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Umbulelo omkhulu nakubazali bam.

DECLARATION

I declare that **The non-cancellation groups of certain groups which are split extensions of a finite abelian group by a finite rank free abelian group** is my own work and that all sources I have used or quoted have been indicated and acknowledged by means of complete reference.

The non-cancellation groups of certain groups which are split extensions of a finite abelian group by a finite rank free abelian group

by

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Dissertation submitted in fulfillment of the requirements for the degree of Master of Science in the Department of Mathematics and Applied Mathematics, University of the Western Cape.

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CONTENTS

ABSTRACT

The groups we consider in this study belong to the class $\mathcal{X}_0$ of all finitely generated groups with finite commutator subgroups. We shall eventually narrow down to the groups of the form $T \rtimes_w \mathbb{Z}^n$ for some $n \in \mathbb{N}$ and some finite abelian group T. For a $\mathcal{X}_0$-group H, we study the *non-cancellation set*, $\chi(H)$, which is defined to be the set of all isomorphism classes of groups K such that $H \times \mathbb{Z} \cong K \times \mathbb{Z}$. For $\mathcal{X}_0$-groups H, on $\chi(H)$ there is an abelian group structure [38], defined in terms of embeddings of K into H, for groups K of which the isomorphism classes belong to $\chi(H)$. If H is a nilpotent $\mathcal{X}_0$-group, then the group $\chi(H)$ is the same as the Hilton-Mislin (see [10]) genus group $\mathcal{G}(H)$ of H. A number of calculations of such Hilton-Mislin genus groups can be found in the literature, and in particular there is a very nice calculation in article [11] of Hilton and Scevenels. The main aim of this thesis is to compute non-cancellation (or genus) groups of special types of $\mathcal{X}_0$-groups such as mentioned above. The groups in question can in fact be considered to be direct products of metacyclic groups, very much as in [11]. We shall make extensive use of the methods developed in [30] and employ computer algebra packages to compute determinants of endomorphisms of finite groups.

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CESSION

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Key words and phrases

1. Automorphism

2. Determinant of an endomorphism

3. Finite abelian group

4. Finitely generated group

5. Finite rank free group

6. Group action

7. Matrix

8. Nielsen equivalence

9. Non-cancellation

10. Semi direct product

CHAPTER 1

INTRODUCTION

In this work we shall be studying the non-cancellation sets $\chi(G)$, where G is in the family of groups which are finitely generated with finite commutator subgroups, denoted by $\mathcal{X}_0$. In particular, we focus on groups in $\mathcal{X}_0$ of the form $T \rtimes_w K$, called semi-direct products. The non-cancellation set $\chi(G)$ is the set of all isomorphism classes $[H]$ of groups H such that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$.

In Chapters 1 to 7 we give an exposition of the known theory, giving our own proofs and examples in several cases. (See, for instance, proofs of 3.7, 3.10, results in Chapter 4, 5.19, 6.19, and 7.3 etc.) Our major original contribution is in Chapter 8, where we find a condition to compute the non-cancellation group $\chi(H)$, where $H = G(n_1; u) \times G(n_2; u) = (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}) \rtimes_w \mathbb{Z}^2$. First we sketch the broader context in which our theme (i.e. non-cancellation sets $\chi(G)$ for $G \in \mathcal{X}_0$) is located.

There is a very close relationship between the notion of non-cancellation sets and the notion of genus sets. In this introduction we include a brief discussion on localization and the genus set, and show how these are related.

Let P be a set of primes and let P' be the set of natural numbers which are relatively prime to the elements of P. A group G is said to be *P-local* if for each $n \in P'$, the function $g \mapsto g^n$ of G into itself is a bijection (see [36, 1.1]). Let $h : G \to H$ be a group homomorphism. Then h is said to be *P-injective* if, for each $g \in \text{Ker } h$, g is of finite order n for some $n \in P'$. If, for every $x \in H$, there exists an integer $n \in P'$ such that $x^n \in h(G)$, then h is said to be *P-surjective.* The homomorphism h is said to be *P-bijective* if it is both *P-injective* and *P-surjective* (see [36, 1.1]). Let N be a nilpotent group and let N_P be a P-local group which admits a P-isomorphism $N \to N_P$. (This homomorphism is called a *P-localizing homomorphism.*) Then a *P-localization* of a nilpotent group N is a homomorphism $\theta_P : N \to N_P$, where N_P is as above, with the universal property that, given any homomorphism $\theta : N \to M$, with M a P-local group, there exists a unique homomorphism $\phi_P : N_P \to M$ such that $\phi_P \circ \theta_P = \theta$ (see [22], [23], [36]). If $P = \{\}$, i.e. empty, then P-localization is also referred to as *rationalization.*

The theory of P-localization is discussed by Baumslag [2]. Later Ribenboim in [26] developed the construction of a P-localization. In this introduction we seek to observe the relationship between the genus and non-cancellation. For any group G, $\mathcal{F}(G)$ is the set of all isomorphism classes of finite quotient groups of G. The *Pickel genus*, for a finitely generated nilpotent group N, is the set of all isomorphism classes of finitely generated nilpotent groups M such that $M_0 \cong N_0$ (i.e. the groups have isomorphic rationalizations) and $\mathcal{F}(M) = \mathcal{F}(N)$. In [21] Mislin defined a different version of genus of a finitely generated nilpotent group N. The *Mislin genus* $\mathcal{G}(N)$ is the set of all isomorphism classes of finitely generated nilpotent groups M such that, for each prime p, $M_p \cong N_p$ (i.e. M and N have isomorphic p-localizations), where N is a nilpotent group (see [21]).

Pickel in [25] proved that the Pickel genus is finite, so we are able to deduce that the Mislin genus is always finite. The Mislin genus and Pickel genus coincide for finitely generated nilpotent groups having finite commutator subgroups. Warfield in [32] provides us with the result that, for any nilpotent $\mathcal{X}_0$-group M, the Mislin genus $\mathcal{G}(M)$ and the non-cancellation set $\chi(M)$ coincide, i.e. $\mathcal{G}(M) = \chi(M)$.

In [10] Hilton and Mislin define an abelian group structure on the genus set $\mathcal{G}(N)$ of a finitely generated nilpotent group N with finite commutator subgroup. Witbooi in [36] makes the observation that, when localizing non-nilpotent groups, the kernel of the localizing homomorphism may be bigger than what is required. Hence we can generalize the idea of the genus to non-nilpotent groups $G \in \mathcal{X}_0$ by considering non-cancellation rather than localizations. Thus we shall not consider localization and genus for the purposes of this work. The following papers may be consulted for further reading on the notion of genus: Warfield [32], Hilton and Mislin [10], Hilton and Schuck [11], and Hilton and Scevenels [12].

Hilton and Mislin define a group structure on the genus $\mathcal{G}(N)$ for a nilpotent group N in $\mathcal{X}_0$. In [38] Witbooi generalizes the Hilton-Mislin genus group. In his generalization he is inspired by the theorem of Warfield in [32, 3.6], which states that $\chi(N) = \mathcal{G}(N)$ for a nilpotent group N in $\mathcal{X}_0$. He then defines a group structure on the non-cancellation set $\chi(G)$ for a group G which is not necessarily nilpotent. He utilizes the notion of Nielsen equivalence classes (see [39, 2]) of abelian group presentations to impose an action of an abelian group on the set $\chi(G)$. He also makes use of the indices of the subgroups of G and a certain function $\mathbb{Z}_n^*/\pm 1 \to \chi(G)$ to achieve his goal. Casacuberta and Hilton [4] calculated the group $\chi(H^k)$ for $H = \langle a, b \mid a^n = 1,\ bab^{-1} = a^u \rangle$, but for n a prime and H nilpotent. Hilton and Schuck [11] dealt with a general nilpotent case. Fransman and Witbooi [6] and Scevenels and Witbooi [30] treated the case where n is a prime power and H is not necessarily nilpotent. Witbooi [41] eventually calculated $\chi(H^k)$ for all possible values of n, u and k. Witbooi further develops results on induced morphisms such as $\chi(G) \to \chi(G \times H)$, $\chi(H) \to \chi(H^k)$, $\chi(G) \to \chi(G/F)$ and $\chi(H) \to \chi(H^k/F)$ in [38] and [40].

In Chapter 2 we discuss a number of elementary results that are used directly or indirectly in this chapter and the subsequent chapters. Our emphasis in Chapter 2 will however be on finitely generated abelian groups. We include definitions of the order of an element, a generating subset, a finitely generated group, automorphism, torsion group, torsion-free group, direct products, characteristic subgroup, and free abelian group. We discuss in detail the properties of direct products since the thesis is on direct factor cancellation, and since we use these properties to understand the construction of semi-direct products in Chapter 5. We prove that a direct product $H \times K$ is abelian if and only if both the groups H and K are abelian. Since every cyclic group is abelian, we show that if H and K are cyclic with relatively prime orders then $H \times K$ is cyclic. We then show that the direct product $G_1 \times G_2 \times \cdots \times G_n$ of n groups is cyclic on conditions similar to the case of a direct product $H \times K$. The other results on direct products that we include are meant for the construction of semidirect products in Chapter 5. Immediately after defining free abelian groups, we give some basic properties of these groups that are relevant to this work. We first note that when an abelian group F is free on a subset X, then the function $X \to F$ is injective, and that F is generated by X. In the construction of direct products and semidirect products we note some similarities with the properties of free abelian groups. Other important properties are: every abelian group is a homomorphic image of a free abelian group; a subgroup H of a free abelian group F on X is also free on a subset Y with $|Y| \leq |X|$; and when F_1 and F_2 are free abelian on X_1 and

X_2, respectively, we show that F_1 and F_2 are isomorphic if and only if the respective cardinalities $|X_1|$ and $|X_2|$ are equal. For a finitely generated abelian group G with torsion subgroup T_G, we show that the quotient group G/T_G is torsion-free, and we also prove that every finitely generated torsion-free abelian group is free, and hence the quotient group G/T_G is free. We look at some fundamental theorems of finite abelian and finitely generated abelian groups. Similarly to free abelian groups, we show that a finite abelian group can be written as a direct product of its cyclic subgroups. The basis theorem for finite abelian groups ensures this fact. The fundamental theorem of finite abelian groups states that every finite abelian group has a unique decomposition into its primary cyclic subgroups. We conclude this chapter with finitely generated abelian groups. The corresponding basis theorem for these groups states that every finitely generated abelian group is a direct sum of cyclic subgroups. The fundamental theorem of finitely generated abelian groups states that every finitely generated abelian group can be written uniquely as the direct product of its primary and infinite cyclic subgroups. These fundamental theorems, we note, are similar to some combination of the properties of the free abelian groups. This is no surprise since one of those properties states that every abelian group is a homomorphic image of a free abelian group.

Chapter 3 deals with the epimorphisms from a free abelian group onto a finite group. In particular, we study the epimorphisms from the free abelian group $\mathbb{Z}^k$ onto a finite abelian group A, and we denote the set of these epimorphisms by $E_k(A)$, $k \in \mathbb{N}$. We note that this set $E_k(A)$ is non-empty if and only if the rank (see Definition 3.1) of A is less than or equal to k. We pay particular attention to the finite abelian group $\mathbb{Z}_n$. Given any homomorphism $f : \mathbb{Z}^k \to \mathbb{Z}_n^k$ we obtain an endomorphism $f_J : \mathbb{Z}_n^k \to \mathbb{Z}_n^k$ such that $f = f_J \circ \eta$, where $\eta : \mathbb{Z}^k \to \mathbb{Z}_n^k$ is the obvious epimorphism. We discuss the determinant of an endomorphism of a finite abelian group, and give an example to illustrate how to compute such a determinant. There is also a proposition whose proof demonstrates how these determinants are useful. For a finite abelian group A of rank $k \in \mathbb{N}$ we define $a = (a_1, a_2, ..., a_k)$ to be a *good* k-tuple of elements of A (see [30]) in terms of the d_i, where d_i is the order of $\langle a_i \rangle$. In conclusion, we define a homomorphism $f_{(a,n)} = f_n : \mathbb{Z}^k \to A$, with $a = (a_1, a_2, ..., a_k)$, in terms of a basis of $\mathbb{Z}^k$. We then show that this homomorphism f_n is an epimorphism if n is relatively prime to d, where d is the order of a_k.

In Chapter 4 we continue with the work started in the preceding chapter, in that we study further the epimorphisms $f_n : \mathbb{Z}^k \to A$ of free abelian groups $\mathbb{Z}^k$ onto finite abelian groups A. In addition to these epimorphisms we introduce the notion of a Nielsen equivalence relation, which is defined as follows: the epimorphisms f and g are said to be Nielsen equivalent if and only if there is an automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}^k)$ such that $f = g \circ \alpha$. We show that Nielsen equivalence is an equivalence relation. For any other epimorphism $g' \in E_k(A)$ equivalent to f, we say that g' is an element of the equivalence class $[f]$ of f. As in the previous chapter, we denote the set of all equivalence classes of the epimorphisms from the free abelian group $\mathbb{Z}^k$ onto a finite abelian group A by $E_k^{\sim}(A)$. We note how these epimorphisms in $E_k(A)$ can induce bijections between equivalence classes. We prove that, for an epimorphism $f \in E_k(A)$, $f \sim f_n$, where f_n is as defined in Chapter 3. When we let A be a finite abelian group, $a = (a_1, a_2, ..., a_k)$ a good ordered k-tuple elements of A, and d the order of a_k, we then prove the following result: there exists an epimorphism $\lambda : A \to \mathbb{Z}_d^k$ such that post-composition with λ is a bijection $\lambda_* : E_k^{\sim}(A) \to E_k^{\sim}(\mathbb{Z}_d^k)$.

We prove that $f_n \sim f_m$ if and only if there exist $u \in \mathbb{Z}^*$ and $r \in \mathbb{Z}$ such that $m = un + rd$. We prove also that the function which sends an integer n to f_n induces a bijection between the group of units $\mathbb{Z}_d^*/\pm 1$ and the set of equivalence classes $E_k^{\sim}(A)$.

In Chapter 5 we continue with the work commenced in Chapter 2 on direct products. Recall that we are particularly interested in split extensions of a finite group by a free abelian group. The semidirect product involves group actions, so we define the notion of a group action and list some of its basic properties, such as the isotropy subgroup. We define split extensions and semidirect products. We further observe ways of defining split extensions in terms of the Second Isomorphism theorem and exact sequences. We define an operation that further distinguishes direct products and semidirect products. We also define semidirect products $G \rtimes_w H$ in terms of the action $w : H \to G$, where H is a free abelian group and G is a finite group. The group $G(A; u_1, ..., u_k) = A \rtimes_w \mathbb{Z}^k$ where $w : \mathbb{Z}^k \to \mathrm{Aut}(A)$ is an action, and $w(z_1, \cdots, z_k)$ acts on A as follows: $a \mapsto (u_1^{z_1} \cdots u_k^{z_k})a$. We also prove the following: $G(d; u) \cong G(d; v)$ if and only if $u \equiv v \bmod d$ or $uv \equiv 1 \bmod d$; and $T \rtimes_v \mathbb{Z}^{k+1} \cong (T \rtimes_w \mathbb{Z}^k) \times \mathbb{Z}$, where $v : \mathbb{Z}^{k+1} \to \mathrm{Aut}(T)$ is an action.

In Chapter 6 we continue with the discussion on finitely generated groups. Our emphasis in this chapter is on finitely generated groups having finite commutator subgroups. We also introduce the notion of non-cancellation sets. We first define the commutator subgroup G' of a finitely generated group G. This commutator subgroup G' will be significant throughout this study, in particular we require that G' be finite. Consequently, we define the class $\mathcal{X}_0$ as the class of all finitely generated groups having finite commutator subgroups. More importantly, for a group G in $\mathcal{X}_0$, we define the set $\chi(G)$ as the set of all isomorphism classes $[H]$ of groups H such that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$. We mention briefly what we mean by the centre, centralizer, and normalizer of a group. We introduce a natural number $n(G) = n_1 n_2 n_3$, where n_1 is the exponent of the torsion subgroup T_G, n_2 is the exponent of the group $\mathrm{Aut}(T_G)$ and n_3 is the exponent of the torsion subgroup of the centre of G. We show that, for a finitely generated abelian group G, T_G is a finite normal subgroup of G. We include some properties of a $\mathcal{X}_0$-group G, viz: if $H \leq G$ then $H \in \mathcal{X}_0$; if F is normal in G, then $G/F \in \mathcal{X}_0$; if L is a group such that $L \times \mathbb{Z} \cong G \times \mathbb{Z}$, then $L \in \mathcal{X}_0$; if $H \in \mathcal{X}_0$, then $G \times H \in \mathcal{X}_0$. Eventually we prove that a group G is a $\mathcal{X}_0$-group if and only if G is a split extension group of a finite group by a finite rank free abelian group. We also note that there is a subgroup H of an infinite $\mathcal{X}_0$-group G such that the index $|G : H|$ is finite. We include results which tell us when a $\mathcal{X}_0$-group H is an element of $\chi(G)$. We also discuss the trivial cases of the set $\chi(G)$.

The main focus of Chapter 7 is to define a group structure on $\chi(G)$, where $G \in \mathcal{X}_0$, the set of all isomorphism classes $[H]$ of groups $H \in \mathcal{X}_0$ such that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$. In Chapter 6 we define $\chi(G)$ with the group G being an element of the class $\mathcal{X}_0$ of all finitely generated groups having finite commutator subgroups. We are particularly interested in the groups $G \in \mathcal{X}_0$, where $G = T \rtimes_w \mathbb{Z}^k$, T is a finite group, $\mathbb{Z}^k$ is a finite rank free abelian group, and $w : \mathbb{Z}^k \to \mathrm{Aut}(T)$ is an action of the group $\mathbb{Z}^k$ onto the automorphism group of T. This subclass of $\mathcal{X}_0$ is denoted by $\mathcal{K}$. In defining the group structure on $\chi(T \rtimes_w \mathbb{Z}^k)$ we employ the function $\theta : \mathbb{Z}_d^*/\pm 1 \to \chi(G)$, where we replace w by f_n and $n(G)$ by d, the order of a_k. We also utilize the Nielsen equivalence relation as in Chapter 4, the indices of subgroups of G and the embeddings $\delta : G \to L$ such that $L \times \mathbb{Z} \cong G \times \mathbb{Z}$. The finite group T is the torsion subgroup of G and is considered to be abelian. We use the notion of the Nielsen equivalence relation to show that when the finite indices of the subgroups H and K of $G = T \rtimes_w \mathbb{Z}^k$ are such that $|G : H| \equiv \pm|G : K| \bmod d$ then H and K are isomorphic. In this case both indices are relatively prime to d. We prove that the function $\theta : \mathbb{Z}_d^*/\pm 1 \to \chi(G)$ is an epimorphism and also show that there is a transitive action of $\mathbb{Z}_d^*/\pm 1$ on the set $\chi(G)$. This action equips $\chi(G)$ with a group structure. The challenge now is to compute $\chi(G)$. For a start we may want to find conditions where the group epimorphism θ is an isomorphism. Recall that a homomorphism is a monomorphism if and only if its kernel is trivial (see [28]). This then compels us to investigate the description of the elements of the kernel of θ. We eventually found that for $s \in \mathbb{Z}$ and d the multiplicative order of u modulo n in $G(n; u)$, that the residue class $\bar{s}$ modulo d is an element of Ker θ if and only if $\bar{s}$ satisfies certain conditions in terms of determinants of the endomorphisms of $\mathrm{Im}(w)$. Also for $n = n_1 n_2$ and $d_i = \mathrm{ord}_{n_i}\, u$, we show that whenever $s \equiv 1 \bmod d_1$ and $s \equiv -1 \bmod d_2$ then $\bar{s} \in$ Ker θ. A more general result for $n = n_1 n_2 \ldots n_l$ is also given.

In Chapter 8 we calculate $\chi(H) = \chi(G(n_1; u) \times G(n_2; u))$ where $H = G(n_1; u) \times G(n_2; u) = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \rtimes_w \mathbb{Z}^2$ for a special case $\gcd(n_1, n_2) = 1$, where $w : \mathbb{Z}^2 \to \mathrm{Aut}(T)$ is an action of $\mathbb{Z}^2$ on $T = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. We recall that the normalizer of $A = \mathrm{Im}(w) \in \mathrm{Aut}(T)$ is defined as $N = N_{\mathrm{Aut}(T)}(A) = \{\lambda \in \mathrm{Aut}(T) \mid \lambda\beta\lambda^{-1} \in A \text{ for all } \beta \in A\}$. For $\lambda \in N_{\mathrm{Aut}(T)}(A)$, the function $\theta_\lambda : A \to A$ defined by $\theta_\lambda : \beta \mapsto \lambda\beta\lambda^{-1}$ is an inner automorphism of A. The centralizer of A in $\mathrm{Aut}(T)$ is defined as $C = C_{\mathrm{Aut}(T)}(A) = \{\lambda \in \mathrm{Aut}(T) \mid \lambda\beta = \beta\lambda \text{ for all } \beta \in A\}$. We define the NC-property as follows: Let T be a finite abelian group and let A be a subgroup of Aut (T). The pair (T, A) is said to be an NC-pair or to have the NC-property if the following condition holds:

$$N_{\mathrm{Aut}(T)}(A) = C_{\mathrm{Aut}(T)}(A).$$

Equivalently, if for every $\lambda \in N_{\mathrm{Aut}(T)}(A)$ the inner automorphism θ_λ of A defined by the rule $\theta_\lambda : a \mapsto \lambda a \lambda^{-1}$ is the identity automorphism of A. We show how in the presence of the NC-property, it is easy to calculate $\chi(H)$ for H as mentioned above.

CHAPTER 2

BASICS OF FINITELY GENERATED ABELIAN GROUPS

This chapter lays a foundation for the work in the forthcoming chapters. We shall be dealing with the non-cancellation sets of split extensions of a finite abelian group by a finite rank free abelian group. We thus pay particular attention to finite abelian groups and free abelian groups. Since the investigation is about the cancellation of $\mathbb{Z}$ as a direct factor in the isomorphism $G \times \mathbb{Z} \cong H \times \mathbb{Z}$, we also spend some time dealing with basic results on direct products.

Definition 2.1: [28, 2.27] Let G be a group and $X \subseteq G$.

(a) The intersection of all subgroups of G which contain X is called the *subgroup of G generated by X* and is denoted by $\langle X \rangle$.

(b) A group G is said to be *finitely generated* if it has a generating subset which is finite, and

(c) G is said to be *cyclic* if $G = \langle x \rangle$ for some $x \in G$.

Remark 2.2: [15] [34] Let G be a group and let g be any element of G.
(a) The smallest $r \in \mathbb{N}$ which is such that $g^r = e$ where e is the identity element of G is said to be an *order* of g. If no such r exists, we say that g is of infinite order. The *exponent* of a group G is the least common multiple (if it exists) of the orders of the elements of G.

(b) If g has infinite order, then $\langle g \rangle$ is an infinite cyclic group. If g has order n, then $\langle g \rangle$ has n elements.

(c) A *cyclic* group is abelian.

(d) A subgroup of a cyclic group is cyclic.

(e) A *primary cyclic group* is a cyclic group of prime power order.

Definition 2.3: [27, p.25] Let G be a group. An *automorphism* of G is an isomorphism from G to G. The set of all automorphisms of G is denoted by $\mathrm{Aut}(G)$.

Remark 2.4: [15, 8.11] The set of automorphisms of a group G is itself a group under the composition of maps.

In this study we are particularly concerned with the non-cancellation sets of split extensions (see Chapter 5). Let us first recall some basics of direct products as the construction of split extensions and semidirect products is a modification of the direct products. We know that the cartesian product of the set X and the set Y is another set $X \times Y$, which is the set of all the ordered pairs (x, y) with $x \in X$ and $y \in Y$. When X and Y are finite sets then $X \times Y$ is a finite set and $|X \times Y| = |X|.|Y|$. Now, for any groups H and K, the set $H \times K$ acquires the structure of a group when we define multiplication as $(h, k)(h', k') = (hh', kk')$ for all $h, h' \in H$ and $k, k' \in K$. From the above definition of multiplication the group axiom of closure is satisfied. The associativity follows since for $h, h_1, h_2 \in H$ and $k, k_1, k_2 \in K$; $(h, k)[(h_1, k_1)(h_2, k_2)] = (hh_1h_2, kk_1k_2) = [(h, k)(h_1, k_1)](h_2, k_2)$. Now, $(1, 1)(h, k) = (h, k) = (h, k)(1, 1)$, therefore $(1, 1)$ is the identity element of $H \times K$. The inverse of $H \times K$ is (h^{-1}, k^{-1}) since $(h, k)(h^{-1}, k^{-1}) = (1, 1) = (h^{-1}, k^{-1})(h, k)$.

Definition 2.5: [28, 2.31] For any groups H and K, the group $H \times K$ is called the *direct product* of H and K.

Remark 2.6: [28, 2.10] (i) If $\phi : G \to H$ is an injective homomorphism then $G \cong \phi(G)$, and for every subgroup K of G, $K \cong \phi(K)$.
(ii) G can be embedded in H if and only if G is isomorphic to a subgroup of H.

Proposition 2.7: [28, Exercise 26] *If $\phi : G \to H$ is a homomorphism and G is abelian then $Im(\phi)$ is abelian.*

Proof: Let ϕ be a homomorphism, and G be abelian. Then for any $g_1, g_2 \in G$; $g_1 g_2 = g_2 g_1$. Now, by assumption we have $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. Then $\phi(g_1)\phi(g_2) = \phi(g_1 g_2) = \phi(g_2 g_1) = \phi(g_2)\phi(g_1)$. $\square$

In [28, 3.38] we have that if one of the subgroups H and K of a group G is normal then $HK = \{hk \mid h \in H \text{ and } k \in K\}$ is also a subgroup of G. If both subgroups are normal in G then $HK \trianglelefteq G$, that is HK is also a normal subgroup of G.

Remark 2.8: [28, 8.2] The groups H and K are normal subgroups of G such that $G = HK$ and $H \cap K = 1$ if and only if $G \cong H \times K$.

Lemma 2.9: [28, 2.33] *The group $H \times K$ has subgroups $H \times 1 = \{(h, 1) \mid h \in H\} \cong H$ and $1 \times K = \{(1, k) \mid k \in K\} \cong K$. Every element of $H \times K$ is expressible as the product of elements of $H \times 1$ and $1 \times K$. Furthermore, every element of $H \times 1$ commutes with every element of $1 \times K$ and $(H \times 1) \cap (1 \times K) = 1$.*

Proof: Let $\phi : H \to H \times K$ be defined by $\phi : h \mapsto (h, 1)$ for all $h \in H$. Then for $h_1, h_2 \in H$, $\phi(h_1h_2) = (h_1h_2, 1) = (h_1, 1)(h_2, 1) = \phi(h_1)\phi(h_2)$. Also for $\phi(h_1) = \phi(h_2)$, then $(h_1, 1) = (h_2, 1)$ which means $h_1 = h_2$, therefore ϕ is a monomorphism. This implies that $H \cong \phi(H) = H \times 1$ (see Remark 2.6) and so $H \times 1 \leq H \times K$. Similarly, $K \cong 1 \times K \leq H \times K$.

For every $h \in H$ and $k \in K$, $(h, 1)(1, k) = (h, k) = (1, k)(h, 1)$ and $(H \times 1) \cap (1 \times K) = \{(1, 1)\} = 1$. $\square$

Remark 2.10: [28, 75, 76] (a) $H \times K$ is abelian if and only if H and K are both abelian.

(b) If $H \cong J$ and $K \cong L$ then $(H \times K) \cong (J \times L)$.

Lemma 2.11: [28, 2.34] *Suppose that G has subgroups H and K such that every element of G is expressible as a product hk with $h \in H$ and $k \in K$, every element of H commutes with every element of K and $H \cap K = 1$. Then $G \cong H \times K$.*

Proof: Every element $g \in G$ is uniquely expressible as a product of an element of H and an element of K. For suppose that $g = hk = h'k'$ with $h, h' \in H$ and $k, k' \in K$. Then $(h')^{-1}h = k'k^{-1} \in H \cap K = 1$, and so we must have $h = h'$ and $k' = k$. Therefore we may define a map

$$\phi : G \to H \times K$$

by

$$\phi : hk \to (h, k)$$

for all $h \in H$ and $k \in K$. By the uniqueness property we discussed above, ϕ is well-defined. It also follows that ϕ is injective, and surjectivity is clear. Therefore ϕ is in fact a bijective map.

Next we show that ϕ is a homomorphism and thus an isomorphism. Now consider $h_1, h_2 \in H$ and $k_1, k_2 \in K$ and let $g_1 = h_1k_1$ and $g_2 = h_2k_2$. Now by hypothesis we have that $k_1h_2 = h_2k_1$. Then $\phi(g_1g_2) = \phi(h_1k_1h_2k_2) = \phi(h_1h_2k_1k_2) = (h_1h_2, k_1k_2) = \phi(h_1k_1)\phi(h_2k_2) = \phi(g_1)\phi(g_2)$ as required. That is we have shown that $G \cong H \times K$ according to the given hypothesis. $\Box$

We can, as expected, extend the definition of direct products of two groups to the direct product of any finite number of groups. Let n be a natural number, and let $G_1, G_2, ..., G_n$ be any n groups (not necessarily distinct). Then $G_1 \times G_2 \times ... \times G_n$ is the set of all ordered n-tuples $(g_1, g_2, ..., g_n)$ with $g_i \in G_i$ for $i = 1, ..., n$. This set, as in Definition 2.5 above, is given the structure of a group called the direct product of $G_1, G_2, ..., G_n$ by defining multiplication of n-tuples component wise.

Proposition 2.12: [15, 13.1] *If G and H are both cyclic finite groups and their orders have no common divisor greater than 1, then $G \times H$ is cyclic.*

Proof: Suppose that G has order n and $G = \langle x \rangle$; and also that H has order m and $H = \langle y \rangle$. Now if $(x, y) \in G \times H$ has order k, then $(x, y)^k = 1 = (x^k, y^k)$. It follows that n divides k since $x^k = 1$ and also that m divides k since $y^k = 1$. Since n and m are relatively prime, it follows that the sets of prime divisors of n and m are disjoint. It follows from this, after writing each of n and m as a product of prime powers, that nm divides k. However, $(x, y)^{nm} = (x^{nm}, y^{nm}) = 1$, and so k divides nm. Thus $k = nm$ is the order of (x, y). From the first part of the proof of Lemma 2.11 it follows that $|G \times H| = nm$. Thus $G \times H = \langle (x, y) \rangle$. $\Box$

By induction we can prove the following more general result.

Corollary 2.13: [15] *Let $n_1, n_2, ..., n_s$ be any sequence of integers each of which is greater than 1, such that the greatest common divisor of any pair n_i, n_j is 1. Let G_i be a cyclic group of order n_i; $1 \leq i \leq s$. Then the group $G_1 \times G_2 \times ... \times G_s$ is cyclic of order $n_1 n_2 ... n_s$.*

Lemma 2.14: [28, 3.11] *Suppose that $G = H \times K$. Define maps $\pi_1 : G \to H$ and $\pi_2 : G \to K$ by $\pi_1 : (h,k) \mapsto h$ and $\pi_2 : (h,k) \mapsto k$ for all $(h,k) \in G$. Then π_1 and π_2 are surjective homomorphisms, called the projections of G onto H and onto K respectively. Ker $\pi_1 = 1 \times K$ and Ker $\pi_2 = H \times 1$.*

Definition 2.15: [28] Let H be a subgroup of a group G, $H \leq G$. Let $\alpha \in \mathrm{Aut}(G)$. If $\alpha(h) \in H$ for every $h \in H$ and $\alpha \in \mathrm{Aut}(G)$, then H is $\mathrm{Aut}(G)$-invariant, and H is called a *characteristic subgroup* of G, denoted by H char G.

Notation 2.16 [28, 8.3] Let H and K be subgroups of G with one subgroup normal so that $HK \leq G$ and $HK = KH$. Moreover if H and K are both normal subgroups of G then $HK \trianglelefteq G$. It then follows that if $G_1, G_2, ..., G_n$ are normal subgroups of G then the product $G_1 G_2 ... G_n$ does not depend on the ordering of factors. The notation $\prod_{i=1}^{n} G_i$ is sometimes used for this product, and thus $\prod_{i=1}^{n} G_i \trianglelefteq G$. We sometimes denote the direct product of groups $G_1, G_2, ..., G_n$ by $\mathrm{Dr}\prod_{i=1}^{n} G_i$ instead of $G_1 \times G_2 \times ... \times G_n$.

We now define a free abelian group and look at some of its basic properties.

Definition 2.17: [27, 2.1] An abelian group F which contains a set $X, X \subseteq F$, is said to be *free abelian* on X, if for every abelian group G and function $f : X \to G$, there is a unique extension to a homomorphism of F into G such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & F \arrow[dl, "f'"] \\ & G. & \end{tikzcd}$$

The set X is called a *basis* of F, and the *cardinality* of X, $|X|$, is called the *rank* of F.

Most of the following properties are discussed in detail in [27].

Remark 2.18: (cf. [27]) (a) Let F_1 and F_2 be free abelian groups on sets X_1 and X_2 respectively. $F_1 \cong F_2$ if and only if $|X_1| = |X_2|$.
(b) The function $i : X \to F$ is injective.
(c) If X is a non-empty set, there exists an abelian group F and a function $\sigma : X \to F$ such that F is free abelian on X and $F = \langle \mathrm{Im}(\sigma) \rangle$.
(d) If F is a free abelian group on a subset X, then F is generated by X. ([29, 11.5])
(e) If F is a free abelian group on a subset X, then F is the direct product of the infinite cyclic subgroups $\langle x \rangle, x \in X$. [27]

Proposition 2.19: [27] *Let G be an abelian group and X a subset of G. Assume that for each nonzero element g of G, there exists a unique subset $\{x_1, x_2, \cdots, x_s\}$ of s distinct elements of X and a sequence $l_1, l_2, \cdots, l_s$ of nonzero integers such that $g = l_1x_1 + l_2x_2 + \cdots + l_sx_s$. Then G is free abelian on X.*

Proposition 2.20: [27] [29] *Every abelian group is a homomorphic image of a free abelian group.*

In order to deal with non-cancellation later on, we require a thorough understanding of the structure of finite abelian groups. Thus we list some important structural results.

Theorem 2.21: [15, 14.11] *The Fundamental Theorem of Finite Abelian Groups: Every finite abelian group G has a unique decomposition in the form*

$$C_{n_1} \times C_{n_2} \times ... \times C_{n_r}$$

where C_{n_i} is a primary cyclic group (see Remark 2.2) and $n_1 n_2 ... n_r = |G|$.

Theorem 2.22: [29, 9.27] *Fundamental Theorem of Finitely Generated Abelian Groups: Every finitely generated abelian group G is a direct sum of primary and infinite cyclic groups, and the number of summands of each kind depends only on G.*

Using Corollary 2.13, from Theorem 2.21 we can derive the following (well-known) decomposition theorem for abelian groups.

Theorem 2.23: *Given any nontrivial finite abelian group G of rank k, there exists a unique sequence $n_1, n_2, \cdots, n_k$ of integers such that:*

(i) $G \cong C_{n_1} \times C_{n_2} \times C_{n_3} \times \cdots \times C_{n_k}$,

(ii) $n_{i+1} | n_i$ *for each* $1 \leq i \leq k-1$.

(The numbers $n_1, n_2, \cdots, n_k$ are called the invariant factors of G.)

Proof: By Theorem 2.21 we can write G in the form:

$$G \cong \prod_{p\in D}[C_{n_1(p)} \times C_{n_2(p)} \times \cdots \times C_{n_k(p)}],$$

where D is the set of all prime divisors of $|G|$, each $n_{i(p)}$ is a power of p (it may be $p^0 = 1$) and $n_{1(p)} \geq n_{2(p)} \geq \cdots \geq n_{k(p)}$. Now for each i, we let $n_i = \prod_{p\in D} n_{i(p)}$. $\square$

Proposition 2.24: [18] *Let G be a finitely generated abelian group and let T_G be the subset consisting of all elements of finite order in G. Then T_G is a finite normal subgroup of G, $T_G \trianglelefteq G$, and the quotient G/T_G is free.*

Proof: Let G be a finitely generated abelian group. Let T_G be the set of all elements of finite order in G. Let $g_1, g_2 \in G$ be elements of T_G with finite orders m and n respectively. Let l be the lowest common multiple of m and n, then $l(g_1 \pm g_2) = lg_1 \pm lg_2 = 0$. This implies that $g_1 \pm g_2$ are of finite orders dividing l. Therefore $T_G \leq G$. Since G is abelian then every subgroup of G is normal in G (see [28, 3.5]), thus $T_G \trianglelefteq G$. Now T_G is finite since a finitely generated torsion abelian group is finite [27].

Next, we show that the quotient group G/T_G is torsion-free. Let $\overline{x}$ be an element of G/T_G such that $m\overline{x} = 0$ for some integer $m \neq 0$. Then for any representative x of $\overline{x}$ in G, we have $mx \in T_G$ and hence $qmx = 0$ for $q \in \mathbb{Z}$ and $q \neq 0$. Then $x \in T_G$, so $\overline{x} = 0$ and this implies G/T_G is torsion-free. Consequently, G/T_G is free since it is a finitely generated torsion-free abelian group. $\square$

CHAPTER 3

PRESENTATIONS OF FINITE ABELIAN GROUPS

The main results of this chapter are based on the work in [30] where the authors dealt with the presentations of modules over a principal ideal domain. In this study we deal with these presentations for a special case where the commutative ring R is taken as the ring of integers $\mathbb{Z}$ and the $\mathbb{Z}$-modules are finite abelian groups. In this chapter we thus study the epimorphisms from a free abelian group $\mathbb{Z}^k$ onto a finite abelian group, say A with cardinality k. Many authors define presentations in terms of generators and relators, see for instance [16] and [27] for more details. From Proposition 2.20, we have that every abelian group is a homomorphic image of a free abelian group. A free presentation of an abelian group G is an epimorphism from a free abelian group F, $\alpha : F \to G$. If $R = \mathrm{Ker}(\alpha)$, then $R \leq F$ and $F/R \cong G$ by the First Isomorphism Theorem. Thus for our purposes we shall consider presentations in terms of these epimorphisms just mentioned. We also include some work on the concept of the determinants of endomorphisms as these determinants shall prove useful in proving some fundamental results later in this thesis. We supply alternative proofs for Lemmas 3.7 and 3.10.

Definition 3.1: [38] Let A be a non-trivial finite abelian group. The *Prüfer rank* of A is the least of the cardinalities of generating subsets of A. For brevity we shall refer to it as the rank of a finitely generated abelian group.

Notation 3.2: 1. For k a positive integer, the set of all epimorphisms from the free abelian group $\mathbb{Z}^k$ onto the finite abelian group A will be denoted by $E_k(A)$.

2. For a finite abelian group A, let d be the lowest common multiple of the orders of the invariant factors of A.

Remark 3.3: The set $E_k(A)$ is non-empty if and only if A can be generated as an abelian group by some finite subset of cardinality less than or equal to k. For if the rank of A is greater than k, then there is no epimorphism $\mathbb{Z}^k \to A$.

Item 3.4: The determinant of an endomorphism: Consider the ring of integers $\mathbb{Z}$. Let d and k be positive integers. Let f be an endomorphism of $\mathbb{Z}^k$ and g be an automorphism of $\mathbb{Z}^k$. The determinant of f is an element of $\mathbb{Z}$, denoted by det f, and the determinant of g is a unit of $\mathbb{Z}$. Similarly, let f' be an endomorphism of $\mathbb{Z}_d^k$ and g' be an automorphism of $\mathbb{Z}_d^k$, where $\mathbb{Z}_d$ is a ring of integers modulo d. The determinant of f' is an element of $\mathbb{Z}_d$ and the determinant of g' is a unit of $\mathbb{Z}_d$. (See also the discussion in [31, 1.10]).

For any finite nontrivial abelian group A, let d be the smallest order of the invariant factors. Let $\alpha : A \to A$ be define by $a \mapsto da$. Then the image J of α is a fully invariant subgroup of A (i.e. for any endomorphism h of A, we have $h(J) \subseteq J$). Let $\beta : A \to A/J$ be the canonical epimorphism. Then $A/J \cong \mathbb{Z}_d^k$ for some $k \in \mathbb{N}$. Now given any $h \in \mathrm{End}(A)$, there exists a unique $h_J \in \mathrm{End}(A/J)$ such that $\beta \circ h = h_J \circ \beta$. The *determinant of* h is now defined as the element det $h_J \in \mathbb{Z}_d$.

We now give an example to illustrate how to compute the determinant of an endomorphism.

Example 3.5: Let an endomorphism $h : \mathbb{Z}_{12} \times \mathbb{Z}_6 \to \mathbb{Z}_{12} \times \mathbb{Z}_6$ be defined by $h : (x, y) \mapsto (2x, \overline{x} + y)$ where $\overline{x}$ is the reduction of x modulo 6.

To compute the determinant of h, $\det(h)$, we form the following diagram

$$\begin{tikzcd}
\mathbb{Z}_{12} \times \mathbb{Z}_6 \arrow[r, "h"] \arrow[d, "\beta"'] & \mathbb{Z}_{12} \times \mathbb{Z}_6 \arrow[d, "\beta"] \\
\mathbb{Z}_6 \times \mathbb{Z}_6 \arrow[r, "h_1"'] & \mathbb{Z}_6 \times \mathbb{Z}_6
\end{tikzcd}$$

so that there exists a unique endomorphism h_1, defined as

$$h_1 : \mathbb{Z}_6 \times \mathbb{Z}_6 \to \mathbb{Z}_6 \times \mathbb{Z}_6$$

$$h_1 : (\overline{x}, y) \mapsto (2\overline{x}, \overline{x} + y).$$

Then we obtain the matrix M of h_1 with respect to the basis $\{(1,0),(0,1)\}$ as follows

$h_1(1,0) = (2,1)$ and $h_1(0,1) = (0,1)$

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore $\det(h) = \det(h_1) = \det(M) = \overline{2} \in \mathbb{Z}_6$. $\square$

Proposition 3.6: [38, 3.1] *Let B be any non-trivial finite abelian additive group of rank k, and let d be as in Notation 3.2. For some positive integer k, suppose that we have epimorphisms $g, h \in E_k(B)$ and an endomorphism $\phi : \mathbb{Z}^k \to \mathbb{Z}^k$ such that $g = h \circ \phi$. The cokernel of ϕ, coker (ϕ), is a finite group and $|\mathrm{coker}(\phi)|$ is relatively prime to d. (The cokernel of a group homomorphism $l : G \to H$ of abelian groups is the quotient group $H/l(G)$.)*

Proof: Let $B = \langle b_1 \rangle \oplus ... \oplus \langle b_k \rangle$. There exists an epimorphism $\alpha : B \to \mathbb{Z}_d^k$. For elements $(x_1, ..., x_k)$ of $\mathbb{Z}^k$, the reduction modulo d of coordinates yields a homomorphism $\mu : \mathbb{Z}^k \longrightarrow \mathbb{Z}_d^k$ defined by $\mu : (x_1, ..., x_k) \longmapsto (\overline{x_1}, ..., \overline{x_k})$. This homomorphism is such that there exist homomorphisms g' and h' making the following diagram commutative, where $g' : \mathbb{Z}_d^k \to \mathbb{Z}_d^k$ is defined by $g' : (\overline{x_1}, ..., \overline{x_k}) \mapsto (\overline{x_1}, ..., \overline{x_k})$ and the homomorphism h' is defined in a similar way.

$$\begin{tikzcd}
\mathbb{Z}^k \arrow[rr, "\phi"] \arrow[dr, "\alpha\circ g"] \arrow[dd, "\mu"'] & & \mathbb{Z}^k \arrow[dl, "\alpha\circ h"'] \arrow[dd, "\mu"] \\
& \mathbb{Z}_d^k & \\
\mathbb{Z}_d^k \arrow[ur, "g'"] \arrow[rr, "\Phi"'] & & \mathbb{Z}_d^k \arrow[ul, "h'"']
\end{tikzcd}$$

Since g' and h' are epimorphisms and $\mathbb{Z}_d^k$ is finite, then g' and h' are isomorphisms. Let $\Phi = (h')^{-1} \circ g'$. Then Φ is an isomorphism. Then the determinant of Φ, det Φ, is a unit of the finite abelian group $\mathbb{Z}_d$. Now, since det Φ is the residue class of the integer det ϕ, then det ϕ is relatively prime to d. Finally, the absolute value of det ϕ, $|\det\phi|$, is exactly equal to $|\text{coker}\phi|$. $\square$

Note that in the lemma below we use the notation as in Item 3.4.

Lemma 3.7: [30, 2.1] *Let k be a positive integer. Suppose that $f, h \in \text{Hom}(\mathbb{Z}^k, \mathbb{Z}_d^k)$, and suppose that $\alpha \in \text{Hom}(\mathbb{Z}_d^k, \mathbb{Z}_d^k)$ such that $\alpha \circ h = f \circ \beta$ for some automorphism β of $\mathbb{Z}^k$. Then there exists $u \in \mathbb{Z}^*$ such that*

$$\det(\alpha)\det(h_J) = \overline{u}\ \det(f_J) \text{ in } \mathbb{Z}_d.$$

Proof: By the definition of α, we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{Z}^k \arrow[r, "\beta"] \arrow[d, "h"'] & \mathbb{Z}^k \arrow[d, "f"] \\ \mathbb{Z}_d^k \arrow[r, "\alpha"'] & \mathbb{Z}_d^k, \end{tikzcd}$$

that is $\alpha \circ h = f \circ \beta$. The units of group $\mathbb{Z}$ are $\mathbb{Z}^* = \{1, -1\}$, and since β is an automorphism, $\beta \in \mathrm{Aut}(\mathbb{Z}^k)$, it has determinant ± 1, that is $\det(\beta) = \pm 1$. Since the diagram is commutative and the elements of $\mathbb{Z}_d^k$ are elements of $\mathbb{Z}^k$ reduced modulo d, then $\det(\alpha) = n$, where n is a unit of the finite abelian group $\mathbb{Z}_d$.

Given f and h, there are morphisms $f_J : \mathbb{Z}_d^k \to \mathbb{Z}_d^k$ and $h_J : \mathbb{Z}_d^k \to \mathbb{Z}_d^k$ for which we have the following commutative square:

$$\begin{tikzcd} \mathbb{Z}_d^k \arrow[r, "\beta'"] \arrow[d, "h_J"'] & \mathbb{Z}_d^k \arrow[d, "f_J"] \\ \mathbb{Z}_d^k \arrow[r, "\alpha"'] & \mathbb{Z}_d^k \end{tikzcd}$$

The assertion of the lemma now follows by multiplicativity of determinants. $\square$

Definition 3.8: [30, 2.2] Let A be a finite abelian group. Let $a = (a_1, ..., a_k)$ be an ordered k-tuple of elements of A , k a positive integer. For each a_i, let d_i be an element of $\mathbb{Z}$ such that the subgroup $\langle d_i \rangle$ of $\mathbb{Z}$ is the annihilator of the cyclic abelian group $\langle a_i \rangle \leq A$. Then a is said to be *good* if the following conditions are satisfied.

1. There exists an integer m, $1 \leq m \leq k$, such that $\langle A \rangle = \langle a_1 \rangle \oplus ... \oplus \langle a_m \rangle$ and $a_i = 0$ if and only if $m < i \leq k$.

2. d_i is a multiple of d_{i+1} for every $i < k$.

Definition 3.9: [30, 2.3] Let A be a finite abelian group. Let $a = (a_1, ..., a_k)$ be a good ordered k-tuple of elements of A. Then for every $n \in \mathbb{Z}$ we define $f_{(a,n)} : \mathbb{Z}^k \to A$ to be the unique function such that

$$f_{(a,n)}(e_i) = \begin{cases} a_i & \text{if } \ i < k \\ na_k & \text{if } \ i = k \end{cases}$$

where $e_1, ..., e_k$ is the standard basis of $\mathbb{Z}^k$. In most cases we shall suppress the k-tuple a, writing f_n instead of $f_{(a,n)}$.

Lemma 3.10: [30, 2.4] *Let A be a finite abelian group and $a = (a_1, ..., a_k)$ be a good ordered k-tuple of elements of A. Let $d_k \in \mathbb{Z}$ such that $\langle d_k \rangle$ is the annihilator of $\langle a_k \rangle$ (That is d_k is the order of a_k). If n is relatively prime to d_k, then f_n is an epimorphism.*

Proof: Let $f_n : \mathbb{Z}^k \to A$ be as in Definition 3.9. The homomorphism property follows from freeness of $\mathbb{Z}$ on $\{e_i \ : \ i = 1, \cdots, k\}$.

By the way f_n is defined, every element of A has a preimage written $f^{-1}(e_i) \in \mathbb{Z}^k$ for $i \leq k$ except when $(n, d_k) \neq 1$. If $(n, d_k) > 1$ then the conditions of Definition 3.9 may not be satisfied, that is, we may have an element a_f in $a = (a_1, ..., a_k)$ which does not have a preimage in $\mathbb{Z}^k$. Therefore f_n is an epimorphism if $(n, d_k) = 1$. $\square$

CHAPTER 4

THE NIELSEN EQUIVALENCE RELATION

This chapter is a continuation of the work commenced in Chapter 3. We introduce the Nielsen equivalence relation, which we define in 4.1. This concept will play a crucial role when we discuss the phenomenon of non-cancellation sets in Chapters 6 and 7. We define the Nielsen equivalence relation on the sets $E_k(A)$ as in Notation 3.2 and prove a number of results, such as: for $f \in E_k(A)$ then $f \sim f_n$ ($n \in \mathbb{Z}$ and f_n as in Definition 3.9 and Lemma 3.10). We also observe how these epimorphisms induce bijections between equivalence classes. We provided the proofs in this chapter, except for Proposition 4.2 which is simply quoted. Furthermore we conclude with an interesting observation, Remark 4.8.

Definition 4.1: [38] [39] Suppose f, g are epimorphisms, $f, g \in E_k(A)$ (see Notation 3.2). Then f and g are said to be *Nielsen equivalent*, $f \sim g$, if and only if there is an automorphism $\alpha : \mathbb{Z}^k \to \mathbb{Z}^k$ such that $f = g \circ \alpha$.

We now show that Nielsen equivalence is an equivalence relation.

1. For $f \in E_k(A), f = f \circ 1$, that is $f \sim f$, which means that $\sim$ is reflexive.
2. For $f, g \in E_k(A)$, $f \sim g$ implies that $f = g \circ \alpha$, then $g = f \circ \alpha^{-1} = f \circ \psi$ ($\psi \in \mathrm{Aut}(\mathbb{Z}^k)$), that is $g \sim f$, which means $\sim$ is symmetric.
3. For $f, g, h \in E_k(A)$, if $f \sim g$ and $g \sim h$ then $f = g \circ \alpha$ and $g = h \circ \beta$ so that $f = (h \circ \beta) \circ \alpha = h \circ (\beta \circ \alpha) = h \circ \delta$ ($\beta, \delta \in \mathrm{Aut}(\mathbb{Z}^k)$), which implies that $f \sim h$, that is $\sim$ is transitive.

Therefore Nielsen equivalence is an equivalence relation.

At this point we quote, without proof, a theorem on counting Nielsen equivalence classes of presentations of a finite group.

Proposition 4.2 [33, Corollary 3.3] *Let A be a finite abelian group of rank k. Let $\nu(A)$ denote the number of Nielsen equivalence classes of epimorphisms $\mathbb{Z}^k \to A$, and let d be the greatest common divisor of the invariant factors of A. Then*

$$\nu(A) = \begin{cases} \frac{\phi(d)}{2} & \text{if } d > 2 \\ 1 & \text{if } d \leq 2. \end{cases}$$

Notation 4.3: [39] The set of Nielsen equivalence classes of the elements of the set $E_k(A)$ is denoted by $E_k^{\sim}(A)$ and the class of the epimorphism $f \in E_k(A)$ is denoted by $[f]$.

Lemma 4.4: [30, 2.4] *Let A be a finite abelian group and $(a_1, ..., a_k)$ be a good ordered k-tuple of elements of A. Let d be the order of a_k. Then every member of $E_k(A)$ is equivalent to some f_n for an n which is relatively prime to d.*

Proof: Since the determinant of an automorphism of $\mathbb{Z}^k$ must be ± 1, it follows that if n and m are relatively to d, then $f_n \sim f_m$ only if $n \equiv \pm m \mod d$. By Proposition 4.2, it follows that the functions f_n (for $(n, d) = 1$) represent all the Nielsen equivalence classes. $\square$

Lemma 4.5: [30, 2.5] *Let A be a finite abelian group and let $(a_1, ..., a_k)$ be a good ordered k-tuple of elements of A. Let d be as in Notation 3.2. There exists an epimorphism $\lambda : A \to \mathbb{Z}_d^k$ such that post-composition with λ is a bijection*

$$\lambda_* : E_k^{\sim}(A) \to E_k^{\sim}(\mathbb{Z}_d^k).$$

Proof: Similar to Lemma 4.4, the proof follows by consideration of the determinants of automorphisms of $\mathbb{Z}^k$, together with Proposition 4.2. $\square$

Let us give explicit representatives for $E_k^{\sim}(A)$.

Lemma 4.6: [30, 2.6] *Let A be a finite abelian group and let $(a_1, ..., a_k)$ be a good ordered k-tuple elements of A. Let $d \in \mathbb{N}$ be the order of a_k. For integers n and m which are relatively prime to d, we have that $f_n \sim f_m$ if and only if there exists $u \in \mathbb{Z}^*$ and $r \in \mathbb{Z}$ such that $m = un + rd$.*

Proof: Let $A \cong \mathbb{Z}_d^k$, where d is as in Notation 3.2. Suppose $f_n \sim f_m$, then there exists $\alpha \in \mathrm{Aut}(\mathbb{Z}^k)$ such that $f_n = f_m \circ \alpha$ by Definition 4.1.

$$\begin{array}{ccc} \mathbb{Z}^k & \xrightarrow{\alpha} & \mathbb{Z}^k \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_m} \\ \mathbb{Z}_d^k & \xrightarrow[\beta=1]{} & \mathbb{Z}_d^k \end{array}$$

Let $u \in \mathbb{Z}^* = \{1, -1\}$ and let $J = \langle d \rangle$ and so $rd \in J$ and $\det(f_m)_J = m,\ \ \det(f_n)_J = n$. Then by Lemma 3.7, we have

$\det(\beta)\ \det(f_m)_J = \overline{u}\ \det(f_n)_J,$ (where $\overline{u}$ is the modulo d residue class.)

Therefore $\overline{m} = \overline{u}\ \overline{n}$.

Therefore $m = un + rd$, for some $r \in \mathbb{Z}$.

The converse follows similarly as in Lemma 4.4. $\square$

The results of this section can now be consolidated in the following theorem.

Theorem 4.7: [30, 2.7] *Let A be a finite abelian group. Let $(a_1, ..., a_k)$ be a good ordered k-tuple of elements of A. Let d be as in Notation 3.2. The function which sends an element $n \in \mathbb{Z}$ to f_n induces a bijection $\mathbb{Z}_d^*/\pm 1 \cong E_k^{\sim}(A)$.*

We conclude this section with the following observation and example.

Remark 4.8 Consider an abelian group $A = \langle a_1 \rangle \oplus \langle a_2 \rangle$.

For any endomorphism h with $\det(h) = \pm 1$, the results of this section implies that $f_1 \circ h \sim f$. Let us consider a special case and verify the existence of an automorphism ζ to replace h, i.e. an isomorphism $\zeta : \mathbb{Z}^2 \to \mathbb{Z}^2$ such that $f_1 \circ \zeta = f_1 \circ h$. Let us consider the case $d = 11$ with both a_1 and a_2 having order equal to 11. Take h to be the following matrix:

$$h = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix}.$$

Then $\det(h) = 45 = 1 + 4 \cdot 11$. The matrix ζ will have to be of the form

$$\zeta = \begin{bmatrix} 6 + 11a & -3 + 11b \\ 3 + 11c & 6 + 11d \end{bmatrix}.$$

A MAPLE computation gives

$$a = -1, \; b = 6, \; c = 0, \; d = -4.$$

$\square$

CHAPTER 5

SEMI-DIRECT PRODUCTS AND SPLIT EXTENSIONS

Chapter 5 is about the construction of semi-direct products. We observe a direct connection between semi-direct product groups and split extension groups. Another important concept in this work is that of *group action* and its properties [28]. We conclude this chapter with recent results from [35] and [37]. We supply an alternative proof to Proposition 5.19 as well as some modifications to the proof of Theorem 5.20.

Definition 5.1: [28] (a) We say that the group G *acts* (on the right) on a non-empty set X if there is a function $X \times G \to X$ defined by $(x, g) \mapsto xg$ for every $x \in X$ and $g \in G$, satisfying the following properties

$x1_G = x$ for every $x \in X$;

$x(gh) = (xg)h$ for every $x \in X$ and for every $g, h \in G$.

(b) The *orbit* of x, is the set $\{xg : g \in G\} \subseteq X$.

(c) If the group G acts on the set X, then the *action* is said to be *transitive* if it has just one orbit.

(d) If the group G acts on the set X, then for any $x \in X$, the *isotropy subgroup* or *stabilizer* of x is the subgroup $\mathrm{Stab}_G(x) = \{g \in G : xg = x\}$ of G.

(e) Conjugation produces an action of a group on itself. The respective orbits are the sets $\{gxg^{-1} \mid g \in G\}$ and the isotropy subgroup becomes the set $\{g \in G \mid gxg^{-1} = x\}$.

Now we discuss the construction of semidirect products and also observe the connection with the split extensions.

Definition 5.2: [28] Let T and Q be groups. Then we say that the group Q acts on the group T if there is a unique element $t^q \in T$ for each $q \in Q$ and for each $t \in T$, such that for every $t, t_1, t_2 \in T$ and $q, q_1, q_2 \in Q$:

$t^1 = t$,

$(t^{q_1})^{q_2} = t^{q_1 q_2}$, and

$(t_1 t_2)^q = t_1^q t_2^q$.

Theorem 5.3: [28] [29] *Let T and Q be groups. Let Q act on T. Then the map $w_q : T \to T$ defined by $w_q : t \mapsto t^q$, for every $q \in Q$, is an automorphism of T. Now the function $w : Q \to \mathrm{Aut}(T)$ defined by $w : q \mapsto w_q$ is a homomorphism. The homomorphism w is itself referred to as the action (of Q on T).*

Definition 5.4: [28] Let T and Q be groups. The set $T \rtimes_w Q$ of all ordered pairs (t, q), $t \in T$ and $q \in Q$, with the binary operation $(t, q)(t_1, q_1) = (tt_1^q, qq_1)$ is a *semi-direct product* of T by Q. Here t_1^q means $w_q(t_1)$.

Remark 5.5: [29, 7.9] We use the binary operation in Definition 5.4 to show that the semi-direct product $G = T \rtimes_w Q$, with $w : Q \to \mathrm{Aut}(T)$ an action, is a group . Let T and Q be groups. We first show that the multiplication is associative. Take $t, t_1, t_2 \in T$ and $q, q_1, q_2 \in Q$. Then we have $(t, q)[(t_1, q_1)(t_2, q_2)] = (t, q)(t_1 t_2^{q_1}, q_1 q_2) = [t(t_1 t_2^{q_1})^q, qq_1 q_2] = (tt_1^q t_2^{qq_1}, qq_1 q_2)$

and $[(t,q)(t_1,q_1)](t_2,q_2) = [tt_1^q, qq_1](t_2,q_2) = (tt_1^q t_2^{qq_1}, qq_1q_2)$. And this shows that multiplication is associative.

Take $(1,1) \in T \rtimes_w Q$, then $(1,1)(t,q) = (t^1, 1.q) = (t,q)$ and $(t,q)(1,1) = (t.1^q, q.1) = (t,q)$; this then means $(1,1)$ is an identity element of $G = T \rtimes_w Q$.

And the inverse of (t,q) is $((t^{q^{-1}})^{-1}, q^{-1})$ because $(t,q)((t^{q^{-1}})^{-1}, q^{-1}) = (t.t^{-1}, 1) = (1,1)$ and $((t^{q^{-1}})^{-1}, q^{-1})(t,q) = ((t^{q^{-1}})^{-1}.t^{q^{-1}}, 1) = (t^{-q^{-1}}.t^{q^{-1}}, 1) = (1,1)$.

Therefore, $G = T \rtimes_w H$ is a group.

Definition 5.6: [34] If Q and T are subgroups of a group G such that

(i) $T \trianglelefteq G$,

(ii) $TQ = G$, and

(iii) $T \cap Q = 1$,

then we say that G is an *internal semi-direct product* of T by Q.

Then the binary operation in Definition 5.4 becomes $(t,q)(t_1,q_1) = (tt_1^q, qq_1) = (tqt_1q^{-1}, qq_1)$.

Remark 5.7: [29] Suppose $G = T \rtimes_w Q$ is an internal semi-direct product. Then every element g of G can be written uniquely as $g = tq$ where $t \in T$ and $q \in Q$.

Proposition 5.8: [28] *A group $G \cong T \rtimes_w Q$ where w is an action $w : Q \to \mathrm{Aut}(T)$ of Q on T if and only if T and Q are subgroups of G such that $G = TQ$, $T \trianglelefteq G$ and $T \cap Q = 1$.*

Proof: See [28, 9.13] and [29]. $\square$

For the purposes of this thesis we are interested in the semi-direct products of a finite abelian group by a finite rank free abelian group (Free Abelian groups are discussed in Chapter 2). Hence we define the following class:

Definition 5.9: [38] [30] The class $\mathcal{K}$ is the class of all groups of the form $T \rtimes_w \mathbb{Z}^k$, for some $k \in \mathbb{N}$ and some finite group T.

In 5.14 and 5.15 we shall see why the semi-direct products are also called the split extenions.

Remark 5.10: [15] [27] There is a very close relationship between semi-direct products and split extensions of groups. We give a brief account of how this relationship is developed. Let G be the semi-direct product of T by H, then by the Second Isomorphism Theorem we have

$$G/T = TH/T \cong H/H \cap T = H/\{1\} \cong H.$$

In this case G is called an extension of T by H, which we now formally define as follows.

Definition 5.11: [15, 20.1] [27, 11.1] A group G is an *extension* of T by H if G has a normal subgroup T such that the quotient group G/T is isomorphic to H.

The notion of a group extension can also be explained in terms of exact sequences.

Definition 5.12: [24] [34] The following sequence of group homomorphisms is said to be an *exact sequence* at B if $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$.

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

The sequence above is said to be a *short exact sequence* at B if $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$ **and** α is a monomorphism with β an epimorphism.

In simpler terms, by a group extension it is meant how a group can be constructed from a normal subgroup and its quotient group. By a *group extension* of T by H is meant a short exact sequence of groups and homomorphisms,

$$T \xrightarrow{\mu} G \xrightarrow{\epsilon} H$$

with $\mathrm{Im}(\mu) = \mathrm{Ker}(\epsilon) = K$ say, and μ a monomorphism and ϵ an epimorphism. Now for $K \cong T$ and $G/K \cong H$, then G is called an *extension* of T by H.

Remark 5.13: [27] We make use of the following example to show that the extension of T by H always exists [27]. For an example, we form a semi-direct product $G = T \rtimes_w H$ corresponding to a homomorphism $w : H \to \mathrm{Aut}(T)$. For $t \in T, h \in H$, define $\mu(t) = (t, 1)$ and $\epsilon(t, h) = h$. Then

$$T \xrightarrow{\mu} G \xrightarrow{\epsilon} H$$

is an extension of T by H.

Definition 5.14: [27, p.304] A group extension

$$T \xrightarrow{\mu} G \xrightarrow{\epsilon} H$$

is said to be a *split extension* if there exists a transversal function $\tau : H \to G$ such that $\epsilon \circ \tau$ is the identity map.

Remark 5.15: [27] Next we show, using this example, that every split extension is a semi-direct product extension. Now, suppose that

$$T \xrightarrow{\mu} G \xrightarrow{\epsilon} H$$

splits via a homomorphism $\tau : H \to G$. Write $X = G^{\epsilon\tau}$ so that for $x \in G$ then $x^{\epsilon\tau} \in X$. Now, since $\epsilon \circ \tau = \tau\epsilon = 1$, we have $(x^{-\epsilon\tau}x)^{\epsilon} = x^{-\epsilon}x^{\epsilon} = 1$ so that $x^{-\epsilon\tau}x \in \mathrm{Ker}(\epsilon) = M$, M a normal subgroup (by The First Isomophism Theorem) of G. Then $G = XM$, and moreover, $X \cap M = 1$ since $x^{\epsilon\tau} \in M$ implies that $1 = (x^{\epsilon\tau})^{\epsilon} = x^{\epsilon}$. Hence $G = M \rtimes X \cong T \rtimes H$ (see Remark 2.10 and Definition 5.6), and this proves our assertion that every split extension is a semi-direct product extension.

Remark 5.16: Let G be a finitely generated abelian group and T_G be the subgroup of elements of finite order in G. Then $T_G \trianglelefteq G$ by Proposition 2.24 and G/T_G is torsion-free and free also by Proposition 2.24. Also we have that $T_G \cap G/T_G = \{1\}$ and $G = T_G.G/T_G$. Then by Definition 5.6, G is a split extension of T_G by G/T_G.

Notation 5.17: [35] (a) Let A be a finite abelian group and $u_1, u_2, ..., u_k$ be a finite sequence of integers all of which are relatively prime to the exponent of A (see

Remark 2.2). The group $G(A; u_1, ..., u_k) = A \rtimes_w \mathbb{Z}^k$ where $w : \mathbb{Z}^k \to \mathrm{Aut}(A)$ is an action, and $w(z_1, \cdots, z_k)$ acts on A as follows: $a \mapsto (u_1^{z_1} \cdots u_k^{z_k})a$. A short-hand notation of $G(\mathbb{Z}_d; u_1, ..., u_k)$ is $G(d; u_1, ..., u_k)$. The subgroup of $\mathbb{Z}_d^*$ generated by the residue classes of the integers $u_1, ..., u_k$ will be denoted by $R(d; u_1, ..., u_k)$.

(b) For $n, u \in \mathbb{N}$ with $\gcd(n, u) = 1$, by $G(n; u)$ we denote the group $H = \mathbb{Z}_n \rtimes_w \mathbb{Z}$ where $w : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}_n)$ is the action such that $w(r) : t \mapsto u^r t$.

Proposition 5.18: [37, 3.1] *Let d be the exponent of a finite abelian group A. Let $u_1, u_2 \in \mathbb{Z}$ be relatively prime to d. Then $G(d; u_1) \cong G(d; u_2)$ if and only if $u_1 \equiv u_2 \bmod d$ or $u_1 u_2 \equiv 1 \bmod d$.*

Proposition 5.19: [35, 0.3] *Let us consider the group $\mathbb{Z}_d$ and let $\{u_1, ..., u_k\}$ be a set of integers which are relatively prime to the positive integer d. Then $G(d; u_1, ..., u_k, 1) \cong G(d; u_1, ..., u_k) \times \mathbb{Z}$.*

Proof: We adopt the Notation in 5.17. Now $G = G(d; u_1, ..., u_k) = \mathbb{Z}_d \rtimes_w \mathbb{Z}^k$ with $w : \mathbb{Z}^k \to \mathrm{Aut}(\mathbb{Z}_d)$ an action; and similarly $H = G(d; u_1, ..., u_k, 1) = \mathbb{Z}_d \rtimes_v \mathbb{Z}^{k+1}$ with $v : \mathbb{Z}^{k+1} \to \mathrm{Aut}(\mathbb{Z}_d)$ also an action. This implies that the images $w(\mathbb{Z}^k)$ and $v(\mathbb{Z}^{k+1})$ coincide. By [35, 3.4] there exists an automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}^{k+1})$ such that $w = v \circ \alpha$ since w is generated by fewer than $k + 1$ elements. This implies that $v \sim w$ by Definition 4.1. Let $K = \mathbb{Z}_d \rtimes_{v_1} \mathbb{Z}^{k+1}$ with the respective action $v_1 : \mathbb{Z}^{k+1} \to \mathrm{Aut}(\mathbb{Z}_d)$. Clearly $v_1 \sim v$. Now suppose that $H \cong K$. Then the respective torsion subgroups T_H and T_K are isomorphic and the torsion-free subgroups $\mathbb{Z}^{k+1}$ are also isomorphic by [28, 8.24]. But $T_G = T_K$ and $\mathbb{Z}^k \times \mathbb{Z} \cong \mathbb{Z}^{k+1}$ and also $w \sim v_1$ (see below Definition 4.1). Thus $H \cong G \times \mathbb{Z}$ since $\cong$ is an equivalence relation by [28, Exercise 20]. Therefore $G(d; u_1, ..., u_k, 1) \cong G(d; u_1, ..., u_k) \times \mathbb{Z}$. □

Theorem 5.20: [35, 4.1] *Let T be a finite abelian group. Let $w : \mathbb{Z}^k \to \mathrm{Aut}(T)$ and $v : \mathbb{Z}^{k+1} \to \mathrm{Aut}(T)$ be homomorphisms such that their images $w(\mathbb{Z}^k)$ and $v(\mathbb{Z}^{k+1})$ coincide. Then $T \rtimes_v \mathbb{Z}^{k+1} \cong (T \rtimes_w \mathbb{Z}^k) \times \mathbb{Z}$.*

Proof: Let $q : \mathbb{Z}^{k+1} \to \mathbb{Z}^k$ be any projection map. Let $w_1 : \mathbb{Z}^{k+1} \to \mathrm{Aut}(T)$ be the homomorphism such that $w_1 = w \circ q$. Let H be the common image of w and v. Then H is a finite abelian group, because $\mathrm{Im}(v)$ and $\mathrm{Im}(w)$ are abelian, (see Proposition 2.7). Also $H = w_1(\mathbb{Z}^{k+1})$. Clearly v and w_1 belong to $E_{k+1}(\mathrm{Aut}(T))$ since T is finite abelian, and hence they are Nielsen equivalent. Thus there exists an automorphism $h : \mathbb{Z}^{k+1} \to \mathbb{Z}^{k+1}$ such that $w_1 = v \circ h$ (see Definition 4.1). Let $G_0 = T \rtimes_v \mathbb{Z}^{k+1}$ and $G_1 = T \rtimes_{w_1} \mathbb{Z}^{k+1}$. Then we obtain an isomorphism $G_1 \to G_0$ by taking an identity map on the torsion subgroups and the map h on the torsion-free quotients. To reach the desired conclusion we make use of the following result: Let $u_1, u_2, ..., u_k$ be a sequence of integers all of which are relatively prime to a positive integer d (see Notation 5.17), then $G(d; u_1, ..., u_k, 1) \cong G(d; u_1, ..., u_k) \times \mathbb{Z}$ by Proposition 5.19. By the definition of H above and since $q(\mathbb{Z}^{k+1}) = \mathbb{Z}^k$, $w_1(\mathbb{Z}^{k+1}) = w(\mathbb{Z}^k)$. This implies $G_1 \cong (T \rtimes_w \mathbb{Z}^k) \times \mathbb{Z}$ by Proposition 5.19. Then it follows, since $w_1 \sim v$ and $G_0 \cong G_1$, that $v(\mathbb{Z}^{k+1}) = w(\mathbb{Z}^k)$ and $T \rtimes_v \mathbb{Z}^{k+1} \cong (T \rtimes_w \mathbb{Z}^k) \times \mathbb{Z}$. $\square$

CHAPTER 6

NON-CANCELLATION OF SPLIT EXTENSION GROUPS

In this chapter we define what we mean by a non-cancellation set. We discuss the non-cancellation phenomenon for finitely generated groups with finite commutator subgroups. We choose these particular groups because a group structure on the non-cancellation set is defined for finitely generated groups whose commutator subgroups are finite. Our main results therefore are on the non-cancellation phenomenon of split extension groups. The main results are from Witbooi's papers [35], [37], [38] and [39]. We present a minor modification of the proof of Theorem 6.18, giving a little more detail than in the original source. We also supplied Example 6.19.

Definition 6.1: [15, Ex. 7.5] (a) Let G be a group. Then the *centre* of G, denoted by $Z(G)$, is the set of all elements z which commute with every element g of G.

$$Z(G) = \{z \in G \mid zg = gz \ \forall \ g \in G\}$$

[15, 10.24] (b) Let H be any subgroup of a group G. The *centralizer* of H in G is denoted by $C_G(H)$ and is defined as follows:

$$C_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\}$$

(We note that for any $H \leq G$, $Z(G) \leq C_G(H) \leq G$.)

[28, 3.55] (c) Let $H \leq G$. The *normalizer* of H in G is

$$N_G(H) = \{g \in G \mid g^{-1}Hg = H\}.$$

We recall the definition of a commutator subgroup and some of its basic properties.

Definition 6.2: [28, 3.46] The *commutator* of an ordered pair g_1, g_2 of elements of a group G is the element $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2 \in G$.

Remark 6.3: [28, 3.47] If $g_1, g_2 \in G$ then $[g_1, g_2]^{-1} = [g_2, g_1]$ and $[g_1, g_2] = 1$ if and only if g_1 and g_2 commute.

Definition 6.4: [28, 3.48] Let $H, K \leq G$. Let $[H, K] = \langle [h, k] \mid h \in H,\ k \in K \rangle$, a subgroup of G. The *commutator subgroup* or *the derived group* of G, denoted by $[G, G]$ or G', is the subgroup generated by all commutators of G, that is

$$[G, G] = G' = \langle [g_1, g_2] \mid g_1, g_2 \in G \rangle.$$

Example 6.5: [15] *A group G is abelian if and only if $G' = 1$.*

Proof: If G is abelian then for every $g_1, g_2 \in G$, we have $g_1g_2 = g_2g_1$. It then follows by Remark 6.3 and Definition 6.4 that $G' = 1$.
Conversely, if $G' = \langle [g_1, g_2] \mid g_1, g_2 \in G \rangle = 1$, then $[g_1, g_2] = 1$ and by Remark 6.3, G is abelian. $\square$

Notation 6.6: [38, Section 1] By $\mathcal{X}_0$ we shall mean the class of all finitely generated groups having finite commutator subgroups.

Another important concept in this work is that of indices. Hence we state and prove the following results.

Definition 6.7: Recall that if H is a subgroup of G, then the index of H in G is the number of cosets of H in G, and is denoted by $|G:H|$.

Lemma 6.8: [15, 5.13] *Let H and K be subgroups of a finite group G with H a subgroup of K. Then $|G:H| = |G:K||K:H|$.*

Proof: $|G:H| = |G|/|H| = (|G|/|K|)(|K|/|H|) = |G:K||K:H|$ $\square$

Proposition 6.9: [38, 2.5] *Let G be any infinite $\mathcal{X}_0$-group, and let m be any natural number. Then there is a subgroup H of G such that $|G:H| = m$.*

Proof: Let M be any subgroup of the free abelian group G/T_G such that $|G/T_G : M| = m$. Let $\pi : G \to G/T_G$ be the canonical epimorphism. Then the subgroup $\pi^{-1}(M)$ of G has index m in G. $\square$

Proposition 6.10: [38] [27] *If G is a $\mathcal{X}_0$-group, then T_G, the subset of all elements of finite order in G, is a finite normal subgroup of G.*

Remark 6.11: T_G is called the torsion radical of G.

Remark 6.12: (a) [38, 2] If H is any subgroup of a $\mathcal{X}_0$-group G, then H is a $\mathcal{X}_0$-group.

(b) [6, 2] If $G \in \mathcal{X}_0$ and F is a normal subgroup of G, then $G/F \in \mathcal{X}_0$.

(c) [38, 2] If G and H are $\mathcal{X}_0$-groups, then $G \times H$ is also a $\mathcal{X}_0$-group. For the ring of integers $\mathbb{Z}$, $G \times \mathbb{Z}$ is a $\mathcal{X}_0$-group.

(d) [38, 2] If $G \in \mathcal{X}_0$ and H is a group such that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$, then H is a $\mathcal{X}_0$-group.

(e) [38, 2] A group G is a $\mathcal{X}_0$-group if and only if G is an extension of a finite group by a finite rank free abelian group. (Extensions are discussed in Chapter 5)

The proofs of 6.12 may be found in [8].

Definition 6.13: [38, 1] The set of all the isomorphism classes $[H]$ of groups H such that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$ is called a *non-cancellation* set and is denoted by $\chi(G)$.

Notation 6.14: [39] For a positive integer k, consider the finite rank free abelian group $\mathbb{Z}^k$, and let T be a finite abelian group. The class $\mathcal{K}^{(k)}$ is the class of all groups G which are semi-direct products of the form $G = T \rtimes_w \mathbb{Z}^k$. By $\mathcal{K}$ we mean the union of the classes of $\mathcal{K}^{(k)}$ (These groups are dealt with in Chapter 5). The homomorphism $w : \mathbb{Z}^k \to \mathrm{Aut}(T)$ is an action of $\mathbb{Z}^k$ on T. The subgroup $\mathrm{Im}(w) = w(\mathbb{Z}^k)$ of $\mathrm{Aut}(T)$ can be seen to be an abelian group by Proposition 2.7.

The conclusion of Remark 6.12(e) is that split extensions of the form $T \rtimes_w \mathbb{Z}^k$ for T finite (which are also called semi-direct products) are $\mathcal{X}_0$-groups. In other words the $\mathcal{K}$-groups are also $\mathcal{X}_0$-groups. We point out this fact because from now on we shall discuss the properties and the group structure of the non-cancellation set of groups G in $\mathcal{K}$.

Notation 6.15: Henceforth we shall often work with a finite abelian group T and an action $w : \mathbb{Z}^k \to \mathrm{Aut}(T)$ on T for some $k \in \mathbb{N}$. We assume that d is the order of a_k for some good k-tuple $(a_1, a_2, \cdots, a_k)$ of $\mathrm{Im}(w)$. The group $T \rtimes_w \mathbb{Z}^k$ will be denoted by G.

Proposition 6.16: [39, 2.2] *Let $G = T \rtimes_w \mathbb{Z}^k$ be as in Notation 6.15 and let $H = T \rtimes_v \mathbb{Z}^k$ for some action v. Then H is isomorphic to G if and only if the following condition holds. There exist automorphisms $h_0 : T \to T$ and $h_1 : \mathbb{Z}^k \to \mathbb{Z}^k$ such that the following identity holds for every $t \in T$ and every $z \in \mathbb{Z}^k$,*

$$[h_0 \circ w(z)](t) = [v(h_1(z))](h_0(t)).$$

Proof: See [37, 2.2].

In particular then we have the following.

Corollary 6.17: [39, 2.3] *Consider a finite abelian group T and let G and H be groups, given as $G = T \rtimes_w \mathbb{Z}^k$ and $H = T \rtimes_v \mathbb{Z}^k$ respectively. If $H \cong G$, then the subgroups Im(v) and Im(w) of Aut(T) are conjugate in Aut(T).*

Theorem 6.18: [39, 2.4] *Let T be a finite abelian group, and $G = T \rtimes_w \mathbb{Z}^k$ and $H = T \rtimes_v \mathbb{Z}^k$ be $\mathcal{K}$-groups. Then $H \times \mathbb{Z} \cong G \times \mathbb{Z}$ if and only if the subgroups Im(v) and Im(w) are conjugate in Aut(T).*

Proof: Let $w, v : \mathbb{Z}^k \to \mathrm{Aut}(T)$ be actions of $\mathbb{Z}^k$ on T. By Corollary 6.17, we have that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$ implies that $\mathrm{Im}(v)$ and $\mathrm{Im}(w)$ are conjugate in $\mathrm{Aut}(T)$.

Conversely, suppose there exists $h \in \mathrm{Aut}(T)$ such that $h(\mathrm{Im}(v))h^{-1} = \mathrm{Im}(w)$. By Theorem 5.20, there is an epimorphism $w_1 : \mathbb{Z}^{k+1} \to \mathrm{Im}(w)$ such that $G \times \mathbb{Z} = T \rtimes_{w_1} \mathbb{Z}^{k+1}$. Similarly, there is an epimorphism $v_1 : \mathbb{Z}^{k+1} \to \mathrm{Im}(v)$ such that $H \times \mathbb{Z} = T \rtimes_{v_1} \mathbb{Z}^{k+1}$. Let $\tau : \mathrm{Im}(w) \to \mathrm{Im}(v)$ be the isomorphism defined by $\tau : \alpha \mapsto h\alpha h^{-1}$. We then apply the notion of the Nielsen equivalence on epimorphisms $(\tau \circ w_1) : \mathbb{Z}^{k+1} \to \mathrm{Im}(v)$ and $v_1 : \mathbb{Z}^{k+1} \to \mathrm{Im}(v)$. Recall that $v : \mathbb{Z}^k \to \mathrm{Aut}(T)$ and hence the $\mathrm{rank}(\mathrm{Im}(v)) \leq k$ by Notation 6.14. Now we know that every member of $E_{k+1}(\mathrm{Im}(v))$ is equivalent to $f_n : \mathbb{Z}^{k+1} \to \mathrm{Im}(v)$ by Lemma 4.4; and since the rank of $\mathrm{Im}(v)$ is less than $k+1$, then $(\tau \circ w_1)$ and v_1 are Nielsen equivalent. Therefore there exists $\alpha \in \mathrm{Aut}(\mathbb{Z}^{k+1})$ such that by Proposition 6.16, since $hv_1(\alpha(\mathbb{Z}^{k+1}))h^{-1} = hv_1(\mathbb{Z}^{k+1})h^{-1} = w_1(\mathbb{Z}^{k+1})$, the automorphisms α and h constitute an isomorphism $G \times \mathbb{Z} \to H \times \mathbb{Z}$. $\square$

We are now ready to present an example of non-cancellation.

Example 6.19: Let $H = G(49; 3)$ and let $L = G(49; -2)$. The order of 3 in $\mathbb{Z}_{49}^*$ is 42, and $-2 \equiv 3^5 \bmod 49$. By Theorem 6.18 it follows that $H \times \mathbb{Z} \cong L \times \mathbb{Z}$. However, by Proposition 5.18, H is not isomorphic to L since -2 is not congruent to 3 or -3 modulo 49. $\square$.

The above example demonstrates that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$ does not imply that H is isomorphic to G. This set $\chi(G)$ of a group G measures the extent to which the infinite cyclic group $\mathbb{Z}$ cannot be cancelled as a common direct factor given an isomorphism $H \times \mathbb{Z} \cong G \times \mathbb{Z}$. However, there are cases when the cancellation can be performed; such as when G is a finite group. A few other instances where cancellation is permissible are described in the literature. We mention some such cases below.

Theorem 6.20: [7, Theorem 5] *Let G be any $\mathcal{X}_0$-group and B any finite group. If H is any group such that $G \times B \cong H \times B$ then $G \cong H$.*

Theorem 6.21: [8, Theorem 2.11] *If G is a finite group and H is any group then $H \times \mathbb{Z} \cong G \times \mathbb{Z}$ if and only if $H \cong G$.*

Another theorem that allows us to cancel the infinite cyclic group $\mathbb{Z}$, under certain conditions, is the following theorem which is due to Hirshon.

Proposition 6.22: [14] *If G and H are groups such that $H \times \mathbb{Z} \cong G \times \mathbb{Z} \times \mathbb{Z}$, then $H \cong G \times \mathbb{Z}$.*

Theorem 6.23: [35, 4.2] *Let $G = T \rtimes_w \mathbb{Z}^k$ be a group which is the semidirect product, with T a finite group and $w : \mathbb{Z}^k \to \mathrm{Aut}(T)$ an action and $k > 1$. If the image $w(\mathbb{Z}^k)$ of w in $\mathrm{Aut}(T)$ can be generated by a subset of fewer than k elements, then $\chi(G)$ is trivial.*

Proof: Let H be any group such that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$. We need to show that $H \cong G$. Let $\{a_1, a_2, ..., a_{k-1}\}$ be a generating subset for $w(\mathbb{Z}^k)$; and for the free abelian group $\mathbb{Z}^{k-1}$, let $\{e_1, e_2, ..., e_{k-1}\}$ be the standard $\mathbb{Z}$-basis (see Definition 2.17). Let $v : \mathbb{Z}^{k-1} \to w(\mathbb{Z}^k)$ be the homomorphism defined by $v : e_i \mapsto a_i$. Then $v(\mathbb{Z}^{k-1}) = w(\mathbb{Z}^k)$. Thus, by Theorem 5.20, we have $G \cong G_0 \times \mathbb{Z}$, where $G_0 = T \rtimes_v \mathbb{Z}^{k-1}$. But then $H \times \mathbb{Z} \cong G \times \mathbb{Z} \cong G_0 \times \mathbb{Z} \times \mathbb{Z}$. Thus $H \cong G_0 \times \mathbb{Z} \cong G$ by Proposition 6.22. $\square$

CHAPTER 7

GROUP STRUCTURE OF THE NON-CANCELLATION SET OF A SPLIT EXTENSION GROUP

In this chapter we continue with the non-cancellation set $\chi(G)$ for $G \in \mathcal{X}_0$, as introduced in Chapter 6, as the set of all isomorphism classes $[H]$ of the groups $H \in \mathcal{X}_0$ such that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$. The emphasis in this chapter will be on the discussion of a group structure on the non-cancellation set of a $\mathcal{K}$-group, i.e. a split extension of a finite group T by a finite rank free abelian group $\mathbb{Z}^k$ which is a subclass of the class $\mathcal{X}_0$ (see Definitions 5.4, 5.9 and 5.14 and Remark 6.12). In defining the group structure on $\chi(G)$ we shall also make use of the notions of an action, exact sequences and indices of embeddings. The main discussions are based on the papers [30], [35], [38], [39] and [41]. We supplied the proofs of Theorem 7.3 and Proposition 7.5. The proof of Proposition 7.6 is an adaptation of [39, Theorem 2.6].

In [38] it is shown how to impose a group structure on $\chi(G)$ for a $\mathcal{X}_0$-group G, in terms of the indices of embeddings of groups in $\chi(G)$. In particular, there will eventually be an epimorphism

$$\theta : \mathbb{Z}^*_{n(G)} \to \chi(G), \tag{$*$}$$

with $n(G)$ as defined in [38].

In [39] it is shown that if G is of the form $T \rtimes_w \mathbb{Z}^k$ for some finite group T, then instead of the epimorphism (*) above, one can replace $n(G)$ by a smaller integer d, where d is the greatest common divisor of the orders of the invariant factors of the abelian group $\mathrm{Im}(w) \subseteq \mathrm{Aut}(T)$. In this chapter we shall give a detailed derivation of the group structure on $\chi(G)$ for groups belonging to the class $\mathcal{K}$.

Notation 7.1: 1. We continue with the notation as in Notation 6.15.

2. Consider the function f_n as in Definition 3.9 (and here we let $A = \mathrm{Im}(w)$, for w as in Notation 6.15). For each integer n relatively prime to d, let $G_{(n)} = T \rtimes_{f_n} \mathbb{Z}^k$ be a split extension group.

3. H^k denotes the direct product of k copies of a group H.

Proposition 7.2: [30, 3.2] *Let T be a finite abelian group. Consider two epimorphisms $u, v : \mathbb{Z}^k \to \mathrm{Aut}(T)$. Then $T \rtimes_u \mathbb{Z}^k \cong T \rtimes_v \mathbb{Z}^k$ if and only if there exists an automorphism α of T such that $\alpha v(z)\alpha^{-1} = w(z)$ for each $z \in \mathbb{Z}^k$.*

Theorem 7.3: (cf [39, Theorem 2.5]) *Let G be as in Notation 6.15. Let H and K be subgroups of G such that the indices $|G : H|$, $|G : K|$ are finite and relatively prime to d and such that $H \times \mathbb{Z} \cong G \times \mathbb{Z} \cong K \times \mathbb{Z}$. If $|G : H| \equiv \pm |G : K| \bmod d$ then $H \cong K$.*

Proof: By assumption H and K are groups of the form $H \cong T \rtimes_\lambda \mathbb{Z}^k$ and $K \cong T \rtimes_\epsilon \mathbb{Z}^k$ for some $\lambda, \epsilon \in E_k(\mathrm{Im}(w))$. By Lemma 4.4 and since $|G : H| \equiv \pm |G : K| \bmod d$, λ and ϵ are equivalent to $f_n : \mathbb{Z}^k \to \mathrm{Im}(w)$ and hence λ and ϵ are Nielsen equivalent. Then there exists an automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}^k)$ such that $\lambda = \epsilon \circ \alpha$ by Definition

4.1. Therefore, by Proposition 6.16 the function $T \rtimes_\lambda \mathbb{Z}^k \to T \rtimes_\epsilon \mathbb{Z}^k$ defined by $(t, z) \mapsto (t, \alpha(z))$ is an isomorphism. $\square$

Remark 7.4: [38, 6.1] [19, 6.8] Suppose that we have groups A, B, and C together with a homomorphism $\beta : A \to C$ and a surjective (group) homomorphism $\gamma : A \to B$. If $\alpha : B \to C$ is a function (between sets) such that $\alpha \circ \gamma = \beta$, then α is a homomorphism. If, moreover, β is surjective, then α is also surjective.

In the paper [30] and later in the paper [38] it is shown how the noncancellation set $\chi(G)$ of a $\mathcal{X}_0$-group is described in terms of mutual embeddings of G and H for different groups H for which $H \times \mathbb{Z} \cong G \times \mathbb{Z}$. In fact it turns out that whenever $H \times \mathbb{Z} \cong G \times \mathbb{Z}$, then H is isomorphic to some subgroup of G. In what follows we make this more explicit, and eventually we describe the group structure on $\chi(G)$ in terms of the indices of the relevant subgroups of $\chi(G)$.

Let us consider the function $\theta : \mathbb{Z}_d^*/\pm 1 \to \chi(G)$ defined by $\theta(\overline{x}) = [H_x]$ where H_x is any subgroup of G of index x having the property that $H_x \times \mathbb{Z} \cong G \times \mathbb{Z}$. Such a group H_x does exist (one can take $G_{(x)}$ for instance). We further assume that $G \in \mathcal{K}$ is infinite since for a finite group K, $\chi(K)$ is trivial (see Theorem 6.21). We shall show that this function θ is firstly well-defined and secondly θ is a surjection. In our proof in 7.6 we consider a function $\mu : X \to \chi(G)$ defined by $\mu : x \mapsto [H_x]$ where X is the set of all the integers which are relatively prime to d.

Proposition 7.5: *Let $X = \{x \in \mathbb{Z} \mid (x, d) = 1\}$ and $G \in \mathcal{K}$. The function $\mu : X \to \chi(G)$ defined by $\mu(x) = [H_x]$ and $|G : H_x| = x$, is a well-defined surjection.*

Proof: Let us define $\mu : X \to \chi(G)$ by $\mu(x) = [H_x]$ where H_x is a subgroup of G with $|G : H_x| = x$ and $[H_x]$ is the isomorphism class of the group H_x. Let us first show that $\mu : X \to \chi(G)$ is well-defined. Let $y \in X$ such that $|G : H_y| = y$. Now if $x \equiv y \mod d$ then $[H_x] \cong [H_y]$ which means μ is well-defined by Theorem 7.3. For every $[H_x] \in \chi(G)$ there is a corresponding $x \in X$, which means μ is surjective. $\square$

Proposition 7.6: [38, 5.2] *Let G be as in Notation 6.15. The function*

$$\theta : \mathbb{Z}_d^*/ \pm 1 \to \chi(G)$$

is a surjection.

Proof: We first show that the function $\theta : \mathbb{Z}_d^*/ \pm 1 \to \chi(G)$ defined by the rule $\theta(\overline{x}) = [H_x]$ is well-defined, where $\overline{x} \in \mathbb{Z}_d^*/ \pm 1$ and $[H_x]$ is an isomorphism class of the group H_x which is such that $H_x \times \mathbb{Z} \cong G \times \mathbb{Z}$, and $|G : H_x| = x$, that is $(x, d) = 1$. Now if $\overline{y} \in \mathbb{Z}_d^*/ \pm 1$, and $(y, d) = 1$ with $|G : H_y| = y$, suppose $x \equiv y \mod d$ then $[H_x] \cong [H_y]$ by Theorem 7.3. Therefore θ is well-defined.
For surjectivity of θ, we consider the following. Let $X = \{x \in \mathbb{Z} \mid (x, d) = 1\}$. Now consider the well-defined surjection $\mu : X \to \chi(G)$ in Proposition 7.5. This function μ factorizes through the reduction modulo d-homomorphism $\epsilon : X \to \mathbb{Z}_d^*/ \pm 1$, which is moreover a surjection, that is there exists a function $\theta : \mathbb{Z}_d^*/ \pm 1 \to \chi(G)$ such that $\mu = \theta \circ \epsilon$. $\square$

We now deal with the derivation of the group structure of the non-cancellation set $\chi(G)$ where $G \in \mathcal{K}$. For $\mathcal{X}_0$-groups in general the derivation may be found in [38].

Proposition 7.7: [30, 3.4] *There is a transitive action of the group $\mathbb{Z}_d^*/\pm 1$ on the set $\chi(G_{(1)})$, given by $\overline{u}.G_{(n)} = G_{(un)}$, for integers n and u that are relatively prime to d.*

It is important to note that for the action described in Proposition 7.7, any two elements of $\chi(G)$ have the same isotropy subgroup due to transitivity and $\mathbb{Z}_d^*/\pm 1$ being abelian. Therefore we can deduce the following Theorem.

Theorem 7.8: [39, 2.6] *Let us consider the split extension $G = G_{(1)}$. The transitive action of $\mathbb{Z}_d^*/\pm 1$ on $\chi(G)$ furnishes the non-cancellation set $\chi(G)$ with a group structure.*

For a nilpotent group G this group structure is the same as the Hilton-Mislin group structure defined on the genus set (see [36]).

Theorem 7.9: [39, 2.7] *Let $G = T \rtimes_w \mathbb{Z}^k$ where T is a finite group and $w : \mathbb{Z}^k \to \mathrm{Aut}(T)$ is an action of $\mathbb{Z}^k$ on T. Suppose that the Prüfer rank of $\mathrm{Im}(w)$ is k. Let $m \in \mathbb{Z}$ and let d be greatest common divisor of the orders of the invariant factors of $\mathrm{Im}(w)$ and suppose that $(m, d) = 1$. The following conditions are equivalent.*

(a) $\overline{m} \in \mathrm{Ker}\ [\mathbb{Z}_d^* \to \chi(G)]$

(b) *There exists an automorphism $\alpha \in \mathrm{Aut}(T)$ such that for the inner automorphism $\tau : v \mapsto \alpha v \alpha^{-1}$ of G, we have $\tau(\mathrm{Im}(w)) = \mathrm{Im}(w)$ and for the automorphism*

$\sigma : \mathrm{Im}(w) \to \mathrm{Im}(w)$ *induced by* τ*, we have det(*σ*)* $= \pm\overline{m}^{\ -1} \in \mathbb{Z}_d^*$.

We include some results on induced morphisms of non-cancellation groups. The following Lemma is used to prove Proposition 7.12.

Lemma 7.10: [28, 3.15] *If* $H \trianglelefteq G$ *and* K *a characteristic subgroup of* H*, then* $K \trianglelefteq G$.

Theorem 7.11: [38, 6.2] *Let* G *and* H *be any* $\mathcal{X}_0$*-groups, and suppose that* G *is infinite. Then the function* $\phi : \chi(G) \to \chi(G \times H)$ *defined by* $\phi : [K] \mapsto [K \times H]$ *is a well-defined epimorphism of groups.*

In [40, 4.1], Witbooi proves a result that when $G \in \mathcal{X}_0$ and F is a finite group then the epimorphism $\phi : \chi(G) \to \chi(G \times F)$ is injective.

Theorem 7.12: [40, 2.1] *Let* F *be a characteristic subgroup of the torsion subgroup of the infinite* $\mathcal{X}_0$*-group* G*. There is a well-defined surjective group homomorphism* $\eta : \chi(G) \to \chi(G/F)$ *such that* $\eta([K]) = [K/F]$.

In [13] Hilton and Witbooi provide another variation of the above result where they deal with the morphisms of the Mislin Genus.

Remark 7.13: Earlier we introduced the $\mathcal{K}$-group $H = G(d; u) = \langle a, b \mid a^d = 1,\ bab^{-1} = a^u \rangle$, and $H = \mathbb{Z}_d \rtimes_f \mathbb{Z}$ with $f : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}_d)$ an action of $\mathbb{Z}$ on

$\mathbb{Z}_d$. Let us briefly discuss the properties of H^k related to H. The group H^k can also be considered to be the group $\mathbb{Z}_d^k \rtimes_w \mathbb{Z}^k$, where $w : \mathbb{Z}^k \to \mathrm{Aut}(\mathbb{Z}_d^k)$ is an action. In 7.6 we showed that $\mathbb{Z}_d^*/\pm 1 \to \chi(G)$ is an epimorphism, and consequently the function $\theta : \mathbb{Z}_d^*/\pm 1 \to \chi(H^k)$ is an epimorphism. Also from 7.11 we can conclude that $\chi(H) \to \chi(H^k)$ is an epimorphism, and the consequence of 7.12 is that $\chi(H^k) \to \chi(H^k/F)$ is an epimorphism, where for the latter, F is a characteristic subgroup of the torsion radical of H^k.

CHAPTER 8

COMPUTING $\chi(G(n_1; u) \times G(n_2; u))$ IN A SPECIAL CASE

In this, the final chapter of this thesis, we identify conditions under which the epimorphism $\theta : \mathbb{Z}_d^*/\pm 1 \to \chi(H)$ mentioned in Proposition 7.6 is a monomorphism, where H is defined as in Notation 8.1. Our result is a contribution towards the general problem of computing the non-cancellation group of a group which is a split extension of a finite abelian group by a finite rank free abelian group as per Definitions 5.4, 5.9 and 5.14.

Notation 8.1: 1. Let k be any natural number and let $S = \{1, 2, 3, ..., k\}$. Fix two finite sequences of elements of $\mathbb{N}$, $n_1, n_2, ..., n_k$ and $u_1, u_2, ..., u_k$ such that for each $i \in S$, u_i is relatively prime to n_i. Let $H_i = G(n_i, u_i)$ (see Notation 5.17), and let $H = H_1 \times H_2 \times ... \times H_k = \mathrm{Dr} \prod_{i \in S} H_i$ (see Notation 2.16). Then H is a $\mathcal{X}_0$-group (by Remark 6.12) with the torsion subgroup $T_H = \mathrm{Dr} \prod_{i \in S} \mathbb{Z}_{n_i}$. Then H is a split extension group $H = T_H \rtimes_w \mathbb{Z}^k$ for some action $w : \mathbb{Z}^k \to \mathrm{Aut}(T_H)$ (see Definitions 5.2 and 5.4, and Theorem 5.3).
Let $A = \mathrm{Im}(w) \leq \mathrm{Aut}(T_H)$. There is an obvious direct decomposition for the abelian group A, as $A = \mathrm{Dr} \prod_{i \in S} \langle w_i \rangle$ where for each $i \in S$, w_i is the automorphism of $\prod_{i \in S} \mathbb{Z}_{n_i}$ defined by $(x_1, x_2, ..., x_k) \mapsto (y_1, y_2, ..., y_k)$ where for each $t \in S$ we have

$$y_t = \begin{cases} x_t & \text{if } t \neq i \\ u_i x_i & \text{if } t = i. \end{cases}$$

2. Let $N = N_{\mathrm{Aut}(T_H)}(A) = \{\lambda \in \mathrm{Aut}(T_H) : \lambda\beta\lambda^{-1} \in A \text{ for all } \beta \in A\}$. That is to say, N is the normalizer of A in $\mathrm{Aut}(T_H)$ (see Definition 6.1). The centralizer of A in $\mathrm{Aut}(T_H)$ is $C = C_{\mathrm{Aut}(T_H)}(A) = \{\lambda \in \mathrm{Aut}(T_H) : \lambda\beta = \beta\lambda \text{ for all } \beta \in A\}$. Now, given any $\lambda \in N$, let $\theta_\lambda : A \to A$ be the function defined by $\theta_\lambda : \beta \to \lambda\beta\lambda^{-1}$. Then θ_λ is an automorphism of A.

3. For each $i \in S$ let d_i be the multiplicative order of u_i modulo n_i and let d be the greatest common divisor of the numbers d_i. (Refer to Definitions 3.8 and 3.9; and also refer to the paragraph above Proposition 3.3 in [30].)

4. For any group G, Inn G denotes the group of all inner automorphisms of G.

5. By $\mathrm{ord}_n u$ we mean the multiplicative order of u modulo n.

Definition 8.2: [42] *The NC Property*: Let T be a finite abelian group and let A be a subgroup of $\mathrm{Aut}(T)$. The pair (T, A) is said to be an NC-pair or to have the NC-property if the following condition holds:

$$N_{\mathrm{Aut}(T)}(A) = C_{\mathrm{Aut}(T)}(A).$$

Equivalently, if for every $\lambda \in N_{\mathrm{Aut}(T)}(A)$ the inner automorphism θ_λ of A defined by the rule $\theta_\lambda : a \mapsto \lambda a \lambda^{-1}$ is the identity automorphism of A.

Theorem 8.3: [42] *Let the groups T and H be as in Notation 8.1, that is $T = \mathrm{Dr}\ \prod_{i\in S} \mathbb{Z}_{n_i}$ and $H = \mathrm{Dr}\ \prod_{i\in S} G(n_i, u_i) = T \rtimes_w \mathbb{Z}^k$ where $w : \mathbb{Z}^k \to \mathrm{Aut}(T)$ is the corresponding action of $\mathbb{Z}^k$ on T. Let d be as in Notation 8.1.*
If $(T, \mathrm{Im}(w))$ is an NC-pair then $\chi(H) \cong \mathbb{Z}_d^*/\pm 1$.

Proof: Let $H = T \rtimes_w \mathbb{Z}^k$ and d be as in Notation 8.1. In Proposition 7.6 it is shown that the function $\theta : \mathbb{Z}_d^*/\pm 1 \to \chi(H)$ is an epimorphism. In Theorem 7.9 the kernel of θ is described in terms of determinants of certain automorphisms of $\mathrm{Im}(w)$.

Let $\overline{s} \in \mathrm{Ker}(\theta)$. We have that $(T, \mathrm{Im}(w))$ is an NC-pair. This implies that for $\lambda \in N_{\mathrm{Aut}(T)}(\mathrm{Im}(w))$ the inner automorphism $\theta_\lambda : a \mapsto \lambda a \lambda^{-1}$ is the identity automorphism of $\mathrm{Im}(w)$ by Definition 8.2. Therefore by Theorem 7.9 $\det(\theta_\lambda) = \pm\overline{s}^{\ -1} = 1$. This implies that $\mathrm{Ker}(\theta) = 1$. Then by [28, 3.10] θ is moreover injective, and thus $\chi(H) \cong \mathbb{Z}_d^*/\pm 1$. $\square$

Lemma 8.4: *Let the groups T and H be as in Notation 8.1, that is $T = \mathrm{Dr}\ \prod_{i\in S} \mathbb{Z}_{n_i}$ and $H = \mathrm{Dr}\ \prod_{i\in S} G(n_i, u_i) = T \rtimes_w \mathbb{Z}^k$ where $w : \mathbb{Z}^k \to \mathrm{Aut}(T)$ is the corresponding action of $\mathbb{Z}^k$ on T. If the gcd$(n_i, n_j) = 1$, whenever $i \neq j$, then*

$$N_{\mathrm{Aut}(T)}(\mathrm{Im}(w)) = C_{\mathrm{Aut}(T)}(\mathrm{Im}(w)).$$

Proof: Since the numbers n_i are pairwise relatively prime, then by Corollary 2.13 the group $T = \mathrm{Dr}\ \prod_{i\in S} \mathbb{Z}_{n_i}$ is also a finite cyclic group. This implies that $\mathrm{Aut}(T)$ is abelian by [28, 4.38]. Then $\mathrm{Inn}(\mathrm{Aut}(T))$ the subgroup of all inner automorphisms of $\mathrm{Aut}(T)$ is trivial. Thus $N_{\mathrm{Aut}(T)}(\mathrm{Im}(w)) = C_{\mathrm{Aut}(T)}(\mathrm{Im}(w))$ by Definition 8.2. $\square$

Corollary 8.5: *Let the groups T and H be as in Notation 8.1, that is $T = \mathrm{Dr}\ \prod_{i\in S} \mathbb{Z}_{n_i}$ and $H = \mathrm{Dr}\ \prod_{i\in S} G(n_i, u_i) = T \rtimes_w \mathbb{Z}^k$ where $w : \mathbb{Z}^k \to \mathrm{Aut}(T)$ is the corresponding action of $\mathbb{Z}^k$ on T. If the gcd$(n_i, n_j) = 1$ whenever $i \neq j$ such that $N_{\mathrm{Aut}(T)}(\mathrm{Im}(w)) = C_{\mathrm{Aut}(T)}(\mathrm{Im}(w))$ then $\chi(H) \cong \mathbb{Z}_d^*/\pm 1$.*

Proof: Since $\gcd(n_i, n_j) = 1$ for $i \neq j$, then $\mathrm{rank}(\mathrm{Im}(w)) \leq k$. But if $\mathrm{rank}(\mathrm{Im}(w)) < k$ then $\chi(G)$ is trivial by Theorem 6.23. The condition $\gcd(n_i, n_j) = 1$ also implies that $(T, \mathrm{Im}(w))$ is an NC-pair by Lemma 8.4. The result follows by Theorem 8.3. $\square$

Example 8.6: Let $k = 2$ in Notation 8.1. We want to compute the non-cancellation group $\chi(H)$ for $H = G(n_1; u) \times G(n_2; u)$ for the case $n_1 = 3^2.41^2$, $n_2 = 61^2$ and $u = 7$. Note that $(n_i, u) = 1$ and $(n_1, n_2) = 1$ and therefore Lemma 8.4 applies. Note that $\langle d_1, d_2 \rangle = \langle d \rangle$ (see section 15 in [5]), where $d = \gcd(d_1, d_2)$.

A MAPLE computation yields:

$d_1 = 4920 = 2^3.3.5.41$,

$d_2 = 3660 = 2^2.3.5.61$,

therefore $d = 2^2.3.5 = 60$.

Thus $\chi(G(n_1, u)) \cong \mathbb{Z}^*_{4920}/\pm 1$,

$\chi(G(n_2, u)) \cong \mathbb{Z}^*_{3660}/\pm 1$, and

$\chi(G) \cong \mathbb{Z}^*_{60}/\pm 1$.

The orders of these groups are, respectively:

$\frac{1}{2}\phi(4920) = \frac{1}{2}(4.2.4.40) = 640$,

$\frac{1}{2}\phi(3660) = \frac{1}{2}(2.2.4.60) = 480$, and

$\frac{1}{2}\phi(60) = \frac{1}{2}(2.2.4) = 8$.

We note the induced epimorphisms

$$\chi(G(n_i, u)) \to \chi(H).$$

The group H can be written as

$$H = \{1, 7, 11, 13, 17, 19, 23, 29\},$$

being a quotient of $\mathbb{Z}_{60}^*$. The group H has no elements of order 8, but the order of 7 is 4. Thus we can write

$$H = \{1, 7, 11, 17\} \oplus \{1, 29\} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2.$$

$\square$

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