TOPOGENOUS ORDERS AND THEIR APPLICATIONS ON LATTICES

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Abstract

In his influential book, [Csá63] Á. Császár developed the well-known theory of syntopogenous structures on a set. His intention was to create a comprehensive framework that simultaneously encompasses the study of topological, proximal, and uniform structures. In the same monograph, he demonstrated independently, along with Pervin [Per62], that every topological space possesses a compatible quasi-uniformity. A similar observation was noted for a uniform space, provided the topological space is completely regular. On the other hand, Herrlich in [Her74a] introduced the concept of "nearness" with the aim of unifying various topological structures.

This Ph.D. thesis aims to investigate topogenous orders and their generalizations, such as quasi-uniformities, syntopogenous structures, on complete lattices which extend and generalize existing literature in this field. We explore the study of quasi-uniformities through the lens of syntopogenous structures, and establish a Galois connection between these two constructs. Furthermore, we provide conditions under which certain Császár structures are order isomorphic to quasi-uniformities on a complete lattice.

As Császár structures are deeply rooted in pointfree topology, our research naturally extends into the realm of frames. We establish a correspondence between pre-nearness and Császár structures. In line with these ideas, we also delve into the relationship between pre-uniformities and entourage quasi-uniformities in the context of frames.

Declaration

I declare that **TOPOGENOUS ORDERS AND THEIR APPLICATIONS ON LATTICES** is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Bakulikira Claude Iragi May 2024

Signed:..

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Contents

Chapter 1

Introduction

In the early sixties, Császár presented his book, *Foundations of general topology* [Csá63]. He aimed to develop a theory that generalizes several branches of pure mathematics. Császár's theory encompasses uniform structures, topological structures, and proximity structures. He demonstrated that proximity structures act as bridge between topological and uniform structures. Furthermore, he examined various properties of syntopogenous spaces including completeness and compactness, and constructed a completion of a syntopogenous structure through the utilization of Cauchy filters.

In the same book, Császár suggested the concept of quasi-uniformity on a set which was already introduced by Nachibin [Nac48] in 1948 under the term semi-uniform structures. Since then, quasi-uniform structures have been extensively explored and documented in the literature for both frames and spaces. For further details, refer to sources such as [Kün92] and [LF82] for insights into spaces, and [Fri86] and [Pic95] for perspectives on frames. Additionally, the readers can also explore the references provided in these works for a comprehensive understanding of the subject matter.

In separate works, Pervin [Per62] and Császár [Csá63] demonstrated that every topological space has a compatible quasi-uniformity and this observation holds true for a uniform space, provided the topological space in question is completely regular. In this context, it becomes apparent that quasi-uniform structures provide an additional perspective on topological structures.

In 1974, Herrlich [Her74a] introduced the concept of nearness structures on a set. His intention was unifying various concepts of topological structures. He also suggested a correspondence between symmetric syntopogenous spaces and nearness spaces.

It is now evident that Császár's structures, which serve as the cornerstone of this Ph.D. thesis, manifest themselves in numerous mathematical structures. In fact, not only are they omnipresent in numerous mathematical spaces but they also hold greater significance in pointfree topology. Indeed, one of the most pleasing features of syntopogenous structures is their remarkable ability to seamlessly bridge the gap between points and open sets, even in the context of their global nature. It is because of this reason that Császár himself considered them as a natural point of departute of the study of pointfree topology. A formal definition of syntopogenous structures in pointfree topology was given by Chung [Chu08] (see also [Chu05] for the case of complete lattices). In [WL95], the authors investigated syntopogenous structures on a complete distributive lattice. They examined the question of cotopology, quasi-uniformity, and T-structures.

Recently, a categorical study of quasi-uniform structures through syntopogenous structures was initiated in [HI19]. This study digs into the concepts of completion and completeness within the categorical framework. Among many discoveries made, it was shown that a quasi-uniformity can be viewed as a collection of families of closure operators on a category.

The classical theory of quasi-uniform structures is presented in terms of entourages [LF82] and conjugate covers [GS72]. While both approaches are well-documented in the context of frames, the latter seems to have received more attention for obvious reasons, see Frith [Fri86] and Picado [Pic95]. Nevertheless, the former approach has also been explored in pointfree topology, see [FHL93a, FHL93b, FHL94].

In their paper, [BP96], Banaschewski and Pultr were the first to axiomatize the concept of nearness with dual objectives. On one hand, the aim was to establish the pointfree counterpart of nearness spaces, as defined by Herrlich [Her74a], on the other hand, the goal was to address the limitations of uniform frames. In fact, nearness frames are uniform frames that lack the star-refinement axiom.

In this dissertation, we explore various questions, with a particular emphasis on the following two primary inquiries:

We introduce a comprehensive theory of quasi-uniformity on general complete lattices, encompassing the Fletcher's et al. theory of entourage quasi-uniformity when applied on frames [FHL93a] on one hand, and, on the other hand, the results presented in [HI19]. In fact, a quasi-uniformity, respectively, a syntopogenous structure on a category $\mathcal{X}$ *supplied with an* $(\mathcal{E}, \mathcal{M})$*-factorization structure for morphism is obtained when, for each object* X *of* $\mathcal{X}$*, a quasi-uniformity, respectively, a syntpogenous structure is defined on the subobject lattice of* X *such that each morphism* $f : X \longrightarrow Y$ *in* $\mathcal{X}$ *is quasi-uniformly respectively syntopogenously continuous on the subobject lattices of* X *and* Y*. Our study of quasi-uniformities is facilitated by the use of the concept of syntopogenous structures.*

Furthermore, while all the structures mentioned above are well-defined in pointfree topology, as far as our current knowledge goes, there is no clear correspondence between nearness structures and Császár structures within the context of frames. In this dissertation, our aim is to address and bridge this gap.

In more detail, this Ph.D. dissertation is structured as follows:

Chapter 2 presents the fundamental definitions and key results that are relevant to the rest of this dissertation.

In Chapter 3 we define and study topogenous orders on complete lattices. We extend numerous notions from topological categories to the realm of general complete lattice theory, including closure and interior operations, which yields several classical results from topological categories as a special case. We prove that topogenous order encompasses both closure and interior operators within lattices. In this context, the results in [HIR16] become special cases. Further, if we narrow down our examination of topogenous orders to frames, our results serve as a pointfree counterpart to the results in [Chu88]. Furthermore, we characterize the so-called " Strict maps" both in frames and in topology. Our methods of proof share similarities with those utilised in [HIR16]. We conclude the chapter with a number of examples highlighting the importance of studying topogenous orders within complete lattices.

In Chapter 4 we use syntopogenous structures to examine quasi-uniformities on complete lattices. We show that the syntopogenous structures (in the sense of our definition) are order isomorphic to a base of a quasi-uniformity on a complete lattice. Moreover, when we consider syntopogenous structures that do not preserve necessarily meets, we establish a Galois connection between quasi-uniformities and syntopogenous structures on a complete lattice. Furthermore, we prove that any $\bigwedge$-structure of a complete lattice determines a base of a transitive quasi-uniformity on the lattice in question. It should be noted that, except for the results in Section 4.4, all our results in this chapter can be restricted to frames. This observation will be explored in the next chapter. We conclude this chapter by establishing a relationship between **SYT**, the category of syntopogenous spaces and **SYNTFrm**, that of syntopogenous frames.

Chapter 5 relates pre-nearness and semi-Császár structures on frames. More precisely, it establishes a correspondence between the category of pre-nearness frames [BP93] and a novel category of semi-Császár structures that we introduce. Moreover, when considering the concept of quasi-uniformities in a frame, we will demonstrate that interpolative Császár structures are in a one-to-one correspondence with the bases of a quasi-uniformity. In concluding the chapter, we build upon the findings of [FHL93b] and [Pic95], to establish a relationship between a base of an entourage quasi-uniformity and a base of pre-uniformities within the context of frames.

In the last chapter, our focus is dedicated to the examination of particular morphisms within a general category equipped with an $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms. This $(\mathcal{E}, \mathcal{M})$-factorization structure permits us to delve into various noteworthy ordering mechanisms and extend the applicability of several established findings. In particular, we define strict maps with respect to two topogenous orders, generalizing "closed maps" relative to two closure operators introduced in [Hol09]. We also define the open maps with respect to two interior operators. Using the topogenous order derived from a functor as introduced [Ira19], we extend the scope of closed maps as defined by G. Castellini and E. Giulli [CG05, CG01].

Some of the main findings in the thesis are being prepared for publication in the following manuscripts:

(1) B. Iragi and D. Holgate. *On Császár orders and Pre-nearness on frames* (Ready for submission);

(2) B. Iragi and D. Holgate. *Topogenous orders and related maps in* **Top** *and in* **Frm** (In preparation);

(3) B. Iragi. *Overview of Császár Orders and Quasi-uniformities on Complete Lattices* (submitted for publication).

Chapter 2

PRELIMINARIES

This chapter provides an overview of some definitions and key results concerning lattices, category, Galois adjunction, and topology that are essential for the subsequent chapters. Many of the results appearing here are widely documented in the literature, and therefore, most of the statements will be presented without formal proofs. As far as frames are concerned, our guiding principle is well-established: whenever a property of a given topological space can be expressed entirely in terms of the lattice of its open sets, it bears significance for us in our exploration of frames.

2.1 Lattice

A partially ordered set, or a **poset**, is a pair $(X, \leq)$ where X is a set and " $\leq$ " a **partial order** on X, that is, " $\leq$ " is reflexive, transitive and antisymmetric. If " $\leq$ " only possesses the first two properties, it is referred to as a **preorder**.

Let Y be a subset of a poset X. An element $x \in X$ is said to be an upper bound (respectively, lower bound) of Y if $y \leq x$ ($y \geq x$) for all $y \in Y$. The **join** of Y is the least upper bound of Y and the **meet** of Y is the greatest lower bound of Y. They are, respectively, denoted $\bigvee Y$ and $\bigwedge Y$. However, if Y has two elements only, say, $Y = \{x, y\}$, the meet and join of Y are written as $\bigwedge Y = x \wedge y$ and $\bigvee Y = x \vee y$, respectively. Similarly, if $Y = \{y_i \mid i \in I\}$, we will write $\bigvee_{i \in I} y_i$ $\left(\bigwedge_{i \in I} y_i\right)$ when we want to denote the join and the meet of the set Y, respectively.

A poset X is said to be a:

- **meet-semilattice (join-semilattice)** if any two elements $x, y \in X$ have meet (join);
- **lattice** if any two elements $x, y \in X$ have a meet and join;

Let X be a lattice. Then X is said to be:

- **distributive** if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds for all $x, y, z \in X$;
- **bounded** if each of its finite subsets has a meet and a join. That is to say X is a lattice with a least (bottom) element denoted by 0_X and a greatest (top) element denoted by 1_X. It should be noted that the subscripts on 0 and 1 can be omitted if it is clear from the context which lattice is being referred to.

- **complete** if each subset of X has a meet and a join. In fact, every complete lattice X is always bounded with

$$0_X = \bigvee \emptyset = \bigwedge X \text{ and } 1_X = \bigwedge \emptyset = \bigvee X$$

- **complemented** if it is bounded and each element $x \in X$ has a complement, that is, an element y with

$$x \vee y = 1_X \text{ and } x \wedge y = 0_X$$

- **Boolean algebra** if it is complemented and distributive.

In a Boolean algebra B, the following distributive law

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\}$$

holds for all $A \subseteq B$, $b \in B$, provided $\bigvee A$ exists.

In a lattice X with 0_X, an element $a \in X$ is said to have a pseudocomplement if there exists a greatest element $x \in X$ satisfying $a \wedge x = 0_X$. We shall denote such an element x by a^*. Equally, we say that a^* is a pseudocomplement of a if

$$x \wedge a = 0_X \Leftrightarrow x \leq a^*$$

for all $x \in X$. Moreover, if the binary meet distributes over arbitrary joins, then a^* also satisfies

$$a^* = \bigvee \{x \in X \mid x \wedge a = 0_X\}.$$

The pseudocomplement of an element if it exists, it is unique. We shall say that a lattice X is **pseudocomplemented** if each of its elements has a pseudocomplement.

Fact: In a bounded distributive lattice each complement is also a pseudocomplement.

Example 2.1.1. For any topological space X, the lattice ΩX, representing the open sets of X, is a pseudocomplemented lattice. For all $B \in \Omega X$, B^* is given by $X \setminus \overline{B}$. For $A \in \Omega X$, we obtain that

$$A \cap B = \emptyset \Leftrightarrow A \subseteq X \setminus B \Leftrightarrow A \subseteq X \setminus \overline{B}$$

The verification of the following properties of pseudocomplement is straightforward:

(P1) $(1_X)^* = 0_X$ and $(0_X)^* = 1_X$;

(P2) $x \leq y \Rightarrow y^* \leq x^*$;

(P3) $x \leq x^{**}$;

(P4) $x^{***} = x^*$;

(P5) $(x \vee y)^* = x^* \wedge y^*$.

Let X and Y be any two lattices. A map $f : X \longrightarrow Y$ is said to be a lattice homomorphism if it preserves the lattice structures, that is,

$$f(x \vee y) = f(x) \vee f(y) \text{ and } f(x \wedge y) = f(x) \wedge f(y)$$

for all $x, y \in X$ and $f(0_X) = 0_Y$ and $f(1_X) = 1_Y$, provided X and Y are bounded.

2.2 Galois connection

Galois connections represent a powerful tool in mathematics. They enable the seamless transition between two distinct mathematical structures. They are generalizations of the correspondence between subgroups and subfields explored in Galois theory.

Definition 2.2.1. *[Ern04]* Let $\mathcal{A} = (A, \leq)$ and $\mathcal{B} = (B, \leq')$ be two posets or simply preordered classes and $f : \mathcal{A} \longrightarrow \mathcal{B}$ and $g : \mathcal{B} \longrightarrow \mathcal{A}$ be two order preserving mappings such that for all $a \in A$ and $b \in B$,

$$f(a) \leq' b \Leftrightarrow a \leq g(b)$$

Then, f and g establish a Galois connection between $\mathcal{A}$ and $\mathcal{B}$. Or, equivalently, the pair of order preserving maps $f : \mathcal{A} \longrightarrow \mathcal{B}$ and $g : \mathcal{B} \longrightarrow \mathcal{A}$ satisfying

$$f(g(y)) \leq y \;\text{ for all } y \in B \;\text{ and } x \leq g(f(x)) \;\text{ for all } x \in A.$$

In such cases, f is said to be the left Galois adjoint of g and g the right Galois adjoint of f. This relationship is commonly denoted by $f \dashv g$ or (f, g) or

$$\mathcal{A} \underset{g}{\overset{f}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}.$$

In fact, the maps f and g exhibit other noteworthy characteristics, including the fact that they uniquely determine each other. Furthermore, the following few lemmas, from D. Dikranjan and W. Tholen [DT95], describe additional aspects of their behavior:

Lemma 2.2.2. *[DT95] Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be a pair of mappings between posets A and B. Then, the following are equivalent:*

(i) $f \dashv g$;

(ii) f *and* g *are order preserving mappings and* $a \leq gf(a)$ *and* $fg(b) \leq b$ *for all* $a \in A$ *and* $b \in B$;

(iii) f *is order preserving mapping and* $g(b) = \max\{a \in A \mid f(a) \leq b\}$ *for all* $b \in B$;

(iv) g *is order preserving mapping and* $f(a) = \min\{a \in A \mid a \leq g(b)\}$ *for all* $a \in A$.

Lemma 2.2.3. *[DT95] Assume $f \dashv g$. Then*

(1) $f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)$, *provided* $\bigvee_{i \in I} a_i$, *exists;*

(2) $g(\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} g(b_i)$, *provided* $\bigwedge_{i \in I} b_i$, *exists.*

Lemma 2.2.4. *[DT95] Let A and B be two posets.*

(1) If arbitrary joins exists in A, then any mapping $f : A \longrightarrow B$ that preserves arbitrary joins has a right adjoint;

(2) If arbitrary meets exists in B, then any mapping $g : B \longrightarrow A$ that preserves arbitrary meets has a left adjoint.

2.3 Category and Functor

The notion of posets and order preserving maps, elucidated above, can be extended to more general frameworks: category and functor. Further, since the concepts of category and functor will be frequently used in this work, we dedicate this section to some concepts of categories and functors that are pertinent to the subsequent discussions in this work. The reader who needs a deeper insight into the topics of this section should consult [AHS90] and many other books of category theory.

A **category** $\mathcal{X}$ is a mathematical structure consisting of the following data:

(1) a class Ob$\mathcal{X}$ whose elements $X, Y, Z, \ldots$ are called $\mathcal{X}$-objects;

(2) for every pair (X, Y) of $\mathcal{X}$-objects, a set $Hom_{\mathcal{X}}(X, Y)$, whose elements will be called morphisms from X to Y;

(3) for every triple of objects X, Y, Z, a composition law

$$Hom_{\mathcal{X}}(X, Y) \times Hom_{\mathcal{X}}(Y, Z) \longrightarrow Hom_{\mathcal{X}}(X, Z)$$

$$(f, g) \longrightarrow g \circ f$$

(4) for every object $X \in \mathcal{A}$, a morphism $1_X \in Hom_{\mathcal{X}}(X, X)$, called the identity on X such that the following conditions hold:

   (1) Associativity law: given a morphisms $f \in Hom_{\mathcal{X}}(X, Y)$, $g \in Hom_{\mathcal{X}}(Y, Z), h \in Hom_{\mathcal{X}}(Z, W)$, the following equation

   $$h \circ (g \circ f) = (h \circ g) \circ f$$

   holds;

   (2) Identity law: for all $f \in Hom_{\mathcal{X}}(X, Y)$,

   $$f \circ 1_X = f \text{ and } 1_Y \circ f = f;$$

   (3) The sets $Hom_{\mathcal{X}}(X, Y)$ are pairwise disjoint.

In this work, we will regularly use the notation $f : X \longrightarrow Y$ in place of $f \in Hom_{\mathcal{X}}(X, Y)$.

Example 2.3.1. *(1) Let* **TOP** *be the category of topological spaces: $Ob_{\mathbf{TOP}}$ is the class of all topological spaces, $Hom_{\mathbf{TOP}}(X, Y)$ is the set of continuous maps from X to Y.*

(2) Let **Haus** *be the category of Hausdorff topological spaces: $Ob_{\mathbf{Haus}}$ is the class of Hausdorff topological spaces. $Hom_{\mathbf{Haus}}(X, Y)$ is the set of all continuous maps from X to Y.*

Let $\mathcal{X}$ and $\mathcal{Y}$ be two categories. The category $\mathcal{Y}$ is said to be a **subcategory** of $\mathcal{X}$ if the following conditions are satisfied:

(1) Ob$\mathcal{Y} \subseteq$ Ob$\mathcal{X}$;

(2) $Hom_{\mathcal{Y}}(X, Y) \subseteq Hom_{\mathcal{X}}(X, Y)$ for every $X, Y \in$ Ob$\mathcal{Y}$;

(3) The composition of morphisms in $\mathcal{Y}$ is induced by the composition of morphisms in $\mathcal{X}$;

(4) The identity morphisms in $\mathcal{Y}$ are identity morphisms in $\mathcal{X}$.

Furthermore, if for any pair (X, Y) of objects of $\mathcal{Y}$, we have $Hom_{\mathcal{Y}}(X, Y) = Hom_{\mathcal{X}}(X, Y)$ then $\mathcal{Y}$ is called a full subcategory of the category $\mathcal{X}$.

Considering the Examples (2.3.1), it is clear that the category **Haus** is a full subcategory of the category **TOP**.

Let $\mathcal{X}$ be a given category. The **dual (opposite) category** of $\mathcal{X}$, denoted by $\mathcal{X}^{op}$, is the category obtained in the following way:

(1) Ob$\mathcal{X}^{op}$=Ob$\mathcal{X}$;

(2) $Hom_{\mathcal{X}^{op}}(X, Y)= Hom_{\mathcal{X}}(Y, X)$ (that means the morphisms of $\mathcal{X}^{op}$ are those of $\mathcal{X}$ pointing in the opposite direction);

(3) $Hom_{\mathcal{X}^{op}}(X, Y) \times Hom_{\mathcal{X}^{op}}(Y, Z) \longrightarrow Hom_{\mathcal{X}^{op}}(X, Z)$ is defined as follows:

$$g^{op} \circ f^{op} = (f \circ g)^{op}$$

for all $f^{op} \in Hom_{\mathcal{X}^{op}}(X, Y)$ and $g^{op} \in Hom_{\mathcal{X}^{op}}(Y, X)$;

(4) The identities in $\mathcal{X}^{op}$ are exactly the same as the ones in $\mathcal{X}$.

As we have just observed, a category $\mathcal{X}$ has two types of data, its objects and its morphisms. Thus, a functor a F from category $\mathcal{X}$ to a category $\mathcal{Y}$ will need to have two components, one that acts on objects, and one that operates on morphisms. Thus,

Given any two categories $\mathcal{X}$ and $\mathcal{Y}$, a **covariant functor** (respectively, **contravariant functor)** $F : \mathcal{X} \longrightarrow \mathcal{Y}$ comprises the following data:

(1) A mapping $X \longmapsto FX : Ob\mathcal{X} \longrightarrow Ob\mathcal{Y}$;

(2) for every pair (X, Y) of $\mathcal{X}$-objects, a mapping $f \longmapsto F(f)$: $Hom_{\mathcal{X}}(X, Y) \longrightarrow Hom_{\mathcal{Y}}(FX, FY)$ such that the following axioms hold:

(a) for any $\mathcal{X}$-objects X, $F(1_X) = 1_{FX}$;

(b) for any $f \in Hom_{\mathcal{X}}(X, Y)$, $g \in Hom_{\mathcal{Y}}(Y, Z)$ we have: $F(g \circ f) = F(g) \circ F(f)$ (respectively, $F(g \circ f) = F(f) \circ F(g)$).

It should be clear to the reader that $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a covariant functor if and only if $F : \mathcal{X}^{op} \longrightarrow \mathcal{Y}$ is a contravariant functor.

There are plenty of adjoint situations between two posets/ categories in various branches of mathematics. In the next section, we present the canonical place where this occurs the most and which will be extensively utilized throughout this work.

Pseudofunctor

Let $\mathcal{X}$ be a general category. $P : \mathcal{X} \longrightarrow \mathbf{Pos}$ is a pseudofunctor to the category **Pos** of posets and order preserving maps between them when, to any X in $\mathcal{X}$ it assigns a poset PX and to any morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ it assigns a Galois connection

$$PX \underset{f^*}{\overset{f^o}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} PY$$

that is, for all $a \in PX$ and $b \in PY$, it holds that

$$f^o(a) \leq b \Leftrightarrow a \leq f^*(b).$$

Note: In this thesis, we shall sometimes treat the posets PX merely not as posets but as complete lattices.

The above adjunction has the following direct consequences, which we will need throughout this thesis.

Lemma 2.3.1. *(1) $a \leq f^*(f^o(a))$, for all $a \in PX$;*

(2) $f^o(f^*(b)) \leq b$, for all $b \in PY$;

(3) $f^o(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} f^o(a_i)$ whenever $a_i \in PX$;

(4) $f^*(\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I} f^*(b_i)$ whenever $b_i \in PY$.

Moreover, if, in a certain category, the right adjoint f^* exhibits commutativity with all joins, in view of Proposition (3.3.8), it possesses a right adjoint. We denote it by f_* and it is defined by

$$f_*(a) = \bigvee\{b \in PY \mid f^*(b) \leq a\}$$

for all $a \in PX$. In this instance, we obtain the following alternative diagram:

$$PY \underset{f_*}{\overset{f^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} PX$$

Equivalently,

$$f^*(c) \leq d \Leftrightarrow c \leq f_*(d)$$

for all $c \in PY$ and $d \in PX$. Furthermore, an analogous observation to the statement of Lemma (2.3.1) can also be made in this context.

A more practical and common example of such functor is obtained when the $\mathcal{X} = \mathbf{Set}$ and P taken to be the powerset functor in the following way:

Let X be a set, and consider the powerset PX of X, forming a poset. Let $f : X \longrightarrow Y$ be a function. The functor P establishes the image-preimage adjunction

$$\text{PX} \underset{f^{-1}}{\overset{f^o}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \text{PY}$$

between PX and PY.
In fact, any function $f : X \longrightarrow Y$ between two sets X and Y gives rise to three mappings between PX and PY which we provide below:

$f^o : PX \longrightarrow PY$ which takes $A \subseteq X$ to $f^o(A)$ and $f^{-1} : PY \longrightarrow PX$ which sends $B \subseteq Y$ to its pre-image under f, that is, $f^{-1}(B)$. Then the following diagram

$$PX \underset{f^{-1}}{\overset{f^o}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} PY$$

is a Galois connection between $(PX, \subseteq)$ and $(PY, \subseteq)$.

Further, the fact that the pre-image behaves well with the set operations (union and intersection) ensures that it also has a right adjoint $f_* : PX \longrightarrow PY$. It is given by

$$f_*(A) = \bigcup\{B \subseteq Y \mid f^{-1}(B) \subseteq A\} \tag{2.3.1}$$

In addition to this, the following observation is also clear for any subset A of X

$$\begin{aligned}
Y \setminus f_*(A) &= \bigcap\{Y \setminus B \mid f^{-1}(B) \subseteq A\}; \\
&= \bigcap\{Y \setminus B \mid X \setminus A \subseteq X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)\}; \\
&= \bigcap\{Y \setminus B \mid f^o(X \setminus A) \subseteq Y \setminus B\}; \\
&= f^o(X \setminus A).
\end{aligned}$$

Thus, the expression in (2.3.1) is a equivalent to

$$f_*(A) = Y \setminus f^o(X \setminus A). \tag{2.3.2}$$

Therefore, we obtain another adjunction:

$$PX \underset{f^{-1}}{\overset{f_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} PY$$

which defines a new Galois connection between the posets $(PX, \subseteq)$ and $(PY, \subseteq)$.

Let $\mathcal{X}$ be any category and $P : \mathcal{X} \longrightarrow \mathbf{Pos}$ a pseudofunctor to the category of posets and order preserving maps between. This pseudofunctor P assigns, to any two morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in $\mathcal{X}$, the following successive Galois adjoints

$$\text{PX} \underset{f^*}{\overset{f^o}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \text{PY} \underset{g^*}{\overset{g^o}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \text{PZ}$$

Now, since Galois adjoints are unique, it follows that:

(i) $(g \circ f)^o(a) = g^o(f^o(a))$ for all $a \in X$;

(ii) $(g \circ f)^*(b) = f^*(g^*(b))$ for all $b \in Z$.

Note: Although we have explicitly defined the concept of a pseudofunctor, which will be widely used in this work, we shall often recall its definition whenever clarity requires it.

2.4 Category of Frames and Frame Homomorphisms

One of the categories which will play a significant role in this dissertation is the category **Frm**.

A **frame** L is a complete lattice satisfying the following distributivity law:

$$a \wedge \bigvee C = \bigvee \{a \wedge c \mid c \in C\}$$

for all $a \in L$ and $C \subseteq L$. We will denote by 0_L and 1_L the bottom and top elements of the frame L, respectively. In certain cases, we will omit the subsripts in 0_L and 1_L when clarity does not necessitate it.

A **subframe** M of a frame L is a subset $M \subseteq L$ which is itself a frame under the same operations as L, that is to say, ($\wedge$ and $\bigvee$) as L and both 1_L and 0_L belong to M.

A **frame homomorphism** is a map $h : L \longrightarrow M$ between frames that preserves:

- all joins, that is, $h(\bigvee S) = \bigvee \{h(s) \ : \ s \in S \text{ for any } S \subseteq L\}$; including the **bottom** element $(h(0_L) = 0_M)$;
- finite meet, that is, $h(s \wedge v) = h(s) \wedge h(v)$ for all $s, v \in L$; including the **top** element $(h(1_L) = 1_M)$.

Frames and frame homomorphisms are objects and morphisms of the category **Frm**.

The example motivating the study of frames and frame homomorphisms stems from topology: for every topological space $(X, \Omega X)$, the complete lattice ΩX of open subsets of X is a frame. The infima and suprema ΩX are given by

$$\bigwedge_{i \in I} A_i = int(\bigcap_{i \in I} A_i) \text{ and } \bigvee_{i \in I} A_i = \bigcup_{i \in I} A_i$$

The bottom element of ΩX is $\emptyset$ and the top element is X. A frame constructed in this way is termed **spatial**. It is essential to highlight that not all frames are spatial. For an illustration of frames which are not spatial, we mention the **non-atomic complete Boolean algebras**.

In fact, there exists a **contravariant functorial** relationship between **Top** and **Frm** which we describe below. Given a topological space $(X, \Omega X)$, the lattice of open sets, is a frame and given any continuous map $f : (X, \Omega X) \longrightarrow (Y, \Omega Y)$, the map $\Omega(f) : \Omega Y \longrightarrow \Omega X$ where

$$\Omega f(A) = f^{-1}(A)$$

for all $A \in \Omega Y$, is a frame homomorphism. It turns out that

$$\Omega : \textbf{Top} \longrightarrow \textbf{Frm}$$

is a contravariant functor called the **open functor** from **Top** to **Frm**.

Definition 2.4.1. Let 2 represent the two-point frame $\{0, 1\}$. A point of a frame L is a frame homomorphism $h : L \longrightarrow 2$.

The collection of all the points of the frame L is called the **spectrum** of L. It is denoted by ΣL. For each $a \in L$, put

$$\Sigma_a = \{\nu : L \longrightarrow 2 \mid \nu(a) = 1\}$$

Then, the set $\tau = \{\Sigma_a \mid a \in L\}$ is the **spectral topology** and the pair $(\Sigma L, \tau)$ is a topological space. We shall frequently write ΣL when referring to the topological space $(\Sigma L, \tau)$.

Now, let $h : L \longrightarrow M$ be a frame homomorphism. The mapping

$$\Sigma h : \Sigma M \longrightarrow \Sigma L$$

defined by $(\Sigma h)(\nu) = \nu \circ h$, where ν is a point of M, is a continuous map. This leads again to a contravariant functor

$$\Sigma : \mathbf{Frm} \longrightarrow \mathbf{Top}$$

called **spectrum functor** from **Frm** to **Top**.

We, thus, have the following well-established theorem that permits the transition between the categories **Top** and **Frm** bidirectionally:

Theorem 2.4.2. *The functors* $\Sigma : \mathbf{Frm} \longrightarrow \mathbf{Top}$ *and* $\Omega : \mathbf{Top} \longrightarrow \mathbf{Frm}$ *are adjoint on the right.*

The dual category of **Frm** is $\mathbf{Frm^{op}} = \mathbf{Loc}$, the category of locales and localic maps. In fact, the category **Loc** offers to pointfree topologists an opportunity of reasoning topologically on frames. When they do so, the functors $\Omega : \mathbf{Top} \longrightarrow \mathbf{Frm}$ and $\Sigma : \mathbf{Frm} \longrightarrow \mathbf{Top}$ become covariant.

Among many other examples of frames and frame homomophisms, we mention:

- Every complete Boolean algebra is a frame;
- Every finite distributive lattice is a frame;
- Every lattice homomorphism $h : A \longrightarrow B$ with both A and B finite lattices is a frame homomorphism;
- Every Boolean homomorphism between complete Boolean algebras is a frame homomorphism.

A frame L is said to be **regular** if each $x \in L$ can be written as

$$x = \bigvee\{a \in L \mid a \prec x\}$$

where the notation $a \prec x$ reads a **rather below** x and it is defined by

$$a \prec x \text{ if and only if } a^* \vee x = 1_L$$

Equivalently, $a \prec x$ in case there is an element z, called a separating element, such that

$$x \wedge z = 0_L \text{ and } z \vee a = 1_L.$$

On the other hand, a frame L is called **completely regular** if each $x \in L$ can be written as the join of elements completely below it, that is,

$$x = \bigvee \{a \in L \mid a \prec\prec x\}$$

where the relation $a \prec\prec x$, which reads a **completely below** x, means that there exists a family of elements

$$\{r_i \in L, i \in \mathbb{Q} \cap [0,1]\}$$

such that $r_0 = a$ and $r_1 = x$, $r_i \prec r_j$ whenever $i < j$.

Because of the join-preserving feature, every frame homomorphism $h : L \longrightarrow M$ is always associated with a right adjoint $h_* : M \longrightarrow L$ such that $h(x) \leq y$ holds in M if and only if $x \leq h_*(y)$ holds in L, and for all $x \in L$, $h_*(x)$ is given by

$$h_*(x) = \bigvee \{y \in L \mid h(y) \leq x\}.$$

Hence, the diagram

$$L \underset{h_*}{\overset{h}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} M.$$

is an adjoint pair.

Note that h_* is not, in general, a frame homomorphism, however, it preserves meets. Moreover, we note the following properties of h_* for a given frame homomorphism $h : L \longrightarrow M$:

- h is onto if and only if $h(h_*(x)) = x$ for all $x \in M$;
- h is one-to-one if and only if $h_*(h(y)) = y$ for all $y \in L$;
- because of the adjunction, $h \dashv h_*$, $h(h_*(x)) \leq x$ and $y \leq h_*(h(y))$ for all $y \in M, x \in L$.

The concept of closed and dense frame homomorphisms will play a crucial role in the subsequent discussions. We recall the definition below.

Definition 2.4.3. A frame homomophism $h : L \longrightarrow M$ is said to be:

(1) dense if $h(x) = 0_M$ implies $x = 0_L$, equivalently $h : L \longrightarrow M$ is dense if $h_*(0_M) = 0_L$;

(2) closed if

$$h_*(h(x) \vee y) = x \vee h_*(y)$$

for all $x \in L$ and $y \in M$.

The following well-known lemma, which establishes a connection between dense onto frame homomorphisms, their adjoints, and pseudocomplements, will play a significant role in the subsequent chapters.

Lemma 2.4.4. *Let $h : L \longrightarrow M$ be a dense onto frame homomorphism. Then for all $x \in L, y \in M$, we have that:*

(1) $(h(x))^* = h(x^*)$;

(2) $(h_*(y))^* = h_*(y^*)$.

Proof.

$$(1)\ (h(x))^* = \bigvee\{z \in M \mid z \wedge h(x) = 0\}$$
$$= \bigvee\{h(y) \in M \mid h(y) \wedge h(x) = 0\}, \text{ since } h \text{ is onto}$$
$$= \bigvee\{h(y) \in M \mid h(y \wedge x) = 0\}$$
$$= \bigvee\{h(y) \in M \mid y \wedge x = 0\}, \text{ since } h \text{ is dense}$$
$$= h\big(\bigvee\{y \in L \mid y \wedge x = 0\}\big)$$
$$= h(x^*).$$

$$(2)\ h(h_*(y) \wedge (h_*(y))^*) = h(0) \Rightarrow h(h_*(y)) \wedge h(h_*(y))^* = 0$$
$$\Rightarrow y \wedge h(h_*(y))^* = 0$$
$$\Rightarrow h(h_*(y))^* \leq y^*$$
$$\Rightarrow (h_*(y))^* \leq h_*(y^*)$$

Conversely,

$$h(h_*(y^*) \wedge h_*(y)) = h(h_*(y^*)) \wedge h(h_*(y));$$
$$= y^* \wedge y \ \text{ since } h \text{ is onto;}$$
$$= 0$$

Thus, $h(h_*(y^*) \wedge h_*(y)) = 0$ and since h is dense, $h_*(y^*) \wedge (h_*(y) = 0$ and $h_*(y^*) \leq (h_*(y))^*$.

□

Following [DN10], we shall say that a frame homomorphism $h : L \longrightarrow M$ is **nearly open** if it satisfies $(h(x))^* = h(x^*)$. In other words all dense and onto frame homomorphisms are nearly open.

2.5 Compactness and Compactification

There are several different ways of characterizing compactness of a topological space. For obvious reasons, the one using open covers can be easily generalized to frames. Let A be a subset of L. Then A is called a cover of L if $\bigvee A = 1_L$. $B \subseteq A$ is called a subcover of A if B covers L also. A frame L is said to be **compact** if each of its covers has a finite subcover.

Definition 2.5.1. A dense and onto frame homomorphism $h : M \longrightarrow L$ is called a **compactification** of the frame L if M is a compact and regular frame.

2.6 Ideals

A subset I of a frame L is called an **ideal** if it satisfies the following:

(i) $0_L \in I$;

(ii) $x, y \in I$ implies $x \vee y \in I$; that is; I is closed under finite joins;

(iii) $x \leq y$ and $y \in I$ implies $x \in I$; that is; I is a downset.

Dualizing the axioms of an ideal gives rise to the concept of a **filter** in a frame. However, it is important to note that this concept will not be utilized in the context of this work.

The set of all ideals within the frame L is denote by $\mathcal{J}L$. This is itself a frame ordered by the set inclusion. In fact, it is also a compact frame. As far as downsets are concerned, there is a notation

$$\downarrow A = \{b \in L \mid \text{there exists } a \in A \text{ with } b \leq a\}$$

for $A \subseteq L$. For $x \in L$, the **principal ideal** $\downarrow \{x\}$ is written as $\downarrow x$ and $\downarrow: L \longrightarrow \mathcal{J}L$ is a frame homomorphism which is also right adjoint to the join map $\bigvee : \mathcal{J}L \longrightarrow L$. Moreover, the join map $\bigvee$ is always dense and onto. Thus, the pair $(\bigvee, \mathcal{J}L)$ forms a compactification of the frame L if $\mathcal{J}L$ is regular.

Note: Although the notion of an $(\mathcal{E}, \mathcal{M})$ factorization will be frequently mentioned in early chapters, we ask the reader to bear with us. The notion will only be formally defined in the last chapter as that is where we will need it the most.

Chapter 3

OVERVIEW OF TOPOGENOUS ORDERS ON COMPLETE LATTICES

3.1 Motivation and Settings

As highlighted in the introductory chapter, the concept of syntopogenous structures on a set [Csá63] generalizes that of topological, proximal and uniform structures simultaneously.

Syntopogenous structures are families of topogenous structures on a set. As one of the aims of this work is to define and investigate syntopogenous structures on complete lattices, we dedicate this chapter to developing an understanding of topogenous structures within the context of complete lattices. Let's start by revisiting the definition of topogenous orders on a given set.

Definition 3.1.1. *[Csá63]* Let X be a non-empty set, and let $\triangleleft$ be an order relation on PX, the powerset of X, such that:

(1) $\emptyset \triangleleft \emptyset$ and $X \triangleleft X$;

(2) $A \triangleleft B \Rightarrow A \subseteq B$ for all $A, B \in PX$;

(3) $A \subseteq C \triangleleft B \subseteq D \Rightarrow A \triangleleft D$ for all $A, B, C, D \in PX$;

(4) *(a)* if $A \triangleleft B$ and $C \triangleleft D$ then $A \cap C \triangleleft B \cap D$ for all $A, B, C, D \in PX$;

(b) if $A \triangleleft B$ and $C \triangleleft D$ then $A \cup C \triangleleft B \cup D$ for all $A, B, C, D \in PX$;

(5) $A \triangleleft B \Rightarrow X \setminus B \triangleleft X \setminus A$, for all $A, B \in PX$.

We shall refer to the binary relation $\triangleleft_X$ on a set X as a topogenous order on X if it satisfies (1)-(4). Such a relation is termed perfect topogenous order provided, in addition, (b) holds for arbitrary union, and it is called symmetric if it satisfies (5). Consequently, the pair $(X, \triangleleft_X)$ is called a topogenous space (respectively, perfect topogenous space, symmetric topogenous space) provided that $\triangleleft_X$ is a topogenous order (respectively, perfect topogenous order, symmetric topogenous order) on the set X.

Definition 3.1.2. *[Csá63]* If $(X, \triangleleft_X)$ and $(Y, \triangleleft_Y)$ are topogenous spaces, a function $f : (X, \triangleleft_X) \longrightarrow (Y, \triangleleft_Y)$ is said to be a topogenous map if

$$A \triangleleft_Y B \Rightarrow f^{-1}(A) \triangleleft_X f^{-1}(B) \tag{3.1.1}$$

for all $A, B \subseteq Y$.

If the function $f : (X, \triangleleft_X) \longrightarrow (Y, \triangleleft_Y)$ satisfies (3.1.1), we shall also say that it is topogenously continuous.

We use the symbol **TopG** (respectively, **PTopG**, **STopG**) to denote the categories of topogenous spaces (respectively, perfect topogenous spaces, symmetric topogenous spaces) and topogenously continuous maps between them. In [Chu88], Chung proved the following:

TOP, the category of topological spaces and continuous maps is isomorphic to the category **PTopG**.

Császár motivated the study of topogenous orders on a topological space (X, τ) by introducing the following order:

$$A \triangleleft B \Leftrightarrow A^o \subseteq B$$

for all $A, B \subseteq X$.

Example 3.1.1. *(1) [Fla72]* Let X represent the set of real numbers and ϵ any positive number. The order relation $<_\epsilon$ defined by

$$A <_\epsilon B \Leftrightarrow supA + \epsilon \leq inf(X \setminus B)$$

for all $A, B \subseteq X$, is a topogenous order on X.

(2) Let (X, d) be a metric space and as in the previous example, ϵ any positive number. For all $A, B \subseteq X$, define,

$$A \triangleleft B \Leftrightarrow A^\epsilon \subseteq B$$

where

$$A^\epsilon = \{x \in X \mid d(x, A) \leq \epsilon\}$$

is a topogenous order on (X, d).

In 1968, E. Čech [Čec68] introduced the study of closure operators on a given set X. These operators are maps $k : PX \longrightarrow PX$, where PX represents the power set of X, and they satisfy the following axioms: groundedness, extensiveness and order preservation. Idempotent closure operators which were also introduced in the same paper, were intensively discussed in [Sie34] and in numerous other books. In fact, closure operators have several pleasing applications in various branches of mathematics, including data analysis [SWW98], quantum mechanics [Aer99] and [Pir98]. As mentionned in [Šla22], the set theoretic closure operators may be generalized by considering them on partially ordered sets instead of the Boolean algebra PX. These generalized closure operations which are extensive, order preserving and idempotent have strong relationship with Galois connection and has been extensively studied in categorical topology yielding satisfactory results. For further details, we refer the reader to the book [DT95] of Dikranjan and Tholen.

Historically, the study of closure operator, also interior operator on abstract categories requires the notion of $(\mathcal{E}, \mathcal{M})$-factorization stuctures and that of subobjects. More precisely, a closure (interior) operation on a category $\mathcal{X}$ equipped with $(\mathcal{E}, \mathcal{M})$-factorization stucture is obtained when, for each object X of $\mathcal{X}$, a closure (interior) operator is given on a subobject lattice such that each $\mathcal{X}$-morphism $f : X \longrightarrow Y$ is continuous with respect to closure (interior) operator on subobject lattices of X and Y. Therefore, exploring the behavior of a closure (interior) operator within an abstract category involves studying the properties of closure (interior) operator on the subobject lattice in the relevant category. To substantiate the points made above, the reader is referred to [DT95], and [Vor00] for closure and interior operators, respectively.

Recently, the study of topogenous order on a general category emerged in [HIR16]. In the paper, a topogenous order $\triangleleft$ on a category $\mathcal{X}$ with a well defined $(\mathcal{E}, \mathcal{M})$-factorization stucture for morphisms is defined on a subobject lattice of an object X of $\mathcal{X}$. This definition guarantees that every morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ is topogenously continuous on the subobject lattices of X and Y.

In summary, investigating closure (interior) operator and topogenous orders on a category entails examining the behavior of closure (interior) operator and topogenous orders on the subobject lattices of objects within that category. It is worth emphasizing that, in each case, these subobject lattices are always assumed to be complete. It is clear that the concept of a **complete lattice** plays a central role in the study of topogenous orders, including closure and interior operators, even at the categorical level.

In fact, this is not surprising to us, as indicated by the authors in [HIR16]. They highlighted that the concepts of $(\mathcal{E}, \mathcal{M})$-factorization stuctures and that of subobjects are not essential in the study of topogenous orders on a category.

The existence of a pseudofunctor $F : \mathcal{X} \longrightarrow \mathbf{Pos}$, which maps each object X in $\mathcal{X}$ to a poset FX, and to every morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ an adjoint pair

$$FX \underset{f^*}{\overset{f^o}{\rightleftarrows}} FY \quad (f^o \dashv f^*)$$

is sufficient.

In this present chapter, we aim to define and study topogenous orders on general complete lattices. We extend numerous notions from topological categories to the realm of general complete lattice theory, including closure and interior operators, which yields several classical results from topological category as a special case. We demonstrate that topogenous orders encompass both closure and interior operators within lattices. In this context, the results in [HIR16] appears as a special case. Moreover, if we narrow down our examination of topogenous orders to frames, our results serve as a pointfree counterpart to the findings in [Chu88]. Our methods of proof share similarities with those utilised in [HIR16]. We also characterise the so-called strict maps both in frames and in topology. We conclude the chapter with a number of illustrative examples, highlighting the importance of studying topogenous within complete lattices.

3.2 Closure and Interior Operations on Complete Lattices

We begin this section by presenting some well-known facts about closure and interior operations on complete lattices which are relevant to this chapter and the subsequent ones. While closure and interior operations are dually order isomorphic when defined on the same lattice, we believe it is important to study both notions independently since we will use them separately at times. Regarding closure, we establish a bijective correspondence between closure systems and idempotent closure operations on a lattice. In the case of non-idempotent closure operations, we obtain a Galois connection between closure systems and closure operations on a lattice. We conclude this section with a few illustrative examples.

Definition 3.2.1. Let X be a complete lattice and let $k_X : X \longrightarrow X$ be a mapping on X. Then k is called a closure operation on X if it satisfies the following conditions:

(K1) expansive, that is, $x \leq k_X(x)$ for all $x \in X$;

(K2) order preserving, that is, $x \leq y \Rightarrow k_X(x) \leq k_X(y)$ for all $x, y \in X$.

Moreover, a closure operation on a complete lattice X is said to be:

(K3) idempotent if $k_X(k_X(x)) = k_X(x)$ for all $x \in X$;

(K4) additive if $k(x \vee y) = k(x) \vee k(y)$ for all $x, y \in X$;

(K5) grounded if $k(0_X) = 0_X$.

We represent the collection of all the closure operations on the lattice X by $C(X)$. It is pre-ordered with the following relation: $k \leq k'$ provided $k(y) \leq k'(y)$ for all $y \in X$. The symbol $idC(X)$, $adC(X)$ and $gC(X)$ will be used to denote the collections of all the idempotent, additive and grounded closure operations on the lattice X, respectively.

Given a complete lattice X and an idempotent closure operation k on X, the pair (X, k_X) is called a closure system. It is important not to confuse this with a closure space. A closure space should be understood as a pair (X, k_X) where X is generally a non-ordered set, and k is a closure operation in the usual sense. That is, a closure operation on the Boolean lattice $(P(X), \subseteq)$ satisfying the axioms all of Definition (3.2.1).

Definition 3.2.2. *[Rom08]* Let X be a complete lattice. A subset S of X is called a $\bigwedge$-structure, or closure system if S is closed under arbitrary meet in X, meaning that $A \subseteq S$ implies $\bigwedge A \in S$.

As described in [Rom08], one compelling aspect of $\bigwedge$-structure lies in their ability to characterize closed sets of a closure operator which is a widely used concept in mathematics.

Any subset of a complete lattice X induces a closure operation on X and this closure operation is idempotent if the subset under consideration is closed under meet.

Proposition 3.2.1. *Let X be any complete lattice and $S \subseteq X$. Then for all $x \in X$,*

$$k^S(x) = \bigwedge\{y \in S \mid x \leq y\} \tag{3.2.2}$$

is a closure operation on X. In addition, if S is a $\bigwedge$-structure in X then k^S is idempotent.

Proof. Evidently, k^S is a closure operation on X. To prove k is idempotent, one uses $(K1)$ to show that $k(a) \leq k(k(a))$, on one hand, on the other hand, $k(k(x)) = \bigwedge\{y \in S \mid k(x) \leq y\} \leq k(x)$, since $k(x)$ is defined in terms of meet. Thus, $k(k(x)) = k(x)$.

□

For a complete lattice X, we use the symbol $\mathsf{MS}(X)$ to denote the collection of all the $\bigwedge$-structures on X. Ordered with the set inclusion, $\mathsf{MS}(X)$ can be seen to be a complete lattice. The arbitrary meets and arbitrary joins are given by the set theoretic union and intersection giving suprema and infima. The least element of $\mathsf{MS}(X)$ is the set containing the bottom element of X and the top element is the lattice itself, X.

Now let $\pi : \mathsf{MS}(X) \longrightarrow \mathsf{C}(X)$ be the map defined by $\pi(S) = k^S$ where

$$k^S(a) = \bigwedge\{x \in S \mid a \leq x\}$$

for all $S \in \mathsf{MS}(X)$. A restriction of π on $\mathsf{idC}(X)$ will also be denoted by Υ. Further, we denote by $\xi : \mathsf{C}(X) \longrightarrow \mathsf{MS}(X)$ the map defined by $\xi(k) = S^k$, where

$$S^k = \{x \in X \mid x = k(x)\}$$

for all $k \in \mathsf{C}X$. The restriction of ξ on $\mathsf{idC}(X)$ will also be denoted by ς.

Theorem 3.2.3. *The* $\pi : \mathsf{MS}(X) \longrightarrow \mathsf{C}(X)$ *and* $\xi : \mathsf{C}(X) \longrightarrow \mathsf{MS}(X)$ *are (Galois) joint between* $(\mathsf{MS}(X), \subseteq)^{op}$ *and* $(\mathsf{C}(X), \leq)$ *with* π *to the right and* ξ *to the left.*

Proof. It is easy to check that the maps π and ξ are order preserving. Let X be a complete lattice and S a closure system. By Proposition (3.2.1), k^S is an idempotent closure operation on X. Further, S^k is a meet structure: if $A \subseteq S^k$ then $\bigwedge A \leq a$ for all a $\in A$, and since k preserves order, we obtain $k(\bigwedge A) \leq k(a) = a$ for all $a \in A$. In particular, $k(\bigwedge A) \leq \bigwedge A$. Also, by $(K1)$, $A \leq k(\bigwedge A)$. Thus $\bigwedge A = k(\bigwedge A)$ and so $\bigwedge A \in S^k$.

Let $S \in \mathsf{MS}(X)$. If $a \in S$ then $a \in \{x \in S \mid a \leq x\}$ which implies that $k^S(a) \leq a$ and so $k^S(a) = a$ by $(K1)$ and $a \in S^{k^S}$. Conversely, $a \in S^{k^S}$ implies $a = \bigwedge\{x \in S \mid a \leq x\} \in S$, since S is a meet structure. Hence $\xi(\pi(S)) = S^{k^S} = S$, on the other hand,

$$\begin{aligned} k^{S^k}(a) &= \bigwedge\{x \in S^k \mid a \leq x\}; \\ &= \bigwedge\{x \in X \mid a \leq x = k(x)\}. \end{aligned}$$

But if $x = k(x)$ then $a \leq x \Rightarrow k(a) \leq k(x) = x$ and $k(a) \leq x \Rightarrow a \leq k(a) \leq x$. Thus, $a \leq x \Leftrightarrow k(a) \leq x$, so $k^{S^k}(a) = \bigwedge\{x \in S^k \mid k(a) \leq x\}$. This shows that, in general, $a \leq k(a) \leq k^{S^k}(a)$, that is, $k \leq \pi(\xi(k))$.

□

Corollary 3.2.4. *There is a bijective correspondence between* $\mathsf{MS}(X)$ *and* $\mathsf{idC}(X)$.

Proof. By Theorem (3.2.3), the mappings Υ and ς restrict to $\varsigma : \mathsf{MS}(X) \longrightarrow \mathsf{idC}(X)$ and $\Upsilon : \mathsf{idC}(X) \longrightarrow \mathsf{MS}(X)$. By the same theorem, $k \leq \varsigma(\Upsilon(k))$. On the other hand, if k is idempotent, then $k(a) = k(k(a))$ means that $k(a) \in S^k$ and $k(a) \in \{x \in S^k \mid a \leq x\}$ thus $k^{S^k}(a) \leq k(a)$ and $\varsigma(\Upsilon(k)) \leq k$. Therefore $k^{S^k} = k$

□

The following example provides more insight into the connection between closure and closure systems on a lattice.

Example 3.2.1. The lattice $X = \mathbb{N} \cup \{\infty\}$ is complete with $n \leq \infty$ for all $n \in \mathbb{N}$. Define $k(n) = n + 1$. Note $\infty + 1 = \infty$. Then $k(n)$ is a closure operation on X, since $n \leq k(n)$ and $n \leq m \Rightarrow k(n) \leq k(m)$. $S^k = \{n \in X \mid n = n+1\} = \{\infty\}$ and $k^{S^k}(a) = \bigwedge\{n \in S^k \mid a \leq n\} = \infty$ for all $a \in X$.

For the rest of the section, we provide a number of examples on closure operations on lattices.

(1) There is one well-known additive and idempotent closure operation on a frame called nucleus. Let L be a frame. A nucleus on L is a mapping $j : L \longrightarrow L$ such that:

   (i) $a \leq j(a)$;
   (ii) *if* $a \leq b \Rightarrow j(a) \leq j(b)$;
   (iii) $j(a \wedge b) = j(a) \wedge (b)$;
   (iv) $j(a) = j(j(a))$.

   Definition 3.2.5. Let L be a frame. An element s in L is called a prime element if for all $a, b \in L$, $a \wedge b \leq s$ implies $a \leq s$ or $b \leq s$.

(2) The following lemma provides an other example of a closure operation on a lattice.

   Lemma 3.2.6. *Let L be a frame. For all $a \in L$, put*

$$k_L(a) = \bigwedge\{s \in L \mid s \text{ is prime and } a \leq s\}$$

   Then $k_L(a)$ is an additive and idempotent closure operation on L.

   Proof. $(K1)$ is clear. For $(K2)$, if $a \leq b$ then $\{s \in L \mid s$ is a prime and $a \leq s\} \subseteq \{s \in L \mid s$ is prime and $b \leq s\}$, thus in particular, $\bigwedge\{s \in L \mid s$ is a prime and $a \leq s\} \leq \bigwedge\{s \in L \mid s$ is prime and $b \leq s\}$ and this gives $k(a) \leq k(b)$. Further, let $a, b \in L$, $k(a \wedge b) = \bigwedge\{s \in L \mid s$ is a prime and $a \wedge b \leq s\}$=$\bigwedge\{s \in L \mid s$ a is prime and $a \leq s$ or $b \leq s\}$=$k(a \wedge b)$, that is k is additive. Lastly, for idempotency, one easily checks that $k(k(a)) = \bigwedge\{s \in L \mid s$ is a prime and $k(a) \leq s\} \leq k(a)$ and that by $(K1)$, $k(x) \leq k(k(x))$ always holds. Thus, $k(x) = k(k(x))$.

Definition 3.2.7. Let X be a complete lattice and let $i_X : X \longrightarrow X$ be a mapping on X. Then i_X is called an interior operation on X if it satisfies the following axioms:

(I1) contractive $i_X(x) \leq x$ for all $x \in X$;

(I2) order preserving $x \leq y \Rightarrow i_X(x) \leq i_X(y)$, for all $x, y \in X$;

In addition, an interior operation on a complete lattice X is classified as:

(I3) idempotent if $i_X(i_X(x)) = i_X(x)$ for all $x \in X$;

(I4) additive if $i(x \vee y) = i(x) \vee i(y)$ for all $x \in X$;

(I5) grounded if $i(1_X) = 1_X$.

The set comprising all the interior operations on a lattice X will be denoted by the symbol $INT(X)$. It is pre-ordered with the following relation: $i \leq i'$ if and only if $i(x) \leq i'(x)$ for all $x \in X$. Additionally, we utilize the symbols $idINT(X)$, $adINT(X)$ and $gINT(X)$ to represent the collections of all the idempotent, additive and grounded interior operations on the lattice X, respectively.

Proposition 3.2.2. *Let X be a complete lattice and N a subset of X. Then the function i_X^N, which associates to each $x \in X$,*

$$i_X^N(x) = \bigvee\{y \in N \mid y \leq x\}$$

is an interior operation on the lattice X. Furthermore, if N satisfies the condition of being closed under arbitrary joins ($A \subseteq N$ implies $\bigvee A \in N$), then i^N is idempotent.

Proof. It is straight forward to see that i_X^N is an interior operation on X. For idempotency, $i_X^N(i_X^N(x)) = \bigvee\{p \in N \mid p \leq i_X^N(x)\} \geq i_X^N(x)$, since N is join preserving, and by, $(I1)$, $i_X^N(i_X^N(x)) \leq i_X^N(x)$.

□

We are now ready to introduce the central concept of this chapter.

3.3 Topogenous Orders on Complete Lattices

We define and study topogenous orders on complete lattices. We explore their fundamental properties. One of the key insights in this section is that topogenous orders respecting arbitrary meets correspond nicely to closure, and the topogenous orders which respect arbitrary joins to interior operations. The findings in this section encompass the work of Holgate et al. as documented in [HIR16]. The examples we furnish at the end of the chapter substantiate this assertion. Furthermore, when we extend our consideration to frames rather than just complete lattices, our results offer a pointfree counterpart to the work of Chung in [Chu88], where the study focused on topogenous spaces.

Definition 3.3.1. Let X be a complete lattice. A topogenous order on X is an order relation $\lhd_X$ on X fulfiling the following axioms:

(T1) $x \lhd_X y \Rightarrow x \leq y$ for all $x, y \in X$;

(T2) $x \leq w \triangleleft_X z \leq y \Rightarrow x \triangleleft_X y$, for all $x, y, z, w \in X$.

The set of all topogenous orders on the lattice X is denoted by $TORD(X)$, and it is preordered in the following manner: $\triangleleft \subseteq \triangleleft^{'}$ if and only if $(x \triangleleft y \Rightarrow x \triangleleft^{'} y)$ for all $x, y \in X$. Clearly $\subseteq$ is a partial order on $TORD(X)$.

Furthermore, we shall say that a topogenous order on a lattice X respects meets if for all $x, y, a, b \in L$, $x \triangleleft a$ and $y \triangleleft b$ implies $x \wedge y \triangleleft a \wedge b$.

One of the reasons why we find it interesting to work with topogenous orders is that they subsume both closure and interior operators.

3.3.1 Topogenous Orders Which Respect Joins or Interior Operation on Complete Lattices

We show that the join respecting topogenous orders correspond nicely to interior operations on a complete lattice. As a consequence of this, the idempotent interior operations are precisely the interpolative topogenous orders.

(T3) Let X be a given complete lattice and $\triangleleft_X$ a topogenous order on X. For all $S \subseteq X$, if $s \triangleleft_X y$ for all $s \in S$ then $\bigvee S \triangleleft_X y$.

(T4) $x \triangleleft_X y$ implies there exists z in X such that $x \triangleleft_X z \triangleleft_X y$.

Considering the formulas $(T3)$ and $(T4)$, we obtain two families of topogenous orders and accordingly two types of subcategories of $TORD(X)$:

- $\bigvee -TORD(X)$ the class of all topogenous orders which respect join.
- $INTORD(X)$ the class of all topogenous orders which interpolate.

Lemma 3.3.2. *Let X be a complete lattice and $\triangleleft \in TORD(X)$. The assignment $i = \{i_X : X \longrightarrow X\}$ given by*

$$i^{\triangleleft}_X(x) = \bigvee \{y \in X \mid y \triangleleft x\} \tag{3.3.3}$$

for all $x \in X$ defines an interior operation on X.

Proposition 3.3.1. *$\bigvee -TORD(X)$ is order isomorphic to $INT(X)$ with the inverse assignments defined by*

$$i^{\triangleleft}_X(x) = \bigvee \{y \mid y \triangleleft x\} \text{ and } x \triangleleft^i_X y \Leftrightarrow x \leq i(y)$$

for all $x, y \in X$.

Proof. $(I1)$ and $(I2)$ follow from Lemma (**??**). On the other hand, $(T1)$ follows from $(I1)$ and $(T2)$ from $(I2)$. The maps $i \longrightarrow \triangleleft^i$ and $\triangleleft \longrightarrow i^{\triangleleft}$ are order preserving: indeed take $\triangleleft, \triangleleft' \in \bigvee$-TORD$(X)$ such that $\triangleleft \subseteq \triangleleft'$. Then $\{z \in X \mid z \triangleleft x\} \subseteq \{z \in X \mid z \triangleleft' y\}$. In particular, $\bigvee\{z \in X \mid z \triangleleft x\} \leq \bigvee\{z \in X \mid z \triangleleft' y\}$. Hence $i^{\triangleleft}_X(x) \leq i^{\triangleleft'}_X(x)$ for all $x \in X$. For the converse, if $i, i' \in$ INT(X) and $i \leq i'$ then $x \leq i(a) \Rightarrow x \leq i_X(a) \leq i'_X(a) \Rightarrow x \leq i'_X(a)$ showing that $\triangleleft^i \subseteq \triangleleft^{i'}$. Lastly, $x \triangleleft^{i^{\triangleleft}} y$ is equivalent to $x \triangleleft y$. On one hand, if $i \in INT(X)$, and $a \in X$, then $i^{\triangleleft^i}(a) = \bigvee\{b \mid b \triangleleft^i_X a\} = \bigvee\{b \mid b \leq i(a)\} = i(a)$. On the other hand, if $a \triangleleft^{i^{\triangleleft}}_X b$ then $a \leq i^{\triangleleft}_X(b) = \bigvee\{t \mid t \triangleleft b\} \Rightarrow a \triangleleft b$, where the implication follows since $\triangleleft$ preserves arbitrary joins.

$\square$

We denote by $\bigvee$-INTORD(X) the collection of all topogenous orders that respect arbitrary joins and are interpolative.

Corollary 3.3.3. *$\bigvee$-INTORD(X)=IdINT(X)*

Proof. Let $i \in INT(X)$ and $x \in X$. If $i(x) = i(i(x))$, then

$$\begin{aligned} y \lhd^i x \quad &\Rightarrow \quad y \leq i(x) = i(i(x)) \\ &\Rightarrow \quad y \leq i(i(x)) \leq i(x) \\ &\Rightarrow \quad y \leq i(z) \leq i(x);\ \text{where } z = i(x) \\ &\Rightarrow \quad y \lhd^i z \lhd^i x \end{aligned}$$

Therefore $\lhd^i$ interpolates. If $\lhd$ interpolates, then $x \lhd y \Rightarrow\ \exists\, z \in X$ such that $x \lhd z \lhd y \Rightarrow x \lhd z \leq i^{\lhd}(y) \Rightarrow x \lhd i^{\lhd}(y)$; Thus, $\{b \in X \mid b \lhd y\} \subseteq \{a \in X \mid a \lhd i^{\lhd}(y)\}$ implying that $i^{\lhd}_X(y) = \bigvee\{b \in X \mid b \lhd y\} \leq \bigvee\{a \in X \mid a \lhd i^{\lhd}(y)\} = i^{\lhd}_X(i^{\lhd}_X(y))$, and this shows that $i^{\lhd}$ is idempotent.

$\square$

Definition 3.3.4. Let $\lhd, \lhd' \in TORD(X)$. The topogenous order $\lhd_X \circ \lhd'_X$ defined by

$$x \lhd_X \circ \lhd'_X y \ \text{ if } \exists\ z \in X :\ x \lhd_X z \lhd'_X y$$

for all $x, y, z \in X$ is called the composition of topogenous orders. Clearly, $\lhd$ interpolates if $\lhd \circ \lhd = \lhd$.

Proposition 3.3.2. *Let $\lhd, \lhd' \in \bigvee$-INTORD(X) and $x, y \in X$. Then $x \lhd_X \circ \lhd'_X y \Leftrightarrow x \leq i^{\lhd}_X(i^{\lhd'}_X(y))$.*

Proof. On one hand, if $\lhd, \lhd' \in \bigvee$-INTORD(X) and $x \lhd_X \circ \lhd'_X y$ for all $x, y \in X$. Then there exists $z \in X$ such that $x \lhd_X z \lhd'_X y$ and since both $\lhd, \lhd'$ preserve joins, we get $x \leq i^{\lhd}_L(z)$ and $z \leq i^{\lhd'}_L(y)$. Hence $x \leq i^{\lhd}_X(i^{\lhd'}_X(y))$.

Conversely, let $x \leq i^{\lhd}_X(i^{\lhd'}_X(y))$ then by setting $z = \lhd'_X(y)$, we obtain $x \lhd_X i^{\lhd}(i^{\lhd'}_X(y))$ which is equivalent to $x \lhd_X \circ \lhd'_X y$. This completes the proof.

$\square$

Confining our examination to frames, in the next lemma, we prove that the interior operation defined in (3.3.3) is a frame homomorphism. This result will be instrumental in establishing a connection between the interior operation and symmetric topogenous orders.

Lemma 3.3.5. *Let X be a frame and $\lhd$ a meet respecting topogenous order on X such that $0_X \lhd\ 0_X$ and $1_X \lhd\ 1_X$. The interior operation defined in (3.3.3) is a frame homomorphism provided the interior operation in question is additive.*

Proof. Clearly, $i^{\triangleleft}$ preserves both the top and the bottom elements, that is, $i^{\triangleleft}_X(0_X) = 0_X$ and $i^{\triangleleft}_X(1_X) = 1_X$. To conclude the proof, we only need to show that $i^{\triangleright}$ preserves finite meets, that is, we need prove that $i^{\triangleleft}_X(a \wedge b) = i^{\triangleleft}_X(a) \wedge i^{\triangleleft}_X(b)$ for all $a, b \in X$. Since for any interior operation

$$i^{\triangleleft}_L(a \wedge b) \leq i^{\triangleleft}_X(a) \wedge i^{\triangleright}_X(b)$$

always holds, it suffices to show the other inequality. To this end and for $a, b, z \in X$, let

$$z \leq i^{\triangleleft}_X(a) \wedge i^{\triangleleft}_X(b).$$

Then $z \leq i^{\triangleleft}_X(a)$ and $z \leq i^{\triangleleft}_X(b)$. Thus, by definition of $i^{\triangleleft}$, we get

$$z \leq \bigvee\{x \mid\ x \triangleleft a\} \text{ and } z \leq \bigvee\{y \mid\ y \triangleleft b\}.$$

It follows, from the last two inequalities, that $z \leq \bigvee\{x \mid\ x \triangleleft a\} \wedge \bigvee\{y \mid\ y \triangleleft b\}$. Hence, putting $t = \bigvee\{x \mid\ x \triangleleft a\}$, we obtain

$$\begin{aligned} z \leq t \wedge \bigvee\{y \mid y \triangleleft b\} \quad &\Rightarrow \quad z \leq \bigvee\{t \wedge y \mid y \triangleleft b\}; \\ &\Rightarrow \quad z \leq \bigvee\{y \wedge \bigvee\{x \mid x \triangleleft a\} \mid y \triangleleft b\}; \\ &\Rightarrow \quad z \leq \bigvee\{\bigvee\{x \wedge y \mid x \triangleleft a\} \mid y \triangleleft b\}; \\ &\Rightarrow \quad z \leq \bigvee\{x \wedge y \mid (x \triangleleft a) \text{ and } (y \triangleleft b)\}; \\ &\Rightarrow \quad z \leq \bigvee\{g \in X \mid g \triangleleft (a \wedge b)\}; \\ &\Rightarrow \quad z \leq i^{\triangleleft}_X(a \wedge b). \end{aligned}$$

This means that $i^{\triangleleft}_X(a) \wedge i^{\triangleleft}_X(b) \leq i^{\triangleleft}_X(a \wedge b)$ and together with the other inequality, we have shown that $i^{\triangleleft}$ is a frame homomorphism.

□

Definition 3.3.6. Let X be a frame and $\triangleleft$ a topogenous order on X. We shall say that $\triangleleft$ is symmetric if for all $a, b \in X$, $x \triangleleft y \Rightarrow y^* \triangleleft x^*$. Where $(*)$ stands for pseudocomplement.

The next proposition relates interior operation (join respecting topogenous orders) and symmetric topogenous orders on the frame X.

Proposition 3.3.3. *Let X be a frame, i an interior operation on X. If i preserves pseudocomplement, then $x \triangleleft^i y \Rightarrow y^* \triangleleft^i x^*$ for all $x, y \in X$.*

Proof. Let i be an interior operation on X. Then for $x, y \in X$, we have

$$\begin{aligned} x \triangleleft^i y \quad &\Rightarrow \quad i(x) \leq x \leq i(y) \leq y; \\ &\Rightarrow \quad y^* \leq i(y)^* \leq x^* \leq i(x)^*; \\ &\Rightarrow \quad y^* \leq i(x^*) \ \text{ since } i \text{ preserves } ()^*; \\ &\Rightarrow \quad y^* \triangleleft^i x^*. \end{aligned}$$

□

3.3.2 Topogenous Orders Which Respect Meets or Closure Operation on Complete Lattices

We simultaneously examine topogenous orders and closure operations on a complete lattice. We prove that topogenous orders which respect meets are order isomorphic to closure operations, and hence the idempotent closure operations are the interpolative ones.

(T5) For all $A \subseteq X$, if $x \lhd_X a$ for all $a \in A$ then $x \lhd_X \bigwedge A$.

In fact, $(T5)$ defines a new type of subcategory of TORD(X). It is denoted by $\bigwedge$-TORD(X) and it is the class of all topogenous orders which respect arbitrary meet on X. Applying analogous reasoning to Lemma (3.3.2) and Proposition (3.3.1) leads to the following conclusions:

Lemma 3.3.7. *Let X be a complete lattice and $\lhd \in$ TORD(X). The assignment $k_X^{\lhd} : X \longrightarrow X$ given by*

$$k_X^{\lhd}(x) = \bigwedge\{y \in X \mid x \lhd y\}$$

is a closure operation on X.

Proposition 3.3.4. *$\bigwedge$-TORD(X) is order isomorphic to $C(X)$ with the inverse correspondences defined by*

$$k^{\lhd}(x) = \bigwedge\{y \mid x \lhd y\} \text{ and } x \lhd^k y \Leftrightarrow k_X(x) \leq y$$

for all $x \in X$.

Denoting by $\bigwedge$-INTORD(X) the collection of all interpolative topogenous orders which preserve meet, we also obtain the following corollary,

Corollary 3.3.8. *$\bigwedge$-INTORD(X) is equivalent to IdC(X)*

which is also not hard to prove.

Definition 3.3.9. Let X be a complete lattice. An element $x \in X$ is said to be $\lhd$-strict if $x \lhd x$.

It is clear that if $\lhd \in \bigvee -TORD(X)$, then the $\lhd$-strict elements are equivalent to the open ones in a lattice while if $\lhd \in \bigwedge -TORD(X)$, then the $\lhd$-strict elements are equivalent to the closed elements.

Definition 3.3.10. *(1)* $\bigwedge$-$aTORD(X)$: The collection of all topogenous orders in $\bigwedge$-$TORD(X)$ satisfying

$$x \lhd y \text{ and } a \lhd b \Rightarrow x \vee a \lhd y \vee b$$

for all $x, y, a, b \in X$.

(2) $\bigwedge$-$gTORD(X)$ the collection of all topogenous orders in $\bigwedge$-$TORD(X)$ satisfying

$$0_X \lhd 0_X$$

Proposition 3.3.5. *Let X be a complete lattice. The following equivalences hold:*

(1) *$\bigwedge$-gTORD(X) $\cong$ gC(X)*

(2) $\bigwedge$*-aTORD(X)* $\cong$ *adC(X)*

Lemma 3.3.11. *Let* X *be a complete lattice and* $\{\lhd_i,\ i \in I\} \subseteq TORD(X)$. *Define* $\mathcal{L}$ *by*

$$\mathcal{L} = \bigcup \{\lhd_i,\ i \in I\} \tag{3.3.4}$$

Then $\mathcal{L}$ *is a topogenous order on* X. *Furthermore,* $\mathcal{L}$ *satisfies* $(T3)$ *and* $(T5)$ *provided each* $\lhd$ *does.*

Similarly

Lemma 3.3.12. *Let* X *be a complete lattice and* $\{\lhd^i_X,\ i \in I\} \subseteq TORD(X)$. *Define* $\mathcal{K}$ *by*

$$\mathcal{K} = \bigcap \{\lhd_i,\ i \in I\}$$

Then $\mathcal{K}$ *is a topogenous order on* X. *In addition,* $\mathcal{K}$ *satisfies* $(T3)$ *and* $(T5)$ *provided each* $\lhd$ *does.*

We recall that $INT(X)$, $C(X)$ and $TORD(X)$ are pointwise ordered collections of all the interior operations, closure operations and topogenous orders on the complete lattice X, respectively.

The following schematic representation provides a summary of our discussions above. It gives a more general description of the diagram in ([HIR16]) obtain for on a general category.

$$\bigwedge -TORD(X) \cong^{op} \mathrm{C(X)} \rightleftarrows \mathrm{TORD(X)} \rightleftarrows \mathrm{INT(X)} \cong \bigvee -TORD(X)$$

3.3.3 Topogenously Continuous Maps

We extend the well known notion of continuous maps from classical topogenous orders (in the sense of Császár) to our more general settings. Having shown that topogenous orders comprise both closure and interior operators, it is natural to first transfer the concept of a continuous map from the classical closure and interior operators to the general setting we provided.

In order to achieve our goal, throughout the remainder of this section, we shall consider a general category $\mathcal{X}$ and a pseudofunctor P described in the introductory chapter:

Let $\mathcal{X}$ be a general category and $P : \mathcal{X} \longrightarrow$ **Pos** a pseudofunctor to the category of partially ordered sets and order preserving maps between them which to any morphism $f : X \longrightarrow Y$ in $\mathcal{X}$, we have a Galois connection

$$PX \underset{f^*}{\overset{f^o}{\rightleftarrows}} PY \quad (\perp)$$

k-Continuous Morphism

Definition 3.3.13. Let $k = \{k_{PX} : PX \longrightarrow PX\}, X \in \mathcal{X}$ be a family of endomaps on each $PX,\ X \in \mathcal{X}$. An $\mathcal{X}$-morphism $f : X \longrightarrow Y$ is said to be k-continuous if

$$f^o(k_X(x)) \leq k_Y(f^o(x)) \tag{3.3.5}$$

for every $x \in PX$.

Definition 3.3.14. A family $k = \{k_{PX} : PX \longrightarrow PX\}_{X\in\mathcal{X}}$ is said to be a closure operator on $\mathcal{X}$ if each k_{PX} is a closure operation on PX and every $\mathcal{X}$-morphism is k-continuous.

Thanks to the Galois connection, the above continuity condition can also be equivalently expressed in terms of the right adjoint as shown below.

Proposition 3.3.6. *Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{X}$ and k a closure operator on $\mathcal{X}$. Then f is k-continuous if and only if:*

$$k_X(f^*(y)) \leq f^*(k_Y(y)) \tag{3.3.6}$$

for each $y \in PY$.

Proof. Let $f : X \longrightarrow Y$ be k-continuous. Then for each $y \in PY$, we have that $f^o(k_X(f^*(y)) \leq k_Y(f^o(f^*y)) \leq k_Y(y)$. It follows that $f^o(k_X(f^*(y)) \leq k_Y(y)$ and $k_X(f^*(y)) \leq f^*(k_Y(y))$ as required.

On the other hand, assume (3.3.6) holds, then for each $x \in P(X)$, $x \leq f^*(f^o(x))$ implies that $k_X(x) \leq k_X(f^*(f^o(x))) \leq f^*(k_X(f^o(x)))$ giving $k_X(x) \leq f^*(k_X(f^o(x)))$ and $f^o(k_X(x)) \leq k_Y(f^o(x))$.

□

Let k be a closure operator on $\mathcal{X}$. An element $x \in X$ is said to be closed (with respect to k) if $k(x) = k$.

Moreover, if one replaces the inequalities in (3.3.5) and in (3.3.6) by equalities, the well know notion of k-closed and k-open morphisms are obtained. These maps have been widely studied within the context of categories equipped with $(\mathcal{E}, \mathcal{M})$-factorization structure and important results have been obtained both in algebra and in topology. See [DT95] for reference.

Definition 3.3.15. Let k be any closure operator on the category $\mathcal{X}$. A morphism $f : X \longrightarrow Y$ is called k-closed if:

$$f^o(k_X(x) \cong k_Y(f^o(x));$$

for all $x \in PX$.

k-open if

$$k_X(f^*(y)) \cong f^*(k_Y(y))$$

for all $y \in PY$.

Remark 3.3.16. *Let $f : X \longrightarrow Y$ be any morphism in $\mathcal{X}$, $x \in PX$ and $y \in PY$. The following equivalent formulation of k-closed and k-open will be needed. f is said to be:*
k-closed if and only if

$$k_Y(f^o(x)) \leq f^o(k_X(x))$$

for all $x \in PX$.
k-open if and only if

$$f^*(k_Y(y)) \leq k_X(f^*(y))$$

for all $y \in PY$.

i-Continuous Morphism

Definition 3.3.17. Let $i = \{i_{PX} : PX \longrightarrow PX\}$ be a family of endomaps on each PX, $X \in \mathcal{X}$. An $\mathcal{X}$-morphism $f : X \longrightarrow Y$ is said to be i-continuous if

$$f^*(i_Y(y)) \leq i_X(f^*(y)) \tag{3.3.7}$$

for every $y \in PY$.

Definition 3.3.18. A family $i = \{i_{PX} : PX \longrightarrow PX\}_{X \in \mathcal{X}}$ is said to be an interior operator on $\mathcal{X}$ if each i_{PX} is an interior operation on PX and every $\mathcal{X}$-morphism is i-continuous.

As in the case of closure operator, interior operator can be expressed in terms of the right adjoint (right to f^*).

Proposition 3.3.7. *Let $f : X \longrightarrow Y$ be any morphism in the category $\mathcal{X}$ and i an interior operator on $\mathcal{X}$. If f^* preserves all joins, then f is i-continuous if and only if*

$$i_Y(f_*(x)) \leq f_*(i_X(x)) \tag{3.3.8}$$

for all $x \in PX$

Proof. If $f^*(i_Y(y)) \leq i_X(f^*(y))$ holds for all $y \in PY$ then $f^*(i_Y(f_*(x)) \leq i_X(f^*(f_*(x))$ for all $x \in PX$. Since f^* is adjoint to the left, we obtain $i_X(f^*(f_*(x)) \leq i_X(x))$ which gives $f^*(i_Y(f_*(x)) \leq i_X(x)$ and $i_Y(f_*(x)) \leq f_*(i_X(x))$ as needed.

For the converse, if $i_Y(f_*(x)) \leq f_*(i_X(x))$ holds for all $x \in PX$ then $f_*(i_X(f^*(y)) \geq i_Y(f_*(f^*(y))$. Since f_* is adjoint to the right, it follows that $i_Y(f_*(f^*(y)) \geq i_Y(y)$ and $f_*(i_X(f^*(y) \geq i_Y(y)$. Thus $f^*(i_Y(y)) \leq i_X(f^*(y))$, that is, f is i-continuous.

□

Like in the case of closure operator, by replacing the inequalities in (3.3.7) and (3.3.8) by equalities the so-called i-open and i-closed morphisms are obtained. These morphisms are well defined and examined in [Ass19] within the context of categories equipped with $(\mathcal{E}, \mathcal{M})$-factorization structure. See also references cited for further exploration.

Definition 3.3.19. Let $f : X \longrightarrow Y$ be any morphism in $\mathcal{X}$ and i an interior operator on $\mathcal{X}$. Then f is said to be: i-open if

$$f^*(i_Y(y) \cong i_X(f^*(y))$$

for every $y \in PY$.

Moreover, if f^* preserves all joins, then f is i-closed if

$$i_Y(f_*(x)) \cong f_*(i_X(x))$$

for all $x \in PX$

Remark 3.3.20. *Let $f : X \longrightarrow Y$ be any morphism in $\mathcal{X}$, $x \in PX$, $y \in PY$. If f^* preserves all joins, we also have the following equivalent formulation to the i-open (i-closed) property:*

(a) f is i-open if and only if

$$i_X(f^*(y)) \leq f^*(i_Y(y))$$

for all $y \in PY$

(b) f is i-closed if and only if

$$f_*(i_X(x)) \leq i_Y(f_*(x))$$

for all $x \in PX$

We provide a few examples of interior operators. We only choose those which cannot be directly translated to closure operators via complementation. Details about many more examples on interior operators can be found in [Ass19] and references therein.

Note: For consistency with the upcoming sections, we use h for frame homomorphisms, even though we could have used f^*.

Example 3.3.1. Let $\mathcal{X}$=**Frm** be the category of frames and frame homomorphisms. For each $L \in$ **Frm**, let $\prec$ symbolize the rather below relation. The map $i_L^{\prec} : L \longrightarrow L$ given by

$$i_L^{\prec}(x) = \bigvee\{b \mid b \prec x\}$$

defines an interior operation on L associated with the rather below relation. Futhermore, since every frame homomorphism $h : L \longrightarrow M$ preserve $\prec$, it follows that

$$h(i_L(x)) \leq i_L(h(x))$$

for all $x \in L$. Thus, the family

$$i = \{i_L^{\prec} \mid L \in \mathbf{Frm}\} \tag{3.3.9}$$

is an interior operator on **Frm** associated with the rather below. Note that if L is a regular frame then the interior operator defined in (3.3.9) is the discrete operator. Moreover, it should also be noted that one could replace $\prec$ with the completely below relation ($\prec\!\prec$) and obtain analogous results.

Although the notion of pre-uniformity is utilized in the next example, it will be formally discussed in Chapter 5.

Example 3.3.2. *[BO94, BP93]* Given a frame L and $\mathcal{U}$ a preuniformity on L. The map $i_L : L \longrightarrow L$ given by

$$i_L^{\triangleleft}(a) = \bigvee\{x \in L \mid x \triangleleft a\}$$

where $x \triangleleft a$ means $Ax \leq a$ for some $A \in \mathcal{U}$ is an interior operation on L induced by the preuniformity $\mathcal{U}$. Furthermore if $h : L \longrightarrow M$ is a uniform homomorphism between pre-uniform frames, we also have that

$$h(i_L^{\triangleleft}(x)) \leq i_M^{\triangleleft}(h(x))$$

Thus, the family $i = \{i_L \mid L \in \mathbf{PUniFrm}\}$ is an interior operator on pre-uniform frames. Note that **PUniFrm** stands for the category of pre-uniform frame and uniform frame homomorphisms.

The following proposition establishes a nice connection between onto frame homomorphisms, their right adjoints, and the interior operator on frames.

Proposition 3.3.8. *Let L and M be not necessarily regular frames and $h : M \longrightarrow L$ an onto frame homomorphism. Further, let $r : L \longrightarrow M$ be the right adjoint of h. Then, for all $x \in L$,*

$$i_L(x) = h(i_M(r(x)))$$

defines an interior operation on L for which the frame homomorphism $h : M \longrightarrow L$ is continuous. Furthermore, i_L is additive and idempotent, provided i_M satisfies the same properties.

Proof. *(1)* *(I1)* $i_L(x) = h(i_M(r(x))) \leq h((r(x))) = x$, since h is onto. Thus $i_L(x) \leq x$ for all $x \in L$.

(I2) For all $x, y \in L$, if $x \leq y$ giving $r(x) \leq r(y)$ then $i_M((r(x))) \leq i_M((r(y)))$. Applying h both sides, we obtain $h(i_M((r(x)))) \leq h(i_M((r(y))))$. Hence $i_L(x) \leq i_L(y)$.

Lastly, since r is a right adjoint, $x \leq r(h(x))$ then $h(i_M(x)) \leq h(i_M(r(h(x)))) = (h(i_M(r)))h(x) = i_L(h(x))$. Therefore h is i-continuous.

(2) If i_M is additive, then

$$\begin{aligned} i_L(x \wedge y) &= h(i_M(r(x \wedge y))) \\ &= h(i_M(r(x) \wedge r(y))) \text{ since } r \text{ is right adjoint} \\ &= h(i_M(r(x))) \wedge h(i_M(r(y))) \text{ since } i_M \text{ is additive} \\ &= i_L^h(x) \wedge i_L^h(x) \end{aligned}$$

That is i_L^h is additive.

If i_M is idempotent, then

$$\begin{aligned} i_L(i_L(x)) &= h(i_M(r(h(i_M(r(x)))))) \\ &\leq i_L(h(r(h(i_M(r(x)))))) \\ &= i_L(h(i_M(r(x)))) \\ &\geq h(i_M(i_M(r(x))) \\ &= h(i_M(r(x)) = i_L^h(x). \end{aligned}$$

Thus i_L^h is idempotent. This completes the proof.

□

Remark 3.3.21. It is important to note that if either L or M is a regular frame, the interior operator defined above reduces to the discrete interior operator.

We are now ready to formulate the definition of topogenously continuous morphism within a general category:

Definition 3.3.22. A topogenous order on $\mathcal{X}$ is a family

$$\triangleleft = \{\triangleleft_{PX}, X \in \mathcal{X}\}$$

such that each $\triangleleft_{PX}$ is a topogenous order on PX and each $\mathcal{X}$-morphism $f : X \longrightarrow Y$ satisfies

$$x \triangleleft_Y y \Rightarrow f^*(x) \triangleleft_X f^*(y) \tag{3.3.10}$$

for all $x, y \in PY$.

When (3.3.10) holds, we say that $f : X \longrightarrow Y$ is topogenously continuous or that it is $\triangleleft$-continuous. The properties of the Galois connections permit us to express the above $\triangleleft$-continuity condition in the following equivalent manner:

Proposition 3.3.9. *A morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ is topogenously continuous if and only if*

$$f^o(x) \triangleleft_Y y \Rightarrow x \triangleleft_X f^*(y) \tag{3.3.11}$$

for all $x \in PX$ and $y \in PY$.

Proof. If (3.3.11) holds, then, since $f^o(f^*(b)) \leq b$ for all $b \in PY$

$$\begin{aligned} b \triangleleft_Y y &\Rightarrow f^o(f^*(b)) \leq b \triangleleft_Y y; \\ &\Rightarrow f^o(f^*(b)) \triangleleft_Y y; \text{ by (T2)} \\ &\Rightarrow f^*(b) \triangleleft_X f^*(y). \text{ Hence (T3)}. \end{aligned}$$

Conversely, if (3.3.10) holds and since for all $x \in PX$, $x \leq f^*(f^o(x))$,

$$\begin{aligned} f^o(x) \triangleleft_Y y &\Rightarrow x \leq f^*(f^o(x)) \triangleleft_X f^*(y); \\ &\Rightarrow x \triangleleft_X f^*(y). \end{aligned}$$

□

Furthermore,

Proposition 3.3.10. *If f^* commutes with all joins, we say that a morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ is topogenously continuous if and only if*

$$x \lhd_Y f_*(y) \Rightarrow f^*(x) \lhd_X y \tag{3.3.12}$$

for all $y \in PX$ and $x \in PY$.

Proof. *If (3.3.12) holds, then*

$$\begin{aligned} a \lhd_Y y &\Rightarrow a \lhd_Y y \leq f_*(f^*(y)); \\ &\Rightarrow a \lhd_Y f_*(f^*(y)); \\ &\Rightarrow f^*(a) \lhd_X f^*(y). \end{aligned}$$

Conversely if $(T3)$ holds and since $f^*(f_*(a)) \leq a$ for all $a \in PX$ then $y \lhd_Y f_*(a) \Rightarrow f^*(y) \lhd_X f^*(f_*(a)) \leq a$ and by $(T2)$, we obtain $f^*(y) \lhd_X a$.

□

Proposition 3.3.11. *Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{X}$. Let i be an interior operator on $\mathcal{X}$ and $\lhd$ a topogenous order on $\mathcal{X}$. If f^* commutes with all joins, then:*

(i) f is i-continuous if f is $\lhd^i$-continuous;

(ii) f is $\lhd$-continuous if f is $i^{\lhd}$-continuous;

Proof. (i) If $f : X \longrightarrow Y$ is i-continuous then for all $x, y \in PY$

$$\begin{aligned} x \lhd^i_Y y &\Rightarrow x \leq i(y); \\ &\Rightarrow f^*(x) \leq f^*(i_X(y)) \leq i_Y(f^*(y)); \\ &\Rightarrow f^*(x) \leq i_Y(f^*(y)); \\ &\Rightarrow f^*(x) \lhd^i_X f^*(y). \end{aligned}$$

Conversely if $f : X \longrightarrow Y$ is $\lhd$-continuous then for all $x \in P(Y)$

$$\begin{aligned} f^*(i^{\lhd}_Y(x)) &= f^*(\bigvee\{y \in X \mid y \lhd x\}); \\ &= \bigvee\{f^*(y) \mid y \in X, y \lhd x\}; \\ &\leq \bigvee\{f^*(y) \in Y \mid f^*(y) \lhd f^*(x)\}; \\ &\leq \bigvee\{z \in Y \mid z \lhd f^*(x)\} = i^{\lhd}_X(f^*(x)). \end{aligned}$$

□

Remark 3.3.23. In view of Proposition (3.3.11), if i and $\lhd$ are, respectively interior operator and $\bigvee$-respecting topogenous orders on $\mathcal{X}$, then $\mathcal{X}$ has i-continuous morphisms if and only if it has $\lhd^i$-continuous morphisms. This argument holds true since, by Proposition (3.3.1) $i = i^{\lhd^i}$.

The following proposition can be proven in a similar manner:

Proposition 3.3.12. *Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{X}$. Let k be a closure operator on $\mathcal{X}$ and $\lhd$ a topogenous order on $\mathcal{X}$. Then*

(i) If f is k-continuous then f is $\lhd^k$-continuous;

(ii) If f is $\lhd$-continuous then f is $k^{\lhd}$-continuous.

Remark 3.3.24. As in Remark (3.3.23), Proposition (3.3.12) means that if k and $\lhd$ are, respectively closure operator and $\bigwedge$-respecting topogenous order on $\mathcal{X}$, then $\mathcal{X}$ has k-continuous morphisms if and only if it has $\lhd^k$-continuous morphisms. To substantiate this, one only needs to see that, by Proposition (3.3.4), $k = k^{\lhd^k}$.

We have shown that topogenous orders provide a general setting in which one can study closure and interior operations at the same time on one hand, whilist on the other hand, they facilitate an extension of well known results from closure and interior operations. In the next section, our intention is to characterize the so-called strict maps in a general category, with a particular emphasis on two categories, namely **Frm** and **Top**.

3.4 Strict Morphisms in a Category

Definition 3.4.1. Let $\mathcal{X}$ be a general category and $P : \mathcal{X} \longrightarrow$ **Pos** a pseudofunctor to the category of partially ordered sets and order preserving maps which to any morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ assigns a Galois adjoint:

$$PX \underset{f^*}{\overset{f^o}{\rightleftarrows}} PY \qquad (\bot)$$

Let $\lhd$ be a topogenous order on $\mathcal{X}$. Then an $\mathcal{X}$-morphism $f : X \longrightarrow Y$ is said to be a strict morphism,or $\lhd$-strict, if

$$f^o(a) \lhd_Y b \Leftrightarrow a \lhd_X f^*(b) \tag{3.4.13}$$

for all $a \in PX$ and $b \in PY$.

If $\lhd$ respects arbitrary joins, the $\lhd$-strict morphisms are precisely the i-open morphisms. While if $\lhd$ respects arbitrary meets, $\lhd$-strict morphisms are the k-closed morphisms. We address this in the coming two propositions.

Proposition 3.4.1. *Let $\mathcal{X}$ be a general category and $\lhd$ a topogenous order on $\mathcal{X}$. If $\lhd$ respects arbitrary joins, then a morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ is $\lhd$-strict if and only if f is i-open, that is, for all $y \in PY$, $f^*(i^{\lhd}(y)) = i^{\lhd}(f^*(y))$.*

Proof. Let $\triangleleft$ be a topogenous order that respects arbitrary joins and f^* a right adjoint to f^o. Then if f is $\triangleleft$-strict, we have

$$\begin{aligned} x \leq f^o(i_Y(y)) &\Leftrightarrow f^o(x) \leq i_Y(y); \\ &\Leftrightarrow f^o(x) \triangleleft_Y y; \\ &\Leftrightarrow x \triangleleft_X f^*(y); \\ &\Leftrightarrow x \leq i_X(f^*(y)). \end{aligned}$$

Conversely, if $f^*(i^{\triangleleft}(y)) = i^{\triangleleft}(f^*(x))$ holds, then

$$\begin{aligned} x \triangleleft_X f^*(y) &\Leftrightarrow x \leq (i^{\triangleleft}_X(f^*(y)); \\ &\Leftrightarrow x \leq f^*(i^{\triangleleft}_Y(y)); \\ &\Leftrightarrow f(x) \leq i^{\triangleleft}_Y(y); \\ &\Leftrightarrow f(x) \triangleleft_Y y. \end{aligned}$$

Similarly, taking into account the fact that if $\triangleleft$ respects arbitray meets, $x \triangleleft^k y$ is equivalent to $k(x) \leq y$, we also prove the following:

Proposition 3.4.2. *Let $\mathcal{X}$ be a general category and $\triangleleft$ a topogenous order on $\mathcal{X}$, If $\triangleleft$ respects arbitrary meets then a morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ is $\triangleleft$-strict if and only if f is k-closed, that to say, for any $x \in PX$, $f^o(k^{\triangleleft}(x)) = k^{\triangleleft}(f^o(x))$.*

These stricts maps were introduced and discussed in [HIR16] on categories equipped with $(\mathcal{E}, \mathcal{M})$ structures for morphisms. In fact, the notion of strict morphisms we present here is a natural way of transferring the concept of strict morphisms to our general context.

3.4.1 Characterization and Behavior of Strict Morphisms in a Category

We dedicate this section to characterizing and explaining the behavior of strict maps in a general category.

Proposition 3.4.3. *Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{X}$ and $\triangleleft$ a topogenous order on $\mathcal{X}$. Then f is $\triangleleft$-strict if and only if f^o and f^* preserve $\triangleleft$.*

Proof. If f is strict, then for all $a, b \in X$, $c \in Y$;

$$\begin{aligned} a \triangleleft b &\Rightarrow a \triangleleft b \leq f^*(f^o(b)), \text{ since } f^* \text{ is left adjoint;} \\ &\Rightarrow a \triangleleft f^*(f^o(b)), \text{ by } (T2); \\ &\Rightarrow f^o(a) \triangleleft f^o(b). \end{aligned}$$

Thus f^o preserves the order. Conversely, if f^o preserves the order, then

$$\begin{aligned} a \triangleleft f^*(c) &\Rightarrow f^o(a) \triangleleft f^o(f^*(c)) \leq c; \\ &\Rightarrow f^o(a) \triangleleft c. \end{aligned}$$

Further, by the continuity condition, f^* always preserves the order. For the converse, assume f^* preserves $\triangleleft$

$$\begin{aligned} f^o(a) \triangleleft c \quad &\Rightarrow \quad f^*(f^o(a)) \triangleleft f^*(c); \\ &\Rightarrow \quad a \leq f^*(f^o(a)) \triangleleft f^*(c); \\ &\Rightarrow \quad a \triangleleft f^*(c). \end{aligned}$$

□

In view of Proposition (3.4.3) if $\mathcal{X}$ is any category and $\triangleleft$ a topogenous order on $\mathcal{X}$, a morphism in $\mathcal{X}$ will be said to be $\triangleleft$-strict if both the left and the right adjoint preserve $\triangleleft$.

Proposition 3.4.4. *Let $\triangleleft$ be a topogenous order on the category $\mathcal{X}$. The assertions in the following statements are always true.*

(a) The class of all $\triangleleft$-morphisms are closed under isomorphisms.

(b) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are $\triangleleft$-strict then $g \circ f$ is $\triangleleft$-strict.

(c) If $g \circ f$ is $\triangleleft$-strict and f is surjective, that is, $f^o(f^*(x)) = x$ then g is $\triangleleft$-strict.

(d) If $g \circ f$ is $\triangleleft$-strict and g is a monomorphism, that is, $g^*(g^o(y)) = y$ then f is $\triangleleft$-strict.

Proof. (a) is clear.
(b) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are $\triangleleft$-strict then

$$\begin{aligned} a \triangleleft_X (g \circ f)^*(b) = f^*(g^*(b)) \quad &\Leftrightarrow \quad f^o(a) \triangleleft_Y g^*(b); \\ &\Leftrightarrow \quad (g^o \circ f^o)(a) = (gf)^o(a) \triangleleft_Y b. \end{aligned}$$

(c) If $f^o(f^*(x)) = x$, then

$$\begin{aligned} x \triangleleft g^*(y) \quad &\Rightarrow \quad f^*(x) \triangleleft_Y f^*(g^*(y)) = (gf)^*(y); \\ &\Rightarrow \quad (gf)^o(f^*(x)) = g^o(f^o(f^*(x))) \triangleleft_Z y; \\ &\Rightarrow \quad g^o(x) \triangleleft_Z y. \end{aligned}$$

Similarly if $g^*(g^o(y)) = y$, we can prove that f is $\triangleleft$-strict.

□

As we indicated earlier, the purpose of this section is not only to extend the notion of strict morphisms, initially defined on a category equipped with an $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms, to our general context, but also to characterize these strict morphisms both in the category of frames and that of topological spaces.

3.4.2 Strict Morphisms in Top and Frm

In this section, our intention is to characterize strict morphisms within the category of frames and frame homomorphisms as well as within the category of topological spaces and continuous maps. It's important for the reader to note that the concept of strict morphisms exhibits favorable behavior in topology. More specifically, these maps correspond neatly to closed and open morphisms with respect to closure and interior operators, respectively.

In the context of frames, however, strict morphisms do not provide favorable features, as we will show later in this section. Nevertheless, we have established a relationship with nearly open and closed frame homomorphisms. We believe that achieving a nicer correspondence with other existing maps in the literature (in the context of frames/locales) may require incorporating the concept of sublocales. This still needs a further investigation.

Expression of Strict Maps in Top

Let $\mathcal{X} = \mathbf{Top}$, the category of topological spaces and continuous maps between them and P, the powerset functor. To each continous map $f : X \longrightarrow Y$ the powerset functor assigns the image-preimage adjunction

$$PX \underset{f^{-1}}{\overset{f^{o}}{\rightleftarrows}} PY \qquad (\bot)$$

between PX and PY, the power set lattices of X and Y, respectively. We also have that

$$f^o(A) \subseteq B \Leftrightarrow A \subseteq f^{-1}(B)$$

for all $A \subseteq X$ and $B \subseteq Y$. Consider the following topogenous order on **Top**:

$$A \lhd B \Leftrightarrow \bar{A} \subseteq B \Leftrightarrow A \subseteq C \subseteq B \tag{3.4.14}$$

for some closed $C \subseteq X$, $X \in \mathbf{Top}$.

Proposition 3.4.5. *The following statements are true for any map $f : X \longrightarrow Y$:*

(i) f^{-1} preserves $\lhd$ if and only f is continous;

(ii) if f is continuous then f^o preserves $\lhd$ if and only if f is closed.

Proof. (i) If f^{-1} preserves $\lhd$, then A closed implies $A \lhd A \Rightarrow f^{-1}(A) \lhd f^{-1}(A) \Rightarrow f^{-1}(A)$ is closed. Hence f is continuous. Conversely, if f is continuous, then:

$\bar{A} \subseteq B \Rightarrow \overline{f^{-1}(A)} \subseteq f^{-1}(\bar{A}) \subseteq f^{-1}(B) \Rightarrow \overline{f^{-1}(A)} \subseteq f^{-1}(B)$. Therefore, f^{-1} preserves the order.

(ii) The forward implication is clear. For the backward, if f is closed, then

$$\begin{aligned} A \lhd B \quad &\Rightarrow \quad \text{there exists a closed set } C \text{ such that } A \subseteq C \subseteq B; \\ &\Rightarrow \quad f^o(A) \subseteq f^o(C) \subseteq f^o(B) \ \text{ with } f^o(C) \text{ closed;} \\ &\Rightarrow \quad f^o(A) \lhd f^o(B). \end{aligned}$$

$\square$

Moreover, if we consider the topogenous order:

$$A \lhd B \Leftrightarrow A \subseteq B^o \tag{3.4.15}$$

we can show, as in Proposition (3.4.5), that:

Proposition 3.4.6. *For any map $f : X \longrightarrow Y$, the following are equivalent:*

(i) f^{-1} preserves $\lhd$ if and only f is continous;

(ii) if f is continuous then f preserves $\lhd$ if and only if f is open.

In general,

Proposition 3.4.7. *If k is any closure operator on* **Top** *and $\lhd$ the topogenous order defined in analogy to (3.4.14). Then f is $\lhd$-strict if and only if f is k-closed.*

Proof. If f is $\lhd$-strict then

$$\begin{aligned} k(A) \subseteq k(A) \quad &\Rightarrow \quad A \lhd k(A); \\ &\Rightarrow \quad f^o(A) \lhd f^o(k(A)) \text{ since } f \text{ is strict;} \\ &\Rightarrow \quad k(f^o(A)) \subseteq f^o(k(A)). \end{aligned}$$

Conversely, if f is k-closed

$$\begin{aligned} k(A) \lhd B \Rightarrow k(A) \subseteq B \quad &\Rightarrow \quad f^o(k(A)) \subseteq f^o(B); \\ &\Rightarrow \quad k(f^o(A)) \subseteq f^o(k(A)) \subseteq f^o(B); \\ &\Rightarrow \quad f^o(A) \lhd f^o(B); \end{aligned}$$

That is to say f is strict.

$\square$

Furthermore, since for every continuous map $f : X \longrightarrow Y$, the inverse image f^{-1} commutes with all unions, it has a right adjoint f_* defined by

$$f_*(A) = \bigcup\{B \in PY \mid f^{-1}(B) \subseteq A\}$$

Equivalently

$$f_*(A) = Y \setminus f(X \setminus A)$$

for all $A \subseteq X$. Thus, we have the following alternative adjunction

$$\text{PY} \underset{f_*}{\overset{f^{-1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \text{PX}$$

between PX and PY and also,

$$f^{-1}(B) \subseteq A \Leftrightarrow B \subseteq f_*(A)$$

for all $B \subseteq Y$ and $A \subseteq X$.

Consider again the topogenous order in (3.4.14).

Proposition 3.4.8. *Let $f : X \longrightarrow Y$ be a continuous map and $\triangleleft$ a topogenous order on* **Top**. *Then f is $\triangleleft$-strict if and only if f is open.*

Proof. For the forward implication, we have

$$\begin{aligned} A \text{ open} &\Rightarrow X \setminus A \triangleleft X \setminus A; \\ &\Rightarrow f_*(X \setminus A) \triangleleft f_*(X \setminus A); \\ &\Rightarrow Y \setminus f^o(A) \triangleleft Y \setminus f^o(A); \\ &\Rightarrow f^o(A) \text{ is open.} \end{aligned}$$

For the backward implication, we obtain

$$\begin{aligned} A \triangleleft B &\Rightarrow X \setminus B \subseteq X \setminus \overline{A} \subseteq X \setminus A, \text{ where } X \setminus \overline{A} \text{ is open;} \\ &\Rightarrow f^o(X \setminus B) \subseteq f^o(X \setminus \overline{A}) \subseteq f^o(X \setminus A), \text{ where } f^o(X \setminus \overline{A}) \text{ is open;} \\ &\Rightarrow Y \setminus f^o(X \setminus A) \subseteq Y \setminus f^o(X \setminus \overline{A}) \subseteq Y \setminus f^o(X \setminus B), \text{ where } Y \setminus f^o(X \setminus \overline{A}) \text{ is closed;} \\ &\Rightarrow f_*(A) \triangleleft f_*(B). \end{aligned}$$

□

Similarly, by considering the topogenous order in (3.4.15), we can establish the following proposition:

Proposition 3.4.9. *Let $f : X \longrightarrow Y$ be a continuous map and $\triangleleft$ a topogenous order on* **Top**. *Then f is $\triangleleft$-strict if and only if f is closed.*

Proposition 3.4.10. *Let $f : X \longrightarrow Y$ be a continuous map, k a closure operator on* **Top** *and $\triangleleft$ the topogenous order defined in analogy to* (3.4.14). *Then f is $\triangleleft$-strict if and only if f is k-open.*

Proof. On one hand, if f is $\triangleleft$-strict then

$$\begin{aligned} f^{-1}(A) \triangleleft k(f^{-1}(A)) &\Rightarrow f_*(f^{-1}(A)) \triangleleft f_*(k(f^{-1}(A))), \text{ since } f \text{ is strict;} \\ &\Rightarrow A \subseteq f_*(f^{-1}(A)) \triangleleft f_*(k(f^{-1}(A))), \text{ since } f^{-1} \dashv f_*; \\ &\Rightarrow A \triangleleft f_*(k(f^{-1}(A))), \text{ by } (T2); \\ &\Rightarrow k(A) \subseteq f_*(k(f^{-1}(A))), \text{ Definition of (3.4.14);} \\ &\Rightarrow f^{-1}(k(A)) \subseteq k(f^{-1}(A)), \text{ again since } f^{-1} \dashv f_*. \end{aligned}$$

On the other hand, since $A \triangleleft B \Rightarrow k(A) \subseteq B$ and if we assume that $k(f_*(A)) \subseteq f_*(B)$, then

$$\begin{aligned} f^{-1}(k(f_*(A)) &\subseteq k(f^{-1}(A)) \;\; f \text{ is } k\text{-open} \\ &\subseteq k(A) \;\; (f^{-1} \dashv f_*); \\ &\subseteq B. \end{aligned}$$

Thus, $f^{-1}(k(f_*(A))) \subseteq B \Rightarrow k(f_*(A)) \subseteq f_*(B)$ and

$$f_*(A) \triangleleft f_*(B)$$

showing that f is $\triangleleft$-strict.

□

Expression of Strict Maps in Frm

In order to understand the notion of strict maps in frames, we have to walk away from the powerset lattice of a space and consider its "open set lattice". More explicitly, let X be a space. We consider its lattice of open sets ΩX instead of PX. Each continuous map $f : X \longrightarrow Y$ gives rise to the Galois adjunction:

$$\Omega Y \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \Omega X \qquad (f^{-1} \dashv f_*,\ \bot)$$

between ΩX and ΩY. That means, for all $A \in \Omega X$, $B \in \Omega Y$:

$$\begin{aligned} B \subseteq f_*(A) &\Leftrightarrow f^{-1}(B) \subseteq A; \\ &\Leftrightarrow X \setminus A \subseteq X \setminus f^{-1}(B) = f^{-1}(Y \setminus B); \\ &\Leftrightarrow f^o(X \setminus A) \subseteq Y \setminus B; \\ &\Leftrightarrow B \subseteq Y \setminus f^o(X \setminus A). \end{aligned}$$

Thus

$$f_*(A) = (Y \setminus f^o(X \setminus A))^o = Y \setminus \overline{f^o(X \setminus A)} \tag{3.4.16}$$

It should be clear to the reader that the topogenous order $A \lhd B \Leftrightarrow A \subseteq B^o$ is no longer useful in this context as we are exclusively working with open subsets.

Now consider the alternative topogenous order: $\overline{A} \subseteq B \Leftrightarrow A^* \cup B = X$, where A^* stands for the pseudocomplement of A in the lattice ΩX viewed as a frame. Given that, by continuity condition, f^{-1} always preserves $\lhd$, strictness in this context will mean that f_* preserves $\lhd$. Indeed, using the expression (3.4.16), we have:

$$\begin{aligned} \overline{A} \subseteq B &\Rightarrow \overline{f_*(A)} \subseteq f_*(B); \\ &\Rightarrow \overline{Y \setminus \overline{f^o(X \setminus A)}} \subseteq Y \setminus \overline{f^o(X \setminus B)}; \\ &\Rightarrow \overline{f^o(X \setminus B)} \subseteq \overline{Y \setminus \overline{f^o(X \setminus A)}} = \overline{f^o(X \setminus A)}^o; \\ &\Rightarrow \overline{f^o(X \setminus B)} \subseteq \overline{f^o(X \setminus A)}^o. \end{aligned}$$

Proposition 3.4.11. *Let $f : X \longrightarrow Y$ be a continuous map. Then:*

(i) f is open and closed implies f is strict;

(ii) f is strict then A clopen implies $\overline{f^o(A)}$ is also clopen.

Remark 3.4.2. In general for any continuous map $f : X \longrightarrow Y$

(i) Closed does not always imply strict;

(ii) Open does not always imply strict.

Counter examples*:*
For (i), let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a constant function with $f^o(x) = 0$, $\forall x \in \mathbb{R}$.
$f^o(A) = \{0\}$ is closed for any $A \subseteq \mathbb{R}$, $A \neq \emptyset$, and $f^o(\emptyset) = \emptyset$ is closed.

$f_*(A) = \mathbb{R} \setminus \overline{f^o(X \setminus A)} = \mathbb{R} \setminus \{0\} \ \ \forall A \neq \mathbb{R}$.
$\overline{f_*(A)} = \overline{\mathbb{R} \setminus \{0\}} = \mathbb{R}$.
Thus $f_*(A)$ is not topogenously below $f_*(B)$ for all $A, B \subseteq \mathbb{R}$.

For (ii), let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}, f^o(x) = arctan(x)$. Then f is open but $f_*(\emptyset) = \mathbb{R} \setminus \overline{f^o(X \setminus \emptyset)} = \mathbb{R} \setminus [\pi/2, \pi/2]$ and $\overline{\emptyset} \subseteq \emptyset$, yet $\overline{f_*(\emptyset)} = \mathbb{R} \setminus (\pi/2, \pi/2) \nsubseteq \mathbb{R} \setminus [\pi/2, \pi/2] = f_*(\emptyset)$.

In particular, if $f : X \longrightarrow Y$ is $\lhd$-strict then $\emptyset \lhd \emptyset \Rightarrow f_*(\emptyset) \lhd f_*(\emptyset) \Rightarrow \overline{f^o(X)} \subseteq \overline{f^o(X)}^o$ and $\overline{f^o(X)}$ is clopen. Hence, if f is not dense then Y is disconnected. As a matter of fact, no $f : X \longrightarrow \mathbb{R}$ that is not dense can be strict.

As we have already indicated earlier, one of the fascinating features of Á Császár orders is they dont rely on points in most cases. This fact make them interesting to pointfree topologists. In fact, our discussions in the previous paragraph show that the topogenous order $A \lhd B \Rightarrow \overline{A} \subseteq B$ has a well established meaning in pointfree topology.

[Pic00] Taking into account the pseudocomplement A^* in ΩX, that is, $X \setminus \overline{A}$, we have

$$\overline{A} \subseteq B \Rightarrow A^* \cup B = X$$

equivalently

$$X = A^* \cup B \equiv A \prec B$$

which is the well known rather below relation. In the context of general frames, define $a \prec b$ if $a^* \vee b = 1$. In this case, the rather below relation serves as the most compelling example to motivate our study of topogenous orders on frames. Besides, frame homomorphisms for which the right adjoint preserves the rather below relations appear as best examples of strict frame homomorphisms.

To start, we recall that each frame homomorphism $h : L \longrightarrow M$ is associated with a right adjoint $h_* : M \longrightarrow L$, hence the following diagram

$$L \underset{h_*}{\overset{h}{\rightleftarrows}} M \qquad (h \dashv h_*)$$

always holds in the category **Frm**. This also means that for all $x \in L$, $y \in M$, $h(x) \leq y$ is equivalent to $x \leq h_*(y)$.

Definition 3.4.3. Let $\lhd$ be a topogenous order on **Frm**. A frame homomorphism $h : L \longrightarrow M$ is called $\lhd$-strict if

$$h(x) \lhd y \Rightarrow x \lhd h_*(y)$$

for all $x \in L$ and $y \in M$.

Since frame homomorphisms always preserve the order, the following proposition characterises strict frames homomorphisms.

Proposition 3.4.12. *Let $h : L \longrightarrow M$ be a frame homomorphism and h_* its right adjoint. Let $\triangleleft$ be a topogenous order on* **Frm**. *Then h is $\triangleleft$-strict if and only if h_* preserves $\triangleleft$, that is,*

$$x \triangleleft_M y \Rightarrow h_*(x) \triangleleft_L h_*(y)$$

for all $x, y \in M$.

Proof. Similar to the one of Proposition (3.4.3).

□

In view of Proposition (3.4.12), strict frame homomorphisms are those frame homomorphisms for which the right adjoint preserves the topogenous order.

The next proposition, whose proof follows a computational approach similar to that of Proposition (3.4.4), outlines the fundamental properties of these maps.

Proposition 3.4.13. *Let $\triangleleft$ be a topogenous order on* **Frm** *and $L \xrightarrow{h} M \xrightarrow{g} L$ be strict frame homomorphisms. Then:*

(i) $g \circ h$ is $\triangleleft$-strict ;

(ii) if $g \circ h$ is $\triangleleft$-strict and h is surjective then g is $\triangleleft$-strict;

(iii) if $g \circ h$ is $\triangleleft$-strict and g is one-one then h is $\triangleleft$-strict;

Proof. (i) if $h : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are $\triangleleft$-strict, then

$$\begin{aligned} a \triangleleft (g \circ h)_*(b) &= h_*(g_*(b)); \\ &\Rightarrow h(a) \triangleleft g_*(b); \\ &\Rightarrow g(h(a)) \triangleleft b; \\ &\Rightarrow (g \circ h)(a) \triangleleft b. \end{aligned}$$

(ii) If h is surjective, that is, $h(h_*(a)) = a$ for all $a \in X$, then

$$\begin{aligned} g(a) \triangleleft b &\Rightarrow g(h(h_*(a))) \triangleleft b; \\ &\Rightarrow h_*(a) \triangleleft (gh)_*(b) = h_*(g_*(b)); \\ &\Rightarrow h(h_*(a)) \triangleleft h(h_*(g_*(b))), \text{ since } h \text{ preserves } \triangleleft; \\ &\Rightarrow a \triangleleft g_*(b). \end{aligned}$$

Similarly one shows that if g is one-to-one, that is, $g_*(g(b)) = b$ for all $b \in Y$ then $h(a) \triangleleft b \Rightarrow a \triangleleft h_*(b)$

□

Proposition 3.4.14. *Let $h : L \longrightarrow M$ be a one-to-one frame homomorphism that preserves pseudocomplement. Then*

$$x \prec y \Leftrightarrow h(x) \prec h(y)$$

for all $x, y \in L$.

Proof. The forward implication follows immediately for any frame homomorphism from $h(x^*) \leq h(x)^*$. For the backward implication, since $x \prec y \Leftrightarrow x^* \vee y = 1$, we have

$$\begin{aligned} h(x) \prec h(y) &\Rightarrow h(x)^* \vee h(y) = 1_M; \\ &\Rightarrow h(x^*) \vee h(y) = 1_M; \\ &\Rightarrow h(x^* \vee y) = 1_M; \\ &\Rightarrow h_*(h(x^* \vee y)) = h_*(1_M); \\ &\Rightarrow x^* \vee y = 1_L; \\ &\Rightarrow x \prec y. \end{aligned}$$

□

Proposition 3.4.15. *Let $h : L \longrightarrow M$ be a dense and onto frame homomorphism. Then*

$$h_*(a) \prec h_*(b) \Rightarrow a \prec b \tag{3.4.17}$$

for all $b, a \in M$. Furthermore, if h_* preserves finite join, the other implication also holds.

Proof.

$$\begin{aligned} (\Rightarrow)\ \ h_*(a) \prec h_*(b) &\Rightarrow h_*(a)^* \vee h_*(b) = 1; \\ &\Rightarrow h_*(a^*) \vee h_*(b) = 1; \\ &\Rightarrow h(h_*(a^*)) \vee h(h_*(b)) = 1; \\ &\Rightarrow a^* \vee b = 1; \\ &\Rightarrow a \prec b. \end{aligned}$$

($\Leftarrow$) $a \prec b$ implies there exists a separating element $c \in L$ such that $a \wedge c = 0_L$ and $c \vee b = 1_L$. This in turn implies that $h_*(a) \wedge h_*(c) = h_*(0_L)$ and $h_*(c) \vee h_*(b) = h_*(1_M) = 1_L$, since by assumption h_* preserves finite join. By denseness of h, that is, $h_*(0_M) = 0_L$, we obtain $h_*(a) \wedge h_*(c) = 0_L$ and $h_*(c) \vee h_*(b) = h_*(1_M) = 1_L$. In consequence $h_*(a) \prec h_*(b)$

□

We conclude this section by the following proposition which shows that the strict frame homomorphisms are intricately connected with closed frame homomorphisms:

Proposition 3.4.16. *Let $h : L \longrightarrow M$ be a frame homomorphism. If h preserves pseudocomplements, then h closed implies h is $\prec$-strict.*

Proof. If h is closed, for all $x \in L$ and $y \in M$, we have

$$\begin{aligned} h(x) \prec y &\Leftrightarrow h(x)^* \vee y = 1_M; \\ &\Leftrightarrow h(x^*) \vee y = 1_M; \\ &\Rightarrow h_*(h(x^*) \vee y) = h_*(1_M) \text{ since } h \text{ is closed;} \\ &\Leftrightarrow x^* \vee h_*(y) = 1_L; \\ &\Leftrightarrow x \prec h_*(y). \end{aligned}$$

□

3.5 More Examples of Topogenous Orders

(I). Let $\mathcal{X} = \mathbf{Grp}$ be the category of groups and group homomorphisms and P the pseudo-functor which assigns to any group homomorphism $f : G \longrightarrow F$ the following adjuction

$$\mathrm{P(G)} \underset{f^{-1}}{\overset{f^o}{\underset{\perp}{\rightleftarrows}}} \mathrm{P(F)}$$

between $P(G)$ and $P(F)$, where $P(G)$ and $P(F)$ represent complete lattices of all subgroups of the groups G and F, respectively.

Consider the order relation $\triangleleft$ defined on $P(G)$ as follows:

$$G_1 \triangleleft G_2 \Leftrightarrow G_1 \subseteq N \subseteq G_2$$

with N a normal subgroup of G.

Then $\triangleleft$ is a topogenous order on $P(G)$. Moreover, since f^{-1} preserves normal subgroups, it follows that $f^{-1}(G_1) \triangleleft f^{-1}(G_2)$ whenever $G_1 \triangleleft G_2$. Thus the family

$$\triangleleft = \{\triangleleft_G, G \in \mathbf{Grp}\}$$

is a topogenous order on **Grp**.

Further, for any subgroup E of the group G, the set

$$i_G(E) = \bigvee\{F \triangleleft G \mid F \leq E\}$$

where $F \leq E$ means that F is a subgroup of E, is an idempotent interior operation on $L(G)$. Furthermore, since

$$f^{-1}(i_G(E)) \leq i_G(f^{-1}(E)) \tag{3.5.18}$$

the family

$$i = \{i_G, G \in \mathbf{Grp}\}$$

is an interior operator on **Grp**.

(II). **Topogenous Order on the Lattice of Fibers of Topological Functors**:

Using Skula's modification of a topological space, we define a topogenous order on the lattice of fibers of topological functors.

Definition 3.5.1. Let (X, τ) be a topological space. The **Skula modification** of X is the space (X, τ_S) where τ_S is the topology generated by the sets in τ and their complements, that is, τ_S given by

$$\tau_S = \{\cup_{i \in I} C_i \mid C_i \in \tau \text{ or } X \setminus C_i \in \tau\}.$$

Definition 3.5.2. For a given functor $F : \mathcal{A} \longrightarrow \mathcal{X}$, let Ini$F$ and Fin$F$ denote the classes of all F-initial and F-final morphisms in $\mathcal{A}$ respectively. F is called a fibration if every $g : X \longrightarrow FA$ has F-initial (F-cartesian) lifting, that is, there is exists a morphism $f \in \text{Ini}F$ with $Ff = g$. Dually F is called a cofibration if every morphism $g' : FB \longrightarrow Y$ has F-final (F-co-cartesian) lifting, that is, there exists a morphism $h \in \text{Fin}F$ with $Fh = g'$.

For a given $X \in \text{Ob}(\mathcal{X})$, $F^{-1}X$:=$\{A \in \text{Ob}(\mathcal{A}) \mid \ FA = X\}$. When F is a fibration any morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ induces a functor $f^*(-) : F^{-1}Y \longrightarrow F^{-1}X$ and for any $B \in F^{-1}Y$, $f^*(B) = A$ where A is the domain of the chosen F-Cartesian lifting of $f : X \longrightarrow FB$. Dually, when F is a cofibration, $f : X \longrightarrow Y$ gives rise to a functor $f^o(-) : F^{-1}X \longrightarrow F^{-1}Y$ and for any $A \in F^{-1}X$, $f^o(A) = B$ where B is the codomain of the chosen F-co-Cartesian lifting of $f : FA \longrightarrow Y$. Thus for a fibration and cofibration functor $F : \mathcal{A} \longrightarrow \mathcal{X}$, any morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ assigns an adjoint pair

$$F^{-1}X \underset{f^*}{\overset{f^o}{\rightleftarrows}} F^{-1}Y. \quad (\bot)$$

As an illustrative example of the above scenario, let F be the forgetful functor from **Top**, the category of topological spaces and continuous maps between them, to **Set**, that of sets and functions. For any set X, $F^{-1}X$ is identified with the lattice of all topologies on X and for any function $f : X \longrightarrow Y$, $f^o(\tau_X) = \lambda_Y$ is the lagest topology on $F^{-1}Y$ such that $f : (X, \tau) \longrightarrow (Y, \lambda_Y)$ is continuous. Dually, given any λ_Y, $f^*(\lambda_Y) = \tau_X$ is the smallest topology on X for which is $f : (X, \tau) \longrightarrow (Y, \lambda_Y)$ is continuous.

In this instance F : **Top** $\longrightarrow$ **Set** can be viewed as being both a fibration and cofibration so that for each function $f : X \longrightarrow Y$ and $\tau \in F^{-1}X$, $\sigma \in F^{-1}Y$ with $F^{-1}X$ and $F^{-1}Y$ being fibers of topologies of X and Y, respectively, we have $f^o(\tau) \subseteq \sigma \Leftrightarrow \tau \subseteq f^{-1}(\sigma)$, that is to mean we have a Galois connection

$$F^{-1}X \underset{f^{-1}}{\overset{f^o}{\rightleftarrows}} F^{-1}Y \ . \quad (\bot)$$

Proposition 3.5.1. *Let $\lhd$ be a given order relation on the lattice $F^{-1}X$. For all $\tau, \lambda \in F^{-1}X$, defined $\lhd$ by*

$$\tau \lhd \lambda \ \text{ if and only if } \lambda \subseteq \lambda_S \subseteq \tau$$

where S is the Skula modification of λ. Then $\lhd$ is a topogenous order on $F^{-1}X$ for which each $f : X \longrightarrow Y$ is $\lhd$-continuous.

Proof. For (T_1), $\tau \lhd \lambda \Rightarrow \lambda \subseteq \tau$, by definition. For (T_2), let $\tau, \tau', \lambda, \lambda'$ be in $F^{-1}X$, if

$$\begin{aligned} \tau' \subseteq \tau \lhd \lambda \subseteq \lambda' \ &\Rightarrow \ \tau' \subseteq \tau \subseteq \lambda \subseteq \lambda_S \subseteq \lambda' \\ &\Rightarrow \ \tau' \subseteq \lambda_S \subseteq \lambda' \\ &\Rightarrow \ \tau' \lhd \lambda' \end{aligned}$$

$(T3)$, let $f : X \longrightarrow Y$ be fuction and $F^{-1}X$, $F^{-1}Y$ the lattices of topologies on X and Y respectively. If $\tau \subseteq \tau_S \subseteq \lambda$, then $f^{-1}(\tau) \subseteq f^{-1}(\tau_S) \subseteq f^{-1}(\lambda)$. Since inverse image preserves arbitrary union and $f^{-1}(Y \setminus C_i) = X \setminus f^{-1}(C_i)$, it follows that $f^{-1}(\tau_S)$ is a Skula modification of $f^{-1}(\tau)$ and $f^{-1}(\tau) \lhd f^{-1}(\lambda)$.

□

(III) More on Topogenous Orders on Frames

(1) Let $L \in \mathbf{Frm}$ and $x, y \in L$. Define $x \lhd y$ if and only if $x \leq y$. Then $\lhd = \{\leq_L \mid L \in \mathbf{Frm}\}$ is a topogenous order on **Frm**.

(2) Recall from [Dub19] that an element a in a frame L is called **compact** if, for any $S \subseteq L$, the condition $a \leq \bigvee E$ implies $a \leq \bigvee S$ for some finite $S \subseteq E$. We denote by $\mathfrak{L}(L)$ the set of compact elements in L. The frame L is said to be **algebraic** if each of its elements is the join of compact elements below it. Furthermore, if for all $a, b \in \mathfrak{L}(L)$, $a \wedge b \in \mathfrak{L}(L)$, then L is said to have the finite intersection property (FIP) on compact elements. A compact algebraic frame with FIP is called a **coherent** frame. Moreover, a frame homomorphism between algebraic frames is called a **coherent map** if it maps compact elements to compact elements. The category of algebraic frames and coherent maps between them is denoted by **AFrm**.

Now let $L \in \mathbf{AFrm}$ and $a, b \in L$. Put

$$a \lhd_L b \Leftrightarrow a \leq c \leq b$$

for some compact element c. Then $\lhd$ is a topogenous order on L. Further, since coherent maps between algebraic frames preserve compact elements, it follows that $h(a) \lhd h(b)$ whenever $a \lhd b$. Therefore, the family

$$\lhd = \{\lhd_L \mid L \in \mathbf{AFrm}\}$$

is a topogenous order on **AFrm**.

(IV) Topogenous Order on the Frame of Ideals

We consider a topogenous order on the frame of ideals. For the join-preserving topogenous orders we define an interior operator associated with it. We also lift the topogenous order on a frame to its frame of ideals. That is, given a topogenous order $\lhd$ on a frame L, we defined a new topogenous order $\lhd^\bullet$ on $\mathcal{J}L$. It is shown that $\lhd^\bullet$ interpolates provided $\lhd$ does.

Proposition 3.5.2. *Let L be a frame and $\mathcal{J}L$ the frame of ideals of L. Consider the following order relation:*

$$I \lhd_{\mathcal{J}L} J \Leftrightarrow \bigvee I \in J$$

for all $I, J \in \mathcal{J}L$. Then $\lhd_{\mathcal{J}L}$ is a topogenous order on $\mathcal{J}L$.

Proof. $(T1)$ For all $I, J \in \mathcal{J}L, I \lhd_{\mathcal{J}L} J \Rightarrow \bigvee I \in J \Leftrightarrow a \in J \ \ \forall\, a \in I \Leftrightarrow I \subseteq J$.

$(T2)$ For all $I, I', J, J' \in \mathcal{J}L$, we put,

$$\begin{aligned} I' \subseteq I \triangleright_{\mathcal{J}L} J \subseteq J' &\Leftrightarrow \bigvee J' \leq \bigvee I \in J \subseteq J'; \\ &\Rightarrow \bigvee I' \in J' \text{ since } J' \text{ is a an ideal ;} \\ &\Rightarrow I' \triangleleft_{\mathcal{J}L} J' \end{aligned}$$

□

The next proposition provides an interior operator on frames associated with $\triangleleft_{\mathcal{J}L}$.

Proposition 3.5.3. *Let L be a frame and $\mathcal{J}L$ the lattice of ideals of L and let $\triangleleft \in \bigvee -TORD(\mathcal{J}L)$. The assignement $i = \{i^{\triangleleft}_{\mathcal{J}L} : \mathcal{J}L \longrightarrow \mathcal{J}L\}$ given by*

$$i^{\triangleleft}_{\mathcal{J}L}(J) = \bigvee \{I \in \mathcal{J}L \mid I \triangleleft J\}$$

is an interior operation on $\mathcal{J}L$.

Proposition 3.5.4. *Let L be any frame and $\mathcal{J}L$ the lattice of all the ideals of L. Let $\triangleleft$ be a topogenous order on L. For all $J, I \in \mathcal{J}L$, define*

$$I \triangleleft^{\bullet} J \Leftrightarrow \forall\, x \in I,\ \exists\, y \in J, x \triangleleft y. \tag{3.5.19}$$

(1) $\triangleleft^{\bullet}$ is a topogenous order on $\mathcal{J}L$;

(2) $\triangleleft^{\bullet}$ interpolates provided $\triangleleft$ does.

Proof. Note that for all $x, y \in L$, $x \triangleleft y \Leftrightarrow \downarrow x \triangleleft^{\bullet} \downarrow y$: Indeed, $x \triangleleft y \Rightarrow \downarrow x \triangleleft^{\bullet} \downarrow y$ since $a \in \downarrow x \Rightarrow a \leq x \triangleleft y \Rightarrow a \triangleleft y \in \downarrow y$. On the other hand, if $\downarrow x \triangleleft^{\bullet} \downarrow y$ then $x \triangleleft y$, since $x \in \downarrow x \Rightarrow \exists\, b \in \downarrow y$ with $x \triangleleft b \leq y \Rightarrow x \triangleleft y$.

(1) let I, J be in $\mathcal{J}L$ then $I \triangleleft^{\bullet} J$ if and only if for all $x \in I$, there exists $y \in J$ such that $x \triangleleft y$ but by $x \triangleleft y$ implies $x \leq y$ and since J is ideal, $x \in J$. Thus $I \subseteq J$.

For $(T2)$ If for all $I, J, J', I' \in \mathcal{J}L$, $I \subseteq I' \triangleleft^{\bullet} J' \subseteq J$ then $x \in I$ implies $x \in I'$ and so by definition of $\triangleleft^{\bullet}$, there exists $y \in J'$ such that $x \triangleleft y$ but $J' \triangleleft J$, so $y \in J$ and $I \triangleleft J$.

(2) If $\triangleleft$ interpolates then for all $I, J \in \mathcal{J}L$, $I \triangleleft^{\bullet} J$ implies for all $x \in I$ there exists $y \in J$ such that $x \triangleleft y$ and since $\triangleleft$ interpolates, there is a $b \in J' \in \mathcal{J}L$ with $x \triangleleft b \triangleleft y$ and $I \triangleleft^{\bullet} J' \triangleleft^{\bullet} J$.

□

Chapter 4

SYNTOPOGENOUS STRUCTURES AND QUASI-UNIFORMITIES ON COMPLETE LATTICES

The concept of syntopogenous structures, since its inception in the 1960s, has been explored by numerous researchers, each approaching it with various objectives: In frames, for instance, Chung [Chu08, Chu01] introduced the concept of H-complete weak syntopogenous frames. Using the notion of convergence of filters and that of filter traces of strict extensions, he provided a characterization of the H-completions of regular weak syntopogenous frames. In [WL95], the authors investigated the unification of co-topology, quasi-uniformities and T-structures on complete distributive lattices. They established the general theory of syntopogenous structures on complete distributive lattices and generalized the corresponding theory in both general and fuzzy topology. They examined connectedness in this type of syntopogenous structures and naturally addressed the connectedness of fuzzy syntopogenous structures. Recently, syntopogenous structures were successfully utilized in [Ira19] to investigate quasi-uniformities in the framework of general categories and significant results were obtained.

To our knowledges, except the two papers aforementioned, no further study of syntopogenous structures has been done as far as syntopogenous structures on complete lattices is concerned. Given the current surge of interest in complete lattices (frames), it seems necessary for us to complete the work done in [Chu08, Chu01] and [WL95] and extend the findings of [Ira19]. In actual fact, the findings presented in this chapter subsume a substantial portion of existing results in the literature: syntopogenous structures have been used to study quasi-uniformities on a general category where the primitive concept has been a "complete lattice" [Ira19]. Indeed, when studying syntopogenous structures on a general category $\mathcal{X}$ supplied with an $(\mathcal{E}, \mathcal{M})$ factorisation structure, the investigation revolves around the behavior of these orders within the subobject lattice of an object X. In this context each $\mathcal{X}$-morphism $f : X \longrightarrow Y$ is considered to be syntopogenously continuous between the subobject lattice of the objects X and Y. Furthermore, all these lattices are always assumed to be complete. Therefore, it is natural to extend the results from [Ira19] to our more generalized framework. This chapter is partially motivated by this imperative.

In concrete terms, we use syntopogenous structures to examine quasi-uniformities on complete lattices. We show that meet-preserving syntopogenous structures are order isomorphic to a base of a quasi-uniformity on a complete lattice. Consequently, a quasi-uniformity appears as a family of closure operations on a complete lattice. Moreover, when we consider syntopogenous structures that do not necessarily preserve meets, we establish a Galois adjunction between quasi-uniformities and syntopogenous structures on a complete lattice. Furthermore, we prove that any meet-structure of a complete lattice determines a transitive quasi-uniformity on the lattice in question. This enables us to characterize all transitive quasi-uniformities compatible with the lattice.

It should be noted that, apart from the results in Section 4.5, all our findings in this chapter can be confined to frames. We close the chapter by establishing a relationship between the category of syntopogenous spaces and that of syntopogenous frames using the so-called open functor.

4.1 Syntopogenous Spaces

Definition 4.1.1. Let X be a set and $\mathcal{S}_X$ a family of topogenous orders on PX. Then $\mathcal{S}_X$ is called a syntopogenous structure on X if it adheres to the following axioms:

(S1) For any $\triangleleft, \triangleleft' \in \mathcal{S}_X$ there exists $\triangleleft'' \in \mathcal{S}_X$ such that $\triangleleft \subseteq \triangleleft''$ and $\triangleleft' \subseteq \triangleleft''$. In other words, $\mathcal{S}_X$ is a directed family of topogenous orders on PX;

(S2) if $\triangleleft \in \mathcal{S}_X$ there exits $\triangleleft_1 \in \mathcal{S}_X$ such that for all $A, B \subseteq X$, if $A \triangleleft B$ then there exists $C \subseteq X$ with $A \triangleleft_1 C \triangleleft_1 B$.

The pair $(X, \mathcal{S}_X)$ is called a syntopogenous space if $\mathcal{S}_X$ is a syntopogenous structure on the set X.

Definition 4.1.2.

(b) A map $f : (X, \mathcal{S}_X) \longrightarrow (Y, \mathcal{S}_Y)$, where $(X, \mathcal{S}_X)$ and $(Y, \mathcal{S}_Y)$ are syntopogenous spaces, is syntopogenously continuous if for every $\triangleleft_Y \in \mathcal{S}_Y$ there exists $\triangleleft_X \in \mathcal{S}_X$ such that for all $A, B \in X$, $A \triangleleft_Y B$, then $f^{-1}(A) \triangleleft_X f^{-1}(B)$.

Syntopogenous spaces and syntopogenously continuous maps are the objects and morphisms of the category **SYN**.

It is less tedious to verify that the category **SYN** subsumes various topology-related categories including **Unif**, **Prox**, and **ToP**. These categories correspond to uniform spaces and uniformly continuous maps, proximity spaces and proximal maps, and topological spaces and continuous maps, respectively. More precisely, it can be observed that all of these categories are isomorphic with full subcategories of **SYN**.

Now, let $(X, \mathcal{S})$ be a syntopogenous space, where $\mathcal{S}$ is a collection of orders represented by

$$\mathcal{S} = \{\triangleleft_i \mid i \in I\} \tag{4.1.1}$$

If the collection $\mathcal{S}$ defined in (4.1.1) consists of a single relation, then $(X, \mathcal{S})$ is called a topological space. Moreover, it attains the status of a proximity space when each $\triangleleft \in \mathcal{S}$ satisfies axiom (5) of Definition (3.1.1). On the other hand, if axiom (4b) of Definition (3.1.1) holds for an infinite union in a proximity space $(X, \mathcal{S})$, it is characterized as a uniform space.

In his Ph.D. thesis [Chu88], Chung proved that the category **Prox** is isomorphic to the category **STopG**, the category of symmetric topogenous orders.

Fact: Given a syntopogenous space $(X, \mathcal{S})$. The family

$$\tau_{\mathcal{S}} = \{A \subseteq X \mid \forall a \in A, \{a\} \triangleleft_i A \text{ for some } \ i \in I\} \tag{4.1.2}$$

is a topology on $(X, \mathcal{S})$. It is the topology induced by $\mathcal{S}$.

Proposition 4.1.1. *Consider a syntopogenous space* $(X, \mathcal{S})$*. For each* $A \in \tau_{\mathcal{S}}$*,*

$$A = \bigcup \big\{B \in \tau_{\mathcal{S}} \mid B \triangleleft_{\mathcal{S}} A\big\} \tag{4.1.3}$$

where $\triangleleft_{\mathcal{S}} = \bigcup \mathcal{S}$

With this new notation, the equation (4.1.2) can be written as

$$\tau_{\mathcal{S}} = \{A \subseteq X \mid \forall a \in A, \{a\} \triangleleft_{\mathcal{S}} A\}$$

Proof. Before proving that (4.1.3) holds, we first note that for all $B \subseteq X$, $B^o = \{x \in X \mid \{x\} \triangleleft_{\mathcal{S}} B\} \in \tau_{\mathcal{S}}$:

$$\begin{aligned} x \in B^o \quad &\Rightarrow \quad \{x\} \triangleleft_{\mathcal{S}} B; \\ &\Rightarrow \quad \exists\, C \subseteq X \mid \{x\} \triangleleft_{\mathcal{S}} C \triangleleft_{\mathcal{S}} B. \end{aligned}$$

and $C \triangleleft_{\mathcal{S}} B \Rightarrow C \subseteq B^o$ since for all $x \in C, \{x\} \subseteq C \triangleleft_{\mathcal{S}} B \Rightarrow \{x\} \triangleleft_{\mathcal{S}} B$. Thus

$$\begin{aligned} x \in B^o \quad &\Rightarrow \quad \{x\} \triangleleft_{\mathcal{S}} C \triangleleft_{\mathcal{S}} B^o; \\ &\Rightarrow \quad \{x\} \triangleleft_{\mathcal{S}} B^o. \end{aligned}$$

Now, to show (4.1.2), on one hand, it is clear that $\bigcup \big\{B \in \tau_{\mathcal{S}} \mid B \triangleleft_{\mathcal{S}} A\big\} \subseteq A$, since $B \triangleleft_{\mathcal{S}} A \Rightarrow B \subseteq A$. On the other hand if $x \ \in A$ then $\{x\} \triangleleft_{\mathcal{S}} A$, by definition of $A \in \tau_{\mathcal{S}}$. So, there exists C such that $\{x\} \triangleleft_{\mathcal{S}} C \triangleleft_{\mathcal{S}} A$ and $\{x\} \subseteq C^o \subseteq C \subseteq A$. But, by the previous argument, $C^o \in \tau_{\mathcal{S}}$, so $A \subseteq \bigcup \big\{B \in \tau_{\mathcal{S}} \mid B \triangleleft_{\mathcal{S}} A\big\}$ and this completes the proof.

□

Remark 4.1.3. Let $\mathcal{S}$ be a syntopogenous structure on X. For all $A, B \subseteq X$,

$$\begin{aligned} A \triangleleft_{\mathcal{S}} B \quad &\Rightarrow \quad \exists\, i \mid A \triangleleft_i B; \\ &\Rightarrow \quad \exists\, C, \exists\, j \mid A \triangleleft_j C \triangleleft_j B; \\ &\Rightarrow \quad A \triangleleft_{\mathcal{S}} C \triangleleft_{\mathcal{S}} B. \end{aligned}$$

Also, given $\lhd \in \mathcal{S}$;

$$\begin{aligned} A \lhd B &\Rightarrow A \lhd_{\mathcal{S}} B; \\ &\Rightarrow \exists\, C \mid A \lhd_{\mathcal{S}} C \lhd_{\mathcal{S}}; B; \\ &\Rightarrow \exists\, C,\ \exists\, i, j \mid A \lhd_i C \lhd_j B; \\ &\Rightarrow \exists\, C,\ \exists\, k, \lhd_j \cup \lhd_i \subseteq \lhd_k \ and\ A \lhd_k C \lhd_k B. \end{aligned}$$

Thus, in view of $(S1)$ of Definition (4.1.1), $(S2)$ is equivalent to saying that $\lhd_{\mathcal{S}}$ is interpolative.

It is important to emphasize that Császár's axioms are rooted in set algebra and, in most instances, do not necessitate the use of individual points of the space. This observation underscores that syntopogenous structures serve as a point of departure for the concept of "pointfree topology".

4.2 Syntopogenous Structures on Complete Lattices

We investigate the notion of syntopogenous structures on a general complete lattice. While our definition extends beyond [WL95] and, to some extent [HI19], it also encapsulates the work of Chung [Chu08] as a special case.

In the context of a syntopogenous space $(X, \mathcal{S})$, the syntopogenous structure on X is defined using topogenous orders on $P(X)$. These orders always constitute a sublattice of $P(X) \times P(X)$. Since $P(X)$ is a complete lattice, and in particular, a frame (pointless structure), we aim to axiomatize the concept of syntopogenous structures on general lattices motivated by Definition (3.4.3).

Definition 4.2.1. Let X be any complete lattice. A syntopogenous structure $\mathcal{S}_X$ on X is a family

$$\mathcal{S}_X = \{\lhd_i \mid i \in I\}$$

of topogenous orders on X which satisfy the following three axioms:

($\mathcal{S}1$) each $\lhd_i \in \mathcal{S}_X$ is a meet-preserving topogenous order on X;

($\mathcal{S}2$) $\mathcal{S}_X$ is directed, meaning that for each $\lhd_i, \lhd_j \in \mathcal{S}_X$ there exists $\lhd_k \in \mathcal{S}_X$ such that $\lhd_i \cup \lhd_j \subseteq \lhd_k$;

($\mathcal{S}3$) For every $\lhd_i \in \mathcal{S}_X$ there is $\lhd_j \in \mathcal{S}_X$ such that for all $x, y \in X$, if $x \lhd_i y$, then there exists $z \in X$ such that $x \lhd_j z \lhd_j y$.

The pair $(X, \mathcal{S}_X)$ where X is a lattice and $\mathcal{S}_X$ is a syntopogenous structure on X is called a syntopogenous lattice. Moreover, we shall say that a syntopogenous structure on a lattice X is interpolative if each $\lhd \in \mathcal{S}$ interpolates.Let X be a complete lattice. We represent the collection of all syntopogenous structures on X as $\mathsf{SYNT}(X)$.

4.3 Quasi-uniformities on Complete Lattices

The general theory of quasi-uniformities on a set is usually presented in terms of entourages or cover-like approach. While the former has been extensively studied in the context of spaces, the latter has been demonstrated to be of greater interest to pointfree topologists. As a justification of this statement see, for instance, ([Fri86],[PP11]).

In this section, we delve into the examination of (entourage) quasi-uniformities on complete lattices. Within the framework of frames, P. Fletcher, W. Hunsaker, and W. Lindgren in [FHL93b] provided an initial foray into this topic. Diverging from the approach of the authors in ([FHL93b]), who introduced the study of quasi-uniformities through the concept of entourages on frames, our focus is on the interplay between quasi-uniformities and syntopogenous structures on lattices.

Definition 4.3.1. Let X be a complete lattice, and $\mathcal{U}$ a family of maps from X into X. Then $\mathcal{U}$ is called a quasi-uniformity on X if the following axioms hold:

(Q1) Each $U \in \mathcal{U}$ is an order preserving map $U : X \longrightarrow X$ with $id_X \leq U$, that is, $x \leq U(x)$ for all $x \in X$;

(Q2) For all $U, V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $W \leq U \wedge V$ (order taken pointwise in X);

(Q3) For all $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ with $V \circ V \leq U$;

(Q4) If $U \in \mathcal{U}$ and $U \leq V$ then $V \in \mathcal{U}$.

We denote by $\mathsf{Q}(X)$ the collection of all the quasi-uniformities on X. The pair $(X, \mathcal{U})$, where X is a complete lattice and $\mathcal{U}$ a quasi-uniformity on X will be called a quasi-uniform lattice. In most instances, our focus will be on manipulating the base of a quasi-uniformity. Hence, the collection adhering to axioms $(Q1)$ through $(Q3)$ is identified as the base of the quasi-uniformity. We denote by $\mathbf{B}(X)$ the collection of all bases of a quasi-uniformity $\mathcal{U}$ on X.

We are now ready to set out one of the significant discoveries in this chapter: establishing an equivalence between a syntopogenous structure and a base of a quasi-uniformity on a lattice.

4.4 Nexus Between Quasi-uniformities and Syntopogenous Structures on a Complete Lattice

We establish a connection between syntopogenous structures and quasi-uniformities on a complete lattice X. If we consider topogenous orders which respect meets, we demonstrate that a syntopogenous structure on a complete lattice forms a base of a quasi-uniformity. In other words, the syntopogenous structure and a basis of a quasi-uniformity are isomorphic. However, for topogenous orders that do not respect meets, we establish a Galois adjunction between syntopogenous structures and quasi-uniformities on a lattice.

Let $\geq$ be a binary relation on $\mathsf{SYNT}(X)$ define $\mathcal{S}_X \geq \mathcal{S}'_X$ if for all $\triangleleft_X \in \mathcal{S}_X$ there exists $\triangleleft'_X \in \mathcal{S}'_X$ such that $\triangleleft_X \subseteq \triangleleft'_X$. Thus $'' \geq ''$ is a preorder on $\mathsf{SYNT}(X)$. Furthermore, define $\mathcal{B} \geq \mathcal{B}'$ if for all $u \in \mathcal{B}$ there is $v \in \mathcal{B}'$ with $v \leq u$. Evidently $'' \geq ''$ is a preorder on $\mathsf{B}(X)$. Note that when speaking of $\mathbf{B}(X)$, we will always assume it is ordered in the similar manner as $\mathsf{Q}(X)$

Proposition 4.4.1. *Let $\mathsf{B}(X)$ be the collection of bases of quasi-uniformity on the lattice X. For $\mathcal{B} \in \mathsf{B}(X)$ define*

$$\mathcal{S}^{\mathcal{B}} = \{\lhd^u \mid u \in \mathcal{B}\} \quad where \quad x \lhd_X^u y \Leftrightarrow u(x) \leq y.$$

Then $\mathcal{S}^{\mathcal{B}}$ is a syntopogenous structure on X. Futhermore, the map $\mathbf{F} : \mathsf{B}(X) \longrightarrow \mathsf{SYNT}(X)$, taking each $\mathcal{B}$ to $\mathcal{S}^{\mathcal{B}}$ is order preserving.

Proof. *(S1)* (a) Since $id_X \leq U$, $x \lhd^U y \Rightarrow x \leq U(x) \leq y \Rightarrow x \leq y$;

(b) Since each U is order preserving, then $x \leq y \lhd^U z \leq y \Rightarrow U(x) \leq U(y) \leq z \leq y \Rightarrow x \lhd^U y$;

(c) $x \lhd^U y_i \quad \forall i \in I \Rightarrow U(x) \leq y_i \quad \forall i \in I \Rightarrow U(x) \leq \bigwedge_{i \in I} y_i \Rightarrow x \lhd^U \bigwedge_{i \in I} y_i$ and $\lhd^U$ is meet preserving.

(S2) Note that if $U \leq V$ then $x \lhd^V y \Rightarrow U(x) \leq V(x) \leq y \Rightarrow x \lhd^U y$. Hence given $\lhd^U, \lhd^V \in \mathcal{S}^{\mathcal{U}}$, since $U \wedge V \in \mathcal{U}$, we obtain $\lhd^U \cup \lhd^V \subseteq \lhd^{U \wedge V} \in \mathcal{S}^{\mathcal{U}}$;

(S3) Given $\lhd^U \in \mathcal{S}^{\mathcal{U}}$ there is $V \in \mathcal{U}$ with $V \circ V \leq U$. Then $x \lhd^U y \Rightarrow U(x) \leq y$. Thus $V(V(x)) \leq U(x) \leq y$ and $x \lhd^V V(x) \lhd^V y$.

Let $\mathbf{F}$ be the function mapping $\mathcal{B} \mapsto \mathcal{S}^{\mathcal{B}}$ and let $\mathcal{B}', \mathcal{B} \in \mathcal{S}^{\mathcal{B}}$ with $\mathcal{B}' \geq \mathcal{B}$ then for all $\lhd^{B'} \in \mathcal{S}^{\mathcal{B}}$ there is $B \in \mathcal{B}$ with $B \leq B'$, thus $\lhd^B \in \mathcal{S}^{\mathcal{B}}$ with $\lhd^{B'} \subseteq \lhd^B$ that is to mean $\mathbf{F}(\mathcal{B}') = \mathcal{S}^{\mathcal{B}'} \geq \mathcal{S}^{\mathcal{B}} = \mathbf{F}(\mathcal{B})$.

□

Proposition 4.4.2. *Let $\mathcal{S} \in \mathsf{SYNT}(X)$ and $\lhd \in \mathcal{S}$. The assignment*

$$\mathcal{B}^{\mathcal{S}} = \{U^{\lhd} \mid \lhd \in S\} \text{ with } U_X^{\lhd}(x) = \bigwedge \{y \in X \mid x \lhd y\}$$

defines a base of a quasi-uniformity $\mathcal{U}$. Moreover the map $\mathbf{G} : \mathsf{SYNT}(X) \longrightarrow \mathsf{B}(X)$, taking each $\mathcal{S}$ to $\mathcal{B}^{\mathcal{S}}$, is order preserving.

Proof. *(Q1)* Clearly $U^{\lhd} : X \longrightarrow X$ is order preserving and since, by $(T1)$, $x \lhd y \Rightarrow x \leq y$, it follows that $x \leq U(x)$.

(Q2) Consider $U^{\lhd_1}$ and $U^{\lhd_2} \in \mathcal{U}^{\mathcal{S}}$ for some $\lhd \in \mathcal{S}$. Then using $(S2)$, there is $\lhd_3$ with $\lhd_1 \cup \lhd_2 \subseteq \lhd_3$ and so for any $x \in X$, $\{y \in X \mid x \lhd_1 y\} \subseteq \{y \in X \mid x \lhd_3 y\}$ giving $U^{\lhd_3}(x) \leq U^{\lhd_1}(x)$. Similarly, it can be shown that $U^{\lhd_3} \leq U^{\lhd_2}$ and so $U^{\lhd_3} \leq U^{\lhd_1} \wedge U^{\lhd_2}$.

(Q3) In view of $(S3)$, given $\lhd \in \mathcal{S}$ there is $\lhd' \in \mathcal{S}$ such that $x \lhd y$ implies there is $z \in X$ with $x \lhd' z \lhd' y$. Then $U^{\lhd'}(x) \leq z \lhd' y$ and so $U^{\lhd'}(x) \lhd' y$. This gives that $\{y \mid x \lhd y\} \subseteq \{y \mid U^{\lhd}(x) \lhd' y\}$ and hence $U^{\lhd'}(U^{\lhd'}(x)) \leq U^{\lhd}(x)$, that is, $U^{\lhd'} \circ U^{\lhd'} \leq U^{\lhd}$ as needed.

Let $\mathbf{G}$ be the function mapping $\mathcal{S} \mapsto \mathcal{B}^{\mathcal{S}}$ and let $\mathcal{S}, \mathcal{S}' \in \mathsf{SYNT}(X)$ with $\mathcal{S} \geq \mathcal{S}'$ then for all $\lhd \in \mathcal{S}$ there exists $\lhd' \in \mathcal{S}'$ with $\lhd \subseteq \lhd'$ giving for all $U^{\lhd} \in \mathcal{B}^{\mathcal{S}}$ there is $V^{\lhd} \in \mathcal{B}^{\mathcal{S}'}$ with $V^{\lhd'} \leq U^{\lhd}$ and $\mathbf{G}(\mathcal{S}) = \mathcal{B}^{\mathcal{S}} \geq \mathcal{B}^{\mathcal{S}'} = \mathbf{G}(\mathcal{S}')$.

□

In consequence of Propositions (4.4.1) and (4.4.2), the statement of the following theorem holds:

Theorem 4.4.1. *Let* $\mathsf{B}(X)$ *be the collection of all bases of a uniformity on the lattice* X *and* $\mathsf{SYNT}(X)$ *that of all the syntopogenous structures on* X*. Then* $\mathsf{B}(X)$ *and* $\mathsf{SYNT}(X)$ *are order isomorphic, that is,*

$$\mathsf{B}(X) \cong \mathsf{SYNT}(X)$$

Proof. As per Propositions (4.4.1) and (4.4.2), $\mathcal{S}^{\mathcal{U}}$ and $\mathcal{U}^{\mathcal{S}}$ constitute syntopogenous structures and bases of a quasi-uniformity on X, respectively. In addition, the mappings $\mathsf{SYNT}(X) \longrightarrow \mathsf{B}(X)$ and $\mathsf{B}(X) \longrightarrow \mathsf{SYNT}(X)$ are order preserving. The final step is to prove that these mappings are inverses of each other. Indeed,

$$\begin{aligned} U^{\triangleleft^U}(x) &= \bigwedge\{a \in X \mid x \triangleleft^U a\}; \\ &= \bigwedge\{a \in X \mid U(x) \leq a\}; \\ &= U(x). \end{aligned}$$

On the other hand,

$$\begin{aligned} x \triangleleft^{U^{\triangleleft}} y &\Leftrightarrow U^{\triangleleft}(x) \leq y; \\ &\Leftrightarrow \bigwedge\{a \mid x \triangleleft a\} \leq y; \\ &\Leftrightarrow x \triangleleft y, \ \text{ since } \triangleleft \in \bigwedge\text{-TORD(X)}. \end{aligned} \tag{4.4.4}$$

□

The culmination of our discussions are summarized in the following theorem which affirms that a syntopogenous structure and a base of a quasi-uniformity on a complete lattice are fundamentally equivalent. In other words, the theorem asserts that a syntopogenous structure is a base of quasi-uniformities on a complete lattice.

If $\mathcal{U} \in \mathsf{B}(X)$, then $\mathcal{U}' = \{U : X \longrightarrow X \mid U$ is order preserving and there exists $V \in \mathcal{U}$ with $V \leq U\}$ is a quasi-uniformity on X. Denote by $Q(X)$ the collection of all quasi-uniformities on X. We have the following Galois correspondence between $\mathsf{B}(X)$ and $\mathsf{Q}(X)$.

Theorem 4.4.2.

$$Q(X) \underset{()'}{\overset{id}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} B(X)$$

between $\mathsf{Q(X)}$ *and* $\mathsf{B(X)}$

Proof.

$$\begin{aligned} \mathcal{U}' \geq^{op} \mathcal{V} &\Leftrightarrow \forall\, V \in \mathcal{V} \ \exists \ U \in \mathcal{U}', U \leq V; \\ &\Leftrightarrow \forall\, V \in \mathcal{V} \ \exists \ U \in \mathcal{U}, U \leq V; \\ &\Leftrightarrow \mathcal{U} \leq \mathcal{V}. \end{aligned}$$

□

Theorem (4.4.1) implies that a syntopogenous structure on a complete lattice is essentially the same as the base of a quasi-uniformity. Furthermore, if we remove the requirement that every $\mathcal{S} \in SYNT(X)$, every $\lhd_X \in \mathcal{S}$ is respects meets, the reverse implication in equation (4.4.4) will not hold and, as a result, the isomorphism obtained in Theorem (4.4.1) fails. In such instances, we establish a Galois connection between $B(X)$ and a subcollection of $\mathsf{SYNT}(X)$ in which topogenous orders need not respect meet. This pivotal insight is central to the next theorem.

We use $SYNT(X)$ to denote the collection of all syntopogenous structures on the complete lattice X such that for each $\mathcal{S} \in SYNT(X)$, every $\lhd_X \in \mathcal{S}$ need not respect meet. We also use the letters F and G to denote the extension of the mappings $\mathbf{F}$ and $\mathbf{G}$ on $SYNT(X)$, respectively.

Theorem 4.4.3. *The pair of mappings $F : B(X) \longrightarrow SYNT(X)$ and $G : SYNT(X) \longrightarrow B(X)$ form a Galois connection*

$$PQ(X) \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} SYNT(X)$$

between $B(X)$ and $SYNT(X)$

Proof.

$$\begin{aligned}
\mathcal{S} \geq \mathcal{S}^{\mathcal{U}} \quad &\Leftrightarrow \quad \forall \lhd \in \mathcal{S}\ \exists \lhd^U \in \mathcal{S}^{\mathcal{U}},\ \lhd \subseteq \lhd^U; \\
&\Leftrightarrow \quad \forall \lhd \in \mathcal{S}\ \exists\ U \in \mathcal{U},\ \lhd \subseteq \lhd^U; \\
&\Leftrightarrow \quad \forall \lhd \in \mathcal{S}\ \exists\ U \in \mathcal{U},\ (x \lhd y \Rightarrow U(x) \leq y); \\
&\Leftrightarrow \quad \forall \lhd^U \in \mathcal{U}^{\mathcal{S}}\ \exists\ U \in \mathcal{U},\ (U(x) \leq \bigwedge\{y \mid x \lhd y\}); \\
&\Leftrightarrow \quad \forall \lhd^U \in \mathcal{U}^{\mathcal{S}}\ \exists\ U \in \mathcal{U},\ U \leq U^{\lhd}; \\
&\Leftrightarrow \quad \mathcal{U}^{\mathcal{S}} \geq \mathcal{U}.
\end{aligned}$$

□

4.5 Closure and Interior Operations Induced by a Quasi-uniformity

Like in the classical case, in this section, we explore the notion of closure and interior operations induced by the quasi-uniformity in a complete lattice.

The general theory tells us that each quasi-uniform space $(X, \mathcal{U})$ is always associated with two other quasi-uniformities [FHL94]: the conjugate quasi-uniformity which we denote by $\tilde{\mathcal{U}}$ and the join uniformity denoted by $\mathcal{U}^* = \mathcal{U} \vee \tilde{\mathcal{U}}$. Additionally, these three quasi-uniformities induce three distinct topologies, denoted as $\Im(\mathcal{U}), \Im(\tilde{\mathcal{U}})$ and $\Im(\mathcal{U}^*)$. For instance,

$$\Im(\mathcal{U}) = \{A \subseteq X \mid \text{for each } a \in A \text{ there exists } U \in \mathcal{U} \text{ with } U(a) \subseteq A.\}$$

If $\Im(\mathcal{U}) = \Im$ where $\Im$ is a given topology on X then $\mathcal{U}$ is said to be compatible with $\Im$. One also says that $(X, \Im)$ admits $\mathcal{U}$. In consequence of this, we also have

$$\tilde{A} = \bigcap\{U^{-1}(A) : U \in \mathcal{U}\} \text{ and } A^o = \{x \mid \text{ there exists } U \in \mathcal{U} \text{ with } U(a) \subseteq A\}$$

respectively, the closure and interior operators associated with the quasi-uniformity $\mathcal{U}$.

All the concepts mentioned above are highly applicable in the realm of lattice theory and more importantly within frames, as we shall explore in Section 4.9. Therefore, in accordance with Proposition (3.3.12) and Theorem (4.4.1), we can derive an interior and closure operator associated with a quasi-uniformity $\mathcal{U}$ on a given lattice X.

Proposition 4.5.1. *Let X be a complete lattice and $\mathcal{U}$ be a quasi-uniformity on X. Then*

(1) The assignment $k = \{k_X : X \longrightarrow X,\ with\ X\ \ a\ \ lattice\}$ given by

$$k^{\mathcal{U}}(x) = \bigwedge\{U(x) : U \in \mathcal{U}\}$$

is an idempotent closure operation on X. Dually,

(2) the assignment $i = \{i_X : X \longrightarrow X,\ \ with\ \ X\ \ a\ \ lattice\}$ given by

$$i^{\mathcal{U}}_X(x) = \bigvee\{p \in X \mid U(p) \leq x\ \ for some\ U \in \mathcal{U}\}$$

is an idempotent interior operation on X.

4.6 Transitive Quasi-Uniformity Determined by a Meet Structure in Lattice

In Chapter 2, we highlighted the significance of a $\bigwedge$-structure in a complete lattice, as it plays an important role by characterizing closed subsets of a complete lattice. Thus, we devote this section to the study of quasi-uniformities induced by a $\bigwedge$-structure in a complete lattice.

Definition 4.6.1. Let X be a complete lattice and $\mathcal{B}_X$ a base of a quasi-uniformity $\mathcal{U}$ on X. Then $\mathcal{B}_X$ is said to be transitive if for every $U \in \mathcal{B}_X$, $U \circ U = U$.

A quasi-uniformity is called transitive if it has a transitive base. The collection of all transitive quasi-uniformities on the complete lattice X will be denoted by $\mathsf{TQ}(X)$. The forthcoming proposition, which bears resemblance with Proposition 3.2 in [HI19], relates interpolative syntopogenous structures and transitive bases of a quasi-uniformity.

Proposition 4.6.1. $\mathsf{INTSYNT(X)}$ *is isomorphic to* $\mathsf{TQ}(X)$

Proof. Let $\mathcal{S} \in \mathsf{INTSYNT(X)}$ and $\triangleleft \in \mathcal{S}_X$ for all $x, y \in X$, if $x \triangleleft y$ then there is $a \in X$ such that $x \triangleleft a \triangleleft y$ which implies $U^{\triangleleft}(x) \leq a \triangleleft y$ and by $(T2)$, we get $U^{\triangleleft}(x) \leq y$. Thus $\bigwedge\{t \mid U^{\triangleleft}(x) \triangleleft t\} \leq \bigwedge\{g \mid x \triangleleft g\}$ which means that $U^{\triangleleft}(U^{\triangleleft}(x)) \leq U^{\triangleleft}(x)$. Further, using $(PQ(1))$, we also have that $U^{\triangleleft}(x) \leq U^{\triangleleft}(U^{\triangleleft}(x))$. Hence $U^{\triangleleft}(x) = U^{\triangleleft}(U^{\triangleleft}(x))$.

On the other hand, let $\mathcal{B}$ be a base of a transitive quasi-uniformity and $x \triangleleft^U y$ for some $U \in \mathcal{U}$. Clearly $U(x) = U(U(x)) \triangleleft^U U(x) \leq y$ and hence $U \triangleleft^U U(x) \triangleleft^U y$.

Theorem 4.6.2. *Let X be a complete lattice and let $\mathcal{M} \subseteq \mathsf{MS}(X)$ with $\mathcal{M} \neq 0$. If $\mathcal{M}$ is closed under meets then*

$$\mathcal{B}^{\mathcal{M}} = \{U^M_X \mid M \in \mathcal{M}\}$$

where $U_X^M : X \longrightarrow X$ *and*

$$U_X^M(x) = \bigwedge\{y \in X \mid x \triangleleft y\}$$

is a base of a transitive quasi-uniformity $\mathcal{U}$ *on* X.

Proof. $(Q1)$ is clear. For $(Q2)$, let $M, N \in \mathcal{M}$ and $x \in X$. *Then*

$$\begin{aligned}(U^M \wedge U^N)(x) &= U^M(x) \wedge U^N(x);\\ &= \bigwedge\{y \in M \mid x \leq y\} \wedge \bigwedge\{z \in N \mid x \leq z\};\\ &= \bigwedge\{c \in M \cup N \mid x \leq c\};\\ &= U^{N\cup M}(x).\end{aligned}$$

Now since $M, N \subseteq M \cup N$ then $M \cup N \in \mathcal{M}$ and so $U^M \wedge U^N \in \mathcal{B}^{\mathcal{M}}$.

(iii) Lastly, for $(PQ3)$ note that if $x \in M$ then $U^M(x) = x$ since M is a $\bigwedge$-structure, $U^M(x) = \bigwedge\{y \in M \mid x \leq y\} \in M$. Hence $U^M(U^M(x)) = U^M(x)$ for any $x \in X$ and $U^M \circ U^M = U^M$. Thus $\mathcal{B}_X^{\mathcal{M}}$ is a transitive base for a quasi-uniformity on X.

Theorem (4.6.2) is highly significant as it aids in the construction of quasi-uniformities generated by idempotent closure operators on a complete lattices.

4.7 Quasi-Uniformly and Syntopogenously Continuous Maps

We extend the well-established classical concept of syntopogenously and quasi-uniformly continuous maps to our general framework. To accomplish this, we revisit the notion of pseudofunctor, which establishes the context within which we will operate in this section: let $\mathcal{X}$ be a general category and $P : \mathcal{X} \longrightarrow$ **Pos** a pseudofunctor to the category of posets and order preserving maps, which to every morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ assigns an adjoint pair:

$$\mathrm{PX} \underset{f^*}{\overset{f^o}{\rightleftarrows}} \mathrm{PY} \quad (\bot)$$

Now,

Quasi-Uniformly Continuous Maps

Definition 4.7.1. A quasi-uniformity on $\mathcal{X}$ is a family

$$\mathcal{U} = \{\mathcal{U}_{PX} \mid X \in \mathcal{X}\}$$

such that each $\mathcal{U}_{PX}$ is a quasi-uniformity on PX (in the sense of Definition (4.2.1)) and for each morphism $f : X \longrightarrow Y \in \mathcal{X}$ and each $U \in \mathcal{U}_{PX}$ there is $V \in \mathcal{U}_{PX}$ such that

$$U(f^*(y)) \leq f^*(V(y)) \tag{4.7.5}$$

holds for all $y \in PY$. When (4.7.5) holds, we also say that f is quasi-uniformly continuous.

Remark 4.7.2. If $\mathcal{X} \cong \mathbf{Pos}$, every morphism in $\mathcal{X}$ coincides with the right adjoint, that is, $f = f^*$ and if furthermore, we assume that f^* commutes with all joins, in view of Lemma (2.2.4) f^* has a right adjoint. Thus, we have the following other adjuction

$$Y \underset{f_*}{\overset{f^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} X$$

*Of course, as we have already mentioned and without loss of generality, we remind the reader that in certain instances, we will have to consider the objects in the category **Pos** to be special posets such as complete lattices.*

The most common situation where this occurs is when $\mathcal{X} \cong \mathbf{Frm}$. Each frame homomorphism $f : X \longrightarrow Y$ is associated with a right adjoint such that we have the adjuction $f \dashv f_*$ between the frames X and Y.

Bearing in mind that in **Frm** f is just the same as f_*, we shall say that $f : X \longrightarrow Y$ (frame homomorphism in this case) is quasi-uniformly continuous if for every $U \in \mathcal{V}_Y$ there exists $V \in \mathcal{U}_X$ such that

$$U(f^*(y)) \leq f^*(V(y)) \tag{4.7.6}$$

holds for all $y \in Y$.

At a certain juncture, we will need a category in which the objects are the quasi-uniform lattices and morphisms are the quasi-uniformly continuous maps. We will represent this category with the symbol **QUNLatt**.

Syntopogenously Continuous Maps

Definition 4.7.3. A syntopogenous structure on a category $\mathcal{X}$ is a family

$$\mathcal{S} = \{\mathcal{S}_{PX}, \mid X \in \mathcal{X}\}$$

such that each $\mathcal{S}_{PX}$ is a syntopogenous structure on PX and for each $f : X \longrightarrow Y$ in $\mathcal{X}$ and each $\triangleleft_Y \in \mathcal{S}_{PY}$ there exits $\triangleleft_X \in \mathcal{S}_{PX}$ such that

$$x \triangleleft_Y y \Rightarrow f^*(x) \triangleleft_X f^*(y)$$

for all $x, y \in PY$.

In this case, we say that f is syntopogenously continuous. In accordance with Remark (4.7.2), if $\mathcal{X} \cong \mathbf{Pos}$ the syntopogenously condition implies that for every $\triangleleft_X \in \mathcal{S}_X$ there exists $\triangleleft_Y \in \mathcal{S}_Y$ such that

$$x \triangleleft_X y \Rightarrow f^*(x) \triangleleft_Y f^*(y)$$

for all $x, y \in Y$.
Thanks to Galois adjoint, the continuity condition in Definition (4.7.3) can be alternatively expressed as follows:

Proposition 4.7.1. *Let $\mathcal{S}$ be a syntopogenous structure on $\mathcal{X}$. A morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ is said to be syntopogenously continuous if and only if for all $\triangleleft_Y \in \mathcal{S}_Y$ there exists $\triangleleft_X \in \mathcal{S}_X$ such that*

$$f^o(x) \triangleleft_Y y \Rightarrow x \triangleleft_X f^*(y)$$

for all $x \in X$ and $y \in Y$.

Proposition 4.7.2. *Suppose that $\mathcal{S}$ is a syntopogenous structure on $\mathcal{X}$ and $f : X \longrightarrow Y$ a morphism in $\mathcal{X}$ such that f^* preserves all joins. Then f is syntopogenously continuous if and only if for all $\triangleleft_Y \in \mathcal{S}_Y$ there exists $\triangleleft_X \in \mathcal{S}_X$ such that*

$$x \triangleleft_Y f_*(x) \Rightarrow f^*(y) \triangleleft_X y$$

for all $x \in X$ and $y \in Y$.

We use the symbol **SYNTLatt** to denote the category of syntopogenous lattices and syntopogenously continuous maps.

The Proposition (4.7.3) below relates the quasi-uniformly continuous maps and syntopogenously continuous morphisms in any general category $\mathcal{X}$.

Proposition 4.7.3. *Let $\mathcal{S}^{\mathcal{U}}$ be the syntopogenous structure induced by a quasi-uniformity $\mathcal{U}$ and $\mathcal{U}^{\mathcal{S}}$ the quasi-uniformity induced by a syntopogenous structure $\mathcal{S}$ on the category $\mathcal{X}$, respectively. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{X}$ such that f^* preserves all joins. Then*

(1) $x \triangleleft^V_Y y \Rightarrow f^*(x) \triangleleft^U_X f^*(y)$

(2) $x \triangleleft^U_Y f_*(z) \Rightarrow f^*(x) \triangleleft^V_X z$

for all $x, y \in Y$ and $z \in X$.

Proof. Let $f : X \longrightarrow Y$ be an $\mathcal{X}$-morphism. If

$$U(f^*(y)) \leq f^*(V(y)) \tag{4.7.7}$$

holds for all $y \in Y$. Then for all $x, z \in X$

$$\begin{aligned} x \triangleleft^V_Y y &\Leftrightarrow V(x) \leq y; \\ &\Rightarrow f^*(V(x)) \leq f^*(y); \\ &\Rightarrow U(f^*(x)) \leq f^*(V(x)) \leq f^*(y) \text{ by (4.7.7)}; \\ &\Rightarrow U(f^*(x)) \leq f^*(y); \\ &\Leftrightarrow f^*(x) \triangleleft^U_X f^*(y). \end{aligned}$$

$$\begin{aligned} x \triangleleft^V f_*(z) &\Rightarrow V(x) \leq f_*(z); \\ &\Rightarrow f^*(V(x)) \leq f^*(f_*(z)) \leq z; \\ &\Rightarrow f^*(V(x)) \leq z; \\ &\Rightarrow U(f^*(x)) \leq f^*(V(x)) \leq z \text{ by (4.7.7)}; \\ &\Rightarrow U(f^*(x)) \leq z \Leftrightarrow f^*(x) \triangleleft^U z. \end{aligned}$$

□

4.8 Example

(1) Let $\mathcal{X} = \mathbf{Grp}$ represent the category of Groups and group homomorphisms between them. Each group homomorphism $f : G \longrightarrow G'$ induces an adjunction between the lattices of subgroups of the groups G and G':

$$\mathrm{L(G)} \underset{f^{-1}}{\overset{f^{o}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{L(G')}$$

Set

$$\mathcal{S} = \{\lhd_G \mid G \in \mathbf{Grp}\}$$

where for all subgroups $H_1, H_2 \subseteq G$ we define $H_1 \lhd_G H_2$ if there exists a normal subgroup N in G with $H_1 \subseteq N \subseteq H_2$. Then $\mathcal{S}$ is a simple syntopogenous structure on $\mathbf{Grp}$.

One can also express this syntopogenous structure as a structure on a complete lattice: indeed, let $L(G)$ be a complete lattice of subgroups of G and consider $\lhd_G$ as an order relation on $L(G)$. Then, $(L(G), \lhd_G)$ is a syntopogenous lattice. This means that one can express the $\lhd_G$ either with reference to G or $L(G)$.
Thus, for any group G we have all possible $\lhd_i$ on G. We can also think of these as $\lhd_i$ on $L(G)$. Then, if $\mathcal{S} = \{\lhd_i\}$ we have that $(L(G), \mathcal{S})$ is a syntopogenous lattice.

(2) Let $\mathcal{X} = \mathbf{Top}$, the category of topological spaces and continuous maps. Each continuous map $f : X \longrightarrow Y$ induces the image/preimage adjunction between the powerset lattices as shown below

$$\mathrm{PX} \underset{f^{-1}}{\overset{f^{o}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{PY}$$

Put

$$\mathcal{S} = \{\lhd_X \mid X \in \mathbf{Top}\}$$

where, for all $A, B \subseteq X$, we define

$$A \lhd_X B \Leftrightarrow A \subseteq C \subseteq B$$

for some $C \in \tau$. Then, $\mathcal{S}$ is a single syntopogenous structure on $\mathbf{Top}$.

As in the previous example, one can think of this syntopogenous structure as a structure on PX. So, considering $\lhd_X$ as an order relation on PX, then $(PX, \lhd_X)$ is a sytopogenous lattice. Therefore, given a topological space X, we have all $\lhd_i$ on X. We can also think of these orders as $\lhd_i$ on PX. Hence, if $\mathcal{S} = \{\lhd_i\}$ we have that $(PX, \mathcal{S})$ is a syntopogenous lattice.

4.9 Syntopogenous Structures in Pointfree Topology

As we have already observed in the previous chapters, Császár orders are used as a foundational framework in pointfree topology. They are utilized to define lattice structures that provide a way of studying topological spaces in a more abstract way. These orders are fundemantal in developing the theory of frames, which is one of the key notion in pointfree topology.

In the realm of spaces, if one reinforces the axioms of a syntopogenous space in certain ways, not only does it yield a quasi-uniform space but also numerous other topological spaces among which we may list proximity and uniform spaces. Thus, syntopogenous structures appear as a general approach to study all these other structures concomitantly.

In pointfree topology the same observations have been made. Apart from Chung [Chu01] who formally defined a syntopogenous structure on a frame, the syntopogenous structures have been extensively studied in the context of pointfree topology under the guise of uniform structures and proximity structures. This means that like in the classical case, syntopogenous structures on frames encompass uniform structures, proximity structures even quasi-uniformities. Thus, the category **SYNTFrm** contains **ProxFrm** (see [Fri86]) and **UniFrm** (see [PP11, Pic95]) as subcategories. As result of this, there is an embedding from **UniFrm** into **SYNTFrm** and from **ProxFrm** into **SYNTFrm**. As we shall see below, Frith [Fri86] established a correspondence between **ProxFrm** and **UniFrm**.

On the other hand, if we focus our discussions on frames rather than complete lattices, we obtain a frame counter part of the results in ([HI19]). In this context, we would say that a quasi-uniformity is a family of closure operations on a frame and that a syntopogenous structure is a base of a quasi-uniformity on a frame.

To provide a better understanding of Császár's orders within a frame, we start by departing from the lattice of open sets and then generalize this approach to all frames.

Given a topological space X, let us now consider, ΩX the lattice of open sets, instead of PX. As in the previous chapter, each continuous map $f : X \longrightarrow Y$ leads to the following alternative adjunction:

$$\Omega\, \mathrm{X} \underset{f_*}{\overset{f^{-1}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \Omega \mathrm{Y}$$

where, as before, f_* is the dual image and is given by

$$f_*(A) = \bigcup\{B \subseteq \Omega Y \mid f^{-1}(B) \subseteq A\}.$$

Now put

$$\mathcal{S} = \{\triangleleft_X \mid X \in \textbf{Top}\}$$

where, for all $A, B \in \Omega X$,

$$A \triangleleft B \Leftrightarrow \overline{A} \subseteq B \Leftrightarrow A^* \cup B = X$$

is a syntopogenous structure on **Top**. Further, speaking of ΩX as a complete lattice of open subsets of X, let $\triangleleft_X$ be an order relation on ΩX. Then the pair $(\Omega X, \triangleleft_X)$ is a syntopogenous lattice which in this case we could also call syntopogenous frame. Hence, $\triangleleft_X$ can also be expressed with reference to ΩX.

Taking into account the fact that ΩX is an abstraction of frames and that the topogenous order $A \triangleleft_{\Omega X} B \Leftrightarrow \overline{A} \subseteq B$ is a general concept applied to any frames, we can now extend the concept of syntopogenous structures on frames. It is important to acknowledge that the initial introduction of this concept to frames is due to [Chu01]:

Let L be a given frame and $\mathcal{S}_L$ a family of topogenous orders on L such that:

(1) $\mathcal{S}_L$ satisfies $(S1)$ and $(S2)$ of Definition (4.2.1) and ;

(2) $(S3)$ each $x \in L$ can be written as

$$x = \bigvee \{y \in L \mid y \lhd_{\mathcal{S}_L} x\}$$

with $\lhd_{\mathcal{S}_L} = \bigcup \{\lhd \in \mathcal{S}_L\}$.

Then the pair $(L, \mathcal{S}_L)$ is called a syntopogenous frame.

Let $(M, \mathcal{S}_M)$ be an a syntopogenous frame. Then a frame homomorphism $h : L \longrightarrow M$ is said to be syntopogenously continuous if for any $\lhd_L \in \mathcal{S}_L$ there exists $\lhd_M \in \mathcal{S}_M$ such that for all $x, y \in L$, $x \lhd_L y \Rightarrow h(x) \lhd_M h(y)$.

The syntopogenous frame homomorphisms and syntopogenous frames correspond to the morphisms and objects in the category **SYNTFrm**.

We now embark on clarifying the relationship between the categories **SYNTFrm** and **SYT** using the open functor. The spectrum functor still needs more terminology before it is established.

Lemma 4.9.1. *Let $(X, \mathcal{S})$ be a syntopogenous space. Put $\Omega X = \tau_{\mathcal{S}}$ and let $\Omega\mathcal{S}$ be given by*

$$\Omega\mathcal{S} = \{\lhd_i \cap \{\tau_{\mathcal{S}} \times \tau_{\mathcal{S}}\} \mid \lhd_i \in \mathcal{S}\}$$

Then $(\Omega X, \Omega\mathcal{S})$ is a syntopogenous frame.

Proof. By Proposition (4.1.2), $\tau_{\mathcal{S}}$ is a topology. Further, $(S1)$ is clear. $(S3)$ also follows from Proposition (4.1.2). To conclude, the proof, we must show that $(S2)$ holds. For all $A, B \in \tau_{\mathcal{S}}$, $A \lhd B \Rightarrow \exists \lhd_1 \in \mathcal{S}, \exists\, C \subseteq X$ with $A \lhd_1 C \lhd_1 B$ which holds since Proposition (4.1.2), $C \in \tau_{\mathcal{S}}$.

□

Now, if $f : (X, \mathcal{S}) \longrightarrow (Y, \mathcal{S}')$ is a syntopogenously continuous map, then the map $\Omega f : (\Omega Y, \Omega\mathcal{S}') \longrightarrow (\Omega X, \Omega\mathcal{S})$ such that for all $A, B \in \Omega Y$, and $\lhd' \in \mathcal{S}'$, there exists $\lhd \in \mathcal{S}$ with $f^{-1}(A) \lhd f^{-1}(B)$ whenever $A \lhd' B$ is a syntopogenous frame homomorphism.

Thus, the open functor $\Omega : \mathbf{SYT} \longrightarrow \mathbf{SYNTFrm}$ maps each syntopogenous space $(X, \mathcal{S})$ to the syntopogenous frame $(\Omega X, \Omega\mathcal{S})$ and to each syntopogenously continuous map $f : (X, \mathcal{S}) \longrightarrow (Y, \mathcal{S}')$ to the syntopogenously continuous frame homomorphism $\Omega f : (\Omega Y, \Omega\mathcal{S}') \longrightarrow (\Omega X, \Omega\mathcal{S})$

Proximity

Definition 4.9.2. A proximity on a frame L is a binary relation $\lhd$ on $L \times L$ such that

(P1) $\lhd$ is a sublattice of $L \times L$

(P2) $x \lhd y \Rightarrow x \leq y$ for all $x, y \in L$;

(P3) if $a \leq x \lhd y \leq b$ then $a \lhd b$ for all $a, b \in L$;

(P4) $x \lhd y \Rightarrow y^* \lhd x^*$ for all $x, y \in L$;

(P5) $\lhd$ interpolates;

(P6) each element $x \in L$ can be written as $x = \bigvee\{a \in L \mid a \lhd x\}$.

The pair $(L, \lhd)$ where $\lhd$ is a proximity on L is called a proximity (or proximal) frame. Let $(L, \lhd)$ and $(M, \lhd)$ be proximity frames. A frame homomorphism $h : L \longrightarrow M$ is a proximity frame homomorphism if

$$x \lhd_L y \Rightarrow h(x) \lhd_M h(y)$$

for all $x, y \in L$. Proximity frames and proximity frame homomorphisms are the objects and morphisms of the category **ProxFrm**.

The following two propositions which establish a relationship between uniform frames and proximal frames are due to Frith ([Fri86]):

Proposition 4.9.1. *Let $(L, \mathcal{A})$ be a uniform frame. For all $x, y \in L$, define $a \lhd_{\mathcal{A}} b$ if and only if $Aa \leq b$ for some $A \in \mathcal{A}$. Then the pair $(L, \lhd_{\mathcal{A}})$ is a proximal frame.*

Proposition 4.9.2. *Consider $(L, \mathcal{A})$ a proximal frame. There exists a compatible uniform structure $\mathcal{A}^{\lhd}$ such that $\mathcal{A}$ induces $\lhd$.*

The proofs of the above two propositions are available in ([Fri86])

Let $L \in$ **UniFrm**, put

$$\mathcal{S}_{(L,\mathcal{A})} = \{\lhd_{\mathcal{A}}\}$$

where, for all $a, b \in L$, $a \lhd_{\mathcal{A}} b$ if and only if $Aa \leq b$ for some $A \in \mathcal{A}$. Then $\mathcal{S}_{(L,\mathcal{A})}$ is a syntopogenous structure on uniform frames.

We note that the notion of quasi-uniformities we introduced in this chapter is compatible with the entourage quasi-uniformities on frames by Fletcher et al. in [FH91, FHL93b]. This will be discussed further in the next chapter.

Chapter 5

PRE-NEARENESS AND CSÁSZÁR STRUCTURES IN FRAMES

As we have already seen in chapter two, Császár developed the theory of syntopogenous structure on a set. He aimed to establish a comprehensive framework that encompasses the study of topological, proximal, and uniform structures simultaneously. Shortly thereafter, in [Her74a], Herrlich introduced the concept of "nearness" with the idea of unifying various concepts of topological structures on a set. He further suggested a correspondence between symmetric syntopogenous spaces and nearness spaces. While all of these structures are well-defined in pointfree topology, to the best of our knowledge, there is no clear correspondence between nearness structures and Császár structures within the context of frames. In this chapter, we intend to bridge this gap. We will establish a correspondence between the category of pre-nearness frames [BP93] and a novel category of semi-Császár structures that we introduce. Moreover, when considering the concept of quasi-uniformities in a frame, we will demonstrate that interpolative Császár structures are in a one-to-one correspondence with the bases of a quasi-uniformity. Building upon the findings of [FHL93b] and [Pic95], we conclude the chapter by establishing a relationship between a base of an entourage quasi-uniformity and a base of pre-uniformities within the framework of frames.

5.1 Nearness Spaces

In [Her74a] Herrlich introduced the concept of "nearness of an arbitrary collection of sets $\mathcal{B}$". It is typically denoted by $\mathcal{B} \in \mu$ or $\mu\mathcal{B}$ for some $\mathcal{B} \in \mu$; which mean "$\mathcal{B}$ is near". His aim was to bring together different types of topological structures.

There exist several equivalent approaches to axiomatize the category of nearness spaces and nearness maps between them [Her74b]. In this part of our work, we mention two, which we intend to link with Császár structures in the context of pointfree topology. Of course, it should be noted that the approach using covers will be often more intriguing and frequently utilized given the context of pointfree topology in which we are working.

Definition 5.1.1. *[Her74b]* Let X be a given set, and μ a non-empty collection of covers of X. The pair (X, μ) is called a nearness space if the following axioms hold:

(N1) if $\mathcal{A} \in \mu, \mathcal{B} \subseteq P(X)$ and $\mathcal{A} \leq \mathcal{B}$ then $\mathcal{B} \in \mu$;

(N2) if $\mathcal{A}, \mathcal{B} \in \mu$ then $\mathcal{A} \wedge \mathcal{B} \in \mu$;

(N3) if $\mathcal{A} \in \mu$ then

$$int_\mu \mathcal{A} = \{int_\mu A \mid A \in \mathcal{A}\} \in \mu$$

where for each $A \subseteq X$,

$$int_\mu A = \{x \in X \mid \{A, X \setminus x\} \in \mu\}$$

Citing [Pic95], we say that the pair (X, μ) is a pre-nearness space if μ satisfies only axiom $(N1)$ of Definition (5.1.1).

Definition 5.1.2. Let (X, μ) and (Y, ν) be two nearness (pre-nearness) spaces. A function $f : X \longrightarrow Y$ is called a nearness (pre-nearness) map $f : (X, \mu) \longrightarrow (Y, \nu)$ from (X, μ) to (Y, ν) if $f^{-1}(A) \in \mu$ whenever $A \in \nu$.

Pre-nearness spaces and pre-nearness maps are objects and morphisms of the category **PNear**. This category includes **Near**, which is the category of nearness spaces and nearness maps, as full subcategory.

Herrlich proposed in [Her74b] an equivalent way of axiomatizing the notion of nearness spaces. In this alternative approach, the members of μ are not necessarily covers. They are simply subsets of the power set PX.

Definition 5.1.3. Let X be a set, and $\mu \subseteq P^2X \subseteq P(PX)$. The pair (X, μ) is called a nearness space if the following axioms hold:

(N'1) $\bigcap \mathcal{A} \neq \emptyset \Rightarrow \mathcal{A} \in \mu$;

(N'2) $\mathcal{A} \notin \mu, \mathcal{B} \notin \mu \Rightarrow \mathcal{A} \cup \mathcal{B} \notin \mu$; where

$$\mathcal{A} \cup \mathcal{B} = \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$$

(N'3) $\emptyset \in \mathcal{A} \Rightarrow \mathcal{A} \notin \mu$;

(N'4) $\mathcal{B} \in \mu,\ \forall A \in \mathcal{A}\ \exists\ B \in \mathcal{B},\ B \subseteq \overline{A} \Rightarrow \mathcal{A} \in \mu$; where

$$x \in \overline{A} \Leftrightarrow \{x, A\} \in \mu$$

Definition 5.1.4. Let (X, μ) and (Y, η) be nearness spaces. Then $f : (X, \mu) \longrightarrow (Y, \eta)$ is called a nearness map if and only if $f : X \longrightarrow Y$ is a function and the following condition is satisfied:

$$\text{if } \mu\mathcal{B}, \text{ then } \eta(f\mathcal{B}),$$

where $f\mathcal{B} = \{f(B) \mid B \in \mathcal{B}\}$.

Proposition 5.1.1. *The correspondence*

$$\mathcal{A} \in \tilde{\mu} \Leftrightarrow \{X \setminus A \mid A \in \mathcal{A}\} \notin \mu \tag{5.1.1}$$

provides an alternative description of a nearness on a set X.

Proof. Let μ be a nearness on a set X as defined in Definition (5.1.1). We show that $\tilde{\mu}$ satisfies Definition 5.1.1.
$(\tilde{N}1)$,
$\{X \setminus A \mid A \in \mathcal{A}\} \notin \mu \Rightarrow \bigcap\{X \setminus A \mid A \in \mathcal{A}\} = \emptyset \Rightarrow \bigcup \mathcal{A} = X$, which implies that $\tilde{\mu}$ is a family of covers of X.
For $(\tilde{N}2)$,

$$\begin{aligned} \mathcal{A}, \mathcal{B} \in \tilde{\mu} &\Rightarrow \{X \setminus A \mid A \in \mathcal{A}\}, \{X \setminus B \mid B \in \mathcal{B}\} \notin \mu; \\ &\Rightarrow \{X \setminus A \cup X \setminus B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \notin \mu; \\ &\Rightarrow \{X \setminus (A \cap B) \mid A \in \mathcal{A}, B \in \mathcal{B}\} \notin \mu; \\ &\Rightarrow \mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \in \tilde{\mu}. \end{aligned}$$

$(\tilde{N}3)$

$$\begin{aligned} X \in \mathcal{A} &\Rightarrow \emptyset \in \{X \setminus A \mid A \in \mathcal{A}\}; \\ &\Rightarrow \{X \setminus A \mid A \in \mathcal{A}\} \notin \mu; \\ &\Rightarrow \mathcal{A} \in \tilde{\mu}. \end{aligned}$$

$(\tilde{N}4)$ $\mathcal{A} \in \tilde{\mu} \Rightarrow \{X \setminus A \mid A \in \mathcal{A}\} \notin \mu$, so if $\forall A \in \mathcal{A}, \exists\, B \in \mathcal{B}$ with

$$X \setminus B \subseteq \overline{X \setminus A}$$

then $\{X \setminus B \mid B \in \mathcal{B}\} \notin \mu$, that is, $\mathcal{B} \in \tilde{\mu}$.

In accordance with reference [Pic95], we note that if μ is a nearness on X, then int_μ is an idempotent interior operation on $P(X)$. It is the interior operation associated with μ and satisfies the following axioms:

(i) $int_\mu(A \cap B) = int_\mu(A) \cap int_\mu(B)$ for all $A, B \subseteq X$;

(ii) $x \in int_\mu(X \setminus \{y\})$ if and only if $y \in int_\mu(X \setminus \{x\})$

Therefore, any nearness structure on a set X gives rise to a topology satisfying (ii). Sets endowed with such topology are symmetric topological spaces or R_o-spaces. In fact, they form a full subcategory of **Top** denoted by $\mathbf{R_o Top}$. Speaking of $\tilde{\mu}$, we note the following

$$\begin{aligned} x \in A^o &\Leftrightarrow x \notin \overline{X \setminus A}; \\ &\Leftrightarrow \{\{x\}, X \setminus A\} \notin \mu; \\ &\Leftrightarrow \mathcal{A}_x = \{X \setminus \{x\}, A\} \in \tilde{\mu}; \\ (*) \quad &\Leftrightarrow \{x\} \subseteq_{\tilde{\mu}} A; \\ &\quad\ \exists\, \mathcal{A} \in \tilde{\mu},\ \mathcal{A}\{x\} \subseteq A. \end{aligned}$$

Where $(*)$ holds since

$$\begin{aligned}(\Rightarrow)\mathcal{A}_x \in \tilde{\mu}\ , \mathcal{A}_x\{x\} &= \bigcup\{U \in \mathcal{A}_x \mid \{x\} \cap U \neq \emptyset\};\\ &= \bigcup\{U \in \mathcal{A}_x \mid x \in U\};\\ &= A \subseteq A.\end{aligned}$$

$(\Leftarrow)$ $\exists \mathcal{A} \in \tilde{\mu}$, $\bigcup\{U \in \mathcal{A} \mid x \in U\} \subseteq A$ then for $U \in \mathcal{A}$,

(i) $x \in U \Rightarrow U \subseteq A$;

(ii) $x \notin U \Rightarrow X \setminus \{x\} \supseteq U$.

From (i) and (ii), we obtain $\mathcal{A}_x \in \tilde{\mu}$.

Herrlich pointed out that $(N3)$ can easily be dropped, and he also noted that replacing $\overline{A}$ with A or dropping $(N2)$ still leads to an interesting concept. However, he was not pursuing such generality.

5.2 Category of Pre-nearness Frames

In pointfree topology, the concept of nearness frame was axiomatized by Banaschewski and Pultr [BP96] with a dual purpose: on one hand, to obtain the pointfree counterpart of the nearness spaces, on the another hand, to overcome the limitations of uniform frames. As we shall see shortly, nearness frames are uniform frames without the star-refinement axiom.

Let L be a frame, and let A be its subset. Then A is called a cover of L if $\bigvee A = 1_L$. The collection of all covers of L is denoted by $\text{cov}(L)$. It is preordered as follows: a cover A refines a cover B, written $A \leq B$, if

$$\text{for each } a \in A \text{ there exists } b \in B \text{ such that } a \leq b.$$

Note that for any $A, B \in \text{cov}(L)$:

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\} \text{ and } A \vee B = A \cup B$$

Further, for each cover A of L and each element $x \in L$, the element Ax given by

$$Ax = \bigvee\{a \mid a \in A, x \wedge a \neq 0\}$$

and is called the star of a in A. In addition to this, if A is a cover of L, its star, denoted by A^*, is the set

$$A^* = AA = \{Aa \mid a \in A\}$$

which is itself a cover since $A \leq A^*$. Furthermore, we shall say that a cover A star refines a cover B, written as $A \leq^* B$ if $AA \leq B$.

Now, let $\mathcal{A}$ be a subcollection of covers, and $a, b \in L$, we write

$$a \lhd_{\mathcal{A}} b$$

and read "a $\mathcal{A}$-strongly below b" if there is $A \in \mathcal{A}$ with $Aa \leq b$.

Given a frame L and $\mathcal{A} \subseteq cov(L)$. We consider the following axioms:

(N1) if $A \in \mathcal{A}$ and $A \leq B$ then $B \in \mathcal{A}$;

(N2) if $A, B \in \mathcal{A}$ then $A \wedge B \in \mathcal{A}$;

(N3) for all $A \in \mathcal{A}$ there exists $B \in \mathcal{A}$ such that $BB \leq A$;

(N4) $\mathcal{A}$ is admissible if each element $a \in L$ can be written as the join of all elements $b \triangleleft_{\mathcal{A}} a$, that is,

$$a = \bigvee \{b \in L \mid b \triangleleft_{\mathcal{A}} a\}$$

A collection $\mathcal{A}$ of covers of the frame L is called a pre-nearness [Pic95] on L if it satisfies $(N1)$ and $(N2)$, that is to say a pre-nearness is a filter of covers on L. $\mathcal{A}$ is, further called, a pre-uniformity on L if it satisfies $(N1)$-$(N3)$ [BP93], a nearness if it satisfies $(N1)$, $(N2)$ and $(N4)$ [BP96], a uniformity if it fulfills $(N1) - (N4)$ [BP96]. The pair $(L, \mathcal{A})$ is called pre-nearness frame, pre-uniform frame, nearness frame, uniform frame, respectively if $\mathcal{A}$ is a pre-nearness, pre-uniform, nearness, uniform structure on L, respectively.

Speaking of pre-nearness within a frame L, we denote the collections of all pre-nearnesses on the frame L by $PN(L)$ and the order on $cov(L)$ can be nicely inherited here.

The next two Lemmas will be useful in the sequel:

Lemma 5.2.1. *[Pul84] In a uniform (nearness) frame L, the relation $x \triangleleft_{\mathcal{A}} y$ implies $x \prec y$*

Proof. If $x \triangleleft_{\mathcal{A}} y$ then $Ax \leq y$ for some $A \in \mathcal{A}$ where as before, Ax is given by

$$Ax = \bigvee \{a \in A \mid a \wedge x \neq 0\}. \text{ Now let } \quad b = \bigvee \{a \in A \mid a \wedge x = 0\} = x^*$$

Then $b \wedge x = 0$ and $Ax \vee b = \bigvee A = 1_L$. Thus also $y \vee b = 1_L$. So, since $x \wedge b = 0$ and $y \vee b = 1_L$, it follows that $x \prec y$.

□

In view of the above lemma, every nearness frame is a regular frame.

Lemma 5.2.2. *[Pul84] For any frame L, $A, B \in cov(L)$ and $x, y \in L$, the following hold:*

(i) $(A \wedge B)x \leq Ax \wedge Bx$;

(ii) $A(Ax) \leq Ax$.

The uniformly below relation has a number of useful features, of which we mention a few by means of the following proposition:

Proposition 5.2.1. *(1)* $a \triangleleft_{\mathcal{A}} b \Rightarrow a \prec b \Rightarrow a \leq b$;

(2) $a \leq x \triangleleft_{\mathcal{A}} b \leq y \Rightarrow a \triangleleft_{\mathcal{A}} y$.

In fact the uniformly below relation can be extended to the arbitrary meet and join as follows: for all $A \in \mathcal{A}$

$$\begin{aligned} x \triangleleft_A y_i \forall i \quad &\Rightarrow \quad Ax \leq y_i \ \forall i; \\ &\Rightarrow \quad Ax \leq \bigwedge y_i; \\ &\Rightarrow \quad x \triangleleft_A \bigwedge y_i. \end{aligned}$$

$$
\begin{aligned}
x_i \lhd_{\mathcal{A}} y \forall i \quad &\Rightarrow \quad \forall\, i\; \exists\, A_i \in \mathcal{A} \mid A_i x_i \leq y;\\
&\Rightarrow \quad A_i \in \mathcal{A} \mid \bigvee (A_i x_i) = A_i \bigvee x_i \leq y;\\
&\Rightarrow \quad \bigvee x_i \lhd_{\mathcal{A}} y.
\end{aligned}
$$

Let $(L, \mathcal{A})$ and $(M, \mathcal{B})$ be pairs of pre-nearness frames, pre-uniform frames, nearness frames, uniform frames. A frame homomorphism $f : L \longrightarrow M$ is uniformly continuous if

$$f(A) = \{f(a) \mid a \in A\} \in \mathcal{B} \text{ whenever } A \in \mathcal{A}.$$

We denote by **PNFrm** the category of pre-nearness frames and uniformly continuous maps between them and by **PUniFrm**, **NFrm** and **UniFrm** the full subcategories of pre-uniform frames, nearness frames and uniform frames, respectively.

All the above categories have counterparts in realm of spaces, see Section 5.1 and [Pic95] for more details.

5.3 Category of Császár Structures

Recall from Chapter 3 Lemma (3.3.12) that if $\mathcal{L} = \{\lhd_i \mid i \in I\}$ is a family of topogenous orders on a frame L, the order relation $\lhd_{\mathcal{L}}$ given by

$$\lhd_{\mathcal{L}} = \bigcup \{\lhd_i \mid i \in I\} \tag{5.3.2}$$

is also a topogenous order on L.

Now, let $\mathcal{L}$ be a family of topogenous order and consider the following axioms:

(L1) Each $\lhd \in \mathcal{L}$ respects:

- (•) binary meet: $a \lhd b$ and $a \lhd c$ implies $a \lhd b \wedge c$;
- (•) arbitrary join: $a \lhd b$ for all $a \in A$ implies $\bigvee A \lhd b$;

(L2) $\mathcal{L}$ is directed: for each $\lhd_1, \lhd_2 \in \mathcal{L}$ there is $\lhd_3 \in \mathcal{L}$ such that $\lhd_1 \cup \lhd_2 \subseteq \lhd_3$;

(L3) admissible, that is, if (5.3.2) holds then

$$a = \bigvee \{b \in L \mid b \lhd_{\mathcal{L}} a\}$$

for any $a \in L$.

The collection $\mathcal{L}$ is called:

- a semi-Császár structure if it satisfies $(L1)$ and $(L2)$;
- a Császár structure if it is, furthermore, admissible.

The pair $(L, \mathcal{L})$ is called a semi-Császár (Császár) frame if $\mathcal{L}$ is a semi-Császár (Császár) structure on L. Note the concept of Császár frame have been used in the literature by [Chu05].

We shall further say that $\mathcal{L}$ is interpolative if each $\triangleleft \in \mathcal{L}$ interpolates, that is, for each $\triangleleft \in \mathcal{L}, x, y \in L, x \triangleleft y$ implies there exists $z \in L$ with $x \triangleleft z \triangleleft y$.

Speaking of Császár structures on a given frame L, we use the symbol $CS(L)$ to denote the collection of all the Császár structures on L. It is preordered in the following manner: let $\mathcal{L}, \mathcal{L}' \in CSC(L)$. Put $\mathcal{L} \sqsupset \mathcal{L}'$ if for all $\triangleleft \in \mathcal{L}$ there exists $\triangleleft' \in \mathcal{L}'$ such that $\triangleleft \subseteq \triangleleft'$.

Definition 5.3.1. Let $(L, \mathcal{A})$ and $(M, \mathcal{M})$ be semi-Császár (Császár) frames. A frame homomorphism $f : L \longrightarrow M$ is continuous (with respect to the Császár structure) if for each $\triangleleft_L \in \mathcal{L}$ there exists $\triangleleft_M \in \mathcal{M}$ such that

$$f(x) \triangleleft_M f(y) \text{ whenever } x \triangleleft_L y$$

for all $x, y \in L$. This is equivalent to saying that

$$x \triangleleft_{\mathcal{L}} y \Rightarrow f(x) \triangleleft_{\mathcal{M}} f(y)$$

for all $x, y \in L$.

We use the symbol **CSFrm** to denote the category of semi-Császár frames and continuous frame homomorphisms between them and by **CFrm**, the full subcategory of **CFrm** of Császár frames and continuous (with respect to the Császár structures) frame homomorphisms.

5.4 Correspondence Between the Categories SCFrm and PN-Frm

We establish a functorial correspondence between the categories of pre-nearness frames and that of semi-Császár frames.

Proposition 5.4.1. *Let $(L, \mathcal{L})$ be a semi-Császár frame. Set*

$$\mathcal{A}^{\mathcal{L}} = \{A \subseteq L \mid \exists \triangleleft \in \mathcal{L}, \bigvee A^{\triangleleft} = 1_L\}$$

where

$$A^{\triangleleft} = \{x \in L \mid \exists\, y \in A,\ x \triangleleft y\}.$$

The pair $(L, \mathcal{A}^{\mathcal{L}})$ is a pre-nearness frame. In addtion, the map taking each $\mathcal{L}$ to $\mathcal{A}^{\mathcal{L}}$ is order preserving.

Proof. Firstly, note that $\mathcal{A}^{\mathcal{L}} \subseteq cov(L)$: Indeed, if $A \in \mathcal{A}^{\mathcal{L}}$ then, since $\forall\, x \in A^{\triangleleft}\ \exists\, a \in A$ with $x \leq a$, it follows that

$$\bigvee A^{\triangleleft} \leq \bigvee A \text{ and } \bigvee A = 1$$

Thus $\mathcal{A}^{\mathcal{L}}$ is a collection of covers.

Assume $A \in \mathcal{A}^{\mathcal{L}}$ and $A \leq B$. Since $\forall x \in A^{\triangleleft}\ \exists\ a \in A$ with $x \triangleleft a$, but since $A \leq B, \exists\ b \in B$ with $x \triangleleft a \leq b$ which implies that $x \triangleleft a$. Thus, by $(T2)$, we obtain $x \triangleleft b$ and so $A^{\triangleleft} \subseteq B^{\triangleleft}$ and $1 = \bigvee B^{\triangleleft}$ showing that $B \in \mathcal{A}^{\mathcal{L}}$. Thus if $A \in \mathcal{A}^{\mathcal{L}}$ and $A \leq B$ then $B \in \mathcal{A}^{\mathcal{L}}$ and so $(N1)$ holds.

Next, let $A, B \in \mathcal{A}^{\mathcal{L}}$, there exist $\triangleleft_1, \triangleleft_2 \in \mathcal{L}$ with $\bigvee A^{\triangleleft_1} = \bigvee B^{\triangleleft_2} = 1_L$. Now pick $\triangleleft \in \mathcal{L}$ with $\triangleleft_1 \cup \triangleleft_2 \subseteq \triangleleft$ and $\triangleleft$ respects meets. Then $A^{\triangleleft_1} \subseteq A^{\triangleleft}$ and $B^{\triangleleft_2} \subseteq B^{\triangleleft}$ and if $x \in A^{\triangleleft}$, there exits $a \in A$ with $x \triangleleft a$ and

$$\begin{aligned}
x &= x \wedge \bigvee B^{\triangleleft} \\
&= x \wedge \bigvee \{y \in L \mid \exists\ \ b \in B, y \triangleleft b\} \\
&= \bigvee \{x \wedge y \mid \exists\ \ b \in B, y \triangleleft b\} \\
&\leq \bigvee \{z \wedge y \mid \exists\ \ a \in A\ \ \exists\ \ b \in B, z \triangleleft a \text{ and } y \triangleleft b\} \\
&= \bigvee \{z \wedge y \mid z \in A^{\triangleleft} \text{ and } y \in B^{\triangleleft}\}
\end{aligned}$$

But $\bigvee \{x \mid x \in A^{\triangleleft}\} = 1_L$ and so $\bigvee \{z \wedge y \mid z \in A^{\triangleleft} \text{ and } y \in B^{\triangleleft}\} = 1$.

In order to prove that $A \wedge B \in \mathcal{A}^{\mathcal{L}}$, we must show that $\bigvee (A \wedge B)^{\triangleleft} = 1$, that is,

$$\bigvee (A \wedge B)^{\triangleleft} = \bigvee \{w \in L \mid \exists\, a \in A, b \in B\ , w \triangleleft a \wedge b\} = 1.$$

If $z \in A^{\triangleleft}$ and $y \in B^{\triangleleft}$, $\exists\ a \in A, b \in B$ with $z \triangleleft a$ and $y \triangleleft b$. Thus $z \wedge y \triangleleft a$ and $z \wedge y \triangleleft b$. Hence since $\triangleleft$ preserves meet, we conclude that $z \wedge y \triangleleft a \wedge b$. Thus

$$1 = \bigvee \{z \wedge y \mid y \in B^{\triangleleft}, z \in A^{\triangleleft}\} \leq \bigvee \{w \mid \exists\, a \in A,\ \exists\, b \in B, w\ \triangleleft a \wedge b\}$$

and, $\bigvee (A \wedge B)^{\triangleleft} = 1$.

Thus $\mathcal{A}^{\mathcal{L}}$ satisfies $(N2)$ and all together shows that $\mathcal{A}^{\mathcal{L}}$ is a prenearness structure on L.

Let $\mathcal{L}, \mathcal{L} \in \mathrm{SC}(L)$ such that $\mathcal{L} \sqsupset \mathcal{L}$. For each $\triangleleft \in \mathcal{L}$, there exists $\triangleleft' \in \mathcal{L}'$ with $\triangleleft \subseteq \triangleleft'$. This gives for all $A^{\triangleleft} \in \mathcal{A}^{\mathcal{L}}$ there exists $A^{\triangleleft'} \in \mathcal{A}^{\mathcal{L}'}$ with $A^{\triangleleft} \subseteq A^{\triangleleft'}$ and this proves that $\mathcal{A}^{\mathcal{L}} \sqsupset \mathcal{A}^{\mathcal{L}'}$ and so the map taking $\mathcal{L}$ to $\mathcal{A}^{\mathcal{L}}$ is order preserving.

$\square$

In what follows, we denote by φ the map taking each $\mathcal{L}$ to $\mathcal{A}^{\mathcal{L}}$. In fact for any semi-Császár structure $\mathcal{L}$, $\varphi(\mathcal{L}) = \mathcal{A}^{\mathcal{L}}$ denote the prenearness induced by the semi-Császár structure $\mathcal{L}$. Besides, the assignment $(L, \mathcal{L}) \longrightarrow (L, \varphi(\mathcal{L}))$ is functorial. We prove this point in the subsequent proposition.

Proposition 5.4.2. *Let $(L, \mathcal{L})$ and $(L, \mathcal{M})$ be semi-Császár frames such that the map $f : (L, \mathcal{L}) \longrightarrow (M, \mathcal{M})$ is a Császár frame homomorphism, Then $f : (L, \varphi(\mathcal{L})) \longrightarrow (M, \varphi(\mathcal{M}))$ is a uniform frame homomorphism.*

Proof. Let $A \in \mathcal{A}^{\mathcal{L}}$ and consider

$$f(A) = \{f(a) \mid a \in A\} \subseteq M$$

There exists $\triangleleft \in \mathcal{L}$ with $\bigvee A^{\triangleleft} = 1_L$. Now, since f is continuous there exists $\triangleleft' \in \mathcal{M}$ such that for all $a, b \in L$, $a \triangleleft b \Rightarrow f(a) \triangleleft' f(b)$. Thus, we obtain:

$$f(A^{\triangleleft}) = \{f(x) \mid \exists\, a \in A, x \triangleleft a\} \subseteq f(A)^{\triangleleft'} = \{y \mid \exists\, f(a) \in f(A), y \triangleleft' f(a)\}$$

Moreover, since f is a frame homomorphism, thence

$$1_M = f(1_L) = f\big(\bigvee A^{\triangleleft}\big) = \bigvee f\big(A^{\triangleleft}\big) \leq \bigvee f\big(A\big)^{\triangleleft}.$$

□

In view of Proposition (5.4.2) if $f : (L, \mathcal{L}) \longrightarrow (M, \mathcal{M})$ is a **SCFrm**-morphism then $f : (L, \varphi(\mathcal{L})) \longrightarrow (M, \varphi(\mathcal{M}))$ is a **PNFrm** morphism.

In this case, we have a functor ψ: **PNFrm** $\longrightarrow$ **SCFrm**

Proposition 5.4.3. *Let $(L, \mathcal{A})$ be a pre-nearness frame. Put*

$$\mathcal{L}^{\mathcal{A}} = \{\triangleleft_A : A \in \mathcal{A}\}$$

where

$$x \triangleleft_A y \Leftrightarrow Ax \leq y \text{ and } Ax = \bigvee\{a \in L \mid a \wedge x \neq 0\}.$$

The pair $(L, \mathcal{L})$ is a semi-Császár frame. Lastly the map taking each $\mathcal{A}$ to $\mathcal{L}^{\mathcal{A}}$ is order preserving.

Proof. Since A is a cover, $x \leq Ax$, so $x \triangleleft_A y \Rightarrow x \leq y$. Further, since $x \leq y \Rightarrow Ax \leq Ay$, we obtain $x \leq y \triangleleft_A z \leq w \Rightarrow Ax \leq Ay \leq x \leq w$ giving $x \triangleleft_A w$. Thus is a $\triangleleft_A$ is topogenous order.

If $x \triangleleft_A y$ and $x \triangleleft_B z$ for $A, B \in \mathcal{A}$ and $x, y, z \in L$. Then since $x \triangleleft_{A \wedge B} y$ and $x \triangleleft_{A \wedge B} z$, we get $(A \wedge B)x \leq y$ and $(A \wedge B)x \leq z$. Thus $(A \wedge B)x \leq y \wedge z$ and $x \triangleleft_{A \wedge B} y \wedge z$. Therefore, it follows from the above arguments and Proposition 5.2.1 that $(L1)$ is true.

Note that if $A \leq B$ then $Ax \leq Bx$ for all $x \in L$. Hence $Bx \leq y \Rightarrow Ax \leq y$ giving $x \triangleleft_B y \Rightarrow x \triangleleft_A y$. In particular, if $\triangleleft_A, \triangleleft_B \in \mathcal{L}^{\mathcal{A}}$ then since $\mathcal{A}$ is a pre-nearness, $A \wedge B \in \mathcal{A}$ and $\triangleleft_A \cup \triangleleft_B \subseteq \triangleleft_{A \wedge B}$. Hence $(L2)$ holds and this completes the proof.

Let $\mathcal{A}, \mathcal{A}' \in PN(L)$ such that $\mathcal{A} \leq \mathcal{A}'$. Then for all $A \in \mathcal{A}$ there exists $B \in \mathcal{A}'$ with $A \leq B$. Hence $\triangleleft_A \in \mathcal{L}^{\mathcal{A}}$ with $\triangleleft_A \subseteq \triangleleft_B$. This shows that $\mathcal{L}^{\mathcal{A}} \leq \mathcal{L}^{\mathcal{A}'}$ and the map $\mathcal{A}$ to $\mathcal{L}^{\mathcal{A}}$ preserves order.

□

In the sequel, we use the symbol ψ for the map taking each $\mathcal{A}$ to $\mathcal{L}^{\mathcal{A}}$. Thus, this being said, given any pre-nearness structure $\mathcal{A}$ on the frame L, $\psi(\mathcal{A}) = \mathcal{L}^{\mathcal{A}}$ denotes a semi-Császár structure induced by the pre-nearness $\mathcal{A}$. In addition to this, the correspondence $(L, \mathcal{A}) \longrightarrow (L, \psi(\mathcal{A}))$ is functorial:

Proposition 5.4.4. *Let $(L, \mathcal{A})$ and $(M, \mathcal{B})$ be pre-nearness frames such that the map $f : (L, \mathcal{A}) \longrightarrow (M, \mathcal{B})$ is a uniform frame homomorphism. Then $f : (L, \psi(\mathcal{A})) \longrightarrow (M, \psi(\mathcal{B}))$ is a semi-Császár frame homomorphism.*

Proof. Let $f : (L, \mathcal{A}) \longrightarrow (M, \mathcal{B})$ be a morphism in **PNFrm** and consider $A \in \mathcal{A}$. Then for all $x, y \in L$:

$$\begin{aligned} x \lhd_A y &\Rightarrow Ax \leq y; \\ &\Rightarrow f(Ax) \leq f(y); \\ &\Rightarrow f(A)f(x) \leq f(y); \\ &\Rightarrow f(x) \lhd_{f(A)} f(y). \end{aligned}$$

Where the third implication follows since

$$\begin{aligned} f(A)f(x) &= \bigvee\{f(a) \in f(A) \mid f(a) \wedge f(x) \neq 0\}; \\ &= \bigvee\{f(a) \in f(A) \mid f(a \wedge x) \neq 0\}; \\ &= f\big(\bigvee\{a \in A \mid f(a \wedge x) \neq 0\}\big); \\ &\leq f\big(\bigvee\{a \in A \mid a \wedge x \neq 0\}\big); \\ &= f(Ax). \end{aligned}$$

Therefore the assignment $(L, \mathcal{A}) \longrightarrow (L, \psi(\mathcal{A}))$ is indeed functorial.

□

We shall represent the functor from **SCFrm** to **PNFrm** by φ

By Proposition (5.4.4), if $f : (L, \mathcal{A}) \longrightarrow (M, \mathcal{B})$ is a morphism in **PNFrm** then $f : (L, \psi(\mathcal{A})) \longrightarrow (M, \psi(\mathcal{B}))$ is a morphism in **SCFrm**. It also follows from Proposition (5.4.1) to Proposition (5.4.4), that the functors ψ and φ are well-defined.

Possible Composition ψ and φ

Proposition 5.4.5. *For any pre-nearness frame, $id_L : (L, \mathcal{A}^{\mathcal{L}^{\mathcal{A}}}) \longrightarrow (L, \mathcal{A})$ is a uniform homomorphism.*

Proof. Let $\mathcal{A}$ be a pre-nearness on L and pick $B \in \mathcal{A}^{\mathcal{L}^{\mathcal{A}}}$. Then there is $A \in \mathcal{A}$ with $\bigvee B^{\lhd_A} = 1_L$. For any $a \in A$,

$$a = a \wedge \bigvee B^{\lhd_A} = \bigvee\{a \wedge x \mid \exists\, x \lhd_A b, \text{for some } b \in B\}.$$

If $a \neq 0$ then there exist $x \in B^{\lhd_A}$ with $a \wedge x \neq 0$ and $x \lhd_A b$ for some $b \in B$. But then $a \leq Ax \leq b$ giving $a \leq b$ and hence $A \leq B$. Since $\mathcal{A}$ is a pre-nearness it follows that $B \in \mathcal{A}$.

□

Proposition 5.4.6. *For any semi-Császár frame, $id_L : (L, \mathcal{L}^{\mathcal{A}^{\mathcal{L}}}) \longrightarrow (L, \mathcal{L})$ is Császár frame homomorphism.*

Proof. Let $x \lhd_A y$ for $\lhd_A \in \mathcal{L}^{\mathcal{A}^{\mathcal{L}}}$. Then there is $\lhd \in \mathcal{L}$ with $\bigvee A^{\lhd} = 1$ and $Ax \leq y$. Now, $\forall\, b \in A^{\lhd}\ \exists\, a \in A$ with $b \lhd a$. If $b \wedge a \neq 0$ then $x \wedge a \neq 0$ and $b \lhd x \lhd a \leq Ax \leq y$. Hence $b \wedge x \lhd y$. Thus, it follows $(L1)$ that

$$x = x \wedge \bigvee\{b \mid b \in A^{\lhd}\} = \bigvee\{x \wedge b \mid b \in A^{\lhd}\} \lhd y$$

□

In view of Propositions (5.4.5) and (5.4.6), we notice that while both ψ and φ are well-defined and they can compose but do not give a canonical result, that is we couldn't get an equivalence or Galois connection, as expected. Consequently, the results of our discussions in this section are encapsulated in the following theorem.

Theorem 5.4.1. *The functors ψ and φ satisfy*

$$\mathbf{SCFrm} \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \mathbf{PNFrm}\,,$$

with $\varphi(L, \mathcal{L}) = (L, \mathcal{A}^{\mathcal{L}})$, $\psi(L, \mathcal{A}) = (L, \mathcal{L}^{\mathcal{A}})$ and

*(1) $\varphi(\psi(\mathcal{A})) \leq \mathcal{A}$ for all $\mathcal{A} \in$ **PNFrm**;*

*(2) $\psi(\varphi(\mathcal{L})) \leq \mathcal{L}$ for all $\mathcal{L} \in$ **SCFrm**.*

Proposition 5.4.7. *If $\mathcal{A}$ is pre-uniformity rather than a nearness then the reverse inequality holds in Theorem (5.4.1) (1), that is to say, $\varphi(\psi(\mathcal{A})) = \mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ is a pre-uniformity, recall that

$$D \in \mathcal{A}^{\mathcal{L}^{\mathcal{A}}} \Leftrightarrow \exists\, C \in \mathcal{A}, \bigvee D^{\lhd_C} = 1_L$$

where

$$D^{\lhd_C} = \{x \in L \mid \exists\, d \in D, x \lhd_C d\}.$$

Now let $A \in \mathcal{A}$, since $\mathcal{A}$ is a pre-uniformity, there exists $B \in \mathcal{A}$ such that $BB \leq A$, thus for each $x \in B, \exists\, a \in A$, $Bx \leq a$ and so

$$B \subseteq A^{\lhd_B} = \{x \mid \exists\, a \in A, x \lhd_B a\} \text{ and } \bigvee A^{\lhd_B} = 1_L.$$

This shows that $A \in \mathcal{A}^{\mathcal{L}_{\mathcal{A}}}$ and $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{L}^{\mathcal{A}}}$, as required.

□

Proposition 5.4.8. *Let $\mathcal{A}$ be a pre-uniformity on the frame L. If $x \lhd_A y$ then there exists $B \in \mathcal{A}$ with $x \lhd_B Bx \lhd_B y$.*

Proof. Let $\mathcal{A}$ is a pre-uniformity on the frame L and $A \in \mathcal{A}$. Then there exits $B \in \mathcal{A}$ such that

$$\begin{aligned} x \lhd_A y &\Rightarrow Ax \leq y; \\ &\Rightarrow BBx \leq Ax \leq y; \\ &\Rightarrow Bx \leq BBx \leq Ax \leq y; \\ &\Leftrightarrow x \lhd_B Bx \lhd_B y. \end{aligned}$$

□

Fact: The relation $\lhd_A$ can be extended to arbitrary meets and arbitrary joins.

Proof. Let L be a frame and A be any cover on L. Then

$$\begin{aligned} x \lhd_A y_i \,\forall i &\Rightarrow Ax \leq y_i \,\forall i; \\ &\Rightarrow Ax \leq \bigwedge y_i; \\ &\Rightarrow x \lhd_A \bigwedge y_i. \end{aligned}$$

$$\begin{aligned} x_i \lhd_A y &\Rightarrow Ax_i \leq y \,\forall i; \\ &\Rightarrow \bigvee (Ax_i); \\ &\Rightarrow A(\bigvee x_i) \leq y; \\ &\Rightarrow \bigvee x_i \lhd_A y \end{aligned}$$

□

Every pre-nearness (pre-uniformity) $\mathcal{A}$ determines an interior operator on the category **PN-Frm** (**PUniFrm**).

Proposition 5.4.9. *Let $\mathcal{A}$ be a pre-nearness (pre-uniformity) on the frame L. The assignment $i = \{i_L : L \longrightarrow L;\ L \in$ **PNFrm**$\}$ given by*

$$i^{\lhd}_L(x) = \bigvee \{y \in L \mid y \lhd_{\mathcal{A}} x\}$$

is an interior operator on **PNFrm** *(***PUniFrm***).*

Proof. It is clear that $i^{\lhd}$ is an interior operation in L. Now let $f : (L,\mathcal{U}) \longrightarrow (M,\mathcal{V})$ be a morphism in **PNFrm** (**PUniFrm**) and $x \in L$. Then

$$\begin{aligned} f(i^{\lhd}_L(x)) &= f\big(\bigvee\{y \in L \mid y \lhd x\}\big); \\ &= \bigvee\{f(y) \mid y \in L, y \lhd x\} \text{ since } f \text{ is a frame homomorphism;} \\ &\leq \bigvee\{f(y) \in M \mid f(y) \lhd f(x)\} \text{ since } f \in \textbf{PFrm } (\textbf{PUniFrm}); \\ &\leq \bigvee\{z \in M \mid z \lhd f(x)\} = i^{\lhd}_M(f(x)). \end{aligned}$$

□

Some of our proofs can be streamlined by employing the following notations.

Let L be a given frame, $\triangleleft$ a topogenous order on L, and A, B covers of L. Define

$$A \triangleleft B \Leftrightarrow \ \forall\, a \in A, \exists\, b \in B, a \triangleleft b$$

then

$$A^{\triangleleft} = \{b \in L \mid \exists\, a \in A, b \triangleleft a\} \triangleleft A$$

We, thus have the following proposition:

Proposition 5.4.10. $\bigvee A^{\triangleleft} = 1_L \Leftrightarrow \exists\, B \in cov(L), B \triangleleft A$

Proof. The forward implication follows since $B = A^{\triangleleft} \triangleleft A$ and for the backward, we then have that $B \triangleleft A \Rightarrow B \subseteq A^{\triangleleft}$, so if B is a cover, $\bigvee A^{\triangleleft} = 1$.

□

Based on Proposition (5.4.10) we obtain the following new description of $\mathcal{A}^{\mathcal{L}}$:

$$\mathcal{A}^{\mathcal{L}} = \{A \in cov(L) \mid \exists \triangleleft \in \mathcal{L}\ \exists\, B \in cov(L), B \triangleleft A\}.$$

Also, from $B \triangleleft A \Rightarrow B \subseteq A^{\triangleleft}$ and $\bigvee A^{\triangleleft} = 1$. It follows that

$$A^{\triangleleft} = \max\{B \subseteq L \mid B \triangleleft A\}.$$

In [Her74a] Herrlich suggested a correspondence between symmetric syntopogenous spaces and nearness spaces. This correspondence can be adopted for frames as follows.

Theorem 5.4.2. *Let L be a frame and $\mathcal{L}$ a semi-Császár structure on L:*

$$\mathcal{L} \longrightarrow \tilde{\mathcal{A}}^{\mathcal{L}} = \{A \subseteq L \mid \forall a \in A\ \exists\, b_a \in L, \exists \triangleleft_a \in \mathcal{L}, b_a \triangleleft_a a \text{ and } \{b_a \mid a \in A\} \in cov(L)\}.$$

Then $\mathcal{L} \longrightarrow \tilde{\mathcal{A}}^{\mathcal{L}}$ is a pre-nearness structure on L. Equivalently

$$\mathcal{L} \longrightarrow \tilde{\mathcal{A}}^{\mathcal{L}} = \{A \subseteq L \mid \forall a \in A\ \exists\, b_a \in L, b_a \triangleleft_{\mathcal{L}} a, \text{and } \{b_a\} \triangleleft_{\mathcal{L}} A \in cov(L)\}$$

The next two propositions associate $\tilde{\mathcal{A}}^{\mathcal{L}}$ and $\mathcal{A}^{\mathcal{L}}$

Proposition 5.4.11. *For any frame L and a semi-Császár structure on L, we have*

$$A \in \mathcal{A}^{\mathcal{L}} \Leftrightarrow \exists \triangleleft \in \mathcal{L}\ \exists\, B = \{b_a \mid a \in A\} \in cov(L) \text{ with } B \triangleleft A$$

Proof. $(\Rightarrow)$ Given $\triangleleft \in \mathcal{L}$ and $B \in cov(L)$, $B \triangleleft A$ then for each $a \in A$, set $b_a = \bigvee\{b \in B \mid b \triangleleft a\}$. Then since $\triangleleft$ respects joins, it follows that $b_a \triangleleft a$ and $B' = \{b_a \mid a \in A\} \triangleleft A$. The other implication is quite clear.

□

Note: The correspondence in Proposition (5.4.11) can be equivalently expressed as

$$A \in \mathcal{A}^{\mathcal{L}} \Leftrightarrow \exists\, B \in cov(L) \mid B \lhd A$$

Proposition 5.4.12. *Given a frame L and $\mathcal{L}$ a semi-Császár structure on L, we may write:*

$$A \in \tilde{\mathcal{A}}^{\mathcal{L}} \Leftrightarrow \exists\, B = \{b_a \mid a \in A\} \lhd A \in cov(L) \text{ with } B \lhd_{\mathcal{L}} A$$

Proof. ($\Rightarrow$) Given a frame L and $\mathcal{L}$ a semi-Császár structure on L, then $\exists\, B \in cov(L)$, $B \lhd_{\mathcal{L}} A$.

□

The reverse implication in Proposition (5.4.12) does not seem to hold. Hence in general, $\tilde{\mathcal{A}}^{\mathcal{L}} \neq \mathcal{A}^{\mathcal{L}}$.

5.5 Quasi-Uniformities and Interpolative Semi-Császár Structures on a Frame

This section focuses on the establishment of a correspondence between quasi-uniformities as developed in the previous chapter and interpolative Császár structures on a given frame L.

We make a slight modification to the definition of Semi-Császár Structures on a frame in Section 5.3, by asking that each $\mathcal{L}$ interpolates and assuming that $\lhd_{\mathcal{L}}$ respects arbitrary meet, that is,

$$\forall\, A \subseteq L, \text{ if } x \lhd_{\mathcal{L}} a \text{ for all } a \in A \text{ then } x \lhd_{\mathcal{L}} \bigwedge A,$$

we obtain a new class of semi-Császár structures on L.
We use $ISC(L)$ to represent . the collection of all the interpolative semi-Császár structures on L such that each $\lhd \in \mathcal{L}$ preserves arbitrary meets. This collection is preodered in the same manner as $SC(L)$. Additionally, as discussed in the previous chapter, $Q(L)$ and $B(L)$ refer to the collection of all quasi-uniformities and bases of a uniformity on L, respectively. With this information, we can adapt the results from Chapter Two to this section in the following way:

Proposition 5.5.1. *Let $B(L)$ be the collection of bases of a quasi-uniformity on a frame L. For each $\mathcal{U}_L \in B(L)$ define*

$$\mathcal{L}_L^{\mathcal{U}} = \{\lhd_L^u \mid u \in \mathcal{U}_L\} \quad \textit{with} \quad x \lhd_L^u y \;\Leftrightarrow\; u(x) \leq y.$$

Then $\mathcal{L}_L^{\mathcal{U}}$ is an interpolating semi-Császár structure on L. It is the semi-Császár structure associated with the bases of a quasi-uniformity $\mathcal{U}_L$. Futhermore, the map $\pi : B(L) \longrightarrow ISC(L)$, taking each $\mathcal{U}_L$ to $\mathcal{L}^{\mathcal{U}}$, is order preserving.

Proposition 5.5.2. *Let $\mathcal{L} \in ISC(L)$ and $\lhd \in \mathcal{L}$. The assignment*

$$\mathcal{U}^{\mathcal{L}} = \{u^{\lhd} \mid \lhd \in \mathcal{L}\} \text{ with } u_L^{\lhd}(x) = \bigwedge\{y \in L \mid x \lhd y\}$$

defines a bases of a quasi-uniformity $\mathcal{U}$. It is the base of a quasi-uniformity induced by the semi-Császár structure $\mathcal{L}$ on L. Moreover, the map $\phi : ISC(L) \longrightarrow B(L)$, taking each $\mathcal{L}_L \longrightarrow \mathcal{U}_L^{\mathcal{L}}$, is order preserving.

The proof of the above two propositions closely resemble those of Propositions (4.4.1) and (4.4.2), respectively. In fact, both propositions illustrate that an interpolative semi-Császár structure can serve as base of a quasi-uniformity on a frame. This is due to the fact that the mappings π and ϕ satisfy $\pi(\mathcal{U}) = \mathcal{U}^{\mathcal{L}}$ and $\phi(\mathcal{L}) = \mathcal{L}^{\mathcal{U}}$ and that they are also inverse to each other, that is to means for all $x, y \in L$, $u \in \mathcal{U}$ and $\triangleleft \in \mathcal{L}$, it holds that $U^{\triangleleft^U}(x) = U(x)$ and $x \triangleleft^{U^{\triangleleft}} y \Leftrightarrow x \triangleleft y$.

It is worth observing that when considering the collection $ISC(L)$ as defined above and eliminating the condition that each $\triangleleft \in \mathcal{L}$ preserves meets, the isomorphism between $B(L)$ and $ISC(L)$ fails. Denoting by $ISC^*(L)$ the collection of all interpolative semi-Császár structures such that $\triangleleft \in \mathcal{L}$ does not respect meets, we establish a Galois connection between $ISC^*(L)$ and $B(L)$.

Proposition 5.5.3. *Let $ISC^*(L)$ denote the collection of all the interpolative semi-Császár structures which do not preserve meets, and $B(L)$ the base of a quasi-uniformity on L. Then there is a Galois connnection between $ISC^*(L)$ and $B(L)$.*

Proof. The proof of this proposition is of the same flavour as the one of Theorem (4.4.3)

□

5.6 Entourage Quasi-Uniformities and Covering Pre-Uniformities

In the classical context, the concept of quasi-uniformity is achieved by dropping the symmetry axiom from the set of axioms of uniformity. The purpose of this section is to exhibit an entourage quasi-uniformity base that arises from that the entourage uniformity base developed in [FH91].

We aim to demonstrate that this theory of entourage quasi-uniformity base is equivalent to the theory of covering pre-uniformity base, which is obtained by excluding the admissibility property from the covering uniformity base. Furthermore, we also establish a connection between our general concept of quasi-uniformity introduced in Chapter 4 and the entourage quasi-uniformity introduced by Fletcher et al [FHL94].

It is important to note that besides Fletcher et al. numerous other authors have also delved into the study of entourage quasi-uniformities within pointfree topology. In [Pic95], J. Picado introduced the Weil quasi-uniformities and demonstrated an equivalence between these and the covering quasi-uniformities proposed by Frith [Fri86]. In [FHL93a], Fletcher, Hunsaker, and Lindgren established a connection between entourage quasi-uniformities and Frith's covering quasi-uniformities. In this context, the symbols **QUNFrm**, **QWUFrm** and **QUNiFrm** denote the category of covering quasi-uniformities and the uniform homomorphisms between them, the category of Weil quasi-uniformities and Weil uniform homomorphisms between them, and the category of entourage quasi-uniformities and quasi-uniform frame homomorphisms between them, respectively. Importantly, these categories are shown to be equivalent.

In order to achieve this exploration, we shall need the notion of U-small element introduced in [FH91, FHL93b] and further used by [Pic00].

For a frame L, F will denote the collection of all order-preserving maps from $L \longrightarrow L$. We define $\leq$, $\wedge$, and $\vee$ pointwise on F. Then $(F, \leq, \wedge, \vee)$ is a frame.

Definition 5.6.1. *[FHL93b, Pic00]* Let L be a frame and $U : L \longrightarrow L$ any function. An element $x \in L$ is said to be U-small, if $x \wedge y \neq 0$ implies $x \leq U(y)$.

Let $S_U = \{x \in L \mid x \text{ is } U\text{-small}\}$ and for each $a \in L$, we denote by

$$S_U(a) = \bigvee\{b \in S_U \mid a \wedge b \neq 0\}.$$

If U is an order preserving map such that join of S_U is the top element of L, that is, if S_U covers L, then U is said to be a $\triangle$-map.

Definition 5.6.2. *[FHL93b]* A frame quasi-uniformity base on a frame L is a non-empty collection $\mathcal{B}$ of Δ-maps such that:

(EQ1) for every $U \in \mathcal{B}_L$ there exists $V \in \mathcal{B}_L$ such that $V \circ V \leq U$;

(EQ2) for all $U, V \in \mathcal{B}_L$ there is a join homomorphism $W : L \longrightarrow L$ such that $W \leq U \wedge V$.

Let L be a frame and $\mathcal{B}$ an entourage quasi-uniformity base on L. The entourage quasi-uniformity $\mathcal{U}$ on L generated by $\mathcal{B}$ is the collection

$$\mathcal{U} = \{V \in F \text{ such that there is } U \in \mathcal{B} \text{ with } U \leq V\}.$$

Let L be a frame and let $\mathcal{B}$ be a base for a frame quasi-uniformity on L. In accordance with Fletcher et al. [FHL93b], if for every $U \in \mathcal{B}$ and for every $x, y \in L$, we have that

$$U(x) \wedge y = 0 \text{ if and only if } U(y) \wedge x = 0$$

then $\mathcal{B}$ is a base for a frame uniformity on L. The members of $\mathcal{B}$ are called entourages.

Definition 5.6.3. *(1)* An entourage quasi-uniform frame is a pair $(L,\mathcal{U})$ where $\mathcal{U}$ is an entourage quasi-uniformity on the frame L.

(2) Let $(L,\mathcal{U})$ and $(M,\mathcal{V})$ be entourage quasi-uniform frames. An entourage quasi-uniform frame homomorphism is a frame homomorphism $h : L \longrightarrow M$ such that for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $V \circ h \leq h \circ U$.

We denote the category of entourage quasi-uniformities and entourage quasi-uniform frame homomorphisms between them by **EQFrm**.

For any collection $\mathcal{U}$ of entourage quasi-uniformities of the frame L and for all $x, y \in L$, the relation $x \triangleleft^{\mathcal{U}} y$ means that there exists $U \in \mathcal{U}$ such that $U(x) \leq y$. It is evident that $x \triangleleft^{\mathcal{U}} y$ implies $x \leq y$. In fact, we have:

Proposition 5.6.1. *If $\mathcal{U}$ is a quasi-uniformity on the frame L and $x, y \in L$, the relation $x \triangleleft^{\mathcal{U}} y$, which means that $U(x) \leq y$ for some $U \in \mathcal{U}$, satisfies the following axioms:*

(i) it is a sublattice of $L \times L$;

(ii) for all $x, y \in L$, $x \leq y \triangleleft^{\mathcal{U}} z \leq y$ implies $x \triangleleft^{\mathcal{U}} y$;

(iii) $\triangleleft^{\mathcal{U}}$ interpolates, that is, for all $x, y \in L$, $x \triangleleft^{\mathcal{U}} y$ implies there is $z \in L$ such that $x \triangleleft^{\mathcal{U}} z \triangleleft^{\mathcal{U}} y$;

(iv) *for any morphism* $h:(L,\mathcal{U})\longrightarrow(M,\mathcal{V})$ *in* **EQFrm** *if* $x \triangleleft^{\mathcal{U}} y$ *then* $h(x) \triangleleft^{\mathcal{V}} h(y)$ *for all* $x,y\in L$.

The following Lemma will be essential in proving the next proposition.

Lemma 5.6.4. *If* U *is a* $\triangle$*-map, then* $x \leq U(x)$ *for all* $x \in L$.

Proof. Let U be a $\triangle$-map, for all $x \in L$, $x = x \wedge \bigvee\{y \in L \mid y \text{ is } U\text{-small}\} = \bigvee\{x \wedge y \mid y \text{ is } U\text{-small and } x\wedge y \neq 0\} \leq U(x)$.

□

Theorem 5.6.5. *Let* $\mathcal{U} \in EQ(L)$ *with* $\mathcal{B}_{\mathcal{U}}$ *its base. Then* $\mathcal{U} \in Q(L)$

Proof*(PQ1)* If $U \in \mathcal{U}$ implies there is $V \in \mathcal{B}_{\mathcal{U}}$ with $V \leq U$, then $x \leq V(x) \leq U(x)$, for all $x \in L$. Thus $x \leq U(x)$ for all $x \in L$.

(PQ2) If $U, U' \in \mathcal{U}$ implies there is $V, V' \in \mathcal{B}_u$ with $V \leq U$ and $V' \leq U'$, then By $(EQ2)$, there is a join-homomorphism W and $Z \in \mathcal{B}_U$ with $Z \leq W \leq V \wedge V' \leq U \wedge U'$ and so $U \wedge U' \in \mathcal{U}$;

(PQ3) If $U \in \mathcal{U}$ implies there is $V \in \mathcal{B}_U$ with $V \leq U$, then, by $(EQ1)$, there is $W \in \mathcal{B}_U$ with $W \circ W \leq U$ and clearly $W \in \mathcal{U}$;

(PQ4) Obvious.

□

In view of the aforementioned proposition, it becomes evident that the Fletcher's entourage quasi-uniformities are a special case within our broader framework of quasi-uniformities as expounded in Chapter 4, that is to say $Q(L) \subseteq EQ(L)$. Unfortunately, we have not been able to establish that $EQ(L)$ is contained in $Q(L)$. We suspect that this discrepancy may arise from an incompatibility between our understanding of U-small elements and and our concept of $U^{\triangleleft}$. Further investigation is needed to address this issue.

Nevertheless, Theorem (5.6.5) asserts that our general quasi-uniformity framework, in Chapter 4, when restricted to frames, encompasses Fletcher et al.'s entourage quasi-uniformities and in consequence the Weil and covering quasi-uniformities.

5.6.1 Quasi-Uniform Frame- Homomorphism

In this section, we establish a connection between syntopogenously continuous frame homomorphisms and quasi-uniformly continuous frame homomorphisms:

Definition 5.6.6. Let $(L,\mathcal{U})$ and $(M,\mathcal{V})$ be two entourage quasi-uniform frames. A frame homomorphism $h: L \longrightarrow M$ is called an entourage quasi-uniform frame homomorphism if for all $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $V \circ h \leq h \circ U$.

The following proposition shares a similar essence to Proposition (4.7.3).

Proposition 5.6.2. *Let $h : L \longrightarrow M$ be a frame homomorphism and $h_* : M \longrightarrow L$ its right adjoint. Let $\mathcal{U}$ and $\mathcal{V}$ be entourage quasi-uniformities on the frames L and M, respectively. If h is a quasi-uniformly continuous frame homomorphism, then the following hold:*

(1) $x \triangleleft^U y \Rightarrow h(x) \triangleleft^V h(y)$

(2) $x \triangleleft^U h_*(z) \Rightarrow h(x) \triangleleft^V z$

for all $x, y \in L$ and $z \in M$.

Proof. The proof of the above proposition is similar to the one of Proposition (4.7.3).

□

5.6.2 Quasi-Uniformity Which Determines a Frame

Proposition 5.6.3. *Let $\mathcal{U}$ be a quasi-uniformity on frames. The assignment $i = \{i_L : L \longrightarrow L,\ L \in \mathbf{Frm}\}$ given by*

$$i_L^{\mathcal{U}}(x) = \bigvee\{a \in L \mid U(a) \leq x \text{ for some } U \in \mathcal{U}\}$$

is an idempotent interior operator on frames. It is the interior operator associated with the quasi-uniformity $\mathcal{U}$.

Proof. $(I1)$ is clear. For $(I2)$, let $\mathcal{U}$ be a quasi-uniformity on the frame L and $x, y \in L$ such that $x \leq y$, then $\{a \in L \mid U(a) \leq x\} \subseteq \{b \in L \mid U(b) \leq y\}$. Thus, in particular $\bigvee\{a \in L \mid U(a) \leq x\} \leq \bigvee\{b \in L \mid U(b) \leq y\}$ and $i_L^{\mathcal{U}}(x) \leq i_L^{\mathcal{U}}(y)$.

For the continuity condition, let $h : L \longrightarrow M$ be a quasi-uniform frame hommorphism, with $\mathcal{U}$ and $\mathcal{V}$ being quasi-uniformities on L and M, respectively. Then, for every $U \in \mathcal{U}$, there is $V \in \mathcal{V}$ such that

$$\begin{aligned} h(i_L^{\mathcal{U}}(x)) &= h\big(\bigvee\{t \mid U(t) \leq x\}\big); \\ &= \bigvee\{h(t) \mid t \in L\ , U(t) \leq x\}; \\ &\leq \bigvee\{a \in M \mid V(h(a)) \leq x\} = i_M^{\mathcal{V}}(h(x)). \end{aligned}$$

□

Definition 5.6.7. We shall say that a quasi-uniformity $\mathcal{U}$ on **Frm** is compatible with an interior operation i if

$$i_L(x) = \bigvee\{a \in L \mid U(a) \leq x \text{ for some } U \in \mathcal{U}\}$$

for all $a \in L$

Considering interior as a special topology, in the classical context, Definition (5.6.7) highlights the idea of a quasi-uniformity $\mathcal{U}$ being compatible with a topology or that of a quasi-uniformity being generated by a topology. In frames the topology is inherent. So, we need a clear understanding of what it means for a frame to be compatible with a quasi-uniformity, that is, what it signifies for a frame to align with a quasi-uniformity.

In the previous section we have recalled that a quasi-uniformity on a space is always associated with two other quasi-uniformities and that each of these three induces a topology. In order to give an adequate theory of quasi-uniformities for frames, P. Fletcher, W. Hunsaker and W. Lindgren [FHL94] constructed for each quasi-uniformity $\mathcal{U}$ on a frame L, a conjugate quasi-uniformity $\tilde{\mathcal{U}}$ such that the join of $\mathcal{U}$ and $\tilde{\mathcal{U}}$ is a uniformity on L. Furthermore, they also constructed two subframes of L which correspond to $\Im(\mathcal{U})$ and $\Im(\tilde{\mathcal{U}})$ in the spatial case.

Definition 5.6.8. *[FHL94]* Let L be a frame and $\mathcal{U}$ a quasi-uniformity on L. The frame of $\mathcal{U}$ is a collection denoted by $L(\mathcal{U})$ and given by

$$L(\mathcal{U}) = \{a \in L \mid a = \bigvee\{b \in L \mid U(b) \leq a \text{ for some } U \in \mathcal{U}\}\} \tag{5.6.3}$$

Clearly, the expression (5.6.3) tells us that a frame of quasi-uniformity $\mathcal{U}$ on a frame L is a family of idempotent interior operations on L.

Proposition 5.6.4. *Let L be a given frame and $\mathcal{U}$ a quasi-uniformity on L. Then $\mathcal{U}$ determines L or $\mathcal{U}$ is compatible with L if and only if $L(\mathcal{U}) = L$.*

By Proposition (5.6.4), a frame L is said to be compatible with a quasi-uniformity $\mathcal{U}$ if the collection of all the idempotent interior operations induced by $\mathcal{U}$ generates L.

In the following section, we establish a nexus between the concept of a pre-uniformity base and that of an entourage quasi-uniformity base. The results presented in this final section of this chapter are a restriction of those in the paper [FH91] which explores the concept of entourage uniformity on frames. Our main motivation for studying these results is to align them with our findings in this chapter (refer to the diagram at the bottom of the chapter).

5.6.3 Base of Covering Pre-Uniformity

In this section we present a base covering pre-uniformity derived from the base covering uniformity defined in [FH91].

Definition 5.6.9. Let L be a frame. A collection $\mathcal{A}$ of covers of L is called a covering pre-uniformity base on L if:

(i) For any $A, B \in \mathcal{A}$, there exists $C \in \mathcal{A}$ such that $C \leq A \wedge B$;

(ii) For any $A \in \mathcal{A}$ there is $B \in \mathcal{A}$ such that $B^* \leq A$.

Furthermore, if $\mathcal{A}$ is admissible, then we obtain the covering uniformity base by Fletcher et et al in [FH91].

We say that a frame is uniformizable (pre-uniformizable) if it admits a uniformity (pre-uniformity) base. Now let L be a frame and $\mathcal{A}$ a covering pre-uniformity base on L. The covering pre-uniformity λ on L generated by $\mathcal{A}$ is the collection of all covers of L that refine some cover in $\mathcal{A}$. More specifically, the covering pre-uniformity λ generated by $\mathcal{A}$ is the collection

$$\lambda = \{B \in cov(L) \mid \exists\ A\ \in \mathcal{A} \text{ such that } A \leq B\}$$

As we already said, a covering pre-uniform frame is a pair (L, λ) where L is a frame, and λ a pre-uniformity on L. The remainder of this section is dedicated to establishing an equivalence between **EQFrm** and **PUniFrm**, the category of covering pre-uniformities and uniformly continuous maps defined in section one.

5.6.4 Correspondence Between EQFrm and PUniFrm

Proposition 5.6.5. *Let $\mathcal{A}$ be a covering pre-uniformity base on a frame. For each $A \in \mathcal{A}$, define*

$$U_A : L \longrightarrow L \text{ by } U_A(x) = Ax.$$

Then

$$\mathcal{U}^{\mathcal{A}} = \{U_A : A \in \mathcal{A}\}$$

is an entourage quasi-uniformity base on L.

Proof. It is clear that U_A is an order-preserving map. Similarly, $x \leq U(x)$ for each $x \in L$. Now, let $B \in \mathcal{A}$ such that $B^* \leq A$. Let $x \in L$ and set $q = Bx$. Further, let $H = \{b \in B \mid b \wedge q \neq 0\}$ and $T = \{a \in A \mid a \wedge x \neq 0\}$. Then $U_B \circ U_B(x) = \bigvee H$ and $U_A(x) = \bigvee T$. Let $b \in H$. There exists a $d \in B$ such that $b \wedge d \neq 0$ and $d \wedge x \neq 0$, and there exists an $a \in A$ such that $b \vee d \leq a$. Since $a \wedge x \neq 0$, $A \in T$. Evidently $b \leq a$. It follows that $\bigvee S \leq \bigvee T$ and so $U_B \circ U_B \leq U_A$.

Let $U_A, U_B \in \mathcal{U}$. Pick $C \in \mathcal{A}$ such that $C \leq A \wedge B$, by Lemma (5.2.2), it follows that $U_C \leq U_A \wedge U_B$. It suffices to show that U_C is a join homomorphism. Let $\{x_i, i \in I\} \subseteq L$, be a collection of elements of L. Then

$$\begin{aligned} U_C(\bigvee_{i\in I} x_i) &= C \bigvee_{i\in I} x_i; \\ &= \bigvee\{c \in C \mid c \wedge (\bigvee_{i\in I} x_i) \neq 0\}; \\ &= \bigvee\{c \in C \mid c \wedge x_i \neq 0 \text{ for some } i \in I\}; \\ &= \bigvee_{i\in I} Cx_i; \\ &= \bigvee_{i\in I} U_C(x_i). \end{aligned}$$

□

In the sequel if α is a covering pre-uniformity base on a frame L, then $\mathcal{U}(\alpha)$ denotes the entourage quasi-uniformity for which $\{U_A \mid A \in \alpha\}$ is a base.

Proposition 5.6.6. *Let* $\mathcal{U}$ *be an entourage quasi-uniformity base on a frame* L. *For each* $U \in \mathcal{U}$, *let* A_U *be the cover of all the* U*-small sets and put*

$$\mathcal{A}^{\mathcal{U}} = \{A_U \mid U \in \mathcal{U}\}$$

Then $\mathcal{A}^{\mathcal{U}}$ *is a covering pre-uniformity base on* L.

Proof. Let $A_U, B_V \in \mathcal{A}^{\mathcal{U}}$ and let $W \in \mathcal{U}$ such that $W \leq U \wedge V$. Let $a \in A_W$. Then a is a W-small set. Assume $x \in L$ and $x \wedge a \neq 0$, then $a \leq W(x) \leq U(x)$ and similarly $a \leq V(x)$. Thus $a \in A_U \cap A_V$ and so $A_W \leq A_U \wedge A_V$.

Further, let $A_U \in \mathcal{A}^{\mathcal{U}}$ and let $V \in \mathcal{U}$ such that $V \circ V \leq U$ holds. Let $S = \{x_\alpha \mid \alpha \in \Lambda\}$ be a subset of A_V such that for each $\alpha, \beta \in \Lambda, x_\alpha \wedge x_\beta \neq 0$. Let $x = \bigvee S$ and assume that $a \in L$ such that $a \wedge x \neq 0$. There exists $\beta \in \Lambda$ such that $x_\beta \wedge a \neq 0$ and so $x_\beta \leq V(a)$. Therefore $V(x_\beta) \leq V \circ V(a) \leq U(a)$. Let $\alpha \in \Lambda$. Then $x_\beta \wedge x_\alpha \neq 0$ and so $x_\alpha \leq g(x_\beta) \leq U(a)$. It follows that $x \leq U(a)$ which shows that x is a U-small element and $A_V^* \leq A_U$.

□

Along these lines if ω stands for an entourage quasi-uniformity base for a frame L, then $\mu(\omega)$ denotes the covering pre-uniformity for which $\{A_U \mid U \in \omega\}$ is a base.

Proposition 5.6.7. *Let* L *be a pre-uniformizable frame, let* α *be a covering pre-uniformity on* L *and let* ω *be an entourage quasi-uniformity on* L. *Then* $\alpha = \mu(\mathcal{U}(\alpha))$ *and* $\omega = \mathcal{U}(\mu(\omega))$

Proof. For the proof, see [FH91]

□

The correspondence $(L, \alpha) \longrightarrow (L, \mathcal{U}(\alpha))$ is functorial:

Proposition 5.6.8. (L, λ) *and* (M, μ) *be covering uniform frames and let* $h : L \longrightarrow M$ *be a uniform frame homomorphism. Then* $h : (L, \mathcal{U}(\lambda)) \longrightarrow (M, \mathcal{U}(\mu))$ *is an entourage uniform frame homomorphism.*

Proof. Let $U_A \in \mathcal{U}(\lambda)$ where $A \in \lambda$. Since h is a uniform frame homomorphism, $h(A) \in \mu$ and thus $U_{h(A)} \in \mathcal{U}(\mu)$. Now, let $x \in L$. Then $U_{h(A)} \circ h(x) = \bigvee\{h(a) \mid a \in A, h(a) \wedge h(x) \neq 0\}$ and $h \circ U_A(x) = h(\bigvee\{a \in A \mid a \wedge x \neq 0\})$. Since h is a uniform frame homomorphism, we have for each $x, a \in L$ that $h(a) \wedge h(x) \neq 0$ implies $a \wedge x \neq 0$. Therefore, combining all these facts leads to $U_{h(A)} \circ h \leq h \circ U_A$.

□

Proposition 5.6.9. *Let* $(L, \mathcal{U})$ *and* $(M, \mathcal{V})$ *be entourage quasi-uniform frames and let* $h : L \longrightarrow M$ *be an entourage quasi-uniform frame homomorphism. Then* $h : (L, \mu(\mathcal{U})) \longrightarrow (M, \nu(\mu))$ *is a uniform frame homomorphism.*

Proof. Let h be an entourage quasi-uniform frame homomorphism. Let $A \in \mu(\mathcal{U})$ and let $D \in \mu(\mathcal{U})$ such that $D^* \leq A$. In view of Proposition (5.6.8) it follows that $U_D \in \mathcal{U}$ and so there exists $V \in \mathcal{V}$ such that $V \circ h \leq h \circ U_D$. There exists $B \in \mu(V)$ such that $U_B \leq V$. Thus $U_B \circ h \leq h \circ U_D$. In order to show that $B \leq h(A)$, we let $b \in B$. Then there exists $d \in D$ such that $b \wedge h(d) \neq 0$. Note that $b \leq Bh(d) = U_B \circ h(d) \leq h \circ U_D(d) = h(Dd)$. But there exists $a \in A$ such that $Dd \leq a$; hence $b \leq h(a)$.

□

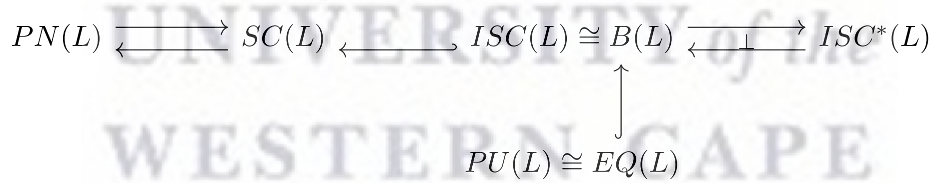
From the discussion in this section (Proposition (5.6.5) to (5.6.9)), there exists a natural functor from the category **EQFrm** to the category **PUniFrm**. In summary, we have the following theorem. It is an extension of the last theorem of [FH91].

Theorem 5.6.10. *The category* **PUniFrm** *is isomorphic with the category* **EQFrm**.

Throughout this chapter, the following preordered collections have been used for a given frame L:

- $PN(L)$ the collection of all the pre-nearnesses on the frame L;
- $PU(L)$ the collection of all the pre-uniformities on the frame L;
- $SC(L)$ the collection of all the semi-Császár structures on the frame L;
- $ISC(L)$ the collection of all the interpolative semi-Császár structures on the frame L such that each $\triangleleft \in \mathcal{L}$ preserves arbitrary meets;
- $ISC^*(L)$ the collection of all the interpolative semi-Császár structures on the frame L such that each $\triangleleft \in \mathcal{L}$ need not to preserve meets;
- $B(L)$ the collection of all the bases of a quasi-uniformity on the frame L;
- $EQ(L)$ the collection of all the entourage quasi-uniformities on the frame L.

Thinking of all structures within a frame L, we have the following schematic diagram which summarizes our discussions in this chapter:



Chapter 6

OTHER TOPOGENOUS ORDERS AND RELATED MAPS

6.1 Introduction

In this chapter, our focus is dedicated to the examination of particular mappings within a general category equipped with an $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms. This $(\mathcal{E}, \mathcal{M})$-factorization structure enables us to delve into various noteworthy ordering mechanisms and extend the applicability of several established findings. In particular, we define strict maps with respect to two topogenous orders, encompassing "closed maps" relative to two closure operators introduced in [Hol09]. We also define the open maps with respect to two interior operators.

Besides, using the topogenous order derived from a functor as introduced in [Ira19], we extend the scope of closed maps originally defined by G. Castellini and E. Giulli [CG05, CG01]. As previously said, the concepts of $(\mathcal{E}, \mathcal{M})$-factorization and related sub-structures will play a big role in this chapter. Thus, we begin the section by furnishing the definition of $(\mathcal{E}, \mathcal{M})$-factorization of a category and some pertinent insights essential for the objectives of this chapter. The reader wishing to have more details on $(\mathcal{E}, \mathcal{M})$-factorization and sub-structures is encouraged to consult [AHS90].

Definition 6.1.1. Let $\mathcal{X}$ be a given category. A pair of classes $(\mathcal{E}, \mathcal{M})$ of morphisms of $\mathcal{X}$ where $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ a class of monomorphisms, is said to be a factorization structure on $\mathcal{X}$ if:

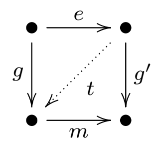
(1) $\mathcal{E}$ and $\mathcal{M}$ are closed under composition with isomorphisms, that is, if $e \in \mathcal{E}$, g is an isomorphism in $\mathcal{X}$ then $g \circ e \in \mathcal{E}$ and if $m \in \mathcal{M}$, g is an isomorphism in $\mathcal{X}$, then $m \circ g \in \mathcal{M}$;

(2) Every morphism $f \in \mathcal{X}$ factors as an $\mathcal{E}$-morphism and an $\mathcal{M}$-morphism, that is, $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$;

(3) $\mathcal{X}$ has a unique $(\mathcal{E}, \mathcal{M})$-diagonalization property, this means that given a commutative diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{e} & \bullet \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \bullet & \xrightarrow[m]{} & \bullet \end{array}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there is a uniquely determined morphism t satisfying $t \circ e = g$ and $m \circ t = g'$ making the following diagram



commute. In this context, we say that every $\mathcal{E}$-morphism e is orthogonal to every $\mathcal{M}$-morphism m and write $e \perp m$.

If the statements of the above definition hold, we also say that the category $\mathcal{X}$ is $(\mathcal{E}, \mathcal{M})$-structured.

The following proposition, due to J. Adámek, H. Herrlich and G. G. Strecker [AHS90], outlines the properties and characteristics of the $\mathcal{E}$-morphisms and $\mathcal{M}$-morphisms in any category $\mathcal{X}$.

Proposition 6.1.1. *Let $\mathcal{X}$ be an $(\mathcal{E}, \mathcal{M})$-structured category and $Iso(\mathcal{X})$ the class of isomorphisms in $\mathcal{X}$, then the following hold:*

(1) $\mathcal{E} \cap \mathcal{M} = Iso(\mathcal{X})$;

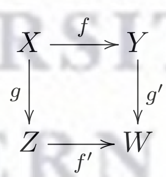
(2) If $g \circ f \in \mathcal{M}$ then $f \in \mathcal{M}$;

(3) If $g \circ f \in \mathcal{E}$ then $g \in \mathcal{E}$;

(4) $\mathcal{E}$ and $\mathcal{M}$ are closed under composition;

(5) $\mathcal{M}$ is closed under intersections;

(6) $\mathcal{M}$ is closed under pullbacks, that is, for any pullback diagram



if $f' \in \mathcal{M}$ then $f \in \mathcal{M}$

Remark 6.1.2. *(1) The $(\mathcal{E}, \mathcal{M})$-factorization of a morphism is unique up to isomorphim.*

(2) In an $(\mathcal{E}, \mathcal{M})$-factorization system, the classes $\mathcal{E}$ and $\mathcal{M}$ determine each other:

(a) $\mathcal{E} = \{e \in \mathcal{X} \mid \forall m \in \mathcal{M} \mid e \perp m\}$;

(b) $\mathcal{M} = \{m \in \mathcal{X} \mid \forall e \in \mathcal{E} \mid e \perp m\}$.

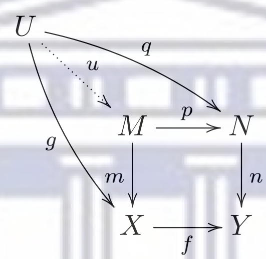
In what follows, we shall provide a brief review of the concept of sub-objects, as it will be necessary for the upcoming sections. Sub-objects are categorical generalizations of mathematical structures. They are consistently described by special morphisms, which can be thought of as generalized inclusion maps. To put it more formally, given an object $X \in \mathcal{X}$, the collection of morphisms representing subobjects of X is denoted by sub$X$. This collection is defined by:

$$subX = \{m \in \mathcal{M} \mid m : M \longrightarrow X\} \tag{6.1.1}$$

Let $m : M \longrightarrow X$ and $n : N \longrightarrow X$ be any two morphisms in sub$X$, we write $m \leq n$ if and only if there exists a morphism $j : M \longrightarrow N$ in $\mathcal{X}$ such that $n \circ j = m$. It is evident that $\leq$ is a reflexive and transitive binary relation on sub$X$, thereby rendering sub$X$ a preordered class. Furthermore, if m and n also satisfy $n \leq m$, we can prove that they are isomorphic. Symbolically, this is expressed as $m \cong n$.

Additionally, since $\cong$ is an equivalence relation on sub$X$, the quotient class of sub$X$ under $\cong$ forms a partially ordered set. This structure also amounts to a complete lattice with the standard symbols for joins (if they exist) and meets, namely $\vee$, $\bigvee$ and $\wedge$, $\bigwedge$, respectively. The least element and the greatest element of sub$X$ are denoted by $0_X : 0_X \longrightarrow X$ and $1_X : X \longrightarrow X$, respectively.

Definition 6.1.3. The category $\mathcal{X}$ is said to have $\mathcal{M}$-pullbacks if for each $\mathcal{X}$-morphism $f : X \longrightarrow Y$ and for each $n \in$ subY, a pullback diagram



exists in $\mathcal{X}$.

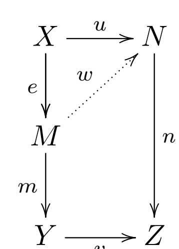
The morphism m, which is uniquely determined up to isomorphism, is then named the inverse image of n under f and is denoted by $f^{-1}(n) : f^{-1}(N) \longrightarrow X$.

The concept of right $\mathcal{M}$-factorization is essential for manipulating the images and inverse images of subobjects effectively.

Proposition 6.1.2. *[DT95] Assume that $\mathcal{X}$ have $\mathcal{M}$-pullback and that for every $\mathcal{X}$-morphism $f : X \longrightarrow Y$, $f^{-1}(-)$ have a left adjoint. Then there exist morphisms e, m in $\mathcal{X}$ satisfying:*

(a) $f = m \circ e$ with $m : M \longrightarrow Y$ in $\mathcal{M}$ and whenever;

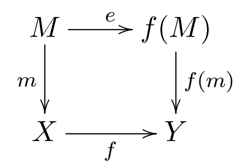
(b) (Diagonalization) whenever one has the following commuting diagram



in $\mathcal{X}$ with $n \in \mathcal{M}$, there is a uniquely determined morphism $w : M \longrightarrow N$ such that $n \circ w = v \circ m$ and $w \circ e = u$.

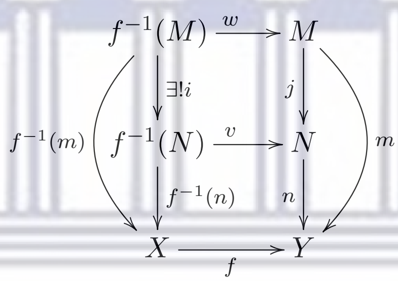
Definition 6.1.4. A right $\mathcal{M}$-factorization of morphism $f : X \longrightarrow Y$ is any factorization $f = m \circ e$ such that the axioms (a) and (b) of Proposition (6.1.2).

Definition 6.1.5. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{X}$ and $m : M \longrightarrow X$ a morphism in subX, the image of m under f, denoted by $f(m) : f(M) \longrightarrow Y$, is defined as the $\mathcal{M}$-component of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ m$ as shown in the diagram below:



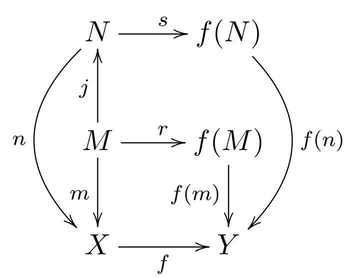
Lemma 6.1.6. *Let $\mathcal{X}$ be a category with $\mathcal{M}$-pullbacks such that every morphism $f : X \longrightarrow Y$ has a right $\mathcal{M}$-factorization. Then maps $f^{-1}(-) : subY \longrightarrow subX$ and $f(-)$: $subX \longrightarrow subY$ are order preserving maps.*

Proof. To show $f^{-1}(-) :$ sub$Y \longrightarrow$ sub$X$ is order preserving, consider $m, n \in$ sub$Y$ such that $m \leq n$. Then there exist t with $m = n \circ t$. We then have the following diagram

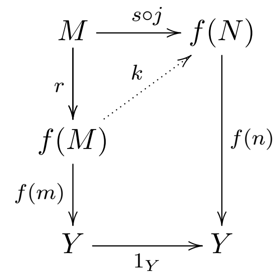


Clearly, since $n \circ v$=$f \circ f^{-1}(n)$ and $m \circ w = f \circ f^{-1}(m)$ then by pullback property the unique morphism $i : f^{-1}(M) \longrightarrow f^{-1}(N)$ exists with $f^{-1}(m) = f^{-1}(n) \circ i$, that is, $f^{-1}(m) \leq f^{-1}(n)$.

Next, let $m, n \in$ subX with $m \leq n$. Then there exists a morphism j such that $m = n \circ j$. Hence, the following diagram:



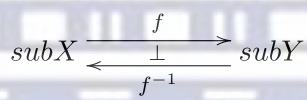
we obtain the following commuting diagram



Now since $f(M)$ and $f(N) \in$ subY, from right $\mathcal{M}$-factorization properties, there exist a unique morphism $k : f(M) \longrightarrow f(N)$ such that $f(m) = f(n) \circ k$ and $f(m) \leq f(n)$.

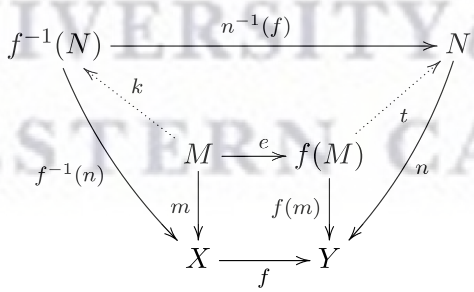
□

Proposition 6.1.3. *For every morphism $f : X \longrightarrow Y$ in $\mathcal{X}$, the pair $(f(-), f^{-1}(-))$ forms a Galois connection between subX and subY with $f^{-1}(-)$ being the right, that is,*



Proof. Note that, since from Lemma (6.1.6) both $f^{-1}(-)$ and $f(-)$ are order preserving maps, it suffices to establish that for every $m \in$ subX, $n \in$ subY, $f(m) \leq n$ if and only if $m \leq f^{-1}(n)$

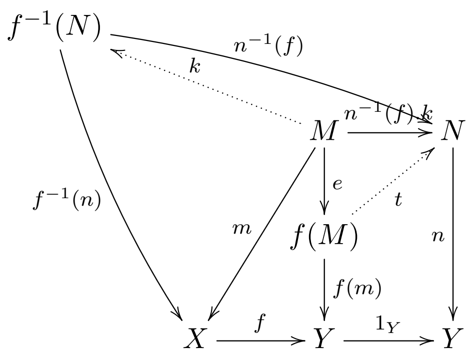
For the forward implication, we assume that $f(m) \leq n$. Then there exists an arrow $t : f(M) \longrightarrow N$ with $f(m) = n \circ t$. Now consider the following diagram



Since $f \circ f^{-1}(n) = n \circ n^{-1}(f)$, the universel property of pullbacks asserts the existence of a unique arrow $k : f(M) \longrightarrow f^{-1}(N)$ such that $m = k \circ f^{-1}(n)$, that is $m \leq f^{-1}(n)$.

For the backward implication, we let $m \leq f^{-1}(n)$, then there exists an arrow $k : M \longrightarrow$

$f^{-1}(N)$ such that $f^{-1}(n) \circ k = m$ as in the triangle at the left of the diagram below.



Since $f(m) \circ e = n \circ n^{-1}(f) \circ k$, by diagoalization property, there exists an arrow $t : f(M) \longrightarrow N$ such that $f(m) = n \circ t$, i.e $f(m) \leq n$. □

The aforementioned adjunction yields the following beneficial consequences:

(1) $m \leq f^{-1}(f(m))$ and $f(f^{-1}(n)) \leq n$;

(2) $f(\bigvee_{i \in I} m_i) \cong \bigvee_{i \in I} f(m_i)$, if $\bigvee_{i \in I} m_i$ exists;

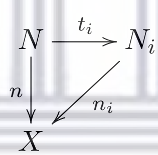
(3) $f^{-1}(\bigwedge_{i \in I} n_i) \cong \bigwedge_{i \in I} f^{-1}(n_i)$;, if $\bigwedge_{i \in I} n_i$ exists;

(4) if $f \in \mathcal{M}$, then $f^{-1}(f(m)) = m$ for all $m \in \mathrm{Sub}X$;

(5) if $f \in \mathcal{E}$ and $\mathcal{E}$ is stable under pullbacks then $n = f(f^{-1}(n))$ for all $n \in \mathrm{Sub}Y$.

Moreover, if the category $\mathcal{X}$ has $\mathcal{M}$-pullbacks, then the preordered class sub$X$ has binary meets for any $X \in \mathcal{X}$.

Definition 6.1.7. A category $\mathcal{X}$ is said to have $\mathcal{M}$-intersetions if for every family $(n_i)_{i \in I}$ in sub$X$, the following multiple pullback diagram



commutes, that is to mean $n = n_i \circ t_i$. This implies also the existence of the join $\bigvee$ of subobjects, in particular, for each object $Y \in \mathcal{X}$, the least subobject $o_Y : OY \longrightarrow Y$ exists.

In consequence, a category $\mathcal{X}$ will be called $\mathcal{M}$-complete if it has $\mathcal{M}$-pullbacks and $\mathcal{M}$-intersections.

For the rest of the chapter, we assume the category $\mathcal{X}$ is equipped with an $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms and that it is $\mathcal{M}$-complete.

As previously indicated in the section, the purpose of this chapter is two fold: firstly, we intend to extend the concept closed maps with respect to two closure operators [Hol09]. This is achieved by introducing the notion of strict maps with respect to two topogenous orders. Secondly, we explore the topogenous orders induced by a faithful functor and employ them to examine specific maps, some of which encompass closed maps introduced in [CG05, CG01], as a paticular case. Additionally, we define an interior operator induced by a functor and demonstrate, as is the case in the classical scenario, it corresponds to the strict map with respect to the topogenous order induced by a faithful functor.

6.2 Strict Morphisms with Respect to Two Topogenous Orders

Starting with a category equipped with an $(\mathcal{E}, \mathcal{M})$-factorization system, Holgate [Hol09] defined the notion of closed maps with respect to two closure operators.

In this section we aim to establish a comprehensive understanding of strict and final maps with respect to two topogenous orders. This perspective incorporates Holgate's definition of closed maps with respect to two closure operators as a special case. This generalization, which deals with strict maps in the context of two orders, enables us to precisely determine the criteria for a morphism within a category to be considered open with respect to two interior operators.

Definition 6.2.1. Consider two topogenous orders $\lhd, \lhd'$ on $\mathcal{X}$. Let $f : X \longrightarrow Y$ be a morphism within the category $\mathcal{X}$. Within this context, we say that f is $\lhd, \lhd'$-continuous if:

(1) $f(m) \lhd'_Y n \Rightarrow m \lhd_X f^{-1}(n)$ for all $m \in subX$ and $n \in subY$;

(2) $q \lhd' n \Rightarrow f^{-1}(q) \lhd f^{-1}(n)$ for all $n, q \in$ subY.

Since a morphism $f : X \longrightarrow Y$ in a category is $\lhd$-strict, respectively $\lhd$-final if $f(m) \lhd_Y n \Leftrightarrow m \lhd_X f^{-1}(n)$ respectively if $q \lhd n \Leftrightarrow f^{-1}(q) \lhd f^{-1}(n)$ for all $m \in$ subX and $n, q \in$ subY and also since f is assumed to be $\lhd$-continous in $\mathcal{X}$, the important implications are $f(m) \lhd_Y n \Leftarrow m \lhd_X f^{-1}(n)$ respectively $q \lhd n \Leftarrow f^{-1}(q) \lhd f^{-1}(n)$.

This fact motivates the following definitions of $\lhd$-strict and $\lhd$-final morphisms with respect to two topogenous orders:

Definition 6.2.2. Let $\lhd, \lhd'$ be two topogenous orders on $\mathcal{X}$ with respect to $\mathcal{M}$. Let $f : X \longrightarrow Y$ represent a morphism within $\mathcal{X}$. Then f is called:

(1) $\lhd, \lhd'$-strict if and only if $f(m) \lhd'_Y n \Leftarrow m \lhd_X f^{-1}(n)$ for all $m \in$ subX and $n \in$ subY;

(2) $\lhd, \lhd'$-final if and only if $q \lhd' n \Leftarrow f^{-1}(q) \lhd f^{-1}(n)$ for all $q, n \in$ subY.

Our main motivation for defining these maps is to offer another way of expressing closed and final maps with respect to two closure operators in terms of topogenous orders.

Proposition 6.2.1. *[Hol09] Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{X}$:*

(1) If both $\lhd, \lhd' \in \bigwedge$-TORD $(\mathcal{X}, \mathcal{M})$, then $f : X \longrightarrow Y$ is $\lhd, \lhd'$-continuous if and only if $f(c^{\lhd}_X(m)) \leq c^{\lhd'}_Y(f(m))$ for all $m \in subX$

(2) If both $\lhd, \lhd' \in \bigvee$-TORD $(\mathcal{X}, \mathcal{M})$, then $f : X \longrightarrow Y$ is $\lhd, \lhd'$-continuous if and only if $f^{-1}(i^{\lhd}_Y(n)) \leq i^{\lhd'}_X(f^{-1}(n))$ for all $n \in subY$

Proposition 6.2.2. *Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{X}$.*
If both $\lhd, \lhd' \in \bigwedge$-TORD $(\mathcal{X}, \mathcal{M})$, then $f : X \longrightarrow Y$ is $\lhd, \lhd'$-strict if and only if

$$f(k^{\lhd}_X(m)) \geq k^{\lhd'}_Y(f(m))$$

for all $m \in subX$

Proof. Let f be $\lhd, \lhd'$-strict. Then

$$\begin{aligned} k_Y^{\lhd'}(f(m)) \leq n &\Leftrightarrow f(m) \lhd'_Y n \\ &\Leftarrow m \lhd_X f^{-1}(n) \\ &\Leftrightarrow k_X^{\lhd}(m) \leq f^{-1}(n) \\ &\Leftrightarrow f(k_X^{\lhd}(m)) \leq n. \end{aligned}$$

Conversely if $k_Y^{\lhd'}(f(m)) \leq f(k_X^{\lhd}(m))$. Then

$$\begin{aligned} m \lhd f^{-1}(n) &\Leftrightarrow k_X^{\lhd}(m) \leq f^{-1}(n) \\ &\Leftrightarrow f(k_X^{\lhd}(m)) \leq n \\ &\Leftarrow k_Y^{\lhd'}(f(m)) \leq n \\ &\Leftarrow f(m) \lhd'_Y n \end{aligned}$$

□

Furthermore, if $\lhd, \lhd' \in \bigvee$-TORD$(\mathcal{X}, \mathcal{M})$, one can also establish the following proposition:

Proposition 6.2.3. *Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{X}$. If both $\lhd, \lhd' \in \bigvee$-TORD $(\mathcal{X}, \mathcal{M})$, then $f : X \longrightarrow Y$ is $\lhd, \lhd'$-strict if and only if*

$$f^{-1}(i_Y^{\lhd}(n)) \geq i_X^{\lhd'}(f^{-1}(n))$$

for all $n \in subY$.

The $\lhd, \lhd'$-strict morphisms have the following features:

Proposition 6.2.4. *Let $\lhd, \lhd', \lhd''$ be topogenous orders on $\mathcal{X}$ with respect to $\mathcal{M}$ and $f : X \longrightarrow Y$, $g : X \longrightarrow Y$ morphisms in $\mathcal{X}$, then:*

(1) If f is $\lhd, \lhd'$-strict and g is $\lhd', \lhd''$-strict then $g \circ f$ is $\lhd, \lhd''$-strict;

(2) If gf is $\lhd, \lhd'$-strict and $g \in \mathcal{M}$ then f is $\lhd, \lhd'$-strict;

(3) If gf is $\lhd, \lhd'$-strict and $f \in \mathcal{E}$ then g is $\lhd, \lhd'$-strict.

The proof depends on the following observations.

$$\begin{aligned} \text{If } g \in \mathcal{M} \text{ then } g^{-1}(g(n)) &= n \\ \text{If } f \in \mathcal{E} \text{ then } f(f^{-1}(n)) &= n. \end{aligned}$$

Proof.

$$\begin{aligned} (1)\ m \lhd'' (gf)^{-1}(n) = f^{-1}(g^{-1}(n)) &\Rightarrow f(m) \lhd_Y g^{-1}(n) \\ &\Rightarrow (fg)(m) \lhd_Z n. \end{aligned}$$

$$
\begin{aligned}
(2)\ \ m \triangleleft'_X f^{-1}(n) = f^{-1}(g(g^{-1}(n)) &\Rightarrow (gf)(m) \triangleleft_Z g(n)\\
&\Rightarrow (g(f((m))) \triangleleft_Z g(n)\\
&\Rightarrow f(m) \triangleleft_Z g(n).
\end{aligned}
$$

$$
\begin{aligned}
(3)\ \ m \triangleleft'_X g^{-1}(n) = f^{-1}(m) &\Rightarrow' f^{-1}(g^{-1}(n)) = (gf)^{-1}(m) \triangleleft_Z g(n)\\
&\Rightarrow (gf)(f^{-1}(m))) \triangleleft_Z n\\
&\Rightarrow g(m) \triangleleft_Z n.
\end{aligned}
$$

□

Simlary, we have the following behavior for the $\triangleleft, \triangleleft'$-final morphisms

Proposition 6.2.5. *Let $\triangleleft, \triangleleft', \triangleleft''$ be topogenous orders on $\mathcal{X}$ with respect to $\mathcal{M}$ and $f : X \longrightarrow Y, g : Y \longrightarrow Z$ morphisms in $\mathcal{X}$, then*

(1) If f is $\triangleleft, \triangleleft'$-final and g $\triangleleft', \triangleleft''$-final then $g \circ f$ is $\triangleleft, \triangleleft''$-final;

(2) If gf is $\triangleleft, \triangleleft'$-final and $g \in \mathcal{M}$ then f is $\triangleleft, \triangleleft'$-final;

(3) If gf is $\triangleleft, \triangleleft'$-final and $f \in \mathcal{E}$ then g is $\triangleleft, \triangleleft'$-final.

Proof. for any $n, m \in$ subY then

$$
\begin{aligned}
(gf)^{-1}(m) \triangleleft (gf)^{-1}(m) &\Rightarrow f^{-1}(g^{-1}(m)) \triangleleft f^{-1}(g^{-1}(n))\\
&\Rightarrow g^{-1}(m) \triangleleft' g^{-1}(n)\\
&\Rightarrow m \triangleleft'' n.
\end{aligned}
$$

$$
\begin{aligned}
f^{-1}(m) \triangleleft f^{-1}(n) &\Rightarrow f^{-1}(m) \triangleleft f^{-1}(g^{-1}(g(n)))\\
&\Rightarrow f^{-1}(m) \triangleleft (gf)^{-1}(g(n))\\
&\Rightarrow (gf)(f^{-1}(m)) \triangleleft g(n)\\
&\Rightarrow g(m) \triangleleft g(n)\\
&\Rightarrow m \triangleleft' n.
\end{aligned}
$$

$$
\begin{aligned}
m \triangleleft' n &\Leftarrow (gf)^{-1}(m) \triangleleft (gf)^{-1}(n)\\
&\Leftarrow f^{-1}(g^{-1}(m)) \triangleleft f^{-1}(g^{-1}(n))\\
&\Leftarrow g^{-1}(m) \triangleleft g^{-1}(n).
\end{aligned}
$$

□

6.3 Topogenous Order Induced by a Faithful $\mathcal{M}$-Fibration and Related Maps

Let $F : \mathcal{A} \longrightarrow \mathcal{X}$ be a given functor. Following [CG05, CG01], we adopt the following notations. The pair (X, m) with X an object of $\mathcal{A}$ and $m : M \longrightarrow FX$ in $\mathcal{M}$ will be called an F-subobject of X. We also use $subFX$ to denote the class of all F-subobjects of X. In the sequel we will refer to m instead of (X, m) for an F-subobject when no-confusion is likely to arise.

Definition 6.3.1. *[CG05, CG01]* An F-closure operator k of $\mathcal{A}$ (with respect to $(\mathcal{E}, \mathcal{M})$) is a family of functions

$$\{k_X^F : subFX \longrightarrow subFX \mid X \in \mathcal{A}\}$$

such that the following axioms hold:

$(K1)$ $m \le k_X^F(m)$ for all $m \in subFX$;

$(K2)$ $m \le n \Rightarrow k_X^F(m) \le k_X^F(n)$ for all for all $m, n \in subFX$;

$(K3)$ for each $\mathcal{A}$-morphism $f : X \longrightarrow Y$, $Ff(k_X^F(m)) \le k_Y^F(Ff(m))$ for all $m \in subFX$.

Remark 6.3.2. As in the classical case, under condition $(K2)$ of the above definition, $(K3)$ is equivalent to the following: given a morphism $f : X \longrightarrow Y$ in $\mathcal{A}$ and F-subobject n of Y,

$$k_X^F(Ff)^{-1}(n) \le (Ff)^{-1}(k_Y^F(n))$$

Definition 6.3.3. An F-interior operator i of $\mathcal{A}$ (with respect to $(\mathcal{E}, \mathcal{M})$) is a family of maps

$$\{i_X^F : subFX \longrightarrow subFX | X \in \mathcal{A}\}$$

such that the following axioms hold:

$(I1)$ $i_X^F(m) \le m$ for all $m \in subFX$;

$(I2)$ $m \le n \Rightarrow i_X^F(m) \le i_X^F(n)$ for all for all $m, n \in subFX$;

$(I3)$ for each $\mathcal{X}$-morphism $f : X \longrightarrow Y$, $(Ff)^{-1}(i_Y^F(n)) \le i_X^F(Ff)^{-1}(n)$ for all n in $subFY$.

Replacing the inequalities with " $\cong$ " in $(I3)$ and $(K3)$ leads to two classes of morphisms, namley the k^F-closed and i^F-open morphisms:

Definition 6.3.4. *[CG01]* Let k^F be a closure operator induced by F on $\mathcal{A}$. An $\mathcal{A}$-morphism $f : X \longrightarrow Y$ is said to be k^F-closed if for every F-subobject m of F,

$$Ff(k_X^F(m)) \cong k_Y^F(Ff(m))$$

for all $m \in$ subFX.

Definition 6.3.5. Let i^F be the interior operator induced by F on $\mathcal{A}$. An $\mathcal{A}$-morphism $f : X \longrightarrow Y$ is said to be i^F-open if for every F-subobject m of F,

$$(Ff)^{-1}(i_Y^F(n)) \cong i_X^F(Ff)^{-1}(n)$$

for all $n \in$ subFY.

Recall from [DT95] that for an $\mathcal{M}$-fibration $F : \mathcal{A} \longrightarrow \mathcal{X}$, $(\mathcal{E}_F, \mathcal{M}_F)$ where $\mathcal{E}_F = F^{-1}\mathcal{E} = \{e \in \mathcal{X} | Fe \in \mathcal{E}\}$ and $\mathcal{M}_F = F^{-1}\mathcal{M} \bigcap$ IniF, with IniF the class of F-initial morphisms is a factorization system in $\mathcal{A}$ and $\mathcal{M}$-subobject properties in $\mathcal{X}$ are inherited by $\mathcal{M}_F$-subobjects in $\mathcal{A}$. In particular:

(1) $\mathcal{A}$ has $\mathcal{M}_F$-pullbacks if $\mathcal{X}$ has $\mathcal{M}$-pullback;

(2) $\mathcal{A}$ is $\mathcal{M}_F$-complete if $\mathcal{X}$ is $\mathcal{M}$-complete;

(3) The $\mathcal{M}_F$-image and $\mathcal{M}_F$-inverse image are obtained by initially lifting the $\mathcal{M}$-image and $\mathcal{M}$-inverse image.

In consequence, we obtain: $Ff^{-1}(n) = (Ff)^{-1}Fn$ and $(Ff)(Fm) = Ff(m)$ for any $f \in \mathcal{A}$ and suitable subobjects n and m.

Lemma 6.3.6. *Let $F : \mathcal{A} \longrightarrow \mathcal{X}$ be a faithful $\mathcal{M}$-fibration.*

(1) for any $X \in \mathcal{A}$, subX and subFX are order equivalent with the inverse assignment λ_X :subX$\longrightarrow$ subFX and δ_X: subFX $\longrightarrow$ subX defined as follows:

$\lambda_X(m) = Fm$ and $\delta_X(n) = q$ with $Fq = n$ and $q \in$ IniF.

(1) Any morphism $f : X \longrightarrow Y$ in $\mathcal{A}$ gives rise to the following functors:

$f(-)$: subX $\longrightarrow$ subY, $f^{-1}(-)$: subY$\longrightarrow$ subX

$Ff(-)$: subFX $\longrightarrow$ subFY, $(Ff)^{-1}(-)$: subFY $\longrightarrow$ subFX.

We then have for any subobjects m, n, q, p the following:

(1) $\lambda_Y(f(m)) = (Ff)(\lambda_X(m))$;

(2) $f(\lambda_X(n)) = \delta_Y(Ff)(n)$;

(3) $f^{-1}(\delta_Y(p)) = \delta_X((Ff)^{-1}(p))$;

(4) $\lambda_Y(f^{-1}(q)) = (Ff)^{-1}(\lambda_Y(q))$.

Proposition 6.3.1. *[HI19] Let $F : \mathcal{A} \longrightarrow \mathcal{X}$ be a faithful $\mathcal{M}$-fibration and $\triangleleft$ be a topogenous order on $\mathcal{X}$ with respect to $\mathcal{M}$. Define $m \triangleleft^F_X n \Leftrightarrow Fm \triangleleft_{FX} \lambda_X(n)$. Then $\triangleleft^F$ is a topogenous order on $\mathcal{A}$ with respect to $\mathcal{M}_F$.*

Proof. We only prove the continuity condition as $(T1)$ and $(T2)$ can be easly seen to be satisfied. To this end, let $f : X \longrightarrow Y$ be a morphism in $\mathcal{A}$ and $f(m) \triangleleft^F_Y n$. Then

$$\begin{aligned} Ff(m) \triangleleft \lambda_Y(n) \Rightarrow & (Ff)Fm \triangleleft_{FY} \lambda_Y(n) \\ \Rightarrow & Fm \triangleleft_{FX} (Ff)^{-1}(\lambda_Y(n)) \\ \Rightarrow & Fm \triangleleft_{FX} \lambda_Y f^{-1}(n) \\ \Leftrightarrow & m \triangleleft^F_X f^{-1}(n). \end{aligned}$$

In consequence of the above-mentioned proposition, the following result arises:

Proposition 6.3.2. *(1) If* $\triangleleft \in \bigvee$*-TORD, then* $m \triangleleft_X^F n \Leftrightarrow \delta_X(k_{FX}^{\triangleleft}(Fm)) \leq n$.
(2) If $\triangleleft \in \bigwedge$*-TORD, then* $m \triangleleft_X^F n \Leftrightarrow m \leq \delta_X(i_{FX}^{\triangleleft}(\lambda_X(n)))$.

Proof. *(1) If* $\triangleleft \in \bigvee$*-TORD, then*

$$\begin{aligned} m \triangleleft_X^F n &\Leftrightarrow Fm \triangleleft_{FX} \lambda_X(n) \\ &\Leftrightarrow k_{FX}^{\triangleleft}(Fm) \leq \lambda_X(n) \\ &\Leftrightarrow \delta_X(k_{FX}^{\triangleleft}(Fm)) \leq \delta_X(\lambda_X(n)) \\ &\Leftrightarrow \delta_X(k_{FX}^{\triangleleft}(Fm)) \leq n \end{aligned}$$

(2) If $\triangleleft \in \bigwedge$*-TORD, then*

$$\begin{aligned} m \triangleleft_X^F n &\Leftrightarrow Fm \triangleleft \lambda_X(n) \\ &\Leftrightarrow Fm \leq i_{FX}^{\triangleleft}(\lambda_X(n)) \\ &\Leftrightarrow \delta_X(Fm) \leq \delta_X(i_{FX}^{\triangleleft}(\lambda_X(n))) \\ &\Leftrightarrow m \leq \delta_X(i_{FX}^{\triangleleft}(\lambda_X(n))). \end{aligned}$$

□

Definition 6.3.7. Let $\triangleleft^F$ be a topogenous order on $\mathcal{A}$. A morphism $f : X \longrightarrow Y$ is $\triangleleft^F$-strict if

$$f(m) \triangleleft_X^F n \Leftrightarrow m \triangleleft_X^F f^{-1}(n)$$

for all $n \in$ subFX and $n \in$ subFY.

Just like in the classical case, if f is $\triangleleft^F$-strict, then $Ff(-)$ maps $\triangleleft^F$-strict objects to $\triangleleft^F$-strict objects.

The following proposition demonstrates that $\triangleleft^F$-strict morphisms behave exactly the same like the $\triangleleft$ strict ones (strict in the classical case):

Proposition 6.3.3. *(1) The class of the* $\triangleleft^F$*-strict morphisms are closed under composition and contains all isomorphisms;*

(2) $\triangleleft^F$*-strict morphisms are left cancellable with respect to* $\mathcal{M}_F$*;*

(3) $\triangleleft^F$*-strict morphisms are right cancellable with respect to* $\mathcal{M}_F$*.*

As we have already observed, topogenous orders provide a general language in which we can easily study closure and interior operators simultaneously. In the next two propositions, we demonstrate that if $\triangleleft^F$ preserves meets, then $\triangleleft^F$-strict morphisms are precisely the $k^{\triangleleft}$-closed morphisms. Similarly, if $\triangleleft^F$ preserves joins, the $\triangleleft^F$-strict morphisms correspond to the $i^{\triangleleft}$-open ones. In other words, we are expressing the definitions of k^F and i^F in terms of orders.

Proposition 6.3.4. *Let* $f : X \longrightarrow Y$ *be a morphism in* $\mathcal{A}$ *and* $\triangleleft^F$ *a meet-preserving topogenous order on* $\mathcal{A}$ *with respect to* $\mathcal{M}_F$*. Then* f *is* $\triangleleft^F$*-strict if and only if* $(Ff)(k_{FX}^{\triangleleft}(m)) = k_{FY}^{\triangleleft}(Ff(m))$

Proof. On one hand if $(Ff)(k^{\triangleleft}_{FX}(m)) = k^{\triangleleft}_{FY}(Ff(m))$. Then

$$\begin{aligned} m \triangleleft^F_X f^{-1}(n) &\Leftrightarrow Fm \triangleleft_{FX} \lambda_X(f^{-1}(n)) \\ &\Leftrightarrow k^{\triangleleft}_{FX}(Fm) \leq \lambda_X(f^{-1}(n)) \\ &\Leftrightarrow k^{\triangleleft}_{FX}(Fm) \leq (Ff)^{-1}(\lambda_X(n)) \\ &\Leftrightarrow Ff(k^{\triangleleft}_{FX}(Fm)) \leq \lambda_X(n) \\ &\Leftrightarrow k^{\triangleleft}_{FY}(Ff(m)) \leq \lambda_Y(n) \\ &\Leftrightarrow Ff(m) \triangleleft \lambda_Y(n) \\ &\Leftrightarrow f(m) \triangleleft^F_Y n. \end{aligned}$$

On the other hand if f is $\triangleleft^F$-strict. Then

$$\begin{aligned} Ff(k^{\triangleleft}_X(m)) \leq n &\Leftrightarrow k^{\triangleleft}_{FX}(m) \leq (Ff)^{-1}(n) \\ &\Leftrightarrow m \triangleleft (Ff)^{-1}(n) \\ &\Leftrightarrow Ff(m) \triangleleft n \\ &\Leftrightarrow k^{\triangleleft}_{FY}(Ff(m)) \leq n. \end{aligned}$$

□

Proposition 6.3.5. *If $f : X \longrightarrow Y$ is a morphism in $\mathcal{A}$ and $\triangleleft^F$ a join-preserving a topogenous order on $\mathcal{A}$ with respect to $\mathcal{M}_F$. Then f is $\triangleleft^F$ strict if and only if $(Ff)^{-1} i^{\triangleleft}_{FY}(n) = i^{\triangleleft}_{FX}(Ff)^{-1}(n)$*

Proof. If $(Ff)^{-1}(i^{\triangleleft}_{FY}(n)) = i^{\triangleleft}_{FX}((Ff)^{-1}(n))$ then for the left hand implication, we have

$$\begin{aligned} m \triangleleft^F_X f^{-1}(n) &\Leftrightarrow Fm \triangleleft \lambda_X f^{-1}(n) \\ &\Leftrightarrow Fm \leq i^{\triangleleft}_X(\lambda_X(f^{-1}(n))) \\ &\Leftrightarrow Fm \leq i^{\triangleleft}_X(Ff)^{-1}(\lambda_Y(n)) \\ &\Leftrightarrow Fm \leq (Ff)^{-1}(i^{\triangleleft}_{FY}\lambda_Y(n)) \\ &\Leftrightarrow (Ff)(Fm) \leq i^{\triangleleft}_{FY}(\lambda_Y(n)) \\ &\Leftrightarrow Ff(m) \leq i^{\triangleleft}_{FY}(\lambda_Y(n)) \\ &\Leftrightarrow Ff(m) \triangleleft \lambda_Y(n) \\ &\Leftrightarrow f(m) \triangleleft^F_X n. \end{aligned}$$

□

Moreover if f is $\triangleleft^F$-strict, then the reverse implication can be easily proved in the similar manner as the one in Proposition 6.3.4.

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