ULTRAFILTERS AND COMPACTIFICATIONS

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Abstract

Ultrafilters and Compactifications

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In this thesis, we construct the ultrafilter space of a topological space using ultrafilters as points, study some of its properties and describe a method of generating compactifications through the ultrafilter space. As part of investigating some properties of the ultrafilter space, we show that the ultrafilter space forms a monad in the category of topological spaces. Furthermore, we show that rendering the ultrafilter space suitably separated results in a generation of separated compactifications which coincide with some well-known compactifications. When the ultrafilter space is rendered T_0 or sober, the resulting compactification is a stable compactification. Rendering the ultrafilter space T_2 or Tychonoff results in the Stone-$\check{C}$ech compactification.

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Declaration

I declare that *Ultrafilters and Compactifications* is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Full Name: Mbekezeli Sibahle Nxumalo Date: 13/03/2020

Signed:..................................

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Contents

List of Abbreviations and Notations

(X, T)	Topological space
T_{dis}	Discrete Topology
T_{pat}	Patch Topology
T_{tr}	Trivial Topology
T_{usu}	Usual Topology
$\overline{A}$	Closure of a set A on a given topological space
$\overline{A}^P = X$	A is dense in the patch topology of X
$(\mathcal{CS})_X$	Collection of compact and saturated sets on X
N_x	Neighbourhood of $x \in X$
$\mathcal{U}_x$	Collection of neighbourhoods of $x \in X$
$A_{\mathcal{G}}$	Set of all limit points of an ultrafilter $\mathcal{G}$ on a topological space X
$C(X)$	Collection of all bounded continuous real-valued functions on X
Set	Category whose objects are sets and morphisms are functions
Top	Category whose objects are topological spaces and morphisms are continuous functions
$\mathbf{Top_0}$	Subcategory of **Top** whose objects are T_0 spaces and morphisms are continuous functions
$\mathbf{Top_1}$	Subcategory of **Top** whose objects are T_1 spaces and morphisms are continuous functions
Haus	Subcategory of **Top** whose objects are Hausdorff spaces and morphisms are continuous functions
Tych	Subcategory of **Top** whose objects are Tychonoff spaces and morphisms are continuous functions

Top$_{Sob}$ Subcategory of **Top** whose objects are sober spaces and morphisms are continuous functions

Comp$_1$ Subcategory of **Top** whose objects are compact T_1 spaces and morphisms are continuous functions

CHaus Subcategory of **Top** whose objects are compact Hausdorff spaces and morphisms are continuous functions

Chapter 0

Introduction

Ultrafilters, compact spaces, and separation axioms play a central role in this thesis. For any set X, a filter on X is usually contemplated as a family of subsets of X which is closed under supersets and finite intersections. The origin of filters is traced back to Henri Cartan in 1937. They are one of the mathematical tools used for describing convergence in topological spaces and are ordered by set inclusion. With this order, an ultrafilter is a maximal filter.

Ultrafilters have a number of applications in mathematics such as proving the Tychonoff's Theorem which states that a non-empty product space is compact if and only if each factor space is compact, defining the asymptotic cone of a group in geometric group theory, and constructing ultraproducts and ultrapowers in model theory. For the existence of more ultrafilters on infinite sets, we can make use of the Kuratowski-Zorn's postulate which asserts that each inductive set (where each ordered subset has an upper bound) has at least one maximal element.

During mid to late nineteenth century, when mathematicians began to understand and specify essential properties of the real line, a compact space was known as a closed and bounded interval of the real line. The Heine-Borel Theorem states that any collection of open intervals covering such interval has a finite subcollection of open intervals that still cover the closed interval. This theorem can be generalized to arbitrary topological spaces by defining a compact space as a topological space in which every open cover of has a finite subcover covering. The duality between closed sets and open sets leads to a characterization of compact spaces with closed sets which involves the Finite Intersection Property (FIP) - a family of subsets of a topological space has the FIP if and only if the intersection of any finite subcollection from that family is non-empty. This characterization is stated as follows: A topological space is compact if and only if each family of closed sets that has the FIP has a non-empty intersection.

Compact spaces are regarded as important spaces in general topology since they behave like finite sets, which are way easier to understand and work with than uncountable sets which are common in topology. In search for more compact spaces, mathematicians resorted in a way of making non-compact spaces to be compact through a process of embedding a non-compact space as a dense subspace of some compact space. This process is referred to as compactification. One of the reasons to study compactification is that it is ordinarily simpler to have a non-compact space as a subspace of a compact space, thus letting you use all of the tools available in the compact setting.

It is widely known that a converging sequence in a metric space has a unique limit. This is one of the nice properties missed by arbitrary topological spaces. In order to recover this and many other properties, topologists devised various separation axioms. These axioms use topological means to distinguish distinct points and disjoint sets, and are mostly denoted with the letter "T" after the German *Trennungsaxiom*, which means "separation axioms". Here is the list of some of the popular separation axioms. [Wi] Let (X, T) be a topological space.

1. (X, T) is said to be T_0 if for each distinct elements x, y of X, there is N_x not containing y, and vice versa.

2. (X, T) is said to be T_1 if for each distinct elements x, y of X, there are N_x and N_y such that $y \notin N_x$ and $x \notin N_y$.

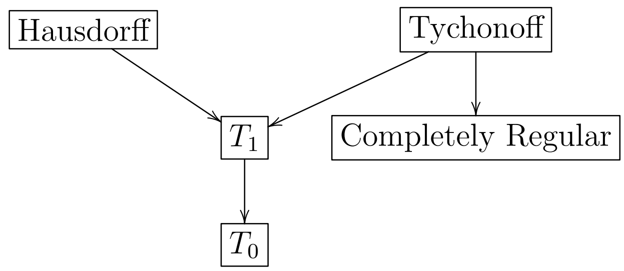
3. (X, T) is said to be T_2 or Hausdorff if for each distinct elements x, y of X, there are disjoint N_x and N_y.

4. (X, T) is said to be completely regular if, whenever A is a closed set in X and $x \notin A$, there is a continuous function $f : X \longrightarrow [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$.

5. (X, T) is said to be Tychonoff or $T_{3\frac{1}{2}}$ if it is T_1 and completely regular.

A topological space satisfying atleast one of these separation axioms is said to be separated.

These separation axioms are sorted as follows:



A remarkable relationship between Hausdorff spaces and Tychonoff spaces is that each compact Hausdorff space is Tychonoff.

Salbany, in [Sa], generated a topological space (called the ultrafilter space) using ultrafilters as elements. This space turned out to be a compactification of an arbitrary topological space and is seldom separated. With the fact that many compactifications are separated, Salbany devised ways of making this space a separated compactification. He achieved this through taking separated reflections of the ultrafilter space. He considered T_0, T_1 and T_2 reflections. This resulted in a number of separated compactifications which coincide with some well-known compactifications. His work is a contribution to the study of compactifications. This thesis is an extension of his work. We show that one can form a monad using the ultrafilter space. We also provide a general approach to the construction of these separated compactifications which allows one to generate more separated compactifications. Sober compactification as well as Tychonoff compactification are some of the separated compactifications that are generated through this approach. These compactifications coincide with some standard compactifications. In addition to these contributions, this thesis accounts for more than three quarters of original proofs.

This thesis is organized as follows. The first chapter introduces a construction of the ultrafilter space of a topological space using ultrafilters as points. We show that the ultrafilter space is one of the compact topological spaces which are seldom separated. Furthermore, we give a construction of a retraction of the ultrafilter space.

The second chapter discusses compactifications as well as separated reflections of topological spaces. The chapter begins with a brief introduction of the notion of categories. We introduce reflective subcategories which are important in deriving the concept of separated reflections.

In the first section, we consider three examples of compactifications: the Alexandroff one-point compactification, the Wallman compactification as well as the Stone-$\check{C}$ech compactification. These compactifications are limited to certain topological spaces. The Alexandroff one-point compactification is limited to locally compact Hausdorff spaces, while the Wallman compactification and the Stone-$\check{C}$ech compactification are limited to T_1 spaces and Tychonoff spaces, respectively. The Stone-$\check{C}$ech compactification is of interest since it is known to be functorial and is the compact Hausdorff reflection of a Tychonoff space. In the last section, we consider T_0, sober, T_1, T_2 and Tychonoff reflections and make use of quotient spaces to construct T_0, T_1, T_2 and Tychonoff reflections.

In the third chapter, we introduce a notion of monads and their algebras and create a monad on **Top** using the ultrafilter space. This monad is called the ultrafilter space monad. The algebras for the ultrafilter space monad are essentially bitopological Salbany stably compact spaces.

The last chapter gives separated compactifications of topological spaces which arise through rendering the ultrafilter space suitably separated. We show that taking the reflector for some reflective subcategory $\mathcal{C}$ of **Top** such that the retraction constructed in Chapter 2 exists for $X' \in \mathcal{C}$, results in a number of separated compactifications. Each resulting compactification coincides with some well-known compactification. Rendering the ultrafilter space T_0 and T_2 results in a stable compactification and the Stone-$\check{C}$ech compactification, respectively. When the ultrafilter space is rendered sober, the resulting compactification coincides with the T_0 compactification. In the case of taking the Tychonoff reflection of the ultrafilter space, the resulting compactification coincides with the Stone-$\check{C}$ech compactification. Rendering the ultrafilter space T_1 results in a compact space having some properties similar to the Wallman compactification. It still remains unclear whether or not this compact space is always Hausdorff.

This thesis is meant for a reader with some knowledge of general topology and category theory. Our chapters are numbered according to their order of appearance in the thesis. The same rule holds for sections in chapters and for formal statements, i.e., propositions, theorems, lemmas and definitions, in sections.

We shall use the words space and topological space interchangeably, and we shall frequently write X to represent a topological space whenever the underlying topology is clear from the context.

Chapter 1

Ultrafilters

The aim of this chapter is to construct the ultrafilter space of a topological space using ultrafilters as points and find some of its properties that shall be frequently used in some parts of this thesis. We shall begin by recalling some basic concepts from the theory of ultrafilters, then give a construction of the ultrafilter space with its properties, and later construct a retraction of the ultrafilter space.

1.1 Introduction to Ultrafilters

Ultrafilters are popular in the literature of general topology. We shall give some basic definitions and a few simple, largely known basic results from the theory of ultrafilters. For further readings, we advise the reader to consult [Bo, Th, Wi].

Definition 1.1.1. *[MM] A non-empty family $\mathcal{F}$ of subsets of a non-empty set X is called a filter on X if it satisfies the following conditions:*

1. $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$;
2. $A, B \in \mathcal{F}$ *implies* $A \cap B \in \mathcal{F}$ *and*
3. $A \subseteq B \subseteq X$ *and* $A \in \mathcal{F}$ *implies* $B \in \mathcal{F}$.

Example 1.1. A. For x in a topological space X, the family $\mathcal{U}_x$ is a filter on X.

Filters on the same set may be compared by simple set inclusion.

Definition 1.1.2. *[MM] If* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are filters on a set* X *such that* $\mathcal{F}_1 \subseteq \mathcal{F}_2$*, then we call* $\mathcal{F}_2$ *a refinement of* $\mathcal{F}_1$*. We say that* $\mathcal{F}_2$ *is finer than* $\mathcal{F}_1$*.*

For our work, the preceding definition is adequate for defining an ultrafilter on a set.

Definition 1.1.3. *[Wi] Let* X *be a set. A filter* $\mathcal{G}$ *is called an ultrafilter on* X *if, whenever* $\mathcal{H}$ *is a filter on* X *and* $\mathcal{H}$ *is a refinement of* $\mathcal{G}$*, then* $\mathcal{G} = \mathcal{H}$*.*

In transit to discussing some properties of ultrafilters, we give the following three results.

Definition 1.1.4. *[ASP] An inductive ordered set is an ordered set in which every chain has an upper bound.*

Lemma 1.1.5. *[Bo] Let* $\Phi(X)$ *be a set of all filters on a non-empty set* X*, and* $\leq$ *be the set inclusion relation on* $\Phi(X)$*. Then* $\Phi(X)$ *is inductive.*

Remark: A proof for the preceding lemma can be found in most books of topology such as [ASP].

In light of the preceding lemma, we make use of the Kuratowski-Zorn's postulate to assert that every filter is contained in some ultrafilter.

Corollary 1.1.6. *[Wi] Every filter* $\mathcal{F} \in \Phi(X)$ *is contained in some ultrafilter.*

Next, we prove some characterizations of ultrafilters which are sometimes used in defining an ultrafilter on a set X.

Theorem 1.1.7. *[Th, Wi] For a filter* $\mathcal{G}$ *on a non-empty set* X*, the following statements are equivalent:*

1. $\mathcal{G}$ *is an ultrafilter on* X*.*
2. *For each* $A, B \subseteq X$*,* $A \cup B \in \mathcal{G}$ *implies* $A \in \mathcal{G}$ *or* $B \in \mathcal{G}$*.*
3. $A \cap F \neq \emptyset$*, for all* $F \in \mathcal{G}$*, implies* $A \in \mathcal{G}$*.*
4. *For each* $A \subseteq X$*, either* $A \in \mathcal{G}$ *or* $X \backslash A \in \mathcal{G}$*.*

Proof: $(1 \Rightarrow 2)$: [Th] Let A and B be subsets of X such that $A \cup B \in \mathcal{G}$, but $A \notin \mathcal{G}$ and $B \notin \mathcal{G}$. If $A = \emptyset$, then $A \cup B = B$. Thus $B \in \mathcal{G}$ and we are done. Suppose that both A and

B are non-empty. Let $\mathcal{E} = \{Y \subseteq X : Y \cup A \in \mathcal{G}\}$. We show that $\mathcal{E}$ is a filter on X. Observe that $\mathcal{E}$ is non-empty, since $B \in \mathcal{E}$. Furthermore, $\emptyset \notin \mathcal{E}$, otherwise $\emptyset \cup A = A \in \mathcal{G}$ which is not possible. Let $C, D \in \mathcal{E}$. Therefore, $C \cup A \in \mathcal{G}$ and $D \cup A \in \mathcal{G}$. Because $\mathcal{G}$ is an ultrafilter, $(C \cup A) \cap (D \cup A) \in \mathcal{G}$. Now, $(C \cap D) \cup A = (C \cup A) \cap (D \cup A) \in \mathcal{G}$. Therefore $C \cap D \in \mathcal{E}$. Finally, if $C \in \mathcal{E}$ and $C \subseteq D \subseteq X$, then $C \cup A \in \mathcal{G}$. It follows that $B \cup A \in \mathcal{G}$, since $\mathcal{G}$ is an ultrafilter. Thus $B \in \mathcal{E}$. Hence $\mathcal{E}$ is a filter on X. Observe that $\mathcal{G} \subseteq \mathcal{E}$. However, if $D \in \mathcal{E}$, then only $D \cup A \in \mathcal{G}$, not D. So, $\mathcal{E} \neq \mathcal{G}$. Contradicting the fact that $\mathcal{G}$ is an ultrafilter on X. Thus $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

$(2 \Rightarrow 3)$: Let $A, F \subseteq X$ and $A \cap F \neq \emptyset$, for all $F \in \mathcal{G}$. It is clear that A and F are non-empty. Also, $F \subseteq A \cup F$ and $A \cup F \neq \emptyset$. Because $\mathcal{G}$ is a filter on X, it follows that $A \cup F \in \mathcal{G}$. By hypothesis, $A \in \mathcal{G}$ or $F \in \mathcal{G}$.

$(3 \Rightarrow 4)$: Let $A \subseteq X$. Then either A meets each $F \in \mathcal{G}$ or $X \backslash A$ does, it cannot be both since $\emptyset \notin \mathcal{G}$. If $A \cap F \neq \emptyset$, for each $F \in \mathcal{G}$, then $A \in \mathcal{G}$. If $(X \backslash A) \cap F \neq \emptyset$, for each $F \in \mathcal{G}$, then $(X \backslash A) \in \mathcal{G}$.

$(4 \Rightarrow 1)$: [Wi] If $\mathcal{G}$ is not an ultrafilter on X, then there is $\mathcal{G}'$ finer than $\mathcal{G}$ such that $\mathcal{G}' \neq \mathcal{G}$, i.e., there is $A \in \mathcal{G}$ but $E \notin \mathcal{G}$. It follows that $(X \backslash A) \notin \mathcal{G}$ otherwise $(X \backslash A) \cap A = \emptyset \in \mathcal{G}'$ which is not possible since $\mathcal{G}'$ is a filter on X. Thus $A \notin \mathcal{G}$ and $(X \backslash A) \notin \mathcal{G}$. ■

Some ultrafilters are generated by a single element. Enroute to the characterization of such ultrafilters, we present the following definition.

Definition 1.1.8. *[Pa] A filter $\mathcal{F}$ on a set X is principal if there exists a non-empty set $S \subseteq X$, such that $\mathcal{F} = \{A \subseteq X : S \subseteq A\}$. Otherwise, $\mathcal{F}$ is non-principal.*

The preceding definition implies that $\bigcap \mathcal{F} \neq \emptyset$, for a principal filter $\mathcal{F}$. When $\bigcap \mathcal{F} = \emptyset$, $\mathcal{F}$ is usually referred to as a free filter. [Pa] A Fréchet filter $\mathcal{F}_r = \{A \subseteq X : X \backslash A \ is\ finite\}$ on an infinite set X is an example of a free filter since $\bigcap \mathcal{F}_r = \emptyset$. If a filter $\mathcal{F}$ is free, then for each $x \in X$ there must be a set $A_x \in \mathcal{F}$ such that $x \notin A_x$. Therefore, the set $X \backslash \{x\}$ contains A_x, so $X \backslash \{x\} \in \mathcal{F}$. Any cofinite set is the intersection of sets of the form $X \backslash \{x\}$, so all cofinite sets must therefore be in $\mathcal{F}$. Therefore, $\mathcal{F}$ contains the Fréchet filter. Furthermore, if a filter $\mathcal{F}$ contains the Fréchet filter, then $\bigcap \mathcal{F} \subseteq \mathcal{F}_r = \emptyset$. Which means that $\mathcal{F}$ is free.

Next, we give a characterization of principal ultrafilters.

Proposition 1.1.9. *[Wi] Let X be a set and let $\mathcal{G}$ be an ultrafilter on X. $\mathcal{G}$ is principal if and only if there is $x \in X$ such that $\mathcal{G} = \{S \subseteq X : x \in S\}$.*

Proof: Let $\mathcal{G}$ be an ultrafilter on X. If $\mathcal{G}$ is principal, then there is a non-empty set $S \subseteq X$ such that $\mathcal{G} = \{A \subseteq X : S \subseteq A\}$. It suffices to show that S is a singleton set. Suppose that $S = \{x, y\}$, for some distinct elements $x, y \in X$. Because $\mathcal{G}$ is an ultrafilter, either $\{x, y\} \in \mathcal{G}$ or $X \backslash \{x, y\} \in \mathcal{G}$. The latter is not possible since $\{x, y\} \subsetneq X \backslash \{x, y\}$. Furthermore, $\{x, y\} = \{x\} \cup \{y\}$ implies that either $\{x\} \in \mathcal{G}$ or $\{y\} \in \mathcal{G}$, in which both cases are not possible. Thus S must be a singleton set, say $S = \{x\}$. Thus $\mathcal{G} = \{A \subseteq X : x \in A\}$. Conversely, let $\mathcal{G} = \{S \subseteq X : x \in S\}$ for some $x \in X$. Then $\mathcal{G} = \{S \subseteq X : \{x\} \subseteq S\}$, making $\mathcal{G}$ principal. ■

When such $\mathcal{G}$ exists, we say that $\mathcal{G}$ is the principal ultrafilter generated by x, and denote it by $\dot{x}$. It is clear that $\bigcap \dot{x} \neq \emptyset$.

[Wi] An ultrafilter containing a given filter need not be unique. Indeed, consider a filter $\mathcal{F}$ of all sets containing a non-empty set $A \subseteq X$. Then, for each $x \in A$, the principal ultrafilter generated by x contains $\mathcal{F}$. As a result, we shall say 'an' ultrafilter.

In the case where $\mathcal{G}$ is an ultrafilter on X and $\bigcap \mathcal{G} = \emptyset$, $\mathcal{G}$ shall be referred to as a free ultrafilter on X. The existence of free ultrafilters is guaranteed by the Kuratowski-Zorn's postulate. The following result shows that every infinite set has a free ultrafilter.

Proposition 1.1.10. *Let X be an infinite set. Then there exists a free ultrafilter on X.*

Proof: Because X is infinite, then X has a Fréchet filter $\mathcal{F}_r$. It follows that there is an ultrafilter $\mathcal{G}$ on X containing $\mathcal{F}_r$. Clearly, $\mathcal{G}$ is free. ■

Corollary 1.1.11. *Every ultrafilter on a finite set X is principal.*

Next, we want to characterize compact and Hausdorff spaces using ultrafilters. To achieve this, we start by discussing the concept of convergence for an ultrafilter on a topological space. We define converging filters and cluster points of filters from the study of filters.

Definition 1.1.12. *[Wi] Let X be a topological space. A filter $\mathcal{F}$ on X converges to a point $x \in X$ if $\mathcal{F}$ is a refinement of $\mathcal{U}_x$. We say that x is a limit point of $\mathcal{F}$.*

Definition 1.1.13. *[Wi] Let X be a topological space. A point $x \in X$ is a cluster point of a filter $\mathcal{F}$ if $x \in \bigcap\{\overline{F} : F \in \mathcal{F}\}$.*

Ultrafilters converge to their cluster points. This is shown by the following proposition.

Proposition 1.1.14. *[Bo] Let X be a topological space. An ultrafilter $\mathcal{G}$ on X converges to a point x if and only if x is its cluster point.*

Proof: Suppose that $\mathcal{G}$ converges to $x \in X$. Then $F \in \mathcal{G}$ for each $F \in \mathcal{U}_x$. So, for all $U \in \mathcal{G}$ and for each $F \in \mathcal{U}_x$, $F \cap U \in \mathcal{G}$. This means that $F \cap U \neq \emptyset$ for all $F \in \mathcal{U}_x$ and any $U \in \mathcal{G}$. Therefore $x \in \overline{U}$, for each $U \in \mathcal{G}$. Thus $x \in \bigcap_{U \in \mathcal{G}} \overline{U}$. Conversely, let $F \in \mathcal{U}_x$. Since $x \in \overline{U}$ for each $U \in \mathcal{G}$, then $U \cap F \neq \emptyset$ for any $U \in \mathcal{G}$. By Theorem 1.1.5, $F \in \mathcal{G}$ and so $\mathcal{U}_x \subseteq \mathcal{G}$. ■

Proposition 1.1.15. *[Bo] A filter $\mathcal{F}$ on a topological space X converges to a point $x \in X$ if and only if every ultrafilter which is finer than $\mathcal{F}$ converges to x.*

Proof: Let $\mathcal{G}$ be an ultrafilter on X and $\mathcal{F}$ a filter on X such that $\mathcal{F} \subseteq \mathcal{G}$. Since $\mathcal{F}$ converges to x for some $x \in X$, then $N_x \in \mathcal{F}$ for each $N_x \in \mathcal{U}_x$, implying that $N_x \in \mathcal{G}$ for every $N_x \in \mathcal{U}_x$. It follows that $\mathcal{G}$ converges to x. On the otherhand, if $N_x \in \mathcal{U}_x$ but $N_x \notin \mathcal{F}$ where $\mathcal{F}$ is a filter on X, then the sets of the form $(X \backslash N_x) \cap F$, for each $F \in \mathcal{F}$, form a filterbase for some filter $\mathcal{V}$ finer than $\mathcal{F}$. Such a filter can be extended to an ultrafilter $\mathcal{G}$ which is finer than $\mathcal{F}$. But then every ultrafilter finer than $\mathcal{F}$ converges to x, so $N_x \in \mathcal{G}$, which is impossible to have both N_x and $X \backslash N_x$ in $\mathcal{G}$. Thus $U_x \in \mathcal{F}$. ■

Proposition 1.1.16. *Suppose that X is a topological space and $\mathcal{G}$ an ultrafilter on X. Then each of the following holds.*

1. *If C is a closed subset of X and $C \in \mathcal{G}$, then $A_{\mathcal{G}} \subseteq C$.*
2. *If $x \in A_{\mathcal{G}}$, then $\overline{\{x\}} \subseteq A_{\mathcal{G}}$.*
3. *$A_{\mathcal{G}}$ is closed.*

Proof: (1) Let $x \in A_{\mathcal{G}}$, then $N_x \cap C \neq \emptyset$ for any $N_x \in \mathcal{G}$. So $x \in \overline{C} = C$.
(2) Let $y \in \overline{\{x\}}$. Then $x \in N_y$ for each $N_y \in \mathcal{U}_y$. Therefore each $N_y \in \mathcal{U}_x$. Thus $y \in A_{\mathcal{G}}$.
(3) Let $x \in \overline{A_{\mathcal{G}}}$. Then, $A_{\mathcal{G}} \cap N_x \neq \emptyset$, for each $N_x \in \mathcal{U}_x$. It follows that $\overline{U} \cap N_x \neq \emptyset$, for any $U \in \mathcal{G}$ and for all N_x. Observe that $\overline{U} \in \mathcal{G}$, for each $U \in \mathcal{G}$. Since $\mathcal{G}$ is an ultrafilter, each N_x belongs to $\mathcal{G}$. Thus x is a limit point of $\mathcal{G}$, i.e., $x \in A_{\mathcal{G}}$. ■

A principal ultrafilter $\dot{x}$ on X converges to x. Indeed, every set containing x must be in $\dot{x}$. This applies to all neighbourhoods of x.

Proposition 1.1.17. *Let X be a topological space. Then for each $x \in X$, $A_{\dot{x}} = \overline{\{x\}}$.*

Proof: Let $y \in A_{\dot{x}}$. Then $N_y \in \dot{x}$ for each $N_y \in \mathcal{U}_y$. Clearly, $x \in N_y$, i.e., $y \in \overline{\{x\}}$. The other way follows from statement 2 of Proposition 1.1.16. ■

We explore a behaviour of ultrafilters in compact and Hausdorff spaces.

Theorem 1.1.18. *[Bo, Wi] Assume X is a topological space and $\mathcal{G}$ an ultrafilter on X. Then the following hold:*

1. *X is Hausdorff if and only if $A_{\mathcal{G}}$ has at most one element.*

2. *X is compact if and only if $A_{\mathcal{G}}$ has at least one element.*

Proof: (1) [Wi] Suppose that X is Hausdorff but $A_{\mathcal{G}} = \{x, y\}$ where $x \neq y$. Then $N_x, N_y \in \mathcal{G}$, for each N_x and for every N_y. So $N_x \cap N_y \neq \emptyset$, for all N_x and for each N_y, which is impossible since X is Hausdorff, so we must then have $x = y$. Conversely, suppose that X is not Hausdorff. Then there are points $x, y \in X$, $x \neq y$, such that $N_x \cap N_y \neq \emptyset$, for each N_x and any N_y. Now, the collection $\{N_x \cap N_y : N_x \in \mathcal{U}_x, N_y \in \mathcal{U}_y\}$ is a filterbase for some filter $\mathcal{F}$. Extend this filter to an ultrafilter $\mathcal{G}$. Therefore x and y are both limit points of $\mathcal{G}$. Hence $A_{\mathcal{G}}$ has more than 1 elements.

(2) Suppose that $A_{\mathcal{G}}$ is empty. Then, for each $x \in X$, $x \notin A_{\mathcal{G}}$, that is, $x \notin \bigcap\{\overline{U} : U \in \mathcal{G}\}$. Therefore $x \in \bigcup_{U \in \mathcal{G}}(X \backslash \overline{U})$. Now,

$$\bigcup_{U \in \mathcal{G}}(X \backslash \overline{U}) = \bigcup_{x \in X}\{x\} = X$$

Put $\mathcal{C} = \{X \backslash \overline{U} : U \in \mathcal{G}\}$. Since X is compact, there are $U_1, U_2, ., U_n \in \mathcal{G}$ such that $\bigcup_{i=1}^{n}(X \backslash \overline{U_i}) = X$. Therefore $\bigcap_{i=1}^{n} \overline{U_i} = \emptyset$, which implies $\bigcap_{i=1}^{n} U_i = \emptyset$. This is impossible because $\mathcal{G}$ is an ultrafilter and ultrafilters are closed under finite intersections. Hence $A_{\mathcal{G}}$ is not empty.

[Wi] Conversely, suppose that $\mathcal{J} = \{U_i : i \in I\}$ is an open cover of X with no finite subcover. Then $\bigcap_{i=1}^{n}(X \backslash U_i) \neq \emptyset$. The collection $\{(X \backslash U_i) : 1 \leq i \leq n\}$ is a filterbase for some filter $\mathcal{F}$. Extend this filter to an ultrafilter $\mathcal{G}$. By hypothesis, there is $x \in X$ such that $x \in A_{\mathcal{G}}$. It is clear that $x \in U$ for some $U \in \mathcal{J}$. Since $U \in \mathcal{U}_x$, $U \in \mathcal{G}$. But, by construction, $X \backslash U \in \mathcal{F} \subseteq \mathcal{G}$. It is impossible to have both U and $X \backslash U$ belonging to $\mathcal{G}$, so we have a contradiction. Thus $\mathcal{J}$ must have a finite subcover. ■

The following corollary is an immediate result of the preceding theorem.

Corollary 1.1.19. *[Bo] A topological space X is compact Hausdorff if and only if $A_{\mathcal{G}}$ has exactly one element.*

Example 1.1. B. Example of a Hausdorff but non-compact space: Consider an infinite space X equipped with T_{dis}. Let $\mathcal{G}$ be a free ultrafilter on X. If $A_{\mathcal{G}} \neq \emptyset$, then there is $x \in A_{\mathcal{G}}$ such that $N_x \in \mathcal{G}$ for each $N_x \in \mathcal{U}_x$. But $\mathcal{F}_r \subseteq \mathcal{G}$, so by Proposition 1.1.12, each N_x belongs to $\mathcal{F}_r$. Since X is discrete, we have that $\{x\} \in \mathcal{F}_r$, i.e., $X \backslash \{x\}$ is finite, which is impossible because X is infinite. Thus $A_{\mathcal{G}} = \emptyset$. This shows that X is not compact, but Hausdorff.

We close this section by looking at how ultrafilters behave in mappings.

Proposition 1.1.20. *[Wi] Let $f : X \longrightarrow X'$ be a function between two sets X and X', and $\mathcal{G}$ an ultrafilter on X. Then the image of $\mathcal{G}$ under f is an ultrafilter on X'.*

Proof: Let $\mathcal{G}$ be an ultrafilter on X. If $\mathcal{F}$ is a filter on X' such that $f(\mathcal{G}) \subseteq \mathcal{F}$, but $f(\mathcal{G}) \neq \mathcal{F}$, then there is $A \in \mathcal{F}$ such that $A \notin f(\mathcal{G})$. Therefore $f^{-1}(A) \notin \mathcal{G}$. By hypothesis, $X \backslash (f^{-1}(A)) \in \mathcal{G}$. Using the fact that $f(\mathcal{G}) \subseteq \mathcal{F}$, which implies $f^{-1}(f(\mathcal{G})) \subseteq f^{-1}(\mathcal{F})$, it follows that $X \backslash (f^{-1}(A)) \in f^{-1}(\mathcal{F})$. Therefore, $f(X \backslash (f^{-1}(A))) \in f(f^{-1}(\mathcal{F})) \subseteq \mathcal{F}$, implying $f(X \backslash (f^{-1}(A))) \in \mathcal{F}$. But then $f(X \backslash (f^{-1}(A))) \subseteq X' \backslash A$, so $X' \backslash A \in \mathcal{F}$, which is impossible, otherwise $\emptyset \in \mathcal{F}$. Thus $\mathcal{H} = f(\mathcal{G})$ and $f(\mathcal{G})$ is an ultrafilter on X'. ■

1.2 Construction of the Ultrafilter Space

In this section we present a construction of the ultrafilter space of an arbitrary topological space, as it was developed in [Sa].

For a topological space X, let $\mathcal{U}(X)$ denote the set of all ultrafilters on X. We would like to consider the set $\mathcal{U}(X)$ as a topological space, and its elements as points in that space. Thus, it would be natural to henceforth denote points of $\mathcal{U}(X)$ (which happen to be ultrafilters on X) by lowercase letters such as p, q, etc.

Definition 1.2.1. *[Sa] Let X be a topological space and suppose that $A \subseteq X$. Then we define $A^* = \{p \in \mathcal{U}(X) : A \in p\}$.*

Before we construct a topology on $\mathcal{U}(X)$, we outline the following properties of sets A^*.

Proposition 1.2.2. *[Sa] For a topological space X, the following statements hold:*

1. $\emptyset^* = \emptyset$.

2. $X^* = \mathcal{U}(X)$.

3. $A^* \subseteq B^*$ *if and only if* $A \subseteq B$.

4. $(A \cup B)^* = A^* \cup B^*$.

5. $(A \cap B)^* = A^* \cap B^*$.

6. $A \cap B = \emptyset$ *if and only if* $(A \cap B)^* = \emptyset$.

Proof: (1) We have that $\emptyset^* = \{p \in \mathcal{U}(X) : \emptyset \in p\} = \emptyset$.
(2) Each $p \in X^*$ automatically belongs to $\mathcal{U}(X)$. Conversely, let $p \in \mathcal{U}(X)$. Since p is an ultrafilter, we have that $X \in p$. Therefore $p \in X^*$. Hence $X^* = \mathcal{U}(X)$, as required.
(3) Let $p \in A^*$, then $A \in p$. It follows that $B \in p$, since $A \subseteq B$ and p is an ultrafilter. Hence $p \in B^*$. Conversely, let $x \in A$. Then $\dot{x} \in A^*$, which, by hypothesis, implies that $\dot{x} \in B^*$. Thus $x \in B$.
(4) Choose $p \in (A \cap B)^*$. Then, $A \cap B \in p$. We know that both A and B are supersets of $A \cap B$, so they both belong to p. Therefore, $p \in A^*$ and $p \in B^*$. Thus $p \in A^* \cap B^*$. Conversely, let $p \in A^* \cap B^*$. Therefore $p \in A^*$ and $p \in B^*$. Now, $A \in p$ and $B \in p$. Since p is an ultrafilter and $A \cap B \in p$, it follows that $p \in (A \cap B)^*$. Hence $(A \cap B)^* = A^* \cap B^*$.
(5) If $p \in (A \cup B)^*$, then $A \cup B \in p$. Since p is an ultrafilter, it follows that $A \in p$ or $B \in p$. Therefore, $p \in A^*$ or $p \in B^*$, implying that $p \in A^* \cup B^*$. Conversely, if $p \in A^* \cup B^*$, then $p \in A^*$ or $p \in B^*$. But $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so in either case, $A \cup B \in p$. By (3), $p \in (A \cup B)^*$.
(6) Let $A, B \subseteq X$ with $A \cap B = \emptyset$. If $(A \cap B)^* \neq \emptyset$, then there is $p \in \mathcal{U}(X)$ such that $p \in (A \cap B)^*$. It is clear that $(A \cap B) \in p$. But then $A \cap B = \emptyset$, so $\emptyset \in p$, which is impossible

since p is an ultrafilter. Thus $(A \cap B)^* = \emptyset$. On the other hand, if $(A \cap B)^* = \emptyset$, then no $p \in \mathcal{U}(X)$ contains $A \cap B$. Therefore $A \cap B = \emptyset$. ■

Proposition 1.2.3. *For a topological space X, $\mathcal{U}(X)\backslash A^* = (X\backslash A)^*$.*

Proof: Let $p \in \mathcal{U}(X)\backslash A^*$, then $p \notin A^*$. Clearly, $A \notin p$ and $X\backslash A \in p$. Therefore $p \in (X\backslash A)^*$. Conversely, if $p \in (X\backslash A)^*$, then $X\backslash A \in p$. So, $A \notin p$. Thus $p \notin A^*$. Therefore $p \in \mathcal{U}(X)\backslash A^*$. Hence $\mathcal{U}(X)\backslash A^* = (X\backslash A)^*$. ■

Next, we endow the set $\mathcal{U}(X)$ with a topology.

Proposition 1.2.4. *[Sa] Let X be a topological space. The collection $\mathcal{B} = \{G^* : G \in T\}$ is a base for some topology on $\mathcal{U}(X)$.*

Proof: Considering the collection $\mathcal{B} = \{G^* : G \in T\}$, it can easily be seen that $\bigcup_{G^* \in \mathcal{B}} G^* = \mathcal{U}(X)$ since $\mathcal{U}(X) = X^*$. Furthermore, if $B_1^*,\ B_2^* \in \mathcal{B}$, with $p \in B_1^* \cap B_2^*$, then by statement 5 of Proposition 1.2.2, $p \in (B_1 \cap B_2)^*$. Because $B_1^*,\ B_2^* \in \mathcal{B}$ implies $B_1, B_2 \in T$ and T being a topology implies $A \cap B \in T$, it follows that $(B_1 \cap B_2)^* \in \mathcal{B}$. Set $B_3 = B_1 \cap B_2$, we have $p \in B_3^*$ and $B_3 \in T$, which implies $B_3^* \in \mathcal{B}$ and $B_3^* \subseteq B_1^* \cap B_2^*$. Thus $\mathcal{B} = \{G^* : G \in T\}$ is a base for some topology on $\mathcal{U}(X)$. ■

Remark: We shall use $\mathcal{U}(T)$ to denote the topology generated by the collection $\mathcal{B} = \{G^* : G \in T\}$ and call $(\mathcal{U}(X), \mathcal{U}(T))$ the ultrafilter space of a topological space X. We shall only use $\mathcal{U}(X)$ if the underlying topology is clear from the context.

Example 1.2. A. Consider the set $X = \{a, b, c\}$ equipped with T_{dis}. The ultrafilters on X are: $p = \{\{a\}, \{a, b\}, \{a, c\}, X\}$, $q = \{\{b\}, \{a, b\}, \{b, c\}, X\}$ and $r = \{\{c\}, \{a, c\}, \{b, c\}, X\}$. The collection $\mathcal{U}(X) = \{p, q, r\}$ together with

$$\mathcal{U}(T_{dis}) = \{\{a\}^*, \{b\}^*, \{c\}^*, \{a, b\}^*, \{b, c\}^*, \{a, c\}^*, X^*, \emptyset\}$$

where $\{a\}^* = \{p\}$, $\{b\}^* = \{q\}$, $\{c\}^* = \{r\}$, $\{a, b\}^* = \{p, q\}$, $\{a, c\}^* = \{p, r\}$, $\{b, c\}^* = \{q, r\}$, $X^* = \mathcal{U}(X)$ and $\emptyset^* = \emptyset$, form the ultrafilter space of X.

Example 1.2. B. When X is infinite and discrete, then X has a free ultrafilter, say q. Observe that $\{q\} \notin \mathcal{U}(T_{dis})$ since there is no basic open set containing q. So, $\mathcal{U}(T)$ need not be discrete.

Example 1.2. C. Endow a set X with T_{tr}. If $\mathcal{G} \in \mathcal{U}(T_{tr})$ and $\mathcal{G} \neq \emptyset$, then we can choose $p \in \mathcal{G}$. Therefore, we can find $G^* \in \mathcal{B}$ such that $p \in G^* \subseteq \mathcal{G}$. Therefore, $G \in p$, so $G \neq \emptyset$. Since $G \in T$, we have that $G = X$, which implies that $G^* = X^* = \mathcal{U}(X) = \mathcal{G}$. Therefore, $\mathcal{U}(T_{tr})$ is trivial.

1.3 Properties of the Ultrafilter Space

In this section we look at some basic properties of the ultrafilter space $\mathcal{U}(X)$ of a topological space X.

Proposition 1.3.1. *[Sa] Let X be a topological space. In $\mathcal{U}(X)$, every basic open set of the form G^*, where $G \in T$, is compact.*

Proof: [Sa] Let G^* be an open set in $\mathcal{U}(X)$, where $G \in T$. Consider a family $\mathcal{F} = \{F_i^* : X \backslash F_i \in T\}$ such that $\{F_i^* \cap G^* : i \in I\}$ has the FIP. Therefore, $(\bigcap_{i=1}^n F_i)^* \cap G^* = ((\bigcap_{i=1}^n F_i) \cap G)^* \neq \emptyset$. It follows from statement 6 of Proposition 1.2.2 that $(\bigcap_{i=1}^n F_i) \cap G \neq \emptyset$. So, the collection $\{F_i \cap G : i \in I\}$ has the FIP. The collection of sets of the form $F_i \cap G$ form a filterbase for some filter $\mathcal{F}$ on X. Extend this filter to an ultrafilter $p \in \mathcal{U}(X)$. Therefore $F_i \cap G \in p$, for all $i \in I$. So, $p \in F_i^*$ and $G \in p$, for each i. Hence $p \in \bigcap_i (F_i)^* \cap G^*$. ■

Proposition 1.3.2. *Suppose that X is a topological space. Then compact open subsets of $\mathcal{U}(X)$ form a basis for $\mathcal{U}(T)$.*

Proof: Let $\mathcal{G} \in \mathcal{U}(T)$ and $p \in \mathcal{G}$. Clearly, there is B^*, where $B \in T$, such that $p \in B^* \subseteq \mathcal{G}$. Because every basic open subset of $\mathcal{U}(X)$ is compact (Proposition 1.3.1), it follows that B^* is compact and open. ■

The ultrafilter space $\mathcal{U}(X)$ is locally compact.

Definition 1.3.3. *[GHK$^+$03] A topological space X is locally compact if for each $x \in X$ and each N_x, there is an open set H and compact set Q such that $x \in H \subseteq Q \subseteq N_x$.*

Examples of locally compact sets include compact Hausdorff spaces. In fact, if $x \in X$ and we consider N_x, then using the fact that every compact Hausdorff space is regular, $x \notin X \backslash N_x$ implies the existence of disjoint open sets G and H such that $x \in G$ and $X \backslash N_x \subseteq H$. Therefore $x \in G \subseteq (X \backslash H) \subseteq N_x$, and since every closed subset of a Hausdorff space is compact, we have that $X\backslash H$ is compact.

Proposition 1.3.4. *[Sa] Let X be a topological space. Then $\mathcal{U}(X)$ is locally compact.*

Proof: Let $p \in \mathcal{U}(X)$, and let $\mathcal{O} \in \mathcal{U}(T)$ such that $p \in \mathcal{O}$. Then, by Proposition 1.3.2, there is $G \in T$ such that $p \in G^* \subseteq \mathcal{O}$, where G^* is compact and open. ■

The ultrafilter space is also compact. We show this in two steps.

Lemma 1.3.5. *Let X be a topological space. If $\mathcal{F}$ is an ultrafilter on $\mathcal{U}(X)$, then $\{A \subseteq X : A^* \in \mathcal{F}\} \in \mathcal{U}(X)$.* *Proof:* Let $p = \{A \subseteq X : A^* \in \mathcal{F}\}$. We start by showing that p is a filter on X. Observe that $\emptyset \notin p$ since $\emptyset = \emptyset^* \notin \mathcal{F}$. Let $A, B \in p$. Then $A^*, B^* \in \mathcal{F}$. Therefore $A^* \cap B^* = (A \cap B)^* \in \mathcal{F}$. Thus $A \cap B \in p$. Finally, let $A \in p$ and $A \subseteq B \subseteq X$. Then $A^* \in \mathcal{F}$. It is clear that $B^* \in \mathcal{F}$. Thus $B \in p$. Now, let $A, B \subseteq X$ such that $A \cup B \in p$. Then $(A \cup B)^* = A^* \cup B^* \in \mathcal{F}$. Therefore, $A^* \in \mathcal{F}$ or $B^* \in \mathcal{F}$. Thus $A \in p$ or $B \in p$, making $p \in \mathcal{U}(X)$. ■

Proposition 1.3.6. *[Sa] Let X be a topological space. Then $\mathcal{U}(X)$ is compact.*

Proof: Let $\mathcal{F}$ be an ultrafilter on $\mathcal{U}(X)$. We show that $\{A \subseteq X : A^* \in \mathcal{F}\} \in A_{\mathcal{F}}$. Let $p = \{A \subseteq X : A^* \in \mathcal{F}\}$. Suppose that there is $N_p \notin \mathcal{F}$. Then there is $B \in T$ such that $p \in B^* \subseteq N_p$ but $B^* \notin \mathcal{F}$. So, $(X\backslash B)^* \in \mathcal{F}$. Therefore $X\backslash B \in p$ and $B \in p$, which is a contradiction. Thus $N_p \in \mathcal{F}$. Hence $\mathcal{U}(X)$ is compact. ■

The ultrafilter space is seldom separated. The following counter-example shows that $\mathcal{U}(X)$ is not T_0 for some topological space X.

Example 1.3. A. Consider a space (X, T_{tr}). We have already shown in Example 1.2. C. that $\mathcal{U}(T_{tr})$ is a trivial topology. Thus $\mathcal{U}(X)$ is not T_0.

However, $\mathcal{U}(X)$ is Hausdorff provided that X is discrete.

Proposition 1.3.7. *[Sa] If X is discrete, then $\mathcal{U}(X)$ is Hausdorff.*

Proof: Suppose that $p, q \in \mathcal{U}(X)$ and $p \neq q$. Then, there is a non-empty set $A \subseteq X$ such that $A \in p$ and $A \notin q$. It is clear that $X\backslash A \in q$. But X is discrete, so $X\backslash A$ and A are open in X. Therefore, $p \in A^*$ and $q \in (X\backslash A)^*$ where both A^* and $(X\backslash A)^*$ are open in $\mathcal{U}(X)$. It follows from the statement 6 of Proposition 1.2.2. that $A^* \cap (X\backslash A)^* = (A \cap (X\backslash A))^* = \emptyset$. ■

For any topological space X and $x \in X$, let $\eta_X : X \longrightarrow \mathcal{U}(X)$ be a map defined by $\eta_X(x) = \dot{x}$. It is easy to see that η_X is a function. In fact, for any two elements x, y of X, if $\eta_X(x) \neq \eta_X(y)$, then there is $A \subseteq X$ such that $A \in \eta_X(x)$ but $A \notin \eta_X(y)$. Therefore, $x \in A$ and $y \notin A$. Thus $x \neq y$.

We close this section by investigating the map η_X and further consider a relationship between two ultrafilter spaces.

Proposition 1.3.8. *Let X be a topological space. Then $\eta_X : X \longrightarrow \mathcal{U}(X)$ is injective.*

Proof: Let $x, y \in X$ and suppose that $\eta_X(x) = \eta_X(y)$. Clearly $\{x\} \in \eta_X(x)$ which implies $\{x\} \in \eta_X(y)$. It follows that $y \in \{x\}$, which implies $x = y$. ■

Observe that, for each injective continuous map $f : X \longrightarrow X'$, $\overline{f(A)} = \overline{f(B)}$ implies $\overline{A} = \overline{B}$ for each $A, B \subseteq X$. Indeed, if $x \in \overline{A}$, then $f(x) \in f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(B)}$, so $x \in \overline{f^{-1}(f(B))} = \overline{B}$. Therefore $\overline{A} \subseteq \overline{B}$. Similar calculation will show that $\overline{B} \subseteq \overline{A}$. Thus $\overline{A} = \overline{B}$. As a result, η_X has the following property.

Proposition 1.3.9. *Let X be a topological space. If for each $x, y \in X$, $\overline{\{\eta_X(x)\}} = \overline{\{\eta_X(y)\}}$, then $\overline{\{x\}} = \overline{\{y\}}$.*

The map η_X is not onto for an infinite space X. This is because an infinite set X has free ultrafilters which, in general, are not generated by a single element of X.

We show that η_X is a continuous function.

Lemma 1.3.10. *Let X be a topological space. Then, for each $G \subseteq X$, $G = \eta_X^{-1}(G^*)$.*

Proof: Let $x \in G$, then $G \in \eta_X(x)$, implying that $\eta_X(x) \in G^*$. Thus $x \in \eta_X^{-1}(G^*)$. On the other hand, if $x \in \eta_X^{-1}(G^*)$, then $\eta_X(x) \in G^*$ implying that $G \in \eta_X(x)$. Thus $x \in G$. Hence $G = \eta_X^{-1}(A^*)$. ■

Proposition 1.3.11. *[Sa] Let X be a topological space. Then η_X is continuous.*

Proof: Let $G^* \in \mathcal{U}(T)$. Because $\eta_X^{-1}(G^*) = G$ and $G \in T$, it follows that $\eta_X^{-1}(G^*) \in T$. ■

We note that the equality $\eta_X(A) = A^*$ doesn't always hold.

Example 1.3. B. Consider a free ultrafilter p on X. Let A be a cofinite subset of X. Then $A \in p$, so that $p \in A^*$. But p is not a principal ultrafilter on X, so, $p \neq \eta_X(x)$ for any $x \in A$.

Thus $p \notin \eta_X(A)$.

We show that η_X is an embedding.

Definition 1.3.12. *[Wi] For topological spaces X and X', X is embedded in X' by a function $f : X \longrightarrow X'$ if f is a homeomorphism between X and some subspace of X'.*

Lemma 1.3.13. *Suppose that X is a topological space. Then $\eta_X(A) = \eta_X(X) \cap A^*$ for each $A \subseteq X$.*

Proof: Let $A \subseteq X$ and $p \in \eta_X(A)$. Then, there is $x \in A$ such that $\eta_X(x) = p$. It is clear that $A \in \eta_X(x)$ which implies $\eta_X(x) \in A^*$. Therefore, $p \in A^*$. Thus $\eta_X(A) \subseteq \eta_X(X) \cap A^*$. Conversely, Let $p \in \eta_X(X) \cap A^*$. Since $\eta_X(X) = \{\eta_X(x) : x \in X\}$, we have that $p = \eta_X(x)$ for some $x \in X$. But also $p \in A^*$ which implies $A \in p$, so $A \in \eta_X(x)$. By definition of $\eta_X(x)$, $x \in A$. Therefore, $p = \eta_X(x) \in \eta_X(A)$. ■

Proposition 1.3.14. *[Sa] Let X be a topological space. Then $\eta_X : X \longrightarrow \mathcal{U}(X)$ is an embedding.*

Proof: Observe that $\eta_X' : X \longrightarrow \eta_X(X)$ induced by η_X is bijective and continuous. To show that $\eta_X' : X \longrightarrow \eta_X(X)$ is open, let $A \in T$. We must show that $\eta_X'(A)$ is open in $\eta_X(X)$. Observe that $\eta_X(A) = \eta_X'(A)$. Because $\eta_X(A) = \eta_X(X) \cap A^*$ and A^* is open in $\mathcal{U}(X)$, we have that $\eta_X(A)$ is open in $\eta_X(X)$. ■

Proposition 1.3.15. *Let X be a topological space. Then $\eta_X(X)$ is dense in $\mathcal{U}(X)$.*

Proof: It suffices to show that $\mathcal{U}(X) \subseteq \overline{\eta_X(X)}$. Let $p \in \mathcal{U}(X)$ and choose N_p. Then there is $B \in T$ such that $p \in B^* \subseteq N_p$. Therefore, $B \in p$. Clearly, $B \neq \emptyset$, so there is $x \in B$ such that $\eta_X(x) \in B^*$. We have that $B^* \cap \eta_X(X) \neq \emptyset$. Thus $N_p \cap \eta_X(X) \neq \emptyset$. ■

Not only is $\eta_X(X)$ a dense subset of $\mathcal{U}(X)$, it is also dense in the patch topology of $\mathcal{U}(X)$. To prove this, we introduce the following useful concepts and results.

Definition 1.3.16. *[No] Let X be a topological space. A subset K of X is saturated if $K = \bigcap\{G \subseteq X : G \in T \text{ and } K \subseteq G\}$.*

Proposition 1.3.17. *Let X be a topological space. If $\mathcal{K} \in (\mathcal{CS})_{\mathcal{U}(X)}$ and $p \notin \mathcal{K}$, then there is G^*, where $G \in T$, such that $\mathcal{K} \subseteq G^*$ and $p \notin G^*$.*

Proof: [Sa] Let $\mathcal{K} \in (\mathcal{CS})_{\mathcal{U}(X)}$, and $p \notin \mathcal{K}$. Then there is an open set which contains $\mathcal{K}$ but not p. By compactness of $\mathcal{K}$, there are open sets $G_i, 1 \leq i \leq n$, such that $\mathcal{K} \subseteq \bigcup_{i=1}^n G_i^* \subseteq \mathcal{U}(X)\backslash\{p\}$. Let $G = \bigcup_{i=1}^n G_i$. Then $\mathcal{K} \subseteq G^*$ and $p \notin G^*$. ■

Definition 1.3.18. *[Sa] For a topological space X, the co-compact topology T_K has as a basis for the open sets, sets of the form $X\backslash K$ where K is compact and saturated.*

Definition 1.3.19. *[GHK+03] The patch topology T_P on a topological space X is the smallest topology containing both the original topology and the co-compact topology.*

Proposition 1.3.20. *[Sa] For any topological space X, $\eta_X(X)$ is a dense subspace of $(\mathcal{U}(X),\mathcal{U}(T)_P)$.*

Proof: Let $p \in \mathcal{U}(X)$ and $\mathcal{G}$ be a neighbourhood of p in $(\mathcal{U}(X),\mathcal{U}(T)_P)$. Then, there is $\mathcal{H} \in \mathcal{U}(T)_P$ such that $p \in \mathcal{H} \subseteq \mathcal{G}$. It follows that there is $\mathcal{H}' \in \mathcal{U}(T) \cup \mathcal{U}(T)_K$ such that $p \in \mathcal{H}' \subseteq \mathcal{H}$. Now, $\mathcal{H}' \in \mathcal{U}(T)$ and $\mathcal{H}' \notin \mathcal{U}(T)_K$, or $\mathcal{H}' \notin \mathcal{U}(T)$ and $\mathcal{H}' \in \mathcal{U}(T)_K$, or $\mathcal{H}' \in \mathcal{U}(T)$ and $\mathcal{H}' \in \mathcal{U}(T)_K$.

Case 1: $\mathcal{H}' \in \mathcal{U}(T)$ and $\mathcal{H}' \notin \mathcal{U}(T)_K$: It follows from Proposition 1.3.15 that $\overline{\eta_X(X)}^P = \mathcal{U}(X)$.

Case 2: $\mathcal{H}' \notin \mathcal{U}(T)$ and $\mathcal{H}' \in \mathcal{U}(T)_K$: Observe that there is $\mathcal{Q}$, where $\mathcal{Q} = \mathcal{U}(X)\backslash\mathcal{K}$ for some $\mathcal{K} \in (\mathcal{CS})_{\mathcal{U}(X)}$, such that $p \in \mathcal{Q} \subseteq \mathcal{H}'$. Clearly, $p \notin \mathcal{K}$, so by Proposition 1.3.17, there is G^*, where $G \in T$, such that $\mathcal{K} \subseteq G^*$ and $p \notin G^*$. Thus $X\backslash G \in p$. Because $X\backslash G \neq \emptyset$, there is $x \in X\backslash G$ such that $\eta_X(x) \in (X\backslash G)^* = \mathcal{U}(X)\backslash G^* \subseteq \mathcal{U}(X)\backslash\mathcal{K} \subseteq \mathcal{H}'$. Hence $\mathcal{G} \cap \eta_X(X) \neq \emptyset$.

Case 3: $\mathcal{H}' \in \mathcal{U}(T)$ and $\mathcal{H}' \in \mathcal{U}(T)_K$: [Sa] Since $\mathcal{H}' \in \mathcal{U}(T)$, there is H^*, where $H \in T$, such that $p \in H^* \subseteq \mathcal{H}'$. Because $\mathcal{H}' \in \mathcal{U}(T)_K$, there is $\mathcal{Q}$, where $\mathcal{Q} = \mathcal{U}(X)\backslash\mathcal{K}$ for some $\mathcal{K} \in (\mathcal{CS})_{\mathcal{U}(X)}$, such that $p \in \mathcal{Q} \subseteq \mathcal{H}'$. Observe that $p \in \mathcal{H}'\backslash\mathcal{K}$. So, by Proposition 1.3.17, there is R^*, where $R \in T$, such that $\mathcal{K} \subseteq R^*$ and $p \notin R^*$. Hence $X\backslash R \in p$, so that $H \cap (X\backslash R) \neq \emptyset$ since $H \in p$. Choose $x \in H\backslash R$. Then $\eta_X(x) \in H^*\backslash R^* \subseteq H^*\backslash \mathcal{K} \subseteq \mathcal{H}'$. Thus $\mathcal{G} \cap \eta_X(X) \neq \emptyset$. Hence $\overline{\eta_X(X)}^P = \mathcal{U}(X)$. ■

We show that the converse of Proposition 1.3.7 is true. Enroute to that, we introduce the following lemma.

Lemma 1.3.21. *[Sa] If X is a finite topological space, then $\eta_X : X \longrightarrow \mathcal{U}(X)$ is a homeomorphism.*

Proof: Observe that each $p \in \mathcal{U}(X)$ is a principal ultrafilter. Thus η_X is onto and $\eta_X(A) = A^*$, for all $A \subseteq X$, making η_X open. Hence η_X is a homeomorphism. ■

Proposition 1.3.22. *[Sa] Let X be a topological space. If $\mathcal{U}(X)$ is T_1, then X is discrete.*

Proof: [Sa] If X is finite and $\mathcal{U}(X)$ is T_1, then by Lemma 1.3.21, X is homeomorphic to $\mathcal{U}(X)$ under η_X. So, X is T_1. Because every finite, T_1 space is discrete, it follows that X is discrete. Similar argument applies when X is finite and $\mathcal{U}(X)$ is T_1. Furthermore, suppose that $\mathcal{U}(X)$ is T_1 but X is infinite and not discrete, then there is a non-empty subset A of X such that $A \in T$ and $X \backslash A \notin T$. It is clear that $A \neq \overline{A}$, i.e., there exists $x \in X$ which is in $\overline{A}$ but not in A. Therefore, $(N_x \backslash \{x\}) \cap A \neq \emptyset$, for each $N_x \in \mathcal{U}_x$. The sets of the form $(N_x \backslash \{x\}) \cap A \neq \emptyset$ form a filterbase for some filter $\mathcal{F}$ which contains all such sets. Extend this filter to an ultrafilter q. Observe that $\dot{x} \neq q$. We show that either $\dot{x} \in H^*$ or $q \in G^*$ for all open neighbourhoods G^*, H^* of $\dot{x}$ and q, respectively. This will show that $\mathcal{U}(X)$ is not a T_1 space, contradicting the hypothesis. Let $\dot{x} \in G^*$ where $G \in T$, and let $q \in H^*$ where $H \in T$, then $G \in \mathcal{U}_x$. Because $\mathcal{U}(X)$ is locally compact, there is $Q \in T$ and compact set $\mathcal{K}$ such that $\dot{x} \in Q^* \subseteq \mathcal{K} \subseteq G^*$. If $q \in \mathcal{K}$, then $q \notin Q^*$, which implies $Q \notin q$. Therefore $X \backslash Q \in q$. But then $Q \in \dot{x}$, so $(Q \backslash \{x\}) \cap A \neq \emptyset$. Therefore $(X \backslash Q) \cap (Q \backslash \{x\}) \cap A \neq \emptyset$ which is impossible. Thus $q \in \mathcal{K}$ so that $q \in G^*$. Therefore $\mathcal{U}(X)$ is not T_1. ■

Proposition 1.3.23. *[Sa] Suppose that X and X' are topological spaces, $p \in \mathcal{U}(X)$, and $f : X \longrightarrow X'$ is a continuous function. Then*

1. *$\mathcal{U}(f)(p) = \{A \subseteq X' : f^{-1}(A) \in p\}$ is an ultrafilter on X'.*

2. *$\mathcal{U}(f)$ is continuous.*

Proof: The proof for (1) follows from Proposition 1.1.17. We show that $\mathcal{U}(f)$ is continuous. Let $A \in T'$. Then A^* is open in $\mathcal{U}(X')$. Choose $p \in (\mathcal{U}(f))^{-1}(A^*)$. Therefore, $\mathcal{U}(f)(p) \in A^*$, which implies $A \in \mathcal{U}(f)(p)$, i.e., $p \in (f^{-1}(A))^*$ where $(f^{-1}(A))^*$ is open in $\mathcal{U}(X)$. Let $q \in (f^{-1}(A))^*$. Then $f^{-1}(A) \in q$, so $A \in \mathcal{U}(f)(q)$, i.e., $\mathcal{U}(f)(q) \in A^*$. We then have $q \in (\mathcal{U}(f))^{-1}(A^*)$. Therefore, $(f^{-1}(A))^* \subseteq (\mathcal{U}(f)^{-1})(A^*)$. Thus $\mathcal{U}(f)$ is continuous. ■

Proposition 1.3.24. *[Sa] Suppose that $f : (X, T) \longrightarrow (X', T')$ and $g : (X', T') \longrightarrow (X'', T'')$ are continuous functions, then*

1. $\mathcal{U}(g \circ f) = \mathcal{U}(g) \circ \mathcal{U}(f)$.

2. $\mathcal{U}(id_X) = id_{\mathcal{U}(X)}$.

Proof: (1) Let $q \in \mathcal{U}(X)$. Observe that

$$\begin{aligned} A \in \mathcal{U}(g)(\mathcal{U}(f)(q)) &\Leftrightarrow g^{-1}(A) \in \mathcal{U}(f)(q) \\ &\Leftrightarrow f^{-1}(g^{-1}(A)) \in q \\ &\Leftrightarrow (g \circ f)^{-1} \in q \\ &\Leftrightarrow A \in \mathcal{U}(g \circ f)(q) \end{aligned}$$

for each $A \subseteq X$.

(2) Consider the identity map id_X of a set X, then for each $p \in \mathcal{U}(X)$, $\mathcal{U}(id_X)(p) = \{A \subseteq X : (id_X)^{-1}(A) \in p\} = \{A \subseteq X : A \in p\} = p = id_{\mathcal{U}(X)}(p)$. ■

Proposition 1.3.25. *Let X and X' be topological spaces, and $f : X \longrightarrow X'$ be a continuous function. Then the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X' \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_{X'}} \\ \mathcal{U}(X) & \xrightarrow[\mathcal{U}(f)]{} & \mathcal{U}(X') \end{array}$$

Proof: We must show that $\eta_{X'} \circ f = \mathcal{U}(f) \circ \eta_X$. Let $x \in X$, then $\eta_{X'}(f(x)) = \{A \subseteq X' : f(x) \in A\}$. So,

$$\begin{aligned} \mathcal{U}(f)(\eta_X(x)) &= \{A \subseteq X' : f^{-1}(A) \in \eta_X(x)\} \\ &= \{A \subseteq X' : x \in f^{-1}(A)\} \\ &= \{A \subseteq X' : f(x) \in A\} \\ &= \eta_{X'}(f(x)). \blacksquare \end{aligned}$$

1.4 Retraction of the Ultrafilter Space

This section serves to construct a retraction of the ultrafilter space of a topological space X. We shall start by defining a retraction of a topological space, then give some properties of retractions, and later find a retraction of the ultrafilter space.

Definition 1.4.1. *[GL] A retract of a topological space X is a topological space X' such that there are two continuous maps $s : X' \longrightarrow X$ (the section) and $r : X \longrightarrow X'$ (the retraction) such that $r \circ s = id_{X'}$.*

One says that r is a retraction of X and s is a section of X.

Example 1.4. A. Consider a topological space X and a singleton set $\{y\} \subseteq X$. Define a map $r : X \longrightarrow \{y\}$ by $r(x) = y$. r is trivially continuous. Let $s : \{y\} \longrightarrow X'$ be an inclusion map (i.e., $s(y) = y$). Then, $r(s(y)) = r(y) = y = id_{\{y\}}$. Thus $\{y\}$ is a retract of X with s a section and r a retraction map.

We give some properties of retraction maps.

Proposition 1.4.2. *[GL] Every retraction $r : X \longrightarrow X'$ is surjective.*

Proof: [GL] Let s be some associated section. r is surjective, since $s(x)$ is an element y such that $r(y) = x$. ■

Proposition 1.4.3. *[AHS] Let $f : X \longrightarrow X'$, $g : X' \longrightarrow X''$ be continuous maps. If $g \circ f : X \longrightarrow X''$ is a retraction, then so is g.*

Proof: Given a continuous function h with $(g \circ f) \circ h = id_{X''}$, we have $g \circ (f \circ h) = id_{X''}$. Thus g is a retraction. ■

Proposition 1.4.4. *Let $e : X \longrightarrow X'$ and $r : X' \longrightarrow X$ be functions such that $r \circ e = 1_X$, $A \subseteq X$ and $B \subseteq X'$. Then:*

1. $e(A) \subseteq r^{-1}(A)$
2. $e^{-1}(B) \subseteq r(B)$

Proof: (1) Let $A \subseteq X$ and let $y \in e(A)$, then there is $x \in A$ such that $e(x) = y$. Therefore $r(e(x)) = r(y)$. But then $r(e(x)) = x$, so $x = r(y) \in A$. Thus $y \in r^{-1}(A)$.
(2) Let $B \subseteq X'$. If $x \in e^{-1}(B)$, then $e(x) \in B$, implying that $r(e(x)) \in r(B)$. Because $r(e(x)) = x$, $x \in r(B)$. ■

Corollary 1.4.5. *For any $A \subseteq X$, $A = e^{-1}(r^{-1}(A))$.*

Proof: Let $A \subseteq X$. It follows that $e^{-1}(r^{-1}(A)) \subseteq r(r^{-1}(A)) \subseteq A$. The converse follows since

$r(e(A)) = A$, implying that $A \subseteq (re)^{-1}(A)$. ■

We shall make use of Salbany stably compact spaces to construct a retraction of $\mathcal{U}(X)$. So, it is important to introduce the concept of Salbany stably compact spaces. We start by introducing supersober spaces.

Definition 1.4.6. *[Sa] A topological space X is supersober if it is compact and for every ultrafilter p on X, $A_p = \overline{\{x\}}$ for some $x \in X$.*

Example 1.4. B. Consider a space X equipped with T_{tr}. X is compact and for each ultrafilter p of X, $A_p = X$. But X is the closure of its singleton sets, so $A_p = \overline{\{x\}}$ for each $x \in X$. Thus X is supersober.

We define a Salbany stably compact space.

Definition 1.4.7. *[Sa] A locally compact and supersober space is called Salbany stably compact.*

Example 1.4. C. Every compact Hausdorff space is an example of a Salbany stably compact space. Indeed, each compact Hausdorff space X is locally compact and, for $p \in \mathcal{U}(X)$, $A_p = \{x\}$ for some unique $x \in X$. Because in a Hausdorff space singleton sets are closed sets, we have that $A_p = \overline{\{x\}}$, making X supersober.

The ultrafilter space of a topological space X is an example of a Salbany stably compact space which is not always Hausdorff.

Proposition 1.4.8. *[Sa] Let X be a topological space. Then $\mathcal{U}(X)$ is Salbany stably compact.*

Proof: [Sa] It is clear that $\mathcal{U}(X)$ is compact and locally compact. Let $\mathcal{F}$ be an ultrafilter of $\mathcal{U}(X)$. It follows that $p = \{A \subseteq X : A^* \in \mathcal{F}\}$ is a limit point of $\mathcal{F}$. It is easy to see that $\overline{\{p\}} \subseteq A_{\mathcal{U}}$. We must show that $A_{\mathcal{F}} \subseteq \overline{\{p\}}$. If $q \in A_{\mathcal{F}}$ but $q \notin \overline{\{p\}}$, then there exists $H \in T$ such that $q \in H^*$ and $H^* \cap \{p\} = \emptyset$, i.e. $p \notin H^*$. Therefore $p \in (X \backslash H)^*$ which implies $(X \backslash H) \in p$. Hence $(X \backslash H)^* \in \mathcal{F}$, which is a contradiction since $H^* \in \mathcal{F}$. Therefore $A_{\mathcal{F}} = \overline{\{p\}}$ for some $p \in \mathcal{U}(X)$. Thus $\mathcal{U}(X)$ is Salbany stably compact. ■

A retract of a Salbany stably compact space is Salbany stably compact.

Proposition 1.4.9. *[Sa] Let $e : X \longrightarrow X'$ be a continuous function and $r : X' \longrightarrow X$ be another continuous function such that $r \circ e = 1_X$. If X' is Salbany stably compact, then so is*

X.

Proof: X is locally compact: $[Ju, Sa]$ Let $x \in X$ and consider N_x. Then, there is $G \in T$ such that $x \in G \subseteq N_x$. It follows that $e(x) \in e(G) \subseteq r^{-1}(G)$. Since r is continuous, $r^{-1}(G)$ is an open neighbourhood of $e(x)$. So, by local compactness of X', there is an open set H and a compact set K in X' such that $e(x) \in H \subseteq K \subseteq r^{-1}(G)$. Observe that $r(e(x)) = x \in r(H) \subseteq r(K) \subseteq G$ and $x \in e^{-1}(H) \subseteq e^{-1}(K) \subseteq G \subseteq N_x$ (since $e^{-1}(r^{-1}(G)) = G$). Because $e^{-1}(K) \subseteq r(K)$, it follows that $x \in e^{-1}(H) \subseteq r(K) \subseteq G \subseteq N_x$. Clearly $r(K)$ is compact. Thus X is locally compact.

X is supersober: [Sa] Compactness of X follows since r is surjective and continuous. Let p be an ultrafilter on X. Then, by Proposition 1.3.23, $q = \{A \subseteq X' : e^{-1}(A) \in p\}$ is an ultrafilter on X'. Since X' is supersober, it follows that $A_q = \overline{\{x\}}$ for some $x \in X'$. We show that $A_p = \overline{\{r(x)\}}$. Let $y \in A_p$, then each $N_y \in p$, so $e(y) \in e(N_y) \in e(p)$. Since $e(N_y) \subseteq r^{-1}(N_y)$ and $e(p)$ is an ultrafilter on X' (by Proposition 1.1.17.), it follows that $r^{-1}(N_y) \in e(p)$. Thus $e(y) \in r^{-1}(N_y) \in e(p)$. Now, we have that $r^{-1}(N_y) \in q$ since $N_y = e^{-1}(r^{-1}(N_y)) \in p$. Therefore $e(y) \in A_q$, which implies $r(e(y)) \in r(\overline{\{x\}}) \subseteq \overline{r(\{x\})} = \overline{\{r(x)\}}$. Thus $y = r(e(y)) \in \overline{\{r(x)\}}$. On the other hand, if $y \in \overline{\{r(x)\}}$, then for each N_y, $r(x) \in N_y$. Now, $x \in r^{-1}(N_y)$. It follows that $r^{-1}(N_y) \in q$, since $x \in A_q$ and $r^{-1}(N_y) \in \mathcal{U}_x$. Therefore, $e^{-1}(r^{-1}(N_y)) \in p$. But then $N_y = e^{-1}(r^{-1}(N_y))$, so $N_y \in p$. Thus $y \in A_p$. ■

We construct a retraction of $\mathcal{U}(X)$.

Proposition 1.4.10. *[Sa] Let X be a Salbany stably compact space. Then there is a retraction $r_X : \mathcal{U}(X) \longrightarrow X$ satisfying $r_X \circ \eta_X = 1_X$.*

Proof: [Sa] Since X is supersober, for each $p \in \mathcal{U}(X)$, $A_p = \overline{\{x\}}$ for some $x \in X$. If $p = \eta_X(x)$ for some $x \in X$, then $A_p = \overline{\{x\}}$. So, for $p \in \eta_X(X)$, define $r_X : \mathcal{U}(X) \longrightarrow X$ by $r_X(p) = x$, where $p = \eta_X(x)$. For $p \in \mathcal{U}(X) \backslash \eta_X(X)$, let $r_X(p) = x$, where x is any element of X such that $\overline{\{x\}} = A_p$. Observe that r_X is a function and $r_X(\eta_X(x)) = r_X(p) = x$, for each $x \in X$. Thus $r_X \circ \eta_X = 1_X$. For continuity, let $A \in T$ and $p \in r_X^{-1}(A)$. Observe that $r_X(p) = x$ for some $x \in X$ such that $A_p = \overline{\{x\}}$. Now, since X is locally compact, there exists an open set $G \subseteq X$ and a compact set $K \subseteq X$ such that $r_X(p) \in G \subseteq K \subseteq A$. Because $r_X(p)$ is a limit point of p, we have that $G \in p$. Therefore $p \in G^*$.To show that $G^* \subseteq r_X^{-1}(A)$, let $t \in G^*$. Observe that

$r_X(t) = y$ for some $y \in X$ such that $A_t = \overline{\{y\}}$. Because $t \in G^* \subseteq K^*$, we have $K \in t$. Since K is compact, t converges to some point in K, say z, i.e., $z \in A_t$. Now, $K \cap A_t \neq \emptyset$ which implies that $A \cap A_t \neq \emptyset$. Therefore $A \cap \overline{\{y\}} \neq \emptyset$. But $y = r_X(t)$, so $A \cap \overline{\{r_X(t)\}} \neq \emptyset$. Now, $z \in A$ and $z \in \overline{\{r_X(t)\}}$. So, $r_X(t) \in A$ since A is a neighbourhood of z. Therefore $t \in r_X^{-1}(A)$. Thus r_X is continuous. Hence r_X is a retraction of $\mathcal{U}(X)$. ■

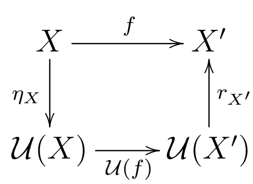
Because compact Hausdorff spaces are Salbany stably compact, we have the following result.

Proposition 1.4.11. *[Sa] If X is a compact Hausdorff space, then there is a unique retraction $r_X : \mathcal{U}(X) \longrightarrow X$ such that $r_X \circ \eta_X = 1_X$.*

Proof: Recall that a compact Hausdorff space is supersober and locally compact, so, it follows that X is Salbany stably compact. By Proposition 1.4.10, there is a retraction $r_X : \mathcal{U}(X) \longrightarrow X$ such that $r_X \circ \eta_X = 1_X$. Uniqueness follows from the fact that X is Hausdorff so, continuous functions agreeing on a dense subspace $\eta_X(X)$ of $\mathcal{U}(X)$ will also agree on $\mathcal{U}(X)$. ■

Proposition 1.4.12. *[Sa] Let $f : X \longrightarrow X'$ be a continuous function from a topological space X to a Salbany stably compact space X'. Then there exists a continuous map, not necessarily unique, $F : \mathcal{U}(X) \longrightarrow X'$ such that $F \circ \eta_X = f$.*

Proof: [Sa] We have $\mathcal{U}(f) : \mathcal{U}(X) \longrightarrow \mathcal{U}(X')$ and $\eta_X : X \longrightarrow \mathcal{U}(X)$. Let $r_{X'} : \mathcal{U}(X') \longrightarrow X'$ be such that $r_{X'} \circ \eta_{X'} = 1_{X'}$. We start by showing that the following diagram commutes:



That is, we must show that $r'_X \circ \mathcal{U}(f) \circ \eta_X = f$. It follows that $r'_X \circ \mathcal{U}(f) \circ \eta_X = r'_X \circ \eta'_X \circ f$, since $\mathcal{U}(f) \circ \eta_X = \eta'_X \circ f$ (by Proposition 1.3.26). Thus $r'_X \circ \mathcal{U}(f) \circ \eta_X = 1'_X \circ f = f$. Let $F = r_{X'} \circ \mathcal{U}(f)$. It follows that $F \circ \eta_X = f$. ■

Chapter 2

Compactifications and Separated Reflections

This chapter consists of two sections. The first section discusses the concept of compactification which is well-known in general topology. The second section discusses some ways of making non-separated spaces separated.

Some notions from category theory shall appear in some parts of the sections of this chapter as well as the other chapters after this one. We start by outlining them.

Definition 2.1. *[AHS] A category $\mathcal{A}$ is a quadruple $= (\mathcal{O}, hom, id, \circ)$ consisting of*

1. *a class $\mathcal{O}$, whose members are called $\mathcal{A}$-objects,*
2. *for each pair (A, B) of $\mathcal{A}$-objects, a set $hom_{\mathcal{A}}(A, B)$, whose members are called $\mathcal{A}$-morphisms from A to B,*
3. *for each $\mathcal{A}$-object A, a morphism $id_A : A \longrightarrow A$, called the $\mathcal{A}$-identity on $\mathcal{A}$, and*
4. *a composition law associating with each $\mathcal{A}$-morphism $f : A \longrightarrow B$ and each $\mathcal{A}$-morphism $g : B \longrightarrow C$ an $\mathcal{A}$-morphism $g \circ f : A \longrightarrow C$, called the composite of f and g, subject to the following conditions:*
   - *(a) composition is associative; i.e., for morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : C \longrightarrow D$, the equation $h \circ (g \circ f) = (h \circ g) \circ f$ holds,*
   - *(b) $\mathcal{A}$-identities act as identities with respect to composition; i.e., for $\mathcal{A}$-morphisms $f : A \longrightarrow B$, we have $id_B \circ f = f$ and $f \circ id_A = f$,*

(c) the sets $hom_{\mathcal{A}}(A, B)$ are pairwise disjoint.

Definition 2.2. *[AHS] A category $\mathcal{B}$ is said to be a subcategory of a category $\mathcal{A}$ provided that the following conditions are satisfied:*

1. *for each $B \in \mathcal{B}$, $B \in \mathcal{A}$,*
2. *for each $B, B' \in \mathcal{B}$, $hom_{\mathcal{B}}(B, B') \subseteq hom_{\mathcal{A}}(B, B')$,*
3. *for each $\mathcal{B}$-object B, the $\mathcal{A}$-identity on B is the $\mathcal{B}$-identity on B,*
4. *the composition law in $\mathcal{B}$ is the restriction of the composition law in $\mathcal{A}$ to the morphisms of $\mathcal{B}$.*

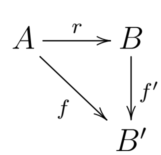
Definition 2.3. *[AHS] Objects A and B on a category $\mathcal{A}$ are said to be isormophic if there is an $\mathcal{A}-$morphism $f : A \longrightarrow B$ such that $g \circ f = id_A$ and $f \circ g = id_B$, for some $\mathcal{A}-$morphism $g : B \longrightarrow A$.*

When two objects are isomorphic, we say that they are the same in a sense that they share mathematical properties. In **Top**, isomorphism is equivalent to homeomorphism.

Definition 2.4. *[AHS] A functor $\mathcal{F}$ is a function between two categories $\mathcal{A}$ and $\mathcal{B}$ that assigns to each $\mathcal{A}$-object A a $\mathcal{B}$-object $\mathcal{F}(A)$, and to each $\mathcal{A}$-morphism $f : A \longrightarrow A'$ a $\mathcal{B}$-morphism $\mathcal{F}(f) : \mathcal{F}(A) \longrightarrow \mathcal{F}(A')$, in such a way that*

1. *$\mathcal{F}$ preserves composition, i.e., $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ whenever $f \circ g$ is defined, and*
2. *$\mathcal{F}$ preserves identity morphisms, i.e., $\mathcal{F}(id_A) = id_{\mathcal{F}(A)}$ for each $\mathcal{A}$-object A.*

Definition 2.5. *[AHS] Let $\mathcal{B}$ be a subcategory of a category $\mathcal{A}$. We say that $\mathcal{B}$ is a reflective subcategory of $\mathcal{A}$ if for each $A \in \mathcal{A}$, there is a $\mathcal{B}$-object B and a $\mathcal{B}-$morphism $r : A \longrightarrow B$ with the following universal property: for any $\mathcal{A}$-morphism $f : A \longrightarrow B'$ from A into some $\mathcal{B}$-object B', there exists a unique $\mathcal{B}$-morphism $f' : B \longrightarrow B'$ such that the triangle*



commutes.

The pair (B, r) is called the $\mathcal{B}-$reflection of A. We will use the name $\mathcal{B}$-reflection for the object B rather than for the pair (B, r). Furthermore, $\mathcal{B}$ is called a reflective subcategory of $\mathcal{A}$ provided that each $\mathcal{A}-$object has a $\mathcal{B}-$reflection.

Proposition 2.6. *[AHS] Let $\mathcal{B}$ be a reflective subcategory of a category $\mathcal{A}$. $\mathcal{B}$-reflections are unique up to an isomorphism.*

Proof: [AHS] If C is another $\mathcal{B}$ reflection of A with $\mathcal{B}$ reflection map r', then there is $k : B \longrightarrow C$ such that $k \circ r = r'$, similarly there is $k' : C \longrightarrow B$ such that $k' \circ r' = r$. Therefore $k \circ (k' \circ r') = (k \circ k') \circ r' = r' = id_C \circ r'$ so that by the uniqueness requirement in the definition of $\mathcal{B}$ reflection, $k \circ k' = id_C$. Analogously, one can see that $k' \circ k = id_B$ so that k is an isomorphism. ■

Remark: The preceding result is the reason why we speak about the $\mathcal{B}$-reflection of A rather than a $\mathcal{B}$-reflection.

Definition 2.7. *[AHS] Let $\mathcal{B}$ be a reflective subcategory of $\mathcal{A}$, and for each $\mathcal{A}$-object A let $r_A : A \longrightarrow B_A$ be a $\mathcal{B}$-reflection morphism. A functor $R : \mathcal{A} \longrightarrow \mathcal{B}$ such that*

1. *$R(A) = B_A$ for each $\mathcal{A}-$object A, and*

2. *for each $\mathcal{A}$-morphism $f : A \longrightarrow A'$, the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{r_A} & R(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle R(f)} \\ A' & \xrightarrow{r_{A'}} & R(A') \end{array}$$

is called a reflector for $\mathcal{B}$.

2.1 Compactifications

Compact spaces are one of the most important classes of topological spaces. There are many spaces which are not compact, and so the best we can hope for in general is that a given space can be embedded into a compact space. It is of interest to study the process of "compactification" that is, the process of embedding a given space as a dense subset of some compact space [Wi]. It is thus the main concern of this section.

For the sake of formality, we define compactification.

Definition 2.1.1. *[En] A compactification of a topological space X is a couple (X', c), where X' is a compact space and $c : X \longrightarrow X'$ is an embedding of X as a dense subspace of X'.*

Remark: For some compactifications, X is not a subspace of X'. When we encounter that, we shall identify X as $c(X)$. This is because X and $c(X)$ are homeomorphic and to topologists, they are the same. We shall give a remark whenever this identification has occurred.

Some authors, such as [Wi], require X' to be Hausdorff. This serves as a significant contribution to the theory of compact Hausdorff spaces which are also important classes of topological spaces. We omitted Hausdorffness because we are interested in making non-compact spaces compact, so we treat Hausdorff compactifications as particular types of compactifications.

Among well-known compactifications of topological spaces are the Alexandroff one-point compactification, the Wallman compactification and the Stone-Čech compactification. Unlike the Alexandroff one-point compactification and the Wallman compactification, the Stone-Čech compactification is of interest since it is functorial, [HMT]. In the first two subsections of this section, we shall introduce the Alexandroff one-point compactification and the Wallman compactification, respectively, and reserve a detailed construction for the Stone-Čech compactification in the third subsection.

2.1.1 Alexandroff One-point Compactification

The simplest sort of compactification of a topological space is made by adjoining a single point which doesn't belong to that space. This procedure is familiar in analysis, for in function theory the complex sphere is constructed by adjoining a single point, ∞, to the Euclidean plane and specifying that the neighbourhoods of ∞ are the complements of bounded subsets of the plane, [En].

In this subsection, we introduce the Alexandroff one-point compactification for locally compact Hausdorff spaces. For the rest of this subsection, X is assumed to be non-compact. We advise a reader to consult [Wy] for further readings and proofs.

Definition 2.1.1.1. *[Wy] Let X and X' be topological spaces and $f : X \longrightarrow X'$ be a continuous function. The pair (X', f) is said to be a one-point compactification of X provided (X', f) is a compactification of X and $X' \backslash X$ is a singleton set.*

Remark: In most cases, f is usually an inclusion map.

Example 2.1. A. The set $[0, 1] \subseteq \mathbb{R}$ together with an inclusion map i from $[0, 1)$ to $[0, 1]$ is an example of one-point compactification of $[0, 1)$.

The construction of the Alexandroff one-point compactification for a non-compact space X is as follows: Let X be a locally compact Hausdorff space and let $X^+ = X \cup \{\infty\}$ where $\infty \notin X$. Define the topology on X^+ by $T^+ = T \cup \{G \subseteq X^+ : X^+ \backslash G \textit{ is a closed compact subset of } X\}$. The set X^+ together with the inclusion map $i : A \longrightarrow X^+$ form a compactification of a non-compact space X. We call this compactification the Alexandroff one-point compactification of X. This compactification is an example of a Hausdorff compactification.

2.1.2 Wallman Compactification

We devote this subsection in introducing a compactification of T_1 spaces known as the Wallman compactification. This compactification is usually constructed using closed ultrafilters as points. We advise a reader to consult [En] for further readings and proofs.

[En] The Wallman compactification of a T_1-space was introduced, in 1938, by Wallman as follows: Let $\mathcal{P}_X$ be a class of subsets of a topological space X which is closed under finite intersections and finite unions. A $\mathcal{P}_X$-filter on X is a collection $\mathcal{F}$ of non-empty elements of $\mathcal{P}_X$ with the following properties:

1. $\mathcal{F}$ is closed under finite intersections and
2. $P_1 \in \mathcal{F}$, $P_1 \subseteq P_2 \in \mathcal{P}_X$ implies $P_2 \in \mathcal{F}$.

A $\mathcal{P}_X-$ultrafilter is a $\mathcal{P}_X$-filter $\mathcal{F}$ satisfying the condition that, whenever $P \in \mathcal{P}_X$ and $P \cap F \neq \emptyset$ for each $F \in \mathcal{F}$, then $P \in \mathcal{F}$. When $\mathcal{P}_X$ is a class of closed sets of X, then the $\mathcal{P}_X-$filters are called closed filters and $\mathcal{P}_X-$ ultrafilters are called closed ultrafilters. In our text, we consider

$\mathcal{P}_X$ as a class of closed sets of X.

Let wX be the collection of closed ultrafilters on X. For each open set U of X, define $U^\bullet = \{\mathcal{U} \in wX : F \subseteq U \text{ for some } F \in \mathcal{U}\}$. The collection $\{U^\bullet : U \in T\}$ forms a base for open sets of some topology on wX, denoted by wT. Furthermore, the set wX together with a map $w : X \longrightarrow wX$ defined by $w(x) = \{A \in \mathcal{P} : x \in A\}$, form a T_1 compactification of a T_1 space X. This compactification is known as the Wallman compactification.

2.1.3 Stone-Čech Compactification

We have noticed in the preceding two subsections how one can compactify locally compact Hausdorff spaces as well as T_1 spaces. In this subsection, we aim to compactify Tychonoff spaces. We achieve this through studying the concept of the Stone-Čech compactification βX.

Definition 2.1.3.1. *[AL] Let X be Tychonoff. The Stone-Čech compactification of X is a Hausdorff compactification $(\beta X, e)$ with the property that: For each continuous function $f : X \longrightarrow X'$, where X' is compact and Hausdorff, there is a unique continuous map $g : \beta X \longrightarrow X'$ such that $g \circ e = f$.*

The Stone-Čech compactification was first introduced in 1937 by Stone and Čech. There are many ways of constructing βX. We consider the embedding of a Tychonoff space X into product copies of the unit interval $[0, 1]$ as the standard construction of βX. This construction can be found in most standard textbooks of general topology (see e.g., S. Willard [Wi]). The standard construction of βX is as follows: $[En, Wi, Wy]$ Let X be a Tychonoff space. Take the range of each $f \in C(X)$ as a closed bounded interval I_f in $\mathbb{R}$. Define a map $s : X \longrightarrow \Pi\{I_f : f \in C(X)\}$ by $s(x)$ is equal to the point in $\Pi\{I_f : f \in C(X)\}$ whose $f-$th coordinate is the real number $f(x)$.

Remark: We recall from [Wi] that, for a topological space X,

1. A map $f : X \longrightarrow \Pi X_\alpha$ is continuous if and only if $\pi_\alpha \circ f$, where π_α is a projection map, is continuous for each $\alpha \in A$;

2. If for each $\alpha \in A$, $f_\alpha : X \longrightarrow X_\alpha$, then the *evaluation map* $e : X \longrightarrow \Pi X_\alpha$ induced by the collection $\{f_\alpha : \alpha \in A\}$ is defined as follows: for each $x \in X$, $e(x)$ is the point in

ΠX_α whose $\alpha-$th coordinate is the real number $f(x)$ for each $\alpha \in A$, and

3. The evaluation map $e : X \longrightarrow \Pi X_\alpha$ is an embedding if and only if X has the weak topology T_{weak} (the topology on X for which the sets $f_\alpha^{-1}(U_\alpha)$, for $\alpha \in A$, and U_α open in X_α, form a subbase) given by the functions f_α and, whenever $x \neq y \in X$, then for some $\alpha \in A$, $f_\alpha(x) \neq f_\alpha(y)$.

From this remark, we get that s is an evaluation map, and we have the following result.

Theorem 2.1.3.2. *[Wy] Let X be a Tychonoff space. Then the following statements hold:*

1. *The map $s : X \longrightarrow \Pi\{I_f : f \in C(X)\}$ is an embedding.*

2. *$\overline{s(X)}$ is compact Hausdorff.*

3. *The pair $(\overline{s(X)}, s)$ is a Hausdorff compactification of X.*

4. *For each continuous map $h : X \longrightarrow X'$ where X' is compact and Hausdorff, there is a unique continuous map $k : \overline{s(X)} \longrightarrow X'$ such that $k \circ s = h$.*

Proof: [Wi, Wy] (1) Let $x, y \in X$ such that $x \neq y$. Then $y \notin N_x$ for some N_x. Since X is completely regular, there is $f \in C(X)$ such that $f(x) = 0$ and $f(y) = 1$, implying that $f(x) \neq f(y)$. We must show that X has the weak topology given by elements of $C(X)$. Claim: The weak topology on X generated by $C(X)$ equals the given topology T on X: Let $G \in T$ and choose $x \in G$. Observe that $X \backslash G$ is closed. Since X is completely regular, there is $f \in C(X)$ such that $f(x) = 0$ and $f(X \backslash G) = 1$. Note that $(-\frac{1}{2}, 1)$ is open is $\mathbb{R}$ and so $f^{-1}(-\frac{1}{2}, 1) \in T_{weak}$. Because $f(X \backslash G) = 1$ and $f(G) \subseteq [0, 1)$, we have that $f^{-1}(-\frac{1}{2}, 1) = G$. Thus $T \subseteq T_{weak}$. On the other hand, if $U \in T_{weak}$ and $x \in U$, then there exists $f_1, ..., f_n \in C(X)$ and $U_1, ..., U_n \in T_{us}$ such that $x \in \bigcap_{i=1}^n f_i^{-1}(U_i) \subseteq U$. Because each f_i is continuous, we have that $\bigcap_{i=1}^n f_i^{-1}(U_i) \in T$. Therefore $U \in T$. Thus $T_{weak} \subseteq T$. As a result, $T = T_{weak}$. Hence e is an embedding.

(2) Because every closed subset of $(\mathbb{R}, T_{us})$ is compact and Hausdorff, it follows that each I_f is compact and Hausdorff. By the Tychonoff Theorem, the product $\Pi\{I_f : f \in C(X)\}$ is compact and Hausdorff. Therefore the closed subspace $\overline{s(X)}$ of $\Pi\{I_f : f \in C(X)\}$ is compact

and Hausdorff.

(3) This follows from (1) and (2).

(4) Let $h : X \longrightarrow X'$ be a continuous map, where X' is compact and Hausdorff. We can embed X' into $\Pi\{I_g : g \in C(X')\}$ by an evaluation map s'. Let $\pi_f : \Pi\{I_f : f \in C(X)\} \longrightarrow I_f$ and $\pi'_g : \Pi\{I_g : g \in C(X')\} \longrightarrow I_g$ be projection maps. Define a map $F : \Pi\{I_f : f \in C(X)\} \longrightarrow \Pi\{I_g : g \in C(X')\}$ as follows: If $t \in \Pi\{I_f : f \in C(X)\}$ and $g \in C(X')$, then define $[F(t)]_{(g)} = t_{g \circ h}$. Observe that $(\pi'_g \circ F)(t) = \pi_{g\circ h}(t)$. For each $g \in C(X')$, the map $\pi'_g \circ F : \Pi\{I_f : f \in C(X)\} \longrightarrow I_{g\circ h}$ is continuous. It follows that F is continuous. Now, for $x \in X$, we have

$$\begin{aligned}(\pi'_g \circ F \circ s)(x) &= (\pi'_g \circ F)(s(x)) \\ &= \pi_{g\circ h}(s(x)) = (\pi_{g\circ h} \circ s)(x) \\ &= (g \circ h)(x) = g(h(x)) \\ &= (\pi'_g \circ s')(h(x)) = (\pi'_g \circ s' \circ h)(x).\end{aligned}$$

Because π'_g is surjective, we have that $F \circ s = s' \circ h$. Since $s'(X')$ is compact and $\overline{s'(X')}$ is Hausdorff, we have that $s'(X') = \overline{s'(X')}$. But $F(s(X)) \subseteq s'(h(X)) \subseteq s'(X')$, so $\overline{F(s(X))} \subseteq \overline{s'(X')} = s'(X')$. On the other hand, because $s(X)$ is dense in $\overline{s(X)}$ and F is continuous, we have that $F(s(X))$ is dense in $F(\overline{s(X)})$ and thus $F(\overline{s(X)}) \subseteq s'(X')$, implying that F maps $\overline{s(X)}$ strictly to $\overline{s'(X')}$ and hence to X'. Now, let F' be a restriction of F into $\overline{s(X)}$, and define $k : \overline{s(X)} \longrightarrow X'$ by $(s')^{-1} \circ F'$. Such a map exists since $X' = s'(X')$ and F' takes values in $s'(X')$. It is clear that k is continuous. Choose $x \in X$, then $(k \circ s)(x) = ((s')^{-1} \circ F' \circ s)(x) = ((s')^{-1} \circ F \circ s)(x) = h(x)$, showing that $k \circ s = h$. Because any two continuous functions agreeing on a dense subspace of a Hausdorff space are necessarily equal, it follows that k is unique. ■

Remark: In some parts of the preceding theorem, we identified X as $s(X)$.

The combination of statements (3) and (4) of the preceding theorem gives the following result.

Corollary 2.1.3.3. *For a Tychonoff space X, the pair $(\overline{s(X)}, s)$ is the Stone-Čech compactification of X.*

From Statement (4) of Theorem 2.1.3.2, we can deduce that **CHaus** is a reflective subcategory of **Tych** and the Stone-Čech compactification βX is the compact Hausdorff reflection of $X \in$ **Tych**.

2.2 Separated Reflections

The goal of this section is to construct separated reflections of topological spaces.

In the following subsections, we shall give constructions of separated reflections. Our focus will be on T_0, sober, T_1, T_2 and Tychonoff reflections. The concepts of quotient spaces and quotient maps shall be frequently used in constructing some of the separated reflections.

Definition 2.2.1. *[Wi] Let (X, T) be a topological space, X' be any non-empty set and $f : X \longrightarrow X'$ be an onto mapping. A topology $T' = \{G \subseteq X' : f^{-1}(G) \in T\}$ on X' is called the quotient topology induced on X' by f. When X' is given such a quotient topology, it is called a quotient space of X and the inducing map is called a quotient map.*

It follows from the preceding definition that for a quotient map $f : X \longrightarrow X'$, with X' a quotient space, a set $A \subseteq X'$ is open(closed) in X' if and only if $f^{-1}(A)$ is open(closed) in X.

In our text, we shall omit the subscript R from each equivalence class $[x]_R$ if X is the only space involved in a problem and an equivalence relation R is the only equivalence relation that has been defined on X.

2.2.1 T_0 Reflection

The first separated reflection to consider is the T_0 reflection of a topological space. The construction of the T_0 reflection was first described in detail by Stone, [AL].

We can derive a definition of the T_0 reflection of a topological space from the concept of $\mathcal{B}-$reflections given in Definition 2.5.

Definition 2.2.1.1. *[CM] A continuous map $r : X \longrightarrow X'$ from a topological space X to a T_0 space X', is a T_0 reflection map if for each continuous map $e : X \longrightarrow X''$ from X to a T_0 space X'', there is a unique continuous map $f : X' \longrightarrow X''$ such that $f \circ r = e$. The set X' is called the T_0 reflection of X.*

Just like $\mathcal{B}-$reflections, T_0 reflection maps are unique up to a homeomorphism, so we speak about the T_0 reflection rather than a T_0 reflection.

We give the following important characterization of T_0 spaces.

Proposition 2.2.1.2. *[Wi] A topological space is T_0 if and only if for each $x, y \in X$, $\overline{\{x\}} = \overline{\{y\}}$ implies that $x = y$.*

Proof: Let X be T_0 and $x \neq y$, then there exists N_x not containing y or there is N_y not containing x. For $y \notin N_x$, we get that $x \notin \overline{\{y\}}$. Because $x \in \overline{\{x\}}$, we have $\overline{\{x\}} \neq \overline{\{y\}}$. Similarly, we can easily show that $x \notin N_y$ implies $\overline{\{x\}} \neq \overline{\{y\}}$. Conversely, suppose that $\overline{\{x\}} = \overline{\{y\}}$ implies $x = y$. If $x \neq y$, then there must exist N_y not containing x, or there must exist N_x not containing y, otherwise $y \in \overline{\{x\}}$ and $x \in \overline{\{y\}}$. Which is a contradiction. ■

Proposition 2.2.1.3. *[Wi] Let $\sim$ be an equivalence relation on X given by $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$, for each $x, y \in X$. The quotient space $X/\sim$ is the T_0 reflection of X and $r : X \longrightarrow X/\sim$, given by $x \mapsto [x]$, is the T_0 reflection map.*

Proof: Let $r : X \longrightarrow X/\sim$, given by $x \mapsto [x]$, denote the quotient map and $X/\sim$ the quotient space. It is clear that r is surjective and continuous. The quotient space $X/\sim$ is T_0: We start by showing that r is open.

Lemma 2.2.1.4. *[Sl] The quotient map $r : X \longrightarrow X/\sim$ is open.*

Proof: Let A be an open subset of X. It suffices to show that $r^{-1}(r(A)) = A$. It is obvious that $A \subseteq r^{-1}(r(A))$. Now, let $x \in r^{-1}(r(A))$, then $[x] \in r(A)$. Therefore $[x] = [y]$ for some $y \in A$. We have that $\overline{\{x\}} = \overline{\{y\}}$, implying that $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Thus $x \in A$. Therefore $A = r^{-1}(r(A))$. By definition of quotient topology, $r(A)$ is open in $X/\sim$. ■

Now, let $x, y \in X$ with $[x] \neq [y]$. Then $x \nsim y$ which implies that there is an open set U containing x and not y. Assume $x \in U$ and $y \notin U$. Then $r(U)$ is an open set containing $[x]$ and not containing $[y]$. Thus $X/\sim$ is T_0. We show that $X/\sim$ is the T_0 reflection of X. Choose any continuous function $f : X \longrightarrow X'$ where X' is T_0. Now, $x \sim y \Rightarrow \overline{\{f(x)\}} = \overline{\{f(y)\}}$. Because X' is T_0, it follows that $f(x) = f(y)$. So, f has a property that $f(x) = f(y)$ whenever $x \sim y$ in X. Define $f' : (X/\sim) \longrightarrow X'$ by $f'([x]) = f(x)$. It is clear that f' is a function. We must show that f' is continuous and unique. For continuity: let U be an open subset of X'.

Then $f^{-1}(U)$ is open in X. Observe that

$$\begin{aligned}(f')^{-1}(U) &= \{[x] : f(x) \in U\} \\ &= \{[x] : x \in f^{-1}(U)\} = r(f^{-1}(U)).\end{aligned}$$

Now,

$$\begin{aligned}r^{-1}(r(f^{-1}(U))) &= \{x : [x] \in r(f^{-1}(U))\} \\ &= \{x : [x] \in (f')^{-1}(U)\} = \{x : f(x) \in U\} \\ &= f^{-1}(U).\end{aligned}$$

Because r is a quotient map, $(f')^{-1}(U)$ is open in X/R. Thus f' is continuous. For uniqueness of f', suppose that f'' is another continuous function from $X/\sim$ to X' satisfying $f'' \circ r = f$. Pick $[x] \in X/\sim$, then $f''([x]) = f''(r(x)) = f(x) = f'([x])$. Thus $f'' = f'$. ■

Example 2.2. **A.** Consider a space (X, T_{tr}). The T_0 reflection of X is the singleton set $X_0 = \{X\}$ equipped with the trivial topology on X_0.

We shall use X_0 and e_{0X} to denote the T_0 reflection and the T_0 reflection map of X, respectively.

Proposition 2.2.1.5. *[Sl] Let X_0 be the T_0 reflection of a topological space X and $e_{0X} : X \longrightarrow X_0$ be the reflection map. Then there is a section $s_{0X} : X_0 \longrightarrow X$, i.e. $e_{0X} \circ s_{0X} = 1_{X_0}$.*

Proof: [Sl] Because e_{0X} is surjective, it follows that there is an injective map, say $s_{0X} : X_0 \longrightarrow X$ such that $e_{0X} \circ s_{0X} = 1_{X_0}$. For continuity, let $A \in T$. Observe that $A = e_{0X}^{-1}(e_{0X}(A))$, where $e_{0X}(A)$ is open in X_0. Now, $s_{0X}^{-1}(A) = s_{0X}^{-1}(e_{0X}^{-1}(e_{0X}(A)))$. By Corollary 1.4.5, $s_{0X}^{-1}(A) = e_{0X}(A)$. Thus $s_{0X}^{-1}(A)$ is open in X_0. Clearly, $s_{0X} : X_0 \longrightarrow X$ is a section. ■

Proposition 2.2.1.6. *[Sa] If X is a Salbany stably compact space, then so is X_0.*

Proof: [Sa] Because s_{0X} is a section and e_{0X} is the T_0-reflection of X such that $e_{0X} \circ s_{0X} = 1_{X_0}$, by Proposition 1.4.9, X_0 is Salbany stably compact. ■

Proposition 2.2.1.7. *[BEL] Let $f : X \longrightarrow X'$ be a continuous function between spaces X and X'. Then there is a unique continuous function $g : X_0 \longrightarrow X'_0$ satisfying $g \circ e_{0X} = e_{0X'} \circ f$.*

Proof: Consider the continuous funtions $e_{0X} : X \longrightarrow X_0$ and $e_{0X'} \circ f : X \longrightarrow X'_0$. By Definition 2.2.1.1, there is a unique continuous function $g : X_0 \longrightarrow X'_0$ such that diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle e_{0X}}\downarrow & & \downarrow{\scriptstyle e_{0X'}} \\ X_0 & \xrightarrow[g]{} & X_0' \end{array}$$

commutes. ■

Denote X_0 and the unique continuous function g described in the preceding proposition by $e_0(X)$ and $e_0(f)$, respectively. Observe that $e_0(f)([x]) = [f(x)]$.

Proposition 2.2.1.8. *$e_0 : \boldsymbol{Top} \longrightarrow \boldsymbol{Top}_0$ given by*

$$X \mapsto e_0(X)$$

$$f \mapsto e_0(f)$$

is a reflector for $\boldsymbol{Top}_0$.

Proof: It is clear that $\mathbf{Top}_0$ is a reflective subcategory of $\mathbf{Top}$ and e_{0X} is the $\mathbf{Top}_0-$reflection morphism. Observe that $e_0(X) \in \mathbf{Top}_0$ for each $X \in \mathbf{Top}$, and for a continuous function $f : X \longrightarrow X'$ where $X' \in \mathbf{Top}$, we have that $e_0(f)$ is a continuous function. Now, choose a continuous function $h : X' \longrightarrow X''$, where $X'' \in \mathbf{Top}$, and select $x \in X$, then

$$\begin{aligned}(e_0(h) \circ e_0(f))([x]) &= e_0(h)(e_0(f)([x])) \\ &= e_0(h)([f(x)]) = [h(f(x))] = e_0(h \circ f)([x]).\end{aligned}$$

Moreover, $e_0(id_X)([x]) = [id_X(x)] = [x] = id_{e_0(X)}([x])$, for each $X \in \mathbf{Top}$. Thus e_0 is a functor. ■

2.2.2 Sober Reflection

Related to T_0 spaces are sober spaces. Our aim is to construct the sober reflection of a topological space. We shall start by introducing the concept of sober spaces and its useful properties, and later construct the sober reflection.

Enroute to introducing sober spaces, we define irreducible sets.

Definition 2.2.2.1. *[GHK+03, Ho] An arbitrary non-empty subset A of a topological space X is said to be irreducible provided that $A \subseteq B \cup C$ for closed subsets B and C, implies $A \subseteq B$ or $A \subseteq C$.*

We give some useful properties of irreducible sets.

Proposition 2.2.2.2. *[ZH] Let X be a topological space. Then*

1. *$A \subseteq X$ is irreducible if and only if $\overline{A}$ is irreducible.*

2. *Each closure of a singleton set is irreducible.*

3. *If $A \subseteq X$ is irreducible, then each non-empty open subset of A is dense in A.*

4. *If X' is another space, f a continuous map between X and X', and A an irreducible subset of X, then $f(A)$ is an irreducible subset of X'.*

Proof: (1) Let $A \subseteq X$ be irreducible and let B and C be closed subsets of X such that $\overline{A} \subseteq B \cup C$. Then, $A \subseteq B \cup C$. Because A is irreducible, it follows that $A \subseteq B$ or $A \subseteq C$. Therefore, $\overline{A} \subseteq B$ or $\overline{A} \subseteq C$ since both B and C are closed. Thus $\overline{A}$ is irreducible. Conversely, let A_1, A_2 be closed sets such that $A \subseteq A_1 \cup A_2$. Then $\overline{A} \subseteq \overline{A_1} \cup \overline{A_2}$. It follows that $\overline{A} \subseteq \overline{A_1} = A_1$ or $\overline{A} \subseteq \overline{A_2} = A_2$. Thus $A \subseteq A_1$ or $A \subseteq A_2$.
(2) Let A and B be closed sets of X such that $\overline{\{x\}} \subseteq A \cup B$. Then $x \in A$ or $x \in B$. Because A and B are closed, we have that $\overline{\{x\}} \subseteq A$ or $\overline{\{x\}} \subseteq B$.
(3) Let U be a non-empty open subset of A. Let $x \in A$ and choose an open subset V of A containing x. Then $V = A \cap G$, for some $G \in T$. If $V \cap U = \emptyset$, then $A \cap G \cap U = \emptyset$. Because $U = A \cap H$, for some $H \in T$, we have that $A \cap (G \cap H) = \emptyset$. Therefore $A \subseteq (X \backslash G) \cup (X \backslash H)$. Since A is irreducible and both $X \backslash G$ and $X \backslash H$ are closed in X, it follows that $A \subseteq X \backslash G$ or $A \subseteq X \backslash H$. $A \subseteq X \backslash G$ is not possible. Furthermore, $A \subseteq X \backslash H$ implies $H \subseteq X \backslash A$, which is impossible since $A \cap H \neq \emptyset$. Thus $V \cap U \neq \emptyset$, implying that U is dense in A.
(4) Let $A \subseteq X$ be irreducible and let B and C be closed subsets of X' such that $f(A) \subseteq B \cup C$. Then, $A \subseteq f^{-1}(B) \cup f^{-1}(C)$. It follows that $A \subseteq f^{-1}(B)$ or $A \subseteq f^{-1}(C)$. Therefore, $f(A) \subseteq B$ or $f(A) \subseteq C$. ■

Next, we define sober spaces.

Definition 2.2.2.3. *[BEL] A topological space X is sober if for each closed and irreducible set $C \subseteq X$, there exists a unique $x \in X$ such that $\overline{\{x\}} = C$.*

We show that the following implication holds:

$$Hausdorff \Rightarrow\ Sober \Rightarrow\ T_0$$

Proposition 2.2.2.4. *[GHK+03] For a topological space X, each of the following statements holds:*

1. *If X is T_2 then X is sober.*

2. *If X is sober then X is T_0.*

Proof: (1) We show that the irreducible closed subspaces are precisely the singleton subspaces. Let C be an irreducible closed subspace of X such that $C = \overline{\{x, y\}}$, where x and y are distinct elements of X. Then, by hypothesis, there are disjoint open sets N_x, N_y. Now, $C = (C\backslash N_x) \cup (C\backslash N_y)$ and both $C\backslash N_x$ and $C\backslash N_y$ are closed sets with $C \subsetneq (C\backslash N_x)$ and $C \subsetneq (C\backslash N_y)$, contradicting the assumption that C is irreducible. Thus $x = y$. Because every closure of a singleton set is irreducible, it follows that in T_2 spaces, irreducible closed subspaces are the singleton subspaces. Thus X is sober.
(2) Let $x, y \in X$ with $\overline{\{x\}} = \overline{\{y\}}$. Because both $\overline{\{x\}}$ and $\overline{\{y\}}$ are irreducible closed sets, it follows that $x = y$. ■

The following example shows that not every sober space is Hausdorff.

Example 2.2. B. Let $X = \{0, 1\}$ and $T = \{\emptyset, X, \{1\}\}$. X is sober, but not Hausdorff. Indeed, there are no disjoint neighbourhoods of 0 and 1 in X. However, $\{0\}$ and $\{0, 1\}$ are closed and irreducible subsets of X such that $\{0\} = \overline{\{0\}}$ and $\{0, 1\} = \overline{\{1\}}$. This makes X a sober space.

There is a relationship between sober spaces and supersober spaces.

Proposition 2.2.2.5. *[GHK+] Every T_0 and supersober space X is sober.*

Proof: [GHK+] Let C be a closed and irreducible subset of X. Then

$$\mathcal{G} = \{B \subseteq C : B\ is\ a\ non-empty\ open\ subset\ of\ C\}$$

is a filterbase for some filter $\mathcal{F}$. Extend this filter to an ultrafilter p on X. Since X is supersober and T_0, $A_p = \overline{\{x\}}$ for some unique $x \in X$. We must show that $C = \overline{\{x\}}$. Clearly, $A_p \subseteq C$ since

$C \in p$ and C is closed. Therefore $\overline{\{x\}} \subseteq C$. On the other hand, let $y \in C$ and consider N_y. Then $C \cap N_y \neq \emptyset$. There is a filter $\mathcal{G}$ finer than p which contains sets of the form $C \cap N_y$. But p is an ultrafilter, so both C and N_y must be in p. Therefore, $y \in A_p = \overline{\{x\}}$. Hence $C = \overline{\{x\}}$. ■

We focus on constructing the sober reflection of a space X.

Proposition 2.2.2.6. *[BEL] Let $S(X)$ be the collection of all irreducible closed subsets of a space X. For each closed set F in X, set $\widetilde{F} = \{C \in S(X) : C \subseteq F\}$. Then the following conditions hold:*

1. $\widetilde{\emptyset} = \emptyset$;
2. $\widetilde{X} = S(X)$;
3. $\bigcap_i \widetilde{F_i} = \widetilde{\bigcap_i F_i}$;
4. $\widetilde{A} \cup \widetilde{B} = \widetilde{(A \cup B)}$.

Proof: (1) There is no irreducible closed subset of X that contains the empty set. Thus, $\widetilde{\emptyset} = \emptyset$.
(2) Because every element of $S(X)$ is a subset of X, it follows that $\widetilde{X} = S(X)$.
(3) Let $B \in \bigcap_i \widetilde{F_i}$. Then $B \in \widetilde{A_i}$ for each i, which implies that $B \subseteq A_i$ for each i. Therefore, $B \subseteq \bigcap_i A_i$. So, $B \in \widetilde{\bigcap_i F_i}$. On the other hand, if $B \in \widetilde{\bigcap_i F_i}$, then $B \subseteq \bigcap_i A_i$, which implies that $B \subseteq A_i$ for each i. Therefore, $B \in \widetilde{A_i}$ for each i. Thus $B \in \bigcap_i \widetilde{F_i}$. Hence $\bigcap_i \widetilde{F_i} = \widetilde{\bigcap_i F_i}$.
(4) Let $C \in \widetilde{A} \cup \widetilde{B}$. Then $C \in \widetilde{A}$ and $C \in \widetilde{B}$. Therefore $C \subseteq A$ or $C \subseteq B$. Thus $C \subseteq A \cup B$. On the other hand, if $C \in \widetilde{(A \cup B)}$, then $C \subseteq A \cup B$. But $C \in S(X)$ and both A and B are closed, so $C \subseteq A$ or $C \subseteq B$. Therefore $C \in \widetilde{A}$ or $C \in \widetilde{B}$. Thus $C \in \widetilde{A} \cup \widetilde{B}$. Hence $\widetilde{A} \cup \widetilde{B} = \widetilde{(A \cup B)}$. ■

Corollary 2.2.2.7. *[BEL] The collection $\mathcal{G} = \{\widetilde{F} : F \text{ is closed in } X\}$ forms a topology on $S(X)$.*

Proof: It is clear that both $\emptyset$ and $S(X)$ belong to $\mathcal{G}$. Furthermore, because every finite union of closed sets is closed, it follows that for each $\widetilde{A}, \widetilde{B} \in \mathcal{G}$, $\widetilde{A} \cup \widetilde{B} = \widetilde{(A \cup B)} \in \mathcal{G}$. Lastly, because an arbitrary intersection of closed sets is closed, it follows that $\bigcap_i \widetilde{F_i} = \widetilde{\bigcap_i F_i} \in \mathcal{G}$. Hence $\mathcal{G} = \{\widetilde{F} : F \textit{ is closed in } X\}$ forms a topology on $S(X)$. ■

Remark: We denote the topology, formed by $\mathcal{G}$, by $S(T)$, and we call $(S(X), S(T))$ the sobrification of X.

Proposition 2.2.2.8. *[EL] Define a map $\upsilon_X : X \longrightarrow S(X)$ by $\upsilon_X(x) = \overline{\{x\}}$ for each $x \in X$. Then the following statements hold:*

1. *υ_X is well-defined.*
2. *υ_X is continuous.*
3. *$\upsilon_X(X)$ is a dense subspace of $S(X)$.*
4. *If X is sober, then υ_X is a homeomorphism.*

Proof: (1) Let $x, y \in X$ such that $x = y$, then $\overline{\{x\}} = \overline{\{y\}}$. Thus $\upsilon_X(x) = \upsilon(y)$.
(2) Let $\widetilde{A} \subseteq S(X)$. We have that $\upsilon_X^{-1}(\widetilde{A}) = \{x : \overline{\{x\}} \in \widetilde{A}\} = \{x : \overline{\{x\}} \subseteq A\}$. Because X is closed, we have that $\{x : \overline{\{x\}} \subseteq A\} = A$. Thus υ_X is continuous.
(3) Let $\mathbb{A}$ be a closed set containing $\upsilon_X(X)$. We must show that $S(X) = \mathbb{A}$. Observe that there is a collection $\mathcal{Q}$ of closed sets in X such that $\mathbb{A} = \bigcap\{\widetilde{F} : F \in \mathcal{Q}\}$. Choose $A \in S(X)$. We show that $A \subseteq F$ for each $F \in \mathcal{Q}$. Let $x \in A$. Then $\overline{\{x\}} \in \upsilon_X(X) \subseteq \mathbb{A}$. Therefore $\overline{\{x\}} \subseteq F$ for each $F \in \mathcal{Q}$. Thus $x \in F$, making $A \subseteq F$ for each F. Thus $A \in \mathbb{A}$. Hence $S(X) = \mathbb{A}$ and $\upsilon_X(X)$ is a dense subspace of $S(X)$.
(4) υ_X is surjective: Let $A \in S(X)$. Because X is sober, it follows that there is a unique $x \in X$ such that $A = \overline{\{x\}} = \upsilon_X(x)$. υ_X is injective: Let $x, y \in X$ such that $\overline{\{x\}} = \overline{\{y\}}$. Clearly $x = y$. υ_X is closed: Let A be a closed subset of X. Because υ_X is bijective, we have that $\widetilde{A} = \{\upsilon_X(x) \in S(X) : \upsilon_X(x) \in \widetilde{A}\} = \upsilon_X(A)$. Thus υ_X is closed. Continuity follows from (1). Hence υ_X is a homeomorphism. ■

In addition to the previous result, one can assert that $A \subseteq X$ is irreducible, whenever $\widetilde{A}$ is. Indeed, if B and C are closed subsets of X such that $A \subseteq B \cup C$, then $\widetilde{A} \subseteq \widetilde{B} \cup \widetilde{C}$. Because $\widetilde{A}$ is irreducible, it follows that $\widetilde{A} \subseteq \widetilde{B}$ or $\widetilde{A} \subseteq \widetilde{C}$. Clearly, $A \subseteq B$ or $A \subseteq C$. Thus A is irreducible.

We gather our results to show that $S(X)$ is the required sober reflection of a space X.

Lemma 2.2.2.9. *[BEL] Let X be a topological space. Then $S(X)$ is sober.*

Proof: Let $\widetilde{A}$ be an irreducible closed subset of $S(X)$. It is clear that $A \in S(X)$. Now, we claim that $\widetilde{A} = \overline{\{A\}}$: Because $\widetilde{A}$ is closed, it follows that $\overline{\{A\}} \subseteq \widetilde{A}$. Let $B \in \widetilde{A}$. Then $B \subseteq A$. Suppose that $A \notin N_B$, for some N_B. Then $A \in S(X) \backslash N_B = \widetilde{F}$, for some closed set $F \subseteq X$. We have that $A \subseteq F$, implying that $B \subseteq F$. Therefore $B \in S(X) \backslash N_B$, which is impossible. Thus $B \in \overline{\{A\}}$. Hence $\widetilde{A} = \overline{\{A\}}$. If we choose a closed and irreducible set $B \subseteq X$ such that $\widetilde{A} = \overline{\{B\}}$, we get that $\widetilde{A} = \overline{\{B\}} = \widetilde{B}$. Therefore $A = B$. Thus A is unique. Hence $S(X)$ is sober. ■

Proposition 2.2.2.10. *[GD] Let X be a topological space. Then the sobrification of X is the sober reflection of X and υ_X is the reflection map.*

Proof: Let $f : X \longrightarrow X'$ be a continuous map where X' is sober. Define $g : S(X) \longrightarrow X'$ by $A \mapsto \upsilon_{X'}^{-1}(\overline{f(A)})$. g is well-defined: Because X' is sober, it follows that $\upsilon_{X'}$ is a homeomorphism. Thus the function $\upsilon_{X'}^{-1}$ exists. Let $A, B \in S(X)$ such that $A = B$. Then, $f(A) = f(B)$ which implies that $\overline{f(A)} = \overline{f(B)}$. It is clear that $\overline{f(A)} \in S(X')$. Therefore, $\upsilon_{X'}^{-1}(\overline{f(A)}) = \upsilon_{X'}^{-1}(\overline{f(B)})$. g is continuous: Let $A \subseteq X'$ be a closed set. Observe that

$$\begin{aligned}
g^{-1}(A) &= \{B \in S(X) : g(B) \in A\} \\
&= \{B \in S(X) : \upsilon_{X'}^{-1}(\overline{f(B)}) \in A\} = \{B \in S(X) : \overline{f(B)} \in \upsilon_{X'}(A)\} \\
&= \{B \in S(X) : \overline{f(B)} \in \widetilde{A}\} \\
&= \{B \in S(X) : \overline{f(B)} \subseteq A\} = \{B \in S(X) : f(B) \subseteq A\} \\
&= \{B \in S(X) : B \subseteq f^{-1}(A)\} = \widetilde{f^{-1}(A)}.
\end{aligned}$$

Because $\widetilde{f^{-1}(A)}$ is closed, it follows that $g^{-1}(A)$ is closed. Thus g is continuous. To show that $g \circ \upsilon_X = f$, let $x \in X$, then $g(\upsilon_X(x)) = g(\overline{\{x\}}) = \upsilon_{X'}^{-1}(\overline{\{f(x)\}}) = f(x)$. For uniqueness of g, suppose that g' is another continuous map from $S(X)$ to X' satisfying $g' \circ \upsilon_X = f$. Choose $A \in S(X)$, then $g(A) \in X'$ and $g'(A) \in X'$. If $g(A) \neq g'(A)$, then, without loss of generality, $g'(A) \in X' \backslash N_{g(A)}$ for some $N_{g(A)}$ (this is because X' is T_0). Therefore $A \notin S(X) \backslash g^{-1}(N_{g(A)})$ and $A \in S(X) \backslash (g')^{-1}(N_{g(A)})$. Because $S(X) \backslash g^{-1}(N_{g(A)})$ and $S(X) \backslash (g')^{-1}(N_{g(A)})$ are closed in $S(X)$, $S(X) \backslash g^{-1}(N_{g(A)}) = \widetilde{F}$ for some closed set $F \subseteq X$ and $S(X) \backslash (g')^{-1}(N_{g(A)}) = \widetilde{C}$ for some closed set $C \subseteq X$. Therefore $A \subsetneq F$ and $A \subseteq C$. So there is $x \in A$ such that $x \in C$ but $x \notin F$. Now, $\overline{\{x\}} \notin (g')^{-1}(N_{g(A)})$ and $\overline{\{x\}} \in g^{-1}(N_{g(A)})$. Therefore $g'(\overline{\{x\}}) \notin N_{g(A)}$ and $g(\overline{\{x\}}) \in N_{g(A)}$. But $g(\overline{\{x\}}) = f(x) = g'(\overline{\{x\}})$, so $g'(\overline{\{x\}}) \in N_{g(A)}$, which is a contradiction.

Thus $g(A) = g'(A)$, and $g = g'$. ■

Denote the sober reflection of X by X_S and the reflection map by e_{SX}. Using an argument similar to the argument used in Proposition 2.2.1.7, we get that for each continuous function $f : X \longrightarrow X'$ between two topological spaces X and X', there is a unique continuous function $e_S(f) : X_S \longrightarrow X'_S$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle e_{SX}}\downarrow & & \downarrow{\scriptstyle e_{SX'}} \\ X_S & \xrightarrow[e_S(f)]{} & X'_S \end{array}$$

This leads to the following result whose proof follows the sketch of the proof provided in Proposition 2.2.1.8, and shall be omitted.

Proposition 2.2.2.11. *[GD]* $e_S : \textbf{\textit{Top}} \longrightarrow \textbf{\textit{Top}}_{Sob}$ *given by*

$$X \mapsto e_S(X)$$

$$f \mapsto e_S(f)$$

is a reflector for $\textbf{\textit{Top}}_{Sob}$.

The following result shows a relationship between the sobrification and the T_0 reflection of a space X.

Proposition 2.2.2.12. *If* X_0 *is sober, then* $X_0 = X_S$.

Proof: We must find an isomorphism between X_0 and X_S. Consider the surjective continuous functions $e_0 : X \longrightarrow X_0$ and $e_S : X \longrightarrow X_S$. It follows that there is a unique continuous function $g : X_0 \longrightarrow X_S$ such that $g \circ e_0 = e_S$. Because X_0 is sober, there is a unique continuous function $h : X_S \longrightarrow X_0$ such that $h \circ e_S = e_0$. Therefore $h \circ g \circ e_0 = e_0$. But $e_0 = 1_{X_0} \circ e_0$, so $h \circ g \circ e_0 = 1_{X_0} \circ e_0$. Also, because e_0 is surjective, we have that $h \circ g = 1_{X_0}$. On the other hand, $g \circ h \circ e_S = e_S = 1_{X_S} \circ e_S$, which implies $g \circ h = 1_{X_S}$, since e_S is surjective. Clearly, g is an isomorphism. Thus $X_0 = X_S$, as required. ■

2.2.3 T_1 Reflection

In this section, the T_1 reflection of a topological space X is constructed.

We start by giving the following useful characterization of T_1 quotient spaces.

Proposition 2.2.3.1. *[Wi] Let X be a topological space and R any equivalence relation defined on X. Then X/R is T_1 if and only if each equivalence class $[x]$ on R is closed in X.*

Proof: Let $r : X \longrightarrow X/R$ be a quotient map. We have that $r^{-1}(\{[x]\}) = [x]$, for each $x \in X$. Suppose that each equivalence class is closed in X. If $x, y \in X$ such that $[x] \neq [y]$, then $x \notin [y]$ and $y \notin [x]$. Therefore $x \in X \backslash r^{-1}(\{[y]\})$ and $y \in X \backslash r^{-1}(\{[x]\})$ and both $X \backslash r^{-1}(\{[y]\})$ and $X \backslash r^{-1}(\{[x]\})$ are open in X. Observe that $X \backslash r^{-1}(\{[y]\}) = r^{-1}((X/R) \backslash \{[y]\})$ and $X \backslash r^{-1}(\{[x]\}) = r^{-1}((X/R) \backslash \{[x]\})$. Therefore $[x] \in (X/R) \backslash \{[y]\}$ and $[y] \in (X/R) \backslash \{[x]\}$. It is clear that both $(X/R) \backslash \{[y]\}$ and $(X/R) \backslash \{[x]\}$ are open in X/R. Thus X/R is T_1. Conversely, suppose that X/R is T_1. Then each set $\{[x]\}$ is closed in X/R. Because $r^{-1}(\{[x]\}) = [x]$, it follows that $[x]$ is closed in X. ■

From the definition of the T_0 reflection provided in Definition 2.2.1.1, the T_1 reflection can be defined in a similar way except that both X' and X'' must be T_1. In this case, X' is called the T_1 reflection of X and r the T_1 reflection map. Just like T_0 reflection maps, T_1 reflection maps are unique up to a homeomorphism.

To construct the T_1 reflection of a topological space X, let

$$\mathcal{C} = \{S \subseteq X \times X : S \textit{ is an equivalence relation and for all } x \in X, [x]_S \textit{ is closed in } X\}$$

where $[x]_S$ is the class of $x \in X$ under S. Observe that $\mathcal{C}$ is non-empty since the trivial relation $R = X \times X \in \mathcal{C}$. As the intersection of any family of equivalence relations is again an equivalence relation, $R = \bigcap\{S \in \mathcal{C}\}$ is well-defined. We have $[x]_R = \bigcap\{[x]_S : S \in \mathcal{C}\}$, for each $x \in X$. Since the intersection of closed sets is closed, we get that each $[x]_R$ is closed in X.

Proposition 2.2.3.2. *X/R is the T_1 reflection of a topological space X and $r : X \longrightarrow X/R$ is the reflection map.*

Proof: Because each $[x]_R$ is closed in X, it follows that X/R is T_1. Let $f : X \longrightarrow X'$ be a continuous function from X to a T_1 space X'. Observe that $R_f = \{(x, y) \in X \times X : f(x) = f(y)\}$ is an equivalence relation. Since X' is T_1 and f is continuous, it follows that $\{f(x)\}$ is closed in X', which implies $f^{-1}(\{f(x)\})$ is closed in X. As $[x]_R = f^{-1}(\{f(x)\})$, $R_f \in \mathcal{C}$. Therefore $R \subseteq R_f$ and if $[x]_R = [y]_R$, we have that $f(x) = f(y)$. Thus $g : X/R \longrightarrow X'$ given by $[x]_R \mapsto f(x)$ is well defined and satisfies $g \circ r = f$. The rest of the proof is similar to the last part of the proof of Proposition 2.2.1.3 (Continuity and uniqueness of g). Thus X/R is the T_1 reflection of X with r being the T_1 reflection map. ■

Denote the T_1 reflection map of X by e_{1X} and the T_1 reflection by X_1. We have the following result whose proof follows the sketch of the proof of Proposition 2.2.1.8.

Proposition 2.2.3.3. *e_1 : **Top** $\longrightarrow$ **Top**$_1$ given by*

$$X \mapsto X_1$$

$$f \mapsto e_1(f)$$

*is a reflector for **Top**$_1$.*

2.2.4 T_2 Reflection

From Definition 2.2.1.1, we define the T_2 reflection map as the continuous map $r : X \longrightarrow X'$ from a topological space X to a T_2 space X', such that for each continuous map $e : X \longrightarrow X''$ from X to a T_2 space X'', there is a unique continuous map $f : X' \longrightarrow X''$ such that $f \circ r = e$. The space X' is called the T_2 reflection of X and r the T_2 reflection map. It is clear that T_2 reflection maps are unique up to a homeomorphism.

Proposition 2.2.4.1. *[vM] Let X be a topological space. Define $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$ for each continuous map f from X to a Hausdorff space X'. Let $X/\sim$ be a quotient space and $r : X \longrightarrow X/\sim$ a quotient map. Then $X/\sim$ is the T_2 reflection of X and r is the reflection map.*

Proof: Observe that each continuous function $f : X \longrightarrow X'$ to a Hausdorff space X' has a property that, $f(x) = f(y)$ whenever $x \sim y$, for all $x, y \in X$. Following argument used in the last part of the proof of Proposition 2.2.1.3, we get that, there is a unique continuous function $h : (X/\sim) \longrightarrow X'$ satisfying $h \circ f = r$. Now, let $x, y \in X$ such that $[x] \neq [y]$. Then $f(x) \neq f(y)$ for some continuous function $f : X \longrightarrow X'$, where X' is Hausdorff. Hausdorffness of X' implies that there are disjoint $N_{f(x)}$ and $N_{f(y)}$. Therefore, $h^{-1}(N_{f(x)})$ and $h^{-1}(N_{f(y)})$ are disjoint open subsets of $X/\sim$ containing $[x]$ and $[y]$, respectively. Thus $X/\sim$ is Hausdorff. Hence $X/\sim$ is the T_2 reflection of X. ■

We denote the T_2 reflection map of X by e_{2X} and the T_2 reflection by X_2.

Proposition 2.2.4.2. *e_2 : **Top** $\longrightarrow$ **Haus** given by*

$$X \mapsto X_2$$

$$f \mapsto e_2(f)$$

*is a reflector for **Haus**.*

2.2.5 Tychonoff Reflection

The aim of this section is to construct the Tychonoff reflection of a topological space. Enroute to our goal, we outline some useful results of zero-sets.

Definition 2.2.5.1. *[St16] Let X be a topological space and let $A \subseteq X$. A is called a zero-set if there is a continuous function $f : X \longrightarrow [0,1]$ such that $A = f^{-1}(\{0\})$.*

Each zero-set is a closed set. Indeed, since $\{0\} \subseteq [0,1]$ is closed and f is continuous, we have that, $A = f^{-1}(\{0\})$ is closed in X. As a result, zero-sets are preserved by continuous pre-images.

Proposition 2.2.5.2. *[St16] Let X be a topological space. Then X is completely regular if and only if the zero-sets of X form a base for the closed sets of X.*

Proof: [Wy] Let F be a closed subset of X. We must show that there is a collection $\mathcal{C}$ of zero-sets of X such that $\bigcap \mathcal{C} = F$. If $F = X$, then $\mathcal{C} = \{F\}$ and we are done. Let F be a proper subset of X and let $x \in X \backslash F$. Then there is a continuous function $f_x : X \longrightarrow [0,1]$ such that $f_x(x) = 1$ and $f_x(F) = \{0\}$. Let $G_x = \{y \in X : f_x(y) = 0\}$, and let $\mathcal{C} = \{G_x : x \notin F\}$. Because for each $x \notin F$, F is a zero-set of X by f_x, it follows that $F \in \mathcal{C}$. Therefore $F \subseteq \bigcap \mathcal{C}$. Let $y \notin F$. Then $f_y(y) = 1$. Therefore $y \notin \bigcap \mathcal{C}$, otherwise $f_y(y) = 0$ which is not possible. Thus $\mathcal{C} = \{F\}$ and $\mathcal{C}$ is a base for closed sets of X. Conversely, let $x \in X$ and F be a closed subset of X not containing x. By hypothesis, $x \notin f^{-1}(\{0\})$ for some $f \in C(X)$. Therefore, $f(x) \neq 0$. Define $g(y) = \dfrac{f(y)}{f(x)}$, then g is continuous and $g^{-1}(\{0\}) = f^{-1}(\{0\})$, i.e., f and g vanishes at the same points. Therefore $g(F) = \{0\}$ and $g(y) = 1$. Hence X is completely regular. ■

We define the Tychonoff reflection of a topological space the same way as we have defined the previous separated reflections except that in Definition 2.2.1.1, T_0 is replaced with Tychonoff. Our main reference for a construction of the Tychonoff reflection is [EL]. For any topological space X, define an equivalence relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$ for any $f \in C(X)$. Denote the set of equivalence classes by X_T, and let $e_{TX} : X \longrightarrow X_T$ be the surjective map assigning to each point of X its equivalence class. For each $f \in C(X)$, define a map $e_{TX}(f) : X_T \longrightarrow [0,1]$ by $e_{TX}(f)([x]) = f(x)$, for each $x \in X$. Equip X_T with a topology T_T whose closed sets are of the form $\bigcap[e_{TX}(f)^{-1}(\{0\}) : f \in H]$, where $H \subseteq C(X)$. We have the following result.

Proposition 2.2.5.3. *Let X be a topological space. Then X_T is Tychonoff.*

Proof: It suffices to show that the map $e_{TX}(f) : X_T \longrightarrow [0,1]$ is continuous. Let $H \subseteq C(X)$ and choose $f \in H$. $e_{TX}(f)$ is well-defined since $[x] = [y]$ implies that $f(x) = f(y)$ for each $f \in H$ and each $x, y \in X$. The proof for showing that T_T is a topology relies on a careful application of pre-images. $e_{TX}(f)$ is continuous: Let A be a closed subset of $[0,1]$. Because every compact Hausdorff space is completely regular, it follows that $A = \bigcap[r^{-1}(\{0\}) : r \in K]$, where $K \subseteq C([0,1])$. Now, $f^{-1}(A) = \bigcap[(r \circ f)^{-1}(\{0\}) : r \in K]$. We have $e_{TX}(f^{-1}(A)) = \bigcap[e_{TX}((r \circ f)^{-1}(\{0\})) : r \circ f \in K]$. Observe that $e_{TX}(f^{-1}(A)) = \{[x] : x \in f^{-1}(A)\} = e_{TX}(f)^{-1}(A)$. Similarly, $e_{TX}((r \circ f)^{-1}(\{0\})) = e_{TX}(r \circ f)^{-1}(\{0\})$. Thus $e_{TX}(f)^{-1}(A)$ is closed in X_T. Hence $e_{TX}(f)$ is continuous. Now, we show that X_T is Tychonoff. X_T is completely regular: Because each $e_{TX}(f)^{-1}(\{0\})$ is a zero-set of X_T, and each closed subset of X_T is an intersection of such zero-sets, it follows that X_T is completely regular. X_T is T_1: Let $[x], [y] \in X_T$ such that $[x] \neq [y]$. Then $f(x) \neq f(y)$ for some $f \in C(X)$. But $[0,1]$ is T_1 so there are $N_{f(x)}, N_{f(y)} \subseteq [0,1]$ such that $f(x) \notin N_{f(y)}$ and $f(y) \notin N_{f(x)}$. It follows that $e_{TX}(f)^{-1}(N_{f(x)}) \in \mathcal{U}_{[x]}$ and $e_{TX}(f)^{-1}(N_{f(y)}) \in \mathcal{U}_{[y]}$ with $[x] \notin e_{TX}(f)^{-1}(N_{f(y)})$ and $[y] \notin e_{TX}(f)^{-1}(N_{f(x)})$. Thus X_T is T_1. Hence X_T is Tychonoff. ∎

Proposition 2.2.5.4. *Let X be a topological space. Then the map $e_{TX} : X \longrightarrow X_T$ is continuous.*

Proof: Let A be a closed subset of X_T, then $A = \bigcap[e_{TX}(f)^{-1}(\{0\}) : f \in K]$, where $K \subseteq C(X)$.

Therefore

$$\begin{aligned}(e_{TX})^{-1}(A) &= \{x \in X : [x] \in A\}\\ &= \{x \in X : [x] \in e_{TX}(f)^{-1}(\{0\}),\ for\ each\ f \in K\}\\ &= \{x \in X : f(x) = 0, for\ each\ f \in K\} = f^{-1}(\{0\}).\end{aligned}$$

Therefore $(e_{TX})^{-1}(A)$ is a zero-set of X. Clearly $(e_{TX})^{-1}(A)$ is closed in X. Thus e_{TX} is continuous. ■

Proposition 2.2.5.5. *[EL] Let X be a topological space. Then X_T is the Tychonoff reflection of X and e_{TX} the reflection map.*

Proof: Let $g : X \longrightarrow X'$ be a continuous function and X' a Tychonoff space. Denote $h : X_T \longrightarrow X'$ by $[x] \mapsto g(x)$ for each $x \in X$. h is well-defined: Let $x, y \in X$ such that $h([x]) \neq h([y])$. Then $g(x) \neq g(y)$. Because X' is Tychonoff (particularly T_1), $\{g(x)\}$ and $\{g(y)\}$ are closed in X' and $g(x) \notin \{g(y)\}$. It follows that there is $f' \in C(X')$ such that $f'(g(x)) = 1$ and $f'(\{g(y)\}) = \{0\}$. Now, $g(y) \in (f')^{-1}(\{0\})$ and $g(x) \notin (f')^{-1}(\{0\})$. We have that $y \in g^{-1}((f')^{-1}(\{0\}))$ and $x \notin g^{-1}((f')^{-1}(\{0\}))$. But $(f' \circ g) \in C(X)$ and $(f' \circ g)(x) \neq (f' \circ g)(y)$, so $[x] \neq [y]$. Thus h is well-defined. h is continuous: Let A be a closed subset of X'. It follows that $A = \bigcap[k^{-1}(\{0\}) : k \in J]$, where $J \subseteq C(X')$. Therefore $g^{-1}(A) = \bigcap[(k \circ g)^{-1}(\{0\}) : k \circ g \in J]$. We have $e_{TX}(g^{-1}(A)) = \bigcap[e_{TX}((k \circ g)^{-1}(\{0\}) : k \circ g \in J]$. But $e_{TX}((k \circ g)^{-1}(\{0\}) = e_{TX}(k \circ g)^{-1}(A)$ and $e_{TX}(g^{-1}(A)) = \{[x] : g(x) \in A\} = h^{-1}(A)$, so $h^{-1}(A)$ is closed in X_T. Thus h is continuous. Definition of h implies that $h \circ e_{TX} = g$. For uniqueness of h, suppose that h' is another continuous function from X_T to X' satisfying $h' \circ e_{TX} = g$. Pick $[x] \in X_T$, then $h'([x]) = h'(e_{TX}(x)) = g(x) = h([x])$. Thus $h' = h$. Hence e_{TX} is the reflection map and X_T the Tychonoff reflection of X. ■

Argument similar to that of Proposition 2.2.2.12 shows that X_2 and X_T coincide whenever X_2 is Tychonoff.

Proposition 2.2.5.6. *If X_2 is Tychonoff, then $X_2 = X_T$.*

Furthermore, the arguments used in Proposition 2.2.1.7 and Proposition 2.2.1.8 can easily show that the following result holds.

Proposition 2.2.5.7. *e_T : **Top** $\longrightarrow$ **Tych** given by*

$$X \mapsto X_T$$

$$f \mapsto e_T(f)$$

*is a reflector for **Tych**.*

Chapter 3

Monad Induced by Ultrafilter Space

This chapter has two sections. In the first section, we introduce the concept of monads, in full generality, and in the second section we introduce the ultrafilter space monad.

3.1 Monads

Monads were first introduced by Roger Godemont in 1958 who referred to them as "standard constructions". They are often used to provide and explore a different view of universal algebra. In general, any monad can be obtained as a monad associated to an adjunction, although different adjunctions may yield the same monad [HST, AHS]. This section serves to introduce the notions of monads and their algebras.

Definition 3.1.1. *[AHS] Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ and $\mathcal{G} : \mathcal{A} \longrightarrow \mathcal{B}$ be functors. A natural transformation $\omega : \mathcal{F} \longrightarrow \mathcal{G}$ is a function that assigns to each $\mathcal{A}$-object A a $\mathcal{B}$-morphism $\omega_A : \mathcal{F}(A) \longrightarrow \mathcal{G}(A)$ in such a way that the following condition holds: for each $\mathcal{A}$-morphism $f : A \longrightarrow A'$, the square*

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\omega_A} & \mathcal{G}(A) \\ {\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(A') & \xrightarrow[\omega_{A'}]{} & \mathcal{G}(A') \end{array}$$

commutes.

Definition 3.1.2. *[AHS] A monad on a category $\mathcal{A}$ is a triple (T, η, μ) consisting of a functor $T : \mathcal{A} \longrightarrow \mathcal{A}$ and natural transformations $\eta : id_{\mathcal{A}} \longrightarrow T$ and $\mu : T \circ T \longrightarrow T$ such that the* *following diagrams commute:*

$$\begin{array}{ccc} T(T(T(X))) & \xrightarrow{\mu_{T(X)}} & T(T(X)) \\ {\scriptstyle T(\mu_X)}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ T(T(X)) & \xrightarrow[\mu_X]{} & T(X) \end{array} \quad \text{and} \quad \begin{array}{ccccc} T(X) & \xrightarrow{\eta_{T(X)}} & T(T(X)) & \xleftarrow{T(\eta_X)} & T(X) \\ & {\scriptstyle id_{T(X)}}\searrow & \downarrow{\scriptstyle \mu_X} & \swarrow{\scriptstyle id_{T(X)}} & \\ & & T(X) & & \end{array} ,$$

for all objects $X \in \mathcal{A}$.

We shall use $T^2(X)$ and $T^3(X)$ to denote $T(T(X))$ and $T(T(T(X)))$, respectively.

We give some examples of monads.

For the following example, we recall that the power-set functor $\mathcal{P}$ is defined by

$$\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$$
$$A \mapsto \mathcal{P}(A)$$
$$f \mapsto \mathcal{P}(f)$$

where $\mathcal{P}(A)$ is the set of all subsets of A and for each $B \subseteq A$, $\mathcal{P}(f)(B)$ is the image $f(B)$ of B under f.

Example 3.1. **A.** [AHS] In **Set**, the triple $(\mathcal{P}, \eta, \mu)$ where $\mathcal{P}$ is the power-set functor and $\eta : id_{\mathbf{Set}} \longrightarrow \mathcal{P}$ together with $\mu : \mathcal{P}^2 \longrightarrow \mathcal{P}$ are given by $\eta_X : X \longrightarrow \mathcal{P}(X)$, $x \mapsto \{x\}$ and $\mu_X : \mathcal{P}^2(X) \longrightarrow \mathcal{P}(X)$, $\{A_i : i \in I\} \mapsto \bigcup_{i \in I} A_i$, respectively, is a monad called the power-set monad. In fact,

1. η is a natural transformation since η_X is well-defined for any $X \in \mathbf{Set}$, and

$$\begin{aligned} (\mathcal{P}(f) \circ \eta_X)(x) &= \mathcal{P}(f)(\eta_X(x)) \\ &= \mathcal{P}(f)(\{x\}) = f(\{x\}) = \{f(x)\} \\ &= \eta_X(f(x)) \end{aligned}$$

   for each $x \in X$.

2. μ_X is a well-defined function for every $X \in \mathbf{Set}$, and we have

$$\begin{aligned}
\mathcal{P}(f)(\mu_X(\{A_i : i \in I\})) &= \mathcal{P}(f)\Big(\bigcup_{i\in I} A_i\Big) \\
&= f(\Big(\bigcup_{i\in I} A_i\Big)) = \bigcup_{i\in I} f(A_i) \\
&= \mu_X(\{f(A_i) : i \in I\}) \\
&= \mu_X\Big(\mathcal{P}(f)(\{A_i : i \in I\})\Big) \\
&= \mu_X\Big(\mathcal{P}\Big(\mathcal{P}(f)(\{A_i : i \in I\})\Big)\Big) \\
&= \mu_X(\mathcal{P}^2(f)(\{A_i : i \in I\})).
\end{aligned}$$

Thus μ is a natural transformation.

3. The diagrams

$$\begin{array}{ccc}
\mathcal{P}^3(X) & \xrightarrow{\mu_{\mathcal{P}(X)}} & \mathcal{P}^2(X) \\
{\scriptstyle \mathcal{P}(\mu_X)}\downarrow & & \downarrow{\scriptstyle \mu_X} \\
\mathcal{P}^2(X) & \xrightarrow[\mu_X]{} & \mathcal{P}(X)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
\mathcal{P}(X) & \xrightarrow{\eta_{\mathcal{P}(X)}} & \mathcal{P}^2(X) & \xleftarrow{\mathcal{P}(\eta_X)} & \mathcal{P}(X) \\
& {\scriptstyle id_{\mathcal{P}(X)}}\searrow & \downarrow{\scriptstyle \mu_X} & \swarrow{\scriptstyle id_{\mathcal{P}(X)}} & \\
& & \mathcal{P}(X) & &
\end{array}$$

commute. Indeed, for the diagram on the left, let $\{\mathbb{A}_i : i \in I\} \in \mathcal{P}^3(X)$, then

$$\mu_X(\mathcal{P}(\mu_X)(\{\mathbb{A}_i : i \in I\})) = \mu_X\Big(\{\mu_X(\mathbb{A}_i) : i \in I\}\Big) = \bigcup_{i\in I} \mu_X(\mathbb{A}_i).$$

But we also have $\mu_X(\mu_{\mathcal{P}(X)}(\{\mathbb{A}_i : i \in I\})) = \mu_X\Big(\bigcup_{i\in I} \mathbb{A}_i\Big) = \bigcup_{i\in I} \mu_X(\mathbb{A}_i)$. Therefore $\mu_X \circ \mu_{\mathcal{P}(X)} = \mu_X \circ \mathcal{P}(\mu_X)$. For the diagram on the right, we start by showing that the diagram

$$\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{\eta_{\mathcal{P}(X)}} & \mathcal{P}^2(X) \\
& {\scriptstyle id_{\mathcal{P}(X)}}\searrow & \downarrow{\scriptstyle \mu_X} \\
& & \mathcal{P}(X)
\end{array}$$

commutes. Let $A \in \mathcal{P}(X)$, then $\mu_X(\eta_{\mathcal{P}(X)}(A)) = \mu_X(\{A\}) = \bigcup A = A = id_{\mathcal{P}(X)}(A)$. Finally, we show that the diagram

$$\begin{array}{ccc}
\mathcal{P}^2(X) & \xleftarrow{\mathcal{P}(\eta_X)} & \mathcal{P}(X)\ , \\
{\scriptstyle \eta_X}\downarrow & \swarrow{\scriptstyle id_{\mathcal{P}(X)}} & \\
\mathcal{P}(X) & &
\end{array}$$

commutes. Let $A \in \mathcal{P}(A)$, then

$$\mu_X(\mathcal{P}(\eta_X)(A)) = \mu_X(\eta_X(A)) = \mu_X(\{A\}) = \bigcup A = A = id_{\mathcal{P}(X)}.$$

Example 3.1. B. [St] In Category **Set**, the triple $(\mathcal{U}, \eta, \mu)$, where

(a) $\mathcal{U} : \mathbf{Set} \longrightarrow \mathbf{Set}$ is given by:

$$X \mapsto \mathcal{U}(X)^{1}$$
$$f \mapsto \mathcal{U}(f)$$

(b) $\eta : id_{\mathbf{Set}} \longrightarrow \mathcal{U}$ is described as $\eta_X : X \longrightarrow \mathcal{U}(X)$, $x \mapsto \{A \subseteq X : x \in A\}$, and

(c) $\mu : \mathcal{U}^2 \longrightarrow \mathcal{U}$ is given by $\mu_X : \mathcal{U}^2(X) \longrightarrow \mathcal{U}(X)$, $\mathcal{G} \mapsto \{A \subseteq X : A^* \in \mathcal{G}\}$

is a monad called the ultrafilter monad. Indeed,

1. It follows from Proposition 1.3.24 that $\mathcal{U}$ is a functor.

2. From Proposition 1.3.25, we have that η is a natural transformation.

3. μ is a natural transformation since for each $\mathcal{G} \in \mathcal{U}^2(X)$, $\mu_X(\mathcal{G}) = \{A \subseteq X : A^* \in \mathcal{G}\}$ is an ultrafilter on X (by lemma 1.2.2.5), and we have that the diagram

$$\begin{array}{ccc} \mathcal{U}^2(X) & \xrightarrow{\mu_X} & \mathcal{U}(X) \\ {\scriptstyle \mathcal{U}^2(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{U}(f)} \\ \mathcal{U}^2(X') & \xrightarrow[\mu_{X'}]{} & \mathcal{U}(X') \end{array}$$

commutes because,

$$\mathcal{U}(f)(\mu_X(\mathcal{G})) = \{A \subseteq X' : f^{-1}(A) \in \mu_X(\mathcal{G})\} = \{A \subseteq X' : (f^{-1}(A))^* \in \mathcal{G}\}.$$

But,

$$\begin{aligned} (f^{-1}(A))^* &= \{p \in \mathcal{U}(X) : f^{-1}(A) \in p\} \\ &= \{p \in \mathcal{U}(X) : A \in \mathcal{U}(f)(p)\} \\ &= \{p \in \mathcal{U}(X) : \mathcal{U}(f)(p) \in A^*\} \\ &= \big(\mathcal{U}(f)\big)^{-1}(A^*), \end{aligned}$$

[1] $\mathcal{U}(X)$ is the collection of all ultrafilters on a set X.

so, $\mathcal{U}(f)(\mu_X(\mathcal{G})) = \{A \subseteq X' : \big(\mathcal{U}(f)\big)^{-1}(A^*) \in \mathcal{G}\}$. On the other hand we have,

$$\begin{aligned}
\mu_{X'}(\mathcal{U}^2(f)(\mathcal{G})) &= \mu_{X'}(\mathcal{U}((\mathcal{U}(f)(\mathcal{G}))) \\
&= \mu_{X'}(\{\mathbb{A} \subseteq \mathcal{U}(X') : \mathcal{U}(f)^{-1}(\mathbb{A}) \in \mathcal{G}\}) \\
&= \{A \subseteq X' : A^* \in \{\mathbb{A} \subseteq \mathcal{U}(X') : \mathcal{U}(f)^{-1}(\mathbb{A}) \in \mathcal{G}\}\} \\
&= \{A \subseteq X' : \mathcal{U}(f)^{-1}(A^*) \in \mathcal{G}\} = \mathcal{U}(f)(\mu_X(\mathcal{G})).
\end{aligned}$$

4. $\mu_X \circ \mu_{\mathcal{U}(X)} = \mu_X \circ \mathcal{U}(\mu_X)$ since for each $\mathcal{G} \in \mathcal{U}^3(X)$, we have

$$\begin{aligned}
\mu_X(\mu_{\mathcal{U}(X)}(\mathcal{G})) &= \{A \subseteq X : A^* \in \mu_{\mathcal{U}(X)}(\mathcal{G})\} \\
&= \{A \subseteq X : (A^*)^* \in \mathcal{G}\}
\end{aligned}$$

and

$$\begin{aligned}
\mu_X(\mathcal{U}(\mu_X)(\mathcal{G}) &= \{A \subseteq X : A^* \in \mathcal{U}(\mu_X(\mathcal{G}))\} \\
&= \{A \subseteq X : \mu_X^{-1}(A^*) \in \mathcal{G}\} = \{A \subseteq X : (A^*)^* \in \mathcal{G}\} \\
&= \mu_X(\mu_{\mathcal{U}(X)}(\mathcal{G})).
\end{aligned}$$

Also, $\mu_X \circ \eta_{\mathcal{U}(X)} = id_{\mathcal{U}(X)}$ because for any $p \in \mathcal{U}(X)$,

$$\begin{aligned}
\mu_X(\eta_{\mathcal{U}(X)}(p)) &= \mu_X(\{\mathbb{A} \subseteq \mathcal{U}(X) : p \in \mathbb{A}\}) \\
&= \{A \subseteq X : A^* \in \{\mathbb{A} \subseteq \mathcal{U}(X) : p \in \mathbb{A}\}\} \\
&= \{A \subseteq X : p \in A^*\} = \{A \subseteq X : A \in p\} \\
&= p = id_{\mathcal{U}(X)}(p)
\end{aligned}$$

We also have $\mu_X \circ \mathcal{U}(\eta_X) = id_{\mathcal{U}(X)}$. In fact,

$$\begin{aligned}
\mu_X(\mathcal{U}(\eta_X)(p)) &= \{A \subseteq X : A^* \in \mathcal{U}((\eta_X)(p))\} \\
&= \{A \subseteq X : \eta_X^{-1}(A^*) \in p\} = \{A \subseteq X : A \in p\} \\
&= p = id_{\mathcal{U}(X)}(p)
\end{aligned}$$

for each $p \in \mathcal{U}(X)$.

We close this section by introducing a notion of algebras for monads. We shall also find algebras for each monad presented in the preceding examples.

Definition 3.1.3. *[St] Let (T, η, μ) be a monad on category $\mathcal{A}$. An algebra for this monad is an $\mathcal{A}$ object X, together with n $\mathcal{A}$ morphism $v : T(X) \longrightarrow X$ such that the following diagrams commute:*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & T(X) \\ & {\scriptstyle id_X}\searrow & \downarrow{\scriptstyle v} \\ & & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T^2(X) & \xrightarrow{Tv} & T(X) \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle v} \\ T(X) & \xrightarrow[v]{} & X \end{array}$$

To find algebras for the power-set monad, we recall the following definitions.

Definition 3.1.4. *[GL] A reflexive, antisymmetric and transitive binary relation $\leq$ is called a partial order.*

Definition 3.1.5. *[GL] A set X together with a partial order $\leq$ is called a partially ordered set.*

Definition 3.1.6. *[GL] A partially ordered set $(X, \leq)$ is complete if every subset of A has a least upper bound.*

Example 3.1. C. [AHS] Algebras for the power-set monad are essentially complete ordered sets X with a given operation $\bigvee$ which takes a subset of X to its least upper bound.

Example 3.2. D. [St] The pair (X, r_X) where X is a compact and Hausdorff space, is an algebra for the ultrafilter monad on **Set**. Indeed, the function $r_X : \mathcal{U}(X) \longrightarrow X$ exists. Hausdorffness of X implies that, for each $p \in \mathcal{U}(X)$, $r_X(p) = x$ for some unique $x \in X$ such that $A_p = \{x\}$. The following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{U}(X) \\ & {\scriptstyle id_X}\searrow & \downarrow{\scriptstyle r_X} \\ & & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{U}^2(X) & \xrightarrow{\mathcal{U}(r_X)} & \mathcal{U}(X) \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle r_X} \\ \mathcal{U}(X) & \xrightarrow[r_X]{} & X \end{array}$$

Commutativity of the diagram on left follows from the construction of r_X. For the diagram on the right, let $\mathcal{G} \in \mathcal{U}^2(X)$. Then $\mathcal{U}(r_X)(\mathcal{G}) \in \mathcal{U}(X)$. Because X is compact and Hausdorff, we have that $\mathcal{U}(r_X)(\mathcal{G})$ converges to a unique limit, say x. Now, $N_x \in \mathcal{U}(r_X)(\mathcal{G})$ for each N_x. Therefore, $r_X^{-1}(N_x) \in \mathcal{G}$. It is true that $r_X^{-1}(N_x) \subseteq (N_x)^*$. Because $\mathcal{G}$ is an ultrafilter on $\mathcal{U}(X)$, $(N_x)^* \in \mathcal{G}$. Definition of μ_X implies that $N_x \in \mu_X(\mathcal{G})$. Therefore, $\mu_X(\mathcal{G})$ converges to x. Clearly $r_X(\mathcal{U}(r_X)(\mathcal{G})) = \mu_X(\mathcal{G})$. Hence the diagram on the right commutes.

3.2 Ultrafilter Space Monad

We dedicate this subsection in introducing a monad induced by the ultrafilter space. This monad is precisely the ultrafilter monad under a new setting (which is **Top**).

Proposition 3.2.1. *[BGJM] The ultrafilter space forms a monad on* ***Top****.*

Proof: Because for each continuous function f between two topological spaces X and $X', \mathcal{U}(f)$ is continuous and both $\mathcal{U}(X)$ and $\mathcal{U}(X')$ are topological spaces, we have that $\mathcal{U} : \mathbf{Top} \longrightarrow \mathbf{Top}$ is a functor. Furthermore, $\eta : id_{\mathbf{Top}} \longrightarrow \mathcal{U}$ is a natural transformation since each η_X is continuous. We show that μ_X is continuous: Observe that

$$\begin{aligned}\mu_X^{-1}(A^*) &= \{\mathcal{G} \in \mathcal{U}^2(X) : \mu_X(\mathcal{G}) \in A^*\} \\ &= \{\mathcal{G} \in \mathcal{U}^2(X) : A \in \mu_X(\mathcal{G})\} \\ &= \{\mathcal{G} \in \mathcal{U}^2(X) : A^* \in \mathcal{G}\} \\ &= (A^*)^*\end{aligned}$$

for any $A \subseteq X$. So, if $A^* \in \mathcal{U}(T)$, then $\mu_X^{-1}(A^*) \in \mathcal{U}^2(T)$. Thus μ_X is continuous.
Hence the triple $(\mathcal{U}, \eta, \mu)$ is a monad on **Top**. ■

We shall call this monad the ultrafilter space monad.

Algebras for the ultrafilter space monad were proved in [BGJM] where they were referred to as bitopological quasi stably compact spaces with bicontinuous maps. Our aim is to describe the algebras for the ultrafilter space monad in terms of Salbany stably compact spaces.

Definition 3.2.2. *[BGJM] A topological space X is said to be weakly sober if, for each closed and irreducible set $A \subseteq X$, there is $x \in X$, not necessarily unique, such that $A = \overline{\{x\}}$.*

Definition 3.2.3. *[BGJM] A topological space X is said to be stable if compact saturated subsets of X are closed under finite intersections.*

Definition 3.2.4. *[BGJM] A topological space X is said to be quasi stably compact if it is compact, locally compact, stable and weakly sober.*

Lemma 3.2.5. *For each topological space X, $K \subseteq X$ is saturated if and only if $K = \{x \in X : \overline{\{x\}} \cap K \neq \emptyset\}$.*

Proof: If $y \in K$ but $\overline{\{y\}} \cap K = \emptyset$, then $K \subseteq X \backslash \overline{\{y\}}$. Since K is saturated, we have that $y \in X \backslash \overline{\{y\}}$, which is impossible. Thus $\overline{\{y\}} \cap K \neq \emptyset$. If $x \in X$ and $\overline{\{x\}} \cap K \neq \emptyset$, then there is $y \in X$ such that $y \in \overline{\{x\}} \cap K$. Therefore $x \in N_y$ for each $N_y \in \mathcal{U}_y$ and $y \in G$ for each open set G containing K. We get that $x \in \bigcap\{G \subseteq X : G \in T, K \subseteq G\}$. But K is saturated, so $x \in K$. Hence $K = \{x \in X : \overline{\{x\}} \cap K \neq \emptyset\}$. For the converse, observe that $x \in \bigcap\{G \subseteq X : G \in T, K \subseteq G\}$ implies that $\overline{\{x\}} \cap K \neq \emptyset$, for all $x \in X$. Thus $x \in K$. ■

Lemma 3.2.6. *Every supersober space X is stable.*

Proof: Let $A, B \in (\mathcal{CS})_X$. $A \cap B$ is saturated: It suffices to show that $\overline{\{x\}} \cap (A \cap B) \neq \emptyset$ implies $x \in A \cap B$, for each $x \in X$. Observe that $\overline{\{x\}} \cap (A \cap B) \neq \emptyset$ implies that $\overline{\{x\}} \cap A \neq \emptyset$ and $\overline{\{x\}} \cap B \neq \emptyset$. But both A and B are saturated, it follows that $x \in (A \cap B)$. Thus $A \cap B$ is saturated. $A \cap B$ is compact: [GHK+03] Choose $p \in \mathcal{U}(X)$ such that $(A \cap B) \in p$. It follows that $A \in p$ and $B \in p$. Because both A and B are compact, there exists $x \in A$ and $y \in B$ such that $x \in A_p$ and $y \in A_p$, respectively. Since X is supersober, $A_p = \overline{\{z\}}$ for some $z \in X$. It follows that $\overline{\{z\}}$ meets both A and B, i.e., $\overline{\{z\}} \cap A \neq \emptyset$ and $\overline{\{z\}} \cap B \neq \emptyset$. But both A and B are saturated, so $z \in A$ and $z \in B$. Thus $z \in A \cap B$. We have just shown that every ultrafilter containing $A \cap B$ converges to some point in $A \cap B$. Thus $A \cap B$ is compact. Hence X is stable. ■

Proposition 3.2.7. *[BGJM] A topological space X is Salbany stably compact if and only if it is quasi stably compact.*

Proof: The proof follows from the combination of Lemma 3.2.6 and Proposition 2.2.2.5. ■

Definition 3.2.8. *[BGJM] A topological space X is said to be stably compact if it is T_0 and quasi stably compact.*

Remark: It is clear that T_0 Salbany stably compact spaces are stably compact.

Definition 3.2.9. *[BGJM] A continuous function $f : (X, T) \longrightarrow (X', T')$, between stably compact spaces (X, T) and (X', T'), is a proper map if the inverse image, under f, of a compact saturated subset of X' is compact in X.*

Definition 3.2.10. *[GHK+03] A bitopological space is a set X equipped with two topologies, written (X, T, τ).*

Definition 3.2.11. *[GHK+03] A function* $f : (X, T, \tau) \longrightarrow (X', T', \tau')$ *is a bicontinuous function if both* $f : (X, T) \longrightarrow (X', T')$ *and* $f : (X, \tau) \longrightarrow (X', \tau')$ *are continuous.*

Definition 3.2.12. *[BGJM] Let* (X, T, τ) *be a bitopological space. We call* (X, T, τ) *a bitopological quasi stably compact space if*

1. (X, T) *is quasi stably compact;*

2. (X, τ) *is a compact Hausdorff space;*

3. $T \subseteq \tau$*;*

4. *Compact saturated sets in* (X, T) *are closed in* (X, τ)*.*

Proposition 3.2.13. *[BGJM] If a function* $f : (X, T, \tau) \longrightarrow (X', T', \tau')$ *is bicontinuous between bitopological quasi stably compact spaces* (X, T, τ) *and* (X', T', τ')*, then* $f : (X, T) \longrightarrow (X', T')$ *is proper.*

Proof: [BGJM] Let K be a compact and saturated subset of (X', T'). Then K is closed in (X', τ'), and by bicontinuity of f, $f^{-1}(K)$ is closed in (X, τ). Therefore, $f^{-1}(K)$ is compact in (X, τ), hence compact in (X, T). ■

It was shown in [BGJM] that the algebras for the ultrafilter space monad are the bitopological quasi stably compact spaces with bicontinuous maps.

Theorem 3.2.14. *[BGJM] The algebras for the ultrafilter space monad are the bitopological quasi stably compact spaces with bicontinuous maps.*

Now, we can describe the algebras for the ultrafilter space monad in terms of Salbany stably compact spaces.

Corollary 3.2.15. *Algebras for the ultrafilter space monad are essentially Salbany stably compact spaces* (X, T)*, with proper maps, such that there is a compact Hausdorff topology* τ *on* X *satisfying the following conditions:*

1. $T \subseteq \tau$ *and*

2. *compact saturated subsets of* (X, T) *are closed in* (X, τ)*.*

Definition 3.2.16. *[BGJM] A subset U of X is an upset provided $x \in U$ and $x \leq y$ imply $y \in U$.*

Definition 3.2.17. *[BGJM] Let (X, T) be a topological space. Then the specialization order $\leq$ of T is given by $x \leq y$ iff $x \in \overline{\{y\}}$.*

Definition 3.2.18. *[BGJM] Let (X, τ) be a topological space. Denote, by τ^u, the topology of upsets of X with the specialization preorder (reflexive and transitive relation on X).*

Proposition 3.2.19. *Let $X \in$ **Top**. Then, for each $X' \in$ **CHaus**, there is a unique continuous function $F : \mathcal{U}(X) \longrightarrow X'$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{U}(X) \\ {\scriptstyle f}\big\downarrow & \swarrow{\scriptstyle F} & \\ X' & & \end{array}$$

.

Proof: Let $(X', \tau') \in$ **CHaus** and let $T' = (\tau')^u$. By Lemma 5.2 and Remark 5.6 in [BGJM], (X', T', τ') is a bitopological quasi stably compact space. Therefore (X', T', τ') is an algebra for the ultrafilter space monad. This implies that, for each continuous function $f : X \longrightarrow X'$, where $X \in$ **Top**, there is a unique continuous function $F : \mathcal{U}(X) \longrightarrow X'$ such that $F \circ \eta_X = f$. ■

Chapter 4

Salbany's Separated Compactifications

Consider a reflective subcategory $\mathcal{C}$ of **Top**. Recall that the triple $(\mathcal{U}, \eta, \mu)$ is a monad on **Top** and we have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{U}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{U}(f)} \\ X' & \xrightarrow[\eta_{X'}]{} & \mathcal{U}(X') \end{array}$$

for $X, X' \in$ **Top** and $f \in hom_{\mathbf{Top}}(X, X')$. If we have $R : \mathbf{Top} \longrightarrow \mathbf{Top}$ as a reflector, which may be occasionally written as $R : \mathbf{Top} \longrightarrow \mathcal{C}$, and $r_X : id_{Top} \longrightarrow R$ as a natural transformation, then there exists a unique continuous map $f' : RX \longrightarrow X'$, where $X' \in \mathcal{C}$, such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{r_X} & RX \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f'} \\ & & X' \end{array}$$

Because R is a functor, for any $A, B \in$ **Top** and $g \in hom_{\mathbf{Top}}(A, B)$, we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{r_X} & RA \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle R(g)} \\ B & \xrightarrow[r_B]{} & RB \end{array}$$

Therefore, the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & \mathcal{U}(X) & \xrightarrow{r_{\mathcal{U}(X)}} & R\mathcal{U}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{U}(f)} & & \downarrow{\scriptstyle R\mathcal{U}(f)} \\ X' & \xrightarrow[\eta_{X'}]{} & \mathcal{U}(X') & \xrightarrow[r_{\mathcal{U}(X')}]{} & R\mathcal{U}(X') \end{array}$$

commutes. Suppose that there exists a retraction $\alpha : \mathcal{U}(X') \longrightarrow X'$ (i.e., $\alpha \circ \eta_{X'} = 1_{X'}$). We have that $\alpha \circ \mathcal{U}(f) \circ \eta_X = \alpha \circ \eta_{X'} \circ f = f$. If such retraction map α exists, then there is a unique continuous map $h' : R\mathcal{U}(X) \longrightarrow X'$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{U}(X) & \xrightarrow{r_X} & R\mathcal{U}(X) \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle h'} \\ & & X' \end{array}$$

where $h = \alpha \circ \mathcal{U}(f)$. Consequently, we have the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & \mathcal{U}(X) & \xrightarrow{r_{\mathcal{U}(X)}} & R\mathcal{U}(X) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle h} & & {\scriptstyle h'} & \\ X' & & \longleftarrow & & \end{array}$$

We shall refer to this diagram as diagram (1).

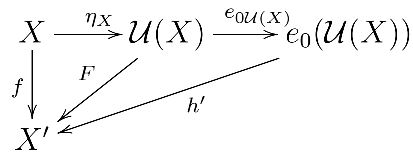
We devote this chapter in considering some examples of reflectors giving rise to diagrams similar to diagram (1) and show that in each example, $R\mathcal{U}(X)$ becomes a separated compactification.

According to the definition of compactification and the combination of Propositions 1.3.6, 1.3.14 and 1.3.15, the ultrafilter space of a topological space X is a compactification and was referred to as Salbany compactification in [BMM]. The separated compactifications in this chapter are named after Salbany simply because they emerged by taking separated reflections of Salbany compactification.

4.1 Salbany's T_0 Compactification

In this section, we aim to show that making $\mathcal{U}(X)$ a T_0 space results in a reflector giving rise to a diagram similar to diagram (1) and we also get a T_0 compactification of a T_0 space X. This compactification has appeared in a number of papers such as [BH] and [BMM], and we shall refer to it as Salbany's T_0 compactification.

Consider the reflector $e_0 : \mathbf{Top} \longrightarrow \mathbf{Top_0}$. Let X' be a stably compact space. Then $X' \in \mathbf{Top}_0$ and the retraction $r_{X'} : \mathcal{U}(X') \longrightarrow X'$ exists. It follows that there is a continuous map $h' : e_0(\mathcal{U}(X)) \longrightarrow X'$ such that the following diagram commutes:



where $F = r_{X'} \circ \mathcal{U}(f)$, as described in Proposition 1.4.12.

Consider the functor $\mathcal{U} : \mathbf{Top} \longrightarrow \mathbf{Top}$. Denote $e_0 \circ \mathcal{U}$ and $e_{0\mathcal{U}(X)} \circ \eta_X$ by β_0 and η_{0X}, respectively. Our next task is to show that $(\beta_0(X), \eta_{0X})$ is a T_0 compactification of a T_0 space X.

Lemma 4.1.1. *The continuous function η_{0X} is injective if and only if X is T_0.*

Proof: Let X be T_0 and choose $x, y \in X$ such that $[\eta_X(x)] = [\eta_X(y)]$. Then $\overline{\{\eta_X(x)\}} = \overline{\{\eta_X(y)\}}$. It follows from Proposition 1.3.9 that $\overline{\{x\}} = \overline{\{y\}}$. But X is T_0, so $x = y$. Thus η_{0X} is injective. Conversely, let $x, y \in X$ such that $x \neq y$. Since η_{0X} is injective, we have that $\eta_{0X}(x) \neq \eta_{0X}(y)$. Therefore, there is an open set $\mathbb{A} \subseteq \beta_0(X)$ such that $\eta_{0X}(x) \in \mathbb{A}$ and $\eta_{0X}(y) \notin \mathbb{A}$. Clearly $x \in \eta_{0X}^{-1}(\mathbb{A})$ and $y \notin \eta_{0X}^{-1}(\mathbb{A})$, where $\eta_{0X}^{-1}(\mathbb{A}) \in T$. Thus X is T_0. ■

Corollary 4.1.2. *For any T_0 space X, η_{0X} is an embedding.*

Proof: Let $(\eta_{0X})' : X \longrightarrow \eta_{0X}(X)$ be a map induced by η_{0X}. It is clear that $(\eta_{0X})'$ is continuous and injective. We must show that it is open. Let $A \in T$. We show that $(\eta_{0X})'(A) = \eta_{0X}(X) \cap e_{0\mathcal{U}(X)}(A^*)$: Let $[p] \in (\eta_{0X})'(A)$, then there is $x \in A$ such that $[\eta_X(x)] = [p]$. Therefore $\eta_X(x) \in A$, implying that $[\eta_X(x)] \in \eta_{0X}(X) \cap e_{0\mathcal{U}(X)}(A^*)$. Conversely, if $[p] \in \eta_{0X}(X) \cap e_{0\mathcal{U}(X)}(A^*)$, then there is $x \in X$ such that $[\eta_X(x)] = [p] \in e_{0\mathcal{U}(X)}(A^*)$. Therefore $\eta_X(x) \in A^*$, implying that $x \in A$. Thus $p = [\eta_X(x)] \in (\eta_{0X})'(A)$. Because $e_{0\mathcal{U}(X)}$ is open, we have that $e_{0\mathcal{U}(X)}(A^*)$ is open in $\beta_0(X)$. Therefore $(\eta_{0X})'(A)$ is open in $(\eta_{0X})(X)$. Thus $(\eta_{0X})'$ is open. Hence η_{0X} is an embedding. ■

Finally, we show that $(\beta_0(X), \eta_{0X})$ is indeed a T_0 compactification of a T_0 space X.

Proposition 4.1.3. *[Sa] Let X be a T_0 space. Then $\beta_0(X)$ is a T_0 compactification of X.*

Proof: Compactness of $\beta_0(X)$ follows from the facts that η_{0X} is surjective and $\mathcal{U}(X)$ is compact.

Because η_X is an embedding and η_{0X} is onto, we have that $\eta_{0X}(X)$ is a dense subspace of $\beta_0(X)$. ■

Salbany's T_0 compactification is a stable compactification.

Definition 4.1.4. *[BH] Let X be a T_0 space. A pair (X', c), where X' is a stably compact space and $c : X \longrightarrow X'$ is an embedding, is a stable compactification of X if $\overline{c(X)}^P = X'$.*

Proposition 4.1.5. *[Sa] For any T_0 space X, $(\beta_0(X), \eta_{0X})$ is a stable compactification of X.*

Proof: Since $e_{0\mathcal{U}(X)}$ is surjective and $\overline{\eta_X(X)}^P = \mathcal{U}(X)$, we have that $\overline{\eta_{0X}(X)}^P = \beta_0(X)$. Also, by Proposition 2.2.1.6, $\beta_0(X)$ is a stably compact space. Thus $(\beta_0(X), \eta_{0X})$ is a stable compactification of X. ■

Proposition 4.1.6. *Let X be a T_0 topological space. Then X is Salbany stably compact if and only if there is a continuous map $v : \beta_0(X) \longrightarrow X$ satisfying $v \circ \eta_{0X} = 1_X$.*

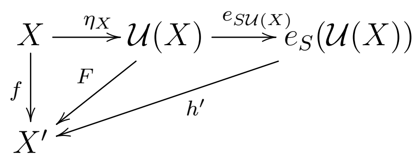
Proof: Suppose that X is a Salbany stably compact space. Observe that, for each $p \in \mathcal{U}(X)$, a unique point $x \in X$ exists such that $A_p = \overline{\{x\}}$. Denote this point by $r_X(p)$. Define a map $v : \beta_0(X) \longrightarrow X$ by $[p] \mapsto r_X(p)$. Such map exists. We show that v is continuous. Let $A \in T$. Then $r_X^{-1}(A) \in \mathcal{U}(T)$. We have $e_{0\mathcal{U}(X)}(r_X^{-1}(A)) = v^{-1}(A)$. Indeed, let $[p] \in e_{0\mathcal{U}(X)}(r_X^{-1}(A))$. Then $p \in r_X^{-1}(A)$ implying that $r_X(p) = v([p]) \in A$. Thus $[p] \in v^{-1}(A)$. Conversely, $[p] \in v^{-1}(A)$ implies that $p \in r_X^{-1}(A)$. Thus $[p] \in e_{0\mathcal{U}(X)}(r_X^{-1}(A))$. Hence, v is continuous. Observe that for each $x \in X$,

$$\begin{aligned} v(\eta_{0X}(x)) &= v([\eta_X(x)]) \\ &= r_X(\eta_X(x)) = x \\ &= id_X(x). \end{aligned}$$

Thus v is the required map. Conversely, let $v : \beta_0(X) \longrightarrow X$ be a continuous map such that $v(\eta_{0X}(x)) = x$. It follows from Proposition 1.4.9 that X is Salbany stably compact. ■

Remark: From the preceding result, we deduce that every Salbany stably compact space X is a retract of $\beta_0(X)$.

When we consider the reflector $e_S : \mathbf{Top} \longrightarrow \mathbf{Top}_{Sob}$ and choose a sober and Salbany stably compact space X', we get that $X' \in \mathbf{Top}_{Sob}$ and a retraction $r_{X'} : \mathcal{U}(X') \longrightarrow X'$ exists. Thus the following diagram commutes and is similar to diagram (1):



Denote $e_S \circ \mathcal{U}$ by β_S. Recall that a T_0 supersober space is sober, so $\beta_0(X)$ is sober and we have the following result.

Proposition 4.1.7. *For each space X, $\beta_0(X) = \beta_S(X)$.*

Proof: This follows from Proposition 2.2.2.12. ■

As a result of the preceding proposition, we deduce that $\beta_S(X)$ is a sober compactification as well as a stable compactification of a T_0 space X.

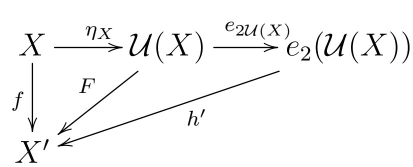
Corollary 4.1.8. *$\beta_S(X)$ is a sober compactification of a T_0 space X.*

Corollary 4.1.9. *$\beta_S(X)$ is a stable compactification of a T_0 space X.*

4.2 Salbany's T_2 Compactification

In this section, we shall show that taking the Hausdorff reflection of $\mathcal{U}(X)$ results in a diagram similar to diagram (1) and we get the Stone-Čech compactification of a Tychonoff space X.

Recall that e_2 : **Top** $\longrightarrow$ **Haus** is a reflector. If X' is compact Hausdorff, then $X' \in$ **Haus** and the retraction $r_{X'} : \mathcal{U}(X') \longrightarrow X'$ exists. So, we get the following commutative diagram.



Denote the functor $e_2 \circ \mathcal{U}$: **Top** $\longrightarrow$ **Haus** and $e_{2\mathcal{U}(X)} \circ \eta_X$ by β_2 and η_{2X}, respectively.

Corollary 4.2.1. *The continuous map h' is unique.*

Proof: Observe that $\eta_{2X}(X)$ is dense in $\beta_2(X)$. Using the fact that continuous functions agreeing on a dense subspace of a Hausdorff space are necessarily equal, it follows that h' is unique. ■

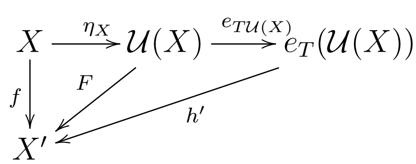
Remark: Uniqueness of h' can also be derived from the fact that X' is an algebra of the ultrafilter space monad (by Proposition 3.2.19). This will make F a unique continuous map, thus making h' unique.

From the preceding proposition, we get that if X is a Tychonoff space, then $\beta_2(X)$ is the compact Hausdorff reflection of X. Therefore the following result holds.

Proposition 4.2.2. *[Sa] Let X be a Tychonoff space. Then $\beta_2(X)$ is the Stone-Čech compactification of X.*

Proof: This follows since a reflector to **CHaus** is essentially unique. ■

If we consider the reflector e_T : **Top** $\longrightarrow$ **Tych** and choose a compact Tychonoff space X', we get that $X' \in$ **Tych** and a retraction $r_{X'} : \mathcal{U}(X') \longrightarrow X'$ exists. Thus the following diagram commutes and is similar to diagram (1):



Denote $e_T \circ \mathcal{U}$ by β_T. Since compact Hausdorff spaces are Tychonoff, it follows that $\beta_2(X)$ is compact and Tychonoff and we have the following result.

Proposition 4.2.3. *For each space X, $\beta_2(X) = \beta_T(X)$.*

Proof: This follows from Proposition 2.2.5.6. ■

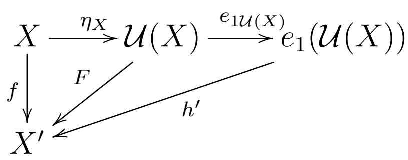
From the preceding result, we get that $\beta_T(X)$ is a Tychonoff compactification of a Tychonoff space X, which in turns coincides with the Stone-Čech compactification $\beta_2(X)$.

Corollary 4.2.4. *$\beta_T(X)$ is the Stone-Čech compactification of a Tychonoff space X.*

4.3 Salbany's T_1 Compactification

In this section we consider the T_1 reflection of $\mathcal{U}(X)$.

Consider the reflector e_1 : **Top** $\longrightarrow$ **Top**$_1$. Let X' be a T_1 Salbany stably compact space. It follows that $X' \in$ **Top**$_1$ and the retraction $r_{X'} : \mathcal{U}(X') \longrightarrow X'$ exists. Thus, the following diagram commutes:



Denote $e_1 \circ \mathcal{U}$ and $e_{1\mathcal{U}(X)} \circ \eta_X$ by β_1 and η_{1X}, respectively. Observe that $\beta_1(X)$ is compact. Unlike in the case of Salbany's T_2 compactification, classification of a space X such that η_{1X} is an embedding remains an open problem. This comes after Salbany, in [Sa], gave an example of a T_1 topological space X such that η_{1X} is not an embedding.

Consider the following examples:

Example 4.3. A. [Sa] Consider $w = \{1, 2, ..\}$ with topology T with finite subsets as basic closed sets. Then $\beta_1(w)$ is a singleton set with its unique topology. Observe that (w, T) is T_1 but η_{1w} is not an embedding.

Example 4.3. B. [Sa] Let (X, T) be a locally compact Hausdorff space. Consider the Alexandroff one-point compactification (X^+, T^+) of (X, T). Then $\beta_1(X^+) = X^+$.

Example 4.3. C. [Sa] Endow the set $X = [0, 1]$ with T_{usu}. Then $\beta_1(X) = X$.

Despite the difficulty of characterizing spaces such that η_{1X} is an embedding, $\beta_1(X)$ has the following properties in common with the Wallman compactification: $\beta_1(X)$ is compact and T_1, $\eta_{1X}(X)$ is a dense subspace of $\beta_1(X)$ and the preceding diagram remains commutative for $X \in \mathbf{Top}_1$ and $X' \in \mathbf{Comp}_1$.

In each of the examples given above, $\beta_1(X)$ is Hausdorff. As noted in [Sa], it still remains unclear whether or not $\beta_1(X)$ is always Hausdorff. However, $\beta_1(X)$ has a relationship with compact Hausdorff spaces.

Definition 4.3.1. *[Sa] A topological space X' is said to be β_1−injective if, whenever a continuous function $f : X \longrightarrow X'$, where X is a topological space, is given, there is a continuous function $F : \beta_1(X) \longrightarrow X'$ such that $F \circ \eta_{1X} = f$.*

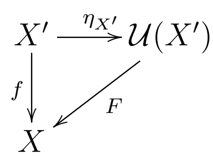
Lemma 4.3.2. *If X is a retract of $\beta_1(X)$, then X is a compact Hausdorff space.*

Proof: [Sa] Let $F : \beta_1(X) \longrightarrow X$ be the retraction map. Observe that a continuous map $F \circ e_{1\mathcal{U}(X)} : \mathcal{U}(X) \longrightarrow X$ exists and satisfies $(F \circ e_{1\mathcal{U}(X)}) \circ \eta_X = 1_X$. Therefore, X is a retract of $\mathcal{U}(X)$. It follows that X is T_1 Salbany stably compact. Now, for each $p \in \mathcal{U}(X)$, there is

$x \in X$ such that $A_p = \overline{\{x\}} = \{x\}$. Hence X is a compact Hausdorff space. ■

Proposition 4.3.3. *[Sa] A topological space X is β_1-injective if and only if X is a compact Hausdorff space.*

Proof: [Sa] Suppose that X is β_1-injective, then $1_X : X \longrightarrow X$ determines $F : \beta_1(X) \longrightarrow X$ such that $F \circ \eta_{1X} = 1_X$. Therefore X is a retract of $\beta_1(X)$. It follows that X is compact and Hausdorff. Conversely, if X is a compact Hausdorff space, then it follows from Proposition 3.2.19 that, for each $f \in hom_{\mathbf{Top}}(X', X)$, there is a unique continuous function $F : \mathcal{U}(X') \longrightarrow X$ such that the following diagram commutes:



By uniqueness of F, we get that the continuous function $h' : \beta_1(X') \longrightarrow X$ satisfying $h' \circ e_{1\mathcal{U}(X')} = F$ is a unique continuous function satisfying $h' \circ \eta_{1X'} = f$. Thus X is β_1-injective.
■

Remark: The proof for the converse of the preceding proposition uses categorical notions and differs from what is given in [Sa].

Conclusion

The work done in this thesis can be summarized as follows:

1. A construction of the ultrafilter space of a topological space X was presented and some properties of the ultrafilter space, including compactness and separability, were investigated.

2. Compactifications and separated reflections were introduced with some examples.

3. The notion of monads and their algebras was presented, and the ultrafilter space monad was introduced.

4. Different separated compactifications were instigated and were compared to some well-known compactifications. These separated compactifications were developed through rendering the ultrafilter space suitably separated.

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