M.A. Petersen, “Minimizing banking risk in a Lévy process setting”, *Journal of Applied Mathematics*, vol. 2007, Article ID 32824, 25 pages, 2007, DOI: 10.1155/2007/32824.

[47] F. Gideon, M.A. Petersen, J. Mukuddem-Petersen, and L. N. P. Hlatshwayo, “Basel III and the net stable funding ratio”, *ISRN Applied Mathematics*, vol. 2013, Article ID 582707, 20 pages, 2013, http://dx.doi.org/10.1155/2013/582707

[48] G.R. Grimmett and D.R. Stirzaker, *Probability and Random Processes*, Oxford University Press, New York, 1992.

[49] F. Gideon, M. A. Petersen, J. Mukuddem-Petersen, and L. N. P. Hlatshwayo, “Bank Liquidity and the Global Financial Crisis”, *Journal of Applied Mathematics*, vol. 2012, Article ID 743656, 27 pages, 2012, http://dx.doi.org/10.1155/2012/743656

[50] D. Hancock, A.J. Laing, J.A. Wilcox, “Bank capital shocks: Dynamic effects and securities, loans and capital”, *Journal of Banking and Finance*, vol. 19, no. 3-4, pp. 661-677, 1995.

[51] H. Hong, J. Huang and D. Wu, “The information content of Basel III liquidity risk measures”, *Journal of Financial Stability*, vol. 15, pp. 91-111, 2014.

[52] M. House, T. Sablik and John R. Walter, “Understanding the New Liquidity Coverage Ratio Requirements”, 2016, https://www.richmondfed.org/-/media/richmondfedorg/publications/research/

[53] J. Hugonnier, E. Morellec, “Bank Capital, Liquid Reserves, and Insolvency Risk”, *Swiss Finance Institute Research*, Paper No. 14-70, August 11, 2016. https://ssrn.com/abstract=2448162orhttp://dx.doi.org/10.2139/ssrn.2448162

[54] R.M. Irani, R. Iyer, R.R. Meisenzahl, J.L. Peydro,“The Rise of Shadow Banking: Evidence from Capital Regulation”, (April 20, 2018). Available at SSRN: https://ssrn.com/abstract=3166219 or http://dx.doi.org/10.2139/ssrn.3166219

[55] P. Jackson and W. Perraudin, “Regulatory implications of credit risk modelling”, *Journal of Banking and Finance*, vol. 24, no. 1-2, pp. 1-14, 2000.

[56] Y. Kabanov, “On the Pontryagin Maximum Principle for SDEs with a Poisson-Type Driving Noise”, *Statistics and Control of Stochastic Processes*, Edited by B. L. Rozovskii, Y. M. Kabanov, and A. N. Shiryaev, World Scientific, River Edge, New Jersey, pp. 173-190, 1997.

[57] J. Kallsen, “Optimal portfolios for exponential Lévy processes”, *Mathematical Methods of Operations Research*, vol. 51, no.3, pp. 357-374, 2000.

[58] F.C. Klebaner, *Introduction To Stochastic Calculus With Applications*, 2nd edition, Imperial College Press, 57 Shelton Street, Covent Garden, London WC2H 9HE, 1998.

[59] M. Kohlmann, “Optimality conditions in optimal control of jump processes extended abstract”, *In: Proceedings in Operations Research.* (Sixth Annual Meeting, Deutsch. Gesellsch. Operations Res., Christian-Albrechts-Univ., Kiel, 1977) Physica, Würzburg, Germany, vol. 7, pp. 48-57, 1977.

[60] C-F. Lee, A. Lee and J. Lee, “Handbook of Quantitative Finance and Risk Management”, *Springer US*, 2010 edition, 2010.

[61] H. Leland, “Risky Debt, Bond Covenants and Optimal Capital Structure”, *Journal of Finance*, vol. 49, pp. 1213-1252, 1994.

[62] D. Li and W.L. Ng, “Optimal Dynamic Portfolio Selection: Multiperiod MeanVariance Formulation”, *Mathematical Finance*, vol 10, no. 3, pp. 387-406, 2000.

[63] G. Ma, Y. Xiandong, and L. Xi, “China’s evolving reserve requirements. Special Issue:The Renminbi as an International Currency: The next Installment of China’s

Economic Reforms?", *Journal of Chinese Economic and Business Studies*, vol. 11, no. 2, pp. 117-137, 2013, DOI:10.1080/14765284.2013.789682.

[64] F.S. Mishkin, *The economics of money banking and financial markets (7th edition)*, Boston, USA: Addison-Wesley Series, 2004.

[65] J. Mukuddem-Petersen, M.A. Petersen, "Optimizing Asset and Capital Adequacy Management in Banking", *Journal of Optimization Theory and Applications*, vol. 137, no. 1, pp. 205-230, 2008.

[66] G.E. Muller, P.J. Witbooi, "An Optimal Portfolio and Capital Management Strategy for Basel III Compliant Commercial Banks", *Journal of Applied Mathematics*, vol. 2014, Article ID 723873, 11 pages, 2014. doi:10.1155/2014/723873

[67] L. Pellizzon and S. Schaefer, "Pillar 1 vs pillar 2 under risk management". In: Carey, M., Stulz, R. (eds.) NBER Conference on Risks of Financial Institutions, 2004.

[68] M.A. Petersen, B. De Waal, J. Mukuddem-Petersen and M.P. Mulaudzi, "Subprime Residential Mortgage Funding and Liquidity Risk", *Quantitative Finance*, vol. 14, no. 3, pp. 545-555, 2012.
http://dx.doi.org/10.1080/14697688.2011.637076

[69] G. Plantin, "Shadow Banking and Bank Capital Regulation", *Review of Financial Studies*, vol. 28, no. 1, pp. 146-175, 2015.

[70] Philip E. Protter, *Stochastic Integration and Differential Equations*, 2nd edition, Springer-Verlag Berlin Heidelberg, New York, 2005.

[71] B. Øksendal, *Stochastic differential equations: An Introduction with Applications*, Springer-Verlag, Heidelberg, New York, 2000.

[72] B. Øksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, Springer-Verlag Berlin Heidelberg, Second edition, 2007.
http://dx.doi.org/10.1007/978-3-540-69826-5

[73] R. Repullo, "Capital Requirements, Market Power and Risk-Taking in Banking", *Journal of Financial Intermediation*, vol. 13, no.2, pp. 156-182, 2004.

[74] S. Roshan, *Stable and Operational Deposits in Liquidity Coverage Ratio*, https://www.linkedin.com/pulse/stable-operational-deposits-liquidity- [9 February 2016]

[75] N. Salidjanova, "China's Foreign Exchange Reserves and Holdings of U.S. Securities", *USCC Economic Issue Brief*, No. 2, https://www.uscc.gov/sites/default/files/Research/USCC20Economic20Issue20Brief_China27s20FX20Reserves20and20Treasury20Holdings.pdf [21 March 2014]

[76] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.

[77] S. Tang, X. Li, "Necessary Conditions for Optimal Control of Stochastic Systems with Random Jumps", *SIAM Journal on Control and Optimization*, vol. 32, no. 5, pp. 1447-1475, 1994.

[78] G.J. van Schalkwyk and P.J. Witbooi, "An Optimal Strategy for Liquidity Management in Banking", *Applied Mathematical Sciences*, vol. 11, no. 6, pp. 275 - 297, 2017.

[79] G.J. van Schalkwyk and P.J. Witbooi, "A model for bank reserves versus treasuries under Basel III", *Applied Stochastic Models in Business and Industry*, vol. 33, Issue 2, pp. 237-247, 2017.

[80] G.J. van Schalkwyk and P.J. Witbooi, "A Benchmark Capital Management problem under the Basel III Framework", *Preprint*.

[81] O. Vasicek, "An Equilibrium Characterisation of the Term Structure", *Journal of Financial Economics*, vol.5, no. 2, pp. 177-188, 1977.

[82] A. Walther, "Jointly Optimal Regulation of Bank Capital and Liquidity", *Journal of Money, Credit and Banking*, vol. 48, no. (2-3), pp. 415448, 2016.

[83] W. Whitesell, "Interest rate corridors and reserves", *Journal of Monetary Economics*, vol. 53, no. 6, pp. 1177-1195, 2006.

[84] P.J. Witbooi, G.J. van Schalkwyk, G.E. Muller, "An optimal investment strategy in bank management", *Mathematical Methods in the Applied Sciences*, vol. 34, no. 13, pp. 1606-1617, 2011.

[85] Y. Zeng Y, Li Z, and J. Liu, "Optimal Strategies of Benchmark and Mean-Variance Portfolio Selection Problems for insurers", *Journal of Industrial and Management Optimization*, vol. 6, no. 3, pp. 483-496, 2010.

[86] Y. Zeng and Z. Li, "Asset-liability management under benchmark and mean-variance criteria in a jump diffusion market", *Journal of Systems Science and Complexity*, vol. 24, no. 2, pp. 317-327, 2011.

[87] Y. Zeng, Z. Li, and H. Wu, "Optimal portfolio selection in a Lévy market with uncontrolled cash flow and only risky assets", *International Journal of Control*, vol. 86, no.3, pp. 426-437, 2013.

[88] X.Y. Zhou XY and D. Li, "Continuous time mean-variance portfolio selection: A stochastic LQ framework", *Applied Mathematics and Optimization*, vol. 42, pp. 19-33, 2000.

[89] Barclays Africa Group Limited - Basel III Pillar 3 Disclosure as at 30 September 2016. `https://www.barclaysafrica.com/content/dam/barclays-africa/bagl/pdf/sens/2016/jul-dec/basel-iii-pillar-3-disclosure-30-september.pdf` [30 September 2016]

[90] Barclays Africa Group Limited - Basel III Pillar 3 Disclosure as at 31 March 2017. `https://www.barclaysafrica.`

com/content/dam/barclays-africa/bagl/pdf/sens/2017/jan-jun/
basel-iii-pillar-3-disclosure-as-at-31-march-released-31-may.pdf