THE DETERMINATION OF DIFFRACTED WAVE FIELDS BY AN ANNULUS ACCORDING TO BRAUNBEK'S METHOD

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CONTENTS

CHAPTER 1

INTRODUCTION

(1.1) Rigorous diffraction theory : a summary

The behaviour of monochromatic scalar waves, harmonic in time, is governed by the Helmholtz equation (2.1.1).

In chapter 2 the existence of a unique solution (2.2.3a) for the Dirichlet problem (2.2.1a), is proved for the case where U is continuous inside and vanishes outside a finite portion of the X-Y-plane and satisfies the Sommerfeld radiation conditions (2.1.10) and (2.1.11) at infinity. An analogous proof that the solution (2.2.3b) satisfies the requirements of the Neumann problem (2.2.1b) is readily obtained. Luneburg (1944) gave an existence proof for the Dirichlet problem under more general conditions, but it contains errors and and requires elucidation (see Appendix).

The diffracted wave of a finite aperture in an infinite screen is related to the scattered wave of a finite screen congruent to the aperture by Babinet's theorem. The term "theorem" is used in preference to "principle", as this statement can be formulated and proved in the framework of boundary value problems. The mathematical formulation of Babinet's theorem is proved in paragraph (2.4).

The half-plane diffraction formulae derived in chapter 3 and summarized in table 3.2 , were obtained from Sommerfeld (1954). When applied to electromagnetic waves, these formulae lead to table 3.3, essentially the same as the table derived by Frahn (1959). A uniqueness theorem applicable to Sommerfeld's solution is given by Jones (1953).

(1.2) The determination of diffracted waves by surface field approximations

According to Hönl, Maue and Westpfahl (1961) there are three approaches to boundary value problems of scattering and diffraction. One of these entails obtaining the field, by means of an integral representation, in terms of the value of the field and/or its normal derivative on the scatterer. These values are the solutions of integral equations on the surface of the scatterer. Various approximations of the surface field and its normal derivative form the basis of an important group of methods used to determine scattered and diffracted wave fields.

Suppose a scalar harmonic wave $u_0 e^{-i\omega t}$ is incident on an infinite screen S with a finite aperture A. The field behind the screen is given by (2.2.3a), (2.2.3b) or a combination of these two equations:

$$u(\underline{R}) = -\tfrac{1}{2}\iint_{\infty}^{\infty} [U(\xi,\eta)\frac{\partial G}{\partial z} + U_n(\xi,\eta)G]\, d\xi d\eta \,. \qquad ---(1.2.1)$$

According to physical optics the field behind the screen is found by replacing the surface field and its normal derivative in the above mentioned three equations by their geometrical optics values. This means that U and U_n are respectively replaced by u_0 and $\partial u_0/\partial n$ on A and by zero on S. The approximation thus obtained from (1.2.1) forms the basis of Kirchhoff diffraction theory, while the approximations obtained from (2.2.3a) and (2.2.3b) are sometimes refered to as the Rayleigh-Sommerfeld diffraction formulae (see Goodman (1968)). Braunbek (1950) refers to the latter formulae as the weaker and better Kirchhoff approximations respectively.

Equation (1.2.1) with Kirchhoff boundary values has been proved to be a rigorous solution of a so-called saltus (Sprungwert) problem (see Hönl, Maue and Westpfahl (1961)). Kirchhoff's theory has also been shown by Wolf and Marchand (1966) to provide an exact solution of a boundary value problem somewhat different from the above. Gomez-Reino Carnota and Vences Benito (1977) proved a uniqueness theorem for the solution of this problem. Kirchhoff's theory makes no distinction between sound soft $(U = 0)$ and sound hard $(U_n = 0)$ screens.

Braunbek (1950) advanced a method in which U and U_n are replaced by their values obtained from Sommerfeld's exact theory of the half-plane, as if the screen had a locally straight edge. This is a reasonable assumption if the wave length is small in comparison with the dimensions of the aperture (see Bouwkamp (1954)). Braunbek applied this method to the scattering of a plane wave by a disc. He compared these results with the numerical values of the exact solution as calculated by Meixner and Fritze (1949), illustrating the superiority of his method over that of Kirchhoff's.

In an asymptotic method developed by Westpfahl and Witte (1967), the diffracted field of a plane wave by a circular aperture is given in the far region by a series of descending powers in ka, a being the radius of the circle. For small angles of diffraction the solutions of Kirchhoff and Braunbek are identical to the first term in this series. The solutions of Kirchhoff and Braunbek differ for wide angles of diffraction, the latter still constituting the first term in the above series.

According to Jones (1964), the originator of Braunbek's method for the electromagnetic case was Macdonald (1913), who approximated the surface current on a convex body by the current the external field would induce on an infinite plane occupying the position of a tangent plane. Du Plessis (1976) improved this method by using the current that would be induced on a so-called "representative" sphere rather than a tangent plane.

Frahn (1959) applied Braunbek's method to the problem of electromagnetic scattering from a perfectly conducting circular disc. He compared the numerical values predicted by this procedure in the far and near regions with those obtained from the rigorous solution by Andrejewski (1953).

Westpfahl and Witte (1971) extended their method to the diffraction of an electromagnetic plane wave by a circular aperture. Again the main term of the asymptotic series solution was found to be identical to the solution obtained by Frahn (1959). Another asymptotic procedure yielding Braunbek's solution as a first approximation was advanced by Saltykov (1973).

(1.3) Braunbek's method for the diffraction of plane waves by an annular aperture

Braunbek (1950) applied his method to the scattering of a scalar plane wave by a sound hard circular disc. By virtue of Babinet's theorem this problem is equivalent to the diffraction by an infinite sound soft screen with a circular aperture. In paragraph (4.1) the diffracted far fields of both sound soft and sound hard screens with annular apertures are derived using the same asymptotic approximations as Braunbek.

In paragraph (4.2) the application by Frahn (1959) of Braunbek's method to the case of vector diffraction by a circular aperture, is generalized to include the case of vector diffraction by an annular aperture. However, in this thesis the near field and transmission coefficient of the annular aperture are not derived.

The procedure followed in paragraph (4.3) to derive the diffracted scalar field on the Z-axis is similar to that of Bouwkamp (1954) who reported on Braunbek's method as applied to scattering by a disc. The results obtained are slightly more accurate than those of Braunbek. See (4.3.8) in this connection.

CHAPTER 2

INTEGRAL SOLUTIONS OF THE HELMHOLTZ EQUATION IN HALF-SPACE

(2.1) The scalar Helmholtz equation

We wish to find solutions of the scalar Helmholtz equation

$$\nabla^2 u + k^2 u = 0 \qquad \text{--- (2.1.1)}$$

in the region G for which either u or its normal derivative $\partial u/\partial n$ assumes a prescribed value on an infinite plane surface, the X-Y-plane in fig.2.1.1. An integral solution is obtained by applying Green's second identity to $G-g$, where G is the region interior to the hemisphere in fig.2.1.1 and g is the region interior to the sphere centred at the point P with coordinates (x,y,z). Let Q be the point (ξ,η,ζ) in-

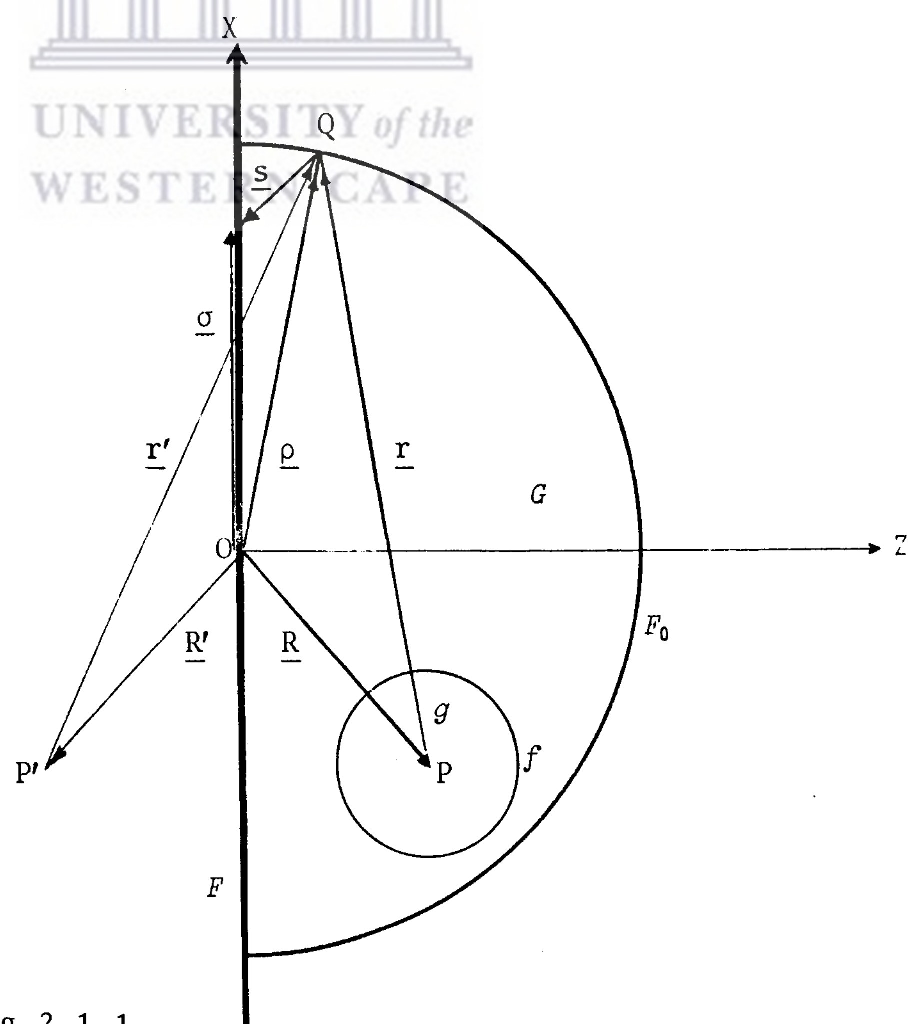

fig.2.1.1

terior to or on the surface of $G-g$, then according to Green's second identity

$$\iiint_{G-g} [u\nabla^2 v - v\nabla^2 u]d\tau = - \oiint_{F\cup F_0\cup f} [u\partial v/\partial n - v\partial u/\partial n]df. \qquad \text{--- (2.1.2)}$$

The unit normal vector $\underline{n}$ on the surfaces F, F_0 and f in fig.2.1.1 is directed towards the interior of $G-g$. The variables of integration are (ξ,η,ζ).

If v is any function that satisfies the Helmholtz equation in the region $G-g$, the left hand side of equation (2.1.2) vanishes and one may write

$$\oiint_f [u\partial v/\partial n - v\partial u/\partial n]df = -\left(\iint_F + \iint_{F_0}\right)[u\partial v/\partial n - v\partial u/\partial n]df. \qquad \text{--- (2.1.3)}$$

Two solutions of $u(\underline{R})$ are obtained by a method basically the same as that of Luneburg (1944). Let

$$v = G + G', \qquad \text{--- (2.1.4)}$$

where G' satisfies the Helmholtz equation inside G. On applying Green's second identity to g with the functions u and G', we find that

$\oiint_f [u\partial G'/\partial n - G'\partial u/\partial n]df = 0$, and hence (2.1.3) becomes

$$\oiint_f [u\partial G/\partial n - G\partial u/\partial n]df = -\left(\iint_F + \iint_{F_0}\right)[u\partial v/\partial n - v\partial u/\partial n]df. \qquad \text{--- (2.1.5)}$$

Let $G = e^{ikr}/2\pi r$ where $r = \underline{PQ}$. The function G clearly satisfies (2.1.1) in any region excluding P. On f $\partial u/\partial n = \partial u/\partial r$ and the left hand side (2.1.5) can be written in the form

$$\frac{\varepsilon}{2\pi} e^{ik\varepsilon} \int_0^{2\pi}\int_0^{\pi} [(-1/\varepsilon + ik)u - \partial u/\partial r] \sin\gamma\, d\gamma\, d\phi,$$

where ε is the radius of g and γ and ϕ are spherical polar coordinates.

We now assume that u is of the class C_2 in G, hence u is bounded and if we take the limit as $\varepsilon \to 0$ in the expression above, it reduces to $-2u(\underline{R})$. Now (2.1.5) becomes

$$2u(\underline{R}) = \left(\iint_F + \iint_{F_0}\right)[u\partial v/\partial n - v\partial u/\partial n]df, \qquad \text{--- (2.1.6)}$$

where $\underline{R} = \underline{OP}$.

Two solutions are obtained from (2.1.6) by applying the method of images. This method consists of alternatively setting

$$G' = -e^{ikr'}/2\pi r' \qquad \text{--- (2.1.7a)}$$

and

$$G' = e^{ikr'}/2\pi r' \qquad \text{--- (2.1.7b)}$$

in (2.1.6). In (2.1.7a) and (2.1.7b) $\underline{r} = \underline{OP'}$, P′ being the mirror image of P in the X-Y-plane. On the plane $r = r'$ and $\zeta = 0$ and therefore

$$\partial v/\partial n = \partial(e^{ikr}/2\pi r \mp e^{ikr'}/2\pi r')/\partial\zeta$$

$$= [-z/r \mp z/r](-1/r^2 + ik/r)e^{ikr}/2\pi$$

$$= -2\ \partial G/\partial z \ \text{ or } \ 0.$$

For the cases (2.1.7a) and (2.1.7b) therefore the equations

$$\left.\begin{aligned} v &= 0 \\ \partial v/\partial n &= -2\ \partial G/\partial z \end{aligned}\right\} \qquad \text{---(2.1.8a)}$$

and

$$\left.\begin{aligned} v &= 2G \\ \partial v/\partial n &= 0 \end{aligned}\right\} \qquad \text{---(2.1.8b)}$$

respectively hold on the X-Y-plane. Substitution of (2.1.8a) and (2.1.8b) into (2.1.6) gives

$$u(\underline{R}) = -\iint_F u\frac{\partial G}{\partial z}df + \tfrac{1}{2}\iint_{F_0}[u\partial v/\partial n - v\partial u/\partial n]df \qquad \text{---(2.1.9a)}$$

$$u(\underline{R}) = -\iint_F \frac{\partial u}{\partial\zeta}G\,df + \tfrac{1}{2}\iint_{F_0}[u\partial v/\partial n - v\partial u/\partial n]df \qquad \text{---(2.1.9b)}$$

From (2.1.9a) and (2.1.9b) it is clear that a knowledge of u and $\partial u/\partial n$ on F is not sufficient to find the value of $u(\underline{R})$ in the region $z>0$, but that the behaviour of u and $\partial u/\partial n$ must also be known on an arbitrary hemisphere. Sufficient for the vanishing of the integrals over a hemisphere F_0 of infinite radius are the so-called radiation conditions of Sommerfeld

$$|u\rho| < C \qquad \text{--- (2.1.10)}$$

$$\rho|\partial u/\partial \rho - iku| \to 0 \qquad \text{--- (2.1.11)}$$

uniformly with respect to direction as $\rho \to 0$. (For the proof see Luneburg (1944).) These conditions give expression to the physical requirement that there cannot be any contribution to the field from infinity.

The derivation of (2.1.9a) and (2.1.9b) is based on the assumption that the divergence theorem may be applied to the vector fields $u\nabla v$ and $v\nabla u$. According to Kellogg (1929) the continuity of $u\nabla v$ and $v\nabla u$ and their partial derivatives and the existence of the volume integrals of $div(u\nabla v)$ and $div(v\nabla u)$ in the closed regoin $G-g$ are sufficient to guarantee the validity of this theorem. If u is of class C_2 in the region $z>0$, these conditions are met.

(2.2) Uniqueness and existence of solutions of the scalar Helmholtz equation

We now give a rigorous formulation of the two boundary value problems to be considered.

(a) The Dirichlet Problem (First Boundary Problem): Find a $u(\underline{R})$ which in the region $z>0$ is of class C_2 and satisfies $(\nabla^2+k^2)u = 0$. On the X-Y-plane it is required that

$$u(x,y,0) = U(x,y), \qquad \text{--- (2.2.1a)}$$

where U is continuous.

(b) The Neumann Problem (Second Boundary Problem):
Find a $u(\underline{R})$ which in the region $z>0$ is of class C_2 and satisfies $(\nabla^2+k^2)u=0$. On the X-Y-plane it is required that

$$\partial u(x,y,0)/\partial z = U_n(x,y), \qquad \text{--- (2.2.1b)}$$

where U_n is continuous.

From paragraph 2.1 it is clear that if these problems have solutions satisfying the conditions

$$\left.\begin{aligned} \lim_{\rho\to\infty} \iint_{F_0} [u\partial G/\partial n - G\partial u/\partial n]df = 0 \\ \lim_{\rho\to\infty} \iint_{F_0} [u\partial G'/\partial n - G'\partial u/\partial n]df = 0 \end{aligned}\right\}, \qquad \text{--- (2.2.2)}$$

they will be unique and respectively given by

$$u(\underline{R}) = -\iint_{-\infty}^{\infty} U(\xi,\eta)\frac{\partial G}{\partial z}\, d\xi\, d\eta \qquad \text{--- (2.2.3a)}$$

and

$$u(\underline{R}) = -\iint_{-\infty}^{\infty} U_n(\xi,\eta)\, G\, d\xi\, d\eta . \qquad \text{--- (2.2.3b)}$$

Note that in (2.1.9a) and (2.1.9b) the integrals over the hemisphere vanish due to (2.2.2) as $v = G + G'$.

Solutions of the Dirichlet and Neumann problems in the region $z<0$ are respectively obtained by substituting $\partial G/\partial|z|$ for $\partial G/\partial z$ in (2.2.3a) and defining $U_n = -\partial u(\xi,\eta,0)/\partial z$ in (2.2.3b).

The question now arises as to which conditions the prescribed values U and U_n should respectively satisfy for the functions defined by (2.2.3a) and (2.2.3b) to be the solutions of the Dirichlet and Neumann problems satisfying the additional equations (2.2.2). For the Dirichlet problem a sufficient condition is given by

Theorem 2.2

If $U(x,y) = 0$ for $x^2+y^2 > D^2$ --- (2.2.4)
and continuous for $x^2+y^2 \leqslant D^2$,
the function $u(R)$ defined by (2.2.3a) will be a solution of the Dirichlet problem subject to the conditions (2.2.2).

Proof:

The function U(R) defined by (2.2.3a) will be of class C_2 in the region $z>0$. This is a consequence of the continuity of $U(x,y)$ and the fact $\partial G(\underline{r})/\partial z$ is of class C_2 if $z>0$.
On substituting u in (2.1.1) according to (2.2.3a) and interchanging the operations of differentiation and integration, it follows directly that the Helmholtz equation is satisfied.

To prove that the assumed boundary values are attained, we write (2.2.3a) in the form

$$u(x,y,z) = -z\int_0^{2\pi}\int_0^{D} U(\rho\cos\phi,\rho\sin\phi)\frac{1}{r}\frac{dG}{dr}\rho\, d\rho\, d\phi\,, \qquad \text{--- (2.2.5)}$$

where $r^2 = (x-\rho\cos\phi)^2 + (y-\rho\sin\phi)^2 + z^2$.
In (2.2.5) integration with respect to ρ is terminated at $\rho = D$ owing to (2.2.4).

It is easy to show that $r^2 \geqslant (\sqrt{x^2+y^2} - \rho)^2 + z^2$ which is positive if $x^2+y^2>D^2$. For these values of x and y equation (2.2.4) and therefore (2.2.1a) follow by setting $z=0$ in (2.2.5).

If $x^2+y^2<D^2$ we may assume without loss of generality that $x = y = 0$. In this case $r^2 = \rho^2 + z^2$ and (2.2.5) can be written in the form

$$u(0,0,z) = -z\left(\int_0^{2\pi}\int_z^{\sqrt{d^2+z^2}} + \int_0^{2\pi}\int_{\sqrt{d^2+z^2}}^{\sqrt{D^2+z^2}}\right) U(\sqrt{r^2-z^2}\cos\phi,\sqrt{r^2-z^2}\sin\phi)\frac{dG}{dr}dr\, d\phi\,,$$

where d is an arbitrary number between 0 and D. Seeing that $r>0$ in the second integral, one may set $z=0$ and so obtain

$$\lim_{z\to 0} u(0,0,z) = \lim_{z\to 0} -z\int_0^{2\pi}\int_z^{\sqrt{d^2+z^2}} U(\sqrt{r^2-z^2}\cos\phi,\sqrt{r^2-z^2}\sin\phi)\frac{dG}{dr}dr\, d\phi$$

$$= \lim_{z\to 0} -z\int_z^{\sqrt{d^2+z^2}} \Theta(\sqrt{r^2-z^2})\frac{dG}{dr}dr\,,$$

where $\Theta(\rho) = \int_0^{2\pi} U(\rho\cos\phi,\rho\sin\phi)\, d\phi\,.$

Because of the continuity of U(x,y) the function $\Theta(\sqrt{r^2-z^2})$ in the above mentioned integral can be approximated to an arbitrary degree of accuracy by $\Theta(0) = 2\pi U(0,0)$ by taking d sufficiently small. Let

$|\Theta(\sqrt{r^2-z^2}) - 2\pi U(0,0)| < \varepsilon(d)$, then

$$\left|\int_z^{\sqrt{d^2+z^2}} \Theta(\sqrt{r^2-z^2})\frac{dG}{dr}dr - 2\pi\int_z^{\sqrt{d^2+z^2}} U(0,0)\frac{dG}{dr}dr\right| < \varepsilon(d)\,|G(\sqrt{d^2+z^2})-G(z)|$$

$$\left|-z\int_z^{\sqrt{d^2+z^2}} \Theta(\sqrt{r^2-z^2})\frac{dG}{dr}\,dr + U(0,0)\left[\frac{ze^{ik\sqrt{d^2+z^2}}}{\sqrt{d^2+z^2}} - e^{ikz}\right]\right| < \frac{\varepsilon(d)}{2\pi}\left[\frac{z}{\sqrt{d^2+z^2}}+1\right]$$

Equation (2.2.1a) follows by letting z tend to zero, bearing in mind that d is arbitrary.

To prove that equation (2.2.2) is satisfied, we write (2.2.3a) in the form

$$u(\xi,\eta,\zeta) = -\int_o^{2\pi}\int_o^{D} U(\sigma cos\theta,\sigma sin\theta)\frac{\partial G(s)}{\partial\zeta}\sigma d\sigma\, d\theta\,,$$ where

$\underline{s} = \underline{\rho} - \underline{\sigma}$ as in fig.2.1.1. For (ξ,η,ζ) on F_0

$$\partial u/\partial n = -\partial u/\partial\rho = \int_o^{2\pi}\int_o^{D} U(\sigma cos\theta,\sigma sin\theta)\frac{\partial^2 G}{\partial\rho\partial\zeta}\sigma d\sigma d\theta$$

The integrand in the first equation (2.2.2) can therefore be written in the form

$$\int_o^{2\pi}\int_o^{D} U(\underline{\sigma})\left[\frac{\partial G(s)}{\partial\zeta}\,\frac{\partial G(r)}{\partial\rho} - G(r)\,\frac{\partial^2 G(s)}{\partial\rho\partial\zeta}\right]\sigma d\sigma\, d\theta$$ or

$$\int_o^{2\pi}\int_o^{D} U(\underline{\sigma})F(\underline{s},\underline{\rho},\underline{r},)\sigma d\sigma\, d\theta$$ where

$$F(\underline{s},\underline{\rho},\underline{r}) = \frac{\zeta}{\rho}\left[\frac{dG(s)}{ds}\,\frac{dG(r)}{dr}\,\frac{\partial s}{\partial\rho}\,\frac{\partial r}{\partial\rho} - G(r)\left\{\frac{dG(s)}{ds}\,\frac{\partial^2 s}{\partial\rho^2} - \frac{d^2G(s)}{ds^2}\left(\frac{\partial s}{\partial\rho}\right)^2\right\}\right].$$

Now $\dfrac{dG(t)}{dt} = (ik - 1/t)G(t)$

$$\frac{d^2G(t)}{dt^2} = (-k^2 - 2ik/t + 2/t^2)G(t)$$

$$\frac{\partial r}{\partial\rho} = \partial\{(\underline{\rho}-\underline{R}).(\underline{\rho}-\underline{R})\}^{\frac{1}{2}}/\partial\rho$$

$$= \{(\underline{\rho}-\underline{R}).(\underline{\rho}-\underline{R})\}^{-\frac{1}{2}}(\underline{\rho}-\underline{R}).\frac{\partial\underline{\rho}}{\partial\rho}$$

$$= \frac{\underline{r}.\underline{\rho}}{r\,\rho}.$$

Furthermore

$$\frac{\partial s}{\partial \rho} \equiv \partial\{(\underline{\sigma}-\underline{\rho}).(\underline{\sigma}-\underline{\rho})\}^{\frac{1}{2}}/\partial\rho$$

$$= -\frac{\underline{s}\cdot\underline{\rho}}{s\rho},$$

$$\frac{\partial^2 s}{\partial \rho^2} = \underline{s}\cdot\underline{\rho}(s+\rho\frac{\partial s}{\partial\rho})/s^2\rho^2 - (\underline{s}\cdot\frac{\partial\underline{\rho}}{\partial\rho} + \underline{\rho}\cdot\frac{\partial\underline{s}}{\partial\rho})/s\rho$$

$$= -(\underline{s}\cdot\underline{\rho})^2/s^3\rho^2 + 1/s.$$

$$\therefore F(\underline{s},\underline{\rho},\underline{r}) = (\zeta/\rho)G(s)G(r)[(ik-1/s)(ik-1/r)\{-(\underline{s}.\underline{\rho})(\underline{r}.\underline{\rho})/sr\rho^2\}$$
$$-(ik-1/s)\{1/s-(\underline{s}.\underline{\rho})^2/s^3\rho^2\}-(-k^2-2ik/s+2/s^2)(\underline{s}.\underline{\rho})^2/s^2\rho^2].$$

As $\rho\to\infty$, $\underline{r}\to\underline{\rho}$, $\underline{s}\to-\underline{\rho}$ and

$$\rho^2F(\underline{s},\underline{\rho},\underline{r}) \to (\zeta/\rho)(e^{2ik\rho}/4\pi^2)[(ik-1/\rho)^2 + (k^2+2ik/\rho-2/\rho^2)]$$
$$\to 0.$$

Similarly

$$\rho^2F(\underline{s},\underline{\rho},\underline{r}') \to 0.$$

Equations (2.2.2) can now be written in the form

$$\lim_{\rho\to\infty} \int_0^{2\pi}\int_0^{\frac{1}{2}\pi} \left[\int_0^{2\pi}\int_0^{D} U(\underline{\sigma})F(\underline{s},\underline{\rho},\underline{r})\sigma d\sigma d\theta\right]\rho^2 \sin\gamma d\gamma d\phi = 0$$

with r replaced by r′ in the second equation. From the above it is evident that these equations are true.

(2.3) Total reflection of scalar plane waves by an infinite plane screen

Suppose a monochromatic plane wave $u_0(z)\,e^{-i\omega t}$ is incident on an infinite screen in the X-Y-plane. This normally incident field causes an excitation on the screen which is the source of a secondary field, the so-called reflected field. We assume that this field has the same time dependence as the incident field and is, in view of symmetry, independent of x and y.

The total field (incident plus reflected field) U satisfies the wave equation

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0. \qquad ---(2.3.1)$$

The total field has a time dependence $e^{-i\omega t}$ and hence u, the time-independent part of U, satisfies the Helmholtz equation (2.1.1) where $k = \omega/c$.

In view of the independence of u on x and y the Helmholtz equation reduces to

$$\frac{d^2u}{dz^2} = -k^2u. \qquad ---(2.3.2)$$

Two cases will be treated, namely sound soft and sound hard screens on which the equations

$$u = 0 \qquad ---(2.3.3s)$$

and

$$\frac{\partial u}{\partial n} = 0 \qquad ---(2.3.3h)$$

respectively hold.

The most general solution of (2.3.2) is given by

$$u = Ae^{ikz} + Be^{-ikz},$$

where A and B are complex constants.
The term Ae^{ikz} represents the incident wave and hence $Ae^{ikz} = u_0$ for the region $z \leqslant 0$. The boundary conditions (2.3.3s) and (2.3.3h) require that B=-A and B=A respectively on the left hand side of the screen.
The term Be^{-ikz} represents the a disturbance propagated in the direction of the negative z-axis. The same energy is associated with this reflected field as with the incident field. Consequently no field is transmitted across the screen.

The value of A may be taken as 1 without the loss of generality. Under these conditions the solutions of the Helmholtz equation are respectively given by

$$u = e^{ikz} - e^{ik|z|} \qquad ---(2.3.4s)$$

and

$$u = e^{ikz} - \frac{z}{|z|}\, e^{ik|z|}. \qquad ---(2.3.4h)$$

(2.4) Babinet's theorem for scalar plane waves

This theorem expresses the relationship between solutions of the Helmholtz equation with boundary values given on two infinitely thin complementary screens in a plane. Two screens are complementary if one is sound soft and the other is sound hard, and when put together, they cover an infinite plane completely without any overlap.

Suppose the plane wave of paragraph (2.3) is incident on an infinite screen in the X-Y-plane from which a finite part has been removed. The total field u is now regarded as the sum of the incident field u_0, the reflected field u_r and a diffracted field u_d.

$$u = u_0 + u_r + u_d, \qquad \text{---(2.4.1)}$$

where $u_0 = e^{ikz}$ and $u_r = -e^{ik|z|}$ or $-\frac{z}{|z|}e^{ik|z|}$ depending on whether the screen is sound soft or sound hard.

Consider next a finite screen complementary to the above screen. The total field u will in this case be regarded as the sum of the incident field u_0 and a scattered field u_s arising from the excitation of the finite screen.

$$u = u_0 + u_s \qquad \text{---(2.4.2)}$$

Babinet's theorem asserts that

$$u_s = -\frac{z}{|z|}u_d \qquad \text{---(2.4.3)}$$

for complementary screens.

We shall prove Babinet's theorem for an infinite sound soft screen S with finite aperture A and its complementary sound hard screen A with infinite aperture S. The procedure for an infinite sound hard screen and it's finite sound soft complement is analogous.

An integral representation is obtained for u_d by assuming that the total field and hence the diffracted field is continuous in the closed plane region A, and that u_d satisfies the radiation conditions (2.2.2). Since u_d vanishes on S due to equations (2.3.3s) and (2.4.1), the conditions of theorem 2.2 are met and u_d is given by (2.2.3a). In the region $z \neq 0$ therefore

$$u_d(\underline{R}) = -\iint_A u_d(\xi,\eta,0)\frac{\partial G}{\partial|z|}\,d\xi\,d\eta \,. \qquad ---(2.4.4)$$

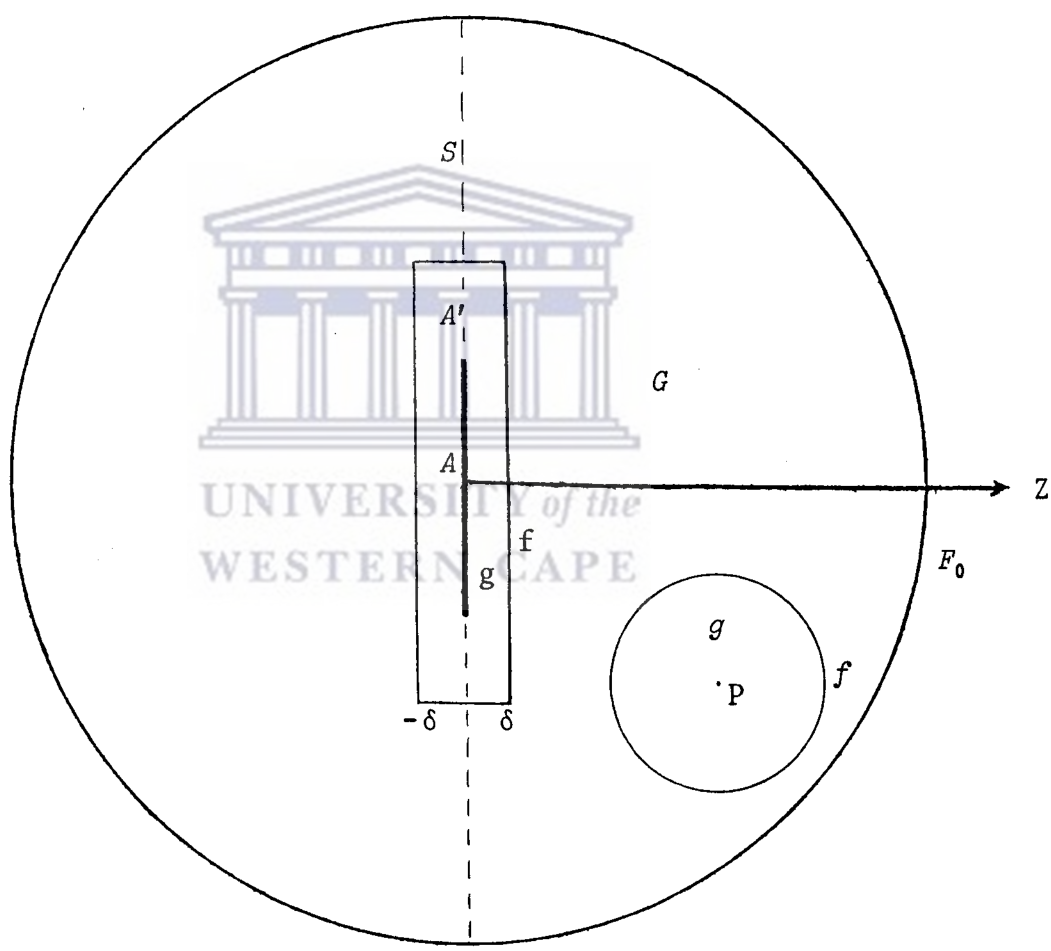

fig.2.4.1

An integral representation for u_s is obtained by applying Green's second identity to the region G-g-g in fig.2.4.1 and following the procedure used in deriving (2.1.6). By choosing $v = \frac{1}{2}G = e^{ikr}/4\pi r$ and assuming that u_s satisfies the radiation conditions (2.2.2), where F_0 is now a complete sphere, it follows that

$$u_s(\underline{R}) = \tfrac{1}{2}\iint_f [u_s \partial G/\partial n - G\partial u_s/\partial n] df \qquad \text{---(2.4.5)}$$

As in the case of equations (2.1.9a) and (2.1.9b) the derivation of (2.4.5) is based on the assumption that u_s is of class C_2 in the closed region G-g-g.

From it's definition in paragraph (2.1) it follows that on the parts of f perpendicular to the Z-axis $\partial/\partial n = \pm\, \partial/\partial\zeta$, depending on whether $\zeta \gtrless 0$. Therefore $\partial G/\partial n = \pm\partial G/\partial\zeta = \mp\,\partial G/\partial z$. Because of the symmetry of $G(\xi,\eta,\zeta)$ and $\partial G(\xi,\eta,\zeta)/\partial z$ with respect to ζ, (2.4.5) can be written in the form

$$u_s(\underline{R}) = -\tfrac{1}{2}\Big(\iint_A + \iint_{A'}\Big)\Big[\{u_s(\xi,\eta,\delta) - u_s(\xi,\eta,-\delta)\}\frac{\partial G}{\partial z}(\xi,\eta,\delta)$$
$$+\{\frac{\partial u_s}{\partial\zeta}(\xi,\eta,\delta) - \frac{\partial u_s}{\partial\zeta}(\xi,\eta,-\delta)\}\, G(\xi,\eta,\delta)\Big] d\xi\, d\eta + \varepsilon, \qquad \text{---(2.4.6)}$$

where A and A' together constitute the part of the X-Y-plane inside g and ε is an integral which vanishes as $\delta \to 0$.

From (2.4.6) it is clear that non-trivial solutions of u_s exist only if u_s and/or it's normal derivative is discontinuous. The nature of the discontinuity will determine whether one or more solutions are possible. We define u_s and $\partial u_s/\partial n$ on the right and left hand sides of A by

$$u_s(x,y,\pm 0) = \lim_{\delta\to 0} u_s(x,y,\pm\delta)$$

and a similar equation for $\partial u_s/\partial n$. From (2.3.3h) and (2.4.2) it follows that

$$\partial u_s(x,y,\pm 0)/\partial z = -ik \quad \text{on } A. \qquad \text{---(2.4.7)}$$

It will now be assumed that u_s and $\partial u_s/\partial n$ behave in such a way that the integral over A' in (2.4.6) vanishes as $\delta \to 0$. This analytical requirement, known as an edge condition, ensures the uniqueness of the field and expresses the fact that the edge cannot be a source of energy. In this regard see Jones (1964).

By letting $\delta \to 0$, it follows from (2.4.6), (2.4.7) and the edge condition that

$$u_s(\underline{R}) = -\tfrac{1}{2} \iint_A [u_s(\xi,\eta,+0) - u_s(\xi,\eta,-0)] \frac{\partial G}{\partial z}(\xi,\eta,0)\,d\xi\,d\eta .$$

Clearly u_s is anti-symmetric with respect to the X-Y-plane.

$$\therefore \quad u_s(\underline{R}) = -\iint_A u_s(\xi,\eta,+0)\frac{\partial G}{\partial z}\,d\xi\,d\eta . \qquad \text{---(2.4.8)}$$

From the above it is clear that if a solution of the Dirichlet problem exists in the region $z>0$ satisfying the conditions of theorem 2.2 as well as the edge condition, then it will be given by (2.4.8). To prove that u_s as defined by (2.4.8) is indeed a solution of the problem above, it must now be shown to satisfy the conditions of theorem 2.2 as well as the edge condition.

From (2.4.8) it follows that u_s is anti-symmetric in z and continuous everywhere except on A. Hence

$$u_s = 0 \quad \text{on } S, \qquad \text{---(2.4.9)}$$

fulfilling one of the conditions of theorem 2.2.
Differentiation of (2.4.8) shows that $\partial u_s/\partial z$ is symmetric with respect to z. Using these results it is easily verified that u_s satisfies the edge condition.

Equation (2.4.8) also defines the unique solution of the Dirichlet problem in the region $z<0$ which satisfies the radiation and edge conditions and is continuous on the left hand side of A.

Assuming that $\partial u/\partial z$ is continuous across the aperture, it follows from (2.4.1) that

$$u_d(x,y,+0)/\partial z = u_d(x,y,-0)/\partial z + 2ik \quad \text{on } A.$$

From (2.4.4) it follows that $\partial u_d/\partial z$ is anti-symmetric,

$$\therefore \quad \partial u_d(x,y,\pm 0)/\partial z = \pm ik \quad \text{on } A. \qquad \text{---(2.4.10)}$$

The proof of Babinet's theorem now depends on the validity of the following assumption:

There exists a unique function u(x,y,z) of class C_2 satisfying the Helmholtz equation and radiation conditions in the region $z < 0$, such that u is continuous on *A*, $u = 0$ on *S* and $\partial u/\partial z = -ik$ on *A*.

This assumtion is a corollary of the theorem that the integral equation

$$\iint_A U(\xi,\eta) \frac{\partial^2 G}{\partial z^2}(\xi,\eta,0,x,y,0)\, d\xi d\eta = -2\pi ik \qquad ---(2.4.11)$$

has a unique solution for U(x,y), (x,y) on *A*. Equation (2.4.11) was obtained from (2.4.4) by differentiation. In their proof of Babinet's principle in two dimensions Baker and Copson (1950) take the validity of a similar theorem for granted.

(2.5) Electromagnetic waves and the vector Helmholtz equation

Suppose a monochromatic linearly polarized electromagnetic field, harmonic in time, is perpendicularly incident on an infinite screen *S* with finite aperture *A* in a homogeneous isotropic medium. We assume that the total field will have the same time dependence. The ℓ^{th} component of electric and magnetic field strengths can therefore be written in the form

$$E_\ell = E_\ell^h e^{-i\omega t}$$
$$H_\ell = H_\ell^h e^{-i\omega t}, \qquad ---(2.5.1)$$

where the 1-, 2- and 3-components are respectively along the X-, Y- and Z-axes.

Let the incident electric field strength be given by $E e^{ikz} \delta_{\ell 2}$ where the Z-axis is perpendicular to the screen, $\delta_{\ell m}$ is the Kronecker delta, $k = \omega/c$ and $c = \sqrt{\epsilon\mu}$. The incident magnetic field vector can be obtained from Maxwell's equations for harmonic waves:

$$\varepsilon_{\ell mn} \partial_m E_n^h = i\omega\mu H_\ell^h$$
$$\varepsilon_{\ell mn} \partial_m H_n^h = -i\omega\mu H_\ell^h. \qquad ---(2.5.2)$$

In (2.5.2) $(\partial_1,\partial_2,\partial_3) = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ and $\varepsilon_{\ell mn}$ is the permutation tensor. For the incident wave the magnetic field strength is therefore given by $-\frac{kE}{\omega\mu} e^{ikz}\delta_{\ell 1} = -\sqrt{\epsilon/\mu}\, E\, e^{ikz}\delta_{\ell 1}$.

For an arbitrary field let

$$E_\ell = E\, e_\ell$$
$$H_\ell = \sqrt{\epsilon/\mu}\, E\, h_\ell \qquad \text{---(2.5.3)}$$

where e_ℓ and h_ℓ are dimensionless quantities. Then the incident fields are given by

$$e^{(i)} = e^{ikz}\delta_{\ell 2}$$
$$h^{(i)} = -e^{ikz}\delta_{\ell 1} \qquad \text{---(2.5.4)}$$

and Maxwell's equations become

$$\varepsilon_{\ell mn}\partial_m e_n = ikh_\ell$$
$$\varepsilon_{\ell mn}\partial_m h_n = -ike_\ell . \qquad \text{---(2.5.5)}$$

By eliminating h_ℓ and e_ℓ alternatively from (2.5.5) the vector Helmholtz equations

$$\nabla^2 e_\ell + k^2 e_\ell = 0$$
$$\nabla^2 h_\ell + k^2 h_\ell = 0 \qquad \text{---(2.5.6)}$$

are obtained.

In order to obtain unique solutions for (2.5.6) in the half-space $z>0$, more knowledge about the nature of the electromagnetic field is required on the boundaries. By assuming that the screen is infinitely thin and perfectly conducting we have

$$e_1 = e_2 = 0 \qquad \text{on } S. \qquad \text{---(2.5.7)}$$

Assuming that the field behind the screen complies with the radiation condition (2.2.2), it follows from theorem 2.2 that this field has 1- and 2-components given by

$$e_\ell(\underline{R}) = -\iint_A e_\ell(\xi,\eta,0)\frac{\partial G}{\partial z}\, d\xi\, d\eta \qquad (\ell=1,2). \qquad \text{---(2.5.8)}$$

A representation for e_3 is obtained by observing that the equation

$$\partial_\ell e_\ell = 0 \qquad \text{---(2.5.9)}$$

is a consequence of Maxwell's equations (2.5.5) wherever e_ℓ is of class C_2. Bearing in mind that the normal derivative in (2.2.3b) is actually the limit of $\partial u(\xi,\eta,\zeta)/\partial\zeta$ as $\zeta \to 0$, the function u may be regarded to be of class C_2 on the plane $z = 0$. From (2.5.7) and (2.5.9) therefore

$$\frac{\partial e_3}{\partial\zeta} = \frac{\partial e_1}{\partial\xi} + \frac{\partial e_2}{\partial\eta} = 0 \quad \text{on } S. \qquad \text{---(2.5.10)}$$

Application of (2.2.3b) yields for $z>0$

$$e_3(\underline{R}) = -\iint_A \frac{\partial e_3}{\partial\zeta}(\xi,\eta,+0)G\,d\xi\,d\eta \qquad \text{---(2.5.11)}$$

$$= \iint_A \left[\frac{\partial e_1}{\partial\xi}(\xi,\eta,+0) + \frac{\partial e_2}{\partial\eta}(\xi,\eta,+0)\right]G\,d\xi\,d\eta$$

$$= \iint_A \left[e_1\frac{\partial G}{\partial x} + e_2\frac{\partial G}{\partial y}\right]d\xi\,d\eta + I, \qquad \text{---(2.5.12)}$$

where

$$I = \iint_A \left[\frac{\partial}{\partial\xi}(e_1G) + \frac{\partial}{\partial\eta}(e_2G)\right]d\xi\,d\eta. \qquad \text{---(2.5.13)}$$

Differentiation (2.5.12) yields

$$\frac{\partial I}{\partial z} = \frac{\partial e_3}{\partial z} - \iint_A \left(e_1\frac{\partial^2 G}{\partial z\,\partial x} + e_2\frac{\partial^2 G}{\partial z\,\partial y}\right)d\xi\,d\eta$$

$$= \frac{\partial e_3}{\partial z} - \frac{\partial}{\partial x}\iint_A e_1\frac{\partial G}{\partial z}\,d\xi\,d\eta - \frac{\partial}{\partial y}\iint_A e_2\frac{\partial G}{\partial z}\,d\xi\,d\eta$$

$$= 0,$$

where (2.5.8) and (2.5.9) were used.

From (2.5.13) and $\partial I/\partial z = 0$ it follows that

$$\iint_A \left[\frac{\partial}{\partial\xi}(e_1G_r) + \frac{\partial}{\partial\eta}(e_2G_r)\right]d\xi\,d\eta = 0, \quad \text{where} \quad G_r = \frac{\partial G}{\partial r}.$$

But from (2.5.13)

$$\frac{\partial I}{\partial x} = -\frac{x}{r}\iint_A \left[\frac{\partial}{\partial\xi}(e_1G_r) + \frac{\partial}{\partial\eta}(e_2G_r)\right]d\xi\,d\eta,$$

hence $\partial I/\partial x = 0$ and similarly $\partial I/\partial y = 0$.

From (2.5.12) it follows that I must satisfy the Helmholtz equation, hence $I = 0$.

The electric field in the region z>0 can now be written the form

$$
\left.\begin{aligned}
e_\ell(\underline{R}) &= -\iint_A e_\ell G_3 \, d\xi \, d\eta \quad (\ell = 1,2) \\
e_3(\underline{R}) &= \iint_A (e_1 G_1 + e_2 G_2) \, d\xi \, d\eta ,
\end{aligned}\right\} \qquad \text{---(2.5.14)}
$$

where $G_\ell = \partial G/\partial x_\ell = -\partial G/\partial \xi_\ell$. ---(2.5.15)

By applying Maxwell's equations (2.5.5) to (2.5.14) an integral representation of the magnetic field is obtained which is valid in the region z>0:

$$
\left.\begin{aligned}
h_1(\underline{R}) &= \frac{1}{ik}\iint_A [e_1 G_{12} + e_2(G_{22}+G_{33})] d\xi \, d\eta \\
h_2(\underline{R}) &= \frac{-1}{ik}\iint_A [e_1(G_{11}+G_{33}) + e_2 G_{12}] d\xi \, d\eta \\
h_3(\underline{R}) &= \frac{1}{ik}\iint_A [e_1 G_{23} - e_2 G_{13}] d\xi \, d\eta ,
\end{aligned}\right\} \qquad \text{---(2.5.16)}
$$

where $G_{\ell m} = \dfrac{\partial^2 G}{\partial x_\ell \partial x_m}$. ---(2.5.17)

(2.6) Babinet's theorem for electromagnetic plane waves

Suppose the incident wave of paragraph (2.5) impinges on a perfectly conducting screen, situated on the X-Y-plane. Assuming that the total field is independent of x and y, the electric field will be given by

$$e_\ell = A_\ell e^{ikz} + B_\ell e^{-ikz} \qquad \text{---(2.6.1)}$$

The boundary condition (2.5.7) demands that $A_1 = -B_1$ and $A_2 = -B_2$. From (2.5.9) and (2.6.1) it follows that $A_3 = B_3 = 0$. Hence

$$e_1 = A_1(e^{ikz} - e^{-ikz})$$

$$e_2 = A_2(e^{ikz} - e^{-ikz})$$

$$e_3 = 0.$$

The terms involving the factors e^{ikz} and e^{-ikz} respectively denote disturbances travelling in the positive and negative

Z-direction. In the region $z<0$, the terms involving e^{ikz} must be equal to the incident wave. It follows that $A_1 = 0$ and $A_2 = 1$ in this region. Energy considerations as in paragraph 2.3 lead to the conclusion that the field vanishes behind the screen. For every value of z one may therefore write

$$e_\ell = e_\ell^{(i)} + e_\ell^{(r)}, \qquad ---(2.6.2)$$

where

$$e^{(r)} = -e^{ik|z|}\,\delta_{\ell 2}. \qquad ---(2.6.3)$$

The magnetic field is obtained by applying Maxwell's equations (2.5.5) to (2.6.2). Thus

$$h_\ell = h_\ell^{(i)} + h_\ell^{(r)}, \qquad ---(2.6.4)$$

where

$$h^{(r)} = \frac{z}{|z|}\,e^{ik|z|}\,\delta_{\ell 1}. \qquad ---(2.6.5)$$

If the screen is now perforated as in paragraph 2.5, the total electric field can be written in the form

$$e_\ell = e_\ell^{(i)} + e_\ell^{(r)} + e_\ell^{(d)}, \qquad ---(2.6.6)$$

where $e_\ell^{(d)}$ satisfies the radiation conditions (2.2.2) on both sides of the screen. As e_ℓ and $e_\ell^{(i)} + e_\ell^{(r)}$ satisfy the boundary conditions (2.5.7), it follows from (2.6.6) that (2.5.7) applies to $e_\ell^{(d)}$. Integral expressions for $e_1^{(d)}$ and $e_2^{(d)}$ in the region $z \neq 0$ can therefore be obtained from (2.5.8) by substituting $|z|$ for z. An expression for $e_3^{(d)}$ is found by observing that (2.5.11) was obtained from (2.2.3b) by putting $U_n(\xi,\eta) = \partial e_3(\xi,\eta,+0)/\partial\zeta$. In (2.5.11) appropriate changes in sign have to be made if it is to apply to the region $z<0$. Hence for $z \neq 0$,

$$\left.\begin{aligned} e_\ell^{(d)}(x,y,\pm|z|) &= \mp \iint_A e_\ell^{(d)}(\xi,\eta,+0)\frac{\partial G}{\partial z}\,d\xi\,d\eta \qquad (\ell = 1,2) \\ e_3\,(x,y,\pm|z|) &= \mp \iint_A \frac{\partial e_3^{(d)}}{\partial \zeta}(\xi,\eta,+0)\,G\,d\xi\,d\eta\,. \end{aligned}\right\} \qquad ---(2.6.7)$$

For the complementary problem where A is the screen and S the infinite aperture, the field $e_\ell^{(s)}$ is defined by

$$e_\ell = e_\ell^{(i)} + e_\ell^{(s)}, \qquad \text{---(2.6.8)}$$

where $e_\ell^{(s)}$ satisfies (2.2.2). Bearing in mind that

$$e_\ell = \delta_{\ell 2} + e_\ell^{(s)} = 0, \quad (\ell = 1,2) \quad \text{on } A \qquad \text{---(2.6.9)}$$

and

$$\frac{\partial e_3^{(s)}}{\partial \zeta} = -\frac{\partial e_1^{(s)}}{\partial \xi} - \frac{\partial e_2^{(s)}}{\partial \eta} = 0 \quad \text{on } A \qquad \text{---(2.6.10)}$$

we obtain the following representation of $e_\ell^{(s)}$ from (2.4.6) after replacing u_s by $e_\ell^{(s)}$:

$$\left.\begin{aligned} e_\ell^{(s)}(\underline{R}) &= -\iint_A \frac{\partial e_\ell^{(s)}}{\partial \zeta}(\xi,\eta,+0)G\, d\xi\, d\eta \qquad (\ell = 1,2) \\ e_3^{(s)}(\underline{R}) &= -\iint_A e_3^{(s)}(\xi,\eta,+0)\frac{\partial G}{\partial z}\, d\xi\, d\eta\,. \end{aligned}\right\} \qquad \text{---(2.6.11)}$$

Babinet's theorem for electromagnetic waves states that

$$\left.\begin{aligned} \underline{e}^{(s)} &= \frac{z}{|z|}\,\underline{h}^{(d)} \\ \underline{h}^{(s)} &= -\frac{z}{|z|}\,e^{(d)} \end{aligned}\right. \qquad \text{---(2.6.12)}$$

These two equations are of course not independent of one another. Any one can be obtained from the other by utilizing Maxwell's equations (2.5.5).

The first equation of (2.6.12) is now proved by assuming that there exists a unique source free vector field $\underline{u}$ satisfying the Helmholtz equation in the region $z>0$ as well as the radiation and edge conditions and, in addition, having the properties:

$$\left.\begin{aligned} u_1 &= 0 \\ u_2 &= -1 \\ \partial u_3/\partial z &= 0 \end{aligned}\right\} \quad \text{on } A \qquad \text{---(2.6.13)}$$

and

$$\left.\begin{aligned} \partial u_1/\partial z &= 0 \\ \partial u_2/\partial z &= 0 \\ u_3 &= 0. \end{aligned}\right\} \quad \text{on } S \qquad \text{---(2.6.14)}$$

For the electric field $\underline{e}^{(s)}$ the equations (2.6.13) follow from (2.6.9) and (2.6.10). Equations (2.6.14) can be derived from the anti-symmetry and continuity properties of the quantities involved.

By assuming that h_1 and h_2 are continuous across the aperture A and keeping in mind that $h_1^{(d)}$ and $h_2^{(d)}$ are anti-symmetric, it it follows that $h_1^{(d)} = 0$ and $h_2^{(d)}(x,y,\pm 0) = \mp 1$ on A. The third equation of (2.6.14) is satisfied by $h_3^{(s)}$, because $\partial_\ell h_\ell = 0$. According to Maxwell's equations (2.5.5)

$$\left.\begin{aligned} ikh_3^{(d)} &= \frac{\partial e_2^{(d)}}{\partial x} - \frac{\partial e_1^{(d)}}{\partial y} = 0 \\ \frac{\partial h_1^{(d)}}{\partial z} &= \frac{\partial h_3^{(d)}}{\partial x} - ike_2^{(d)} = 0 \\ \frac{\partial h_2^{(d)}}{\partial z} &= \frac{\partial h_3^{(d)}}{\partial y} + ike_1^{(d)} = 0 \end{aligned}\right\} \quad \text{on } S.$$

Hence $\underline{e}^{(s)}$ and $\underline{h}^{(d)}$ satisfy the same boundary conditions and by the uniqueness assumption the first equation of (2.6.12) follows.

CHAPTER 3

SOMMERFELD'S SOLUTION FOR THE DIFFRACTION OF PLANE WAVES BY A HALF-PLANE

(3.1) The scalar case

We now consider the same problem as in paragraph (2.2), except that the part of the screen for which $x < 0$, is removed. The resultant stationary field is independent of y (see fig. 3.1.1) and therefore satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0. \qquad \text{---(3.1.1)}$$

In addition to this equation the boundary conditions (2.3.3s) or (2.3.3h) hold on the screen.

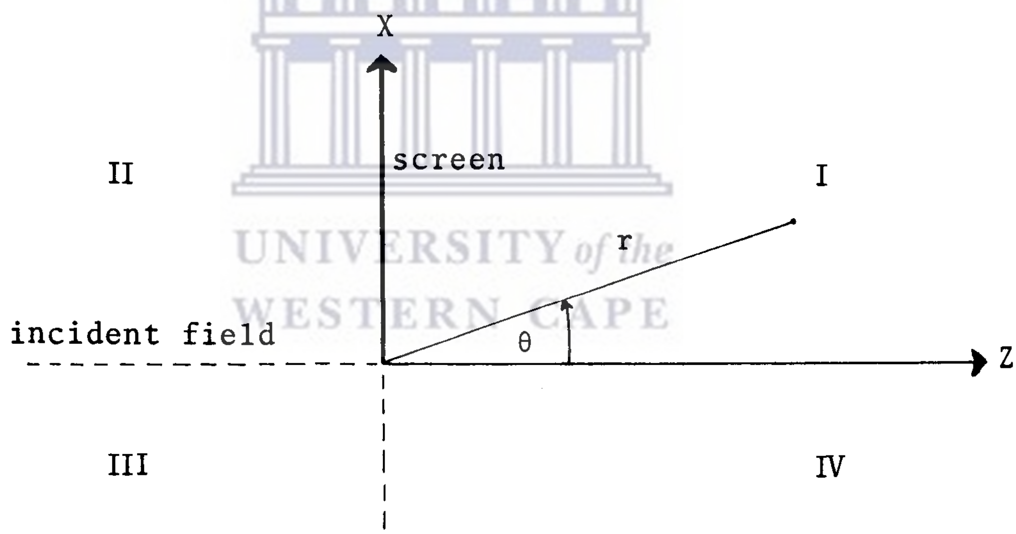

fig.3.1.1

Sommerfeld (1896) solved this problem by modifying the method of images. this entails writing the solution in the forms

$$u(r,\theta) = v(r,\theta) - v(r,\theta+\pi) \qquad \text{---(3.1.2s)}$$

and

$$u(r,\theta) = v(r,\theta) + v(r,\theta+\pi) \qquad \text{---(3.1.2h)}$$

in the sound soft and sound hard cases respectively. The variables r and θ are defined by

$z = r\cos\theta$

$x = r\sin\theta$,

where the domain of θ is still to be determined. Due to (2.3.3s) and (2.3.3h) the function $v(r,\theta)$ must of course respectively satisfy

$$v(r,\theta_0) = v(r,\theta_0+\pi) \qquad \text{---(3.1.3s)}$$

and

$$\partial v(r,\theta_0)/\partial\theta = -\partial v(r,\theta_0+\pi)/\partial\theta\ , \qquad \text{---(3.1.3h)}$$

where θ_0 is the value of θ on the screen. Equation (3.1.3h) was obtained from (3.1.2h), the equation

$$\partial/\partial z = \cos\theta\, \partial/\partial r - r^{-1}\sin\theta\, \partial/\partial\theta$$

and the boundary condition $\partial u(r,\theta)/\partial z\,|_{\theta=\theta_0} = 0$.

By separating the variables in (3.1.1), the solution $u = A\,e^{i(k_1x+k_3z)}$ is found, where k_1 and k_3 are real and $k_1^2+k_3^2=k^2$. This can also be written in the form $u = A\,e^{ikr\cos(\theta-\alpha)}$, which defines the incident wave when A=1 and α=0. The function

$$u = \int_K A(\alpha,\theta)\, e^{ikr\cos\alpha} d\alpha \qquad \text{---(3.1.4)}$$

is a general solution from which particular solutions can be found by a suitable choice of $A(\alpha,\theta)$ and K. For example if $A(\alpha,\theta)$ is a complex function of α possesing a first order pole with residue $\frac{1}{2\pi i}$ at $\alpha=\theta$ and K a closed path in the complex α-plane enclosing no singularities except for the above mentioned pole, the integral (3.1.4) reduces to the incident wave.

The function $A(\alpha,\theta)$ and the curve K can now be chosen in such a way that the integral (3.1.4) satisfies the conditions (3.1.3s) and (3.1.3h). This is accomplished by defining

$$v(r,\theta) = \frac{1}{2\pi}\int_K \frac{e^{\frac{1}{2}i\alpha}\, e^{ikr\cos\alpha}}{e^{\frac{1}{2}i\alpha} - e^{-\frac{1}{2}i\theta}}\, d(\tfrac{1}{2}\alpha) \qquad \text{---(3.1.5)}$$

where K is the path indicated in fig.3.1.2, $-2\pi<\theta<2\pi$ and $\theta\neq 0$.

α-plane

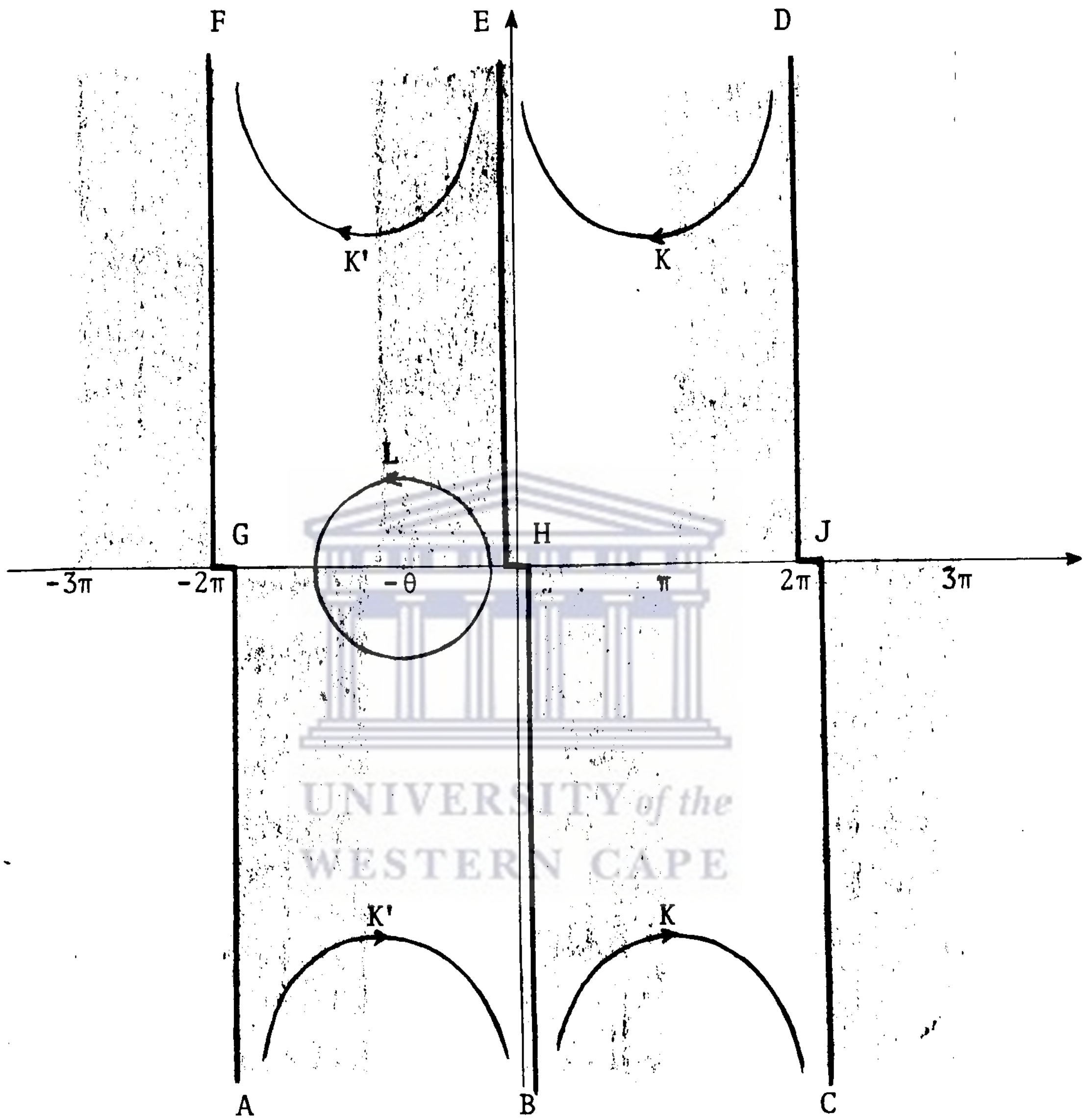

fig.3.1.2

First of all it must be proved that the infinite integral (3.1.5) exists. The shadowed regions in fig.3.1.2 are those where the real part of ik$r\cos\alpha$ is negative. When the imaginary part of α tends to plus or minus infinity in these areas, the integral can be approximated to an arbitrary degree of accuracy by a constant times e^{-s} where s is the variable of integration. In the case $r=0$ the convergence of the integral (3.1.5) is not so obvious. On putting $e^{\frac{1}{2}i\alpha} = z$, (3.1.5) can be written in the form

$$v(0,\theta) = \lim_{a\to\infty} \frac{1}{2\pi i}\left(\int_{a}^{ae^{i\pi}} + \int_{a^{-1}e^{i\pi}}^{a^{-1}} \right) \frac{dz}{z-b}$$

$$= \lim_{a\to\infty} \frac{1}{2\pi i}\left(\log\frac{ae^{i\pi}-b}{a-b} + \log\frac{a^{-1}-b}{a^{-1}e^{i\pi}-b}\right)$$

i.e.

$$v(0,\theta) = \tfrac{1}{2}. \qquad ---(3.1.6)$$

Secondly the function $v(r,\theta)$ posesses the property

$$v(r,\theta) + v(r,\theta-2\pi) = e^{ikr\cos\theta} \qquad ---(3.1.7)$$

which is proved by showing that $v(r,\theta) + v(r,\theta-2\pi)$ is identical to the right hand side of (3.1.5) with K replaced by the closed path L in fig.3.1.2. This identity can be proved by noting that

$$v(r,\theta-2\pi) = \frac{1}{2\pi}\int_K \frac{e^{\frac{1}{2}i\alpha}\ e^{ikr\cos\alpha}}{e^{\frac{1}{2}i\alpha}-e^{-\frac{1}{2}i(\theta-2\pi)}}\ d(\tfrac{1}{2}\alpha)$$

$$= \frac{1}{2\pi}\int_{K'} \frac{e^{\frac{1}{2}i\alpha}\ e^{ikr\cos\alpha}}{e^{\frac{1}{2}i\alpha}-e^{-\frac{1}{2}i\theta}}\ d(\tfrac{1}{2}\alpha).$$

(See fig.3.1.2 where K and K' are defined.)

Therefore

$$v(r,\theta) + v(r,\theta-2\pi) = \frac{1}{2\pi}\left(\int_K + \int_{K'}\right) \frac{e^{\frac{1}{2}i\alpha}\ e^{ikr\cos\alpha}}{e^{\frac{1}{2}i\alpha}-e^{-\frac{1}{2}i\theta}}\ d(\tfrac{1}{2}\alpha)$$

The path in the above integral may be extended to include CD and FA in fig.3.1.2 because of the periodicity of 4π of the integrand and the fact that AF can be brought into coincidence with CD by a translation of 4π along the real axis. The closed path thus completed may now be deformed into L without the integral changing it's value. The integrand has only one singularity in the region $-2\pi < \alpha < 2\pi$, namely a first order pole with residue $-e^{ikr\cos\theta}$ at $\alpha = -\theta$. The equation (3.1.7) is obtained by applying Cauchy's integral formula.

Thirdly the proof of the properties (3.1.3s) and (3.1.3h) follows from the representation of $v(r,\theta)$ by a Fresnel integral.

(3.2) The scalar solution in tems of Fresnel integrals

The path K in the definition of $v(r,\theta)$ may be deformed into two paths BHJC and DJHE in fig.3.1.2. The parts of these integrals along the real axis are equal, but of opposite sign if $\theta > 0$. In this case

$$v(r,\theta) = \frac{1}{2\pi}\left(\int_{BE} + \int_{DC}\right) \frac{e^{\frac{1}{2}i\alpha} \; e^{ikr\cos\alpha}}{e^{\frac{1}{2}i\alpha} - e^{-\frac{1}{2}i\theta}} \, d(\tfrac{1}{2}\alpha) \,. \qquad ---(3.2.1)$$

If $\theta < 0$, the pole on the real axis is circled once and $e^{ikr\cos\theta}$ must be added to the above expression. Another method of finding the value of $v(r,\theta)$ when $\theta < 0$, is obtained by writing

$$v(r,\theta-2\pi) = \frac{1}{2\pi}\left(\int_{AF} + \int_{EB}\right) \frac{e^{\frac{1}{2}i\alpha} \; e^{ikr\cos\alpha}}{e^{\frac{1}{2}i\alpha} - e^{-\frac{1}{2}i\theta}} \, d(\tfrac{1}{2}\alpha)$$

$$= \frac{-1}{2\pi}\left(\int_{BE} + \int_{DC}\right) \frac{e^{\frac{1}{2}i\alpha} \; e^{ikr\cos\alpha}}{e^{\frac{1}{2}i\alpha} - e^{-\frac{1}{2}i\theta}} \, d(\tfrac{1}{2}\alpha)$$

and applying (3.1.7).

Equation (3.2.1) which holds for $\theta > 0$ can be simplified by writing the integral along DC as an integral along EB in which α is replaced by $\alpha + 2\pi$. One may therefore write

$$v(r,\theta) = \frac{1}{4\pi}\int_{-i\infty}^{i\infty} f(\alpha,\theta)\, e^{ikr\cos\alpha}\, d\alpha$$

$$= \frac{1}{4\pi}\int_{0}^{i\infty} [f(\alpha,\theta) + f(-\alpha,\theta)]\, e^{ikr\cos\alpha}\, d\alpha,$$

where

$$f(\alpha,\theta) = e^{\frac{1}{2}i\alpha}\left(\frac{1}{e^{\frac{1}{2}i\alpha}-e^{-\frac{1}{2}i\theta}} - \frac{1}{e^{\frac{1}{2}i\alpha}+e^{-\frac{1}{2}i\theta}}\right)$$

$$= \frac{2e^{\frac{1}{2}i(\alpha-\theta)}}{e^{i\alpha} - e^{-i\theta}}.$$

Now

$$f(\alpha,\theta) + f(-\alpha,\theta) = \frac{2e^{-\frac{1}{2}i\theta}\,[\,e^{\frac{1}{2}i\alpha}(e^{-i\alpha}-e^{-i\theta}) + e^{-\frac{1}{2}i\alpha}(e^{i\alpha}-e^{-i\theta})]}{(e^{i\alpha}-e^{-i\theta})(e^{-i\alpha}-e^{-i\theta})}$$

$$= \frac{2e^{\frac{1}{2}i\theta}[\,e^{-\frac{1}{2}i\alpha}+e^{\frac{1}{2}i\alpha} - e^{-i\theta}(e^{\frac{1}{2}i\alpha}+e^{-\frac{1}{2}i\alpha})]}{e^{i\theta} - e^{i\alpha} - e^{-i\alpha} + e^{-i\theta}}$$

$$= \frac{e^{\frac{1}{2}i\theta}(e^{\frac{1}{2}i\alpha} + e^{-\frac{1}{2}i\alpha})(1-e^{-i\theta})}{\cos\theta - \cos\alpha}$$

$$= \frac{-4i\cos\frac{1}{2}\alpha\,\sin\frac{1}{2}\alpha}{\cos\alpha - \cos\theta}.$$

Therefore

$$v(r,\theta) = -\frac{i}{\pi}\sin\tfrac{1}{2}\theta\int_{0}^{i\infty}\frac{e^{ikr\cos\alpha}\cos\frac{1}{2}\alpha}{\cos\alpha - \cos\theta}\, d\alpha. \qquad ---(3.2.2)$$

The integral (3.2.2) is readily evaluated by firstly differentiating with respect to r; then integrating with respect to α and finally integrating with respect to r. From (3.2.2)

$$\partial[v(r,\theta)e^{-ikr\cos\theta}]/\partial r = \frac{k}{\pi}\sin\tfrac{1}{2}\theta\int_{0}^{i\infty} e^{ikr(\cos\alpha-\cos\theta)}\cos\tfrac{1}{2}\alpha\; d\alpha$$

$$= \frac{k}{\pi}\sin\tfrac{1}{2}\theta\, e^{2ikr\sin^2\frac{1}{2}\theta}\int_{0}^{i\infty} e^{-2ikr\sin^2\frac{1}{2}\alpha}\cos\tfrac{1}{2}\alpha\; d\alpha$$

$$= \frac{2ik}{\pi r}\sin\tfrac{1}{2}\theta\, e^{2ikr\sin^2\frac{1}{2}\theta}\int_{0}^{\infty} e^{i\pi\tau^2} d\tau,$$

$$---(3.2.3)$$

where $\tau = -i(2kr/\pi)^{\frac{1}{2}}\sin\frac{1}{2}\alpha$.

The integral $\int_0^\infty e^{i\pi\tau^2}\,d\tau$ can be evaluated by considering the related integral $\oint_C e^{-\pi z^2}\,dz$ where C is the closed contour in fig.3.2.1. The integral vanishes because the integrand is analytic.

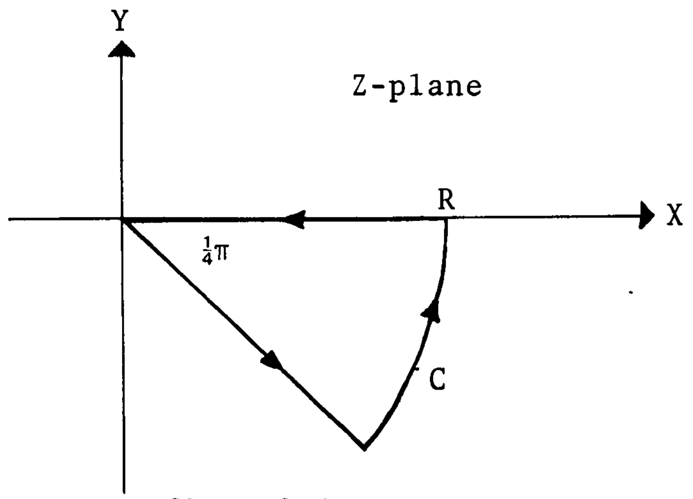

fig.3.2.1

$$\therefore\ \int_R^0 e^{-\pi x^2}\,dx + e^{-\frac{1}{4}i\pi}\int_0^R e^{i\pi r^2}\,dr + iR\int_{-\frac{1}{4}\pi}^0 e^{-\pi R^2(\cos 2\theta + i\sin 2\theta)}e^{i\theta}\,d\theta = 0. \qquad \text{---(3.2.4)}$$

The absolute value of the last term is smaller than or equal to

$$R\int_{-\frac{1}{4}\pi}^{0} e^{-\pi R^2\cos 2\theta}\,d\theta = R\int_0^{\frac{1}{2}\pi} e^{-\pi R^2\sin\phi}\,d\phi$$

$$\leq \tfrac{1}{2}R\int_0^{\frac{1}{2}\pi} e^{-2R^2\phi}\,d\phi\,,$$

where Jordan's inequality $2\pi \leq \pi\sin\phi$ (see Copson (1970)) was used. Hence the last term $\to 0$ as $R\to\infty$. Also

$$\int_0^\infty e^{-\pi x^2}\,dx = \left[\int_0^\infty\int_0^\infty e^{-\pi(x^2+y^2)}\,dx\,dy\right]^{\frac{1}{2}}$$

$$= \left[\int_0^\infty\int_0^{\frac{1}{2}\pi} e^{-\pi r^2}\,r\theta\,d\theta\,dr\right]^{\frac{1}{2}}$$

$$= \tfrac{1}{2}.$$

Hence from (3.2.4) we get

$$e^{-\frac{1}{4}i\pi}\int_0^\infty e^{i\pi r^2}\,dr = \tfrac{1}{2}. \qquad \text{---(3.2.5)}$$

From (3.2.3) and (3.2.5) it follows that

$$\partial[v(r,\theta)e^{-ikr\cos\theta}]/\partial r = i\left(\frac{k}{2\pi r}\right)^{\frac{1}{2}} \sin\tfrac{1}{2}\theta\, e^{\frac{1}{4}i\pi} e^{2ikr\sin^2\frac{1}{2}\theta} .$$

Integrate with respect to r:

$$v(r,\theta)e^{-ikr\cos\theta} = -\left(\frac{k}{2\pi}\right)^{\frac{1}{2}} \sin\tfrac{1}{2}\theta\, e^{-\frac{1}{4}i\pi}\int r^{-\frac{1}{2}} e^{2ikr\sin^2\frac{1}{2}\theta} dr ,$$

or

$$v(r,\theta) = e^{-\frac{1}{4}i\pi} e^{ikr\cos\theta}\int_{-\infty}^{T} e^{i\pi\tau^2} d\tau , \qquad \text{---(3.2.6)}$$

where $\int_{-\infty}^{T} e^{i\pi\tau^2} d\tau$ is an integral of the Fresnel type and

$$T = \sqrt{2kr\pi}\sin\tfrac{1}{2}\theta . \qquad \text{---(3.2.7)}$$

The arbitrary function of θ arising from the integration was chosen to be zero because for $r=0$ the above expression yields $v(0,\theta) = \frac{1}{2}$ which is in accordance with equation (3.1.6). (This can be verified by setting $r=0$ and $\theta=\pi$ in (3.2.2).)

The equation (3.2.6) has been derived from (3.2.1) which holds for $0<\theta<2\pi$. When $-2\pi<\theta<0$, the constant 1 has to be added to the expression obtained above for $v(r,\theta)e^{-ikr\cos\theta}$. This implies a new choice for the arbitrary function of θ. In order to yield $v(0,\theta) = \frac{1}{2}$ as (3.1.6) requires, the function referred to has to be chosen equal to -1. (This can be verified by adding $e^{ikr\cos\theta}$ to the expression obtained for $v(r,\theta)$ in (3.2.2) and then setting $r=0$, $\theta=-\pi$.)

Although v was left undefined at the singularities ($\theta=\pm 2n\pi$; $n=1,2,3...$) of the integrand in (3.1.5), the derived representation (3.2.6), obtained by deforming the path of integration, is meaniningful at these points. We therefore take (3.2.6) to be the definition of $v(r,\theta)$ for all values of r and θ.

To ensure that $u(r,\theta)$ as defined by (3.1.2s) and (3.1.2h) exhibits the correct asymtotic behaviour, namely that it tends to zero in region I, $e^{ikr\cos\theta} \mp e^{-ikr\cos\theta}$ in region II and $e^{ikr\cos\theta}$ in regions III and IV, the domain of θ must be restricted to the interval $[-\frac{3}{2}\pi, \frac{1}{2}\pi]$.

The properties (3.1.3s) and (3.1.3h) follow readily from the definition (3.2.6) and the expression for the derivative

$$\partial v(r,\theta)/\partial\theta = -e^{-\frac{1}{4}i\pi}e^{ikr\cos\theta}\left[e^{i\pi T^2}\left(\frac{kr}{2\pi}\right)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta + ikr\sin\theta\int_{-\infty}^{-T} e^{i\pi\tau^2}\,d\tau\right]. \qquad ---(3.2.8)$$

Clearly, by taking $\theta_0 = \frac{1}{2}\pi$, it follows from (3.2.6) and (3.2.8) that

$$v(r,\tfrac{1}{2}\pi) = v(r,\tfrac{3}{2}\pi) = e^{-\frac{1}{4}i\pi}\int_{-\infty}^{-\sqrt{kr/\pi}} e^{i\pi\tau^2}\,d\tau \qquad ---(3.2.9s)$$

and

$$v(r,\tfrac{1}{2}\pi)/\partial\theta = -\partial v(r,\tfrac{3}{2}\pi)/\partial\theta = -e^{-\frac{1}{4}i\pi}\left[\tfrac{1}{2}\sqrt{kr/\pi}\,e^{ikr} + ikr\int_{-\infty}^{-\sqrt{kr/\pi}} e^{i\pi\tau^2}\,d\tau\right]. \qquad ---(3.2.9h)$$

The conditions (3.1.3s) and (3.1.3h) are also satisfied if θ_0 is taken to be $-\frac{3}{2}\pi$:

$$v(r,-\tfrac{3}{2}\pi) = v(r,-\tfrac{1}{2}\pi) = e^{-\frac{1}{4}i\pi}\int_{-\infty}^{\sqrt{kr/\pi}} e^{i\pi\tau^2}\,d\tau \qquad ---(3.2.10s)$$

and

$$\partial v(r,-\tfrac{3}{2}\pi)/\partial\theta = -\partial v(r,-\tfrac{1}{2}\pi)/\partial\theta = e^{-\frac{1}{4}i\pi}\left[\tfrac{1}{2}\sqrt{kr/\pi}\,e^{ikr} - ikr\int_{-\infty}^{\sqrt{kr/\pi}} e^{i\pi\tau^2}\,d\tau\right]. \qquad ---(3.2.10h)$$

Using the definitions

$$\Phi(kr) = 2e^{-\frac{1}{4}i\pi}\int_{-\infty}^{-\sqrt{kr/\pi}} e^{i\pi\tau^2}\,d\tau \qquad ---(3.2.11)$$

and

$$\Psi(kr) = \frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi kr}}\,e^{ikr}, \qquad ---(3.2.12)$$

equations (3.2.9s) and (3.2.9h) become

$$v(r,\tfrac{1}{2}\pi) = v(r,\tfrac{3}{2}\pi) = \tfrac{1}{2}\Phi(kr) \qquad ---(3.2.13)$$

and

$$\partial v(r,\tfrac{1}{2}\pi)/\partial\theta = -\partial v(r,\tfrac{3}{2}\pi)/\partial\theta = \tfrac{1}{2}ikr[\Psi(kr) - \Phi(kr)] \qquad ---(3.2.14)$$

respectively.

From (3.2.5) and the fact that $e^{i\pi\tau^2}$ is an even function, it follows that

$$e^{-\frac{1}{4}i\pi}\left(\int_{-\infty}^{-T} + \int_{-\infty}^{T}\right)e^{i\pi\tau^2}d\tau = 1. \qquad ---(3.2.15)$$

Equations (3.2.10s) and 3.2.10h) together with (3.2.11), (3.2.12) and (3.2.15) yield

$$v(r,-\tfrac{1}{2}\pi) = 1 - \tfrac{1}{2}\Phi(kr) \qquad ---(3.2.16)$$

and

$$\partial v(r,-\tfrac{1}{2}\pi)/\partial\theta = ikr[1 - \tfrac{1}{2}\Phi(kr) + \tfrac{1}{2}\Psi(kr)]. \qquad ---(3.2.17)$$

By making use of the definitions (3.1.2s) and (3.1.2h), the equations (3.2.13), (3.2.14), (3.2.16) and (3.2.17) and keeping in mind that $\partial/\partial z = -r^{-1}\sin\theta\,\partial/\partial\theta$ holds where $\cos\theta = 0$, the following table can now be drawn up for the values of u and $\partial u/\partial z$ in the X-Z-plane:

	sound soft screen		sound hard screen	
	u_I	$\partial u_I/\partial z$	u_{II}	$\partial u_{II}/\partial z$
screen $(\theta=\frac{1}{2}\pi)$	0	$ik(\Phi - \Psi)$	Φ	0
aperture $(\theta=-\frac{1}{2}\pi)$	$1 - \Phi$	ik	1	$ik(1 - \Phi + \Psi)$

table 3.2

(3.3) Electromagnetic waves

Suppose the incident wave of paragraph 2.5 impinges on a thin, perfectly conducting semi-infinite plane of which the edge includes an angle ψ with the 2-axis as in fig.3.3.1.

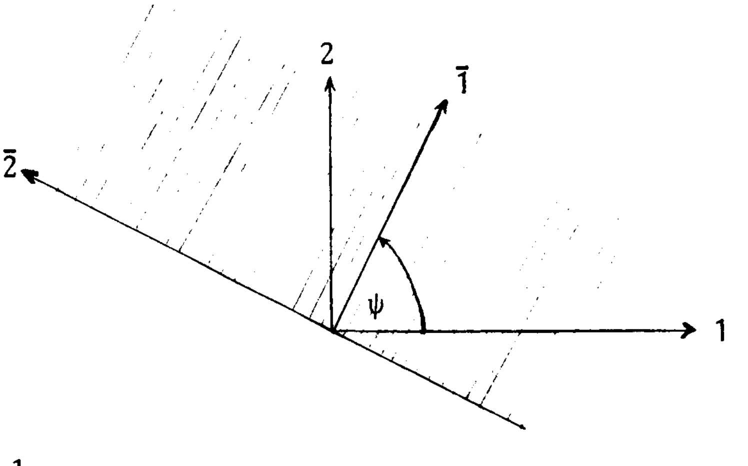

fig.3.3.1

On introducing a new system of axes with the $\bar{2}$-axis along the edge, the incident electric and magnetic fields are respectively given by

$$\bar{e}_\ell^{(i)} = e^{ikz}(\sin\psi\,\delta_{\ell 1} + \cos\psi\,\delta_{\ell 2})$$

$$\bar{h}_\ell^{(i)} = e^{ikz}(-\cos\psi\,\delta_{\ell 1} + \sin\psi\,\delta_{\ell 2}). \quad ---(3.3.1)$$

As the choice of origin along the edge is arbitrary, the fields are independent of $\bar{x}_2$ so that Maxwell's equations (2.5.5) assume the form

$$\left.\begin{aligned} ik\bar{e}_1 &= \bar{\partial}_3\bar{h}_2 \\ ik\bar{e}_2 &= \bar{\partial}_1\bar{h}_3 - \bar{\partial}_3\bar{h}_1 \\ ik\bar{e}_3 &= -\bar{\partial}_1\bar{h}_2 \\ ik\bar{h}_1 &= -\bar{\partial}_3\bar{e}_2 \\ ik\bar{h}_2 &= \bar{\partial}_3\bar{e}_1 - \bar{\partial}_1\bar{e}_3 \\ ik\bar{h}_3 &= \bar{\partial}_1\bar{e}_2. \end{aligned}\right\} \quad ---(3.3.2)$$

From (3.3.2) it is obvious that the total field can be deduced from a knowledge of $\bar{e}_2$ and $\bar{h}_2$. These two fields may be regarded as scalars, respectively satisfying the boundary conditions

$$\bar{e}_2 = 0 \quad ---(3.3.3s)$$

$$\partial\bar{h}_2/\partial n = 0 \quad ---(3.3.3h)$$

on the screen. Note that the boundary condition (3.3.3s) does not apply to $\bar{e}_1$ at the edge, because $\bar{e}_1$ is not tangential to the edge.

By reason of symmetry and the arbitrariness of the position of the origin on the edge, the $\bar{1}$-components of the incident fields do not contribute to $\bar{e}_2$ and $\bar{h}_2$. Making use of table 3.2, it follows that these fields are given by

$$\bar{e}_2 = u_I \cos\psi$$

$$\bar{h}_2 = u_{II} \sin\psi.$$

From Maxwell's equations (3.3.2) it follows that

$$\bar{e}_1 = \partial u_{II}/\partial z \ \sin\psi$$

$$\bar{h}_1 = -\partial u_I/\partial z \ \cos\psi.$$

By transforming to the original coordinates, the following table is obtained:

	e_1	e_2	h_1	h_2
screen	0	0	$-\Phi+\frac{1}{2}\Psi(1+\cos 2\psi)$	$\frac{1}{2}\Psi\sin 2\psi$
aperture	$\frac{1}{2}\Psi\sin 2\psi$	$1-\Phi+\frac{1}{2}\Psi(1-\cos 2\psi)$	-1	0

table 3.3

CHAPTER 4

BRAUNBEK'S METHOD FOR THE DIFFRACTION OF PLANE WAVES BY AN ANNULUS

(4.1) The far field: scalar case

According to Braunbek's method (see introduction) the field u in the region z>0 in fig.4.1.1 is obtained from (2.2.3a) or (2.2.3b) by assigning approximate values to u(x,y,+0) or $\partial u(x,y,+0)/\partial z$.

In the case of an annular aperture there is the added complication of two diffracting edges. An acceptable procedure which assigns unique boundary values to the field at Q(x,y,+0) by Braunbek's method, is to take the parameter r in $\Phi(kr)$ and $\Psi(kr)$ of table 3.2 equal to s, the shortest distance from Q to the nearest edge of the screen. In this way the plane of the screen is divided into two regions , viz. $A_1 \cup S_1$ for which $\rho < \frac{1}{2}(a+b)$ and $A_2 \cup S_2$ for which $\rho > \frac{1}{2}(a+b)$. The relationship between ρ and s is given in the table below.

Position of Q	A_1	A_2	S_1	S_2
ρ =	b + s	a - s	b - s	a + s

table 4.1

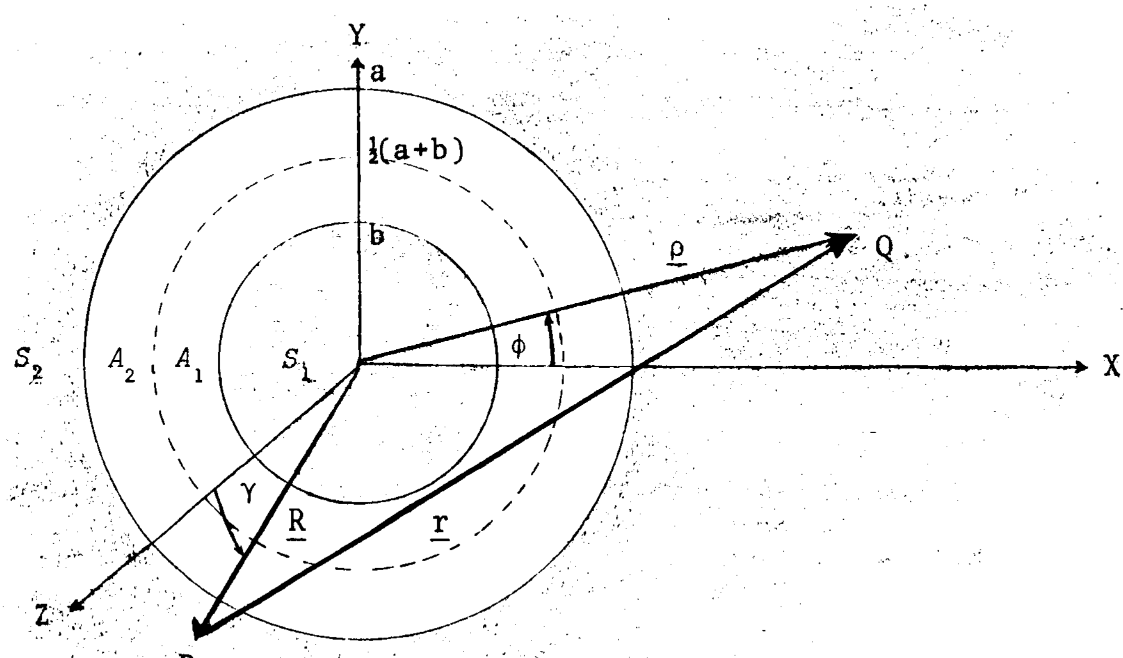

fig.4.1.1

Utilizing table 3.2 and assuming that each of the representations (2.2.3a) and (2.2.3b) are valid for both sound soft and sound hard screens (Their validity has only been proved for the case where the integration is taken over a finite region.), Braunbek's method yields the following values for u in the region z>0:

$$u(\underline{R}) = -z\int_o^{2\pi}\int_b^a [1-\Phi(ks)]\frac{1}{r}\frac{dG}{dr}\rho\, d\rho\, d\phi \qquad ---(4.1.1sa)$$

$$u(\underline{R}) = -ik\int_o^{2\pi}\int_b^a [1-\Phi(ks)+\Psi(ks)]\, G\rho d\rho\, d\phi \qquad ---(4.1.1hb)$$

$$u(\underline{R}) = -ik\int_o^{2\pi}\left(\int_o^b + \int_a^\infty\right)[\Phi(ks)-\Psi(ks)]\, G\rho d\rho\, d\phi - ik\int_o^{2\pi}\int_b^a G\rho d\rho\, d\phi$$

$$---(4.1.1sb)$$

$$u(\underline{R}) = -z\int_o^{2\pi}\left(\int_o^b + \int_a^\infty\right)\Phi(ks)\frac{1}{r}\frac{dG}{dr}\rho\, d\rho\, d\phi - z\int_o^{2\pi}\int_b^a \frac{1}{r}\frac{dG}{dr}\rho\, d\rho\, d\phi .$$

$$---(4.1.1ha)$$

Let

$$\int_o^{2\pi} G d\phi = F(\rho) \qquad ---(4.1.2)$$

and

$$\int_o^{2\pi} \frac{1}{r}\frac{dG}{dr}\, d\phi = F'(\rho), \qquad ---(4.1.3)$$

then it follows from table 4.1 that

$$u(\underline{R}) = -z\int_b^a F'(\rho)\rho d\rho + z\int_o^{\frac{1}{2}(a-b)}\Phi(ks)[(b+s)F'(b+s) + (a-s)F'(a-s)]\, ds$$

$$---(4.1.4sa)$$

$$u(\underline{R}) = -ik\int_b^a F(\rho)\rho d\rho + ik\int_o^{\frac{1}{2}(a-b)}[\Phi(ks)-\Psi(ks)]\, [(b+s)F(b+s) + (a-s)F(a-s)]ds$$

$$---(4.1.4hb)$$

$$u(\underline{R}) = -ik\int_b^a F(\rho)\rho d\rho - ik\int_o^b [\Phi(ks)-\Psi(ks)]\,(b-s)F(b-s)ds$$

$$-ik\int_o^\infty [\Phi(ks)-\Psi(ks)]\,(a+s)F(a+s)ds \quad \text{---(4.1.4sb)}$$

$$u(\underline{R}) = -z\int_b^a F'(\rho)\rho d\rho - z\int_o^b \Phi(ks)\,(b-s)F'(b-s)ds - z\int_o^\infty \Phi(ks)\,(a+s)F'(a+s)ds.$$

$$\text{---(4.1.4ha)}$$

Approximate values of these integrals can be obtained by making a few assumptions in connection with the dimensions of a and b; the position of the point P and the behaviour of the functions Φ and Ψ.

In the first place Braunbek's method is a short wave approximation, hence

$$ka > kb \gg 1 \quad \text{---(4.1.5)}$$

and

$$k(a-b) \gg 1. \quad \text{---(4.1.6)}$$

Secondly the position of the point P will be restricted to a region far from the z-axis and including an angle well in excess of *arcsin*(1/kb) with the z-axis, hence

$$R\sin\gamma \gg a \quad \text{---(4.1.7)}$$

and

$$kb\sin\gamma \gg 1. \quad \text{---(4.1.8)}$$

This means that P is far removed from the shaded region in fig.4.1.2.

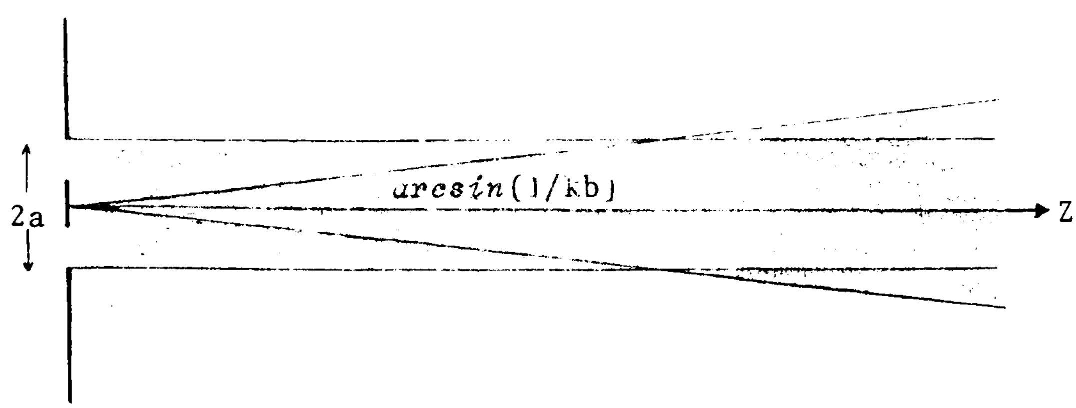

fig.4.1.2

Thirdly $\Phi(ks)$ and $\Psi(ks)$ differ significantly from zero for small values of s only, say for ks<1. For these values of s it follows from (4.1.5) and (4.1.6) that

$$s \ll b < a \qquad \text{---(4.1.9)}$$

and

$$s \ll a-b \qquad \text{---(4.1.10)}$$

Consequently the upper limits of the integrals in equations (4.1.4) may be extended to infinity without significantly altering the values of the integrals. Furthermore, the parameters a±s and b±s may respectively be taken equal to a and b where convenient. Equations (4.1.4) thus become

$$u(\underline{R}) = -z\,[K(F',a,b) - L(\Phi,F',b,a)] \qquad \text{---(4.1.11sa)}$$

$$u(\underline{R}) = -ik[K(F,a,b) - L(\Phi,F,b,a) + L(\Psi,F,b,a)] \qquad \text{---(4.1.11hb)}$$

$$u(\underline{R}) = -ik[K(F,a,b) + L(\Phi,F,a,b) - L(\Psi,F,a,b)] \qquad \text{---(4.1.11sb)}$$

$$u(\underline{R}) = -z\,[K(F',a,b) + L(\Phi,F',a,b)]\,, \qquad \text{---(4.1.11ha)}$$

where

$$K(F,a,b) = \int_b^a F(\rho)\,\rho d\rho \qquad \text{---(4.1.12)}$$

and

$$L(\Phi,F,a,b) = \int_0^\infty \Phi(ks)\,[aF(a+s) + bF(b-s)]\,ds\,. \qquad \text{---(4.1.13)}$$

Without loss of generality the field point P may be assumed to have coordinates $(R\sin\gamma, 0, R\cos\gamma)$. From fig.4.1.1 it follows that the coordinates of the integration point Q are $(\rho\cos\phi, \rho\sin\phi, 0)$. Therefore

$$r^2 = R^2 + \rho^2 - 2R\rho\,\sin\gamma\,\cos\phi\,. \qquad \text{---(4.1.14)}$$

From (4.1.7) it follows that if ρ is of the same order of magnitude as a or smaller, the ineqality

$$\rho^2 \ll \rho R\,\sin\gamma \ll R^2 \qquad \text{---(4.1.15)}$$

holds. Expanding (4.1.14) binomially yields

$$r \approx R - \rho\,\sin\gamma\,\cos\phi\,. \qquad \text{---(4.1.16)}$$

The integral (4.1.13) is evaluated by using (4.1.19) and the following approximation of (4.1.24):

$$J_0\{k(a\pm s)\sin\gamma\} \approx \frac{\cos\{k(a\pm s)\sin\gamma-\frac{1}{4}\pi\}}{(\frac{1}{2}\pi k^3 R^2 \sin^3\gamma)^{\frac{1}{2}}} . \qquad ---(4.1.27)$$

Hence we have

$$L(\Phi,F,a,b) = \frac{e^{ikR}}{R}\int_0^\infty \Phi(ks)\,\frac{a}{\sqrt{\frac{1}{2}\pi A}}\cos(ks\sin\gamma+A-\tfrac{1}{4}\pi)\,ds$$

$$+\frac{e^{ikR}}{R}\int_0^\infty \Phi(ks)\,\frac{b}{\sqrt{\frac{1}{2}\pi B}}\cos(ks\sin\gamma-B+\tfrac{1}{4}\pi)\,ds . \qquad ---(4.1.28)$$

The integrals in (4.1.28) are obtainable in terms of

$$E(\Phi,k,\ell,\alpha) = \int_0^\infty \Phi(ks)\,e^{i(\ell s+\alpha)}\,ds . \qquad ---(4.1.29)$$

Replacing Φ by Ψ and using (3.2.12), equation (4.1.29) becomes

$$E(\Psi,k,\ell,\alpha) = \frac{e^{i(\alpha+\frac{1}{4}\pi)}}{\sqrt{\pi k}}\int_0^\infty s^{-\frac{1}{2}}\,e^{i(k+\ell)s}\,ds$$

$$= \frac{2e^{i(\alpha+\frac{1}{4}\pi)}}{\sqrt{k(k+\ell)}}\int_0^\infty e^{i\pi r^2}\,dr .$$

From (3.2.5) therefore

$$E(\Psi,k,\ell,\alpha) = \frac{i\,e^{i\alpha}}{\sqrt{k(k+\ell)}} . \qquad ---(4.1.30)$$

In order to find an expression for $E(\Phi,k,\ell,\alpha)$ it is useful to observe from (3.2.11) and (3.2.12) that

$$\frac{\partial\Phi(ks)}{\partial k} = is\Psi(ks). \qquad ---(4.1.31)$$

Hence from (4.1.29), (4.1.30) and (4.1.31) we have

$$\frac{\partial E(\Phi,k,\ell,\alpha)}{\partial k} = \frac{\partial E(\Psi,k,\ell,\alpha)}{\partial \ell}$$

$$= -ike^{i\alpha}(k+\ell)^{-\frac{3}{2}}.$$

The approximation (4.1.16) may be used in the evaluation of the integrals (4.1.12) and (4.1.13), because where ρ is much larger than a, the functions Φ(ks) and Ψ(ks) approximate zero. From the definition of G and (4.1.16) it follows that

$$G = \frac{e^{ikr}}{2\pi r} \approx \frac{e^{ikr}}{R} \frac{e^{-ik\rho \sin\gamma \cos\phi}}{2\pi} \qquad \text{---(4.1.17)}$$

and

$$\frac{1}{r}\frac{dG}{dr} \approx \frac{ike^{ikR}}{R^2} \frac{e^{-ik\rho \sin\gamma \cos\phi}}{2\pi} . \qquad \text{---(4.1.18)}$$

Equations (4.1.2) and (4.1.3) may therefore be approximated by

$$F(\rho) = \frac{e^{ikR}}{R} J_0(k\rho \sin\gamma) \qquad \text{---(4.1.19)}$$

and

$$F'(\rho) = \frac{ik}{R} F(\rho), \qquad \text{---(4.1.20)}$$

where

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iz\cos\phi} d\phi \qquad \text{---(4.1.21)}$$

is the Bessel function of order zero.

Because of the relationship $J_1(z) = \frac{1}{z}\int_0^z \zeta J_0(\zeta)d\zeta$ between the Bessel functions, it follows from (4.1.19) that (4.1.12) can be written in the form

$$K(F,a,b) = \frac{e^{ikR}}{kR\sin\gamma}[aJ_1(A) - bJ_1(B)], \qquad \text{---(4.1.22)}$$

where $A = ka\sin\gamma$ and $B = kb\sin\gamma$. ---(4.1.23)

In the region where (4.1.8) holds the Bessel functions can be expanded asymptotically:

$$J_0(z) \approx (\tfrac{1}{2}\pi z)^{-\frac{1}{2}}\cos(z-\tfrac{1}{4}\pi) \qquad \text{---(4.1.24)}$$

$$J_1(z) \approx (\tfrac{1}{2}\pi z)^{-\frac{1}{2}}\sin(z-\tfrac{1}{4}\pi). \qquad \text{---(4.1.25)}$$

From (4.1.22) and (4.1.25),

$$K(F,a,b) = \frac{e^{ikR}[\sqrt{a}\,\sin(A-\tfrac{1}{4}\pi) - \sqrt{b}\,\sin(B-\tfrac{1}{4}\pi)]}{(\tfrac{1}{2}\pi k^3R^2\sin^3\gamma)^{\frac{1}{2}}} \qquad \text{---(4.1.26)}$$

$$\therefore\ E(\Phi,k,\ell,\alpha) = -\frac{ie^{i\alpha}}{2\ell}\int\{\frac{k}{\ell}(1+\frac{k}{\ell})^3\}^{-\frac{1}{2}}\,d(\frac{k}{\ell}) \qquad \text{---(4.1.32)}$$

The integral (4.1.32) is found by the substitution $tan\theta = \sqrt{k/\ell}$ if $\ell>0$ and by $cosh\theta = \sqrt{-k/\ell}$ if $\ell<0$. In both cases the result is $2\sqrt{k/(k+\ell)} + C$. The integration constant C is determined by using the fact that according to (3.2.11) and (4.1.29), $E(\Phi,\infty,\ell,\alpha) = 0$.

$$\therefore\ E(\Phi,k,\ell,\alpha) = \frac{ie^{i\alpha}}{\ell}(1-\frac{k}{\sqrt{k(k+\ell)}}).$$

From (4.1.30) therefore

$$\ell\, E(\Phi,k,\ell,\alpha) = ie^{i\alpha} - k\,E(\Psi,k,\ell,\alpha). \qquad \text{---(4.1.33)}$$

From (4.1.28) and (4.1.29) it follows that

$$L(\Phi,F,a,b) = \frac{\frac{1}{2}e^{ikR}}{(\frac{1}{2}\pi kR^2\, sin\gamma)^{\frac{1}{2}}}[\sqrt{a}\,E(\Phi,k,k\,sin\gamma,A-\tfrac{1}{4}\pi) + \sqrt{a}\,E(\Phi,k,-k\,sin\gamma,-A+\tfrac{1}{4}\pi)$$
$$+\sqrt{b}\,E(\Phi,k,k\,sin\gamma,-B+\tfrac{1}{4}\pi) + \sqrt{b}\,E(\Phi,k,-k\,sin\gamma,B-\tfrac{1}{4}\pi)].$$
$$\text{---(4.1.34)}$$

On using (4.1.33) and (4.1.26) in (4.1.34) we find:

$$L(\Phi,F,a,b) = -K(F,a,b) - \frac{\frac{1}{2}e^{ikR}}{(\frac{1}{2}\pi kR^2\, sin^3\gamma)^{\frac{1}{2}}}\times$$
$$[\sqrt{a}\,E(\Psi,k,k\,sin\gamma,A-\tfrac{1}{4}\pi) - \sqrt{a}\,E(\Psi,k,-k\,sin\gamma,-A+\tfrac{1}{4}\pi)$$
$$+\sqrt{b}\,E(\Psi,k,k\,sin\gamma,-B+\tfrac{1}{4}\pi) - \sqrt{b}\,E(\Psi,k,-k\,sin\gamma,B-\tfrac{1}{4}\pi)].$$
$$\text{---(4.1.35)}$$

From (4.1.30) and the identities

$$\sqrt{1+sin\gamma} \pm \sqrt{1-sin\gamma} = \sqrt{2(1\pm cos\gamma)}, \qquad \text{---(4.1.36)}$$

it follows that (4.1.35) can be written in the form

$$K(F,a,b) + L(\Phi,F,a,b) = \frac{i\,sec\ e^{ikR}}{(\pi k^3R^2\,sin^3\gamma)^{\frac{1}{2}}}[C(a,b)\sqrt{1-cos\gamma} - iS(a,b)\sqrt{1+cos\gamma}],$$
$$\text{---(4.1.37)}$$

where

$$C(a,b) = \sqrt{a}\,\cos(A-\tfrac{1}{4}\pi) + \sqrt{b}\,\cos(B-\tfrac{1}{4}\pi)$$
$$S(a,b) = \sqrt{a}\,\sin(A-\tfrac{1}{4}\pi) - \sqrt{b}\,\sin(B-\tfrac{1}{4}\pi). \qquad \text{---}(4.1.38)$$

Replacing Φ by Ψ in (1.4.34) and applying (4.1.30) and the identities (4.1.36) we find that $L(\Psi,F,a,b)$ can also be expressed in terms of $C(a,b)$ and $S(a,b)$:

$$L(\Psi,F,a,b) = \frac{i\,\sec\gamma\; e^{ikR}}{(\pi k^3 R^2 \sin\gamma)^{\frac{1}{2}}}\left[\,C(a,b)\sqrt{1+\cos\gamma} - i\,S(a,b)\sqrt{1-\cos\gamma}\,\right]. \qquad \text{---}(4.1.39)$$

In the expressions (4.1.11) for the field the values of $K(F,a,b)$, $L(\Phi,F,a,b)$ and $L(\Psi,F,a,b)$ are given by (4.1.26), (4.1.37) and (4.1.39). From (4.1.12), (4.1.13) and (4.1.20) we have that

$$z\,K(F',a,b) = z\,K(\tfrac{ik}{R}F,a,b) = ik\cos\gamma\,K(F,a,b)$$
$$z\,L(\Phi,F',a,b) = z\,L(\Phi,\tfrac{ik}{R}F,a,b) = ik\cos\gamma\,L(\Phi,F,a,b)$$
$$z\,L(\Psi,F',a,b) = z\,L(\Psi,\tfrac{ik}{R}F,a,b) = ik\cos\gamma\,L(\Psi,F,a,b).$$

Substituting these values into (4.1.11) and utilizing (4.1.37), (4.1.39) and the identities

$$\sqrt{1\pm\cos\gamma} - \sin\gamma\,\sqrt{1\mp\cos\gamma} = \pm\cos\gamma\,\sqrt{1\pm\cos\gamma}\,,$$

it is found that (4.1.11sa) and (4.1.11sb) yield identical results, as do (4.1.11ha) and (4.1.11hb):

$$u(\underline{R}) = -\frac{e^{ikR}\left[\,C(a,b)\sqrt{1-\cos\gamma} + i\,S(a,b)\sqrt{1+\cos\gamma}\,\right]}{(\pi k R^2 \sin^3\gamma)^{\frac{1}{2}}} \qquad \text{---}(4.1.40s)$$

$$u(\underline{R}) = \frac{e^{ikR}\left[\,C(a,b)\sqrt{1-\cos\gamma} - i\,S(a,b)\sqrt{1+\cos\gamma}\,\right]}{(\pi k R^2 \sin^3\gamma)^{\frac{1}{2}}}. \qquad \text{---}(4.1.40h)$$

Both (4.1.40s) and (4.1.40h) yield for the square of the magnitude of the field:

$$|u(\underline{R})|^2 = \frac{a[1 - \cos\gamma\, \sin(2ka\, \sin\gamma)] + b[1 - \cos\gamma\, \sin(2kb\, \sin\gamma)]}{\pi k R^2 \sin^3\gamma}$$

$$+ \frac{2\sqrt{ab}\, [\sin\{k(a+b)\sin\gamma\} - \cos\gamma\, \cos\{k(a-b)\sin\gamma\}]}{\pi k R^2 \sin^3\gamma} .$$

---(4.1.41)

The result (4.1.41) reduces to equation II(13) of Braunbek (1950) if b=0, i.e. if the annulus becomes a circle.

Substituting (4.1.26) into (4.1.11hb) or (4.1.11sb) and ignoring the terms containing L, the better Kirchhoff approximation (see introduction) is obtained:

$$u(\underline{R}) = - \frac{ike^{ikR}[\sqrt{a}\sin(A-\frac{1}{4}\pi) - \sqrt{b}\sin(B-\frac{1}{4}\pi)]}{(\frac{1}{2}\pi k^3 R^2 \sin^3\gamma)^{\frac{1}{2}}} .$$

---(4.1.42)

Multiplying (4.1.42) by $\cos\gamma$ produces the weaker Kirchhoff solution. This is evident from equations (4.1.11) and the relationship between F and F'. Note that the solutions of Kirchhoff and Braunbek are the same for small values of γ.

(4.2) The far field: electromagnetic case

With P and Q respectively the points

$$\underline{R} = (x_\ell) = (R\sin\gamma\cos\theta, R\sin\gamma\sin\theta, R\cos\theta)$$

and

$$\underline{\rho} = (\xi_\ell) = (\rho\cos\phi, \rho\sin\phi, 0)$$

in fig.4.1.1, the derivatives (2.5.15) and (2.5.17) of G are given by

$$G_\ell = \frac{1}{r}\frac{dG}{dr}(x_\ell - \xi_\ell)$$

$$G_{\ell m} = \frac{1}{r^2}\left(\frac{d^2G}{dr^2} - \frac{1}{r}\frac{dG}{dr}\right)(x_\ell - \xi_\ell)(x_m - \xi_m) + \frac{1}{r}\frac{dG}{dr}\delta_{\ell m} .$$

In the far field region as depicted in fig.4.1.2 the derivatives of G are approximated, utilizing (4.1.17) and (4.1.18), as follows:

$$(G_\ell) = ikG\,(\sin\gamma\cos\theta,\ \sin\gamma\sin\theta, \cos\theta) \qquad \text{---(4.2.1)}$$

$$[G_{\ell m}] = -k^2G\begin{bmatrix} \sin^2\gamma\cos^2\theta & \sin^2\gamma\sin\theta\cos\theta & \sin\gamma\cos\gamma\cos\theta \\ \sin^2\gamma\sin\theta\cos\theta & \sin^2\gamma\sin^2\theta & \sin\gamma\cos\gamma\sin\theta \\ \sin\gamma\cos\gamma\cos\theta & \sin\gamma\cos\gamma\sin\theta & \cos^2\gamma \end{bmatrix}$$

---(4.2.2)

Applying the approximations (4.2.1) to the solutions (2.5.14) we obtain:

$$e_1 = -ik\cos\gamma\iint_A e_1G\,d\xi\,d\eta$$

$$e_2 = -ik\cos\gamma\iint_A e_2G\,d\xi\,d\eta \qquad \text{---(4.2.3)}$$

$$e_3 = -\tan\gamma(\cos\theta\,e_1 + \sin\theta\,e_2).$$

If one assumes that the operations carried out to obtain (2.5.14) remain valid when integration takes place over the complete X-Y-plane, the magnetic field can be obtained by the same procedure:

$$h_1 = -ik\cos\gamma\iint_{-\infty}^{\infty} h_1G\,d\xi\,d\eta$$

$$h_2 = -ik\cos\gamma\iint_{-\infty}^{\infty} h_2G\,d\xi\,d\eta \qquad \text{---(4.2.4)}$$

$$h_3 = -\tan\gamma(\cos\theta\,h_1 + \sin\theta\,h_2)$$

By applying Maxwell's equations (2.5.5) we now obtain h_ℓ and e_ℓ respectively from (4.2.3)and (4.2.4) in terms of the values of e_ℓ and h_ℓ on the plane z=0. In the differentiation the approximations (4.2.2) apply. We find:

$$h_1 = -\tan\gamma\sin\gamma\sin\theta(\cos\theta\,e_1 + \sin\theta\,e_2) - \cos\gamma\,e_2$$

$$h_2 = \cos\gamma\,e_1 + \tan\gamma\sin\gamma\cos\theta(\cos\theta\,e_1 + \sin\theta\,e_2)$$

$$h_3 = \sin\gamma(-\sin\theta\,e_1 + \cos\theta\,e_2)$$

---(4.2.5)

and

$$e_1 = \cos\gamma\sin\gamma\sin\theta(\cos\theta\, h_1 + \sin\theta\, h_2) + \cos\gamma\, h_2$$

$$e_2 = -\cos\gamma\, h_1 - \tan\gamma\sin\gamma\cos\theta(\cos\theta\, h_1 + \sin\theta\, h_2)$$

$$e_3 = \sin\gamma(\sin\theta\, h_1 - \cos\theta\, h_2).$$

---(4.2.6)

In (4.2.5) e_1 and e_2 are given by (4.2.3) and in (4.2.6) h_1 and h_2 are given by (4.2.4).

Maxwell's equations, applied to (4.2.5) and (4.2.6), yield the original results (4.2.3) and (4.2.4) respectively. Equations (4.2.5) and (4.2.6) are not independent either; the first two equations of (4.2.6) are obtainable from the corresponding equations in (4.2.5) by solving for e_1 and e_2.

Comparing figures 3.3.1 and 4.1.1 we see that $\psi = \phi$ for $\rho > \frac{1}{2}(a+b)$ and $\psi = \phi+\pi$ for $\rho < \frac{1}{2}(a+b)$. In table 3.3 the angle 2ψ may therefore be equated to 2ϕ. The first two equations of (4.2.3) and (4.2.4) can now respectively be written in the form:

$$e_1(\underline{R}) = -ik\cos\gamma\int_0^{2\pi}\int_b^a \tfrac{1}{2}\Psi(ks)\sin 2\phi\, G\rho\, d\rho\, d\phi$$

$$e_2(\underline{R}) = -ik\cos\gamma\int_0^{2\pi}\int_b^a [1 - \Phi(ks) + \tfrac{1}{2}\Psi(ks)(1-\cos 2\phi)]G\rho\, d\rho\, d\phi$$

---(4.2.7)

and

$$h_1(\underline{R}) = -ik\cos\gamma\int_0^{2\pi}\left(\int_0^b + \int_a^\infty\right)[-\Phi(ks) + \tfrac{1}{2}\Psi(ks)(1+\cos 2\phi)]G\rho\, d\rho\, d\phi$$
$$+ik\cos\gamma\int_0^{2\pi}\int_b^a G\rho\, d\rho\, d\phi$$

$$h_2(\underline{R}) = -ik\cos\gamma\int_0^{2\pi}\left(\int_0^b + \int_a^\infty\right)\tfrac{1}{2}\Psi(ks)\sin 2\phi\, G\rho\, d\rho\, d\phi.$$

---(4.2.8)

$$e_1(\underline{R}) = -\tfrac{1}{2}ik\cos\gamma\ L(\Psi,F_s,b,a)$$

$$e_2(\underline{R}) = -ik\cos\gamma[K(F,a,b) - L(\Phi,F,b,a) + \tfrac{1}{2}L(\Psi,F,b,a) - \tfrac{1}{2}L(\Psi,F_c,b,a)]$$

$$h_1(\underline{R}) = ik\cos\gamma[K(F,a,b) + L(\Phi,F,a,b) - \tfrac{1}{2}L(\Psi,F,a,b) - \tfrac{1}{2}L(\Psi,F_c,a,b)]$$

$$h_2(\underline{R}) = -\tfrac{1}{2}ik\cos\gamma\ L(\Psi,F_s,a,b).$$

---(4.2.12)

In the electromagnetic case the approximation (4.1.17) must be replaced by

$$G \approx \frac{e^{ikR}}{R}\,\frac{e^{-ik\rho\sin\gamma\cos(\phi-\theta)}}{2\pi}\,. \qquad \text{---(4.2.13)}$$

From (4.1.2) and (4.2.13) it follows that (4.1.19), derived for the scalar case, remains valid:

$$F(\rho) = \frac{e^{ikR}}{R}\,J_0\,(k\rho\sin\gamma). \qquad \text{---(4.2.14)}$$

From (4.2.9), (4.2.13) and the properties of Bessel functions, it follows that

$$F_s(\rho) = -\frac{e^{ikR}}{R}\sin 2\theta\,J_2\,(k\rho\sin\gamma)$$

---(4.2.15)

$$F_c(\rho) = -\frac{e^{ikR}}{R}\cos 2\theta\,J_2\,(k\rho\sin\gamma)\,,$$

where

$$\sin 2\theta\,J_2(z) = -\frac{1}{2\pi}\int_0^{2\pi}\sin 2\phi\,e^{-iz\cos(\phi-\theta)}d\phi$$

and ---(4.2.16)

$$\cos 2\theta\,J_2(z) = -\frac{1}{2\pi}\int_0^{2\pi}\cos 2\phi\,e^{-iz\cos(\phi-\theta)}d\phi\,.$$

By applying the asymptotic expansion $J_2(z) \approx -J_0(z)$ and the result (4.2.14), equations (4.2.15) become

$$F_s(\rho) = \sin 2\theta\,F(\rho)$$

---(4.2.17)

$$F_c(\rho) = \cos 2\theta\,F(\rho)\,.$$

To evaluate the above integrals (4.1.2) is used as well as the following related functions:

$$\int_0^{2\pi} G\, sin2\phi\, d\phi = F_s(\rho)$$

$$\int_0^{2\pi} G\, cos2\phi\, d\phi = F_c(\rho). \qquad \text{---(4.2.9)}$$

With the aid of table 4.1 equations (4.2.7) and (4.2.8) can respectively be written in the form:

$$e_1(\underline{R}) = -\tfrac{1}{2}ik\cos\gamma \int_0^{\frac{1}{2}(a-b)} \Psi(ks)[(b+s)F_s(b+s) + (a-s)F_s(a-s)]ds$$

$$e_2(\underline{R}) = -ik\cos\gamma \int_b^a F(\rho)\rho d\rho + ik\cos\gamma \int_0^{\frac{1}{2}(a-b)} \Phi(ks)[(b+s)F(b+s)+(a-s)F(a-s)]ds$$

$$-\tfrac{1}{2}ik\cos\gamma \int_0^{\frac{1}{2}(a-b)} \Psi(ks)[(b+s)\{F(b+s) - F_c(b+s)\} + (a-s)\{F(a-s) - F_c(a-s)\}]ds \qquad \text{---(4.2.10)}$$

and

$$h_1(\underline{R}) = ik\cos\gamma[\int_0^b \Phi(ks)(b-s)F(b-s)ds + \int_0^\infty \Phi(ks)(a+s)F(a+s)ds]$$

$$-\tfrac{1}{2}ik\cos\gamma \int_0^b \Psi(ks)(b-s)\{F(b-s) + F_c(b-s)\}ds$$

$$-\tfrac{1}{2}ik\cos\gamma \int_0^\infty \Psi(ks)(a+s)\{F(a+s) + F_c(a+s)\}ds + ik\cos\gamma \int_b^a F(\rho)\, d\rho$$

$$h_2(\underline{R}) = -ik\cos\gamma[\int_0^b \Psi(ks)(b-s)F_s(b-s)\, ds + \int_0^\infty \Psi(ks)(a+s)F_s(a+s)\, ds]. \qquad \text{---(4.2.11)}$$

As in the scalar case, the expressions (4.2.10) and (4.2.11) are approximated by extending certain limits of integration to infinity and respectively replacing a±s and b±s by a and b where convenient. In terms of the definitions (4.1.12) and (4.1.13) therefore:

Substitution of (4.2.17) into (4.1.13) yields

$$L(\Phi,F_s,a,b) = \sin 2\theta\, L(\Phi,F,a,b)$$
$$L(\Phi,F_c,a,b) = \cos 2\theta\, L(\Phi,F,a,b). \qquad \text{---(4.2.18)}$$

Substituting (4.2.18) into (4.2.12) and utilizing the third equations of (4.2.3) and (4.2.4) leads to expressions for e_ℓ and h_ℓ which can be written in the following form:

$$e_1(\underline{R}) = -ik\cos\gamma\,\sin\theta\cos\theta\,L(\Psi,F,b,a)$$

$$e_2(\underline{R}) = ik\cos\gamma[K(F,b,a) + L(\Phi,F,b,a) - \sin^2\theta\,L(\Psi,F,b,a)]$$

$$e_3(\underline{R}) = -ik\sin\gamma\,\sin\theta\,[K(F,b,a) + L(\Phi,F,b,a) - L(\Psi,F,b,a)]$$

$$h_1(\underline{R}) = ik\cos\gamma[K(F,a,b) + L(\Phi,F,a,b) - \cos^2\theta\,L(\Psi,F,a,b)]$$

$$h_2(\underline{R}) = -ik\cos\gamma\,\sin\theta\cos\theta\,L(\Psi,F,a,b)$$

$$h_3(\underline{R}) = -ik\sin\gamma\,\cos\theta\,[K(F,a,b) + L(\Phi,F,a,b) - L(\Psi,F,a,b)]\,. \qquad \text{---(4.2.19)}$$

Hence if (4.1.37), (4.1.38) and (4.1.39) are used in (4.2.19) it follows that:

$$e_1(\underline{R}) = \frac{\sin\theta\cos\theta\, e^{ikR}[C(a,b)\sqrt{1+\cos\gamma} + iS(a,b)\sqrt{1-\cos\gamma}]}{(\pi kR^2\sin\gamma)^{\frac{1}{2}}}$$

$$e_2(\underline{R}) = \frac{e^{ikR}C(a,b)(\sin\gamma\,\sin^2\theta\,\sqrt{1+\cos\gamma} - \sqrt{1-\cos\gamma})}{(\pi kR^2\sin^3\gamma)^{\frac{1}{2}}} + \frac{e^{ikR}iS(a,b)(\sin\gamma\,\sin^2\theta\,\sqrt{1-\cos\gamma} - \sqrt{1+\cos\gamma})}{(\pi kR^2\sin^3\gamma)^{\frac{1}{2}}}$$

$$e_3(\underline{R}) = -\frac{\sin\ \ e^{ikR}[C(a,b)\sqrt{1-\cos\gamma} - iS(a,b)\sqrt{1+\cos\gamma})}{(\pi kR^2\sin\gamma)^{\frac{1}{2}}}$$

$$h_1(\underline{R}) = \frac{e^{ikR}C(a,b)(\sin\gamma\cos^2\theta\sqrt{1+\cos\gamma} - \sqrt{1-\cos\gamma})}{(\pi kR^2\sin^3\gamma)^{\frac{1}{2}}}$$

$$-\frac{e^{ikR}iS(a,b)(\sin\gamma\cos^2\theta\sqrt{1-\cos\gamma} - \sqrt{1+\cos\gamma})}{(\pi kR^2\sin^3\gamma)^{\frac{1}{2}}}$$

$$h_2(\underline{R}) = \frac{\sin\theta\cos\theta\, e^{ikR}[C(a,b)\sqrt{1+\cos\gamma} - iS(a,b)\sqrt{1-\cos\gamma}]}{(\pi kR^2\sin\gamma)^{\frac{1}{2}}}$$

$$h_3(\underline{R}) = -\frac{\cos\theta\, e^{ikR}[C(a,b)\sqrt{1-\cos\gamma} + iS(a,b)\sqrt{1+\cos\gamma}]}{(\pi kR^2\sin\gamma)^{\frac{1}{2}}}. \qquad ---(4.2.20)$$

Note that if equations (4.2.20) are substituted into (4.2.5) and (4.2.6), the result is again (4.2.20). The assumption made in the derivation of (4.2.4) therefore appears to be reasonable.

After some manipulation it follows from (4.2.20) that the magnitudes of $\underline{e}$ and $\underline{h}$ are identical to that of the scalar field given by (4.1.41).

(4.3) The field on the Z-axis: scalar case

If the point P in fig.4.1.1 is on the Z-axis, integration with respect to ϕ in equations (4.1.1) reduces to multiplication by 2π. Integration by parts of terms containing $\Phi(ks)$ and use of (3.2.5) and (3.2.11) cause the first terms of (4.1.1sa) and (4.1.1hb) and the last terms of (4.1.1sb) and (4.1.1ha) to cancel, hence:

$$u(0,0,z) = -2\pi z\int_b^a \frac{\partial\Phi(ks)}{\partial\rho}G\,d\rho \qquad ---(4.3.1sa)$$

$$u(0,0,z) = -\int_b^a [\frac{\partial\Phi(ks)}{\partial\rho} + \frac{ik\rho\Psi(ks)}{r}]e^{ikr}d\rho \qquad ---(4.3.1hb)$$

$$u(0,0,z) = \Phi(kb)e^{ikz} + (\int_o^b + \int_a^\infty)[\frac{\partial\Phi(ks)}{\partial\rho} + \frac{ik\rho\Psi(ks)}{r}]e^{ikr}d\rho \qquad ---(4.3.1sb)$$

$$u(0,0,z) = \Phi(kb)e^{ikz} + 2\pi z\left(\int_0^b + \int_a^\infty\right) \frac{\partial\Phi(ks)}{\partial\rho} G\, d\rho .$$

---(4.3.1ha)

Transformation of the variable of integration to s according to table 4.1 and bearing in mind that $\partial\Phi(ks)/\partial s = ik\Psi(ks)$, gives:

$$u(0,0,z) = -ikz\int_0^{\frac{1}{2}(a-b)} \Psi(ks)\left[\frac{e^{ik\sqrt{(b+s)^2+z^2}}}{\sqrt{(b+s)^2+z^2}} - \frac{e^{ik\sqrt{(a-s)^2+z^2}}}{\sqrt{(a-s)^2+z^2}}\right] ds$$

---(4.3.2sa)

$$u(0,0,z) = -ik\int_0^{\frac{1}{2}(a-b)} \Psi(ks)\left[1 + \frac{b+s}{\sqrt{(b+s)^2+z^2}}\right]e^{ik\sqrt{(b+s)^2+z^2}} ds$$

$$+ik\int_0^{\frac{1}{2}(a-b)} \Psi(ks)\left[1 - \frac{a-s}{\sqrt{(a-s)^2+z^2}}\right]e^{ik\sqrt{(a-s)^2+z^2}} ds$$

---(4.3.2hb)

$$u(0,0,z) = \Phi(kb)e^{ikz} - ik\int_0^{b} \Psi(ks)\left[1 - \frac{b-s}{\sqrt{(b-s)^2+z^2}}\right]e^{ik\sqrt{(b-s)^2+z^2}} ds$$

$$+ik\int_0^{\infty} \Psi(ks)\left[1 + \frac{a+s}{\sqrt{(a+s)^2+z^2}}\right]e^{ik\sqrt{(a+s)^2+z^2}} ds$$

---(4.3.2sb)

$$u(0,0,z) = \Phi(kb)e^{ikz} - ikz\int_0^{b} \Psi(ks) \frac{e^{ik\sqrt{(b-s)^2+z^2}}}{\sqrt{(b-s)^2+z^2}} ds$$

$$+ ikz\int_0^{\infty} \Psi(ks) \frac{ik\sqrt{(a+s)^2+z^2}}{\sqrt{(a+s)^2+z^2}} ds .$$

---(4.3.2ha)

The field at the origin can be obtained directly from equations (4.1.1sa) and (4.1.1ha) by application of theorem 2.2. The same results follow by setting z=0 either in (4.3.1sa) and (4.3.1ha) or in (4.3.2sa) and (4.3.2ha). To find the field at the origin by means of (4.3.2hb) or (4.3.2sb) the variable of integration is transformed to $\tau = \sqrt{2ks/\pi}$. On using (3.2.5), (3.2.11) and (3.2.12) it follows that:

$$u(\underline{0}) = 0 \qquad \text{---(4.3.3sa)}$$

$$u(\underline{0}) = \sqrt{2}\, e^{ikb}[1 - \Phi\{k(a-b)\}] \qquad \text{---(4.3.3hb)}$$

$$u(\underline{0}) = \Phi(kb) - \sqrt{2}\, e^{ika} \qquad \text{---(4.3.3sb)}$$

$$u(\underline{0}) = \Phi(kb). \qquad \text{---(4.3.3ha)}$$

Where comparison is possible by setting b=0, these results are in agreement with those of Bouwkamp (1954). However, for obvious reasons, the field at the centre of a circular aperture cannot be obtained from (4.3.3sa). It can be found by setting b=0 in (4.1.1sa) and applying theorem 2.2.

The main contribution to the integrals in equations (4.3.3) come from the neighbourhood of s=0. For small values of s we have:

$$\left.\begin{aligned} R_a^2 &= a^2 + z^2 \\ R_b^2 &= b^2 + z^2 \\ \sin\alpha &= a/R_a \\ \sin\beta &= b/R_b\,, \end{aligned}\right\} \qquad \text{---(4.3.4)}$$

then

$$\left.\begin{aligned} &\frac{a \pm s}{\sqrt{(a\pm s)^2 + z^2}} \approx \sin\alpha \\ &e^{ik\sqrt{(a\pm s)^2+z^2}} \approx e^{ikR_a}\, e^{\pm iks\,\sin\alpha}. \end{aligned}\right\} \qquad \text{---(4.3.5)}$$

All the integrals in equations (4.3.3) are thus reduced to linear combinations of the following integrals:

$$I(\pm a, d) = \int_0^d \Psi(ks)\, e^{\pm iks\,\sin\alpha}\, ds. \qquad \text{---(4.3.6)}$$

Using (3.2.5), (3.2.11) and (3.2.12) we find that

$$I(\pm a,d) = \frac{2e^{\frac{1}{4}i\pi}}{k\sqrt{1\pm sin\alpha}} \int_0^{\sqrt{kd(1\pm sin\alpha)/\pi}} e^{i\pi r^2} dr$$

$$= \frac{i\,[1 - \Phi\{kd(1\pm sin\alpha)\}]}{k\sqrt{1\pm sin\alpha}}. \qquad \text{---(4.3.7)}$$

Substituting (4.3.4) into equations (4.3.2) and using (4.3.7) yields:

$$u(0,0,z) = e^{ikR_b}\sqrt{1-sin\beta}\,[1 - \Phi\{\tfrac{1}{2}k(a-b)(1+sin\beta)\}]$$
$$-e^{ikR_a}\sqrt{1+sin\alpha}\,[1 - \Phi\{\tfrac{1}{2}k(a-b)(1-sin\alpha)\}] \qquad \text{---(4.3.8sa)}$$

$$u(0,0,z) = e^{ikR_b}\sqrt{1+sin\beta}\,[1 - \Phi\{\tfrac{1}{2}k(a-b)(1+sin\beta)\}]$$
$$-e^{ikR_a}\sqrt{1-sin\alpha}\,[1 - \Phi\{\tfrac{1}{2}k(a-b)(1-sin\beta)\}] \qquad \text{---(4.3.8hb)}$$

$$u(0,0,z) = \Phi(kb)e^{ikz} + e^{ikR_b}\sqrt{1-sin\beta}\,[1 - \Phi\{kb(1-sin\beta)\}]$$
$$-e^{ikR_a}\sqrt{1+sin\alpha} \qquad \text{---(4.3.8sb)}$$

$$u(0,0,z) = \Phi(kb)e^{ikz} + e^{ikR_b}\sqrt{1+sin\beta}\,[1 - \Phi\{kb(1-sin\beta)\}]$$
$$- {}^{ikR_a}\sqrt{1-sin\alpha}\,. \qquad \text{---(4.3.8ha)}$$

Note that equations (4.3.3) can be obtained from equations (4.3.8) by setting z=0.

Equation II(6) of Braunbek (1950) follows from (4.3.8sb) by setting b=0 or from (4.3.8sa) by setting b=0 and $\Phi(\frac{1}{2}ka) \approx 0$.

Kirchhoff's approximations are produced by ignoring Φ and Ψ in equations (4.1.1) and integrating. The weaker and better Kirchhoff solutions respectively follow from (4.1.1sa) and (4.1.1sb) or (4.1.1ha) and (4.1.1hb):

$$u(0,0,z) = e^{ikR_b} - e^{ikR_a} \qquad \text{---(4.3.9a)}$$

$$u(0,0,z) = cos\beta\, e^{ikR_b} - cos\alpha\, e^{ikR_a}. \qquad \text{---(4.3.9b)}$$

APPENDIX

According to Luneburg (1944) the condition (2.2.4) in theorem 2.2 may be weakened to read:

U(x,y) is sectionally continuous in the X-Y-plane. Outside a circle with centre at the origin U(x,y) is continuous and has continuous derivatives such that

$$|U(x,y)| < \frac{B}{\sqrt{x^2+y^2}}$$

$$\left|\frac{\partial U(x,y)}{\partial x}\right| < \frac{B}{\sqrt{x^2+y^2}} \qquad ---(1)$$

$$\left|\frac{\partial U(x,y)}{\partial x}\right| \quad \frac{B}{\sqrt{x^2+y^2}}\,.$$

His proof of the theorem is now reproduced, but in the notation used in paragraph (2.2) of this thesis.

Let the polar coordinates (t,ψ) be defined by

$$\underline{r} = (t\cos\psi, t\sin\psi, -z). \qquad ---(2)$$

Then (2.2.3a) can be written in the form:

$$u(x,y,z) = -z\int_0^{2\pi}\int_0^{\infty} U(x+t\cos\psi, y+t\sin\psi)\,\frac{1}{r}\,\frac{dG}{dr}\,t\,dt\,d\psi. \qquad ---(3)$$

Defining

$$\Theta(x,y,t) = \int_0^{2\pi} U(x+t\cos\psi, y+t\sin\psi)\,d\psi\,, \qquad ---(4)$$

it follows that

$$u(x,y,z) = -z\left(\int_z^{zT} + \int_{zT}^{\infty}\right)\Theta(x,y,\sqrt{r^2-z^2})\,\frac{dG}{dr}\,dr. \qquad ---(5)$$

On using the inequalities

$$|\Theta(x,y,t)| < \frac{B'}{t} \qquad ---(6)$$

and

$$\left|\frac{dG}{dr}\frac{dr}{dt}\right| \leqslant \frac{1+kD}{2\pi D^2 t} \quad \text{for } r \geqslant D, \qquad ---(7)$$

the absolute value of the second integral in (5) is found to be smaller than

$$z\int_{z\sqrt{T^2-1}}^{\infty}(B''/t^2)\,dt \;=\; \frac{B''}{\sqrt{T^2-1}}\;, \qquad \text{---(8)}$$

where B' and B" are constants independent of z and T. From (5) and (8) one may therefore conclude that

$$\left|u(x,y,z) + \int_1^T \Theta(x,y,z\sqrt{s^2-1})\frac{\partial}{\partial s}\left(\frac{e^{ikzs}}{2\pi s}\right) ds\,\right| < \frac{B''}{\sqrt{T^2-1}}\;. \qquad \text{---(9)}$$

By letting $z \to 0$, it follows that

$$\left|u^*(x,y,0) + \left(\frac{1}{T}-1\right)U(x,y)\right| < \frac{B''}{\sqrt{T^2-1}}\;, \qquad \text{---(10)}$$

where $u^*(x,y,0) = \lim_{z\to o} u(x,y,z)$.

Seeing that (10) holds for any value of T, we have that $\lim_{z\to o} u(x,y,z) = U(x,y)$, which completes the first part of the proof. In order to complete his proof Luneburg had to show that u as defined by (2.2.3a) also satisfies the radiation conditions, in this thesis formulated by (2.2.2). As regards this part of the proof he says:

"It remains to be shown that u also satisfies the conditions (45.14) and (45.141). If the function f(x,y) is zero outside a certain finite domain these conditions follow directly from the fact that the kernel

$$K = -\frac{z}{2\pi r}\frac{\partial}{\partial r}\left(\frac{1}{r}e^{ikr}\right)$$

satisfies these conditions. For functions f(x,y) which satisfy only the conditions (45.13) one has to proceed in a manner similar to the above by considering first a finite domain of integration and then estimating the rest."

[In the above (45.14) and (45.141) are the radiation conditions, equations (45.13) are the equations (1) and f(x,y) = U(x,y).]

This merely outlines a procedure by means of which the proof may conceivably be concluded.

The above proof is not rigorous, because B" is not independent of z and T. From (7) it follows that (8) only holds for $zT > D$. Letting $z \to 0$ in (9) is therefore not permissible. In addition the inequality (6) does not hold for all values of t. From (1) we have:

$$|U(x+t\cos\psi, y+t\sin\psi)| < \frac{B}{\sqrt{x^2+y^2+t^2+2t(x\cos\psi+y\sin\psi)}} \qquad \text{---(11)}$$

The number inside the square root is larger than or equal to

$$x^2+y^2+t^2-2t\sqrt{x^2\cos^2\psi+y^2\sin^2\psi+2xy\cos\psi\sin\psi}$$

$$= x^2+y^2+t^2-2t\sqrt{x^2+y^2-(x\sin\psi-y\cos\psi)^2}$$

$$\geqslant (t-R)^2, \qquad \text{---(12)}$$

where $R^2 = x^2+y^2$. ---(13)

By restricting the point (t,ψ) to the region outside the annulus $R-\delta \leqslant t \leqslant R+\delta$, the variable t will satisfy the inequality $R-t-\delta > 0$ or $t-R-\delta > 0$. Therefore $(R-t)(R-\delta) > \delta t$ or $(t-R)(R+\delta) > \delta t$. In both cases

$$|t-R| > \frac{\delta t}{R-\delta}. \qquad \text{---(14)}$$

The inequality (6) follows from (4), (11), (12) and (14).

Luneburg's proof can be made rigorous as follows:

From (5) we can deduce the inequality

$$|u(x,y,z) - U(x,y)| \leqslant \left|z\int_D^\infty \Theta(x,y,\sqrt{r^2-z^2})\frac{dG}{dr}dr\right|$$

$$+\left|U(x,y) + z\int_z^D \Theta(x,y,\sqrt{r^2-z^2})\frac{dG}{dr}dr\right|. \qquad \text{---(15)}$$

By choosing $D = \delta - R$, the first term on the right hand side of (15) will be smaller than or equal to

$$\left|z\int_{R-\delta}^{R+\delta} \Theta(x,y,\sqrt{r^2-z^2})\frac{dG}{dr}dr\right| + \left|z\int_{R+\delta}^{\infty} \Theta(x,y,\sqrt{r^2-z^2})\frac{dG}{dr}dr\right|$$

$$< \frac{zC(1+kD)}{2\pi D^2}\log\left(\frac{R+\delta}{R-\delta}\right) + \frac{B''}{\sqrt{T^2-1}}. \qquad \text{---(16)}$$

The inequalities (7) and (8) as well as the fact that Θ is sectionally continuous and therefore smaller than a constant C, have been used in the last step. The value of D was set equal to zT and consequently the expression (16) can be made arbitrarily small by choosing z small enough.

The second term on the right hand side of (15) has already been shown in paragraph (2.2) to vanish as $z \to 0$.

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ABSTRACT

In this thesis a short wave approximation, the method of W Braunbek, is used to determine the diffracted fields (acoustic and electromagnetic) of plane harmonic waves by an annular aperture.

Integral representations of the rigorous diffracted field in terms of the surface field and its normal derivative are derived. Babinet's theorem is proved for acoustic as well as electromagnetic plane harmonic incident waves. A derivation of Sommerfeld's solution for the diffraction of plane harmonic waves by a half-plane is included.

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