Measurements of Edge Uncolourability in Cubic Graphs

by
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Dedication

I humbly dedicate this thesis to all who have contributed, in their own way, to my ability to do mathematics.

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I would like to thank: my parents, for their continued support and encouragement; my wife, for her love and respect; my supervisor, for his time and mentorship. I hope this thesis brings each of you much pride and joy.

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Most importantly, all thanks are due to Allah, who has granted me the blessings mentioned above, and the ability to be thankful to them and to Him.

Declaration

I hereby declare that this thesis is my own work, that it has not been submitted for any degree or examination at any other academic institution, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Imran Allie   August 2020

Signed: ....

Publications arising from this thesis

- I. Allie, *Oddness to resistance ratios in cubic graphs*, Discrete Math. 342 (2019), pp. 387 - 391.

- I. Allie, *3-Critical subgraphs of snarks*, Submitted (2019).

Abstract

The history of the pursuit of uncolourable cubic graphs dates back more than a century. This pursuit has evolved from the slow discovery of individual uncolourable cubic graphs such as the famous Petersen graph and the Blanusa snarks, to discovering infinite classes of uncolourable cubic graphs such as the Louphekine and Goldberg snarks, to investigating parameters which measure the uncolourability of cubic graphs. These parameters include resistance, oddness and weak oddness, flow resistance, among others. In this thesis, we consider current ideas and problems regarding the uncolourability of cubic graphs, centering around these parameters. We introduce new ideas regarding the structural complexity of these graphs in question. In particular, we consider their 3-critical subgraphs, specifically in relation to resistance. We further introduce new parameters which measure the uncolourability of cubic graphs, specifically relating to their 3-critical subgraphs and various types of cubic graph reductions. This is also done with a view to identifying further problems of interest. This thesis also presents solutions and partial solutions to long-standing open conjectures relating in particular to oddness, weak oddness and resistance.

Contents

Chapter 1

Introduction

The question of the colourability of cubic graphs has long been the subject of major consideration in graph theory. To this day, this question is still largely considered. The reason is that many major theorems and conjectures in graph theory are easily solved for graphs which are not specifically both cubic and uncolourable. That is to say, cubic graphs which cannot have its edges assigned one of three colours such that no two adjacent edges are assigned the same colour. Many of these conjectures are still open, on which we will further elaborate. First we present an overview of the proverbial hunting of the snark.

In 1880, Tait proved that the Four Colour Theorem is equivalent to the statement that no uncolourable cubic graph is planar[41]. It was in proving this statement that the study of these uncolourable cubic graphs was initiated. The now famous Petersen graph, which is ubiquitous as an opportune counter-example to many seemingly true results in graph theory, was the first such graph to be discovered. The Petersen graph is in fact the smallest uncolourable cubic graph, with order 10. This was in 1898, by Julius Petersen [34]. It wasn't until 1946 that the next two known uncolourable cubic graphs was discovered by Blanusa [4]. These are now known as the Blanusa snarks, both of which have order 18. Descartes (a pseudonym of Tutte) [10] and Szekeres [40] soon after discovered another uncolourable cubic graph each. Up until 1975, these were the only known uncolourable cubic graphs.

In between this time, unrelated to the pursuit of uncolourable cubic graphs, Vizing published what we know today as Vizing's theorem [45, 46]. The theorem

states that any graph with maximum degree Δ, can have its edges properly coloured with Δ colours or $\Delta + 1$ colours. The terms class one graph and class two graph are used, respectively. The term colourable is also used for class one, while the term uncolourable is used for class two.

In 1975, Isaacs [22] discovered the first and second infinite classes of cubic class two graphs. In fact, the previously known cubic class two graphs of Blanusa, Descartes, and Szekeres, are members of one of these infinite classes, thereafter dubbed the BDS class. Thus the pursuit of these seemingly rare cubic class two graphs had evolved. Gardner then coined the term *snarks* in 1976 in reference to cubic class two graphs, after the mysterious and elusive object of the poem "The Hunting of the Snark" by Lewis Carroll. Gardner specifically used the term snarks to refer to 'non-trivial' cubic class two graphs. Although, there is no clear and obvious definition of what 'nontrivial' means, it is a discussion to which we briefly contribute in this thesis. Therefore, some authors use the term snarks to refer to all cubic class two graphs. Here, we will continue to use the term snark to refer to any cubic class two graph.

The natural progression after the discovery of infinite classes of snarks then tended towards methods of snark generation. In [13], Fiol proposed a method based on boolean algebras. Loupekhine's snarks [23] and Goldberg's snarks [18, 19] can be constructed by this method. In [27], Kochol presented a method which he called superposition. The methods of generation and construction then lead to questions of triviality and reduction.

A simple property of a snark to be considered trivial for example, is whether it contains a cycle of length 3 or not. This is because such a 3-cycle can simply be truncated to a cubic vertex, and the resultant graph would still be a snark. Such a truncation is an example of a type of reduction. Many other types of reduction have been considered; see for instance [6, 33, 31]. Most notably, for our purposes, are the reductions defined in [38] by Steffen. We note also that snarks for which it is easy to discover reductions to smaller snarks, are generally considered less complex than snarks for which such reductions are more difficult to discover. This is because easily reducible snarks are considered to contain more triviality.

In the greater effort to work towards finding solutions to the aforementioned major conjectures, and deepen our understanding of snarks, researchers began to introduce parameters which reflect the complexity of the structure of snarks, or how far they are from being class one. These parameters have been termed *measurements*

of uncolourability, and have been used to gain new insights into snark complexity, as well as to find partial solutions to open problems. Some of these parameters include: the resistance of a snark, defined as the minimum number of edges that can be removed from a graph to render a colourable graph; the vertex resistance of a snark, defined as the minimum number of vertices that can be removed from a graph to render a colourable graph; the oddness of a graph, defined as the minimum number of odd components in any 2-factor of the graph; and the weak oddness of a graph, defined as the minimum number of odd components in any even factor of the graph. Oddness and weak oddness are grouped along with these parameters since it is not difficult to see that cubic graphs have 2-factors or even factors with no odd components if and only if the graph is class one. Other parameters have also been introduced, see [12] for a survey on measurements of uncolourability.

On the other hand, much research has been done regarding flows in the context of snarks. See for example, [17, 16, 32, 36, 39]. In fact, it is relatively easy to show that a cubic graph is colourable if and only it admits a nowhere zero 4-flow. Máčajova and Škoviera further showed that snarks coincide in terms of their criticality with regard to 3-edge-colourings and their criticality with regard to nowhere zero 4-flows. In view of this close relationship between 4-flows and 3-edge-colourings, another parameter which measures uncolourability of snarks was introduced in [12] by Fiol et al. That is, the flow resistance of a snark, defined as the minimum number of zero edges in a 4-flow of the snark. Also in [12], it was conjectured that resistance is always greater than or equal to flow resistance. In this thesis we provide some insights on this conjecture, and prove that the resistance is bounded by two times the flow resistance of snark.

As alluded to above, the reason that snarks are the subject of consideration in graph theory is due to the fact that many major open conjectures are easily solvable for non-snarks. We present some examples of these open problems. In [42], Tutte formulated the 5-Flow Conjecture which states that every bridgeless graph admits a nowhere zero 5-flow. In [15], the Berge-Fulkerson Conjecture was published, which states that every bridgeless cubic graph has six 1-factors such that every edge is in precisely two of them. In [11], the Fan-Raspaud Conjecture was published, which states that every bridgeless cubic graph has three 1-factors such that no edge is in each of them. In [35], the Cycle Double Cover Conjecture was published, which states that every bridgeless graph admits a cycle double cover. Many partial solutions have been presented to each these problems. These partial solutions are generally presented in terms of parameters which measure the complexity of the

snark, as previously mentioned. In other words, a critical understanding of snarks and uncolourability opens up an understanding of many problems.

In Chapter 2, we present preliminary definitions and results which will be used at various points in this thesis. These include all the necessary basic graph theoretical definitions, building towards recently introduced ideas. This chapter will also present in general the current topography of the question of the colourability of cubic graphs, at least in areas of relevance to this thesis. Where possible, we will present proofs, some of which are our own.

In Chapter 3, we attempt to further our understanding of the structure of class two cubic graphs. We do this by introducing ideas relating to their 3-critical subgraphs, or as we will call them, minimal conflicting subgraphs. 3-critical graphs are subcubic graphs with edge chromatic number 4 such that the removal of any one edge renders the graph to have an edge chromatic number of 3. We consider how the minimal conflicting subgraphs of a graph relate to its possible minimal colourings (colourings with the least number of conflicting edges). We characterise the resistance in terms of its minimal conflicting subgraphs, and discuss ideas which are consequent to the ones introduced. Similarly, we also characterise what we call *the critical subgraph* of a snark in terms of its minimal conflicting subgraphs. Furthermore, we briefly discuss hypo-Hamiltonicity in the context of snarks. The reason for the focus on hypo-Hamiltonicity is because hypo-Hamiltonian snarks can be considered as the closest in structure to being cubic class one. This was reasoned and remarked by Steffen in [37]. In addition, there have been other somewhat surprising deductions between hypo-Hamiltonicity and snarks, in particular amongst smaller graphs in both classes. We prove a new result on hypo-Hamiltonian snarks which may strengthen the idea that hypo-Hamiltonian snarks may be regarded as the ones closest to being class one.

In Chapter 4, we consider vertex reductions of snarks (as defined by Steffen in [38]). That is, the removal of two vertices and subsequent addition of edges to restore 3-regularity. These reductions can be further classified into different types. That is, when the two vertices in question are not adjacent, when they are adjacent, and when they are adjacent and edges are added back in a particular manner. We introduce new parameters using these types of reductions which can be regarded as measurements of uncolourability of snarks, or even as measurements of triviality of snarks given the previous comments made. Furthermore, we present new insightful results concerning these parameters. In particular, we state one particular conjecture relating resistance to one of the new parameters. If indeed the conjecture is true, we

show that it can be used to prove a long-standing open conjecture regarding weak oddness and resistance. The conjecture in question states that the weak oddness of a cubic graph is bounded by two times the resistance. We also show that it can be used to prove that the flow resistance of a snark is less than or equal to its resistance, as alluded to previously.

In Chapter 5, we disprove a long standing open conjecture on cubic class two graphs which relates resistance to oddness. The conjecture states the oddness of a cubic graph is bounded by two times the resistance of a graph. The chapter is devoted simply to the presentation of a class of graphs which disproves this conjecture. Across chapters 4 and 5, we relate back to ideas introduced in Chapter 3.

In Chapter 6, we conclude with a brief discussion.

Chapter 2

Preliminaries

In this chapter, we present necessary background and preliminary results required throughout this thesis. We begin with a few basic graph theoretic definitions, after which the focus builds towards results and definitions specifically relating to snarks. These include minimal edge colourings as well as measurements of uncolourability. Definitions and results are augmented with examples where required.

2.1 Basic graph theoretic concepts

We begin with the most basic definition, that of a graph. We generalise this basic definition to formally cater for the idea of 'dangling edges'. The idea of edges which are associated with just one vertex, or 'dangling edges', is useful for our purposes. In this view, we define *semi-graphs*.

Definition 2.1.1. A *semi-graph* G is a pair $G = (V, E)$ which consists of a set of vertices $V = V(G)$ and a set of edges $E = E(G) \subseteq [V]^2 \cup [V]^1$. If E contains no elements from $[V]^1$ then we call G a *graph*. The 2-element subsets of V in E are called *edges*. The 1-element sets are called *semi-edges*. We denote the semi-edge $\{u\}$ as $[u]$ and the edge $\{u, v\}$ as $[u, v]$.

Example 2.1.2. A semi-graph containing semi-edges.
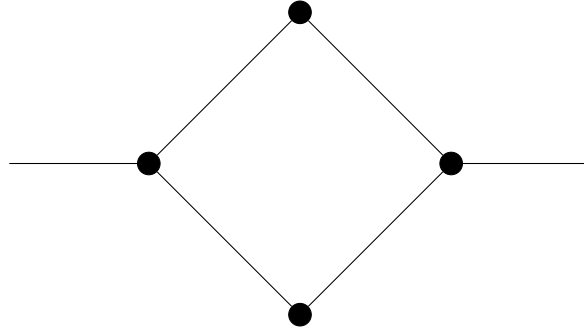
If two edges or semi-edges share a vertex then it is said that they are *adjacent*. That is, if $e_1, e_2 \in E$, then they are adjacent if $e_1 \cap e_2 \neq \varnothing$. Vertices u and v are said to be adjacent if there exists an edge $e = [u, v]$. If a vertex is contained in an edge or semi-edge then it is said that the vertex and edge or semi-edge are *incident*.

This thesis only considers graphs for which the vertex set is finite. Going forward, we may refer to the set of edges and semi-edges together simply as the set of edges, for convenience. This will only be done where the context is clear.

We now define various types of sequences of vertices, after which we define some more basic graph concepts.

Definition 2.1.3. Let $G = (V, E)$ be a semi-graph.

(a) A *walk* is a sequence of vertices $v_0, v_1, ..., v_k$ such that $[v_i, v_{i+1}] \in E$ for every $i \in \{0, 1, \ldots, k-1\}$. k is called the *length* of the walk.

(b) A *trail* is a walk in which every edge $[v_i, v_{i+1}]$ is distinct for $i \in \{0, 1, \ldots, k-1\}$.

(c) A *path* is a trail in which every vertex in $\{v_0, v_1, \ldots, v_{k-1}\}$ is distinct.

(d) A *cycle* is a path in which $v_k = v_0$. An *n-cycle* is a cycle with n vertices.

Remark 2.1.4. Even though a walk is defined in terms of vertices, we may think of it as containing edges as well. Note then that a semi-edge cannot be contained in a walk.

Let $G = (V, E)$ be a semi-graph. The *degree* of a vertex $v \in V$ is the number of edges or semi-edges incident to v. G is *regular* if every vertex in V has the same degree. If each vertex has degree d then it is said that G has degree d. G is *cubic* if G has degree 3. G is *subcubic* if every vertex in V has degree 3 or less. The number of vertices in G is called the *order* of G. The number of edges in the smallest cycle contained in G is called the *girth* of G.

Let $G = (V, E)$ be a semi-graph. Define a relation $\sim$ on V by $x \sim y$ if and only if there exists a path from x to y. If $x \sim y$ then x and y are said to be *connected*. It is not hard to see that $\sim$ is an equivalence relation. Each equivalence class $[x]$ is called a *component* of G (note that the notation $[x]$ in this context is not referring to a semi-edge). If a graph has more than one component then it is *disconnected*, otherwise it is *connected*. A *bridge* is an edge $e \in G$ such that $G - e$ contains one more component than G. A *cut-set* is a set of edges S such that $G - S$ contains more components than G. G is k-edge-connected if k is the minimum size of any cut-set of G. G is cyclically k-edge-connected if k is the minimum size of any cut-set S of G, such that each resulting component in $G - S$ contains a cycle.

Some commonly used operations on edges and vertices will be utilised in this thesis. These are specifically: subdivision of edges; and suppression of vertices of degree 2.

Definition 2.1.5. Let G be a subcubic semi-graph. Let x be a vertex in G with degree 2 and incident edges $[x, y]$ and $[x, z]$.

(a) A *subdivision* of edge $[u, v] \in G$ is the removal of $[u, v]$ and consequent addition of vertex w, and edges $[u, w]$ and $[w, v]$.

(b) A *suppression* of vertex $x \in G$ is the removal of $[x, y]$ and $[x, z]$, and consequent addition of edge $[y, z]$.

Furthermore, in this thesis we will also be dealing with the joining of two semi-edges to form an edge, the splitting of an edge into two semi-edges, and the joining of a semi-edge to a vertex to form an edge. Formally, we define joins and splits in graphs as follows.

Definition 2.1.6. Let G be a semi-graph.

(a) A *join* between two semi-edges $[u]$ and $[v]$ is the removal of semi-edges $[u]$ and $[v]$, and the addition of edge $[u, v]$. A *join* between semi-edge $[u]$ and vertex v is the removal of semi-edge $[u]$ and the addition of edge $[u, v]$.

(b) The *splitting* of an edge $[u, v]$ is the removal of the egde $[u, v]$ and the addition of two semi-edges $[u]$ and $[v]$.

We say that a semi-graph G' is a subgraph of G if $V(G') \subseteq V(G)$, $[u, v] \in E(G')$ implies that $[u, v] \in E(G)$, $[u] \in E(G')$ implies that $[u] \in E(G)$ or that there is an edge $[u, v] \in E(G)$, and for every vertex $u \in V(G')$ the degree of u in G is greater than or equal to the degree of u in G'. For a semi-graph G, the subgraph *induced* by a set of vertices $V' \subseteq V(G)$ is the subgraph containing vertex set V' and all edges and semi-edges in G which are incident to vertices in V'. Similarly, the subgraph induced by a set of edges $E' \subseteq E(G)$ is the subgraph containing edge set E' and all vertices in G which are incident to edges in E'.

We note the following definition which is pertinent for our purposes: a k-factor of a graph G is a spanning subgraph of G in which every vertex has degree k.

2.2   Definitions and concepts relating to snarks

Snarks are cubic graphs defined in terms of edge colourings.

Definition 2.2.1. Let G be a semi-graph. A *k-edge-colouring*, f, of G is a map $f : E \longrightarrow \{1, \ldots, k\}$. f is a *proper* k-edge-colouring of G if for any two adjacent edges $e_1, e_2 \in G$, we have that $f(e_1) \neq f(e_2)$.

In 1964, Vizing published his now famous theorem, which relates colourings to the maximum degree of a graph. By Vizing's theorem [5, Theorem 6.2], if G is a graph and f is a proper colouring then the smallest possible value of k is Δ or $\Delta + 1$, where Δ is the maximum degree of any vertex in G. It is easy to see that Vizing's theorem can be extended to semi-graphs as well, since a semi-edge never needs to be coloured the same colour as any one of its adjacent edges in a Δ-edge-colouring. If the smallest possible value of k is Δ, then it is said that G is *class one*, *Δ-edge-colourable*, or *colourable*. Otherwise it is said that G is *class two*, or *uncolourable*.

Inevitably then, when colouring a cubic or subcubic class two graph with a 3-edge-colouring, there will be a conflict. That is, some vertex will have incident edges mapped to the same colour. Thus we formally define a snark.

Definition 2.2.2. A *snark* is an uncolourable bridgeless cubic graph.

Any disconnected class two graph is made up of components at least one of which is class two, and the rest of which may be class one. Each component on its own can be thought of as either a class one graph or a class two graph. Thus for our purposes, we need only consider connected graphs. It is worth noting at this point that the snark with smallest order is the famous Petersen graph, with 10 vertices. The fact that no snark has order less than 10 is used implicitly in this thesis.

In this thesis, for 3-edge-colourings and 4-edge-colourings of cubic graphs, we will use colour sets $\{1, 2, 3\}$ and $\{0, 1, 2, 3\}$, respectively. Given a k-edge-colouring f, the set $f^{-1}(i)$ is called a colour class. We will always let $|f^{-1}(0)|$ be the minimum order of all the colour classes.

Definition 2.2.3. Let G be a cubic semi-graph. A vertex $v \in G$ is *conflicting* with regard to a 3-edge-colouring f of G, if more than one of the edges or semi-edges adjacent to v are mapped to the same colour. An edge is *conflicting* with regard to a minimal 4-edge-colouring f of G if $f(e) = 0$.

As mentioned previously, this thesis will briefly consider hypo-Hamiltonicity in the context of snarks. First, we present the definition of Hamiltonicity.

Definition 2.2.4. A semi-graph G is *Hamiltonian* if G contains a cycle C such that C contains every vertex in G.

It is simple matter to prove that no snark can be Hamiltonian. However, closely related to the property of Hamiltonicity is the property of hypo-Hamiltonicity. This property is rare in itself, just as are snarks. Intriguingly, and possibly surprisingly, these two rare properties overlap significantly, with many snarks, especially small ones, being hypo-Hamiltonian. A formal definition of a hypo-Hamiltonian graph follows.

Proposition 2.2.5. *Let G be a snark. Then G is not Hamiltonian.*

Proof. Since any cubic graph must have an even number of vertices, it is possible to colour the edges of a Hamiltonian cycle of a Hamiltonian cubic graph alternatively with two colours and the remaining edges with a third colour, to obtain a proper 3-edge-colouring. Thus no snark can be Hamiltonian. □

Definition 2.2.6. A semi-graph G is *hypo-Hamiltonian* if G is not Hamiltonian, and $G - v$ is Hamiltonian for every vertex $v \in G$.

There exists infinitely many hypo-Hamiltonian snarks [14]. In fact, Steffen showed that there exists hypo-Hamiltonian snarks of every even order $n \geq 92$. Even though the proof of this result is based on an erroneous result of Fiorini [14], the result still holds, as the mistake in [14] was fixed by Goedgebeur and Zamfirescu in [17]. Indeed, they determined all numbers n which are the order of a hypo-Hamiltonian snark. Steffen also proved that the removal of any two vertices in a hypo-Hamiltonian snark renders a 3-edge-colourable graph [38, 39]. Also, every such snark is cyclically 4-edge-connected and has girth 5 [33]. 4-edge-connectivity and girth 5 is generally used as the restriction of triviality in the study of snarks. Hypo-Hamiltonian snarks have also been studied in connection with the famous Cycle Double Cover Conjecture [7], which as mentioned is a conjecture easily solvable for all graphs except snarks.

The connection between these two properties is in fact so evident, especially in smaller snarks, that Steffen was prompted to suggest that hypo-Hamiltonian snarks can be regarded as those closest to class one cubic graphs [38]. In the next chapter of this thesis we prove a result which we feel points to the affirmation of this statement.

Let G be a strictly subcubic class one graph with no semi-edges and let f be a proper 3-edge-colouring of G. Let H be the subgraph of G consisting of only the edges in G which are coloured a and b by f, and their incident vertices, where $a, b \in \{1, 2, 3\}$. It is easy to see that H is then a collection of disconnected cycles and paths. Each cycle in H has even length and edges are coloured alternately with a and b. Each path has 2 terminal vertices and edges are coloured a and b alternately. We refer to these cycles and paths as $a - b$ cycles and $a - b$ paths, respectively.

These $a - b$ cycles and $a - b$ paths are examples of Kempe chains. Kempe chains were first introduced by Alfred Kempe in his attempt to prove the Four Colour Theorem [26]. Kempe chains have proven to be useful when considering problems relating to snarks and edge colourings, and have been used extensively by previous researchers (see for example [1, 2]).

Example 2.2.7. The diagram depicts a subcubic class one graph G with a proper 3-edge-colouring. In the first diagram the 1-2 cycles and paths are highlighted, and in the second diagram the 1-3 cycles and paths are highlighted.
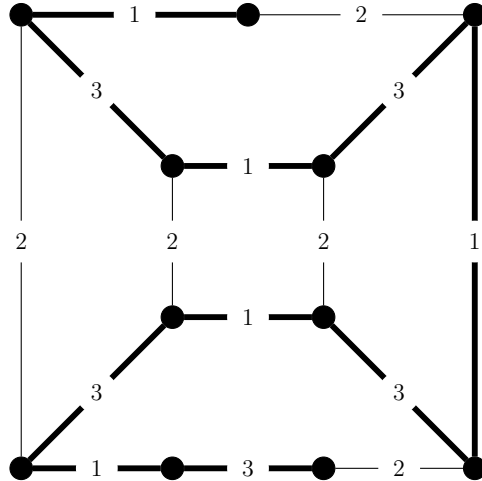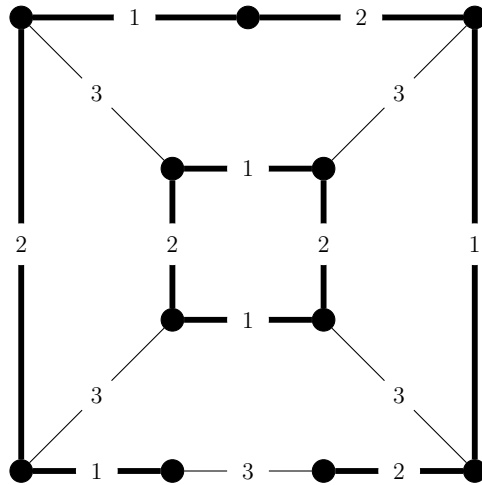
The Parity Lemma is another useful tool in proving whether a cubic graph is 3-edge-colourable or not. It has been used extensively by previous researchers and has been presented in different ways. Essentially, it says that given a proper 3-edge colouring of a graph with no semi-edges, any cut-set must have edges coloured in parity. We present a version of the lemma. Given its significance in our discussion, we present our own proof.

Lemma 2.2.8 (The Parity Lemma). [22, 24] *Let G be a cubic semi-graph with m semi-edges and let f be a proper 3-edge-colouring of G. If m_i equals the number of semi-edges coloured i by f for $i = \{1, 2, 3\}$, then*

$$m_1 \equiv m_2 \equiv m_3 \equiv m \mod 2.$$

Proof. Let G be a 3-edge-colourable cubic semi-graph with m semi-edges. Let f be a proper 3-edge-colouring of G. Let m_i be the number of semi-edges coloured i by f where $i \in \{1, 2, 3\}$. Then $m = m_1 + m_2 + m_3$. If m_i is even, then we join pairs of semi-edges coloured i to form $m_i/2$ edges each coloured i. If m_i is odd, then we join pairs of semi-edges coloured i to form $(m_i - 1)/2$ edges each coloured i. Let G' be the resulting graph, for which f is a proper 3-edge-colouring. If it is not the case that $m_1 \equiv m_2 \equiv m_3 \equiv m \mod 2$, then we are left with essentially two cases to consider for G'.

(i) G' is cubic and has one semi-edge which is coloured 1. Let v be the vertex incident to the semi-edge. v then has two other incident edges, coloured 2 and 3. Since v is not contained in a $1 - 2$ cycle, it must be a terminal vertex in a $1 - 2$ path. This path must contain another terminal vertex. However, every other vertex in G' has three incident edges coloured 1, 2 and 3. Thus no other vertex in G' is a terminal vertex in a $1 - 2$ path, a contradiction.

(ii) G' has two semi-edges which are coloured 1 and 2. Let u and v be the two vertices incident to the two semi-edges coloured 1 and 2, respectively. u then has two other incident edges, coloured 2 and 3. v then has two other incident edges, coloured 1 and 3. Since u is not contained in a $1 - 3$ cycle, it must be a terminal vertex in a $1-3$ path. This path must contain another terminal vertex. However, every other vertex in G' either has three incident edges coloured 1, 2 and 3, or has two incident edges coloured 1 and 3. Thus no other vertex in G' is a terminal vertex in a $1 - 3$ path, a contradiction.

Therefore, $m_1 \equiv m_2 \equiv m_3 \equiv m \mod 2$. $\square$

As a point of interest, an immediate consequence of the Parity Lemma is that any cubic graph containing a bridge is uncolourable. For a more general version of this Lemma which utilises Boole colourings, see [12].

2.3   Definition of k-flows and basic results

A k-flow is an assignment of non-negative integers less than k to the edges of a graph, combined with a direction from either vertex to the other, such that the sum of values flowing into a vertex equals the sum of values flowing out of a vertex. A nowhere zero k-flow is such an assignment, except that the integer 0 is not assigned to any edge of the graph.

Often when dealing with edge colourings, numbers are used as symbols instead of 'colours', a convention to which we adhere. If one thinks of a nowhere zero 4-flow and a proper 3-edge colouring as we describe it in this thesis, both these types of assignments of integers to edges utilise the set of integers $\{1, 2, 3\}$, suggesting that there may be a close relationship between them. As it turns out, these two types of assignments are indeed closely related. Specifically, it has been proven that a cubic graph admits a nowhere zero 4-flow if and only if it admits a proper 3-edge colouring [47].

We present formal definitions.

Definition 2.3.1. Let G be a graph. An *orientation* of G is an assignment of one of two possible directions to each edge in G, denoted as $D(G)$. If edge $[u, v]$ is assigned direction from u to v then it is said to have *tail* at u and *head* at v. Given $D(G)$, we let $E^-(v)$ and $E^+(v)$ denote the edges incident to vertex v with tail at v and with head at v, respectively.

In diagrams, we will portray direction with an arrow, with the head of the arrow at the head of the direction. Note also that when there is no confusion, we will denote an orientation simply as D instead of $D(G)$.

Definition 2.3.2. Let G be a graph.

(a) A *n-flow* of G is a pair (D, ϕ) where D is an orientation of G, ϕ is a function

defined by $\phi : E(G) \longrightarrow \mathbb{N}$ such that $0 \le \phi(e) < n$, and

$$\sum_{e \in E^-(v)} \phi(e) = \sum_{e \in E^+(v)} \phi(e)$$

for each $v \in G$.

(b) A *modular n-flow* is a pair (D, ϕ) where D is an orientation of G, ϕ is a function defined by $\phi : E(G) \longrightarrow \mathbb{Z}_n$, and

$$\sum_{e \in E^-(v)} \phi(e) = \sum_{e \in E^+(v)} \phi(e)$$

for each $v \in G$.

(c) The *support* of a (modular) n-flow (D, ϕ) is the set $\{e : \phi(e) \ne 0\}$, denoted by $\text{supp}(D, \phi)$. If $E(G) = \text{supp}(D, \phi)$ then we call (D, ϕ) a *nowhere zero (modular) n-flow*. We refer to edges in $G - \text{supp}(D, \phi)$ as *zero edges*.

Given a nowhere zero modular n-flow, if we reverse the direction of an edge e and replace the flow value by $-\phi(e)$, then we obtain another nowhere-zero modular n-flow on G. Hence, if G admits a nowhere-zero modular n-flow for a given D, then G admits a nowhere zero modular 4-flow for any other orientation D' of G. Thus when thinking of modular n-flows, the assignment of integers should be regarded as more essential than the assignment of a direction. We will use this fact implicitly in this thesis.

Due to Theorem 2.2 in [35], we also have the following results which are useful for our purposes. Theorem 2.2 in [35] was derived from results proven in [42, 43, 44].

Theorem 2.3.3. *Let G be a bridgeless graph. There exists an n-flow (D, ϕ) of G with k zero edges if and only if there exists a modular n-flow (D', ϕ') of G with k zero edges.*

We present our own proof for the aforementioned result regarding 3-edge-colourings and nowhere zero 4-flows, which utilises Theorem 2.3.3.

Proposition 2.3.4. *Let G be a cubic graph. Then G admits a 3-edge-colouring if and only if it admits a nowhere zero 4-flow.*

Proof. Let G be a cubic graph and f be a proper 3-edge-colouring of G. Consider the $1 - 3$ cycles in G with regard to f. We assign directions to these edges such that the head of each edge is incident to the tail of another. Then for each vertex v, the sum of integers flowing in minus the sum of values flowing out equals 2 or -2. Since all remaining edges are coloured 2, any direction on each of these edges results in a nowhere zero modular 4-flow of G, using the assignment of values from the 3-edge-colouring. By Theorem 2.3.3, G admits a nowhere zero 4-flow.

Let (D, ϕ) be a nowhere zero 4-flow of G. Note that at every vertex in G the number of incident edges with odd flow is even. Since the only even flow number is 2, it follows that the set of edges with flow number 2 is a 1-factor of G. Let F be the complementary 2-factor and let C be a component of F. Considering just the edges in C, the sum of flows at every vertex in C is either 2 or -2. Also, no two adjacent vertices in C can have the same sum of flows. Therefore, the sum of flows at vertices in C alternate between 2 and -2. It follows that C has even order. F is then 2-edge-colourable. Therefore, G is 3-edge colourable. □

Example 2.3.5. The diagram illustrates how a nowhere zero modular 4-flow can be derived from a proper 3-edge-colouring, as described in the proof of Proposition 2.3.4.
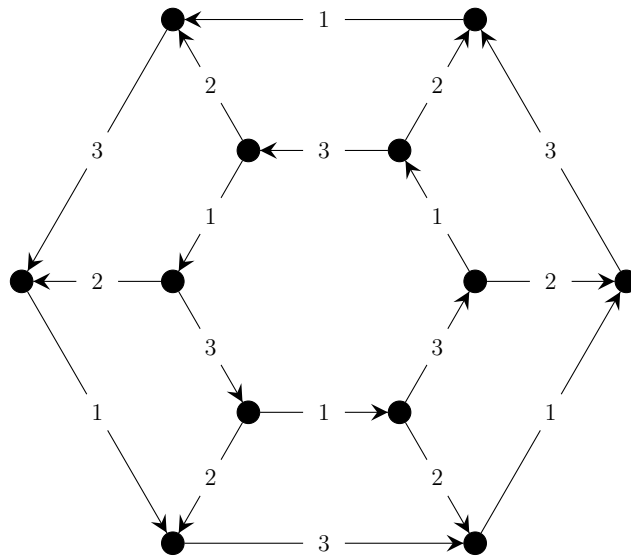
2.4   Measurements of uncolourability

The research presented in this thesis predominantly centres around parameters commonly known as *measurements of uncolourability*. Most parameters which have been previously studied and can be grouped in this category, have been done so in the particular context of snarks. While there exists many such parameters in mathematical literature which can be grouped in this category, our focus is predominantly on the following: resistance, vertex resistance, oddness, weak oddness and flow resistance.

   We present formal results and brief summaries of notable results involving these parameters.

Definition 2.4.1. Let G be a subcubic semi-graph. The *resistance* of G, denoted as $r(G)$, is defined as the $\min\{|f^{-1}(i)| : f$ is a proper 4-edge-colouring of G$\}$. That is, the minimum number of edges that can be removed from a graph such that the resulting graph is 3-edge-colourable.

Example 2.4.2. At least two edges need to be removed from the Petersen graph so that it is 3-edge-colourable. The diagram displays such an example of two edges being removed. The Petersen graph has resistance 2.
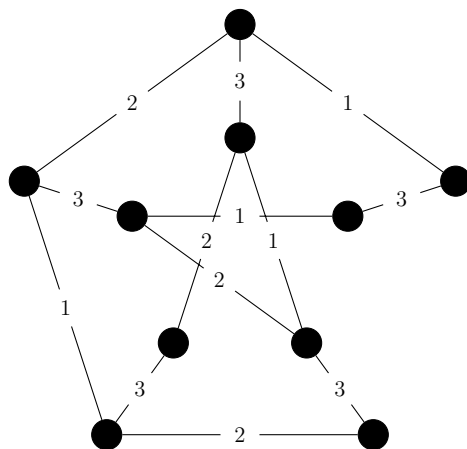

Definition 2.4.3. Let G be a subcubic semi-graph. The *vertex resistance* of G, denoted as $r_v(G)$, is defined as the minimum number of vertices to be removed from G so that the resulting graph admits a proper 3-edge-colouring.

Example 2.4.4. At least two vertices need to be removed from the Petersen graph so that it is 3-edge-colourable. The diagram displays such an example of two vertices being removed. The Petersen graph has vertex resistance 2.
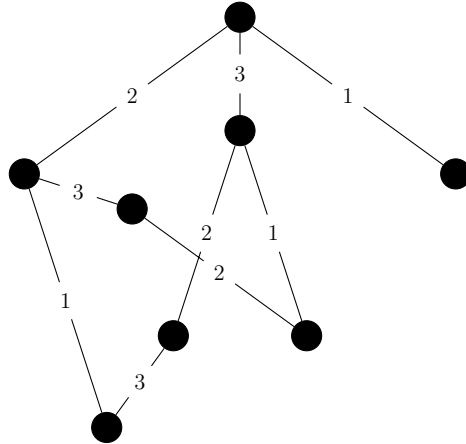

Definition 2.4.5. Let G be a bridgeless cubic graph. The *oddness* of G, denoted as $\omega(G)$, is defined as the $\min\{o(\mathcal{O}) : \mathcal{O}$ is a 2-factor of G$\}$.

Example 2.4.6. Any 2-factor of the Petersen graph must have at least 2 odd cycles. Such a 2-factor is represented by the thicker edges in the diagram. The Petersen graph has oddness 2.
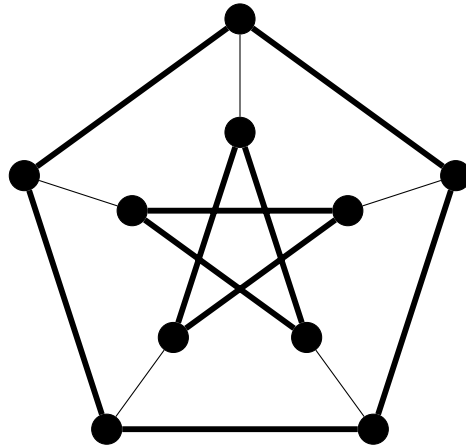
Definition 2.4.7. Let G be a bridgeless cubic graph. The *weak oddness* of G, denoted as $\omega'(G)$, is defined as the $\min\{o(\mathcal{O}) : \mathcal{O}$ is an even factor of G$\}$.

Example 2.4.8. Any even factor of the Petersen graph must have at least 2 odd cycles or isolated vertices. Such an even factor is represented by the thicker edges in the diagram. The Petersen graph has weak oddness 2.
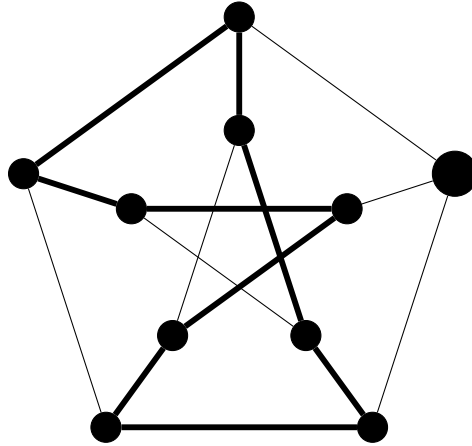
Definition 2.4.9. Let G be a cubic graph. The *flow resistance* of G, denoted as $r_f(G)$, is defined as the $\min\{|E(G) - \text{supp}(D, \phi)| : (D, \phi)$ is a $4 - $flow on G$\}$.

Example 2.4.10. Any modular 4-flow of the Petersen Graph must contain at least one zero edge. Such a modular 4-flow is displayed in the figure. The Petersen graph has flow resistance 1.
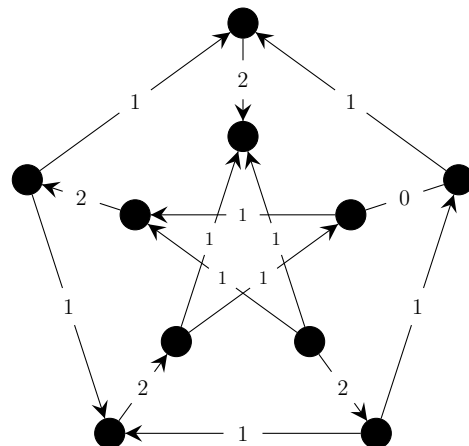
The primary reason for these parameters being considered as measurements of uncolourability, is due to the results highlighted in Proposition 2.4.11. While some of the parameters are defined in such a way that the statement of Proposition 2.4.11 applies, for others it is a consequence of the definition.

Proposition 2.4.11. *Let G be a bridgeless cubic graph. Then*

(i) $r(G) = 0$ if and only if G is 3-edge-colourable.

(ii) $r_v(G) = 0$ if and only if G is 3-edge-colourable.

(iii) $\omega(G) = 0$ if and only if G is 3-edge-colourable.

(iv) $\omega'(G) = 0$ if and only if G is 3-edge-colourable.

(v) $r_f(G) = 0$ if and only if G is 3-edge-colourable.

Proof.

(i) This follows straight from Definition 2.4.1.

(ii) This follows straight from Definition 2.4.3.

(iii) Let $\omega(G) = 0$. Let $\mathcal{O}$ be a 2-factor of G with 0 odd components. Each component in $\mathcal{O}$ is then an even cycle. Each of these cycles can have their edges coloured alternately with 1 and 2. The remaining edges in G can be coloured 3, so that G is coloured properly with three colours. Therefore G is 3-edge-colourable. Let G instead be 3-edge-colourable and let f be a proper 3-edge-colouring. All the edges coloured 1 and 2 by f together from a collection of even cycles which span the vertex set of G. Thus, they form a 2-factor with 0 odd components. Therefore $\omega(G) = 0$.

(iv) Let $\omega'(G) = 0$. Let $\mathcal{O}$ be an even factor of G with 0 odd components. $\mathcal{O}$ must also be a 2-factor, since it cannot contain any isolated vertices. By the same argument as (ii), G is 3-edge-colourable. Let G instead be 3-edge-colourable and let f be a proper 3-edge-colouring. Also as in (ii), we are able to find an even factor of G with 0 odd components. Therefore $\omega'(G) = 0$.

(v) This follows directly from Proposition 2.3.4. $\qquad\square$

Consequent to Proposition 2.4.11, we have that if any of the aforementioned measurements of uncolourability equal 0, then they all equal 0. Interestingly, this can also be said for when any of the measurements equal 2, with the proviso that we exclude flow resistance from consideration [12]. In fact, it is impossible for a snark to have resistance, vertex resistance, oddness or weak oddness equal to 1. This is easily comprehendible for oddness and weak oddness, since these parameters cannot equal any odd number at all, due to the fact that cubic graphs necessarily have even order. However, it is not at all obvious for resistance. From our investigations, at this point we would suggest that snarks with resistance 2 should be viewed as a special subclass of snarks which should be the focus of their own study.

Proposition 2.4.12. *Let G be a snark. Then $r(G) > 1$.*

Proof. Assume that $r(G) = 1$. Then there exists a 3-edge-colouring f of G with just one conflicting vertex, say v. Then the edges incident to v are all coloured the same colour, or two are coloured the same and a second colour is used for the third edge. We remove v, leaving behind the three incident edges which are now semi-edges. Then the resultant graph has three semi-edges, essentially coloured either with 1,1 and 1, or 1,1 and 2. Either way we have a contradiction by the Parity Lemma. Therefore, $r(G) > 1$. $\qquad\square$

In the next section we will be equipped to present our own proof that if one of the measures equal 2 then they all equal 2. For now, we first present a known result about vertex resistance and resistance. This result is perhaps counter intuitive, and states that vertex resistance equals resistance for all subcubic graphs. This result was proved by Kochol in [27]. Here, we present our own proof, and a useful corollary.

Theorem 2.4.13. [27] *Let G be a subcubic graph. Then $r_v(G) = r(G)$.*

Proof. Let f be a 3-edge-colouring of G with $r_v(G)$ conflicting vertices. There are essentially two possibilities of the colours of the edges incident to a conflicting vertex. In the first case, exactly two incident edges are coloured with some colour, say a, and the other edge (if it exists) with some colour, say b. In the second case, three incident edges are coloured with some colour, say c. For all vertices which has the first case with regard to f, we can colour one of the two edges coloured a with 0 instead. The vertex is then no longer conflicting. f is now a 4-edge-colouring and has conflicting vertices only as in the second case, if any. For a vertex as in the second case, at least one of the $c - a$ paths which initiate from such a conflicting vertex must terminate at some other vertex in G. The $c - a$ path will either terminate at a vertex which

is incident to an edge coloured with 0, or at some other conflicting vertex with all three incident edges having the same colour. If we swap the colours on this $c - a$ path, then the edges incident to the terminal vertices are coloured the same as in the first case, in which case we appropriately colour one of the edges with 0, or one of the incident edges is already coloured with 0. We now have a 4-edge-colouring of G with $r_v(G)$ conflicting edges. Therefore, $r_v(G) \geq r(G)$.

Let f be a proper 4-edge-colouring of G with $r(G)$ conflicting edges. For each conflicting edge, which is coloured 0, we may instead colour it properly with regard to exactly one of its incident vertices, in which case the other vertex becomes conflicting. We now have a 3-edge-colouring of G with $r(G)$ conflicting vertices. Therefore, $r(G) \geq r_v(G)$.

Therefore, $r_v(G) = r(G)$. $\square$

It is interesting that not only is $r_v(G) = r(G)$ for a snark G, but each edge in the set of $r(G)$ conflicting edges with regard to a 4-edge-colouring is incident to a unique conflicting vertex in some 3-edge-colouring of G. Thus the following corollary follows directly from the proof of Theorem 2.4.13.

Corollary 2.4.14. *Let G be a bridgeless cubic graph. There exists a 3-edge-colouring of G with $r(G)$ conflicting vertices $\{v_1, \ldots, v_r\}$ if and only if there exists a 4-edge-colouring of G with $r(G)$ conflicting edges $\{e_1, \ldots, e_r\}$, such that each e_i is incident to v_i, and not incident to any other v_j where $i \neq j$ and $i, j \in \{1, \ldots, r\}$.*

Due to Theorem 2.4.13 we will, for the most part, no longer make mention of vertex resistance. Furthermore, due to Corollary 2.4.14 we will not consider 3-edge-colourings with conflicting vertices, we will only consider proper 3-edge-colourings or proper 4-edge-colourings. More specifically, we will only be concerned with 4-edge-colourings where the order of the smallest colour class equals resistance. Such a 4-edge-colouring (or possibly 3-edge-colouring if resistance equals 0) is called a *minimal* 4-edge-colouring, or simply a *minimal colouring*, a termed coined by Steffen in [37].

The following result relates the three measurements oddness, weak oddness and resistance. This result is pertinent, and will be used throughout this thesis.

Proposition 2.4.15. *Let G be a snark. Then $\omega(G) \geq \omega'(G) \geq r(G)$.*

Proof. Every 2-factor of G is also an even factor of G. Therefore, $\omega(G) \geq \omega'(G)$. Now, let $\mathcal{O}$ be an even factor of G with $\omega'(G)$ odd components. From each odd component or isolated vertex in $\mathcal{O}$, we remove one vertex and incident edges. Each even component in $\mathcal{O}$ can have edges properly coloured alternatively with 1 and 2. The same can be done with the odd cycles, which are now not cycles since vertices have been removed. The remaining edges can be coloured 3, so that G is now properly coloured. Therefore, $\omega'(G) \geq r_v(G) \geq r(G)$. $\qquad\square$

2.5 Colourings with minimal conflicts

As mentioned, we will not consider 3-edge-colourings of snarks. We will only consider proper 4-edge-colourings of snarks such that resistance equals the number of conflicting edges. That is, as previously mentioned, minimal colourings. The following properties of minimal colourings are also attributed to Steffen in [37]. These properties of minimal colourings are used throughout this thesis.

Let f be a minimal colouring of a snark G. We define the set H_i for $i \in \{1, 2, 3\}$, as follows.

$$H_i := \{e \in G \mid f(e) = 0 \text{ and } e \text{ has two adjacent edges coloured i by } f\}.$$

We note as well that $H_1 \cup H_2 \cup H_3 = f^{-1}(0)$. That is, every conflicting edge in G with regard to f is contained in exactly one of H_1, H_2 or H_3.

Lemma 2.5.1. [38] *Let f be a minimal colouring of a snark G. Then $|H_1| \equiv |H_2| \equiv |H_3| \mod 2$.*

Proof. For each edge $[u, v] \in H_1$, subdivide $[u, v]$ with vertex w and add semi-edge $[w]$. Edges $[u, w]$ and $[w, v]$ can then be coloured properly, so that $[w]$ can be coloured properly with 1. Similarly, we do the same with each edge in H_2 and H_3, colouring the added semi-edges with 2 and 3 respectively. For each $i \in \{1, 2, 3\}$, we have that $|H_i|$ in G is equal to the number of semi-edges now coloured i. That $|H_1| \equiv |H_2| \equiv |H_2| \mod 2$ follows directly from the Parity Lemma. $\qquad\square$

Lemma 2.5.2. [38] *Let f be a minimal colouring of a snark G. Let a, b and c be distinct elements in $\{1, 2, 3\}$. Then for each $[u, v] \in H_a$, there exists a $b - c$ path from u to v.*

Proof. Consider the $b - c$ path starting from u. Assume that this path terminates at w which is distinct from v. If we swap the colours on this $b-c$ path, then $[u, v]$ is now properly colourable with one of b or c. Also, no conflict arises at any other vertex in G. However, we now have a proper 4-edge-colouring of G with less conflicting edges than f, a contradiction. Therefore, the $b - c$ path initiating at u must terminate at v. $\square$

Henceforth, for any $[u, v] \in H_a$, we will refer to the $b - c$ path from u to v together with $[u, v]$ as the cycle *induced by the conflicting edge* $[u, v]$, where a, b and c be distinct elements in $\{1, 2, 3\}$.

Lemma 2.5.3. [38] *Let f be a minimal colouring of a snark G. Let C_1 and C_2 be two distinct cycles induced by conflicting edges in G with regard to f. Then C_1 and C_2 are disjoint.*

Proof. Let C_1 and C_2 be induced by e_1 and e_2 respectively and let $e_1, e_2 \in H_a$. Then $C_1 - e_1$ and $C_2 - e_2$ are both $b - c$ paths, and all the edges incident to C_1 and C_2 are coloured a. Therefore, C_1 and C_2 must be disjoint since they are distinct.

Now, let $e_1 \in H_a$ and $e_2 \in H_b$. Assume that there exists some $e \in C_1 \cap C_2$. Then $f(e) = c$. Let $e = [u, v]$ and $e_1 = [u_1, v_1]$ and assume that e is the first such edge contained in both induced cycles on the $b - c$ path initiating at u_1. Assume as well that this $b-c$ path contains vertex u. Let $e_2 = [u_2, v_2]$. There exists some $a-c$ path which initiates at u and terminates at e, say at vertex u_2. We now swap the colours on the $b - c$ path from u_1 to u, and swap the $a - c$ colours on the $a - c$ path from u to u_2. Edges e_1 and e_2 are now properly colourable. u is now either a conflicting vertex with 3 incident edges all coloured c, or we can colour e with 0 so that neither u or v are conflicting vertices. We now have a colouring with less conflicts than f, a contradiction. Therefore, $C_1 \cap C_2 = \emptyset$. This completes the proof. $\square$

Remark 2.5.4. Let $[u, v] \in H_1$ be a conflicting edge with regard to a minimal colouring f of a snark G. Let C be the cycle induced by $[u, v]$ and let $[v, w]$ be adjacent to $[u, v]$ in C. If we swap the colours of $[u, v]$ and $[v, w]$, then G is still properly coloured. Thereby, we note that we are able to shift the conflicting edge to any other edge in C.

Example 2.5.5. The diagram depicts three conflicting edges and their induced cycles, in a snark G. G contains one conflicting edge in each of H_1, H_2 and H_3, displaying the parity described in Lemma 2.5.1. By Lemma 2.5.3, these three cycles are all disjoint.
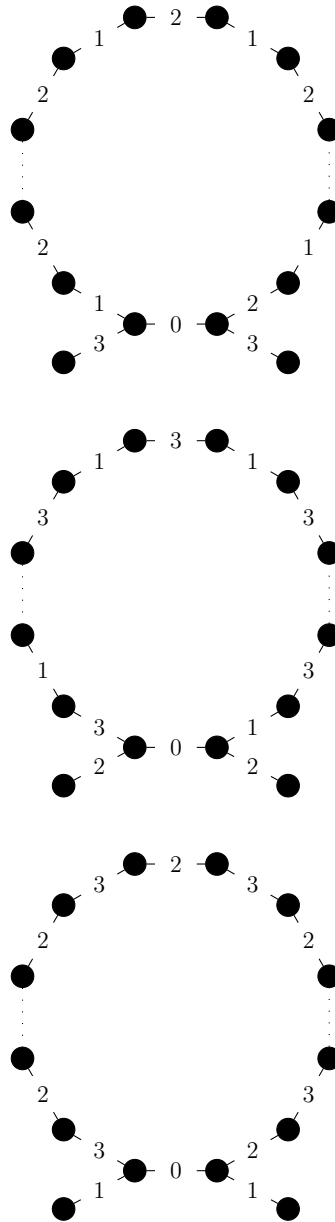
To complete this chapter, we present our proof that if one of resistance, oddness or weak oddness equals two, then they all coincide.

Proposition 2.5.6. [12] *Let G be a snark such that $r(G) = 2$, $\omega(G) = 2$ or $\omega'(G) = 2$. Then $r(G) = \omega(G) = \omega(G) = 2$.*

Proof. Let $\omega(G) = 2$. Then $\omega'(G) \leq 2$, and we know that $\omega'(G) > 1$. Therefore, $\omega(G) = 2$ implies $\omega'(G) = 2$.

Let $\omega'(G) = 2$. Then $r(G) \leq 2$, and by Proposition 2.4.12 we know that $r(G) > 1$. Therefore, $\omega'(G) = 2$ implies $r(G) = 2$.

Let $r(G) = 2$ and let f be a minimal 4-edge-colouring of G. By Lemma 2.5.1, let the two conflicting edges be contained in H_a. Then the $b - c$ cycles induced by the two conflicting edges, and the other $b - c$ cycles in G, form a 2-factor of G with two odd components. Therefore, $\omega(G) \geq 2$ which implies that $\omega(G) = 2$. Therefore, $r(G) = 2$ implies $\omega(G) = 2$. This completes the proof. $\square$

Chapter 3

3-Critical subgraphs of snarks

In this chapter we delve into the structure of snarks by introducing concepts relating to their 3-critical subgraphs, or as we will call them, minimal conflicting subgraphs. We consider how the minimal conflicting subgraphs relate to its possible minimal colourings. We fully characterise the resistance of a snark in terms of its minimal conflicting subgraphs. We also define what we call the critical subgraph of a snark, and similarly characterise this subgraph in terms of the snark's minimal conflicting subgraphs. In addition, we introduce new parameters which can be thought of as measuring uncolourability, as well as discuss ideas and problems which are consequent to the concepts introduced in this chapter.

3.1   Defining minimal conflicting subgraphs

The resistance of a snark informs how many edges need to be removed from the snark in order to render it a 3-edge-colourable graph. A natural consideration is then to ask: which are the edges that could be removed to render 3-edge-colourability. Or essentially, which are the edges that are contributing to the uncolourability of the snark, and which are redundant in this regard. The Petersen graph, for example, can be considered the most primitive snark. It is also edge-transitive. Every edge in the Petersen graph is essentially contributing to its uncolourability. Other snarks however may contain edges, or entire subgraphs, which if removed, do not affect the resistance. That is to say, the removal of these subgraphs do not affect the uncoloura-

bility of the graph. Other subgraphs, however, may themselves be uncolourable and thus be contributing to the overall uncolourability of the graph. This idea was introduced in [12] where the term *conflicting zone* was used. We opt for the term *conflicting subgraph.*

Definition 3.1.1. Let G be a snark. A *conflicting subgraph* of G is a subgraph of G which does not admit a proper 3-edge-colouring.

With a view to further understand what makes a cubic graph class two, we extend on this idea by defining *minimal conflicting subgraphs.* The essential idea is to isolate from the graph only that which is uncolourable, or contributing to the graph's uncolourability.

Definition 3.1.2. Let G be a snark. A *minimal conflicting subgraph* of G is a conflicting subgraph M of G such that $M - e$ is colourable for every $e \in M$. The *maximal conflicting subgraph* of G, denoted by M_G, is the subgraph of G induced by the edge set

$$E(M_G) = \bigcup \{E(M) \mid M \text{ is a minimal conflicting subgraph of } G\}.$$

Definition 3.1.3. Let G be a snark. The *conflict-cut set* of G is the set

$$C_G = \{e \in E(G) \mid e \notin M_G \text{ and } e \text{ is adjacent to some edge } e' \text{ in } M_G\}.$$

Definition 3.1.4. Let G be a snark. The *buffer subgraph* of G is the set

$$B_G = \{e \in E(G) \mid e \notin M_G \cup C_G\}.$$

It is easy to see that the definition of a minimal conflicting subgraph coincides with the definition of a 3-critical graph (see definition below), in that a minimal conflicting subgraph of a snark can be thought of as a 3-critical subgraph of a snark. Many authors have studied 3-critical graphs, see for instance [8, 9].

Definition 3.1.5. A subcubic graph G is *3-critical* if G is uncolourable and $G - e$ is colourable for every $e \in G$.

If the 3-critical subgraphs represent only that which is essentially not 3-edge-colourable, then the buffer subgraph represents that which is essentially redundant in contributing to the uncolourability of the cubic graph.

We list some properties of 3-critical subgraphs, or minimal conflicting subgraphs, of subcubic graphs. Proposition 3.1.6 refers to known properties of 3-critical graphs in general, to some of which we provide our own proofs.

Proposition 3.1.6. *Let M be a 3-critical graph. The following statements are true.*

(i) $r(M) = 1$ *and every edge $e \in M$ is conflicting in some minimal colouring of M.*

(ii) *M is strictly subcubic.*

(iii) *M is bridgeless.*

(iv) *Every vertex in M has degree two or three.*

(v) *Every vertex in M has at least two neighbours of degree three.*

Proof.

(i) This follows on directly from the definition of a 3-critical graph.

(ii) By Proposition 2.4.12, any snark has resistance greater than 1. Therefore, if M is cubic then M cannot be 3-critical, since at least two edges have to be removed in order to render a 3-colourable graph.

(iii) Assume that M contains a bridge, say e. Let f be a minimal colouring of M such that e is the only conflicting edge with regard to f. Let e_0 and e_1 be the two edges adjacent to e in one component of $M - e$, and e_2 and e_3 be the two edges adjacent to e in the other component of $M - e$. Let $f(e_0) = f(e_2) = a$, $f(e_1) = b$ and $f(e_3) = c$, where a, b and c are distinct in $\{1, 2, 3\}$. The $b - c$ path initiating with e_1 must terminate in the same component of $M - e$ as e_1. If we swap the colours on this path then e_1 is coloured with c, and e can be coloured b instead of being conflicting. We then have a proper 3-edge-colouring of M, a contradiction. Therefore, M is bridgeless.

(iv) Assume that $v \in M$ has degree one with incident edge e. Since e has only two other adjacent edges, there exists no minimal colouring of M where e is conflicting, a contradiction. Therefore, v has degree two or three.

(v) The is the case of Vizing's Adjacency Lemma for the case of 3-critical graphs [46]. The lemma states that for a vertex v with adjacent vertex u, v is adjacent to at least $4 - d(u) + 1$ vertices, where $d(u)$ denotes the degree of u. $\qquad\square$

Proposition 3.1.7 refers to properties of minimal conflicting subgraphs which are pertinent for our purposes.

Proposition 3.1.7. *Let G be a bridgeless cubic graph. The following statements are true.*

(i) *The distance between any two edge-disjoint minimal conflicting subgraphs of G is at least one.*

(ii) *Every conflicting subgraph in G contains a minimal conflicting subgraph.*

Proof.

(i) This follows on directly from Proposition 3.1.6 (iv).

(ii) Let M be a conflicting subgraph of G. Choose an edge $e \in M$. We check e by considering $r(M - \{e\})$. If $r(M - \{e\}) \neq 0$ then remove e from M. If $r(M - \{e\}) = 0$ then leave M as is and mark e as checked. Continue checking edges in M until every edge is checked. Once every edge is checked, M is then a minimal conflicting subgraph. □

Example 3.1.8. Graph G presented here is a 3-critical graph which will feature prominently in this thesis.
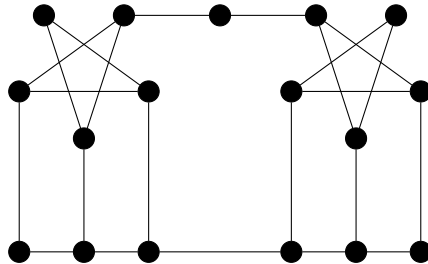
Example 3.1.9. Graph G presented here is a 3-critical graph which will feature in this thesis. It is the Petersen graph with one vertex removed.
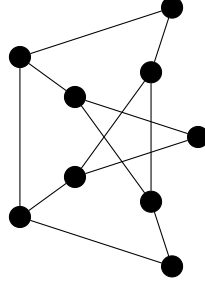
3.2   Minimal conflicting subgraphs and resistance

We begin our investigation into these minimal conflicting subgraphs. We consider their existence relative to conflicting edges in minimal colourings. Note that although our primary interest is in cubic graphs, some results are applicable to subcubic graphs as well, and are stated as such.

Proposition 3.2.1. *Let G be a subcubic class two graph and let f be a minimal colouring of G. For each conflicting edge e with regard to f, there exists at least one minimal conflicting subgraph M which contains e and contains no other conflicting edge with regard to f.*

Proof. Let f be a minimal colouring of G and let $R = \{e_1, \ldots, e_r\}$ be the set of conflicting edges with regard to f. For each $i \in \{1, \ldots, r\}$ let $M_i = \{e_i\}$ and conduct the following process. Choose an edge e not contained in $M_i \cup R$ which is adjacent to some edge in M_i. Add edge e to M_i. While $r(M_i) = 0$, we keeping adding such edges. Since $r(G - (R - \{e_i\}))$ must equal 1, we know that eventually we will have $r(M_i) = 1$. If $r(M_i) = 1$ then M_i is a conflicting subgraph which contains no conflicting edge with regard to f besides e_i. By Proposition 3.1.7 (ii), M_i contains a minimal conflicting subgraph. Since $r(M_i - e_i) = 0$, this minimal conflicting subgraph must contain e_i. This completes the proof. □

For convenience, we introduce a parameter which counts the number of distinct minimal conflicting subgraphs of a snark.

Definition 3.2.2. *Let G be a subcubic class two graph. By $s(G)$ we denote the number of distinct minimal conflicting subgraphs contained in G.*

For a snark G it is clear from Proposition 3.2.1 that $r(G) \leq s(G)$. Furthermore, there exists no function of $r(G)$ which upper bounds $s(G)$. The flower snarks and Louphekine snarks represent counter-examples to this idea. Each of the graphs in each of these classes have resistance 2. However, the order of the graphs can be arbitrarily large. Let G be some graph in one of these classes with order n. Now, the removal of any vertex v in G leaves behind a distinct minimal conflicting subgraph not containing v, and containing every other vertex in G. Therefore, $s(G) \geq n$. Since n can be arbitrarily large, there exists no function of $r(G)$ which bounds $s(G)$ for any snark G.

While no such bound exists, there does exist an essential relationship between resistance and minimal conflicting subgraphs. We will prove a theorem which provides much insight on this relationship, and will also equip us to formally present a characterisation of the resistance of a graph in terms of the its minimal conflicting subgraphs. First, we present a necessary definition.

Definition 3.2.3. Let G be a subcubic class two graph with minimal conflicting subgraphs $M_1, \ldots, M_s$. A *representative conflicting subset* of G is a set $R = \{e_1, \ldots, e_s\} \subseteq E(G)$ such that each $e_i \in M_i$ for $i \in \{1, \ldots, s\}$.

Remark 3.2.4. Since some e_i may be contained in more than one minimal conflicting subgraph, there may exist representative conflicting subsets with varying order.

A cubic graph G may have resistance $r(G)$, but given that information there is no way of knowing which combination of $r(G)$ edges may be removed from G in order to render colourability. Besides equipping us to characterise the relationship between resistance and minimal conflicting subgraphs, the following theorem is further significant in that it shows us that we can choose the conflicting edges of a minimal colouring, by simply selecting a combination of edges from each minimal conflicting subgraph, as long as this is done minimally.

Theorem 3.2.5. *Let G be a subcubic class two graph and let R be a representative conflicting subset of G. Then $G - R$ is 3-edge-colourable. Moreover, if $|R|$ is minimal then $r(G) = |R|$.*

Proof. Let $\mathcal{M} = \{M_1, \ldots, M_s\}$ be the set of all minimal conflicting subgraphs in G. Note that no M_i in $\mathcal{M}$ is a subgraph of $G - R$. Assume now that $G - R$ is not

3-edge-colourable. Then $G - R$ contains some minimal conflicting subgraph M' by Proposition 3.1.7 (ii). But M' is also contained in G, which is a contradiction since M' is not contained in $\mathcal{M}$. Therefore $G - R$ is 3-edge-colourable.

Let $|R|$ be minimal. Since $G - R$ is 3-edge-colourable, we have that $r(G) \leq |R|$. Assume that $r(G) < |R|$ and let f be a minimal colouring of G. Let R' be the conflicting vertices with regard to f. By Proposition 3.2.1, every element in R' is contained in some minimal conflicting subgraph of G. If R' is not a representative conflicting subset then there exists some minimal conflicting subgraph $M' \subset G$ which contains no conflicting edges with regard to f. In which case, we have a minimal conflicting subgraph of G which is properly coloured by f using just three colours, which is a contradiction. If R' is a representative conflicting subset, then the minimality of $|R|$ is contradicted since $|R'| = r(G) < |R|$. Therefore, $r(G) = |R|$. $\quad\square$

Remark 3.2.6. Using Theorem 3.2.5, we are now able to explicitly characterise the relationship between the resistance of a cubic graph and its minimal conflicting subgraphs. The resistance of a graph is equal to the minimum number of non-empty intersections of minimal conflicting subgraphs, such that the union of these intersections itself has nonempty intersection with each minimal conflicting subgraph. This is because we can select one edge from each of these nonempty intersections to form a representative conflicting subset of minimal order.

Theorem 3.2.7. *Let G be a subcubic class two graph and let $\mathcal{M} = \{M_1, \ldots, M_s\}$ be the set of all minimal conflicting subgraphs in G. Let $\mathcal{J}$ be the set of all non-empty intersections of one or more elements of $\mathcal{M}$. Then*

$$r(G) = \min\{r | \text{there exists } J_1, \ldots, J_r \in \mathcal{J} \text{ with } (J_1 \cup \cdots \cup J_r) \cap M_i \neq \varnothing$$

$$\text{for every } M_i \in \mathcal{M}\}.$$

Proof. This result follows directly from Theorem 3.2.5 and Remark 3.2.6 $\quad\square$

3.3 Defining and characterising critical edges

With Theorem 3.2.7 we recognise that if there exists some edge e which is contained in exactly one minimal conflicting subgraph M, but M has non-empty intersection with some other minimal conflicting subgraph M', then e may not be conflicting

in any minimal colouring of G. Another way of saying this is, if $e \in M$ where M is a minimal conflicting subgraph of G, then it is not necessarily the case that $r(G-e) = r(G) - 1$. Equivalently, we could say that there does not necessarily exist some minimal colouring of $H \subset G$ which can be extended to a minimal colouring of G. As such, it is possible to have a minimal colouring of a subgraph $H \subset G$ with say, r_1 conflicting edges, such that a minimal extension has r_2 further conflicting edges, but $r_1 + r_2 > r(G)$. What is also clear is that if every minimal conflicting subgraph of G is disjoint, then $r(G) = s(G)$. Consequent to this discussion, we define the following.

Definition 3.3.1. Let G be a subcubic class two graph. The *critical subgraph* of G, denoted by K_G, is the subgraph of G induced by the edge set

$$E(K_G) = \{e \in G \mid f(e) = 0 \text{ for some minimal colouring } f \text{ of } G\}.$$

As we did with resistance, we will explicitly characterise the critical subgraph in terms of the minimal conflicting subgraphs.

Theorem 3.3.2. *Let G be a subcubic class two graph and let $\mathcal{M} = \{M_1, \ldots, M_s\}$ be the set of all minimal conflicting subgraphs in G. Let $\mathcal{J}$ be the set of all non-empty intersections of one or more elements of $\mathcal{M}$. Then*

$$K_G = \bigcup \{J_1 \cup \cdots \cup J_{r(G)} \mid J_1, \ldots, J_{r(G)} \text{ is a collection of } r(G) \text{ elements of } \mathcal{J}$$

$$\text{such that } (J_1 \cup \cdots \cup J_{r(G)}) \cap M_i \neq \varnothing \text{ for every } M_i \in \mathcal{M}\}.$$

Proof. For any collection of $r(G)$ elements of $\mathcal{J}$ such that $(J_1 \cup \cdots \cup J_{r(G)}) \cap M_i \neq \varnothing$ for every $M_i \in \mathcal{M}$, it is clear that every edge in $J_1 \cup \cdots \cup J_{r(G)}$ is conflicting in some minimal colouring of G, by Remark 3.2.6. Therefore, the union of all such unions $J_1 \cup \cdots \cup J_{r(G)}$ is contained in K_G.

Let f be a minimal colouring of G. Let $R = \{e_1, \ldots, e_{r(G)}\}$ be the set of conflicting edges with regard to f. Then R must be a representative conflicting subset of minimal order, by Theorem 3.2.5. Therefore, we can find $J_1, \ldots, J_{r(G)} \in \mathcal{J}$ such that $e_1 \in J_1, \ldots, e_{r(G)} \in J_{r(G)}$, and $(J_1 \cup \cdots \cup J_{r(G)}) \cap M_i \neq \varnothing$ for every $M_i \in \mathcal{M}$, by Theorem 3.2.7. Therefore, K_G is contained in the union of all such unions $J_1 \cup \cdots \cup J_{r(G)}$. This completes the proof. $\square$

Example 3.3.3. Each of the Venn diagrams depict six minimal conflicting subgraphs in a snark G with $r(G) = 3$. In each of the first three diagrams, we may remove from the graph any edge contained in each shaded area, to render a 3-edge-colourable graph, by Theorem 3.2.5. The fourth Venn diagram depicts the union of the shaded areas of the first three diagrams. This depicts the critical subgraph of G.
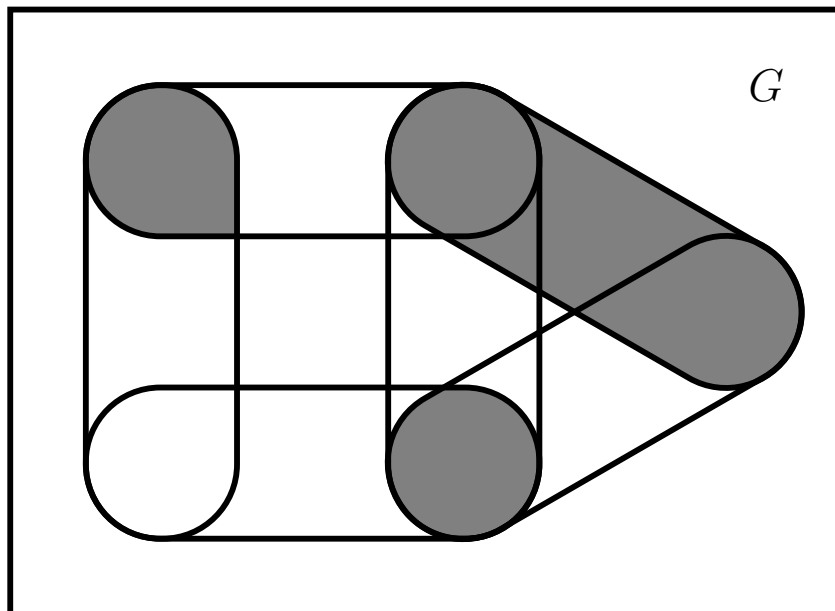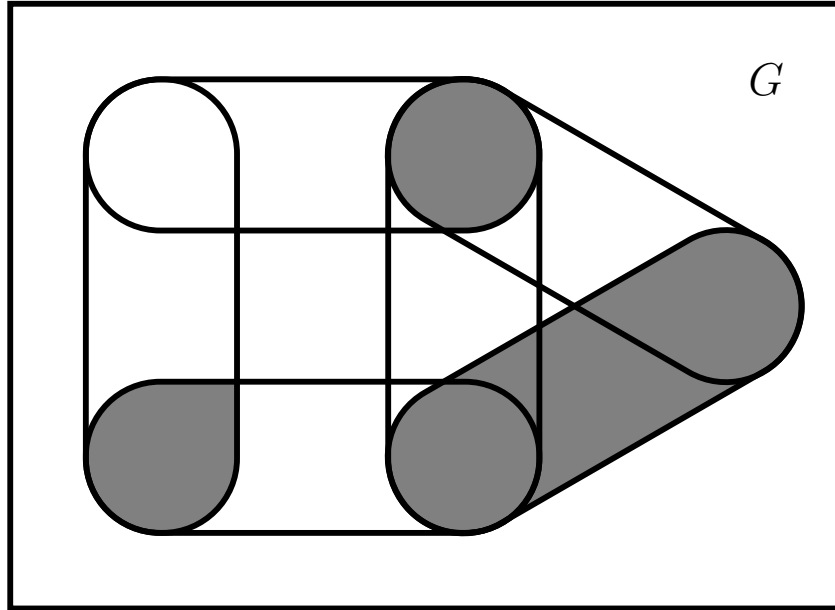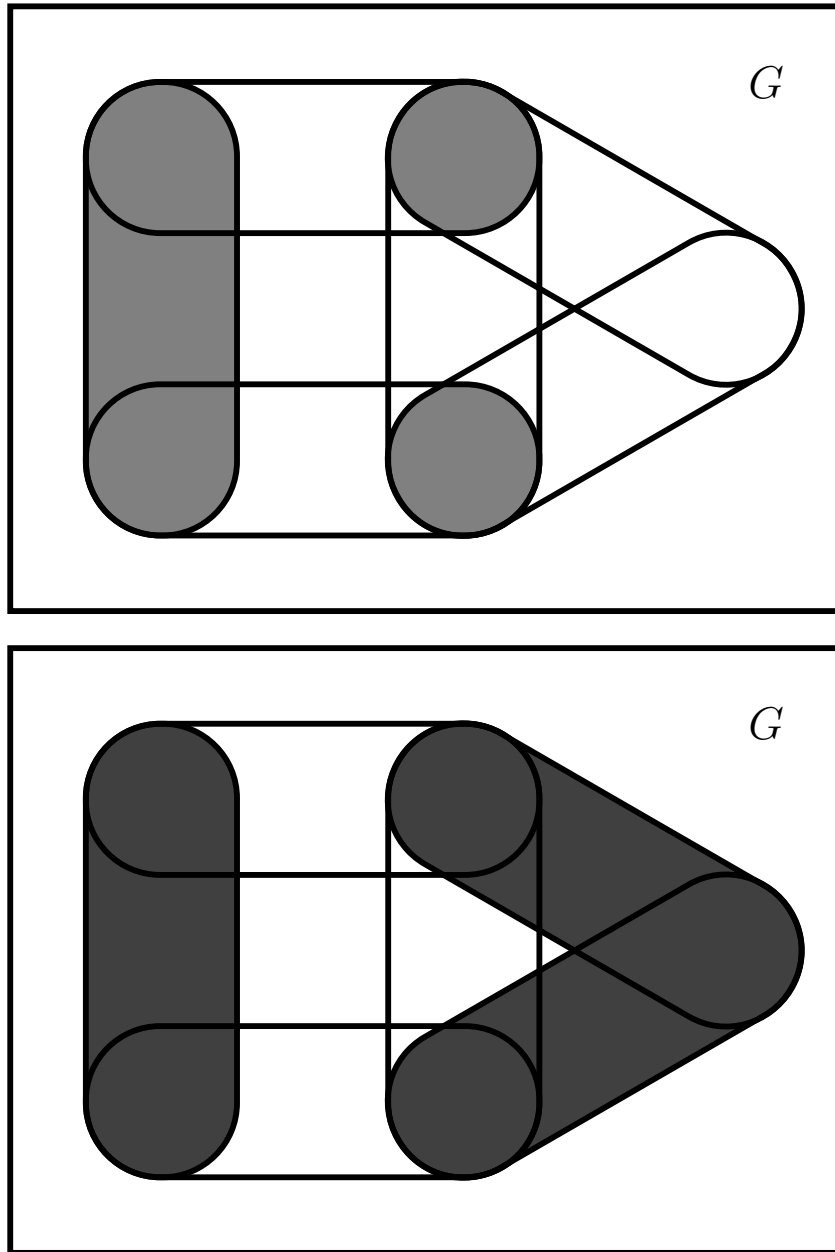
It is clear that $K_G \subseteq M_G$. We present a simple example where K_G is a strict subgraph of M_G, and another where $K_G = M_G$.

Example 3.3.4. The graph G depicted consists of two identical overlapping minimal conflicting subgraphs M_1 and M_2 as in Example 3.1.8. Thus $M_G = G$ and $\mathcal{J} = \{M_1, M_2, M_1 \cap M_2\}$. Therefore, $r(G) = 1$ and $K_G = M_1 \cap M_2$ by Theorem 3.3.2. That is, $K_G \subset M_G$. $M_1 \cap M_2$ is represented by the thicker edges. Any edge in $M_1 \cap M_2$ is itself a representative conflicting subset of minimal order. Thus, even though $K_G = M_1 \cap M_2$ is itself 3-edge-colourable, any minimal colouring of G must contain a conflicting edge in $K_G = M_1 \cap M_2$.
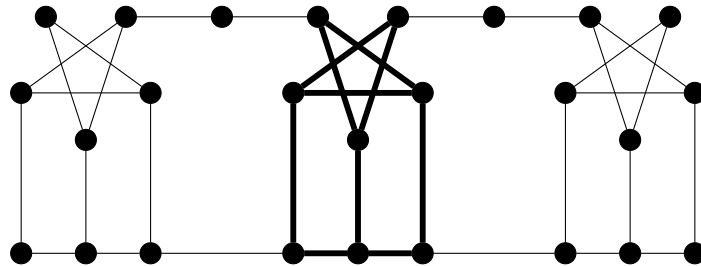
Example 3.3.5. The snark G depicted consists of three identical non-overlapping minimal conflicting subgraphs M_1, M_2 and M_3 as in Example 3.1.9. Thus $M_G \subset G$ and $\mathcal{J} = \{M_1, M_2, M_3\}$. Therefore, $r(G) = 3$ and $K_G = M_1 \cup M_2 \cup M_3$ by Theorem 3.3.2. That is, $K_G = M_G$. $M_1 \cup M_2 \cup M_3$ is represented by the thicker edges. Any set of three edges, one each from M_1, M_2 and M_3, is a representative conflicting subset.
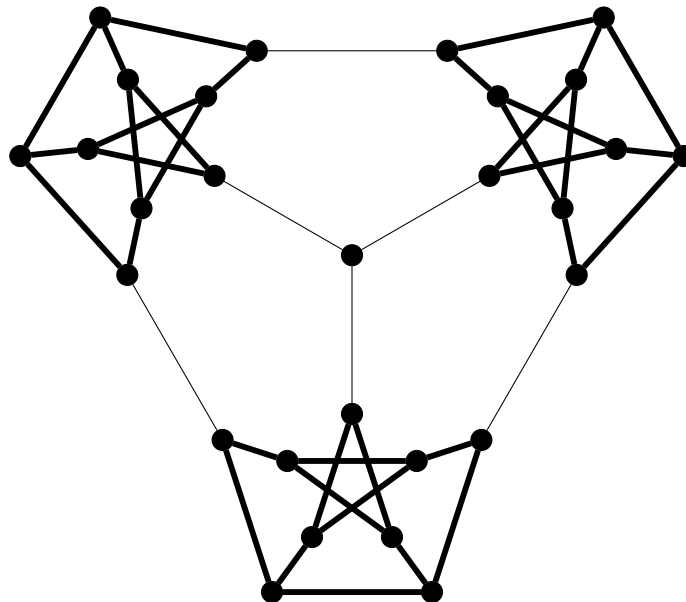
3.4   Other measurements and problems

We introduce the following two parameters which we claim can be regarded as measuring the uncolourability of subcubic graphs. We then briefly consider cases where minimal conflicting subgraphs are not disjoint, which appears to be typically the case in smaller snarks. To facilitate this, we define *clusters*.

Definition 3.4.1. Let G be a subcubic class two graph.

(a) The *conflicting ratio* of G, denoted by $m(G)$, is defined as

$$m(G) = \frac{|E(M_G)|}{|E(G)|}.$$

(b) The *critical ratio* of G, denoted by $k(G)$, is defined as

$$k(G) = \frac{|E(K_G)|}{|E(G)|}.$$

The conflicting ratio measures the proportion of the graph which is essentially contributing to it being uncolourable. The critical ratio on the other hand measures the proportion of the graph which is potentially conflicting in a minimal colouring of the graph.

Definition 3.4.2. Let G be a subcubic class two graph. Let $\mathcal{M} = \{M_1, \ldots, M_m\}$ be a collection of minimal conflicting subgraphs in G.

(i) $\mathcal{M}$ is a *cluster* of minimal conflicting subgraphs if for every $i \in \{1, \ldots, m\}$ there exists some $j \in \{1, \ldots, m\}$ such that $M_i \cap M_j \neq \varnothing$, and $M \cap M_i = \varnothing$ for any other minimal conflicting subgraph $M \notin \mathcal{M}$.

(ii) $\mathcal{M}$ is a *dense cluster* if it is a cluster and $\bigcap M_i \neq \varnothing$. If $\mathcal{M}$ is not dense then we say it is *sparse*.

(iii) $\mathcal{M}$ is a *densely sparse cluster* if $\mathcal{M}$ is sparse cluster such that for every $i, j \in \{1, \ldots, m\}$, we have that $M_i \cap M_j \neq \varnothing$.

Proposition 3.4.3. *Let G be a snark. If $m(G) = 1$ then G consists entirely of one sparse cluster of minimal conflicting subgraphs.*

Proof. Since the distance between any two clusters must be at least one, that one edge is then not in any minimal conflicting subgraph. Thus $m(G) = 1$ implies that G consists entirely of one cluster of minimal conflicting subgraphs. Assume the cluster is dense. Then there exists some e which is contained in every minimal conflicting subgraph of G. $G - \{e\}$ must then contain no conflicting subgraph. However, G is then a cubic graph with resistance one, which is impossible. Therefore, G consists entirely of one sparse cluster of minimal conflicting subgraphs. $\square$

The nature of sparse clusters is intriguing. In particular, the difference between sparse clusters and densely sparse clusters. A sparse cluster, we suspect, is densely sparse if and only if it is cubic. In Example 3.3.4 we overlapped two graphs as from Example 3.1.8. If we similarly overlap more such graphs on either end, we would have a sparse cluster which is not densely sparse and is not cubic. On the other hand, the Petersen graph is a densely sparse cluster which is cubic. If a sparse cluster is strictly subcubic, then it possibly allows for the addition of edges without adding more minimal conflicting subgraphs so that the conflicting ratio could be less than one. Furthermore, for any densely sparse cluster, it appears from our investigations as if we can always find a representative conflicting subset of order 2. This leads us to formulate the following conjectures.

Conjecture 3.4.4. *Let G be a snark. If $k(G) = 1$ then $r(G) = 2$.*

Conjecture 3.4.5. *Let G be a snark. If $m(G) = 1$ then $r(G) = 2$.*

Since $k(G) = 1$ implies that $m(G) = 1$, it is clear that Conjecture 3.4.4 is true if Conjecture 3.4.5 is true.

3.5 Hypo-Hamiltonian snarks and critical edges

A snark G is said to be *bicritical* if $G - \{u, v\}$ is colourable for any two vertices $u, v \in G$. Steffen showed that if a snark G is hypo-Hamiltonian then G is bicritical [38]. Nedela and Skoviera [33] showed further that every cubic bicritical graph has girth at least 5 and is cyclically 4-edge-connected. Thus as Steffen remarked in [37], we may consider hypo-Hamiltonian cubic class two graphs, to represent the cubic

class two graphs which are in a sense closest to cubic class one graphs. We feel that the following result on hypo-Hamiltonian snarks asserts this claim.

Theorem 3.5.1. *Let G be a hypo-Hamiltonian snark. Then $K_G = G$.*

Proof. Let G be a hypo-Hamiltonian snark and let $[u, v] \in E(G)$. Let C be a Hamiltonian cycle in $G - v$ and let w be another neighbour of v. Colour all edges in C alternatively with 1 or 2, except at vertex w in C where one incident edge in C is coloured 2, and the other is coloured 0. Colour every other edge in $G - [u, v]$ with 3, except for $[v, w]$ which can be properly coloured with 1. Thus we have a proper colouring of $G - [u, v]$ with just one conflicting edge (see Figure 3.1). Since $r(G) = 2$, $[u, v]$ must be conflicting in some minimal colouring of G. Therefore, $K_G = G$. $\square$

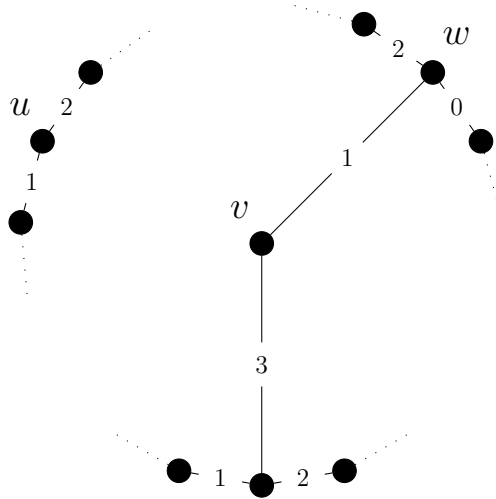Figure 3.1: A proper 4-edge-colouring of $G - v$ with one conflicting edge. The 3-coloured chordal edges and the alternatively coloured 1-2 edges in C are not depicted in the diagram.

Since hypo-Hamiltonian snarks are bicritical, we note that this implies that every minimal conflicting subgraph in a hypo-Hamiltonian snark contains all but one vertex. We also note that this does not imply that $G - v$ is a minimal conflicting subgraph for every vertex v in a hypo-Hamiltonian snark G. It could be that $G - v$ contains an edge which is not conflicting in any minimal 3-edge-colouring of $G - v$.

Chapter 4

Reducing resistance in snarks

In this chapter, we introduce various related parameters, each of which can be considered as measuring the uncolourability of a cubic graph. These measures are related to snark reductions as introduced in [38]. That is, the removal of two vertices and subsequent addition of edges to restore 3-regularity. Vertices may or may not be adjacent, and edges which are added back can be done so in various ways. We prove insightful results on these parameters, as well as present a significant conjecture relating these parameters to resistance. The conjecture is significant in the consequences of it being true. We prove that if the conjecture is true, then we are able to prove two previously stated conjectures. The first conjecture is regarding the relationship between weak oddness and resistance, which states that the weak oddness of a graph is bounded by two times the resistance. The second conjecture is regarding flow resistance, which states that resistance is greater than flow resistance.

4.1   Measures and results relating to reductions

Since many different types of snark reductions have been previously considered by various authors, it is necessary for us to formalise our definition of a reduction. These types of reductions were presented in [38].

Definition 4.1.1. Let G be a cubic graph.

(i) Let u and v be vertices in G. A *vertex reduction* of G is a graph obtained by the removal of vertices u and v and their incident edges, and subsequent addition of edges to restore 3-regularity. It is said that G has been *vertex reduced* on u and v.

(ii) Let $[u, v]$ be an edge in G. A *1-reduction* of G is a graph obtained by the removal of vertices u and v and their incident edges, and subsequent addition of edges to restore 3-regularity. It is said that G has been *1-reduced* on $[u, v]$.

(iii) Let $[u, v]$ be an edge in G. An *edge reduction* of G is the graph obtained by the removal of vertices u and v and their incident edges, and subsequent addition of an edge between the remaining two neighbours of u and an edge between the remaining two neighbours of v. It is said that G has been *edge reduced* on $[u, v]$.

Example 4.1.2. An example of a vertex reduction, 1-reduction and edge reduction.
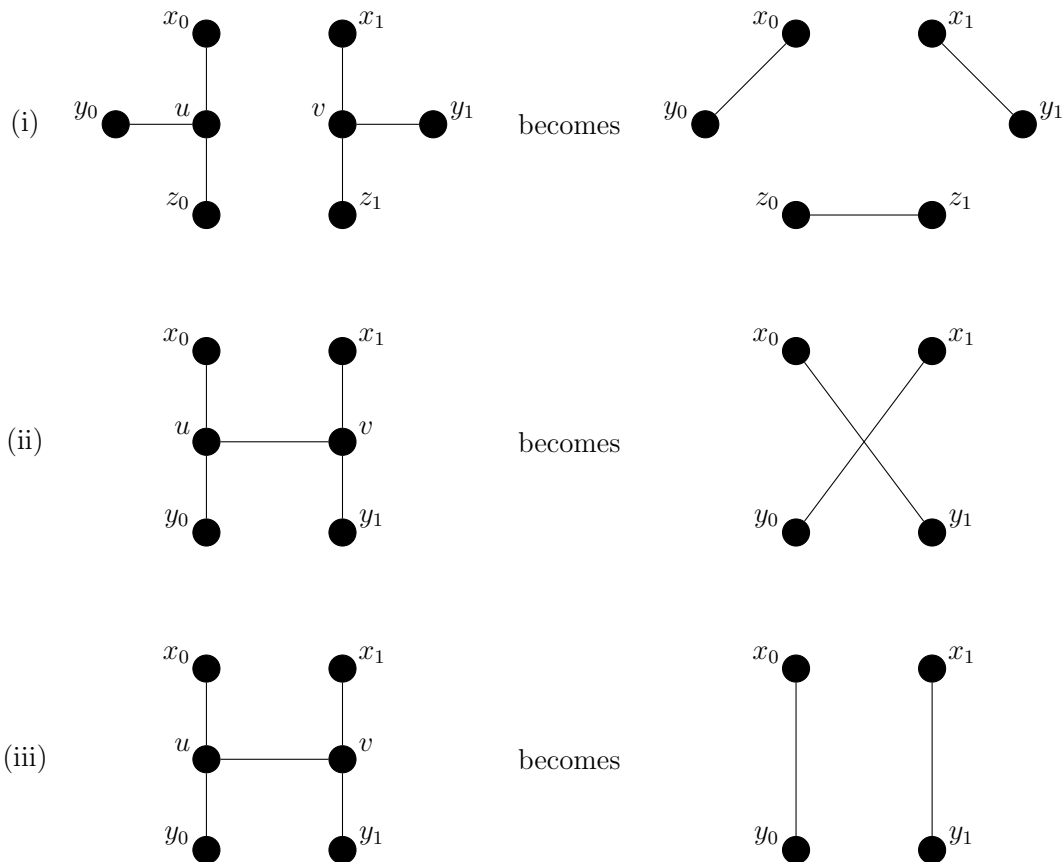
Remark 4.1.3. It is important to note that the addition of edges to restore 3-regularity can be done in various ways. 1-reductions involve the adding back of two edges in possibly three different ways, while vertex reductions which are not 1-reductions involve the adding back of three edges in possibly fifteen different ways. It is not always possible to add back edges in three or fifteen ways for 1-reductions or vertex-reductions respectively, as we do not consider graphs with double edges and loops for our purposes. For example, there is only way to 1-reduce a cubic graph on two adjacent vertices which are part of a triangle.

In [38], Steffen classifies snarks into three different categories. The classes are: snarks that are vertex reducible to colourable graphs only; snarks that are vertex reducible to uncolourable graphs only; and snarks that are vertex reducible to either colourable or uncolourable graphs. In fact, Steffen strongly motivates for a conjecture which states that the Petersen graph is the only snark which is vertex reducible only to a colourable graph. Here, we are interested in the effect of vertex reductions not just on colourability, but specifically on resistance as well.

Firstly, we consider whether it is always possible to reduce the resistance of a graph via a vertex reduction. As it turns out, this is always possible. Recall that the set H_i for $i \in \{1, 2, 3\}$ is defined as

$$H_i := \{e \in G \mid f(e) = 0 \text{ and } e \text{ has two adjacent edges coloured i by } f\}.$$

Theorem 4.1.4. *Let G be a snark. If $r(G) \neq 3$ then there exists a vertex reduction G' of G such that $r(G') = r(G) - 2$. If $r(G) = 3$ then there exists a vertex reduction G' of G such that $r(G') = 2$.*

Proof. Let $r(G) \neq 3$ and let f be a minimal colouring of G. Then for some $a \in \{1, 2, 3\}$, $|H_a| \geq 2$. Let $e_0, e_1 \in H_a$ such that v_0 and v_1 are incident to e_0 and e_1 respectively. Let x_0, y_0, x_1, y_1, z_0 and z_1 be the vertices in G such that $f([v_0, x_0]) = f([v_1, x_1]) = a$, $f([v_0, y_0]) = f([v_1, y_1]) = b$ and $f([v_0, z_0]) = f([v_1, z_1]) = 0$. If we remove v_0 and v_1 from G, we can add back edges $[x_0, x_1], [y_0, y_1]$ and $[z_0, z_1]$ to form G'. We may then colour these edges as $f([x_0, x_1]) = a, f([y_0, y_1]) = b$ and $f([z_0, z_1]) = b$. f now has two less conflicting edges in G', therefore $r(G') = r(G) - 2$. See Figure 4.1. Note that if G' contains loops or double edges, then we may opt for adding back the edges in a different way, or if necessary, we may also opt to first shift one or both of the conflicting edges along their induced cycles. It is clear that we can ensure G' contains no loops or double edges.

Let $r(G) = 3$. Let $e_0 \in H_a$ and $e_1 \in H_b$. Let x_0, y_0, x_1, y_1, z_0 and z_1 be the

vertices in G such that $f([v_0, x_0]) = f([v_1, x_1]) = a$, $f([v_0, y_0]) = f([v_1, y_1]) = b$ and $f([v_0, z_0]) = f([v_1, z_1]) = 0$. If we remove v_0 and v_1 from G, we can add back edges $[x_0, x_1], [y_0, y_1]$ and $[z_0, z_1]$. We may then colour these edges as $f([x_0, x_1]) = a, f([y_0, y_1]) = b$ and $f([z_0, z_1]) = 0$. f now has one less conflicting edge in G', therefore $r(G') = r(G) - 1$. See Figure 4.1. As before, we can ensure that G' contains no loops or double edges. $\qquad \square$

While it is always possible to reduce the resistance via a vertex reduction, it is also clear that the resistance cannot decrease by more than two. This is since for any minimally coloured snark, a vertex can be adjacent to at most one conflicting edge. Thus in the class of snarks which can be reduced to colourable graphs, each snark must have resistance 2.

Whether it is always possible to reduce resistance via a 1-reduction proves to be much more complicated. In our investigations, this seems to always be possible even for an edge reduction, except in cases where resistance equals 2. When resistance equals 2, then there exists an edge reduction which is colourable if and only if there exists a 2-factor with two odd components distance 1 apart [38]. This is similar to saying that there exists an edge reduction which is colourable if and only there exists a minimal colouring with conflicting edges distance 1 apart.

If we consider the case of resistance 2, defined by two disjoint minimal conflicting subgraphs with a large buffer subgraph in between, then it is clearly impossible to find a 1-reduction which reduces resistance. However, in our investigations we have not found a case in which we cannot find two 1-reductions, or even edge reductions, which reduce resistance to zero. Thus, an interesting alternate question to ask is: how many 1-reductions (edge reductions) are necessary to reduce a snark to a colourable graph?

As mentioned previously, there are possibly up to three different ways to add edges back in 1-reductions. We will associate a referencing notation with each of these three ways. Even though these notations are associated abstractly, its purpose is that we may henceforth be able to distinguish between them. Let $[u, v] \in G$. We assign *references* to the neighbours of u and v as follows. Let r_0 and r_1 reference the other two vertices adjacent to u, and s_0 and s_1 reference the other two vertices adjacent to v. When 1-reducing on $[u, v]$, we first remove the vertices u and v. Now, we use subscripts to distinguish between the different types. By 1-reducing G on $[u, v]_1$, we mean that an edge is added between reference points r_0 and r_1, and an
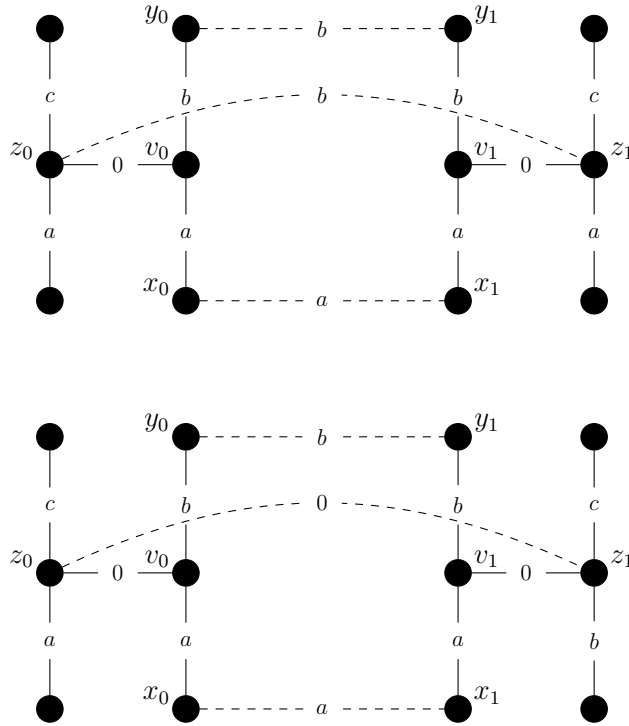
Figure 4.1: The above diagram depicts the two cases referred to in the proof of Theorem 4.1.4.

edge is added between reference points s_0 and s_1. As is clear, this is the same as edge-reducing G on $[u, v]$. By 1-reducing G on $[u, v]_2$, we mean that an edge is added between reference points r_0 and s_0, and an edge is added between reference points r_1 and s_1. By 1-reducing G on $[u, v]_3$, we mean that an edge is added between reference points r_0 and s_1, and an edge is added between reference points r_1 and s_0. Similarly, in the case of vertex reductions, we may reduce on $\{u, v\}_x$ where instead $x \in \{1, \ldots, 15\}$.

Now, we may consider 1-reducing G on a set of edges. Consider the ordered set

$$S = \{[u_0, v_0]_{x_0}, [u_1, v_1]_{x_1}, \ldots, [u_m, v_m]_{x_m}\}$$

with $x_i \in \{1, 2, 3\}$ for each i, such that any two vertices from two distinct elements in S are distance greater than 1 apart. By reducing G on S, we mean reducing G on $[u_0, v_0]_{x_0}$, then reducing the resulting graph on $[u_1, v_1]_{x_1}$, continuing in this way

until we have reduced on $[u_m, v_m]_{x_m}$ to get G_S. Similarly, we may vertex reduce G on a set of pairs of vertices. That is, S would be of the form

$$S = \{\{u_0, v_0\}_{x_0}, \{u_1, v_1\}_{x_1}, \ldots, \{u_m, v_m\}_{x_m}\}$$

with $x_i \in \{1, \ldots, 15\}$. Such a set S we call a *vertex reducible set of G*, or a *1-reducible set of G* as in the former case. Given the condition that any two vertices from any two distinct elements in S are distance greater than 1 apart, it is easy to see that the order of elements in S is in fact irrelevant to G_S. That is, $G_{S'} = G_S$ for any permutation S' of elements of S.

Henceforth, we will refer only to reductions on sets, with $|S| = 1$ in the single case. We say that G_S is a *set vertex reduction* of G on the vertex reducible set S. In the case of 1-reductions, we say that G_S is a *set 1-reduction* of G on the 1-reducible set S. In the case of edge reductions we say that G_S is a *set edge reduction* of G on the 1-reducible set S. Where context is clear, we will sometimes use S to denote the set of all vertices referenced in S as well.

From these set reductions, we are now able to define the following parameters. These parameters can be thought of as measuring how far G is from being colourable.

Definition 4.1.5. Let G be a snark. We define the following notations.

(i) $v_r(G) = \min\{|S| : G_S \text{ is a set vertex reduction of } G \text{ and } G_S \text{ is colourable}\}$.

(ii) $1_r(G) = \min\{|S| : G_S \text{ is a set 1-reduction of } G \text{ and } G_S \text{ is colourable}\}$.

(iii) $e_r(G) = \min\{|S| : G_S \text{ is a set edge reduction of } G \text{ and } G_S \text{ is colourable}\}$.

Following on directly from the definition, it is clear that $e_r(G) \geq 1_r(G) \geq v_r(G)$. The following theorem relates $v_r(G)$ to $r(G)$.

Theorem 4.1.6. *Let G be a snark. Then $v_r(G) = \lceil \frac{r(G)}{2} \rceil$.*

Proof. Let S be the vertex reducible set of G such that $|S| = v_r(G)$. The result follows directly from Theorem 4.1.4. $\square$

For any minimal conflicting subgraph of a snark, it is not difficult to prove that there exists an edge reduction which results in that particular minimal conflicting subgraph no longer existing. As we mentioned earlier though, reducing resistance

via a 1-reduction is quite complicated. This is because it is possible that an edge reduction may simultaneously create a new minimal conflicting subgraph. However, it seems intuitive that there must be a set of edge reductions or 1-reductions, less than resistance in total, which reduces the snark to a colourable graph. Our investigations support this hypothesis. Thus, we conjecture the following.

Conjecture 4.1.7. *Let G be a snark. Then $r(G) \geq 1_r(G)$.*

Conjecture 4.1.8. *Let G be a snark. Then $r(G) \geq e_r(G)$.*

Since $e_r(G) \geq 1_r(G)$, Conjecture 4.1.7 is true if Conjecture 4.1.8 is true. This conjecture has significant consequences if true, as we highlight in each of the following two sections.

4.2 Insights on weak oddness

The first significant consequence of Conjecture 4.1.7 or Conjecture 4.1.8 being true relates to weak oddness and resistance. First, we present a necessary result relating weak oddness to 1-reductions, which is at the crux of the matter. As it turns out, the weak oddness of a bridgeless cubic graph G is greater than the weak oddness of a 1-reduction of G by at most two. The content of this section can also be viewed in our publication in Discrete Mathematics, see [3].

Proposition 4.2.1. *Let G be a snark. If a G' is a 1-reduction of G, then $\omega'(G) \leq \omega'(G') + 2$.*

Proof. Let $\mathcal{O}$ be an even factor of G' with a minimal number of odd components. If neither of the two added edges in G' are contained in a cycle in $\mathcal{O}$, then $\mathcal{O}$ along with the two removed vertices from G as isolated vertices, form an even factor of G with two more odd components than $\mathcal{O}$ in G'.

If just one of the added edges is contained in a cycle C in $\mathcal{O}$, say edge $[u, v]$, then we may adjust $\mathcal{O}$ to be an even factor of G. We do this by adjusting C to contain the path between the vertices u and v, instead of $[u, v]$. If the distance from u to v in G is equal to 3, then the adjusted $\mathcal{O}$ in G has the same number of odd components as $\mathcal{O}$ in G'. If the distance from u to v in G is equal to 2, then we need to add an isolated vertex to the adjusted $\mathcal{O}$ in G (the isolated vertex would be the vertex

removed which is not contained in the shortest path from u to v in G). In this case, the adjusted $\mathcal{O}$ in G has at most two more odd components than $\mathcal{O}$ in G'.

Let both the added edges, say edges $[u_0, v_0]$ and $[u_1, v_1]$, be contained in cycles in $\mathcal{O}$. Say $[u_0, v_0] \in C_0$ and $[u_1, v_1] \in C_1$, where C_0 and C_1 are not necessarily distinct. We may adjust $\mathcal{O}$ to be an even factor of G by removing $[u_0, v_0]$ and $[u_1, v_1]$ from $C_0 \cup C_1$, and adding instead two disjoint paths of length two edges from G, such that the adjusted $C_0 \cup C_1$ now contains either exactly one cycle or exactly two cycles. Thus, the adjusted $\mathcal{O}$ is an even factor of G with at most two more odd components than $\mathcal{O}$ in G'.

Therefore, $\omega'(G) \leq \omega'(G') + 2$. $\qquad\square$

Theorem 4.2.3 below highlights the consequence of either Conjecture 4.1.7 or Conjecture 4.1.8 being true. It allows us to prove a Conjecture (see [12]) relating weak oddness to resistance which states that the weak oddness of a cubic graph is bounded by two times its resistance. Theorem 4.2.3 only refers to Conjecture 4.1.7, as we recall that Conjecture 4.1.8 implies Conjecture 4.1.7.

Conjecture 4.2.2. [12] *Let G be a bridgeless cubic graph. Then $\omega'(G) \leq 2r(G)$.*

Theorem 4.2.3. *Let G be a bridgeless cubic graph. If Conjecture 4.1.7 is true, then $\omega'(G) \leq 2r(G)$.*

Proof. Assume that Conjecture 4.1.7 is true. If G is not a snark, then the result is trivial. Let G be a snark. Then we can perform a sequence of $1_r(G)$ 1-reductions on G to get a series of graphs $G_1, G_2, \ldots, G_{1_r(G)}$ such that $G_{1_r(G)}$ is colourable. Then $\omega'(G_{1_r(G)}) = 0$. By Proposition 4.2.1, we have that $\omega'(G) \leq \omega'(G_1) + 2 \leq \omega'(G_2) + 4 \leq \cdots \leq \omega'(G_{1_r(G)}) + 2(1_r(G)) \leq 2r(G)$. Therefore, $\omega'(G) \leq 2r(G)$. $\qquad\square$

4.3 Insights on flow resistance

Recall that $r_f(G)$ denotes the flow resistance of a graph G. As has been alluded to previously, 4-flows are closely related to edge-colourings in cubic graphs. This relationship is given further credit by Conjecture 4.1.7, in that even though it has seemingly nothing to do with 4-flows, if it is true then we are able to prove Conjecture 4.3.1 below. Conjecture 4.3.1 was stated in [12].
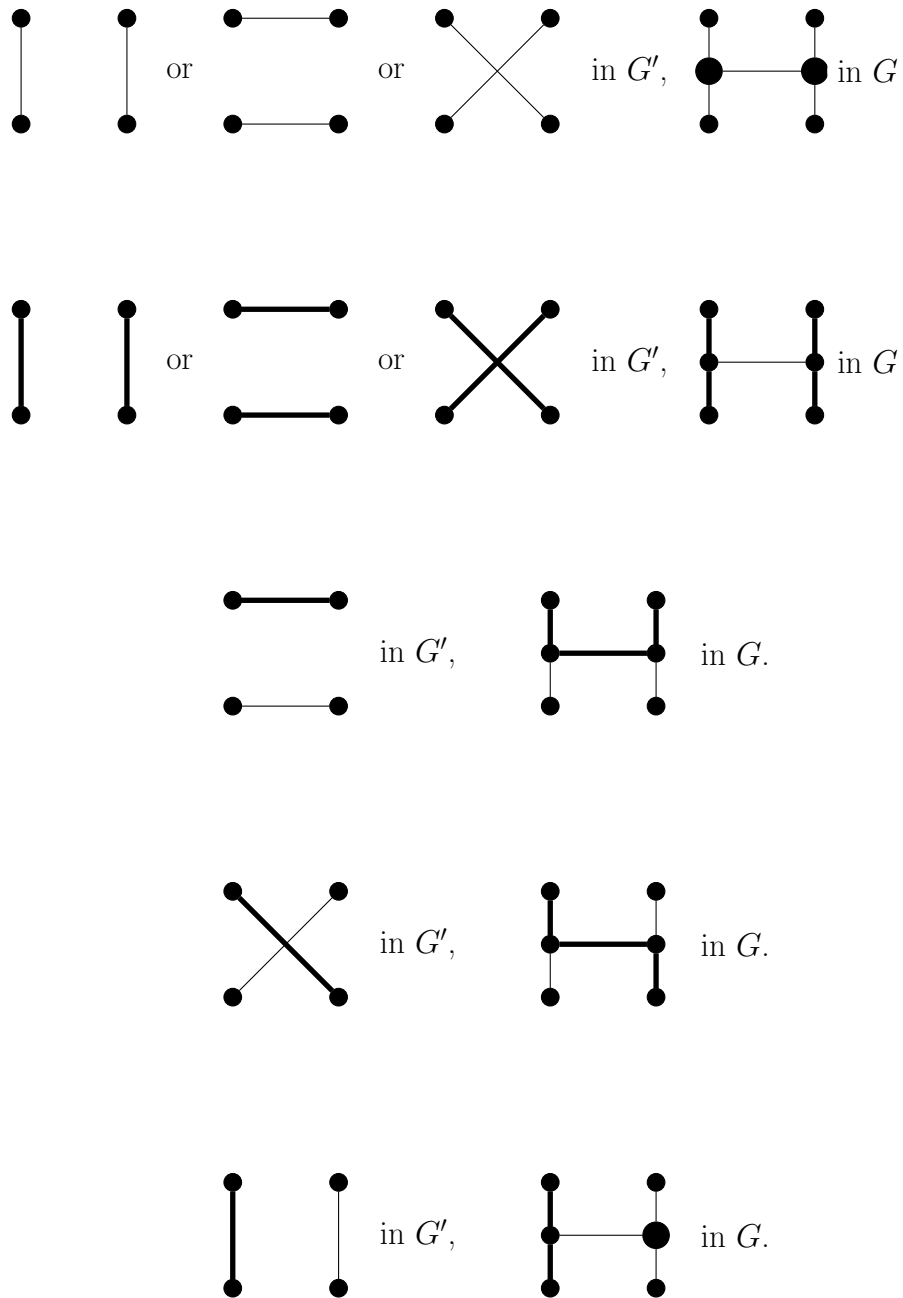
Figure 4.2: A diagramatical representation of the proof of Proposition 4.2.1. The thicker edges and vertices represent edges and vertices contained in an even factor as part of cycles or as isolated vertices, respectively.

Conjecture 4.3.1. *Let G be a snark. Then $r(G) \geq r_f(G)$.*

We are also able to prove an upper bound of resistance in terms of flow resistance as follows, further emphasising the close relationship between 4-flows and edge colourings in cubic graphs.

Proposition 4.3.2. [Steffen, 2018] *Let G be a snark. Then $2r_f(G) \geq r(G)$.*

Proof. Let G be a snark and let (D, ϕ) be a 4-flow of G with $r_f(G)$ zero edges. Let $e = [u, v]$ be a zero edge in G. Remove e from G and suppress vertices u and v. Let the resultant graph be G'. We prove the result by induction on $r_f(G)$.

Let $r_f(G) = 1$. Vertices u and v are both incident to two edges with the same flow number in $G - e$, with the tail of one edge meeting the head of the other edge. It is then easy to see that G' admits a nowhere-zero 4-flow and is thus 3-edge-colourable by Proposition 2.3.4. Furthermore, it is easy to see that G has a 3-edge-colouring with 2 conflicting vertices.

Let $r_f(G) > 1$. Then $r_f(G') + 1 \leq r_f(G)$. By induction hypothesis, $2r_f(G') \geq r(G')$. Since $r_v(G') = r(G')$, G' admits a 3-edge colouring with at most $2r_f(G')$ conflicting vertices. Then G admits a 3-edge colouring with at most $2r_f(G') + 2 \leq 2(r_f(G))$ conflicting vertices. Therefore, $2r_f(G) \geq r_v(G) = r(G)$. $\square$

Now, we show that for a snark, the number of 1-reductions required to render a 3-edge-colourable graph is greater than the flow resistance of a snark. This will equip us to prove another significant consequence of Conjecture 4.1.7 being true as mentioned.

**Proposition 4.3.3.** *Let G be a snark. Then $1_r(G) \geq r_f(G)$.*

Proof. Let G be 1-reduced $1_r(G)$ times in order to render a colourable cubic graph, G', which now also admits a nowhere zero 4-flow.

Let (D, ϕ) be a nowhere zero 4-flow of G. In Figure 4.3, we cover every case of reversing a 1-reduction in G', while retaining the 4-flow (D, ϕ) by relaxing it to be a modular 4-flow. As we can see, when we reverse any 1-reduction, we never need to add more than one zero edge to the modular 4-flow. We can then apply Theorem 2.3.3 to get a 4-flow with the same amount of zero edges. Therefore, $1_r(G) \geq r_f(G)$. $\square$

In Theorem 4.3.4, we bring everything together to easily show that $2r_f(G) \geq r(G) \geq r_f(G)$ if Conjecture 4.1.7 is true.

Theorem 4.3.4. *Let G be a snark. If Conjecture 4.1.7 is true, then $2r_f(G) \geq r(G) \geq r_f(G)$.*

Proof. $2r_f(G) \geq r(G)$ by Proposition 4.3.2. By Conjecture 4.1.7 and Proposition 4.3.3, we have that $r(G) \geq 1_r(G) \geq r_f(G)$. $\square$
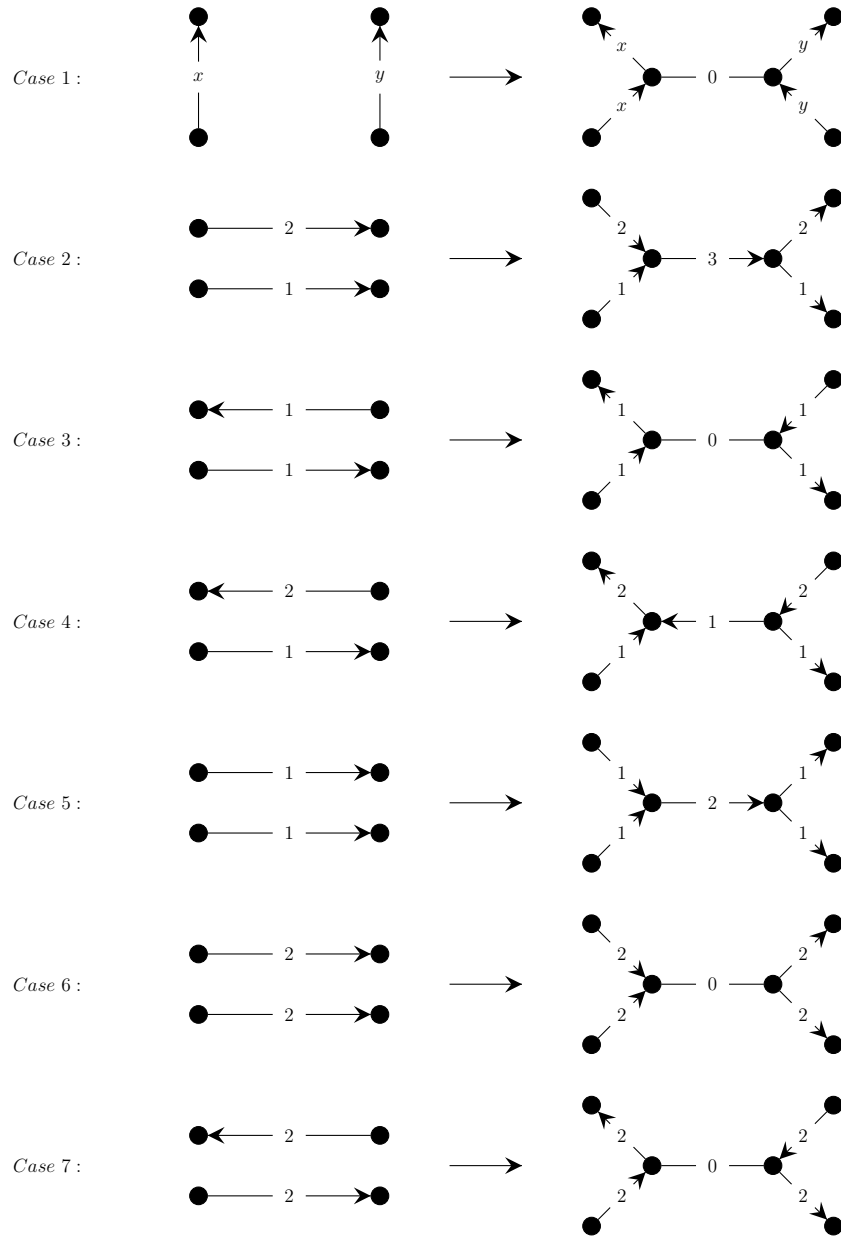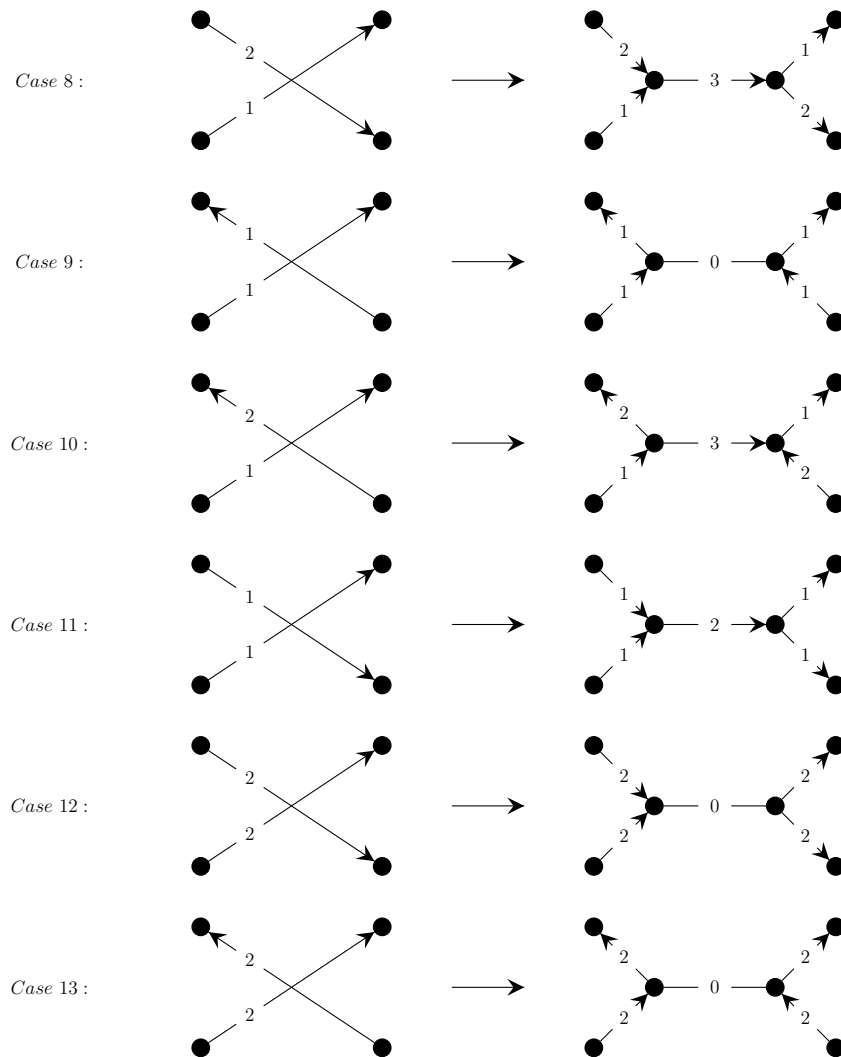
Figure 4.3: These are essentially all 13 cases which need to be considered with regard to Proposition 4.3.3. For each case, the diagram on the left reflects graph G' and the diagram on the right reflects graph G. As we can see, in each case there need only be a maximum of one more zero edge in the 4-flow in the original graph G than in G'.

Chapter 5

Oddness to resistance ratios

This chapter is devoted to disproving a conjecture which states that $\omega(G) \leq r(G)$ for a snark G. We do this by presenting a class of snarks, in which each next instance of the class has greater oddness, but each instance of the class has weak oddness 4 and resistance 3. In presenting this class, we also answer problems posed in [12] and [29]. These problems speak to the possible existence of cyclically k-edge-connected snarks with differing weak oddness and oddness, for $k > 2$. Thus far, all known snarks for which oddness and weak oddness differ are cyclically 2-edge-connected. We also highlight the minimal conflicting subgraphs in our class of snarks, and note that in each instance of the class we have three disjoint minimal conflicting subgraphs and increasing buffer subgraphs. Consequently, the problem is posed about whether the conjectured relationship between oddness and resistance exists in snarks which do not contain a buffer subgraph. The content of this chapter can also be viewed in our publication in Discrete Mathematics, see [3].

5.1   Pertinent semi-graphs and their properties

The conjecture we disprove in this chapter is formally stated as follows.

Conjecture 5.1.1. [12] *Let G be a bridgeless cubic graph. Then $\omega(G) \leq 2r(G)$.*

We present a class of snarks, each instance of which serves as a counter-example

to Conjecture 5.1.1, except for the initial instances which are the smallest in size. Furthermore, from the existence of this class of snarks, we are able to prove that the ratio of oddness to resistance can in fact be arbitrarily large. That is to say, not only is it not true that $\omega(G) \leq 2r(G)$ for any bridgeless cubic graph, there in fact exists no constant k for which it is true for all bridgeless cubic graphs that $\omega(G) \leq kr(G)$.

Each instance of the counter-example class of snarks we will present is created by joining particular semi-graphs which contain semi-edges. These semi-graphs are subgraphs of the Petersen graph, and we present them below. The semi-graph in Figure 5.1 is the Petersen graph with 2 adjacent vertices removed and has four semi-edges, and the semi-graph in Figure 5.2 is the Petersen graph with one vertex removed and has 3 semi-edges.
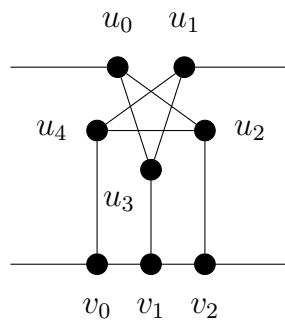
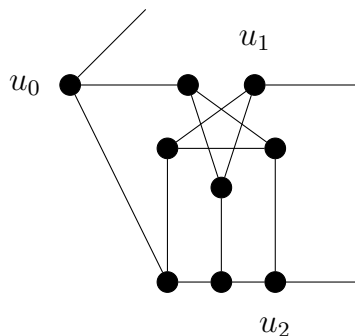Figure 5.1: The Petersen graph with two adjacent vertices removed.

Figure 5.2: The Petersen graph with one vertex removed.

The pertinent properties of the semi-graphs in Figures 5.1 and 5.2 are listed and proved in Lemmas 5.1.2 and 5.1.4. It is these properties which allow us to build the aforementioned class of snarks such that it serves our purposes of disproving Conjecture 5.1.1, and proving that the ratio of oddness to weak oddness, and thus resistance as well, can be arbitrarily large.

Lemma 5.1.2. *Let G be the semi-graph in Figure 5.1. The following statements are true.*

(i) There exists no Hamiltonian path from $u_0(u_1)$ to $v_2(v_0)$ nor from $u_0(v_2)$ to $u_1(v_0)$.

(ii) There exists a Hamiltonian path from $u_0(u_1)$ to $v_0(v_2)$.

(iii) G is Hamiltonian.

(iv) The girth of G is 5.

(v) In a proper 3-edge-colouring f of G, $f([u_0]) = f([v_0])$ and $f([u_1]) = f([v_2])$.

Proof.

(i) It is easy to check every possible path from, and to, the said vertices to see that none of them are Hamiltonian.

(ii) The paths $v_0, v_1, v_2, u_2, u_4, u_1, u_3, u_0$ and $v_2, v_1, v_0, u_4, u_2, u_0, u_3, u_1$ are Hamiltonian paths.

(iii) The cycle $v_2, v_1, v_0, u_4, u_1, u_3, u_0, u_2$ is Hamiltonian.

(iv) It is easy to check that G contains no cycle of order less than 5, and that G contains a cycle of order 5.

(v) Suppose that $f([u_0]) \neq f([v_0])$. By the Parity Lemma we have that $f([u_1]) \neq f([v_2])$ and that two colours are used to colour the four semi-edges. We add vertex w and join $[u_0]$ and $[v_0]$ to w to form edges $[w, u_0]$ and $[w, v_0]$. Similarly, we add vertex z and edges $[z, u_1]$ and $[z, v_2]$. Now, we add edge $[w, z]$ and colour it with the third colour. The resultant graph is then 3-edge-colourable and is identical to the Petersen graph, a contradiction. Therefore, $f([u_0]) = f([v_0])$ which implies that $f([u_1]) = f([v_2])$. □

Remark 5.1.3. Let G be a graph which contains the semi-graph G' as in Figure 5.1. Let C be a cycle in a 2-factor of G which contains from G' the semi-edges $[v_0]$ and $[u_1]$. It is easy to establish through exhaustive search that it is impossible for C, or any other cycle in the 2-factor, say C', to contain both the other two semi-edges $[u_0]$ and $[v_2]$ from G'. This is because it is impossible for C alone, or C and C', to contain all vertices in G' or leave behind vertices which form a cycle. Given this, as well as Lemma 5.1.2 (i) and 5.1.2 (iv), the only possibility for C is that it contains exactly one other vertex from G', so that the remaining vertices of G' may form a 5-cycle as part of the 2-factor of G. Thus, the remaining vertices of G' add one odd component to the 2-factor of G. Crucially, and by the same argument, the same applies if C contained instead semi-edges $[v_0]$ and $[v_2]$, $[u_0]$ and $[v_2]$, or $[u_0]$ and $[u_1]$. Therefore, if a cycle from a 2-factor traverses through G', that is, enter on the left (right) side and exit on the right (left) side, then the 2-factor must contain an odd component entirely contained in G' as well.

Lemma 5.1.4. *Let G be the semi-graph in Figure 5.2. The following statements are true.*

(i) *There exists no Hamiltonian path from u_0 to u_1, from u_0 to u_2, or from u_1 to u_2.*

(ii) *G is Hamiltonian.*

(iii) *The girth of G is 5.*

(iv) *G is not 3-edge-colourable.*

Proof.

(i) It is easy to check every possible path from, and to, the said vertices to see that none of them are Hamiltonian.

(ii) The Petersen graph is commonly known to be hypo-Hamiltonian. Therefore, G must contain a Hamiltonian cycle since it is the Petersen graph with one vertex removed.

(iii) It is easy to check that G contains no cycle of order less than 5, and that G contains a cycle of order 5.

(iv) We know that no cubic graph can have vertex resistance exactly 1. We know that the Petersen graph has vertex resistance greater than 0. Therefore, G is not 3-edge-colourable. $\qquad\square$

Remark 5.1.5. Similar to Remark 5.1.3, let G be a graph which contains the semi-graph G' as in Figure 5.2. Let C be a cycle in a 2-factor of G which contains from G' two of the three semi-edges $[u_o]$, $[u_1]$ or $[u_2]$. Given Lemma 5.1.4 (i) and 5.1.4 (iii), C must contain exactly two other vertices of G', so that the remaining vertices of G' form a 5-cycle as part of the 2-factor. Thus, the remaining vertices add one odd component to the 2-factor. Essentially, if a cycle from a 2-factor of G traverses through G', then the 2-factor must contain an odd component entirely contained in G' as well.

Henceforth, we will refer to the semi-graph in Figure 5.1 as X and the semi-graph in Figure 5.2 as Y. If a graph G contains X (or Y), we will say that it contains an *instance* of X (or Y). A graph may contain a number of instances of X or Y. Note that for convenience, we will not consider Y to contain an instance of X.

5.2 Snark with weak oddness 4 and oddness 6

The question of whether there exists a snark with weak oddness 4 and oddness 6 was posed in [29]. As the first instance of our class of graphs, we present such a graph. The graph we present also answers another question posed in [29], about the existence of a cyclically k-edge-connected graph with differing oddness and weak oddness for $k > 2$. The graph we present is cyclically 3-edge-connected.
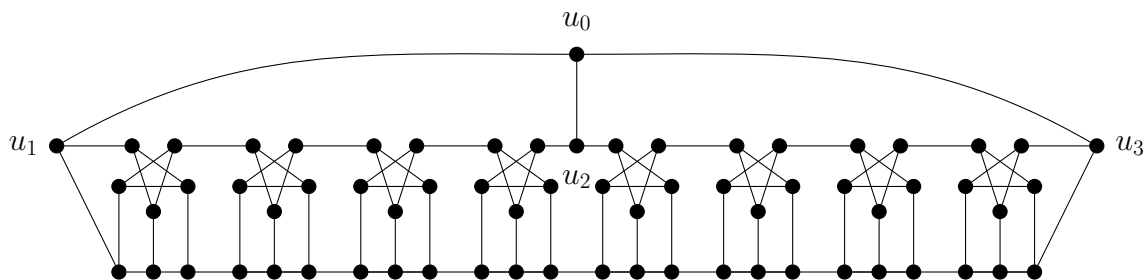

Figure 5.3: 3-edge-connected snark with oddness 6 and weak oddness 4.

Theorem 5.2.1. *There exists a snark G such that $r(G) = 3$, $\omega(G) = 6$ and $\omega'(G) = 4$.*

Proof. Let G be the graph in Figure 5.3. G contains two instances of Y and each of these instances on their own are uncolourable. G also contains six instances of X. Two of these instances are adjacent to vertex u_2. By the Parity Lemma and Lemma 5.1.2 (v), the combined two instances (with semi-edges joined) including vertex u_2, is another uncolourable subgraph of G. Therefore, $r(G) \geq 3$. It can easily be checked that the removal of u_1, u_2 and u_3 renders a 3-edge-colourable graph. Thus, $r(G) = 3$ which implies that $\omega'(G) \geq 4$. Taking the Hamiltonian cycle of the combined two instances of X adjacent to u_2 along with u_2 (Lemma 5.1.2 (ii)); the Hamiltonian cycle of each other instance of X and Y; and the singular vertex u_0, we have an even factor of G with 4 odd components. Therefore, $\omega'(G) = 4$.

Let $\mathcal{O}$ be a 2-factor of G. There exists a cycle $C \in \mathcal{O}$ which contains vertex u_0. C traverses through at least one instance of Y and three instances of X. This is the case if C contains $[u_0, u_1]$ and $[u_0, u_2]$, or $[u_0, u_3]$ and $[u_0, u_2]$. Without loss of generality, we assume that C contains $[u_0, u_1]$ and $[u_0, u_2]$. By Remarks 5.1.3 and 5.1.5, for each traversal of C through an instance of X or Y, $\mathcal{O}$ contains an additional odd component. Thus, $\mathcal{O}$ has at least 4 odd components. We note that it is impossible for C, as a component of $\mathcal{O}$, to contain any vertices from the other instance of Y as well. Thus, by Lemma 5.1.4 and Remark 5.1.5, the presence of the other instance of Y adds at least one more odd component to $\mathcal{O}$. Therefore, $\omega(G) \geq 6$. If we let C traverse through exactly three instances of X and one of Y, then taking the Hamiltonian cycle of each other instance of X and Y yields a 2-factor with exactly 6 odd components. Therefore, $\omega(G) = 6$. $\qquad \square$

5.3 Class of cyclically 3-edge-connected snarks

Following on from this, it is simple to construct a snark with an arbitrarily large oddness to weak oddness ratio. Consider the graph $G(a, b)$ in Figure 5.4. This graph is the same as the graph in Figure 5.3 except we have arbitrarily increased the number of instances of X (a instances on the left side of the diagram and b instances on the right side). It is easy to note that this graph is cyclically 3-edge-connected.

Theorem 5.3.1. *For each integer k such that $k \geq 1$, there exists a cyclically 3-edge-connected snark G such that $\omega(G) \geq k\omega'(G)$.*

Proof. Consider the graph $G(a, b)$ as in Figure 5.4. Just as in Theorem 5.2.1 for the graph in Figure 5.3, we can see that $r(G(a, b)) = 3$ and $\omega'(G(a, b)) = 4$.
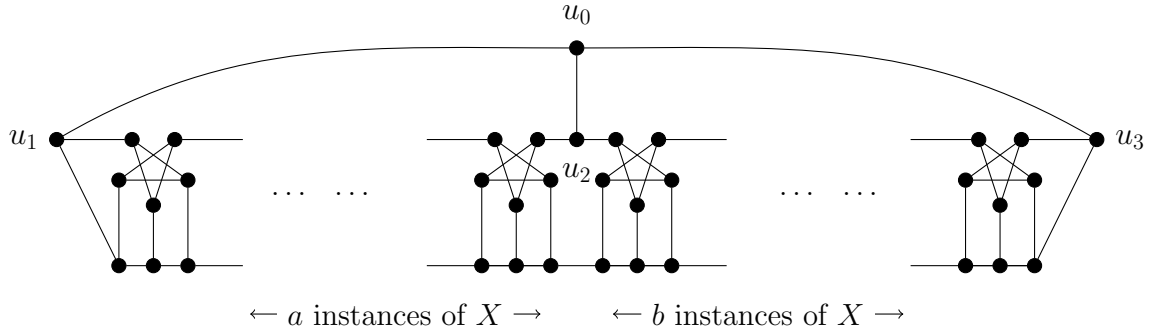
Figure 5.4: Graph $G(a, b)$

Let k be an integer such that $k \geq 1$ and let $\mathcal{O}$ be a 2-factor of $G(a, b)$. There exists a cycle $C \in \mathcal{O}$ which contains vertex u_0. C traverses through at least $min\{a, b\}$ instances of X. By Remark 5.1.3, $\mathcal{O}$ therefore has at least $min\{a, b\}$ odd components. Therefore, $\omega(G(a, b)) \geq min\{a, b\}$. We may choose a and b such that $min\{a, b\} \geq 4k = \omega'(G(a, b))k$ which then implies that $\omega(G(a, b)) \geq \omega'(G(a, b))k$. $\qquad \square$

Consequently, since $\omega'(G) \geq r(G)$ for any bridgeless cubic graph G, we are able to disprove Conjecture 5.1.1 as follows.

Corollary 5.3.2. *For each integer k such that $k \geq 1$, there exists a cyclically 3-edge-connected bridgeless cubic graph G such that $\omega(G) \geq kr(G)$.*

Proof. This follows on from Theorem 5.3.1 and the fact that $\omega'(G) \geq r(G)$ for any bridgeless cubic graph G. $\qquad \square$

5.4   Class of cyclically 4-edge-connected snarks

Furthermore, our graphs are easily adjustable to ensure that they are cyclically 4-edge-connected. Thus we present a cyclically 4-edge-connected cubic graph with resistance 3, weak oddness 4 and oddness 6 in Figure 5.5.

Also as before, we may simply increase the instances of X in order to increase the oddness whilst keeping resistance equal to 3 and weak oddness equal to 4.
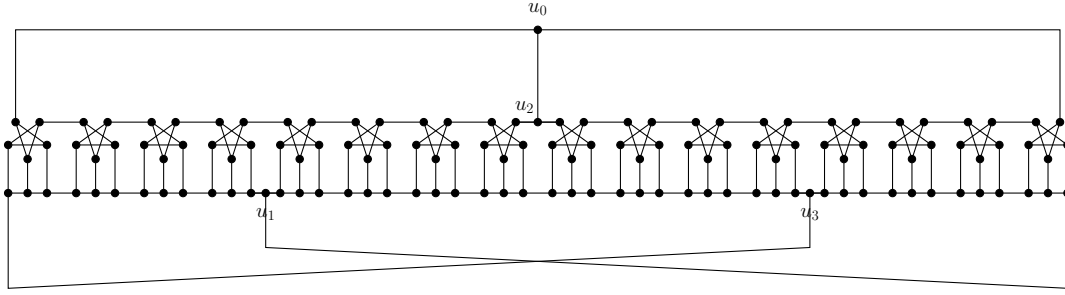
Figure 5.5: 4-edge-connected snark with oddness 6 and weak oddness 4.

Theorem 5.4.1. *For each integer k such that $k \geq 1$, there exists a cyclically 4-edge-connected snark G such that $\omega(G) \geq k\omega'(G)$.*

Proof. Let G' be the graph in Figure 5.5. Similar to the proof of Theorem 5.2.1, for each $i \in \{1, 2, 3\}$, u_i combined with its two adjacent instances of X is an uncolourable subgraph of G'. These three subgraphs are disjoint, therefore $r(G') \geq 3$. Also, the removal of each u_i renders a colourable graph. Thus, $r(G') = 3$. Taking the Hamiltonian cycles of these three subgraphs, the Hamiltonian cycles of each other instance of X, and the isolated vertex u_0, we have an even factor with 4 odd components. Thus, $\omega'(G') = 4$.

Let k be an integer such that $k \geq 1$. It is easy to see that if we add more instances of X to G' as we did in Figure 5.4, then the resistance and weak oddness does not change. Let G be the graph obtained from G' by adding more instances of X in this way, such that a cycle C containing vertex u_0 in a 2-factor $\mathcal{O}$ of G, must traverse through a minimum of $4k$ instances of X. As before, we then have at least $4k$ odd components in $\mathcal{O}$. Therefore, $\omega(G) \geq 4k = \omega'(G)k$. $\qquad \square$

Corollary 5.4.2. *For each integer k such that $k \geq 1$, there exists a cyclically 4-edge-connected snark G such that $\omega(G) \geq kr(G)$.*

Proof. This follows on from Theorem 5.4.1 and the fact that $\omega'(G) \geq r(G)$ for any snark G. $\qquad \square$

5.5   Considering the minimal conflicting subgraphs

Although we have disproved Conjecture 5.1.1, it may still hold under certain conditions. Given any 3-critical graph, which we recall is necessarily strictly subcubic and has resistance equal to 1, it is always possible to join vertices of degree 2 and if necessary add a single vertex to obtain a cubic graph with resistance 2. This graph then has oddness 2. Therefore, we may be inclined to think that a minimal conflicting subgraph in a cubic graph contributes to the oddness of the graph, but only with one or two odd cycles in a 2-factor.

The graphs supporting the proof of Corollary 5.4.2 each have exactly 3 disjoint minimal conflicting subgraphs. See Figure 5.6. For each instance of the class, we are adding an instance of graph X only to the buffer subgraph. This then necessarily increases the ratio of oddness to resistance, whilst keeping resistance constant. This poses the question of whether it is possible to increase oddness and keep weak oddness constant other than by adding to the buffer subgraph only. This further poses the question of whether Conjecture 5.1.1 holds for graphs with no buffer subgraph.
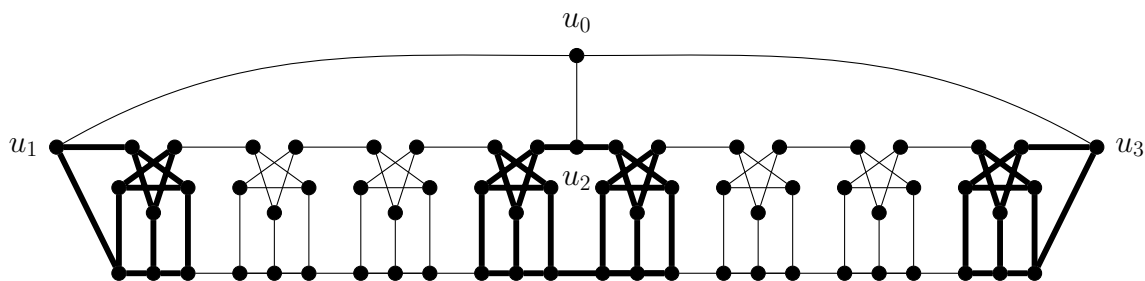

Figure 5.6: Graph with oddness 6 and weak oddness 4. Each component of connected thicker edges represents a minimal conflicting subgraph. There are three disjoint minimal conflicting subgraphs.

Any addition to the maximal conflicting subgraph of a graph, by adding new minimal conflicting subgraphs, would intuitively increase both the resistance and oddness of the graph. This is in line with results presented in chapter 3. Thus we support the following conjecture, which is a refined version of Conjecture 5.1.1.

Conjecture 5.5.1. *Let G be a snark which contains no buffer subgraph. Then $\omega(G) \leq 2r(G)$.*

Chapter 6

Conclusions

Chapter 4 of this thesis relates the much investigated idea of snark reductions with newly presented parameters measuring edge uncolourability, with potential significant consequences in the form of Conjecture 4.1.7. Recall that Conjecture 4.1.7 states that $r(G) \geq 1_r(G)$ for any snark G. It is our contention that ideas produced in Chapter 3 may potentially lead to a proof of this Conjecture 4.1.7. Roughly, we think that an edge reduction which reduces resistance should always exist in some cluster of minimal conflicting subgraphs. Intuitively, it would be highly surprising if this was not the case. Indeed, Conjecture 3.4.5 and Conjecture 3.4.4 also represents a fresh new insight into snarks of resistance 2. Recall that Conjecture 3.4.4 states that if every edge in a snark is conflicting in some minimal colouring then the snark has resistance 2, Conjecture 3.4.5 states that if every edge in a snark is contained in a minimal conflicting subgraph of the snark then the snark has resistance 2. As briefly mentioned in the thesis, we feel that it may be worthwhile to research the snarks of resistance 2 in isolation.

If Conjecture 4.1.7 proves to be true, then the contrast of outcomes regarding the conjectures by previous authors about oddness and weak oddness proves quite interesting and counter-intuitive. To recall, these conjectures stated that oddness and weak oddness are bounded by two times resistance. That it is interesting and counter-intuitive is especially true since the existence of cubic graphs with differing oddness and weak oddness was in question for a long time. We feel that this also adds further interest to Conjecture 5.5.1, and as such also adds interest to ideas relating to minimal conflicting subgraphs and buffer subgraphs. Recall that Conjecture 5.5.1

states that the oddness of a snark is bounded by two times its resistance if the snark contains no buffer subgraph. One also wonders whether there exists other cubic graphs with arbitrarily large ratio differences between oddness and weak oddness, other than those presented in Chapter 5, or whether such large ratio differences between the two said parameters are dependent on the inclusion of instances of graph X. We raise this especially since graph X is a subgraph of the Petersen Graph, which is always at the crux of all discussions involving snarks.

The ideas and results presented in this thesis represent a combination of different types of developments in this line of research regarding class two cubic graphs. Furthermore, the potential for further development by deeper research into known, and not yet defined so-called measurements of uncolourability, is emphasised. This emphasis is mostly by way of motivation of new conjectures, which as we have shown represent interesting new insights and significant consequences if proved to be true.

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