MAXIMAL LEFT IDEALS AND IDEALIZERS IN MATRIX RINGS

by

ANDREW FRANSMAN

UNIVERSITY of the
WESTERN CAPE

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PREFACE

The main objective of this work is to give a detailed discussion of a paper by Stone [5]. Numerous examples are provided in order to clarify concepts and results as far as possible.

In Chapter 1 we supply all the basic tools which will be needed later on.

Chapter 2 deals with a characterization of the maximal ideals of $M_n(R)$. Moreover, once we know the maximal ideals of the base ring R, we can exactly tell the form of the maximal ideals of $M_n(R)$. We also provide alternative visualizations of D(A:u) in the $M_n(R)$-module R^n, in R^n/A^n and finally in the module $M_n(R)$.

In Chapter 3 the focus is mainly on idealizers and contractions. We use the concept of the idealizer to find a connection between $M_n(A)$ and D(A:u). We also show that a contraction of any maximal ideal in $M_n(R)$ is maximal in R, provided that R is left quasi-duo.

The emphasis in Chapter 4 is on necessary and sufficient conditions for the equality of maximal ideals D(M:u) and D(M:v). It is most interesting to note importance of the role of the idealizer in this regard.

In Chapter 5 we give discussion of how the property of conjugacy of ideals is propagated in matrix rings.

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ANDREW FRANSMAN
BELLVILLE
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CONTENTS

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CHAPTER 1

PRELIMINARIES

In this chapter we supply all the necessary definitions as well as the required results needed in this work. All the notation and terminology will also be explained carefully.

§1 DEFINITIONS AND NOTATION

R will always denote a ring with identity and $M_n(R)$ will denote the ring of nxn matrices over R. As usual the ring of integers, the ring of integers modulo n and the field of rational numbers will be denoted by Z, Z_n and Q respectively. R[x] will denote the ring of polynomials in the indeterminate x. The constant term of any polynomial $f \in R[x]$ will be denoted by const(f).

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Ideal (or module) will always mean left ideal (or module). In order to simplify notation we shall adopt the convention M,N , M/N, etc. in stead of $_RM$, $_RN$, $_RM/N$, etc., for left R-modules. It will however always be evident from the context, to which ring R we are referring.

Max(R) will denote the collection of all maximal left ideals of R. M and N will be generic symbols for maximal left ideals.

The elements of R^n will be thought of as nx1 columns which are normally written as the transposed of rows; i.e. $u = (u_1, \ldots, u_n)'$. For a matrix X we shall let X_i denote the

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i-th row; whenever needed, X will be denoted by its entries x_{ij}; i.e. $X = [x_{ij}]$. e_{ij} denotes the matrix having 1 in the (i,j)-position and 0 elsewhere. e_i denotes the nx1 column with 1 in the i-th position and 0 elsewhere.

Normally mappings will be written on the left except in the cases of Proposition 1.12 and 1.15. R will be considered as a subring of $M_n(R)$ via the natural embedding $r \to \mathrm{diag}(r,\ldots,r)$.

If a and b are integers, then their greatest common divisor is denoted by (a,b). $a|b$ will mean a divides b or b is multiple of a.

Let C and D be arbitrary categories. Then a covariant functor $F : C \to D$ is a category equivalence in case there is a covariant functor $G : D \to C$ and natural isomorphisms $GF \simeq 1_C$ and $FG \simeq 1_D$. A functor G with this property (also a category equivalence) is called an inverse equivalence of F. Two categories are *equivalent* in case there exists a category equivalence from one to the other. In this case we write $C \approx D$.

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1.1 Definition

Two rings R and S are *Morita equivalent* in case their categories $_RM$ and $_SM$ are equivalent. The equivalence is referred to as a *Morita equivalence*.

1.2 Definition

M_1 is called a *maximal submodule* of M if for every submodule M_2 of M such that $M_1 \subset M_2 \subset M$ it follows that $M_1 = M_2$ or $M_2 = M$.

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For $M \in \text{Max}(R)$, $\text{End}(R/M)$ will denote the ring of all R-endomorphisms of R/M.

1.3 Definition

For a left ideal A of R, $I(A) = \{r \in R: Ar \subset A\}$ is called the *idealizer* of A in R.

1.4 Definition

A ring which is isomorphic to an nxn matrix ring over a division ring is referred to as a *simple artinian ring*.

1.5 Definition

The *center* C of a ring R is defined as the set $C = \{x \in R : xa = ax$ for every $a \in R\}$.

1.6 Definition

A set of elements of a ring which is closed under multiplication of its elements is called a *multiplicative subset* of R.

If A and B are sets, then the relative complement of B in A is denoted by $A - B$. The number of elements of A is denoted by card(A). If R is a commutative ring and $M \in \text{Max}(R)$ we shall let q_M stand for card(R/M). For u, $v \in R^n - M^n$ we write $u \equiv v \pmod{M}$ if and only if $u_i - v_i \in M$ for each $i = 1,\ldots,n$.

1.7 Definition

A ring R is called *semi-local* if it has a finite number of maximal ideals.

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$GL_n(R)$ will denote the set of nxn invertible matrices with entries from R.

The phrases *for each*, *for all* and *for every* will all have the same interpretation. The symbol □ will be used to indicate the end of a proof or well-known result.

§2 RESULTS NEEDED

The following results are well-known and their proofs can be found in many standard text-books; e.g. [1] and [2].

1.8 Proposition

If $M \in Max(R)$, *then* $(R/M)^n$ *is a simple* $M_n(R)$*-module.* □

1.9 Proposition

A left R*-module* T *is simple if and only if* $T \simeq R/M$ *for some maximal left ideal* M *of* R. □

1.10 Proposition

If M *is a maximal submodule of* R *and if* $x \in R-M$, *then* $M+Rx=R$. □

1.11 Proposition

If $M \in Max(R)$, *then* End(R/M) *is a division ring.* □

1.12 Proposition

If $f, g \in End(R/M)$, *then* $f+g \in End(R/M)$ *and* $fg \in End(R/M)$ *where addition and multiplication is defined by*

$(r+M)(f+g) = (r+M)f + (r+M)g$ *and*

$(r+M)fg = ((r+M)f)g.$ □

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1.13 Proposition

Let R *be a commutative ring,* S *a non-empty multiplicative subset of* R *with* $0 \notin S$ *and let* T *be the set of non-zero divisors of* R. *If* S *is a subset of* T, *then we can construct fractions* r/s *with denominators in* S *as follows. We define a relation on the product set* RXS *by setting* $(r,s) \sim (r',s')$ *if and only if* $rs' = sr'$. *Then*

1.13.1 $\sim$ *defines an equivalence relation on* RXS;

1.13.2 *if we write* r/s *for the equivalence class containing* (r,s) *and if we define addition and multiplication by the rules* $r/s + r'/s' = (rs' + sr')/ss'$ *and* $(r/s).(r'/s') = rr'/ss'$, *then the set* $S^{-1}R$ *of equivalence classes forms a ring, called the ring of fractions with denominators in S, under these operations;*

1.13.3 R *can be considered as a subring of* $S^{-1}R$. □

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1.14 Proposition

For all $m,n \geq 1$, $M_m(M_n(R)) \simeq M_{mn}(R)$. □

The next result is due to Fitting [3] and was also proved by Goldie [4].

1.15 Proposition

If $M \in Max(R)$, *then* $I(M)/M \simeq End(R/M)$.

Proof

Our aim is to define a ring isomorphism from I(M)/M onto End(R/M) by associating an element of I(M)/M with an R-endomorphism of R/M. This is achieved as follows. For

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$x+M \in I(M)/M$ let f be defined by the rule $f : x+M \to g_x$, where $g_x : r+M \to rx+M$ for any $r+M \in R/M$. We claim that $g_x \in End(R/M)$ and that f is the required ring isomorphism.

$g_x \in End(R/M)$. g_x is well-defined, for if $r+M = r'+M$, then $r-r' \in M$. But $x \in I(M)$ and so $(r-r')x \in M$; i.e. $rx-r'x \in M$. Hence $rx+M=r'x+M$ and so $(r+M)g_x = (r'+M)g_x$. g_x is an R-endomorphism, for if $r+M$, $r'+M \in R/M$ and $a \in R$, then $((r+M)+(r'+M))g_x = ((r+r')+M)g_x = (r+r')x+M = (rx+r'x)+M = (rx+M) + (r'x+M) = (r+M)g_x + (r'+M)g_x$ and $(a(r+M))g_x = (ar+M)g_x = (ar)x+M = a(rx)+M = a(rx+M) = a((r+M)g_x)$. Thus $g \in End(R/M)$, as required.

f is a ring isomorphism. f is well-defined, for suppose that $x+M = y+M$ where $x,y \in I(M)$. Then $x = y+m$ for some $m \in M$. Let $r \in R$. Then $(r+M)g_x = rx+M = ry+rm+M = ry+M = (r+M)g_y$ and hence $g_x = g_y$. Thus $(x+M)f = g_x = g_y = (y+M)f$. f is a ring homomorphism. Let $x+M$, $y+M \in I(M)/M$ and let g_x and g_y be the corresponding R-endomorphisms. Then by Proposition 1.12 g_x+g_y and g_xg_y are R-endomorphisms. Now let $r+M \in R/M$. Then $(r+M)(g_x+g_y) = (r+M)g_x + (r+M)g_y = (rx+M) + (ry+M) = (rx+ry)+M = r(x+y)+M = (r+M)g_{x+y}$ and $(r+M)g_xg_y = ((r+M)g_x)g_y = (rx+M)g_y = (rx)y+M = r(xy)+M = (r+M)g_{xy}$. Therefore $g_x+g_y = g_{x+y}$ and $g_xg_y = g_{xy}$. But then it follows that $((x+M) + (y+M))f = ((x+y)+M)f = g_{x+y} = g_x + g_y = (x+M)f + (y+M)f$ and $((x+M)(y+M))f = (xy+M)f = g_{xy} = g_xg_y = (x+M)f(y+M)f$. Thus we have established that f is indeed an R-homomorphism. f is one-to-one, for if $(x+M)f = (y+M)f$, then $g_x = g_y$. Therefore $x+M = (1+M)g_x = (1+M)g_y = y+M$. Finally we see that f is onto, for given any $g \in End(R/M)$ such that $g : 1+M \to x+M$ for some $x \in I(M)$. Then g is the required image of x+M under f. Hence f is a ring isomorphism. Thus $I(M)/M \simeq End(R/M)$. □

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The remaining three results will be useful in the construction of examples.

1.16 Proposition

Let R = Z[x], n *a positive integer and* p *a prime number. Then*

1.16.1 A = {f ∈ R : const(f) ∈ nZ} *is an ideal of* R;

1.16.2 M = {f ∈ R : const(f) ∈ pZ} *is a maximal ideal of* R.

Proof

1.16.1 A is non-empty, since the zero polynomial lies in A. Let f, g ∈ A and let const(f) = an and const(g) = bn. Then const(f−g) = const(f) − const(g) = an−bn = (a−b)n ∈ nZ and hence f−g ∈ A. Let f ∈ R, g ∈ A with const(f) = c and const(g) = an. Then const(fg) = const(f).const(g) = can ∈ nZ. Therefore fg ∈ A and hence A is an ideal of R.

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1.16.2 Let N be an ideal of R such that $M \subsetneq N$. Then there exists a polynomial g ∈ N such that g ∉ M. Put $g = b + \sum_{i=1}^{n} a_i x^i$ and let $f = a + \sum_{i=1}^{n} a_i x^i \in M$. Then (a,b) = 1 and so there exist integers r and s such that rb+sa=1. Now $rg+sf = rb + \sum_{i=1}^{n} ra_i x^i + sa + \sum_{i=1}^{n} sa_i x^i = rb+sa + \sum_{i=1}^{n} (r+s)a_i x^i = 1 + \sum_{i=1}^{n} (r+s)a_i x^i$. But we also have that $x^i \in M$ for all i > 0, because p ∈ M and $-p + x^i \in M$ imply that $p-p+x^i \in M$; i.e. $x^i \in M$. Thus $\sum_{i=1}^{n} (r+s)a_i x^i \in N$. However, since f,g ∈ N, it follows that rg+sf ∈ N and hence 1 ∈ N. Therefore N=R and so M is a maximal ideal of R. □

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In the proof of the next result we use similar arguments than those in the previous one. However, the main reason for its inclusion is that it is a non-commutative ring and as such it provides us with a large collection of maximal left ideals which will turn out to be rather useful later on.

1.17 Proposition

Let R = $M_2(Z)[x]$, n *a positive integer and* p *a prime number. Then*

1.17.1 $A = \left\{f \in R : \text{const}(f) \in \begin{bmatrix} nZ & Z \\ nZ & Z \end{bmatrix}\right\}$ *is an ideal of* R;

1.17.2 $M = \left\{f \in R : \text{const}(f) \in \begin{bmatrix} pZ & Z \\ pZ & Z \end{bmatrix}\right\}$ *is a maximal ideal of* R;

1.17.3 $I(M) = \left\{g \in R : \text{const}(g) \in \begin{bmatrix} Z & Z \\ pZ & Z \end{bmatrix}\right\}$.

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Proof

1.17.1 A is non-empty since the zero polynomial lies in it.

Let $f,g \in A$. Put $\text{const}(f) = \begin{bmatrix} na & c \\ nb & d \end{bmatrix}$ and $\text{const}(g) = \begin{bmatrix} na' & c' \\ nb' & d' \end{bmatrix}$.

Then $\text{const}(f-g) = \text{const}(f) - \text{const}(g) = \begin{bmatrix} n(a-a') & c-c' \\ n(b-b') & d-d' \end{bmatrix} \in \begin{bmatrix} nZ & Z \\ nZ & Z \end{bmatrix}$. Thus $f-g \in A$. Let $f \in R$, $g \in A$ and suppose that $\text{const}(f) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $\text{const}(g) = \begin{bmatrix} na' & c' \\ nb' & d' \end{bmatrix}$. Then $\text{const}(fg) = \text{const}(f).\text{const}(g) = \begin{bmatrix} n(aa'+cb') & ac'+cd' \\ n(ba'+db') & bc'+dd' \end{bmatrix} \in \begin{bmatrix} nZ & Z \\ nZ & Z \end{bmatrix}$. Hence $fg \in A$. Therefore A is an ideal of R.

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1.17.2 Let N be an ideal of R such that $M \subsetneq N$. Then there exists a polynomial $g \in N-M$, say $g = \begin{bmatrix} a & c \\ b & d \end{bmatrix} + \sum_{i=1}^{n} a_i x^i$. So at least one of a or b is not a multiple of p; suppose it is a. Then there exist integers r and s such that $ra + sp = 1$. Now since $h = \begin{bmatrix} 0 & c \\ 0 & d-r \end{bmatrix} \in M$, it follows that $g' = g - h \in N$; i.e.

$g' = \begin{bmatrix} a & 0 \\ b & r \end{bmatrix} + \sum_{i=1}^{n} a_i x^i \in N-M$. Since $f = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} + \sum_{i=1}^{n} a_i x^i \in M$,

it follows that $\begin{bmatrix} r & 0 \\ -b & a \end{bmatrix} g' + \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} f$

$$= \begin{bmatrix} r & 0 \\ -b & a \end{bmatrix}\begin{bmatrix} a & 0 \\ b & r \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} r & 0 \\ -b & a \end{bmatrix} a_i x^i + \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} a_i x^i$$

$$= \begin{bmatrix} ra & 0 \\ 0 & ra \end{bmatrix} + \begin{bmatrix} sp & 0 \\ 0 & sp \end{bmatrix} + \sum_{i=1}^{n} \begin{bmatrix} r+s & 0 \\ -b & a+s \end{bmatrix} a_i x^i$$

$$= \begin{bmatrix} ra+sp & 0 \\ 0 & ra+sp \end{bmatrix} + k, \text{ say}$$

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$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k$. Now for each $i > 0$ it follows that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x^i \in M$,

since for example $\begin{bmatrix} p & 0 \\ p & 1 \end{bmatrix} \in M$ and $\begin{bmatrix} -p & 0 \\ -p & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x^i \in M$ and

hence their sum, which equals $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x^i \in M$. Thus $k \in N$.

However, since both $f, g' \in N$, it follows that $\begin{bmatrix} r & 0 \\ -b & a \end{bmatrix} g' + \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} f \in N$.

Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in N$ and so $N=R$. Thus M is a maximal ideal R.

The other case is proven similarly.

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1.17.3 Let $g \in R$ and let f be any polynomial of M. Suppose that $\text{const}(g) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $\text{const}(f) = \begin{bmatrix} pa' & c' \\ pb' & d' \end{bmatrix}$. Then

$fg \in M$ if and only if $\text{const}(fg) = \begin{bmatrix} pa' & c' \\ pb' & d' \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} =$

$\begin{bmatrix} paa'+c'b & pa'c+c'd \\ pb'a+d'b & pb'c+d'd \end{bmatrix} \in \begin{bmatrix} pZ & Z \\ pZ & Z \end{bmatrix}$. Hence $paa'+c'b,\ pb'a+d'b \in pZ$ for all $a', c', b', d' \in Z$. Thus $b \in pZ$ and so $I(M) = \left\{g \in R : \text{const}(g) \in \begin{bmatrix} Z & Z \\ pZ & Z \end{bmatrix}\right\}$. □

1.18 Proposition

Let p be a prime number and let $A = \{(a,b) \in Z^2 : p \mid a+b\}$,

$B = \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_2(Z) : p \mid a(x+y)+b(z+w), \text{ where } (a,b) \in A\right\}$ *and*

$C = \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_2(Z) : p \mid (x+y)-(z+w)\right\}$. *Then*

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1.18.1 B=C;

1.18.2 B *and* C *are subrings of* $M_2(Z)$.

Proof

1.18.1 Let $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in B$. Then $p \mid a(x+y)+b(z+w)$ for all $(a,b) \in A$. In particular, if we choose $a=1$ and $b=-1$, then $a+b=0$, which is certianly divisible by p; i.e. $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in C$ and hence $B \subset C$.

For the converse we let $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in C$ be an arbitrary element.

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Let $(a,b) \in A$. Then there exists an integer k such that $a=kp-b$. Hence $a(x+y)+b(z+w) = kp(x+y)-b(x+y)+b(z+w)=kp(x+y)-b((x+y)-(z+w))$. But by hypothesis $p|(x+y)-(z+w)$ and hence $p|kp(x+y)-b((x+y)-(z+w))$; i.e. $p|a(x+y)+b(z+w)$. Thus $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in B$ and so $C \subset B$. Hence $B=C$.

1.18.2 Since we have just proved that B=C, it suffices to show the subring condition for one of B or C only, say for C. C is non-empty, because $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in C$. Let $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and $Y = \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}$ be elements of C. Then $X-Y = \begin{bmatrix} x-x' & y-y' \\ z-z' & w-w' \end{bmatrix}$. Now $(x-x')+(y-y')-((z-z')+(w-w'))=x+y-(x'+y')-(z+w)+z'+w'=((x+y)-(z+w))-((x'+y')-(z'+w'))$ and since $X,Y \in C$, it follows that p divides the above difference; i.e. $X-Y \in C$. Next we see that $XY = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix} = \begin{bmatrix} xx'+yz' & xy'+yw' \\ zx'+wz' & zy'+ww' \end{bmatrix}$. Now we have that $(xx'+yz')+(xy'+yw')-(zx'+wz'+zy'+ww')=(x-z)x'+(x-z)y'+y(z'+w')-w(z'+w')=(x-z)(x'+y')+(y-w)(z'+w')$. However, since $X,Y \in C$, it follows that $x+y=kp+z+w$ and $x'+y'=k'p+z'+w'$. Hence $(x-z)(x'+y')+(y-w)(z'+w')=(kp-y+w)(k'p+z'+w')+(y-w)(z'+w')$ $=kpk'p+k'p(w-y)$, which is certainly divisible by p; i.e. $XY \in C$. Therefore C is a subring, as required. □

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CHAPTER 2

THE MAXIMAL LEFT IDEALS OF $M_n(R)$

In this chapter we give a characterization of the maximal ideals of $M_n(R)$. In fact, the main result (Proposition 2.7) tells us exactly how to find all the maximal ideals of $M_n(R)$ once the maximal ideals of R are known. We also provide alternative visualizations of D(A:u) in the $M_n(R)$-module R^n, in R^n/A^n and finally in the module $M_n(R)$.

§3 A CHARACTERIZATION OF THE MAXIMAL LEFT IDEALS OF $M_n(R)$

Let A be a left ideal of R, let $u=(u_1,\ldots,u_n)' \in R^n$ and consider the $M_n(R)$-linear maps

$$M_n(R) \overset{f}{\to} R^n \overset{g}{\to} (R/A)^n \simeq R^n/A^n,$$

defined for $X \in M_n(R)$, $v = (v_1,\ldots,v_n)' \in R^n$ by $f(X)=Xu$ and $g(v)=(v_1+A,\ldots, v_n+A)'$; i.e. g is the natural surjection mod A. Let $X \in \ker(gof)$. Then $(gof)(X) = (A,\ldots,A)'$. Thus $g(Xu)= (X_1u+A, \ldots, X_nu+A)' = (A,\ldots,A)'$ and hence $X_iu+A=A$ for each $i=1,\ldots,n$; i.e. $X_iu \in A$ for each $i=1,\ldots,n$. But then we also have that $Xu = (X_1,\ldots, X_i,\ldots, X_n)u = (X_1u,\ldots, X_iu,\ldots,X_nu)' \in A^n$. Thus

$$\ker(gof) = \{X \in M_n(R) : X_iu \in A,\ i=1,\ldots,n\}$$
$$= \{X \in M_n(R) : Xu \in A^n\}.$$

We adopt the notation $D(A:u) = \ker(gof)$.

2.1 Proposition

D(A:u) *is a proper left ideal of* $M_n(R)$ *for any* $u \in R^n-A^n$.

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Proof

Consider any $X,Y \in D(A:u)$. Then $(X-Y)u = Xu-Yu \in A^n$ and so we have that $X-Y \in D(A:u)$. Since A is a left ideal of R, it follows that $X^*u^* \in A^n$ for any $X^* \in M_n(R)$ and $u^* \in A^n$. So suppose that $X \in M_n(R)$ and $Y \in D(A:u)$. Then $Yu \in A^n$ and hence by the above observation we have that $(XY)u = X(Yu) \in A^n$. Therefore it follows that $XY \in D(A:u)$ and hence $D(A:u)$ is a left ideal of $M_n(R)$. Suppose next that $u \notin A^n$. Then there exists $u_i \in R$ such that $u_i \notin A$. Let X be the matrix having the entry 1 in the (1,i) position and zero's elsewhere. Then

$$Xu = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} (u_1,\ldots, u_i,\ldots, u_n)' = (0,\ldots,u_i,\ldots,0)' \notin A^n.$$

Therefore $X \notin D(A:u)$ and so we indeed have that $D(A:u)$ is a proper ideal of $M_n(R)$. □

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2.2 Proposition

If $M \in \text{Max}(R)$ *and if* $u \in R^n - M^n$*, then the following hold.*

2.2.1 gof, *as defined above, is onto;*

2.2.2 $M_n(R)/D(M;u) \simeq (R/M)^n$;

2.2.3 $D(M:u)$ *is a maximal left ideal of* $M_n(R)$.

Proof

2.2.1 Let $(u_1+A,\ldots, u_n+A)' \in (R/A)^n$. Put $u=(u_1,\ldots,u_n)'$ and let $X=I_n$, the nxn identity matrix of $M_n(R)$. Then $(gof)(X) = g(Xu) = g((u_1,\ldots,u_n)') = (u_1+A,\ldots, u_n+A)'$ and hence it follows that gof is onto.

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2.2.2 Since $(R/M)^n \simeq R^n/M^n$, we aim to prove that $M_n(R)/D(M:u) \simeq R^n/M^n$. Define a map $f : M_n(R)/D(M:u) \to R^n/M^n$ by the rule $f : X+D(M:u) \to Xu+M^n$. f is well-defined, for if $x+D(M:u) = Y+D(M:u)$, then $X-Y \in D(M:u)$ and hence we have that $(X-Y)u \in M^n$; i.e. $Xu-Yu \in M^n$. Therefore $Xu+M^n = Yu+M^n$ and so $f(X+D(M:u)) = f(Y+D(M:u))$. f is an $M_n(R)$-linear map. Let $X+D(M:u)$, $Y+D(Mu) \in M_n(R)/D(M:u)$. Then $f((X+D(M:u)) + (Y+D(M:u))) = f((X+Y) + D(M:u)) = (X+Y)u+M^n = (Xu+Yu) + M^n = (Xu+M^n) + (Yu+M^n) = f(X+D(M:u)) + f(Y+D(M:u))$. Let $Y \in M_n(R)$ and $X+D(M:u) \in M_n(R)/D(M:u)$. Then $f(Y(X+D(M:u))) = f(YX+D(M:u)) = (YX)u+M^n = Y(Xu)+M^n = Y(Xu+M^n) = Yf(X+D(M:u))$. f is one-to-one. Suppose that $X+D(M:u) \in \ker f$. Then $Xu \in M^n$ and so we have that $X \in D(M:u)$. Therefore it follows that $X+D(M:u) = D(M:u)$, the zero submodule of $M_n(R)/D(M:u)$ and so f is one-to-one. f is onto. Since R^n/M^n is a simple $M_n(R)$-module, it follows by Proposition 1.9 that M^n is a maximal submodule of R^n. By hypothesis $u \in R^n-M^n$ and hence by Proposition 1.10 we have that $M_n(R)u + M^n = R^n$. So let $v+M^n \in R^n/M^n$ be given. Then there exists a matrix $X_v \in M_n(R)$ such that $X_vu+w=v$, for some $w \in M^n$. Hence $f(X_v+D(M:u)) = X_vu+M^n = v-w+M^n = v+M^n$. Thus f is onto. Hence we conclude that $M_n(R)/D(M:u) \simeq R^n/M^n$.

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2.2.3 By 2.2.2 above $M_n(R)/D(M:u) \simeq R^n/M^n$. However $R^n/M^n \simeq (R/M)^n$ and hence $M_n(R)/D(M:u) \simeq (R/M)^n$. Since $(R/M)^n$ is a simple $M_n(R)$-module, it follows that $M_n(R)/D(M:u)$ is also simple and so by Proposition 1.9 $D(M:u)$ is a maximal left ideal of $M_n(R)$. □

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2.3 Example

Let $X = \begin{bmatrix} x_{11} & \cdots & x_{1i} & \cdots & x_{1n} \\ \vdots & & \vdots & & \vdots \\ x_{n1} & \cdots & x_{ni} & \cdots & x_{nn} \end{bmatrix} \in D(O:e_i)$. Then we have that

$Xe_i = (x_{1i},\ldots,x_{ni})' = (0,\ldots,0)'$ and hence $x_{1i} = \ldots = x_{ni} = 0$.

Thus $D(O:e_i) = \begin{bmatrix} R & \cdots & 0 & \cdots & R \\ \vdots & & \vdots & & \vdots \\ R & \cdots & 0 & \cdots & R \end{bmatrix}$, where the zero's appear in

the i-th column. Next we assert $D(O:e_i)$ is a maximal left ideal of $M_n(R)$ if and only if R is a division ring. Suppose that R is a division ring. Then O and R are the only left ideals of R and hence $D(O;e_i)$ is indeed a maximal left ideal of $M_n(R)$. For the converse we suppose that $D(O:e_i)$ is a maximal left ifeal of $M_n(R)$. Let $x \in R$ such that $x \neq 0$. Then, since $1 \in R$, we have that Rx is a left ideal of R such that $Rx \neq 0$.

Hence $D(O:e_i) = \begin{bmatrix} R & \cdots & 0 & \cdots & R \\ \vdots & & \vdots & & \vdots \\ R & \cdots & 0 & \cdots & R \end{bmatrix} \subsetneqq \begin{bmatrix} R & \cdots & Rx & \cdots & R \\ \vdots & & \vdots & & \vdots \\ R & \cdots & Rx & \cdots & R \end{bmatrix}$. But since

$D(O:e_i)$ is maximal we conclude that $Rx=R$. Hence there exists $x' \in R$, $x' \neq 0$, such that $x'x=1$. Similarly as above,it can be proved that $Rx'=R$. So there exists $x'' \in R$, $x'' \neq 0$, such that $x''x'=1$. However $x''=x''1 = x''(x'x) = (x''x')x = 1x = x$ and so x' is the multiplicative inverse of x. Therefore R is a division ring.

2.4 Example

Let K be any field, M=O and $u=(1,-1)'$. Consider any

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$X = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in D(0:u)$. Then it follows that $\begin{bmatrix} a & c \\ b & d \end{bmatrix}(1,-1)' = \begin{bmatrix} a-c \\ b-d \end{bmatrix} = (0,0)'$. Thus a=c and b=d. Therefore D(0:u)

$$= \left\{\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in M_2(K) \ : \ a-c=0 \text{ and } b-d=0\right\}$$

$$= \left\{\begin{bmatrix} a & a \\ b & b \end{bmatrix} \in M_2(K) \ : \ a,b \in K\right\}.$$

Indeed, D(0:u) is a maximal left ideal of $M_2(K)$. For suppose that $D(0:u) \subsetneqq N$, for some left ideal N of $M_2(K)$. Then there exists an element $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in N - D(0:u)$. Therefore $a-c \neq 0$ or $b-d \neq 0$. Suppose that $a-c \neq 0$ and $b-d=0$. In this case $\begin{bmatrix} a & c \\ b & b \end{bmatrix} \in N$ and it is also clear that $a \neq 0$ or $c \neq 0$, say $a \neq 0$. Since $\begin{bmatrix} 0 & 0 \\ b & b \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix}$ and $\begin{bmatrix} c & c \\ 0 & 0 \end{bmatrix}$ are elements of D(0:u), they also lie in N. Therefore $\begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & c \\ b & b \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ b & b \end{bmatrix} \in N$. Hence $\begin{bmatrix} a & c \\ a & a \end{bmatrix} = \begin{bmatrix} a & c \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a & a \end{bmatrix} \in N$ and so $\begin{bmatrix} a-c & 0 \\ a & a \end{bmatrix} = \begin{bmatrix} a & c \\ a & a \end{bmatrix} - \begin{bmatrix} c & c \\ 0 & 0 \end{bmatrix} \in N$. However, since $(a-c)^{-1}$ and a^{-1} exist, it follows that $\begin{bmatrix} (a-c)^{-1} & 0 \\ -(a-c)^{-1} & a^{-1} \end{bmatrix}\begin{bmatrix} a-c & 0 \\ a & a \end{bmatrix} \in N$; i.e. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in N$. Thus $N = M_2(K)$ and hence D(0:u) is a maximal ideal of $M_2(K)$. The other cases are proven similarly.

2.5 Example

Let $R = Z_{15}$, $M = \bar{5}Z_{15}$ and $u = (\bar{0},\bar{1})'$. Then $D(M:u) = \begin{bmatrix} Z_{15} & M \\ Z_{15} & M \end{bmatrix}$, which is certainly a maximal left ideal of $M_2(R)$.

2.6 Example

In R=Z let M=pZ, where p is a prime number, and let u=(1,-1)'.

Consider any $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D(pZ:u)$. Then $Xu = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (1,-1)' =$

$\begin{bmatrix} a-b \\ c-d \end{bmatrix} \in (pZ)^2$. Thus a-b, c-d $\in$ pZ; i.e. $a \equiv b(\text{mod}p)$ and $c \equiv d(\text{mod}p)$.

Hence $D(pZ:u) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z) : a \equiv b(\text{mod}p) \text{ and } c \equiv d(\text{mod}p) \right\}$.

Moreover, D(pZ:u) is a maximal left ideal of $M_2(Z)$. For let A be a left ideal of $M_2(Z)$ such that $D(pZ:u) \subsetneq A$ and suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in A - D(pZ:u)$. Then $a \not\equiv b(\text{mod}p)$ or $c \not\equiv d(\text{mod}p)$; i.e. $p \nmid a-b$ or $p \nmid c-d$, say $p \nmid c-d$. Then there are integers r and s such that r(c-d)+sp = 1. Now $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$

$$\begin{bmatrix} r(c-d)+sp & 0 \\ 0 & r(c-d)+sp \end{bmatrix} = \begin{bmatrix} r(c-d) & 0 \\ 0 & r(c-d) \end{bmatrix} + \begin{bmatrix} sp & 0 \\ 0 & sp \end{bmatrix} =$$

$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} c-d & 0 \\ 0 & c-d \end{bmatrix} + \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$. Our next aim is to show that the above sum is an element of A. This can be seen as follows. $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \in D(pZ:u)$ and so we have that

$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \in A$, since D(pZ:u) is contained in A. On the other hand we see that $\begin{bmatrix} c-d & 0 \\ 0 & c-d \end{bmatrix}$ can be expressed as follows,

$$\begin{bmatrix} c-d & 0 \\ 0 & c-d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -d & -d \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -c & -d \end{bmatrix}.$$

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Now $\begin{bmatrix} -d & -d \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ c & c \end{bmatrix} \in D(pZ:u)$ and since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in A$ we also have that $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in A$ and $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in A$; i.e. $\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \in A$ and $\begin{bmatrix} 0 & 0 \\ -c & -d \end{bmatrix} \in A$. Hence $\begin{bmatrix} c-d & 0 \\ 0 & c-d \end{bmatrix} \in A$ and so $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} c-d & 0 \\ 0 & c-d \end{bmatrix} \in A$. Therefore we have that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in A$ and so $A=M_2(Z)$. This means that $D(pZ:u)$ is indeed a maximal ideal of $M_2(Z)$. The other case is proved in a similar manner.

2.7 Proposition

The collection of $D(M:u)$, *for* $M \in Max(R)$ *and* $u \in R^n - M^n$, *gives all the maximal left ideals of* $M_n(R)$.

Proof

From 2.2.3 we have seen that for $M \in Max(R)$ and $u \in R^n-M^n$, $D(M:u)$ is a maximal left ideal of $M_n(R)$. We shall therefore only show that every maximal ideal of $M_n(R)$ has this form. So let M' be such an ideal of $M_n(R)$. Then by Proposition 1.9 $M_n(R)/M'$ is a simple $M_n(R)$-module. By the Morita-equivalence between R and $M_n(R)$, it follows that $M_n(R)/M' \simeq E^n$, where E is a simple left R-module. Thus, again by Proposition 1.9, it follows that $E \simeq R/M$ for some $M \in Max(R)$. We therefore have an isomorphism f from $M_n(R)/M'$ to R^n/M^n built up as follows:

$$M_n(R)/M' \to E^n \to (R/M)^n \to R^n/M^n.$$

Suppose $f(1+M') = u+M^n$. Then we assert that $M' = D(M:u)$. Indeed, if $X \in D(M:u)$, then $Xu \in M^n$ and therefore $f(X+M') = Xu+M^n = M^n$. But since f is an isomorphism, it follows that $X+M' = M'$; i.e. $X \in M'$. Thus $D(M:u) \subset M'$. Since $D(M:u)$ is

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maximal as well, equality follows. □

It may happen that $D(A:u)$ is maximal even though A is not maximal in R. The following example illustrates this point.

2.8 Example

Let $R=Z$, $A=4Z$ and $u=(2,0) \notin A^2$. Then A is not maximal in Z. However

$$D(A:u) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z) : \begin{bmatrix} a & b \\ c & d \end{bmatrix}(2,0)' \in A^2\right\}$$

$$= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z) : 2a \in 4Z \text{ and } 2c \in 4Z\right\}$$

$$= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z) : a \in 2Z \text{ and } c \in 2Z\right\}$$

$$= \begin{bmatrix} 2Z & Z \\ 2Z & Z \end{bmatrix},$$ which is indeed a maximal ideal of $M_2(Z)$.

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§4 ALTERNATIVE VISUALIZATIONS OF D(A:u)

In order to construct alternative visualizations of $D(A:u)$ in any $M_n(R)$, we make use of the following two results.

2.9 Proposition

Let F *be a submodule of the left* R*-module* E *and for* $x \in E$ *let* $(F:x) = \{r \in R : rx \in F\}$. *Then*

2.9.1 $(F:x)$ *is a left ideal of* R;

2.9.2 $(F:x)$ *is proper if and only if* $x \notin F$.

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Proof

2.9.1 Let $r, r' \in (F:x)$. Then $(r-r')x = rx - r'x \in F$ and so $r-r' \in (F:x)$. Let $r \in R$ and $a \in (F:x)$. Then $(ra)x = r(ax) = rb$ for some $b \in F$. But F is an R-module and so $rb \in F$. Thus $ra \in (F:x)$ and hence $(F:x)$ is a left ideal of R.

2.9.2 Suppose that $(F:x)$ is a proper ideal of R. Then $1 \notin (F:x)$ and so $1x = x \notin F$. Conversely, if $x \notin F$ then $1x \notin F$ and hence $1 \notin (F:x)$. Thus $(F:x)$ is proper. □

2.10 Proposition

If F *is a maximal submodule of the* R*-module* E *and if* $x \in E-F$, *then*

2.10.1 $(F:x) \in \mathrm{Max}(R)$;

2.10.2 $R/(F:x) \simeq E/F$.

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Proof

2.10.1 By the previous result $(F:x)$ is a left ideal of R. Suppose that I is an ideal of R such that $(F:x) \subsetneq I$, where $x \in E-F$. Then there exists $r \in I$ such that $rx \notin F$. Since F is a maximal submodule of E it follows by Proposition 1.10 that

$$F + Rx = E \qquad \text{..... (i)}$$

But then there exists $a \in F$ such that $a+rx=x$. Thus $(1-r)x = x-rx = a \in F$. Therefore $1-r \in (F:x) \subsetneq I$. So $1-r=r' \in I$ and hence $1=r+r' \in I$. Thus $I=R$, which proves the maximality of $(F:x)$.

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2.10.2 Define a map $f : R/(F:x) \rightarrow E/F$ by the rule $f : r+(F:x) \rightarrow rx+F$. f is well-defined, for if $r+(F:x) = r'+(F:x)$, then $r-r' \in (F:x)$. Hence $(r-r')x = rx-r'x \in F$. Therefore $rx+F = r'x+F$; i.e. $f(r+(F:x)) = f(r'+(F:x))$. f is an R-linear map. Given any $r+(F:x)$, $r+(F:x) \in R/(F:x)$. Then $f((r+(F:x)) + (r'+(F:x))) = f((r+r')+(F:x)) = (r+r')x+F = (rx+r'x)+F = (rx+F)+(r'x+F) = f(r+(F:x))+f(r'+(F:x))$. Also if $r \in R$ and $r'+(F:x) \in R/(F:x)$, then $f(r(r'+(F:x))) = f(rr:+(F:x)) = (rr')x+F = r(r'x+F) = rf(r'+(F:x))$. f is one-to-one. Let $a = r+(F:x) \in \ker f$. Then $f(a) = F$. Thus $rx+F = F$ and so $rx \in F$. But then it follows that $r \in (F:x)$ and hence $a = (F:x)$, the zero of $R/(F:x)$. Thus $\ker f = 0$ and so f is one-to-one. f is onto, for suppose that $y+F \in E/F$. Then by (i) above $y=b+rx$ for some $b \in F$, $r \in R$. Thus $f(r+(F:x)) = rx+F = y-b+F = y+F$. Therefore the map f defined above is an R-isomorphism; i.e. $R/(F:x) \simeq E/F$. □

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2.11 Corollary

If $M \in \mathrm{Max}(R)$ *and if* $u \in R-M$, *then in* $M_1(R)=R$ *we have* $D(M:u) = (M:u)$.

Proof

Let $X \in (M:u)$. Then $X \in R = M_1(R)$ such that $Xu \in M = M^1$. Thus $X \in D(M:u)$ and so $(M:u) \subset D(M:u)$. Since both $(M:u)$ and $D(M:u)$ are maximal ideals, equality follows. □

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2.12 Example

In Proposition 1.17 choose n=4 and p=2. Thus

$A = \left\{f \in R : \text{const}(f) \in \begin{bmatrix} 4Z & Z \\ 4Z & Z \end{bmatrix}\right\}$ and

$M = \left\{f \in R : \text{const}(f) \in \begin{bmatrix} 2Z & Z \\ 2Z & Z \end{bmatrix}\right\}$ where $R = M_2(Z)[x]$. Now

since M is a maximal ideal of R and since $A \subset M$, it follows that F = M/A is a maximal R-submodule of E = R/A. Let $f^* = f+A$, where $\text{const}(f) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. We assert that $(F:f^*) = \left\{r \in R : \text{const}(r) \in \begin{bmatrix} Z & 2Z \\ Z & 2Z \end{bmatrix}\right\}$. Let therefore $r \in (F:f^*)$ and assume that $\text{const}(r) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $r(f+A) \in F$ and hence $rf+A \in F = M/A$; i.e. $rf \in M$. But then it follows that $\text{const}(rf) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in \begin{bmatrix} 2Z & Z \\ 2Z & Z \end{bmatrix}$; i.e. $\begin{bmatrix} b & a+b \\ d & c+d \end{bmatrix} \in \begin{bmatrix} 2Z & Z \\ 2Z & Z \end{bmatrix}$. Therefore $b,d \in 2Z$ and hence $\text{const}(r) \in \begin{bmatrix} Z & 2Z \\ Z & 2Z \end{bmatrix}$ and the assertion follows. The proof that $(F:f^*)$ is a maximal ideal of R proceeds along the same lines as the one in Proposition 1.17 and is therefore omitted.

2.13 Example

Let E=Z/6Z and let F=3Z/6Z be Z-modules. Let $x=5+6Z \notin F$. Then F is a maximal submodule of E. Moreover, (F:x)=3Z, for if $r \in (F:x)$, then $r(5+6Z) \in 3Z/6Z$. Therefore $5r+6Z \in 3Z/6Z$. Hence $5r \in 3Z$ and so $r \in 3Z$. Thus $(F:x) \subset 3Z$. But by 2.10.1 (F:x) is a maximal ideal of Z and so (F:x) = 3Z. Indeed, $Z/(F:x) = Z/3Z \simeq (Z/6Z)/(3Z/6Z) = E/F$.

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2.14 Example

In Example 2.6 we have seen that $M = D(pZ:(1,-1)') =$ $\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z) \ : \ a \equiv b (\text{mod} p) \text{ and } c \equiv d (\text{mod} p)\right\}$ is a maximal left ideal of $R = M_2(Z)$. Now let $u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R-M$. Consider any $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in (M:u)$. Then $Xu \in M$. Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} =$ $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \in M$; i.e. $p|a$ and $p|c$. Hence $X \in \begin{bmatrix} pZ & Z \\ pZ & Z \end{bmatrix}$ and so $(M:u) = \begin{bmatrix} pZ & Z \\ pZ & Z \end{bmatrix} = D(M:u)$ in $M_1(R)$.

In view of the preceding discussion we are now able to give three alternative visualizations of $D(A:u)$ in any $M_n(R)$, for any left ideal A of R and $u \in R^n$.

2.15 Proposition

$D(A:u) = (A^n:u)$ *computed in the* $M_n(R)$*-module* R^n.

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Proof

For $X \in D(A:u)$ it follows that $Xu \in A^n$. So regarding F as being A^n and E as being the $M_n(R)$-module R^n, we indeed have that $X \in (A^n:u)$. Hence $D(A:u) \subset (A^n:u)$. Conversely, if $X \in (A^n:u)$, then $Xu \in A^n$. Thus $X \in D(A:u)$ and so $(A^n:u) \subset D(A:u)$. Hence $D(A:u) = (A^n:u)$. □

2.16 Proposition

$D(A:u) = (0 : u+A^n)$ *computed in the* $M_n(R)$*-module* R^n/A^n.

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Proof

Let $X \in D(A:u)$. Then $Xu \in A^n$. So $X(u+A^n) = Xu+A^n = A^n = A^n/A^n$, the zero submodule of R^n/A^n. Hence $X \in (0 : u+A^n)$. Conversely, if $X \in (0 : u+A^n)$, then $X(u+A^n) \in 0 = A^n/A^n$. Therefore $Xu+A^n \in A^n/A^n$ and so $Xu \in A^n$; i.e. $X \in D(A:u)$. Thus $D(A:u) = (0 : u+A^n)$. □

2.17 Proposition

Let U be the nxn matrix having u down the first column and zero's elsewhere. Then $D(A:u) = (M_n(A):U)$ in the module $M_n(R)$.

Proof

Suppose that $X \in D(A:u)$. Then $X_i u \in A$ for each $i = 1,\dots,n$.

Therefore $XU = (X_1,\dots,X_n)' \begin{bmatrix} u_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ u_n & 0 & \dots & 0 \end{bmatrix}$

$$= (X_1,\dots,X_n)' \begin{bmatrix} & 0 & \dots & 0 \\ u & \vdots & & \vdots \\ & 0 & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} X_1u & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ X_nu & 0 & \dots & 0 \end{bmatrix} \in M_n(A). \quad \text{Hence } X \in (M_n(A):U)$$

and so $D(A:u) \subset (M_n(A):U)$. For the converse we let $X \in (M_n(A):U)$. Then it follows that $XU \in M_n(A)$; i.e.

$\begin{bmatrix} X_1u & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ X_nu & 0 & \dots & 0 \end{bmatrix} \in M_n(A)$. Thus $X_i u \in A$ for each $i=1,\dots,n$; i.e.

$X \in D(A:u)$. Therefore $(M_n(A):U) \subset D(A:u)$ and so combining the above inclusions, equality follows. □

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2.18 Corollary

For $n=1$, $\mathrm{Max}(R) = \{(M:u) : M \in \mathrm{Max}(R),\ u \in R-M\}$.

Proof

Let $M \in \mathrm{Max}(R)$ and let $u \in R-M$. Then by Corollary 2.11 $D(M:u) = (M:u)$ computed in the $M_1(R)$-module R. But by Proposition 2.7 all the maximal ideals of $M_1(R)=R$ are of this form. Thus $\mathrm{Max}(R)$ is as predicted. □

Consider the following example.

2.19 Example

Let $R = M_3(Z_9)$, $N = \bar{3}Z_9$ and $u = (\bar{1},\bar{0},\bar{8})' \in Z_9^3 - N^3$. Put $M' = D(N : (\bar{1},\bar{0},\bar{8})')$ and let $Y = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{bmatrix} \in M'$. Then

$Yu = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{bmatrix} (\bar{1},\bar{0},\bar{8})' = \begin{bmatrix} \bar{x}_1 + \bar{8}\bar{x}_3 \\ \bar{y}_1 + \bar{8}\bar{y}_3 \\ \bar{z}_1 + \bar{8}\bar{z}_3 \end{bmatrix} \in N^3$. So by Proposition 2.7 $M' = \left\{ \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{bmatrix} \in M_3(Z_9) : \bar{x}_1+\bar{8}\bar{x}_3,\ \bar{y}_1+\bar{8}\bar{y}_3,\ \bar{z}_1+\bar{8}\bar{z}_3 \in N \right\}$

is a maximal ideal of $M_3(Z_9)$. Let $X = \begin{bmatrix} \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{2} & \bar{0} \\ \bar{0} & \bar{3} & \bar{0} \end{bmatrix} \in M_3(Z_9)-M'$.

Then $Xu = \begin{bmatrix} \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{2} & \bar{0} \\ \bar{0} & \bar{3} & \bar{0} \end{bmatrix} (\bar{1},\bar{0},\bar{8})' = (\bar{1},\bar{0},\bar{0})' \in R^3-N^3$. Consider any

$Y = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{bmatrix} \in (M':X)$. Then $YX = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{bmatrix} \begin{bmatrix} \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{2} & \bar{0} \\ \bar{0} & \bar{3} & \bar{0} \end{bmatrix} \in M'$

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and $\begin{bmatrix} \bar{x}_1 & \bar{x}_1+\bar{2}\bar{x}_2+\bar{3}\bar{x}_3 & \bar{0} \\ \bar{y}_1 & \bar{y}_1+\bar{2}\bar{y}_2+\bar{3}\bar{y}_3 & \bar{0} \\ \bar{z}_1 & \bar{z}_1+\bar{2}\bar{z}_2+\bar{3}\bar{z}_3 & \bar{0} \end{bmatrix} \in M'$. Therefore we have

$$\bar{x}_1,\ \bar{y}_1,\ \bar{z}_1 \in N \qquad \ldots\ldots \text{(i)}$$

But then it follows that $Y(Xu) = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{bmatrix} (\bar{1},\bar{0},\bar{0})' =$

$(\bar{x}_1,\ \bar{y}_1,\ \bar{z}_1) \in N^3$ and hence $Y \in D(N:Xu)$. Thus $(M':X) \subset D(N:Xu)$.

But from (i) above we indeed have that $(M':X) = \begin{bmatrix} N & Z_9 & Z_9 \\ N & Z_9 & Z_9 \\ N & Z_9 & Z_9 \end{bmatrix}$,

which is a maximal ideal of $M_3(Z_9)$. Also, since $Xu \in R^3-N^3$, it follows that $D(N:Xu)$ is also a maximal ideal of $M_3(Z_9)$.

Thus $(M':X) = \begin{bmatrix} N & Z_9 & Z_9 \\ N & Z_9 & Z_9 \\ N & Z_9 & Z_9 \end{bmatrix} = D(N:Xu)$.

The preceding example is a motivation for the following general result.

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2.20 Proposition

Let $R = M_n(S)$ *for some ring* S. *If* N *is a maximal ideal of* S, $u \in S^n-N^n$, $M' = D(N:u)$ *and* $X \in R-M'$, *then*

2.20.1 M' *is a maximal ideal of* R;

2.20.2 (M':X) *is a maximal ideal of* R;

2.20.3 (M':X) = D(N:Xu).

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Proof

2.20.1 Since N is a maximal ideal of S and $u \in S^n - N^n$ it follows by Proposition 2.7 that D(N:u) is a maximal ideal of $M_n(S)$; i.e. M' is a maximal ideal of R.

2.20.2 Since M' is a maximal R-submodule of the R-module R such that $X \in R-M'$, we invoke Proposition 2.10 to obtain the required result.

2.20.3 Let $Y \in (M':X)$. Then $YX \in M' = D(N:u)$. Therefore $Y(Xu) = (YX)u \in N^n$ and so it follows that $Y \in D(N:Xu)$; i.e. $(M':X) \subset D(N:Xu)$. But since (M':X) is a maximal ideal of R it follows that $(M':X) = D(N:Xu)$. □

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CHAPTER 3

IDEALIZERS AND CONTRACTIONS

The focus in this chapter is mainly on idealizers and contractions. We use the concept of the idealizer to find a connection between $M_n(A)$ and $D(A:u)$. We in fact show that a contraction of any maximal ideal in $M_n(R)$ is maximal in R, provided that R is left quasi-duo. On the other hand, if R is an integral domain with K its field of fractions, then no maximal left ideal of $M_n(K)$ contracts to a maximal left ideal of $M_n(R)$.

§5 IDEALIZERS

3.1 Example

Since $B = \bar{6}Z_{12}$ is an ideal of Z_{12} it follows that

$A = \begin{bmatrix} B[x] & Z_{12}[x] \\ B[x] & Z_{12}[x] \end{bmatrix}$ is a left ideal of $M_2(Z_{12}[x])$. Hence

$$I(A) = \begin{bmatrix} Z_{12}[x] & Z_{12}[x] \\ \bar{6}Z_{12}[x] & Z_{12}[x] \end{bmatrix}.$$

3.2 Proposition

The following hold for a left ideal A *of* R.

3.2.1 I(A) *is a subring of* R.

3.2.2 I(A) *is the largest subring of* R *in which* A *sits as a a two-sided ideal.*

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Proof

3.2.1 Since $A \subset I(A)$, it follows that $I(A)$ is non-empty. Let $x,y \in I(A)$ and let $a \in A$. Then $a(x-y) = ax-ay \in A$ and so it follows that $A(x-y) \subset A$. Hence $x-y \in I(A)$. We also have that $a(xy) = (ax)y \in A$, because $ax \in A$ and $y \in I(A)$. Thus $I(A)$ is a subring of R.

3.2.2 Since $ar \in A$ for every $a \in A$ and $r \in I(A)$, it follows that A is a right ideal of I(A). However, by hypothesis A is a left ideal of R and hence also of I(A). So A is a two-sides ideal of I(A). Next we let B be any subring of R such that A is a two-sided ideal of B. Let $b \in B$ be given. Then $Ab \subset A$ and so it follows that $b \in I(A)$; i.e. $B \subset I(A)$ and the result follows. □

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3.3 Corollary

I(A)=R *if and only if* A *is a two-sided ideal of* R. □

3.4 Proposition

Let $M \in Max(R)$. *Then the following hold.*

3.4.1 I(M)/M *is a division ring;*

3.4.2 M *is also a maximal ideal of* I(M).

Proof

3.4.1 By Proposition 1.15 $I(M)/M \simeq End(R/M$ and by Proposition 1.11 End(R/M) is a division ring. Therefore I(M)/M is a division ring.

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3.4.2 Suppose that N is a left ideal of I(M) such that $M \subsetneq N$. Then there exists an element $x \in N$ such that $x \notin M$. Therefore x+M is a non-zero element of I(M)/M, which is a division ring, by the first part. So there exists a non-zero element y+M of I(M)/M such that (y+M)(x+M) = 1+M. So yx+m=1, for some $m \in M$. However, since $x,m \in N$ and since N is a left ideal of I(M), it follows that $yx+m \in N$; i.e. $1 \in N$. Thus N=I(M) and hence M is a maximal ideal of I(M). □

3.5 Proposition

If $M \in Max(R)$, *then the following hold.*

3.5.1 $M_n(M)$ *is a left ideal of* $M_n(R)$;

3.5.2 $I(M_n(M)) = M_n(I(M))$.

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Proof

3.5.1 $M_n(M)$ is non-zero, since the zero matrix lies in it. Since M is closed under addition of its elements as well as under multiplication of the elements of R from the left, it follows that $M_n(M)$ is indeed a left ideal of $M_n(R)$.

3.5.2 Let $X \in M_n(I(M))$ and suppose that $X = [x_{ij}]$. Then for all i,j = 1,...,n it follows that $x_{ij} \in I(M)$ and hence $Mx_{ij} \subset M$. Consider any nxn matrix $[m_{ij}] \in M_n(M)$. Then

$$[m_{ij}][x_{ij}] = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix},$$

where each c_{ij} is a sum of products of elements of M and I(M)

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(in that order). Therefore each $c_{ij} \in M$ and hence $[m_{ij}][x_{ij}] \in M_n(M)$. But this means that $X \in I(M_n(M))$ and so $M_n(I(M)) \subset I(M_n(M))$. For the converse we suppose that $X = [x_{ij}] \in I(M_n(M))$. Let $m \in M$ be arbitrary. Define for each pair of indices i and j an nxn matrix $M_{ij} = [m]$ having the entry m in the (i,j)-position and zero's elsewhere. So it follows that $M_{ij} \in M_n(M)$ for each i,j=1 ,..., n. But since $X \in I(M_n(M))$ we indeed have that $M_{ij}X \in M_n(M)$ for all i,j=1 ,..., n. However, $M_{ij}X$ is an nxn matrix having the entry mx_{ij} in (i,j)-position and zero's elsewhere. Therefore $mx_{ij} \in M$ for each i,j=1 ,..., n. So each entry $x_{ij} \in I(M)$ and hence it follows that $X \in M_n(I(M))$; i.e. $I(M_n(M)) \subset M_n(I(M))$. Combining the above inclusions, equality follows. □

3.6 Corollary

If $M \in Max(R)$, *then the following hold:*

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3.6.1 $I(M_n(M))/M_n(M) = M_n(I(M))/M_n(M) \simeq M_n(I(M)/M)$;

3.6.2 $I(M_n(M))/M_n(M)$ *is a simple artinian ring.*

Proof

3.6.1 In view of Proposition 3.5, $I(M_n(M) = M_n(I(M))$ and so $I(M_n(M))/M_n(M) = M_n(I(M))/M_n(M)$. In order to prove the required ring isomorphism, we define a map

$f : M_n(I(M))/M_n(M) \rightarrow M_n(I(M)/M)$ by the rule

$f : [a_{ij}] + M_n(M) \rightarrow [a_{ij}+M]$. f is well-defined, for suppose that $[a_{ij}] + M_n(M) = [b_{ij}] + M_n(M)$. Then $[a_{ij}] - [b_{ij}] \in M_n(M)$ and so $[a_{ij} - b_{ij}] \in M_n(M)$; i.e. $a_{ij} - b_{ij} \in M$ for each

$i,j=1,\dots,n$. Thus $a_{ij}+M = b_{ij}+M$ for each $i,j=1,\dots,n$; i.e. $[a_{ij}+M] = [b_{ij}+M]$ and so $f([a_{ij}] + M_n(M)) = f([b_{ij}] + M_n(M))$.

f is a ring homomorphism, for suppose that $a = [a_{ij}] + M_n(M)$ and $b = [b_{ij}] + M_n(M)$. Then we have that $f(a+b) = f([a_{ij}] + [b_{ij}] + M_n(M))$ $= f([a_{ij}+b_{ij}] + M_n(M)) = [a_{ij} + b_{ij} + M] = [a_{ij}+M] + [b_{ij}+M] =$ $f([a_{ij}] + M_n(M)) + f([b_{ij}] + M_n(M)) = f(a) + f(b)$ and

$$f(ab) = f\left(\begin{bmatrix} \sum_{i=i}^{n} a_{1j}b_{j1} & \cdots & \sum_{j=1}^{n} a_{1j}b_{jn} \\ \vdots & & \vdots \\ \sum_{j=1}^{n} a_{nj}b_{j1} & \cdots & \sum_{j=1}^{n} a_{nj}b_{jn} \end{bmatrix} + M_n(M)\right)$$

$$= \begin{bmatrix} \sum_{j=1}^{n} a_{1j}b_{j1}+M & \cdots & \sum_{j=1}^{n} a_{1j}b_{jn}+M \\ \vdots & & \vdots \\ \sum_{j=1}^{n} a_{nj}b_{j1}+M & \cdots & \sum_{j=1}^{n} a_{nj}b_{jn}+M \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+M & \cdots & a_{1n}+M \\ \vdots & & \vdots \\ a_{n1}+M & \cdots & a_{nn}+M \end{bmatrix}\begin{bmatrix} b_{11}+M & \cdots & b_{1n}+M \\ \vdots & & \vdots \\ b_{n1}+M & \cdots & b_{nn}+M \end{bmatrix} = f(a)f(b).$$

f is one-to-one, for if $a = [a_{ij}] + M_n(M) \in \ker f$, then $f(a) = [a_{ij}+M] = M_n(M)$. So for each $i,j=1,\dots,n$ it follows that $a_{ij}+M = M$; i.e. $a_{ij} \in M$. Thus $[a_{ij}] \in M_n(M)$ and hence $a = M_n(M)$, the zero element of $M_n(I(M))/M_n(M)$. f is onto, for let $b \in M_n(I(M)/M)$ be given. Then there exist n^2 elements $b_{ij} \in I(M)$ such that $b = [b_{ij}+M]$. So $a = [b_{ij}] + M_n(M)$ is the required element in $M_n(I(M))/M_n(M)$ such that $f(a)=b$. It follows that the rings under discussion are indeed isomorphic.

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3.6.2 By 3.4.1 it follows that $I(M)/M$ is a division ring and by definition 1.4 it in turn follows that $M_n(I(M)/M)$ is a simple artinian ring. Thus $I(M_n(M))/M_n(M)$, being isomorphic to $M_n(I(M)/M)$, is also a simple artinian ring. □

§6 A CONNECTION BETWEEN $M_n(A)$ AND $D(A:u)$

3.7 Proposition

Let A *be a left ideal of* R. *Then a left ideal of* $M_n(R)$ *contains* A *if and only if it contains* $M_n(A)$.

Proof

Suppose that the left ideal A of R is contained in the left ideal I of $M_n(R)$. Then $\begin{bmatrix} a & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & a \end{bmatrix} \in I$ for any $a \in A$. Now I

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is of the form $M_n(B)$, where B is a left ideal of R such that B contains A. So let $[a_{ij}]$ be any element of $M_n(A)$. Then it follows that each $a_{ij} \in A \subset B$. Hence $[a_{ij}] \in M_n(B) = I$ and so $M_n(A) \subset I$. For the converse we suppose that $M_n(A)$ is contained in the left ideal I of $M_n(R)$. Let $a \in A$. Then since $\begin{bmatrix} a & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & a \end{bmatrix} \in M_n(A) \subset I$, it follows that $A \subset I$ and the result follows. □

3.8 Proposition

If A *is a left ideal of* R, *then* $B = \{r \in R : rR \subset A\}$ *is a two-sided ideal of* R.

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Proof

$0 \in B$ and so we have that B is non-empty. Let $a,b \in B$ and let $x \in R$. Then $(a-b)x = ax-bx \in A$ and so $a-b \in B$. By definition B is a right ideal of R. It remains to show that it is also a left ideal of R. So let $x,y \in R$ and let $b \in B$. Then $(xb)y = x(by) = xa$ for some $a \in A$. But since A is a left ideal of R, it follows that $xa \in A$; i.e. $xb \in B$. Thus B is a left ideal of R and the result follows. □

3.9 Definition

If A is a left ideal of R, then $B = (A:R) = \{r \in R : rR \subset A\}$ is called the *transporter ideal of* A.

3.10 Example

Let $A = \begin{bmatrix} Z & 4Z \\ Z & 4Z \end{bmatrix}$ and let $R = M_2(Z)$. Let $r = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in (A:R)$ and $r' = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in R$. Then $rr' \in A$ and so the equations $ay+bw = 4s$ and $cy+dw = 4t$ must hold for all $y, w, s, t \in Z$. So in particular, if we first choose $y=0$ and $w=1$ and then $y=1$ and $w=0$, we see that $a, b, c, d \in 4Z$; i.e. $r \in M_2(4Z)$. Hence $\left(\begin{bmatrix} Z & 4Z \\ Z & 4Z \end{bmatrix} : M_2(Z)\right) = M_2(4Z)$.

We can now say precisely how $M_n(A)$ is related to $D(A:u)$.

3.11 Proposition

Let A *be a left ideal of* R, C *the center of* R *and* B *the transporter ideal of* A. *Then it follows that*

$M_n(A \cap C) \subset M_n(A \cap B) \subset \bigcap_u D(A:u) \subset M_n(A)$, *where the intersection*

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is taken over all $u \in R^n$.

Proof

$M_n(A \cap C) \subset M_n(A \cap B)$.

Let $X = [a_{ij}] \in M_n(A \cap C)$. Now for each $i,j=1, \ldots, n$ it follows that $a_{ij} \in A \cap C$. So $a_{ij} \in A$ and $a_{ij}r = ra_{ij}$ for each $r \in R$. But since A is a left ideal of R, it follows that $ra_{ij} \in A$ and so $a_{ij}r \in A$. Thus $a_{ij} \in B$ and hence $a_{ij} \in A \cap B$; i.e. $X \in M_n(A \cap B)$, proving the required inclusion.

$M_n(A \cap B) \subset \bigcap_u D(A:u)$.

Let $X = (X_1, \ldots, X_i, \ldots, X_n)' \in M_n(A \cap B)$ and suppose that $X_i = [x_{i1} \ldots x_{in}]$ for $i=1, \ldots, n$. Let $u \in R^n$. Then $X_i u = x_{i1}u_1 + \ldots + x_{in}u_n \in A$, because $x_{i1}, \ldots, x_{in} \in A \cap B \subset B$. Thus $X \in D(A:u)$ for every $u \in R^n$ and hence it follows that $X \in \bigcap_u D(A:u)$. Therefore $M_n(A \cap B) \subset \bigcap_u D(A:u)$, where u range over all the elements of R^n.

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$\bigcap_u D(A:u) \subset M_n(A)$.

Let $X = (X_1, \ldots, X_j, \ldots, X_n)' \in \bigcap_u D(A:u)$, where $X_j = [x_{j1} \ldots x_{jn}]$ for $j=1, \ldots, n$. Then for each $u \in R^n$ it follows that $Xu \in A^n$. So in particular for $u = e_i$ and $i=1, \ldots, n$, we have that $Xe_i \in A^n$; i.e. $X_j e_i \in A$ for each $i,j=1, \ldots, n$. So if we fix j and let i range over all the indices $i=1, \ldots, n$, then it follows that $x_{j1}, \ldots, x_{jn} \in A$. If we now let j range over all the indices from 1 to n, it follows that $x_{ij} \in A$. Hence $X \in M_n(A)$ and so the required inclusion follows. □

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3.12 Proposition

For a left ideal A *of* R, *the following hold.*

3.12.1 $M_n(A) \subset D(A:u)$ *if and only if each* $u_i \in I(A)$;

3.12.2 $M_n(A) = \bigcap_u D(A:u)$, *where the intersection is taken over all* $u \in R^n$, *if and only if* A *is two-sided.*

Proof

3.12.1 Suppose that $M_n(A) \subset D(A:u)$. Let $a \in A$ be given. Let X be the matrix of $M_n(A)$ having the entry a in the (i,i)-position and zero's elsewhere. Then $X \in D(A:u)$ and so $Xu \in A^n$; i.e. $au_i \in A$. Hence $u_i \in I(A)$. Since for each i we can construct such a matrix X, it indeed follows that each $u_i \in I(A)$. For the converse we suppose that each $u_i \in I(A)$. Let $X \in M_n(A)$ and assume that $X_i = [x_{i1} \dots x_{in}]$. Then for each i=1 ,..., n it follows that $x_{i1}, \dots, x_{in} \in A$. But by hypothesis we have that $u_1, \dots, u_n \in I(A)$ and hence it follows that $X_i u = [x_{i1} \dots x_{in}](u_1, \dots, u_n)' = x_{i1}u_1 + \dots + x_{in}u_n \in A$ for each i=1 ,..., n. Thus $Xu \in A^n$ and so $X \in D(A:u)$; i.e. $M_n(A) \subset D(A:u)$.

3.12.2 Suppose that $M_n(A) \subset \bigcap_u D(A:u)$, where u range over all the elements of R^n. Let $a \in A$ and $r \in R$ be given. If X is the matrix of $M_n(A)$ having the entry a in the (1,1)-position and zero's elsewhere and if $u = (r, 0, \dots, 0)'$, then in particular for this choice of u, it follows that $Xu \in A^n$; i.e. $(ar, \dots, 0)' \in A^n$. Hence $ar \in A$, proving that A is a right ideal of R. But since A is a left ideal of R by hypothesis,

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it follows that A is a two-sided ideal of R. Suppose conversely that A is a two-sided ideal of R. Since, by Proposition 3.11, we have already shown that $\bigcap_u D(A:u) \subset M_n(A)$, it suffices to show that $M_n(A) \subset \bigcap_u D(A:u)$ only. So let $X \in M_n(A)$ and let $u \in R^n$. As before, let the rows X_i of X be denoted by $X_i = [x_{i1} \ldots x_{in}]$. Now, using the hypothesis that A is also a right ideal of R, we indeed have that $X_i u = x_{i1}u_1 + \ldots + x_{in}u_n \in A$ for each i=1 ,..., n; i.e. $X \in D(A:u)$. But the element $u \in R^n$ was chosen arbitrarily and so $X \in D(A:u)$ for each $u \in R^n$; i.e. $X \in \bigcap_u D(A:u)$. Thus $M_n(A) \subset \bigcap_u D(A:u)$, as required. □

In view of the preceding result it may well happen that $M_n(A)$ fails to equal $\bigcap_u D(A:u)$ if we dispense with the condition that A be a two-sided ideal of R. The following counter-example illustrates this point.

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3.13 Counter-example

Consider the left ideal $A = \begin{bmatrix} Z & 0 \\ Z & 0 \end{bmatrix}$ of $M_2(Z)$. Let $X \in \bigcap_u D(A:u)$.

Put $X = \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix}$ and $u = \begin{bmatrix} \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \\ \begin{bmatrix} u_1' & u_2' \\ u_3' & u_4' \end{bmatrix} \end{bmatrix}$. Then it follows that

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\begin{bmatrix} u_1' & u_2' \\ u_3' & u_4' \end{bmatrix} \in \begin{bmatrix} Z & 0 \\ Z & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} + \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}\begin{bmatrix} u_1' & u_2' \\ u_3' & u_4' \end{bmatrix} \in \begin{bmatrix} Z & 0 \\ Z & 0 \end{bmatrix},$$

for all $u_i, u_i' \in Z$, where i=1,2,3,4. Hence the following equations must hold for all $u_2, u_4, u_2', u_4' \in Z$.

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$a_1u_2 + a_2u_4 + b_1u_2' + b_2u_4' = 0$

$a_3u_2 + a_4u_4 + b_3u_2' + b_4u_4' = 0$

$c_1u_2 + c_2u_4 + d_1u_2' + d_2u_4' = 0$

$c_3u_2 + c_4u_4 + d_3u_2' + d_4u_4' = 0.$

So in particular for $u_2=1$ and $u_4 = u_2' = u_4' = 0$, it follows that $a_1 = a_3 = c_1 = c_3 = 0$. Similarly one can prove that all the other entries of X are zero. Hence X is the zero matrix of $\bigcap_u D(A:u)$. Thus $\bigcap_u D(A:u) = 0 \neq M_2(A)$.

§7 CONTRACTIONS

In this section we are concerned about the question of whether maximal ideals of $M_n(R)$ lie over and thus contract to maximal ideals of R. We in fact provide a necessary and sufficient condition (see Proposition 3.25) for such a contraction to hold. Some of the following results, each of which is preceded by an appropiate example, would be helpful in this regard.

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3.14 Example

Let $R = M_2(Z_8[x])$ and let $B = \bar{2}Z_8$. Since B[x] is a maximal ideal of $Z_8[x]$ it follows that $M = \begin{bmatrix} B[x] & Z_8[x] \\ B[x] & Z_8[x] \end{bmatrix}$ maximal left ideal of R. Moreover, $I(M) = \begin{bmatrix} Z_8[x] & Z_8[x] \\ B[x] & Z_8[x] \end{bmatrix}$. Let $u = \begin{bmatrix} x^2 & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix} \in I(M)-M$. Consider any $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \in (M:u)$.

Then we have that $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}\begin{bmatrix} x^2 & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix} \in M$ and so $\begin{bmatrix} f_1x^2 & \bar{0} \\ f_3x^2 & \bar{0} \end{bmatrix} \in M$.

Thus $f_1x^2, f_3x^2 \in B[x]$. Suppose next that $f_1 = \sum_{i=0}^{m} a_ix^i$. Then

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it follows that $f_1x^2 = \sum_{i=0}^{m} a_i x^{i+2}$. Thus each $a_i \in B$; i.e. $f_1 \in B[x]$. The proof that $f_3 \in B[x]$ follows in a similar manner. Thus $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \in M$ and hence $(M:u) \subset M$. Since we are dealing with maximal ideals, equality follows; i.e. $(M:u) = M$ for $u \in I(M)-M$.

3.15 Proposition

For $M \in \mathrm{Max}(R)$, $(M:u) = M$ *if and only if* $u \in I(M)-M$.

Proof

Suppose that $(M:u) = M$ and assume that $u \notin I(M)-M$. Then we are left with two possibilities, namely $u \in M$ or $u \in R-I(M)$. If $u \in M$, then since M is a left ideal of R, $Ru \subset M$; i.e. $R \subset (M:u) = M$ and so $R=M$, an obvious contradiction. On the other hand, $u \in R-I(M)$ would also lead to a contradiction, since $(M:u) = M$ implies that $Mu \subset M$; i.e. $u \in I(M)$, by its definition. Hence $u \in I(M)-M$. For the converse we suppose that $u \in I(M)-M$. Then $Mu \subset M$ and so it follows that $M \subset (M:u)$. But since M is a maximal ideal of R and $u \in R-M$, it follows that $(M:u) = M$. □

3.16 Example

Let $R = Z_8[x]$, $A = \bar{4}Z_8[x]$, $B = \bar{2}Z_8[x]$, n=2 and $u = (\bar{3}x,\bar{0})' \in R^2$. Then $A \subset B$. Now $D(A:u) = \begin{bmatrix} \bar{4}Z_8[x] & Z_8[x] \\ \bar{4}Z_8[x] & Z_8[x] \end{bmatrix}$ and

$D(B:u) = \begin{bmatrix} \bar{2}Z_8[x] & Z_8[x] \\ \bar{2}Z_8[x] & Z_8[x] \end{bmatrix}$. Thus $D(A:u) \subset D(B:u)$.

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3.17 Proposition

If $A \subset B$ *are ideals of* R *and if* $u \in R^n$, *then* $D(A:u) \subset D(B:u)$.

Proof

Let $X \in D(A:u)$. Then $Xu \in A^n \subset B^n$, since $A \subset B$. Hence $X \in D(B:u)$ and so $D(A:u) \subset D(B:u)$. □

3.18 Example

Let $R = Z_{12}$, $A_1 = \bar{2}Z_{12}$, $A_2 = \bar{3}Z_{12}$ and $u = (\bar{5},\bar{0})'$. Then the following hold. $A_1 \cap A_2 = \bar{6}Z_{12}$, $D(A_1:u) = \begin{bmatrix} \bar{2}Z_{12} & Z_{12} \\ \bar{2}Z_{12} & Z_{12} \end{bmatrix}$,

$D(A_2:u) = \begin{bmatrix} \bar{3}Z_{12} & Z_{12} \\ \bar{3}Z_{12} & Z_{12} \end{bmatrix}$ and $D(A_1:u) \cap D(A_2:u) = \begin{bmatrix} \bar{6}Z_{12} & Z_{12} \\ \bar{6}Z_{12} & Z_{12} \end{bmatrix} =$

$D(A_1 \cap A_2:u)$.

3.19 Proposition

If $A = \bigcap_i A_i$ *is the intersection of a collection of ideals of* R, *then* $D(A:u) = \bigcap_i D(A_i:u)$.

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Proof

Let $A = \bigcap_i A_i$ be an intersection of left ideals of R, where $i \in I$, for some index set I. Suppose that $X \in D(A:u)$. Then for each $j \in \{1,\ldots,n\}$, it follows that $X_j u \in A = \bigcap_i A_i$. Thus $X_j u \in A_i$ for each $i \in I$; i.e. $Xu \in A_i^n$ for each $i \in I$. Hence $X \in D(A_i:u)$ for each $i \in I$; i.e. $X \in \bigcap_i D(A_i:u)$. Therefore $D(A:u) \subset \bigcap_i D(A_i:u)$. Conversely, let $X \in \bigcap_i D(A_i:u)$. Then $X \in D(A_i:u)$ for each $i \in I$. Hence $X_j u \in A_i$ for each $j \in \{1,\ldots,n\}$ and for each $i \in I$. Thus $X_j u \in \bigcap_i A_i = A$ for each $j \in \{1,\ldots,n\}$. Hence $X \in D(A:u)$ and so $\bigcap_i D(A_i:u) \subset D(A:u)$. By the above inclusions the equality follows. □

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3.20 Example

Let $R = M_2(Z)$, $A = \begin{bmatrix} 4Z & 0 \\ 4Z & 0 \end{bmatrix}$, n=2 and let

$u = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right)' \in R^2 - A^2$. Then $u_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$u_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. Let $X = \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix} \in D(A:u)$.

Then the following equations must hold.

$\begin{bmatrix} a_1 - b_1 & a_2 \\ a_3 - b_3 & a_4 \end{bmatrix} = \begin{bmatrix} 4r & 0 \\ 4r' & 0 \end{bmatrix}$ and $\begin{bmatrix} c_1 - d_1 & c_2 \\ c_3 - d_3 & c_4 \end{bmatrix} = \begin{bmatrix} 4s & 0 \\ 4s' & 0 \end{bmatrix}$, for all

$r, r', s, s' \in Z$. Hence $a_1 \equiv b_1 (\text{mod}4)$, $a_3 \equiv b_3 (\text{mod}4)$, $c_1 \equiv d_1 (\text{mod}4)$, $c_3 \equiv d_3 (\text{mod}4)$ and $a_2 = a_4 = c_2 = c_4 = 0$. Therefore

$$D(A:u) = \left\{ \begin{bmatrix} \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & 0 \\ c_3 & 0 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix} \in M_2(R) : a_i \equiv b_i (\text{mod}4) \text{ and } c_i \equiv d_i (\text{mod}4), \text{ for } i = 1,3 \right\}.$$

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Next we consider any $X \in D(A:u) \cap R$. Then X is of the form

$X = \begin{bmatrix} \begin{bmatrix} 4a & 0 \\ 4b & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 4a & 0 \\ 4b & 0 \end{bmatrix} \end{bmatrix}$, where $a,b \in Z$. Thus

$$D(A:u) \cap R = \left\{ \begin{bmatrix} \begin{bmatrix} 4a & 0 \\ 4b & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 4a & 0 \\ 4b & 0 \end{bmatrix} \end{bmatrix} \in M_2(R) : a,b \in Z \right\}.$$

Furthermore, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in (A:u_1)$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in A$ and

so $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} 4Z & 0 \\ 4Z & 0 \end{bmatrix}$; i.e. $a,c \in 4Z$ and $b=d=0$. Thus

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$(A:u_1) = \begin{bmatrix} 4Z & 0 \\ 4Z & 0 \end{bmatrix}$. On the other hand, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in (A:u_2)$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \in A$; i.e. $\begin{bmatrix} -a & 0 \\ -c & 0 \end{bmatrix} \in \begin{bmatrix} 4Z & 0 \\ 4Z & 0 \end{bmatrix}$. Thus $a,c \in 4Z$ and so $(A:u_2) = \begin{bmatrix} 4Z & Z \\ 4Z & Z \end{bmatrix}$. But then we have that $L = (A:u_1) \cap (A:u_2) = \begin{bmatrix} 4Z & 0 \\ 4Z & 0 \end{bmatrix}$. Now regarding L as a subring of $M_2(R)$ under the embedding

$$\begin{bmatrix} 4a & 0 \\ 4b & 0 \end{bmatrix} \to \begin{bmatrix} \begin{bmatrix} 4a & 0 \\ 4b & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 4a & 0 \\ 4b & 0 \end{bmatrix} \end{bmatrix},$$

we observe that $D(A:u) \cap R = (A:u_1) \cap (A:u_2)$.

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3.21 Proposition

For A *an ideal of* R *and* $u \in R^n$, $D(A:u) \cap R = \bigcap_{i=1}^{n} (A:u_i)$.

Proof

Let $X \in D(A:u) \cap R$. Then X is of the form

$$X = \begin{bmatrix} r & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & r & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & r \end{bmatrix}, \text{ for some } r \in R \text{ such that}$$

$$\begin{bmatrix} r & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & r & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & r \end{bmatrix} (u_1, \ldots, u_i, \ldots, u_n)' \in A^n.$$ Hence

$(ru_1, \ldots, ru_i, \ldots, ru_n)' \in A^n$; i.e. $ru_i \in A$ for each $i=1, \ldots, n$. Thus $r \in (A:u_i)$ for each $i=1, \ldots, n$; i.e. $r \in \bigcap_{i=1}^{n} (A:u_i)$. However, regarded as an element of $M_n(R)$, r

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has the form $\begin{bmatrix} r & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & r \end{bmatrix} = X$. Hence $X \in \bigcap_{i=1}^{n} (A:u_i)$ and so $D(A:u) \cap R \subset \bigcap_{i=1}^{n} (A:u_i)$. For the converse inclusion we let $r \in \bigcap_{i=1}^{n} (A:u_i)$. Then $r \in (A:u_i)$ for each $i=1, \ldots, n$. However, regarded as an element of $M_n(R)$, r has the form $\begin{bmatrix} r & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & r \end{bmatrix}$.

Hence $ru = \begin{bmatrix} r & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & r \end{bmatrix}(u, \ldots, u_n)' = (ru_1, \ldots, ru_n)' \in A^n$. Thus $r \in R$ such that $ru \in A^n$; i.e. $r \in D(A:u) \cap R$. Thus $\bigcap_{i=1}^{n} (A:u_i) \subset D(A:u) \cap R$. Therefore $D(A:u) \cap R = \bigcap_{i=1}^{n} (A:u_i)$. □

3.22 Corollary

If $M \in \text{Max}(R)$ *is a two-sided ideal of* R, *then* $D(M:u)$ *contracts to* M.

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Proof

Since M is a two-sided ideal of R, it follows that $I(M) = R$. Now $u = (u_1, \ldots, u_n)' \in R^n - M^n$ by hypothesis. So there is at least one $u_i \notin M$. Therefore $u_i \in R-M$; i.e. $u_i \in I(M)-M$. Thus by Proposition 3.15 it follows that $(M:u_i) = M$ for such ones. On the other hand for $j \neq i$, we have that $(M:u_j) = R$, since these u_i's are in M. So we see by Proposition 3.21 that $D(M:u) \cap R = \bigcap_{i=1}^{n} (M:u_i) = M$, since the intersection contains at least one M as a member. Therefore $D(M:u)$ contracts to M. □

3.23 Definition

3.23.1 A ring R is called *left duo* if every left ideal of R is two-sided.

3.23.2 A ring R is called *left quasi-duo* if every maximal left ideal of R is two-sided.

3.23.3 A ring R is called a *local ring* if it has a unique maximal left ideal.

3.24 Examples

3.24.1 Every left duo ring is left quasi-duo.

3.24.2 $R = \left\{\begin{bmatrix} a & b \\ o & a \end{bmatrix} : a,b \in Z\right\}$ is a left duo ring. We first show that R is a ring. Let $\begin{bmatrix} a & b \\ o & a \end{bmatrix}, \begin{bmatrix} x & y \\ O & x \end{bmatrix} \in R$. Then we have that $\begin{bmatrix} a & b \\ O & a \end{bmatrix} + \begin{bmatrix} x & y \\ O & x \end{bmatrix} = \begin{bmatrix} a+x & b+y \\ O & a+x \end{bmatrix} \in R$ and $\begin{bmatrix} a & b \\ O & a \end{bmatrix}\begin{bmatrix} x & y \\ O & x \end{bmatrix} = \begin{bmatrix} ax & ay+bx \\ O & ax \end{bmatrix} \in R$. The other ring properties follows from the fact that R is a subset of $M_2(Z)$. In order to prove that R is indeed left duo, we observe that the ideals of R are all of the type

$A = \left\{\begin{bmatrix} a & b \\ O & a \end{bmatrix} \in R : a \in I,\ b \in J \text{ where I and ideals of Z such that } I \subset J\right\}$. We assert that each such A is a left ideal of R. A is non-empty, because $\begin{bmatrix} O & O \\ O & O \end{bmatrix} \in A$. Let $\begin{bmatrix} a & b \\ O & a \end{bmatrix}, \begin{bmatrix} x & y \\ O & x \end{bmatrix} \in A$. Then there exist ideals I and J of Z such that $I \subset J$ with $a,x \in I$

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and $b,y \in J$. Thus $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} = \begin{bmatrix} a-x & b-y \\ 0 & a-x \end{bmatrix} \in A$, because $a-x \in I$ and $b-y \in J$. Let $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in R$ and let $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \in A$. Then $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} = \begin{bmatrix} ax & ay+bx \\ 0 & ax \end{bmatrix} \in A$, because $ax \in I$ and $ay+bx \in J$ (since $x \in I \subset J$). Thus A is a left ideal of R, as asserted. Moreover, A is also a right ideal of R. For suppose that $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in R$ and $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix} \in A$. Then we have that $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} xa & xb+ya \\ 0 & xa \end{bmatrix} \in A$, since the ideals I and J of Z are two-sided. So each left ideal of R is also a right ideal and hence R is indeed a left duo ring.

3.24.3 The ring R of 2x2 lower triangular matrices over a division ring D is a left quasi-duo ring which is not left duo.

Let $R = \begin{bmatrix} D & 0 \\ D & D \end{bmatrix}$, for some division ring D. Then the left ideals of R are $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} D & 0 \\ D & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$ and $A_5 = R$. A_2 is the only maximal left ideal of R. Moreover since $A_2R = \begin{bmatrix} D & 0 \\ D & 0 \end{bmatrix}\begin{bmatrix} D & 0 \\ D & D \end{bmatrix} \subset \begin{bmatrix} D & 0 \\ D & 0 \end{bmatrix} = A_2$, it follows that A_2 is also a right ideal of R. Thus R is left quasi-duo. However, since $A_4R = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} D & 0 \\ D & D \end{bmatrix} \subset \begin{bmatrix} 0 & 0 \\ D & D \end{bmatrix} \neq A_4$, it follows that the left ideal A_4 is not a right ideal; i.e. R is not a left duo ring.

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3.24.4 Any field is a local ring.

3.24.5 Z_9 is a local ring, because $\bar{3}Z_9$ is its only maximal ideal.

3.24.6 Example 3.24.3 above is also an example of a local ring.

3.25 Proposition

For $n \geq 2$, *every maximal left ideal of* $M_n(R)$ *contracts to a maximal left ideal of* R *if and only if* R *is left quasi-duo.*

Proof

Suppose that R is not left quasi-duo. Then there exists a maximal left ideal M of R such that M is not two-sided. Let $r \in R-I(M)$. Then $u = (1,r,\ldots,0)' \in R^n-M^n$ and so by Proposition 3.21 $D(M:u) \cap R = (M:1) \cap (M:r) = M \cap (M:r)$. By Proposition 3.15 $(M:r) \neq M$ and so $M \cap (M:r) \subsetneq M$; for if $M \cap (M:r) = M$, then $M \subsetneq (M:r) \neq R$, which is obviously a contradiction since M is a maximal ideal of R. Hence the contraction $D(M:u) \cap R$ is not maximal. For the converse we suppose that R is left quasi-duo. Let M' be any maximal left deal of $M_n(R)$. Then by Proposition 2.7 M' is of the form $D(M:u)$ for some $M \in Max(R)$ and $u \in R^n-M^n$. But by hypothesis M is a two-sided ideal of R. We can therefore apply Corollary 3.22 to see that $D(M:u)$ contracts to M; i.e. M' contracts to the maximal ideal M of R. □

3.26 Proposition

The contraction of a maximal ideal of $M_n(R)$ *is always an intersection of maximal ideals of* R.

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Proof

Let M' be a maximal ideal of $M_n(R)$. Then by Proposition 2.7 $M' = D(M:u)$ for some $M \in Max(R)$ and $u \in R^n - M^n$. But by Proposition 3.21 we have the following contraction of M', namely $M' \cap R = D(M:u) \cap R = (M:u_1) \cap \ldots \cap (M:u_n)$. Since $u \in R^n - M^n$, it follows that some $u_i \notin M$. For such u_i's we have by Corollary 2.18 that $(M:u_i)$ is a maximal ideal of R. On the other hand, if $u_j \in M$ for $j \neq i$, then $(M:u_j) = R$. So in any case we have $M' \cap R$ is an intersection of maximal ideals of R, since $(M:u_i) \cap R = (M:u_i)$, which is maximal. □

3.27 Example

Since 0 is a maximal ideal of Q, the rational field of Z, we have that each $D(0:e_i)$ is a maximal ideal of $M_n(Q)$. Now we have

$$D(0:e_i) \cap M_n(Z) = \begin{bmatrix} Q & \ldots & 0 & \ldots & Q \\ \vdots & & \vdots & & \vdots \\ Q & \ldots & 0 & \ldots & Q \end{bmatrix} \cap M_n(Z) = \begin{bmatrix} Z & \ldots & 0 & \ldots & Z \\ \vdots & & \vdots & & \vdots \\ Z & \ldots & 0 & \ldots & Z \end{bmatrix},$$

which is not a maximal ideal of $M_n(Z)$.

In the above example we have noticed that the maximal left ideals $D(0:e_i)$ of $M_n(Q)$, where Q is the rational field of Z, do not contract to maximal ideals of $M_n(Z)$. It is therefore natural to investigate whether this behaviour is typical. We in fact look at a more general situation: Let R, S and $S^{-1}R$, be as in Proposition 1.13. For A' an ideal of $S^{-1}R$, we let $A = A' \cap R$ denote its contraction to R. We are now able to prove the following result.

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3.28 Proposition

Let P' *be a prime ideal of* $S^{-1}R$ *and let* $u = (u_1/s_1, u_2/s_2, \ldots, u_n/s_n)' \in (S^{-1}R)^n$. *If some entry of* u *is not in* P'*, then*

3.28.1 $D(P':u)$ *is a proper left ideal of* $M_n(S^{-1}R)$;

3.28.2 *for* $s = s_1s_2 \ldots s_n$, $D(P:su)$ *is a proper left ideal of* $M_n(R)$;

3.28.3 *the contraction of* P' *to* $M_n(R)$ *is* $D(P:su)$.

Proof

3.28.1 The proof is similar to the one in Proposition 2.1.

3.28.2 We remark that s is a unit of $S^{-1}R$; indeed $s^{-1} = 1/s_1s_2 \ldots s_n = (1/s_1)(1/s_2) \ldots (1/s_n)$. Thus $s \notin P'$, for if $s \in P'$, then $1 = s^{-1}s \in P'$ and so $P' = S^{-1}R$ which is a contradiction, because some $u_i/s_i \notin P'$ by hypothesis. It therefore follows that $s(u_i/s_i) \notin P'$, for if not the case, it would then mean that $s^{-1}(s(u_i/s_i)) \in P'$; i.e. $u_i/s_i \in P'$, which is a contradiction. Now $s(u_i/s_i) = (s_1/1) \ldots (s_i/1)(u_i/s_i) \ldots (s_n/1)$ $= (s_1/1) \ldots (s_iu_i/s_i) \ldots (s_n/1) = (s\ /1) \ldots (u_i/1) \ldots (s_n/1)$, since $s_iu_i/s_i = u_i/1$. By Proposition 1.13 R can be considered as a subring of $S^{-1}R$ and so $(s_1/1) \ldots (u_i/1) \ldots (s_n/1)$ is indeed the element $s_1 \ldots u_i \ldots s_n \in R$. So $s(u_i/s_i) \in R$, but not in P'. Thus $s(u_i/s_i) \notin P' \cap R = P$. Hence $su \in R^n - P^n$ and so by Proposition 2.1 $D(P:su)$ is a proper left ideal of $M_n(R)$.

3.28.3 Let $X \in D(P:su)$. Then for each $i=1,\ldots,n$ $X_i(su) \in P = P' \cap R$. But since R is commutative, it follows that each $s(X_iu) \in P'$. Since P' is a prime ideal and $s \notin P'$, it

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follows that each $X_i u \in P'$. Hence $X \in D(P':u)$. But since we are concerned about those X's in $M_n(R)$ only, it follows that $D(P:su) \subset D(P':u) \cap M_n(R)$. The converse inclusion follows even without the primality assumption. For let $X \in D(P':u) \cap M_n(R)$. Then $X_i u \in P'$ for each $i=1,\ldots,n$. However, since $s \in R \subset S^{-1}R$, it follows that $s(X_i u) \in P'$. But R is commutative and so $X_i(su) \in P'$. On the other hand we have that
$su = (s_1/1) \ldots (s_n/1)(u_1/s_1,\ldots, u_n/s_n)' = ((s_1u_1/s_1 \ldots s_n/1),\ldots, (s_1/1 \ldots s_nu_n/s_n))' = ((u_1/1 \ldots s_n/1),\ldots, (s_1/1 \ldots u_n/1))' = (u_1 \ldots s_n ,\ldots, s_1 \ldots u_n)' \in R^n$. However, since each entry in X_i lies inside R, it is therefore evident that $X_i(su) \in R$. Hence $X_i(su) \in P' \cap R = P$ for each $i=1,\ldots, n$; i.e. $X \in D(P:su)$ and so $D(P':u) \cap M_n(R) \subset D(P:su)$. Therefore $D(P':u) \cap M_n(R) = D(P:su)$, as required. □

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3.29 Proposition

Let R *be an integral domain,* S *the set of non-zero elements of* R *and* K *its field of fractions. Then no maximal left ideal of* $M_n(K)$ *contracts to a maximal left ideal of* $M_n(R)$.

Proof

Since O is the only maximal ideal of K, we see by Proposition 2.7 that all the maximal left ideals of $M_n(K)$ have the form $D(O:u)$, where $u \in K^n - O^n$. On the other hand, since O is also a prime ideal of K, we can invoke Proposition 3.28 to obtain $D(O:u) \cap M_n(R) = D(O:su)$. However, since R is an integral domain, O is not a maximal ideal of R. So again by Proposition 2.7 $D(O:su)$ cannot be a maximal ideal of $M_n(R)$. Hence no

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maximal left ideal of $M_n(K)$ contracts to a maximal left ideal of $M_n(R)$. □

3.30 Remark

The integral domain R, regarded as a subring of K trivially has the property described in Proposition 3.29, since the maximal ideal 0 of K contracts to the non-maximal ideal 0 of R. However, in the matrix ring case the non-zero maximal ideals of $M_n(K)$ all contract to non-zero, non-maximal ideals of $M_n(R)$.

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CHAPTER 4

EQUALITY OF D(M:u) AND D(M:v)

We wish to know under which circumstances it so happens that D(M:u) equals D(M:v) for u, v $\in R^n - M^n$. We provide necessary and sufficient conditions for such equalities. It is interesting to note the importance of the role of the idealizer in this regard. In the second part of the chapter we attempt to count the number of maximal ideals of $M_n(R)$ in the case where R is a commutative ring.

§8 NECESSARY AND SUFFICIENT CONDITIONS FOR D(M:u) TO EQUAL D(M:v)

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4.1 Example

Consider the maximal ideal $M = \bar{3}Z_6$ of Z_6. Let n=2 and let $u = (\bar{2},\bar{5})'$, $v = (\bar{2},\bar{2},)' \in Z_6^2 - M^2$. Then $u \equiv v(\mathrm{mod}M)$. Let $\begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \in D(M:u)$. Then $\begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}(\bar{2},\bar{5})' \in M^2$ and hence it follows that $\bar{2}\bar{a} + \bar{5}\bar{b} \in M$ and $\bar{2}\bar{c} + \bar{5}\bar{d} \in M$. However, $\bar{2}\bar{a} + \bar{5}\bar{b} = \overline{2a} + \overline{5b} + \overline{3b} = \overline{2a} + \overline{2b}$ and similarly we have that $\overline{2c} + \overline{5d} = \overline{2c} + \overline{2d}$; i.e. $\bar{2}\bar{a} + \bar{2}\bar{b}$, $\bar{2}\bar{c} + \bar{2}\bar{d} \in M$. Thus $\begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \in D(M:v)$ and so in view of the maximality of the ideals, it follows that D(M:u) = D(M:v).

4.2 Proposition

If $u \equiv v(\mathrm{mod}M)$, *then* D(M:u) = D(M:v).

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Proof

Suppose that $u \equiv v(\mathrm{mod}M)$. Then for each $i=1,\ldots,n$ there exists $m_i \in M$ such that $u_i = v_i + m_i$. Let $X \in D(M:v)$. Then $X_i v \in M$ for each $i=1,\ldots,n$. Thus $X_i u = X_i u_1 + \ldots + X_i u_n = X_i(v_1+m_1) + \ldots + X_i(v_n+m_n) = X_i v_1 + \ldots + X_i v_n + m' = X_i v+m'$, where $m' = X_i m_1 + \ldots + X_i m_n$. Since $m' \in M$ and $X \in D(M:v)$ we have that $X_i v+m' \in M$; i.e. $X_i u \in M$. Hence $X \in D(M:u)$ and so $D(M:v) \subset D(M:u)$. Since we are dealing with maximal ideals, equality follows. □

4.3 Example

Take p=2 in Proposition 1.16. Then we have the maximal ideal $M = \{f \in Z[x]: \mathrm{const}(f) \in 2Z\}$ of $Z[x]$. Now since $Z[x]$ is commutative, $I(M) = Z[x]$. Let $n=3$, $u = (0, 1-x^2, 2x^3)'$, $v = (2,3-x,x^2)'$ and $c = 5+x^5 \in I(M)-M$. Then $v-uc = (2,3-x,x^2)' - (0,5-5x^2+x^5-x^7, 10x^3+2x^8)' = (2,-2-x+5x^2-x^5+x^7, x^2-10^3-2x^8)' \in M^3$.

Let $X \in D(M:u)$, say $X = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix}$. Then we have that $Xu \in M^3$; i.e. the elements $f_2-f_2x^2+2f_3x^3$, $f_5-f_5x^2+2f_6x^3$ and $f_8-f_8x^2+2f_9x^3$ are all in M. But this will hold only if $\mathrm{const}(f_2)$, $\mathrm{const}(f_5)$, $\mathrm{const}(f_8) \in 2Z$. Hence $D(M:u) = \begin{bmatrix} Z[x] & M & Z[x] \\ Z[x] & M & Z[x] \\ Z[x] & M & Z[x] \end{bmatrix}$

On the other hand, $\begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \in D(M:v)$ if and only if $2f_1+3f_2-f_2x+f_3x^2$, $2f_4+3f_5-f_5x+f_6x^2$, $2f_7+3f_8-f_8x+f_9x^2 \in M$, i.e. $3\mathrm{const}(f_2)$, $3\mathrm{const}(f_5)$, $3\mathrm{const}(f_8) \in 2Z$. But since $(2,3)=1$, we have that $\mathrm{const}(f_2)$, $\mathrm{const}(f_5)$, $\mathrm{const}(f_8) \in 2Z$. Thus

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$$D(M:v) \quad = \quad \begin{bmatrix} Z[x] & M & Z[x] \\ Z[x] & M & Z[x] \\ Z[x] & M & Z[x] \end{bmatrix} = D(M:u).$$

4.4 Proposition

If $u,v \in R^n - M^n$ *and* $v \equiv uc(\text{mod}M)$ *for some* $c \in I(M)$, *then* $D(M:u) = D(M:v)$.

Proof

Since we are dealing with maximal ideals it suffices to prove one inclusion only. Let $X \in D(M:u)$. Then each $X_i u \in M$. By hypothesis there exists $m \in M^n$ such that $v=uc+m$. Therefore $X_i v = X_i(uc) + X_i m = (X_i u)c + X_i m \in M$, since $X_i u$, $X_i m \in M$ and $c \in I(M)$. Thus $X \in D(M:v)$. Hence $D(M:u) \subset D(M:v)$ and so $D(M:u) = D(M:v)$, by the observation at the beginning of the proof. □

4.5 Proposition

$v \equiv uc(\text{mod}M)$ *for some* $c \in I(M)-M$ *if and only if* $u \equiv vc(\text{mod}M)$ *for some* $c \in I(M)-M$.

Proof

Suppose that $v \equiv uc(\text{mod}M)$ for some $c \in I(M)-M$. Then for each $i=1,\ldots,n$, $v_i=u_ic+m$ for some $m \in M$. We also have that the coset $c+M$ is invertible in the division ring $I(M)/M$; i.e. there exist elements $c' \in I(M)-M$ and $m' \in M$ such that $cc'=1+m'$. So for each $i=1,\ldots,n$ we have that $u_i = u_i1 = u_i(cc'-m') = u_icc' - u_im' = (v_i-m)c' - u_im' = v_ic' - mc' - u_im' = v_ic' + m''$, where $m'' = -mc' - u_im' \in M$. Thus $u \equiv vc'(\text{mod}M)$, where $c' \in I(M) - M$.

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By interchanging the roles of u and v, it is clear that the converse statement follows similarly. □

4.6 Remark

We observe that if A is a left ideal of R and D(A:u) = D(A:v), then the n-tuples u and v must behave alike (with respect to A) at each coordinate.

4.7 Proposition

Let D(A:u) = D(A:v) *and let* $i \in \{1,\ldots,n\}$. *Then*

4.7.1 $u_i \in A$ *if and only if* $v_i \in A$;

4.7.2 $u_i \in I(A)-A$ *if and only if* $v_i \in I(A)-A$

4.7.3 $u_i \in R-I(A)$ *if and only if* $v_i \in R-I(A)$.

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Proof

In view of Remark 4.6 above it suffices to prove each statement for the coordinates u_i only, since the proofs concerning the v_i's would proceed along the same lines.

4.7.1 Let $u_i \in A$. Then $e_{1i} = \begin{bmatrix} 0 & \ldots & 1 & \ldots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \ldots & 0 & \ldots & 0 \end{bmatrix} (u_1,\ldots, u_i,\ldots,u_n)'$

$= (u_i,\ldots, 0,\ldots, 0)' \in A^n$. Therefore $e_{1i} \in D(A:u) = D(A:v)$ and hence $e_{1i}v \in A^n$. But, as above, $e_{1i}v = (v_i,\ldots, 0,\ldots,0)'$ and so it follows that $v_i \in A$.

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4.7.2 Let $u_i \in I(A)-A$. Then $u_i \notin A$, and so by 4.7.1 above $v_i \notin A$. Let $a \in A$. Then $ae_{1i}u = a(u_i,\ldots,0)' = (au_i,\ldots,0)' \in A^n$, since $a \in A$ and $u_i \in I(A)$. Thus $ae_{1i} \in D(A:u) = D(A:v)$. Hence $ae_{1i}v \in A^n$ and so $(av_i,\ldots,0)' \in A^n$. Thus $av_i \in A$; i.e. $v_i \in I(A)-A$.

4.7.3 Let $u_i \in R-I(A)$. Then $u_i \notin I(A)$. By the definition of the idealizer it is clear that $u_i \notin A$ and so, again by 4.7.1 above, $v_i \notin A$. If, however, $v_i \in I(A)-A$, then by 4.7.2 above it follows that $u_i \in I(A)-A$, which would obviously contradict the hypothesis. Thus $v_i \notin I(A)-A$. Since we have also seen that $v_i \notin A$, it follows that $v_i \in R-I(A)$, as required. □

4.8 Example

Take n=2 and p=3 in Proposition 1.17. Then $M = \left\{f \in M_2(Z)[x]: \text{const}(f) \in \begin{bmatrix} 3Z & Z \\ 3Z & Z \end{bmatrix}\right\}$ is a maximal ideal of $R = M_2(Z)[x]$ and

$I(M) = \left\{g \in M_2(Z)[x]: \text{const}(g) \in \begin{bmatrix} Z & Z \\ 3Z & Z \end{bmatrix}\right\}$. Let c, u_1, u_2, v_1 and v_2 be polynomials in R such that $\text{const}(c) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$,

$\text{const}(u_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\text{const}(u_2) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$,

$\text{const}(v_1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\text{const}(v_2) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Put $u=(u_1,u_2)'$ and $v=(v_1,v_2)'$. Then $\text{const}(v_1-u_1c) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ and

$\text{const}(v_2-u_2c) = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$. We assert that $D(M:u) = D(M:v)$.

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Let us therefore consider any $X = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \in D(M:u)$ and suppose that $const(f_1) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $const(f_2) = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, $const(f_3) = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$ and $const(f_4) = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$. From $Xu \in M^2$ we have that $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} =$

$\begin{bmatrix} a_1-2b_1 & 0 \\ a_3-2b_3 & 0 \end{bmatrix} \in \begin{bmatrix} 3Z & Z \\ 3Z & Z \end{bmatrix}$ and similarly it follows that

$\begin{bmatrix} c_1-2d_1 & 0 \\ c_3-2d_3 & 0 \end{bmatrix} \in \begin{bmatrix} 3Z & Z \\ 3Z & Z \end{bmatrix}$. Hence $a_i-2b_i \equiv 0(mod3)$ and $c_i-2d_i \equiv 0(mod3)$, for $i=1,3$. However, by adding the respective congruences $3b_i \equiv 0(mod3)$ and $3d_i \equiv 0(mod3)$, the above congruences reduce to $a_i+b_i \equiv 0(mod3)$ and $c_i+d_i \equiv 0(mod3)$ for $i=1,3$. This imply that $X \in D(M:v)$, i.e. $D(M:u) \subset D(M:v)$. But since $D(M:u)$ and $D(M:v)$ are both maximal, eqaulity follows and the assertion is proved.

In the above example we note that $u_1,u_2,v_1,v_2 \in I(M)$, $c \in I(M)-M$ and $v \equiv uc(modM)$. Indeed we now have the following result.

4.9 Proposition

If each u_i *and* v_i *is in* $I(M)$, *then* $D(M:u) = D(M:v)$ *if and only if* $v \equiv uc(modM)$ *for some* $c \in I(M)-M$.

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Proof

If u and v are in M^n, then each u_i and each v_i is in M and hence in I(M). The necessary and sufficiency conditions are all satisfied, since $D(M:u) = M_n(R) = D(M:v)$ and for c we can choose the value 1. So we may assume that $u \notin M^n$. In order to prove the required result, we firstly assume that $D(M:u) = D(M:v)$. We distinguish between two types of $u_i \in I(M)$, namely $u_i \in I(M)-M$ and then $u_i \in M$. For $u_i \in I(M)-M$ there exists $w_i \in I(M)-M$ such that $u_i w_i + M = 1+M = w_i u_i + M$, since I(M)/M is a division ring. So there exist elements $m_i, m_i' \in M$ such that $u_i w_i = 1+m_i$ and $w_i u_i = 1+m_i'$. Let k be a fixed integer such that $u_k \in I(M)-M$. Then by 4.7.2 $v_k \in I(M)-M$. Thus $c = w_k v_k \in I(M)-M$. Now $v_k - u_k c = v_k - u_k(w_k v_k) = v_k - (u_k w_k)v_k = v_k - (1+m_k)v_k = v_k - v_k - m_k v_k = -m_k v_k \in M$. Now let j be another index such that $u_j \in I(M)-M$ and let $X = w_k e_{1k} - w_j e_{1j}$.

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$$\text{Thus } Xu = \begin{bmatrix} 0 & \dots & w_k & \dots & -w_j & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{bmatrix} (u_1,\dots,u_k,\dots,u_j,\dots,0)' =$$

$$\begin{bmatrix} w_k u_k - w_j u_j \\ \vdots \\ 0 \end{bmatrix} \in M^n.$$ Hence $X \in D(M:u) = D(M:v)$. Therefore

$Xv \in M^n$ and so $w_k v_k - w_j v_j \in M$; i.e. $c - w_j v_j \in M$, or $c = w_j v_j + m_j''$, for some $m_j'' \in M$. Now we have that $v_j - u_j c = v_j - u_j(w_j v_j + m_j'') = v_j - u_j w_j v_j - u_j m_j'' = v_j - (1+m_j)v_j - u_j m_j'' = -m_j v_j - u_j m_j'' \in M$, since $v_j \in I(M)$ and $m_j'' \in M$. Finally, if $u_j \in M$, then by 4.7.1 $v_j \in M$. So in any case $v_j - u_j c \in M$. We have therefore succeeded in proving that $v_j - u_j c \in M$ for each index j; i.e. $v \equiv uc(\text{mod}M)$, where $c \in I(M)-M$ is constructed as above.

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The converse was proved in Proposition 4.4, without the idealizer assumption on u and v. □

4.10 Example

In Proposition 1.16 choose n=2 and p=2. Then $M = \{f \in R : \text{const}(f) \in 2Z\}$ is a maximal ideal $R = Z[x]$ and $I(M) = Z[x]$. Let $u = (1-x^2, -1+x)'$, $v = (3-2x, 1+x^3)'$ and $c = 5+x$. Now $v_1-u_1c = 3-2x-(5+x-5x^2-x^3) = -2-3x+5x^2+x^3 \in M$ and $v_2-u_2c = 1+x^3-(-5+4x+x^2) = 6-4x-x^2+x^3 \in M$. So $v \equiv uc(\text{mod}M)$, $c \in I(M)-M$ and each u_i, $v_i \in I(M)$. Let $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \in D(M:u)$.

Then $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}(1-x^2, -1+x)' = \begin{bmatrix} f_1-f_2-f_1x^2+f_2x \\ f_3-f_4-f_3x^2+f_4x \end{bmatrix} \in M^2$. Thus

$\text{const}(f_1-f_2)$, $\text{const}(f_3-f_4) \in 2Z$; i.e. $\text{const}(f_1) \equiv \text{const}(f_2)(\text{mod}2)$ and $\text{const}(f_3) \equiv \text{const}(f_4)(\text{mod}2)$. So $D(M:u) = \left\{\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \in M_2(R) :\right.$ $\left.\text{const}(f_1) \equiv \text{const}(f_2)(\text{mod}2) \text{ and } \text{const}(f_3) \equiv \text{const}(f_4)(\text{mod}2)\right\}$.

On the other hand, if $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \in D(M:v)$, then

$\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}(3-2x,1+x^3)' = \begin{bmatrix} 3f_1+f_2-2f_1x+f_2x^3 \\ 3f_3+f_4-2f_3x+f_4x^3 \end{bmatrix} \in M^2$. Thus

$\text{const}(3f_1+f_2)$, $\text{const}(3f_3+f_4) \in 2Z$; i.e. $3\text{const}(f_1) + \text{const}(f_2) \in 2Z$ and $3\text{const}(f_3) + \text{const}(f_4) \in 2Z$; i.e. $3\text{const}(f_1) \equiv -\text{const}(f_2)(\text{mod}2)$ and $3\text{const}(f_3) \equiv -\text{const}(f_4)(\text{mod}2)$. Now if we add the congruence equations $-2\text{const}(f_1) \equiv 2\text{const}(f_2)(\text{mod}2)$ and $-2\text{const}(f_3) \equiv 2\text{const}(f_4)(\text{mod}2)$ to the appropriate ones above, we obtain $\text{const}(f_1) \equiv \text{const}(f_2)(\text{mod}2)$ and $\text{const}(f_3) \equiv \text{const}(f_4)(\text{mod}2)$.

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Hence $D(M:v) = \left\{ \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \in M_2(R) : \ const(f_1) \equiv const(f_2)(mod2)\right.$

$\left. \text{and } const(f_3) \equiv const(f_4)(mod2) \right\} = D(M:u).$

4.11 Remarks

4.11.1 M can be considered as a subset of $M_n(R)$ via the natural embedding of R in $M_n(R)$. So M generates the left ideal $M_n(M)$. By 3.12.1 we can restate Proposition 4.9 as follows. If $D(M:u)$ and $D(M:v)$ contain M, then they are equal if and only if $v \equiv uc(modM)$ for some $c \in I(M)-M$.

4.11.2 It may seem that the idealizer assumptions in Proposition 4.9 push everything inside $I(M)$, in which case we may as well assume initially that M is a two-sided ideal. However, the ideal $D(M:u)$ is still being calculated in $M_n(R)$. In fact, in Example 4.8 all the u_i are in $I(M)$, but $D(M:u)$ possesses an element having none of its entries in $I(M)$, namely

the element $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$ with $const(f_1) = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$,

$const(f_2) = \begin{bmatrix} 4 & 6 \\ 5 & 3 \end{bmatrix}$, $const(f_3) = \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix}$ and

$const(f_4) = \begin{bmatrix} 16 & -5 \\ 7 & 8 \end{bmatrix}$.

It is sometimes not so easy to compute the idealizer of a left ideal. We are now able to describe the idealizer of $D(M:u)$ in $M_n(R)$ whenever u behaves nicely enough with respect to M.

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4.12 Corollary

If each $u_i \in I(M)$, *then the idealizer of* D(M:u) *is given by* $I(D(M:u)) = \{X \in M_n(R): Xu \equiv uk(\text{mod}M)$ for some $k \in I(M)\}$.

Proof

Let $X \in M_n(R)$ such that $Xu \equiv uk(\text{mod}M)$ for some $k \in I(M)$. Consider any $Y \in D(M:u)$. Then $(YX)u = Y(Xu) \equiv Y(uk)(\text{mod}M) = (Yu)k(\text{mod}M)$ and so $(YX)u - (Yu)k \in M^n$. However, since $Yu \in M^n$ and $k \in I(M)$, it follows that $(Yu)k \in M^n$. But then we have that $(YX)u \in M^n$. Thus $YX \in D(M:u)$ and so $X \in I(D(M:u))$. Conversely we suppose that $X \in I(D(M:u))$, but $X \notin D(M:u)$ itself. Then by 2.20.3 it follows that $D(M:Xu) = (D(M:u):X)$ and by Proposition 3.15 $(D(M:u):X) = D(M:u)$. Hence $D(M:Xu) = D(M:u)$. Now by Proposition 4.9, with $v = Xu$, we have that $Xu \equiv uk(\text{mod}M)$ for some $k \in I(M)-M$. On the other hand, if $X \in D(M:u)$, then $Xu \in M^n$ and hence $Xu \equiv u.0(\text{mod}M)$. □

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4.13 Remark

In the case of a matrix ring over a commuative ring Corollary 4.12 says that the idealizer of D(M:u) consists of all matrices X which act on u like scalar multiplication modM; i.e. those X's which have u as an eigenvector modM.

4.14 Example

Let K be any commutative field. Consider any element

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$$X = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{j1} & \cdots & a_{jj} & \cdots & a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \in I(D(O:e_j)).$$ Then by Corollary 4.12

$Xe_j \equiv e_j k(\text{mod}O)$; i.e. $(a_{1j},\ldots, a_{jj},\ldots, a_{nj})' - (0,\ldots,k,\ldots,0)' = (a_{1j},\ldots, a_{jj}-k,\ldots, a_{nj})' = (0,\ldots,0,\ldots,0)'$. Thus $a_{jj}=k$ and

$a_{ij}=0$ for $i\neq j$ and so $I(D(O:e_j)) = \begin{bmatrix} K & \cdots & O & \cdots & K \\ \vdots & & \vdots & & \vdots \\ K & \cdots & K & \cdots & K \\ \vdots & & \vdots & & \vdots \\ K & \cdots & O & \cdots & K \end{bmatrix}$; i.e.

$I(D(O:e_j))$ consists of all matrices whose j-th column is zero off the diagonal. For the special case n=2 and j=1 we recover the well-known result $I(D(O:e_1)) = I\left(\begin{bmatrix} O & K \\ O & K \end{bmatrix}\right) = \begin{bmatrix} K & K \\ O & K \end{bmatrix}$.

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4.15 Example

Let $R=Z$, $M=2Z$, $n=2$ and $u=(1,0)'$. Then $D(M:u) = \begin{bmatrix} 2Z & Z \\ 2Z & Z \end{bmatrix}$ and

$I(M) = Z$. Suppose that $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in I(D(M:u))-D(M:u)$.

Then $\begin{bmatrix} x & y \\ z & w \end{bmatrix}(1,0)' \equiv (1,0)'k(\text{mod}2Z)$; i.e. $(x,z)' \equiv (k,0)'(\text{mod}2Z)$.

Hence $x-k \in 2Z$ and $z-0.k \in 2Z$. So by choosing $k=1 \in I(M)-M$,

we see that $X = \begin{bmatrix} 2a+1 & y \\ 2b & w \end{bmatrix}$. On the other hand, for $X \in D(M:u)$ we choose $k=0$ and so in this case $x=x-1.0 \in 2Z$ and $z \in 2Z$, in any case. Thus $I(D(2Z:(1,0)')) = \{X \in M_2(Z) : X(1,0)' \equiv (1,0)'k(\text{mod}2Z)$

where $k=1$ or $0\} = \begin{bmatrix} Z & Z \\ 2Z & Z \end{bmatrix}$, which is indeed the case.

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4.16 Corollary

If M *is two-sided, then* D(M:u) = D(M:v) *if and only if* $v \equiv uc(\text{mod}M)$ *for some* $c \in R-M$.

Proof

Since M is two-sided, I(M)=R and the result follows by Proposition 4.9. □

4.17 Corollary

If all u_i *and* v_i *are central in* R *(or even just central* modM), *then* D(M:u) = D(M:v) *if and only if* $v \equiv uc(\text{mod}M)$ *for some* $c \in I(M)-M$.

Proof

If x is central in R, then xr=rx for every $r \in R$. So in particular we have that mx=xm for every $m \in M$. Thus $x \in I(M)$. So the central elements u_i and v_i are therefore in I(M) and hence by Proposition 4.9, the result follows. On the other hand, if u_i and v_i are central modM, we have that $u_ix-xu_i \in M$ for each $x \in R$. So in particular for $m \in M$, $mu_i=u_im+m' \in M$. Thus $u_i \in I(M)$. Similarly it follows that $v_i \in I(M)$. Therefore, again by Proposition 4.9, the result follows. □

4.18 Corollary

D(M:u) = D(M:e_i) *if and only if* $u_i \in I(M)-M$ *and* $u_k \in M$ *for* $k \neq i$.

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Proof

Suppose that $D(M:u) = D(M:e_i)$. Now we have that

$D(M:e_i) = \begin{bmatrix} R & \cdots & M & \cdots & R \\ \vdots & & \vdots & & \vdots \\ R & \cdots & M & \cdots & R \end{bmatrix}$. By choosing $X \in D(M:u)$ suitably,

we are now able to prove that $u_k \in M$ for $k \neq i$, e.g. if X is the matrix having the entry 1 in the (1,k)-position (with $k \neq i$) and zero's elsewhere, then $u_k \in M$. On the other hand, for $m \in M$ let X be the matrix having the entry m in the (1,i)-position and zero's elsewhere. So it follows that $Xu = (0,\ldots,mu_i,\ldots,0)' \in M^n$; i.e. $mu_i \in M$ for every $m \in M$. Thus $u_i \in I(M)$. Moreover we have that $u_i \notin M$, otherwise it follows that $u \in M^n$ and hence $D(M:u) = M_n(R)$, an obvious contradiction. Thus $u_k \in M$ for $k \neq i$ and $u_i \in I(M)-M$. For the converse we suppose that $u_k \in M$ for $k \neq i$ and let $u_i \in I(M)-M$. Put $v = e_i = (0,\ldots,1,\ldots,0)'$. If we now choose $c=u_i$ and then interchange the roles of u and v in Proposition 4.9, it follows that $u-vc = (u_1,\ldots, u_i-1.u_i,\ldots,u_n)' = (u_1,\ldots,0,\ldots,u_n)' \in M^n$; $u \equiv vc(\text{mod}M)$. Since each u_i, $v_i \in I(M)$, it follows by Proposition 4.9 that $D(M:v) = D(M:u)$; i.e. $D(M:e_i) = D(M:u)$, as required. □

4.19 Example

Let n=2 and p=3 in Proposition 1.17. Then

$M = \left\{f \in R: \text{ const}(f) \in \begin{bmatrix} 3Z & Z \\ 3Z & Z \end{bmatrix}\right\}$ is a maximal ideal of

$R = M_2(Z)[x]$ and $I(M) = \left\{g \in R: \text{const}(g) \in \begin{bmatrix} Z & Z \\ 3Z & Z \end{bmatrix}\right\}$. Let

$u = (u_1,u_2)'$ where u_1 and u_2 are polynomials in R such that

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$\mathrm{const}(u_1) = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $\mathrm{const}(u_2) = \begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}$. Then $u_1 \in I(M)-M$, $u_2 \in M$, $i=2$ and $k=1\neq 2$. Now $D(M:e_1) = \begin{bmatrix}M & R\\ M & R\end{bmatrix} = \left\{\begin{bmatrix}f_1 & f_2\\ f_3 & f_4\end{bmatrix} \in M_2(R) : \mathrm{const}(f_1), \mathrm{const}(f_3) \in \begin{bmatrix}3Z & Z\\ 3Z & Z\end{bmatrix}\right\}$. Consider any $X = \begin{bmatrix}f_1 & f_2\\ f_3 & f_4\end{bmatrix} \in D(M:u)$ and suppose that

$\mathrm{const}(f_1) = \begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}$, $\mathrm{const}(f_2) = \begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix}$,

$\mathrm{const}(f_3) = \begin{bmatrix}c_1 & c_2\\ c_3 & c_4\end{bmatrix}$ and $\mathrm{const}(f_4) = \begin{bmatrix}d_1 & d_2\\ d_3 & d_4\end{bmatrix}$. Since $Xu \in M^2$, it follows that $\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix} =$

$\begin{bmatrix}a_1+3b_1 & b_1\\ a_3+3b_3 & b_3\end{bmatrix} \in \begin{bmatrix}3Z & Z\\ 3Z & Z\end{bmatrix}$ and similarly we have that

$\begin{bmatrix}c_1+3d_1 & d_1\\ c_3+3d_3 & d_3\end{bmatrix} \in \begin{bmatrix}3Z & Z\\ 3Z & Z\end{bmatrix}$. Hence a_1+3b_1, a_3+3b_3, c_1+3d_1, $c_3+3d_3 \in 3Z$ and so it follows that $a_1, a_3, c_1, c_3 \in 3Z$; i.e. $\mathrm{const}(f_1), \mathrm{const}(f_3) \in \begin{bmatrix}3Z & Z\\ 3Z & Z\end{bmatrix}$. Thus $X \in D(M:e_1)$ and so $D(M:u) \subset D(M:e_1)$. But since we are dealing with maximal ideals, equality follows.

4.20 Corollary

If K *is a commutative field, then* $D(O:u) = D(O:v)$ *in* $M_n(K)$ *if and only if* $u=cv$ *for some* $c\neq 0$ *in* K.

Proof

Since u and v are non-zero, it follows that $u,v \in R=I(O)$. Now by Proposition 4.9 $D(O:u) = D(O:v)$ if and only if $u \equiv vc \pmod{O}$ for some $c \in R-O$; i.e. $u_i-cv_i=0$ for some $c\neq 0$; i.e. $u=cv$ for some $c\neq 0$. □

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4.21 Example

Let $K=Z_5$, $v=(\bar{2},\bar{0})'$ and $c=\bar{3}$. Then $u=(\bar{1},\bar{0})'$ and so $u=cv$. Thus

$$D(O:u) = \begin{bmatrix} O & Z_5 \\ O & Z_5 \end{bmatrix} = D(O:v).$$

4.22 Remark

When n=1, Proposition 4.9 says that for $u,v \in I(M)$, $(M:u) = (M:v)$ if and only if $u \equiv vc(\mathrm{mod} M)$ for some $c \in I(M)-M$. However, we shall see in Corollary 4.26 that the restriction on u and v is not necessary for the equivalence.

In the next three results we adopt the following notation. Let S be a ring, $R = M_n(S)$, N a maximal left ideal of S and $w = (w_1,\ldots,w_n)' \in S^n-N^n$. Let $M' = D(N:w)$ in R and let $X = [x_{ij}]$ and Y be in R. In $I_S(N)$ and $I_R(M')$ it is understood that the subscript indicates the ring in which the idealizer is being computed.

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4.23 Proposition

If each $w_i \in I_S(N)$ *and each* $x_{ij} \in I_S(N)$ *and* $(M':X) = (M':Y)$, *then* $X \equiv YC(\mathrm{mod} M')$ *for some* $C \in I_R(M')-M'$.

Proof

By 2.20.3 $(M':X) = D(N:Xw)$ and so by hypothesis $D(N:Xw) = D(N:Yw)$. The hypotheses also gaurantee that the entries of Xw are in $I_S(N)$. Thus by Proposition 4.7 the entries of Yw are in $I_S(N)$. Hence we can invoke Proposition 4.9 to find $k \in I_S(N)-N$ such that

$$Xw \equiv Ywk(\mathrm{mod} N) \qquad \ldots\ldots 4.23.1$$

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Now since $I_S(N)/N$ is a division ring, it follows that for each $w_i \notin N$ there exist $y \in I_S(N)$ and $n_i \in N$ such that $y_i w_i = 1+n_i$. For each $i=1,\ldots,n$ we define c_i as follows.

$$c_i = \begin{cases} 0 & \text{if } w_i \in N, \\ w_i k y_i & \text{if } w_i \notin N. \end{cases}$$

Let C be the diagonal matrix $C = \text{diag}(c_1,\ldots,c_n)$. Then

$$Cw-wk = \begin{bmatrix} c_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & c_n \end{bmatrix}(w_1,\ldots,w_n)' - (w_1,\ldots,w_n)'k$$

$$= (c_1w_1,\ldots,c_nw_n)' - (w_1k_1,\ldots,w_nk)'$$

$$= (c_1w_1-w_1k,\ldots,c_nw_n-w_nk)' \qquad \ldots\ldots \; 4.23.2$$

Now each entry in 4.23.2 is in N, for if $w_i \in N$, then $c_i=0$ and hence $c_iw_i-w_ik = -w_ik \in N$ since $k \in I_S(N)$. On the other hand, if $w_i \notin N$, then $c_iw_i = w_iky_iw_i = w_ik(1+n_i) = w_ik + w_ikn_i$. Thus $c_iw_i - w_ik = w_ikn_i \in N$, since N is a left ideal. But then it means that

$$Cw - wk \in N^n \qquad \ldots\ldots \; 4.23.3$$

i.e. $Cw \equiv wk(\text{mod}N)$. Hence $YCw - Ywk = Y(Cw-wk) \in N^n$. But by 4.23.1 above $Xw-Ywk \in N^n$ and so by combining these results it follows that $Xw-YCw = (Xw-Ywk) - (YCw-Ywk) \in N^n$; i.e. $Xw \equiv YCw(\text{mod}N)$. Now $(X-YC)w \in N^n$; i.e. $X-YC \in D(N:w) = M'$. Thus $X \equiv YC(\text{mod}M')$. It remains to show that $C \in I_R(M')-M'$. We proceed as follows. Let $Z \in M' = D(N:w)$. Then $ZCw-Zwk = Z(Cw-wk) \in N^n$, because $Cw \equiv wk(\text{mod}N)$; i.e. $ZCw \equiv Zwk(\text{mod}N)$. But $Zwk \in N^n$, because $Zw \in N^n$ and $k \in I_S(N)$. Thus $ZCw-Zwk+Zwk \in N^n$; i.e. $ZCw \in N^n$ and so $ZC \in D(N:w) = M'$. Thus $C \in I_R(M')$. Finally we have by hypothesis that some $w_i \notin N$

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and so for them we have that $w_i k \notin N$, since $k \in I_S(N)$. Thus $wk \notin N^n$. But then it means that $C \notin M'$. For if $C \in M'$, then $Cw \in N^n$ and so together with 4.23.3 it follows that $wk = Cw-(Cw-wk) \in N^n$, which is an obvious contradiction. □

4.24 Example

Let $S=Z$, $R=M_2(S)$, $N=3Z$, $w=(1,1)'$, $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$.

Then $M' = D(3Z:(1,1)') = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R : a+b \equiv 0(\text{mod}3) \text{ and } c+d \equiv 0(\text{mod}3) \right\}$. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in (M':X)$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \in M'$; i.e. $a-b \equiv 0(\text{mod}3)$ and $c-d \equiv 0(\text{mod}3)$. Thus $(M':X) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R : a \equiv b(\text{mod}3) \text{ and } c \equiv d(\text{mod}3) \right\}$. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in (M':Y)$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4a-b & 3a \\ 4c-d & 3c \end{bmatrix} \in M'$;

$7a-b \equiv 0(\text{mod}3)$ and $7c-d \equiv 0(\text{mod}3)$. But $-6a \equiv 0(\text{mod}3)$ and $-6c \equiv 0(\text{mod}3)$ and hence by adding the respective congruences we get $a \equiv b(\text{mod}3)$ and $c \equiv d(\text{mod}3)$. Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in (M':X)$ and hence $(M':X) = (M':Y)$. So all the hypotheses of Proposition 4.23 hold.

We next assert that $I_R(M') = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in R : x+y \equiv z+w(\text{mod}3) \right\}$.

Let $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in I_R(M')$ and let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M'$. Then

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix} \in M'$. Thus $a(x+y)+b(z+w) = (ax+bz)+(ay+bw) \equiv 0(\text{mod}3)$ and similarly $c(x+y)+d(z+w) \equiv 0(\text{mod}3)$.

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Since the above congruences hold for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M'$, it follows that they indeed hold for all $a,b,c,d \in Z$ subject to the conditions $a+b \equiv 0 (\text{mod}3)$ and $c+d \equiv 0(\text{mod}3)$; i.e. $3|a+b$ and $3|c+d$. Let $A = \{(a,b) \in Z^2 : 3|a+b\}$. Then

$$I_R(M') = \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in R : 3|a(x+y)+b(z+w) \text{ and } 3|c(x+y)+d(z+w), \text{ where } (a,b), (c,d) \in A\right\} = \left\{\begin{bmatrix} a & y \\ z & w \end{bmatrix} \in R : 3|a(x+y)+b(z+w), \text{where } (a,b) \in A\right\} \cap \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in R : 3|c(x+y)+d(z+w), \text{ where } (c,d) \in A\right\} =$$

$$\left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in R : 3|(x+y)-(z+w)\right\} \cap \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in 3|(x+y)-(z+w)\right\} \overset{*}{}$$

$$= \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in R : 3|(x+y)-(z+w)\right\} = \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in R : x+y \equiv z+w (\text{mod}3)\right\},$$

where * follows from Proposition 1.18 with p=3. Thus our assertion is proved.

Now let $C = \begin{bmatrix} 1 & -3 \\ 8 & -4 \end{bmatrix}$. Then $C \in I_R(M')$, since $1-3-(8-4) = -6$ is divisible by 3. But $C \notin M'$, because 3 does not divide (1-3). Thus $C \in I_R(M')-M$. Finally we see that $X-YC = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} -$

$$\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 28 & -24 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -27 & 24 \\ 1 & -4 \end{bmatrix} \in M',$$

because $-27+24 = -3$ and $1-4 = -3$, which are both divisible by 3.

4.25 Corollary

If N *is a two-sided ideal of* S, *then* $(M':X) = (M':Y)$ *if and only if* $X \equiv YC(\text{mod}M')$ *for some* $C \in I(M')-M'$.

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Proof

Let N be a two-sided ideal of S and suppose that (M':X) = (M':Y). Then $I_S(N)$ = S and so all the hypotheses of Proposition 4.23 are satisfied. Thus $X \equiv YC(\text{mod} M')$ for som $C \in I_R(M')-M'$. For the converse we suppose that $X \equiv YC(\text{mod} M')$ for some $C \in I_R(M')-M'$, where M' = D(N:w) with $w \in S^n-N^n$. Then $X \equiv YC(\text{mod}(D(N:w)))$ and so $(X-YC)w \in N^n$; i.e. Xw = YCw+u for some $u \in N^n$. Let $Z \in$ (M':Y) = (D(N:Yw). Our aim is to show that $Z \in$ (M':X). Now we have that ZXw = Z(YCw+u) = Z(YC)w+Zu. Since $Z \in R$ and since N is an ideal of S, it follows that $Zu \in N^n$. It remains to show that $Z(YC)w \in N^n$. Now since $Z \in$ (M':Y) by assumption, we have that $ZYw \in N^n$ and so $ZY \in D(N:w) = M'$. However $C \in I_R(M')$ and hence we have that $(ZY)C \in M' = D(N:w)$; i.e. $(ZY)Cw \in N^n$; i.e. $Z(YC)w \in N^n$, which is what we intended to prove. Hence $ZXw \in N^n$ and so $Z \in D(N:Xw) = (M':X)$. Thus $(M':Y) \subset (M':X)$. But since these ideals are both maximal, equality follows. □

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As was remarked in 4.22 we shall now see that for the case n=1 in Proposition 4.9 we may dispense with the idealizer restrictions on u and v, namely that u and v be in I(M), in order for the equivalence to hold.

4.26 Corollary

If R *is a matrix ring over a commutative (or local or left quasi-duo) ring, then in* R, (M:u) = (M:v) *if and only if* $v \equiv uc(\text{mod} M)$ *for some* $c \in I(M)-M$.

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Proof

If $R = M_n(S)$, where S is a commutative (or local or left quasi-duo) ring, then every left ideal in S is two-sided. Let M be a maximal ideal of R. Then by Proposition 2.7 $M = D(N:w)$, where N is an ideal of S and $w \in S^n - N^n$. If we now let M'=M, X=u and Y=v in Corollary 4.25, then it follows that $(M:u) = (M:v)$ if and only if $u \equiv vc(\text{mod} M)$ for some $c \in I(M)-M$ in $M_1(R) = R$. □

§9 A COUNTING PRINCIPLE

In this section we make an attempt to count the number of maximal left ideals of $M_n(R)$, where R is a commutative ring. We first consider the special case where n=2 and R is a commutative field.

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4.27 Proposition

If K *is a commutative field, then*

4.27.1 $\text{Max}(M_2(K)) = \{D(0:u) : u=(0,1)' \text{ or } u=(1,c)', c \in K\}$;

4.27.2 $\text{card}(\text{Max}(M_2(K))) = \text{card}(K)+1$.

Proof

4.27.1 By Proposition 2.7 the maximal ideals of $M_2(K)$ are of the form $D(0:(0,c)')$, $D(0:(c,0)')$ and $D(0:(c,d)')$ where $c,d \neq 0$.

But $D(0:(0,c)') = D(0:(0,1)')$, for if $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in D(0:(0,c)')$,

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then $\begin{bmatrix} x & y \\ z & w \end{bmatrix}(0,c)' = (0,0)'$; i.e. $(yc,wc)' = (0,0)'$; i.e. $yc = wc = 0$. Now since $c \neq 0$, it follows that $y=w=0$; i.e. $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in D(0:(0,1)')$. Thus $D(0:(0,c)') \subset D(0:(0,1)')$ and since we are dealing with maximal ideals, equality follows. As above, it also follows that $D(0:(c,0)') = D(0:(1,0)')$. On the other hand, $D(0:(c,d)') = D(0:(1,c^{-1}d)')$, for if $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in D(0:(c,d)')$, then $\begin{bmatrix} x & y \\ z & w \end{bmatrix}(c,d)' = (0,0)'$. Thus $xc+yd=0$ and $zc+wd=0$. Since K is a commutative field and $c \neq 0$, it follows that $x+yc^{-1}d=0$ and $z+wc^{-1}d=0$; i.e. $\begin{bmatrix} x & y \\ z & w \end{bmatrix}(1,c^{-1}d)' = (0,0)'$ or $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in D(0:(1,c^{-1}d)')$. Hence $D(0:(c,d)') \subset D(0:(1,c^{-1}d)'$ and since we are dealing with maximal ideals, equality follows. Hence the maximal ideals of $M_2(K)$ are $D(0:(0,1)')$, $D(0:(1,0)')$ and $D(0:(1,c)')$, where $c \in K$ and $c \neq 0$. Thus $Max(M_2(K)) = \{D(0:u) : u=(1,0)'$ or $u=(1,c), c \in K\}$.

4.27.2 The map $f : Max(M_2(K)) \to K \cup \{\alpha\}$ defined by $f(D(0:1)')=\alpha$ and $f(D(0:(1,c)'))=c$, is a bijection. f is well-defined, for if $D(0:(1,c)') = D(0:(1,d)')$, then by Corollary 4.20 $(1,c)' = k(1,d)'$, for some $k \neq 0$ in K; i.e. $(1,c)' = (k,kd)'$. Thus $k=1$ and so $c=kd=d$. Hence $f(D(0:(1,c)')) = c = d = f(D(0:(1,d)'))$. Also $D(0:(0,1)')$ is mapped onto the unique element α and so we have that f is well-defined. f is one-to-one, since $D(0:(0,1)')$ is mapped onto α and if $f(D(0:(1,c)')) = f(D(0:(1,d)'))$, then $c=d$. Thus $D(0:(1,c)') = D(0:(1,d)')$. f is onto, since α is the image of $D(0:(0,1)')$ under f and given

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any $c \in K$, it follows that $M_c = D(0:(1,c)')$ is maximal left ideal of $M_2(K)$. So c is the image of M_c under f. Thus f is a bijection and so $\mathrm{card}(\mathrm{Max}(M_2(K))) = \mathrm{card}(K)+1$. □

4.28 Remark

By the preceding result we see that the maximal ideals of $M_2(K)$ are indexed by $(0,1)'$ and $(1,c)'$ for $c \in K$. Similarly for $n=3$, etc. the maximal left ideals of $M_3(K)$, etc. are indexed by $(0,0,1)'$, $(0,1,a)'$ and $(1,b,c)'$ for $a,b,c \in K$. If we let $q = \mathrm{card}(K)$, then for $n=2,3$, etc. it follows that the maximal left ideals of $M_2(K)$, $M_3(K)$, etc. are respectively $\sum_{i=0}^{1} q^i$, $\sum_{i=0}^{2} q^i$, etc. So in general $M_n(K)$ has $\sum_{i=0}^{n-1} q^i$ maximal left ideals.

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4.29 Example

Let $K=Z_3$. Then the maximal left ideals of $M_2(Z_3)$ are

$$D(0:(\bar{0},\bar{1})') = \begin{bmatrix} Z_3 & 0 \\ Z_3 & 0 \end{bmatrix},\ D(0:(\bar{1},\bar{0})') = \begin{bmatrix} 0 & Z_3 \\ 0 & Z_3 \end{bmatrix},$$

$$D(0:(\bar{1},\bar{1})') = \left\{\begin{bmatrix} \bar{x} & \bar{y} \\ \bar{z} & \bar{w} \end{bmatrix} \in M_2(Z_3) : \bar{x}+\bar{y}=\bar{0} \text{ and } \bar{z}+\bar{w}=\bar{0}\right\} \text{ and}$$

$$D(0:(\bar{1},\bar{2})') = \left\{\begin{bmatrix} \bar{x} & \bar{y} \\ \bar{z} & \bar{w} \end{bmatrix} \in M_2(Z_3) : \bar{x}+\bar{2}\bar{y}=\bar{0} \text{ and } \bar{z}+\bar{2}\bar{w}=\bar{0}\right\}.$$ Thus

$\mathrm{card}(M_2(Z_3)) = 4 = 3+1 = \mathrm{card}(Z_3)+1$.

Recalling that for a commuatative ring R, q_M denotes $\mathrm{card}(R/M)$ for $M \in \mathrm{Max}(R)$, we now have the following result.

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4.30 Proposition

Let R *be a commutative ring. Then* $M_n(R)$ *has* $\sum_M \sum_{i=0}^{n-1} q_M^i$ *maximal left ideals, where the outside sum is taken over* $M \in \text{Max}(R)$.

Proof

If M and N are distinct maximal ideals of R, then since they are two-sided, we see by Corollary 3.22 that the maximal ideals of $M_n(R)$ lying over M are all distinct from those lying over N. For M fixed, the map $f : M_n(R) \to M_n(R/M)$ defined by

$$f : \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \to \begin{bmatrix} a_{11}+M & \cdots & a_{1n}+M \\ \vdots & & \vdots \\ a_{n1}+M & \cdots & a_{nn}+M \end{bmatrix},$$ sets up a one-to-one

correspondence between those maximal left ideals of $M_n(R)$ lying over M and the maximal left ideals of $M_n(R/M)$. This can be seen as follows. Let $f : D(M:u) \to D(M/M:(1+M^n))$. Then f is well-defined, for if $D(M:u) = D(M:v)$, then since R is commutative, M is two-sided and so by Corollary 4.16 $v \equiv uc(\text{mod}M)$ for some $c \in R-M$. Thus for each $i=1,\ldots,n$ there exists $m \in M$ such that $v_i = u_i c+m$. From this we assert that $D(M/M : u+M^n) =$

$D(M/M : v+M^n)$, for if $X = \begin{bmatrix} a_{11}+M & \cdots & a_{1n}+M \\ \vdots & & \vdots \\ a_{n1}+M & \cdots & a_{nn}+M \end{bmatrix} \in D(M/M : u+M^n)$, then

$$X(v+M^n) = \begin{bmatrix} a_{11}+M & \cdots & a_{1n}+M \\ \vdots & & \vdots \\ a_{n1}+M & \cdots & a_{nn}+M \end{bmatrix}(v_1+M,\ldots,v_n+M)' =$$

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$$\begin{bmatrix} a_{11}+M & \cdots & a_{1n}+M \\ \vdots & & \vdots \\ a_{n1}+M & \cdots & a_{nn}+M \end{bmatrix}((u_1+M)(c+M),\ldots,(u_n+M)(c+M))' =$$

$$(c+M) \begin{bmatrix} a_{11}+M & \cdots & a_{1n}+M \\ \vdots & & \vdots \\ a_{n1}+M & \cdots & a_{nn}+M \end{bmatrix}(u_1+M,\ldots,u_n+M)' \in (c+M)(M/M,\ldots, M/M)',$$

since $X \in D(M/M : u+M^n)$. However, since $(c+M)(m+M) = cm+M = M$, it follows that $(c+M)(M/M,\ldots,M/M)' = (M/M,\ldots,M/M)' = (M/M)^n$ and hence $X \in D(M/M : v+M^n)$. Thus $D(M/M : u+M^n) \subset D(M/M : v+M^n)$ and since we are dealing with maximal ideals, equality follows and the assertion is proved. Thus $f(D(M:u)) = f(D(M:v))$.

f is one-to-one, for if $f(D(M:u)) = f(D(M:v))$, then $D(M/M : u+M^n) = D(M/M : v+M^n)$. By Corollary 4.20 $u+M^n = (c+M)(v+M^n)$ for some $c \notin M$; i.e. $(u_1+M,\ldots,u_n+M)' = (c+M)(v_1+M,\ldots,v_n+M)' = ((c+M)(v_1+M),\ldots,(c+M)(v_n+M))' = (cv_1+M,\ldots,cv_n+M)'$. Thus $u_1+M = cv_1+M,\ldots,u_n+M = cv_n+M$; i.e. $u_i - cv_i \in M$ for each $i=1,\ldots,n$; i.e. $u \equiv cv \pmod M$, where $c \in R-M$. Hence by Corollary 4.16 $D(M:u) = D(M:v)$, as required. Finally we see that f is onto, for given any maximal ideal $D(M/M : u+M^n)$ of $M_n(R/M)$, then $u+M^n \neq M^n$. Thus $u \notin M^n$ and so $D(M:u)$ is the required maximal ideal of $M_n(R)$ which is mapped onto $D(M/M : u+M^n)$. In view of the above bijection and since $M_n(R/M)$ has $\sum_{i=0}^{n-1} q_M^i$ maximal left ideals by Remark 4.28, it follows that there is the same amount of maximal left ideals of $M_n(R)$ lying over M. Thus $M_n(R)$ has exactly $\sum_M \sum_{i=0}^{n-1} q_M^i$ maximal left ideals. □

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4.31 Corollary

4.31.1 *The sum above is infinite unless* R *is semi-local and each residual field is finite.*

4.31.2 *In particular, if* m *is a positive integer, then* $M_n(Z_m)$ *has* $\sum_{p|m} \sum_{i=0}^{n-1} p^i = \sum_{p|m}(p^n-1)/(p-1)$ *maximal left ideals, where* p *is a prime number.*

Proof

4.31.1 If the sum is infinite we are done. If R is semi-local, let $M_1,\ldots,M_k$ be its maximal ideals. Then R/M_i is a field for each $i=1,\ldots,k$. By assumption $\text{card}(R/M_i) = q_{M_i}$ is finite. Hence by Proposition 4.30 above $M_n(R)$ has $s = \sum_{i=1}^{k} M_i \sum_{j=0}^{n-1} q_{M_i}^j$ maximal left ideals, which is obviously a finite number.

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4.31.2 Z_m has one maximal ideal for each prime p dividing m. So the total number of maximal ideals are $\sum_{p|m} \sum_{i=0}^{n-1} p^i =$

$$\sum_{p|m} (1+p+p^2+\ldots+p^{n-1}) = \sum_{p|m} (p^n-1)/(p-1).$$ □

4.32 Examples

4.32.1 Let $R = Z_6$ and let $n=2$. Then the maximal ideals of R are $M = \bar{2}Z_6$ and $N = \bar{3}Z_6$. Now $Z_6/M \simeq Z_2$ and $Z_6/N \simeq Z_3$. Thus $q_M = 2$ and $q_N = 3$, which are the prime divisors of 6. Also $\sum_{i=0}^{1} q_M^i = 1+2 = 3$ and $\sum_{i=0}^{1} q_N^i = 1+3 = 4$ and so according

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to Proposition 4.30 $M_2(Z_6)$ should have $\sum_{M\in Max(Z_6)} \sum_{i=0}^{1} q_M^i = 3+4=7$

maximal left ideals. Moreover, if we calculate the maximal ideals by using the formula in 4.31.2 with n=2, m=6 and p=2 and 3, we get $\sum_{p=2,3|6} \sum_{i=0}^{1} p^i = (2^2-1)/(2-1) + (3^2-1)/(3-1) = 3+4=7$,

which agrees with the number obtained above. Indeed the maximal left ideals of $M_2(Z_6)$ are

$$D(\bar{2}Z_6 : (\bar{1},\bar{1})') = \left\{\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix} \in M_2(Z_6) : \bar{a}+\bar{b},\ \bar{c}+\bar{d} \in 2Z_6\right\};$$

$$D(\bar{2}Z_6 : (\bar{1},\bar{0})') = \begin{bmatrix}\bar{2}Z_6 & Z_6\\ \bar{2}Z_6 & Z_6\end{bmatrix};\ D(\bar{2}Z_6 : (\bar{0},\bar{1})') = \begin{bmatrix}Z_6 & \bar{2}Z_6\\ Z_6 & \bar{2}Z_6\end{bmatrix};$$

$$D(\bar{3}Z_6 : (\bar{1},\bar{1})') = \left\{\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix} \in M_2(Z_6) : \bar{a}+\bar{b},\ \bar{c}+\bar{d} \in 3Z_6\right\};$$

$$D(\bar{3}Z_6 : (\bar{1},\bar{0})') = \begin{bmatrix}\bar{3}Z_6 & Z_6\\ \bar{3}Z_6 & Z_6\end{bmatrix};\ D(\bar{3}Z_6 : (0,1)') = \begin{bmatrix}Z_6 & \bar{3}Z_6\\ Z_6 & \bar{3}Z_6\end{bmatrix} \text{ and}$$

$$D(\bar{3}Z_6 : (\bar{2},\bar{1})') = \left\{\begin{bmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{bmatrix} \in M_2(Z_6) : \bar{a}+\bar{2}\bar{b},\ \bar{c}+\bar{2}\bar{d} \in \bar{3}Z_6\right\}.$$

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4.32.2 We observe that we can also apply the formula in 4.31.2 to Example 4.29 to get the $\sum_{3|3} \sum_{i=0}^{1} 3^i = 1+3 = 4$ maximal left ideals of $M_2(Z_3)$.

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CHAPTER 5 •

CONJUGATE IDEALS

Our main objective in this chapter is to investigate how the property of conjugacy is propagated to matrix rings; i.e. if M is conjugate to N in R, does it imply that D(M:u) is conjugate to D(N:v) in $M_n(R)$? We also study the seemingly easier question, namely for a given maximal ideal M of R, are all the D(M:u) conjugate to one another in $M_n(R)$?

We recall the following well-known result.

5.1 Proposition

If p *is a unit of* R, *then the map* $i_p : R \to R$ *defined by* $i_p : r \to prp^{-1}$ *is an automorphism which is called an inner automorphism.* □

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In view of the above result we now also have the following easily proved result.

5.2 Proposition

If p *is a unit of* R *and if* B *is a left ideal of* R, *then* $i_p(B) = pBp^{-1} = \{pbp^{-1} : b \in B\}$ *is a left ideal of* R. □

5.3 Definition

We say that two left ideals A and B are *conjugate* if $A = i_p(B) = pBp^{-1}$ for some unit p of R and we then write $A \sim B$.

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5.4 Proposition

The relation $\sim$ defined above is an equivalence relation on the collection of left ideals of R.

Proof

$\sim$ is reflexive since $A = 1A1^{-1}$ for any left ideal A of R. $\sim$ is symmetric, for if $A \sim B$, then $A = pBp^{-1}$ for some unit p of R. But then we have that $B = p^{-1}Ap = p^{-1}A(p^{-1})^{-1}$; i.e. $B \sim A$, because p^{-1} is also a unit of R. $\sim$ is transitive, for if $A \sim B$ and $B \sim C$, then there are units p and q such that $A = pBp^{-1}$ and $B = qCq^{-1}$. Thus $A = p(qCq^{-1})p^{-1} = pqCq^{-1}p^{-1} = pqC(pq)^{-1}$; i.e. $A \sim C$, since the product of two units is again a unit. This proves that $\sim$ defines indeed an equivalence relation on the collection of left ideals of R. □

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5.5 Remarks

5.5.1 Since we are dealing with left ideals, we can also say that $A \sim B$ if and only if $A = Bp$ for some unit p of R. Whenever it is convenient, we shall use this definition in stead.

5.5.2 If A and B are two-sided ideals of R, then $A \sim B$ if and only if $A = B$; i.e. when dealing with two-sided ideals, conjugacy means equality. This holds since $Bp = B$, where p is a unit of R.

5.6 Proposition

Let A *and* B *be left ideals of* R *such that* $A \sim B$, *say* $A = Bp$ *for some unit* p *of* R. *Then* $A = B$ *if and only if* $p, p^{-1} \in I(A)$.

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Proof

Since A is a two-sided ideal of I(A) and in view of Remark 5.5.2 above, if suffices to show that B is a two-sided ideal of I(A). Moreover, since B is a left ideal by hypothesis, we need only to show that it is also a right ideal of I(A). Now since $A = Bp$ for some unit p of R, we also have that $B = Ap^{-1}$. So let $b \in B$ and $x \in I(A)$. Then there exists $a \in A$ such that $bx = ap^{-1}x = ap^{-1}xpp^{-1} \in Ap^{-1} = B$, because $x, p^{-1}, p \in I(A)$ and $a \in A$. Thus B is a right ideal of I(A) and the result follows.□

5.7 Example

Consider the left ideal $A = \begin{bmatrix} 2Z & 0 \\ 2Z & 0 \end{bmatrix}$ of $R = M_2(Z)$. Then

$I(A) = \begin{bmatrix} Z & 0 \\ Z & Z \end{bmatrix}$.

5.7.1 Let $p = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. Then p is a unit of R. Indeed

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$p^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. Let $B = Ap$. Then $B \sim A$. Moreover, $B = A$, for if $x \in B$, then $x = \begin{bmatrix} 2a & 0 \\ 2b & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 2b & 0 \end{bmatrix} \in A$

and so $B \subset A$. On the other hand, if $x \in A$, then

$x = \begin{bmatrix} 2a & 0 \\ 2b & 0 \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 2b & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \in Ap = B$; i.e. $A \subset B$.

Thus $A = B$.

5.7.2 Let $p = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Then p is a unit of R and

$p^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Let $B = Ap$. Then again we have that $B \sim A$.

Consider any $x \in B$. Then $x = \begin{bmatrix} 2a & 0 \\ 2b & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2a & 4a \\ 2b & 4b \end{bmatrix}$;

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i.e. $B = \begin{bmatrix} 2Z & 4Z \\ 2Z & 4Z \end{bmatrix} \neq A$.

We observe therefore that in Example 5.7.1 both p and p^{-1} are in I(A) and so the equality of A and B follows. However, in Example 5.7.2 neither p nor p^{-1} lies in I(A) and hence $A \neq B$.

5.8 Proposition

If A *and* B *are conjugate left ideals of* R *such that one of them is maximal, then so is the other.*

Proof

Let $A \sim B$, say $A = Bp$ for some unit p of R and suppose that A is maximal. Let N be any left ideal of R such that $B \subsetneq N$. Then there exists an element x in N such that $x \notin B$. Thus $xp \notin A$. But $xp \in Np$. So $A \subsetneq Np$. But since A is maximal, it follows that $Np = R$. So there exists $n \in N$ such that $np = 1$ and hence $p^{-1} = n \in N$. Thus $1 = pp^{-1} \in N$ and so $N = R$. Therefore B is maximal as well. □

5.9 Example

Let $R = M_2(Z)$, $M = D(3Z:(1,1)') = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R : 3|a+b \text{ and } 3|c+d\right\}$ and $p = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Then p is a unit of R and by Proposition 2.7 M is a maximal ideal of R. Let $X \in Mp$.

Then $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -2a+b \\ c & -2c+d \end{bmatrix}$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M$;

i.e. $3|a+b$ and $3|c+d$. Thus $b = 3k-a$ and $d = 3k'-c$ for some $k,k' \in Z$. However, $-2a+b = -2a+3k-a = -3a+3k \in 3Z$ and similarly

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$-2c+d \in 3Z$. So any element of Mp is of the form

$$\begin{bmatrix} a & 3a' \\ c & 3c' \end{bmatrix} \in \begin{bmatrix} Z & 3Z \\ Z & 3Z \end{bmatrix} = D(3Z:(0,1)') = N, \text{ say.} \quad \text{Thus } Mp \subset N.$$

However, by Proposition 5.8 Mp is a maximal ideal of R and since N is obviously also a maximal ideal of R, it follows that N = Mp.

5.10 Proposition

If M *and* N *are conjugate maximal ideals of* R, *then* R/M *and* R/N *are isomorphic (simple) left* R-*modules.*

Proof

Since M and N are maximal, R/M and R/N are indeed simple left R-modules. Suppose next that M = Np for some unit p of R. Define a map $f : R/M \to R/N$ by the rule $f : r+Np \to rp^{-1}+N$. Then f is well-defined, for if $r+Np = r'+Np$, then there exists $n \in N$ such that $r = r'+np$. Thus $rp^{-1} = r'p^{-1}+n$ and so $rp^{-1}+N = r'p^{-1}+N$. f is an R-homomorphism. Let $r,r' \in R$. Then $f((r+Np)+(r'+Np)) = f(r+r'+Np) = (r+r')p^{-1}+N = rp^{-1}+r'p^{-1}+N = rp^{-1}+N+r'p^{-1}+N = f(r+Np)+f(r'+Np)$. Also $f(r(r'+Np)) = f(rr'+Np) = (rr')p^{-1}+N = r(r'p^{-1})+N = r(r'p^{-1}+N) = rf(r'+Np)$. f is one-to-one, for if $f(r+Np) = f(r'+Np)$, then $rp^{-1}+N = r'p^{-1}+N$. So there exists $n \in N$ such that $rp^{-1} = r'p^{-1}+n$. Thus by postmultiplying by p we get than $r = r'+np$ and hence $r+Np = r'+Np$. f is onto, for if $r+N \in R/N$, then the element $rp+Np$ is mapped by f onto it. Thus f is the required R-isomorphism. □

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5.11 Example

For any field K, $D(O:e_i) = \begin{bmatrix} K & \dots & O & \dots & K \\ \vdots & & \vdots & & \vdots \\ K & \dots & O & \dots & K \end{bmatrix}$ is a maximal ideal of $M_n(K)$. Let P be the invertible nxn elementary matrix interchanging the i-th and the j-th columns; i.e. $P = [e_1 \dots e_j \dots e_i \dots e_n]$, where e_j appears in i-th and e_i in the j-th column respectively. We assert that $D(O:e_i) = D(O:e_j)P$. By Proposition 5.8 $D(O:e_j)P$ is also a maximal ideal and hence it suffices to prove one inclusion only. So let $Y \in D(O:e_j)P$. Then $Y = XP$ for some $X \in D(O:e_j)$. Now $XP = X\,[e_1 \dots e_j \dots e_i \dots e_n] = [Xe_1 \dots Xe_j \dots Xe_i \dots Xe_n] = [x^1 \dots x^j \dots x^i \dots x^n]$, where the superscripts denote the columns of X. But since $X \in D(O:e_j)$, it means that the j-th column x^j of X consists of zero entries only; i.e. $Y = XP = [x^1 \dots O \dots x^i \dots x^n] \in D(O:e_i)$. Thus $D(O:e_j)P \subset D(O:e_i)$ and the assertion follows.

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5.12 Definition

If M is a maximal ideal of R such that all D(M:u) are conjugate to one another in $M_n(R)$, then M is called a *c.p. ideal*.

5.13 Proposition

If M *and* N *are conjugate maximal left ideals of* R *with* $N = pMp^{-1}$ *for a unit* p *of* R *and if* $u \in R^n - N^n$, *then* $D(N:u) = D(M:up)$.

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Proof

$up \notin M^n$, for if not, then $u_i p \in M$ for each $i=1,\ldots,n$. Thus $p(u_i p)p^{-1} \in N$; i.e. $pu_i \in N$. But since p is a unit and N is a left ideal of R, it follows that $u_i = p^{-1}pu_i \in N$ for each $i=1,\ldots,n$; i.e. $u \in N^n$, which is a contradiction. By Proposition 2.1 $D(M:up)$ and $D(N:u)$ are proper ideals of $M_n(R)$ and by Proposition 2.7 they are maximal. Therefore it suffices to prove that $D(N:u) \subset D(M:up)$. So let $X \in D(N:u)$. Then for each $i=1,\ldots,n$ it follows that $X_i u \in N$. Thus $X_i u = pmp^{-1}$ for some $m \in M$. Hence $X_i up = pm \in M$ for each $i=1,\ldots,n$ Therefore $X \in D(M:up)$ and required inclusion follows. Thus $D(N:u) = D(M:up)$. □

5.14 Example

Let $R = Z_9$ and let $M = \bar{3}Z_9$. Then the units of R are $\bar{1},\bar{2},\bar{4},\bar{5},\bar{7}$ and $\bar{8}$. If $u = (\bar{1},\bar{2},\bar{8})'$ and $p = \bar{2}$, then $up = (\bar{2},\bar{4},\bar{7})'$. Also, since R is commutative $N = Np = M$. Let $X \in D(N:(\bar{1},\bar{2},\bar{8})')$,

say $X = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{bmatrix}$. Then $Xu \in N^3 = M^3$ and so the

following conditions hold; $\bar{a}_1 + \bar{2}\bar{a}_2 + \bar{8}\bar{a}_3 \in M$, $\bar{b}_1+\bar{2}\bar{b}_2+\bar{8}\bar{b}_3 \in M$, $\bar{c}_1+\bar{2}\bar{c}_2+\bar{8}\bar{c}_3 \in M$. But then it follows that $\bar{2}(\bar{a}_1+\bar{2}\bar{a}_2+\bar{8}\bar{a}_3) = \bar{2}\bar{a}_1+\bar{4}\bar{a}_2+\bar{7}\bar{a}_3 \in M$, and similarly $\bar{2}\bar{b}_1+\bar{4}\bar{b}_2+\bar{7}\bar{b}_3 \in M$ and $\bar{2}\bar{c}_1+\bar{4}\bar{c}_2+\bar{7}\bar{c}_3 \in M$.

Thus $\begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{bmatrix}(\bar{2},\bar{4},\bar{7})' \in M^3$; i.e. $X \in D(M:(\bar{2},\bar{4},\bar{7})')$.

Therefore $D(N:(\bar{1},\bar{2},\bar{8})') = D(M:(\bar{2},\bar{4},\bar{7})') = D(M:(\bar{1},\bar{2},\bar{8})'\bar{2})$.

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5.15 Proposition

If $P \in GL_n(R)$, *then* $PD(M:u)P^{-1} = D(M:Pu)$.

Proof

As before, it suffices to prove one inclusion only. Let $X \in PD(M:u)P^{-1}$. Then $X = PYP^{-1}$ for some $Y \in D(M:u)$. Now $XP = PY$ and so $X(Pu) = (XP)u = (PY)u = P(Yn) \in M^n$, since $Yu \in M^n$. Therefore $X \in D(M:Pu)$ and hence $PD(M:u)P^{-1} \subset D(M:Pu)$. Therefore $PD(M:u)P^{-1} = D(M:Pu)$, as required. □

5.16 Example

Let $R = M_2(Z)$, $M = \begin{bmatrix} 2Z & Z \\ 2Z & Z \end{bmatrix}$ and let

$$P = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \in GL_2(R). \quad \text{Then } P^{-1} = P.$$

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Let $u = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right)' \in R^2 - M^2$. Then

$$Pu = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}.$$

Consider any $X \in D(M:Pu)$. Then $XPu \in M^2$, i.e.

$$\begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a_1 & 0 \\ a_3 & 0 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ b_3 & 0 \end{bmatrix} \\ \begin{bmatrix} c_1 & 0 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ d_3 & 0 \end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix} a_1+b_1 & 0 \\ a_3+b_3 & 0 \end{bmatrix} \\ \begin{bmatrix} c_1+d_1 & 0 \\ c_3+c_3 & 0 \end{bmatrix} \end{bmatrix} \in M^2.$$ This reduces to $a_i+b_i \equiv 0 \pmod 2$ and

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$c_i+d_i \equiv 0 \pmod 2$, for $i=1,3$. Thus

$$D(M:Pu) = \left\{ \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix} \in M_2(R) \ : \ a_i+b_i \equiv 0 \pmod 2, \ c_i+d_i \equiv 0 \pmod 2 \right.$$

$\left. \text{for } i=1,3 \right\}$. On the other hand, if $X \in D(M:u)$, then

$$Xu = \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ c_3 & c_4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a_1-b_1 & 0 \\ a_3-b_3 & 0 \end{bmatrix} \\ \begin{bmatrix} c_1-c_1 & 0 \\ c_3-d_3 & 0 \end{bmatrix} \end{bmatrix} \in M^2;$$

i.e. $a_i \equiv b_i \pmod 2$, $c_i \equiv d_i \pmod 2$ for $i=1,3$. Hence

$$D(M:u) = \left\{ \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix} \in M_2(R) \ : \ a_i \equiv b_i \pmod 2, \ c_i \equiv d_i \pmod 2 \right.$$

$\left. \text{for } i=1,3 \right\}$ Let us finally consider any $Y \in PD(M:u)P^{-1}$. Then $Y = PXP^{-1}$ for some $X \in D(M:u)$. Thus

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$$Y = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix},$$

$$= \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} -c_1 & -c_2 \\ -c_3 & -c_4 \end{bmatrix} & \begin{bmatrix} -d_1 & -d_2 \\ -d_3 & -d_4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} -b_1 & -b_2 \\ -b_3 & -b_4 \end{bmatrix} \\ \begin{bmatrix} -c_1 & -c_2 \\ -c_3 & -c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix},$$

subject to $a_i \equiv b_i \pmod 2$ and $c_i \equiv d_i \pmod 2$ for $i=1,3$; i.e. $a_i-b_i \equiv 0 \pmod 2$ and $c_i-d_i \equiv 0 \pmod 2$ for $i=1,3$; i.e. $a_i+(-b_i) \equiv 0 \pmod 2$ and $-(c_i+(-d_i)) \equiv 0 \pmod 2$ for $i=1,3$. So $a_i+(-b_i) \equiv 0 \pmod 2$ and $(-c_i)+d_i \equiv 0 \pmod 2$ for $i=1,3$. This means that $Y \in D(M:Pu)$, and so it follows that $D(M:Pu) = PD(M:u)P^{-1}$.

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5.17 Remark

We note that showing that M is c.p is equivalent to show that for any $u \in R^n - M^n$ we get $D(M:u) \sim D(M:e_1)$, since all $D(M:u)$ should be conjugate to one another and obviously $e_1 \in R^n - M^n$. Therefore since $PD(M:u)P^{-1} = D(M:Pu)$ by Proposition 5.15, we therefore have to show that $D(M:e_1) = PD(M:u)P^{-1} = D(M:Pu)$ and so by Corollary 4.18 it would therefore be sufficient to show the existence of $P \in GL_n(R)$ such that $P_1u \in I(M)-M$ and $P_iu \in M$ for $i \neq 1$; i.e. for $i \geq 2$; i.e. to find an invertible matrix whose first row "pushes" u into the idealizer of M (but not into M) and whose other rows "push" u into M. On the other hand, to show conjugacy by writing $D(M:u) = PD(M:e_1)P^{-1} = D(M:Pe_1)$, it would be sufficient to show that any u is congruent modM to a column of an invertible matrix, because $Pe_1 = P^1$, the first column of P.

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5.18 Proposition

Let $u \in R^n - M^n$. *If* $v \equiv Puc$ (modM) *for some* $P \in GL_n(R)$ *and* $c \in I(M)-M$, *then* $D(M:u) \sim D(M:v)$.

Proof

By Proposition 5.15 $PD(M:u)P^{-1} = D(M:Pu)$ and by Proposition 4.2 $D(M:Pu) = D(M:v)$. Therefore $D(M:v) = PD(M:u)P^{-1}$ and hence $D(M:u) \sim D(M:v)$. □

5.19 Remark

In particular, if v is a permutation of the entries of u in

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Proposition 5.18, then $v = Pu$, with P a product of row-interchanging matrices and hence $D(M:v) = D(M:Pu) = PD(M:u)P^{-1}$; i.e. $D(M:v) \sim D(M:u)$.

5.20 Proposition

5.20.1 *If some* $u_i \in I(M)-M$, *or*

5.20.2 *if some* u_i *is congruent* modM *to unit of* R,

then $D(M:u) \sim D(M:e_i)$.

Proof

By Remark 5.19 we may let $i=1$ in either case.

5.20.1 Since M is a maximal ideal of R and since $u_1 \notin M$, there exist elements $b \in R$ and $m \in M$ such that $bu_1+m=1$. Then for $i=2,\ldots,n$ we have that $(u_ib)u_1-u_i = u_i(bu_1)-u_i = u_i(1-m)-u_i = u_i-u_im-u_i = -u_im \in M$. Let X be the nxn matrix having $(0,u_2b,u_3b,\ldots,u_nb)'$ as its first column and zero's elsewhere and let I denote the nxn identity matrix. Then

$$X^2 = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ u_2b & 0 & \ldots & 0 \\ u_3b & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ u_nb & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \ldots & 0 \\ u_2b & 0 & \ldots & 0 \\ u_3b & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} = 0 \text{ and } P = X + I \text{ is}$$

invertible, since $P(I-X) = (X+I)(I-X) = X-X^2+I-X =$

$$X+I-X = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ u_2b & 1 & 0 & \ldots & 0 \\ u_3b & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ u_nb & 0 & 0 & \ldots & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ u_2b & 0 & 0 & \ldots & 0 \\ u_3b & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ u_nn & 0 & 0 & & 0 \end{bmatrix} = I;$$

i.e. $P^{-1} = I-X$. Furthermore, $Pe_1u_1 \equiv u(\text{mod}M)$, because $Pe_1u_1-u =$

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$$(X+I)e_1u_1-u = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ u_2b & 1 & 0 & \dots & 0 \\ u_3b & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ u_nb & 0 & 0 & \dots & 1 \end{bmatrix} (1,0,0,\dots,0)'u_1-(u_1,u_2,u_3,\dots,u_n)'$$

$= (1,u_2b,u_3b,\dots,u_nb)'u_1-(u_1,u_2,u_3,\dots,u_n)' =$

$(u_1,u_2bu_1,u_3bu_1,\dots,u_nbu_1)' - (u_1,u_2,u_3,\dots,u_n)' =$

$(0,(u_2b)u_1-u_2,\ (u_3b)u_1-u_3,\dots,\ (u_nb)u_1-u_n)' \in M^n$, since

$(u_ib)u_1-u_i \in M$ for each $i=2,3,\dots,n$. Hence $Pe_1u_1 \equiv u(\text{mod}M)$.

Thus, since $u_1 \in I(M)-M$ we have from Proposition 5.18 that

$D(M:u) \sim D(M:e_1)$.

5.20.2 Let u_1 be a unit of R and let P be the nxn matrix having u as its first column, the other diagonal elements unity and zero's elsewhere. Then $P \in GL_n(R)$, because

$$\begin{bmatrix} u_1 & 0 & \dots & 0 \\ u_2 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ u_n & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} u_1^{-1} & 0 & \dots & 0 \\ -u_2u_1^{-1} & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -u_nu_1^{-1} & 0 & \dots & 1 \end{bmatrix} = I. \quad \text{Now } Pe_1 = \begin{bmatrix} u_1 & 0 & \dots & 0 \\ u_2 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ u_n & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} =$$

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$(u_1,u_2,\dots,u_n)' = u$. Thus by Proposition 5.18 with c=1, it follows that $D(M:u) \sim D(M:e_1)$. □

5.21 Example

Let R=Z and let M=5Z. Then I(M)=Z. Let $u=(3,1,0)'$, then $u_1=3$, $u_2=1$, $u_3=0$. Now $(-3)3+10=1$ and so $b = -3$ and $m = 10 \in 5Z$. Also $u_2bu_1-u_2 = 1(-3)3-1 = -10 \in 5Z$ and $u_3bu_1-u_3 = 0 \in 5Z$.

Let $X = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $P = X+I = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$p^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Now let $X \in D(5Z : (3,1,0)')$ and suppose that

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$X = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Then $Xu = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}(3,1,0)' =$

$\begin{bmatrix} 3a_1+a_2 \\ 3b_1+b_2 \\ 3c_1+c_2 \end{bmatrix} \in (5Z)^3$. Thus $D(5Z : (3,1,0)') =$

$$\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \in M_3(Z) : 3a_1+a_2 \in 5Z,\ 3b_1+b_2 \in 5Z,\ 3c_1+c_2 \in 5Z\right\}.$$

Also $D(5Z:e_1) = \begin{bmatrix} 5Z & Z & Z \\ 5Z & Z & Z \\ 5Z & Z & Z \end{bmatrix}$ and so if $Y \in PD(5Z:e_1)P^{-1}$, then

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5a_1 & a_2 & a_3 \\ 5b_1 & b_2 & b_3 \\ 5c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 5a_1 & a_2 & a_3 \\ -15a_1+5b_1 & -3a_2+b_2 & -3a_3+b_3 \\ 5c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 5a_1+3a_2 & a_2 & a_3 \\ -15a_1+5b_1-9a_2+3b_2 & -3a_2+b_2 & -3a_3+b_3 \\ 15c_1+3c_2 & c_2 & c_3 \end{bmatrix}.$$ However

$3(5a_1+3a_2)+a_2 = 15a_1+10a_2 \in 5Z$, $3(-15a_1+5b_1-9a_2+3b_2)+(-3a_2+b_2) =$
$-45a_1+15b_1-30a_2+10b_2 \in 5Z$ and $3(15c_1+3c_2)+c_2 = 45c_1+10c_2 \in 5Z$
and hence $Y \in D(5Z:(3,1,0)')$. Thus $D(5Z:(3,1,0)') = PD(5Z:e_1)P^{-1}$;
i.e. $D(5Z:(3,1,0)') \sim D(5Z:e_1)$.

5.22 Proposition

If M *and* N *are conjugate maximal left ideals of* R *and some* u_i *satisfies* 5.20.1 *or* 5.20.2 *and some* v_j *satisfies* 5.20.1 *or* 5.20.2 *(with respect to* N*), then* $D(M:u) \sim D(N:v)$.

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Proof

Say M=Np for some unit p of R. By Proposition 5.20 it suffices to show that $D(M:e_1) \sim D(N:e_1)$, which in turn is equal to $D(N:e_1p)$, by Proposition 5.13. Let P = diag(p,1,...,1), which is certainly invertible because $P^{-1} = \text{diag}(p^{-1},1,\ldots,1)$.

Moreover, P satisfies $Pe_1 = \begin{bmatrix} p & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}(1,0,\ldots,)' = (p,0,\ldots,0)' =$

$(1,0,\ldots,0)'p = e_1p$. Thus by Proposition 5.13 $D(N:e_1) = D(M:e_1p) = D(M:Pe_1) = PD(M:e_1)P^{-1}$, by Proposition 5.15. Hence $D(M:e_1) \sim D(N:e_1)$, as required. □

5.23 Example

Let $M = D(3Z:(1,1)')$ and $N = D(3Z:(0,1'))$ be as in Example 5.9. Then $M \sim N$; in fact $N = M\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Let $u = \left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)'$

and $v = \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)'$. Let $X \in D(M:u)$. Then Xu =

$$\begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a_1 & -a_1+a_2 \\ a_3 & -a_3+a_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & -c_1+c_2 \\ c_3 & -c_3+c_4 \end{bmatrix} \end{bmatrix} \in M^2.$$ Hence

$3 \mid a_1+(-a_1+a_2) = a_2$. Similarly it follows that $3 \mid a_4$, $3 \mid c_2$ and $3 \mid c_4$. Thus $D(M:u) = \begin{bmatrix} \begin{bmatrix} Z & 3Z \\ Z & 3Z \end{bmatrix} & M_2(Z) \\ \begin{bmatrix} Z & 3Z \\ Z & 3Z \end{bmatrix} & M_2(Z) \end{bmatrix}$. On the other hand,

$$X \in D(N:v), \text{ then } Xv = \begin{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} & \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} & \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{bmatrix} =$$

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$$\begin{bmatrix}\begin{bmatrix}-a_1+b_2 & a_2\\ -a_3+b_4 & a_4\end{bmatrix}\\ \begin{bmatrix}-c_1+d_2 & c_2\\ -c_3+d_4 & c_4\end{bmatrix}\end{bmatrix} \in N^2.$$ Hence $a_2, a_4, c_2, c_4 \in 3Z$ and so

$$D(N:v) = \begin{bmatrix}\begin{bmatrix}Z & 3Z\\ Z & 3Z\end{bmatrix} & M_2(Z)\\ \begin{bmatrix}Z & 3Z\\ Z & 3Z\end{bmatrix} & M_2(Z)\end{bmatrix}.$$ Thus $D(M:u) = D(N:v)$ in

$M_2(M_2(Z))$ and hence they are equivalent. □

5.24 Remark

It is interesting to note that the previous example actually tells us more than what we expected, namely, for given maximal conjugate left ideals M and N of R, it is possible that in $M_n(R)$ we obtain equality of $D(M:u)$ and $D(N:v)$.

5.25 Proposition

Every two-sided maximal left ideal is c.p.

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Proof

Let M be a two-sided maximal left ideal of R. The $I(M) = R$, and so for any two maximal ideals $D(M:u)$ and $D(M:v)$ of $M_n(R)$ some $u_i \notin M$ and some $v_i \notin M$; i.e. some $u_i \in I(M)-M$ and some $v_i \in I(M)-M$. So by Proposition 5.22 with $N = M$, it follows that $D(M:u) \sim D(M:v)$; i.e. M is c.p. □

5.26 Proposition

Let M *be a maximal ideal of* R. *Then*

5.26.1 $M \subset D(M:u)$ *if and only if each* $u_i \in I(M)$;

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5.26.2 *all the maximal left ideals of* $M_n(R)$ *which contain* M *are conjugate, even if* M *is not c.p.*

Proof

5.26.1 Suppose that $M \subset D(M:u)$. Let $m \in M$ be given. Then

$$X = \text{diag}(m,\ldots,m) = \begin{bmatrix} m & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & m \end{bmatrix} \in D(M:u) \text{ and so } Xu \in M^n; \text{ i.e.}$$

$$\begin{bmatrix} m & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & m \end{bmatrix}(u_1,\ldots,u_n) \in M^n; \text{ i.e. } mu_i \in M \text{ for each } i=1,\ldots,n.$$

Thus each $u_i \in I(M)$. For the converse we suppose that each $u_i \in I(M)$. Let $m \in M$. Then, regarded as an element of $M_n(R)$,

$$m = \begin{bmatrix} m & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & m \end{bmatrix} \text{ and so } mu = \begin{bmatrix} m & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & m \end{bmatrix}(u_1,\ldots,u_n)' =$$

$(mu_1,\ldots,mu_n)' \in M^n$, since each $u_i \in I(M)$. Hence $m \in D(M:u)$ and so $M \subset D(M:u)$.

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5.26.2 Let $D(M:u)$ and $D(M:v)$ be maximal ideals of $M_n(R)$ such that $M \subset D(M:u)$ and $M \subset D(M:v)$. Then by 5.26.1 above each $u_i, v_i \in I(M)$. However, not all $u_i, v_i \in M$, for otherwise it would mean that $D(M:u) = D(M:v) = M_n(R)$, an obvious contradiction. So by 5.20.1 it follows that $D(M:u) \sim D(M:e_i) \sim D(M:v)$. Thus $D(M:u) \sim D(M:v)$. □

5.27 Proposition

If M *and* N *are two-sided non-conjugate (i.e. non-equal) maximal left ideals of* R, *then any proper ideals* $D(M:u)$ *and* $D(M:v)$ *are non-conjugate in* $M_n(R)$.

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Proof

Suppose that the proper ideals D(M:u) and D(M:v) are conjugate in $M_n(R)$. Then there exists $P \in GL_n(R)$ such that D(M:u) = $PD(N:v)P^{-1}$ = D(N:Pv), by Proposition 5.15. However, by Corollary 3.26 D(M:u) contract to M and D(N:Pv) contracts to N. This is a contradiction, because $M \neq N$ by hypothesis. Thus D(M:u) and D(N:v) are non-conjugate. □

5.28 Corollary

If R *is a local ring, then all the maximal left ideals of* $M_n(R)$ *are conjugate.*

Proof

Since R is a local ring it has a unique maximal left ideal M, which is two-sided. So by Proposition 5.25 M is c.p. and hence all the maximal left ideals D(M:u) of $M_n(R)$ are conjugate. □

5.29 Corollary

If K *is a field, then all the maximal left ideals of* $M_n(K)$ *are conjugate.*

Proof

Since K is a local ring, the result follows by Corollary 5.28. □

5.30 Example

Let $K = Z_3$ and let $R = M_2(Z_3)$. Then by Example 4.30 the maximal left ideals of $M_2(Z_3)$ are $A_1 = \begin{bmatrix} Z_3 & 0 \\ Z_3 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & Z_3 \\ 0 & Z_3 \end{bmatrix}$

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$A_3 = \left\{\begin{bmatrix}\bar{x} & \bar{y}\\ \bar{z} & \bar{w}\end{bmatrix} \in M_2(Z_3) : \bar{x}+\bar{y} = \bar{0} \text{ and } \bar{z}+\bar{w} = \bar{0}\right\}$ and

$A_4 = \left\{\begin{bmatrix}\bar{x} & \bar{y}\\ \bar{z} & \bar{w}\end{bmatrix} \in M_2(Z_3) : \bar{x}+\bar{2}\bar{y} = \bar{0} \text{ and } \bar{z}+\bar{2}\bar{w} = \bar{0}\right\}$. Now we have the following equivalences (In each case we prove one inclusion only, since the ideals under discussion are all maximal).

$A_1 \sim A_2$:

Let $X \in A_2p$, where p is the unit $\begin{bmatrix}\bar{0} & \bar{1}\\ \bar{1} & \bar{0}\end{bmatrix}$. Then

$X = \begin{bmatrix}\bar{0} & \bar{x}\\ \bar{0} & \bar{y}\end{bmatrix}\begin{bmatrix}\bar{0} & \bar{1}\\ \bar{1} & \bar{0}\end{bmatrix} = \begin{bmatrix}\bar{x} & \bar{0}\\ \bar{y} & \bar{0}\end{bmatrix} \in A_1$ and so $A_2p \subset A_1$.

Thus $A_1 = A_2p$.

$A_1 \sim A_3$:

Let $X \in A_3p$ where p is the unit $\begin{bmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{1}\end{bmatrix}$. Then

$X = \begin{bmatrix}\bar{x} & \bar{y}\\ \bar{z} & \bar{w}\end{bmatrix}\begin{bmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{1}\end{bmatrix} = \begin{bmatrix}\bar{x} & \bar{x}+\bar{y}\\ \bar{z} & \bar{z}+\bar{w}\end{bmatrix}$. But since $\begin{bmatrix}\bar{x} & \bar{y}\\ \bar{z} & \bar{w}\end{bmatrix} \in A_3$ we

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indeed have that $\bar{x}+\bar{y} = \bar{0}$ and $\bar{z}+\bar{w} = \bar{0}$. Thus $X = \begin{bmatrix}\bar{x} & \bar{0}\\ \bar{z} & \bar{0}\end{bmatrix} \in A_1$ and hence $A_1 = A_3p$.

$A_1 \sim A_4$:

Let $X \in A_4p$ where p is the unit $\begin{bmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{2}\end{bmatrix}$. Then

$X = \begin{bmatrix}\bar{x} & \bar{y}\\ \bar{z} & \bar{w}\end{bmatrix}\begin{bmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{2}\end{bmatrix} = \begin{bmatrix}\bar{x} & \bar{x}+\bar{2}\bar{y}\\ \bar{z} & \bar{z}+\bar{2}\bar{w}\end{bmatrix}$. However $\begin{bmatrix}\bar{x} & \bar{y}\\ \bar{z} & \bar{w}\end{bmatrix} \in A_4$

and so $\bar{x}+\bar{2}\bar{y} = \bar{0}$ and $\bar{z}+\bar{2}\bar{w} = \bar{0}$. Therefore $X = \begin{bmatrix}\bar{x} & \bar{0}\\ \bar{y} & \bar{0}\end{bmatrix} \in A_1$ and so $A_4p \subset A_1$. Thus $A_1 = A_4p$.

Now since $\sim$ is an equivalence relation, it follows that all the maximal left ideals of $M_2(Z_3)$ are conjugate.

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The final result shows that the c.p. property propagates itself.

5.31 Proposition

If $M \subset R$ *is a c.p. ideal and* $u \in R^n - M^n$, *then* $D(M:u)$ *is a c.p. ideal of* $M_n(R)$.

Proof

By Proposition 1.14 $M_m(M_n(R)) \simeq M_{mn}(R)$. Let $D(M:u)$ be a maximal ideal of $M_n(R)$. Then, as in Proposition 2.20, we have that $D(D(M:u):U) = D(M:Uu)$, where $U \in M_n(R)^m - D(A:u)^m$. But since M is c.p. it follows that $D(M:Uu) \sim D(M:Vu)$, say. But $D(M:Vu) = D(D(M:u):V)$ and hence $D(D(M:u):U) \sim D(D(M:u):V)$; i.e. $D(M:u)$ is c.p. □

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NOTATION AND TERMINOLOGY

Z	the ring of integers
Z_n	the ring of integers modulo n
Q	the field of rational numbers
$R[x]$	the ring of polynomials in the indeterminate x
const(f)	the constant term of a polynomial f of $R[x]$
$GL_n(R)$	the set of all nxn invertible matrices with entries from R
$\in$	is an element of
$\notin$	is not an element of
$\subset$	is a subset of
$\subsetneq$	is a proper subset of
$\simeq$	ring- or R-isomorphism
$\sim$	only in Proposition 1.13 it means an equivalence relation, otherwise its meaning is "is conjugate to"
$(a,b)=1$	a and b are co-prime
$a\mid b$	$b=na$ for some $n \in Z$
$a\nmid b$	$b\neq na$ for every $n \in Z$
$a\equiv b(\mathrm{mod}\,n)$	$n\mid a-b$
$a\not\equiv b(\mathrm{mod}\,n)$	$n\nmid a-b$
$u\equiv v(\mathrm{mod}\,M)$	$u_i-v_i \in M$ for each $i=1,\ldots,n$
$u\not\equiv v(\mathrm{mod}\,M)$	there exists an i such that $u_i-v_i \notin M$
$A-B$	the relative complement of B in A, where A and B are sets

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