CHARACTER TABLES OF SOME GROUPS OF EXTENSION TYPE

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Abstract

The main aim of this mini-thesis is to give a description of some of the basic methods and techniques that have been developed to calculate the character tables of groups of extension type. We restrict our attention to split extensions $\overline{G}$ of the normal subgroup N of $\overline{G}$ by the subgroup G with the property that every irreducible character of N can be extended to an irreducible character of its inertia group in $\overline{G}$. This is particularly true when N is abelian. We are therefore interested in this special case for which Bernd Fischer developed the theory of Fischer matrices based on the Clifford Theory, to calculate the character tables for both split and non-split extensions.

Before the character table can be determined, the conjugacy classes of our group extensions are calculated using the method of coset analysis. As mentioned earlier we concentrate on examples of split extensions $\overline{G}$ in which N is always abelian, that is, either cyclic or elementary abelian.

A brief outline of the classical theory of characters pertinent to this study, is followed by a detailed discussion of the Clifford theory which provides the basis for the theory of Fischer matrices. Some of the properties of these Fischer matrices which make their calculation much easier, are also given.

In our final chapter, we give four examples illustrating the use of both the classical theory as well as the Fischer matrices to calculate the character tables of our examples which are all maximal subgroups of their respective groups.

DECLARATION

I declare that *Character Tables Of Some Groups of Extension Type* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Abraham Love Prins December 2002

Signed : ....

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Contents

Chapter 1

INTRODUCTION

The classification of finite simple groups is a landmark of tremendous importance in the development of finite group theory. It states that each finite simple group is isomorphic to exactly one of the following:

- A cyclic group of prime order,
- An alternating group A_n of degree at least 5,
- A group of Lie type,
- One of twenty-six sporadic simple groups.

The form of this result, and in particular the existence of the twenty-six sporadic groups, raises many questions. Subsequent work has focussed on attempts to understand these groups, their maximal subgroups and automorphism groups. The study of maximal subgroups of the sporadic groups is very important to reveal the structure of the sporadic groups themselves.

Since the classification of all finite simple groups, more recent work in group theory has involved methods of calculating character tables of maximal subgroups of finite simple groups. The character tables of all the maximal subgroups are not yet known. Most of these maximal subgroups are extensions of elementary abelian groups, so methods have been developed for calculating the character tables of extensions of elementary abelian groups. A knowledge of the character table of a group provides considerable information about the group, and hence it is of importance in the physical sciences as well as in pure mathematics. Character tables of finite groups can be constructed using various techniques. For example, the Schreier-Sims algorithm, Todd-Coxeter coset enumeration method, the Burnside-Dixon algorithm and various other techniques. However Bernd Fischer presented a powerful and interesting technique for calculating the character tables of group extensions. This technique, which is known as the technique of the Fischer-Clifford matrices, derives its fundamentals from the Clifford theory. If $\overline{G} = N.G$ is an appropriate extension of N by G, the method involves the construction of a nonsingular matrix for each conjugacy class of $\overline{G}/N$. In this mini-thesis, we apply the Fischer-Clifford theory only to split extensions. This technique has also been discussed and used by many other researchers, but applied only to split extensions or to the case when every irreducible character of N can be extended to an irreducible character of its intertia group in $\overline{G}$.

However the same method cannot be used to construct character tables of certain non-split group extensions. For example, it cannot be applied to the non-split extensions of the forms $3^7 \cdot O_7(3)$ and $3^7 \cdot (O_7(3) : 2)$ which are maximal subgroups of Fischer's largest sporadic simple group Fi_{24}' and its automorphism group Fi_{24}, respectively. In an attempt to generalize these methods to such type of non-split group extensions, Ali [1] considered the projective representations and characters and showed how the technique of Fischer-Clifford matrices can be applied to any such type of non-split extensions. However in order to apply this technique, the projective characters of the inertia factors must be known and these can be difficult to determine for some groups. Ali [1] successfully applied the technique of Fischer-Clifford matrices and determined the Fischer-Clifford matrices and hence the character tables of the non-split extensions $3^7 \cdot O_7(3)$ and $3^7 \cdot (O_7(3) : 2)$.

In Chapter 2 we give some preliminary results on group extensions and group characters that will be required in the subsequent chapters. In Section 2.1 we define group extensions and discuss some basic results. In Section 2.2 we discuss the conjugacy classes of group extensions. We briefly discuss the technique of coset analysis for computing the conjugacy classes of group extensions $\overline{G}$ of N by G where N is an abelian normal subgroup of $\overline{G}$. This technique was developed and first used by Moori in [16] , [17] and has since been widely used for computing the conjugacy classes of group extensions. In Section 2.3 we give an example of how the technique of coset analysis is applied to calculate the conjugacy classes of a group of extension type.

Chapter 3 deals with the basic results on representations and characters of finite groups such as Maschke's theorem and its general form, Schur's lemma and Frobenius Reciprocity. Restriction and induction of characters are discussed in great detail. Row and column orthogonality relations for irreducible characters are given. The relation between irreducible characters and conjugacy classes is also discussed.

Chapter 4 is devoted to the study of Clifford theory for ordinary representations of a group $\overline{G}$ and its related consequences which will be required to describe the Fischer-Clifford matrices. In Section 4.1 we study the relationship between characters of a group $\overline{G}$ and its normal subgroup N. We present various sufficient conditions for the extendibility of an irreducible character θ of N to its inertia group $\overline{H}$ in $\overline{G}$. In Section 4.2 we describe the theory of the Fischer-Clifford matrices. If $\overline{G}=N.G$ is an appropriate group extension of N by G , the technique involves the construction of a non-singular matrix for each conjugacy class of $\overline{G}\ /N \cong G$. Then by using these matrices together with the fusion maps and character tables of some subgroups of G which are inertia factors of the inertia groups in $\overline{G}$, we are able to construct the complete character table of $\overline{G}$. In this mini-thesis we apply this technique only to split group extensions. This technique has been discussed and used (mainly to split extensions) in, among many others, Moori and Mpono [19], [20], [21], Mpono [22] and Whitley [26]. This section deals with the properties of the Fischer-Clifford matrices which are helpful in their computations. In particular we study a special case of Fischer-Clifford matrices of an extension $\overline{G} = N.G$ with the property that every irreducible character of N can be extended to an irreducible character of its inertia group in $\overline{G}$.

For notation on the conjugacy classes of elements, we follow the notation used in the ATLAS [3]. All our groups and sets are finite unless otherwise specified. For the accuracy and time-efficiency of data, extensive use was made of the program MAGMA[14], to compute conjugacy classes, centralizers of the representatives of conjugacy classes, character tables and inertia factors of the subgroups N and G of the split extension $\overline{G}$.

Chapter 2

THE CONJUGACY CLASSES OF GROUP EXTENSIONS

In this chapter some basic theory on group extensions is first given in section 2.1 and then a method for finding the conjugacy classes of group extensions is described in section 2.2. In section 2.3 we look at an example due to Whitley[26] to illustrate how the theory developed in section 2.2 is used to calculate the conjugacy classes of the group $2^3 : GL_3(2)$. For section 2.1 , the books by Rotman[24] and Gorenstein[8] were used as references while for section 2.2 we used the works of Whitley[26], Moori[18], Moori and Mpono[15] and Salleh[25].

2.1 Definitions and Basic Results

Definition 2.1.1 *If N and G are groups, an extension of N by G is a group $\overline{G}$ that satisfies the following properties*

1. $N \triangleleft \overline{G}$

2. $\overline{G}/N \cong G$.

We say that $\overline{G}$ is a *split extension* of N by G if $\overline{G}$ contains subgroups N and G_1 with $G_1 \cong G$ such that

1. $N \lhd \overline{G}$

2. $NG_1 = \overline{G}$

3. $N \cap G_1 = \{1_{\overline{G}}\}$

In this case $\overline{G}$ is also called a *semi-direct* product of N by G, and we identify G_1 with G.

Note 1 If $\overline{G}$ is a semi-direct product of N by G, then every $\overline{g} \in \overline{G}$ can be uniquely expressed in the form $\overline{g} = ng$, where $n \in N$ and $g \in G$. Multiplication in $\overline{G}$ satisfies $(n_1g_1)(n_2g_2) = n_1n_2^{g_1}g_1g_2$, where n^g denotes gng^{-1}.

Definition 2.1.2 *The automorphism group of a group G, denoted by Aut(G), is the set of all automorphisms of G under the binary operation of composition.*

If $\overline{G}$ is a *split extension* of N by G , then there is a homomorphism $\theta : G \to \mathrm{Aut}(N)$ given by $\theta_g(n) = gng^{-1} = n^g(n \in N, g \in G)$, where we denote $\theta(g)$ by θ_g. Thus G acts on N, and we say that the extension $\overline{G}$ realizes θ.

Conversely, given any groups N and G, and $\theta : G \to \mathrm{Aut}(N)$, we can define a semi-direct product of N by G that realizes θ as follows. Let $\overline{G}$ be the set of ordered pairs $(n, g)(n \in N, g \in G)$ with multiplication $(n_1, g_1)(n_2, g_2) = (n_1\theta_{g_1}(n_2), g_1g_2)$. Then $\overline{G}$ is a semi-direct product of N by G.

Hence a split extension of N by G is completely described by the map $\theta : G \to \text{Aut}(N)$, that is, it is described by the way G acts on N.

We use the ATLAS [3] notation and let $N.G$ denote an arbitrary extension of N by G. A split extension is denoted by $N : G$ or $N :^{\theta} G$, where $\theta : G \to \text{Aut}(N)$ determines the extension. A non-split extension is denoted by $N \cdot G$.

If $\overline{G}$ is a split extension of N by G, then $\overline{G} = NG = \bigcup_{g\in G} Ng$, so G may be regarded as a right transversal for N in $\overline{G}$ (that is, a complete set of right coset representatives of N in $\overline{G}$). Now suppose $\overline{G}$ is any extension of N by G, not necessarily split. Since $\overline{G}/N \cong G$, there is an epimorphism $\lambda : \overline{G} \to G$ with kernel N. For $g \in G$, define a lifting of g to be an element $\overline{g} \in \overline{G}$ such that $\lambda(\overline{g}) = g$. Then choosing a lifting of each element of G, we get the set $\{\overline{g} : g \in G\}$ which is a transversal for N in $\overline{G}$.

We now show that for a non-split extension $\overline{G}$ of N by G, where N is abelian, G acts on N. This result can be obtained from Rotman[24].

Lemma 2.1.3 *Let $\overline{G}$ be an extension of an abelian group N by G , then there is a homomorphism $\theta : G \to Aut(N)$ such that $\theta_g(n) = \overline{g}n\overline{g}^{-1} (n \in N)$, and θ is independent of the choice of liftings $\{\overline{g} : g \in G\}$.*

Proof: For $a \in \overline{G}$, denote conjugation by a by γ_a. Since N is normal in $\overline{G}, \gamma_a|_N$ is an automorphism of N and the function $\mu : \overline{G} \to \text{Aut}(N)$ defined by $\mu(a) = \gamma_a|_N$ is a homomorphism.

If $a \in N$, then $\mu(a) = 1_N$, since N is abelian. Therefore there is a homomorphism $\mu^* : \overline{G}/N \to \text{Aut}(N)$ defined by $\mu^*(Na) = \mu(a)$.

Now $G \cong \overline{G}/N$ and for any lifting $\{\overline{g} : g \in G\}$, the map $\phi : G \to \overline{G}/N$ defined by $\phi(g) = N\overline{g}$ is an isomorphism. If $\{\overline{h} : h \in G\}$ is another choice of liftings , then $\overline{g}\overline{h}^{-1} \in N$ so that $N\overline{g} = N\overline{h}$.

Therefore the isomorphism ϕ is independent of the choice of liftings. Now let $\theta : G \to \text{Aut}(N)$ be the composite $\mu^* \circ \phi$. If $g \in G$ and $\overline{g} \in \overline{G}$ is a lifting of g , then $\theta(g) = \mu^*(\phi(g)) = \mu^*(N\overline{g}) = \mu(\overline{g}) \in \text{Aut}(N)$, so for $n \in N$, $\theta_g(n) = \mu(\overline{g})(n) = \overline{g}n\overline{g}^{-1} = n^{\overline{g}}$, as required. □

Note 2 Let $\overline{G}$ be an extension of an abelian group N by G. For each $g \in G$ we choose a lifting $\overline{g} \in \overline{G}$, and for convenience we take $\overline{1} = 1$. We identify G with $\overline{G}/N$ under the isomorphism $g \rightarrow N\overline{g}$. Now $\{\overline{g} : g \in G\}$ is a right transversal for N in $\overline{G}$, so every element $h \in \overline{G}$ has a unique expression of the form $h = n\overline{g}$ $(n \in N, g \in G)$, and we have the following relations.

1. $\overline{g}n = n^{\overline{g}}\overline{g}$, where $n \in N$ and $g \in G$.

2. $\overline{g}\overline{h} = f(g,h)\overline{gh}$ for some $f(g,h) \in N$, where $g, h \in G$.

2.2 The Conjugacy Classes of Group Extensions

Let $\overline{G} = N.G$, where N is abelian. Then for each conjugacy class $[g]$ in G with representative $g \in G$, we analyse the coset $N\overline{g}$, where $\overline{g}$ is a lifting of g in $\overline{G}$ and $\overline{G} = \bigcup_{g \in G} N\overline{g}$. To each class representative $g \in G$ with lifting $\overline{g} \in \overline{G}$, we define

$$C_{\overline{g}} = \{x \in \overline{G} : x(N\overline{g}) = (N\overline{g})x\}.$$

Then $C_{\overline{g}}$ being the set stabilizer of $N\overline{g}$ in $\overline{G}$ under the action by conjugation of $\overline{G}$ on $N\overline{g}$, is a subgroup of $\overline{G}$. The following lemmas and their proofs due to Whitley[26] and Moori and Mpono[15] will be required in the next section.

Lemma 2.2.1 $N \lhd C_{\overline{g}}$.

Proof: For any $n \in N$

$$n(N\overline{g})n^{-1} = N\overline{g}n^{-1} = N\overline{g}n^{-1}\overline{g}^{-1}\overline{g} = N\overline{g},$$

the last step following from the fact that $(n^{-1})^{\overline{g}} \in N$ since $N \lhd \overline{G}$.

Hence $N \subseteq C_{\overline{g}}$. From $N \leq C_{\overline{g}} \leq \overline{G}$ and $N \lhd \overline{G}$, we obtain $N \lhd C_{\overline{g}}$. $\square$

Lemma 2.2.2 $C_{\overline{g}}/N = C_{\overline{G}/N}(N\overline{g})$.

Proof: Consider $Nk \in \overline{G}/N$. Then

$$\begin{aligned} Nk \in C_{\overline{G}/N}(N\overline{g}) &\iff Nk(N\overline{g})(Nk)^{-1} = N\overline{g} \\ &\iff NkN\overline{g}Nk^{-1} = N\overline{g} \\ &\iff NkN\overline{g}k^{-1} = N\overline{g} \\ &\iff NkNn\overline{g}k^{-1} = N\overline{g} \quad \forall\, n \in N \\ &\iff Nkn\overline{g}k^{-1} = N\overline{g} \quad \forall\, n \in N \\ &\iff kn\overline{g}k^{-1} \in N\overline{g} \quad \forall\, n \in N \\ &\iff k \in C_{\overline{g}} \\ &\iff Nk \in C_{\overline{g}}/N. \end{aligned}$$

Thus we obtain that $C_{\overline{g}}/N = C_{\overline{G}/N}(N\overline{g})$. □

From the two preceding lemmas, we have that $C_{\overline{g}} = N.C_{\overline{G}/N}(N\overline{g})$. For a lifting $\overline{g} \in \overline{G}$ of $g \in G$, we can identify $C_{\overline{G}/N}(N\overline{g})$ with $C_G(g)$ and write $C_{\overline{g}} = N.C_G(g)$ in general. If $\overline{G} = N : G$ then we can identify $C_{\overline{g}}$ with $C_g = \{x \in \overline{G} : \; x(Ng) = (Ng)x\}$ and in this case we obtain the following corollary.

Corollary 2.2.3 *Let $\overline{G} = N : G$. Then $C_g = N : C_G(g)$.*

Proof: We have already shown in the Lemma 2.2.1 that $N \lhd C_g$. Now we show that $C_G(g) \leq C_g$ and that $N \cap C_G(g) = \{1_G\}$. Let $x \in C_G(g)$. Then we obtain $(Ng)^x = x(Ng)x^{-1} = xNgx^{-1} = Nxgx^{-1} = Ng$. Thus $x \in C_g$ and hence $C_G(g) \leq C_g$. Since $N \cap C_G(g) \leq N \cap G = \{1_G\}$, then we have that $N \cap C_G(g) = \{1_G\}$. This completes the proof. □

The conjugacy classes of $\overline{G}$ will be determined from the action by conjugation of C_g, for each conjugacy class $[g]_G$ of G, on the elements of $N\overline{g}$ or in the case of a split extension on the elements of Ng.

Since $C_g = N : C_G(g)$, we act first N and then act $\{\overline{h} : h \in C_G(g)\}$ on the elements of $N\overline{g}$. Where as usual $\overline{h}$ denotes the lifting of h in $C_G(g)$. The outline of this action is given in two steps by Moori and Mpono [15] as follows:

STEP 1: *The action of N on $N\overline{g}$:*

Let $C_N(\overline{g})$ be the stabilizer of $\overline{g}$ in N. Then for any $n \in N$ we have

$$\begin{aligned}
x \in C_N(n\overline{g}) &\Leftrightarrow x(n\overline{g})x^{-1} = n\overline{g} \\
&\Leftrightarrow xnx^{-1}x\overline{g}x^{-1} = n\overline{g} \\
&\Leftrightarrow n(x\overline{g}x^{-1}) = n\overline{g}, \quad \textit{since } N \textit{ is abelian} \\
&\Leftrightarrow x\overline{g}x^{-1} = \overline{g} \\
&\Leftrightarrow x \in C_N(\overline{g}).
\end{aligned}$$

Thus $C_N(\overline{g})$ fixes every element of $N\overline{g}$. Now let $|C_N(\overline{g})| = k$. Then under the action of N, $N\overline{g}$ splits into k orbits $Q_1, Q_2, ..., Q_k$, where

$$\begin{aligned}
|Q_i| &= [N : C_N(\overline{g})] \\
&= \frac{|N|}{k}, \quad \text{for } i \in \{1, ..., k\}.
\end{aligned}$$

STEP 2: *The action of $\{\overline{h} : h \in C_G(g)\}$ on $N\overline{g}$*

Since the elements of $N\overline{g}$ are now in the orbits $Q_1, ..., Q_k$ from step 1 above, we need only to act $\{\overline{h} : h \in C_G(g)\}$ on the k orbits. Suppose that under this action f_j of the orbits $Q_1, ..., Q_k$ fuse together to form one orbit $\triangle_l$, then the $f_j's$ obtained this way must satisfy
$\sum_j f_j = k$

and we have

$| \triangle_l| = f_j \times \frac{|N|}{k}$

Thus for $x = d_l\overline{g} \ \in \ \triangle_l$, we obtain that

$$\begin{aligned} |[x]_{\overline{G}}| &= |\triangle_l| \times |[g]_G| \\ &= f_j \times \frac{|N|}{k} \times \frac{|G|}{|C_G(g)|} \\ &= f_j \times \frac{|\overline{G}|}{k|C_G(g)|} \end{aligned}$$

and thus we obtain that

$$\begin{aligned} |C_{\overline{G}}(x)| &= \frac{|\overline{G}|}{|[x]_{\overline{G}}|} \\ &= |\overline{G}| \times \frac{k|C_G(g)|}{f_j|\overline{G}|} \\ &= \frac{k|C_G(g)|}{f_j}. \end{aligned}$$

Thus to calculate the conjugacy classes of $\overline{G} = N{\cdot}G$, we need to find the values of k and the $f_j's$ for each class representative $g \in G$. We note that the values of k can be determined from the action of G on N(given in lemma 2.1.3). If $\overline{G} = N : G$ (a split extension) however, we analyse the coset Ng instead of $N\overline{g}$ since in the split case $G \leq \ \overline{G}$. Under the action of N on Ng, we always assume that $g \in Q_1$. Since $C_G(g)$ fixes g, Q_1 does not fuse with any other Q_i. Hence we will always have that $f_1 = 1$. Hence

$$\begin{aligned} k &= \sum_j f_j \\ &= 1 \quad + \quad \sum_m f_m, \end{aligned}$$

where the sum is taken over all m such that $g \notin Q_m$.

We now apply the method described in the Step 1 and Step 2 in the next section.

2.3 The Conjugacy Classes of $2^3 : GL_3(2)$

In this section we give the conjugacy classes of the group $\overline{G} = N : G$ where N is an elementary abelian group of order 8 and $G \cong GL_3(2)$, as calculated by Whitley[26], where G acts naturally on N.

We regard N as the vector space $V_3(2)$ of dimension three over a field of two elements. Let N be generated by $\{e_1, e_2, e_3\}$ with $e_i^2 = 1$ for $1 \leq i \leq 3$, so

$$N = \{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}$$

To determine the conjugacy classes of $\overline{G}$ we analyse the cosets Ng where g is a representative of a class of G. (Note that the extension is split, so $\overline{G} = \bigcup_{g \in G} Ng$). Now

$$|C_{\overline{G}}(x)| = \frac{k.|C_G(g)|}{f_j},$$

where f_j of the k blocks of the coset Ng have fused to give a class of $\overline{G}$ containing x.We need the conjugacy classes of G, so we exhibit them here (obtained from ATLAS [3]).

class	(1A)	(2A)	(3A)	(4A)	(7A)	(7B)
centralizer	168	8	3	4	7	7

Table 1.3.1: The conjugacy table of $GL_3(2)$.

The representatives thus must come from the classes mentioned in the table above:

- $\underline{g = 1_G:}$

For g the identity of G, g fixes all elements of N, so k = 8. Since G is transitive on $N - \{1\}$ under the action of $C_G(g) = G$, we have two orbits with $f_1 = 1$ and $f_2 = 7$, so this coset gives two classes of $\overline{G}$:

$x = 1$, class(1), $|C_{\overline{G}}(x)| = 8 \times 168 = 1344$

$x = e_1$, class(2_1), $|C_{\overline{G}}(x)| = \frac{8\times168}{7} = 192$

- $\underline{g \in (2A):}$

We take

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

with $|C_G(g)| = 8$. The action of g on N is represented by the cycle structure

$(1)(e_1)(e_1e_2e_3)(e_2e_3)(e_2\ e_3)(e_1e_2\ e_1e_3)$, so $k = 4$.

The four orbits of N on Ng are $\{g,\ e_2e_3g\}, \{e_1g,\ e_1e_2e_3g\}, \{e_2g,\ e_3g\}$ and $\{e_1e_2g,\ e_1e_3g\}$.

Now we act

$$C_G(g) = \left\langle \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\rangle$$

on these orbits.

For $eg \in Ng, h \in C_G(g), (eg)^h = e^h g^h = e^h g$ so we obtain the following orbits:
$\{g, e_2e_3g\}^{C_G(g)} = \{g, e_2e_3g\}, \{e_1g, e_1e_2e_3g\}^{C_G(g)} = \{e_1g, e_1e_2e_3g\}, \{e_2g, e_3g\}^{C_G(g)}$
$= \{e_2g, e_3g, e_1e_2g, e_1e_3g\}$

Therefore we get three classes of $\overline{G}$:

$f_1 = 1,\ x = g$, class(2_2), $\quad |C_{\overline{G}}(x)| = 4 \times 8 = 32;$

$f_2 = 1,\ x = e_1g$, class(2_3), $\quad |C_{\overline{G}}(x)| = 32;$

$f_3 = 2,\ x = e_2g$, class(4_1), $\quad |C_{\overline{G}}(x)| = \frac{4\times 8}{2} = 16.$

- $\underline{g \in (3A):}$

We take

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with $|C_G(g)| = 3$. The action of g on N is represented by $(1)(e_1e_2e_3)(e_1\ e_2\ e_3)(e_1e_2\ e_1e_3\ e_2e_3)$, so $k = 2$ which means we must have two blocks. These cannot fuse together under $C_G(g)$, since $g^{C_G(g)} = \{g\}$. Therefore we have two classes of $\overline{G}$, with $f_1 = 1$ and $f_2 = 1$:
$x = g$, class(3_1), $\quad |C_{\overline{G}}(x)| = 2 \times 3 = 6;$

$x = e_1g$, class(6_1), $\quad |C_{\overline{G}}(x)| = 6.$

- $\underline{g \in (4A):}$

We get two classes of $\overline{G}$ once more:

$x = g$, class(4_2), $|C_{\overline{G}}(x)| = 8$;

$x = e_1 g$, class(4_3), $|C_{\overline{G}}(x)| = 8$.

- $g \in (7A)$:

For the class $(7A)$, we have $k = 1$, so each coset has just one class in $\overline{G}$. We thus get the class (7_1) of $\overline{G}$, with centralizer of order 7.

- $g \in (7B)$:

This case works the same as for the previous class and we obtain class (7_2) of $\overline{G}$, with centralizer of order 7.

class of G	(1A)		(2A)			(3A)		(4A)		(7A)	(7B)
class of $\overline{G}$	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(3_1)	(6_1)	(4_2)	(4_3)	(7_1)	(7_2)
centralizer	1344	192	32	32	16	6	6	8	8	7	7

Table 1.3.2: The conjugacy table of $2^3 : GL_3(2)$.

Chapter 3

REPRESENTATIONS AND CHARACTERS

Two ways of approaching representation and character theory are through the use of modules on the one hand (for instance, the approach used by James and Liebeck [10]), and through the classical approach used by Feit[5] for example, on the other hand. Our discussion is along the classical approach and for this purpose we follow the class notes of Moori[18].

We give some basic results on the representations and characters of finite groups in this chapter. In the first section, theorems and lemmas will almost always be stated without proofs. Section 3.2 deals with the relationship between characters of groups and the and characters of their subgroups, while in section 3.3 we shall look at the role of normal subgroups in the calculation of characters of a group. In the last two sections mentioned, only the proofs of the main results (that is those results dealing more directly with the techniques of finding the characters of a group) are given. These proofs are mainly taken from Moori's notes [18].

3.1 Basic Concepts

Definition 3.1.1 *Let G be a group. Let $f : G \to GL_n(F)$ be a homomorphism. Then we say that f is a matrix representation of G of degree n (or dimension n), over the field F.*

If $Ker(f) = \{1_G\}$, then we say that f is a $faithful$ representation of G. In this situation $G \cong Image(f)$, so that G is isomorphic to a subgroup of $GL_n(F)$.

Definition 3.1.2 *Let $f : G \to GL_n(F)$ be a representation of G over the field F. The function $\chi : G \to F$ defined by $\chi(g) = trace(f(g))$ is called the character of f.*

Definition 3.1.3 *If $\phi : G \to F$ is a function from a group G to a field F which is constant on conjugacy classes of G, that is $\phi(g) = \phi(xgx^{-1}), \forall x \in G$, then ϕ is a class function.*

Lemma 3.1.4 *A character is a class function.*

Proof: See [18, Lemma i.4]

Definition 3.1.5 *Two representations $\rho, \phi : G \to GL_n(F)$ are said to be equivalent if there exists an $n \times n$ matrix P over F such that*

$$P^{-1}\rho(g)P = \phi(g), \quad \forall g \in G.$$

Theorem 3.1.6 *Equivalent representations have the same character.*

Proof: See [18,Theorem i.5]

Before defining the concepts of reducibility and irreducibility of representations and characters, we need to say what is meant by a reducible and an irreducible set of matrices. If S is a set of matrices, then S is *reducible* if $\exists\ m,\ k\ \in\ \mathbb{N}$, and $\exists\ P \in\ GL_n(F)$ such that $\forall\ A \in S$ we have

$$P^{-1}AP = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$$

where B is an $m \times m$ matrix, D is a $k \times k$ matrix, C is a $k \times m$ matrix and 0 is the zero matrix. If no such P exists, we say that S is *irreducible.* Furthermore if $C = 0\ \forall\ A \in S$, we say that S is fully reducible and if $\exists\ P \in\ GL_n(F)$ such that

$$P^{-1}AP = \begin{pmatrix} B_1 & 0 & \cdots & \cdots & 0 \\ 0 & B_2 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & B_k \end{pmatrix}, \quad \forall\ A \in S,$$

where each B_i is irreducible, we say S is completely reducible.

Definition 3.1.7 *Let $f : G \rightarrow GL_n(F)$ be a representation of G over F and let $S = \{f(g) : g \in G\}$. We say that f is reducible, fully reducible, or completely reducible if S is reducible, fully reducible, or completely reducible, respectively.*

Definition 3.1.8 *If χ_ρ is a character afforded by a representation ρ of G, then we say that χ_ρ is an irreducible character of G if ρ is an irreducible representation.*

Definition 3.1.9 *Let $\rho : G \rightarrow GL_n(F)$ and $\phi : G \rightarrow GL_m(F)$ be two representations of G over F. Define $\rho + \phi : G \rightarrow GL_{n+m}(F)$ by*

$$(\rho+\phi)(g) = \begin{pmatrix} \rho(g)_{n\times n} & 0_{n\times m} \\ 0_{m\times n} & \phi(g)_{m\times m} \end{pmatrix} = \rho(g) \oplus \phi(g), \quad \forall g \in G.$$

Then $\rho + \phi$ is a representation of G over F, of degree $n + m$.

If χ_1 and χ_2 are the characters of ρ and ϕ respectively and χ is the character of $\rho + \phi$, then for all $g \in G$ we have $\chi(g) = \chi_1(g) + \chi_2(g)$.

Theorem 3.1.10 *(Maschke's theorem) Let G be a finite group. Let f be a representation of G over a field F whose characteristic is either equal to zero or is a prime that does not divide $|G|$. If f is reducible, then f is fully reducible.*

Proof: See [18,Theorem i.6]

Theorem 3.1.11 *(The general form of Maschke's theorem)*
Let G be a finite group and F be a field whose characteristic is either equal to zero or is a prime that does not divide $|G|$. Then every representation of G over F is completely reducible.

Proof: See [5,(1.1)]

Theorem 3.1.12 *(Schur's lemma) Let $\rho : G \rightarrow GL_n(F)$ and $\phi : G \rightarrow GL_m(F)$ be two representations of a group G over a field F. Assume there exists an $m \times n$ matrix P such that $P\rho(g) = \phi(g)P$ for all $g \in G$. Then either $P = 0_{m\times n}$ or P is non-singular so that $\rho(g) = P^{-1}\phi(g)P$ (that is, ρ and ϕ are equivalent representations).*

Proof: See [5,(1.2)]

Definition 3.1.13 *Let G be a finite group and assume that the characteristic of the field F does not divide $|G|$. If ρ and ϕ are two functions from G into F, we define an innerproduct $\langle , \rangle$ by the following rule:*

$$\langle \rho, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \rho(g)\phi(g^{-1}) \ ,$$

where $\frac{1}{|G|}$ stands for $|G|^{-1}$ in F.

Theorem 3.1.14 *The inner product $\langle , \rangle$ is bilinear:*

(i) $\langle \rho_1 + \rho_2, \phi \rangle = \langle \rho_1, \phi \rangle + \langle \rho_2, \phi \rangle$

(ii) $\langle \rho, \phi_1 + \phi_2 \rangle = \langle \rho, \phi_1 \rangle + \langle \rho, \phi_2 \rangle$

(iii) $\langle \ a\rho, \phi \ \rangle = a\langle \rho, \phi \rangle = \langle \rho, a\phi \rangle, \quad \forall a \in F$

and symmetric:

$$\langle \rho, \phi \rangle = \langle \phi, \rho \rangle$$

Proof:

(i)

$$\begin{aligned}
\langle \rho_1 + \rho_2, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} \Big(\rho_1 + \rho_2 \Big)(g) \phi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \Big(\rho_1(g) + \rho_2(g) \Big) \phi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \Big(\rho_1(g)\phi(g^{-1}) + \rho_1(g)\phi(g^{-1}) \Big), \text{F being an additive abelian group} \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_1(g)\phi(g^{-1}) + \frac{1}{|G|} \sum_{g \in G} \rho_2(g)\phi(g^{-1}), \\
&= \langle \rho_1, \phi \rangle + \langle \rho_2, \phi \rangle
\end{aligned}$$

(ii) Similar to (i).

(iii)

$$\begin{aligned}
\langle a\rho, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} \Big(a\rho \Big)(g) \phi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} a \Big(\rho(g) \Big) \phi(g^{-1}) \\
&= a \frac{1}{|G|} \sum_{g \in G} \rho(g) \phi(g^{-1}) \\
&= a \langle \rho, \phi \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle a\rho, \phi \rangle &= \frac{1}{|G|} \sum_{g \in G} \Big(a\rho \Big)(g) \phi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} a\rho(g) \phi(g^{-1})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|G|}\sum_{g\in G}\rho(g)a\phi(g^{-1}), \text{ F being a multiplicative abelian group} \\
&= \frac{1}{|G|}\sum_{g\in G}\rho(g)\Big(a\phi\Big)(g^{-1}) \\
&= \langle\rho, a\phi\rangle
\end{aligned}
$$

To complete the proof, see [18,Theorem i.11]. □

Note 1 If $\rho : G \to GL_n(\mathbb{C})$ is a representation of a group G, then we denote the (i, j) entry of $\rho(g)$ by $\rho_{ij}(g)$. Hence $\rho_{ij}(g)$ is a map from G into $\mathbb{C}$.

For the rest of this chapter we shall mean finite groups when mentioning groups, unless explicit exceptions are made and all representations will be over the field $\mathbb{C}$ of complex numbers.

Theorem 3.1.15 *Let G be a finite group and let ρ and ϕ be two irreducible representations of G.*

(i) If ρ and ϕ are inequivalent, then

$$\langle\rho_{rs}, \phi_{ij}\rangle = 0, \quad \forall i, j, r, \quad \textit{and } s.$$

(ii) $\langle\rho_{rs}, \phi_{ij}\rangle = \frac{\delta_{is}.\delta_{jr}}{deg(\rho)}$.

Proof: See [18, Theorem ii.1]

Theorem 3.1.16 *Let G be a finite group and let ρ and ϕ be two irreducible representations of G, with characters χ_ρ and χ_ϕ.*

(i) If ρ and ϕ are equivalent, then

$$\langle \chi_\rho, \chi_\phi \rangle = 1$$

(ii) If ρ and ϕ are not equivalent, then

$$\langle \chi_\rho, \chi_\phi \rangle = 0$$

(iii) $\langle \chi_\rho, \chi_\rho \rangle = 1$

Proof: See [18,Theorem ii.2]

Theorem 3.1.17 *Two representations of a group G are equivalent if and only if they have the same characters.*

Proof: See [18,Corollary ii.4]

Lemma 3.1.18 *(i) If*

$$\chi = \sum_{i=1}^{k} \lambda_i \chi_i$$

where χ_i are distinct irreducible characters of a group G and λ_i are nonnegative integers, then

$$\langle \chi, \chi \rangle = \sum_{i=1}^{k} \lambda_i^2.$$

(ii) If χ is a character of G, then χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Proof:

(i)

$$\begin{aligned}
\langle \chi, \chi \rangle &= \langle \sum_{i=1}^{k} \lambda_i \chi_i \, , \ \sum_{i=1}^{k} \lambda_i \chi_i \rangle \\
&= \sum_{i=1}^{k} \lambda_i \sum_{i=1}^{k} \lambda_i \langle \chi_i \, , \ \chi_i \rangle \\
&= \sum_{i=1}^{k} \lambda_i^2 \langle \chi_i \, , \ \chi_i \rangle \\
&= \sum_{i=1}^{k} \lambda_i^2
\end{aligned}$$

(ii)By theorem 3.1.16(iii), we have that if χ is irreducible, then $\langle \chi, \chi \rangle = 1$.

For the converse, assume that $\langle \chi \, , \ \chi \rangle = 1$. Let

$$\chi = \sum_{i=1}^{k} \lambda_i \chi_i$$

where χ_i are distinct irreducible characters of G and λ_i are nonnegative integers, then by (i), we have

$$\sum_{i=1}^{k} \lambda_i^2 = \langle \chi \, , \ \chi \rangle \ = 1$$

$$\Leftrightarrow \quad \lambda_i^2 \ = 1, \text{ for some } \quad i = 1, 2, ..., k$$

$$\Leftrightarrow \lambda_i = 1$$

Thus $\chi = \chi_i$ is irreducible. $\square$

Note 2 If C_i is a conjugacy class of G, then

$$C_{i'} = \{\ g \in G : g^{-1} \in C_i\}$$

is also a conjugacy class of G and $C_i = C_{i'}$ if and only if $g \sim g^{-1}$ for all $g \in C_i$.

Theorem 3.1.19 *Let $Irr(G) = \{\chi_1, \chi_2, ..., \chi_k\}$. Then*

(i) $\frac{1}{|G|}\ \sum_{g\in G} \chi_i(g)\chi_j(g) = \delta_{ij}$, *(row orthogonality)*

(ii) $\sum_{s\ =\ 1}^{k} \chi_s(g_i)\chi_s(g_j) = \delta_{ij'}\ |C_G(g_j)|$, *(column orthogonality)*

Proof: See [18, Theorem ii.17]

Theorem 3.1.20 *The number of irreducible characters of a group G equals the number of conjugacy classes of G.*

Proof: See [18, Theorem ii.18]

Proposition 3.1.21 *Let $G = \langle\ x\ \rangle$ be a cyclic group of order n. Let $e^{\frac{2k\pi}{n}i}$ be the n-th roots of unity in $\mathbb{C}$, $k = 0, 1, 2, ..., n-1$. Define $\rho_k : G \to \mathbb{C}^*$ by*

$$\rho_k(x^m) = [e^{\frac{2k\pi}{n}i}\ ]^m.$$

For $k = 0, 1, 2, ..., n-1$, ρ_k defines the n distinct irreducible representations of G.

Proof: We first show that ρ_k is well defined:

Let $x^m = x^{m'}$, where $m = sn + t$, $m' = s'n + t'$, $s, s' \in \mathbb{Z}$ and $t, t' = 0, 1, 2, ..., n-1$. From which we get $x^t = x^{t'} \Rightarrow t = t'$.

If, $[e^{\frac{2k\pi}{n} i}]^m \neq [e^{\frac{2k\pi}{n} i}]^{m'}$, then we have

$$\begin{aligned}
[e^{\frac{2k\pi}{n} i}]^{m-m'} \neq 1 \quad & \Rightarrow \quad [e^{\frac{2k\pi}{n} i}]^{(s-s')n + (t-t')} \neq 1 \\
& \Rightarrow \quad [e^{\frac{2k\pi}{n} i}]^{(s-s')n} \neq 1 \\
& \Rightarrow \quad \rho_k(x^{(s-s')n}) \neq 1 \\
& \Rightarrow \quad \rho_k(x^0) \neq 1 \\
& \Rightarrow \quad [e^{\frac{2k\pi}{n} i}]^0 \neq 1,
\end{aligned}$$

giving a contradiction. Hence ρ_k is well defined.

Next we show that ρ_k is a homomorphism:

$$\begin{aligned}
\rho_k(x^m)\rho_k(x^{m'}) &= \rho_k(x^t)\rho_k(x^{t'}) \\
&= [e^{\frac{2k\pi}{n} i}]^t [e^{\frac{2k\pi}{n} i}]^{t'} \\
&= [e^{\frac{2k\pi}{n} i}]^{t+t'} \\
&= \rho_k(x^{t+t'}) \\
&= \rho_k(x^t . x^{t'}) \\
&= \rho_k(x^m . x^{m'})
\end{aligned}$$

So ρ_k is a homomorphism and hence a representation.

ρ_k is unique:

Let $\rho_k = \rho_{k'}$ with $k, k' \leq n$. Now $\forall g \in \langle x \rangle, \quad g = x^r$ where $r = 0, 1, 2, ..., n-1$. So we have

$$\begin{aligned}
\rho_k(x^r) = \rho_{k'}(x^r) \quad &\Rightarrow \quad [e^{\frac{2k\pi}{n} i}]^r = [e^{\frac{2k'\pi}{n} i}]^r \\
&\Rightarrow \quad e^{(\frac{2k\pi.r}{n} i \; - \; \frac{2k'\pi.r}{n} i)} = 1 \\
&\Rightarrow \quad e^{\frac{2\pi.r}{n} \; (k-k')i} = 1 \\
&\Rightarrow \quad \rho_{(k-k')}(x^r) = 1, \quad \forall \; r = 0, 1, 2, ..., n-1. \\
&\Rightarrow \quad k - k' = 0, \text{ so that } k = k'.
\end{aligned}$$

Lastly we must show that ρ_k is irreducible:

We use lemma 3.1.18(ii).

$$\begin{aligned}
\langle \rho_k, \rho_k \rangle &= \frac{1}{|\langle x \rangle|} \sum_{g \in \langle x \rangle} \rho_k(g)\rho_k(g^{-1}) \\
&= \frac{1}{n} \sum_{g \in \langle x \rangle} \rho_k(gg^{-1}) \\
&= \frac{1}{n} \sum_{g \in \langle x \rangle} \rho_k(1_{\langle x \rangle}) \\
&= \frac{1}{n} \sum_{g \in \langle x \rangle} 1_{\mathbb{C}^*} \\
&= \frac{1}{n} n \\
&= 1.
\end{aligned}$$

Hence ρ_k is irreducible.

This completes the proof of the proposition. □

Definition 3.1.22 *Let $P = (p_{ij})_{m \times m}$ and $Q = (q_{ij})_{n \times n}$ be two matrices. Then the $mn \times mn$ matrix $P \otimes Q$ is defined by*

$$P \otimes Q := (p_{ij}Q) = \begin{pmatrix} p_{11}Q & p_{12}Q & \dots & \dots & \dots & p_{1m}Q \\ p_{21}Q & p_{22}Q & \dots & \dots & \dots & p_{2m}Q \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{m1}Q & p_{m2}Q & \dots & \dots & \dots & p_{mm}Q \end{pmatrix}$$

From this definition, we can show that

$(P \otimes Q)(P' \otimes Q') = (PP') \otimes (QQ') \quad (*)$:

$$\begin{aligned} (P \otimes Q)(P' \otimes Q') &= (\sum_{k=1}^{m} p_{il}Qp'_{li}Q')_{mn \times mn} \\ &= (\sum_{k=1}^{m} p_{il}p'_{li}QQ')_{mn \times mn} \\ &= (PP') \otimes (QQ'). \end{aligned}$$

Definition 3.1.23 *Let T and U be representations of a group G, then the tensor product $T \otimes U$ is defined by:*

$$(T \otimes U)(g) := T(g) \otimes U(g)$$

Theorem 3.1.24 *Let T and U be representations of a group G, then*

(i) $T \otimes U$ is a representation of G.

(ii) if $\chi_{(T \otimes U)}$ is the character afforded by $T \otimes U$ then

$$\chi_{(T \otimes U)} = \chi_T \chi_U$$

Proof: See [18,Theorem iii.1]

Definition 3.1.25 *Let $G = H \times K$ be the direct product of two groups H and K and let $T : H \longrightarrow GL_m(\mathbb{C})$ and $U : K \longrightarrow GL_n(\mathbb{C})$ be representations of H and K, respectively. Since every element $g \in G$, can be expressed uniquely in the form $g = hk$, for some $h \in H$ and some $k \in K$, the direct product $T \times U$ can be defined by*

$$(T \times U)(g) : \; = \; T(h) \otimes U(k)$$

From the uniqueness of $g = hk$ and because representations T and U are well defined, it can be shown that $T \times U$ is well defined. Also for $g = hk$ and $g' = h'k'$ with $h, h' \in H$ and $k, k' \in K$, we have

$$\begin{aligned}(T \times U)(g)(T \times U)(g') &= (T(h) \otimes U(k))(T(h') \otimes U(k')) \\ &= T(h)T(h') \otimes U(k)U(k'), \text{ by } (*) \\ &= T(hh') \otimes U(kk') \\ &= (T \times U)(gg'),\end{aligned}$$

which means $T \times U$ is a homomorphism and therefore a representation.

From definition 3.1.22, we can deduce that for two matrices P and Q, that

$$Trace(P \otimes Q) \; = \; Trace(P).Trace(Q).$$

So we show the following

$$\begin{aligned}\chi_{(\ T\ \times\ U\ )}(g) &= Trace(\ (\ T\ \times\ U\ )(g)\ ) \\ &= Trace(\ T(h)\ \otimes\ U(k)\ ) \\ &= Trace(T(h)).Trace(U(k))\end{aligned}$$

and the next theorem tells us that all the characters of a direct product are constructed in this way.

Theorem 3.1.26 *Let $G = H \times K$ be the direct product of two groups H and K. Then the direct product of any irreducible character of H and any irreducible character of K is an irreducible character of G. Moreover, every irreducible character of G can be constructed in this way.*

Proof: See [18,Theorem iii.2]

Definition 3.1.27 *Let χ be a character of a group G. For $n \in (\mathbb{N} \cup \{0\})$, we define χ^n by*

$$\chi^n(g) :\ =\ (\chi(g))^n, \quad \forall g \in G.$$

If G is a group and H is a subgroup of G, then we can use the irreducible characters of G to find at least some of the characters of H and vice versa. We deal with the methods of doing this in the following section and use the notes of Moori[18] again.

3.2 Restriction and Induction of Characters

Definition 3.2.1 *Let G be a group and H be a subgroup of G. If $\rho : G \longrightarrow GL_n(\mathbb{C})$ is a representation of G, then $(\rho \downarrow H) : H \longrightarrow GL_n(\mathbb{C})$ given by*

$$(\rho \downarrow H)(h) \quad = \quad \rho(h), \quad \forall\ h \in H,$$

is a representation of H. We say that $\rho \downarrow H$ is the restriction of ρ to H. If χ_ρ is the character of ρ, then $\chi_\rho \downarrow H$ is the character of $\rho \downarrow H$. We refer to $\chi_\rho \downarrow H$ as the restriction of χ_ρ to H.

Theorem 3.2.2 *Let G be a group and $H \leq G$. If ψ is a character of H, then there is an irreducible character χ of G such*

$$\langle \chi \downarrow H, \psi \rangle_H \quad \neq \quad 0.$$

Proof: See [18,Theorem iv.1.1].

Theorem 3.2.3 *Let G be a group and $H \leq G$. If*

$$\chi \in Irr(G) \quad \text{and} \quad Irr(H) = \{\psi_1, \psi_2, ..., \psi_r\},$$

then

$$\chi \downarrow H \quad = \sum_{i=1}^{r} \delta_i \psi_i, \text{ where} \quad \delta_i \in (\mathbb{N} \cup \{0\}) \quad \text{and}$$

$$\sum_{i=1}^{r} \delta_i^2 \quad \leq \quad [G:H]\ (**)$$

Moreover, we have equality in $(**)$ if and only if $\chi(g) = 0, \quad \forall g \in (G \setminus H)$.

Proof: Since $\chi \downarrow H$ is a character of H, $\exists \delta_i \in (\mathbb{N} \cup \{0\})$ such that

$$\chi \downarrow H \quad = \quad \sum_{i=1}^{r} \delta_i \psi_i.$$

Now

$$\begin{aligned}
\langle \chi \downarrow H, \chi \downarrow H \rangle_H &= \left\langle \sum_{i=1}^{r} \delta_i \psi_i, \sum_{i=1}^{r} \delta_i \psi_i \right\rangle_H \\
&= \sum_{i=1}^{r} \delta_i^2 \ \langle \psi_i, \psi_i \rangle_H \\
&= \sum_{i=1}^{r} \delta_i^2
\end{aligned}$$

and

$$\langle \chi \downarrow H, \chi \downarrow H \rangle_H = \frac{1}{|H|} \sum_{h \in H} \chi(h).\overline{\chi(h)}.$$

Hence we get

$$\begin{aligned}
\sum_{i=1}^{r} \delta_i^2 &= \frac{1}{|H|} \sum_{h \in H} \chi(h).\overline{\chi(h)} \quad \textit{so that} \\
|H| \sum_{i=1}^{r} \delta_i^2 &= \sum_{h \in H} \chi(h).\overline{\chi(h)} \quad (***)
\end{aligned}$$

From

$$\begin{aligned}
1 &= \langle \chi, \chi \rangle_G \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g).\overline{\chi(g)} \\
&= \frac{1}{|G|} \sum_{g \in H} \chi(g).\overline{\chi(g)} + \frac{1}{|G|} \sum_{g \in (G \setminus H)} \chi(g).\overline{\chi(g)} \\
&= \frac{|H|}{|G|} \sum_{i=1}^{r} \delta_i^2 + \frac{1}{|G|} \sum_{g \in (G \setminus H)} \chi(g).\overline{\chi(g)} \quad \textit{by } (***) \\
&= \frac{|H|}{|G|} \sum_{i=1}^{r} \delta_i^2 + \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2
\end{aligned}$$

we obtain that

$$\frac{|H|}{|G|} \sum_{i=1}^{r} \delta_i^2 = 1 - \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2 \leq 1$$

and therefore

$$\sum_{i=1}^{r} \delta_i^2 \quad \leq \quad \frac{|G|}{|H|} = [G:H]$$

Also

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2 &= 0 \quad \textit{if and only if} \\ |\chi(g)|^2 &= 0 \quad \forall\, g \in (G \setminus H). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in (G \setminus H)} |\chi(g)|^2 &= 0 \quad \textit{if and only if} \\ \chi(g) &= 0 \quad \forall\, g \in (G \setminus H) \end{aligned}$$

and we have the equality in $(**)$. $\square$

We have seen how the irreducible characters of G can be used to find characters of a subgroup H and can now look at a technique of finding the characters of G from the irreducible characters of any subgroup. We start with the following definition.

Definition 3.2.4 *Let H be a subgroup of G. The right transversal of H in G is a set of representatives for the right cosets of H in G.*

The following theorem tells us how a representation of H can be extended to a representation of G.

Theorem 3.2.5 *Let H be a subgroup of G and T be a representation of H of degree n. Extend T to G by $T^0(g) = T(g)$ if $g \in H$ and $T^0(g) = 0_{n \times n}$ if $g \notin H$. Let $\{x_1, x_2, ..., x_r\}$ be a right transversal of H in G. Define $T \uparrow G$ by*

$$
(T \uparrow G)(g) \; := \begin{pmatrix} T^0(x_1 g x_1^{-1}) & T^0(x_1 g x_2^{-1}) & \dots & \dots & \dots & T^0(x_1 g x_r^{-1}) \\ T^0(x_2 g x_1^{-1}) & T^0(x_2 g x_2^{-1}) & \dots & \dots & \dots & T^0(x_2 g x_r^{-1}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ T^0(x_n g x_1^{-1}) & T^0(x_n g x_2^{-1}) & \dots & \dots & \dots & T^0(x_n g x_r^{-1}) \end{pmatrix}
$$

$$
= \quad \Big(T^0(x_i g x_j^{-1})\Big)_{i,j=1,2,\dots,r}, \quad \forall\, g \in G.
$$

Then $T \uparrow G$ is a representation of G of degree nr.

Proof: See [18, theorem iv.2.1].

Definition 3.2.6 *The representation $T \uparrow G$ defined in the previous theorem is said to be induced from the representation T of H. Let ϕ be the character afforded by T. Then the character afforded by $T \uparrow G$ is called the induced character from ϕ and is denoted by ϕ^G. If we extend ϕ to G by $\phi^0(g) = \phi(g)$ if $g \in H$ and $\phi^0(g) = 0$ if $g \notin H$, then*

$$
\begin{aligned}
\phi^G(g) &= Trace\Big((T \uparrow G)(g)\Big) \\
&= \sum_{i=1}^{r} Trace\Big((T^0(x_i g x_i^{-1})\Big) \\
&= \sum_{i=1}^{r} \phi^0(x_i g x_i^{-1})
\end{aligned}
$$

In order to construct a formula to find the induced character, the next two propositions are needed.

Proposition 3.2.7 *If $H \leq G$ and ϕ is a character of H, then ϕ^G is independent of the choice of transversal.*

Proof: See [18, Proposition iv. 2.2].

Proposition 3.2.8 *The values of the induced character are given by*

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1}), \quad g \in G$$

Proof: See [18, Proposition iv.2.3].

The following proposition provides us with a formula to calculate the induced character and the proof is provided by Moori [18, Proposition iv.2.4].

Proposition 3.2.9 *Let $H \leq G$, ϕ be a character of H and $g \in G$. Let $[g]$ denote the conjugacy class containing g.*

(i) If $H \cap [g] = \emptyset$, then $\phi^G(g) = 0$,

(ii) If $H \cap [g] \neq \emptyset$, then

$$\phi^G(g) = |C_G(g)|\sum_{i=1}^{m}\frac{\phi(x_i)}{|C_H(x_i)|},$$

where $x_1, x_2, ..., x_m$ are representatives of classes of H that fuse to $[g]$. (That is $H \cap [g]$ breaks up into m conjugacy classes of H with representations $x_1, x_2, ..., x_m$.)

Proof: By Proposition 3.2.8, we have

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1}).$$

If $H \cap [g] = \emptyset$, then $xgx^{-1} \notin H$ for all $x \in G$, so $\phi^0(xgx^{-1}) = 0 \quad \forall\, x \in G$ and $\phi^G(g) = 0$.

If $H \cap [g] \neq \emptyset$, then as x runs over G, xgx^{-1} covers $[g]$ exactly $|C_G(g)|$ times, so

$$\begin{aligned}
\phi^G(g) &= \frac{1}{|H|} \times |C_G(g)| \sum_{y\in[g]}\phi^0(y) \\
&= \frac{1}{|H|} \times |C_G(g)| \sum_{y\in([g]\cap H)}\phi(y) \\
&= \frac{|C_G(g)|}{|H|} \times \sum_{i=1}^{m}[H : C_H(x_i)].\phi(x_i) \\
&= |C_G(g)|\sum_{i=1}^{m}\frac{\phi(x_i)}{|C_H(x_i)|} \quad \square
\end{aligned}$$

The restriction and induction of characters are related and can be expressed by means of a matrix which we call the Frobenius Reciprocity table. To obtain this relationship, we shall take the route through class functions. We shall use the proof given by Moori [18] for the main result(the Frobenius Reciprocity theorem) in establishing the relationship.

Definition 3.2.10 *Let H be a subgroup of G and ϕ be a class function on H then the induced class function ϕ^G on G is defined by*

$$\phi^G(g) = \frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1}), \quad g \in G$$

where ϕ^0 coincides with ϕ on H and is zero otherwise. Notice that

$$\begin{aligned}
\phi^G(ygy^{-1}) &= \frac{1}{|H|}\sum_{x\in G}\phi^0(xygy^{-1}x^{-1}) \\
&= \frac{1}{|H|}\sum_{x\in G}\phi^0((xy)g(xy)^{-1}) \\
&= \frac{1}{|H|}\sum_{z\in G}\phi^0(zgz^{-1}) \\
&= \phi^G(g).
\end{aligned}$$

Thus ϕ^G is also a class function on G.

Note 3 If $H \leq G$ and ϕ is a class function on G, then $\phi \downarrow H$ is a class function on H.

Theorem 3.2.11 *(Frobenius Reciprocity)*

Let $H \leq G$, ϕ be a class function on H and ψ a class fuction on G. Then

$$\langle\, \phi\,,\, \psi \downarrow H\rangle_H \;=\; \langle\, \phi^G,\, \psi\rangle_G.$$

Proof:

By definition

$$\begin{aligned}
\langle\, \phi^G,\, \psi\rangle_G &= \frac{1}{|G|}\sum_{g\in G}\phi^G(g).\overline{\psi(g)} \\
&= \frac{1}{|G|}\sum_{g\in G}\Big(\frac{1}{|H|}\sum_{x\in G}\phi^0(xgx^{-1})\Big).\overline{\psi(g)}
\end{aligned}$$

$$= \frac{1}{|G|.|H|} \sum_{g \in G} \sum_{x \in G} \phi^0(xgx^{-1}).\overline{\psi(g)} \quad (****)$$

Let $y = xgx^{-1}$. Then as g runs over G, xgx^{-1} runs through G. Also since ψ is a class function on G, $\psi(y) = \psi(xgx^{-1}) = \psi(g)$. Thus by $(****)$ we have

$$\begin{aligned}
\langle \phi^G, \psi \rangle_G &= \frac{1}{|G|.|H|} \sum_{y \in G} \sum_{x \in G} \phi^0(y).\overline{\psi(y)} \\
&= \frac{1}{|G|.|H|} \sum_{x \in G} \Big(\sum_{y \in G} \phi^0(y).\overline{\psi(y)} \Big) \\
&= \frac{1}{|G|.|H|} |G| \sum_{y \in G} \phi^0(y).\overline{\psi(y)} \\
&= \frac{1}{|H|} \sum_{y \in H} \phi(y).\overline{\psi(y)} \\
&= \langle \phi, \psi \downarrow H \rangle_H \quad \square
\end{aligned}$$

Corollary 3.2.12 *Let $H \leq G$. Assume that $Irr(G) = \{\chi_1, \chi_2, ..., \chi_r\}$ and $Irr(H) = \{\psi_1, \psi_2, ..., \psi_s\}$. Suppose that*

$$\begin{aligned}
\chi_j \downarrow H &= \sum_{i=1}^{s} b_{ij} \psi_i \text{ and} \\
\psi_i^G &= \sum_{j=1}^{r} a_{ij} \chi_j, \text{ then} \\
a_{ij} &= b_{ij}, \quad \forall i, j.
\end{aligned}$$

Proof: See [18,Corollary iv.3.2].

Remark 1 (Frobenius Reciprocity table)

Let $H \leq G$. Assume that $Irr(G) = \{\chi_1, \chi_2, ..., \chi_r\}$and $Irr(H) = \{\psi_1, \psi_2, ..., \psi_s\}$, then by the previous corollary we have

$$\chi_j \downarrow H = \sum_{i=1}^{s} a_{ij}\psi_i \quad \text{and}$$
$$\psi_i^G = \sum_{j=1}^{r} a_{ij}\chi_j, \quad \text{then}$$

the matrix $A = (a_{ij})_{sr}$ is called the Frobenius Reciprocity table for G and H.

3.3 Normal Subgroups

In this section we shall look mainly at how the irreducible characters of a quotient group of a group G can be used to find some of the characters of G itself .

In order to justify a definition for the concept $ker(\chi)$, where χ is a character of G , we state lemma 3.3.1 and lemma 3.3.2 and prove the lemma 3.3.2 using the thesis of Whitley [26].

Lemma 3.3.1 *Let χ be a character of a group G afforded by the representation T. Then for $g \in G$, $T(g)$ is similar to a diagonal matrix $diag(e_1, e_2, ..., e_n)$ where each e_i is a complex root of unity. Then $\chi(g) = e_1 + e_2 + ... + e_n$ and $\chi(g^{-1}) = \overline{\chi(g)}$, where $\overline{x}$ denotes the complex conjugate of x.*

Proof: See [9,Lemma 2.15].

Lemma 3.3.2 *Let χ be a character of a group G afforded by the representation T. Then $g \in ker(T)$ if and only if $\chi(g) = \chi(1)$.*

Proof:

Let $n = \chi(1)$, so n is the degree of T. If $g \in ker(T)$ then $T(g) = I_n = T(1)$, where I_n is the

$n \times n$ identity matrix, so $\chi(g) = n = \chi(1)$. Conversely, assume $\chi(g) = \chi(1) = n$. By lemma 3.3.1, $\chi(g) = e_1 + e_2 + ... + e_n$, where each e_i is a complex root of unity. Therefore, $e_1 + e_2 + ... + e_n = n$. But $|e_i| = 1$ for all i, so we must have $e_i = 1 \quad \forall i$. Hence $T(g)$ is similar to $diag(e_1, e_2, ..., e_n) = I_n$, so $g \in ker(T)$. $\square$

Definition 3.3.3 *Let χ be a character of a group G. We define*

$$ker(\chi) = \{g \in G : \chi(g) = \chi(1)\}.$$

We note from lemma 3.3.2 that $ker(\chi)$ is a normal subgroup of G. The next two theorems taken from the Moori-notes[18] will tell us how the normal subgroups of G can be determined from its character table and how we can tell whether G is simple or not.

Theorem 3.3.4 *Let N be a normal subgroup of G. Then there exists irreducible characters $\chi_1, \chi_2, ..., \chi_s$ of G such that*

$$N = \bigcap_{i=1}^{s} ker(\chi_i).$$

Proof: See [18, Theorem v.3].

Theorem 3.3.5 *A group G is simple if and only if $\chi(g) \neq \chi(1)$ for all nontrivial irreducible characters of G and for all non-identity elements g of G.*

Proof: See [18, Theorem v.4].

The following results form the basis for another tool in finding the characters of a group.

Theorem 3.3.6 *Let N be a normal subgroup of G.*

(a) Let $\hat{\chi}$ be a character of G/N and $\chi : G \to \mathbb{C}$ be defined by

$$\chi(g) \quad = \quad \hat{\chi}(gN) \quad for\ g \in G,$$

Then χ is a character of G and χ has the same degree as $\hat{\chi}$.

(b) Let χ be a character of G, $N \leq ker(\chi)$ and $\hat{\chi} : G/N \to \mathbb{C}$ be defined by

$$\hat{\chi}(gN) \quad = \quad \chi(g) \quad for\ g \in G,$$

Then $\hat{\chi}$ is a character of G/N.

(c) In both of the statements above, $\hat{\chi}$ is an irreducible character of G/N if and only if χ is an irreducible character of G.

Proof:

(a) Let $\hat{T}$ be the representation of degree n that affords $\hat{\chi}$ and define $T : G \rightarrow GL_n(\mathbb{C})$ by $T(g) = \hat{T}(gN)$. Then for $g_1, g_2 \in G$,

$$\begin{aligned} g_1 = g_2 &\implies g_1N = g_2N \\ &\implies \hat{T}(g_1N) = \hat{T}(g_2N) \\ &\implies T(g_1) = T(g_2). \end{aligned}$$

So T is well-defined. Also

$$\begin{aligned} T(g_1g_2) &= \hat{T}(g_1g_2N) \\ &= \hat{T}(g_1Ng_2N) \\ &= \hat{T}(g_1N)\hat{T}(g_2N) \\ &= T(g_1)T(g_2) \end{aligned}$$

Hence T is a homomorphism and therefore a representation.

Now $Trace(T(g)) = Trace(\hat{T}(gN)) = \hat{\chi}(gN) = \chi(g)$ for all $g \in G$, so T affords χ. Moreover

$$I_m = T(1) = \hat{T}(N) = I_n$$

and so the degree of χ is the same as that of $\hat{\chi}$.

(b) Let T be the representation that affords χ and define $\hat{T} : G/N \to GL_n(\mathbb{C})$ by $\hat{T}(gN) = T(g)$. Then for $g_1, g_2 \in G$,

$$\begin{aligned} g_1N = g_2N &\implies g_1^{-1}g_2 \in N \leq ker(\chi) = ker(T) \\ &\implies T(g_1^{-1}g_2) = I, \text{the identity matrix} \\ &\implies T(g_1^{-1})T(g_2) = I \\ &\implies T(g_1) = T(g_2) \\ &\implies \hat{T}(g_1N) = \hat{T}(g_2N) \end{aligned}$$

thus $\hat{T}$ is well-defined and

$$\begin{aligned} \hat{T}(g_1Ng_2N) &= \hat{T}(g_1g_2N) \\ &= T(g_1g_2) \\ &= T(g_1)T(g_2) \\ &= \hat{T}(g_1N)\hat{T}(g_2N) \end{aligned}$$

Hence T a representation.

$Trace(\hat{T}(gN)) = Trace(T(g)) = \chi(g) = \hat{\chi}(gN)$ for all $g \in G$, so $\hat{T}$ affords $\hat{\chi}$.

$$\chi|_N = \sum_{i=1}^{t}\langle \chi|_N, \theta_i\rangle\theta_i.$$

But $\langle \chi|_N, \theta_i\rangle = \langle \chi|_N, \theta\rangle$ since θ_i and θ are conjugate and so the proof is complete. $\square$

Definition 4.1.2 *Let $N \lhd \overline{G}$ and $\theta \in Irr(N)$. Then $I_{\overline{G}}(\theta) = \left\{g \in \overline{G} : \theta^g = \theta\right\}$ is the inertia group of θ in $\overline{G}$.*

Since $I_{\overline{G}}(\theta)$ is the stabilizer of θ in the action of $\overline{G}$ on $Irr(N)$, we have that $I_{\overline{G}}(\theta)$ is a subgroup of $\overline{G}$ and $N \subseteq I_{\overline{G}}(\theta)$. Also $[\overline{G} : I_{\overline{G}}(\theta)]$ is the size of the orbit containing θ, so in the formula $\chi|_N = e\sum_{i=1}^{t}\theta_i$, we have $t = [\overline{G} : I_{\overline{G}}(\theta)]$.

As a consequence of Clifford's theorem, we have the following theorem.

Theorem 4.1.3 *Let $N \lhd \overline{G}$, $\theta \in Irr(N)$ and $\overline{H} = I_{\overline{G}}(\theta)$. Then induction to $\overline{G}$ maps the irreducible characters of $\overline{H}$ that contain θ in their restriction to N faithfully onto the irreducible characters of $\overline{G}$ which contain θ in their restiction to N.*

Proof: See [26,Theorem 3.3.2]

Theorem 4.1.3 shows that to find the irreducible characters of $\overline{G}$ that contain θ in their restriction to N, it suffices to find the irreducible characters of $\overline{H} = I_{\overline{G}}(\theta)$ that contain θ in their restriction. If θ can be extended to an irreducible character ψ of $\overline{H}$ (that is, $\psi \in Irr(\overline{H})$ with $\psi|_N = \theta$), then the relevant characters of $\overline{H}$ can be obtained by using the following theorem.

Theorem 4.1.4 *(Gallaghar [6]) With N, $\overline{G}, \theta$ and $\overline{H}$ as above, if θ extends to a character $\psi \in Irr(\overline{H})$ then as β ranges over all irreducible characters of $\overline{H}$ that contain N in their kernel,*

$\beta\psi$ ranges over all irreducible characters of $\overline{H}$ that contain θ in their restriction.

Proof: By definition of $\overline{H}$, θ is the only $\overline{H}$-conjugate of θ, so by Clifford's theorem, $\theta^{\overline{H}}|_N = f\theta$ for some integer f. Comparing degrees, $\theta^{\overline{H}}|_N = [\overline{H} : N]\theta$, so

$$\begin{aligned} \langle \theta^{\overline{H}}, \theta^{\overline{H}} \rangle &= \langle \theta, \theta^{\overline{H}}|_N \rangle \\ &= [\overline{H} : N]. \end{aligned}$$

Now we claim that $\theta^{\overline{H}} = \sum_\beta \beta(1)\beta\psi$, where β runs over all irreducible characters of $\overline{H}$ that contain N in their kernel, or, equivalently, over all irreducible characters of $\overline{H}/N$. Both $\theta^{\overline{H}}$ and $\sum_\beta \beta(1)\beta\psi$ are zero off N because for $g \notin N$, $\theta^{\overline{H}}(g) = 0$ since $xgx^{-1} \notin N \; \forall \; x \in \overline{G}$, and by the column orthogonality for the character table of $\overline{H}/N$ since g does not belong to N, we have

$$\sum_\beta \beta(1)(\beta\psi)(g) = \sum_\beta (\beta(1)\beta(g))\psi(g) = 0.$$

Also

$$\theta^{\overline{H}}|_N = [\overline{H} : N]\theta = (\sum_\beta \beta(1)\beta\psi)|_N$$

because for $g \in N$,

$$\begin{aligned} \sum_\beta \beta(1)\beta(g)\psi(g) &= \sum_\beta (\beta(1))^2.\psi(g) \\ &= [\overline{H} : N]\psi(g) \\ &= [\overline{H} : N]\theta(g). \end{aligned}$$

Therefore $\theta^{\overline{H}} = \sum_\beta \beta(1)\beta\psi$ as claimed. Now

$$\begin{aligned}[\overline{H}:N] &= \langle \theta^{\overline{H}}, \theta^{\overline{H}} \rangle \\ &= \langle \sum_\beta \beta(1)\beta\psi, \sum_\gamma \gamma(1)\gamma\psi \rangle \\ &= \sum_{\beta,\gamma} \beta(1)\gamma(1)\langle \beta\psi, \gamma\psi \rangle.\end{aligned}$$

The diagonal terms contribute at least $\sum \beta(1)^2 = [\overline{H}:N]$ so the $\beta\psi$ are irreducible and distinct. These $\beta\psi$ are all the irreducible constituents of $\theta^{\overline{H}}$, so are all the irreducible characters of $\overline{H}$ that contain θ in their restriction, since for $\phi \in Irr(\overline{H})$, $\langle \phi|_N, \theta \rangle = \langle \phi, \theta^{\overline{H}} \rangle$. □

Note 1 Now suppose $\overline{G}$ is an extension of N by G. If every irreducible character of N can be extended to its inertia group in $\overline{G}$, then by application of theorems 4.1.3 and 4.1.4 the characters of $\overline{G}$ can be obtained as follows:

Let $\theta_1, \theta_2, ..., \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$. For each i, let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and let $\psi_i \in Irr(\overline{H}_i)$ with $\psi_i|_N = \theta_i$. Now each irreducible character of $\overline{G}$ contains some θ_i in its restriction to N by Clifford's theorem, so by theorems 4.1.3 and 4.1.4 we have

$$Irr(\overline{G}) = \bigcup_{i=1}^{t} \left\{ (\beta\psi_i)^{\overline{G}} : \beta \in Irr(\overline{H}_i),\ N \subset ker(\beta) \right\}$$

Hence the characters of $\overline{G}$ fall into blocks, with each block corresponding to an inertia group.

We now quote some results which give sufficient conditions for the irreducible characters of N to be extendible to their respective inertia groups, so that the above method can be used to calculate the characters of $\overline{G}$.

The following result and proof was obtained from Curtis and Reiner ([4, page 353]).

Theorem 4.1.5 *(Mackey's theorem) Suppose that N is a normal subgroup of $\overline{H}$ such that N is abelian and $\overline{H}$ is a semi-direct product of N and H for some $H \leq \overline{H}$. If $\theta \in Irr(N)$ is invariant in $\overline{H}$ (that is, $\theta^h = \theta, \forall\, h \in \overline{H}$) then θ can be extended to a linear character of $\overline{H}$.*

Proof: Since $\overline{H}$ is a semi-direct product, any $h \in \overline{H}$ can be written uniquely as $h = nk, n \in N, k \in H$. Define χ on $\overline{H}$ by $\chi(nk) = \theta(n)$. Since N is abelian, θ has degree 1, hence is linear, and the fact that $\theta = \theta^h$ for all $h \in \overline{H}$ implies that $\theta(n) = \theta(hnh^{-1})$ for all $h \in \overline{H}$. Then if $h_1 = n_1k_1,\ h_2 = n_2k_2$, we have

$$\begin{aligned}
\chi(h_1h_2) &= \chi(n_1k_1n_2k_2) \\
&= \chi(n_1n_2^{k_1}k_1k_2) \\
&= \theta(n_1n_2^{k_1}) \\
&= \theta(n_1)\theta(n_2^{k_1}) \\
&= \theta(n_1)\theta(n_2) \\
&= \theta(n_1n_2) = \chi(h_1)\chi(h_2).
\end{aligned}$$

Therefore χ is a linear character of $\overline{H}$, and $\chi|_N = \theta$. □

Since in all our examples that we will consider, N is abelian and the extension is split, Mackey's theorem will apply. Mackey's theorem is a corollary of a more general result by Karpilovsky [11] which we state without proof.

Theorem 4.1.6 *Let the group $\overline{H}$ contain a subgroup H of order n such that $\overline{H} = NH$ for N normal in $\overline{H}$ and let $\chi \in Irr(N)$ be invariant in $\overline{H}$. Then χ extends to an irreducible character* *of $\overline{H}$ if the following conditions hold:*

1. $(m, n) = 1$ *where* $m = \chi(1)$,

2. $N \cap H \leq N'$ *where* N' *is the derived subgroup of* N.

Another extension theorem which can be found in [7] is the following:

Theorem 4.1.7 *If N is a normal subgroup of $\overline{H}$ and θ is an irreducible character of N that is invariant in $\overline{H}$, then θ is extendable to an irreducible character of $\overline{H}$ if*

$$([\overline{H} : N],\ \tfrac{|N|}{\theta(1)}) \ = \ 1.$$

4.2 Properties of Fischer Matrices

In this section we give some properties of the Fischer matrices. We however need to look at some background material first.

Let $\overline{G}$ be an extension of N by G, with the property that every irreducible character of N can be extended to its inertia group. With the notation of the previous chapter we have that $[Irr(\overline{G}) = \bigcup_{i=1}^{t}\{(\beta\psi_i)^{\overline{G}} : \beta \in Irr(\overline{H_i})$ with $N \subset ker(\beta)\}]$ Now we show how the character table $\overline{G}$ can be constructed using this result. We construct a matrix for each conjugacy class of G (the Fischer matrices). Then the character table of $\overline{G}$ can be constructed using these matrices and the character tables of factor groups of the inertia groups. These constructions of Fischer matrices have been discussed by Salleh [25], List [12] and List and Mahmoud [13].

As previously, let $\theta_1, ..., \theta_t$ be representatives of the orbits of $\overline{G}$ on $Irr(N)$, and let $\overline{H}_i = I_{\overline{G}}(\theta_i)$ and $H_i = \overline{H}_i/N$. Let ψ_i be an extension of θ_i to $\overline{H}_i$. We take $\theta_1 = 1_N$, so $\overline{H}_1 = \overline{G}$ and $H_1 = G$. We consider a conjugacy class $[g]$ of G with representative g. Let $X(g) = \{x_1, ... x_{c(g)}\}$ be representatives of $\overline{G}$-conjugacy classes of elements of the coset $N\overline{g}$. Take $x_1 = \overline{g}$. Let $R(g)$ be a set of pairs (i, y) where $i \in \{1, ..., t\}$ such that H_i contains an element of $[g]$, and y ranges over representatives of the conjugacy classes of H_i that fuse to $[g]$. Corresponding to this $y \in H_i$, let $\{y_{l_k}\}$ be representatives of conjugacy classes of $\overline{H_i}$ that contain liftings of y.

If $\beta \in Irr(\overline{H_i})$ with $N \subset \ker(\beta)$, then β has been lifted from some $\hat{\beta} \in Irr(H_i)$, with $\hat{\beta}(y) = \beta(y_{l_k})$ for any lifting y_{l_k} of y. For convenience we write $\beta(y)$ for $\hat{\beta}(y)$.

Now, using the formula for induced characters given in Proposition 3.2.9., we have

$$\begin{aligned}
(\psi_i\beta)^{\overline{G}}(x_j) &= \sum_{y:(i,y)\in R(g)} \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}(\psi_i\beta)(y_{l_k}) \\
&= \sum_{y:(i,y)\in R(g)} \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k})\hat{\beta}(y) \\
&= \sum_{y:(i,y)\in R(g)} \left(\sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k}) \right) \beta(y)
\end{aligned}$$

By $\Sigma_k{}'$ we mean that we sum over those k for which y_{l_k} is conjugate to x_j in $\overline{G}$. Now we define the Fischer matrix $M(g) = (a^j_{(i,y)})$ with columns indexed by $X(g)$ and rows indexed by $R(g)$ by

$$a^j_{(i,y)} = \sum_k{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|}\psi_i(y_{l_k})$$

Then

$$(\psi_i\beta)^{\overline{G}}(x_j) = \sum_{y:(i,y)\in R(g)} a^j_{(i,y)}\beta(y).$$

The rows of $M(g)$ can be divided into blocks, each block corresponding to an inertia group. Denote the submatrix corresponding to H_i by $M_i(g)$, and let $C_i(g)$ be the fragment of the character table of H_i consisting of the columns corresponding to classes that fuse to $[g]$. Then, by the above relation , the characters of $\overline{G}$ at the classes represented by $X(g)$ obtained from inducing characters of $\overline{H}_i$ are given by the matrix product $C_i(g).M_i(g)$.

We now state a result of Brauer and prove a lemma which will be needed later.

Lemma 4.2.1 *(Brauer) Let A be a group of automorphisms of a group K. Then A also acts on Irr(K) and the number of orbits of A on Irr(K) is the same as that on the conjugacy classes of K.*

Proof: See [8, 4.5.2]

Lemma 4.2.2 *Let A be a group of automorphisms of a group K , so A acts on Irr(K) and on the conjugacy classes of K with the same number of orbits on each by the previous lemma. Suppose we have the following matrix describing these actions:*

$$\begin{array}{c} \\ s_1 \\ s_2 \\ \vdots \\ s_i \\ \vdots \\ s_t \end{array} \begin{array}{c} \begin{array}{ccccccc} 1=l_1 & l_2 & \dots & l_j & \dots & l_t \end{array} \\ \left(\begin{array}{cccccc} 1 & 1 & \dots & 1 & \dots & 1 \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2t} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{it} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{t1} & a_{t2} & \dots & a_{tj} & \dots & a_{tt} \end{array} \right) \end{array}$$

where $a_{ij} = 1$ for $j = 1, ...t$, l_j 's are lengths of orbits A on the conjugacy classes of K,
s_i 's are lengths of orbits of A on Irr(K),
a_{ij} is the sum of s_i irreducible characters of K on the element x_j, where x_j is an element of the orbit of length l_j.
Then the following relation holds for $i, i' \in \{1, ...t\}$:

$$\sum_{j=1}^{t} a_{ij}\overline{a_{i'j}}l_j = |K|s_i\delta_{ii'}$$

.

Proof: Let $\underline{s_i}$ denote the sum of s_i irreducible characters of K, so $\underline{s_i}(x_j) = a_{ij}$. Then

$$< \underline{s_i}, \underline{s_{i'}} >= |K|^{-1}\sum_{j=1}^{t} l_j\underline{s_i}(x_j)\overline{\underline{s_{i'}}(x_j)} = |K|^{-1}\sum_{j=1}^{t} l_j a_{ij}\overline{a_{i'j}}$$

.

But by orthogonality of irreducible characters, $< \underline{s_i}, \underline{s_{i'}} >= \delta_{ii'} s_i$, so

$$\sum_{j=1}^{t} l_j a_{ij} \overline{a_{i'j}} = |K| s_i \delta_{ii'}. \quad \square$$

Now let $M(g) = (a^j_{(i,y)})$ be the Fischer matrix for $\overline{G} = N.G$ at $g \in G$. We present $M(g)$ with corresponding "weights" for columns and rows as follows:

$$\begin{array}{c|cccc}
 & |C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & \dots & |C_{\overline{G}} x_{c(g)}| \\
\hline
|C_{H_1}(g)| & 1 & 1 & \dots & 1 \\
\hline
|C_{H_2}(y)| & a^1_{(2,y)} & a^2_{(2,y)} & \dots & \\
|C_{H_2}(y')| & a^1_{(2,y')} & a^2_{(2,y')} & \dots & \\
\vdots & \vdots & & & \\
\hline
|C_{H_i}(y)| & a^1_{(i,y)} & a^2_{(i,y)} & \dots & \\
\vdots & \vdots & & & \\
\hline
|C_{H_t}(y)| & a^1_{(t,y)} & a^2_{(t,y)} & \dots & \\
\vdots & \vdots & & &
\end{array}$$

The matrix $M(g)$ is divided into blocks (separated by horizontal lines), each corresponding to an inertia group. Note that $a^j_{(1,g)} = 1$ for all $j \in \{1, ..., c(g)\}$. Fischer has shown that $M(g)$ is square and nonsingular(see[13]). In the following propositions and note we give further properties of Fischer matrices.

Proposition 4.2.3 *(column orthogonality)*

$$\sum_{(i,y)\in R(g)} |C_{H_i}(y)| a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|$$

.

Proof: The partial character table of $\overline{G}$ at classes $x_1, ..., x_{c(g)}$ is

$$\begin{bmatrix} C_1(g)M_1(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$$

where $C_i(g), M_i(g)$ are as defined earlier in this section.

By column orthogonality of the character table of $\overline{G}$, we have

$$\begin{aligned}
|C_{\overline{G}}(x_j)|\delta_{jj'} &= \sum_{i=1}^{t} \sum_{\beta_i \in Irr(H_i)} \left(\sum_{y:(i,y)\in R(g)} a^j_{(i,y)} \beta_i(y))(\overline{\sum_{y':(i,y')\in R(g)} a^{j'}_{(i,y')} \beta_i(y')} \right) \\
&= \sum_{i=1}^{t} \sum_{\beta_i \in Irr(H_i)} \left(\sum_{y} a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} \beta_i(y) \overline{\beta_i(y)} + \sum_{y} \sum_{y' \neq y} a^j_{(i,y)} \overline{a^{j'}_{(i,y')}} \beta_i(y) \overline{\beta_i(y')} \right) \\
&= \sum_{i=1}^{t} \left(\sum_{y} a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} \sum_{\beta_i \in Irr(H_i)} \beta_i(y)\overline{\beta_i(y)} + \sum_{y}\sum_{y'\neq y} a^j_{(i,y)} \overline{a^{j'}_{(i,y')}} \sum_{\beta_i \in Irr(H_i)} \beta_i(y)\overline{\beta_i(y')} \right) \\
&= \sum_{i=1}^{t} \left(\sum_{y} a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} |C_{H_i}(y)| + 0 \right) \\
&= \sum_{(i,y)\in R(g)} a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} |C_{H_i}(y)|. \quad \square
\end{aligned}$$

Proposition 4.2.4 *(List [12]) At the identity of G, the matrix M(1) is the matrix with rows equal to orbit sums of the action of $\overline{G}$ on Irr(N) with duplicate columns discarded.*
For this matrix we have $a^j_{(i,1)} = [G : H_i]$, and an orthogonality relation for rows:

$$\sum_{j=1}^{t} a^j_{(i,1)} a^j_{(i',1)} |C_{\overline{G}}(x_j)|^{-1} = \delta_{ii'} |C_{H_i}(1)|^{-1} = \delta_{ii'} |H_i|^{-1}$$

Proof:: The $(i,1), j^{th}$ entry of $M(1)$ is

$$a^j_{(i,1)} = \sum_k \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|} \psi_i(y_{l_k})$$

where we sum over representatives of conjugacy classes of $\overline{H}_i$ that fuse to $[x_j]$ in $\overline{G}$. Therefore $a^j_{(i,1)} = \psi_i^{\overline{G}}(x_j)$. By theorem 4.1.3 $\psi_i^{\overline{G}}$ is an irreducible character of $\overline{G}$, and $< \psi_i^{\overline{G}}|_N, \theta_i >=< \psi_i|_N, \theta_i >= 1$. Therefore, by Clifford's Theorem (Theorem 4.1.1), $\psi_i^{\overline{G}}|_N = \sum_\alpha \chi_\alpha$, where we sum over all $\chi_\alpha \in Irr(N)$ in the orbit containing θ_i. Now $x_j \in N$, and $a^j_{(i,1)} = \sum_\alpha \chi_\alpha(x_j)$. The orthogonality relation follows by Lemma 4.2.2. □

Note 1 If N is an elementary abelian group (which is the case for our calculations), then List[12] has also shown the following for $M(g)$, where $g \neq 1$:

If $\overline{G}$ is a split extension of N by G, then $M(g)$ is the matrix of orbit sums of C_g (as defined in section 2.2) acting on the rows of the character table for a certain factor group of N with duplicate columns discarded.

If the extension is not split, $M(g)$ is the matrix of orbit sums of C_g acting on the rows of the character table with duplicate columns discarded and with each row multiplied by a $p-th$ root of unity where $|N| = p^n$ for some n. It may be that the root of unity for each row is 1.
For these matrices (N elementary abelian, any extension) $a^1_{(i,y)} = \frac{|C_G(g)|}{|C_{H_i}(y)|}$, and we have an orthogonality relation for rows (as a consequence of Lemma 4.2.2.):

$$\sum_{j=1}^{c(g)} m_j a^j_{(i,y)} \overline{a^j_{(i',y')}} = \delta_{(i,y)(i',y')} |C_G(g)||C_{H_i}(y)|^{-1}|N| = \delta_{(i,y)(i',y')} a^1_{(i,y)}|N|$$

where $m_j = [C_g : C_{\overline{G}}(x_j)]$.

(In the notation of section 2.2, m_j is the length of the orbit Δ_l of C_g, so $m_j = \frac{f|N|}{k}$)

The relations given in the above propositions and note will be used later in our calculations of Fischer matrices, so for convenience we list them in a theorem.

Theorem 4.2.5 *For a Fischer matrix $M(g) = (a^j_{(i,y)})$ of $\overline{G} = N.G$ we have the following relations.*

1. $a^j_{(1,g)} = 1$ *for all* $j \in \{1, ..., c(g)\}$.
2. $\sum_{(i,y) \in R(g)} |C_{H_i}(y)| a^j_{(i,y)} \overline{a^{j'}_{(i,y)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|$.
3. *If N is elementary abelian, then* $a^1_{(i,y)} = \frac{|C_G(g)|}{|C_{H_i}(y)|}$, *and*
4. $\sum_{j=1}^{c(g)} m_j a^j_{(i,y)} \overline{a^j_{(i',y')}} = \delta_{(i,y)(i',y')} a^1_{(i,y)} |N|$.

Chapter 5

CHARACTER TABLES OF SOME GROUP EXTENSIONS

In this chapter we show how the theory developed earlier can be used to calculate the character tables of some groups of extension type. We will only deal with split extensions and the examples that we use are taken from the ATLAS [3]. Our first example $2^4 : 15$ is a maximal subgroup of the group $GL(2, 16)$. We now proceed to calculate the character table of this group using classical methods in our first section. We use the notation that was developed earlier.

5.1 The Character Table of $2^4 : 15$

Let $\overline{G}$ be a split extension of N, an elementary abelian two-group of order 16, by G, a cyclic subgroup of $GL(4, 2)$ of order 15. We use the method of **coset analysis** , described in section 2.2 of chapter 2, to calculate the conjugacy classes of $\overline{G}$. G can be generated by the following element of order 15 in G

$$x = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and $N \cong V(4,2)$, the vector space of dimension four over a field of two elements. G , being cyclic, has 15 conjugacy classes each of which consists of a single element which is a power of x. In this example, we thus work with fifteen cosets, namely Nx^j where $j = 0, 1, 2, ..., 14$. For each j we must consider the action of $N \cong \langle e_1, e_2, e_3, e_4 \rangle$ and $C_G(x^j)$ on Nx^j.

Action of N and $C_G(1_G)$ on $N1_G$:

1_G fixes all elements of N so that $C_N(1_G) = N$. That is we have sixteen orbits , Q_s with $s = 1, 2, ..., 16$, each containing one element. Now $C_G(1_G) = G$ so we only need to look at the action of x on N. This action is represented by the cycle structure $(e_1 \quad e_1e_3 \quad e_1e_2e_3 \quad e_1e_2 \quad e_2 \quad e_3)$. So

$$\triangle_1 = \{1\} = Q_1 \quad and \quad \triangle_2 = \bigcup_{s=2}^{16} Q_s.$$

Hence $f_1 = 1$ and $f_2 = 15$. We obtain the following :

$$\begin{aligned} | C_{\overline{G}}(1_G) | &= \frac{k \times |C_G(1_G)|}{f_1} = \frac{16 \times 15}{1} = 240; \\ |[1_G]_{\overline{G}}| &= \frac{f_1 \times |\overline{G}|}{k \times |C_G(1_G)|} = \frac{1 \times 240}{16 \times 15} = 1; \\ |C_{\overline{G}}(e_1)| &= \frac{16 \times 15}{f_2} = 16; \\ |[e_1]_{\overline{G}}| &= \frac{15 \times 240}{16 \times 15} = 15. \end{aligned}$$

Action of N and $C_G(x)$ on Nx :

$C_N(x) = \{1_G\}$. So $|C_N(x)| = k = 1$ and therefore $f = 1$. Also $C_G(x) = G$ so we have $|C_{\overline{G}}(x)| = 15$. In fact $|C_{\overline{G}}(x^j)| = 15$ for all $j = 1, 2, ..., 14$ because the action of x^j is represented by a $15-cycle$ and hence $x^j (j \neq 0)$ fixes only 1_N. We thus have $C_N(x^j) = \{1\}$, $j \neq 0$ and so $k = 1$ and again $f = 1$. With $C_G(x^j) = G, j \neq 0$ we have $|C_{\overline{G}}(x^j)| = 15, \forall\ j = 1, 2, ..., 14$. With that , the conjugacy table of

$\overline{G}$ is completed :

class	(1)	(e_1)	(x)	(x^2)	(x^3)	(x^4)	(x^5)	(x^6)	(x^7)	(x^8)	(x^9)	(x^{10})	(x^{11})	(x^{12})
h_i	1	15	16	16	16	16	16	16	16	16	16	16	16	16
order	1	2	15	15	15	15	15	15	15	15	15	15	15	15
centralizer	240	16	15	15	15	15	15	15	15	15	15	15	15	15

Table 5.1.1.A : **The conjugacy table of** $2^4 : 15$.

class	(x^{13})	(x^{14})
h_i	16	16
order	15	15
centralizer	15	15

Table 5.1.1.B : **The conjugacy table of** $2^4 : 15$(continued).

We use the method of **inducing characters of subgroups** of $\overline{G}$ (discussed in section 2.2)to calculate the **character table** of $\overline{G}$. In this case we will make use of the irreducible characters of N and G. The character table of $N \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is easily calculated from the character table of $\mathbb{Z}_2 = \langle a : a^2 = 1 \rangle$ by using the product of these characters (theorem 2.1.26). We give the character tables of $\mathbb{Z}_2$ and N.

class	(1)	(a)
centralizer	2	2
ψ_1	1	1
ψ_2	1	-1

<u>Table 5.1.2</u> : **The character table of $\mathbb{Z}_2$.**

class	(1)	(e_3e_4)	(e_4)	(e_2e_4)	(e_2)	$(e_2e_3e_4)$	(e_1e_2)	(e_1e_4)	(e_3)	$(e_1e_2e_4)$	(e_1e_3)
h_i	1	1	1	1	1	1	1	1	1	1	1
order	1	2	2	2	2	2	2	2	2	2	2
centralizer	16	16	16	16	16	16	16	16	16	16	16
τ_1	1	1	1	1	1	1	1	1	1	1	1
τ_2	1	1	1	1	1	1	-1	-1	1	-1	-1
τ_3	1	1	1	-1	-1	-1	-1	1	1	-1	1
τ_4	1	1	1	-1	-1	-1	1	-1	1	1	-1
τ_5	1	-1	1	1	1	-1	1	1	-1	1	-1
τ_6	1	-1	1	1	1	-1	-1	-1	-1	-1	1
τ_7	1	-1	1	-1	-1	1	-1	1	-1	-1	-1
τ_8	1	-1	1	-1	-1	1	1	-1	-1	1	1
τ_9	1	-1	-1	-1	1	-1	1	-1	1	-1	1
τ_{10}	1	-1	-1	-1	1	-1	-1	1	1	1	-1
τ_{11}	1	-1	-1	1	-1	1	-1	-1	1	1	1
τ_{12}	1	-1	-1	1	-1	1	1	1	1	-1	-1
τ_{13}	1	1	-1	-1	1	1	1	-1	-1	-1	-1
τ_{14}	1	1	-1	-1	1	1	-1	1	-1	1	1
τ_{15}	1	1	-1	1	-1	-1	-1	-1	-1	1	-1
τ_{16}	1	1	-1	1	-1	-1	1	1	-1	-1	1

<u>Table 5.1.3.A</u> : **The character table of the group 2^4.**

class	(e_2e_3)	(e_1)	$(e_1e_2e_3e_4)$	$(e_1e_2e_3)$	$(e_1e_3e_4)$
h_i	1	1	1	1	1
order	2	2	2	2	2
centralizer	16	16	16	16	16
τ_1	1	1	1	1	1
τ_2	1	-1	-1	-1	-1
τ_3	-1	1	-1	-1	1
τ_4	-1	-1	1	1	-1
τ_5	-1	1	-1	-1	-1
τ_6	-1	-1	1	1	1
τ_7	1	1	1	1	-1
τ_8	1	-1	-1	-1	1
τ_9	1	1	-1	1	-1
τ_{10}	1	-1	1	-1	1
τ_{11}	-1	1	1	-1	-1
τ_{12}	-1	-1	-1	1	1
τ_{13}	-1	1	1	-1	1
τ_{14}	-1	-1	-1	1	-1
τ_{15}	1	1	-1	1	1
τ_{16}	1	-1	1	-1	-1

Table 5.1.3.B : **The character table of the group** 2^4 (continued).

We have seen in proposition 3.1.21 that if $H = \langle x : x^n = 1 \rangle$, then $\rho_k : H \longrightarrow \mathbb{C}^*$ defined by

$$\rho_k(x^m) = [e^{\frac{2k\pi i}{n}}]^m$$

defines n irreducible representations of H. So the character table of $G = \langle x : x^{15} = 1 \rangle$ is completely determined by its representatives of this type. The character table of G is as follows :

class	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
h_i	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
order	1	15	15	15	15	15	15	15	15	15	15	15	15	15	15
$\|C_G(g)\|$	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15
ρ_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ρ_1	1	w^1	w_1^2	w_1^3	w_1^4	w_1^5	w_1^6	w_1^7	w_1^8	w_1^9	w_1^{10}	w_1^{11}	w_1^{12}	w_1^{13}	w_1^{14}
ρ_2	1	w_2	w_2^2	w_2^3	w_2^4	w_2^5	w_2^6	w_2^7	w_2^8	w_2^9	w_2^{10}	w_2^{11}	w_2^{12}	w_2^{13}	w_2^{14}
ρ_3	1	w_3	w_3^2	w_3^3	w_3^4	w_3^5	w_3^6	w_3^7	w_3^8	w_3^9	w_3^{10}	w_3^{11}	w_3^{12}	w_3^{13}	w_3^{14}
ρ_4	1	w_4	w_4^2	w_4^3	w_4^4	w_4^5	w_4^6	w_4^7	w_4^8	w_4^9	w_4^{10}	w_4^{11}	w_4^{12}	w_4^{13}	w_4^{14}
ρ_5	1	w_5	w_5^2	w_5^3	w_5^4	w_5^5	w_5^6	w_5^7	w_5^8	w_5^9	w_5^{10}	w_5^{11}	w_5^{12}	w_5^{13}	w_5^{14}
ρ_6	1	w_6	w_6^2	w_6^3	w_6^4	w_6^5	w_6^6	w_6^7	w_6^8	w_6^9	w_6^{10}	w_6^{11}	w_6^{12}	w_6^{13}	w_6^{14}
ρ_7	1	w_7	w_7^2	w_7^3	w_7^4	w_7^5	w_7^6	w_7^7	w_7^8	w_7^9	w_7^{10}	w_7^{11}	w_7^{12}	w_7^{13}	w_7^{14}
ρ_8	1	w_8	w_8^2	w_8^3	w_8^4	w_8^5	w_8^6	w_8^7	w_8^8	w_8^9	w_8^{10}	w_8^{11}	w_8^{12}	w_8^{13}	w_8^{14}
ρ_9	1	w_9	w_9^2	w_9^3	w_9^4	w_9^5	w_9^6	w_9^7	w_9^8	w_9^9	w_9^{10}	w_9^{11}	w_9^{12}	w_9^{13}	w_9^{14}
ρ_{10}	1	w_{10}	w_{10}^2	w_{10}^3	w_{10}^4	w_{10}^5	w_{10}^6	w_{10}^7	w_{10}^8	w_{10}^9	w_{10}^{10}	w_{10}^{11}	w_{10}^{12}	w_{10}^{13}	w_{10}^{14}
ρ_{11}	1	w_{11}	w_{11}^2	w_{11}^3	w_{11}^4	w_{11}^5	w_{11}^6	w_{11}^7	w_{11}^8	w_{11}^9	w_{11}^{10}	w_{11}^{11}	w_{11}^{12}	w_{11}^{13}	w_{11}^{14}
ρ_{12}	1	w_{12}	w_{12}^2	w_{12}^3	w_{12}^4	w_{12}^5	w_{12}^6	w_{12}^7	w_{12}^8	w_{12}^9	w_{12}^{10}	w_{12}^{11}	w_{12}^{12}	w_{12}^{13}	w_{12}^{14}
ρ_{13}	1	w_{13}	w_{13}^2	w_{13}^3	w_{13}^4	w_{13}^5	w_{13}^6	w_{13}^7	w_{13}^8	w_{13}^9	w_{13}^{10}	w_{13}^{11}	w_{13}^{12}	w_{13}^{13}	w_{13}^{14}
ρ_{14}	1	w_{14}	w_{14}^2	w_{14}^3	w_{14}^4	w_{14}^5	w_{14}^6	w_{14}^7	w_{14}^8	w_{14}^9	w_{14}^{10}	w_{14}^{11}	w_{14}^{12}	w_{14}^{13}	w_{14}^{14}

Table 5.1.4 : **The character table of $\mathbb{Z}_{15}$.**

where for each $k = 1, 2, ..., 14, w_k = e^{\frac{2k\pi i}{15}}$.

We use the formula in proposition 3.2.9 to induce the irreducible characters of N and G to $\overline{G}$. If $\tau \in IrrN$, then

$$\begin{aligned}
\tau^{\overline{G}}(1) &= 240\left(\frac{1\tau(1)}{16}\right) = 15\tau(1) \\
\tau^{\overline{G}}(e_1) &= 16\sum_{g\in N, g\neq 1}\frac{\tau(g)}{16} = \sum_{g\in N, g\neq 1}\tau(g) \\
\tau^{\overline{G}}(x^i) &= 0, \qquad for\ each\ i = 1, 2, ..., 14
\end{aligned}$$

and we obtain the following characters of $\overline{G}$:

class	1	e_1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
h_i	1	15	16	16	16	16	16	16	16	16	16	16	16	16	16	16
order	1	2	15	15	15	15	15	15	15	15	15	15	15	15	15	15
centralizer	240	16	15	15	15	15	15	15	15	15	15	15	15	15	15	15
$\tau_1^{\overline{G}}$	15	15	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\tau_2^{\overline{G}}$	15	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.1.5 : **The induced characters of $\overline{G}$ from N.**

If $\rho \in Irr(G)$, then

$$\begin{aligned}
\rho^{\overline{G}}(1) &= 240\left(\frac{1\cdot\rho(1)}{|C_G(1)|}\right) = \frac{240\rho(1)}{15} = 16\cdot\rho(1) \\
\rho^{\overline{G}}(e_1) &= 0 \\
\rho^{\overline{G}}(x^i) &= 15\left(\frac{\rho(x^i)}{15}\right) = \rho(x^i) \qquad for\ each\ i = 1, 2, ..., 14
\end{aligned}$$

The characters of $\overline{G}$ induced from G are :

class	1	e_1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
h_i	1	15	16	16	16	16	16	16	16	16	16	16	16	16	16	16
order	1	2	15	15	15	15	15	15	15	15	15	15	15	15	15	15
$\|C_{\overline{G}}(\overline{g})\|$	240	16	15	15	15	15	15	15	15	15	15	15	15	15	15	15
$\rho_0^{\overline{G}}$	16	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\rho_1^{\overline{G}}$	16	0	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}	w_{13}	w_{14}
$\rho_2^{\overline{G}}$	16	0	w_2	w_4	w_6	w_8	w_{10}	w_{12}	w_{14}	w_1	w_3	w_5	w_7	w_9	w_{11}	w_{13}
$\rho_3^{\overline{G}}$	16	0	w_3	w_6	w_9	w_{12}	1	w_3	w_6	w_9	w_{12}	1	w_3	w_6	w_9	w_{12}
$\rho_4^{\overline{G}}$	16	0	w_4	w_8	w_{12}	w_1	w_5	w_9	w_{13}	w_2	w_6	w_{10}	w_{14}	w_3	w_7	w_{11}
$\rho_5^{\overline{G}}$	16	0	w_5	w_{10}	1	w_5	w_{10}	1	w_5	w_{10}	1	w_5	w_{10}	1	w_5	w_{10}
$\rho_6^{\overline{G}}$	16	0	w_6	w_{12}	w_3	w_9	1	w_6	w_{12}	w_3	w_9	1	w_6	w_{12}	w_3	w_9
$\rho_7^{\overline{G}}$	16	0	w_7	w_{14}	w_6	w_{13}	w_5	w_{12}	w_4	w_{11}	w_3	w_{10}	w_2	w_9	w_1	w_8
$\rho_8^{\overline{G}}$	16	0	w_8	w_1	w_9	w_2	w_{10}	w_3	w_{11}	w_4	w_{12}	w_5	w_{13}	w_6	w_{14}	w_7
$\rho_9^{\overline{G}}$	16	0	w_9	w_3	w_{12}	w_6	1	w_9	w_3	w_{12}	w_6	1	w_9	w_3	w_{12}	w_6
$\rho_{10}^{\overline{G}}$	16	0	w_{10}	w_5	1	w_{10}	w_5	1	w_{10}	w_5	1	w_{10}	w_5	1	w_{10}	w_5
$\rho_{11}^{\overline{G}}$	16	0	w_{11}	w_7	w_3	w_{14}	w_{10}	w_6	w_2	w_{13}	w_9	w_5	w_1	w_{12}	w_8	w_4
$\rho_{12}^{\overline{G}}$	16	0	w_{12}	w_9	w_6	w_3	1	w_{12}	w_9	w_6	w_3	1	w_{12}	w_9	w_6	w_3
$\rho_{13}^{\overline{G}}$	16	0	w_{13}	w_{11}	w_9	w_7	w_5	w_3	w_1	w_{14}	w_{12}	w_{10}	w_8	w_6	w_4	w_2
$\rho_{14}^{\overline{G}}$	16	0	w_{14}	w_{13}	w_{12}	w_{11}	w_{10}	w_9	w_8	w_7	w_6	w_5	w_4	w_3	w_2	w_1

Table 5.1.6 : **The induced characters of $\overline{G}$ from G.**

where for each $k = 1, 2, ..., 14$, $\quad w_k = e^{\frac{2k\pi i}{15}}$.

Besides the trivial character χ_0, we have another irreducible character of $\overline{G}$ in $\tau_2^{\overline{G}}$, since

$$\langle \tau_2^{\overline{G}}, \rho_2^{\overline{G}} \rangle \quad = \quad 1.$$

For each $i = 1, 2, ..., 14$,

$$\langle \rho_i^{\overline{G}}, \rho_i^{\overline{G}} \rangle \quad = \quad 2.$$

Hence none of these characters are irreducible, but for each i ,

$$\langle \rho_i^{\overline{G}}, \tau_2^{\overline{G}} \rangle \quad = \quad 1.$$

This means that for each $i = 1, 2, ..., 14$, $\rho_i^{\overline{G}}$ is the sum of two irreducible characters of $\overline{G}$ of which one is $\tau_2^{\overline{G}}$. Hence for each i, $\chi_i = \rho_i^{\overline{G}} - \tau_2^{\overline{G}}$ is an irreducible character of $\overline{G}$. With this, we now have all the irreducible characters of $\overline{G}$.

class	1	e_1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
h_i	1	15	16	16	16	16	16	16	16	16	16	16	16	16	16	16
order	1	2	15	15	15	15	15	15	15	15	15	15	15	15	15	15
$\|C_{\overline{G}}(\overline{g})\|$	240	16	15	15	15	15	15	15	15	15	15	15	15	15	15	15
χ_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_1	1	1	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}	w_{13}	w_{14}
χ_2	1	1	w_2	w_4	w_6	w_8	w_{10}	w_{12}	w_{14}	w_1	w_3	w_5	w_7	w_9	w_{11}	w_{13}
χ_3	1	1	w_3	w_6	w_9	w_{12}	1	w_3	w_6	w_9	w_{12}	1	w_3	w_6	w_9	w_{12}
χ_4	1	1	w_4	w_8	w_{12}	w_1	w_5	w_9	w_{13}	w_2	w_6	w_{10}	w_{14}	w_3	w_7	w_{11}
χ_5	1	1	w_5	w_{10}	1	w_5	w_{10}	1	w_5	w_{10}	1	w_5	w_{10}	1	w_5	w_{10}
χ_6	1	1	w_6	w_{12}	w_3	w_9	1	w_6	w_{12}	w_3	w_9	1	w_6	w_{12}	w_3	w_9
χ_7	1	1	w_7	w_{14}	w_6	w_{13}	w_5	w_{12}	w_4	w_{11}	w_3	w_{10}	w_2	w_9	w_1	w_8
χ_8	1	1	w_8	w_1	w_9	w_2	w_{10}	w_3	w_{11}	w_4	w_{12}	w_5	w_{13}	w_6	w_{14}	w_7
χ_9	1	1	w_9	w_3	w_{12}	w_6	1	w_9	w_3	w_{12}	w_6	1	w_9	w_3	w_{12}	w_6
χ_{10}	1	1	w_{10}	w_5	1	w_{10}	w_5	1	w_{10}	w_5	1	w_{10}	w_5	1	w_{10}	w_5
χ_{11}	1	1	w_{11}	w_7	w_3	w_{14}	w_{10}	w_6	w_2	w_{13}	w_9	w_5	w_1	w_{12}	w_8	w_4
χ_{12}	1	1	w_{12}	w_9	w_6	w_3	1	w_{12}	w_9	w_6	w_3	1	w_{12}	w_9	w_6	w_3
χ_{13}	1	1	w_{13}	w_{11}	w_9	w_7	w_5	w_3	w_1	w_{14}	w_{12}	w_{10}	w_8	w_6	w_4	w_2
χ_{14}	1	1	w_{14}	w_{13}	w_{12}	w_{11}	w_{10}	w_9	w_8	w_7	w_6	w_5	w_4	w_3	w_2	w_1
$\chi_{15} = \tau_2^{\overline{G}}$	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5.1.7 : **The character table of $2^4 : 15$**

where for each $k = 1, 2, ..., 14, \quad w_k = e^{\frac{2k\pi i}{15}}$.

We now continue with our second example.

5.2 The Character Table of 11 : 5

Let $\overline{G} = 11 : 5$, be the **semi-direct product** of a cyclic group N of order 11 by a cyclic group G of order 5. This group is a maximal subgroup of the group $GL(2,11)$. Here the **action** of G on N is by **conjugation**.

Let

$$N \quad = \quad \langle b \rangle \quad = \quad \{1, b, b^2, ..., b^{10}\}, \quad b^{11} = 1$$

$$and$$

$$G \quad = \quad \langle a \rangle \quad = \quad \{1, a, a^2, a^3, a^4\}, \quad a^5 = 1$$

For this extension one defines an action of G on N :

$$\theta : G \longmapsto Aut(N) \cong \mathbb{Z}_{10} \quad = \quad \langle \alpha \rangle \quad , \quad \alpha^{10} = 1$$

by

$$\theta : 1 \mapsto 1$$
$$a \mapsto \alpha^2$$
$$a^2 \mapsto \alpha^4$$
$$a^3 \mapsto \alpha^6$$
$$a^4 \mapsto \alpha^8 \ ,$$

where $\alpha : b \mapsto b^2$.

Since $|\overline{G}| = 55$, it is an easy exercise to see that $\overline{G} = 11 : 5$ must be one of the following groups :

$$
\begin{aligned}
\overline{G}_1 &= \langle a,b \mid a^5 = 1 = b^{11}, \quad ab = ba\rangle \\
\overline{G}_2 &= \langle a,b \mid a^5 = 1 = b^{11}, \quad ab = b^3a\rangle \\
\overline{G}_3 &= \langle a,b \mid a^5 = 1 = b^{11}, \quad ab = b^4a\rangle \\
\overline{G}_4 &= \langle a,b \mid a^5 = 1 = b^{11}, \quad ab = b^5a\rangle \\
\overline{G}_5 &= \langle a,b \mid a^5 = 1 = b^{11}, \quad ab = b^9a\rangle
\end{aligned}
$$

After checking, it can be seen that our group is in fact $\overline{G}_3$.

We now immediately proceed to calculate the conjugacy classes of 11 : 5 by coset analysis. Here we analyze the the cosets Ng, where g is representative of a class of G, to determine the conjugacy classes of $\overline{G}$. Firstly, we act N and then $C_G(g)$ on the elements of Ng.The action of N and $C_G(g)$ on Ng is by conjugation. The method of coset analysis is discussed in section 2.2 and we will use the notation likewise. So $|C_{\overline{G}}(x) = \frac{k|C_G(x)|}{f_j}$, where f_j of the k blocks of the coset Ng have fused to give a class of $\overline{G}$ containing x. Also $\overline{G} = \bigcup_{g\in G} Ng$ because $\overline{G}$ is a split extension.

Each of the five classes of G , contains only one element. G is cyclic ,hence $C_G(g)$ =G for all $g \in G$. The set of conjugacy classes of G is $\{[1], [a], [a^2], [a^3], [a^4]\}$. Now we are ready to calculate the conjugacy classes of $\overline{G}$.

<u>•$g = 1 \in G$:</u>

The identity element of G fixes all elements of N. Therefore $C_N(1) = N$ and hence $k = 11 = |N|$. The coset N splits into 11 orbits, $Q_1, Q_2, ..., Q_{11}$,each containing $\frac{|N|}{k} = 1$ element. Thus we obtain :

$Q_1 = \{1\}$, $Q_2 = \{b\}$, $Q_3 = \{b^2\} = Q_4 = \{b^3\}$, $Q_5 = \{b^4\}$, $Q_6 = \{b^5\}$,

$Q_7 = \{b^6\}$, $Q_8 = \{b^7\}$, $Q_9 = \{b^8\}$, $Q_{10} = \{b^9\}$ and $Q_{11} = \{b^{10}\}$.

Now we act $C_G(1) = G$ on the orbits Q_i, $1 \le i \le 11$. We obtain :

$\triangle_1 = Q_1^G = \{1\}$,

$\triangle_2 = \{b\}^G = \{b, b^3, b^4, b^5, b^9\}$ and

$\triangle_3 = \{b^2, b^6, b^7, b^8, b^{10}\}$

Thus, under the action of $C_G(1)$, we obtain three orbits with $f_1 = 1$ and $f_2 = f_3 = 5$ and so this coset gives three classes of $\overline{G}$:

$f_1 = 1$, $x = 1$, $|C_{\overline{G}}(1)| = 11 \times 5 = 55$;

$f_2 = 5$, $x = b$, $|C_{\overline{G}}(b)| = \frac{11\times5}{5} = 11$;

$f_3 = 5$, $x = b^2$, $|C_{\overline{G}}(b^2)| = \frac{11\times5}{5} = 11$.

$\bullet g = a \in G$:

The action of g on N fixes only the identity element. Hence $|C_N(a)| = 1$ and therefore the coset Na has only one class in $\overline{G}$.

Also for the classes $[a^2]$, $[a^3]$and$[a^4]$ we have $k = 1$ and hence the cosets Ng has only class each in $\overline{G}$.

So the conjugacy classes of Na^j ,j = 1,2,3,4 , are as follows :

$|C_{\overline{G}}(a^j)| = 5$, for each j = 1,2,3,4 and

$|[a^j]_{\overline{G}}| = 11$, for each j = 1,2,3,4 .

The process of coset analysis is done and the conjugacy classes of $\overline{G}$ are as follows :

classes	1	a	a^2	a^3	a^4	b	b^2
no. of elements	1	11	11	11	11	5	5
order	1	11	11	11	11	5	5
centralizer	55	5	5	5	5	11	11

Table 5.2.1 : **The conjugacy classes of** 11 : 5.

We immediately proceed to compute the irreducible characters of $\overline{G}$. The methods of lifting and induction of characters are being used to compute the character table of $\overline{G}$. These methods are fully described in sections 3.2 and 3.3 of our text. Now $G \cong \overline{G}/N$ and by theorem 3.3.6 some of the irreducible characters of $\overline{G}$ can be found by lifting the irreducible characters of G to $\overline{G}$. The character table of G is calculated by making use of proposition 3.1.21 ,so our first five irreducible characters of $\overline{G}$ are the liftings χ_i, i = 1,2,...,5, of $\hat{\chi}_i \in IrrG$.

$\overline{G}/N$	K	Ka	Ka^2	Ka^3	Ka^4
classes of G	1	a	a^2	a^3	a^4
no. of elements	1	1	1	1	1
$\hat{\chi}_1$	1	1	1	1	1
$\hat{\chi}_2$	1	w	w^2	w^3	w^4
$\hat{\chi}_3$	1	w^2	w^4	w	w^3
$\hat{\chi}_4$	1	w^3	w	w^4	w^2
$\hat{\chi}_5$	1	w^4	w^3	w^2	w

Table 5.2.2 : **The character table of $\overline{G}/N \cong G$**

where $w = e^{\frac{2\pi i}{5}}$

We still need two more irreducible characters of $\overline{G}$. For this purpose we induce the characters of $\mathbb{Z}_5$ to $\overline{G}$ and exactly two more irreducible characters of $\overline{G}$ are found. So, we are done with the process of finding all the irreducible characters of $\overline{G}$. See the tables below.

classes	1	b	b^2	b^3	b^4	b^5	b^6	b^7	b^8	b^9	b^{10}
no. of elements	1	1	1	1	1	1	1	1	1	1	1
order	1	11	11	11	11	11	11	11	11	11	11
τ_0	1	1	1	1	1	1	1	1	1	1	1
τ_1	1	v	v^2	v^3	v^4	v^5	v^6	v^7	v^8	v^9	v^{10}
τ_2	1	v^2	v^4	v^6	v^8	v^{10}	v	v^3	v^5	v^7	v^9
τ_3	1	v^3	v^6	v^9	v	v^4	v^7	v^{10}	v^2	v^5	v^8
τ_4	1	v^4	v^8	v	v^5	v^9	v^2	v^6	v^{10}	v^3	v^7
τ_5	1	v^5	v^{10}	v^4	v^9	v^3	v^8	v^2	v^7	v	v^6
τ_6	1	v^6	v	v^7	v^2	v^5	v^8	v^3	v^9	v^{10}	v^5
τ_7	1	v^7	v^3	v^{10}	v^6	v^2	v^9	v^5	v	v^8	v^4
τ_8	1	v^8	v^5	v^2	v^{10}	v^7	v^4	v	v^9	v^6	v^3
τ_9	1	v^9	v^7	v^5	v^3	v	v^{10}	v^8	v^6	v^4	v^2
τ_{10}	1	v^{10}	v^9	v^8	v^7	v^6	v^5	v^4	v^3	v^2	v

Table 5.2.3 : **The character table of $\mathbb{Z}_{11}$**

classes	1	a	a^2	a^3	a^4	b	b^2
no. of elements	1	11	11	11	11	5	5
order	1	11	11	11	11	5	5
centralizer	55	5	5	5	5	11	11
τ_0^G	5	0	0	0	0	5	5
τ_1^G	5	0	0	0	0	c	d
τ_2^G	5	0	0	0	0	d	c
τ_3^G	5	0	0	0	0	c	d
τ_4^G	5	0	0	0	0	d	c
τ_5^G	5	0	0	0	0	c	d
τ_6^G	5	0	0	0	0	d	c
τ_7^G	5	0	0	0	0	c	d
τ_8^G	5	0	0	0	0	d	c
τ_9^G	5	0	0	0	0	c	d
τ_{10}^G	5	0	0	0	0	d	c

Table 5.2.4 : **Characters of $\overline{G}$ induced from $\mathbb{Z}_{11}$**

where ,

$$c = v + v^3 + v^4 + v^5 + v^9 ;$$

$$d = v^2 + v^6 + v^7 + v^8 + v^{10}$$

classes	1	a	a^2	a^3	a^4	b	b^2
no. of elements	1	11	11	11	11	5	5
order	1	11	11	11	11	5	5
centralizer	55	5	5	5	5	11	11
χ_1	1	1	1	1	1	1	1
χ_2	1	w	w^2	w^3	w^4	1	1
χ_3	1	w^2	w^4	w	w^3	1	1
χ_4	1	w^3	w	w^4	w^2	1	1
χ_5	1	w^4	w^3	w^2	w	1	1
χ_6	5	0	0	0	0	c	d
χ_7	5	0	0	0	0	d	c

Table 5.2.5 : **The character table of 11 : 5.**

We now continue with our *3rd* example.

5.3 The Character Table of $2^3 : SP(2,2)$

In [23] the conjugacy classes of $2^3 : SP(2,2)$ were obtained using GAP. We use coset analysis to illustrate an alternative method to calculate the conjugacy classes of $2^3 : SP(2,2)$. To compute the character table of $2^3 : SP(2,2)$ we let $G = SP(2,2)$ act by conjugation on $N = 2^3$. We make use of the Fischer-Clifford theory which enables us to compute the irreducible characters of our extension through the use of Fischer Matrices. The material developed in sections 4.1 and 4.2 will be utilised in the process of constructing the character table of the split extension $2^3 : SP(2,2)$. We use the same notation employed in these sections.

Let $\overline{G} = N : G$ where N is an elementary abelian 2-group of order 8 and $G = SP(2,2)$. $SP(2,2)$ is the sympletic group of dimension 2 over $GF(2)$. Also $2^3 : SP(2,2) \cong P(2) : H$.

Let $P(2)$ be generated by $\{e_1, e_2, e_3\}$, where

$$\mathbf{e_1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \textit{ and } \mathbf{e_3} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $e_i^2 = 1$, for $1 \leq i \leq 3$.

Hence $P(2) = \{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}$ where,

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{e_1e_2} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{e_1e_3} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{e_2e_3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{e_1e_2e_3} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $H = \langle \alpha, \beta \rangle = \{1, \alpha, \beta, \beta^2, \alpha\beta, \beta\alpha\}$, where $\alpha^2 = 1 = \beta^3$ and,

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \boldsymbol{\beta^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \alpha\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\beta\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The operation on the elements of $\overline{G}$ is matrix multiplication. Also G acts by conjugation on N.

We will now immediately employ the process of computing the conjugacy classes of $\overline{G}$.Firstly, we calculate the conjugacy classes of H by means of MAGMA:

classes	$(1A)$	$(2A)$	$(3A)$
no. of elements	1	3	2
order	1	2	3
centralizer	6	2	3

Table 5.3.1 : **The conjugacy classes of $G = H$.**

Using the method discussed in chapter2, section2.2, we act N and $C_G(g)$ by conjugation on the cosets Ng where $g \in \{(1A), (2A), (3A)\}$ to compute the conjugacy classes of $\overline{G}$.

- $\underline{g = 1_G:}$

  For g the identity of G , g fixes all the elements of N , so $k = |C_N(1_G)| = 8$. So the coset N splits into eight orbits, Q_i, where $1 \leq i \leq 8$. Each orbit containing $\frac{|N|}{k} = \frac{8}{8} = 1$ element. Under the action of $C_G(1_G) = G$ these orbits are fused as follows :

  $\triangle_1 = \{1\}^G = \{1\} \Longrightarrow f_1 = 1$,

  $\triangle_2 = Q_2^G = \{e_1\}^G = \{e_1, e_2, e_1e_2\} = \{Q_2 \bigcup Q_3 \bigcup Q_5\} \Longrightarrow f_2 = 3$,

  $\triangle_3 = Q_4^G = \{e_3\}^G = \{e_3, e_2e_3, e_1e_2e_3\} = \{Q_4 \bigcup Q_7 \bigcup Q_8\} \Longrightarrow f_3 = 3$,

  $\triangle_4 = Q_6^G = \{e_1e_3\}^G = \{e_1e_3\} = Q_6 \Longrightarrow f_4 = 1$,

  So this coset gives four conjugacy classes of $\overline{G}$ as follows :

  $|C_G(1)| = \frac{8\times6}{1} = 48$, $|[1]_G| = \frac{48}{8\times6} = 1$

  $|C_G(e_1)| = \frac{8\times6}{3} = 16$, $|[e_1]_G| = \frac{3\times48}{8\times6} = 3$

  $|C_G(e_3)| = \frac{8\times6}{3} = 16$, $|[e_3]_G| = \frac{3\times48}{8\times6} = 3$

  $|C_G(e_1e_3)| = 48$, $|[e_1e_3]_G| = 1$

- $\underline{g \in (2A):}$

  We take

$$g = \beta\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  The action of g on N fixes the following elements: $\{1, e_2, e_1e_3, e_1e_2e_3\}$. Therefore $|C_N(g)| = k = 4$ and hence Ng splits into 4 orbits of 2 elements each. Under the action of N on Ng we obtain the following orbits :

$Q_1 = \{g, e_2g\}$, $Q_2 = \{e_1g, e_1e_2g\}$,

$Q_3 = \{e_3g, e_2e_3g,\}$ and $Q_4 = \{e_1e_3g, e_1e_2e_3g\}$.

$C_G(g) = \langle\beta\alpha\rangle = \{1, \beta\alpha\} \Longrightarrow |C_G(g)| = 2$

Under the action of $C_G(g)$ on Ng we obtain the following orbits :

$\triangle_1 = Q_1^{C_G(g)} = Q_1$, $\triangle_2 = Q_2^{C_G(g)} = Q_2$, $\triangle_3 = Q_3^{C_G(g)} = Q_3$ and $\triangle_4 = Q_4^{C_G(g)} = Q_4$.

And so the action on this coset Ng gives us 4 conjugacy classes of $\overline{G}$:

$|C_{\overline{G}}(g)| = \frac{k \times |C_G(g)|}{f_1} = \frac{4\times 2}{1} = 8$

$|[g]_{\overline{G}}| = \frac{f_1 \times |\overline{G}}{k \times |C_G(g)|} = \frac{1\times 48}{4\times 2} = 6$

The computation of the cardinality of the three remaining classes is exactly the same as the class $\triangle_1$ above.

- $\underline{g \in (3A)}$:

We take

$$g = \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The action of g on N fixes the elements $\{1, e_1e_3\}$ of $N \Longrightarrow |C_N(g)| = \text{k} = 2$ and hence Ng

splits into 2 orbits of 4 elements each.

Under the action of N on Ng the following 2 orbits are obtained:

$Q_1 = \{g, e_1g, e_2g, e_1e_2g\}$ and

$Q_2 = \{e_3g, e_1e_3g, e_2e_3g, e_1e_2e_3g\}$

$C_G(g) = \langle\beta\rangle = \{1, \beta, \beta^2\} \Longrightarrow |C_G(g)| = 3$

These orbits can't fuse together under $C_G(g)$, since $g^{C_G(g)} = \{g\}$. Therefore we have two classes of $\overline{G}$, each with $f = 1$:

$x = g$, $class(3_1)$, $|C_{\overline{G}}(x)| = 6$, $|[x]_{\overline{G}}| = 8$

$x = e_3g$, $class(6_1)$, $|C_{\overline{G}}(x)| = 6$, $|[x]_{\overline{G}}|$ =8

x Thus the conjugacy classes of $\overline{G}$ are as follows :

classes of G	$(1A)$				$(2A)$				$(3A)$	
classes of $\overline{G}$	(1)	(2_1)	(2_2)	(2_3)	(2_4)	(4_1)	(4_2)	(2_5)	(3_1)	(6_1)
h_i	1	3	3	1	6	6	6	6	8	8
$\lvert C_{\overline{G}}(x)\rvert$	48	16	16	48	8	8	8	8	6	6

Table 5.3.2 : **The conjugacy classes of** $2^3 : SP(2,2)$.

We proceed to compute the **Fischer Matrices**. The construction of the Clifford-Fischer matrices and the determination of the inertia factors of the classes of G are based on the theory developed in sections 4.1 and 4.2. From the action of G on $Irr(N)$ we obtain the same number of orbits as when G acts on N,(see Lemma 4.2.1). There are 4 orbits , where two have length 3 and the other two orbits length 1. Hence there are 4 inertia groups $\overline{H_i}$ where $i = 1, 2, 3, 4$. The inertia groups are $\overline{H_1} = \overline{H_4} = \overline{G}$ and $\overline{H_2} = \overline{H_3}$, where $[\overline{G} : \overline{H_3}] = [\overline{G} : \overline{H_4}] = 3$. Let $H_i = \overline{H_i}/N$, and the following inertia factors are obtained : $H_1 = H_4 = G \cong S_3$ and $H_2 = H_3 = \langle \beta\alpha \rangle \cong \langle (23) \rangle \leq S_3$. Note that $[G : H_2] = [G : H_3] = 3$ and all the inertia factors are maximal subgroups of G, which were determined through the use of the computer program MAGMA.

See the tables below for the irreducible characters and fusion maps into G of the inertia factors :

classes	$(1A)$	$(2A)$	$(3A)$
no. of elements(h_i)	1	3	2
centralizer	6	2	3
ψ_1	1	1	1
ψ_2	1	-1	1
ψ_3	2	0	-1

Table 5.3.3 : **The character table of $H_1 = H_4 \cong S_3$.**

classes	$(1A)$	$(2A)$
h_i	1	1
centralizer	2	2
ξ_1	1	1
ξ_2	1	-1

Table 5.3.4 : **The character table of $H_2 = H_3$.**

classes of H_2	classes of G
$(1A)$	$(1A)$
$(2A)$	$(2A)$

Table 5.3.5 : **The fusion map of H_2 into G.**

Now to calculate the Fischer matrices we will use the relations of Theorem 4.2.5. Note that all the relations hold, since N is elementary abelian. For every g in Ng, we have the Fischer matrix $M(g)$. For each matrix $M(g)$, we index the columns by the orders of the centralizers of the class representatives of $\overline{G}$ which comes from Ng and the rows by the orders of the centralizers of the class representatives of the inertia factors which fuse to $[g]$ in G. See the discussion after Lemma 4.2.2. Also note that the Fischer matrices $M(g)$ are all square and nonsingular, and that the sizes of these matrices are determined by the number of $\overline{G}$-conjugacy classes of the cosets Ng. Corresponding to the identity element of G, we let

48 16 16 48

$$M(1_G) = \begin{matrix} 6 \\ 2 \\ 2 \\ 6 \end{matrix} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

because the action of G on N delivered four classes of $\overline{G}$, and hence we obtained a 4×4 matrix. By

the first and third relations of Theorem 4.2.5 : $a_1 = a_2 = a_3 = a_4 = 1$; $b_1 = c_1 = 3$ and $d_1 = 1$.Column orthoganality given by the 2^{nd} relation of Theorem 4.2.5, resulted in the following equations :

$$6 + 2|b_2|^2 + 2|c_2|^2 + 6|d_2|^2 = 16$$

$$6 + 6b_2 + 6c_2 + 6d_2 = 0$$

$$6 + 2|b_3|^2|c_3|^2 + 6|d_3|^2 = 16$$

$$6 + 6b_3 + 6c_3 + 6d_3 = 0$$

$$6 + 2|b_4|^2 + 2|c_4|^2 + 6|d_4|^2 = 48$$

$$6 + 6b_4 + 6c_4 + 6d_4 = 0$$

$$6 + 2b_2.b_3 + 2c_2c_3 + 6d_2d_3 = 0$$

$$6 + 2b_2b_4 + 2c_2c_4 + 6d_2d_4 = 0$$

$$6 + 2b_3b_4 + 2c_3c_4 + 6d_3d_4 = 0$$

Solving these equations simultaneously the following matrix is obtained :

$$\begin{matrix} & 48 & 16 & 16 & 48 \end{matrix}$$

$$M(1_G) = \begin{matrix} 6 \\ 2 \\ 2 \\ 6 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & 1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

Similarly, we compute the other Fischer matrices which appear below :

- $g \in (2A)$:

$$
\begin{array}{c} \\ \end{array}
\begin{array}{cc}
 & \begin{array}{cccc} 8 & 8 & 8 & 8 \end{array} \\
M(g) = \begin{array}{c} 2 \\ 2 \\ 2 \\ 2 \end{array} & \left(\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{array} \right)
\end{array}
$$

- $g \in (3A)$:

$$
\begin{array}{cc}
 & \begin{array}{cc} 6 & 6 \end{array} \\
M(g) = \begin{array}{c} 3 \\ 3 \end{array} & \left(\begin{array}{rr} 1 & 1 \\ 1 & -1 \end{array} \right)
\end{array}
$$

We are now ready to determine the character table of $\overline{G}$. There are four inertia factors, hence the characters of $\overline{G}$ are divided into *four blocks.* This process is described as follows : **multiplying rows of the matrix $M(g)$ with sections of the character tables of the inertia factors fusing to the class $[g]$.**

The identity element of G corresponds to :

$$
M(1_G) = \left(\begin{array}{rrrr} 1 & 1 & 1 & 1 \\ 3 & -1 & 1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & 1 & -1 & -1 \end{array} \right)
$$

By multiplying each row of $M(1)$ by the columns in the character tables of the inertia factors which correspond with the classes fusing to 1_G respectively, we obtain the values of the characters of $\overline{G}$ on the $\overline{G}$-classes with representatives $1, e_1, e_3, e_1e_3$:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 & -3 \\ 3 & -1 & 1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 & 3 \\ 3 & -1 & -1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 2 & 2 & -2 & -2 \end{pmatrix}$$

We determine the values of the irreducible characters of $\overline{G}$ corresponding to the class of G with representation $\beta\alpha$ in a similar fashion :

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The values of the characters of $\overline{G}$ corresponding to class of G with representative β are as follows :

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$$

The process of computing the irreducible characters is completed, and the character table is as follows :

classes($\overline{G}$)	(1)	(2_1)	(2_2)	(2_3)	(2_4)	(4_1)	(4_2)	(2_5)	(3_1)	(6_1)
h_i	1	3	3	1	6	6	6	6	8	8
$\|C_{\overline{G}}(x)\|$	48	16	16	48	8	8	8	8	6	6
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1	1	1
χ_3	2	2	2	2	0	0	0	0	-1	-1
χ_4	3	-1	1	-3	1	-1	1	-1	0	0
χ_5		-1	1	-3	-1	1	-1	1	0	0
χ_6	3	-1	-1	3	1	-1	-1	1	0	0
χ_7	3	-1	-1	3	-1	1	1	-1	0	0
χ_8	1	1	-1	-1	1	1	-1	-1	1	-1
χ_9	1	1	-1	-1	-1	-1	1	1	1	-1
χ_{10}	2	2	-2	-2	0	0	0	0	-1	1

Table 5.3.6 : **The Character Table of** $2^3 : SP(2,2)$.

And now we discuss our last split-extension.

5.4 The Character Table of $2^4 : S_5$

We let $\overline{G} = N : G$ where N is an elementary abelian 2-group of order 16 and $G = S_5$. Now $N \cong V(2,4) = < e_1, e_2, e_3, e_4 >$, the vector space of dimension 4 over a field of two elements, and the symmetric group S_5 is generated by (12) and (12345). By identifying (12) and (12345) with

$$g_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

respectively, we can regard S_5 as the subgroup $< g_1, g_2 >$ of $GL(2,4) \cong A_8$, that is, $G = S_5 \cong < g_1, g_2 > \leq GL(2,4)$. Note that $S_5 \leq S_6$ and that S_6 is a maximal subgroup of A_8. The group $< g_1, g_2 >$ then acts naturally on $V(2,4) \cong N$.

To determine the conjugacy classes of G we use the method of Coset Analysis described earlier. We need the conjugacy classes of S_5. These are obtained using MAGMA. We start by taking representatives g from these conjugacy classes and consider the action of N and $C_G(g)$ on the cosets Ng. The conjugacy classes of S_5 is as follows :

classes	$(1A)$	$(2A)$	$(2B)$	$(3A)$	$(4A)$	$(5A)$	$(6A)$
no. of elements	1	10	15	20	30	24	20
$\lvert C_G(g)\rvert$	120	12	8	6	4	5	6

Table 5.4.1 : **The conjugacy classes of** S_5.

We now proceed to compute the conjugacy classes of $2^4 : S_5$:

- $\underline{g = 1:}$

The identity element of G fixes all elements of N, so $k = 16$. Hence Ng splits into 16 orbits, Q_i where $i = 1, 2, ..., 16$. Under the action of $C_G(1) = G$ the orbits of Ng fuse as follows :

$\triangle_1 = \{1\}$

$\triangle_2 = \{e_1\}^G = \{e_1,\ e_2,\ e_3,\ e_4,\ e_1e_2e_3e_4\}$

$\triangle_3 = \{e_1e_2\}^G = \{e_1e_2,\ e_2e_3,\ e_2e_3e_4,\ e_1e_3,\ e_3e_4,\ e_1e_2e_4,\ e_2e_4,\ e_1e_4,\ e_1e_2e_3,\ e_1e_3e_4\}$. Hence

$$f_1 = 1\ ,\ f_2 = 5 \ \text{ and }\ f_3 = 10.$$

Hence we obtained the following classes of $\overline{G}$ from the coset N :

$|C_{\overline{G}}(1)| = \frac{16\times120}{1} = 1920$ *and* $|[1]_{\overline{G}}| = \frac{1\times1920}{16\times120} = 1$;

$|C_{\overline{G}}(e_1)| = \frac{16\times120}{5} = 384$ *and* $|[e_1]_{\overline{G}}| = \frac{5\times1920}{16\times120} = 5$ and

$|C_{\overline{G}}(e_1e_2)| = \frac{16\times120}{10} = 192$ *and* $|[e_1e_2]_{\overline{G}}| = \frac{10\times1920}{16\times120} = 10$

- $\underline{g \in (2A):}$

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The action of g on N fixes the elements $\{1,\ e_2,\ e_3,\ e_1e_4,\ e_2e_3,\ e_1e_2e_4,\ e_1e_3e_4,\ e_1e_2e_3e_4\}$,

therefore k = 8. The coset Ng splits into eight orbits, Q_i where $i = 1, 2, ..., 8$, under the action of N.

$$C_G(g) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\rangle$$

Note that $|C_G(g)| = 12$. Under the action of $C_G(g)$ the orbits of the coset Ng fuse as follows :

$\triangle_1 = Q_1^{C_G(g)} = \{g,\ e_1e_4g\}^{C_G(g)} = Q_1$

$\triangle_2 = Q_2^{C_G(g)} = \{e_1g,\ e_4g\}^{C_G(g)} = Q_2$

$\triangle_3 = Q_3^{C_G}(g) = \{e_2g,\ e_1e_2e_4g\}^{C_G(g)} = \{e_2g,\ e_1e_2e_4g,\ e_3g,\ e_1e_3e_4g,\ e_2e_3g,\ e_1e_2e_3e_4g\} = Q_3 \cup Q_4 \cup Q_7$

$\triangle_4 = Q_5^{C_G(g)} = \{e_1e_2g,\ e_2e_4g\}^{C_G(g)} = \{e_1e_2g,\ e_2e_4g,\ e_1e_3g,\ e_3e_4g,\ e_1e_2e_3g,\ e_2e_3e_4g\} = Q_5 \cup Q_6 \cup Q_8$, where

$$f_1 = 1,\ f_2 = 1,\ f_3 = 3,\ f_4 = 3$$

The action of N and G on the coset Ng give the following classes of $\overline{G}$:

$|C_{\overline{G}}(g)| = \frac{8\times 12}{1} = 96$ *and* $|[g]_{\overline{G}}| = \frac{16\times 120}{8\times 12} = 20$;

$|C_{\overline{G}}(e_1)| = 96$ *and* $|[e_1]_{\overline{G}}| = 20$;

$|C_{\overline{G}}(e_2)| = \frac{8\times 12}{3} = 32$ *and* $|[e_2]_{\overline{G}}| = \frac{3\times 1920}{8\times 12} = 60$;

$|C_{\overline{G}}(e_1e_2)| = 32$ *and* $|[e_1e_2]_{\overline{G}}| = 60$.

- $\underline{g \in (2B)}$:

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The action of g fixes the elements $\{1,\ e_2,\ e_1e_4,\ e_1e_2e_4\}$ of N and hence $k = 4$. Under the action of N on Ng we obtained four orbits, Q_i where $i = 1,2,...,4$, with each orbit containing 4 elements.

$$C_G(g) = \left\langle \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\rangle$$

Note that $|C_G(g)| = 8$. Under the action of $C_G(g)$ on Ng the following orbits fuse as follows :

$\triangle_1 = Q_1^{C_G(g)} = \{g,\ e_1e_4g,\ e_1e_2e_4g,\ e_2g\}^{C_G(g)} = Q_1$

$\triangle_2 = Q_2^{C_G(g)} = \{e_1g,\ e_4g,\ e_2e_4g,\ e_1e_2g\}^{C_G(g)} = \{e_1g,\ e_4g,\ e_2e_4g,\ e_1e_2g,\ e_3g,\ e_2e_3g,\ e_1e_3e_4g,\ e_1e_2e_3e_4g\}$
$= Q_2 \cup Q_3$

$\triangle_3 = Q_4^{C_G(g)} = \{e_1e_3,\ e_3e_4,\ e_1e_2e_3,\ e_2e_3e_4\}^{C_G(g)} = Q_4$

The action of $C_G(g)$ and N on the coset Ng give the following classes of $\overline{G}$:

$|C_{\overline{G}}(g)| = \frac{4\times 8}{1} = 32$ *and* $|[g]_{\overline{G}}| = \frac{1\times 1920}{4\times 8} = 60$

$|C_{\overline{G}}(e_1g)| = \frac{4\times 8}{2} = 16$ *and* $|[e_1g]_{\overline{G}}| = \frac{2\times 1920}{4\times 8} = 120$

$|C_{\overline{G}}(e_1e_3g)| = 32$ *and* $|[e_1e_3]_{\overline{G}}| = 60$

- $\underline{g \in (3A):}$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The action of g fixes the elements $\{1,\ e_1,\ e_4,\ e_1e_4\}$ of N and hence $k = 4$. Therefore, under the action of N, Ng splits into 4 orbits Q_i where $i = 1,2,3,4$ and each orbit containing 4 elements.

$$C_G(g) = \left\langle \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\rangle$$

$|C_G(g)| = 6$. Under the action of $C_G(g)$ the orbits fuse as follows:

$\triangle_1 = Q_1^{C_G(g)} = \{g,\ e_2e_3g,\ e_1e_2e_4g,\ e_1e_3e_4\}^{C_G(g)} = Q_1$
$\triangle_2 = Q_2^{C_G(g)} = \{e_1g,\ e_2e_4g,\ e_3e_4g,\ e_1e_2e_3g\}^{C_G(g)} = Q_2 \cup Q_4$
$\triangle_3 = Q_3^{C_G(g)} = \{e_2g,\ e_3g,\ e_1e_4g,\ e_1e_2e_3e_4g\}^{C_G(g)} = Q_3$
Hence we obtain

$$f_1 = 1,\ f_2 = 2, \text{ and } \ f_3 = 1$$

The action of $C_G(g)$ and N on the coset Ng give us the following classes of $\overline{G}$:
$|C_{\overline{G}}(g)| = \frac{4\times 6}{1} = 24$ and $|[g]_{\overline{G}}| = \frac{1\times 1920}{4\times 6} = 80$;
$|C_{\overline{G}}(e_1g)| = \frac{4\times 6}{2} = 12$ and $|[e_1g]_{\overline{G}}| = \frac{2\times 1920}{4\times 6} = 160$
$|C_{\overline{G}}(e_2g)| = 24$ and $|[e_2g]_{\overline{G}}| = 80$

- $g \in (4A)$:

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The action of g fixes the elements $\{1,\ e_2\}$ of N and hence $k = 2$. Therefore, under the action of N, the coset Ng splits into 2 orbits each containing 8 elements.

$$C_G(g) = \left\langle \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\rangle$$

$|C_G(g)| = 4$. Under the action of $C_G(g)$ the orbits of the coset Ng stay unchanged and are as follows:

$\triangle_1 = Q_1^{C_G(g)} = \{g,\ e_2g,\ e_3g,\ e_1e_3g,\ e_1e_4g,\ e_3e_4g,\ e_1e_2e_3g,\ e_1e_2e_4g\} = Q_1$

$\triangle_2 = Q_2^{C_G(g)} = N \setminus Q_1 = Q_2$

So the coset Ng gives the following classes of $\overline{G}$:

$|C_{\overline{G}}(g)| = 8$ and $|[g]_{\overline{G}}| = 240$

$|C_{\overline{G}}(e_1g)| = 8$ and $|[e_1g]_{\overline{G}}| = 240$

- $g \in (5A)$:

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Here $k = 1$ and hence we obtained only one class of $\overline{G}$, which is as follows:

$\triangle_1 = |C_{\overline{G}}(g)| = 5$ and $|[g]_{\overline{G}}|$ =384.

Note that $|C_G(g)|$ =5.

- $\underline{g \in (6A):}$

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Under action of g on N we obtain that $|C_N(g)| = 2$. Hence the coset Ng splits into 2 orbits,

Q_i where i = 1,2, under the action of N.

$$C_G(g) = \left\langle \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\rangle$$

Note that $|C_G(g)|$ =6. Under the action of $C_G(g)$ the 2 orbits of Ng are unchanged and hence

we obtained the following classes of $\overline{G}$:

$|C_{\overline{G}}(g)| = 12$ and $|[g]_{\overline{G}}| = 160$;

$|C_{\overline{G}}(e_1 g)| = 12$ and $|[e_1 g]_{\overline{G}}| = 160$.

The process of computing the conjugacy classes of $\overline{G}$ is done. See the table below.

classes of G	$(1A)$			$(2A)$				$(2B)$			$(3A)$		
classes of $\overline{G}$	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(2_4)	(4_2)	(2_5)	(4_3)	(4_4)	(3_1)	(6_1)	(6_2)
h_i	1	5	10	20	20	60	60	60	120	60	80	160	80
$\lvert C_{\overline{G}}(\overline{g})\rvert$	1920	384	192	96	96	32	32	32	16	32	24	12	24

Table 5.4.2.A : **The conjugacy classes of $2^4 : S_5$.**

classes of G	$(4A)$		$(5A)$	$(6A)$	
classes of $\overline{G}$	(4_5)	(8_1)	(5_1)	(6_3)	(12_1)
h_i	240	240	384	160	160
$\lvert C_{\overline{G}}(\overline{g})\rvert$	8	8	5	12	12

Table 5.4.2.B : **The conjugacy classes of $2^4 : S_5$ (continue).**

We are now ready to compute the Fischer matrices. From the action of G on $Irr(N)$ we obtain three orbits and determine the inertia groups, $\overline{H_i}$ where i = 1,2,3, from these orbits. The lengths of these orbits are 1, 5 and 10. Note $[G : H_1] = 1$, $[G : H_2] = 5$ and $[G : H_3] = 10$. Hence the inertia factors ,which are maximal subgroups of G, are as follows:

$$H_1 = S_5;\ H_2 = S_4 \text{ and } H_3 = S_3 \times S_2.$$

The inertia factors, character tables of the inertia factors and the fusion maps of the inertia factors into G were all determined through the use of MAGMA. See tables below.

classes	$(1A)$	$(2A)$	$(2B)$	$(3A)$	$(4A)$	$(5A)$	$(6A)$
h_i	1	10	15	20	30	24	20
$\|C_G(g)\|$	120	12	8	6	4	5	6
ψ_1	1	1	1	1	1	1	1
ψ_2	1	-1	1	1	-1	1	-1
ψ_3	4	2	0	1	0	-1	-1
ψ_4	4	-2	0	1	0	-1	1
ψ_5	5	1	1	-1	-1	0	1
ψ_6	5	-1	1	-1	1	0	-1
ψ_7	6	0	-2	0	0	1	0

Table 5.4.3 : **The character table of** $S_5 = H_1$.

classes	$(1A)$	$(2A)$	$(2B)$	$(3A)$	$(4A)$
h_i	1	2	2	3	4
$\|C_{H_2}(h_2)\|$	24	4	8	3	4
Γ_1	1	1	1	1	1
Γ_2	1	-1	1	1	-1
Γ_3	2	0	2	-1	0
Γ_4	3	1	-1	0	-1
Γ_5	3	-1	-1	0	1

Table 5.4.4 : **The character table of** $S_4 = H_2$**.**

classes	(1)	(2A)	(2B)	(2C)	(3A)	(6A)
h_i	1	3	1	3	2	2
$\lvert C_{H_3}(h_3)\rvert$	12	4	12	4	6	6
φ_1	1	1	1	1	1	1
φ_2	1	-1	1	-1	1	1
φ_3	1	1	-1	-1	1	-1
φ_4	1	-1	-1	1	1	-1
φ_5	2	0	2	0	-1	-1
φ_6	2	0	-2	0	-1	1

Table 5.4.5 : **The character table of $S_3 \times S_2 = H_3$.**

S_4	S_5
$(1A)$	$(1A)$
$(2A)$	$(2A)$
$(2B)$	$(2B)$
$(3A)$	$(3A)$
$(4A)$	$(4A)$

Table 5.4.6 : **The fusion map of S_4 into S_5.**

$H_3 = S_3 \times S_2$	$H_1 = S_5$
$(1A)$	$(1A)$
$(2A)$	$(2A)$
$(2B)$	$(2A)$
$(2C)$	$(2B)$
$(3A)$	$(3A)$
$(6A)$	$(6A)$

Table 5.4.7 : **The fusion map of $S_3 \times S_2$ into S_5.**

The Fischer matrices are computed through the use of the relations of Theoerem 4.2.5 (see example 5.3) and are as follows:

- $g = 1_G$:

$$M(1_G) = \begin{array}{c} \\ 120 \\ 24 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 1920 & 384 & 192 \end{array} \\ \left(\begin{array}{ccc} 1 & 1 & 1 \\ 5 & -3 & 1 \\ 10 & 2 & -2 \end{array}\right) \end{array}$$

- $g \in (2A)$:

$$M(g) = \begin{array}{c} \\ 12 \\ 4 \\ 4 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} 96 & 96 & 32 & 32 \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 3 & 3 & -1 & -1 \\ 3 & -3 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{array}\right) \end{array}$$

- $\underline{g \in (2B):}$

$$
\begin{array}{c c}
 & \begin{array}{ccc} 32 & 16 & 32 \end{array} \\
M(g) = \begin{array}{c} 8 \\ 8 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix}
\end{array}
$$

- $\underline{g \in (3A):}$

$$\begin{array}{rc} & \begin{array}{ccc} 24 & 12 & 24 \end{array} \\ M(g) = \begin{array}{c} 6 \\ 3 \\ 6 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \end{array}$$

- $\underline{g \in (4A):}$

$$\begin{array}{rc} & \begin{array}{cc} 8 & 8 \end{array} \\ M(g) = \begin{array}{c} 4 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$$

- $\underline{g \in (5A):}$

$$\begin{array}{rc} & 5 \\ M(g) = 5 & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

- $\underline{g \in (6A):}$

$$\begin{array}{rc} & \begin{array}{cc} 12 & 12 \end{array} \\ M(g) = \begin{array}{c} 6 \\ 6 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$$

We are now ready to compute the character table of $\overline{G}$. There are three inertia factors and hence $Irr(\overline{G})$ are divived into three blocks. Each block of irreducible characters corresponds to an inertia factor group. $Irr(\overline{G})$ are computed from the Fischer matrices and the character tables of the inertia factors. This process dictates to multiplying rows of the matrix $M(g)$ with sections of the character tables of the inertia factors fusing to the class of g in G. This process was fully illustrated in section 5.3. We conlude this section with the character table of $2^4 : S_5$.

classes of G	$(1A)$			$(2A)$				$(2B)$			$(3A)$		
classes of $\overline{G}$	(1)	(2_1)	(2_2)	(2_3)	(4_1)	(2_4)	(4_2)	(2_5)	(4_3)	(4_4)	(3_1)	(6_1)	(6_2)
h_i	1	5	10	20	20	60	60	60	120	60	80	160	80
$\|C_{\overline{G}}(\overline{g})\|$	1920	384	192	96	96	32	32	32	16	32	24	12	24
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1
χ_3	4	4	4	2	2	2	2	0	0	0	1	1	1
χ_4	4	4	4	-2	-2	-2	-2	0	0	0	1	1	1
χ_5	5	5	5	1	1	1	1	1	1	1	-1	-1	-1
χ_6	5	5	5	-1	-1	-1	-1	1	1	1	-1	-1	-1
χ_7	6	6	6	0	0	0	0	-2	-2	-2	0	0	0
χ_8	5	-3	1	3	3	-1	-1	1	-1	1	2	0	-2
χ_9	5	-3	1	-3	-3	1	1	1	-1	1	2	0	-2
χ_{10}	10	-6	2	0	0	0	0	2	-2	2	-2	0	2
χ_{11}	15	-9	3	3	3	-1	-1	-1	1	-1	0	0	0
χ_{12}	15	-9	3	-3	-3	1	1	-1	1	-1	0	0	0
χ_{13}	10	2	-2	4	-4	0	0	2	0	-2	1	-1	1
χ_{14}	10	2	-2	-2	2	2	-2	-2	0	2	1	-1	1
χ_{15}	10	2	-2	2	-2	-2	2	-2	0	2	1	-1	1
χ_{16}	10	2	-2	-4	4	0	0	2	0	-2	1	-1	1
χ_{17}	20	4	-4	2	-2	2	-2	0	0	0	-1	1	-1
χ_{18}	20	4	-4	-2	2	-2	2	0	0	0	-1	1	-1

Table 5.4.8.A : **The character table of $2^4 : S_5$.**

classes of G	$(4A)$		$(5A)$	$(6A)$	
classes of $\overline{G}$	(4_5)	(8_1)	(5_1)	(6_3)	(12_1)
h_i	240	240	384	160	160
$\lvert C_{\overline{G}}(\overline{g})\rvert$	8	8	5	12	12
χ_1	1	1	1	1	1
χ_2	-1	-1	1	-1	-1
χ_3	0	0	-1	-1	-1
χ_4	0	0	-1	1	1
χ_5	-1	-1	0	1	1
χ_6	1	1	0	-1	-1
χ_7	0	0	1	0	0
χ_8	1	-1	0	0	0
χ_9	-1	1	0	0	0
χ_{10}	0	0	0	0	0
χ_{11}	-1	1	0	0	0
χ_{12}	1	-1	0	0	0
χ_{13}	0	0	0	1	-1
χ_{14}	0	0	0	1	-1
χ_{15}	0	0	0	-1	1
χ_{16}	0	0	0	-1	1
χ_{17}	0	0	0	-1	1
χ_{18}	0	0	0	1	-1

Table 5.4.8.B : **The character table of $2^4 : S_5$** (continue).

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