RADICALS AND ANTIRADICALS IN NEAR-RINGS

BY

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To Magdalene, Brandon and Alan.

Contents

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Abstract

This thesis deals with Jacobson type radicals and ideals, which are antiradical in the sense that they are direct sums of minimal left ideals and annihilate the quasi-radical, $Q(N)$. Of central importance are the s-radical denoted by $J_s(N)$ and an antiradical called the socle-ideal, denoted by $Soi(N)$.

In addition to basic definitions and results, Chapter 1 gives a historical account of some aspects of radical and antiradical theories in near-rings.

In Chapter 2 we generalise the notion of s-primitivity, first introduced in [10]. Some of the remaining problems from [10] are settled here. We prove, for example, that if N is a near-ring satisfying the descending chain condition for left ideals, (DCCL), then $J_s(N)$ is the smallest two-sided ideal which contains $Q(N)$. We conclude this chapter with examples and some of the typical radical-like properties satisfied by $J_s(N)$.

$Soi(N)$, which is a generalisation of the Laxton-Machin critical ideal [18], is defined in Chapter 3. If N satisfies the descending chain condition for N-subgroups of N^+ (DCCN), then $Soi(N)$ is uniquely maximal amongst all ideals whose intersection with $Q(N)$ is zero. In the DCCL case, $J_s(N)$ is uniquely minimal amongst all ideals A such that $Soi(N/A) = N/A$. We also prove that $Soi(N)$ is contained in the crux of N, first defined by S.D. Scott [27]. If N has DCCN, then $Soi(N)$ and the crux of N coincide.

Chapter 4 is devoted to the development of a representation theory for antiradicals. We construct the antiradicals $C(\Omega)$ using only the faithful N-group Ω. In the DCCN case $C(\Omega) = Soi(N)$ for any faithful N-group Ω.

In Chapter 5 we consider nil-rigid and other radical-antiradical series in an attempt to find a measure for the non-nilpotence of $J_s(N)$.

It turns out that $J_s(N/C)$ is non-zero and nilpotent for some member C of the nil-rigid series for N. This enables us to give the following decomposition theorem for $J_s(N)$. Let N be a near-ring with identity and satisfying the DCCN. If

$$L_1 = J_0(N) = \{0\} \subset C_1 \subset L_2 \subset C_2 = N$$

is the nil-rigid series for N, then

$$J_s(N) = W \oplus Q(N),$$

where

$$W = J_s(N) \cap \mathrm{Soi}(N) = J_s(N) \cap (\text{the crux of } N).$$

Some future problems connected with the work in this thesis are discussed in Chapter 6.

CHAPTER 1

Preliminaries

1. Basic definitions and results

Definition A right near-ring is a non-empty set N with two binary operations, addition (+) and multiplication (·) such that

(i) N is a group under +. We denote this group, which is not in general commutative, by N^+.

(ii) N is a semi-group under multiplication.

(iii) $(x+y)\cdot z = x\cdot z + y\cdot z$ for all $x,y,z \in N$.

(iv) N is zero symmetric, that is $x\cdot 0 = 0$ for all $x \in N$, where 0 is the identity of N^+.

Similarly one defines a left near-ring in which the left distributive property holds and such that $0\cdot x = 0$ for all $x \in N$. Near-rings throughout this thesis will be right near-rings, and we write xy instead of $x\cdot y$. Examples of near-rings can be found in [23].

Definition An element s of a near-ring N is said to be distributive if $s(x+y) = sx + sy$ for all $x,y \in N$. If N^+ is generated by a set of distributive elements, then N is said to be a distributively generated (d.g.) near-ring.

Definition Let Ω be a group and N a near-ring. Ω is called an N-group if there is a mapping $(n,\omega) \to n\omega$ of $N \times \Omega$ into Ω such that

(i) $(x+y)\omega = x\omega + y\omega$ and

(ii) $(xy)\omega = x(y\omega)$ for all $x,y \in N$ and all $\omega \in \Omega$.

If N has a multiplicative identity 1, then we add the condition that $1\omega = \omega$ for all $\omega \in \Omega$ (i.e. Ω is a unitary N-group). One verifies easily that N^+ is an N-group.

Definition Let S be a subset of the near-ring N and let Δ be a subset of the N-group Ω. Then we define

$$S\Delta = \{s\delta \mid s \in S,\ \delta \in \Delta\}.$$

In particular Δ may be a subset of the N-group N^+. More generally, let $\{S_i\}$, $i = 1,\ldots,k$ be a collection of subsets of N; then $S_1 S_2 \ldots S_k$ denotes the set of all elements of the form $s_1 s_2 \ldots s_k$, with $s_i \in S_i$ for $i = 1,\ldots,k$. If $S_1 = S_2 = \ldots = S_k = S$, then we denote $S_1 S_2 \ldots S_k$ by S^k.

Definition The subgroups Δ of the N-group Ω such that $N\Delta \subseteq \Delta$ are called N-subgroups of Ω.

The definitions of near-ring homomorphisms, endomorphisms and isomorphisms can be found in [23].

Definition A homomorphism ψ of an N-group Ω_1 into an N-group Ω_2 is called an N-homomorphism if $\psi(n\omega) = n\psi(\omega)$ for all $n \in N$ and all $\omega \in \Omega_1$.

Definition The kernel of an N-homomorphism will be called an N-kernel.

It is easily shown [23] that a normal subgroup Δ of an N-group Ω is an N-kernel of Ω if and only if

$$n(\omega+\delta) - n\omega \in \Delta \quad \text{for all } n \in N,\ \omega \in \Omega \text{ and } \delta \in \Delta.$$

The N-kernels of Ω are the "ideals" of Ω in the terminology of Pilz [23].

Remark: If Ω is an N-group and Δ an N-kernel of Ω, then the factor group $\Omega - \Delta$ may be regarded as an N-group under $N \times (\Omega - \Delta) \to \Omega - \Delta$ given by

$$n(\omega+\Delta) = n\omega + \Delta, \qquad n \in N,\ \omega \in \Omega.$$

Definition The N-kernels of N^+ are called left ideals of N. A normal subgroup R of N is called a right ideal of N if $RN \subseteq R$ and the ideals of N are precisely the left ideals which are also right ideals of N.

Definition A left ideal L of N is said to be modular if there exists an $e \in N$ such that $ne - n \in L$ for all $n \in N$. If N has a right multiplicative identity, then, of course, every left ideal of N is modular.

Definition Let $\{\Delta_i\}_{i \in I}$ be a family of N-subgroups of the N-group Ω; then $\sum_{i \in I} \Delta_i$ denotes the N-subgroup of Ω generated by $\bigcup_{i \in I} \Delta_i$.

We state the following results without proof [25].

Proposition 1.1 (a) If $\{\Delta_i\}$, $i \in I$, is a family of N-kernels of the N-group Ω, then $\sum_{i\in I} \Delta_i$ is an N-kernel of Ω.

(b) If Δ_1 is an N-kernel and Δ_2 an N-subgroup of Ω, then the set $\{\delta_1+\delta_2 \mid \delta_1 \in \Delta_1,\ \delta_2 \in \Delta_2\}$ is the N-subgroup of Ω generated by $\Delta_1 \cup \Delta_2$.

(c) If Δ_1 is an N-kernel and Δ_2 an N-subgroup of Ω, then the factor N-groups $(\Delta_1+\Delta_2) - \Delta_1$ and $\Delta_2 - (\Delta_1 \cap \Delta_2)$ are N-isomorphic.

Definition Let Ω be an N-group and $\{\Delta_i\}$, $i \in I$, be a family of N-kernels of Ω such that

(i) $\Omega = \sum_{i\in I} \Delta_i$,

(ii) $\Delta_k \cap \sum_{\substack{i\in I\\ k\neq i}} \Delta_i = \{0\}$ for each $k \in I$,

then we say that Ω is a direct sum of the Δ_i. We write $\Omega = \bigoplus_{i\in I} \Delta_i$ if Ω is a direct sum of the N-kernels Δ_i. One can show that each element of Ω has a unique expression as a finite sum of elements from different Δ_i's.

The following result, due to D.W. Blackett [3], will be used frequently throughout this thesis:

Lemma 1.2 (left distribution over N-kernels) Let $\Omega = \bigoplus_{i\in I} \Delta_i$ be a direct sum of the N-kernels Δ_i. Then $n(\delta_{k_1} + \dots + \delta_{k_s}) = n\delta_{k_1} + \dots + n\delta_{k_s}$, for all $n \in N$ and δ_{k_j} from different Δ_{k_j}.

Definition An N-group Ω possesses a cyclic generator $\omega \in \Omega$ if $N\omega = \Omega$. We say Ω is a cyclic N-group.

Definition [2] A non-zero N-group Ω is said to be

(i) minimal if Ω contains only the trivial N-subgroups $\{0\}$ and Ω;

(ii) of type-2 if Ω is minimal and $N\Omega \neq \{0\}$;

(iii) irreducible if Ω has no proper, non-zero N-kernels;

(iv) of type-1 if Ω is irreducible, $N\Omega \neq \{0\}$ and for each $\omega \in \Omega$, $N\omega = \{0\}$ or $N\omega = \Omega$;

(v) of type-0 if Ω is irreducible and possesses a cyclic generator.

Definition Let Δ_1 and Δ_2 be subsets of an N-group Ω; then $(\Delta_1 : \Delta_2)$ denotes the set of all elements of N which map Δ_2 into Δ_1, that is

$$(\Delta_1 : \Delta_2) = \{n \in N : n\Delta_2 \subseteq \Delta_1\}.$$

Clearly, the set $(0 : \Delta_2)$ is actually a left ideal of N. In particular, $(0 : \omega)$ is a left ideal for each $\omega \in \Omega$ (here we have replaced $\{\omega\}$ by ω). It is called the annihilating left ideal of ω. If Δ_1 is an N-kernel and Δ_2 an N-group, then $(\Delta_1 : \Delta_2)$ is an ideal. In particular $(0 : \Delta_2)$ is called the annihilating ideal of the N-group Δ_2. If $\omega \in \Omega$ is a cyclic generator of Ω, then $(0 : \omega)$ is a modular left ideal of N, and if in addition Ω is of type-0, $(0 : \omega)$ is a modular maximal left ideal of N.

Definition

(i) A near-ring N is called ν-primitive if it has a faithful N-group of type-ν, $\nu = 0,1,2$.

(ii) An ideal A of a near-ring N is ν-primitive if $A = (0 : \Omega)$, where Ω is of type-ν, $\nu = 0,1,2$. Equivalently, A is ν-primitive if the factor near-ring N/A, [23] is ν-primitive.

(iii) $J_\nu(N)$ is the intersection of all ν-primitive ideals of N, $\nu = 0,1,2$. If N has no ν-primitive ideals, then we define $J_\nu(N)$ to be N itself. $J_2(N)$ is called the radical of N. In the ring case

$$J_0(N) = J_1(N) = J_2(N)$$

is just the Jacobson radical of N.

(iv) The quasi-radical Q(N) is the intersection of all modular, maximal left ideals of N. Again, if N has no modular, maximal left ideals, then we define Q(N) to be N.

Clearly, 2-primitive $\Rightarrow$ 1-primitive $\Rightarrow$ 0-primitive and we have

$$J_2(N) \supseteq J_1(N) \supseteq Q(N) \supseteq J_0(N).$$

If N has a multiplicative identity, then $J_2(N) = J_1(N)$.

Definition An element $z \in N$ is called left quasi regular (l.q.r.) if, and only if, the minimal left ideal of N containing all elements of the form

$x - xz$, $x \in N$, coincides with N. A subset of N is called left quasi regular if each element contained in it is l.q.r.. Q(N) is a l.q.r. left ideal of N containing all l.q.r. left ideals of N; see [23, 24].

The definitions of various chain conditions satisfied by near-rings can be found in Pilz [23]. Throughout this thesis "N has DCCI(DCCL,DCCN)" will mean that N satisfies the descending chain condition for ideals (respectively left ideals, N-subgroups of N^+). Similarly, ACCI(ACCL,ACCN) will mean that the ascending chain condition holds for ideals (respectively left ideals, N-subgroups of N^+).

Definition A subset X of a near-ring N is said to be nilpotent if $X^k = 0$ for some positive integer k. X will be called nil if every single element subset of X is nilpotent. Clearly, nilpotency implies nilness.

If N has DCCN, then Q(N) is a nilpotent left ideal containing all nilpotent left ideals of N. In this case $J_0(N)$ is a nilpotent ideal of N containing all nilpotent ideals of N. Proofs of the latter statements can be found in [15,24].

2. Three important lemmas

The following lemmas are essential for much of the work in this thesis and for this reason we discuss them in a separate section. The first and third are well known [10,16]; the second is a generalisation of the ring theoretic result [4].

Lemma 1.3 Let N be a near-ring with DCCL, $N \not\subseteq Q(N)$. Then

$$N - Q(N) = \bigoplus_{i=1}^{k} \Delta_i$$

where each N-kernel Δ_i is of type-0. Furthermore, any N-group of type-0 is N-isomorphic to one of the Δ_i appearing in this decomposition.

The proof of lemma 1.3 was first given for a d.g. near-ring with identity and DCCN [16]. Proof of the above generalisation goes over almost verbatim.

<u>Lemma 1.4</u> Let $\Omega = \bigoplus_{i\in I} \Omega_i$ be an N-group where each N-kernel Ω_i is of type-0. Then any non-trivial N-kernel Δ of Ω is a direct summand of Ω and each of $\Omega - \Delta$ and Δ is N-isomorphic to a direct sum of some of the Ω_i's.

<u>Proof</u> Let $\mathcal{C}$ be the collection of all N-subgroups Γ of Ω which are of the form

$$\Gamma = \Delta \oplus \left(\bigoplus_{t\in T\subseteq I} \Omega_t \right).$$

Partially order $\mathcal{C}$ by

$$\Gamma_1 = \Delta \oplus \left(\bigoplus_{t\in T_1\subseteq I} \Omega_t \right) \leq \Gamma_2 = \Delta \oplus \left(\bigoplus_{t\in T_2\subseteq I} \Omega_t \right)$$

if, and only if, $T_1 \subseteq T_2$. Any chain

$$\Gamma_1 < \Gamma_2 < \dots < \Gamma_k < \dots$$

in $\mathcal{C}$ is bounded above by

$$\bigcup_k \Gamma_k = \Delta \oplus \left(\bigoplus_{t\in\cup T_k} \Omega_t \right) \in \mathcal{C}.$$

Since $\Delta + \Omega_i \in \mathcal{C}$ for some i, $\mathcal{C} \neq \emptyset$ and so, by Zorn's lemma, $\mathcal{C}$ has a maximal element $\Gamma^* = \Delta \oplus (\bigoplus_{u\in U\subseteq I} \Omega_u)$, say. If $\Gamma^* \neq \Omega$, then there exists Ω_i, $i \in I$, such that $\Gamma^* \cap \Omega_i = \{0\}$ because Ω_i is of type-0. But then we would have

$$\Gamma^* < \Delta \oplus \left(\bigoplus_{u\in U} \Omega_u \right) \oplus \Omega_i \in \mathcal{C},$$

contradicting the maximality of Γ^*. Thus $\Gamma^* = \Omega$.

<u>Definition</u> An N-group Γ is said to be a <u>subfactor</u> of the N-group Ω if there exist N-subgroups Δ_1 and Δ_2 of Ω, with Δ_2 an N-kernel of Δ_1 such that Γ is N-isomorphic to the factor N-group $\Delta_1 - \Delta_2$. We write $\Gamma \ll \Omega$ if Γ is a subfactor of Ω.

<u>Lemma 1.5</u> Let N be a <u>finite</u> near-ring with identity and Ω a faithful N-group. If Δ is an N-group of type-0, then $\Delta \ll \Omega$.

The proof of lemma 1.5 is identical with the one given in [16] for d.g. near-rings. In [16], Ω was assumed to be of type-0, but this was not strictly necessary.

3. Radical and antiradical theories in near-rings

It is well known that, unlike in rings, the radical $J_2(N)$ of a near-ring N with DCCN need not be nilpotent [3]. Various studies have been made of this oddity, and one objective has been to try to find a measure of $J_2(N)$'s non-nilpotence; roughly, something that will tell us how far the radical is from being nilpotent. An attempt to this end was made in [10], where the properties of the quasi-radical $Q(N)$ were more closely examined. Not surprisingly, there are many parallels between the internal characterisations of $Q(N)$ and the Jacobson radical in rings. The drawback is that $Q(N)$ is not, in general, a two-sided ideal, and in this connection the following problem was posed in [16]: can $Q(N)$ be a two-sided ideal without being equal to the radical $J_2(N)$? The affirmative answer to this was given in [10] and in the process the notion of s-primitivity and a further Jacobson-type radical, $J_s(N)$, were introduced. $J_s(N)$ is an ideal which contains $Q(N)$ and is contained in $J_2(N)$. If N has DCCN, then $Q(N)$ is a two-sided ideal if, and only if, $J_s(N) = Q(N) = J_0(N)$, [10]. Moreover, if $J_s(N) = \{0\}$, where N is a near-ring with DCCN, then N is a direct sum of minimal left ideals and we have the simplest class of near-ring structures after the class of semi-simple ones. Several outstanding questions remain in connection with the s-radical, $J_s(N)$, and some of these will be answered in the next chapter. For example, we prove that for a large class of near-rings, $J_s(N)$ is the smallest two-sided ideal which contains $Q(N)$. In addition, we generalise the notion of s-primitivity to include all near-rings, with or without a multiplicative identity, and we show that $J_2(N) \supseteq J_1(N) \supseteq J_s(N) \supseteq Q(N)$ in the general case as well. Examples of near-rings with multiplicative identity exist for which $J_2(N) \neq J_s(N)$, [11]. However, there are classes of near-rings, for example certain types of finite Neumann d.g. near-rings [19], for which $J_2(N) = J_s(N) \neq Q(N)$. In this case $J_2(N)$ is nilpotent if, and only if, $Q(N)$ if a two-sided ideal. Because it is in some sense dual to one of the antiradicals introduced in Chapter 4, a discussion of the s-radical $J_s(N)$ merits a place in this thesis.

The antiradical or socle of a ring plays a prominent part in the theory of rings. In the last decade fruitful results were obtained by studying socles in ring modules. For example, it can be shown that a ring module is Artinian if and only if the socles of its factor modules are essential and finitely generated [1] (a submodule S of a module M is called essential if whenever K is a submodule of M such that $K \cap S = \{0\}$, then $K = \{0\}$). For the important classical applications of the socle in rings one needs to go back to the old masters and indeed part of this thesis could well have been inspired by Reinhold Baer's paper [4]. As it is, the antiradicals discussed here arose through purely near-ring theoretic considerations. Baer used an antiradical series to establish a necessary and sufficient condition for the existence of his radical. This antiradical series was defined as follows:

(i) $M_1 = M(R)$, the antiradical of R. That is, M_1 is the sum of all the minimal left ideals of R.

(ii) $M_{\nu+1}$ is a uniquely determined ideal in R such that $M_\nu \subset M_{\nu+1}$ and $M(R/M_\nu) = M_{\nu+1}/M_\nu$.

(iii) If ν is a limit ordinal, M_ν is the union of all ideals M_u for $u < \nu$.

(iv) There exists a smallest ordinal $m = m(R)$ such that $M_m = M_{m+1}$.

Baer proved that if $R = M_m$ for some ordinal m, then the radical of R exists. Furthermore, R is Artinian and hence the Baer radical is nilpotent in this case.

In this thesis we shall use a similarly defined radical-antiradical series in near-rings in an attempt to find a measure for the non-nilpotence of Jacobson-type radicals which contain $Q(N)$. This radical-antiradical series is a special case of a more general series first defined by S.D. Scott [27]. Scott called his series, which is properly ascending, the nil-rigid series, and he showed that it is finite if the near-ring N has DCCN. If the near-ring N has an identity and satisfies the DCCN, then we prove that $J_s(N/C)$ is nilpotent, where C is a certain element in the nil-rigid series for N. But let us start at the beginning.

Antiradical ideals have appeared on the near-ring scene fairly recently [18,19]. Unlike rings, the sum of all minimal left ideals of a near-ring is only a left ideal and not in general a two-sided ideal. We shall call this left ideal the socle of the near-ring. Heatherly [12] considered socle-like structures in near-rings which are not zero-symmetric. In the zero-symmetric case, S.D. Scott [27] studied the socles of tame near-rings. Since, in general, the socle of a near-ring is only a left ideal it seems more profitable to consider <u>two-sided ideals</u> which are <u>antiradicals</u> in the sense that they (a) have socle-like structures and (b) annihilate one or more of the radicals in a near-ring. We note that (b) is a strong condition, and even the socle of a ring is not necessarily antiradical in the above sense. We will give an example to this effect in Chapter 3. However, there are examples of ideals in near-rings which satisfy (a) and (b). One of the best-known amongst these is the <u>critical ideal</u>, first studied by A. Machin [19]. He considered the intersection of all non-zero ideals in a Neumann d.g. near-ring defined on a reduced free group, whose laws were precisely the universal laws of a <u>critical</u> group [21]. Machin discovered this intersection of ideals to be a non-zero ideal and a direct sum of left ideals, each of which was an N-group of type-0. Moreover, this non-zero ideal, which he called the critical ideal, turned out to be a direct summand of the near-ring N and this enabled him to find examples in which the radical $J_2(N)$ split as $J_2(N) = C \oplus Q(N)$, where $C = C^2$ is the critical ideal and $C.Q(N) = \{0\}$. In these examples the defining group was minimal simple, and Machin regarded the ideal C to be the "obstruction" to $J_2(N)$ being nilpotent. Indeed, by factoring out C he obtained a Neumann d.g. near-ring with a nilpotent radical in this case. All this is as far as it goes. Later the notion of the critical ideal was extended to more general d.g. near-rings with identity and satisfying the DCCN and ACCN [18]. This Laxton-Machin critical ideal was defined via a faithful representation on an N-group and the Jordan-Hölder theory played an important rôle in its construction. Again, this more general critical ideal has the antiradical property of being a direct sum of minimal left ideals and annihilating Q(N). In addition it is a direct summand of the near-ring N and hence the radical $J_2(N)$ splits as $J_2(N) = D \oplus L$, where $D^2 = D$ is an ideal contained in the critical ideal and L is a left ideal containing Q(N). It was thought that by successively factoring out critical ideals in the manner of Baer, one would, after a finite number of steps, arrive at a factor near-ring with a nilpotent radical. Thus, if it had worked, the number of successive critical ideals in the chain before a nilpotent radical was obtained would have been some sort of measure for the non-nilpotence of $J_2(N)$. However, the factor near-ring of a near-ring by its critical ideal has zero critical ideal, and near-rings with zero critical ideal and non-nilpotent radical are known to exist [19]. Thus this idea for a measure of the radical's non-nilpotence failed.

Closely connected with antiradicals is the crux of a near-ring N, denoted by Crux(N), first defined by S.D. Scott [27]. The crux has a strong "nil avoidance" property and it annihilates the nilradical [27] of a near-ring. However, it is not an antiradical in our sense as it does not, in general, have a socle-like structure. Scott proved that the crux of N/Crux(N) is zero. Furthermore, if N has DCCI and Crux(N) $\neq$ N, then the nilradical of N/Crux(N) $\neq$ {0}. Also, if the nilradical nil(N) of N is not equal to N, then the crux of N/nil(N) is not zero in the DCCI case. The last two statements were proved by Scott and they led to the construction of his nil-rigid series. Roughly speaking, the nil-rigid series of a near-ring is a properly ascending sequence of ideals which arises through alternately factoring out the nilradical and crux of successive factor near-rings of the near-ring. We shall give the formal definition in Chapter 5, where we will consider connections between the nilpotence of $J_s(N)$ and the nil-rigid series for N. Later [28] Scott put his ideas in a more abstract setting and developed a theory for formation radicals. In this setting his crux emerged as the complementary radical of the Baer lower radical.

We will construct an antiradical which we call the <u>socle-ideal</u> and denote by Soi(N) in Chapter 3. This antiradical, which is a generalisation of the Laxton-Machin critical ideal, has the properties of being a direct sum of minimal left ideals and annihilating Q(N). Furthermore, the socle-ideal is contained in the crux and if the near-ring N satisfies the DCCN, then these two ideals coincide.

In addition we extend the Laxton-Machin [18] representation theory to construct, for the near-ring N satisfying the DCCL, the ideals $C(\Omega)$, where Ω is a faithful N-group. The ideals $C(\Omega)$ are antiradical in the sense that they have socle-like structures and $C(\Omega).Q(N) = \{0\}$ for each faithful Ω. Of these ideals C(N) is the most important. However, if N has DCCN, then we will show that the choice of the faithful N-group Ω is actually irrelevant. This is a generalisation of the Laxton-Machin result and it follows from the fact that $Soi(N) = C(\Omega)$ for any faithful N-group.

In connection with the nilpotence of the s-radical we show that there exists an element L in the nil-rigid series for a near-ring N, with identity and satisfying the DCCN, such that $J_s(\bar{N}) = \bar{W} \oplus Q(\bar{N})$, where $\bar{N} = N/L$ and $\bar{W}$ is contained in the socle-ideal of $\bar{N}$. This is a general form of the Laxton-Machin decomposition of $J_2(N)$ for certain types of Neumann d.g. near-rings. If L is zero, then, of course, $J_s(N) = W \oplus Q(N)$. Whether or not L is zero depends partly on the length of the nil-rigid series of N. We therefore consider it not unreasonable to use the length of nil-rigid series in order to find a measure for the non-nilpotence of the s-radical. How the radical fits into this scheme of things if it is not the s-radical, we have been unable to establish. Indeed, there are many other outstanding questions remaining, and we regard this work as exploratory only.

CHAPTER 2

s-Primitivity

In this chapter we generalise the notion of s-primitivity and, unlike in [10], our near-rings are not assumed to have a multiplicative identity. Our near-rings are by definition zero-symmetric and we will make no attempt to discuss s-primitivity in a more general setting. We reiterate that throughout this thesis our direct sums are direct sums of N-kernels, and $\Omega = \bigoplus_{i\in I} \Omega_i$ will always mean that Ω_i is an N-kernel of Ω for each $i \in I$.

Definition Let Ω be an N-group and $\omega, \omega' \in \Omega$. We say that ω is equivalent to ω', written $\omega \sim \omega'$, if

$$\text{(i)} \quad N\omega = N\omega' \qquad \text{and} \qquad \text{(ii)} \quad (0:\omega) = (0:\omega').$$

It is clear that "$\sim$" is an equivalence relation on Ω. Also, if N has a right identity e, then $e\omega \sim \omega$ for all $\omega \in \Omega$.

Definition An N-group Ω of type-0 is said to be of type-s if for all $\omega \in \Omega$ for which $N\omega \neq \{0\}$ we have

(i) $N\omega = \bigoplus_{i\in I} \Omega_i$, where Ω_i is of type-0 for each $i \in I$;

(ii) there exists an $\omega' \in N\omega$ such that $\omega \sim \omega'$.

We note that the index set I above varies with ω. By a previous remark, (ii) is superfluous if N has a right identity. Moreover, ω' in (ii) has a unique expression of the form $\omega' = \omega_{i_1} + \omega_{i_2} + \dots + \omega_{i_r}$, with ω_{i_j} in different Ω_{i_j}. By the left distribution over the N-kernels Ω_{i_j}, it follows that I is a finite set because $N\omega' = N\omega$. Clearly, type-2 $\Rightarrow$ type-1 $\Rightarrow$ type-s $\Rightarrow$ type-0.

Definition A near-ring N will be called s-primitive if it has a faithful N-group of type-s. An ideal A of N will be called s-primitive if the factor near-ring N/A is s-primitive. An ideal A is thus s-primitive if, and only if, $A = (0:\Omega)$, where Ω is of type-s.

<u>Definition</u> The intersection of all s-primitive ideals of N is called the <u>s-radical</u> of N and we denote it by $J_s(N)$. If N has no s-primitive ideals, then we define $J_s(N)$ to be N.

From the definition of $J_s(N)$ it is clear that $J_1(N) \supseteq J_s(N) \supseteq J_0(N)$, but its connection with Q(N) is not quite so immediate. We prove that $J_s(N)$ contains the quasi-radical and will in the process arrive at another characterisation of the former. We note that if Ω is of type-s and Δ is an N-group of type-0 contained in Ω, then Δ is of type-s.

<u>Definition</u> [23] A left ideal L of N is said to be <u>ν-modular</u>, $\nu \in \{0,1,s,2\}$ if L is modular and $N - L$ is of type-ν.

We see that the 0-modular left ideals of N are precisely the modular, maximal left ideals of N.

<u>Theorem 2.1</u> If A is an s-primitive ideal of N, then A is an intersection of s-modular left ideals of N.

<u>Proof</u> There exists an N-group Ω of type-s such that $A = (0 : \Omega)$. If $N\omega \neq \{0\}$, $\omega \in \Omega$, then for some k, $N\omega = \bigoplus_{i=1}^{k} \Omega_i$ where each Ω_i is of type-0 and there exists an $\omega' \in N\omega$ such that $\omega \sim \omega'$. Now $\omega' = \omega_1 + \dots + \omega_k$, with $\omega_i \in \Omega_i$. Also $N\omega_i = \Omega_i$, $i = 1,\dots,k$ by the left distribution over the N-kernels Ω_i, and because $N\omega = N\omega'$. We have

$$(0 : \omega') = \bigcap_{i=1}^{k} (0 : \omega_i)$$

and so $(0 : \omega')$ is an intersection of 0-modular left ideals. By a remark preceding the theorem, Ω_i is actually of type-s so that $(0 : \omega_i)$ is s-modular for $i = 1,\dots,k$. Thus $(0 : \omega) = (0 : \omega')$ is an intersection of s-modular left ideals. Consequently,

$$A = (0 : \Omega) = \bigcap_{\omega \in \Omega} (0 : \omega) = \bigcap_{\substack{\omega \in \Omega \\ N\omega \neq \{0\}}} (0 : \omega)$$

is an intersection of s-modular left ideals of N.

<u>Corollary</u> $J_s(N) \supseteq Q(N) \supseteq J_0(N)$.

We can now give the following characterisation of $J_s(N)$.

Theorem 2.2 $J_s(N)$ is the intersection of all s-modular left ideals of N.

Proof If $J_s(N) = N$, then we are through. So suppose $J_s(N) \neq N$. Theorem 2.1 tells us that $J_s(N)$ certainly contains the above intersection. On the other hand, if L is an s-modular left ideal of N i.e. $N - L$ is of type-s and L is modular by e, say, then putting $\bar{e} = e + L$ we have

$$\begin{aligned} J_s(N) \subseteq (0 : \bar{e}) &= \{n \in N : ne \in L\} \\ &= \{n \in N : n \in L\} \\ &= L. \end{aligned}$$

Whether, in general, an O-primitive ideal which is an intersection of O-modular left ideals is s-primitive, is not known. We will prove it true for the class of all near-rings satisfying the DCCL. For this purpose we need the following lemma, due to Betsch [2].

Lemma 2.3 If L is an O-modular left ideal and M is a modular left ideal of N, then $L \cap M$ is modular.

Lemma 2.4 If N is a near-ring satisfying the DCCL, then $Q(N)$ is a modular left ideal of N.

Proof Follows from lemma 2.3 and the definition of $Q(N)$.

Theorem 2.5 Let N be a near-ring satisfying the DCCL. If P is an O-primitive ideal of N which is an intersection of O-modular left ideals, then P is s-primitive.

Proof If $Q(N) = N$, then the theorem is trivially true, so we may suppose that $Q(N) \neq N$. We note that since $Q(N)$ is modular and $P \supseteq Q(N)$, P itself is modular, so that the factor near-ring N/P has a right multiplicative identity. By lemma 1.3 we have

$$N - Q(N) = \bigoplus_{i=1}^{k} \Delta_i$$

where each Δ_i is of type-O. Since $P - Q(N)$ is an N-kernel of $N - Q(N)$, lemma 1.4 tells us that the factor near-ring N/P is of the form then

$$q' = q'e = q'l_{j_1} + \dots + q'l_{j_s} \in Q(N) \cap \bigoplus_{j \in J} L_j = \{0\}$$

and we are done.

We note that lemma 2.6 need not be true if N does not have a right identity. We give as an example the following Clay "small" near-ring [5].

Example 2.1 Consider the near-ring $N = (V_4,+,*)$, where $(V_4,+) = \{0,a,b,c\}$ is the four-group with $2a = 2b = 2c$ and the multiplication $*$ is given in the following table:

*	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

N is a distributive near-ring and

$$\begin{aligned} N &= \{0,c\} \oplus \{0,b\} \\ &= \{0,a\} \oplus \{0,c\} \\ &= \{0,b\} \oplus \{0,a\}. \end{aligned}$$

We see that $\{0\} \neq \{0,b\} = J_\nu(N) = Q(N)$ for $\nu = 0,1,2,s$.

Lemma 2.7 Let N be a near-ring satisfying the DCCL. Then any 0-primitive ideal P of N which contains Q(N) is s-primitive.

Proof If $Q(N) = N$, then the lemma is vacuously true since an 0-primitive ideal P cannot be equal to N. So we may assume $Q(N) \neq N$. By theorem 2.5 we need only show that P is an intersection of 0-modular left ideals. From the decomposition of $N - Q(N)$ as a direct sum of N-groups of type-0 and the fact that $P - Q(N)$ is an N-kernel of $N - Q(N)$ we again see that the factor near-ring N/P can be written as $N/P = \bigoplus_{t \in T} \bar{L}_t$, for some finite index set T. Since Q(N) is a modular left ideal and $P \supseteq Q(N)$, N/P has a right identity. Hence by lemma 2.6 Q(N/P) is the zero ideal. Taking inverse images back in N we see that P is an intersection of 0-modular left ideals of N.

Theorem 2.8 If N is a near-ring satisfying the DCCL, then any ideal A of N, $A \nsubseteq N$, which contains $Q(N)$ is an intersection of s-primitive ideals.

Proof By lemma 1.4 and the decomposition of $N - Q(N)$ as a direct sum of N-groups of type-0, we may write $N/A = \bigoplus_{t \in T} \bar{L}_t$, where each $\bar{L}_t$ is an N/A-group of type-0. Since $A \supseteq Q(N)$, N/A has a right identity and is thus a faithful N/A-group. Thus $A = \bigcap_{t \in T} (0 : \bar{L}_t)$ where, of course, each $\bar{L}_t$ is also an N-group of type-0. Hence each $(0 : \bar{L}_t)$ is an 0-primitive ideal of N containing $Q(N)$ and it follows from lemma 2.7 that A is an intersection of s-primitive ideals.

Corollary 1 The s-radical $J_s(N)$ is the smallest ideal of N containing $Q(N)$.

Corollary 2 Every ideal $A \nsubseteq N$ containing $Q(N)$ is an intersection of s-modular left ideals.

Now $J_0(N)$ is a l.q.r. ideal of N which contains all the l.q.r. ideals of N. Hence if $Q(N)$ is a two-sided ideal, then $J_0(N) = Q(N)$. Corollary 1 of the last theorem shows

Theorem 2.9 Let N be a near-ring satisfying the DCCL. Then $Q(N)$ is a two-sided ideal if, and only if, $J_0(N) = Q(N) = J_s(N)$.

Let $\mathcal{M}$ be the set of all ideals of a near-ring N and define the sets $\mathcal{S}$ and $\mathcal{Y}$ as follows:

$\mathcal{S} = \{A \in \mathcal{M} : N/A$ has no non-zero, l.q.r. left ideals$\}$;

$\mathcal{Y} = \{A \in \mathcal{M} : N/A$ has no non-zero, nilpotent left ideals$\}$.

We have

Theorem 2.10

(i) If N is a near-ring satisfying the DCCL, then $J_s(N) = \bigcap_{A \in \mathcal{S}} A$;

(ii) If N is a near-ring satisfying the DCCN, then $J_s(N) = \bigcap_{A \in \mathcal{Y}} A$.

Proof

(i) We may assume $N \nsubseteq Q(N)$. Since $Q(N/J_s(N)) = \{0\}$ it follows that $J_s(N) \in \mathcal{S}$. On the other hand, if $A \in \mathcal{S}$, then $Q(N/A) = \{0\}$ so that A is an intersection of 0-modular left ideals. Thus $A \supseteq Q(N)$ and so by corollary 1 of theorem 2.8 we have $A \supseteq J_s(N)$.

(ii) The proof of (ii) is as above and uses the fact that the quasi-radical is nilpotent in this case.

Our next objective is to find a necessary and sufficient condition for a left ideal to be in $J_s(N)$ similar to the one given in [17] for $J_2(N)$ in the d.g. near-ring case. For this purpose we need the following, which follows immediately from theorem 2.2 and the proof of lemma 1.3.

Lemma 2.11 If N is a near-ring with DCCL and $J_s(N) = 0$, then $N = \bigoplus_{i=1}^{k} L_i$, where each left ideal L_i is an N-group of type-s.

Lemma 2.12 If N is a near-ring which satisfies the DCCL, then $J_s(N) - Q(N)$ is zero or a finite direct sum of N-groups of type-O which are not of type-s.

Proof If $N/J_s(N)$ is not zero, then by lemma 2.11 $\bar{N} = N/J_s(N) = \bigoplus_{i=1}^{k} \bar{L}_i$, where $\bar{L}_i$ is an $\bar{N}$-group of type-s. The Jordan-Hölder theorem now tells us that any O-modular left ideal containing $J_s(N)$ must be s-modular. If L is an O-modular left ideal which is not s-modular, then $N = L + J_s(N)$ so that

$$N - L \overset{N}{\cong} (L + J_s(N) - L)$$
$$\overset{N}{\cong} J_s(N)/J_s(N) \cap L.$$

Thus $J_s(N)/J_s(N) \cap L$ is a type-O N-group which is not of type-s. Since $Q(N) = J_s(N) \cap (\bigcap_{i=1}^{d} L_i)$, for some d, where L_i is O-modular but not s-modular the decomposition claimed will follow.

Theorem 2.13 Let N be a near-ring satisfying the DCCL. Then the left ideal L is contained in $J_s(N)$ if, and only if, $L - L \cap Q(N)$ is zero or a direct sum of N-groups of type-O which are not of type-s.

Proof If $L \subseteq J_s(N)$, then $(L + Q(N)) - Q(N)$ is an N-kernel of $J_s(N) - Q(N)$ and so by lemma 1.4 is a zero or a direct sum of N-groups of type-O which are not of type-s. From the isomorphism $(L + Q(N)) - Q(N) \overset{N}{\cong} L - L \cap Q(N)$ it follows that $L - L \cap Q(N)$ is zero or a direct sum of N-groups of type-O which are not of type-s.

Conversely, suppose $L - L \cap Q(N)$ is a direct sum of N-groups of type-O which are not of type-s. Since $J_s(N) \cap L - L \dot{\cap} Q(N)$ is an N-kernel of $L - L \cap Q(N)$, lemma 1.4 tells us that $(L - L \cap Q(N)) - (J_s(N) \cap L - L \cap Q(N))$ is a direct sum of N-groups of type-0 which are not of type-s. Thus $L - J_s(N) \cap L$ is such a direct sum and consequently $(L + J_s(N)) - J_s(N) \cong L - J_s(N) \cap L$ is a direct sum of N-groups of type-0 which are not of type-s. But $(L + J_s(N)) - J_s(N)$ is an N-kernel of $N - J_s(N)$, which is a direct sum of N-groups of type-s by lemma 2.11. Again by lemma 1.4 it follows that $L + J_s(N) = J_s(N)$ and so $L \subseteq J_s(N)$.

We note that if N satisfies the DCCN, then we have a Laxton-type criterion [17] for a left ideal L to be in $J_s(N)$, namely, that L is an extension of a nilpotent left ideal by a direct sum of N-groups of type-0, which are not of type-s. Of course, in the more general DCCL case, we cannot assume that $L \cap Q(N)$ is nilpotent.

We recall that in the ring case, a necessary and sufficient condition for Baer's upper radical to be equal to his lower radical is for the ring to be right (or left) Artinian. In this case both the upper and lower radicals coincide with the classical radical. One may similarly define "radical ideals" such that, with suitable chain conditions, $J_s(N)$ is "upper" and $J_0(N)$ "lower", or $J_2(N)$ "upper" and $J_s(N)$ "lower", as follows:

<u>Definition</u> A subset A of the near-ring N will be called <u>J_0-radical</u> if

(i) A is an ideal of N;

(ii) $A - A \cap Q(N)$ is zero or a direct sum of N-groups of type-0 which are not of type-s;

(iii) N/A has no non-zero nilpotent ideals.

If N satisfies the DCCN, then $J_s(N)$ is the sum of all ideals of N satisfying (i) and (ii), and $J_0(N)$ is the intersection of all ideals satisfying (i) and (iii). It is clear that $J_0(N)$ and $J_s(N)$ are both J_0-radicals in the DCCN case.

<u>Definition</u> A subset A of the near-ring N is called <u>J_s-radical</u> if

(i) A is an ideal of N;

(ii) $A - A \cap Q(N) = \{0\}$ or a direct sum of N-groups of type-0 which are not of type-2;

(iii) N/A has no non-zero nilpotent left ideals.

If N satisfies the DCCN, then $J_2(N)$ is the sum of all ideals satisfying (i) and (ii), whilst $J_s(N)$ is the intersection of all ideals satisfying (i) and (iii). Also, $J_2(N)$ and $J_s(N)$ are both J_s-radical ideals of N.

It is not our intention to pursue the above observations any further. Suffice it to say that they again point to the highly non-Abelian nature of the near-ring. Furthermore, we will not embark upon a detailed study of the s-radical and s-primitive near-rings. As we pointed out before, the importance of the s-radical in this thesis is simply because of its connection with one of the antiradicals which we will define in the next chapter. Examples of s-primitive near-rings which are not 2-primitive are easy to construct in the general near-ring case [10]. Examples of non-trivial s-primitive d.g. near-rings are harder to come by, almost certainly because such near-rings are more ring-like in structure. We first give an example of a class of d.g. near-rings (which are not rings) for which

$$J_2(N) = J_s(N) \not\subseteq J_0(N).$$

The details of the theory on which this example is based can be found in [19]. In addition, we need the following:

<u>Fact 2.1</u> Let G be a finite, simple group such that every subgroup of G is a direct sum of simple groups. Then G is Abelian of prime order (App. 1).

<u>Example 2.2</u> Let G be an additively written finite group and V a reduced

free group (again written additively) on n generators whose laws are precisely the universal laws of G. We assume that $n \geq |G| - 1$. Let N be the Neumann d.g. near-ring associated with V [19]. N is finite with an identity element. Furthermore, N has a faithful N-group Ω, all of whose N-subgroups are cyclic and there is a one-to-one lattice correspondence between the subgroups of G and the N-subgroups of Ω. Under this correspondence, normal N-subgroups of Ω correspond to normal subgroups of G and subfactors of Ω correspond to subfactors of G. It is clear that every subfactor of Ω is a cyclic N-group. By lemma 1.5 every N-group of type-O is a subfactor of Ω. Let Δ be an N-group of type-s. Then $\Delta \ll \Omega$ so that there exist N-subgroups Γ_1 and Γ_2 of Ω such that $\Delta \overset{N}{\cong} \Gamma_1 - \Gamma_2$. By the lattice correspondence there exists a subfactor $\bar{H}$ of G corresponding to $\Gamma_1 - \Gamma_2$. Since $\Gamma_1 - \Gamma_2$ is of type-O it follows that $\bar{H}$ is simple. If $\bar{K}$ is a subgroup of $\bar{H}$, then $\bar{K}$ corresponds to an N-subgroup $\bar{\Gamma}$ of $\Gamma_1 - \Gamma_2$. Now $\bar{\Gamma}$ is cyclic and since Δ is of type-s $\bar{\Gamma}$ is a direct sum of N-groups of type-O. Consequently, $\bar{K}$ is a direct sum of simple groups. By fact 2.1 this is possible only if $\bar{H}$ is cyclic of prime order. Hence $\Gamma_1 - \Gamma_2$ is of type-2 and we have $J_s(N) = J_2(N)$.

We now give an example of a class of d.g. near-rings in which $J_2(N) \not\subseteq J_s(N) \not\subseteq J_0(N)$. This is a generalisation of an example of a non-trivial s-primitive d.g. near-ring given in [11]. Another generalisation of the example in [11] was given by J. Hall [9]. Hall constructed an s-primitive d.g. near-ring N with A_n, $n \geq 6$ as a faithful N-group of type-s and in which A_{n-1} is the only proper, non-zero N-subgroup of A_n.

<u>Example 2.3</u> Consider the alternating group $A_{\alpha n+1}$, $\alpha > 1$, $n \geq 5$ on the set of symbols $X = \{1,2,\ldots,\alpha n, \alpha n+1\}$. We use the additive notation for the group operation in $A_{\alpha n+1}$. Define the subsets X_i of X by

$$X_i = \{n(i-1)+1, n(i-1)+2, \ldots, in\}$$

for $i = 1,2,\ldots,\alpha$. We note that $\alpha n+1 \notin X_i$ for any i. Let S_{n_i} (A_{n_i}) denote the symmetric (respectively, alternating) group on X_i. For each i and $x \in S_{n_i}$

define a map $\phi_x : A_{\alpha n+1} \to A_{\alpha n+1}$ by $\phi_x : a \to x + a - x$ for all $a \in A_{\alpha n+1}$. Let Φ denote the set of all these mappings. Clearly, Φ is a set of automorphisms of $A_{\alpha n+1}$ which induce automorphisms on A_{n_i} for each $i = 1,\ldots,\alpha$. We note that ϕ_x as defined above is the identity map on A_{n_j} if $i \neq j$. Let N be the d.g. near-ring generated by Φ. We show that N is 0-primitive with $A_{\alpha n+1}$ a faithful N-group of type-0 and $J_2(N) \neq J_3(N)$. For this purpose we characterise the N-subgroups of $A_{\alpha n+1}$.

a) Let $H \neq \{0\}$ be an N-subgroup of $A_{\alpha n+1}$. Then H contains A_{n_i} for some $i = 1,\ldots,\alpha$.

<u>Proof</u> Let $h \in H$, $h \neq 0$ and write $h = h_1 + h_2 + \ldots + h_m$ as a sum of disjoint cycles. Write $h_m = (a_1 \ldots a_r)$ and suppose that $a_r \in X_i$. We will show that $H \supseteq A_{n_i}$. Put $h' = h_1 + \ldots + h_{m-1}$ so that $h = h' + h_m$. Let $b \neq a_r$ be any fixed element of X_i. Then $*(a_r b) + h + (a_r b) - h \in H$. We have the following three cases to consider:

<u>Case 1</u> b does not appear in any of the cycles $h_1,\ldots,h_m$. In this case, from * above,

$$(a_r b) + (a_1 \ldots a_r) + (a_r b) + h' - h' + (a_r a_{r-1} \ldots a_1) \in H.$$

Thus

$$(a_r a_{r-1} b) \in H.$$

If $c \in X_i$, $c \neq b$ or a_r or a_{r-1}, then

$$(a_r bc) + (a_r a_{r-1} b) + (a_r cb) = (a_r c a_{r-1}) \in H.$$

Hence H contains the alternating group on $X_i \cup \{a_{r-1}\}$.

<u>Case 2</u> b appears in some h_j, $j \neq m$.

Without loss of generality we can suppose that b appears in h_{m-1}. Write

$$h = h'' + h_{m-1} + h_m,$$

where $h'' = h_1 + \ldots + h_{m-2}$ and $h_{m-1} = (b_1 \ldots b_t b)$. From * we have

$$(a_r b) + (a_1 \ldots a_r) + (b_1 \ldots b_t b) + (a_r b) + h'' - h'' + (a_r \ldots a_1) + (b\, b_t \ldots b_1)$$
$$= (a_r b) + (b_t a_{r-1}) \in H.$$

If $c \in X_i$ and c is not equal to any one of a_r, b, b_t, a_{r-1} (this is possible because $n \geq 5$)

$$(a_r b c) + (a_r b) + (b_t a_{r-1}) + (a_r c b) + (a_r b) + (b_t a_{r-1}) \in H;$$

that is $(a_r c b) \in H$, and so as in case 1 $H \supseteq A_{n_i}$.

Case 3 b appears in h_m.

Suppose $b = a_i$ for some $i = 1, \ldots, r-1$. Then by *

$$(a_r a_i) + (a_1 \ldots a_i \ldots a_r) + (a_r a_i) + h' - h' + (a_r \ldots a_i \ldots a_1) \in H.$$

If a_i and a_r are adjacent, then an easy calculation shows that we have the situation as in case 1. Hence we may assume that $i \neq 1$ or $r-1$. In this case $(a_r a_i) + (a_{i-1} a_{r-1}) \in H$ and we have the result as in case 2.

b) Let H be an N-subgroup of $A_{\alpha n+1}$ and suppose H contains an element h involving the symbol $\alpha n + 1$. Then H contains the alternating group on the set of symbols $X_i \cup \{\alpha n + 1\}$ for some $i = 1, \ldots, \alpha$.

Proof Write $h = h_1 + \ldots + h_m$ as a sum of disjoint cycles. Let $h_m = (a_1 \ldots a_r\ \alpha n+1)$ and suppose that $a_1 \in X_i$ for some i. Again let $b \neq a_1$ be a fixed element of X_i. Then

$$† (a_1 b) + h + (a_1 b) - h \in H$$

and we have the following three cases:

Case 1 b does not appear in any of the cycles $h_1, \ldots, h_m$. Using † above we see that $(a_1\ \alpha n+1\ b) \in H$. As in case 1 of a) we see that H contains the alternating group on the symbols $X_i \cup \{\alpha n + 1\}$.

Case 2 b appears in h_k, $k \neq m$.

Without loss of generality we may suppose that $k = m - 1$. Write $h = h'' + h_{m-1} + h_m$, where $h'' = h_1 + \ldots + h_{m-2}$ and $h_{m-1} = (b_1 \ldots b_t b)$. From † we have $(a_1 b) + (\alpha n+1\ b_t) \in H$. Suppose $b_t \in X_j$ for some j. We do not assume that X_j is different from X_i. Choose $d \in X_j$ such that $d \neq b_t$, $d \neq a_1$, $d \neq b$. Such a d exists because $n \geq 5$. We have

$$(b_t d) + (a_1 b) + (\alpha n{+}1\ b_t) + (b_t d) + (a_1 b) + (\alpha n{+}1\ b_t) \in H.$$

That is, $(b_t\ \alpha n{+}1\ d) \in H$, and it follows that H contains the alternating group on $X_j \cup \{\alpha n + 1\}$.

Case 3 b appears in h_m.

Suppose $b = a_i$ for some $i = 2,\ldots,r$. Using † we see that

$$(a_1 a_i) + (\alpha n{+}1\ a_{i-1}) \in H.$$

If $i = 2$, then we proceed as in case 1 of b). Otherwise we proceed as in case 2 of b).

From a) it is clear that the A_{n_i} are N-groups of type-2. The alternating group K_i on the set of symbols $X_i \cup \{\alpha n + 1\}$ is a cyclic N-group with generator $(a\ b\ \alpha n{+}1)$ where $a, b \in X_i$. In fact, K_i is an N-group of type-s with A_{n_i} as its only proper non-zero N-subgroup. Consequently, $J_2(N) \neq J_s(N)$. Finally, we find a cyclic generator for $A_{\alpha n+1}$. Let H be an N-subgroup of $A_{\alpha n+1}$ such that H contains the alternating group on the set $\bigcup_{i=1}^{k} X_i$. If H contains a three cycle (a,b,c) with $a,b \in \bigcup_{i=1}^{k} X_i$ and $c \in X_{k+1}$, then H contains the alternating group on $\bigcup_{i=1}^{k+1} X_i$. It is now easy to see that $N\omega = A_{\alpha n+1}$ where

$$\omega = (1\ 2\ n{+}1) + (n{+}2\ n{+}3\ 2n{+}1) + \ldots + ((\alpha-2)n{+}2\ \ (\alpha-2)n{+}3\ \ n(\alpha-1)+1) + (n(\alpha-1){+}2\ \ n(\alpha-1){+}3\ \ \alpha n{+}1).$$

Now $A_{\alpha n+1}$, $\alpha > 1$ is not of type-s since the N-subgroup generated by $(1\ 2) + (n{+}1\ n{+}2)$ contains both A_{n_1} and A_{n_2} as N-kernels so that it is not of type-0 and not expressible as a direct sum of N-groups of type-0. Thus for the near-ring N we have $J_2(N) \neq J_s(N) \neq \mathcal{C}(N) \neq J_0(N) = \{0\}$.

We remark that in his construction Hall [9] considered only the case $\alpha = 1$ with $n \geq 6$.

In conclusion, and for the sake of completeness, we discuss some of the typical radical-like properties which hold for $J_s(N)$. First of all we need the following definition from universal algebra [23]:

Definition A map f which assigns to each near-ring N an ideal f(N) of N is called a radical map if for every pair of near-rings N,N′ we have

(i) $f(N/f(N)) = \{0\}$;

(ii) if $h \in Hom(N,N')$, then $h(f(N)) \subseteq f(h(N))$.

Definition Let f be a radical map. The near-ring N is called

(a) f-semisimple if, and only if, $f(N) = \{0\}$;

(b) f-radical if, and only if, $f(N) = N$.

The proofs of the following are identical with those given in Pilz [23] for $J_\nu(N)$, $\nu = 0,1,2$.

Proposition 2.14 $N \to J_s(N)$ is a radical map.

Corollary If A is an ideal of N, then $J_s(N/A) \supseteq (J_s(N) + A)/A$.

Proposition 2.15 If the ideal A is a direct summand of the near-ring N, then $J_s(N) \supseteq J_s(N) \cap A$.

Proposition 2.16 If $N = \bigoplus_{i \in I} A_i$ of ideals A_i, then

$$J_s(N) \supseteq \bigoplus_{i \in I} J_s(A_i).$$

Now, there is a theorem [23] which states that $J_\nu(A) \subseteq J_\nu(N) \cap A$ if A is an ideal of N and N satisfies the DCCL for $\nu = 1,2$. The proof of this theorem utilises the fact that every ideal of a $J_1(N)$-semisimple near-ring is $J_1(N)$-semisimple, and this in turn follows from the fact that 1-primitive ideals are maximal. It is easy to see that s-primitive ideals are not, in general, maximal ideals and one suspects that things will now start to go wrong. However, we are unable to construct an example of a near-ring in which $J_s(A) \subseteq J_s(N) \cap A$ does not hold.

CHAPTER 3

An antiradical for near-rings

The main aim in this chapter is to construct an antiradical for near-rings which we call the socle-ideal. In general, the crux [27] of a near-ring contains the socle-ideal, but it is not an antiradical as it lacks a socle-like structure. We nevertheless discuss it here because of its connections with the socle-ideal. For example, we show that if N is a near-ring with DCCN, then the above two ideals coincide.

1. The Socle-ideal

Definition Let $\mathfrak{F}$ be the collection of all ideals A of the near-ring N which are of the form $A = \bigoplus_{i\in I} Ne_i$, $e_i \in Ne_i$ for each $i \in I$ such that

(i) each Ne_i is an N-group of type-0 and a left ideal of N;

(ii) $e_i^2 = e_i$ for all $i \in I$ and $e_ie_j = 0$ if $i > j$ for some ordering on I.

If $A = \bigoplus_{i\in I} Ne_i$ and the Ne_i and e_i, $i \in I$ are as above, then we say that $\bigoplus_{i\in I} Ne_i$ is an $\mathfrak{F}$-decomposition of A.

Lemma 3.1 Let $A, B \in \mathfrak{F}$ and suppose that $A = \bigoplus_{i\in I} Ne_i$ and $B = \bigoplus_{i\in J} Nf_i$ are $\mathfrak{F}$-decompositions for A and B respectively. If $A \subseteq B$, then there exists an index set T such that $I \subseteq T$ and $B = \bigoplus_{t\in T} Ne_t$ is an $\mathfrak{F}$-decomposition for B.

Proof By lemma 1.4 A is a direct summand of B, so we may write

$$B = A \oplus \left(\bigoplus_{j\in J'} Nf_j\right) = \bigoplus_{i\in I} Ne_i \oplus \left(\bigoplus_{j\in J'} Nf_j\right)$$

for some $J' \subseteq J$. Consequently, $A.Nf_j = \{0\}$ and so $e_k.f_r = 0$ for all $k \in I$ and $r \in J'$. Put $T = I \cup J'$ and order T as follows. The orderings on I and J' are as in the above $\mathfrak{F}$-decompositions for A and B and $i > j$ for all $i \in I$ and $j \in J'$. It is now clear that $B = \bigoplus_{t\in T} Ne_t$ is an $\mathfrak{F}$-decomposition for B with $e_t = f_t$ for all $t \in J'$.

Theorem 3.2 (a) If $A, B \in \mathfrak{F}$, then $A + B \in \mathfrak{F}$;

(b) If $\mathfrak{F}$ is not empty, then there exists a unique maximal element in $\mathfrak{F}$.

<u>Proof</u> (a) Suppose $A = \bigoplus_{i \in I} Ne_i$ and $B = \bigoplus_{j \in J} Nf_j$ are two $\mathfrak{F}$-decompositions of A and B respectively. We have $A \cap Nf_j = \{0\}$ or $Nf_j \subseteq A$ since the Nf_j are of type-0. By dropping those $Nf_i \subseteq A$, if any, we may write

$$A + B = \bigoplus_{i \in I} Ne_i + \left(\bigoplus_{j \in J'} Nf_j\right) = A + L,$$

where J' is some subset of J and $L = \bigoplus_{i \in J'} Nf_i$ is a left ideal. It is straightforward to show that $A \cap L = \{0\}$ so that

$$A + B = A \oplus L = \bigoplus_{i \in I} Ne_i \oplus \left(\bigoplus_{j \in J'} Nf_j\right),$$

a direct sum. Now $e_i f_j \in AL = \{0\}$ for all $i \in I$ and $j \in J'$ and so by extending the orderings of I and J' to $I \cup J'$ (with $i > j$ for $i \in I$, $j \in J'$) we have $A + B \in \mathfrak{F}$.

(b) Partially order $\mathfrak{F}$ by $A \leqslant B$ if and only if $A \subseteq B$. Consider a chain $A_1 < A_2 < \ldots < A_s < \ldots$ in $\mathfrak{F}$. We first prove that the ideal $A = \bigcup_s A_s$ is an element of $\mathfrak{F}$. By lemma 3.1 we have a chain of ordered sets $I_1 \subset I_2 \subset \ldots \subset I_s \subset I_{s+1} \subset \ldots$ with $A_s = \bigoplus_{i \in I_s} Ne_i$ an $\mathfrak{F}$-decomposition for A_s. Furthermore, for each s and r, $s > r$, the ordering in I_s induces that in I_r in the sense that for $j \in I_r \setminus I_s$, $k \in I_s$, $k > j$. These orderings induce an ordering in $I = \bigcup_s I_s$. Thus $A = \bigoplus_{k \in I} Ne_k$ is an $\mathfrak{F}$-decomposition for A and hence A is an upper bound for the chain. By Zorn's lemma, $\mathfrak{F}$ has a maximal element M, say. If $B \in \mathfrak{F}$, then by (a) $B + M \in \mathfrak{F}$ so that $B \leqslant M$ because M is a maximal element of $\mathfrak{F}$. Consequently, M is the unique maximal element of $\mathfrak{F}$.

<u>Definition</u> The unique maximal element of theorem 3.2 is called the <u>socle-ideal</u> of the near-ring N and we denote it by Soi(N). If $\mathfrak{F}$ is empty, then we define Soi(N) to be the zero ideal.

We note that if $\mathrm{Soi}(N) = \bigoplus_{i \in I} Ne_i$ is an $\mathfrak{F}$-decomposition of Soi(N), then $(0 : e_i)$ is an 0-modular left ideal for each $i \in I$. Hence $Q(N) \subseteq \bigcap_{i \in I} (0 : e_i)$.

<u>Lemma 3.3</u> $Q(N) \cap \mathrm{Soi}(N) = \{0\}$. In particular, if $N = \mathrm{Soi}(N)$, then $Q(N) = \{0\}$.

<u>Proof</u> If $x \in Q(N) \cap Soi(N) \subseteq \bigcap_{i\in I} (0:e_i) \cap \bigoplus_{i\in I} Ne_i$, then

$$x = n_{i_1} e_{i_1} + \dots + n_{i_r} e_{i_r}, \quad i \in I,$$

where we assume that $i_1 < i_2 < \dots < i_r$. We have $0 = xe_{i_1} = n_{i_1} e_{i_1}$ so that $x = n_{i_2} e_{i_2} + \dots + n_{i_r} e_{i_r}$. By induction we obtain $x = 0$.

The above lemma implies that $Soi(N) \cdot Q(N) = Soi(N) \cdot J_0(N) = \{0\}$ so that $Soi(N)$ is an antiradical in the sense of Chapter 1.

<u>Theorem 3.4</u> If N has DCCL, then Soi(N) is a direct summand of N.

<u>Proof</u> The DCCL implies that Soi(N) has an $\mathfrak{F}$-decomposition which is a finite direct sum $Soi(N) = \bigoplus_{i=1}^{k} Ne_i$. Since $(0:e_i)$ is a maximal left ideal, Ne_i is of type-0 and $e_i^2 = e_i$, we have $N = (0:e_i) \oplus Ne_i$ for $i = 1,\dots,k$. Suppose we have shown that

$$N = \bigoplus_{i=1}^{l-1} Ne_i \oplus \bigcap_{i=1}^{l-1} (0:e_i),$$

for $k \geq l \geq 2$. Now $Ne_l \subseteq \bigcap_{i=1}^{l-1} (0:e_i)$ as $e_l e_i = 0$ for $i = 1,\dots,l-1$. Thus

$$Ne_l \oplus \bigcap_{i=1}^{l} (0:e_i) \subseteq \bigcap_{i=1}^{l-1} (0:e_i).$$

If $x \in \bigcap_{i=1}^{l-1} (0:e_i)$, then $x = x_1 + x_2$ where $x_1 = ne_l \in Ne_l$ and $x_2 \in (0:e_l)$. Hence for each $i = 1,\dots,l-1$ we have $0 = xe_i = x_1 e_i + x_2 e_i = (ne_l)e_i + x_2 e_i = x_2 e_i$. Thus $x_2 \in \bigcap_{i=1}^{l-1} (0:e_i)$ and so $x_2 \in \bigcap_{i=1}^{l} (0:e_i)$. Consequently,

$$Ne_l \oplus \bigcap_{i=1}^{l} (0:e_i) = \bigcap_{i=1}^{l-1} (0:e_i)$$

and it follows that

$$N = \bigoplus_{i=1}^{l-1} Ne_i \oplus Ne_l \oplus \bigcap_{i=1}^{l} (0:e_i)$$

$$= \bigoplus_{i=1}^{l} Ne_i \oplus \bigcap_{i=1}^{l} (0:e_i).$$

By induction we have shown that $N = Soi(N) \oplus L$, where $L = \bigcap_{i=1}^{k} (0 : e_i)$.

We note that $Soi(N).L = \{0\}$. Also, thus far we needed to exercise extreme care not to violate the notion of one-sided orthogonality for the cyclic generators of the type-0 summands in Soi(N). For certain classes of near-rings this caution is not necessary.

Theorem 3.5 If Soi(N) has an $\mathcal{Y}$-decomposition which is a finite direct sum $Soi(N) = \bigoplus_{i=1}^{k} Ne_i$, then $Soi(N) = \bigoplus_{i=1}^{k} Nf_i$, where $Nf_i = Ne_i$, $f_i^2 = f_i$ and $f_i f_j = 0$ if $i \neq j$ for $i,j = 1,\ldots,k$.

Proof We first show that $Soi(N) = Soi(N)e$ where $e = e_1 + \ldots + e_k$. By left distribution over the N-kernels Ne_i we see that for $n \in N$, $ne_k = ne_k(e_1 + \ldots + e_k)$ because $e_k e_i = 0$ for $1 \leq i < k$ and $e_k^2 = e_k$. Consequently $Ne_k \subseteq Soi(N)e$. Suppose we have shown that $Ne_i \subseteq Soi(N)e$ for all i such that $k \geq i > l$, then again by left distribution over N-kernels we have for any $n \in N$

$$ne_l = ne_l(e_1 + \ldots + e_k) - ne_l e_k - ne_l e_{k-1} - \ldots - ne_l e_{l+1} .$$

Now $ne_l(e_1 + \ldots + e_k) \in Soi(N)e$ and $ne_l e_j \in Soi(N)e$ for $j = l+1,\ldots,k$ by assumption. Thus $Ne_l \subseteq Soi(N)e$ and by induction we obtain $Soi(N) = \bigoplus_{i=1}^{k} Ne_i \subseteq Soi(N)e$ and so $Soi(N) = Soi(N)e$. Next we show that the mapping ψ: $Soi(N) \to Soi(N)e$ given by $\psi: n \to ne$ for all $n \in Soi(N)$ is an N-automorphism. By the above it is certainly an N-epimorphism. Suppose $\psi(n) = 0$ where $n = n_1 e_1 + \ldots + n_k e_k$; then using left and right distributions over N-kernels we have

$$0 = \psi(n) = (n_1 e_1 + \ldots + n_k e_k)(e_1 + \ldots + e_k)$$

$$= n_1 e_1(e_1 + \ldots + e_k) + n_2 e_2(e_1 + \ldots + e_k) + \ldots + n_k e_k(e_1 + \ldots + e_k)$$

$$= (n_1 e_1 + n_1 e_1 e_2 + \ldots + n_1 e_1 e_k) + (n_2 e_2 + n_2 e_2 e_3 + \ldots + n_2 e_2 e_k) + \ldots + (n_k e_k)$$

because $e_i e_j = 0$ if $i > j$. Now $n_1 e_1$ is the only term on the right hand side which is in Ne_1 and because $\psi(n) = 0$ it follows that $n_1 e_1 = 0$. Hence also $n_1 e_1 e_j = 0$ for $j = 2,\ldots,k$. Thus we have

$$\psi(n) = (n_2 e_2 + \ldots + n_2 e_2 e_k) + \ldots + (n_k e_k) = 0.$$

By induction we deduce that $n = 0$. Now $e \in Soi(N) = Soi(N)e$ and so there exists an $f \in Soi(N)$ such that $\psi(f) = e$. For any $x \in Soi(N)$ we have

$$\psi(xf) = x\psi(f) = xe = \psi(x)$$

and hence $xf = x$ for all $x \in Soi(N)$. Put $f = f_1 + f_2 + \ldots + f_k$, where $f_i \in Ne_i$ for $i = 1,\ldots,k$; then

$$\bigoplus_{i=1}^{k} Ne_i = Soi(N) = Nf = \bigoplus_{j=1}^{k} Nf_j.$$

It is now easy to see that $f_i^2 = f_i$ and $f_i f_j = 0$ if $i \neq j$.

Corollary If N satisfies the DCCL, then $Soi(N) = \bigoplus_{i=1}^{k} Nf_i$, where $\{f_i\}$ is an orthogonal set of idempotents.

We now give a characterisation of $Soi(N)$ for near-rings which satisfy the DCCN. For this purpose we need a generalisation of lemma 1.3 due to S.D. Scott [28].

Lemma 3.6 Let N be a near-ring satisfying the DCCN and L a left ideal such that $L \not\subseteq Q(N)$. Then $L-(L\cap Q(N))$ is a direct sum

$$L-(L\cap Q(N)) = L_1-(L\cap Q(N)) \oplus \ldots \oplus L_k-(L\cap Q(N)),$$

where the L_i are left ideals such that $L_i - L\cap Q(N)$ is of type-0, $i = 1,\ldots,k$. Furthermore, there exists a set $\{e_i + L\cap Q(N)\}$ of cyclic generators of the $L_i-(L\cap Q(N))$ such that $e_i e_j \in L\cap Q(N)$ if $i \neq j$ and $e_i^2 - e_i \in L\cap Q(N)$ for $i,j = 1,\ldots,k$.

Using the above lemma we immediately have

Theorem 3.7 Let N be a near-ring with DCCN and A an ideal of N. Then $A\cap Q(N) = \{0\}$ if, and only if, $A \subseteq Soi(N)$.

Corollary $Soi(N)$ is the unique maximal ideal with zero intersection with the quasi-radical $Q(N)$.

As we have seen, $\mathrm{Soi}(N).Q(N) = \{0\}$ because $\mathrm{Soi}(N) \cap Q(N) = \{0\}$ for any near-ring N. However, example 2.1 shows that Soi(N) is not maximal with respect to annihilating Q(N) even in the finite case. The following is a further connection between the socle-ideal and the quasi-radical:

<u>Theorem 3.8</u> Let N be a near-ring with DCCL and A an ideal of N. Then $\mathrm{Soi}(N/A) = N/A$ if, and only if, $Q(N) \subseteq A$.

<u>Proof</u> If $\mathrm{Soi}(N/A) = N/A$, then by theorem 3.5 N/A has an $\mathcal{G}$-decomposition of the form $N/A = \bigoplus_{i=1}^{k} N\bar{e}_i$, where $\bar{e}_1, \ldots, \bar{e}_k$ is an orthogonal set of idempotents. It is clear that $\bar{e}_1 + \ldots + \bar{e}_k$ is a right identity of N/A. Hence by lemma 2.6 $Q(N/A) = \{0\}$ so that $A \supseteq Q(N)$. Conversely, if $A \supseteq Q(N)$, then by corollary 2 of theorem 2.8, A is an intersection of s-modular left ideals so that $Q(N/A) = \{0\}$. By lemma 1.3

$$N/A = \bigoplus_{i=1}^{k} \bar{L}_i,$$

where $\bar{L}_i$ is an N/A-group of type-0, $i = 1, \ldots, k$. Since $A \supseteq Q(N)$ and Q(N) is a modular left ideal it follows that N/A has a right identity. The decomposition of this right identity as a sum $\bar{e}_1 + \ldots + \bar{e}_k$, $\bar{e}_i \in \bar{L}_i$, $i = 1, \ldots, k$ yields an orthogonal set of idempotents generating the $\bar{L}_i$. By the maximality of the socle-ideal we have $\mathrm{Soi}(N/A) = N/A$.

From the fact that $J_s(N)$ is the smallest ideal of N which contains Q(N) we immediately have

<u>Corollary</u> $J_s(N)$ is the unique smallest ideal amongst all ideals A of N such that $\mathrm{Soi}(N/A) = N/A$.

<u>Example 3.1</u> Consider the ring Z_n of integers modulo n. Write

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$$

as a product of powers of distinct primes. If $m|n$, then the isomorphic copy of the cyclic group of order m contained in Z_n will be denoted by Z_m. Thus

$$Z_n = Z_{p_1^{\alpha_1}} \oplus \dots \oplus Z_{p_r^{\alpha_r}}.$$

Now $Z_{p_i^{\alpha_i - 1}}$ is nilpotent and the unique maximal ideal of Z_n contained in the ideal $Z_{p_i^{\alpha_i}}$, $i = 1,\dots,r$. Hence if $\alpha_i > 1$ for each $i = 1,\dots,r$, then $Soi(N) = \{0\}$. Otherwise, if $\alpha_1 = \alpha_2 = \dots = \alpha_s = 1$ for $s < r$ and $\alpha_i > 1$ for $i = s+1,\dots,r$, then

$$Soi(Z_n) = Z_{p_1} \oplus \dots \oplus Z_{p_s}.$$

We note that if

$$Z_n = Z_{p_1^{\alpha_1}} \oplus \dots \oplus Z_{p_r^{\alpha_r}},$$

$\alpha_i > 1$ for $i = 1,\dots,r$, then

$$Z_n / Z_{p_1^{\alpha_1 - 1}} \cong Z_{p_1} \oplus Z_{p_2^{\alpha_2}} \oplus \dots \oplus Z_{p_r^{\alpha_r}}$$

so that

$$Soi\left(Z_n / Z_{p_1^{\alpha_1 - 1}}\right) \neq \{0\}.$$

Indeed, if we factor out the nilpotent component in any direct sum of the $Z_{p_i^{\alpha_i}}$ we obtain a ring with non-zero socle-ideal.

<u>Example 3.2</u> Consider the near-ring N on the four-group $V_4 = \{0,a,b,c\}$ of example 2.1. As we pointed out, N is a direct sum of left ideals

$$N = \{0,a\} \oplus \{0,c\}, \quad \text{with} \quad a^2 = a,\ c^2 = c.$$

From the multiplication table for N one sees that $a\,c \neq 0$ and $c\,a \neq 0$ and clearly an orthogonal set of idempotents, the elements of which are cyclic generators of the summands in N, does not exist. Furthermore, neither $\{0,a\}$ nor $\{0,c\}$ is a two-sided ideal and since $b^2 = 0$ we are forced to conclude that $Soi(N) = \{0\}$. As we have seen $J_\nu(N) = Q(N) = \{0,b\}$, $\nu = 0,1,2,s$.

<u>Example 3.3</u> [5] Consider the near-ring $N = (Z_6,+,*)$ where + is addition modulo 6 and * is given in the following table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	4	0	0	4	0
2	0	2	0	0	2	0
3	0	0	0	0	0	0
4	0	4	0	0	4	0
5	0	2	0	0	2	0

First of all, $A = \{0,2,4\}$ is an ideal of N. In order to verify this, one needs to consider all elements of the form $n'(n+x) - n'x$ with $n,n' \in N$, $x \in A$. Since the difference of any two products in a row is in A this follows trivially. That it is a right ideal is equally obvious. Moreover, 4 is an idempotent generator of the irreducible N-group A and hence $A \subseteq \mathrm{Soi}(N)$. We now verify that $\{0,3\}$ is an ideal of N. It certainly is a two-sided N-subgroup of N^+. Consider all elements of the form $n'(n+x) - n'n$, where $n',n \in N$, $x \in \{0,3\}$. All these are of the form 0, $\pm n'4 \mp n'1$, $\pm n'5 \mp n'2$, $\pm n'3$ which are all zero. In fact $\{0,3\} = J_\nu(N)$, $\nu = 0,1,2,s$ because $3^2 = 0$. Thus $A = \mathrm{Soi}(N)$ and N decomposes as $N = \mathrm{Soi}(N) \oplus J_\nu(N)$, $\nu = 0,1,2,s$.

Remarks The near-ring in example 3.2 is in fact a ring in which the left socle ($\mathrm{Soc}_L(N)$) = the sum of all minimal left ideals of N = N and the right socle ($\mathrm{Soc}_R(N)$) = the sum of all minimal right ideals of N = $\{0\}$. Baer's antiradical [4] is the right socle, $\mathrm{Soc}_R(N)$ of the ring N and he proves that $\mathrm{Soc}_R(N).J_2(N) = \{0\}$ for any N with DCCL. It is not in general true that $J_2(N).\mathrm{Soc}_R(N) = \{0\}$ even if N is finite as example 3.2 shows. Similarly, $J_2(N).\mathrm{Soc}_L(N) = \{0\}$ though $\mathrm{Soc}_L(N).J_2(N) \neq \{0\}$ in general even in the finite case. Our socle-ideal is more special in that we insist on an idempotent set of generators, which are orthogonal under some ordering. Certainly we have that $J_2(N).\mathrm{Soi}(N) = \{0\}$, because $\mathrm{Soi}(N) \subseteq \mathrm{Soc}_L(N)$. On the other hand, if N is a ring, then $\mathrm{Soi}(N).J_2(N) = \mathrm{Soi}(N).Q(N) = \{0\}$. Of course, if $\mathrm{Soc}_R(N) = \mathrm{Soc}_L(N)$ for the ring N, then either would annihilate $J_2(N)$ from the left and from the right. Even in this case, however, they need not coincide with Soi(N) as the following example shows.

Example 3.4 Consider the Clay "small" near-ring [5], $N = (V_4,+,*)$ where V_4 is the four-group $\{0,a,b,c\}$ written additively and the multiplication $*$ is given in the following table.

*	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

N is a commutative ring and we see easily that

$$Soc_R(N) = Soc_L(N) = N \quad \text{whereas} \quad Soi(N) = \{0,a\} \neq N.$$

2. The crux of a near-ring

The crux of a near-ring N was first defined by S.D. Scott [27]. The main aim in this section is to establish relationships between the crux and the socle-ideal. We begin by giving the definitions due to Scott.

Definition The nil-radical of a near-ring N is the sum of all nil ideals of N. We denote the nil-radical of N by nil(N).

We note that nil(N) is itself a nil ideal of N.

Definition An ideal A of N is said to be rigid if whenever D is an ideal of N contained in A, then $A/D \cap nil(N/D) = \{0\}$.

Definition The sum of all rigid ideals of N is called the crux of N. We denote the crux of N by Crux(N).

In [27] Scott proved that Crux(N) is itself a rigid ideal. Because of its rather strong nil "avoidance" property, Crux(N) annihilates the nil radical of N. We will show that Soi(N) is a rigid ideal. We first prove the following:

Lemma 3.9 Let $L = \bigoplus_{i \in I} Ne_i$ be a left ideal of N, where for each i, Ne_i is a left ideal of N and of type-0 and e_i is an N-generator of Ne_i. Suppose further that for all $i,j \in I$, $e_i^2 = e_i$ and $e_ie_j = 0$ if $i > j$ for some ordering on I. If $B \neq \{0\}$ is an ideal of N contained in L, then $B \in \mathcal{F}$. Moreover, $B = \bigoplus_{i\in J} Ne_i$ for some $J \subseteq I$.

Proof By lemma 1.4 B is a direct summand of L and we may write

$$B \oplus \left(\bigoplus_{j\in J'} Ne_j\right) = \bigoplus_{i\in I} Ne_i = L,$$

for some $J' \subseteq I$. Let $k \in J = I \backslash J'$, the complement of J' in I; then

$$e_k = b + n_{j_1}e_{j_1} + \cdots + n_{j_s}e_{j_s},$$

$b \in B$, $j_r \in J'$ for $r = 1,\ldots,s$. Without loss of generality we may assume $j_1 < j_2 < \cdots < j_s$. If $k < j_1$, then

$$e_k = e_ke_k = be_k + n_{j_1}e_{j_1}e_k + \cdots + n_{j_s}e_{j_s} = be_k,$$

since $e_{j_r}e_k = 0$ for $r = 1,\ldots,s$. Hence $e_k = be_k \in B$ in this case. If $k > j_1$, then

$$0 = e_ke_{j_1} = be_{j_1} + n_{j_1}e_{j_1} + \cdots + n_{j_s}e_{j_s}e_{j_1}$$

$$= be_{j_1} + n_{j_1}e_{j_1}.$$

Consequently, $-be_{j_1} = n_{j_1}e_{j_1} \in B \cap Ne_{j_1} = \{0\}$ and so

$$e_k = b + n_{j_2}e_{j_2} + \cdots + n_{j_s}e_{j_s}.$$

Now say $j_1 < j_2 < \cdots j_t < k < j_{t+1} < \cdots < j_s$. By induction and the above process we see that $e_k = b + n_{j_{t+1}}e_{j_{t+1}} + \cdots + n_{j_s}e_{j_s}$. Hence

$$e_k = e_k^2 = be_k \in B.$$

Finally say $j_1 < \cdots < j_s < k$. By the above process we deduce that $e_k = b \in B$. Thus $e_k \in B$ for all $k \in J$ and hence $B \supseteq \bigoplus_{i\in J} Ne_i$. It follows easily that $B = \bigoplus_{i\in J} Ne_i$ and so $B \in \mathcal{F}$.

Theorem 3.10 For any near-ring N, $\mathrm{Soi}(N) \subseteq \mathrm{Crux}(N)$.

Proof Let $\mathrm{Soi}(N) = \bigoplus_{i\in I} Ne_i$ be an $\mathcal{J}$-decomposition of Soi(N). If A is an ideal contained in Soi(N), consider the ideal $(\mathrm{Soi}(N)/A)\cap \mathrm{nil}(N/A) = B/A$, say, where B is an ideal of N such that $\mathrm{Soi}(N) \supseteq B \supsetneq A$. If A = B, then we are through, so suppose $B \neq A$. Then $B \neq \{0\}$ and hence by lemma 3.9, B has an expression of the form $B = \bigoplus_{i\in J} Ne_i$ for some index set $J \subseteq I$. For any $j \in J$ we have $e_j + A \in \mathrm{nil}(N/A)$, so there exists a positive integer m, depending on j, such that $(e_j+A)^m = e_j^m + A = A$. Thus $e_j = e_j^m \in A$ and so $Ne_j \subseteq A$ for all $j \in J$. Hence A = B, which is a contradiction. Thus A = B and the result is proved.

In [28] Scott proved that if N has DCCN, then $\mathrm{Crux}(N) \cap Q(N) = \{0\}$. In his proof, which involves Zorn's lemma, he uses the fact that for such near-rings N

$$Q(N/T) = (Q(N)+T)/T$$

for any ideal T.

Hence by theorem 3.7 we have

Theorem 3.11 If N is a near-ring with DCCN, then Soi(N) = Crux(N).

Whether the equality Soi(N) = Crux(N) holds in the more general DCCL case is not known. One suspects not, for Soi(N) is strongly rigid in the following sense:

Definition An ideal A of N is said to be strongly rigid if whenever B is an ideal of N contained in A, then

$$A/B \cap J_0(N/B) = \{0\}.$$

Since $\mathrm{nil}(N) \subseteq J_0(N)$ for any near-ring N it follows that every strongly rigid ideal is rigid. The proof of the fact that Soi(N) is strongly rigid is similar to the proof of theorem 3.10 and will be omitted. In the proof one uses the fact that an idempotent cannot be left quasi regular. In the following example we show that Crux(N) and Soi(N) are in general not the same.

Example 3.5 The Sasiada simple, radical ring [6],[7]

We begin by outlining the construction of this well-known ring. Let x and y be two non-commutating indeterminates and let T be the set of all formal power series in x and y with coefficients in Z_2. Let $S \subset T$ be the set of all elements in T with zero constant term. S is an ideal of T and each element of S is a non-unit of T. Furthermore, every element of T not in S is a unit, and it follows that S is the Jacobson radical of T. Now consider S itself as a ring. It can be proved that x is not contained in the ideal generated by $x + yx^2y$ [6]. Using Zorn's lemma select an ideal M of S maximal with respect to exclusion of x and inclusion of $x + yx^2y$. Then the intersection $\bar{M}$ of all non-zero ideals of S/M is not zero since each contains $x + M$. Now S is a radical ring and hence $\bar{M}$ is radical. Since $x + M \in \bar{M}$ it follows that $yx + M . xy + M = yx^2y + M \in \bar{M}$, so that $x + M \in \bar{M}^2$ because $x + yx^2y = x - yx^2y \in M$. Thus $\bar{M}^2 = \bar{M}$.

If $\bar{D} \neq \{0\}$ is an ideal of $\bar{M}$, then the ideal of S/M generated by $\bar{D}$ is $\bar{M}$ itself, as $\bar{M}$ is the minimal ideal of S/M. It follows that $\bar{M}^3 \subseteq \bar{D}$. Hence $\bar{M}^3 = \bar{M} \subseteq \bar{D} \subseteq \bar{M}$. Consequently, $\bar{M}$ is a simple, radical ring. It follows that $\bar{M}$ does not have any $\bar{M}$-groups of type-0, so that $Soi(\bar{M}) = \{0\}$. Now the nil radical of $\bar{M}$ is zero for $x \equiv yx^2y \not\equiv 0$ and one can show by induction that $x \equiv (yx)^n xy^n$ for all $n \geq 1$. If the nil radical of $\bar{M}$ equals $\bar{M}$, then in particular yx is nilpotent modulo M and so $yx^k \equiv 0$ for some k. In this case we would have $x \equiv 0$ which is a contradiction. It now follows that $\bar{M}$ is rigid and hence $Crux(\bar{M}) = \bar{M} \neq Soi(\bar{M})$.

As we mentioned before, Scott [27] proved that $Crux(N/Crux) = \{0\}$ for any near-ring with DCCI. We will prove a similar result for the socle-ideal in the case of a near-ring with DCCL.

Theorem 3.12 Let N be a near-ring satisfying the DCCL and let A be any ideal contained in Soi(N). Write $N = Soi(N) \oplus L$, $Soi(N) = A \oplus L'$ and consider the left ideal $M = L \oplus L'$ as a near-ring. Then $Soi(M) = L'$.

Proof It is easy to show that every left ideal of M is also a left ideal of

N. Furthermore, L' is a two-sided ideal of M. This follows from the fact that $L'L = \{0\}$. It is now clear that $Soi(M) \supseteq L'$. Consider the left ideal $Soi(M) \oplus A$ of N. We show that $Soi(M) \oplus A$ is a two-sided ideal of N. If $n \in Soi(M) \oplus A$ and $x \in N$, then $n = m + a$, $m \in Soi(M)$, $a \in A$ and $x = m' + a'$, $m' \in M$, $a' \in A$. We have

$$nx = (m+a)(m'+a') = mm' + ma' + am' + aa' = mm' + ma' + aa',$$

by left distribution over a direct sum of N-kernels and the fact that $am' = 0$. Thus we see that $nx \in Soi(M) \oplus A$, so that it is an ideal of N. Now $Soi(M)$ has an $\mathcal{Y}$-decomposition with respect to M. By the above, and the fact that A annihilates M, it follows that $Soi(M) \oplus A$ has an $\mathcal{Y}$-decomposition with respect to N. But $Soi(N) = L' \oplus A \subseteq Soi(M) \oplus A$ so that $Soi(N) = Soi(M) \oplus A$ by the maximality of $Soi(N)$. Consequently, $Soi(M) = L'$.

We note that if $\psi: N \to N'$ is an isomorphism from the near-ring N onto the near-ring N', then $\psi(Soi(N)) = Soi(N')$. Thus we have

<u>Corollary 1</u> $Soi(N/A) = Soi(N)/A$ for any ideal A contained in $Soi(N)$.

<u>Proof</u> Any $x \in N$ has a unique expression of the form $x = a + m$, $a \in A$, $m \in M$. Hence there is an isomorphism $\psi: N/A \to M$ given by $\psi(x+A) = m$. It is an easy matter to show that $\psi(Soi(N)/A) = Soi(M) = L'$. By the above remark we obtain $Soi(N/A) = Soi(N)/A$.

<u>Corollary 2</u> $Soi(N/Soi(N)) = \{0\}$.

CHAPTER 4

A representation theory for antiradicals

Throughout this chapter our near-ring N is assumed to have a multiplicative identity and to satisfy the DCCL. Our aim is to construct antiradicals by means of faithful N-groups. The representation theory developed here is based on the Laxton-Machin one [18], in which the main feature is the splitting up of irreducible N-groups into two disjoint classes. Our splitting up of the irreducible N-groups is quite different from the one in [18], but will yield the same antiradical if N satisfies both the ACCN and DCCN. We recall that the N-group Δ is said to be a subfactor of the N-group Ω, written $\Delta \ll \Omega$; if Δ is N-isomorphic to a factor group of an N-subgroup of Ω. We note that if $\Delta \ll \Omega$ and $\Omega \ll \Gamma$, then $\Delta \ll \Gamma$. As before, all our direct sums are direct sums of N-kernels.

1. The antiradicals $C(\Omega)$

Definition Let Ω be a faithful N-group and let $\mathcal{C}$ be the collection of all N-groups Δ such that $\Delta \ll \Omega$. Define $\mathcal{K}$ as follows:

$$\mathcal{K} = \{\Delta \in \mathcal{C} : \Delta \text{ does not have an N-group of type-0 as a direct summand}\}.$$

Clearly, $\mathcal{K}$ is not empty as $\{0\} \in \mathcal{K}$. Observe that every non-cyclic, irreducible N-group which is a subfactor of Ω is in $\mathcal{K}$. Furthermore, $\mathcal{K}$ does not contain any N-group of type-0.

Lemma 4.1 Any cyclic N-group Δ such that $\Delta \ll \Omega$, $\Delta \notin \mathcal{K}$ can be written in the form $\Delta = \bigoplus_{i=1}^{m} \Delta_i \oplus \Delta'$, where each Δ_i is of type-0 and $\Delta' \in \mathcal{K}$.

The proof of the above lemma, which uses the fact that Δ satisfies the descending chain condition on N-kernels, is straightforward and will be omitted.

Definition Let $\mathcal{U}$ be the collection of all cyclic N-subgroups of the faithful N-group Ω. Further, let $\mathcal{W}$ be given by

$$\mathcal{W} = \{\Delta \in \mathcal{C}: \Delta \text{ irreducible and } \Delta \ll \Gamma \in \mathcal{U}\}.$$

Define the subsets G_Ω and B_Ω of $\mathcal{W}$ as follows:

$$G_\Omega = \{\Delta \in \mathcal{W}: \Delta \text{ is of type-0 and } \Delta \text{ is not a subfactor of any } \Delta' \in \mathcal{K} \cap \mathcal{U}\};$$

$$B_\Omega = \{\Delta \in \mathcal{W}: \Delta \notin G_\Omega\} = \mathcal{W} \backslash G_\Omega.$$

It is clear that if $\Delta \in \mathcal{W}$ and Δ is not cyclic, then $\Delta \in B_\Omega$. That is, B_Ω consists of non-cyclic irreducible N-groups and type-0 N-groups Δ such that $\Delta \ll \mathcal{K} \cap \mathcal{U}$.

<u>Definition</u> The subsets $\mathcal{G}_\Omega$ and $\mathcal{B}_\Omega$ are given by

$$\mathcal{G}_\Omega = \{\Delta \in \mathcal{U}: \Delta = \bigoplus_{i \in I} \Delta_i, \text{ where } \Delta_i \in G_\Omega \text{ for each } i \in I\};$$

$$\mathcal{B}_\Omega = \{\Delta \in \mathcal{U}: \Delta \in \mathcal{K} \text{ or } \Delta = \bigoplus_{i \in J} \Delta_i \oplus \Delta', \text{ where } \Delta_i \in B_\Omega \text{ for each } j \in J \text{ and } \Delta' \in \mathcal{K}\}.$$

Since N has DCCL and all the N-groups in $\mathcal{G}_\Omega$ and $\mathcal{B}_\Omega$ are cyclic, all the index sets I and J are finite. When the faithful N-group Ω does not need specific mentioning, then we will denote the above classes by G, B, $\mathcal{G}$ and $\mathcal{B}$, respectively. From lemma 4.1 we have

<u>Property I</u> Any cyclic N-subgroup of Ω is either in $\mathcal{B}_\Omega$ or a direct sum of an element from $\mathcal{B}_\Omega$ and an element from $\mathcal{G}_\Omega$. The summands are, of course, N-kernels of the cyclic N-subgroup.

Property I is assumption B of [18].

<u>Lemma 4.2</u> If Δ is an N-group of type-0 and $\Delta \subseteq \bigoplus_{i \in I} \Delta_i$, then $\Delta \ll \Delta_i$ for some $i \in I$.

<u>Proof</u> If we write, as we may, $\Delta = N\delta$, then δ is uniquely expressible in the form $\delta = \delta_{i_1} + \dots + \delta_{i_r}$ where $\delta_{i_j} \in \Delta_{i_j}$, $i_j \in I$. Using lemma 1.2 we see that $(0 : \delta) = \bigcap_{j=1}^{r} (0 : \delta_{i_j})$, an intersection of left ideals. But $(0 : \delta)$ is a maximal left ideal because Δ is of type-0 and so $(0 : \delta) = (0 : \delta_{i_j})$, for some $j = 1, \dots, r$. The result follows.

Let $\Delta \in \mathcal{B}$ and suppose Δ' is a cyclic N-subgroup of Δ. We will show that $\Delta' \in \mathcal{B}$. If $\Delta' \in \mathcal{K}$, then $\Delta' \in \mathcal{B}$ by definition, so suppose $\Delta' = \bigoplus_{i \in I} \Delta_i \oplus \Delta''$, where each Δ_i is of type-0 and $\Delta'' \in \mathcal{K}$. We must show that each Δ_i is an element of B. Now Δ is an element $\mathcal{B}$ so it is expressible in the form

$$\Delta = \bigoplus_{j \in J} \Gamma_j \oplus \Gamma',$$

where $\Gamma_j \in B$ and $\Gamma' \in \mathcal{K}$. Since $\Delta_i \subseteq \Delta$ lemma 4.2 tells us that $\Delta_i \ll \Gamma'$ or $\Delta_i \ll \Gamma_j$ for some j. In either case $\Delta_i \in B$. Thus we have established

<u>Property II</u> Every cyclic N-subgroup of an element in $\mathcal{B}$ is again in $\mathcal{B}$.

Property II is assumption A in [18].

<u>Definition</u> Let Ω be a faithful N-group and $\mathcal{B}$ and $\mathcal{G}$ classes of cyclic N-subgroups of Ω as defined previously. Now define

$$C(\Omega) = \bigcap_{N\omega \in \mathcal{B}} (0 : N\omega), \qquad L(\Omega) = \bigcap_{\substack{\omega \in \Omega \\ N\omega \in \mathcal{G}}} (0 : \omega).$$

Thus $C(\Omega)$ is an ideal of N whilst the left ideal $L(\Omega)$ is an intersection of maximal left ideals. Hence $Q(N) \subseteq L(\Omega)$.

<u>Lemma 4.3</u> $C(\Omega) \cap L(\Omega) = \{0\}$.

The proof uses the property I and is identical with the one given in [18], lemma 3.

<u>Lemma 4.4</u> $N - L(\Omega)$ is a direct sum of elements from G.

The proof follows from the definition of $L(\Omega)$. From lemma 4.3 and the fact that $Q(N) \subseteq L(\Omega)$ we immediately have

<u>Lemma 4.5</u> (i) $C(\Omega) \cap Q(N) = \{0\}$;

(ii) $C(\Omega) . Q(N) = \{0\}$

<u>Lemma 4.6</u> $C(\Omega)$ is a direct sum of elements from G.

<u>Proof</u> $(C(\Omega) \oplus L(\Omega)) - L(\Omega)$ is an N-kernel of $N - L(\Omega)$ and hence, by lemma 1.4, $C(\Omega) \cong (C(\Omega) \oplus L(\Omega)) - L(\Omega)$ is a direct sum of elements from G.

Lemmas 4.5 and 4.6 imply that $C(\Omega)$ is antiradical in the sense of Chapter 1.

We now fix our attention on the faithful N-group N^+.

Theorem 4.7 $C(N)$ is contained in $Soi(N)$ and is a direct summand of N.

Proof If $N \in \mathcal{K}$, then $N \in \mathcal{B}_N$ so that $C(N) = \{0\}$ in this case. Thus we may assume that $N \notin \mathcal{K}$. By lemma 4.1 N can be written in the form

$$N = \bigoplus_{i=1}^{k} \Delta_i \oplus \Delta,$$

where the left ideals Δ_i are of type-0 for each i and $\Delta \in \mathcal{K}$. The identity 1 of N has a unique expression of the form $1 = f_1 + \dots + f_k + f$, where $f_i \in \Delta_i$, $i = 1,\dots,k$ and $f \in \Delta$. It is clear that $Nf_i = \Delta_i$, $i = 1,\dots,k$, and $\{f_i\}$ is an orthogonal set of idempotents. Thus the left ideal $\bigoplus_{i=1}^{k} \Delta_i$ satisfies the conditions of lemma 3.9 and we need only show that $C(N) \subseteq \bigoplus_{i=1}^{k} \Delta_i$. The maximality of $Soi(N)$ will then imply that $C(N) \subseteq Soi(N)$. Any $n \in C(N)$ can be written as $n = n(f_1 + \dots + f_k + f) = nf_1 + \dots + nf_k + nf = nf_1 + \dots + nf_k$ because $\Delta \in \mathcal{B}_N$ (and hence $C(N).\Delta = \{0\}$). Thus $C(N) \subseteq \bigoplus_{i=1}^{k} \Delta_i$ and we have our result.

Corollary $C(N)$ is a direct sum of minimal left ideals each of which is an element of G_N.

Whether $C(\Omega)$ is contained in $Soi(N)$ for any faithful Ω in the DCCL case is not known. $C(\Omega)$ may well depend on Ω and it is only in the case of near-rings which satisfy the DCCN that we are able to show that the choice of Ω is immaterial as long as it is faithful. This generalises the Laxton-Machin result which was proved for d.g. near-rings with both DCCN and ACCN. We note further that theorem 4.7 need not be true if N does not have an identity. The following example demonstrates this.

Example 4.1 Consider the near-ring $N = (V,+,*)$ of example 2.1. We have seen that $N = \{0,a\} + \{0,c\} = \{0,a\} + \{0,b\}$. We have

$$B_N = \phi, \qquad G_N = \Big\{\{0,a\},\{0,b\},\{0,c\}\Big\}.$$

Hence $C(N) = N$ whereas $Soi(N) = \{0\}$. Note that here, when no identity is present, $C(N)$ contains a nilpotent left ideal.

Theorem 4.8 Each element in $\mathcal{G}_N$ has an isomorphic copy as a direct summand of $Soi(N)$.

Proof By theorem 3.4, $N = Soi(N) \oplus L$, L a left ideal and since $1 \in N$, L is cyclic. Hence by lemma 4.1 we can write

$$N = Soi(N) \oplus L_1 \oplus \ldots \oplus L_d \oplus L_{d+1} \oplus \ldots \oplus L_s \oplus L'.$$

We may assume, without loss of generality, that $L_j \in \mathcal{B}_N$ for $j = 1,\ldots,d$, $L_k \in \mathcal{G}_N$ for $k = d+1,\ldots,s$ and $L' \in \mathcal{K}$. We are going to show that $d = s$ i.e. there are no type-0 terms from $\mathcal{G}_N$ in $L_1 \oplus \ldots \oplus L_s$ (of course, this latter may be empty). Since L and so also L' is an intersection of maximal left ideals, $Q(N) \subseteq L'$ and hence $N - Q(N) \cong Soi(N) \oplus L_1 \oplus \ldots \oplus L_s \oplus L' - Q(N)$. By lemma 1.3 any N-group of type-0, in particular an element in $\mathcal{G}_N$, is N-isomorphic to a summand of $Soi(N)$ or N-isomorphic to L_j for some $j = 1,\ldots,s$. This is because an element from $\mathcal{G}_N$ cannot be a subfactor of $L' \in \mathcal{K}$. We show that $Soi(N) + L_{d+1} + \ldots + L_s$ is an ideal of N with an $\mathcal{Y}$-decomposition. The maximality of $Soi(N)$ will then give the required result. Clearly, $A = Soi(N) \oplus L_{d+1} \oplus \ldots \oplus L_s$ is a left ideal of N. If $n \in N$, then

$$n = a + l_1 + \ldots + l_d + l',$$

where $a \in A$, $l_j \in L_j$, $j = 1,\ldots,d$, $l' \in L'$. Also any $a' \in A$ has an expression of the form $a' = x + l_{d+1} + \ldots + l_s$ where $x \in Soi(N)$, $l_i \in L_i$ for $i = d+1,\ldots,s$. We show that $a'n \in A$. By left and right distribution over N-kernels we obtain

$$a'n = x(a+l_1+\ldots+l_d+l') + (l_{d+1}+\ldots+l_s)a + l_{d+1}l_1 + l_{d+1}l' + \ldots + l_s l'.$$

The first two terms on the right-hand side are in A since $Soi(N)$ is an ideal. If $l_i l_j \neq 0$ for $i = d+1,\ldots,s$, $j = 1,\ldots,d$, then $L_i l_j \neq \{0\}$. Consequently, the mapping $L_i \to L_j l_j, l_j \in L_j$ given by $y \to y l_j$ for all $y \in L_i$ is a monomorphism from L_i into L_j so that $L_i \ll L_j$, contradicting property II. Similarly, we see that $l_i l' = \{0\}$ for $i = d+1,\ldots,s$. Thus $a'n \in A$. That A has an

$\mathcal{G}$-decomposition follows from the fact that A is a direct summand of N and N has an identity.

Now by theorems 4.7 and 4.8 we may write N as

$$N = \mathrm{Soi}(N) \oplus L$$
$$= C(N) \oplus L_1 \oplus \dots \oplus L_r \oplus L_{r+1} \oplus \dots \oplus L_t \oplus L,$$

where $L_i \in G_N$ for $i = 1,\dots,r$ and $L_j \in B_N$ for $j = r+1,\dots,t$. G_N or B_N may, of course, be empty. One shows as in theorem 4.8 that $M = C(N) \oplus L_1 \oplus \dots \oplus L_r$ is a two-sided ideal of N. Let Δ be any element of $\mathcal{B}_N$ and $x \in M$; then

$$x = c + l_1 + \dots + l_r,$$

for $c \in C(N)$, $l_i \in L_i$, $i = 1,\dots,r$. Hence for any $\delta \in \Delta$ we have

$$x\delta = c\delta + l_1\delta + \dots + l_r\delta = l_1\delta + \dots + l_r\delta,$$

since $c\delta = 0$. If $l_i\delta \neq 0$, then $L_i\delta \neq \{0\}$ and so the mapping $L_i \to L_i\delta$ given by $x \to x\delta$ for all $x \in L_i$ is a monomorphism of L_i into Δ. Property II tells us that this is impossible. Hence $x\delta = 0$ and we have

$$M \subseteq \bigcap_{\Delta \in \mathcal{B}} (0 : \Delta) = C(N).$$

Thus we have proved

Theorem 4.9 C(N) is precisely the direct sum of all elements of G_N appearing as summands in Soi(N).

Corollary C(N) contains copies of all elements of G_N in its $\mathcal{G}$-decomposition.

The following is immediate from the proof of theorem 4.8:

Theorem 4.10 If $N = \mathrm{Soi}(N) \oplus L$, then $L \in \mathcal{B}_N$.

Theorem 4.11 $C(N) = N$ if, and only if, every N-group of type-0 is in G_N.

Proof If $C(N) = N$, then by lemma 2.6, $Q(N) = \{0\}$. Hence by lemma 1.3 every N-group of type-0 has an isomorphic copy as a summand in C(N). Theorem 4.9

implies that every N-group of type-0 is in G_N. Conversely, suppose that every N-group of type-0 is in G_N. Write N as $N = C(N) \oplus L$; then

$$N - Q(N) \cong C(N) \oplus L - Q(N)$$

because $Q(N) \subseteq L$ (an intersection of maximal left ideals). Now as in the proof of theorem 4.9 we see that $L \in \mathcal{B}$. Since every N-group of type-0 is in G_N, $B_N = \phi$ so that $L - Q(N)$ cannot be expressed as a direct sum of N-groups of type-0. The only possibility is $L = Q(N) = \{0\}$ so that Soi(N) = N. By theorem 4.9 C(N) = Soi(N) = N.

It is an open question whether or not Soi(N) = C(Ω) generally. Furthermore, we have not been able to establish any connection between C(Ω) and C(N) for general faithful N-groups Ω in the DCCL case. We have also not been able to construct an example of a near-ring in which $C(\Omega) \neq C(N)$ for some faithful N-group Ω. We do, however, have the following, which follows immediately from lemma 4.5 and theorem 3.7:

<u>Theorem 4.12</u> Let N be a near-ring satisfying the DCCN and Ω a faithful N-group. Then $C(\Omega) \subseteq \text{Soi}(N)$ and is a direct summand of N.

As in theorem 4.9, we can now prove

<u>Theorem 4.13</u> If N is a near-ring which satisfies the DCCN and Ω is a faithful N-group, then C(Ω) is precisely the direct sum of all elements of G_Ω which appear as summands of Soi(N).

<u>Theorem 4.14</u> If N is an 0-primitive near-ring with DCCN, then $\text{Soi}(N) \neq \{0\}$.

<u>Proof</u> The faithful N-group Ω of type-0 is in G_Ω and so appears as a summand of $C(\Omega) \subseteq \text{Soi}(N)$. Thus $\text{Soi}(N) \neq \{0\}$.

Let N be a near-ring with DCCN and Ω a faithful N-group. By theorem 4.13 we may write

$$N = C(\Omega) \oplus L_1 \oplus \dots \oplus L_t \oplus L,$$

where $\text{Soi}(N) = C(\Omega) \oplus L_1 \oplus \dots \oplus L_t$. Let $C(\Omega) = L_1' \oplus \dots \oplus L_r'$ and $\text{Soi}(N) = L_1' \oplus \dots \oplus L_r' \oplus L_1 \oplus \dots \oplus L_t$ be $\mathcal{G}$-decompositions of C(Ω) and Soi(N) respectively. We will show that $Soi(N) = C(\Omega)$. For this purpose we need the following:

Lemma 4.15 No L_i is a subfactor of L for $i = 1,\ldots,t$.

Proof Suppose $L_i \ll L$; then there exists an N-subgroup Δ of L and an N-kernel Δ' of Δ such that $L_i \overset{\psi}{\cong} \Delta - \Delta'$. Now there exists an $e_i \in L_i$ such that $Ne_i = L_i$, $e_i^2 = e_i$ for each $i = 1,\ldots,t$. If $\psi\colon e_i \to \delta_i + \Delta'$, $\delta_i \in \Delta$; then

$$\psi(e_i) = \psi(e_i^2) = e_i\psi(e_i) = e_i\delta + \Delta'$$

so that

$$\delta_i - e_i\delta \in \Delta'.$$

But $e_i\delta \in Soi(N) \cap L = \{0\}$. Hence $\delta_i \in \Delta'$ and this implies that $\Delta = \Delta'$, which is a contradiction.

Lemma 4.16 L_i is not a subfactor of $L\omega$ for any $\omega \in \Omega$ and any $i = 1,\ldots,t$.

Proof We assume $L\omega \neq \{0\}$. The mapping $L \overset{\psi}{\to} L\omega$ given by $l \to l\omega$ for all $l \in L$ is an N-homomorphism of L onto $L\omega$ so that $L - Ker\,\psi \cong L\omega$. Since L_i cannot be a subfactor of L by lemma 4.15, the result follows.

Lemma 4.17 Let $\Delta \in \mathcal{K}$ be a cyclic N-subgroup of Ω. Then $\Delta = L\delta$ for some $\delta \in \Delta$.

Proof Let δ be a cyclic N-generator of Δ; then

$$\begin{aligned} N\delta &= (L_1' \oplus \ldots \oplus L_r' \oplus L_1 \oplus \ldots \oplus L_t \oplus L)\delta \\ &= L_1'\delta + \ldots + L_r'\delta + L_1\delta + \ldots + L_t\delta + L\delta. \end{aligned}$$

If $L_i'\delta \neq \{0\}$, then $L_i' \cong L_i'\delta$ so that L_i' is a subfactor of $\Delta \in \mathcal{K}$. But then $L_i' \notin G_\Omega$, which contradicts theorem 4.13. Thus we have $N\delta = L_1\delta + \ldots + L_t\delta + L\delta$. If $L_i\delta \neq \{0\}$, then $L_i\delta$ is of type-0 for $i = 1,\ldots,t$, so we may write

$$N\delta = \bigoplus_{i \in I} L_i\delta + L\delta$$

for some $I \subseteq \{1,\ldots,t\}$. Now $\bigoplus_{i\in I} L_i\delta \cap L\delta$ is an N-kernel of $\bigoplus_{i\in I} L_i\delta$ and so by lemma 1.4 it is a direct sum of N-isomorphic copies of some of the $L_i\delta$. But this implies that $L_i \cong L_i\delta$ is a subfactor of $L\delta$, contradicting lemma 4.16. Hence $\bigoplus_{i\in I} L_i\delta \cap L\delta = \{0\}$ and we have

$$\Delta = N\delta = \bigoplus_{i\in I} L_i\delta \oplus L\delta.$$

Now $\Delta \in \mathcal{K}$ so it cannot have N-groups of type-0 as direct summands. Consequently, $L_i\delta = 0$ for each $i \in I$ and we have $\Delta = L\delta$, as required.

Theorem 4.18 Let N be a near-ring with DCCN and Ω any faithful N-group; then $Soi(N) = C(\Omega)$.

Proof As previously, we may write

$$N = C(\Omega) \oplus L_1 \oplus \ldots \oplus L_t \oplus L,$$

where $Soi(N) = C(\Omega) \oplus L_1 \oplus \ldots \oplus L_t$. Lemmas 4.16 and 4.17 imply that for each i, L_i cannot be a subfactor of any cyclic N-subgroup $\Delta \in \mathcal{K}$ of Ω. If $L_i\omega = \{0\}$ for all $\omega \in \Omega$, then $L_i = \{0\}$ because Ω is faithful. Hence there exists $\omega \in \Omega$ such that $L_i\omega \neq \{0\}$ and so $L_i \cong L_i\omega$ is a subfactor of Ω. From the definition of G_Ω it follows that $L_i \in G_\Omega$.

Theorem 4.13 now tells us that $L_i \subseteq C(\Omega)$ for $i = 1,\ldots,t$ and the result follows.

As we pointed out, theorem 4.18 is a generalisation of the result in [18]. This generalisation is possible partly because our class $\mathcal{B}_\Omega$ is defined differently from the one in [18]. In the more general DCCL case we can prove the following in a manner similar to theorem 4.18.

Theorem 4.19 If N is a near-ring satisfying the DCCL, then $Soi(N) = C(N)$.

Theorem 4.20 If N is a near-ring with DCCN and Ω a faithful N-group, then $C(\Omega) = N$ implies that every N-group of type-0 is in $\mathcal{G}_\Omega$.

The proof of the above theorem is as in theorem 4.11. Whether the converse is true in this case we do not know. It would certainly be true if we could establish that $G_\Omega \subseteq G_N$.

We point out that if N satisfies the DCCN and the Laxton-Machin critical ideal [18] exists, then it coincides with our $C(\Omega)$. In their construction of the critical ideal they assumed that G_Ω was the maximal class of N-groups of type-0 such that properties I and II were valid. We have made no such assumption, and indeed it is not clear whether, in general, our particular construction gives the above maximal class. Of central importance in the construction of $C(\Omega)$ is the set $\mathcal{U} \cap \mathcal{K} = \{\Delta \subseteq \Omega :$ Δ does not have an N-group of type-0 as a direct summand$\}$. We see this in the following:

Theorem 4.21 If N is a near-ring with DCCN and Ω a faithful N-group, then $C(\Omega) = \bigcap_{\Delta \in \mathcal{K} \cap \mathcal{U}} (0 : \Delta)$.

Proof Since $\mathcal{K} \cap \mathcal{U} \subseteq \mathcal{B}$ we certainly have

$$\bigcap_{\Delta \in \mathcal{K} \cap \mathcal{U}} (0 : \Delta) \supseteq \bigcap_{\Delta^* \in \mathcal{B}} (0 : \Delta^*).$$

On the other hand, if $\Delta^* \in \mathcal{B}$, $\Delta^* \notin \mathcal{K}$, then

$$\Delta^* = \bigoplus_{i=1}^{k} \Delta_i \oplus \Delta',$$

where Δ_i is of type-0 for each $i = 1,\ldots,k$ and $\Delta' \in \mathcal{K} \cap \mathcal{U}$. Since $\Delta_i \in B$ there exist $\Gamma_i \in \mathcal{K} \cap \mathcal{U}$ such that $\Delta_i \ll \Gamma_i$, $i = 1,\ldots,k$. Hence

$$\bigcap_{i=1}^{k} (0 : \Gamma_i) \cap (0 : \Delta') \subseteq (0 : \Delta^*)$$

and we see that

$$\bigcap_{\Delta \in \mathcal{K} \cap \mathcal{U}} (0 : \Delta) \subseteq \bigcap_{\Delta^* \in \mathcal{B}} (0 : \Delta^*).$$

Example 4.2 Let N be the 0-primitive d.g. near-ring in example 2.3 with $\alpha = 2$ so that $\Omega = A_{2n+1}$ is a faithful N-group of type-0. Let A_{n_1+1} be the alternating group on the symbols $1,2,\ldots,n,2n+1$, and A_{n_2+1} the alternating group on the symbols $n+1,\ldots,2n,2n+1$. The N-subgroup Δ cyclically generated by $(1\ 2) + (n+1, 2n+1)$ contains A_{n_1} and A_{n_2+1} as N-kernels. However, it is not a direct sum of these. In fact, Δ is in $\mathcal{K} \cap \mathcal{U}$. Similarly, one shows that there is an N-subgroup in $\mathcal{K}\cap\mathcal{U}$ containing A_{n_2} and A_{n_1+1}. Thus

$$G_\Omega = \{A_{2n}, A_{2n+1}\}, \qquad B_\Omega = \{A_{n_1}, A_{n_2}, A_{n_1+1}, A_{n_2+1}\} \cup \{\text{all other irreducibles}\}.$$

Example 4.3 Let $n = 9 \times 32$ and consider the group $\Omega = Z_n$ of integers modulo n. As before, if m divides n, then we denote the cyclic subgroup of order m by Z_m. Let N be the zero symmetric near-ring of all mappings of $Z_n \to Z_n$ which will take each one of the following groups into itself:

$$Z_n - Z_9, Z_9, Z_n - Z_{32}, Z_{32}, Z_{16}, Z_{16} - Z_8, Z_8, Z_4 - Z_2, Z_2, Z_3.$$

Z_9 and Z_{32} are N-kernels of Z_n so that $Z_n = Z_9 \oplus Z_{32}$. Z_n is of course a faithful N-group and we have

$$G = \{Z_9, Z_{32}, Z_3, Z_n - Z_9, Z_n - Z_{32}\},$$

$$B = \{Z_{16} - Z_8, Z_8, Z_4 - Z_2, Z_2\},$$

$$\mathcal{G} = \{Z_9, Z_{32}, Z_9 \oplus Z_{32}, Z_3, Z_3 \oplus Z_{32}\},$$

$$\mathcal{B} = \{Z_{16}, Z_8, Z_4, Z_2, \{0\}\}.$$

$$C(\Omega) = (0 : Z_{16}) = \mathrm{Soi}(N).$$

We see that $C(\Omega) \cap J_s(N)$ = a direct sum of copies of $Z_{32} \not\subseteq C(\Omega)$.

2. Near-rings with DCCN and ACCN

All near-rings in this section are assumed to satisfy the ACCN and DCCN. We begin by giving the Laxton-Machin construction of the critical ideal [18].

Definition Let Δ be an N-group and

$$\Delta = \Delta_0 \supset \Delta_1 \supset \dots \supset \Delta_k = \{0\}$$

be a series of N-subgroups of Δ such that Δ_i is an N-kernel of Δ_{i-1} for $i = 1, \dots, k$. Such a series is called a normal series for Δ. If each subfactor $\Delta_{i-1} - \Delta_i$ is irreducible, then we say that the above series is complete.

<u>Definition</u> [18] Let Ω be a faithful N-group and $\mathcal{U}$ the set of all cyclic N-subgroups of Ω. Further, let $\mathcal{C}$ be the collection of all irreducible N-groups which appear as subfactors of Ω. Then

$$LG = \{\Delta \in \mathcal{C}\colon \Delta \text{ is of type-0}\},$$

$$LB = \mathcal{C} \setminus LG,$$

$$L\mathcal{G} = \{\Delta \in \mathcal{U}\colon \Delta = \bigoplus_{i \in I} \Delta_i \text{ and } \Delta_i \in LG \text{ for each } i\},$$

$$L\mathcal{B} = \{\Delta \in \mathcal{U}\colon \Delta = \Delta_0 \supset \Delta_1 \supset \ldots \supset \Delta_k = \{0\} \text{ is a complete series with } \Delta_{i-1} - \Delta_i \in LB\}.$$

We assume that properties I and II hold for the classes $L\mathcal{G}$ and $L\mathcal{B}$. We note that LG contains only cyclic N-groups.

<u>Fact 4.1</u> [18] Let $L\mathcal{G}_1, L\mathcal{B}_1$ and $L\mathcal{G}_2, L\mathcal{B}_2$ be two pairs of classes of elements of $\mathcal{U}$ which satisfy properties I and II. Then $L\mathcal{G}_1 \cup L\mathcal{G}_2$ and $L\mathcal{B}_1 \cap L\mathcal{B}_2$ also satisfy properties I and II.

We shall assume that LG is the maximal class of N-groups of type-0 such that properties I and II are valid. Let $\mathcal{K}$, B, G, $\mathcal{B}$ and $\mathcal{G}$ be the classes of N-groups as defined at the beginning of this chapter. Then $LG \supseteq G$, $LB \subseteq B$, $L\mathcal{G} \supseteq \mathcal{G}$ and $L\mathcal{B} \subseteq \mathcal{B}$.

<u>Lemma 4.22</u> If Ω is a faithful N-group and Δ a cyclic N-subgroup of Ω, then $\Delta \in \mathcal{B}$ if and only if Δ has a complete series all of whose irreducible subfactors are in B.

<u>Proof</u> Since Δ is a cyclic N-group it satisfies both the ACC and DCC for N-subgroups. If $\Delta \in \mathcal{B}$, then $\Delta = \bigoplus_{i=1}^{k} \Delta_i \oplus \Delta'$ where $\Delta_i \in B$, $i = 1, \ldots, k$ and $\Delta' \in \mathcal{K} \cap \mathcal{U}$. Thus Δ has a complete series of form

$$\Delta = \bigoplus_{i=1}^{k} \Delta_i \oplus \Delta' \supsetneq \bigoplus_{i=1}^{k-1} \Delta_i \oplus \Delta' \supset \ldots \supset \Delta' = \Delta'_r \supset \ldots \supset \Delta'_0 = \{0\}.$$

By definition the subfactors of such a series are in B. Conversely, let

$\Delta = \Delta_k \supset \Delta_{k-1} \supset \dots \supset \Delta_0 = \{0\}$ be a complete series for Δ such that each $\Delta_i - \Delta_{i-1} \in B$. If $\Delta \in \mathcal{K}$, then we are done. So suppose $\Delta = \bigoplus_{i=1}^{m} \Delta_i' \oplus \Delta''$, Δ_i' of type-0, $i = 1,\dots,m$ and $\Delta'' \in \mathcal{K}$. Then

$$\Delta = \bigoplus_{i=1}^{m} \Delta_i' \oplus \Delta'' \supset \bigoplus_{i=1}^{m-1} \Delta_i' \oplus \Delta'' \supset \dots \supset \Delta'' = \Delta_r^* \supset \Delta_{r-1}^* \supset \dots \supset \Delta_0^* = \{0\}$$

is another complete series for Δ. By the Jordan-Hölder theorem the factors of this series are isomorphic to factors in the above series under some pairing. Thus $\Delta_i' \in B$ for each $i = 1,\dots,m$ and consequently $\Delta \in \mathcal{B}$.

<u>Lemma 4.23</u> If $\Delta \in \mathcal{K} \cap \mathcal{U}$, then $\Delta \in L\mathcal{B}$, where Δ is an N-subgroup of the faithful N-group, Ω.

The proof of the lemma is immediate from the definition of the elements in $\mathcal{K}$. It is now clear that

$$\mathcal{B} \supseteq L\mathcal{B} \supseteq \mathcal{K} \cap \mathcal{U}.$$

The critical ideal, Crit(N), is defined as $\text{Crit}(N) = \bigcap_{\Delta \in L\mathcal{B}} (0 : \Delta)$.

The following follows from the definition of $C(\Omega)$ and theorem 4.21:

<u>Theorem 4.24</u> If N is a near-ring which satisfies the DCCN and ACCN, then $C(\Omega) = \text{Crit}(N) = \text{Soi}(N)$.

We now seek a necessary and sufficient condition for $C(\Omega)$ to be non-zero. We will define a certain type of N-group Ω such that $C(\Omega) \neq \{0\}$ if Ω is faithful. For this purpose we need

<u>Definition</u> An N-group of type-0 is called <u>maximal type-0</u> if it is not a proper subfactor of any other N-group of type-0.

We note that if Ω is an N-group and Δ an N-group of type-0 such that $A = (0 : \Omega) \subseteq (0 : \Delta)$, then Ω is a faithful N/A-group and Δ an N/A-group of type-0. In the finite case lemma 1.5 tells us that $\Delta \ll \Omega$. Also, if N is finite, then any prime ideal [24] P of N is 0-primitive, that is, $P = (0:\Delta)$ with Δ an N-group of type-0. Thus P is a <u>minimal prime</u> ideal if, and only if, Δ is maximal type-O if N is finite.

Definition A faithful cyclic N-group $\Omega = \bigoplus_{i=1}^{t} \Omega_i \oplus \Omega'$ with $\Omega' \in \mathcal{K}$ will be called critically faithful if for each i = 1,...,t,

(i) Ω_i is maximal type-O,

(ii) Ω_i is not a subfactor of Ω'.

If Ω is critically faithful, then we say $\Omega = \bigoplus_{i=1}^{t} \Omega_i \oplus \Omega'$ is a critical decomposition of Ω if Ω' and the Ω_i, i = 1,...,t satisfy (i) and (ii) above. If $\Omega = \bigoplus_{i=1}^{t} \Omega_i \oplus \Omega'$ is a critical decomposition of Ω, then $\Omega_i \cong \Omega_j$ implies that

$$\bigoplus_{\substack{i=1 \\ i \neq j}}^{t} \Omega_i \oplus \Omega'$$

is critically faithful. Thus, without loss of generality, we will assume that $\Omega_i \ncong \Omega_j$ if $i \neq j$, throughout our discussion.

Lemma 4.25 Let Ω be a critically faithful N-group with a critical decomposition $\Omega = \bigoplus_{i=1}^{t} \Omega_i \oplus \Omega'$. Further, let Δ be an N-subgroup of Ω such that Ω_s is a subfactor of a complete series for Δ, for some s = 1,...,t. If N is a finite near-ring, then Ω_s is a direct summand of Δ.

Proof Put $A_0 = \Omega'$, $A_n = \Omega_1 \oplus \dots \oplus \Omega_n \oplus \Omega'$ and $\Delta_n = \Delta \cap A_n$ for n = 1,...,t; then

$$\Omega = A_t \supset A_{t-1} \supset \dots \supset A_1 \supset A_0 = \Omega' \supset \{0\}$$

and

$$\Delta = \Delta_t \supset \Delta_{t-1} \supset \dots \supset \Delta_1 \supset \Delta_0 = \Omega' \cap \Delta \supset \{0\}$$

are normal series for Ω and Δ respectively. Thus

$$\begin{aligned} \Delta_n - \Delta_{n-1} &\cong \Delta_n - \Delta \cap A_{n-1} \\ &= \Delta_n - \Delta_n \cap A_{n-1} \\ &\cong (\Delta_n + A_{n-1}) - A_{n-1} \\ &\subseteq A_n - A_{n-1} \cong \Omega_n, \end{aligned}$$

for n = 1,...,t. By the Jordan-Hölder theorem, Ω_s is N-isomorphic to a factor of the second series (after refinement). Since Ω_s is not a subfactor of Ω', this factor can only come from a refinement of

$$\Delta = \Delta_t \supset \dots \supset \Delta_1 \supset \Delta_0 = \Delta \cap \Omega'.$$

Thus by the above

$$\Omega_s \ll \Delta_n - \Delta_{n-1} \ll \Omega_n$$

for some $n = 1,\dots,t$. Since the Ω_i are maximal type-0 and $\Omega_i \not\ll \Omega_j$ if $i \neq j$ it follows that the only possibility is $n = s$ and we have equality in the above equation. Hence $(\Delta_s + A_{s-1}) - A_{s-1} = A_s - A_{s-1}$ and we have $A_s = \Delta_s + A_{s-1}$. Now let ω_s be an N-generator of Ω_s; then

$$\omega_s = \delta + \omega_1 + \dots + \omega_{s-1} + \omega',$$

where $\omega_i \in \Omega_i$, $i = 1,\dots,s-1$, $\delta \in \Delta_s$ and $\omega' \in \Omega'$. Thus

$$-\delta = \omega_1 + \dots + \omega_{s-1} + \omega' - \omega_s.$$

Put

$$Q_s = \bigcap_{\substack{j=1 \\ j\neq s}}^{t} P_j \cap P',$$

where $P_j = (0 : \Omega_j)$ for $j = 1,\dots,t$ and $P' = (0 : \Omega')$. Then the ideal $Q_s \not\subseteq P_s$. For otherwise

$$P_1 \dots P_{s-1} P_{s+1} \dots P_t P' \subseteq Q_s \subseteq P_s.$$

Since P_s is a prime ideal this implies that $P_j \subseteq P_s$ or $P' \subseteq P_s$ for some $j = 1,\dots,t$, $j \neq s$. But P_s is a minimal prime ideal by a previous remark, so that $P' \subseteq P_s$ is the only possibility. However, $P' \subseteq P_s$ if, and only if, $\Omega_s \ll \Omega'$. Since Ω_s is not a subfactor of Ω' it follows that $P' \not\subseteq P_s$, and so we have verified the above claim that $Q_s \not\subseteq P_s$. Now

$$\begin{aligned} Q_s(-\delta) &= Q_s(\omega_1 + \dots + \omega_{s-1} + \omega' - \omega_s) \\ &= Q_s\omega_1 + \dots + Q_s\omega_{s-1} + Q_s\omega' + Q_s(-\omega_s), = Q_s(-\omega_s) \end{aligned}$$

by left definition over a direct sum of N-kernels and the definition of the Q's. Consequently, $Q_s\omega_s = Q_s\delta \subseteq \Delta_s = A_s \cap \Delta$. Now $Q_s\omega_s$ is an N-kernel of $N\omega_s = \Omega_s$ so that $Q_s\omega_s = \Omega_s$ or $Q_s\omega_s = \{0\}$, because Ω_s is of type-0. $Q_s\omega_s = \{0\}$ is, of course, not possible as $Q_s \not\subseteq P_s = (0 : \Omega_s)$ and hence $\Omega_s = Q_s\omega_s \subseteq A_s \cap \Delta \subseteq \Delta$. Putting $\Omega = \Omega_s \oplus \Gamma$, where

$$\Gamma = \bigoplus_{\substack{i=1 \\ i\neq s}}^{t} \Omega_i \oplus \Omega',$$

we see by the modular law that $\Delta = \Omega_s \oplus \Gamma \cap \Delta$.

<u>Theorem 4.26</u> Let N be a finite near-ring. Then $Soi(N) \neq \{0\}$ if and only if N has a representation on a critically faithful group.

<u>Proof</u> Let Ω be a critically faithful N-group with critical decomposition $\Omega = \bigoplus_{i=1}^{m} \Omega_i \oplus \Omega'$, $\Omega' \in \mathcal{K}$. We will show that the N-kernels $\Omega_i \in G_\Omega$ for each $i = 1,\ldots,m$. Suppose $\Omega_i \in B_\Omega$; then Ω_i is a subfactor of some $\Delta' \in \mathcal{K} \cap \mathcal{U}$. Hence there exists an N-subgroup Γ of Δ' and an N-kernel Γ' of Γ such that $\Omega_i \cong \Gamma - \Gamma'$. The normal series $\Gamma \supset \Gamma' \supset \{0\}$ can be refined to a complete series so that Ω_i is a direct summand of Γ by lemma 4.25. But then the N-kernel $\Omega_i \subset \Delta'$ and so we have a normal series $\Delta' \supset \Omega_i \supset \{0\}$ for Δ'. This normal series can be refined to a complete series with Ω_i as a (last) subfactor so that by lemma 4.25 again Ω_i is a direct summand of Δ'. Since $\Delta' \in \mathcal{K}$ we have a contradiction. Conversely, if $Soi(N) \neq \{0\}$, then $N = Soi(N) \oplus L = \bigoplus_{i=1}^{t} L_i \oplus L$, where $\bigoplus_{i=1}^{t} L_i$ is an $\mathcal{Y}$-decomposition for $Soi(N)$. Using lemmas 4.1 and 4.15 it is an easy matter to obtain a critically faithful N-group from N^+.

<u>Example 4.4</u> If $J_0(N) = \{0\}$, then N has a faithful N-group Ω which is a direct sum of N-groups of type-0. Take $\Omega = N - Q(N)$ and drop from this direct sum those type-0 N-groups which are not maximal. Then we obtain a critically faithful N-group with zero $\mathcal{K}$-component.

<u>Example 4.5</u> Let Z_{p^α}, p a prime, be a faithful cyclic N-group. If Z_{p^α} contains a proper, non-zero N-kernel, then $Soi(N) = \{0\}$. Otherwise Z_{p^α} is of type-0, so that $Soi(N) \neq \{0\}$.

CHAPTER 5

Radical-antiradical series for near-rings

1. Nil-rigid series

As we pointed out in Chapter 1, the series discussed here are just special cases of Scott's more general nil-rigid series [27]. We begin by giving a description of Scott's idea.

Let $L_1(N) = nil(N)$ and $C_1(N)$ be the ideal of N containing $L_1(N)$ such that

$$C_1(N)/L_1(N) = Crux(N/L_1(N)).$$

Further, let $L_2(N)$ be the ideal of N containing C_1 such that

$$L_2(N)/C_1(N) = nil(N/C_1(N)).$$

If α is a non-limit ordinal, define $L_\alpha(N)$ to be the ideal of N such that

$$L_\alpha(N)/C_{\alpha-1}(N) = nil(N/C_{\alpha-1}(N))$$

and C_α to be the ideal of N such that

$$C_\alpha(N)/L_\alpha(N) = Crux(N/L_\alpha(N)).$$

If α is a limit ordinal, define

$$C_\alpha(N) = \bigcup_{\beta<\alpha} C_\beta(N)$$

and

$$L_\alpha(N) = \bigcup_{\beta<\alpha} L_\beta(N).$$

Scott [27] shows that if N is a near-ring with DCCI, then the transfinite series

$$\{0\}, L_1(N), C_1(N), L_2(N), C_2(N), \ldots$$

is ascending and only fails to be properly ascending at limit ordinals.

In the case of a near-ring with DCCI, Scott calls this series the nil-rigid series for N. One can similarly define the J_0-socle-ideal series in the case of a near-ring with DCCL. In such a series nil(N) is replaced by $J_0(N)$ and Crux(N) is replaced by Soi(N). However, it is not clear that a J_0-socle-ideal series is properly ascending at non-limit ordinals. In any event, we will assume that the near-ring N satisfies at least the DCCN. In this case nil(N) = $J_0(N)$ and Soi(N) = Crux(N) so that the J_0-socle-ideal series for N is just its nil-rigid series. In [27] Scott proved that if N has DCCN and a right identity, then the nil-rigid series for N is finite and there exists a positive integer α such that $C_\alpha(N) = N$. Using this fact we prove

Theorem 5.1 Let N be a near-ring with DCCN and a right identity. If in the nil-rigid series for N we have $L_\alpha(N) \subset C_\alpha(N) = N$, $\alpha > 1$, then $J_s(N/C_{\alpha-1}(N))$ is a non-zero, nilpotent ideal of $N/C_{\alpha-1}(N)$.

Proof $C_\alpha(N) = N$ implies that $\text{Soi}(N/L_\alpha(N)) = N/L_\alpha(N)$ so that $N/L_\alpha(N)$ is a direct sum of left ideals of $N/L_\alpha(N)$ each of which is an $N/L_\alpha(N)$-group of type-0. By lemma 2.6 $Q(N/L_\alpha(N))$ is zero, so that $L_\alpha(N)$ is an intersection of maximal left ideals of N and hence $L_\alpha(N)/C_{\alpha-1}(N)$ is an intersection of maximal left ideals of $N/C_{\alpha-1}(N)$. Thus

$$\begin{aligned} Q(N/C_{\alpha-1}(N)) &\subseteq L_\alpha(N)/C_{\alpha-1}(N) \\ &= \text{nil}(N/C_{\alpha-1}(N)) \\ &= J_0(N/C_{\alpha-1}(N)) \end{aligned}$$

and theorem 2.9 implies that $J_s(N/C_{\alpha-1}(N))$ is nilpotent. By the strict ascendency of nil-rigid series $L_\alpha(N) \not\subseteq C_{\alpha-1}(N)$. Consequently, $J_s(N/C_{\alpha-1}(N)) \not\subseteq \{0\}$.

We remark that if in the nil-rigid series for N, $\alpha = 1$, then $J_s(N)$ is nilpotent. In fact we have

Theorem 5.2 Let $L_1(N), C_1(N), \ldots, L_\alpha(N), C_\alpha(N) = N$ be the nil-rigid series for N. Then $\alpha = 1$ if, and only if, $J_s(N)$ is nilpotent.

In the above theorem, $J_s(N)$ may, of course, be zero - as indeed is the case in s-primitive near-rings. If $\alpha > 1$, then $C_{\alpha-1}$ does not contain the quasi-radical, for otherwise it would contain $J_s(N)$ by corollary 1 of theorem 2.8. In this case we would have $J_s(N/C_{\alpha-1}(N)) = \{0\}$, contradicting theorem 5.1. In view of the above, it would seem reasonable to define the "nilpotence level" of the s-radical to be $\alpha - 1$ if $C_\alpha = N$. In the special case where $\alpha = 1$ and $J_s(N)$ itself is nilpotent, the s-radical will then have "nilpotence level" zero. However, this needs further scrutiny for $J_s(N/C_{\alpha-1}(N))$ may well coincide with $J_2(/C_{\alpha-1}(N))$ even though $J_2(N) \not\subseteq J_s(N)$. In this case we need to decide what $\alpha - 1$ really measures. Also, as is so often the case in near-rings, there are extreme situations. Take for example the zero symmetric near-ring N of all mappings of Z_4 into itself, which takes Z_2 into Z_2. N is s-primitive with Z_4 a faithful N-group of type-s. The radical $J_2(N)$ is a direct sum of copies of Z_4 and $(J_2(N))^2 = J_2(N)$. Indeed, no proper ideal A of N exists such that $J_2(N/A)$ is nilpotent. One may take the "nilpotence level" of the radical to be infinite in this case. Clearly, $J_2(N/C_{\alpha-1}(N))$ is nilpotent if, and only if, it coincides with $J_s(N/C_{\alpha-1}(N))$, where $C_\alpha = N$, $\alpha > 1$. We give another necessary and sufficient condition for the nilpotence of $J_2(N/C_{\alpha-1}(N))$:

<u>Theorem 5.3</u> Let N be a near-ring with DCCN and a right identity. If in the nil-rigid series for N we have $C_\alpha(N) = N$, $\alpha > 1$, then $J_2(N/C_{\alpha-1}(N))$ is nilpotent if, and only if, $Soi(N/L_\alpha(N)) \cap J_2(N/L_\alpha(N)) = \{0\}$.

<u>Proof</u> Since $C_\alpha(N) = N$, $Soi(N/L_\alpha(N)) = N/L_\alpha(N)$, so that

$$J_2(N/L_\alpha(N)) \cap Soi(N/L_\alpha(N)) = J_2(N/L_\alpha(N)).$$

If $J_2(N/C_{\alpha-1}(N))$ is nilpotent, then $L_\alpha(N)/C_{\alpha-1}(N) = J_2(N/C_{\alpha-1}(N))$, so $N/L_\alpha(N)$ is semi-simple. Hence $J_2(N/L_\alpha(N)) = \{0\}$. Conversely, suppose $Soi(N/L_\alpha(N)) \cap J_2(N/L_\alpha(N)) = \{0\}$ i.e. $J_2(N/L_\alpha(N)) = \{0\}$. Then

$L_\alpha(N) \supseteq J_2(N)$ so that $L_\alpha(N)$ is an intersection of 2-modular left ideals. Hence $L_\alpha(N)/C_{\alpha-1}(N) = J_2(N/C_{\alpha-1}(N))$, and so $J_2(N/C_{\alpha-1}(N))$ is nilpotent.

From the above we see that $Soi(N/L_\alpha(N))$ is in some sense the obstruction to $J_2(N/C_{\alpha-1}(N))$ being nilpotent. In view of this we define the following radical-antiradical series:

Let $L_1' = J_0(N)$ and let U_1 be the ideal of N containing L_1' such that $U_1/L_1' = Soi(N/L_1') \cap J_2(N/L_1')$. Further, let L_2' be the ideal containing U_1 such that $L_2'/U_1 = J_0(N/U_1)$. If α is a non-limit ordinal, let L_α' be the ideal of N containing $U_{\alpha-1}$ such that $L_\alpha'/U_{\alpha-1} = J_0(N/U_{\alpha-1})$ and U_α the ideal of N containing L_α' such that

$$U_\alpha/L_\alpha' = Soi(N/L_\alpha') \cap J_2(N/L_\alpha').$$

If α is a limit ordinal, define $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$ and $L_\alpha' = \bigcup_{\alpha>\beta} L_\beta'$. Thus we obtain an ascending sequence of ideals

$$L_1', U_1, L_2', \ldots, L_\alpha', U_\alpha, \ldots .$$

<u>Definition</u> If N is a near-ring with right identity and DCCN, then we call the above sequence the <u>J_0-U sequence</u> of N.

Similarly we define the following sequence of ideals. Let $L_1^* = J_0(N)$ and let W_1 be the ideal of N containing L_1^* such that

$$W_1/L_1^* = Soi(N/L_1^*) \cap J_3(N/L_1^*).$$

Further, let L_2^* be the ideal containing W_1 such that $L_2^*/W_1 = J_0(N/W_1)$. In the non-limit ordinal case we have $L_\alpha^*/W_{\alpha-1} = J_0(N/W_{\alpha-1})$ and $W_\alpha/L_\alpha^* = Soi(N/L_\alpha^*) \cap J_3(N/L_\alpha^*)$. The limit ordinal cases are as in the J_0-U sequence and we have the following ascending sequence of ideals

$$L_1^*, W_1, L_2^*, \ldots, L_\gamma^*, W_\gamma, \ldots .$$

<u>Definition</u> If N is a near-ring with right identity and DCCN, then we call the above sequence the <u>J_0-W sequence</u> of N.

Before giving examples of radical-antiradical series we note that if $J_2(N)$ is nilpotent, then $J_2(N) \cap Soi(N) = \{0\}$. The converse is not true, as the following example shows. (We use the representation theory of Chapter 4 in our examples.)

<u>Example 5.1</u> Let G be a direct sum of S_n, $n \geq 5$ and a group of order p, p a prime which does not divide n! . Let V be a reduced free group on m generators whose laws are precisely the universal laws of G. We assume that m is at least as great as the minimum number of generators of G. Let N be the Neumann d.g. near-ring [19,22] associated with V; then $\Omega = N^+ - \iota$ is a faithful N-group where the left ideal ι is defined by $V' = V\iota$ and $V - V' \cong G$ (see ref. [18] p.227). Now the N-groups of type-0 which appear as summands of Soi(N) correspond to the subgroup of order p in G so that these are all of type-2. Thus $J_2(N) \cap Soi(N) = \{0\}$. However, the N-subgroup of Ω corresponding to the alternating group A_n is of type-0 but not of type-2. Thus the radical is not nilpotent.

We note that in the above example every N-subgroup of Ω is cyclic [18]. Also, the N-subgroup of Ω corresponding to S_n is in $\mathcal{B}_\Omega$. Since p does not divide n!, the cyclic group of order p is not a subfactor of S_n.

We now give the promised example in which we consider the three radical-antiradical series defined so far.

<u>Example 5.2</u> Consider the group $\Omega = Z_9 + Z_{64}$. Let N be the zero symmetric near-ring of all mappings of Ω into itself which will take each one of the following into itself:

$$\Omega - Z_9,\ \Omega - Z_{64},\ Z_9,\ Z_{64},\ Z_3,\ Z_{32} - Z_{16},\ Z_{16},\ Z_8 - Z_4,\ Z_4,\ Z_2.$$

Z_9 and Z_{64} are N-kernels of Ω. Also, Z_9 is of type-s and Z_{64} of type-0 but not of type-s.

(a) The nil-rigid series for N:

$$\{0\} = L_1 \subset C_1 = (0 : Z_{32}) \subset L_2 = (0 : (Z_{32}-Z_{16}) \bullet Z_{16}) \subset C_2$$

$$= (0 : Z_8) \subset L_3 = (0 : (Z_8-Z_4) \bullet Z_4) \subset C_3 = N.$$

In this case $\alpha = 3$ and we see that $J_s(N/C_2)$ is non-zero and nilpotent. $J_2(N/C_2)$ is not nilpotent since Z_4 is of type-s but not of type-2. In fact, $J_2(N/C_2)$ is a direct sum of copies of Z_4.

(b) The J_0-U series for N:

$$\{0\} = L_1' \subset U_1 = (0 : Z_3 \oplus Z_{32}) \subset L_2' = (0 : Z_3 \oplus Z_{32} - Z_{16} \oplus Z_{16}) \subset U_2$$

$$= (0 : Z_3 \oplus Z_{32} - Z_{16} \oplus Z_8) \subset L_3' = (0 : Z_3 \oplus Z_{32} - Z_{16} \oplus Z_8 - Z_4 \oplus Z_4) \subset U_3$$

$$= (0 : Z_3 \oplus Z_{32} - Z_{16} \oplus Z_8 - Z_4 \oplus Z_2) = L_4'.$$

We note that $J_2(N/U_3)$ is zero and $J_2(N/U_2)$ is not nilpotent.

(c) The J_0-W series for N:

$$\{0\} = L_1^* \subset W_1 = (0 : Z_9 \oplus Z_{32}) \subset L_2^* = (0 : Z_9 \oplus Z_{32} - Z_{16} \oplus Z_{16} \subset W_2$$

$$= (0 : Z_9 \oplus Z_{32} - Z_{16} \oplus Z_8) \subset L_3^* = (0 : Z_9 \oplus Z_{32} - Z_{16} \oplus Z_8 - Z_4 \oplus Z_4) = W_3.$$

We see that $J_s(N/W_2)$ is non-zero and nilpotent.

<u>Remarks</u> There is no subgroup Δ of Ω such that $J_2(N/A)$ is non-zero and nilpotent, where $A = (0 : \Delta)$. Also,

$$L_i^*, L_i' \subset L_i, \qquad W_j, U_j \subset C_j \qquad \text{for } i,j = 1,2,3.$$

These inclusions are no accidents and later we will prove that they always hold for finite near-rings.

Putting $U(N) = J_2(N) \cap \mathrm{Soi}(N)$ and $W(N) = J_s(N) \cap \mathrm{Soi}(N)$ we have the following from theorem 3.12:

<u>Lemma 5.4</u> If N is a near-ring with DCCN, then

(a) $U(N/U(N)) = \{0\}$,

(b) $W(N/W(N)) = \{0\}$.

In what follows $D(N)$ will denote either $U(N)$ or $W(N)$ so that $L_1 \subsetneq D_1 \subsetneq L_2 \subseteq \cdots \subsetneq D_\alpha \subseteq L_{\alpha+1} \subseteq \cdots$ will denote either a J_0-U or a J_0-W sequence.

<u>Theorem 5.5</u> If N is a near-ring with DCCN, then

(a) $D_\alpha = L_{\alpha+1}$ implies $L_{\alpha+1} = D_{\alpha+1}$

(b) $L_\alpha = D_\alpha$ implies $L_{\alpha+1} = D_\alpha$.

<u>Proof</u> (a) By lemma 5.4 we have

$$\mathrm{Soi}((N/L_\alpha)/(D_\alpha/L_\alpha)) \cap J_\nu((N/L_\alpha)/(D_\alpha/L_\alpha)) = \{0\},$$

where $D_\alpha = W_\alpha$ if $\nu = s$, $D_\alpha = U_\alpha$ if $\nu = 2$. Since $(N/L_\alpha)/(D_\alpha/L_\alpha) \cong N/D_\alpha$ we have

$$\begin{aligned} D_{\alpha+1}/L_{\alpha+1} &= \mathrm{Soi}(N/L_{\alpha+1}) \cap J_\nu(N/L_{\alpha+1}) \\ &= \mathrm{Soi}(N/D_\alpha) \cap J_\nu(N/D_\alpha) \\ &= \{0\}. \end{aligned}$$

(b) $J_0((N/D_{\alpha-1})/(L_\alpha/D_{\alpha-1})) = J_0((N/D_{\alpha-1})/\ J_0(N/D_{\alpha-1})) = \{0\}$.
Since $(N/D_{\alpha-1})/(L_\alpha/D_{\alpha-1}) \cong N/L_\alpha$ it follows that $J_0(N/D_\alpha) = J_0(N/L_\alpha) = \{0\}$.
Thus $L_{\alpha+1}/L_\alpha = L_{\alpha+1}/D_\alpha = J_0(N/D_\alpha) = \{0\}$.

We note that if α is a limit ordinal and β is an ordinal less than α, then $\beta + 1 < \alpha$. Since $L_\beta \subseteq D_\beta \subseteq L_{\beta+1}$ we have $L_\alpha = \bigcup_{\beta<\alpha} L_\beta = \bigcup_{\beta<\alpha} D_\beta = D_\alpha$. Hence the J_0-W and J_0-U sequences are properly ascending only at non-limit ordinals. Our aim is to show that these sequences are always finite in the case of near-rings with DCCN. If α is an ordinal and $D_\alpha \neq L_\alpha$ (so α is a non-limit ordinal), then

$$D_\alpha/L_\alpha = L_1'/L_\alpha \oplus \ldots \oplus L_t'/L_\alpha,$$

where L_i'/L_α is an N/L_α-group of type-0, $i = 1,\ldots,t$ and possesses a cyclic generator, $e_i + L_\alpha$, with $e_i^2 - e_i \in L_\alpha$. Similarly, for any ordinal $\gamma > \alpha$ with $D_\gamma \neq L_\gamma$ we have

$$D_\gamma/L_\gamma = L_1/L_\gamma \oplus \ldots \oplus L_s/L_\gamma,$$

where again L_j/L_γ is of type-0 with an idempotent generator, $j = 1,\ldots,s$. Suppose $\psi\colon L_i'/L_\alpha \to L_j/L_\gamma$ is an N-isomorphism of L_i'/L_α onto L_j/L_γ for some i and some j. Further, let $\psi(e_i+L_\alpha) = y + L_\gamma$, $y \in L_j$. Then $y \notin L_\gamma$ because

$e_i + L_\alpha$ is a cyclic generator of L_i'/L_α. But $L_\gamma = e_i y + L_\gamma$, because $e_i \in D_\alpha \subset L_\gamma$, $\gamma > \alpha$. Hence $L_\gamma = e_i(y+L_\gamma) = e_i\psi(e_i+L_\alpha) = \psi(e_i+L_\alpha) = y + L_\gamma$, which is a contradiction. Thus if $D_\alpha \neq D_\gamma$, then the N-groups of type-0 appearing as summands in D_α/L_α cannot be N-isomorphic to those which appear as summands in D_γ/L_γ. If N satisfies the DCCN, then N has, up to isomorphism, only a finite number of distinct N-groups of type-0. In this case there exists a finite ordinal β such that $D_\beta = D_{\beta+1}$. Theorem 5.5 implies that $D_\beta = L_{\beta+1} = D_{\beta+1} = L_{\beta+2} = \ldots$. Thus we have

<u>Theorem 5.6</u> The J_0-U and J_0-W sequences of a near-ring are both finite.

In example 5.2 we have seen that the J_0-U sequence for N terminates at U_3 whereas the J_0-W series for N terminates at L_3^*. In the next section we will show that a J_0-W sequence always terminates at L_α^* for some α if N is finite. It may happen that a J_0-U sequence terminates at L_α' for some α, as the following example shows:

<u>Example 5.3</u> Let $G = Z_{2^6} \oplus Z_{3^7}$ and let N be the zero symmetric near-ring of all mappings of G into G which takes each one of the following groups into itself:

$G - Z_{2^6}$, $G - Z_{3^7}$, Z_{2^6}, $Z_{2^5} - Z_{2^4}$, Z_{2^4}, $Z_8 - Z_4$, Z_4, Z_2, Z_{3^7}, $Z_{3^6} - Z_{3^5}$, Z_{3^5}, $Z_{3^4} - Z_{3^3}$, Z_{27}, $Z_9 - Z_3$, Z_3.

(a) The nil-rigid series for N:

$$L_1 = \{0\} \subset C_1 = (0 : Z_{2^5} \oplus Z_{3^6}) \subset L_2 = (0 : (Z_{2^5} - Z_{2^4}) \oplus Z_{2^4} \oplus (Z_{3^6} - Z_{3^5}) \oplus Z_{3^5})$$

$$\subset C_2 = (0 : Z_8 \oplus Z_{3^4}) \subset L_3 = (0 : (Z_8 - Z_4) \oplus Z_4 \oplus (Z_{3^4} - Z_{3^3}) \oplus Z_{3^3})$$

$$\subset C_3 = (0 : Z_9) \subset L_4 = (0 : (Z_9 - Z_3) \oplus Z_3) \subset C_4 = N.$$

(b) The J_0-U series for N:

$$L_1' = \{0\} \subset U_1 = (0 : Z_{2^5} \oplus Z_{3^6}) \subset L_2' = (0 : (Z_{2^5} - Z_{2^4}) \oplus Z_{2^4} \oplus (Z_{3^6} - Z_{3^5}) \oplus Z_{3^5})$$

$$\subset U_2 = (0 : (Z_{2^5} - Z_{2^4}) \oplus Z_8 \oplus (Z_{3^6} - Z_{3^5}) \oplus Z_{3^4})$$

$$\subset L_3' = (0 : (Z_{2^5} - Z_{2^4}) \oplus (Z_8 - Z_4) \oplus Z_4 \oplus (Z_{3^6} - Z_{3^5}) \oplus (Z_{3^4} - Z_{3^3}) \oplus Z_{27})$$

$\subset U_3 = (0 : (Z_{25}-Z_{24}) \oplus (Z_8-Z_4) \oplus Z_2 \oplus (Z_{36}-Z_{35}) \oplus (Z_{34}-Z_{33}) \oplus Z_9)$

$\subset L_4' = (0 : (Z_{25}-Z_{24}) \oplus (Z_8-Z_4) \oplus Z_2 \oplus (Z_{36}-Z_{35}) \oplus (Z_{34}-Z_{33}) \oplus (Z_{32}-Z_3) \oplus Z_3)$

$= U_4$

(c) The J_0-W series for N:

$L_1^* = \{0\} \subset W_1 = (0 : Z_{25} \oplus Z_{36}) \subset L_2^* = (0 : (Z_{25}-Z_{24}) \oplus Z_{24} \oplus (Z_{36}-Z_{35}) \oplus Z_{35})$

$\subset W_2 = (0 : (Z_{25}-Z_{24}) \oplus Z_8 \oplus (Z_{36}-Z_{35}) \oplus Z_{34})$

$\subset L_3^* = (0 : (Z_{25}-Z_{24}) \oplus (Z_8-Z_4) \oplus Z_4 \oplus (Z_{36}-Z_{35}) \oplus (Z_{34}-Z_{33}) \oplus Z_{27})$

$\subset W_3 = (0 : (Z_{25}-Z_{24}) \oplus (Z_8-Z_4) \oplus Z_4 \oplus (Z_{36}-Z_{35}) \oplus (Z_{34}-Z_{33}) \oplus Z_9)$

$\subset L_4^* = (0 : (Z_{25}-Z_{24}) \oplus (Z_8-Z_4) \oplus Z_4 \oplus (Z_{36}-Z_{35}) \oplus (Z_{34}-Z_{33}) \oplus (Z_9-Z_3) \oplus Z_3)$

$= W_4 \ \ldots$.

We write the three series together in order to compare

(a) $L_1 = \{0\} \subset C_1 \subset L_2 \subset C_2 \subset L_3 \subset C_3 \subset L_4 \subset C_4 = N$;

(b) $L_1' = \{0\} \subset U_1 \subset L_2' \subset U_2 \subset L_3' \subset U_3 \subset L_4' = U_4 = \ldots$;

(c) $L_1^* = \{0\} \subset W_1 \subset L_2^* \subset W_2 \subset L_3^* \subset W_3 \subset L_4^* = W_4 = \ldots$.

We see that the J_0-U and J_0-W series both terminate at an L. Moreover, $J_2(N/C_3) = J_s(N/C_3)$ is nilpotent and non-zero. Also, $J_s(N/W_3)$ is non-zero and nilpotent, $J_s(N/W_3) \neq J_2(N/W_3)$, $W_3 \subset C_3$. Furthermore, $J_2(N/U_3) \neq \{0\}$ is nilpotent and $J_2(N/U_3) = J_s(N/U_3)$.

2. Finite near-rings with identity

Throughout this section our near-rings are assumed to be finite and to have a multiplicative identity.

Theorem 5.7 If $J_0(N/W(N)) = \{0\}$, then $J_s(N) = W(N) = \{0\}$, where $W(N) = Soi(N) \cap J_s(N)$.

Proof Write N as $N = W \oplus L_1 \oplus \ldots \oplus L_k \oplus L$, where $Soi(N) = W \oplus L_1 \oplus \ldots \oplus L_k$, $W = W(N)$

with each L_i an N-group of type-0. Since $J_s(N)$ annihilates L_i, $i = 1,\ldots,k$, each L_i is in fact an N-group of type-s. We have $N/W \cong N' = L_1 \oplus \ldots \oplus L_k \oplus L$ and $J_0(N') = \{0\}$. By theorem 3.12 $Soi(N') = L_1 \oplus \ldots \oplus L_k$. Also the L_i are N'-groups of type-s. Since $J_0(N') = \{0\}$, N' has a faithful representation on a direct sum of N-groups of type-0. We may assume that these are maximal type-0 N-groups. By theorems 4.13, 4.18 and 4.26, these maximal type-0 N-groups have isomorphic copies appearing as summands of $Soi(N')$. That is, N' has a faithful representation on a direct sum of N'-groups of type-s, so that $J_s(N') = \{0\}$. Thus N' is a direct sum of N'-groups of type-s and hence L is a direct sum of N'-groups of type-s. Since W annihilates L it follows that L is a direct sum of N-groups of type-s. Consequently, N itself is a direct sum of N-groups of type-0. By lemma 2.6 we have $J_s(N) = Q(N) = \{0\}$.

<u>Corollary</u> Let $L_1 \subset W_1 \subset L_2 \subset \ldots \subset L_{\alpha-1} \subset W_{\alpha-1} \subset L_\alpha \subset \ldots \subset L_\beta = W_\beta = \ldots$ be the J_0-W sequence for N. If $\alpha > 1$ and $W_{\alpha-1} = L_\alpha$, then $L_{\alpha-1} = W_{\alpha-1}$.

<u>Proof</u> We have $N/W_{\alpha-1} \cong (N/L_{\alpha-1})/(W_{\alpha-1}/L_{\alpha-1})$ and $J_0(N/W_{\alpha-1}) = L_\alpha/W_{\alpha-1} = \{0\}$. Since $W_{\alpha-1}/L_{\alpha-1} = Soi(N/L_{\alpha-1}) \cap J_s(N/L_{\alpha-1})$ it follows from the theorem that $J_s(N/L_{\alpha-1}) = \{0\}$. Hence $W_{\alpha-1} = L_{\alpha-1}$.

<u>Theorem 5.8</u> Let N be a finite near-ring with identity and let $L_1 \subset W_1 \subset \ldots W_{\alpha-2} \subset L_{\alpha-1} = W_{\alpha-1} = \ldots$, $W_{\alpha-2} \neq L_{\alpha-1}$ be the J_0-W series for N. Then $J_s(N/W_{\alpha-2})$ is non-zero and nilpotent.

<u>Proof</u> By the above $Q(N/L_{\alpha-1}) = \{0\}$ and thus $L_{\alpha-1}/W_{\alpha-2}$ is an intersection of maximal left ideals of $N/W_{\alpha-2}$. Thus

$$J_0(N/W_{\alpha-2}) = L_{\alpha-1}/W_{\alpha-2} \supseteq Q(N/W_{\alpha-2}).$$

But $J_0(N/W_{\alpha-2}) \subseteq Q(N/W_{\alpha-2})$ always, and so we have equality. But then by theorem 2.9 $J_s(N/W_{\alpha-2}) = Q(N/W_{\alpha-2})$ and so is nilpotent. If $J_s(N/W_{\alpha-2}) = \{0\}$, then $J_0(N/W_{\alpha-2}) = \{0\}$ i.e. $L_{\alpha-1}/W_{\alpha-2} = \{0\}$, contrary to our assumption.

Let $L_1' \subset D_1 \subset L_2' \subset \ldots \subset D_\alpha \subset L_{\alpha+1}' = D_{\alpha+1} \ldots$ again denote either the J_0-U or the J_0-W series of N. Suppose further that $L_1 \subset C_1 \subset L_2 \subset \ldots \subset C_r = N$ is the nil-rigid series for N. For a finite near-ring with identity we have

<u>Theorem 5.9</u> (i) If for some $k \geq 1$ $D_k \subseteq C_k$, then $L_{k+1}' \subseteq L_{k+1}$;

(ii) if for some $k \geq 1$ $L_k' \subseteq L_k$, then $D_k \subseteq C_k$.

<u>Proof</u> (i) We have $L_{k+1}'/D_k = J_0(N/D_k)$ so that $(L_{k+1}')^\gamma \subseteq D_k$ for some positive integer γ. Since $D_k \subseteq C_k$ it follows that $(L_{k+1}' + C_k)^\gamma \subseteq C_k$ and hence $(L_{k+1}' + C_k)/C_k$ is a nilpotent ideal of N/C_k. That is,

$$(L_{k+1}' + C_k)/C_k \subseteq J_0(N/C_k) = L_{k+1}/C_k$$

and so

$$L_{k+1}' \subseteq L_{k+1}.$$

(ii) There exists a faithful N/L_k'-group Ω' which contains an N/L_k'-isomorphic copy of a faithful N/L_k-group Ω. For Ω' we may take, for example, the Cartesian product of N/L_k by N/L_k' on which N/L_k'-action is defined component-wise. Furthermore, if Δ is an N/L_k-group such that $\Delta \in \mathcal{B}_\Omega$, then $\Delta \in \mathcal{B}_{\Omega'}$ as an N/L_k'-group. Using the fact that

$$C(\Omega) = \mathrm{Soi}(N) = \mathrm{Crux}(N)$$

we have

$$C_k = \bigcap_{\Delta \in \mathcal{B}_\Omega} (0 : \Delta), \quad \Delta \text{ an } N/L_\alpha\text{-group};$$

$$\supseteq \bigcap_{\Delta' \in \mathcal{B}_{\Omega'}} (0 : \Delta'), \quad \Delta' \text{ an } N/L_\alpha'\text{-group};$$

$$= M, \quad \text{where } M/L_\alpha' = \mathrm{Soi}(N/L_\alpha').$$

Since $D_k/L_k' \subseteq M/L_\alpha'$, it follows that $C_k \supseteq D_k$.

<u>Corollary</u> With notation as in the theorem we have $D_k \subseteq C_k$ and $L_{k+1}' \subseteq L_{k+1}$ for all $k \geq 1$.

<u>Proof</u> $L_1 = J_0(N) = L_1'$, and the result follows by induction.

<u>Theorem 5.10</u> Let $L_1 \subset \ldots L_\alpha \subset C_\alpha = N$ be the nil-rigid series and $L'_1 \subset W_1 \subset \ldots \subset L'_\beta = W_\beta = \ldots$ be the J_0-W series for N, $W_{\beta-1} \neq L'_\beta$. Then $\beta \geq \alpha$.

<u>Proof</u> If $\beta < \alpha$, then by the corollary to theorem 5.9 $L'_\beta \subseteq L_\beta \subset L_\alpha$. By theorem 5.8 $J_s(N/W_{\beta-1}) = Q(N/W_{\beta-1}) = L'_\beta/W_{\beta-1}$ so that L'_β is an intersection of maximal left ideals. Hence by theorem 2.8 L_β is an intersection of maximal left ideals. Thus

$$N/L_\beta = \mathrm{Soi}(N/L_\beta) = C_\beta/L_\beta$$

and so $N = C_\beta$, contradicting the strict ascendency of the nil-rigid series for N.

In our examples so far we have seen that $\alpha = \beta$. We strongly suspect that this is always the case for finite near-rings, but we have been unable to prove this.

3. Nilpotent-idempotent sequences for finite near-rings

In what follows we generalise the notion of nil-rigid series in the case of finite near-rings with identity. For this purpose we need the following result due to S.D. Scott. Scott proved this result for more general near-rings with DCCI in terms of the Crux and nil-radical of N.

<u>Fact 5.1</u> If N is a near-ring with DCCN and right identity, then $\mathrm{Soi}(N/J_0(N)) \neq \{0\}$ and $J_0(N/\mathrm{Soi}(N)) \neq \{0\}$ if $\mathrm{Soi}(N) \neq N$.

We now define a sequence of ideals for the finite near-ring N as follows: A_1 is an ideal contained in $J_0(N)$ with $A_1 = \{0\}$ only if $J_0(N) = \{0\}$. Let E_1 be an ideal containing A_1 such that $E_1/A_1 \subseteq \mathrm{Soi}(N/A_1)$ with $E_1 = A_1$ only if $\mathrm{Soi}(N/A_1) = \{0\}$. Further, let A_2 be an ideal containing E_1 such that $A_2/E_1 \subseteq J_0(N/E_1)$ with $A_2 = E_1$ only if $J_0(N/E_1) = \{0\}$. For $\alpha \geq 2$, A_α is an ideal containing $E_{\alpha-1}$ such that $A_\alpha/E_{\alpha-1} \subseteq J_0(N/E_{\alpha-1})$ with $A_\alpha = E_{\alpha-1}$ only if $J_0(N/E_{\alpha-1}) = \{0\}$ and $E_\alpha/A_\alpha \subseteq \mathrm{Soi}(N/A_\alpha)$ with

$E_\alpha = A_\alpha$ only if $\mathrm{Soi}(N/A_\alpha) = \{0\}$. In this way we obtain a sequence

$$A_1 \subseteq E_1 \subseteq \ldots \subseteq A_{\alpha-1} \subseteq E_{\alpha-1} \subseteq A_\alpha \subseteq E_\alpha \subseteq \ldots$$

of ideals. Since N is finite, there exists a k such that

$$A_k = E_k = A_{k+1} = \ldots .$$

The following lemma tells us that the above series actually terminates at N:

Lemma 5.11 (i) If $A_{\alpha-1} = E_{\alpha-1} \neq N$, then $E_{\alpha-1} \neq A_\alpha$, $\alpha > 1$;

(ii) If $E_{\alpha-1} = A_\alpha \neq N$, then $A_\alpha \neq E_\alpha$, $\alpha > 1$.

Proof We only prove (i). By definition, if $E_{\alpha-1}/A_{\alpha-1} = \{0\}$, then $\mathrm{Soi}(N/A_{\alpha-1}) = \{0\}$. Thus if $J_0(N/A_{\alpha-1}) = \{0\}$, also, we would contradict Fact 5.1. Hence $J_0(N/A_{\alpha-1}) \neq \{0\}$, so that $J_0(N/E_{\alpha-1}) \neq \{0\}$. Consequently, we have $A_\alpha/E_{\alpha-1} \neq \{0\}$ by the definition.

Definition We call the sequence $A_1 \subseteq E_1 \subseteq \ldots \subseteq \ldots = N$ a nilpotent-idempotent sequence (N-I sequence for short) of N.

We note that the nil-rigid series for N is a particular N-I sequence for N if N is finite. Furthermore, if $E_{\alpha-1} \subseteq A_\alpha = E_\alpha = N$, then $E_{\alpha-1} = A_\alpha = N$ because N has an identity. Thus an N-I sequence terminates at an E_k. In what follows we adopt the following notation for the sake of convenience:

Notation: $\Delta \in \mathcal{B}_\Omega(N)$ means that $\Delta \subseteq \Omega$, $\Delta \in \mathcal{B}_\Omega$ as an N-group.

As we have seen before, if Ω is a faithful N/A-group, then there exists a faithful N-group Ω' which contains an N-isomorphic copy of every N/A-subgroup of Ω. Furthermore, we have

Fact 5.2 If $\Delta \in \mathcal{B}_\Omega(N/A)$, then $\Delta' \in \mathcal{B}_{\Omega'}(N)$ where $\Delta' \overset{N}{\cong} \Delta$.

It is well known that the nature of the N-groups of type-0 determines whether or not $J_\nu(N)$, $\nu = s,2$, is nilpotent. One might consider the relative sizes of the classes B and G as a crude measure for the non-nilpotence of $J_\nu(N)$, $\nu = s,2$. For example, $\Delta \in G$ may well contain a large number of elements from B. In the ring case this is of course not possible. However, the relative sizes of B and G remain at most a crude measure of non-nilpotency since G may well be empty even though $J_\nu(N)$ is nilpotent. We will now explore this question using the representation theory.

Theorem 5.12 Let $A_1 \subset E_1 \subset \ldots \subset A_r \subset E_r = N$ be an N-I sequence for N. If Δ is an N-group of type-0, then $\Delta \in \mathcal{G}(N)$ or $\Delta \in \mathcal{G}(N/A_i)$ for some $i = 1,\ldots,r$.

Proof If $\Delta \in \mathcal{G}(N)$ or $\Delta \in \mathcal{G}(N/A_1)$, then there is nothing to prove. Now Δ is an N/A_1-group of type-0 because $A_1 \subseteq J_0(N)$. Suppose $\Delta \in \mathcal{B}(N/A_1)$; then $E_1/A_1 \subseteq \bar{C}(\Omega) = \mathrm{Soi}(N/A_1)$ for some faithful N/A_1-group Ω (cf. theorem 4.18). Thus $E_1 \subseteq (0:\Delta)$ and hence Δ is an N/E_1-group of type-0. Consequently, $A_2/E_1 \subseteq J_0(N/E_1)$ annihilates Δ and, as above, we see that Δ is an N/A_2-group of type-0. In general, if $\Delta \in \mathcal{B}(N/A_k)$ for some $k < r$, then we can show that Δ is an N/A_{k+1}-group of type-0. In particular, if $\Delta \in \mathcal{B}(N/A_k)$ for all $k < r$, then Δ is an N/A_r-group of type-0. In this case $\Delta \in \mathcal{G}(N/A_r)$ since $\mathcal{B}(N/A_r) = \{\{0\}\}$.

We note that in the above theorem we have used lemma 1.5. Since the faithful N/A_i-group is immaterial in this case, we have not mentioned it specially. The following theorem is immediate from Fact 5.2:

Theorem 5.13 If Δ is an N/A_j-group of type-0 and $\Delta \in \mathcal{B}(N/A_j)$, then $\Delta \in \mathcal{B}(N/A_k)$ for all $0 < k \leq j$, $j = 1,\ldots,r$.

Theorem 5.14 Let $A_1 \subset E_1 \subset A_2 \subset \ldots \subset E_{r-1} \subset A_r \subset E_r = N$ be the nil-rigid series for N and let Δ be an N-group of type-0; then $\Delta \in \mathcal{G}(N/A_i)$ for exactly one i.

Proof By theorems 5.12 and 5.13 there exists an i such that $\Delta \in \mathcal{G}(N/A_i)$ and $\Delta \in \mathcal{B}(N/A_j)$ if $j < i$. If Δ is an N/A_k-group for some

$k > i$, then Δ is an N/E_{k-1}-group since $A_k \supset E_{k-1}$. But then $\Delta \in \mathcal{B}(N/E_{k-1})$ because $\mathrm{Soi}(N/E_{k-1}) = \{0\}$ by corollary 2 of theorem 3.12 (see also Scott [27], lemma 1.4, Chapter 3). Fact 5.2 now tells us that $\Delta \in \mathcal{B}(N/A_i)$, contradicting our choice of i. Thus Δ is not an N/A_k-group for any $k > i$.

Definition Let $A_1 \subset E_1 \subset \ldots \subset A_r \subset E_r = N$ be the nil-rigid series for N and Δ an N-group of type-0. The unique integer i such that $\Delta \in \mathcal{G}(N/A_i)$ is called the level of Δ. We denote the level of Δ by $\mathrm{lev}(\Delta)$.

Let $L_1 \subset C_1 \subset \ldots \subset L_\alpha \subset C_\alpha = N$ be the nil-rigid series of N and $A_1 \subset E_1 \subset \ldots \subset A_r \subset E_r = N$ any N-I sequence of N. One can show the following as in theorem 5.9:

Lemma 5.15 $A_i \subseteq L_i$ and $E_i \subseteq C_i$ for $i = 1,\ldots,r$. Furthermore, $r \geq \alpha$.

Theorem 5.16 Let Δ be an N-group of type-0 and suppose $\Delta \in \mathcal{G}(N/A_i)$ in some N-I sequence; then $i \geq \mathrm{lev}(\Delta)$.

Proof If $i < \mathrm{lev}(\Delta) = j$, say, then $\Delta \in \mathcal{B}(N/L_i)$ by theorem 5.14. By lemma 5.15, $A_i \subseteq L_j$, so that by Fact 5.2 $\Delta \in \mathcal{B}(N/A_i)$, which is a contradiction.

We give the following examples of levels for various types of N-groups of type-0.

Example 5.4 In each example Δ is of type-0.

(a) If N is a ring, then $\mathrm{lev}(\Delta) = 1$ for any N-group of type-0;

(b) if $\Delta \in \mathcal{G}_{N^+}$, then $\mathrm{lev}(\Delta) = 1$;

(c) if $\Delta \in \mathcal{B}_{N^+}$ and Δ is maximal type-0, then $\mathrm{lev}(\Delta) = 1$.

We state the following somewhat obvious results:

Theorem 5.17 If Ω is of type-s and Δ is an N-subgroup of Ω, then $\mathrm{lev}(\Omega) = \mathrm{lev}(\Delta)$. Conversely, if Ω is of type-0 and for each N-subgroup Δ of Ω we have that $\mathrm{lev}(\Omega) = \mathrm{lev}(\Delta)$, then Ω is of type-s.

Theorem 5.18 If Δ_1, Δ_2 are N-groups of type-0 and $\Delta_1 \ll \Delta_2$ (i.e. Δ_1 is a subfactor of Δ_2), then $\mathrm{lev}(\Delta_1) \geq \mathrm{lev}(\Delta_2)$.

Proof Suppose $\mathrm{lev}(\Delta_2) = j$, i.e. $\Delta_2 \in \mathcal{G}(N/A_j)$; then Δ_1 is an N/A_j-group of type-0. If $\Delta_1 \in \mathcal{G}(N/A_j)$, then $\mathrm{lev}(\Delta_1) = j$ by theorem 5.14, so suppose $\Delta_1 \in \mathcal{B}(N/A_j)$. We have $\Delta_1 \in \mathcal{B}(N/A_k)$ for all $0 < k \leq j$ by theorem 5.13 so that $\mathrm{lev}(\Delta_1) > j$.

In general, given a faithful N-group Ω one can always construct a faithful N-group Γ, containing an isomorphic copy of Ω and such that every N-group of type-0 is a subfactor of Γ. Why is it so important to know whether or not a faithful N-group has every N-group of type-0 as a subfactor? Well, in our examples we constructed nil-rigid series using a finite faithful N-group Ω. The first term in each series is $J_0(N)$, the intersection of the annihilating ideals of all cyclic irreducible subfactors of Ω. If not every N-group of type-0 appeared as a subfactor of Ω, then this intersection may not be $J_0(N)$. Indeed, at each stage we would get ideals distinct from the L_i of a nil-rigid series. Since we assume N to be finite, lemma 1.5 ensures that an N-group of type-0 is a subfactor of any faithful N-group.

Let Ω be a faithful N-group and define T as follows:

$$T = \{\Delta : \Delta \text{ of type-0}, \ \Delta \ll \Omega\}.$$

Let

$$\mathrm{LEV} = \{ i : i = \mathrm{lev}(\Delta), \ \Delta \in T\}.$$

Since LEV is a finite set there exists a maximal element m in LEV.

Theorem 5.19 If $\mathrm{lev}(\Delta)$ is maximal, then Δ is of type-s.

Proof For any subgroup Γ of Δ we have $\mathrm{lev}(\Gamma) \geq \mathrm{lev}(\Delta)$ by theorem 5.18. By the maximality of $\mathrm{lev}(\Delta)$ we have $\mathrm{lev}(\Delta) = \mathrm{lev}(\Gamma)$ and so by theorem 5.17 Δ is of type-s.

Theorem 5.20 Let $L_1 \subset C_1 \subset \ldots \subset L_r \subset C_r = N$ be the nil-rigid series for N. If $\mathrm{lev}(\Delta) = m$ is maximal, then $m = r$. The proof is straightforward and will be omitted.

We note that if $lev(\Delta) = m$ is maximal, then $J_s(N/C_{m-1})$ is non-zero and nilpotent. Accordingly, we have the following:

Definition Let $lev(\Delta) = m$ be maximal; then we define the nilpotency level of $J_s(N)$ to be $m-1$.

If N is a ring then $lev(\Delta) = 1$ for every N-group of type-0, so that the nilpotency level of $J_s(N) = J_2(N)$ is zero. In our examples it was an easy matter to construct the nil-rigid series for N. Indeed, one can always find the nilpotency level of $J_s(N)$ fairly quickly in this manner, provided that the N-group is sufficiently small. However, we have not given an answer to the following question. Given the faithful N-group Ω, is it possible to discover the nilpotency level of $J_s(N)$ without constructing the nil-rigid series for N? That is, can one find the nilpotency level of $J_s(N)$ by simply looking at the irreducible subfactors of Ω? We suggest that the following procedure can be adopted:

Again let T be defined as follows:

$$T = \{\Delta \ll \Omega : \Delta \text{ is of type-0}\}.$$

That is, T consists of all N-groups of type-0. Partially order T by $\Delta_1 \leq \Delta_2$ if and only if Δ_1 is a subfactor of Δ_2. Let $\Delta \in T$ be maximal type-0 and consider the following chain in T:

$$\Delta = \Delta_n > \Delta_{n-1} > \dots > \Delta_1.$$

We may assume that $\Delta_j \geq \Omega > \Delta_{j-1}$, $\Omega \in T$, $j = 2,\dots,n$, implies that $\Delta_j = \Omega$. That is, it is not possible to "refine" this chain any further. There may of course be several such chains starting with the given Δ. In each chain we discover whether or not there exists a $\Omega_j \in \mathcal{K}$ (i.e. an Ω_j which does not have an N-group of type-0 as a direct summand) such that $\Delta_j > \Omega_j > \Delta_{j-1}$. Where these exist, we insert them in the chain and obtain, for example,

$$\Delta = \Delta_n > \Delta_{n-1} > \ldots > \Delta_j > \Omega_j > \Delta_{j-1} > \ldots > \Delta_i > \Omega_i > \Delta_{i-1} > \ldots > \Delta_1.$$

Among all maximal type-0 N-groups Δ we consider a chain with the largest number of Ω_i is inserted. This number will give us the length of the nil-rigid series for N. If Ω is large this procedure may be just as tedious as constructing the nil-rigid series for N. At any rate, either method uses only the faithful N-group Ω.

4. A decomposition theorem for the s-radical

In conclusion, we give a decomposition theorem for the s-radical. For this purpose we need the following:

Lemma 5.21 Let N be a near-ring with DCCN and multiplicative identity. Write $N = Soi(N) \oplus L$. Then $J_s(N).L \subseteq Q(N)$ if, and only if, $J_s(N) \cap L = Q(N)$.

Proof $J_s(N) \cap L = Q(N)$ obviously implies $J_s(N).L \subseteq Q(N)$. On the other hand, if $J_s(N).L \subseteq Q(N)$, then $J_s(N) \cap L$ is nilpotent and hence contained in $Q(N)$. Since $Q(N) \subseteq J_s(N) \cap L$ we have equality.

We note that $J_s(N)$ decomposes as $J_s(N) = W \oplus J_s(N) \cap L$, where $W = J_s(N) \cap Soi(N)$ and $N = Soi(N) \oplus L$.

Lemma 5.22 Let N be a near-ring with DCCN and a multiplicative identity and $W = Soi(N) \cap J_s(N)$. Then $J_s(N/W)$ is nilpotent if, and only if, $J_s(N) = W \oplus Q(N)$.

Proof If $J_s(N) = W \oplus Q(N)$, then by the remark following lemma 5.21 $J_s(N) \cap L = Q(N)$, so that $J_s(N).L \subseteq Q(N)$ also by the lemma. Write N as $N = W \oplus L' \oplus L$, where $W \oplus L' = Soi(N)$. Since every left ideal of $L' \oplus L$ is also a left ideal of N and $Q(N) \subseteq L$, it follows that $Q(N) = Q(L' \oplus L)$. Now $Q(N)(L' \oplus L) \subseteq J_s(N)(L' \oplus L) \subseteq J_s(N).L \subseteq Q(N)$ because $J_s(N)L' = \{0\}$. Hence $Q(N)$ is a two-sided ideal of $L' \oplus L$ and so by theorem 2.9. $J_s(L' \oplus L) = Q(L' \oplus L)$. From the isomorphism $N/W \cong L' \oplus L$ we see that $J_s(N/W)$ is nilpotent. Conversely, suppose that $J_s(N/W)$ is nilpotent. Then $Q(N)$ is a two-sided ideal of $L' \oplus L$ so that $Q(N)$ is a two-sided ideal of L. That is, $Q(N) = J_s(L)$ and thus $L - Q(N)$ is a direct sum of

L-groups of type-s by lemma 2.11. Since $W \oplus L' = \mathrm{Soi}(N)$ annihilates L it follows that $L - Q(N)$ is a direct sum of N-groups of type-s. Consequently, $J_s(N)$ annihilates $L - Q(N)$ and we have $J_s(N).L \subseteq Q(N)$. By lemma 5.21 $J_s(N) \cap L = Q(N)$, and we have the required decomposition for $J_s(N)$.

Lemma 5.23 Let $N = \mathrm{Soi}(N) \oplus L$ be a near-ring with a multiplicative identity and satisfying the DCCN. Put $W = \mathrm{Soi}(N) \cap J_s(N)$. If $J_s(N/\mathrm{Soi}(N))$ is nilpotent, then $J_s(N/W)$ is nilpotent.

Proof Write N as $N = W \oplus L' \oplus L$ with $\mathrm{Soi}(N) = W \oplus L'$. We need only show that $Q(N)$ is a two-sided ideal of $L' \oplus L$. The result will then follow from the isomorphism $N/W \cong L' \oplus L$. Certainly $Q(N)L \subseteq Q(N)$ because $Q(N)$ is a two-sided ideal of L. Also, $Q(N).L' \subseteq J_s(N).L' = \{0\}$ and so $Q(N)(L' \oplus L) \subseteq Q(N)$.

Theorem 5.24 Let N be a near-ring with identity and satisfying the DCCN. Further, let $L_1 \subset C_2 \subset \ldots \subset L_\alpha \subset C_\alpha = N$, $\alpha > 1$, be the nil-rigid series for N. Then $J_s(N/L_{\alpha-1}) = \bar{W} \oplus Q(N/L_{\alpha-1})$, where $\bar{W} = \mathrm{Soi}(N/L_{\alpha-1}) \cap J_s(N/L_{\alpha-1})$.

Proof By theorem 5.1, $J_s(N/C_{\alpha-1})$ is nilpotent and hence by lemma 5.23 $J_s((N/L_{\alpha-1})/\bar{W})$ is nilpotent. The result now follows from lemma 5.22.

Corollary If $\alpha = 2$ and $J_0(N) = \{0\}$, then $J_s(N) = W \oplus Q(N)$, where $W = \mathrm{Soi}(N) \cap J_s(N)$.

Example 5.5 [18,19] Let V be a reduced free group on m generators whose laws are precisely the universal laws of A_n, $n \geqslant 5$. Take m at least as great as the minimum number of generators for A_n. Let N be the Neumann d.g. near-ring associated with V; N is finite with an identity. Now $V - K \cong A_n$ for some normal subgroup K of V and so there is a left ideal L such that $N - L$ is a faithful N-group. Moreover, every N-subgroup of $N - L$ is cyclic and there is a one-to-one lattice correspondence between the subgroups of A_n and the N-subgroups of $N - L$. Under this correspondence the N-kernels of $N - L$ correspond to the normal subgroups of A_n. Consequently, N is 0-primitive with $N - L$ a faithful N-group of type-0. Using Fact 2.1, we again see that $J_2(N) = J_3(N) \nsubseteq Q(N)$. We will obtain decompositions of $J_2(N)$ for specific values of n.

(a) $n = 5$. Since every proper subgroup of A_5 is soluble, $N - L$ is, to within an isomorphism, the only non-minimal N-group of type-0. Now every group of prime order which appears as a subfactor of A_5 is a subfactor of a subgroup which does not have a simple group as a direct summand. Hence by the one-to-one lattice correspondence Soi(N) is a direct sum of copies of $N - L$. Thus the only N/Soi(N)-groups of type-0 are of type-2, and so $J_2(N/\mathrm{Soi}(N))$ is nilpotent. The nil-rigid series for N is $\{0\} = J_0(N) = L_1 \subset C_1 = \mathrm{Soi}(N) \subset L_2 = (0 : \bigoplus_{i \in I} \Delta_i) \subset C_2 = N$, where Δ_i ranges over copies of the proper N-subgroups of $N - L$. The corollary to theorem 5.24 tells us that $J_2(N) = \mathrm{Soi}(N) \oplus Q(N)$. We remark that Machin [19] obtained the above decomposition for any Neumann near-ring N associated with a minimal simple group. Since A_5 is minimal simple, the stated decomposition for $J_2(N)$ follows straight from the Machin theory.

(b) $n = 6$. A_6 is not minimal simple, so that the Machin theory does not apply. The copies of A_5 contained in A_6 are all maximal subgroups of A_6, so that Soi(N) contains direct sums of copies of A_6 and A_5. All other simple groups which appear as subfactors of A_6 are Abelian of prime order. Each 3- and 2×2-cycle is contained in a copy of S_4. Furthermore, every 5-cycle is contained in a subgroup of order 10, which does not have a simple group as a direct summand. Also, each 3×3-cycle is contained in a direct sum of copies of A_3. Thus Soi(N) is precisely a direct sum of copies of A_5 and A_6. It is now clear that the only N/Soi(N)-groups of type-0 are the minimal ones, and hence $J_2(N/\mathrm{Soi}(N)) = J_0(N/\mathrm{Soi}(N))$. The nil-rigid series for N is $\{0\} = J_0(N) \subset \mathrm{Soi}(N) = C_1 \subset L_2 \subset C_2 = N$. Again we see that $J_2(N) = \mathrm{Soi}(N) \oplus Q(N)$.

CHAPTER 6

Some future problems

In Chapter 2 we have shown that the s-radical is the "closest" to being nilpotent amongst all Jacobson-type radicals which contain the quasi-radical. It seems natural therefore to study the non-nilpotence of the radical via this ideal. However, very little is known about the connections between the radical and the s-radical. For example, if N satisfies the DCCN, then $J_2(N)$ contains all nilpotent N-subgroups of N^+ [24]. Is it possible for $J_s(N)$ to contain all nilpotent N-subgroups of N^+ and yet not be equal to $J_2(N)$?

A further problem concerns a density theorem for s-primitive near-rings. Of course, there is a density theorem for more general 0-primitive near-rings due to G. Betsch, but it is hoped that in the s-primitive case such a theorem will take on a particularly nice form.

In view of the corollary to theorem 2.8 we have that if N is a d.g. near-ring with identity and DCCL, then $J_s(N)$ consists of all elements of the form $\sum_i -x_i + q_i r_i + x_i$ for $x_i, r_i \in N$ and $q_i \in Q(N)$. The question that arises here is whether one can use the above result to generalise the notion of left quasi-regularity.

A. Machin [19] first studied a right representation theory for Neumann d.g. near-rings. More recently these results have been extended by R.R. Laxton and M.H. Rahbari to general d.g. near-rings with identity. For example, they have shown that a finite d.g. near-ring with identity is left simple if and only if it is right simple. A right radical theory for d.g. near-rings is still in its infancy, and it is hoped that it will shed some light on the problems remaining from the left representation theory.

It appears that the socle-ideal has not been studied in the ring case at all. It would be of some interest to know whether a Baer-type antiradical series involving the socle-ideal will yield results similar to the ones in [4]. In the ring case one can, of course, develop the notion of a right socle-ideal using right ideals. The natural question to ask here is whether this right socle-ideal coincides with our (left) socle-ideal. The study of the right socle-ideal can perhaps be extended to include d.g. near-rings and developed alongside the right radical theory mentioned before.

The representation theory of Chapter 4 needs further scrutiny, particularly in the Neumann d.g. near-ring case. For example, if the socle-ideal of a finite Neumann d.g. near-ring N is non-zero, then we know that it has a critically faithful N-group. One should like an interpretation of this in terms of variety theory. That is, can one determine whether or not Soi(N) is zero by only considering the associated reduced free group V or the defining group G?

The theory concerning "nilpotency level" that we attempted to develop in Chapter 5 is in many respects inconclusive. For example, we have not been able to find answers to the questions concerning the nilpotency level of $J_2(N)$. Perhaps this problem should be investigated using a different type of radical-antiradical series. Also, it would be of interest to know whether, in the DCCN case, the nil-rigid series for the near-ring N is the shortest nilpotent-idempotent series in N. Recently C.G. Lyons and J.D.P. Meldrum* linked the nilpotence of radicals containing $J_0(N)$ with certain series (called N-series) defined on a faithful N-group Ω. They specifically linked the nilpotency class of such a radical J with the length of a special type of N-series on Ω. Define $\underset{\sim}{L}_J$ as follows:

$\underset{\sim}{L}_J$: $\Omega = L_0$; L_{i+1} is the N-kernel of L_i generated by JL_i.

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* C.G. Lyons and J.D.P. Meldrum, N-series and tame near-rings (submitted)

Of course, there is no guarantee that $\underset{\sim}{L}_J$ is finite i.e. that $\underset{\sim}{L}_J$ is an N-series, in general. In ✱ it is proved that J is nilpotent of class at most n if $\underset{\sim}{L}_J$ is an N-series of length n. It follows trivially that if $J_s(N) = J$ and $\underset{\sim}{L}_J$ is an N-series, then the nil-rigid series for N is $L_1 = J_s(N) \subset C_1 = N$ in the DCCN case. Although N-series are defined on N-groups whereas the nil-rigid series for N is constructed in the near-ring, we nevertheless feel that connections between these two notions are worth investigating.

Finally, a further problem is connected with nil-rigid series in Neumann d.g. near-rings. Can one make a study of nil-rigid series for Neumann d.g. near-rings N by considering the defining group G only? The main obstacle appears to be the following. Let V be a reduced free group and G a finite group whose laws are precisely the universal laws of V. Consider the ideal A in the Neumann d.g. near-ring N, associated with V. Then N/A is a Neumann d.g. near-ring associated with the reduced free group $V - W$, where W is a fully invariant subgroup of V corresponding to A [19, page 21]. It is not clear how one would obtain a new defining group $\bar{G}$ from G such that $V - W$ and $\bar{G}$ have the same universal laws. This may well prove to be a hard problem in variety theory.

Appendix 1

Fact 2.1, page 20. Let G be a finite, simple group every one of whose subgroups is a direct sum of simple groups; then G is Abelian of prime order.

Proof [D.L. Johnson]. Let $\mathcal{X}$ be the class of all finite, non-Abelian simple groups such that $K \in \mathcal{X}$ implies that every subgroup of K is a direct sum of simple groups. Let G be an element of $\mathcal{X}$ such that $|G|$ is minimal and H a proper subgroup of G. Then $H = \bigoplus_{i=1}^{k} H_i$ where H_i is simple for each i. Now every subgroup of H_i is a direct sum of simple groups (inherited from G). Since $|G|$ is minimal $H_i \notin \mathcal{X}$ for each i. Consequently H_i is Abelian of prime order, so that H is Abelian. If P is a p-Sylow subgroup of G, then the normaliser N(P) of P is a proper subgroup of G because G is simple. Hence, by the above, N(P) is Abelian so that N(P) = C(P), where C(P) is the centraliser of P in G. The Burnside normal complement theorem [29, page 137] now tells us that P has a normal complement in G. This contradicts the simplicity of G.

Appendix 2

Index to examples

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