Boundedness and pseudocompactness in pointfree topology

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ABSTRACT

This dissertation is a presentation to generalize boundedness and pseudocompactness in pointfree topology. We first obtain and introduce a boundedness notion for elements of a frame. This is then further inspiration to introduce a definition of bounded frame homomorphism whose domain may be any frame E, not just the frame of open sets of the reals.

Consequently we arrive at a generalization of pseudocompactness which we term:

($E-$**Pseudocompactness Of Frames**)

Where a frame L is E-pseudocompact if any homomorphism with domain E and codomain L is bounded.

After surveying pseudocompactness in both general and pointfree topology, we give our definition of bounded element in a frame and study related properties before introducing bounded frame homomorphisms and E-pseudocompact frames. Various properties of these are studied and compared with classical result of pseudocompactness in topology and frame theory.

Key Words:
Frame, Bounded frame element, bounded frame homomorphism, Pseudocompact space, Pseudocompact frame, E-pseudocompact frame

DECLARATION

I declare that *Boundedness and pseudocompactness in pointfree topology* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Signed: Fatma Alderaz

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Contents

INTRODUCTION

In the late 1940's, Hewitt [22] introduced the notion of pseudocompactness in classical topology, founded on the relationship between two rings of real valued continuous functions. (A space X is pseudocompact if the rings of real valued continuous functions and the rings of bounded, real valued continuous functions on X coincide.) This notion was translated into the context of pointfree topology by Baboolal and Banaschewski [1], in a paper which showed that the Stone-Čech compactification of a completely regular frame is locally connected if and only if the frame is locally connected and pseudocompact.

Already in 1989 Gilmour expressed (in private communication) that the pseudocompactness of frames can be characterized in terms of a cover condition, that is: a frame L is pseudocompact if and only if every completely regular sequence (a_n) in this frame such that $\bigvee a_n = 1$ terminates. This was a central result in [5].

These concepts were developed by a number of topologists to give more related properties and results in frame theory. These ranged from characterizations of pseudocompact frames to related compactness properties and their interaction with other topological properties such as connectedness. For example:

- Banaschewski and Pultr [7] gave some characterizations within the category of completely regular frames.
- In [5], Banaschewski and Gilmour systematically explored the concept in frames, their central idea was to describe the pseudocompact frames

without reference to the (localic) reals.

- Marcus obtained further results regarding the pseudocompactness property, through an investigation of the relationship between pseudocompactness, realcompactness and compactness. He also proved that the compact pseudocompact frames are realcompact [29].

- Walters-Wayland [38] showed that, a completely regular frame is pseudocompact if and only if it admits only paracompact uniformities.

- A comparison between pseudocompactness for frames and other weaker forms of compactness; namely, feeble compactness and countable compactness was presented by Hlongwa in [23]. These ideas were extended by Banaschewski, Holgate and Sioen in [6].

- In Dube and Matutu's studies [17] some external characterizations were established giving necessary and sufficient conditions for a completely regular frame to be pseudocompact.

- On the other hand, in [15] Dube and others demonstrated that not every completely regular pseudocompact frame is spatial. In contrast to the compact regular case, there is a non-spatial completely regular pseudocompact frame.

Boundedness is central to the study of pseudocompactness. Classically, boundedness is a metric but not a topological property. The definition of boundedness for a subset of a topological space was introduced by Lambarinos in 1973 [27]. This was an inspirational idea for Dube to define a bounded quotient frame homomorphism and other variations of the concept, such as almost bounded frames and H-quotients. He then related them to the concept of compactness in pointfree topology in a number of contexts [14]. In this direction, we will compare the bounded quotient frame homomorphism with a new definition of boundedness and bounded frame homomorphism.

Why pseudocompactness?

Historically, the archetypal definition of pseudocompactness in topology is:

A topological space is pseudocompact if all of its continuous real-valued functions are bounded.

Thus pseudocompactness is first and foremost about bounded maps. Nevertheless, in pointfree topology the study of bounded elements in general frames have never been defined before, this is one contribution of this thesis.

The main aim of this thesis is to introduce a new definition of bounded element in a frame and thus provide a generalization of pseudocompactness in terms of bounded frame homomorphisms which in turn depends on bounded elements.

Our method relies heavily on techniques that are developed around the above mentioned bounded elements in a general frame. This is the main contribution which we make and it suggests a direction for future study as well.

Thesis outline

Chapter 1 recounts the relevant definitions pertaining to frame theory and outlines the required background for the ensuing chapters.

Chapter 2 is divided into two sections, the first section is dedicated to pseudocompact spaces, summarising properties which for the most part appeared in [34, 22]. The second section sets out to survey key results regarding the pointfree version of pseudocompactness which has been investigated in many of the aforementioned references .

In **Chapter 3**, the definition of a bounded element in frames is introduced. In the first section, we start with the definition of bounded elements in frames, then we give some examples and properties then end up with a relationship between a bounded element in a Boolean frame with a "Dube bounded" sublocale which maps this frame to the downset of the bounded element.

In the second section, we introduce the notion of E-pseudocompact frames and explore some examples and properties.

Finally, we end with a brief overview of unsolved problems which we have faced in this dissertation.

1. PRELIMINARIES

The history of frame theory goes back to Stone [35], and Wallman in [37], who initially studied topological concepts using lattice theory.

By the end of the 1950's, Ehresmann and Benabou ([18], [9]) considered certain complete lattices with an appropriate distributivity property which deserved to be studied as a generalisation of topological spaces called 'local lattices'.

In a series of papers in the 1960's and 1970's, Dowker and Papert ([12], [11], [13]) introduced the term frame for a local lattice and extended many results of topology to frame theory. More historical information about the concept of frames and their categorical dual which are called **locales** can be found in Johnstone ([25], [26]) and the more recent text by Picado and Pultr [32].

In this chapter a brief introduction to some needed background material on frame theory is given. We concentrate on the definitions, results and properties required for this thesis. Full details can be found in [32, 25] and most of our proofs are taken from these texts.

1.1 Lattices

A binary relation $\leq$ on a set L is called a partial order if it is:

(1) reflexive, $a \leq a$ for all $a \in L$,

(2) antisymmetric, $a \leq b$ and $b \leq a$ implies $a = b$ for all $a, b \in L$, and

(3) transitive $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in L$.

The set L together with the partial order $\leq$ is called a partially ordered set or **poset**.

If A is a subset of a poset L then, an element $b \in L$ is called an **upper bound** (**lower bound**) of A if $a \leq b$ ($a \geq b$), $\forall a \in A$. Further, the **join (meet)** of A is the least upper bound (the greatest lower bound) of A. We denote the join of A by $\bigvee A$, and the meet by $\bigwedge A$. If $A = \{a, b\}$ has only two elements then we write $\bigvee A = a \vee b$ and $\bigwedge A = a \wedge b$.

In addition, a poset L is:

(1) A **meet-semilattice (join-semilattice)** if there exists a meet (join) for any two elements $a, b \in L$.

(2) A **lattice** if there is meet and a join for any two elements in L. A lattice L is called:

(a) **Modular**, if the implication below holds for all elements of L,

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c.$$

(b) **Distributive** if the equality below holds for all elements of L,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

(3) A **bounded lattice** whenever all finite subsets of L have a meet and a join. This means that L is a lattice which has a greatest (top) element 1_L and a least (bottom) element 0_L.

(4) A **complete lattice** if every subset of L has a meet and a join.

Note that every complete lattice L is bounded with

$$0_L = \bigvee \emptyset = \bigwedge L \text{ and } 1_L = \bigwedge \emptyset = \bigvee L.$$

A **complemented lattice** is a bounded lattice, in which every element a has a complement, i.e. an element b such that:

$$a \vee b = 1 \text{ and } a \wedge b = 0.$$

A complemented, distributive lattice is called a **Boolean Algebra**.

A mapping $f : X \longrightarrow Y$ between two posets X, Y is called **monotone** if:

$$f(x) \leq f(y) \text{ wherever } x \leq y.$$

It is called an **isomorphism** if it is bijective and its inverse is monotone as well. Moreover, we say that f is a **lattice homomorphism** if X and Y are lattices and:

$$f(x \vee y) = f(x) \vee f(y) \ , \ f(x \wedge y) = f(x) \wedge f(y), \ \forall x, y \in X.$$

An **adjunction map** is a pair of monotone maps f and g $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ between two posets such that $\forall x \in X$ and $y \in Y$ the relation holds:

$$f(x) \leq y \text{ if and only if } x \leq g(y).$$

Then f is called a **left adjoint** of g and g is called a **right adjoint** of f. Equivalently,the pair f, g is adjoint if $g \circ f$ is above the identity on X and $f \circ g$ is below the identity on Y. General theory tells that:

- adjoints are unique,
- a right (left) adjoint preserves all existing meets (joins),
- a monotone map $f : X \longrightarrow Y$ has a right adjoint g iff $\forall y \in Y$ the right-hand side in the identity below exists and f preserves all such joins:

$$g(y) = \bigvee \{x \mid f(x) \leq y\}.$$

- dually, a monotone map $f : X \longrightarrow Y$ has a left adjoint g iff $\forall y \in Y$ the right-hand side in the identity below exists and f preserves all such meets:

$$g(y) = \bigwedge\{x \mid y \leq f(x)\}.$$

Remark 1.1.1. The adjunction maps in this section are a special case of adjunctions in category theory. We will later see such an example when dealing with frames and topological spaces in Section 1.2.5.

In a lattice L with 0, an element a in L is said to have a **pseudocomplement** if there exists a greatest element x in L such that $a \wedge x = 0$. We denote such an x by a^*. Equivalently, a^* is the pseudocomplement of a if:

$$x \wedge a = 0 \Longleftrightarrow x \leq a^*, \forall x \in L.$$

More formally

$$a^* = \bigvee\{x \in L \mid x \wedge a = 0\}.$$

The lattice L is called **pseudocomplemented** if every element in L has a pseudocomplement. For example every finite distributive lattice is pseudocomplemented.

Note that lattice homomorphisms do not necessarily preserve pseudocomplements. One has obviously $f(a^*) \leq f(a)^*$ if f is monotone, but the other inequality generally need not hold.

Pseudocomplements, if they exist, satisfy the following properties:

1. $a \leq a^{**}$,
2. $a^* = a^{***}$,
3. $a \leq b$ implies $b^* \leq a^*$,
4. $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.
5. $(a \vee b)^* = a^* \wedge b^*$ [De Morgan Law].

1.2 Frames

A frame L is a complete lattice such that for any point $a \in L$ and any set $M \subseteq L$ the following infinite distributive law holds:

$$a \wedge \bigvee M = \bigvee \{a \wedge m \mid m \in M\}.$$

Example 1.2.1. Any complete Boolean algebra is a frame.

If the existence of joins and the above distributive law hold for countable sets $M \subseteq L$ then the bounded lattice L is called a **σ-frame**. This can be generalized to subsets $M \subseteq L$ of cardinality κ and the resulting lattices are called **κ-frames**.

A **subframe** P of a frame L, is a subset $P \subseteq L$ which is a frame under the same operations ($\wedge$ and $\bigvee$) as L, with $0_L, 1_L \in P$.

An **atom** (co-atom) in a frame L is an element $a > 0$ ($a < 1$) such that for each $x \in L$, $a \geq x > 0$ implies that $x = a$ ($a \leq x < 1$ implies that $x = a$). A Boolean algebra is **atomic** if each of its element is a join of atoms.

A frame L is said to be a **zero-dimensional** frame if every element is a join of complemented elements, also L is called a **Boolean frame** if $L = BL$, where BL is the set of all complemented elements of the frame L. As is noted in [32], a frame L which is a Boolean algebra coincides with a Boolean frame.

Lemma 1.2.1. ([32] Proposition II.5.4.2) In a Boolean frame L the following statements are equivalent.

(1) L is atomic,

(2) Each element of L is a meet of co-atoms,

(3) L is isomorphic as frame to the power set of the set X of all atoms of L.

A frame **homomorphism** is a map $h : M \longrightarrow L$ between two frames which preserves:

- All finite meets ($h(x \wedge y) = h(x) \wedge h(y)$ for all $x, y \in M$),
- All arbitrary joins ($h(\bigvee X) = \bigvee\{h(x) \mid x \in X\}$ for any $X \subseteq M$).

Note that such an h is automatically order preserving and preserves both the top ($h(1_M) = 1_L$) and the bottom ($h(0_M) = 0_L$).

A frame homomorphism h is said to be:

- **dense** if $h(a) = 0 \Rightarrow a = 0$,
- **codense** if $h(a) = 1 \Rightarrow a = 1$,
- a **quotient** map if it is onto,
- an **isomorphisim** if it is onto (surjective) and one-to-one (injective).

Because a frame homomorphism h preserves arbitrary joins, h has a **right adjoint** $h_* : L \longrightarrow M$ satisfying the property that $x \leq h_*(y)$ in M if and only if $h(x) \leq y$ in L. For $a \in L$,

$$h_*(a) = \bigvee\{x \in M \mid h(x) \leq a\}.$$

A homomorphism is called **closed** if $h_*(h(x) \vee y) = x \vee h_*(y)$, $\forall x \in M, \forall y \in L$.

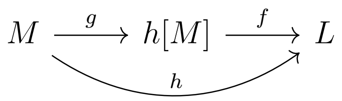
Example 1.2.2. The standard (you may say motivating) initial example of a frame and frame homomorphism is taken from topology. If X is a topological space then the set $\mathcal{O}X$ of all open subsets of X forms a frame ordered by set inclusion. Let $f : X \longrightarrow Y$ be a continuous map between topological spaces X and Y, the map $\mathcal{O}f : \mathcal{O}Y \longrightarrow \mathcal{O}X$ which is given by

$$\mathcal{O}f(U) = f^{-1}(U), \forall U \subseteq Y \text{ with } U \text{ open},$$

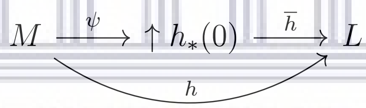
is a frame homomorphism.

In fact $\mathcal{O}$ is a (contravariant) functor from the category **Top** to **Frm**.

Remark 1.2.1. In the category of frames, any frame homomorphism $h : M \longrightarrow L$ has a factorisation $f \circ g$



via the image of M under h with surjective g and injective f described by $g(a) = h(a)$ and $f(h(a)) = h(a)$ for any $a \in M$. A related factorisation gives a useful result which states that every frame homomorphism $h : M \longrightarrow L$ has an onto dense factorisation:



where $\uparrow h_*(0) = \{a \in M \mid a \geq h_*(0)\}$ and ψ is a quotient frame map $x \longmapsto x \vee h_*(0)$ and $\overline{h} : \uparrow h_*(0) \longrightarrow L$ is dense mapping as h.

We note the following lemma which relates surjective homomorphisms, their adjoints and pseudocomplements.

Lemma 1.2.2. For a surjective frame homomorphism $h : M \longrightarrow L$ with $a \in L$ and $h_*(a) = \bigvee\{x \in M \mid h(x) = a\}$. If h is dense surjective, then $h_*(a^*) = (h_*(a))^*$.

1.2.3 Regular and completely regular frames

In a bounded lattice L, an element $a \in L$ is said to be **rather below** $b \in L$, denoted $a \prec b$, if there exists a separating element $c \in L$ such that

$$a \wedge c = 0 \text{ and } c \vee b = 1$$

If L is pseudocomplemented, then this is equivalent to:

$$a \prec b \Leftrightarrow a^* \vee b = 1_L.$$

Then a frame L is called **regular** if for every $b \in L$,

$$b = \bigvee\{a \in L \mid a \prec b\}.$$

Regularity for frames is a **conservative notion**, meaning that for a topological space X, X is regular (as a space) if and only if $\mathcal{O}X$ is regular (as a frame).

Further, for $a, b \in L$, a is said to be **completely below** b, denoted $a \prec\prec b$, if there is a set of elements

$$\{c_r \in L \mid r \in \mathbb{Q} \cap [0,1]\}$$

such that $a = c_0$ and $b = c_1$, $c_p \prec c_r$ when $p < r$. We say that the sequence $\{c_r\}$ **witnesses** the relation $a \prec\prec b$.

A frame L is called **completely regular** in case every b in L is the join of elements completely below it,

$$b = \bigvee\{a \in L \mid a \prec\prec b\}.$$

In a frame L an element a is said to be **way below** b written $a \ll b$ if $b = \bigvee S$ implies $a \leq \bigvee F$ for some finite $F \subseteq S$. Whenever every element a in a frame L can be written,

$$a = \bigvee\{b \mid b \ll a\},$$

then L is called a **continuous frame**.

We mention two useful results concerning the above.

Lemma 1.2.3. In regular frames, any dense frame homomorphism is injective.

Lemma 1.2.4. In a regular (completely regular) frame L,

$$a \ll b \Longrightarrow a \prec b \ (a \prec\!\prec b)$$

1.2.4 Filters and Ideals

A non-empty subset F of a frame L is said to be a **filter** if it is:

(i) an upset ($a \leq b$ and $a \in F \Longrightarrow b \in F$), and

(ii) closed under finite meets ($a, b \in F \Longrightarrow a \wedge b \in F$).

F is a **filter base** if $\uparrow F = \{b \in L \mid \text{there exists } a \in F \text{ with } a \leq b\}$ is a filter on L. (The filter $\uparrow\{a\}$ for $a \in L$ is written as $\uparrow a$.) A filter (base) F is said to:

- be **(completely) regular** if $\forall x \in F$ there is $y \in F$ such that $y \prec x$ ($y \prec\!\prec x$), and
- **cluster** if $\bigvee\{x^* \mid x \in F\} \neq 1$.

Further, a filter F is **prime** if when $a \vee b \in F$, then either $a \in F$ or $b \in F$. It is **completely prime** if

$$\bigvee S \in F, \text{ then } S \cap F \neq \emptyset.$$

The set of all completely prime filters on L is denoted by ΣL.

On the other hand, a non-empty subset I of a frame L is an **ideal** if it is:

(i') a downset ($a \leq b$ and $b \in I \Longrightarrow a \in I$), and

(ii') closed under finite joins ($a, b \in I \Longrightarrow a \vee b \in I$).

The set of all ideals of a frame L is denoted $\mathcal{J}L$. This is itself a frame ordered by set inclusion. As for downsets there is the notation $\downarrow A = \{b \in L \mid \text{there exists } a \in A \text{ with } b \leq a\}$ for $A \subseteq L$. The principle ideal $\downarrow\{a\}$ for $a \in L$ is written as $\downarrow a$ and $\downarrow : L \to \mathcal{J}L$ is a frame homomorphism with left inverse $\bigvee : \mathcal{J}L \to L$ given by $J \longmapsto \bigvee J$.

A **σ-ideal** is an ideal I which is closed under countable joins, i.e, for any countable $X \subseteq I$, $\bigvee X \in I$. The collection of all σ-ideals of L is denoted by $\mathcal{H}L$. Like $\mathcal{J}L$, $\mathcal{H}L$ is a frame ordered by set inclusion. In fact we can view these as functors $\mathcal{J} : \mathbf{Frm} \to \mathbf{Frm}$ and $\mathcal{H} : \sigma\mathbf{Frm} \to \mathbf{Frm}$.

Moreover, if L is a σ-frame then $\mathcal{H}L$ is called the **envelope** of L. Similar to ideals, the relation between a σ-frame L and its frame envelope is a σ-frame homomorphism $L \longrightarrow \mathcal{H}L$ taking each $a \in L$ to its principal ideal $\downarrow a$, while the natural $\mathcal{H}L \longrightarrow L$ is given by a join map.

An ideal I in a frame L is called **proper** if $1_L \notin I$. It is called **σ- proper** if $\bigvee S \neq 1_L$ for every countable $S \subseteq I$, and also called **completely proper** if $\bigvee I \neq 1_L$.

1.2.5 Points of frames

We assume that the reader is familiar with the basics of topological spaces.

A **point** in a frame L is a frame homomorphism $\zeta : L \longrightarrow \mathbf{2}$. ($\mathbf{2}$ is the frame with two elements $0 \leq 1$.) The set of all points in a frame L is denoted ΣL. This is called the **spectrum of** L.

We already used the notation ΣL for the set of completely prime filters on a frame L. This is permitted because there are several equivalent descriptions of the points in a frame – homomorphisms, completely prime filters and prime elements.

By a **prime element** we mean an element p in a frame L which has the property that for any $a, b \in L$, $a \wedge b \leq p$ implies that either $a \leq p$ or $b \leq p$. (Prime elements are also called meet irreducible elements.)

Assume $\zeta : L \longrightarrow \mathbf{2}$ is a point, and put $F = \zeta^{-1}(1)$. Then:

- $a \in F$ and $a \leq b \Longrightarrow 1 = \zeta(a) \leq \zeta(b) \Longrightarrow \zeta(b) = 1$.
- $a, b \in F \Longrightarrow \zeta(a) = 1, \zeta(b) = 1 \Longrightarrow \zeta(a \wedge b) = \zeta(a) \wedge \zeta(b) = 1$.
- $\bigvee S \in F \Longrightarrow \zeta(\bigvee S) = 1 \Longrightarrow \bigvee_{s \in S} \zeta(s) = 1 \Longrightarrow \exists s \in S, \zeta(s) = 1 \Longrightarrow S \cap F \neq \emptyset$.

This correspondence is easily reversed. If F is a completely prime filter on L then $\zeta : L \longrightarrow \mathbf{2}$ is defined by $\zeta(a) = 1$ iff $a \in F$.

These points can be also described as prime (meet irreducible) elements $1 \neq p \in L$. For $\zeta : L \longrightarrow \mathbf{2}$ put

$$p = \bigvee \zeta^{-1}(0).$$

Then $a \wedge b \leq p \Longrightarrow \zeta(a) \wedge \zeta(b) = \zeta(a \wedge b) = 0 \Longrightarrow \zeta(a) = 0$ or $\zeta(b) = 0 \Longrightarrow a \leq p$ or $b \leq p$. Then p is prime. This correspondence is reversed by making $\zeta(a) = 0$ iff $a \leq p$ for prime element p and $a \in L$.

We now define a topology on the set ΣL. For $a \in L$ define

$$\Sigma_a = \{\zeta \in \Sigma L \mid \zeta(a) = 1\} \subseteq \Sigma L$$

It is easy to check that $\{\Sigma_a \mid a \in L\}$ is a topology on ΣL since $\Sigma_a \cap \Sigma_b = \Sigma_{a \wedge b}$ and $\bigcup_{a \in S} \Sigma_a = \Sigma_{\bigvee S}$ with $\Sigma_0 = \emptyset$ and $\Sigma_1 = \Sigma L$. (It is called the spectral topology.)

Now, assume that $h : L \longrightarrow M$ is a frame homomorphism and:

$$\Sigma h : \Sigma M \longrightarrow \Sigma L$$

is defined by:

$$\Sigma h(\zeta) = \zeta \circ h, \forall \zeta \in \Sigma M$$

For each $a \in L$, $(\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)}$, because $\zeta \in (\Sigma h)^{-1}(\Sigma_a) \Leftrightarrow (\Sigma h)(\zeta) \in \Sigma_a \Leftrightarrow (\zeta \circ h)(a) = 1 \Leftrightarrow \zeta(h(a)) = 1 \Leftrightarrow \zeta \in \Sigma_{h(a)}$. This shows that $\Sigma h : \Sigma M \longrightarrow \Sigma L$ is continuous.

In fact Σ : **Frm** $\longrightarrow$ **Top** is a (contravariant) functor which is right adjoint to $\mathcal{O}$: **Top** $\longrightarrow$ **Frm** sending each frame L to the space ΣL and each frame homomorphism to the continuous map $\Sigma h : \Sigma M \longrightarrow \Sigma L$.

For a topological space X define $\epsilon_X : X \longrightarrow \Sigma\mathcal{O}X$ by:

$$\epsilon_X(x)(U) = 1 \ (U \in \mathcal{O}X) \Longleftrightarrow x \in U$$

Note that ϵ_X is continuous since:

$$\epsilon_X^{-1}(\Sigma_U) = \{x \mid \epsilon_X(x) \in \Sigma_U\} = \{x \mid x \in U\} = U, \forall U \in \mathcal{O}X$$

If $h : L \longrightarrow \mathcal{O}X$ is a frame homomorphism, define $h' : X \to \Sigma L$ by $h'(x)(a) = 1 \Longleftrightarrow x \in h(a)$ which is unique with the property that $\Sigma h \circ \epsilon_X = h'$.

$$\begin{array}{ccc} X & \xrightarrow{\epsilon_X} & \Sigma\mathcal{O}X \\ & {\scriptstyle h'} \searrow & \downarrow {\scriptstyle \Sigma h} \\ & & \Sigma L \end{array}$$

On the other hand, define $\eta_L : L \longrightarrow \mathcal{O}\Sigma L$ for a frame L by $\eta_L(a) = \Sigma_a$. Then η_L is a frame homomorphism and we get adjunction identities:

$$\mathcal{O}X \xrightarrow{\eta\mathcal{O}X} \mathcal{O}\Sigma\mathcal{O}X \xrightarrow{\mathcal{O}\epsilon_X} \mathcal{O}X$$

$$\Sigma L \xrightarrow{\epsilon_{\Sigma L}} \Sigma\mathcal{O}\Sigma L \xrightarrow{\Sigma\eta_L} \Sigma L$$

In general $\epsilon_X : X \longrightarrow \Sigma\mathcal{O}X$ is not a bijective map, and if it is then X is called a **sober space**. Similarly, $\eta_L : L \longrightarrow \mathcal{O}\Sigma L$ is not generally a bijection, if it is L is called a **spatial frame**.

A closed set F in a topological space X is called **(join) irreducible** if $F = D_1 \cup D_2$ with D_1,D_2 closed implies $F = D_1$ or $F = D_2$.

Lemma 1.2.5. A space X is sober $\Longleftrightarrow$ each closed irreducible set F is the closure of a unique point.

Lemma 1.2.6. ([32] Propositions II.5.3, II.5.4.3, II.5.4.4)

(1) A frame L is spatial iff each of its elements is a meet of prime elements.

(2) In a Boolean algebra every prime element is a co-atom.

(3) Every element of a spatial Boolean frame is a meet of co-atoms and consequently, a Boolean frame is spatial only if it is atomic.

1.2.6 Frames of reals

There are various equivalent ways to introduce the frame of real numbers. We consider the description which is introduced in [4].

Definition 1.2.1. The frame of reals, denoted $\mathcal{L}(\mathbb{R})$, is the frame generated by all ordered pairs (p, q) where $p, q \in \mathbb{Q}$, subject to the relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.

(R2) $(p, q) \vee (r, s) = (p, s)$ where $p \leq r < q \leq s$.

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.

(R4) $1_{\mathcal{L}(\mathbb{R})} = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

Remark 1.2.2. It follows from (R3) that if $q \leq p$ then $(p, q) = 0$.

Proposition 1.2.1. ([32] Proposition XIV.2.1) The frame $\mathcal{L}(\mathbb{R})$ is completely regular.

Proof. If $p < r < s < q$ then $(r, s) \prec (p, q)$. Considering $\{(u, v) \mid p < u < r < s < v < q$ in $\mathbb{Q}\}$ it is clear that $(r, s) \prec\prec (p, q)$, and then by (R3), it immediately follows that $\mathcal{L}(\mathbb{R})$ is completely regular. □

1.2.7 Compactness and compactification of frames

A **cover** C of a frame L is a subset C of L such that $\bigvee C = 1$. A subset $D \subseteq C$ is a **subcover** of C if $\bigvee D = 1$.

A cover C is **co-completely regular** if for each $c \in C$, $\exists d \in C$ such that $c \prec\prec d$.

A frame L is **compact (countably compact)** if each of its covers (countable covers), admits a finite subcover. Similarly, L is **Lindelöf** if each of its covers admits a countable subcover.

A surjective (quotient map) dense frame homomorphism $h : M \longrightarrow L$ is called a **compactification** of L if M is a compact regular frame.

For completely regular frames, a compact frame K together with a dense frame homomorphism $h : K \longrightarrow L$ is called the Stone-Čech compactification of the frame L if for every dense frame homomorphism with compact domain $\varphi : K' \longrightarrow L$ there is a unique frame homomorphism $\varphi' : K' \longrightarrow K$ such that the following diagram commutes.

$$\begin{array}{ccc} K' & \overset{\varphi'}{\dashrightarrow} & K \\ & \underset{\varphi}{\searrow} & \downarrow h \\ & & L \end{array}$$

The existence of such a compactification for completely regular frames is well established in point free topology.

We note the following results without proof.

Proposition 1.2.2. ([32] Propositions VII.2.2.2, VII.6.3.4)

(1) Each dense frame homomorphism $h : M \longrightarrow L$ is injective if L is compact and M is regular.

(2) Each compact regular frame is spatial.

Note: Compactness in frames is hereditary. This is easy to see because joins in a subframe are exactly as in the larger frame.

1.2.8 Cozero Sets of Frames

An element a in a frame L is said to be a **cozero element** if there is a frame homomorphism:

$$\varphi : \mathcal{L}(\mathbb{R}) \longrightarrow L, \text{ such that } a = \varphi(-,0) \vee \varphi(0,-),$$

where $(-,0) = \bigvee\{(p,0) \mid 0 > p \in \mathbb{Q}\}$ in $\mathcal{L}(\mathbb{R})$ and $(0,-) = \bigvee\{(0,p) \mid 0 < p \in \mathbb{Q}\}$. We write the cozero element a above as $a = coz(\varphi)$ and denote by $CozL$ the set of all cozero elements of L.

The following results show that cozero elements can be characterised without requiring reference to the frame of reals, $\mathcal{L}(\mathbb{R})$.

Lemma 1.2.7. ([32] Propositions XIV 5.2.2 and 6.2.1) In a frame L, if $a \prec\!\prec b$ then there exists $\varphi : \mathcal{L}(\mathbb{R}) \to L$ with $a \leq \varphi(\frac{1}{2},-) \leq \varphi(0,-) \leq b$.

Proof. If $\{c_r \mid r \in \mathbb{Q} \cap [0,1]\}$ witnesses $a \prec\!\prec b$ then define $\varphi : \mathcal{L}(\mathbb{R}) \to L$ by

$$\varphi(x,-) = \bigvee_{x<r\leq 1} c_{1-r} \ \text{ if } \ 0 \leq x < 1$$

$$\varphi(-,x) = \bigvee_{0\leq r<x} c^*_{1-r} \ \text{ if } \ 0 < x \leq 1$$

with $\varphi(x,-) = 1$ if $x < 0$, $\varphi(x,-) = 0$ if $x \geq 1$ and $\varphi(-,x) = 0$ if $x \leq 0$, $\varphi(-,x) = 1$ if $x > 1$.

Similar to the proof of Proposition 2.2.1 below we can show that φ takes the relations (R1) to (R4) to identities in L and is thus a homomorphism. The inequalities $a \leq \varphi(\frac{1}{2},-) \leq \varphi(0,-) \leq b$ follow since

$$a \leq \bigvee_{0\leq s\leq\frac{1}{2}} c_s = \varphi(\frac{1}{2},-) \ \text{ and } \ \varphi(0,-) = \bigvee_{0<s\leq 1} c_s \leq b.$$

□

Proposition 1.2.3. (See [5] or [32] Propositions XIV 6.2.3) For any frame L the following are equivalent for $a \in L$:

(1) $a \in CozL$

(2) $a = \bigvee x_n$ where $x_n \prec\prec$ a for all $n = 1, 2, \ldots$

(3) $a = \bigvee a_n$ where $a_n \prec\prec a_{n+1}$, for all $n = 1, 2, \ldots$

Proof. : $(1 \Rightarrow 2)$ If $a = cos(\varphi)$ for some $\varphi : \mathcal{L}(\mathbb{R}) \longrightarrow L$ then, $a = \varphi(-,0) \vee \varphi(0,-) = \bigvee_{n\in\mathbb{N}} \varphi((-,-\frac{1}{n}) \vee (\frac{1}{n},-))$. Since $(-,-\frac{1}{n}) \vee (\frac{1}{n},-) \prec\prec (-,0) \vee (0,-)$ and any homomorphism preserves $\prec\prec$ which concludes that $x_n = \varphi((-,-\frac{1}{n}) \vee (\frac{1}{n},-)) \prec\prec a$.

$(2 \Rightarrow 3)$: Let $a = \bigvee_{n\in\mathbb{N}} x_n$ as in (2). Now define a_n inductively by:

$$a_1 = x_1,\ a_n \vee x_{n+1} \prec\prec a_{n+1} \prec\prec a,$$

which is possible since $\prec\prec$ interpolates and is stable under binary joins.

$(3 \Rightarrow 1)$: For each n, let $\{c^n_r \mid r \in \mathbb{Q} \cap [0,1]\}$ be an interpolating sequence witnessing $a_n \prec\prec a_{n+1}$. For each $r \in \mathbb{Q} \cap [0,1]$ define:

$$c_r = c^n_{\tau_n(r)} \text{ if } \frac{n-1}{n} \leq r < \frac{n}{n+1}$$

where τ_n is an increasing bijection between $\mathbb{Q} \cap [0,1]$ and $\mathbb{Q} \cap [\frac{n-1}{n}, \frac{n}{n+1}]$, which defines an interpolating sequence $\{c_r \mid r \in \mathbb{Q} \cap [0,1)\}$ between a_1 and a such that $a = \bigvee_{r\in\mathbb{Q}\cap[0,1)} c_r$, and $a_n = c_{1-\frac{1}{n}}$ for each $n \in \mathbb{N}$. Then define φ as follows

$$\varphi(p,q) = \bigvee\{c_r \wedge c^*_s \mid p < s < r < q\}\}$$

It can then be shown that φ is a frame homomorphism and:

$$\varphi(-,0) \vee \varphi(0,-) = 0 \vee \bigvee_{0<s<1} c_s = a.$$

□

The authors in [5] have also shown the following as significant consequences of Proposition 1.2.3 for any frame L:

- $CozL$ is a regular sub σ-frame of L.
- A frame L is completely regular if and only if it is generated by its cozero elements.
- In any completely regular Lindelöf frame L, $a \in L$ is cozero iff it is Lindelöf. (An element $a \in L$ is Lindelöf if for any $S \subseteq L$, $a \leq \bigvee S \Longrightarrow a \leq \bigvee S'$ for some countable $S' \subseteq S$.)

In fact Coz is a functor $\mathbf{Frm} \longrightarrow \mathbf{Reg\sigma Frm}$ which is right adjoint to $\mathcal{H}$ restricted to $\mathbf{Reg\sigma Frm}$. The unit and co-unit are $\downarrow : L \rightarrow Coz\mathcal{H}L$ given by taking any element of frame L to its principle ideal, and $\bigvee : \mathcal{H}CozL \longrightarrow L$ is given by the join map. This adjunction and the fact that it preserves compactness and regularity originally appeared in [33] and [8].

1.2.9 Nearness Frames

In frame theory, the set of all covers of a frame L is denoted by $CovL$. Let $A, B \in CovL$ then, we say that A **refines** B (written $A \leq B$) if for any $a \in A$, there exists $b \in B$ such that $a \leq b$.

Furthermore, we say that A **star refines** B, (written $A \leq^* B$) if $AA \leq B$ with:

$$AA = \{Ax \mid x \in A\} \text{ and } Ax = \bigvee\{s \in A \mid s \wedge x \neq 0\}.$$

Let $\mathcal{A}$ be a system of covers of L, the relation $\lhd_{\mathcal{A}}$ on L is called **uniformly below** and is defined by

$$x \lhd_{\mathcal{A}} y \text{ if and only if } Ax \leq y \text{ for some } A \in \mathcal{A}.$$

This system of covers $\mathcal{A}$ is then called **admissible** if each $x \in L$ is the join of elements uniformly below it:

$$\forall x \in L, \quad x = \bigvee \{y \mid y \lhd_{\mathcal{A}} x\}$$

A **nearness** $\mathcal{N}$ on a frame L is an admissible filter $\mathcal{N}$ in $CovL$. This nearness is called a **uniformity** if for each $A \in \mathcal{N}$ there exists $B \in \mathcal{N}$ such that $B \leq^* A$.

If $\mathcal{A}$ is a uniformity on L then the pair $(L, \mathcal{A})$ is called a uniform frame.

Let $(L, \mathcal{A})$ and $(M, \mathcal{B})$ be uniform frames. A uniform homomorphism $h : (L, \mathcal{A}) \longrightarrow (M, \mathcal{B})$ is a frame homomorphism $h : L \longrightarrow M$ such that:

$$\forall A \in \mathcal{A}, h(A) \in \mathcal{B}.$$

A uniform map $h : M \longrightarrow L$ is called a **surjection** if it is both onto on the underlying frames and the uniformities. A uniform frame L is said to be **complete** if every dense surjection $h : M \longrightarrow L$ is an isomorphism.

A **completion** of a uniform frame L is a dense surjection $M \longrightarrow L$ with M complete.

A nearness frame $(L, \mathcal{N})$ is **totally bounded** if every $A \in \mathcal{N}$ is refined by some finite $B \in \mathcal{N}$. In other word, a nearness frame is totally bounded iff every uniform cover has a finite uniform subcover.

A subset S of a frame L is said to be **locally finite** if there exists a cover C such that each element $c \in C$ meets finitely many elements of S. Then the frame is **paracompact** if every cover has a locally finite refinement.

1.2.10 Normal Frames

A frame L is said to be **normal** if for any $a, b \in L$, if $a \vee b = 1$ then there is $c \in L$ such that:

$$a \vee c = 1 \text{ and } c^* \vee b = 1.$$

A cover A of a frame L is said to be **normal** whenever there exists a sequence of covers $(A_n)_{n\in\mathbb{N}}$ such that $A = A_1$ and $A_{n+1} \leq^* A_n, \forall n$. Then L is called **fully normal** if every cover of it is normal.

Lemma 1.2.8. In a normal frame L the relation $\prec$ is interpolated and coincides with the $\prec\!\prec$ one, which implies that regularity coincides with complete regularity.

Lemma 1.2.9.

- For every cover $\{a_i \mid i \in \mathbb{N}\}$ of a normal frame L there is a cover $\{b_i \mid i \in \mathbb{N}\}$ such that $\forall i$, $b_i \prec a_i$.
- A compact regular frame is normal.

1.2.11 Metrizable Frames

A frame L is **metrizable** if it admits a countably generated uniformity. We collect some useful facts about such frames. The proof of these facts can be found in [17, 36].

Facts 1.2.1.

- Each metrizable frame is fully normal.
- Each metrizable frame is paracompact.
- A metrizable frame is compact iff it is countably compact.
- The quotient of a metrizable frame is metrizable.

2. PSEUDOCOMPACTNESS

As indicated in the introduction, the study of pseudocompactness in a topological space was initiated by Hewitt [22]. He gave one characterisation of the property in terms of the Stone-Čech compactification, and another in terms of the zero sets of continuous real-valued functions.

In [20] Glickberg characterised pseudocompactness via convergence properties of sequences of continuous functions and in terms of sequences of closed neighbourhoods. For instance, a space X is pseudocompact if for any $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, a sequence of non empty open subsets in X such that $\overline{U_i} \cap \overline{U_j} = \emptyset$ whenever $i \neq j$, then $\mathcal{U}$ has a cluster point in X. (Consequently pseudocompact spaces are those spaces in which each locally finite family of open subsets is finite.)

Mardešic and Papic [31] obtained a very elegant characterization of pseudocompactness in terms of a covering property which was discussed by Iseki and Kasahara [24] in which a completely regular space is pseudocompact if and only if every locally finite open covering has a finite subcovering.

Bagley, Connell and McKnight [2] characterised pseudocompact completely regular spaces by means of a convergence property of continuous functions. That is, a space X is pseudocompact if every locally convergent sequence of continuous functions on X, converges uniformly on X. This happens if every sequence of continuous functions which converges uniformly at each point of X, converges uniformly on X.

Stephenson [34] investigated whether subspaces of pseudocompact spaces are pseudocompact, he also presented several product theorems which apply to non completely regular pseudocompact spaces. (For completely regular spaces, these properties related to pseudocompactness were investigated in [2] as mentioned above.)

In the current chapter, we start with an introduction of pseudocompactness in classical topology. Then we turn to the pointfree context and present some results on pseudocmpactness of frames.

2.1 Pseudocompact spaces

In this section we denote the family of continuous functions between topological spaces X and Y by $C(X,Y)$, with $C(X, \mathbb{R})$ denoted by $C(X)$ where $\mathbb{R}$ has the usual topology. Furthermore $C^*(X)$ will denote the family of bounded functions in $C(X)$.

These rings of real valued continuous functions on topological spaces originally received interest because some of their algebraic properties could describe topological properties of the underlying spaces. Also, algebraic techniques applied to these rings might serve as powerful tools for solving topological problems. In the case of completely regular spaces, the rings of all continuous real valued functions are large enough to describe the topology of base spaces.

In [22], Hewitt used a relationship between $C(X)$ and $C^*(X)$ to describe a topological property called pseudocompactness. A space X is **pseudocompact** if the two rings coincide, that is if $C(X) = C^*(X)$.

Before we continue, we establish notation and terminology that will assist in describing such pseudocompact spaces.

If X is a topological space and $f \in C(X)$, the **zero set** $f^{-1}(0)$ will be denoted by $Z(f)$. Put $\mathcal{Z}(X) = \{Z(f) \mid f \in C(X)\}$. The **cozero** set $X - Z(f) = f^{-1}(\mathbb{R} - \{0\})$ is denoted by $Coz(f)$, put $\mathcal{Y}(X) = \{Coz(f) \mid f \in C(X)\}$.

$L(X)$ is the set of all continuous functions f such that $f : X \longrightarrow [0,1]$.

If $\mathcal{B}$ is a collection of sets, then $\widehat{\mathcal{B}}$ is the notation for the set of all finite intersections of elements of $\mathcal{B}$, and $\mathcal{B}$ is said to be **fixed** (**free**) if $\bigcap \mathcal{B} \neq \emptyset$ ($\bigcap \mathcal{B} = \emptyset$).

A filter base $\mathcal{F}$ on a space X is said to be an **open filter base** if and only if every $F \in \mathcal{F}$ is open. An open filter base $\mathcal{F}$ is called **completely regular** if for each $F \in \mathcal{F}$ there exists $F' \in \mathcal{F}$ and a function $f \in L(X)$ such that $f(F') = 0$ and $f(X - F) = 1$.

An open cover $\mathcal{C}$ of a space X is said to be **cocompletely regular** if for each $A \in \mathcal{C}$, $\exists A' \in \mathcal{C}$ and a function $f \in L(X)$ such that f vanishes on A and equal to 1 on $X - A'$. (We should note that a continuous map f vanishes on A whenever $f(A) = 0$.)

We refer to [34], [22] and [39] as references for this part of the thesis.

Recall that a topological space X is called **countably compact** if and only if each of its countable open covers admits a finite subcover. Countable compactness has a number of useful characterisations, of which we mention two:

- A space is countably compact if and only if each of its sequences has a cluster point.
- A T_1-space is countably compact if and only if every infinite subset of it has a cluster point.

Definition 2.1.1. A space X is said to be **pseudocompact** iff every continuous real-valued function on X is bounded, i.e, if $C(X) = C^*(X)$.

Example 2.1.1. Every countably compact space is pseudocompact according to Theorem 17.13 in [39], which says that:

A continuous real-valued function on a countably compact space is bounded.

To see this, suppose that $f : X \longrightarrow \mathbb{R}$ is a continuous map on a countably compact space X, then $\{f^{-1}[(-n,n)] \mid n \in \mathbb{N}\}$ is a countable open cover of X, countable compactness of X implies that $f^{-1}[(-k,k)] = X$ for some $k \in \mathbb{N}$. Thus f is bounded.

It is well known that a topological space X is called **normal** if and only if whenever A and B are disjoint closed sets in X, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$, a space Y is a completely normal space if every subspace of Y is a normal space.

In terms of the normal space definition, it is easy to see that a compact Hausdorff space is normal since every compact Hausdorff space is a T_4-space. (Theorem (17.10) in [39], where a T_4- space is defined as a normal T_1-space.) Furthermore, if X is a metric space, then X is normal.

Urysohn's Lemma says: A space is normal if and only if whenever A_1 and A_2 are non-empty disjoint closed sets in X, there exists a continuous function $f : X \longrightarrow [0,1]$, with $f(A_1) = \{0\}$ and $f(A_2) = \{1\}$. This leads to **Tietze's extension theorem** which states:

If X is a normal topological space and $f : A \longrightarrow \mathbb{R}$ is a continuous map with A a closed set in X, then there is a continuous map $g : X \longrightarrow \mathbb{R}$ with $g(a) = f(a), \forall a \in A$. Furthermore g may be chosen such that $\sup\{|f(a)| \mid a \in A\} = \sup\{|g(x)| \mid x \in X\}$ i.e if f is bounded then g is also bounded. Such g is called a continuous extension of f.

The following Lemma was proved in [22] by using the idea of Tietze's extension theorem.

Lemma 2.1.1. A normal space is pseudocompact if and only if it is countably compact.

The equivalences in the next proposition are proved by Hewitt and Stephenson in [22] and [34]. The result sets out a number of characterisations of pseudocompactness, which have served as an inspiration for many topologists to prove similar result in the point-free context as we can see in the next section.

Note that while pseudocompactness can be defined for any topological space X, it is more meaningful for completely regular topological spaces as these spaces have enough continuous maps to $\mathbb{R}$ for $C(X)$ and $C^*(X)$ to hold significant information about X.

Proposition 2.1.1. For any completely regular topological space X, the following are equivalent:

(A) X is pseudocompact.

(B) For every space Y and function $f \in C(X,Y)$, $f(X)$ is pseudocompact.

(C) For every completely normal space Y and function $f \in C(X,Y)$, $f(X)$ is countably compact.

(D) For every metric space Y and function $f \in C(X,Y)$, $f(X)$ is compact.

(E) For every $f \in C(X)$, $f(X)$ is compact.

(F) For every $f \in C^*(X)$, $f(X)$ is compact.

(G) $f(X)$ is a closed subset of $\mathbb{R}$ for every $f \in C^*(X)$.

(H) For every $f \in C^*(X)$, $\exists x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$.

(I) If $\mathcal{B}$ is a countable subset of $\mathcal{Z}(X)$ and $\emptyset \notin \widehat{\mathcal{B}}$, then $\mathcal{B}$ is fixed.

(J) Every locally finite subset of $\mathcal{Y}(X)$ is finite.

(K) Every countable completely regular filter base on X is fixed.

(L) Every countable co-completely regular cover of X has a finite subcover.

Proof.

(A) $\Rightarrow$ (B) Assume X is pseudocompact and $f \in C(X,Y)$. Consider the factorisation of f through $f(X)$ and let $\phi \in C(f(X))$.

$$\begin{array}{ccccc} X & \xrightarrow{f'} & f(X) & \xrightarrow{\phi} & \mathbb{R} \\ & \searrow^{f} & \downarrow & & \\ & & Y & & \end{array}$$

Since X is pseudocompact $\phi \circ f'$ is bounded. So there exists $(a,b) \subseteq \mathbb{R}$ such that $(\phi \circ f')^{-1}[(a,b)] = X \Longrightarrow f'^{-1}(\phi^{-1}[(a,b)]) = X \Longrightarrow f'(f'^{-1}(\phi^{-1}[(a,b)])) = f(X)$. Since f' is onto this implies that:

$$\phi^{-1}[(a,b)] = f(X).$$

(B) $\Rightarrow$ (C) Follows from Lemma 2.1.1.

(C) $\Rightarrow$ (D) Since any metric space is completely normal and countably compact metric spaces are compact.

(D) $\Rightarrow$ (E) Since $\mathbb{R}$ is a metric space.

(E) $\Rightarrow$ (F) Clear, since $C^*(X) \subseteq C(X)$.

(F) $\Rightarrow$ (G) By the Heine-Borel Theorem, that is, a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

(G) $\Rightarrow$ (H) Let $f \in C^*(X)$ and assume there is no $x_0 \in X$, such that $f(x_0) = \inf_{x \in X} f(x)$. Now put $\inf_{x \in X} f(x) = t$ then t is in the closure of $f(X)$ but not in $f(X)$. This contradicts (G).

(H) $\Rightarrow$ (I) Let $\mathcal{B} = \{Z_n \mid n \in \mathbb{N}\}$ be a family of zero sets such that $\emptyset \notin \widehat{\mathcal{B}}$, but $\bigcap \mathcal{B} = \emptyset$. For each $n \in \mathbb{N}$, put $B_n = \bigcap_{i=1}^{n} Z_i$ then

$$\ldots \subset B_n \subset \ldots \subset B_3 \subset B_2 \subset B_1 \quad \text{and} \quad \bigcap_{n \in \mathbb{N}} B_n = \emptyset.$$

Note that for each $n \in \mathbb{N}$ there exists a function $f_n \in C(X)$ such that $B_n = f_n^{-1}(0)$, where f_n is the finite product of functions corresponding to zero sets Z_i, $i = 1 \ldots n$.

Now define $g_n = \min\{f_n^2, 1\}$ then $g_n^{-1}(0) = B_n$ and consider the function

$$\varphi = \sum_{n=1}^{\infty} 2^{-n} \cdot g_n.$$

Since $\bigcap_{n=1}^{\infty} B_n = \emptyset$, for each $p \in X$ there exists $m \in \mathbb{N}$ with $p \in B_m^c$. Thus

$$\varphi(p) \geq 2^{-m} \cdot g_m(p) > 0$$

and φ is strictly positive. On the other hand, if $p \in B_n$ then

$$g_1 = g_2 = \ldots = g_n(p) = 0$$

and so

$$\varphi(p) = \sum_{k=n+1}^{\infty} 2^{-k} \cdot g_k(p) \leq 2^{-n}.$$

Since $\bigcap \widehat{\mathcal{B}} \neq \emptyset$ it follows that $\inf_{p \in X} \varphi(p) = 0$ and φ is a function which contradicts (H).

(I) $\Rightarrow$ (J) Suppose that $\mathcal{D} = \{C_n \mid n \in \mathbb{N}\}$ is an infinite locally finite system of non empty elements of $\mathcal{Y}(X)$, it follows from the normality of $\mathbb{R}$ that, for each i there is a function $g_i \in L(X)$ which equal to 1 on $X - C_i$ and vanishes on C_i. For each $n \in \mathbb{N}$ and $x \in X$ let $h_n(x) = \inf\{g_i(x) \mid i \leq n\}$, then each $Z(h_n)$ is non empty and contain $Z(h_{n+1})$, and since $\mathcal{D}$ is locally finite, each $h_n \in L(X)$ and $\bigcap\{Z(h_n)\} = \emptyset$, which contradicts (I)

(J) $\Rightarrow$ (K) Let $\mathcal{F} = \{F_n \mid n \in \mathbb{Z}\}$ be a completely regular filter base on X such that $F_{n+1} \subset F_n$. For each $F \in \mathcal{F}$ choose a function $f_F \in L(X)$, which vanishes on $X - F$ and equals to 1 on some set in F. Then (J) implies that there is a point x at which $\{Coz(f_F) \mid F \in \mathcal{F}\}$ is not locally finite. Evidently $x \in \{\overline{F} \mid F \in \mathcal{F}\} = \bigcap \mathcal{F}$.

(K) $\Rightarrow$ (L) Assume that there is a countable co-completely regular cover $\mathcal{U}$ of X which has no finite subcover. Then $\emptyset \notin \{X - \widehat{U \mid U} \in \mathcal{U}\}$. Since for each set $U \in \mathcal{U}$ there is a set, U' such that $\overline{U} \subset U'$, and also $\emptyset \notin \mathcal{V} = \{X - \widehat{\overline{U} \mid U} \in \mathcal{U}\}$. Thus $\mathcal{V}$ is a countable open filter base on X. Consider $G = \bigcap\{X - \overline{U}_i \mid i = 1, ..., s\} \in \mathcal{V}$. For each i there is $U'_i \in \mathcal{U}$ and a function $f_i \in L(X)$ such that f_i vanishes on U_i and equals to 1 on $X - U'_i$. Define $G' = \bigcap\{X - \overline{U'_i} \mid i = 1, ..., s\}$ for some $s \in \mathbb{N}$ and $f = \min\{f_i \mid 1, ..., s\}$, and let g be the function given by $g(x) = 1 - f(x)$. Then $G' \in \mathcal{V}$, $g \in L(X), g(G') = 0$, and $g(X - G) = 1$. Thus $\mathcal{V}$ is free.

(L) $\Rightarrow$ (A) For any arbitrary function $f \in C(X)$. Let $U_n = f^{-1}((-n, n))$,

$\forall n \in \mathbb{N}$, and define $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$. Then $\mathcal{U}$ is a countable co-completely regular cover of X, so (L) implies that there is $k \in \mathbb{N}$ such that $X \subseteq U_k$, thus $f \in C^*(X)$.

□

2.2 Pseudocompact frames

Definition 2.2.1. [5] A frame homomorphism $\varphi : \mathcal{L}(\mathbb{R}) \longrightarrow L$ is called **bounded** if there exist $p, q \in \mathbb{Q}$ such that $\varphi(p, q) = 1_L$. The frame L is said to be **pseudocompact** whenever all frame homomorphisms $\varphi : \mathcal{L}(\mathbb{R}) \longrightarrow L$ are bounded.

It is well established that these properties are conservative. We give the details of the proof below. Remember that $\mathcal{L}(\mathbb{R})$ is generated by pairs (p, q) with $p, q \in \mathbb{Q}$.

Proposition 2.2.1. [5] A topological space X is pseudocompact if and only if the frame $\mathcal{O}X$ is pseudocompact.

Proof. For any topological space X, there is a bijective map

$$\mathbf{Frm}(\mathcal{L}(\mathbb{R}), \mathcal{O}X) \longrightarrow \mathbf{Top}(X, \mathbb{R})$$

taking each φ: $\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{O}X$ to $\bar{\varphi} : X \longrightarrow \mathbb{R}$, where

$$p < \bar{\varphi}(x) < q \Leftrightarrow x \in \varphi(p, q).$$

First note that the points of $\mathcal{L}(\mathbb{R})$ coincide with the points of $\mathbb{R}$. Any homomorphism $\zeta : \mathcal{L}(\mathbb{R}) \longrightarrow \mathbf{2}$ determines $\lambda \in \mathbb{R}$ such that $p < \lambda < q$ iff $1_\mathbf{2} = \zeta(p, q)$. This λ is given by the Dedekind cut (U, V) where:

$$U = \{r \in \mathbb{Q} \mid \zeta(r, q) = 1_\mathbf{2} \text{ for some } q \in \mathbb{Q}\},$$
$$V = \{s \in \mathbb{Q} \mid \zeta(p, s) = 1_\mathbf{2} \text{ for some } p \in \mathbb{Q}\}.$$

Observe that (U, V) is an open Dedekind cut. To check that:

(1) $1_{\mathcal{L}(\mathbb{R})} = \bigvee\{(p,q) \mid p,q \in \mathbb{Q}\} \implies 1_{\mathbf{2}} = \bigvee\{\zeta(p,q) \mid p,q \in \mathbb{Q}\} \implies \exists p$, $q \in \mathbb{Q}$ such that $\zeta(p,q) = 1_{\mathbf{2}}$ and so $p \in U$ and $q \in V$.

(2) To show that V is a upset, let $s' \geq s \in V$ then $(p,s) \leq (p,s')$. Thus, $\zeta(p,s) = 1_{\mathbf{2}} \implies \zeta(p,s') = 1_{\mathbf{2}}$ and $s' \in V$. Similarly U is a downset.

(3) Let $r \in U$ then $(r,q) = \bigvee_{n\in\mathbb{N}}(r+\frac{1}{n}, q)$, since $\zeta(r,q) = 1_{\mathbf{2}} \implies \bigvee_{n\in N}\zeta(r+\dfrac{1}{n}, q) = 1_{\mathbf{2}}$, this implies that $\exists n_0 \in \mathbb{N}$ such that $\zeta(r+\dfrac{1}{n_0}, q) = 1_{\mathbf{2}}$, and similarly for any $s \in V$.

(4) Let $p \leq q$ in $\mathbb{Q}$, then

$\bigvee_{n\in\mathbb{N}}((q-n,q) \vee (p,p+n)) = 1_{\mathcal{L}(\mathbb{R})}$
$\implies \zeta\bigvee_{n\in\mathbb{N}}((q-n,q) \vee (p,p+n)) = 1_{\mathbf{2}}$
$\implies \bigvee_{n\in\mathbb{N}}(\zeta(q-n,q) \vee \zeta(p,p+n)) = 1_{\mathbf{2}}$
$\implies \exists n_0, \zeta(q-n_0,q) \vee \zeta(p,p+n_0) = 1_{\mathbf{2}}$
$\implies \exists n_0, \zeta(q-n_0,q) = 1_{\mathbf{2}}$ or $\zeta(p,p+n_0) = 1_{\mathbf{2}}$
$\implies p \in U$ or $q \in V$.

(5) Let $m \in V \cap U$. Then there exist r and s with $\zeta(r,m) = 1_{\mathbf{2}} = \zeta(m,s)$. Hence $\zeta(r,m) \wedge (m,s) = 1$ but $(r,m) \wedge (m,s) = 0$, a contradiction,

This shows that (U, V) is a Dedekind cut. Taking (U, V) to represent a real number λ, for $p, q \in \mathbb{Q}$ we have that

$$p \leq \lambda \Leftrightarrow p \in U \text{ and } \lambda \leq q \Leftrightarrow q \in V \text{ if and only if } \lambda \in (p,q).$$

We note that $\zeta(p,q) = 1_{\mathbf{2}} \Leftrightarrow p \in U$ and $q \in V$ and so (U, V) determines λ such that $\lambda \in (p,q) \Leftrightarrow \zeta(p,q) = 1_{\mathbf{2}}$ as claimed.

On the other hand, given $\lambda \in \mathbb{R}$, define $\zeta_\lambda : \mathcal{L}(\mathbb{R}) \longrightarrow \mathbf{2}$ by

$$\zeta_\lambda(p,q) = 1_{\mathbf{2}} \Leftrightarrow \lambda \in (p,q)$$

then using the relations R1 to R4 for the frame of reals we observe:

$(R1')$ $\zeta_\lambda(p,q) \wedge \zeta_\lambda(m,n) = 1_{\mathbf{2}} \Leftrightarrow \zeta_\lambda(p,q) = 1_{\mathbf{2}} = \zeta_\lambda(m,n) \Leftrightarrow \lambda \in (p,q)$ and $\lambda \in (m,n) \Leftrightarrow \lambda \in (p \vee m, q \wedge n) \Leftrightarrow \zeta_\lambda(p \vee m, q \wedge n) = 1_{\mathbf{2}}$.

$(R2')$ Consider $p \leq r < s \leq q$. Then $\zeta_\lambda(p,q) = 1_{\mathbf{2}} \Leftrightarrow \lambda \in (p,q) \Leftrightarrow \lambda \in (p,s)$ or $\lambda \in (r,q) \Leftrightarrow \zeta_\lambda(p,s) \vee \zeta_\lambda(r,q) = 1_{\mathbf{2}}$.

$(R3')$ $\zeta_\lambda(p,q) = 1_{\mathbf{2}} \Leftrightarrow \lambda \in (p,q) \Leftrightarrow \exists(r,s)$ such that $p < r < s < q$ with $\lambda \in (r,s) \Leftrightarrow \exists(r,s)$ such that $p < r < s < q$ with $\zeta_\lambda(r,s) = 1_{\mathbf{2}} \Leftrightarrow \bigvee\{\zeta_\lambda(r,s) \mid p < r < s < q\} = 1_{\mathbf{2}}$.

$(R4')$ For any $\lambda \in \mathbb{R}$, $\exists p,q \in \mathbb{Q}$ such that $p < \lambda < q$ then $\zeta_\lambda(p,q) = 1_{\mathbf{2}}$. Thus $1_{\mathbf{2}} = \zeta_\lambda(p,q) \leq \bigvee\{\zeta_\lambda(r,s) \mid r,s \in \mathbb{Q}\} \leq \zeta(1_{\mathbb{R}}) = 1_{\mathbf{2}}$.

Thus we have a frame homomorphism $\zeta_\lambda : \mathcal{L}(\mathbb{R}) \longrightarrow \mathbf{2}$ and the correspondences established above $\zeta \longmapsto \lambda$ and $\lambda \longmapsto \zeta_\lambda$ are inverse to each other. Hence we have a bijective map;

$$\tau : \Sigma\mathcal{L}(\mathbb{R}) \longrightarrow \mathbb{R}$$

satisfying $p < \tau(\zeta) < q \Leftrightarrow \zeta(p,q) = 1 \; \forall p,q \in \mathbb{Q}$. It remains to show that τ is a homeomorphism.

Now, the topology of $\Sigma\mathcal{L}(\mathbb{R})$ is generated by the sets

$$\Sigma_{(p,q)} = \{\zeta \in \Sigma\mathcal{L}(\mathbb{R}) \mid \zeta(p,q) = 1\}$$

and τ maps these to

$$\tau(\Sigma_{(p,q)}) = \{\tau(\zeta) \mid \zeta(p,q) = 1\} = \{\tau(\zeta) \mid p < \tau(\zeta) < q\} = (p,q)$$

the open real intervals which generate $\mathbb{R}$. This induces an isomorphism $\mathcal{O}\Sigma\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{O}\mathbb{R}$ taking each $\Sigma_{(p,q)}$ to (p,q).

Applying the observations above shows that any continuous function $f : X \to \mathbb{R}$ corresponds to a unique homomorphism

$$\varphi : \mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{O}X$$

via:

$$\varphi(p,q) = \{x \in X \mid p < f(x) < q\}.$$

Clearly f is bounded (for some p, q, $p < f(x) < q$ for all $x \in X$) if and only if φ is bounded (for some p, q, $\varphi(p,q) = X$). Hence a space X is pseudocompact if and only if the frame $\mathcal{O}X$ is pseudocompact. □

Proposition 2.2.2. [17] For any pseudocompact frame L and any $a \in L$ with $\uparrow(a \vee a^*)$ pseudocompact, then $\uparrow a^*$ is also pseudocompact.

Proof. Assume that h and g are frame homomorphisms such that:

$$\mathcal{L}(\mathbb{R}) \xrightarrow{h} \uparrow a^* \xrightarrow{g} \uparrow(a \vee a^*)$$

$$\text{where } g \text{ maps } x \longmapsto x \vee a$$

Since $\uparrow (a \vee a^*)$ is pseudocompact, $g \circ h$ is a bounded frame homomorphism. Thus $\exists s \in \mathbb{R}$ such that

$$1_L = g(h(-s,s)) = h(-s,s) \vee a.$$

Now, define a map $f : \mathcal{L}(\mathbb{R}) \longrightarrow L$ by:

$$f(U) = \begin{cases} h(s,\infty) \wedge h(U) \wedge a & , s \notin U \\ h(-s,s) \vee h(U) & , s \in U \end{cases} \tag{2.1}$$

which is a frame homomorphism and the way to check that is similar to the argument used in Proposition 3.4.3.

Now, since L is pseudocompact $\exists r \in \mathbb{R}$ with $f(-r,r) = 1_L$. By the definition of f it follows that $s \in (-r,r)$. So $1_L = f(-r,r) = h(-s,s) \vee h(-r,r) = h(-r,r)$. Therefore h is bounded which implies that $\uparrow a^*$ is pseudocompact.

□

Lemma 2.2.1. [3] For any compact σ-frame, its frame envelope is compact as well.

Proposition 2.2.3. [5] For any completely regular frame L the following are equivalent:

(1) L is pseudocompact

(2) Any sequence $a_0 \prec\prec a_1 \prec\prec a_2 \prec\prec \ldots$ in L with $\bigvee a_n = 1_L$ terminates, that is $a_k = 1_L$ for some k.

(3) $CozL$ is compact

(4) The frame $\mathcal{H}CozL$ is compact

Proof. (1) $\Rightarrow$ (2) For any sequence $a_0 \prec\prec a_1 \prec\prec a_2 \prec\prec \ldots$ in L with $\bigvee a_n = 1_L$, assume that c_n and $\varphi(p,q)$ are defined as in Proposition 1.2.3. We have bounded $\varphi : \mathcal{L}(\mathbb{R}) \longrightarrow L$ since L is a pseudocompact frame. So $\exists p, q \in \mathbb{Q}$ such that $\varphi(p,q) = 1$. We may assume that $q < 1$ and so there is $k \in \mathbb{N}$ with $q < 1 - \frac{1}{k} < 1$ which gives:

$$a_k \geq \bigvee\{c_{\acute{q}} \mid p < \acute{q} < q\} \geq \bigvee\{c^*_{\acute{p}} \wedge c_{\acute{q}} \mid p < \acute{p} < \acute{q} < q\} = \varphi(p,q) = 1$$
$$\Longrightarrow a_k = 1$$

(2) $\Rightarrow$ (1) Assume that, $\varphi : \mathcal{L}(\mathbb{R}) \longrightarrow L$ is a frame homomorphism. Put $a_n = \varphi(-,n)$ then $a_0 \prec\prec a_1 \prec\prec a_2 \prec\prec \ldots$ in L and $\bigvee a_n = 1 \Rightarrow \exists k$ such that $a_k = 1$.

Similarly, put $b_n = \varphi(-n,-)$ and $b_0 \prec\prec b_1 \prec\prec b_2 \prec\prec \ldots$ in L with $\bigvee b_n = 1$. Thus $\exists j$ such that $b_j = 1$. Therefore:

$$1 = a_k \wedge b_j = \varphi(-,k) \wedge \varphi(-j,-) = \varphi(-j,k)$$

Thus L is pseudocompact since φ is a bounded frame homomorphism.

(2) $\Rightarrow$ (3) If $1_L = \bigvee_n \{a_n \mid a_n \in CozL\}$ then by Proposition 1.2.3, for each $n \in \mathbb{N}$ we can find a_{nk} for all $k \in \mathbb{N}$ such that $a_n = \bigvee_{k\in\mathbb{N}} a_{nk}$ where $a_{nk} \prec\prec a_{n(k+1)}$.

Put $c_n = a_{1n} \vee a_{2n} \vee \ldots\ldots \vee a_{nn}$. Then $c_n \prec\!\prec c_{n+1}$ and $\bigvee c_n = 1_L$, hence $c_k = 1_L$ for some k, then $a_1 \vee a_2 \ldots\ldots \vee a_k = 1_L$, from which the compactness follows.

(3) $\Rightarrow$ (4) Since $CozL$ is a sigma frame, by Lemma 2.2.1 $\mathcal{H}CozL$ is compact.

(4) $\Rightarrow$ (1) Any φ: $\mathcal{L}(\mathbb{R}) \longrightarrow L$ lifts through $\mathcal{H}CozL$. Thus there exists $\overline{\varphi} : \mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{H}CozL$ such that $\bigvee \circ \overline{\varphi} = \varphi$. Since $\mathcal{L}(\mathbb{R})$ is completely regular, as was remarked after Proposition 1.2.3, it follows that φ: $\mathcal{L}(\mathbb{R}) \longrightarrow L$ is a map into $CozL$.

Now define $\overline{\varphi}(p, q) \in \mathcal{H}CozL$, for any $p, q \in \mathbb{Q}$ by

$$\overline{\varphi}(p, q) = \downarrow\varphi(p, q) \text{ in } CozL.$$

$$\begin{array}{ccc} \mathcal{L}(\mathbb{R}) & \xrightarrow{\overline{\varphi}} & \mathcal{H}CozL \\ & \searrow^{\varphi} & \downarrow \bigvee \\ & & L \end{array}$$

Now the map

$$\mathcal{L}(\mathbb{R}) \xrightarrow{\varphi} CozL \xrightarrow{\downarrow} \mathcal{H}CozL \xrightarrow{\bigvee} L$$

is a σ-frame homomorphism, and since

$$\bigvee \overline{\varphi}(p, q) = \bigvee \downarrow\varphi(p, q),$$

it follows that

$$\bigvee \bar{\varphi} = \varphi.$$

Because (p, q) generate $\mathcal{L}(\mathbb{R})$, and it is given that $\mathcal{H}CozL$ is compact, we conclude that $\overline{\varphi} : \mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{H}CozL$ is bounded which makes $\bigvee \bar{\varphi} = \varphi$ also bounded. Thus L is pseudocompact.

$\square$

Independently, Clarke and Gilmour [10] have recently presented the pseudocompactness property on sigma frames, with results bearing much similarity to the pseudocompactness characterizations in Proposition 2.1.1. The main difference between them is that pseudocompactness is not always available in sigma frames (as it is in frames).

A related result to those above states that a pseudocompact frame is compact if it is Lindelöf, this conclusion was evolved from the following observations:

A countably generated regular frame is pseudocompact if and only if it is compact and then apply the fact that every frame which admits a countable basis is a Lindelöf one [5].

Lemma 2.2.2. [17] A paracompact normal frame is pseudocompact if and only if it is countably compact.

Proof. ($\Rightarrow$) Suppose $\{a_i \mid i \in \mathbb{N}\}$ is a countable cover of a normal paracompact frame L. Then there exists a cover $\{b_i \mid i \in \mathbb{N}\}$ such that $b_i \prec a_i$ for each $i \in \mathbb{N}$ (Lemma 1.2.9). Now since L is a normal frame, then $b_i \prec\prec a_i$ (Lemma1.2.8) and thus there is a cozero element c_i such that $b_i \prec c_i \prec a_i$. Therefore $\{c_i \mid i \in \mathbb{N}\}$ is a cover of $CozL$. Thus there are finitely many c_i which have join 1_L since $CozL$ compact by pseudocompactness of L. So there are finitely many a_i that cover L therefore L countably compact.

($\Leftarrow$) Obvious.

□

Proposition 2.2.4. [17] The following are equivalent for any frame L:

(1) L is pseudocompact

(2) For any injective map $h : M \longrightarrow L$, M is pseudocompact.

(3) For any injective map $h : M \longrightarrow L$ with M normal and paracompact, M is countably compact.

(4) If M is a metrizable frame with an injective map $h : M \longrightarrow L$, M is compact.

(5) For any composition

$$\mathcal{O}\mathbb{R} \xrightarrow{f} M \xrightarrow{h} L$$

If f is a surjective map and h is an injective map, $f(\mathcal{O}\mathbb{R})$ is compact.

Proof.

(1)⇒ (2) Assume that $a_0 \prec\prec a_1 \prec\prec a_2 \prec\prec \ldots$ is a sequence in M with $\bigvee a_n = 1$. Then $h(a_0) \prec\prec h(a_1) \prec\prec h(a_2) \prec\prec \ldots$ is a sequence on L, and $\bigvee h(a_n) = 1$ and by pseudocompactness of L there exists k with $h(a_k) = 1$. Since h is injective, $a_k = 1$.

(2)⇒ (3) Lemma 2.2.2 shows that M is countably compact.

(3)⇒ (4) Follows since a countably compact metrizable frame M is compact. (By facts 1.2.1.)

(4)⇒ (5) Since $\mathcal{O}\mathbb{R}$ is metrizable then by facts 1.2.1, $f(\mathcal{O}\mathbb{R})$ is metrizable. Then M is compact implies that $f(\mathcal{O}\mathbb{R}) \subseteq M$ is also compact.

(5)⇒ (1) Consider $\phi : \mathcal{O}\mathbb{R} \longrightarrow L$ factoring through its image

$$\mathcal{O}\mathbb{R} \xrightarrow{f} f(\mathcal{O}\mathbb{R}) \xrightarrow{h} L, \qquad \phi = h \circ f$$

with surjective f and injective h. Since $f(\mathcal{O}\mathbb{R})$ is compact and the set $\{f(-n, n) \mid n \in \mathbb{N}\}$ covers $f(\mathcal{O}\mathbb{R})$, then $\exists k \in \mathbb{N}$ such that $h(f(-k, k))) = h(1) = 1_L$ showing that L is pseudocompact.

□

Proposition 2.2.5. [17] For any frame L, the following are equivalent:

(1) L is pseudocompact.

(2) Every locally finite subset of $CozL$ is finite.

(3) Every countable completely regular filter base in L clusters.

(4) Every countable co-completely regular cover of L admits a finite subcover.

Proof.

(1)⇒ (2) Suppose that there exists a countably infinite locally finite set $B \subseteq CozL$ consisting of nonzero elements. Let C be a cover of $CozL$ that finitizes B, i.e. for any $a \in C$, $a \wedge b_n = 0$ for all but finitely many $b_n \in B$. Now for any $n \in \mathbb{N}$ define a_n:

$$a_n = \bigvee\{x \mid x \wedge b_k = 0, \forall k \geq n, x \in C, b_k \in B\}.$$

We have $a_n < a_{n+1}, \forall n \in \mathbb{N}$, $a_n \in CozL$, also $A = \{a_n \mid n \in \mathbb{N}\}$ is a cover of $CozL$. By Proposition 2.2.3 $CozL$ is compact which implies that:

$$\exists k \in \mathbb{N} \text{ such that } a_k = 1.$$

so:

$b_k = b_k \wedge 1$
$= b_k \wedge \bigvee\{x \mid x \wedge b_i = 0, \forall i \geq k, x \in C, b_i \in B\}$
$= \bigvee\{x \wedge b_k \mid x \wedge b_i = 0, \forall i \geq k, x \in C, b_i \in B\}$, but $x \wedge b_k = 0$ always and thus $b_k = 0$ which contradicts the assumption that any element of B is non-zero.

(2)⇒ (3) Let F be a countable completely regular filter base and let y_n be a meet of finitely many elements of F, that is $y_n = x_1 \wedge x_2 \wedge .... \wedge x_n$. It is clear that $y_n \neq 0$ and $y_{n+1} \leq y_n$. For $p \in \mathbb{N}$ find $x_{n_1}, x_{n_2} \dots x_{n_p} \in F$ such that $x_{n_1} \prec\prec x_1, \dots, x_{n_p} \prec\prec x_p$ which implies

$x_{n_1} \wedge x_{n_1} \wedge x_{n_2} \dots \wedge x_{n_p} \prec\prec x_1 \wedge x_2 \wedge \dots x_p = y_p$,

put $m = \max\{n_1, n_2....n_p\}$. Then

$y_m = x_1 \wedge x_2 \wedge \dots \wedge x_m \leq x_{n_1} \wedge \dots \wedge x_{n_p} \prec\prec y_p$,

thus from y_n we can get a subsequence $(y_{m_k})_{k\in\mathbb{N}}$ such that $y_{m_k} \leq y_k$ for each k and :

$\dots \prec\prec y_{m_3} \prec\prec y_{m_2} \prec\prec y_1$.

Therefore there exist cozero elements $c_1, c_2 \dots$ and $d_1, d_2 \dots$ with:

- $\dots \prec y_{m_k} \prec c_k \prec y_{m_{k-1}} \prec \dots \prec y_{m_1} \prec c_1 \prec y_1$, and
- $y_1^* \prec d_1 \prec y_{m_1}^* \prec \dots \prec y_{m_{k-1}}^* \prec d_k \prec y_{m_k}^* \prec \dots$.

For each n, $x_n^* \leq y_n^*$ since $y_n \leq x_n$.

If F does not cluster, then $\bigvee_{n\in\mathbb{N}} y_n^* = 1$ and $D = \{d_n \mid n \in \mathbb{N}\}$ is a cover of $CozL$. To prove that $C = \{c_n \mid n \in \mathbb{N}\}$ is locally finite in $CozL$, pick $n(k), l(k) \in \mathbb{N}$ with $d_k \prec y_{n(k)}^*$ and $c_{l(k)} \prec y_{n(k)}$ for any $k \in \mathbb{N}$.

Since $y_{n(k)}^* \wedge y_{n(k)} = 0$ and c_n decreases, then d_k has non-zero meet with at most $c_1, \dots, c_{l(k)-1}$, thus $C \subseteq CozL$ is locally finite and hence finite (by the hypothesis).

Now put $C = \{c_{q_1}, \dots, c_{q_s}\}$ with $c_{q_1} \geq c_{q_2} \geq \dots \geq c_{q_s}$.

$$\bigvee_{i\in\mathbb{N}} c_i^* \geq \bigvee_{i\in\mathbb{N}} y_{m_i}^* = \bigvee y_n^* = 1.$$

But $\bigvee c_i^* = c_{q_s}^*$, so $c_{q_s}^* = 1$ and then $y_k = 0$ for some k, which contradicts that F is a filter base.

(3)⇒ (4) Suppose that there exists a countable co-completely regular cover C which has no finite subcover, and suppose $S = \{\bigwedge_{x\in F} x^* \mid F \subseteq C, F$ finite$\}$.

Claim: S is a filter base. Let $c_1, \ldots, c_m$ be finitely many elements of C, it must be shown that $(\bigvee_{i=1}^{m} c_i)^* \neq 0$.

Let $c_1^* \wedge c_2^* \wedge \ldots \wedge c_m^* = 0$ then there are $d_1 \ldots d_m \in C$ with $c_i \prec\prec d_i$ and we have:

$c_1 \vee c_2 \ldots \vee c_m \prec\prec d_1 \vee d_2 \ldots \vee d_m$, and $(c_1 \vee c_2 \vee \ldots \vee c_m)^* \vee (d_1 \vee d_2 \vee \ldots \vee d_m) = 1$. Since $(c_1 \vee c_2 \vee \ldots \vee c_m)^* = c_1^* \wedge c_2^* \wedge \ldots \wedge c_m^* = 0$ then $d_1 \vee d_2 \vee \ldots \vee d_m = 1$ which means that C admits a finite subcover, which it does not. Thus S is a filter base.

Now, let $s = x_1^* \wedge x_2^* \wedge \ldots \wedge x_m^*$ be an arbitrary element of S. Choose $y_i \in C$ such that $x_i \prec\prec y_i$ for each i, then $y_i^* \prec\prec x_i^*$ for each i. Therefore $y_1^* \wedge y_2^* \wedge \ldots \wedge y_m^*$ is an element of S which is completely below s. Thus S is a completely regular countable filter base and by assumption S clusters. So there is contradiction against C being a cover since for all $c \in C$, $c^* \in S$ and hence:

$1 \neq \bigvee_{x\in S} x^* \geq \bigvee_{c\in C} c^{**}$.

(4)⇒ (1) If $\{a_n\}$ is a sequence with $a_1 \prec\prec a_2 \prec\prec \ldots$ with $\bigvee a_n = 1$. Then $\{a_n \mid n \in \mathbb{N}\}$ is a countably co-completely regular cover of L and by the hypothesis it has a finite subcover such that $a_{n_1} \vee a_{n_2} \vee \ldots \vee a_{n_k} = 1$ which implies that L is a pseudocompact frame.

□

Lemma 2.2.3. [16] Every frame which has a dense pseudocompact quotient is pseudocompact.

Proof. Suppose $h : M \longrightarrow L$ is a dense surjective frame homomorphism with L pseudocompact. If F is a countable completely regular filter base in M, then $h(F)$ is a countable completely regular filter base in L.

Now by Proposition 2.2.5 since L is pseudocompact it follows:

$$\bigvee\{h(t)^* \mid t \in F\} \neq 1.$$

Hence, we have $h(\bigvee\{t^* \mid t \in F\}) \neq 1$ (because dense frame homomorphisms preserve pseudocomplements) and therefore $\bigvee\{t^* \mid t \in F\} \neq 1$, giving that M is pseudocompact. □

Proposition 2.2.6. [16] For any completely regular frame L, the following are equivalent:

(1) L is pseudocompact.

(2) If M is Lindelöf, for any frame homomorphism $h : M \longrightarrow L$, $\uparrow h_*(0)$ is compact.

(3) If M is hereditarily Lindelöf, for any frame homomorphism $h : M \longrightarrow L$, $h[M]$ is compact.

(4) If M is countably generated, for any frame homomorphism $h : M \longrightarrow L$, $h[M]$ is compact.

Proof. (1)⇒ (2) Given that M is Lindelöf and ψ (given by $\psi(a) = a \vee h_*(0)$) is a closed quotient, $\uparrow h_*(0)$ is Lindelöf.

$$M \xrightarrow{\psi} \uparrow h_*(0) \xrightarrow{\overline{h}} h[M]$$

Now since L is pseudocompact then $h[M]$ is pseudocompact as a subframe of L, also since $\overline{h}$ is dense onto then by Lemma 2.2.3 it follows that $\uparrow h_*(0)$ is pseudocompact. Thus $\uparrow h_*(0)$ is compact.

(2)⇒ (3) $h[M]$ is a Lindelöf frame since it is a quotient of hereditarily Lindelöf M. Now suppose $\varphi : h[M] \longrightarrow L$ is the inclusion map, the compactness of $\uparrow \varphi_*(0)$ implies also $\varphi_*(0) = 0$,

$$h[M] \xrightarrow{\mu} \uparrow \varphi_*(0) \xrightarrow{\overline{\varphi}} L$$

and we have μ dense with $\uparrow\varphi_*(0)$ compact, thus by Proposition 1.2.2 μ is an injective map which makes $h[M]$ compact.

(3)⇒(4) Since any countably generated frame is hereditarily Lindelöf.

(4)⇒(1) Suppose $f : \mathcal{L}(\mathbb{R}) \longrightarrow L$ is frame homomorphism. $\mathcal{L}(\mathbb{R})$ is Lindelöf since it is a countably generated frame, then by the hypothesis, $f[(\mathcal{L}\mathbb{R})]$ is compact. Choose the collection $\{(-n, n) \mid n \in \mathbb{N}\}$ which covers $\mathcal{L}(\mathbb{R})$. Then $\{f(-n, n) \mid n \in \mathbb{N}\}$ is a cover of $f[\mathcal{L}(\mathbb{R})]$ which has a finite subcover. Since the $f(-n, n)$ form an increasing sequence, there is a $k \in \mathbb{N}$ with $f(-k, k) = 1$. Thus L is pseudocompact.

□

3. BOUNDED FRAME ELEMENTS AND E-PSEUDOCOMPACT FRAMES

In general topology, boundedness is typically understood to be a metric or uniform property, not a topological one. In the 1970's, a variety of mathematicians independently presented definitions as candidates for a bounded set in a general topological space. A subspace $A \subseteq X$ of a topological space X was termed:

- **Absolutely bounded** (Gagola and Gemignani 1968 [19]) if A is contained in a member of any directed open cover of X.
- **e-relatively compact** (Hechler 1975 [21]) if any open cover $\mathcal{C}$ of $\overline{A}$ contains a finite subcover of A.
- **Bounded** (Lambrinos 1973 & 1976 [27], [28]) if any open cover $\mathcal{C}$ of X contains a finite subcover of A.

These definitions can all be interpreted as a form of relative compactness, free from the additional constraint of closedness. In fact, it is not difficult to prove that the three definitions above are actually equivalent:

If A is absolutely bounded and $\mathcal{C}$ is an open cover of $\overline{A}$, then $\mathcal{D} = \{\bigcup \mathcal{B} \mid \mathcal{B}$ is a finite subset of $\mathcal{C} \cup \{X \setminus \overline{A}\}\}$ is a directed open cover of X. A is contained in a member of $\mathcal{D}$ implies that A is contained in a finite subcover of $\mathcal{C}$ and A is e-relatively compact.

Clearly if A is e-relatively compact then it is bounded since any open cover of X also covers $\overline{A}$. And further, if A is bounded then it is absolutely bounded since the union of any finite subset of a directed cover is contained in a member of that cover.

Boundedness in the above sense has a number of natural properties, for example the finite unions of bounded sets are bounded; a subset of a bounded set is bounded as well; and A is bounded if and only if $\overline{A}$ is bounded. From our perspective it lends itself to a natural definition in pointfree topology too.

Not very much has been studied about "bounded elements" in pointfree topology. Marcus first studied boundedness in the pointfree context, when he defined the bounded elements in continuous frames as those which are way below the top element of the frame, i.e. a is bounded in L if $a \ll 1_L$. He then used these to define the concept of bounded frame homomorphisms in [30].

Dube followed Lambrinos' approach, introducing bounded sublocales (quotient frame homomorphisms) via covers in [14]. We will see that these notions are closely linked but not quite equivalent in any frame.

In this chapter, we now proceed to define the notion of bounded elements in frames and use it to define bounded frame homomorphisms and thence E-pseudocompactness.

3.1 Bounded frame elements

Definition 3.1.1. An element $a \in L$ is bounded if and only if every cover of L containing a^* has a finite subcover.

The set of all bounded elements of a frame L is denoted by $Bd(L)$.

Remark 3.1.1. (1) Since in any frame L, $a^* = a^{***}$ it follows immediately that a is bounded if and only if a^{**} is bounded. This is in accord with the intuition that in topology a set is bounded if and only if its closure is bounded.

(2) It is clear that a frame L is compact if and only if 1_L is bounded.

Example 3.1.1. (1) An element a of a Boolean frame is bounded iff a is the join of finitely many atoms (see below).

(2) Any $U \in \mathcal{O}\mathbb{R}$ is bounded iff $U \subseteq (-a, a)$ for some $a \in \mathbb{R}$.

(3) In a compact frame M, every element is bounded (since the top is bounded). So a frame is compact if and only if every element is bounded.

Proposition 3.1.1. For any frame L, $Bd(L)$ is an ideal.

Proof.

(1) The bottom element is bounded for any frame L since $0^* = 1$.

(2) $Bd(L)$ is a downset.
Assume that $b \leq a$ with a a bounded element in L, also let $K \in CovL$ with $b^* \in K$. Put $K' = K \cup \{a^*\}$, which gives $K' \in CovL$ such that $a^* \in K'$ which thus has a finite subcover. This ensures $\{k_1, k_2, \ldots, k_n, a^*\}$ with each $k_i \in K$ and $k_1 \vee k_2 \vee \ldots \vee k_n \vee a^* = 1_L$. Since $a^* \leq b^*$, then:

$$k_1 \vee k_2 \vee \ldots \vee k_n \vee b^* = k_1 \vee k_2 \vee \ldots \vee k_n \vee a^* \vee b^* = 1_L.$$

Thus $\{k_1, k_2, \ldots, k_n, b^*\}$ is a finite subcover of K.

(3) If a, b are bounded in L then $a \vee b$ is bounded as well.
Assume $K \in CovL$ with $(a \vee b)^* \in K$. Put $K' = \{k \in K \mid k \neq (a \vee b)^*\}$
$\Longrightarrow (a \vee b)^* \vee \bigvee K' = 1_L$
$\Longrightarrow (a^* \wedge b^*) \vee \bigvee K' = (a^* \vee \bigvee K') \wedge (b^* \vee \bigvee K') = 1_L$.

Thus there exists finite $K_1 \subseteq K'$ and $K_2 \subseteq K'$ such that:

$$a^* \vee \bigvee K_1 = 1_L \text{ and } b^* \vee \bigvee K_2 = 1_L.$$

Now, put $\tilde{K} = K_1 \cup K_2$ which is finite and:

$$a^* \vee \bigvee \tilde{K} = b^* \vee \bigvee \tilde{K} = 1_L \Longrightarrow (a^* \wedge b^*) \vee \bigvee \tilde{K} = (a \vee b)^* \vee \bigvee \tilde{K} = 1_L.$$

Thus $\tilde{K} \cup \{(a \vee b)^*\}$ is a finite subcover of K.

□

As mentioned above, Marcus introduced the notion of a bounded element in [30] specifically in the case of a continuous frame L. In that setting $a \in L$ is defined to be bounded if $a \ll 1_L$. This is closely related to our definition.

Proposition 3.1.2. Let L be a frame and $a \in L$.

(1) If a is bounded then $a \ll 1_L$.

(2) If L is regular then a is bounded if and only if $a \ll 1_L$.

(3) If $\bigvee Bd(L) = 1_L$ then a is bounded iff $a \ll 1_L$.

Proof. (1) If a is bounded and $1_L \leq \bigvee A$, for $A \subseteq L$, then $A \cup \{a^*\}$ is a cover of L and so there exists finite $B \subseteq A \cup \{a^*\}$ which covers L. Since $a \wedge a^* = 0_L$ it must be that $a \leq \bigvee(B \setminus \{a^*\})$ and so $a \ll 1_L$.

(2) Let L be regular and $a \ll 1_L$. Since L is regular, for any cover C of L, $\tilde{C} = \{x \in L \mid x \prec c \text{ for some } c \in L\}$ also covers L. Thus if C is a cover of L with $a^* \in C$, then $\tilde{C}$ is also a cover and $1_L \leq \bigvee \tilde{C}$. So since $a \ll 1_L$ there is a finite $\{c_1, c_2, \ldots, c_n\} \subseteq \tilde{C}$ with $a \leq c_1 \vee c_2 \vee \ldots \vee c_n$. By definition of $\tilde{C}$ there are $\{d_1, d_2, \ldots, d_n\} \subseteq C$ with $c_i \prec d_i$ for each i from 1 to n. This gives

$$a \leq \bigvee_{i=1}^{n} c_i \prec \bigvee_{i=1}^{n} d_i.$$

Thus $a \prec \bigvee_{i=1}^n d_i$ and so $a^* \vee \bigvee_{i=1}^n d_i = 1_L$ and $\{d_1, d_2, \ldots, d_n, a^*\}$ is a finite subcover of C.

(3) Assume that $\bigvee Bd(L) = 1_L$. If $a \ll 1_L$ then $1_L \leq \bigvee Bd(L)$ implies that there is a finite $A \subseteq Bd(L)$ with $a \leq \bigvee A$. But since $Bd(L)$ is an ideal, $\bigvee A \in Bd(L)$ and so $a \in Bd(L)$ too.

□

Proposition 3.1.3. Let X be a topological space. $U \in \mathcal{O}X$ is bounded if and only if $\overline{U}$ is compact.

Proof. If $\overline{U}$ is the closure of U in X, we know that $U^* = X \setminus \overline{U}$.

($\Rightarrow$) Assume U is bounded in $\mathcal{O}X$ and let $\mathcal{C}$ be an open cover of $\overline{U}$ in X. $\mathcal{C} \cup \{U^*\}$ is an open cover of X and since U is bounded there is a finite subcover $\mathcal{C}' \subseteq \mathcal{C} \cup \{U^*\}$. Then $\mathcal{C}' \setminus \{U^*\} \subseteq \mathcal{C}$ is finite with $\overline{U} \subseteq \bigcup(\mathcal{C}' \setminus \{U^*\})$ and $\overline{U}$ is compact.

($\Leftarrow$) Assume $\overline{U}$ is compact, assume also that $\mathcal{K}$ is a cover of $\mathcal{O}X$ containing U^*. Then $\{A \cap \overline{U} \mid A \in \mathcal{K}\}$ is a cover of $\overline{U}$ which has a finite subcover $\{A_i \cap \overline{U} \mid i = 1, 2, \ldots, n\}$. Now $\{A_i \mid n = 1, 2, \ldots, n\} \bigcup \{U^*\}$ is a finite subcover of $\mathcal{K}$. Therefore U is bounded.

□

Proposition 3.1.4. Let L be a Boolean frame. The following are equivalent for an element $a \in L$:

(1) a is the join of finitely many atoms;

(2) a is bounded.

(3) a is compact.

Proof.

(1) ⇒ (2) Let $a = \bigvee S$, where S is a finite set of atoms in L. Let $K \in CovL$ with $a^* \in K$. Now, $s \leq \bigvee(K \setminus \{a^*\})$ for each $s \in S$. Since s is an atom there exists $k_s \in K \setminus \{a^*\}$ with $s \leq k_s$ then:

$$a = \bigvee S \leq \bigvee \{k_s | s \in S\}.$$

So, we have

$$1_L = a \vee a^* \leq a^* \vee \bigvee \{k_s | s \in S\}$$

and

$$\{k_s | s \in S\} \cup \{a^*\} \subseteq K$$

is a finite subcover.

(2) ⇒ (3) If $a \leq \bigvee K$ then $\bigvee K \vee a^* = 1_L$, so $K \cup \{a^*\} \in CovL$. Thus there exists a finite $\grave{K} \subseteq K$ with $a^* \vee \bigvee \grave{K} = 1_L$ then $a = a \wedge (a^* \vee \bigvee \grave{K}) = a \wedge \bigvee \grave{K}$, so $a \leq \bigvee \grave{K}$ and a is compact.

(3) ⇒ (1) Let a be compact, then $\downarrow a = \{x \in L | x \leq a\}$ is a compact Boolean frame. Now, since compact Boolean frames are spatial, a is the join of atoms (Lemma 1.2.6, Lemma1.2.1). Then it follows by compactness that a is the join of finitely many atoms.

□

We say that a filter F on L **clusters** if $\bigvee_{x \in F} x^* \neq 1$ and that F is **convergent** if F intersects every cover of L.

Proposition 3.1.5. Consider the following properties of $a \in L$.

(1) a is bounded.

(2) $a \ll 1$

(3) For all filters F on L, $a \in F \Rightarrow F$ clusters.

(4) For all filters F on L, $a^* \notin F \Rightarrow F$ clusters.

(5) For all prime filters F on L, $a \in F \Rightarrow F$ is convergent.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) and (2) $\Rightarrow$ (5). If L is regular then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) .

Proof.

(1)$\Rightarrow$(2) Shown above.

(2)$\Rightarrow$(3) Assume that $a \in F$ but that F does not cluster, so $\bigvee_{x\in F} x^* = 1$. Since $a \ll 1$ there is a finite $A \subseteq F$ with $a \leq \bigvee_{x\in A} x^*$. But then because $a \in F$, $\bigvee_{x\in A} x^* \in A$, and because A is finite, $\bigwedge A \in F$. However, $(\bigwedge A) \wedge (\bigvee_{x\in A} x^*) = 0$ which is a contradiction. Thus $\bigvee_{x\in F} x^* \neq 1$ and F clusters.

(3)$\Rightarrow$(4) If $a^* \notin F$ then $x \wedge a \neq 0$ for all $x \in F$. (Else $x \wedge a = 0 \Rightarrow x \leq a^*$ giving $a^* \in F$.) Thus $F \cup \{a\}$ is a filter base containing a and so $\bigvee_{x\in F\cup\{a\}} x^* \neq 1$. Obviously then $\bigvee_{x\in F} x^* \neq 1$ and F clusters.

(4)$\Rightarrow$(3) Immediate since $a \in F \Rightarrow a^* \notin F$ because $a \wedge a^* = 0$.

(2)$\Rightarrow$(5) Let F be a prime filter with $a \in F$ and let C be a cover of L. We have to show that $F \cap C \neq \emptyset$. Since $a \ll 1$, $1 \leq \bigvee C \Rightarrow a \leq \bigvee D$ for a finite $D \subseteq C$. Then because $a \in F$ it follows that $\bigvee D \in F$ and then since D is finite and F is prime, $D \cap F \neq \emptyset$.

(3)$\Rightarrow$(2) Let L be regular, and assume (3). We show that $a \ll 1_L$ and together with Proposition 3.1.2 the result follows. If $a \not\ll 1_L$ then there is a cover C of L so that for all finite $A \subseteq C$, $a \not\leq \bigvee A$. Form the set $D = \{d \in L \mid d \prec \bigvee A,\ A \text{ is finite},\ A \subseteq C\}$ and since L is regular, D is a cover of L.

Put $F = \{a \wedge d^* \mid d \in D\}$ and we show that F is a filter base on L.

(a) For each $d \in D$ there is a finite $A_d \subseteq C$ with $d \prec \bigvee A_d$, i.e. $d^* \vee \bigvee A_d = 1_L$. But $a \not\leq \bigvee A_d$ and so $a \wedge d^* \neq 0_L$. (Else

$a = a \wedge 1_L = a \wedge (d^* \vee \bigvee A_d) = (a \wedge d^*) \vee (a \wedge \bigvee A_d) = a \wedge \bigvee A_d \Rightarrow a \leq \bigvee A_d$.)

(b) D is closed under finite joins. If $d_1, d_2 \in D$ then there are finite subsets A_{d_1} and A_{d_2} of C with $d_1 \prec \bigvee A_{d_1}$ and $d_2 \prec \bigvee A_{d_2}$. But then $(d_1 \vee d_2) \prec (\bigvee A_{d_1} \vee \bigvee A_{d_2}) = \bigvee(A_{d_1} \cup A_{d_2})$ and $A_{d_1} \cup A_{d_2}$ is a finite subset of C, so $d_1 \vee d_2 \in D$.

Thus F is closed under finite meet since for $d_1, d_2 \in D$, $(a \wedge d_1^*) \wedge (a \wedge d_2^*) = a \wedge (d_1^* \wedge d_2^*) = a \wedge (d_1 \vee d_2)^* \in F$.

Now $a \in F$ since $\bigvee \emptyset = 0_L \in D$ gives $a = a \wedge 1_L = a \wedge 0_L^* \in F$. So by (3) $\bigvee\limits_{d \in D} (a \wedge d^*)^* \neq 1_L$. But for any $d \in D$, $(a \wedge d^*)^* \geq a^* \vee d^{**} \geq a^* \vee d \geq d$ so it then follows that $\bigvee D \neq 1_L$ contradicting that D is a cover. Hence $a \ll 1_L$ and the result follows.

□

3.2 Bounded sublocales

A commonly adopted generalisation to frames of the topological notion of a subspace is to use surjective frame homomorphisms, which are also termed **sublocales**. Bounded subpaces are thus generalised by considering bounded sublocales, or bounded surjective homomorphisms (quotient maps). In [14] Dube defines such a bounded notion on a quotient frame map as follows.

Definition 3.2.1. A quotient map $h : L \longrightarrow M$ of L is called bounded if for every cover C of L there exists a finite $K \subseteq C$ such that $h[K]$ is a cover of M.

We will refer to this definition of bounded quotient as **D-bounded** to distinguish it from our notion of bounded homomorphism given in Definition 3.3.1 below. The next two propositions show how our definitions of bounded frame elements relate to associated D-bounded closed and open sublocales.

Proposition 3.2.1. An element a in a frame L is bounded if and only if the quotient map $-\vee a^* : L \longrightarrow \uparrow a^*$ is a D-bounded sublocale.

Proof. ($\Rightarrow$) Let a be bounded and C be a cover of L. Put $C' = C \cup \{a^*\}$ then there is a finite $K' \subseteq C'$ with $\bigvee K' = 1_L$. Now if $K = K' \setminus \{a^*\}$ then $\bigvee K \vee a^* = 1_L = 1_{\uparrow a^*}$ and $-\vee a^*$ is D-bounded.

($\Leftarrow$) Let C be a cover of L with $a^* \in C$. By D-boundedness, there is a finite $K \subseteq C$ such that $\bigvee K \vee a^* = 1_{\uparrow a^*} = 1_L$, then $K' = K \cup \{a^*\} \subseteq C$ is finite with $\bigvee K' = 1_L$.

□

Proposition 3.2.2. An element $a \ll 1$ in a frame L if and only if the quotient map $-\wedge a : L \longrightarrow \downarrow a$ is a D-bounded sublocale.

Proof. ($\Rightarrow$) Let $a \ll 1_L$ and C be a cover of L. There is a finite $K \subseteq C$ with $a \leq \bigvee K$. Then $\bigvee K \wedge a = a = 1_{\downarrow a}$ and $-\wedge a$ is D-bounded.

($\Leftarrow$) Let C be a cover of L then by D-boundedness, there is a finite $K \subseteq C$ such that $\bigvee K \wedge a = 1_{\downarrow a} = a$. Thus $a \leq \bigvee K$.

□

3.3 Bounded frame homomorphisms

In general topology, a map $f : X \longrightarrow Y$ is bounded if there is a bounded subspace B of Y with $f(X) \subseteq B$. Using this as our motivation we introduce the following definition for bounded frame homomorphisms.

Definition 3.3.1. A frame homomorphism $f : E \longrightarrow L$ is bounded if and only if $f(b) = 1_L$ for some bounded $b \in E$.

Example 3.3.1.

(1) If E is a compact frame, then every frame homomorphism $h : E \longrightarrow L$ is bounded, since 1_E is bounded.

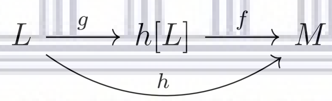
(2) Let X and Y be metric space and suppose $f : X \longrightarrow Y$ is a continuous map then the frame homomorphism $f^{-1}(-) : \mathcal{O}Y \longrightarrow \mathcal{O}X$ is bounded if f is bounded in the usual sense.

(3) Let X and Y be sets and suppose $f : X \longrightarrow Y$ is a function. Then $f^{-1}(-) : P(Y) \longrightarrow P(X)$ is bounded if and only if $f^{-1}(F) = X$ for some finite subset F of Y.

(4) Let M and L be Boolean frames. Then $f : M \longrightarrow L$ is bounded iff $f(s) = 1_L$ where s is a join of finitely many atoms in M.

(5) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous map, the frame homomorphism: $f^{-1}(-) : \mathcal{O}\mathbb{R} \longrightarrow \mathcal{O}\mathbb{R}$ is bounded iff $f^{-1}(-a, a) = \mathbb{R}$ for some $a \in \mathbb{Q}$.

An obvious option is to consider h to be bounded if when taking the factorisation of $h = f \circ g$ through its image produces g a D-bounded sublocale.



We call such h **D-bounded** too, i.e. $h : L \longrightarrow M$ for which any cover C of L contains a finite K such that $h[K]$ covers M.

Proposition 3.3.1. A frame homomorphism $h : L \longrightarrow M$ is D-bounded if it is bounded.

Proof. Let $h : L \longrightarrow M$ be a bounded frame homomorphism, and assume $C \in CovL$, then:

$$\bigvee h(C) = h(\bigvee C) = h(1_L) = 1_M = h(b)$$

for some $b \in Bd(L)$.

Put $\tilde{C} = C \cup \{b^*\}$, then there exists a finite $\tilde{K} \subseteq \tilde{C}$ with $\bigvee \tilde{K} = 1$. Now, let $K = \{k \in \tilde{K} \mid k \neq b^*\}$, so $b \leq \bigvee K \Longrightarrow 1_M = h(b) \leq h(\bigvee K) = \bigvee h(K)$ which is a cover of M. $\square$

While in general if $h : L \longrightarrow M$ is bounded then it is D-bounded, in the absence of additional assumptions on the frames or on $Bd(L)$ it is not possible to extract a generic bounded element from a D-bounded map to show that it is bounded. The most natural element to consider is $h_*(0)^* = (\bigvee\{a \in L \mid h(a) = 0\})^*$, the pseudocomplement of the largest element mapped by h to 0.

Proposition 3.3.2. If $h : L \longrightarrow M$ is bounded then $h_*(0)^*$ is bounded.

Proof. Let $h : L \longrightarrow M$ be bounded with $b \in Bd(L)$ such that $h(b) = 1_M$. Then $h(b^*) \leq h(b)^* = 1^* = 0 \Rightarrow b^* \leq h_*(0) \Rightarrow h_*(0)^* \leq b^{**}$. But since b is bounded, b^{**} is also bounded, and then because $Bd(L)$ is an ideal, $h_*(0)^*$ is bounded too. □

Lemma 3.3.1. If $h : L \longrightarrow M$ with $h(x) = 1$ and $x \prec y$ then $h_*(0)^* \leq y$.

Proof. If $h(x) = 1$ then $h(x^*) \leq h(x)^* = 0 \Rightarrow x^* \leq h_*(0) \Rightarrow h_*(0)^* \leq x^{**}$. If $x \prec y$ then $x^{***} \vee y = x^* \vee y = 1$ and so $x^{**} \prec y$ giving $h_*(0)^* \leq x^{**} \leq y$. □

Proposition 3.3.3. For L a regular frame, if $h : L \longrightarrow M$ is D-bounded then $h_*(0)^*$ is bounded.

Proof. According to Proposition 3.1.2 it suffices to show that $h_*(0)^* \ll 1_L$. Given a cover C of L, because L is regular we form the cover $\tilde{C} = \{x \in L \mid x \prec c \text{ for some } c \in C\}$. Then since h is D-bounded, there exists a finite $K \subseteq \tilde{C}$ such that $h(\bigvee K) = 1_M$.

By the construction of $\tilde{C}$, for each $k \in K$ there is a corresponding $c_k \in C$ with $k \prec c_k$. Then $\bigvee K \prec \bigvee_{k\in K} c_k$ and by Lemma 3.3.1, since $h(\bigvee K) = 1_M$, $h_*(0)^* \leq \bigvee_{k\in K} c_k$. This shows that $h_*(0)^* \ll 1_L$. □

Corollary 3.3.1. If L is Boolean, then $h : L \longrightarrow M$ is bounded iff h is D-bounded iff $(h_*(0))^*$ is bounded.

Proof. Combine Proposition 3.3.1 and Proposition 3.3.3 with the observation that in a Boolean frame $h((h_*(0))^*) = 0 \vee h((h_*(0))^*) = h(h_*(0)) \vee h((h_*(0))^*) = h(h_*(0) \vee (h_*(0))^*) = h(1) = 1$. □

Corollary 3.3.2. In regular frames, if $h : L \longrightarrow M$ is a bounded (hence D-bounded) dense quotient then L is compact.

Proof. L is compact if and only if $1_L \ll 1_L$. If h is D-bounded then by Proposition 3.3.3, $(h_*(0))^*$ is bounded, hence by regularity $(h_*(0))^* \ll 1_L$. But by denseness $h_*(0) = 0_L$ rendering $(h_*(0))^* = 1_L$ and L is thus compact. □

Proposition 3.3.4. If $\bigvee Bd(L) = 1$ then $h : L \longrightarrow M$ is bounded iff h is D-bounded.

Proof. One direction follows from Proposition 3.3.1. For the other, assume that h is D-bounded, then since $\bigvee Bd(L) = 1_L$ there is a finite $A \subseteq Bd(L)$ with $h(\bigvee A) = 1_M$. Since $Bd(L)$ is an ideal and A is finite, $\bigvee A \in Bd(L)$ and h is bounded. □

3.4 E-Pseudocompact Frames

A topological space X is pseudocompact if every real-valued continuous map with domain X is bounded. With a more general definition of bounded map (not only real-valued) we can introduce a more general definition of pseudocompactness.

Definition 3.4.1. Let E be a frame. A frame L is E-pseudocompact if and only if every frame homomorphism $f : E \longrightarrow L$ is bounded.

Example 3.4.1.

(1) $E = \mathcal{O}\mathbb{R}$. A completely regular frame L is an E-pseudocompact frame precisely when L is a pseudocompact frame.

Proof. If $\phi : \mathcal{O}\mathbb{R} \longrightarrow L$ is a bounded frame homomorphism, then $\exists (p, q) \in \mathcal{O}\mathbb{R}$ such that $\phi(p, q) = 1_L$, (p, q) is bounded in $\mathcal{O}\mathbb{R}$. □

(2) Consider $E = \mathcal{O}\mathbb{N}$. A zero dimensional frame L is E-pseudocompact iff BL is finite.

Proof.

(⇒) If L is E-pseudocompact then any frame homomorphism from $E = \mathcal{O}\mathbb{N}$ is bounded and also it is known that a zero dimensional frame is bounded if it is finite.

(⇐) It is clear that L is bounded which means every frame homomorphism h: $\mathcal{O}\mathbb{N} \longrightarrow L$ is bounded.

□

(3) Let A be a sub σ-frame of L, and $E = \mathcal{H}A$ then L is $E-$pseudocompact if A is compact

In general topology, the pseudocompactness property is not very well behaved. It is closed under continuous images but not under products or (closed) subspaces. We conclude by considering a few results of this nature in the point-free setting.

Proposition 3.4.1. If $h : L \longrightarrow M$ is injective and M is E-pseudocompact, then L is E-pseudocompact.

Proof. Consider $f : E \longrightarrow L$, then if M is E-pseudocompact, $h \circ f$ is bounded and there is a bounded element $d \in E$ with $(h \circ f)(d) = 1_M$. Since h is injective, $f(d) = 1_L$ and f is bounded showing that L is E-pseudocompact. □

Proposition 3.4.2. If $h : L \longrightarrow M$ is a co-dense quotient and M is E-pseudocompact, then L is E-pseudocompact.

Proof. To show L is E-pseudocompact, let $g : E \longrightarrow L$ be any frame homomorphism.

$$\begin{array}{ccc} E & \xrightarrow{g} & L \\ & \searrow^{f} & \downarrow h \\ & & M \end{array}$$

Since M is E-pseudocompact, then $f = h \circ g$ is bounded, so $\exists b \in Bd(E)$ such that $h(g(b)) = 1_M$. Because h is codense, $g(b) = 1_L$. □

Pseudocompactness is not closed under (closed) subspaces in general topology but we finish with the following result in that direction.

Proposition 3.4.3. For any $E-$pseudocompact frame L and any $a \in L$ with $\uparrow(a \vee a^*)$ $E-$pseudocompact, then $\uparrow a^*$ is also $E-$pseudocompact.

Proof. Assume that h and g are frame homomorphisms such that:

$$E \xrightarrow{h} \uparrow a^* \xrightarrow{g} \uparrow(a \vee a^*)$$

where g maps $x \longmapsto x \vee a$

We know that $h(s) \vee a = g \circ h(s) = 1_L$ for some bounded $s \in E$ and must show that h is bounded.

Define a map $f : E \longrightarrow L$ by

$$f(x) = \begin{cases} h(s^*) \wedge h(x) \wedge a & , s \wedge x = 0 \\ h(s) \vee h(x) & , s \wedge x \neq 0 \end{cases} \tag{3.1}$$

Since $x \leq s^*$ if $s \wedge x = 0$, then:

$$f(x) = \begin{cases} h(x) \wedge a & , s \wedge x = 0 \\ h(s) \vee h(x) & , s \wedge x \neq 0 \end{cases} \tag{3.2}$$

$f(x)$ defines a frame homomorphism which we may prove by the following:

- $f(0) = h(0) \wedge (a) = a^* \wedge a = 0$, and $f(1) = h(s) \vee h(1) = 1$

- To show $f(x \wedge y) = f(x) \wedge f(y)$ we consider three cases:

  (1) $s \wedge x = 0$ and $s \wedge y = 0$,

$$\begin{aligned} f(x \wedge y) =& [h(x) \wedge h(y)] \wedge (a) \\ =& [h(x) \wedge a] \wedge [h(y) \wedge a] \\ =& f(x) \wedge f(y). \end{aligned}$$

  (2) $s \wedge x \neq 0$ and $s \wedge y \neq 0$ and $x \wedge y \neq 0$

$$\begin{aligned} f(x \wedge y) =& h(s) \vee h(x \wedge y) \\ =& h(s) \vee h(x \wedge y) \\ =& h(s) \vee [h(x) \wedge h(y)] \\ =& [h(s) \vee h(x)] \wedge [h(s) \vee h(y)] \\ =& f(x) \wedge f(y). \end{aligned}$$

  (3) $s \wedge x = 0$ and $s \wedge y \neq 0$

$$\begin{aligned} f(x) \wedge f(y) =& [a \wedge h(x)] \wedge [(h(y) \vee h(s)] \\ =& a \wedge [(h(x) \wedge h(s)) \vee (h(x) \wedge h(y))] \\ =& a \wedge [h(x \wedge s) \vee h(x \wedge y)] \\ =& a \wedge [h(0) \vee h(x \wedge y)] \\ =& a \wedge [a^* \vee h(x \wedge y)] \\ =& (a \wedge a^*) \vee (a \wedge h(x \wedge y)) \\ =& f(x \wedge y) \end{aligned}$$

- To show $f(\bigvee x_i) = \bigvee f(x_i)$, we consider three cases:

(1) $s \wedge x_i = 0, \forall i$

$$\begin{aligned} f(\bigvee x_i) =& h(\bigvee x_i) \wedge a \\ =& [\bigvee h(x_i)] \wedge a \\ =& \bigvee [h(x_i) \wedge a] \\ =& \bigvee f(x_i) \end{aligned}$$

(2) $s \wedge x_i \neq 0$, $\forall i$

$$\begin{aligned} f(\bigvee x_i) =& h(s) \vee h(\bigvee x_i) \\ =& \bigvee [h(s) \vee h(x_i)] \\ =& \bigvee f(x_i) \end{aligned}$$

(3) First consider $f(x \vee y)$, where $s \wedge x = 0$ and $s \wedge y \neq 0$

$$\begin{aligned} f(x) \vee f(y) =& [a \wedge h(x)] \vee [(h(y) \vee h(s)] \\ =& [(h(x)) \vee h(y \vee s)] \wedge [(a \vee h(y \vee s)] \\ =& [h(x \vee y \vee s)] \wedge [1 \vee h(y)] \\ =& h(x \vee y \vee s) \\ =& f(x \vee y) \end{aligned}$$

• What about $f(\bigvee A)$ for $A \subseteq E$?
In general for $A \subseteq E$ put $A = A_1 \cup A_2$, $\bigvee A = \bigvee A_1 \vee \bigvee A_2$ for $\bigvee A_1 \wedge s = 0$ and $\bigvee A_2 \wedge s \neq 0$, by applying (1),(2) (3) above we get:

$$\begin{aligned} f(\bigvee A) =& f(\bigvee A_1 \vee \bigvee A_2) \\ =& f(\bigvee A_1) \vee (f \bigvee A_2)(by(3)) \\ =& \bigvee f(A_1) \vee \bigvee f(A_2)(by(1),(2)) \\ =& \bigvee f(A) \end{aligned}$$

Now, since L is $E-$pseudocompact, there exists bounded $r \in E$ with $f(r) = 1_L$ and without loss of generality we may assume that $r \geq s$. Thus $f(r) = h(s) \vee h(r) = h(r)$; therefore h is bounded which implies the E-pseudocompactness of $\uparrow a^*$. □

FINAL REMARKS

In this thesis, after recalling some classical results regarding pseudocompactness, we offered a new definition of a bounded element in a frame. We have seen many interesting properties of this boundedness notion and used it to define bounded homomorphisms. In particular these new notions are conservative (generalising what is known from topology) and offer some simpler proofs. For example:

- The relationship between a bounded element in frames with its closure in a classical topology (Proposition 3.1.3).
- E-pseudocompact frames provide a generalization of pseudocompact frames, making the study of bounded frame homomorphisms easier, without requiring the frame of reals. (Compare Proposition 3.4.3 and Proposition 2.2.2.)

However, the boundedness notion in frame homomorphisms brings about many open problems of which we will list a few:

- Proposition 3.3.1 shows that Dube's bounded definition ([14]), which is defined by the frame theoretic analogue of subspaces, is implied by Definition 3.3.1 which is defined via bounded elements in frames. We contend that this is a very natural definition for bounded frame homomorphisms.

  The converse of this proposition is an unsolved problem in general. It can be obtained for Boolean frames (see Corollary 3.3.1).

- Chapter 2 highlighted characterisations of pseudocompactness of frames, in many instances by using the cozero part of frame. This has no immediate analogue for E-pseudocompactness since a cozero set is only associated with the reals. None-the-less linking properties of E with properties of E-pseudocompactness will make an interesting further study.

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