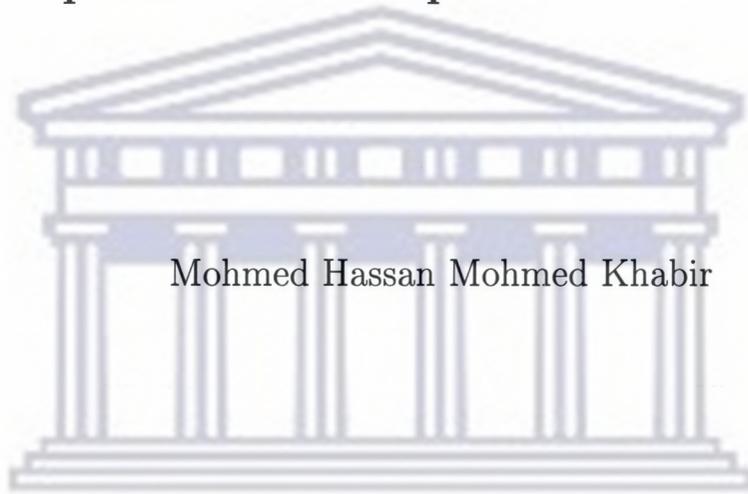


**Numerical singular perturbation approaches based  
on spline approximation methods for solving  
problems in computational finance**



Mohmed Hassan Mohmed Khabir

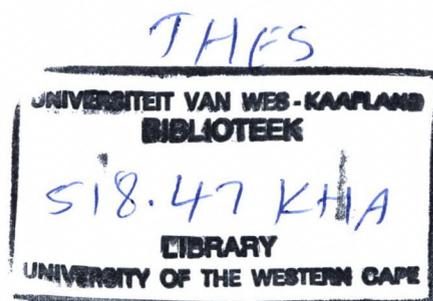
A Thesis submitted in partial fulfillment of the requirements for the degree of Doctor  
of Philosophy in the Department of Mathematics and Applied Mathematics at the  
Faculty of Natural Sciences, University of the Western Cape

Supervisor: Prof. Kailash C. Patidar

May 2011



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# KEYWORDS

Computational Finance

Options Pricing

Black-Scholes Equation

Standard Options

Nonstandard Options

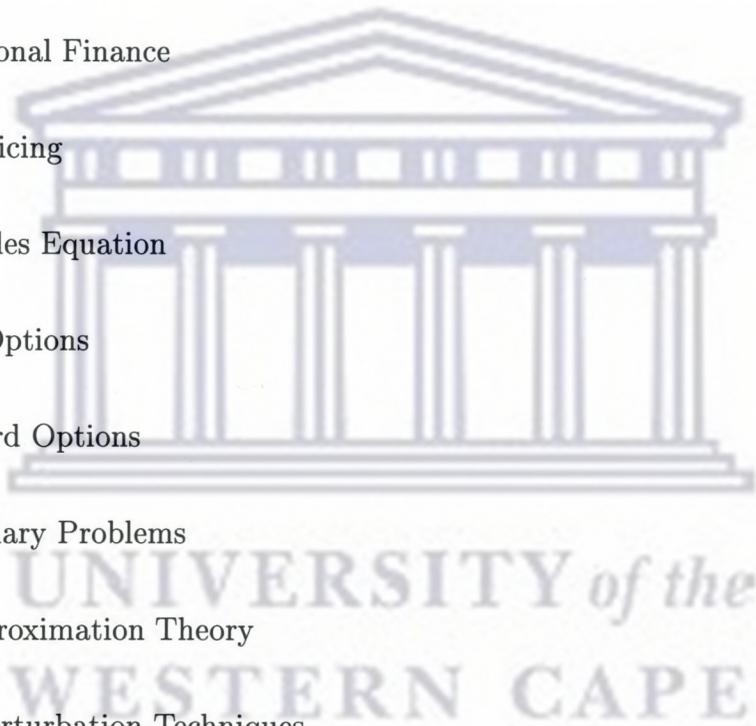
Free Boundary Problems

Spline Approximation Theory

Singular Perturbation Techniques

Numerical Methods

Convergence Analysis



# ABSTRACT

**Numerical singular perturbation approaches based on spline approximation  
methods for solving problems in computational finance**

by

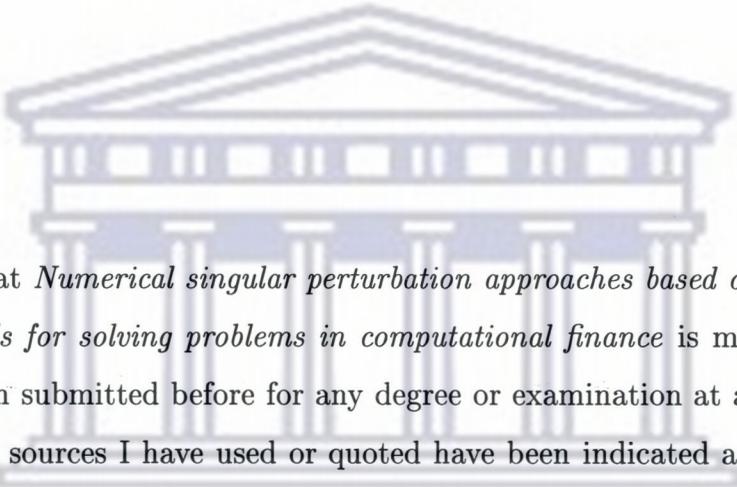
**Mohmed Hassan Mohmed Khabir**

**PhD thesis, Department of Mathematics and Applied Mathematics, Faculty of  
Natural Sciences, University of the Western Cape.**

Options are a special type of derivative securities because their values are derived from the value of some underlying security. Most options can be grouped into either of the two categories: European options which can be exercised only on the expiration date, and American options which can be exercised on or before the expiration date. American options are much harder to deal with than European ones. The reason being the optimal exercise policy of these options which led to free boundary problems. Ever since the seminal work of Black and Scholes [*J. Pol. Econ.* **81(3)** (1973), 637-659], the differential equation approach in pricing options has attracted many researchers. Recently, numerical singular perturbation techniques have been used extensively for solving many differential equation models of sciences and engineering. In this thesis, we explore some of those methods which are based on spline approximations to solve the option pricing problems. We show a systematic construction and analysis of these methods to solve some European option problems and then extend the approach to solve problems of pricing American options as well as some exotic options. Proposed methods are analyzed for stability and convergence. Thorough numerical results are presented and compared with those seen in the literature.

May 2011.

# DECLARATION

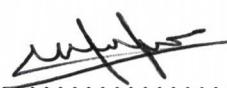


I declare that *Numerical singular perturbation approaches based on spline approximation methods for solving problems in computational finance* is my own work, that it has not been submitted before for any degree or examination at any other university, and that all sources I have used or quoted have been indicated and acknowledged by complete references.

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Mohmed Hassan Mohmed Khabir

May 2011

Signed .....  .....

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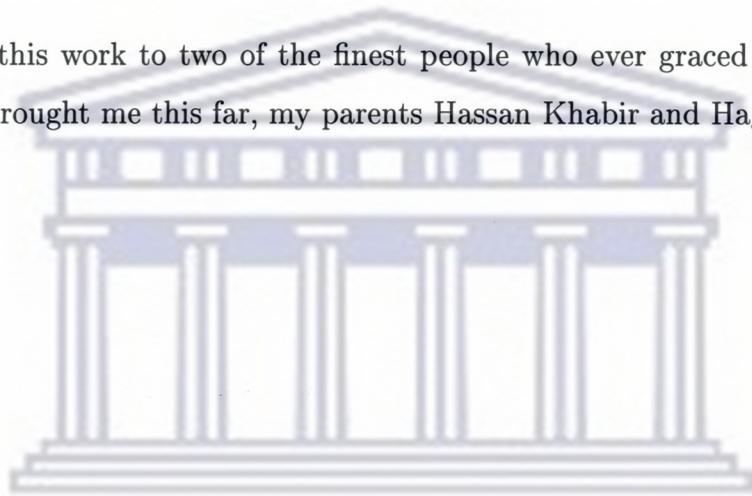
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# DEDICATION

I dedicate this work to two of the finest people who ever graced this world and who naturally brought me this far, my parents Hassan Khabir and Hager Mubarak.



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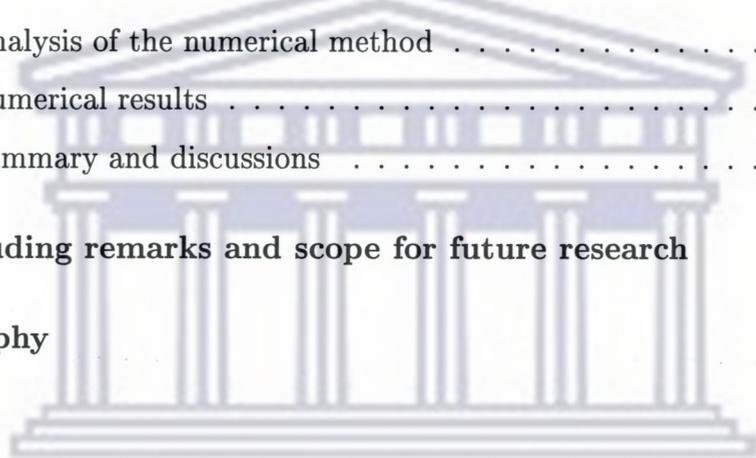
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# List of Publications

Part of this thesis has been submitted in the form of the research papers listed below. We also have some technical reports whose revised form is being submitted to prestigious international journals for publications.

1. M.H.M. Khabir and K.C. Patidar, Spline approximation method to solve an option pricing problem, submitted for publication.
2. M.H.M. Khabir and K.C. Patidar, Spline approximation approach for pricing the barrier options, submitted for publication.
3. M.H.M. Khabir and K.C. Patidar, Spline approximation method for pricing American options, submitted for publication.
4. M.H.M. Khabir and K.C. Patidar, Comparison of some numerical methods for option pricing problems, submitted for publication.
5. M.H.M. Khabir and K.C. Patidar, A B-spline approach to price an option on a dividend paying asset, Report Nr. UWC-MRR 2011/21, University of the Western Cape, 2011.
6. M.H.M. Khabir and K.C. Patidar, Some experiments with cubic splines and high order time integration schemes to solve option pricing problems, Report Nr. UWC-MRR 2011/22, University of the Western Cape, 2011.
7. M.H.M. Khabir and K.C. Patidar, A B-spline approach to solve a partial integro-differential equation model of pricing options, Report Nr. UWC-MRR 2011/23,

University of the Western Cape, 2011.

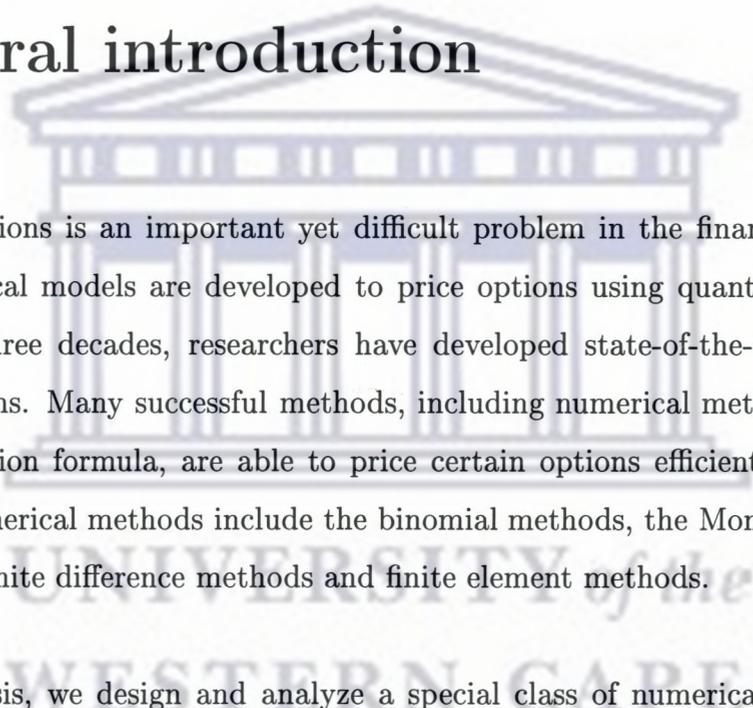
8. M.H.M. Khabir, K.C. Patidar and A.O.M. Sidahmed, Investigation of some numerical methods for option pricing problems, Report Nr. UWC-MRR 2009/11, University of the Western Cape, 2009.



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# Chapter 1

## General introduction



Pricing options is an important yet difficult problem in the finance research. Many mathematical models are developed to price options using quantitative analysis. In the past three decades, researchers have developed state-of-the-art solvers to price these options. Many successful methods, including numerical methods and analytical approximation formula, are able to price certain options efficiently. The recent advanced numerical methods include the binomial methods, the Monte Carlo simulation methods, finite difference methods and finite element methods.

In this thesis, we design and analyze a special class of numerical methods, namely, the spline approximation methods. These methods have been used very widely to solve the problems in Sciences and Engineering. In recent past, they have been used to solve singularly perturbed ordinary and partial differential equations. See for example, [79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 97] and some references therein. Due to the fact that some of these methods were very powerful when applied to the highly stiff problems like those described by singularly perturbed problems, we decided to explore them whether they can be used to solve option pricing problems arising in computational finance.

Before we proceed with designing such methods, below we introduce some basic con-

cepts in the option pricing theory.

## 1.1 Option pricing: A brief overview

The past few decades have witnessed a revolution in the trading of *derivative securities* in world financial markets. A derivative security, or *contingent claim*, is a financial contract whose value at its expiry date  $T$  is completely determined by the prices of an *underlying asset* in a fixed range of times within the interval  $[0, T]$ . An underlying asset to a derivative security can be any financial asset, such as shares, stocks or bonds, or a commodity such as an agricultural product or mineral. Derivative securities can be divided into three classes: options, forwards and futures, and swaps. In this thesis, we focus on options.

### Options

An option is the right (but not the obligation) to buy or sell a risky asset at a pre-specified fixed price within a specified period [68]. An option is a financial instrument that allows - amongst other things - to make a bet on rising or falling values of an underlying asset. The underlying asset typically is a stock, or a parcel of shares of a company. Other examples of underlying include stock indices (for example, the Dow Jones Industrial Average), currencies, or commodities. Since the value of an option depends on the value of the underlying asset, options and other related financial instruments are called *derivatives*. An option is a contract between two parties about trading the asset at a certain future time. One party is the *writer*, for example, a bank, who fixes the terms of the option contract and sells the option. The other party is the *holder*, who purchases the option, paying the market price, which is called *premium*. The holder of the option must decide what to do with the rights the option contract grants. The decision will depend on the market situation, and on the type of option.

Options have a limited life time. The *maturity date*  $T$  fixes the time horizon. At

this date the rights of the holder expire, and for later times ( $t > T$ ) the option is worthless. There are two basic types of option: call and put. The call option gives the holder the right to *buy* the underlying for an agreed price  $E$  by the date  $T$ . The put option gives the holder the right to *sell* the underlying for the price  $E$  by the date  $T$ . The previously agreed price  $E$  of the contract is called strike or *exercise price* [141]. It is important to note that the holder is not obligated to *exercise*, that is, to buy or sell the underlying according to the terms of the contract. The holder may wish to close his position by selling the option. In summary, at time  $t$  the holder of the option can choose to

- sell the option at its current market price on some options exchange (at  $t < T$ ),
- return the option and do nothing,
- exercise the option ( $t \leq T$ ), or
- let the option expire worthless ( $t \geq T$ ).

In contrast, the writer of the option has the obligation to deliver or buy the underlying for the price  $E$ , in case the holder chooses to exercise. The risk situation of the writer differs strongly from that of the holder. The writer receives the premium when he issues the option and somebody buys it. This up-front premium payment compensates for the writer's potential liabilities in the future.

Not every option can be exercised at any time  $t \leq T$ . For European options exercise is only permitted at expiration  $T$ . However, American options can be exercised at any time up to and including the expiration date. For options the labels American or European have no geographical meaning. Both of these options are traded in every continent.

In addition to these standard options which are traded on many financial exchanges, there is a huge over-the-counter market in which financial institutions sell a variety

of exotic options tailored to meet the demands of various clients. For example, Asian options feature payoffs which depend on the average price of the underlying asset during the contract. Lookback options depend on the highest or lowest price reached by the underlying asset. There are also a number of different kinds of barrier options. In general, such contracts specify various payoffs if the underlying asset price reaches certain levels. For example, an up-and-out call option is like a standard call provided that the underlying asset price remains below a barrier level for the duration of the contract. Should the barrier level be reached, the contract is canceled and the option's payoff will become zero, i.e., the option will be worthless.

In this thesis, we discuss both of these standard (European and American) options and some non-standard (barrier) options. To this end, below we describe some important concepts which will be useful throughout the thesis.

### Payoff functions

At time  $t = T$ , the holder of a European call option will check the current price  $S = S_T$  of the underlying asset. The holder will exercise the call (buy the stock for the strike price  $E$ ), when  $S > E$ . Then the holder can immediately sell the asset for the spot price  $S$  and makes a gain of  $S - E$  per share. In this situation the value of the option is  $V = S - E$  ignoring transaction costs. In case  $S < E$  the holder will not exercise the option, since then the asset can be purchased on the market for the cheaper price  $S$ . In this case the option is worthless,  $V = 0$ .

In summary, the value  $V(S, T)$  of a call option at expiration date  $T$  is given by

$$V_{\text{call}}(S_T, T) = \begin{cases} 0 & \text{in case } S_T \leq E \text{ (option expires worthless),} \\ S_T - E & \text{in case } S_T > E \text{ (option is exercised).} \end{cases} \quad (1.1.1)$$

Hence

$$V(S_T, T) = \max \{S_T - E, 0\}. \quad (1.1.2)$$

Considering for all possible prices  $S_t > 0$ ,  $\max \{S_t - E, 0\}$  is a function of  $S_t$ . This function is the payoff function. Using the notation  $f^+ := \max \{f, 0\}$ , this payoff can be written in the compact form  $(S_t - E)^+$ . Accordingly, the value  $V(S_T, T)$  of a call at day  $T$  is

$$V_{\text{call}}(S_T, T) = (S_T - E)^+. \quad (1.1.3)$$

For a European put options, exercising only makes sense in case if  $S < E$ . The payoff  $V(S, T)$  of a put at expiration time  $T$  is

$$V_{\text{put}}(S_T, T) = \begin{cases} E - S_T & \text{in case } S_T < E \text{ (option is exercised),} \\ 0 & \text{in case } S_T \geq E \text{ (option is worthless).} \end{cases} \quad (1.1.4)$$

Hence

$$V_{\text{put}}(S_T, T) = \max \{E - S_T, 0\}, \quad (1.1.5)$$

or

$$V_{\text{put}}(S_T, T) = (E - S_T)^+. \quad (1.1.6)$$

The equations (1.1.3), (1.1.6) remain valid for American type options. The payoff function for an American call is  $(S_t - E)^+$  and for an American put  $(E - S_t)^+$  for any  $t \leq T$ .

### Options in the market

The features of the options imply that an investor purchases puts when the price of the underlying is expected to fall, and buys calls when the prices are about to rise. This mechanism inspires speculators.

The value  $V(S, t)$  also depends on other factors. Dependence on the strike  $E$  and the maturity  $T$  is evident. Market parameters affecting the price are the interest rate  $r$ , the volatility  $\sigma$  of the price  $S_t$ , and dividends in case of a dividend-paying asset. The interest rate  $r$  is the risk-free rate, which applies to zero bonds or to other investments that are considered free of risks. The important volatility parameter  $\sigma$  can be defined as standard deviation of the fluctuations in  $S_t$ , and for scaling it is divided by the square root of the observed time period. The larger the fluctuations (represented by large values of  $\sigma$ ) the harder is to predict a future value of the asset. Hence the volatility is a standard measure of risk. Thus the dependence of  $V$  on  $\sigma$  is highly sensitive. We write  $V(S, t, T, E, r, \sigma)$  when the focus is on the dependence of  $V$  on the market parameters.

The units of  $r$  and  $\sigma^2$  are usually mentioned per year. Time is measured in years. Writing  $\sigma = 0.2$  means a volatility of 20%, and  $r = 0.05$  represents an interest rate of 5%. In Table 1.1.1, we list some notations which are used in the rest of the thesis. These notations are standard except for the strike  $E$ , which is sometimes denoted  $X$  or  $K$ .

The time period of interest is  $t_0 \leq t \leq T$ . One might think of  $t_0$  denoting the date when the option is issued and  $t$  as a symbol for “today.” But in this thesis we set  $t_0 = 0$  in the role of “today,” without loss of generality. Then the interval  $0 \leq t \leq T$  represents the remaining life time of the option. The price  $S_t$  is a stochastic process; in real markets, the interest rate  $r$  and the volatility  $\sigma$  vary with time. To keep the models and the analysis simple, we assume  $r$  and  $\sigma$  to be constant on  $0 \leq t \leq T$ . Further we suppose that all variables are arbitrarily divisible and consequently can vary continuously, that is, all variables vary in the set  $\mathbb{R}$  of real numbers.

Table 1.1.1: List of some notations used in the thesis

$t$	current time, $0 \leq t \leq T$
$T$	expiration time, maturity
$r > 0$	risk-free interest rate
$S, S_t$	spot price, current price per share of stock/asset/underlying
$\sigma$	annual volatility
$E$	strike, exercise price per share
$V(S, t)$	value of an option at time $t$ and underlying price $S$

### European and American options as standard options

Derivatives securities have been used quite extensively by investors and financial institutions in past two decades. However, using the most widely accepted financial models, there are many types of securities which can not be priced in closed form. Closed form solutions are available only in few special cases. One example is a European option written on a single underlying asset. The European option valuation formula was derived in Black and Scholes [9] and Merton [117]. The value of a European option satisfies the Black–Scholes equation with appropriately specified final and boundary conditions, see, for example, (Shaw [143], Wilmott *et al.* [155]). For American options, the essential difficulty lies in the problem that they are allowed to be exercised at any time before the expiration day. Such an early exercise right purchased by the holder of the option has changed the problem into a so-called free boundary value problem, since the optimal exercise price prior to the expiration of the option is time-dependent.

As a result of the unknown boundary being part of the solution of the problem, the valuation of American options becomes a nonlinear problem. In the case of American options, analytical expressions for the price have been derived but there are no easily computable explicit formulas available so far. Further complications arise when the pay-off of the derivative security depends on multiple assets or multiple sources of uncertainty. Analytical solutions are often not available for options with path-dependent payoffs and other exotic options. This has necessitated efficient numerical procedures.

A fundamental problem in financial mathematics is the analysis of the American call and put options. These are more difficult to analyze than the corresponding European options in that the American options may be exercised prior to the expiration dates. Mathematically the American options lead to Partial Differential Equations (PDEs) with moving boundaries, which can only rarely be solved exactly. The location of the moving boundary is important since it corresponds to the optimal exercise boundary for the option. The free-boundary problems associated with the American options are generally solved by converting them into a sequence of fixed-boundary problems. Each of these fixed-boundary problems are then solved by some reliable PDE solvers.

The moving boundary approach, that is, the idea of transforming a free-boundary problem to fixed-boundary problems, is exciting and powerful, particularly because solving the fixed-boundary problem is much easier. It is evident from the mechanism of the method that there is considerable scope for generalizing it to other optimal stopping problems, though a significant amount of work in deriving the boundary update equations and establishing convergence might be required.

### **Exotic options as nonstandard options**

Exotic options also called special -purpose options or customer tailored options, implying that each type of exotic options can somehow serve a special purpose which standard options cannot do conveniently or cheaply. Exotic options differ from standard options in at least one aspect. Examples include, a deferred option or forward-start option is an option whose effective starting time is some time in the future after the contract is signed rather than in the present; a compound option which is an option written on a standard option rather than on an underlying asset directly; a spread option which is an option written on the difference between two prices or indices, rather than on one single price or index as in case of standard options, and so on.

The most popular group of exotic options is path-dependent options, which includes Asian or average-price options, barrier options, lookback options, and forward-start options. Another large group of exotic options is correlation options which includes spread options, out-performance options, two-color rainbow options, quanto options, exchange options, basket options, etc. Some other popular exotic options are chooser options or as-you-like options, power options, binary or digital options [162].

Barrier options are probably the oldest of all exotic options. Options with the barrier feature are considered to be the simplest types of path dependent options. Barrier option's distinctive feature lies in the fact that the payoff depends not only on the final price of the underlying asset, but also on whether the asset price has breached (one-touch) some barrier level during the life of the option.

Barrier options can be classified into knock-out and knock-in options. Considering the barrier price  $X$ , the knock-out option can be exercised unless the asset price  $S$  reaches the barrier  $X$  during the day of purchase and expiration day. The knock-in option can be exercised if the asset price  $S$  overtakes the barrier  $X$ . The knock-out options can be classified into "up-and-out" and "down-and-out". The up-and-out option can be exercised unless the asset price  $S$  reaches the barrier  $X$  from below the barrier and the down-and-out option can be done unless the asset price reaches the barrier from above the barrier. Similarly, the knock-in options can be classified into "up-and-in" and "down-and-in" options. The up-and-in option can be exercised if the asset reaches the barrier from below the barrier and the down-and-in option can be done if the asset price reaches the barrier from above the barrier. Barrier options are used widespread, particularly for foreign currency contracts.

There are also a variety of other instruments with similar kinds of contingent payoffs, including capped options, ladder options, and interest rate corridors. The problem can be readily generalized to incorporate early exercise, although we must then find solu-

tions numerically. In principle, barrier features may be applied to any options. The valuation algorithms of the options are almost similar and therefore we discuss only the down-and-out option later in this thesis.

## A simple model for asset pricing

It is often stated that asset prices must move randomly because of the efficient market hypothesis. There are several different forms of this hypothesis with different restrictive assumptions, but they all basically imply two things:

- The past history is fully reflected in the present price, which does not hold any further information,
- Markets respond immediately to any new information about an asset.

Thus the modelling of asset prices is really about modelling the arrival of new information which affects the price. With the two assumptions above, unanticipated changes in the asset price are modeled as a Markov process.

Firstly, we note that the *absolute* change in the asset price is not by itself a useful quantity. With each change in asset price, a return defined to be the change in the price divided by the original value, i.e.,  $\frac{dS}{S}$  (see, e.g. [155]). This relative measure of the change is clearly a better indicator of its size than any absolute measure.

Now suppose that at any time  $t$  the asset price is  $S$ . Let us consider a small subsequent time interval  $dt$ , during which  $S$  changes to  $S + dS$ . How the corresponding return on the asset  $\frac{dS}{S}$  might be modeled? The most common model decomposes this return into two parts. One is a predictable, deterministic and anticipated return akin to the return on money invested in a risk-free bank. It gives a contribution  $\mu dt$  to the return  $\frac{dS}{S}$ , where  $\mu$  is a measure of the average rate of growth of the asset price, also known as the drift. In simple models  $\mu$  is taken to be a constant. In more complicated

models, for example, for exchange rates,  $\mu$  can be a function of  $S$  and  $t$ . The second contribution to  $\frac{dS}{S}$  models the random change in the asset price in response to external effects, such as unexpected news. It is represented by a random sample drawn from a normal distribution with mean zero and adds a turn  $\sigma dX$  to  $\frac{dS}{S}$ . Here the quantity  $dX$  is the sample from a normal distribution. Putting these contributions together, we have the stochastic differential equation

$$\frac{dS}{S} = \sigma dX + \mu dt, \quad (1.1.7)$$

which is the mathematical representation of the simple recipe for generating asset prices (see [155] for further details).

### Role of Itô's lemma

In real life asset prices are quoted at discrete intervals of time. There is thus a practical lower bound for the basic time-step  $dt$  of the random walk (1.1.7). If we use this time-step in practice to value options, we would find that we had to deal with unmanageably large amounts of data. Instead, we set up our mathematical models in the continuous time limit  $dt \rightarrow 0$ ; it is much more efficient to solve the resulting differential equations than it is to value options by direct simulation of the random walk on a practical timescale. In order to do this, Itô's lemma is the most important result about the manipulation of random variables. First we need the following result, with probability 1,

$$dX^2 \rightarrow dt \quad \text{as} \quad dt \rightarrow 0. \quad (1.1.8)$$

Thus the smaller  $dt$  becomes, the more certainly  $dX^2$  is equal to  $dt$ .

Suppose that  $f(S)$  is a smooth function of  $S$  and temporarily assume that  $S$  is stochastic. If we vary  $S$  by a small amount  $dS$  then clearly  $f$  also varies by a small amount provided we are not close to singularities of  $f$ . From the Taylor series expansion we

can write

$$df = \frac{df}{dS}dS + \frac{1}{2} \frac{d^2f}{dS^2}dS^2 + \dots, \quad (1.1.9)$$

where the dots denote a reminder which is smaller than any of the terms we have retained. Now recall that  $dS$  is given by (1.1.7). Squaring it we find that

$$\begin{aligned} dS^2 &= (\sigma SdX + \mu Sdt)^2 \\ &= \sigma^2 S^2 dX^2 + 2\sigma\mu S^2 dt dX + \mu^2 S^2 dt^2. \end{aligned} \quad (1.1.10)$$

We now examine the order of magnitude of each of the terms in (1.1.10). Since

$$dX = \mathcal{O}(\sqrt{dt}), \quad (1.1.11)$$

the first term is the largest for small  $dt$  and dominates the other two terms. Thus, to leading order,

$$dS^2 = \sigma^2 S^2 dX^2 + \dots$$

Since  $dX^2 \rightarrow dt$ , we get

$$dS^2 \rightarrow \sigma^2 S^2 dt.$$

We substitute this into (1.1.9) and return only those terms that are at least as large as  $\mathcal{O}(dt)$ . Using also the definition of  $dS$  from (1.1.7), we find that

$$\begin{aligned} df &= \frac{df}{dS}(\sigma SdX + \mu Sdt) + \frac{1}{2}\sigma^2 S^2 \frac{d^2f}{dS^2}dt \\ &= \sigma S \frac{df}{dS}dX + \left( \mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2f}{dS^2} \right) dt. \end{aligned} \quad (1.1.12)$$

This is Itô's lemma relating the small change in a function of a random variable to the small change in the variable itself.

The result (1.1.12) can be further generalized by considering a function of the random variable  $S$  and of time,  $f(S, t)$ . This entails the use of partial derivatives since

there are now *two* independent variables,  $S$  and  $t$ . We can expand  $f(S + dS, t + dt)$  in a Taylor series about  $(S, t)$  to get

$$df = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}dS^2 + \dots$$

Using expressions (1.1.7) for  $dS$  and (1.1.8) for  $dX^2$  we find that the new expression for  $df$  is

$$df = \sigma S \frac{\partial f}{\partial S}dX + \left( \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt. \quad (1.1.13)$$

## 1.2 The Black-Scholes Equation

We begin this section with a discussion of the concept of arbitrage. One of the fundamental concepts in the theory of option pricing is the absence of arbitrage opportunities, called the *no arbitrage principle*. As an illustrative example [155] of an arbitrage opportunity, suppose the prices of a given stock in Exchanges A and B are listed at \$100 and \$102, respectively. Assuming there is no transaction cost, one can lock in a riskless profit of \$2 per share by buying at \$100 in Exchange A and selling at \$102 in Exchange B. The trader who engages in such a transaction is called an *arbitrager*. If the financial market functions properly, such an arbitrage opportunity cannot occur since traders are well aware of the differential in stock prices and they immediately compete away the opportunity. However, when there is transaction cost, which is a common form of market friction, the small difference in prices may persist. For example, if the transaction costs for buying and selling per share in Exchanges A and B are both \$1.50, then the total transaction costs of \$3 per share will discourage arbitragers.

More precisely, an arbitrage opportunity can be defined as a self-financing trading strategy requiring no initial investment, having zero probability of negative value at expiration, and yet having some possibility of a positive terminal payoff.

Before describing the Black–Scholes analysis which leads to the value of an option we used the following assumptions ( as mentioned in Wilmott *et al.* [155]):

- The asset price follows the lognormal random walk (1.1.7).

Other models do exist, and in many cases it is possible to perform the Black–Scholes analysis to derive a differential equation for the value of an option.

- The risk-free interest rate  $r$  and the asset volatility  $\sigma$  are known functions of time over the life of the option.
- There are no transaction costs associated with hedging a portfolio.
- The underlying asset pays no dividends during the life of the option.

This assumption can be dropped if the dividends are known beforehand. They can be paid either at discrete intervals or continuously over the life of the option.

- There are no arbitrage possibilities.

The absence of the arbitrage opportunities means that all risk-free portfolios must earn the same return.

- Trading of the underlying asset can take place continuously.

This is clearly an idealization, and becomes important in the case of transaction costs.

- Short selling is permitted and the assets are divisible.

By this assumption, we can buy and sell any number (not necessarily an integer) of the underlying asset, and we may sell assets that we do not own.

Suppose that we have an option whose value  $V(S, t)$  depends only on  $S$  and  $t$ . It is not necessary at this stage to specify whether  $V$  is a call or a put; indeed,  $V$  can be the value of a whole portfolio of different options although for simplicity the reader can think of a simple call or put. Using Itô's lemma, equation (1.1.13) can be written as

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \quad (1.2.1)$$

This gives the random walk followed by  $V$ .

Now construct a portfolio consisting of one option and a number  $(-\Delta)$  of the underlying asset. This number is unspecified as yet. The value of this portfolio is

$$\Pi = V - \Delta S. \quad (1.2.2)$$

The jump in the value of this portfolio in one time-step is

$$d\Pi = dV - \Delta dS. \quad (1.2.3)$$

Here  $\Delta$  is held fixed during the time-step; if it were not then  $d\Pi$  would contain terms in  $d\Delta$ . Putting (1.1.7), (1.2.1) and (1.2.2) together, we find that  $\Pi$  follows the random walk

$$d\Pi = \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) dX + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt. \quad (1.2.4)$$

The random component in this random walk can be eliminated by choosing

$$\Delta = \frac{\partial V}{\partial S}. \quad (1.2.5)$$

This results in a portfolio whose increment is wholly deterministic, i.e.,

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (1.2.6)$$

We now appeal to the concepts of arbitrage and supply and demand, with the assumption of no transaction costs. The return on an amount  $\Pi$  invested in riskless assets would see a growth of  $r\Pi dt$  in a time  $dt$ . If the right-hand side of (1.2.6) were greater than this amount, an arbitrager could make a guaranteed riskless profit by borrowing an amount  $\Pi$  to invest in the portfolio. The return for this risk-free strategy would be greater than the cost of borrowing. Conversely, if the right-hand side of (1.2.6) were less than  $r\Pi dt$  then the arbitrager would short the portfolio and invest  $\Pi$  in the bank. Either way the arbitrager would make a riskless, no cost, instantaneous profit. The existence of such arbitrager with the ability to trade at low cost ensures that the return on the portfolio and on the riskless account are more or less equal. Thus, we have

$$r\Pi dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (1.2.7)$$

Substituting (1.2.2) and (1.2.5) into (1.2.7) and dividing throughout by  $dt$  we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (1.2.8)$$

This is the famous Black–Scholes partial differential equation. By deriving the partial differential equation for a quantity, such as an option price, we have made an enormous step towards finding its value. The main aim of this thesis is to find this value by solving the equation. The value of an option should be unique (otherwise, arbitrage possibilities would arise), and so, to find the solution, specifies the behavior of the required solution at some part of the solution domain. This would mostly be achieved by appropriate initial (final) and boundary conditions.

### 1.3 Literature review on numerical methods for pricing standard options

The numerical solutions of several mathematical models arising in financial economics for the valuation of the European options on different types of assets is considered. Most of these models are based on the Black-Scholes partial differential equation. As far as their numerical solutions are concerned, many results are seen in the literature on the numerical discretization of linear Black-Scholes equation.

Brennan and Schwartz [18] were the first to describe finite-difference methods for option pricing. Geske and Shastri [58] compared the efficiency of various finite-difference and other numerical methods for option pricing. Vázquez [152] presented an upwind scheme for solving the backward parabolic partial differential equation problem in the case of European options.

The earliest work on American options is by McKean [114], where a free-boundary problem for the price function and the optimal exercise boundary (the free boundary) is derived. The price function is expressed in terms of the optimal exercise boundary. Moerbeke [120] further extended the analysis and studied the properties of the optimal exercise boundary. Brennan and Schwartz [17], Courtadon [37] and Schwartz [139] developed numerical methods to solve the free-boundary problem.

The most common numerical method for pricing American options is the binomial methods (Cox *et al.* [35]), where the price process of the underlying asset is approximated by a binomial lattice (see, e.g., Muthuraman [121]). Another approach to computing the expectation (as mentioned in [121]) is to represent the price as the sum of the European option price and an early exercise premium (see, Kim [101], Jacka [74], Carr *et al.* [24]) using an integral equation. Huang *et al.* [70] use Richardson extrapolation to solve the integral expression. Ju [78] makes a piece-wise exponential

approximation to the exercise boundary and is able to solve for the price. Geske and Johnson [57] express the price as an infinite series of multivariate cumulative normal functions. Broadie and Detemple [19] provide pricing methods based on lower and upper bounds. Most of these numerical algorithms, results and implementation focus on computing the option price for a given time to expiration and underlying stock price.

In [31] Cho *et al.* considered a free boundary problem arising in the pricing of an American call option. The free boundary represents the optimal exercise price as a function of time before a maturity date. They developed a parameter estimation technique to obtain the optimal exercise curve of an American call option and its price. For the numerical solution of a forward problem, they adopted a time marching finite element method.

Choi and Marozzi [32] considered the valuation of options written on a foreign currency when interest rates are stochastic and the matrix of the diffusion representing the global economy is strongly coercive. They solved the associated variational inequality for the value function numerically by the finite element method. In the European case, a comparison is made with the exact solution. They also presented a corresponding result for the American option.

In [47], Engström and Nordén estimated the value of the early exercise premium in American put option prices using Swedish equity options data. They found the value of the premium as the deviation of the American put price from European put-call parity, and computed a theoretical estimate of the premium. They also used the empirically found premium in a modified version of the control variate approach to value American puts. Their results indicate a substantial value of the early exercise premium, where the premium derived from put-call parity is higher than the theoretical premium.

The approach of Lindset and Lund [109] for the valuation of an American put option

under stochastic interest rates consists of a combination of a Monte Carlo simulation approach for the valuation of Bermudan options and the valuation technique for American options proposed by Geske and Johnson [57].

In [127] Perrakis and Lefoll examined the optimal super-replication of American put options with physical delivery of the underlying asset, such as stock options, by means of a stock-plus-riskless asset portfolio. Their framework of the analysis was the binomial model with proportional transactions costs on stock transactions. They extended the model for European options, originally presented in Merton [116] and Boyle and Vorst [16] and generalized in Bensaid *et al.* [8]. They adapted the optimizing framework of this latter study for put options held by investors and perfectly hedged by a market maker, and to put options written by investors. Furthermore, they showed that a unique optimal super-replicating portfolio exists at every node of the binomial tree for the long option, as well as for the short option when transactions costs are low.

Some of the other popular numerical methods that are used so far for pricing American options are the front-fixing method (Wu and Kwok [156] and Nielsen [122]) and the penalty method (Nielsen [122]). Front-fixing methods apply a non-linear transformation to fix the boundary and solve the resulting non-linear problem. Penalty methods on the other hand eliminate the free-boundary by adding a non-linear penalty term to the PDE. Both these methods boil down to solving a set of non-linear equations, the computational speed and accuracy of which largely depends on the initial guess, the problem size and the underlying non-linear solver used. These methods are not very efficient for pricing American options but they are far more general in their applicability.

The American put option problem is posed either as a linear complementarity problem

(LCP) of the form

$$\begin{aligned} V_\tau &\geq \mathcal{L}V, & V(S, 0) &= \max(E - S, 0), \\ V(S, \tau) &\geq V(S, 0), & (V_\tau = \mathcal{L}V) \wedge (V(S, \tau) = V(S, 0)), \end{aligned} \quad (1.3.1)$$

where  $T$  is the expiry time,  $\tau = T - t$ , and

$$\mathcal{L} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - \delta)S \frac{\partial}{\partial S} - r, \quad (1.3.2)$$

represents the spatial operator, or it can be posed as a free boundary value problem as

$$\begin{aligned} V_\tau &= \mathcal{L}V, & \min\left(E, \frac{rE}{\delta}\right) &= S_f(\tau) \leq S \leq \infty, & 0 \leq \tau \leq T, \\ V(S, 0) &= \max(S - E, 0), \\ \frac{\partial V}{\partial S}(S_f(\tau), \tau) &= -1, \\ V(S_f(\tau), \tau) &= E - S_f(\tau), \\ \lim_{S \rightarrow \infty} V(S, \tau) &= 0. \end{aligned} \quad (1.3.3)$$

The two different formulations lead to different numerical algorithms for the pricing the American options.

The first algorithm to value an American put option was introduced by Brennan and Schwartz [17]. They approximated the partial derivatives by finite differences. Their algorithm is based on transforming a tridiagonal system to a lower bidiagonal system and then solving this system while enforcing the American constraint.

For the American option problem, Forsyth and Vetzal [50] showed that the addition of a penalty term

$$\lambda_i^{j+1} = \frac{1}{\epsilon \Delta \tau} \max(V_i^0 - V_i^{j+1}, 0),$$

to the Black-Scholes inequality, gives

$$\frac{\partial V_i}{\partial \tau} = \mathcal{L}V_i + \lambda_i. \quad (1.3.4)$$

In [122], Nielsen *et al.* proposed the use of

$$\lambda^{j+1} = \frac{\epsilon r E}{V^{j+1} - V^0 + \epsilon},$$

where  $\epsilon$  is related to the tolerance error in the solution. Discretization of Eq. (1.3.4) with an implicit treatment of the penalty term leads to a non-linear system. In [122], the resulting non-linear system is solved using a Newton iteration method.

Ikonen and Toivanen [72] proposed a different technique known as operator splitting for time discretization for solving the linear complementarity problems arising from the pricing of American options. The space discretization is done using a central finite difference scheme. The operator splittings are based on the Crank–Nicolson method and the two-step backward differentiation formula.

Wu and Kwok [156] proposed a transformation  $S = e^y S_f(\tau)$  which turns the unknown free boundary of the American option into a known fixed boundary, after dividing  $S$ ,  $S_f(\tau)$ ,  $V(S, \tau)$  by  $E$  to obtain normalized variable and functions, and the American problem is posed as

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial y^2} + \left[ r - \delta - \frac{\sigma^2}{2} + \frac{S'_f(\tau)}{S_f(\tau)} \right] \frac{\partial V}{\partial y} - rV, \\ V(y, 0) &= 0, \quad y \in (0, \infty), \\ V(0, \tau) &= 1 - S_f(\tau), \quad \frac{\partial V(0, \tau)}{\partial y} = -S_f(\tau), \\ \lim_{y \rightarrow \infty} V(y, \tau) &= 0. \end{aligned}$$

The presence of the term  $\frac{S'_f(\tau)}{S_f(\tau)} \frac{\partial V}{\partial y}$  reveals the non-linear nature of the valuation problem

as exposed by the transformation, and the condition

$$-\frac{\sigma^2}{2} \frac{\partial^2 V(0, \tau)}{\partial y^2} - \left( \delta + \frac{\sigma^2}{2} \right) S_f(\tau) + r = 0,$$

at  $y = 0$  is used to fix the boundary conditions in the numerical procedure.

The use of transformations  $x = \log(S/E)$  and  $\tau = \sigma^2(T - t)/2$  with

$$u(x, \tau) = e^{\hat{\alpha}x + \hat{\beta}\tau} V(S, t)/E, \quad (1.3.5)$$

where  $\hat{\alpha}$  and  $\hat{\beta}$  are defined as

$$\hat{\alpha} = \frac{1}{2} \left( \frac{2(r - \delta)}{\sigma^2} - 1 \right), \quad \hat{\beta} = \frac{1}{4} \left( \frac{2(r - \delta)}{\sigma^2} - 1 \right)^2 + \frac{2r}{\sigma^2},$$

transformed the free boundary problem (1.3.3) to the heat equation

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2}, \quad x_f(\tau) \leq x < \infty, \quad 0 \leq \tau \leq \frac{\sigma^2 T}{2}, \\ u(x, 0) &= g(x, 0), \quad x_f(\tau) \leq x < \infty, \\ u(x_f(\tau), \tau) &= g(x_f(\tau), \tau), \quad g(x, \tau) = e^{\hat{\alpha}x + \hat{\beta}\tau} \max(1 - e^x, 0), \\ u(x, \tau) &\rightarrow 0 \text{ as } x \rightarrow \infty. \end{aligned} \quad (1.3.6)$$

For the heat solution  $u_c(x, \tau)$  of an American call transformed problem upper boundary condition  $u_c(\hat{x}_{f_c}(\tau), \tau) = g_c(\hat{x}_{f_c}(\tau), \tau)$  at any  $\hat{x}_{f_c}(\tau) > x_{f_c}(\tau)$ , Han and Wu [61] proved that based on the strong maximum principle for parabolic equations, the following inequality holds

$$u_c(x, \tau) < g_c(x, \tau), \quad x_{f_c}(\tau) < x < \hat{x}_{f_c}(\tau), \quad (1.3.7)$$

where  $g_c(x, \tau)$  represents the transformed payoff for a call option. On the basis of the

put–call symmetry and using (1.3.7) with the transformation relation (1.3.5) they have

$$u_{x,\tau} < g_{x,\tau}, \quad x_{f_c}(\tau) < x < \hat{x}_{f_c}(\tau), \quad (1.3.8)$$

and they used this inequality as a test condition for determining the location of the free boundary in the numerical computation for an American put option.

Tangman *et al.* [150] described a new finite difference algorithm for the American option problem which is an improvement of the method proposed by Han and Wu [61]. They used an optimal higher-order compact scheme [145] instead of the Crank–Nicolson scheme used in [61]. They set-up the problem in a singularity separating framework given by

$$\begin{aligned} \frac{\partial u_D}{\partial \tau} &= \frac{\partial^2 u_D}{\partial x^2}, \quad x_f(\tau) \leq x < \infty, \\ u_D(x, 0) &= 0, \quad x_f(0) \leq x < \infty, \\ u_D(x_f(\tau), \tau) &= u_D(x_f(\tau), \tau) - u_E(x_f(\tau), \tau), \quad 0 \leq \tau \leq \tau_{\max}, \\ u_D(x, \tau) &\rightarrow 0 \quad \text{as } x \rightarrow \infty, \end{aligned} \quad (1.3.9)$$

where  $u_E$  is the transformed value of a European put option. They noted that the transformed value of the American put option which is given by  $u = u_D + u_E$  is made up of a numerical part  $u_D$  and an analytical part  $u_E$ .

The Black–Scholes model for American put problems take the form of moving-boundary problems. The American early exercise constraint leads to the following model for the

value  $P(S, t)$  of an American put option to sell the asset

$$\begin{aligned}
 \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP &= 0, \quad S > S_f(t), \quad 0 \leq t < T, \\
 P(S, T) &= \max(E - S, 0), \quad S \geq 0, \\
 \frac{\partial P}{\partial S}(S_f, t) &= -1, \\
 P(S_f(t), t) &= E - S_f(t), \\
 \lim_{S \rightarrow \infty} P(S, t) &= 0, \\
 S_f(T) &= E, \\
 P(S, t) &= E - S, \quad 0 \leq S < S_f(t).
 \end{aligned} \tag{1.3.10}$$

Note that, since early exercise is permitted, the value  $P$  of the option must satisfy

$$P(S, t) \geq \max(E - S, 0), \quad S \geq 0, \quad 0 \leq t \leq T. \tag{1.3.11}$$

In the case of the American put option (1.3.10) which involves an unknown boundary, Khaliq *et al.* [100] approximated the model by adding a penalty term yielding a nonlinear partial differential equation on a fixed domain. Specifically, with  $0 < \epsilon \ll 1$ , a small regularization parameter, they considered the initial-boundary value problem

$$\begin{aligned}
 \frac{\partial V_\epsilon}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_\epsilon}{\partial S^2} + rS \frac{\partial V_\epsilon}{\partial S} - rV_\epsilon + \frac{\epsilon C}{V_\epsilon + \epsilon - q(S)} &= 0, \quad S \in [0, S_\infty], \quad t \in [0, T], \\
 V_\epsilon(S, T) &= \max(E - S, 0), \\
 V_\epsilon(0, t) &= E, \\
 V_\epsilon(S_\infty, t) &= 0,
 \end{aligned} \tag{1.3.12}$$

where  $C \geq rE$  is a positive constant and  $q(S) = E - S$ . They discretized the domain  $[0, S_\infty] \times [0, T]$ , and applied an implicit algorithm to (1.3.12) obtained result in a system of nonlinear algebraic equations. They used the well-known  $\theta$ -method with second-order central differencing applied to the diffusion operator and upwind differencing of

the transport term to avoid oscillations due to spatial discretization, and get

$$\begin{aligned} & \frac{V_j^{n+1} - V_j^n}{\Delta t} + \frac{1}{2}\sigma^2 S_j^2 \left[ \theta \frac{\delta_S^2 V_j^{n+1}}{\Delta S^2} + (1 - \theta) \frac{\delta_S^2 V_j^n}{\Delta S^2} \right] \\ & + r S_j \left[ \theta \frac{\Delta_S V_j^{n+1}}{\Delta S} + (1 - \theta) \frac{\Delta_S V_j^n}{\Delta S} \right] - r[\theta V_j^{n+1} + (1 - \theta)V_j^n] \\ & + \theta \frac{\epsilon C}{V_j^{n+1} + \epsilon - q(S_j)} + (1 - \theta) \frac{\epsilon C}{V_j^n + \epsilon - q(S_j)} = 0, \end{aligned} \quad (1.3.13)$$

where  $\delta^2 V v_j^n = V_{j+1}^n - 2V_j^n + V_{j-1}^n$  and  $\Delta_S V_j^n = V_{j+1}^n - V_j^n$ . They treat the penalty term in (1.3.13) explicitly by replacing  $V_j^n$  by  $V_j^{n+1}$  in that term. The corresponding linearly implicit scheme then has a form which does not require a nonlinear iterative solver.

Hon [67] developed another numerical method for solving the Black-Scholes equation for valuation of American options prices. Since his method does not require solving a resultant full matrix, the ill-conditioning problem resulting from using the radial basis functions as a global interpolant can be avoided. He showed that the method is effective in solving problems with free boundary condition. He used the transformation  $S = e^x$  to transform the Black-Scholes equation to

$$\frac{\partial U}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + (r - \frac{1}{2}\sigma^2) \frac{\partial U}{\partial x} - rU = 0, \quad (1.3.14)$$

with terminal condition

$$U(x, T) = \begin{cases} \max \{E - e^x, 0\}, & \text{for put} \\ \max \{e^x - E, 0\}, & \text{for call.} \end{cases} \quad (1.3.15)$$

He approximated the unknown function  $U$  using the quasi-radial basis functions as

$$U(x, t) \simeq \sum_{j=0}^N U_j(t) \Psi_j(x), \quad (1.3.16)$$

where  $U_j$  is the unknown option value at  $x = x_j$  which depends on time  $t$  and  $\Psi_j(x) = \Psi(\|x - x_j\|)$  is a linear combination of the radial basis functions  $\phi(\|x - x_j\|)$ . He used the implicit time integration scheme for the time-discretization to discretize equation (1.3.14) for the valuations of the European and American options. To satisfy the early optimal exercise for the valuation of the American put options, he simply updated, at each time step  $t$ , in the valuation of the European option, the elements of  $U_n$  by  $U_n(i) = \max \{E - e^{x_i}, U_n(i)\}$ .

More relevant numerical works dealing with pricing American options include Allegretto *et al.* [1], Clift and Forsyth [36], Ekström [46], Israel and Rincon [73], Kallast and Kivinukk [99], Kohler [103], Kwok [105], Markolefas [113], Wilmott *et al.* [155], Zvan *et al.* [165], and some of the references there in. Some other works pertaining to the standard options will be discussed further in the respective chapters.

## 1.4 Literature review on numerical methods for pricing nonstandard options

Exotic options are widely used in the field of finance (see Bormetti *et al.* [13]; Joshi [77]; Lasserre *et al.* [107]; Taleb [148] and Zhang [162]). Exotic options are particularly challenging for traditional numerical methods which can perform inaccurately due to the discontinuities in the payoff function (or its derivatives). Large errors may also occur in estimating the hedging parameters e.g., delta, vega, and gamma values, even though the prices appear to be correct. The non-smooth data can further lead to serious degradation in the convergence of the numerical schemes.

Explicit schemes are easy to implement but suffer from stability problems noticed by Heston and Zhou [64]. The fully implicit Backward Euler method may be used to accurately solve the Black-Scholes PDE due to its strong stability properties (see Zvan *et al.* [164], [166]). Pooley *et al.* [126], Giles and Carter [59] utilized a smoothing scheme developed by Rannacher [130] which uses a finite number of steps of the fully implicit Backward Euler method followed by the Crank–Nicolson method.

Goto *et al.* [60] described the valuation scheme of European, barrier, and Asian options of single asset by using radial basis function approximation. The option prices are governed with Black–Scholes equation. They discretized the equation with Crank–Nicolson scheme and then the option price is approximated with the radial basis functions with unknown parameters. They showed that the European and the barrier options, the prices are governed with Black–Scholes equation, but the governing option of the Asian option is different from them.

Arciniega and Allen [5] analyzed the fully implicit and Crank–Nicolson difference schemes for solving option prices. They proved that the error expansions for the difference methods have the correct form for applying Richardson extrapolation to increase the order of accuracy of the approximations. They applied the difference methods to European, American, and down-and-out knock-out call options. Their computational results indicated that Richardson extrapolation significantly decreases the amount of computational work in estimation of option prices.

In [15] Boyle and Tian considered an explicit finite difference approach. They discuss the issue of aligning grid points with barriers by constructing a grid which lies right on the barrier and, if necessary, interpolating to find the option value corresponding to the initial stock price.

Figlewski and Gao [48] illustrated the application of an adaptive mesh technique to the case of barrier options. Their basic idea is to use a fine mesh in regions where it is required (e.g. close to a barrier) and to graph the computed results from this onto a coarser mesh which is used in other regions. This is an interesting approach and would appear to be both quite efficient and flexible, though they only examined a simple case of a down-and-out European call option with a flat, continuously monitored barrier. It also should be pointed out that restrictions are needed to make sure that points on the coarse and fine grids line up.

Zvan *et al.* [166], proposed to use an implicit method which has superior convergence (when the barrier is close to the region of interest) and stability properties as well as offering additional flexibility in terms of constructing the spatial grid. Their method also allows to place grid points either near or exactly on barriers. In particular, they presented an implicit method which can be used for PDE models with general algebraic constraints on the solution. Examples of constraints can include early exercise features as well as barriers. Also in their method, barrier options with or without American constraints can be handled. Either continuously or discretely monitored barriers can also be accommodated, as can time-varying barriers.

For some further reading on Barrier options, the reader may refer to [20, 21, 69, 71, 106, 111, 133, 147, 153, 154].

Some other works related to the nonstandard options would be reviewed inside the chapters.

## 1.5 Outline of the thesis

We have organized the rest of this thesis as follows:

A basic theory and some properties of spline functions are described in Chapter 2. These properties will be very useful in deriving/proving some results in the thesis.

In Chapter 3, we will make a thorough comparison of various numerical methods to solve a typical option pricing problems. This includes, the application of method of lines and cubic spline interpolation to discretize the problem in the spatial direction. For the time integration of the system obtained via method of lines, we have used a number of MATLAB ode solvers whereas for the one obtained by using cubic spline, we use an implicit Euler method. We also present results obtained via B-spline in this chapter. After a thorough comparison, we found that the results obtained by B-spline are more suitable for a number of reasons which are indicated in subsequent chapters. Also it is noteworthy that B-splines have the smallest support size among all splines and therefore, we decided to use them further to solve other option pricing problems.

Chapter 4 deals with a thorough derivation of B-spline for solving problem that price a European option. The method is analyzed for stability and convergence. Several comparative numerical results are presented.

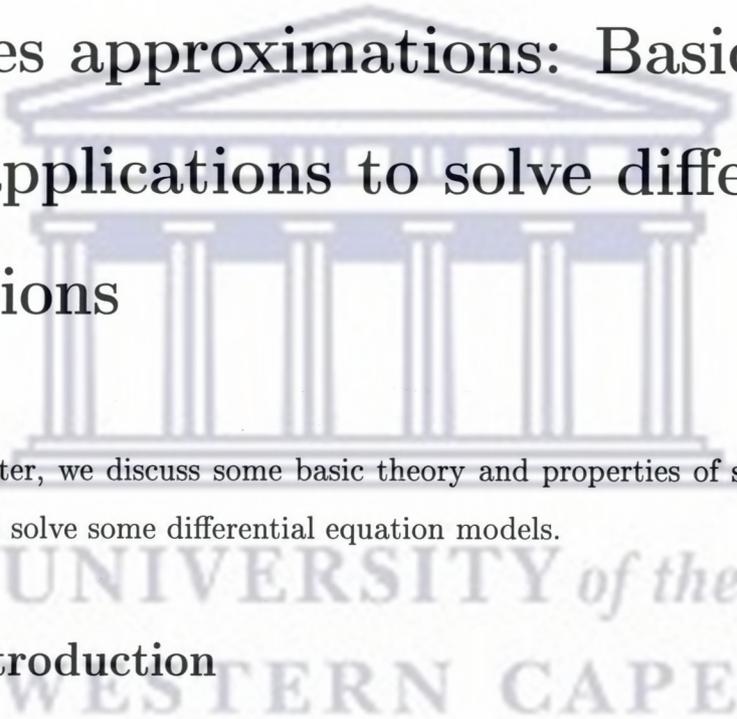
The method presented in Chapter 4 is extended in Chapter 5 to solve American option problems where a different derivation of the method is discussed by reducing the problem to a constant coefficient problem. Then using an update procedure, the American option problem is solved.

In Chapter 6, we extend the B-spline approach to solve a class of exotic options.

Finally, we provide some concluding remarks and scope for future research in Chapter 7.

## Chapter 2

# Splines approximations: Basic theory and applications to solve differential equations

The logo of the University of the Western Cape is a faint watermark in the background. It features a classical building with a pediment and six columns. Below the building, the text 'UNIVERSITY of the WESTERN CAPE' is written in a serif font.

In this chapter, we discuss some basic theory and properties of spline functions that are useful to solve some differential equation models.

### 2.1 Introduction

It is more than 50 years since Schoenberg [135, 136] introduced “spline functions” to the mathematical literature. Schoenberg is generally acknowledged to be the “father” of splines. These functions were named and singled out for special study by him in the middle of the 1940’s. Since 1960 the field of spline interpolation and approximation has grown enormously.

Splines are proved to be very useful and important in various branches of Mathematics such as approximation theory, numerical analysis, numerical treatment of differential and integral equations, and Statistics. Also, they have become useful tools in other domain, for example, Engineering, Biosciences, Chemistry, Physics, Geophysics, Meteorology, Medicine, Business and Social Sciences, Imaging and Visualization, Computer-

aided design and manufacturing, Computer Vision and Robotics, etc. It is well known that interpolating polynomial splines can be derived as the solution of certain mathematical problems.

The rest of the chapter is organized as follows. In Section 2.2 we discuss the interpolation using different type of splines. Some properties of splines, in particular, those of B-splines are mentioned in Section 2.3. Finally, applications of these methods are discussed in Section 2.4.

## 2.2 Interpolation by different splines

### Quadratic B-splines

To describe the quadratic B-splines, we partition the interval  $[a, b]$  into  $N$  finite elements of equal length  $h$  by knots  $x_i$ , such that  $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$ . The set of splines  $\{\phi_{-1}, \phi_0, \dots, \phi_N\}$  forms a basis for functions defined over the problem domain  $[a, b]$ . Quadratic B-splines  $\phi_i(x)$  with the required properties are defined by [128]

$$\phi_i(x) = \frac{1}{h^2} \begin{cases} (x_{i+2} - x)^2 - 3(x_{i+1} - x)^2 + 3(x_i - x)^2, & [x_{i-1}, x_i] \\ (x_{i+2} - x)^2 - 3(x_{i+1} - x)^2, & [x_i, x_{i+1}] \\ (x_{i+2} - x)^2, & [x_{i+1}, x_{i+2}] \\ 0, & \text{otherwise,} \end{cases}$$

where  $h = x_{i+1} - x_i$ ,  $i = -1, 0, \dots, N$ .

The quadratic spline  $\phi_i(x)$  and its first derivative  $\phi'_i(x)$  at the knots are given in Table 2.2.1. An approximate solution  $U_N(x, t)$  to the analytical solution  $U(x, t)$  to a

Table 2.2.1: Values of the quadratic B-splines  $\phi_i(x)$  and its derivatives with knots at different points

$x$	$x_{i-1}$	$x_i$	$x_{i+1}$	$x_{i+2}$
$\phi_i(x)$	0	1	1	0
$h\phi'_i(x)$	0	2	-2	0

differential equation is usually sought in form of an expansion

$$U_N(x, t) = \sum_{i=-1}^N c_i(t)\phi_i(x),$$

where  $c_i(t)$  are time-dependent nodal parameters needs to be determined using a given boundary conditions.

Each spline covers three intervals so that three splines  $\phi_{i-1}(x)$ ,  $\phi_i(x)$ ,  $\phi_{i+1}(x)$  cover each finite element  $[x_i, x_{i+1}]$ . All other splines are zero in this region.

### Cubic B-splines

The region  $[a, b]$  is partitioned into  $N$  finite elements of equal length  $h$  by knots  $x_i$ , such that  $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$ . The cubic B-splines will be used to approximate a solution  $U_N(x, t)$  to the analytical solution  $U(x, t)$  to a differential equation. Thus, an approximation  $U_N(x, t)$  to the analytical solution  $U(x, t)$  can be expressed in terms of the cubic b-splines as

$$U_N(x, t) = \sum_{i=-1}^{N+1} \delta_i(t)\phi_i(x_j), \quad j = 0, 1, 2, \dots, N,$$

where  $\delta_i(t)$  are time dependent parameters to be determined from boundary conditions and collocation form of the differential equation. A cubic B-spline covers four elements

and is defined as

$$\phi_i(x) = \begin{cases} \left(\frac{x-x_{i-2}}{h}\right)^3, & [x_{i-2}, x_{i-1}] \\ 1 + 3\left(\frac{x-x_{i-1}}{h}\right) + 3\left(\frac{x-x_{i-1}}{h}\right)^2 - 3\left(\frac{x-x_{i-1}}{h}\right)^3, & [x_{i-1}, x_i] \\ 1 + 3\left(\frac{x_{i+1}-x}{h}\right) + 3\left(\frac{x_{i+1}-x}{h}\right)^2 - 3\left(\frac{x_{i+1}-x}{h}\right)^3, & [x_i, x_{i+1}] \\ \left(\frac{x_{i+2}-x}{h}\right)^3, & [x_{i+1}, x_{i+2}] \\ 0, & \text{otherwise} \end{cases} \quad (2.2.1)$$

where  $h = x_{i+1} - x_i$ ,  $i = -1, 0, \dots, N + 1$ . So the four cubic B-splines  $\phi_{i-1}(x)$ ,  $\phi_i(x)$ ,  $\phi_{i+1}(x)$ ,  $\phi_{i+2}(x)$  lie in each element. Over the typical element  $[x_i, x_{i+1}]$ , the approximate  $U_N$  is given by

$$U_N(x, t) = \sum_{i=j-1}^{j+2} \delta_i(t) \phi_i(x_j),$$

where  $\phi_i(x)$  act as element shape functions of the element, with  $\delta_i(t)$  as element parameters. This form shows the variation of all contributing cubic B-splines over a single element and is useful for working out the solution inside the element. The values of  $\phi_i(x)$  and its derivatives are shown in Table 2.2.2.

Table 2.2.2: Values of the cubic B-splines  $\phi_i(x)$  and its derivatives with knots at different points

$x$	$x_{i-2}$	$x_{i-1}$	$x_i$	$x_{i+1}$	$x_{i+2}$
$\phi_i(x)$	0	1	4	1	0
$h\phi'_i(x)$	0	-3	0	3	0
$h^2\phi''_i(x)$	0	6	-12	6	0

## Quintic B-splines

We subdivide the interval  $[a, b]$  into subintervals by the set of  $N + 1$  distinct points  $x_i$ ,  $i = 0, 1, \dots, N$ , such that  $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$ .

The construction of the quintic B-spline interpolate  $U_N$  to the analytic solution  $U$  of a differential equation for spaced knots  $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$  can be performed with the help of the 10 additional knots such that

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} \text{ and } x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4} < x_{N+5}.$$

The quintic B-splines  $\phi_i(x)$ ,  $i = -2, -1, \dots, N + 2$ , are defined by

$$\phi_i(x) = \frac{1}{h^5} \begin{cases} (x - x_{i-3})^5, & [x_{i-3}, x_{i-2}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5, & [x_{i-2}, x_{i-1}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5, & [x_{i-1}, x_i] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 - 20(x - x_i)^5, & [x_i, x_{i+1}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 - 20(x - x_i)^5 + 15(x - x_{i+1})^5, & [x_{i+1}, x_{i+2}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 - 20(x - x_i)^5 + 15(x - x_{i+1})^5 - 6(x - x_{i+2})^5, & [x_{i+2}, x_{i+3}] \\ 0, & \text{otherwise} \end{cases}$$

where  $h = x_{i+1} - x_i$ ,  $i = -2, 0, \dots, N + 2$ .

The set of quintic B-splines  $\phi_i(x)$ ,  $i = -2, -1, \dots, N + 2$ , form a basis over the region  $a \leq x \leq b$  [128]. A global quintic B-spline interpolate  $U_N$  to the analytic solution  $U$ , is given by

$$U_N(x, t) = \sum_{i=-2}^{N+2} \delta_i(t) \phi_i(x),$$

where  $\delta_i$  are time-dependent nodal parameters needs to be determined. The values of  $\phi_i(x)$  and its derivatives are shown in Table 2.2.3.

Table 2.2.3: Values of the quintic B-splines  $\phi_i(x)$  and its derivatives with knots at different points

$x$	$x_{i-3}$	$x_{i-2}$	$x_{i-1}$	$x_i$	$x_{i+1}$	$x_{i+2}$	$x_{i+3}$
$\phi_i(x)$	0	1	26	66	26	1	0
$h\phi'_i(x)$	0	5	50	0	-50	-5	0
$h^2\phi''_i(x)$	0	20	40	120	40	20	0

### Sextic B-splines

The region  $[a, b]$  is partitioned into  $N$  finite elements of equal length  $h$  by knots  $x_i$ , such that  $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$ . Let  $\phi_i$ ,  $i = -3, \dots, N + 2$  be the sextic B-splines with both knots  $x_i$  and 12 additional knots outside the region positioned at:

$$x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$$

$$\text{and } x_N < x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4} < x_{N+5} < x_{N+6}.$$

The set of sextic B-splines  $\{\phi_{-3}, \phi_{-2}, \dots, \phi_{N+2}\}$  forms a basis for functions defined over the problem domain  $[a, b]$  ([128]).

The sextic B-splines are defined as

$$\phi_i(x) = \frac{1}{h^6} \begin{cases} (x - x_{i-3})^6, & [x_{i-3}, x_{i-2}] \\ (x - x_{i-3})^6 - 7(x - x_{i-2})^6, & [x_{i-2}, x_{i-1}] \\ (x - x_{i-3})^6 - 7(x - x_{i-2})^6 + 21(x - x_{i-1})^6, & [x_{i-1}, x_i] \\ (x - x_{i-3})^6 - 7(x - x_{i-2})^6 + 21(x - x_{i-1})^6 - 35(x - x_i)^6, & [x_i, x_{i+1}] \\ (x - x_{i+4})^6 - 7(x - x_{i+3})^6 + 21(x - x_{i+2})^6, & [x_{i+1}, x_{i+2}] \\ (x - x_{i+4})^6 - 7(x - x_{i+3})^6, & [x_{i+2}, x_{i+3}] \\ (x - x_{i+4})^6, & [x_{i+3}, x_{i+4}] \\ 0, & \text{otherwise} \end{cases}$$

where  $h = x_{i+1} - x_i$ ,  $i = -3, 0, \dots, N + 2$ .

The sextic B-splines and its first fifth derivatives vanish outside the interval  $[x_{i-3}, x_{i+4}]$ . The values of the sextic B-splines and its principal five derivatives at the knots are listed in Table 2.2.4.

An approximate solution  $U_N(x, t)$  to the analytical solution  $U(x, t)$  to a differential equation is usually sought in form of

$$U_N(x, t) = \sum_{i=-3}^{N+2} \omega_i(t) \phi_i(x),$$

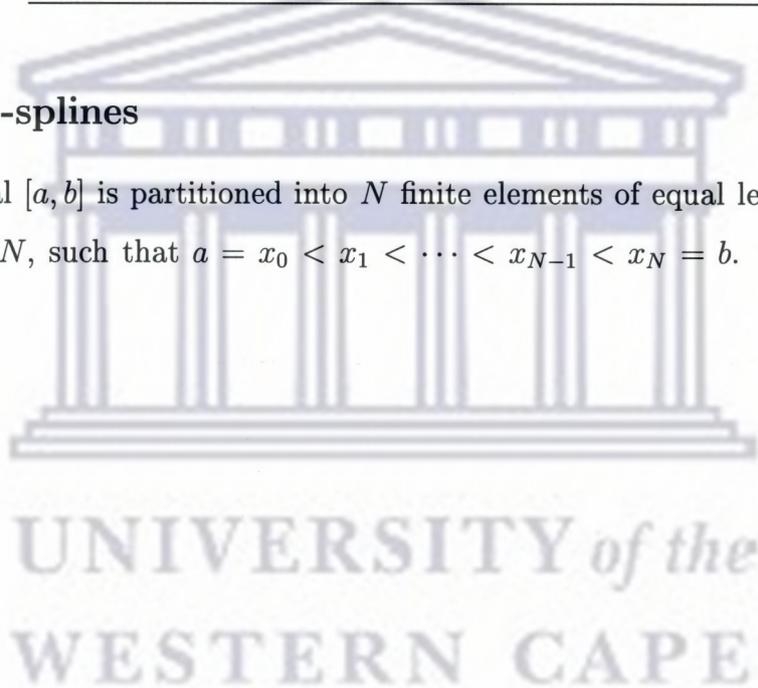
where  $\omega_i$  are time dependent parameters to be determined using a given boundary conditions for the differential equation.

Table 2.2.4: Values of the sextic B-splines  $\phi_i(x)$  and its derivatives with knots at different points

$x$	$x_{i-2}$	$x_{i-1}$	$x_i$	$x_{i+1}$	$x_{i+2}$	$x_{i+3}$
$\phi_i(x)$	1	57	302	302	57	1
$h\phi'_i(x)$	6	150	240	-240	-150	-6
$h^2\phi''_i(x)$	30	270	-300	-300	270	30
$h^3\phi'''_i(x)$	120	120	-960	960	-120	-120
$h^4\phi_i^{(4)}(x)$	360	-1080	720	720	-1080	360
$h^5\phi_i^{(5)}(x)$	720	-3600	7200	-7200	3600	-720

### Septic B-splines

The interval  $[a, b]$  is partitioned into  $N$  finite elements of equal length  $h$  by knots  $x_i$ ,  $i = -0, \dots, N$ , such that  $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$ . The septic B-spline



function  $\phi_i(x)$  at these knots is given by

$$\phi_i(x) = \frac{1}{h^7} \begin{cases} (x - x_{i-4})^7, & [x_{i-4}, x_{i-3}] \\ (x - x_{i-4})^7 - 8(x - x_{i-3})^7, & [x_{i-3}, x_{i-2}] \\ (x - x_{i-4})^7 - 8(x - x_{i-3})^7 + 28(x - x_{i-2})^7, & [x_{i-2}, x_{i-1}] \\ (x - x_{i-4})^7 - 8(x - x_{i-3})^7 + 28(x - x_{i-2})^7 - 56(x - x_{i-1})^7, & [x_{i-1}, x_i] \\ (x_{i+4} - x)^7 - 8(x_{i+3} - x)^7 + 28(x_{i+2} - x)^7 - 56(x_{i+1} - x)^7, & [x_i, x_{i+1}] \\ (x_{i+4} - x)^7 - 8(x_{i+3} - x)^7 + 28(x_{i+2} - x)^7, & [x_{i+1}, x_{i+2}] \\ (x_{i+4} - x)^7 - 8(x_{i+3} - x)^7, & [x_{i+2}, x_{i+3}] \\ (x_{i+4} - x)^7, & [x_{i+3}, x_{i+4}] \\ 0, & \text{otherwise} \end{cases}$$

where  $h = x_{i+1} - x_i$ ,  $i = -3, 0, \dots, N + 3$ , implying that all intervals  $[x_{i-1}, x_i]$  are of equal size. This means that the values of the septic B-spline function  $\phi_i(x)$ , and all its first, second and third derivatives vanish outside the interval  $[x_{i-4}, x_{i+4}]$ . The set of splines  $\{\phi_{-3}, \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \dots, \phi_N, \phi_{N+1}, \phi_{N+2}, \phi_{N+3}\}$  forms a basis for the functions defined over  $[a, b]$ . The values of  $\phi_i(x)$  and its derivatives are shown in Table 2.2.5.

An approximate solution  $U_N(x, t)$  to the analytical solution  $U(x, t)$  to a differential equation is usually sought in form of an expansion of B-splines

$$U_N(x, t) = \sum_{i=-3}^{N+3} \omega_i(t) \phi_i(x_j), \quad j = 0, 1, 2, \dots, N,$$

Table 2.2.5: Values of the septic B-splines  $\phi_i(x)$  and its derivatives with knots at different points

$x$	$x_{i-4}$	$x_{i-3}$	$x_{i-2}$	$x_{i-1}$	$x_i$	$x_{i+1}$	$x_{i+2}$	$x_{i+3}$	$x_{i+4}$
$\phi_i(x)$	0	1	120	1191	2416	1191	120	1	0
$h\phi'_i(x)$	0	7	392	1715	0	-1715	-392	-7	0
$h^2\phi''_i(x)$	0	42	1008	630	-3360	630	1008	42	0
$h^3\phi'''_i(x)$	0	210	1680	-3990	0	3990	-1680	-210	0

where  $\omega_i$  are time dependent parameters to be determined using the given boundary conditions for the differential equation.

## 2.3 Basic properties of splines

We define spaces of polynomial splines and show that there exists a basis consisting of polynomials and truncated power functions. Spline spaces are prototypes of weak Chebyshev spaces. We begin the discussion here with the definition of polynomial splines.

**Definition 2.3.1** *Let points  $a = x_0 < x_1 < \dots < x_k < x_{k+1} = b$  and an integer  $m \geq 1$  be given. We call*

$$S_m(x_1, \dots, x_k) = \{s \in C^{m-1}[a, b] : s|_{[x_i, x_{i+1}]} \in P_m, \quad i = 0, \dots, k\}, \quad (2.3.1)$$

*the space of polynomial splines of degree  $m$  with  $k$  fixed knots  $x_1, \dots, x_k$ , where  $P_m$  is the polynomial of order  $m$ . For a given spline space  $S_m(x_1, \dots, x_k)$ , we always associate further points  $x_{-m} < \dots < x_{-1} < a$  and  $b < x_{k+2} < \dots < x_{k+m+1}$ , where these points may be chosen arbitrarily.*

**Definition 2.3.2** For a given point  $x \in (a, b)$  the function

$$(t - x)_+^m = \begin{cases} 0, & \text{if } t \leq x \\ (t - x)^m, & \text{if } t > x, \end{cases} \quad (2.3.2)$$

is called the truncated power function of degree  $m$  with knot  $x$ .

We now show that there exists a basis of a given spline space consisting of polynomials and truncated power functions.

**Theorem 2.3.1** The set of functions

$$\{1, t, \dots, t^m, (t - x_1)_+^m, \dots, (t - x_k)_+^m\} \quad (2.3.3)$$

forms a basis of  $S_m(x_1, \dots, x_k)$ . In particular, the dimension of  $S_m(x_1, \dots, x_k)$  is  $k + m + 1$ .

**Proof.** It is easy to see that

$$\{1, t, \dots, t^m, (t - x_1)_+^m, \dots, (t - x_k)_+^m\}$$

is a subset of  $S_m(x_1, \dots, x_k)$ . It remains to show that every  $s \in S_m(x_1, \dots, x_k)$  has a unique representation

$$s(t) = \sum_{i=0}^m a_i t^i + \sum_{i=1}^k b_i (t - x_i)_+^m, \quad t \in [a, b]. \quad (2.3.4)$$

Let a spline  $s \in S_m(x_1, \dots, x_k)$  be given. We set

$$p_i(t) = s(t), \quad t \in [x_i, x_{i+1}], \quad i = 0, \dots, k.$$

Then we have  $p_0, \dots, p_k \in P_m$ . Therefore,  $p_0 \in P_m$  has a unique representation

$$p_0(t) = \sum_{i=0}^m a_i t^i, \quad t \in [x_0, x_1].$$

Moreover, since  $s \in C^{(m-1)}[a, b]$ , we have

$$p_1^{(i)}(x_1) = p_0^{(i)}(x_1), \quad i = 0, \dots, m-1.$$

Since  $p_1 - p_0 \in P_m$ , this implies that  $p_1 - p_0$  has a unique representation

$$p_1(t) - p_0(t) = b_1(t - x_1)^m.$$

Therefore, we have

$$s(t) = \sum_{i=0}^m a_i t^i + b_1(t - x_1)_+^m, \quad t \in [x_0, x_2].$$

Proceeding recursively, we finally obtain (2.3.4). This prove Theorem 2.3.1.

The next theorem, due to Schoenberg and Whinteny [137], says that spline spaces are weak Chebyshev subspaces.

For proving this result, we need the following version of the well-known Rolle's Theorem which can be found in standard books on Analysis.

**Theorem 2.3.2** *Let a function  $f \in C^1[a, b]$  and points  $a < t_1 < t_2 < b$  be given such that  $f(t_1) = f(t_2) = 0$ . Then the function  $f'$  has at least one zero in  $(t_1, t_2)$ . If, in addition  $f(t) \neq 0$  for some point  $t \in (t_1, t_2)$ , then  $f'$  has at lest one sign change in  $t_1, t_2$ .*

**Theorem 2.3.3** *The space  $S_m(x_1, \dots, x_k)$  is a  $(k+m+1)$ -dimensional weak Chebyshev subspace of  $C[a, b]$ .*

**Proof.** We will show that every  $s \in S_m(x_1, \dots, x_k)$  has at most  $k + m$  sign changes. Then it follows from Theorem 1.6 in Nürnberger [123] that  $s \in S_m(x_1, \dots, x_k)$  is a weak Chebyshev subspace. Suppose that a spline  $s \in S_m(x_1, \dots, x_k)$  has at least  $k + m + 1$  sign changes. Then it follows from Theorem 2.3.2 that  $s' \in S_{m-1}(x_1, \dots, x_k)$  has at least  $k + m$  sign changes. We consider further derivatives of  $s$  and finally get that  $s^{(m-1)} \in S_1(x_1, \dots, x_k)$  has at least  $k + 2$  sign changes. This is a contradiction, since such a spline of degree one has at most  $k + 1$  sign changes. This proves Theorem 2.3.3.

## Basic properties of B-splines

It is shown that the so-called B-splines form a basis of spline spaces. B-splines are splines which have smallest possible support, in other words, they are zero on a large set. For the evaluation of splines, it is desirable to have basis functions with this property. Moreover, a stable evaluation of B-splines with the aid of a recurrence relation is possible. B-splines form a partition of unity and that the B-spline basis is variation diminishing. Also, we give results on the differentiation and integration of splines.

In this section, we discuss the properties of B-splines. We need the following definition of polynomial splines on  $(-\infty, \infty)$ .

**Definition 2.3.3** *Let points  $x_{-m} < \dots < x_{-1} < a = x_0 < x_1 < \dots < x_k < x_{k+1} = b < x_{k+2} < \dots < x_{k+m+1}$  be given. A function  $s : (-\infty, \infty) \rightarrow \mathbb{R}$  is called a polynomial spline of degree  $m$  with knots  $x_{-m}, \dots, x_{k+m+1}$  if  $s$  has  $m - 1$  continuous derivatives at  $x_i$ ,  $i = -m, \dots, k + m + 1$ , and  $s|_{x_i, x_{i+1}} \in P_m$ ,  $i = -m - 1, \dots, k + m + 1$ , where  $x_{-m-1} = -\infty$  and  $x_{k+m+2} = \infty$ .*

The first result on the existence and uniqueness of splines with certain zero properties is due to Curry and Schoenberg [38, 39].

**Theorem 2.3.4** *For each  $i \in \{-m, \dots, k\}$ , there exists a unique spline  $B_i^m$  of degree*

$m$  with knots  $x_{-m}, \dots, x_{k+m+1}$  such that

$$B_i^m(t) = 0, \quad t \in (-\infty, x_i] \cup [x_{i+m+1}, \infty), \quad (2.3.5)$$

$$B_i^m(t) > 0, \quad t \in (x_i, x_{i+m+1}), \quad (2.3.6)$$

and

$$\int_{x_i}^{x_{i+m+1}} B_i^m(t) dt = 1. \quad (2.3.7)$$

**Proof.** Every spline  $B_i^m$  of degree  $m$  satisfying (2.3.5) has the form

$$B_i^m(t) = \sum_{j=1}^{i+m+1} a_j (t - x_j)_+^m \quad t \in (-\infty, \infty). \quad (2.3.8)$$

It follows from (2.3.5) that

$$\sum_{j=1}^{i+m+1} a_j (t - x_j)^m = 0, \quad t \in [x_{i+m+1}, \infty). \quad (2.3.9)$$

Then by using the binomial theorem, we have

$$\sum_{j=i}^{i+m+1} \sum_{r=0}^{m+1} a_j (-1)^r \binom{m}{r} x_j^r t^{m-r} = 0, \quad t \in [x_{i+m+1}, \infty). \quad (2.3.10)$$

Since the coefficients of the functions  $1, t, \dots, t^m$  must be zero, we get that

$$\sum_{j=i}^{i+m+1} a_j x_j^r = 0, \quad r = 0, \dots, m. \quad (2.3.11)$$

Moreover, it follows from (2.3.7) that

$$\sum_{j=i}^{i+m+1} a_j (x_{i+m+1} - x_j)_+^{m+1} = m + 1. \quad (2.3.12)$$

Again by using the binomial theorem, we obtain

$$\sum_{j=i}^{i+m+1} \sum_{r=0}^{m+1} a_j (-1)^r \binom{m+1}{r} x_j^r x_{i+m+1}^{m+1-r} = m+1. \quad (2.3.13)$$

Equations (2.3.11) and (2.3.13) imply

$$\sum_{j=i}^{i+m+1} a_j x_j^{m+1} = (-1)^{m+1} (m+1). \quad (2.3.14)$$

The determinant corresponding to the linear system of equations (2.3.11) and (2.3.14) is the nonzero Vandermonde determinant. This shows that the unknowns  $a_i, \dots, a_{i+m+1}$  are uniquely determined and that there exists a unique spline  $B_i^m$  of degree  $m$  satisfying (2.3.5) and (2.3.7).

Property (2.3.6) can be easily proved by induction on  $m$  with aid of the subsequent recurrence relation (2.3.29) which is independent of (2.3.6). This proves Theorem 2.3.4.

**Definition 2.3.4** *The spline  $B_i^m$  in Theorem 2.3.4 is called the B-spline of degree  $m$  with support  $[x_i, x_{i+m+1}]$ .*

**Remark 2.3.1** The proof of Theorem 2.3.4 shows that, if  $i \in \{-m, \dots, k\}$ ,  $r \in \{1, \dots, m\}$  and  $s$  is a spline of degree  $m$  with knots  $x_m, \dots, x_{k+m+1}$  satisfying

$$s(t) = 0, \quad t \in (-\infty, x_i] \cup [x_{i+r}, \infty), \quad (2.3.15)$$

then  $s = 0$ . Therefore, we may say that B-splines have “minimal” support.

Curry and Schoenberg [39] proved the following result on the shape of B-splines.

**Theorem 2.3.5** *Let an index  $i \in \{-m, \dots, k\}$  be given. Then for all  $j \in \{1, \dots, m-1\}$ , the spline  $(B_i^m)^{(j)}$  has exactly  $j$  distinct zeros in  $(x_i, x_{i+m+1})$  and it changes sign at these zeros.*

**Proof.** Let an integer  $i \in \{-m, \dots, k\}$  be given. We first show that

$$(B_i^m)^{(j)} \text{ has at least } j \text{ sign changes in } (x_i, x_{i+m+1}), j = 1, \dots, m-1. \quad (2.3.16)$$

Since  $B_i^m(x_i) = B_i^m(x_{i+m+1}) = 0$ , it follows from (2.3.6) and Theorem 2.3.2 that the spline  $(B_i^m)'$  has at least one sign change. By applying Theorem 2.3.2 several times, we see that (2.3.16) holds. Next, we show that

$$(B_i^m)^{(j)} \text{ has only finitely many zeros in } (x_i, x_{i+m+1}), j = 1, \dots, m-1. \quad (2.3.17)$$

Indeed, if  $(B_i^m)^{(j)}$  vanishes on a knot-interval in  $[x_i, x_{i+m+1}]$ , then  $(B_i^m)^{(m-1)}$  vanishes on this interval. But then it is easy to see that the spline  $(B_i^m)^{(m-1)}$  of degree one cannot have  $m-1$  sign changes, which contradicts (2.3.16). Finally, we show that

$$(B_i^m)^{(j)} \text{ has at most } j \text{ distinct zeros in } (x_i, x_{i+m+1}), j = 1, \dots, m-1. \quad (2.3.18)$$

Assume to the contrary that  $(B_i^m)^{(j)}$  has at least  $j+1$  distinct zeros in  $(x_i, x_{i+m+1})$ . Then, since in addition  $B_i^m(x_i) = B_i^m(x_{i+m+1}) = 0$ , it follows from Theorem 2.3.2 that  $(B_i^m)'$  has at least  $j+2$  sign changes. By applying Theorem 2.3.2 several times, we get that  $(B_i^m)^{(m-1)}$  has at least  $m$  sign changes. This is a contradiction since  $(B_i^m)^{(m-1)}$  is a spline of degree one with  $(B_i^m)^{(m-1)}(x_i) = (B_i^m)^{(m-1)}(x_{i+m+1}) = 0$ , and therefore has at most  $m-1$  sign changes. Now, the result follows from (2.3.16) and (2.3.18). This proves Theorem 2.3.5.

## B-Spline basis

It is shown that for a given spline space there exists a basis consisting of B-splines. The result formulated in the next theorem is due to Curry and Schoenberg [39].

**Theorem 2.3.6** *The set of B-splines*

$$\{B_{-m}^m, \dots, B_k^m\} \quad (2.3.19)$$

*forms a basis of  $S_m(x_1, \dots, x_k)$  on  $[a, b]$ .*

**Proof.** We will show that the B-splines  $\{B_{-m}^m, \dots, B_k^m\}$  are linearly independent on  $[a, b]$ . Suppose to the contrary that there exist real numbers  $a_{-m}, \dots, a_k$  such that

$$\sum_{i=-m}^k |a_i| \neq 0, \quad (2.3.20)$$

and

$$\sum_{i=-m}^k a_i B_i^m(t) = 0, \quad t \in [a, b]. \quad (2.3.21)$$

We set  $j = \min\{i \in \{-m, \dots, k\} : a_i \neq 0\}$ . Then by the properties of the B-splines

$$\sum_{i=-m}^k a_i B_i^m(t) = a_j B_j^m(t) \neq 0, \quad t \in [x_j, x_{j+1}]. \quad (2.3.22)$$

This implies that  $j < 0$ , otherwise we get a contradiction to (2.3.22). We set

$$s(t) = \sum_{i=-m}^k a_i B_i^m(t), \quad t \in [-\infty, x_{k+1}]. \quad (2.3.23)$$

Then we have

$$s(t) = 0, \quad t \in (-\infty, x_{-m}] \cup [x_0, x_{k+1}]. \quad (2.3.24)$$

Then it follows from Remark 2.3.1 that  $s = 0$  which implies that

$$\sum_{i=-m}^k a_i B_i^m(t) = 0, \quad t \in [x_{-m}, x_0]. \quad (2.3.25)$$

Since  $j \in \{-m, \dots, -1\}$ , this is a contradiction to (2.3.22). This proves Theorem 2.3.6.

**Remark 2.3.2** It follows from Theorem 2.3.6 that every spline  $s \in S_m(x_1, \dots, x_k)$  has a unique representation

$$s(t) = \sum_{i=-m}^k a_i B_i^m(t) = 0, \quad t \in [a, b]. \quad (2.3.26)$$

This representation has the desirable property that if we have to compute the value  $s(t)$  for some  $t \in [a, b]$ , then only  $m+1$  values of the  $k+m+1$  values  $B_{-m}^m(t), \dots, B_k^m(t)$  are different from zero. (This follows from (2.3.5) in Theorem 2.3.4).

### Recurrence relations

We show that B-splines can be represented as divided differences of truncated power functions and as complex contour integrals. Moreover, it is shown that a stable evaluation of B-splines is possible by using a recurrence relation. Finally, we prove that normalized B-splines form a partition of unity and give results on differentiation and integration of splines.

The first result shows that there is a fundamental relation between B-splines and divided differences.

**Theorem 2.3.7** For all  $t \in (-\infty, \infty)$ ,

$$B_i^m(t) = (-1)^{m+1}(m+1)(t-x)_+^m[x_i, \dots, x_{i+m+1}], \quad (2.3.27)$$

i.e.,  $B_i^m(t)$  is the divided difference of order  $m+1$  of the function  $x \rightarrow (-1)^{m+1}(m+1)(t-x)_+^m$ ,  $x \in (-\infty, \infty)$ , with respect to the knots  $x_i, \dots, x_{i+m+1}$ .

**Proof.** We set

$$s(t) = (-1)^{m+1}(m+1)(t-x)_+^m[x_i, \dots, x_{i+m+1}], \quad t \in (-\infty, \infty),$$

and show that  $s$  satisfies the B-spline properties 2.3.5 and 2.3.6. If  $t \in (-\infty, x_i]$ , then

$$(t - x_r)_+^m = 0, \quad r = i, \dots, i + m + 1,$$

and therefore  $s(t) = 0$ . If  $t \in [x_{i+m+1}, \infty)$ , then

$$(t - x)_+^m = (t - x)^m, \quad x \in [x_i, x_{i+m+1}].$$

Since  $x \rightarrow (t - x)^m$  is a polynomial of degree  $m$ , it follows (see Nürnberger [123]), that  $s(t) = 0$ . This shows that  $s$  satisfies (2.3.5). Moreover, we have

$$\begin{aligned} \int_{x_i}^{x_{i+m+1}} s(t) dt &= \int_{x_i}^{x_{i+m+1}} (-1)^{m+1} (m+1) (t - x)_+^m [x_i, \dots, x_{i+m+1}] dt \\ &= (-1)^{m+1} (t - x)_+^m [x_i, \dots, x_{i+m+1}] \Big|_{t=x_i}^{x_{i+m+1}} \\ &= (-1)^{m+1} (x_{i+m+1} - x_i)^{m+1} [x_i, \dots, x_{i+m+1}] \\ &= x_i^{m+1} [x_i, \dots, x_{i+m+1}] = 1. \end{aligned}$$

This shows that  $s$  satisfies (2.3.7). Then it follows from the proof of Theorem 2.3.4 that  $s = B_i^m$ . This proves Theorem 2.3.7.

The following complex integral representation of B-splines is due to Meinardus [115].

**Theorem 2.3.8** For all  $t \in (-\infty, \infty)$ ,

$$B_i^m(t) = \frac{1}{2\pi i} \int_{C_t} \frac{(m+1)(z-t)^m}{(z-x_j) \cdots (z-x_{j+m+1})} dz, \quad (2.3.28)$$

where  $C_t$  is a simply closed rectifiable curve in the complex plane containing all knots  $x_r$  with  $t \leq x_r \leq x_{j+m+1}$  and no others in its interior, and the integration is carried

out in the positive direction.

**Proof.** For all  $t \in (-\infty, \infty)$ , we denote the right hand side of (2.3.28) by  $I_j^m(t)$ . It follows from the residue theorem that the function  $I_j^m : (-\infty, \infty) \rightarrow \mathbb{R}$  is a spline of degree  $m$  with knots  $x_j, \dots, x_{j+m+1}$ . Since the numerator of the integrand in (2.3.28) is a polynomial of degree  $m$  and the denominator is a polynomial of degree  $m+2$ , by the residue theorem we get that

$$I_j^m(t) = 0, \quad t \in (-\infty, x_j].$$

Moreover, it follows from Cauchy's theorem that

$$I_j^m(t) = 0, \quad t \in [x_{j+m+1}, \infty).$$

Furthermore, we have

$$\begin{aligned} \int_{x_j}^{x_{j+m+1}} I_j^m(t) dt &= -\frac{1}{2\pi i} \int_{C_{x_{j+m+1}}} \frac{(z - x_{j+m+1})^{m+1}}{(z - x_j) \cdots (z - x_{j+m+1})} dz \\ &\quad + \frac{1}{2\pi i} \int_{C_{x_j}} \frac{(z - x_j)^{m+1}}{(z - x_j) \cdots (z - x_{j+m+1})} dz. \end{aligned}$$

Again by Cauchy's theorem the first integral is zero. Therefore, it follows from the residue theorem that

$$\int_{C_{x_j}} \frac{(z - x_j) \cdots (z - x_{j+m}) - (z - x_j)^{m+1}}{(z - x_j) \cdots (z - x_{j+m+1})} dz = 0.$$

Since the numerator of the integral is a polynomial of degree  $m$  and the denominator is a polynomial of degree  $m + 2$ . This implies that

$$\begin{aligned} & \int_{x_j}^{x_{j+m+1}} I_j^m(t) dt \\ &= \frac{1}{2\pi i} \int_{C_{x_j}} \frac{(z - x_j)^{m+1}}{(z - x_j) \cdots (z - x_{j+m+1})} dz \\ &= \frac{1}{2\pi i} \int_{C_{x_j}} \frac{(z - x_j)^{m+1} + (z - x_j) \cdots (z - x_{j+m}) - (z - x_j)^{m+1}}{(z - x_j) \cdots (z - x_{j+m+1})} dz \\ &= \frac{1}{2\pi i} \int_{C_{x_j}} \frac{1}{(z - x_{j+m+1})} dz = 1. \end{aligned}$$

Therefore, it follows from Theorem 2.3.4 that  $I_j^m = B_j^m$ . This proves Theorem 2.3.8.

As has been shown by Meinardus [115], the subsequent results on B-splines can be derived from the representation (2.3.28) by using a simple decomposition technique for the rational integrand. Furthermore, by using (2.3.28) it is easy to compute derivatives of B-splines with respect to the knots. For example, if  $m \geq 2$  and  $r \in \{j, \dots, j + m + 1\}$ , then for all  $t \in (-\infty, \infty)$ ,

$$\frac{\partial B_j^m(t)}{\partial x_r} = \frac{1}{2\pi i} \int_{C_t} \frac{(m+1)(z-t)^m}{(z-x_j) \cdots (z-x_{r-1})(z-x_r)^2(z-x_{r+1}) \cdots (z-x_{j+m+1})} dz.$$

This expression can be reduced to B-splines of degree  $m - 1$ .

The next result, due to de Boor [10] and Cox [34], shows that B-splines can be evaluated with the aid of a recurrence relation.

**Theorem 2.3.9** *If  $m \geq 2$ , then for all  $t \in (-\infty, \infty)$ ,*

$$B_i^m(t) = \frac{m+1}{m} \left( \frac{t - x_i}{x_{i+m+1} - x_i} B_i^{m-1}(t) + \frac{x_{i+m+1} - t}{x_{i+m+1} - x_{i+1}} B_{i+1}^{m-1}(t) \right). \quad (2.3.29)$$

**Proof.** Let  $m \geq 2$  and  $t \in (-\infty, \infty)$  be given. We set  $f_1(x) = (t - x)$  and  $f_2(x) = (t - x)_+^{m-1}$  for all  $x \in (-\infty, \infty)$ . Then it follows (see Nürnberger [123] for further details), that

$$\begin{aligned}
 B_i^m(t) &= (-1)^{m+1}(m+1)(f_1 f_2)[x_i, \dots, x_{i+m+1}] \\
 &= (-1)^{m+1}(m+1) \sum_{r=i}^{i+m+1} f_1[x_i, \dots, x_r] f_2[x_r, \dots, x_{i+m+1}] \\
 &= (-1)^{m+1}(m+1)(f_1[x_i] f_2[x_i, \dots, x_{i+m+1}] + \\
 &\quad + f_1[x_i, x_{i+1}] f_2[x_{i+1}, \dots, x_{i+m+1}]) \\
 &= (-1)^{m+1} \frac{m+1}{m} (m f_2[x_{i+1}, \dots, x_{i+m+1}] + \\
 &\quad + (t - x_i) \frac{m f_2[x_{i+1}, \dots, x_{i+m+1}] - m f_2[x_i, \dots, x_{i+m}]}{x_{i+m+1} - x_i}) \\
 &= \frac{m+1}{m} \left( \frac{t - x_i}{x_{i+m+1} - x_i} (-1)^m m f_2[x_i, \dots, x_{i+m}] + \right. \\
 &\quad \left. + \frac{x_{i+m+1} - t}{x_{i+m+1} - x_i} (-1)^m m f_2[x_{i+1}, \dots, x_{i+m+1}] \right) \\
 &= \frac{m+1}{m} \left( \frac{t - x_i}{x_{i+m+1} - x_i} B_i^{m-1}(t) + \frac{x_{i+m+1} - t}{x_{i+m+1} - x_i} B_{i+1}^{m-1}(t) \right).
 \end{aligned}$$

This proves Theorem 2.3.9.

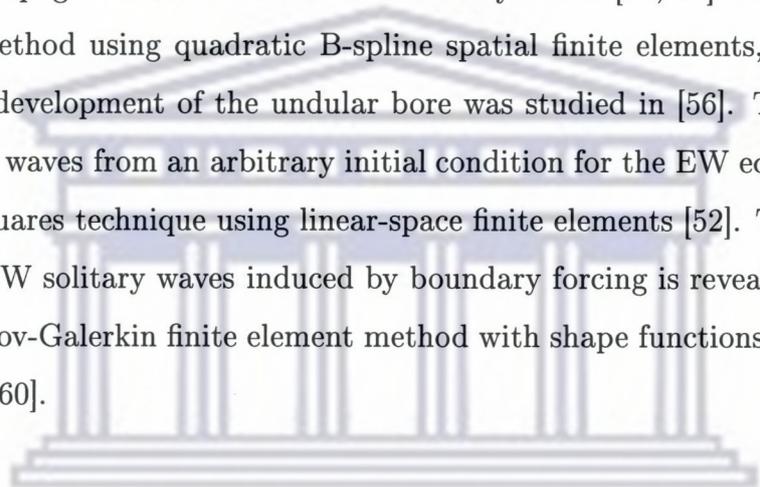
## 2.4 Applications of splines approximation methods

In this thesis, we develop a numerical method based on the B-spline collocation approach. The specific splines that are used to solve the option pricing problems considered in this thesis are described in the respective chapters. Below we provide literature review on some of the works (apologies for any omissions which is due to space limitations and completely unintentional) that use splines to solve the differential equation models. Such methods have become interesting and very promising in solving partial differential equations, due to their flexibility in practical applications.

A lot of work has been done using B-splines in other fields of sciences and engineering. The B-spline functions are used as window functions to construct a reproducing kernel function in the reproducing kernel methods and meshfree particle methods [7, 28, 29, 30, 108, 110, 158]. B-splines are also used as basis functions in the finite element methods [2, 3, 66, 104]. A variant of B-splines method has been successfully applied to solve singular perturbation problems [23, 89, 90, 94, 95, 96, 97, 98, 131].

In the field of nonlinear partial differential equations, nonlinear dispersive wave equations exhibit fascinating solutions such as solitary waves and solitons. Existence of such solutions has been source of intense interest. Solution of those equations is not analytically available in general. There are many different examples of these type of equations, each modelling several different physical problems, for example, many researchers concentrate on the equal width (EW) equation whose solutions exhibits soliton like solutions. Main properties of those solutions are that solitary waves propagate in one direction with constant speed without changing its shape and that the solitary waves pass through one another and emerge unaltered in shape. B-splines are applied to find the numerical solutions of those equations in order to develop an understanding of the nonlinear phenomena, see, for example, [3, 40, 41, 42, 45, 119, 129, 132, 140, 159, 163].

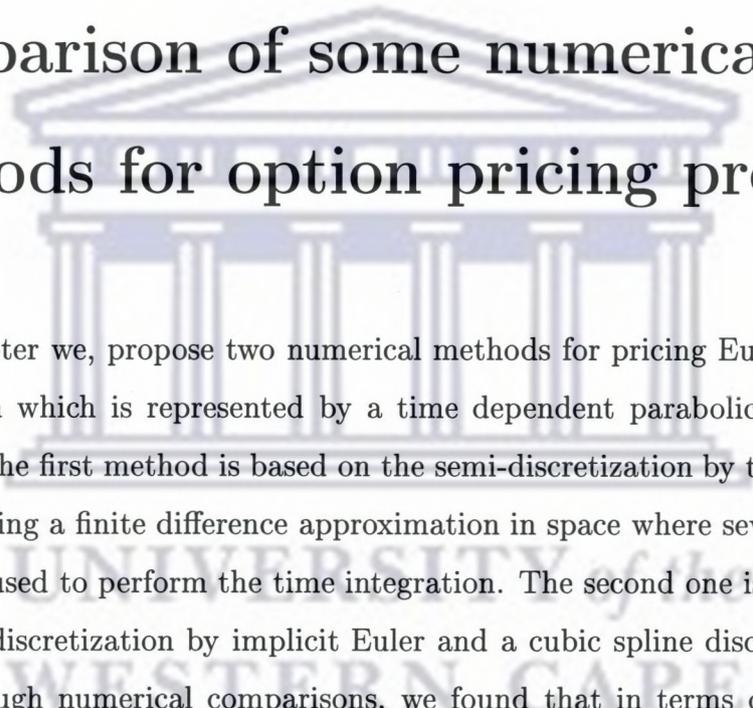
The use of various degree of the B-splines in getting the numerical solution of some partial differential equations are shown to provides easy and simple algorithms [52, 53, 54, 55, 56]. Various forms of finite element method incorporated with the B-splines as shape functions have been presented to give smooth solutions and these functions guarantee continuity of approximating functions at the mesh points up to one less degree of B-splines. Cubic B-spline Galerkin finite element method is applied to EW equation to model propagation and interaction of solitary waves [52, 53]. By using the Petrov-Galerkin method using quadratic B-spline spatial finite elements, motion of solitary waves and development of the undular bore was studied in [56]. The development of the solitary waves from an arbitrary initial condition for the EW equation is examined via least squares technique using linear-space finite elements [52]. The development of a train of EW solitary waves induced by boundary forcing is revealed by implementation of Petrov-Galerkin finite element method with shape functions taken as quadratic B-splines [160].



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## Chapter 3

# Comparison of some numerical methods for option pricing problems



In this chapter we, propose two numerical methods for pricing European option pricing problem which is represented by a time dependent parabolic partial differential equation. The first method is based on the semi-discretization by the Method of Lines and then using a finite difference approximation in space where several MATLAB ode solvers are used to perform the time integration. The second one is based on the temporal semi-discretization by implicit Euler and a cubic spline discretization in space. After thorough numerical comparisons, we found that in terms of applicability, the approach based on splines is more flexible than the one based on the method of line.

### 3.1 Introduction

Financial mathematics is a branch of applied mathematics that assesses the risk and value of various financial instruments. Banks, companies, and other institutions mitigate their risk through financial instruments known as derivatives, that derive their value from some underlying asset. These derivatives are often represented by differential equations. However, equations that arise from pricing and modeling can be very complex, and thus leading to the necessity of numerical methods.

The specific derivatives that we are interested to discuss in this chapter are options. An option is a security giving its holder the right to buy or sell an asset, subject to certain conditions, within a specified period of time. If the option for buying the asset, it is called a *Call option* whereas if it is for selling the asset, it is a *Put option*. These options are mainly classified as standard and non-standard options. From these classes, we choose a standard option, namely, European put options to study in this chapter. From the definition of the European option, which states that, a European option can be exercised only on the expiration date, we see that the holder of option has the right without obligation to transact, so the option has some positive value.

Numerical methods in option valuation have been investigated by many researchers. The numerical approaches vary from finite element discretizations [49, 124] to finite difference approximations [155]. A finite-difference scheme often employed is the Crank-Nicolson (CN) scheme (see [155]). The CN scheme employs a classical trapezoidal formula for time integration and second-order central difference formulas for discretization of asset derivatives.

Brennan and Schwartz [18] were the first to describe finite-difference methods for option pricing. Geske and Shastri [58] compared the efficiency of various finite-difference and other numerical methods for option pricing. Vázquez [152] presented a upwind scheme for solving the backward parabolic partial differential equation problem in the case of European options.

Second-order L-stabilized time integration schemes have been proposed by Chawla *et al.* [25]. Chawla *et al.* [26] presented high-accuracy finite-difference methods for the Black-Scholes equation in which they employed the fourth-order L-stable time integration schemes (LSIMP) developed in Chawla *et al.* [27] and the well-known Numerov method for discretization in the asset direction. They compared the computational effi-

ciency of their LSIMP-NUM schemes with the CN and Douglas schemes by considering valuation of European options and American options via the linear complementarity approach.

Company *et al.* [33] constructed a finite difference scheme and the numerical analysis of its solution for a nonlinear Black-Scholes partial differential equation modelling stock option prices in the realistic case when transaction costs arising in the hedging of portfolios are taken into account.

The method of lines is an interesting numerical method for solving partial differential equations. The idea is to semi-discretize the PDE into a system of continuous and interdependent ODEs, which can then be solved by using efficient time integration schemes. However, this method is only suitable for certain classes of partial differential equations, namely initial value problems (IVPs). The pricing of the European options meets this criteria because of its structure in time. An example of an unsuitable partial differential equation would be the standard Laplace equation which does not have any such initial conditions. Our IVP is solved using the MATLAB ode suite [142].

After we study the method of lines, we discuss another class of numerical methods, namely, a cubic spline interpolation. In terms of applicability, the approach based on splines is more flexible than the one based on method of lines.

The rest of the chapter is organized as follows. In Section 3.2, we describe an option pricing problem and show how to reduce it to a simple parabolic problem. The numerical methods are constructed in Section 3.3. Comparative numerical results are presented in Section 3.5 whereas in Section 3.6 we summarize the main outcomes.

## 3.2 Problem description

The value of a European option satisfies the Black–Scholes equation with appropriately specified final and boundary conditions, see, for example, ([143],[155]):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad 0 < S < \infty, 0 < t \leq T. \quad (3.2.1)$$

The parabolic equation (3.2.1) has boundary conditions

$$V(0, t) = V_0(t), \quad V(\infty, t) = V_T(S), \quad (3.2.2)$$

and a final payoff condition

$$V(S, T) = V_T(S), \quad (3.2.3)$$

for given  $V_0(t)$ ,  $V(\infty, t)$  and  $V_T(S)$ .

In the above,  $V = V(S, t)$  denotes the value of a European put option, where  $S$  is the value of the underlying asset at time  $t$ ,  $\sigma$  is the volatility of the underlying asset;  $E$  is the exercise price;  $r$  is the interest rate and  $T$  is the expiry time  $T$ .

We reduce the above problem to a simple parabolic problem.

Note that Black and Scholes had proposed the backwards parabolic equation model (3.2.1) for the valuation of European options with the final condition at  $t = T$

$$V(S, T) = \max(E - S, 0). \quad (3.2.4)$$

The boundary condition at  $S = 0$  satisfies

$$V(S, t) = Ee^{-r(T-t)} - S, \quad (3.2.5)$$

and the boundary condition at  $S = +\infty$  satisfies

$$V(S, t) = 0. \quad (3.2.6)$$

The use of log transformation transforms the Black–Scholes equation to a standard diffusion equation. With the transformations

$$S = Ee^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V(S, t) = E \exp \left[ -\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau \right] u(x, \tau), \quad (3.2.7)$$

and setting  $k = 2r/\sigma^2$  the Black–Scholes equation (3.2.1) is transformed into

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 < \tau \leq \frac{1}{2}\sigma^2 T. \quad (3.2.8)$$

The final condition (3.2.4) is transformed to the initial condition

$$u(x, 0) = f(x) = \max \left( \exp \left[ \frac{1}{2}(k-1)x \right] - \exp \left[ \frac{1}{2}(k+1)x \right], 0 \right) \quad (3.2.9)$$

and the boundary conditions (3.2.5) and (3.2.6) are transformed to

$$u_{-\infty}(\tau) = \exp \left[ \frac{1}{2}(k-1)x_{-\infty} + \frac{1}{4}(k+1)^2\tau \right] \exp \left( -\frac{2r\tau}{\sigma^2} \right), \quad (3.2.10)$$

and

$$u_{\infty}(\tau) = 0. \quad (3.2.11)$$

In next section we explain two different approaches to solve the above reduced problem.

### 3.3 Solving option pricing problem by method of lines

The method of lines (MOL) is used to solve diffusion equations by reducing the problem to an IVP. This is done by introducing approximations for the  $x$ -derivatives, and using initial value methods to solve the resulting problem. The basic idea behind the

MOL is to replace the spatial (boundary-value) derivatives in the PDE with algebraic approximations. Once this is done, the spatial derivatives are no longer stated explicitly in terms of the spatial independent variables. Thus, in effect, only the initial-value variable, typically time in a physical problem, remains. In other words, with only one remaining independent variable, we have a system of ODEs that approximate the original PDE. Once formulating the approximating system of ODEs is done, we can apply any integration algorithm for initial-value ODEs to compute an approximate numerical solution to the PDE. Thus, one of the salient features of the MOL is the use of existing, and generally well-established, numerical methods for IVPs for ODEs.

To proceed with, first we discretize the domain. The infinite interval  $-\infty < x < \infty$  is replaced by a finite interval  $x_{-\infty} \leq x \leq x_{\infty}$ . The end values  $x_{-\infty} = x_{\min} < 0$  and  $x_{\infty} = x_{\max} > 0$  should be chosen in such a way that for  $S_{\min} = Ee^{x_{-\infty}}$ ,  $S_{\max} = Ee^{x_{\infty}}$  and the interval  $S_{\min} \leq S \leq S_{\max}$ , a sufficient smooth approximation can be obtained. Then for a suitable integer  $n$ , the step length in  $x$  is defined by  $\Delta x = h = (x_{\infty} - x_{-\infty})/n$ .

To illustrate the procedure, we carry out the following steps (see [65] for further details) for the diffusion equation (3.2.8).

The first step is to evaluate the equation at  $x = x_i$ . This gives

$$u_{\tau}(x_i, \tau) = u_{xx}(x_i, \tau), \quad 0 \leq \tau \leq \frac{1}{2}\sigma^2 T. \quad (3.3.1)$$

Introducing the centered difference approximation for the spatial derivative, we obtain

$$u_{\tau}(x_i, \tau) = \frac{u(x_{i+1}, \tau) - 2u(x_i, \tau) + u(x_{i-1}, \tau)}{h^2} + \mathcal{O}(h^2). \quad (3.3.2)$$

Dropping the truncation error term, we obtain

$$\frac{d}{d\tau}u_i(\tau) = \frac{u_{i+1}(\tau) - 2u_i(\tau) + u_{i-1}(\tau)}{h^2}, \quad 1 \leq i \leq n-1, \quad (3.3.3)$$

where  $u_i(\tau)$  is the resulting approximation for  $u(x_i, \tau)$ .

Combining all the above steps, we see that the solution to  $u_i(\tau)$  is the solution to the following IVP:

$$f(x) = \max \left( \exp \left[ \frac{1}{2}(k-1)x \right] - \exp \left[ \frac{1}{2}(k+1)x \right], 0 \right), \quad (\text{the initial value}) \quad (3.3.4a)$$

$$\left\{ \begin{array}{l} u_0 = u_{-\infty}(\tau), \quad (\text{the left boundary value}) \\ \left( \frac{du}{d\tau} \right)_1 = \frac{1}{h^2} (u_2 - 2u_1 + u_0), \\ \left( \frac{du}{d\tau} \right)_2 = \frac{1}{h^2} (u_3 - 2u_2 + u_1), \\ \vdots \\ \left( \frac{du}{d\tau} \right)_{n-2} = \frac{1}{h^2} (u_{n-1} - 2u_{n-2} + u_{n-3}), \\ \left( \frac{du}{d\tau} \right)_{n-1} = \frac{1}{h^2} (u_n - 2u_{n-1} + u_{n-2}), \\ u_n = u_{\infty}(\tau). \quad (\text{the right boundary value}). \end{array} \right. \quad (3.3.4b)$$

Solving the above problem, we obtain the approximation for  $u(x_i, \tau)$ .

Collecting the  $u_i$ 's together (excluding the left and the right boundary values), equation (3.3.4b) can be written in a vector form as

$$\frac{d}{dt}\mathbf{u}(t) = \mathbf{C}\mathbf{u}. \quad (3.3.5)$$

where

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n-1}(t) \end{pmatrix} \quad (3.3.6)$$

and

$$\mathbf{C} = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & 0 \\ & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}. \quad (3.3.7)$$

The initial condition  $u(x, 0) = f(x)$  now takes the form

$$\mathbf{u}(0) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{pmatrix}. \quad (3.3.8)$$

Equations (3.3.5)-(3.3.8) represents a standard IVP. Furthermore, we can see that the system is strictly diagonally dominant and hence non-singular. This guarantees the uniqueness of the solution. We can now use a wide variety of IVP solvers to solve the system for  $\mathbf{u}$  and recover the solution by using the transformation (3.2.7) back to  $V(S, t)$ . To this end, in this work, we have used MATLAB solvers *ode45*, *ode15s* and *ode23s*.

### 3.4 Solving option pricing problem by cubic splines

In this section, we present a numerical method which is based on implicit Euler for temporal semi-discretization and then the use of a cubic spline for the discretization in space. We consider a two-dimensional grid as follows: Let  $\Delta\tau$  and  $\Delta x$ , be the mesh step-sizes in the  $\tau$  and  $x$ -directions. The step-size in  $\tau$ -direction is given by  $\Delta\tau = \tau_{\max}/m$  with  $\tau_{\max} = \frac{1}{2}\sigma^2 T$  where  $m$  is an integer. The calculation of the step-size for the  $x$ -discretization is done as in the previous section where the method of lines was applied. Note that the equidistant grid is defined in terms of  $x$  and  $\tau$ , and not for  $S$  and  $t$ . Transforming the  $(x, \tau)$ -grid via the transformation in (3.2.7) back to the  $(S, t)$ -plane, leads to a nonuniform grid with unequal distances of the grid lines  $S = S_i = Ee^{x_i}$ . The actual error is then controlled via the numbers  $n$  and  $m$  of grid lines.

#### Time semi-discretization

Now for temporal discretization, we use finite difference technique with uniform step-size  $\Delta\tau$ , for discretizing equation (3.2.8) and obtain the following system of linear ordinary differential equations:

$$u^0 = f(x), \quad -\infty < x < \infty, \quad (3.4.1a)$$

$$\frac{u^{m+1} - u^m}{\Delta\tau} = u_{xx}^{m+1}, \quad -\infty < x < \infty, \quad \tau > 0, \quad (3.4.1b)$$

with the boundary conditions,

$$u^{m+1}(x_{-\infty}) = u_{-\infty}(\tau^{m+1}), \quad u^{m+1}(x_{\infty}) = u_{\infty}(\tau^{m+1}), \quad (3.4.1c)$$

where  $u^{m+1}$  is the solution of Eq.(3.4.1) at  $(m+1)^{th}$  time level. Here  $u^m = u(x, \tau^m)$ ,  $\Delta\tau$  is the time step-size and the superscript  $m$  denotes  $m^{th}$  time level, i.e.,  $\tau^m = m\Delta\tau$ .

At time level  $m = 0$ , we can rewrite Eq.(3.4.1) as

$$u^0 = f(x), \quad -\infty < x < \infty, \quad (3.4.2a)$$

$$\delta u_{xx}^1 + u^1 = u^0, \quad -\infty < x < \infty, \quad \tau > 0, \quad (3.4.2b)$$

with the boundary conditions,

$$u^1(x_{-\infty}) = u_{-\infty}(\tau^1), \quad u^1(x_{\infty}) = u_{\infty}(\tau^1), \quad (3.4.2c)$$

where  $\delta = -\Delta\tau$ .

The same can be done at all levels. Then at each of these levels, we will use cubic spline approximations to solve the problem in spatial direction. This is explained below.

### Spatial discretization

In this section, we describe the derivation of the cubic spline, in general, as well as in context of our problems.

#### Cubic spline in general

Suppose we have  $n + 1$  points  $x_0, x_1, \dots, x_n$  in the segment  $[a, b]$  which satisfy a grid  $a = x_0 < x_1 < \dots < x_n = b$ . These points are called knots. The points  $x_0$  and  $x_n$  are called end (boundary) knots. The grid above is called uniform if a distance between every two neighboring knots is the same ([144]).

A function  $S(x)$  given on segment  $[a, b]$  is called a spline of type  $p + 1$  (degree  $p$ ) if this function consists of piecewise polynomial which are  $p - 1$  times continuously differentiable on every segment  $\Delta_j = [x_j, x_{j+1}]$ ,  $j = 0, 1, \dots, n - 1$ , that is, we can write

$S(x)$  in the form

$$S(x) = S_j(x) = \sum_{k=0}^p a_k^{(j)}(x - x_j)^k, \quad j = 0, 1, \dots, n - 1, \quad (3.4.3)$$

where  $S(x) \in C^{p-1}[a, b]$ . The condition  $S(x) \in C^{p-1}[a, b]$  means that the function  $S(x)$  and its derivatives  $S'(x), S''(x), \dots, S^{p-1}(x)$  at the points  $x_1, x_2, \dots, x_{n-1}$  are continuously differentiable. There is a separate cubic polynomial for each interval, each with its own coefficients:

$$S_j(x) = a_0^{(j)} + a_1^{(j)}(x - x_j) + a_2^{(j)}(x - x_j)^2 + a_3^{(j)}(x - x_j)^3. \quad (3.4.4)$$

Note that the index  $(j)$  of coefficient  $a_k^{(j)}$  indicates for every partial segment  $\Delta_j$  a system of numbers of the function  $S(x)$  (see, e.g., [144]).

Given a function  $y(x)$  defined on  $[a, b]$  and a set of knots  $a = x_0 < x_1 < \dots < x_n = b$ , a cubic spline interpolant,  $S$ , for  $y(x)$  is a function that satisfies the following conditions ([22]):

- (a)  $S$  is a cubic polynomial denoted by  $S_j$  on the subinterval  $[x_j, x_{j+1}]$  for  $j = 0, 1, \dots, n - 1$ ,
- (b)  $S(x_j) = y(x_j)$  for  $j = 0, 1, \dots, n$ ,
- (c)  $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$  for  $j = 0, 1, \dots, n - 2$ ,
- (d)  $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$  for  $j = 0, 1, \dots, n - 2$ ,
- (e)  $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$  for  $j = 0, 1, \dots, n - 2$ ,
- (f) one of the following set of end (boundary) conditions is satisfied
  1.  $S''(x_0) = S''(x_n) = 0$ , (free or natural boundary),
  2.  $S'(x_0) = y'(x_0)$  and  $S'(x_n) = y'(x_n)$ , (clamped boundary).

When the free boundary conditions occur, the spline is called a natural spline, and it approximately takes the shape of a long elastic rod if forced to go through the data points. In general clamped splines are more accurate approximations since they include more information about the function.

Why do we need the end conditions? In each interval we need to find 4 coefficients to specify the cubic polynomials, and we have  $n$  intervals. We therefore have a total of  $4n$  unknowns to find. The conditions (b) give  $n + 1$  independent equations, and the conditions (c), (d) and (e) give  $3 \times (n - 1)$  independent equations. So we have  $4n$  unknowns and  $4n - 2$  equations. There are two missing equations, and that is why the end (boundary) conditions (f) are required. The conditions (b) are called the interpolation conditions, and the conditions (c), (d) and (e) are called the continuity conditions.

Now we derive the equation for  $S_j(x)$  on the interval  $[x_j, x_{j+1}]$ . First we define the numbers  $z_j = S''(x_j)$ . These  $z_j$  exist for  $0 \leq j \leq n$  and satisfy

$$\lim_{x \rightarrow x_j^-} S''(x) = z_j = \lim_{x \rightarrow x_j^+} S''(x), \quad (1 \leq j \leq n - 1), \quad (3.4.5)$$

because  $S''(x)$  is continuous at each interior knots [102].

Since  $S_j(x)$  is a cubic polynomial on  $[x_j, x_{j+1}]$ ,  $S''(x)$  is a linear function satisfying  $S''_j(x_j) = z_j$  and  $S''_j(x_{j+1}) = z_{j+1}$  and therefore it is given by the straight line between  $z_j$  and  $z_{j+1}$ , i.e.,

$$S''_j(x) = \frac{z_j}{h_j}(x_{j+1} - x) + \frac{z_{j+1}}{h_j}(x - x_j), \quad (3.4.6)$$

where  $h_j = x_{j+1} - x_j$ . Integrating twice, we obtain

$$S_j(x) = \frac{z_j}{6h_j}(x_{j+1} - x)^3 + \frac{z_{j+1}}{6h_j}(x - x_j)^3 + C(x - x_j) + D(x_{j+1} - x), \quad (3.4.7)$$

where  $C$  and  $D$  are the constant of integration. The interpolation conditions  $S_j(x_j) = y_j$  and  $S_j(x_{j+1}) = y_{j+1}$  can be imposed on  $S_j$  to determine  $C$  and  $D$ ; where we use the notation  $y(x_j) = y_j$ . This gives

$$S_j(x) = \frac{z_j}{6h_j}(x_{j+1} - x)^3 + \frac{z_{j+1}}{6h_j}(x - x_j)^3 + \left(\frac{y_{j+1}}{h_j} - \frac{z_{j+1}h_j}{6}\right)(x - x_j) + \left(\frac{y_j}{h_j} - \frac{z_jh_j}{6}\right)(x - x_j). \quad (3.4.8)$$

To determine  $z_1, z_2, \dots, z_{n-1}$ , we use the continuity conditions for  $S'$ . At the interior knots  $x_j$ , we should have  $S'_{j-1}(x_j) = S'_j(x_j)$ . Equation (3.4.8) at  $x = x_j$  gives

$$S'_j(x_j) = -\frac{h_j}{3}z_j - \frac{h_j}{6}z_{j+1} - \frac{y_j}{h_j} + \frac{y_{j+1}}{h_j}, \quad (3.4.9)$$

and

$$S'_{j-1}(x_j) = \frac{h_{j-1}}{6}z_{j-1} + \frac{h_{j-1}}{3}z_j - \frac{y_{j-1}}{h_{j-1}} + \frac{y_j}{h_{j-1}}. \quad (3.4.10)$$

The continuity condition therefore implies

$$h_{j-1}z_{j-1} + 2(h_j + h_{j-1})z_j + h_jz_{j+1} = \frac{6}{h_j}(y_{j+1} - y_j) - \frac{6}{h_{j-1}}(y_j - y_{j-1}), \quad (3.4.11)$$

where  $1 \leq j \leq n - 1$ . It then gives a system of  $n - 1$  linear equations for the  $n + 1$  unknowns  $z_0, z_1, \dots, z_n$ . We can set  $z_0 = 0$  and  $z_n = 0$  corresponds to placing simple supports at the end [1], and solve the resulting system of equations to obtain  $z_1, z_2, \dots, z_{n-1}$ . The resulting spline function is called a natural cubic spline [102]. The linear system of equations (3.4.11) with  $z_0 = 0$  and  $z_n = 0$  is symmetric, tridiagonal, diagonally



where  $S \approx u$ . We substitute  $z_j$  in equations (3.4.11) and obtain

$$\begin{aligned} \frac{1}{\delta} h_{j-1} [f_{j-1} - u_{j-1}] + \frac{2}{\delta} (h_j + h_{j-1}) [f_j - u_j] + \frac{1}{\delta} h_j [f_{j+1} - u_{j+1}] \\ = \frac{6}{h_j} u_{j+1} - \frac{6}{h_j} u_j - \frac{6}{h_{j-1}} u_j + \frac{6}{h_{j-1}} u_{j-1}, \end{aligned} \quad (3.4.15)$$

which upon simplifications leads to

$$\begin{aligned} \left[ \frac{-h_{j-1}}{\delta} - \frac{6}{h_{j-1}} \right] u_{j-1} + \left[ \frac{-2(h_j + h_{j-1})}{\delta} + \frac{6}{h_j} + \frac{6}{h_{j-1}} \right] u_j + \left[ \frac{-h_j}{\delta} - \frac{6}{h_j} \right] u_{j+1} \\ = -\frac{h_{j-1}}{\delta} f_{j-1} - \frac{-2(h_j + h_{j-1})}{\delta} f_j - \frac{h_j}{\delta} f_{j+1}. \end{aligned}$$

Multiplying by  $-\delta$ , we have for  $1 \leq j \leq n-1$ :

$$\begin{aligned} \left[ h_{j-1} + \frac{6\delta}{h_{j-1}} \right] u_{j-1} + \left[ 2(h_j + h_{j-1}) - \frac{6\delta}{h_j} - \frac{6\delta}{h_{j-1}} \right] u_j + \left[ h_j + \frac{6\delta}{h_j} \right] u_{j+1} \\ = h_{j-1} f_{j-1} + 2(h_j + h_{j-1}) f_j + h_j f_{j+1}. \end{aligned} \quad (3.4.16)$$

By choosing a uniform mesh spacing  $h$ , equation (3.4.16) becomes

$$\begin{aligned} \left[ h + \frac{6\delta}{h} \right] u_{j-1} + \left[ 4h - \frac{12\delta}{h} \right] u_j + \left[ h + \frac{6\delta}{h} \right] u_{j+1} \\ = h f_{j-1} + 4h f_j + h f_{j+1}, \end{aligned} \quad (3.4.17)$$

or

$$\gamma_j^- y_{j-1} + \gamma_j^c y_j + \gamma_j^+ y_{j+1} = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}, \quad (3.4.18)$$

where

$$\gamma_j^- = h + \frac{6\delta}{h},$$

$$\gamma_j^c = 4h - \frac{12\delta}{h},$$

$$\gamma_j^+ = h + \frac{6\delta}{h},$$

$$q_j^- = h,$$

$$q_j^c = 4h,$$

$$q_j^+ = h.$$

Equation (3.4.18) gives a system of  $n-1$  linear equations for the unknowns  $u_1, u_2, \dots, u_{n-1}$  with  $u_0 = u_{-\infty}(\tau)$  and  $u_n = u_{\infty}(\tau)$  of the form

$$\mathbf{A}\mathbf{u} = \mathbf{q}, \tag{3.4.19}$$

where

$$\mathbf{A} = \begin{bmatrix} \gamma_1^c & \gamma_1^+ & & & \\ \gamma_2^- & \gamma_2^c & \gamma_2^+ & & \\ & \gamma_3^- & \gamma_3^c & \gamma_3^+ & \\ & & \ddots & \ddots & \ddots \\ & & & \gamma_{n-2}^- & \gamma_{n-2}^c & \gamma_{n-2}^+ \\ & & & & \gamma_{n-1}^- & \gamma_{n-1}^c \end{bmatrix}, \tag{3.4.20}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix}, \tag{3.4.21}$$

and

$$\mathbf{q} = \begin{bmatrix} q_1^- f_0 + q_1^c f_1 + q_1^+ f_2 - \gamma_1^- u_0 \\ q_2^- f_1 + q_2^c f_2 + q_2^+ f_3 \\ q_3^- f_2 + q_3^c f_3 + q_3^+ f_4 \\ \vdots \\ q_{n-2}^- f_{n-3} + q_{n-2}^c f_{n-2} + q_{n-2}^+ f_{n-1} \\ q_{n-1}^- f_{n-2} + q_{n-1}^c f_{n-1} + q_{n-1}^+ f_n - \gamma_{n-1}^+ u_n \end{bmatrix}. \quad (3.4.22)$$

We can see that the system is strictly diagonally dominant and hence non-singular. Hence this method applied to the problem above using a basis of cubic spline has a unique solution. It should be noted that at each time level we solve the system (3.4.19) to get the solution of equation (3.2.8).

### 3.5 Numerical simulations and results

In this section, we present some numerical results for the solution of Black-Scholes equation describing European put option. The values  $V(S, t)$  can be interpreted as a piece of surface over the subset  $S > 0$ ,  $0 \leq t \leq T$  of the  $(S, t)$ -plane. We use the following parameters for numerical simulations:

Expiration date	$T = 0.5$ (year)
Exercise price	$E = 10.0$
Risk free interest rate	$r = 0.05$
Volatility	$\sigma = 0.2$
Number of equations	$m = 100$

#### 3.5.1 Numerical results using method of lines

Figure 3.5.1 illustrates the character of this surface for the European put option for the fixed values of  $E$ ,  $T$ ,  $r$  and  $\sigma$ . Through Figure 3.5.2, we explain that the European put option can take values above the lower bound  $Ee^{-r(T-t)} - S$ . For small values of  $S$  the value  $V$  approaches its lower bound.

In Table 3.5.1 we have tabulated the comparative results; it consists of the exact, Quasi-RBFs and MOL solutions for the European put option.

In Table 3.5.2 we have tabulated the exact solution, B-spline solution and solution obtained by method of lines along with MATLAB solver *ode45* for a European put option. Note that here we only put results for B-splines. We compute results using B-splines with the parameters given above along with  $\Delta t = 10^{-5}$  and  $\Delta x = 0.005$ . The actual error is controlled via the numbers  $n$  and  $m$ .

In Table 3.5.3 we have tabulated the exact solution and those obtained by using method of lines along with MATLAB solvers *ode45*, *ode15s* and *ode23s*.

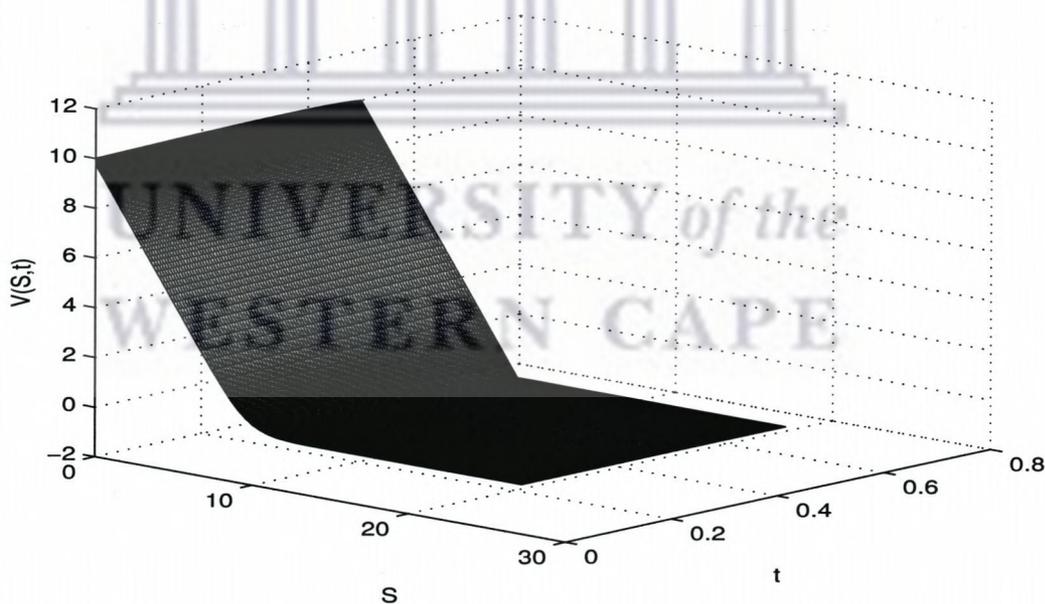


Figure 3.5.1: Values of European put option obtained by using method of lines for  $T = 6/12$ ,  $E = 10$ ,  $r = 0.05$ ,  $\sigma = 0.20$  with  $\Delta x = 0.05$ ,  $x \in (-10, 1)$

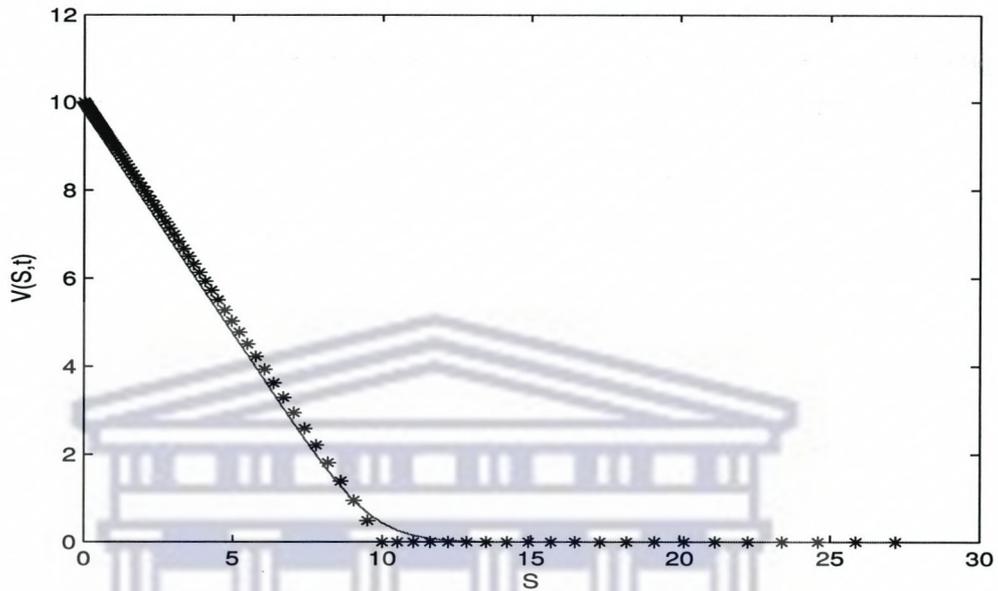


Figure 3.5.2: Values of European put option at  $t = 0$  using method of lines for  $T = 6/12$ ,  $r = 0.05$ ,  $\sigma = 0.20$  with  $\Delta x = 0.05$ . The curve with '\*' shows payoff whereas the solid curve represents the value of the option.

Table 3.5.1: Comparison between the exact solution, Quasi-RBF solution [67] and solution obtained by method of lines along with MATLAB solver *ode45* for a European put option for two different space step-sizes

$S$	Exact solution	Quasi-RBF solution [67]	MOL solutions	
			$\Delta x = 0.01$	$\Delta x = 0.005$
2.00	7.7531	7.7531	7.7531	7.7531
4.00	5.7531	5.7531	5.7531	5.7531
6.00	3.7532	3.7532	3.7532	3.7532
7.00	2.7568	2.7568	2.7569	2.7568
8.00	1.7987	1.7988	1.7988	1.7987
9.00	0.9880	0.9881	0.9881	0.9880
10.00	0.4420	0.4420	0.4416	0.4419
11.00	0.1606	0.1606	0.1607	0.1606
12.00	0.0483	0.0483	0.0484	0.0484
13.00	0.0124	0.0124	0.0124	0.0124
14.00	0.0028	0.0028	0.0028	0.0028
15.00	0.0006	0.0006	0.0006	0.0006
16.00	0.0001	0.0001	0.0001	0.0001

Table 3.5.2: Comparison between the exact solution, B-spline solution and solution obtained by method of lines along with MATLAB solver *ode45* for a European put option for two different space step-sizes

$S$	Exact solution	B-spline solution	MOL solutions	
			$\Delta x = 0.01$	$\Delta x = 0.005$
2.00	7.7531	7.7531	7.7531	7.7531
4.00	5.7531	5.7531	5.7531	5.7531
6.00	3.7532	3.7532	3.7532	3.7532
7.00	2.7568	2.7568	2.7569	2.7568
8.00	1.7987	1.7987	1.7988	1.7987
9.00	0.9880	0.9880	0.9881	0.9880
10.00	0.4420	0.4419	0.4416	0.4419
11.00	0.1606	0.1606	0.1607	0.1606
12.00	0.0483	0.0484	0.0484	0.0484
13.00	0.0124	0.0124	0.0124	0.0124
14.00	0.0028	0.0028	0.0028	0.0028
15.00	0.0006	0.0006	0.0006	0.0006
16.00	0.0001	0.0001	0.0001	0.0001

Table 3.5.3: Comparison between the exact solution and solution obtained by method of lines along with different MATLAB solvers for the European put option.

$S$	Exact solution	MOL solutions with $\Delta x = 10^{-3}$		
		<i>ode45</i>	<i>ode15s</i>	<i>ode23s</i>
2.00	7.7531	7.7531	7.7531	7.7531
4.00	5.7531	5.7531	5.7531	5.7531
6.00	3.7532	3.7532	3.7532	3.7532
7.00	2.7568	2.7568	2.7568	2.7569
8.00	1.7987	1.7987	1.7987	1.7987
9.00	0.9880	0.9880	0.9880	0.9880
10.00	0.4420	0.4419	0.4419	0.4419
11.00	0.1606	0.1606	0.1606	0.1606
12.00	0.0483	0.0484	0.0484	0.0483
13.00	0.0124	0.0124	0.0124	0.0124
14.00	0.0028	0.0028	0.0028	0.0028
15.00	0.0006	0.0006	0.0006	0.0006
16.00	0.0001	0.0001	0.0001	0.0001

### 3.5.2 Numerical results using cubic spline

Figure 3.5.3 illustrates the character of this surface for the European put option for the fixed values of  $E$ ,  $T$ ,  $r$  and  $\sigma$ . Through Figure 3.5.4, we explain that the European put option can take values above the lower bound  $Ee^{-r(T-t)} - S$ . For small values of  $S$  the value  $V$  approaches its lower bound.

In Table 3.5.4, we have tabulated the comparative results. It consists of the exact, B-spline and cubic spline solutions for the European put option for  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ , with  $\Delta t = 10^{-5}$  and  $\Delta x = 0.005$ . Note that the results obtained by cubic spline and B-spline are exactly the same. In Table 3.5.5 we have tabulated the exact, B-spline and cubic spline solutions for the European put option for  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ , with  $\Delta t = 10^{-5}$  and  $\Delta x = 0.008$ . Also the results obtained by cubic spline and B-spline are exactly the same. The actual error is controlled via the numbers  $n$  and  $m$ .

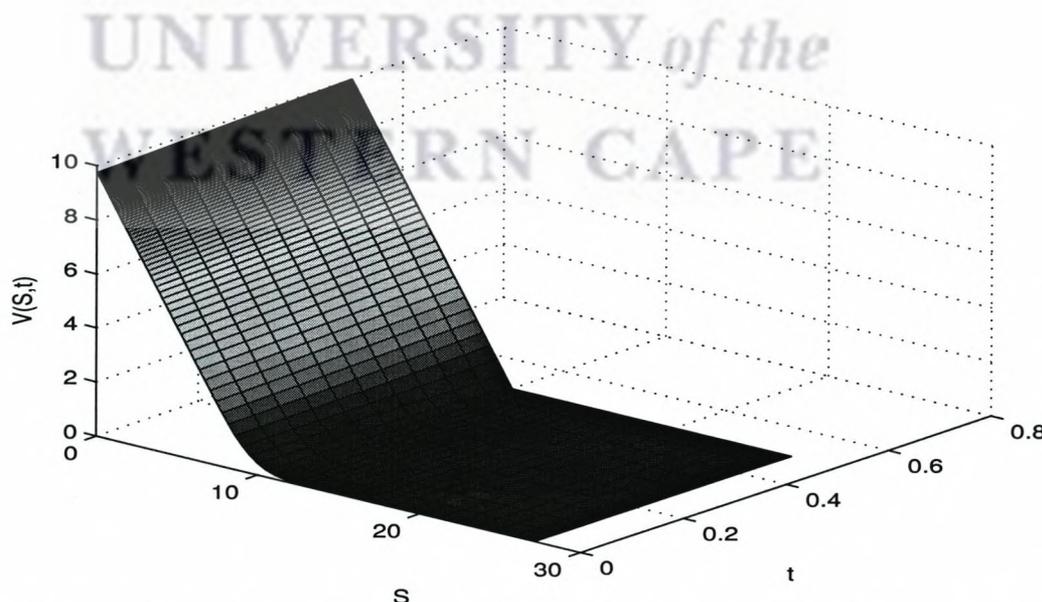


Figure 3.5.3: Values of European put option obtained by using cubic spline for  $T = 6/12$ ,  $E = 10$ ,  $r = 0.05$ ,  $\sigma = 0.20$  with  $\Delta\tau = 0.001$ , and  $\Delta x = 0.05$ ,  $x \in (-10, 1)$

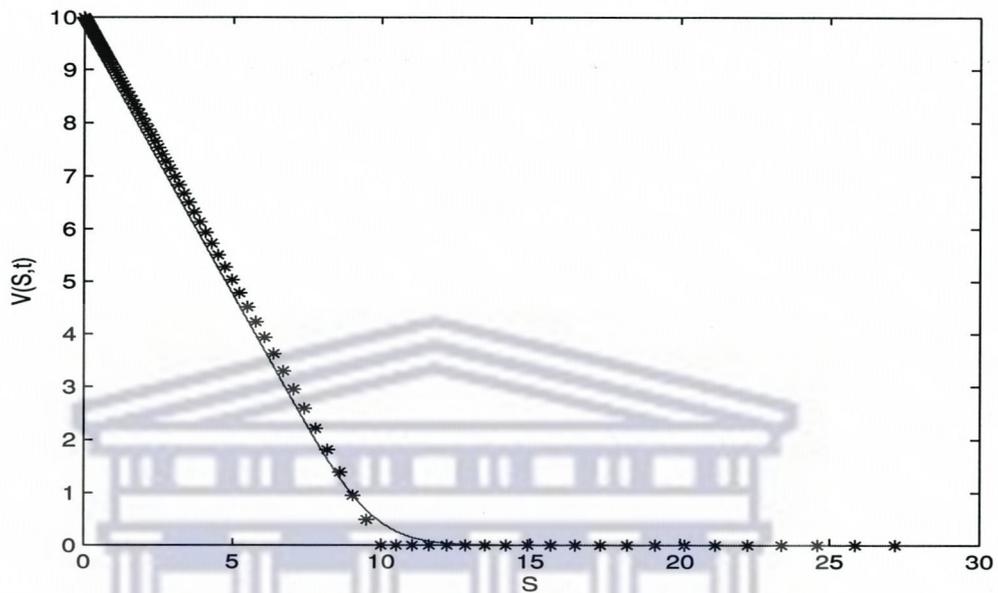


Figure 3.5.4: Values of European put option at  $t = 0$  using cubic spline for  $T = 6/12$ ,  $r = 0.05$ ,  $\sigma = 0.20$  with  $\Delta\tau = 0.001$ , and  $\Delta x = 0.05$ . The curve with '\*' shows payoff whereas the solid curve represents the value of the option.

Table 3.5.4: Comparison between the exact, B-spline and the cubic spline solutions for the European put option for  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ . With  $\Delta x = 0.0050$  and  $\Delta t = 10^{-5}$ .

$S$	Exact solution	B-spline solution	Cubic spline solution
2.00	7.7531	7.7531	7.7531
4.00	5.7531	5.7531	5.7531
6.00	3.7532	3.7532	3.7532
7.00	2.7568	2.7568	2.7568
8.00	1.7987	1.7987	1.7987
9.00	0.9880	0.9880	0.9880
10.00	0.4420	0.4419	0.4419
11.00	0.1606	0.1606	0.1606
12.00	0.0483	0.0484	0.0484
13.00	0.0124	0.0124	0.0124
14.00	0.0028	0.0028	0.0028
15.00	0.0006	0.0006	0.0006
16.00	0.0001	0.0001	0.0001

Table 3.5.5: Comparison between the exact, B-spline and the cubic spline solutions for the European put option for  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ . With  $\Delta x = 0.008$  and  $\Delta t = 10^{-5}$ .

$S$	Exact solution	B-spline solution	Cubic spline solution
2.00	7.7531	7.7531	7.7531
4.00	5.7531	5.7531	5.7531
6.00	3.7532	3.7532	3.7532
7.00	2.7568	2.7568	2.7568
8.00	1.7987	1.7987	1.7987
9.00	0.9880	0.9880	0.9880
10.00	0.4420	0.4418	0.4418
11.00	0.1606	0.1606	0.1606
12.00	0.0483	0.0483	0.0483
13.00	0.0124	0.0124	0.0124
14.00	0.0028	0.0028	0.0028
15.00	0.0006	0.0006	0.0006
16.00	0.0001	0.0001	0.0001

### 3.6 Summary and discussions

In this chapter, we studied two classes of numerical methods for pricing European option pricing problem which is represented by a time dependent parabolic partial differential equation. The first method is based on the semi-discretization by the Method of Lines and then using a finite difference approximation in space where several MATLAB ode solvers are used to perform the time integration. The second one is based on the temporal semi-discretization by implicit Euler and a cubic spline discretization in space. As it is seen from the tabular results, in each case we obtained the results that can be compared with those seen in the literature.

## Chapter 4

# B-spline approximation method for pricing European options

In this chapter, we construct a numerical method based on spline approximations to solve a nonlinear Black-Scholes partial differential equation modelling European option pricing problem on a single asset. We use the classical Euler implicit method for the time-discretization and a B-spline collocation method for the spatial discretization. The method is shown to be unconditionally stable and accurate of order  $\mathcal{O}((\Delta x)^2 + \Delta \tau)$ . The computational performance of the proposed scheme is compared with those obtained by using a scheme based on the quasi-radial basis functions.

### 4.1 Introduction

It is very important for financial institutions operating in the over-the-counter market to have accurate models in order to determine what price to charge for these individually-tailored contracts and how they hedge the risk exposure arising from their sale. Options can be used, for instance, to hedge assets and portfolios in order to control the risk due to movements in the share price. In an idealized financial market the price of a European option can be obtained as the solution of the celebrated Black-Scholes equation [9, 117]. This equation also provides a hedging portfolio that

replicates the contingent claim. The Black-Scholes equation has been derived under quite restrictive assumptions. In recent years, nonlinear Black-Scholes equations have been derived in order to model, transaction costs arising in the hedging of portfolios [6, 43], feedback effects due to large traders [51, 125, 138], and incomplete markets [62].

As far as their numerical solutions are concerned, a few results are seen in the literature on the numerical discretization of linear Black-Scholes equation. The numerical approaches vary from finite element discretizations [49, 124] to finite difference approximations [155]. A finite-difference scheme often employed is the Crank-Nicolson (CN) scheme (see [155]). The CN scheme employs a classical trapezoidal formula for time integration and second-order central difference formulas for discretization of asset derivatives. Second-order L-stabilized time integration schemes have been proposed by Chawla *et al.* [25]. Chawla *et al.* [26] presented high-accuracy finite-difference methods for the Black-Scholes equation in which they employed the fourth-order L-stable time integration schemes (LSIMP) developed in Chawla *et al.* [27] and the well-known Numerov method for discretization in the asset direction. They compared the computational efficiency of their LSIMP-NUM schemes with the CN and Douglas schemes by considering valuation of European options and American options via the linear complementarity approach. Company *et al.* [33] constructed a finite difference scheme and the numerical analysis of its solution for a nonlinear Black-Scholes partial differential equation modelling stock option prices in the realistic case when transaction costs arising in the hedging of portfolios are taken into account.

Hon [67] developed a numerical method for solving the Black-Scholes equation for valuation of American options where he has used the concept of quasi-interpolation and radial basis functions (RBFs) approximation.

In this chapter, we develop a numerical method based on the B-spline collocation approach to solve European option problems. Such methods have become interesting

and very promising in solving partial differential equations, due to their flexibility in practical applications. A lot of work has been done using B-splines in other fields of sciences and engineering. The B-spline functions are used as window functions to construct a reproducing kernel function in the reproducing kernel methods and meshfree particle methods [7, 28, 29, 30, 108, 110, 158]. B-splines are also used as basis functions in the finite element methods [2, 3, 66, 104].

The rest of the chapter is organized as follows. In Section 4.2, we describe the option pricing problem. The numerical method is constructed in Section 4.3. This method is analyzed for convergence in Section 4.4. Comparative numerical results are presented in Section 4.5 whereas Section 4.6 deals with some concluding remarks and scope for future research.

## 4.2 Problem description

The value of a European option satisfies the Black–Scholes equation with appropriately specified final and boundary conditions, see, for example, [143, 155]. We denote its value by  $V = V(S, t)$ , where  $S$  is the current value of the underlying asset and  $t$  is the time. The variables  $S$  and  $t$  are independent. The value of the option also depends on the volatility of the underlying asset  $\sigma$ , the exercise price  $E$ , the expiry time  $T$ , and the risk-free interest rate  $r$ . With these notations, the governing Black-Scholes equation for  $V(S, t)$  on a single asset is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (4.2.1)$$

The domain of the independent variables  $S, t$  is  $(0, \infty) \times (0, T)$ . For a European put option exercising only make sense in case  $S < E$ . The payoff  $V(S, T)$  of a put at

expiration time  $T$  is

$$V(S, T) = \begin{cases} E - S & \text{for } S < E \text{ (option is exercised)} \\ 0 & \text{for } S \geq E \text{ (option is worthless),} \end{cases}$$

and hence the final condition at  $t = T$  is

$$V(S, T) = \max(E - S, 0), \quad (4.2.2)$$

the boundary condition at  $S = 0$  satisfies ([141])

$$V(S, t) = Ee^{-r(T-t)} - S, \quad (4.2.3)$$

and the boundary condition at  $S = +\infty$  satisfies

$$V(S, t) = 0. \quad (4.2.4)$$

In order to construct our numerical method, we first reduce the above problem to a standard diffusion equation. The use of log transformation transform the Black-Scholes equation as a standard diffusion equation. With the transformations

$$S = Ee^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V(S, t) = E \exp \left[ -\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau \right] u(x, \tau), \quad (4.2.5)$$

and setting  $k = 2r/\sigma^2$  the Black-Scholes equation (4.2.1) is transformed into

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 < \tau < \frac{1}{2}\sigma^2 T. \quad (4.2.6)$$

The final condition (4.2.2) is transformed to

$$u(x, 0) = u^0(x) = \max \left( \exp \left[ \frac{1}{2}(k-1)x \right] - \exp \left[ \frac{1}{2}(k+1)x \right], 0 \right) \quad (4.2.7)$$

whereas the boundary conditions (4.2.3) and (4.2.4) are transformed to

$$u_{-\infty}(\tau) = \exp \left[ \frac{1}{2}(k-1)x_{-\infty} + \frac{1}{4}(k+1)^2\tau \right] \exp \left( -\frac{2r\tau}{\sigma^2} \right), \quad (4.2.8)$$

and

$$u_{\infty}(\tau) = 0, \quad (4.2.9)$$

respectively. We solve problem (4.2.6)-(4.2.9) using B-spline for space discretization and implicit Euler for time discretization as described in the next section and then recover the solution of the original problem (4.2.1)-(4.2.4) using (4.2.5).

### 4.3 Construction of the numerical method

Our numerical method is based on B-spline for the discretization in space and the finite difference techniques for the temporal one. We consider a two-dimensional grid as follows: Let  $\Delta\tau$  and  $\Delta x$ , be the mesh step-sizes in the  $\tau$  and  $x$ -directions. The step-size in  $\tau$ -direction is given by  $\Delta\tau = \tau_{\max}/m$  with  $\tau_{\max} = \frac{1}{2}\sigma^2T$  where  $m$  is an integer. The calculation of the step-size for the  $x$ -discretization is little complicated. The infinite interval  $-\infty < x < \infty$  must be replaced by a finite interval  $x_{-\infty} \leq x \leq x_{\infty}$ . Here the end values  $x_{-\infty} = x_{\min} < 0$  and  $x_{\infty} = x_{\max} > 0$  should be chosen in such a way that for  $S_{\min} = Ee^{x_{-\infty}}$ ,  $S_{\max} = Ee^{x_{\infty}}$  and the interval  $S_{\min} \leq S \leq S_{\max}$ , a sufficient quality of approximation is obtained. Then for a suitable integer  $n$  the step length in  $x$  is defined by  $\Delta x = (x_{\infty} - x_{-\infty})/n$ . This defines a two-dimensional uniform grid. Note that the equidistant grid is defined in terms of  $x$  and  $\tau$ , and not for  $S$  and  $t$ . Transforming the  $(x, \tau)$ -grid via the transformation in (4.2.5) back to the  $(S, t)$ -plane, leads to a nonuniform grid with unequal distances of the grid lines  $S = S_i = Ee^{x_i}$ . The actual error is then controlled via the numbers  $n$  and  $m$  of grid lines.

Now for temporal discretization, we use finite difference technique with uniform step-size  $\Delta\tau$ , for discretizing equation (4.2.6) and obtain the following system of linear

ordinary differential equations:

$$u^0 = u^0(x), \quad -\infty < x < \infty, \quad (4.3.1a)$$

$$\frac{u^{m+1} - u^m}{\Delta\tau} = u_{xx}^{m+1}, \quad -\infty < x < \infty, \quad \tau > 0 \quad (4.3.1b)$$

with the boundary conditions,

$$u^{m+1}(x_{-\infty}) = u_{-\infty}(\tau^{m+1}), \quad u^{m+1}(x_{\infty}) = u_{\infty}(\tau^{m+1}), \quad (4.3.1c)$$

where  $u^{m+1}$  is the solution of Eq.(4.2.6) at  $(m+1)$ th time level. Here  $u^m = u(x, \tau^m)$ ,  $\Delta\tau$  is the time step-size and the superscript  $m$  denotes  $m$ th time level, i.e.,  $\tau^m = m\Delta\tau$ .

We can rewrite Eq.(4.3.1b) as

$$-\delta u_{xx}^{m+1} + u^{m+1} = u^m, \quad (4.3.2)$$

where  $\delta = \Delta\tau$ .

The spatial discretization is done as follows. Given  $n$  distinct knots  $x_1 < x_2 < \dots < x_n$  in the open interval  $(a, b)$  and an integer  $k \geq 1$ , let  $S^k(x)$  be the space of functions of class  $C^{k-1}$  over  $[a, b]$  which coincide with polynomials of degree at most  $k$  on each interval  $[x_j, x_{j+1}]$ , for  $1 \leq j \leq n$ , with  $x_0 = a$  and  $x_{n+1} = b$ . The space  $S^k(x)$  is called the space of splines of degree  $k$  [134]. A function of the form

$$B_j^{(0)}(x) = \begin{cases} 1, & x \in [x_j, x_{j+1}] \\ 0, & x \notin [x_j, x_{j+1}] \end{cases}$$

is called a B-spline function of degree zero defined on segment  $[a, b]$ . The B-spline function of degree  $k \geq 1$  defined on segment  $[a, b]$  is constructed by the recurrent

relation

$$B_j^{(k)}(x) = \frac{x - x_j}{x_{j+1} - x_j} B_j^{(k-1)}(x) + \frac{x_{j+k+1} - x}{x_{j+k+1} - x_{j+1}} B_{j+1}^{(k-1)}(x). \quad (4.3.3)$$

To adopt the above approach for our case, let the approximate solution of problem (4.3.2) be given in the form of the B-spline. To proceed with, we subdivide the interval  $[a, b]$ , and we choose uniformly distributed mesh points represented by  $\omega = \{x_0, x_1, x_2, \dots, x_n\}$ , such that  $x_0 = a$ ,  $x_n = b$  and  $h$  is the uniform spacing between two mesh points. We then define the cubic B-spline for  $i = 1, 2, \dots, n$  as

$$B_i(x) = \begin{cases} \left(\frac{x-x_{i-2}}{h}\right)^3, & \text{if } x \in [x_{i-2}, x_{i-1}], \\ 1 + 3\left(\frac{x-x_{i-1}}{h}\right) + 3\left(\frac{x-x_{i-1}}{h}\right)^2 - 3\left(\frac{x-x_{i-1}}{h}\right)^3, & \text{if } x \in [x_{i-1}, x_i], \\ 1 + 3\left(\frac{x_{i+1}-x}{h}\right) + 3\left(\frac{x_{i+1}-x}{h}\right)^2 - 3\left(\frac{x_{i+1}-x}{h}\right)^3, & \text{if } x \in [x_i, x_{i+1}], \\ \left(\frac{x_{i+2}-x}{h}\right)^3, & \text{if } x \in [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise.} \end{cases} \quad (4.3.4)$$

We introduce four additional knots as  $x_{-2} < x_{-1} < x_0$  and  $x_{n+2} > x_{n+1} > x_n$ . From equation (4.3.4) we see that each of the functions  $B_i(x)$  are twice continuously differentiable on the entire real line. Also

$$B_i(x_j) = \begin{cases} 4, & \text{if } i = j, \\ 1, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2, \end{cases} \quad (4.3.5)$$

and that  $B_i(x) = 0$  for  $x \geq x_{i+2}$  and  $x \leq x_{i-2}$ . Furthermore, we can show that

$$B_i'(x_j) = \begin{cases} 0, & \text{if } i = j, \\ \pm \frac{3}{h}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2, \end{cases} \quad (4.3.6)$$

and

$$B_i''(x_j) = \begin{cases} -\frac{12}{h^2}, & \text{if } i = j, \\ \frac{6}{h^2}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2. \end{cases} \quad (4.3.7)$$

Let  $\Omega = \{B_{-1}, B_0, B_1, \dots, B_{n+1}\}$ . The functions in  $\Omega$  are linearly independent on  $[a, b]$ .

Now we define

$$S(x) = \sum_{i=-1}^{n+1} c_i B_i(x), \quad (4.3.8)$$

where  $c_i$  are unknown real coefficients and  $B_i(x)$  are cubic B-spline functions. Here we have introduced two extra splines  $B_{-1}$  and  $B_{n+1}$  to force  $S(x)$  to satisfy the boundary conditions. Then let  $S(x)$  satisfy the equation (4.3.2). We have

$$LS(x_j) = f(x_j), \quad 0 \leq x_j \leq n, \quad f(x_j) = u^m(x_j), \quad (4.3.9)$$

where  $Lu^{m+1} \equiv -\delta u_{xx}^{m+1} + u^{m+1}$ , therefore

$$-\delta \sum_{i=-1}^{n+1} c_i B_i''(x_j) + \sum_{i=-1}^{n+1} c_i B_i(x_j) = f_j, \quad f_j = f(x_j).$$

By solving the above equation and noting that the support of the function  $B_i(x)$  is the segment  $[x_{i-2}, x_{i+2}]$ , we have

$$\begin{aligned} & c_{j-1}(-\delta B_{j-1}''(x_j) + B_{j-1}(x_j)) + c_j(-\delta B_j''(x_j) + B_j(x_j)) \\ & + c_{j+1}(-\delta B_{j+1}''(x_j) + B_{j+1}(x_j)) = f_j, \quad \forall j = 0, 1, \dots, n, \end{aligned} \quad (4.3.10)$$

by using equations (4.3.5) and (4.3.7) we get

$$(h^2 - 6\delta)c_{j-1} + (4h^2 + 12\delta)c_j + (h^2 - 6\delta)c_{j+1} = h^2 f_j, \quad \forall j = 0, 1, \dots, n. \quad (4.3.11)$$

The given boundary conditions (4.2.8) and (4.2.9) becomes

$$c_{-1} + 4c_0 + c_1 = u_{-\infty}(\tau_0), \quad (4.3.12)$$

and

$$c_{n-1} + 4c_n + c_{n+1} = 0. \quad (4.3.13)$$

Equations (4.3.11), (4.3.12) and (4.3.13) lead to an  $(n+3) \times (n+3)$  tridiagonal system with  $(n+3)$  unknowns  $c_{-1}, c_0, \dots, c_{n+1}$ . By eliminating  $c_{-1}$  from the first equation of (4.3.11) and (4.3.12), we get

$$36\delta c_0 = f_0 h^2 - (h^2 - 6\delta)u_{-\infty}(\tau_0). \quad (4.3.14)$$

Similarly, eliminating  $c_{n+1}$  from the last equation of (4.3.11) and (4.3.13), we get

$$36\delta c_n = f_n h^2. \quad (4.3.15)$$



## 4.4 Analysis of the numerical method

For the sake of simplicity, let us denote  $L_1 \equiv -\frac{\partial^2}{\partial x^2}$ . As we have mentioned earlier, we discretize the time variable by implicit Euler with uniform step-size  $\Delta\tau$ , therefore we rewrite (4.3.1) as

$$u^0 = u^0(x), \quad -\infty < x < \infty, \quad (4.4.1a)$$

$$(I + \Delta\tau L_1)u^{m+1} = u^m, \quad -\infty < x < \infty, \quad \tau > 0, \quad (4.4.1b)$$

$$u^{m+1}(x_{-\infty}) = u_{-\infty}(\tau^{m+1}), \quad u^{m+1}(x_{\infty}) = u_{\infty}(\tau^{m+1}), \quad (4.4.1c)$$

which gives semi-discrete approximations  $u^m(x)$ , at time levels  $\tau_m = m\Delta t$ , to the exact solution  $u(x, \tau)$  of (4.2.6). The stability of (4.4.1) follows from the maximum principle for the operator  $I + \Delta\tau L_1$ , because

$$\|(I + \Delta\tau L_1)^{-1}\|_{\infty} \leq \frac{1}{1 + \bar{b}\Delta\tau}. \quad (4.4.2)$$

The local truncation error of the time semi-discretization method (4.4.1) is given by  $e_{m+1} = u(\tau^{m+1}) - \hat{u}^{m+1}$ , where  $\hat{u}^{m+1}$  is the solution of

$$(I + \Delta\tau L_1)\hat{u}^{m+1}(x) = u(x, \tau^m), \quad -\infty < x < \infty, \quad \tau > 0, \quad (4.4.3a)$$

$$\hat{u}^{m+1}(x_{-\infty}) = u_{-\infty}(\tau^{m+1}), \quad \hat{u}^{m+1}(x_{\infty}) = u_{\infty}(\tau^{m+1}). \quad (4.4.3b)$$

This error measures the contribution at each time step to the global error of the time semi-discretization which is defined as  $E_m \equiv u(x, \tau^m) - u^m(x)$ .

Now we show that the following accuracy results hold:

**Lemma 4.4.1** (*Local error estimate*) *If*

$$\left| \frac{\partial^i}{\partial t^i} u(x, \tau) \right| \leq C_0, \quad -\infty < x < \infty, \quad 0 < \tau < \frac{1}{2}\sigma^2 T, \quad 0 \leq i \leq 2, \quad (4.4.4)$$

then the local error satisfies

$$\|e_{m+1}\|_{\infty} \leq C_0(\Delta\tau)^2, \quad (4.4.5)$$

where  $C_0$  is a positive constant independent of  $\Delta\tau$ .

**Proof.** Since the function  $\hat{u}^{m+1}$  satisfies

$$(I + \Delta\tau L_1)\hat{u}^{m+1}(x) = u(x, \tau^m),$$

and as the solution of (4.2.6) is smooth enough, we have

$$u(\tau^m) = u(\tau^{m+1}) + \Delta\tau L_1 u(\tau^{m+1}) + \int_{\tau^m}^{\tau^{m+1}} (\tau^m - s) \frac{\partial^2 u}{\partial \tau^2}(s) ds = (I + \Delta\tau L_1)u^{m+1}(x) + \mathcal{O}(\Delta\tau^2).$$

Then,  $e_{m+1}$  is the solution of a boundary value problem of type

$$(I + \Delta\tau L_1)e_{m+1} = \mathcal{O}(\Delta\tau^2),$$

$$e_{m+1}(x_{-\infty}) = e_{m+1}(x_{\infty}) = 0,$$

and now (4.4.5) follows when applying the stability result (4.4.2).

**Theorem 4.4.1** (*Global error estimate*). *Under the hypotheses of Lemma 4.4.1, we have*

$$\|E_m\|_{\infty} \leq C_0\Delta\tau, \quad \forall m \leq \frac{\sigma^2 T}{2\Delta\tau}. \quad (4.4.6)$$

**Proof.** Using the local error estimate up to the  $m$ th time level given by Lemma (4.4.1), we get the following global error estimate at  $(m + 1)$ th time level

$$\begin{aligned}
 \|E_{m+1}\|_{\infty} &= \left\| \sum_{l=1}^m e_l \right\|, \quad m \leq \frac{\sigma^2 T}{2\Delta\tau} \\
 &\leq \|e_1\|_{\infty} + \|e_2\|_{\infty} + \dots + \|e_m\|_{\infty} \\
 &\leq C_1(m\Delta\tau)\Delta\tau \quad \text{using (4.4.5)} \\
 &\leq C_1 \left( \frac{1}{2}\sigma^2 T \right) \Delta\tau \quad \text{since } m\Delta\tau \leq \frac{1}{2}\sigma^2 T \\
 &= C_0\Delta\tau. \tag{4.4.7}
 \end{aligned}$$

Therefore the time semi-discretization process is uniformly convergent of order one. □

Now we prove that the B-spline collocation method convergent of order two in the spatial domain. To proceed with, we first prove the following lemma:

**Lemma 4.4.2** *The B-splines  $B_{-1}, B_0, B_1, \dots, B_{n+1}$  defined in Eq.(4.3.4), satisfy the inequality*

$$\sum_{i=-1}^{n+1} |B_i(x)| \leq 10, \quad x_{-\infty} \leq x \leq x_{\infty}.$$

**Proof.** We know that

$$\left| \sum_{i=-1}^{n+1} B_i(x) \right| \leq \sum_{i=-1}^{n+1} |B_i(x)|.$$

At any node  $x_i$ , we have

$$\sum_{i=-1}^{n+1} |B_i| = |B_{i-1}| + |B_i| + |B_{i+1}| = 6 < 10.$$

Also we have

$$|B_i(x)| \leq 4 \quad \text{and} \quad |B_{i-1}(x)| \leq 4, \quad x_{i-1} \leq x \leq x_i.$$

Similarly,

$$|B_{i-2}(x)| \leq 1 \quad \text{and} \quad |B_{i+1}(x)| \leq 1, \quad x_{i-1} \leq x \leq x_i.$$

Therefore, for any point  $x_{i-1} \leq x \leq x_i$ , we have

$$\sum_{i=-1}^{n+1} |B_i(x)| = |B_{i-2}| + |B_{i-1}| + |B_i| + |B_{i+1}| \leq 10.$$

□

**Theorem 4.4.2** *Let  $S(x)$  be the approximation from the space of cubic splines  $S^3(\omega)$  to the solution  $\hat{u}^{m+1}(x)$  of the semi-discrete boundary value problem (4.4.3) at the  $(m+1)$ th time level. If  $f(x) \in C^2[x_{-\infty}, x_{\infty}]$ , then the uniform error estimate is given by*

$$\|\hat{u}^{m+1}(x) - S(x)\|_{\infty} \leq Mh^2,$$

where  $M$  is a positive constant independent of  $h$ .

**Proof.** To estimate the error  $\|\hat{u}^{m+1} - S(x)\|_{\infty}$ , let us assume that  $Y_n$  be the unique spline interpolant from  $S^3(\omega)$  to the solution  $\hat{u}^{m+1}(x)$  of our semi-discrete boundary value problem (4.4.3). If  $f(x) \in C^2[x_{-\infty}, x_{\infty}]$ , then  $\hat{u}^{m+1}(x) \in C^4[x_{-\infty}, x_{\infty}]$ , and it follows from the de Boor-Hall error estimates ([11]) that

$$\|D^j(\hat{u}^{m+1}(x) - Y_n)\|_{\infty} \leq \zeta_j h^{4-j}, \quad j = 0, 1, 2, \quad (4.4.8)$$

where  $\zeta_j$ 's are constants independent of  $h$  and  $m$ .

Let

$$Y_n(x) = \sum_{i=-1}^{n+1} b_i B_i(x). \quad (4.4.9)$$

It is clear from the estimates (4.4.8) that

$$|LS(x_i) - LY_n(x_i)| = |f(x_i) - LY_n(x_i) + L\hat{u}^{m+1}(x_i) - L\hat{u}^{m+1}(x_i)| \leq \lambda h^2, \quad (4.4.10)$$

where

$$\lambda = [\delta\zeta_2 + \zeta_0 h^2]. \quad (4.4.11)$$

Also

$$LS(x_i) = L\hat{u}^{m+1}(x_i) = f(x_i).$$

Let

$$LY_n(x_i) = \hat{f}_n(x_i), \quad \forall i$$

and

$$\hat{f}^n = (\hat{f}_n(x_0), \hat{f}_n(x_1), \dots, \hat{f}_n(x_n))^T.$$

Now from system (4.3.16) and (4.4.10), it is clear that the  $i$ th component  $[A(x^N - y^N)]_i$  of  $A(x^N - y^N)$ , where  $y^N = (b_0, b_1, \dots, b_N)^T$ , satisfies the inequality

$$|[A(x^N - y^N)]_i| = h^2 |f_i - \hat{f}_i| \leq \lambda h^4. \quad (4.4.12)$$

Since  $(Ax^N)_i = h^2 f(x_i)$  and  $(Ay^N)_i = h^2 \hat{f}_i(x_i)$ ,  $\forall i = 1, 2, 3, \dots, n$ . Also  $(Ax^N)_0 = h^2 f(x_0) - (h^2 - 6\delta)u_{-\infty}(\tau)$  and  $(Ay^N)_0 = h^2 \hat{f}_n(x_0) - (h^2 - 6\delta)u_{-\infty}(\tau)$ . But the  $i$ th component  $[A(x^N - y^N)]$  is the  $i$ th equation

$$(h^2 - 6\delta)\eta_{i-1} + (4h^2 + 12\delta)\eta_i + (h^2 - 6\delta)\eta_{i+1} = \xi_i, \quad 1 \leq i \leq n-1, \quad (4.4.13)$$

where  $\eta_i = b_i - c_i$ ,  $-1 \leq i \leq n+1$ , and  $\xi_i = h^2[f(x_i) - \hat{f}_n(x_i)]$ ,  $1 \leq i \leq n-1$ . Obviously  $|\xi_i| \leq \lambda h^4$ .

Let  $\xi = \max_{1 \leq i \leq n-1} |\xi_i|$ , consider  $\eta = (\eta_{-1}, \eta_0, \dots, \eta_{n+1})^T$ , and then define  $\varrho_i = |\eta_i|$

and  $\tilde{\varrho}_i = \max_{1 \leq i \leq n} |\varrho_i|$ . Eq. (4.4.13) then becomes

$$(4h^2 + 12\delta)\eta_i = \xi_i + (6\delta - h^2)(\eta_{i-1} + \eta_{i+1}), \quad 1 \leq i \leq n - 1. \quad (4.4.14)$$

Taking absolute value and simplifying, we have

$$(4h^2 + 12\delta)\varrho_i \leq \xi + 2\tilde{\varrho}(6\delta - h^2). \quad (4.4.15)$$

Therefore,

$$(4h^2 + 12\delta)\varrho_i \leq \xi + 2\tilde{\varrho}(6\delta - h^2) \leq \xi + 2\tilde{\varrho}(6\delta + h^2).$$

In particular,

$$(4h^2 + 12\delta)\tilde{\varrho} \leq \xi + 2\tilde{\varrho}(6\delta + h^2). \quad (4.4.16)$$

Solving for  $\tilde{\varrho}$ , we obtain

$$2h^2\tilde{\varrho} \leq \xi \leq \lambda h^4,$$

which gives

$$\tilde{\varrho} \leq \frac{1}{2}\lambda h^2. \quad (4.4.17)$$

Now to estimate  $\varrho_{-1}$ ,  $\varrho_0$ ,  $\varrho_n$  and  $\varrho_{n+1}$ , we observe that the first equation of the system  $A(x^N - y^N) = h^2(f^n - \hat{f}^n)$  where  $f^n = (f_0, f_1, \dots, f_n)$  yields  $36\delta\eta_0 = h^2(f_0 - \hat{f}_0)$ , which gives

$$\varrho_0 \leq \frac{\lambda h^4}{36\delta}. \quad (4.4.18)$$

Similarly, we obtain

$$\varrho_n \leq \frac{\lambda h^4}{36\delta}. \quad (4.4.19)$$

Now  $\varrho_{-1}$  and  $\varrho_{n+1}$  can be evaluated using the boundary conditions given by Eqs. (4.3.12) and (4.3.13) (therefore note that  $\eta_{-1} = (0 - 4\eta_0 - \eta_1)$  and  $\eta_{n-1} = (-4\eta_n - \eta_{n-1})$ ) as

$$\varrho_{-1} \leq \frac{\lambda h^4}{9\delta} + \frac{1}{2}\lambda h^2 \quad (4.4.20)$$

and

$$\varrho_{n+1} \leq \frac{\lambda h^4}{9\delta} + \frac{1}{2}\lambda h^2. \quad (4.4.21)$$

Using (4.4.11), it is easy to see that there exists a constant  $\tilde{C}$  such that

$$\varrho = \max_{-1 \leq i \leq n+1} \{\varrho_i\} \leq \tilde{C}h^2. \quad (4.4.22)$$

The above inequality enables us to estimate  $\|S(x) - Y_n(x)\|_\infty$ , and hence  $\|\hat{u}^{m+1}(x) - S(x)\|_\infty$ . In particular, we will have

$$S(x) - Y_n(x) = \sum_{i=-1}^{n+1} (c_i - b_i)B_i(x). \quad (4.4.23)$$

Thus

$$|S(x) - Y_n(x)| = \max |c_i - b_i| \sum_{i=-1}^{n+1} |B_i(x)|. \quad (4.4.24)$$

Since

$$\sum_{i=-1}^{n+1} |B_i(x)| \leq 10, \quad x_{-\infty} \leq x \leq x_\infty, \quad (\text{using Lemma 4.4.2}). \quad (4.4.25)$$

Combining Eqs. (4.4.22), (4.4.24) and (4.4.25), we see that

$$\|S - Y_n\|_\infty \leq 10\tilde{C}h^2. \quad (4.4.26)$$

Moreover,

$$\|\hat{u}^{m+1} - Y_n\|_\infty \leq \zeta_0 h^4$$

and

$$\|\hat{u}^{m+1}(x) - S(x)\|_\infty \leq \|\hat{u}^{m+1}(x) - Y_n\|_\infty + \|Y_n - S(x)\|_\infty.$$

This implies that

$$\|\hat{u}^{m+1}(x) - S(x)\|_\infty \leq Mh^2, \quad (4.4.27)$$

where  $M = 10\tilde{C} + \zeta_0 h^2$ .

□

We have therefore proved the following main result.

**Theorem 4.4.3** *Let  $u(x, \tau)$  be the solution of problem (4.2.6) and  $S(x, \tau^m)$  be the B-spline collocation approximation from the space  $S^3(\omega)$  to the solution  $u(x, \tau^m)$ . If  $f(x, \tau^m) \in C^2[x_{-\infty}, x_{\infty}]$ , then under the hypotheses of Theorems 4.4.1 and 4.4.2, the error estimate is given by*

$$\|u(x, \tau^m) - S(x)\|_{\infty} \leq \tilde{M}(\Delta\tau + h^2), \quad (4.4.28)$$

where  $\tilde{M}$  is independent of mesh parameters.

**Proof.** The proof is accomplished by using the results from theorems 4.4.1 and 4.4.2.

□

**Remark 4.4.1** To determine the functional relationship between the two step-sizes used in the numerical simulation, we apply the conventional von-Neumann stability analysis for the system (4.3.11). Using

$$c_j^m = \varepsilon^m \exp(i\beta jh), \quad i = \sqrt{-1}, \quad (4.4.29)$$

along with (4.3.11), where  $\varepsilon$  is the growth factor and  $\beta$  is the mode number, we obtain at  $m$ th time level

$$\alpha c_{j-1}^{m+1} + \tilde{\alpha} c_j^{m+1} + \alpha c_{j+1}^{m+1} = c_{j-1}^m + 4c_j^m + c_{j+1}^m, \quad \forall j = 0, 1, \dots, n, \quad (4.4.30)$$

where

$$\alpha = 1 - r_1, \quad \tilde{\alpha} = 4 + r_2, \quad r_1 = 6 \frac{\Delta\tau}{h^2}, \quad r_2 = 2r_1.$$

Using Eq. (4.4.29) and the recurrence relation (4.4.30), we get

$$\varepsilon^{m+1} [\alpha \exp(-i\beta h) + \tilde{\alpha} + \alpha \exp(i\beta h)] = \varepsilon^m [\exp(-i\beta h) + 4 + \exp(i\beta h)], \quad (4.4.31)$$

which implies that

$$\varepsilon = \frac{3 - 2 \sin^2(\frac{\beta h}{2})}{3 - 2 \sin^2(\frac{\beta h}{2}) + 2r_1 \sin^2(\frac{\beta h}{2})}. \quad (4.4.32)$$

Clearly,  $0 < \varepsilon \leq 1$  for all  $r_1 > 0$  and all  $\beta$ . Therefore, the proposed numerical method is unconditionally stable.

## 4.5 Numerical results

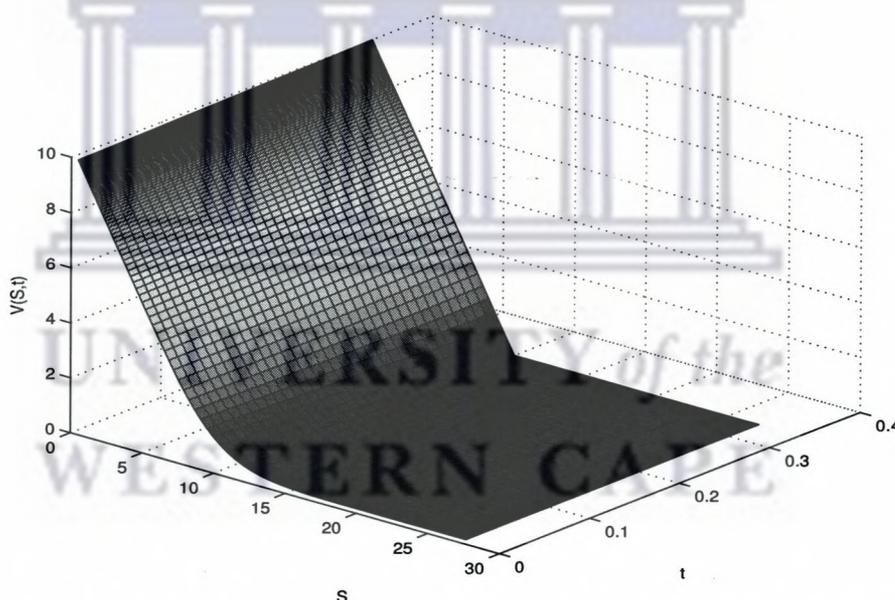


Figure 4.5.1: European put option: numerical solution obtained via B-spline, for  $T = 4/12$ ,  $E = 10$ ,  $r = 0.06$ ,  $\sigma = 0.45$  with  $\Delta\tau = 0.001$ , and  $\Delta x = 0.05$ ,  $x \in (-10, 1)$

In this section, we present some numerical results for the solution of Black-Scholes equation describing European put option. The values  $V(S, t)$  can be interpreted as a piece of surface over the subset  $S > 0$ ,  $0 \leq t \leq T$  of the  $(S, t)$ -plane.

Figure 4.5.1 illustrates the character of this surface for the European put option for the fixed values of  $E$ ,  $T$ ,  $r$  and  $\sigma$ . Through Figure 4.5.2, we explain that the European

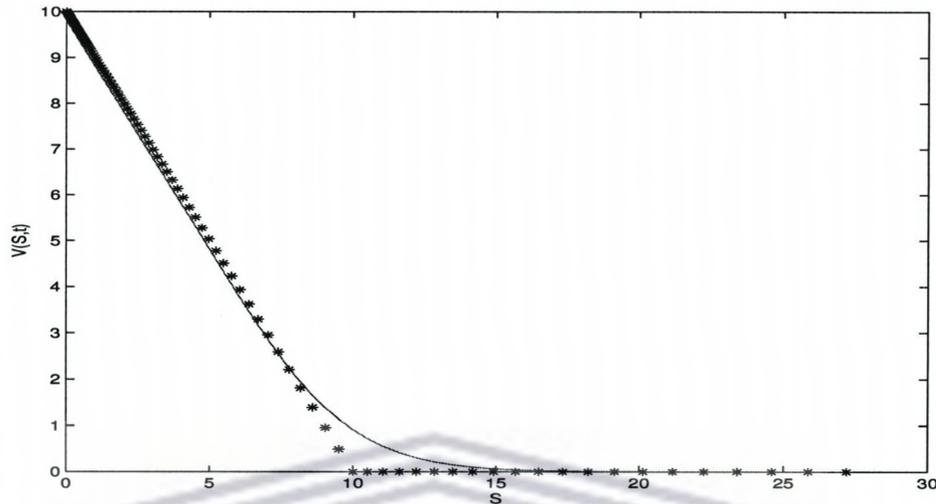


Figure 4.5.2: Values of a European put option at  $t = 0$  for  $T = 4/12$ ,  $r = 0.06$ ,  $\sigma = 0.45$  with  $\Delta\tau = 0.001$ , and  $\Delta x = 0.05$ . The curve with '\*' shows payoff whereas the solid curve represents the value of the option.

Table 4.5.1: Comparison between the exact, Quasi-RBFs and B-spline solutions for the European put option.  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ . With  $\Delta x = 0.0050$  and  $\Delta t = 10^{-5}$ .

$S$	Exact solution	Quasi-RBF solution	B-spline solution
2.00	7.7531	7.7531	7.7531
4.00	5.7531	5.7531	5.7531
6.00	3.7532	3.7532	3.7532
7.00	2.7568	2.7568	2.7568
8.00	1.7987	1.7988	1.7987
9.00	0.9880	0.9881	0.9880
10.00	0.4420	0.4420	0.4419
11.00	0.1606	0.1606	0.1606
12.00	0.0483	0.0483	0.0484
13.00	0.0124	0.0124	0.0124
14.00	0.0028	0.0028	0.0028
15.00	0.0006	0.0006	0.0006
16.00	0.0001	0.0001	0.0001

put option can take values above the lower bound  $Ee^{-r(T-t)} - S$ . For small values of  $S$  the value  $V$  approaches its lower bound.

In Table 4.5.1, we have tabulated the comparative results. It consists of the exact,

Table 4.5.2: Comparison between the exact and B-spline solutions for the European put option.  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ . With  $\Delta x = 0.0080$  and  $\Delta t = 10^{-5}$ .

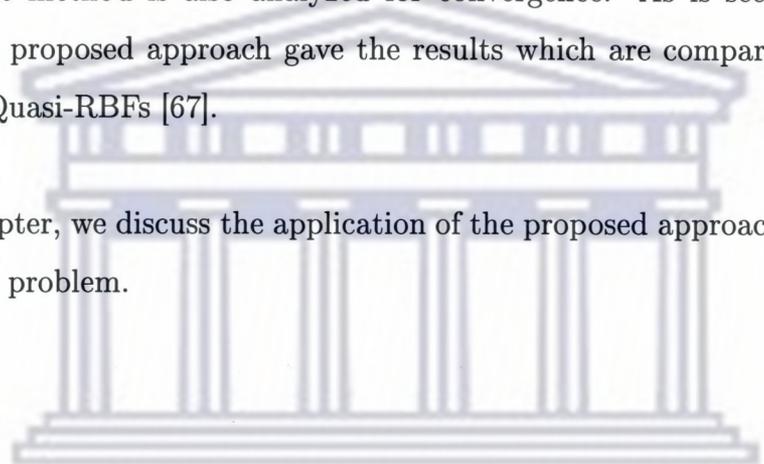
$S$	Exact solution	B-spline solution
2.00	7.7531	7.7531
4.00	5.7531	5.7531
6.00	3.7532	3.7532
7.00	2.7568	2.7568
8.00	1.7987	1.7987
9.00	0.9880	0.9880
10.00	0.4420	0.4418
11.00	0.1606	0.1606
12.00	0.0483	0.0483
13.00	0.0124	0.0124
14.00	0.0028	0.0028
15.00	0.0006	0.0006
16.00	0.0001	0.0001

Quasi-RBFs and B-spline solutions for the European put option for  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ , with  $\Delta t = 10^{-5}$  and  $\Delta x = 0.005$ . In Table 4.5.2 we have tabulated the exact and B-spline solutions for the European put option for  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ , with  $\Delta t = 10^{-5}$  and  $\Delta x = 0.008$ . The actual error is controlled via the numbers  $n$  and  $m$ . Using Matlab 2009b, with  $\Delta t = 10^{-5}$  and  $\Delta x = 0.008$ , the CPU time that our code took on a 32 bit machine running UBUNTU linux was 66.96 seconds whereas this code took only 33.2 seconds on a 64 bit Window machine. One of the works that we could found in the literature where CPU time was calculated was that of Hon [67], where the computations were performed on a SUN Sparc workstation by using FORTRAN 77 with double precision. Using,  $\Delta t = 0.005$  and  $\Delta x = 0.004$  his code took about 12 seconds whereas the same can be done by our code using only 0.2849 seconds.

## 4.6 Summary and discussions

A numerical method is developed to solve the nonlinear Black-Scholes partial differential equation modelling European option pricing on a single asset. This method is based on the implicit Euler method for the temporal discretization and the B-spline collocation method in the spatial direction on a uniform mesh. Applying the von-Neumann stability analysis, we found that the proposed method is unconditionally stable. The method is also analyzed for convergence. As is seen from the tabular results, the proposed approach gave the results which are comparable with those obtained by Quasi-RBFs [67].

In next chapter, we discuss the application of the proposed approach to solve an American option problem.



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## Chapter 5

# B-spline approximation method for pricing American options

The problem of pricing an American option can be cast as a partial differential equation described by the famous Black-Scholes equation. Analytical solutions of this Black-Scholes model for pricing American options problems are seldom available and hence such derivatives must be priced by stable and efficient numerical techniques. The troublesome factor in pricing American options is the existence of an optimal exercise boundary.

In this chapter, we construct a numerical method based on spline approximations to solve a nonlinear Black-Scholes partial differential equation modelling American put option price on a single asset. The method is shown to be unconditionally stable and accurate of order  $\mathcal{O}((\Delta x)^2 + \Delta \tau)$ . The computational performance of the proposed method is compared with other methods seen in the literature. Furthermore, procedurally we solve a European option pricing problem and then use an update procedure, therefore, we also give comparative numerical results obtained for this problem so that the update procedure is pre-justified.

In this chapter, we first use a different transformation to solve the European option pricing problem and then use an update procedure to solve the problem of pricing American options.

## 5.1 Introduction

Most options can be grouped into either of two categories: European options which can be exercised only on their expiration date, and American options which can be exercised on or before their expiration date. In practice, most options are American. American options are much harder to deal with than European ones. The problem is that it may be optimal to use (exercise) the option before the final expiry date. This optimal exercise policy will affect the value of the option, and the exercise policy needs to be known when solving the PDE. In view of this fact, the Black-Scholes equation for American options results in a free boundary value problem.

For European options, the analytic solution is relatively easier to obtain. However, pricing American options is a challenging numerical task. First of all there is no closed form and exact solution to this problem. The basic property of an American option is the early exercise feature of the option. Hence, at any time, there is specific value of the asset price that divides the asset domain into the early exercise region, where the option should be exercised, and the continuation region, where the option should be held. Therefore, the early exercise feature gives an additional constraint that the value of an American option must be greater than or equal to its payoff; this constraint requires special treatment. The American option pricing problem can be posed either as a linear complementarity problem (LCP) or as a free boundary value problem. These two different formulations have led to a number of different methodologies for solving American options. Below we review some of the works.

Probably the first algorithm to value an American option was introduced by Bren-

nan and Schwartz [17]. The convergence of their finite difference method was proved by Jaillet *et al.* [75]. Soon after the work of Brennan and Schwartz [17], Cox *et al.* [35] introduced the binomial method for solving American options. The convergence of this method is proved by Amin and Khanna [4]. Since then many other versions of binomial parameters have been proposed, for example, the one given in Hull [68]. Boyle [14] gave a trinomial model for option pricing which is similar to the binomial method, but gives an accurate value faster than the binomial one. Kim [101], Jacka [74] and Carr *et al.* [24] provided integral formulas which express the value of American option as the sum of corresponding European option and integral function of free boundary. Then they use recursive numerical algorithm to solve for optimal exercise boundary and option price. Algorithms that solve the discrete linear complementarity problem at spatial grid points have been suggested in [12, 44].

In [31] Cho *et al.* considered a free boundary problem arising in the pricing of an American call option. The free boundary represents the optimal exercise price as a function of time before a maturity date. They developed a parameter estimation technique to obtain the optimal exercise curve of an American call option and its price. For the numerical solution of a forward problem, they adopted a time marching finite element method.

Choi and Marozzi [32] considered the valuation of options written on a foreign currency when interest rates are stochastic and the matrix of the diffusion representing the global economy is strongly coercive. They solved the associated variational inequality for the value function numerically by the finite element method. In the European case, a comparison is made with the exact solution. They also presented a corresponding result for the American option.

In [47], Engström and Nordén estimated the value of the early exercise premium in American put option prices using Swedish equity options data. They found the value

of the premium as the deviation of the American put price from European put-call parity, and computed a theoretical estimate of the premium. They also used the empirically found premium in a modified version of the control variate approach to value American puts. Their results indicate a substantial value of the early exercise premium, where the premium derived from put-call parity is higher than the theoretical premium.

Front-fixing method [122, 156] and the penalty method [50, 100, 122] for pricing American options are widely used by researchers in the past. Front-fixing methods apply a non-linear transformation to fix the boundary and solve the resulting non-linear problem. Penalty methods on the other hand eliminate the free-boundary by adding a non-linear penalty term to the PDE. Both these methods boil down to solving a set of non-linear equations, the computational speed and accuracy of which largely depends on the initial guess, the problem size and the underlying non-linear solver used. These methods are not very efficient for pricing American options but they are far more general in their applicability.

Some other popular numerical methods for pricing American option problems are the method of lines [118], compact finite difference methods [149, 150, 161], adaptive Monte Carlo simulations [112], operator splitting [72], etc.

The rest of the chapter is organized as follows. In Section 5.2, we describe the option pricing problem for American puts. The numerical method is constructed and analyzed for convergence in Section 5.3. Comparative numerical results are presented in Section 5.4 whereas Section 5.5 deals with some concluding remarks and scope for future research.

## 5.2 Problem description

In this chapter, we are concerned with the numerical valuation of American put option that satisfies the Black-Scholes equation which is actually used in real markets to obtain the current theoretical option value. The governing equation for American put problems take the form of free-boundary problems. The American early exercise constraint leads to the following model for the value  $V(S, t)$  of an American put option to sell the underlying asset

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV &= 0, \quad S > S_f(t), \quad 0 \leq t < T \quad (5.2.1) \\ V(S, T) &= \max(E - S, 0), \quad S \geq 0, \\ \frac{\partial V}{\partial S}(S_f, t) &= -1, \\ V(S_f(t), t) &= E - S_f(t), \\ \lim_{S \rightarrow \infty} V(S, t) &= 0, \\ S_f(T) &= E, \\ V(S, t) &= E - S, \quad 0 \leq S < S_f(t), \end{aligned}$$

where  $S_f(t)$  represents the free (and moving) boundary. Since early exercise is permitted, the value  $V$  of the option must satisfy

$$V(S, t) \geq \max(E - S, 0), \quad S \geq 0, \quad 0 \leq t \leq T. \quad (5.2.2)$$

In the above,  $S$  denotes the market price of the underlying asset,  $\sigma$  is the volatility of the underlying asset,  $E$  is the exercise price,  $T$  is the expiry time, and  $r$  is the risk-free interest rate.

The essential difficulty in solving the above problem lies in the fact that the early exercise right purchased by the holder of the option has changed the problem into a so-called free boundary value problem. The optimal exercise price prior to the expira-

tion of the option is time-dependent. As a result of the unknown boundary being part of the solution of the problem, the valuation of American options becomes a nonlinear problem.

### 5.3 Computation of the American put options and analysis of the numerical method

Before we proceed, it is worth mentioning here that to solve the problem of pricing American put options we firstly solve a corresponding option pricing problem for European puts and then use an update procedure. To this end then we construct and analyze the numerical method to solve European options and then we will explain the update procedure.

The governing Black-Scholes equation for  $V(S, t)$  on a single asset is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (5.3.1)$$

The domain of the independent variables  $S, t$  is  $(0, \infty) \times (0, T]$ . The final condition at  $t = T$  is given by the maximum payoff valuation

$$V(S, T) = \max(E - S, 0). \quad (5.3.2)$$

The boundary condition at  $S = 0$  satisfies ([141])

$$V(S, t) = Ee^{-r(T-t)} - S, \quad (5.3.3)$$

and the boundary condition at  $S = +\infty$  satisfies

$$V(S, t) = 0. \quad (5.3.4)$$

For numerical applications, we transform the time variable  $t$  to

$$\tau = T - t, \quad (5.3.5)$$

the domain still  $(0, \infty) \times (0, T]$  (i.e, we convert the final-boundary value problem to an initial-boundary value problem), we consider the initial-boundary value problem

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV, \\ V(S, \tau = 0) &= \max(E - S, 0), \\ V(0, \tau) &= Ee^{-r\tau}, \\ V(S_\infty, \tau) &= 0. \end{aligned} \quad (5.3.6)$$

A simple transformation

$$S = e^x, \quad (5.3.7)$$

changes equation (5.3.6) to

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + (r - \frac{1}{2}\sigma^2) \frac{\partial U}{\partial x} - rU, \quad (5.3.8a)$$

$$U(x, \tau = 0) = U^0(x) = \max(E - e^x, 0), \quad (5.3.8b)$$

$$U(x_{-\infty}, \tau) = U_{-\infty}(\tau) = Ee^{-r\tau}, \quad (5.3.8c)$$

$$U(x_\infty, \tau) = U_\infty(\tau) = 0. \quad (5.3.8d)$$

In view of the time and space transformations mentioned in (5.3.5) and (5.3.7), the expiration time  $t = T$  is determined in the “new” time by  $\tau = 0$ , and  $t = 0$  is transformed to  $\tau = T$ . The new time variable  $\tau$  represents the remaining life time of the option. And the original domain of the half strip  $S > 0, 0 \leq t \leq T$  belonging to (5.3.1) becomes the strip

$$-\infty < x < \infty, \quad 0 \leq \tau \leq T, \quad (5.3.9)$$

on which we are going to approximate a solutions  $U(x, \tau)$  to (5.3.8). After this we again apply the transformations mentioned in (5.3.5) and (5.3.7) to derive the value of the option  $V(S, t)$  from  $U(x, \tau)$ .

It should be noted that the Black-Scholes equation in the version (5.3.1) has variable coefficients  $S^j$  with powers matching the order of the derivative with respect to  $S$ . That is, the relevant terms in (5.3.1) are of the type

$$S^j \frac{\partial^j V}{\partial S^j}, \quad j = 0, 1, 2.$$

Linear differential equations with such terms are known as Euler's differential equations; their analysis motivates the transformation  $S = e^x$ . The transformed version in equation (5.3.8a) has constant coefficients, which simplifies implementing numerical algorithms.

### 5.3.1 Construction of the numerical method

Our numerical method is based on B-spline for the discretization in space and the finite difference techniques for the temporal one. We consider a two-dimensional grid as follows: Let  $\Delta\tau$  and  $\Delta x$ , be the mesh step-sizes in the  $\tau$  and  $x$ -directions. The step-size in  $\tau$ -direction is given by  $\Delta\tau = T/m$ , where  $m$  is an integer. In order to avoid technical complications, let us accept that, since we are going to restrict any numerical scheme to a finite mesh, we may as well restrict the problem to a finite interval. That is, the infinite interval  $-\infty < x < \infty$  must be replaced by a finite interval  $x_{-\infty} \leq x \leq x_{\infty}$ . The end values  $x_{-\infty} = x_{\min} < 0$  and  $x_{\infty} = x_{\max} > 0$  should be chosen in such a way that for  $S_{\min} = e^{x_{-\infty}}$ ,  $S_{\max} = e^{x_{\infty}}$  and the interval  $S_{\min} \leq S \leq S_{\max}$ , a sufficient smooth approximation is obtained. Then for a suitable integer  $n$  the step length in  $x$  is defined by  $\Delta x = (x_{\infty} - x_{-\infty})/n$ .

In the computations, we choose  $x_{\min} = -2.0$  and  $x_{\max} = 3$  and 5 so that the

range for the stock  $S$  is sufficiently large to satisfy the boundary conditions (5.3.3) and (5.3.4). Transforming the  $(x, \tau)$ -grid back to the  $(S, t)$ -plane, leads to a nonuniform grid with unequal distances of the grid lines  $S = S_i = e^{x_i}$ . The actual error is then controlled via the numbers  $n$  and  $m$  of grid lines.

To discretize (5.3.8) in the temporal discretization, we use an implicit finite difference technique with uniform step-size  $\Delta\tau$  and obtain the following system of linear ordinary differential equations:

$$\frac{1}{2}\sigma^2 \frac{\partial^2 U^{m+1}}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial U^{m+1}}{\partial x} - \left(r + \frac{1}{\Delta\tau}\right) U^{m+1} = -\frac{U^m}{\Delta\tau}. \quad (5.3.10)$$

We rewrite (5.3.10) as

$$a \frac{\partial^2 U^{m+1}}{\partial x^2} + b \frac{\partial U^{m+1}}{\partial x} - g U^{m+1} = f^m, \quad (5.3.11)$$

where

$$a = \frac{1}{2}\sigma^2, \quad b = \left(r - \frac{1}{2}\sigma^2\right), \quad g = \left(r + \frac{1}{\Delta\tau}\right), \quad \text{and} \quad f^m = -\frac{U^m}{\Delta\tau}.$$

The spatial discretization is done as follows. Given  $n$  distinct knots  $x_1 < x_2 < \dots < x_n$  in the open interval  $(a, b)$  and an integer  $k \geq 1$ , let  $S^k(x)$  be the space of functions of class  $C^{k-1}$  over  $[a, b]$  which coincide with polynomials of degree at most  $k$  on each interval  $[x_j, x_{j+1}]$ , for  $1 \leq j \leq n$ , with  $x_0 = a$  and  $x_{n+1} = b$ . The space  $S^k(x)$  is called the space of splines of degree  $k$  [134]. With the above spatial mesh, we recall from Chapter 4, that a B-spline is a function of the form:

$$B_j^{(0)}(x) = \begin{cases} 1, & x \in [x_j, x_{j+1}] \\ 0, & x \notin [x_j, x_{j+1}] \end{cases}$$

It is a function of degree  $k \geq 1$  defined on segment  $[a, b]$  is constructed by the recurrent relation

$$B_j^{(k)}(x) = \frac{x - x_j}{x_{j+1} - x_j} B_j^{(k-1)}(x) + \frac{x_{j+k+1} - x}{x_{j+k+1} - x_{j+1}} B_{j+1}^{(k-1)}(x). \quad (5.3.12)$$

Let

$$\Omega := \{B_{-1}, B_0, B_1, \dots, B_{n+1}\}.$$

The functions in  $\Omega$  are linearly independent on  $[a, b]$ .

Now suppose the approximate solution is given by

$$S(x) = \sum_{i=-1}^{n+1} c_i B_i(x), \quad (5.3.13)$$

where  $c_i$  are unknown real coefficients and  $B_i(x)$  are cubic B-spline functions defined by

$$B_i(x) = \begin{cases} \left(\frac{x-x_{i-2}}{h}\right)^3, & \text{if } x \in [x_{i-2}, x_{i-1}], \\ 1 + 3\left(\frac{x-x_{i-1}}{h}\right) + 3\left(\frac{x-x_{i-1}}{h}\right)^2 - 3\left(\frac{x-x_{i-1}}{h}\right)^3, & \text{if } x \in [x_{i-1}, x_i], \\ 1 + 3\left(\frac{x_{i+1}-x}{h}\right) + 3\left(\frac{x_{i+1}-x}{h}\right)^2 - 3\left(\frac{x_{i+1}-x}{h}\right)^3, & \text{if } x \in [x_i, x_{i+1}], \\ \left(\frac{x_{i+2}-x}{h}\right)^3, & \text{if } x \in [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise.} \end{cases} \quad (5.3.14)$$

Since it is required that the approximate solution  $S(x)$  satisfies the given problem (5.3.1) at mesh points  $\omega$  as well as boundary conditions at  $x = x_0$  and  $x = x_n$ , we

have therefore introduced two extra splines  $B_{-1}$  and  $B_{n+1}$  to force  $S(x)$  to satisfy the boundary conditions.

To illustrate how to apply the B-spline formula given by (5.3.13) for solving the option pricing model, we let  $S(x)$  satisfy the equation (5.3.11), i.e.,

$$LS(x_j) = f^m(x_j), \quad 0 \leq j \leq n, \quad (5.3.15)$$

where  $LU^{m+1} \equiv aU_{xx}^{m+1} + bU^{m+1} - gU^{m+1}$ . Thus at time level  $m = 0$ , we have

$$a \sum_{i=-1}^{n+1} c_i B_i''(x_j) + b \sum_{i=-1}^{n+1} c_i B_i'(x_j) - g \sum_{i=-1}^{n+1} c_i B_i(x_j) = f_j^0, \quad f_j^0 = f^0(x_j).$$

By solving this equation and noting that the support of the function  $B_i(x)$  is the segment  $[x_{i-2}, x_{i+2}]$ , we have

$$\begin{aligned} c_{j-1} [aB_{j-1}''(x_j) + bB_{j-1}'(x_j) - gB_{j-1}(x_j)] + c_j [aB_j''(x_j) + bB_j'(x_j) - gB_j(x_j)] \\ + c_{j+1} [aB_{j+1}''(x_j) + bB_{j+1}'(x_j) - gB_{j+1}(x_j)] = f_j^0, \quad \forall j = 0, 1, \dots, n. \end{aligned} \quad (5.3.16)$$

Now we note from (5.3.14) that

$$B_i(x_j) = \begin{cases} 4, & \text{if } i = j, \\ 1, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2, \end{cases} \quad (5.3.17)$$

$B_i(x) = 0$  for  $x \geq x_{i+2}$  and  $x \leq x_{i-2}$ ,

$$B'_i(x_j) = \begin{cases} 0, & \text{if } i = j, \\ \pm \frac{3}{h}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2, \end{cases} \quad (5.3.18)$$

and

$$B''_i(x_j) = \begin{cases} \frac{-12}{h^2}, & \text{if } i = j, \\ \frac{6}{h^2}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2. \end{cases} \quad (5.3.19)$$

By using equations (5.3.17)-(5.3.19), we get

$$\begin{aligned} & \left[ a \left( \frac{6}{h^2} \right) + b \left( \frac{-3}{h} \right) - g \right] c_{j-1} + \left[ a \left( \frac{-12}{h^2} \right) + b(0) - 4g \right] c_j \\ & + \left[ a \left( \frac{6}{h^2} \right) + b \left( \frac{3}{h} \right) - g \right] c_{j+1} = f_j^0, \quad \forall j = 0, 1, \dots, n. \end{aligned} \quad (5.3.20)$$

The boundary conditions (5.3.8c) and (5.3.8d) becomes

$$c_{-1} + 4c_0 + c_1 = Ee^{-r\tau}, \quad (5.3.21)$$

and

$$c_{n-1} + 4c_n + c_{n+1} = 0. \quad (5.3.22)$$



$$\beta_0 = (-36a + 12bh) \text{ and } \gamma_0 = 6bh,$$

$$\alpha = 6a - 3bh - gh^2,$$

$$\beta = -12a - 4gh^2,$$

$$\gamma = 6a + 3bh - gh^2,$$

$$\alpha_n = -6bh, \text{ and } \beta_n = (-36a - 12bh).$$

We can see that the system is strictly diagonally dominant and hence non-singular. So we can solve the system for  $c_0, c_1, \dots, c_n$  and substitute into the boundary equations (5.3.21) and (5.3.22) to obtain  $c_{-1}$  and  $c_{n+1}$ . At each time level we solve (5.3.27) and recover the solution via (5.3.5) and (5.3.7).

### 5.3.2 Convergence analysis of the numerical method

As we have mentioned earlier, we discretized the time variable by implicit Euler with uniform step-size  $\Delta\tau$ , therefore we get the following system of linear ordinary differential equations:

$$U^0 = U^0(x), \quad -\infty < x < \infty, \quad (5.3.28a)$$

$$\frac{U^{m+1} - U^m}{\Delta\tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 U^{m+1}}{\partial x^2} - \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial U^{m+1}}{\partial x} + rU^{m+1} = 0, \quad (5.3.28b)$$

$$U^{m+1}(x_{-\infty}) = U_{-\infty}(\tau^{m+1}), \quad U^{m+1}(x_{\infty}) = U_{\infty}(\tau^{m+1}), \quad (5.3.28c)$$

which gives semi-discrete approximations  $U^m(x)$ , at time levels  $\tau_m = m\Delta\tau$ , to the exact solution  $U(x, \tau)$  of (5.3.8).

For the sake of simplicity, let us denote

$$L_x \equiv -\frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} - \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial}{\partial x} + rI,$$

so that we rewrite (5.3.28) as

$$U^0 = U^0(x), \quad -\infty < x < \infty, \quad (5.3.29a)$$

$$(I + \Delta\tau L_x)U^{m+1} = U^m, \quad -\infty < x < \infty, \quad \tau > 0, \quad (5.3.29b)$$

$$U^{m+1}(x_{-\infty}) = U_{-\infty}(\tau^{m+1}), \quad U^{m+1}(x_{\infty}) = U_{\infty}(\tau^{m+1}). \quad (5.3.29c)$$

The stability of (5.3.29) follows from the maximum principle for the operator  $I + \Delta\tau L_x$ , because

$$\|(I + \Delta\tau L_x)^{-1}\|_{\infty} \leq \frac{1}{1 + \tilde{b}\Delta\tau}. \quad (5.3.30)$$

The local truncation error of the time semi-discretization method (5.3.29) is given by

$$e_{m+1} = U(\tau^{m+1}) - \hat{U}^{m+1},$$

where  $\hat{U}^{m+1}$  is the solution of

$$(I + \Delta\tau L_x)\hat{U}^{m+1}(x) = U(x, \tau^m), \quad -\infty < x < \infty, \quad \tau > 0, \quad (5.3.31a)$$

$$\hat{U}^{m+1}(x_{-\infty}) = U_{-\infty}(\tau^{m+1}), \quad \hat{U}^{m+1}(x_{\infty}) = U_{\infty}(\tau^{m+1}). \quad (5.3.31b)$$

This error measures the contribution at each time step to the global error of the time semi-discretization which is defined as

$$E_m \equiv U(x, \tau^m) - U^m(x).$$

Now we show that the following accuracy results hold:

**Lemma 5.3.1** (Local error estimate). *If*

$$\left| \frac{\partial^i}{\partial t^i} u(x, \tau) \right| \leq C_0, \quad -\infty < x < \infty, \quad 0 < \tau < T, \quad 0 \leq i \leq 2, \quad (5.3.32)$$

*then the local error satisfies*

$$\|e_{m+1}\|_\infty \leq C_0(\Delta\tau)^2, \quad (5.3.33)$$

*where  $C_0$  is a positive constant independent of  $\Delta\tau$ .*

**Proof.** Since the function  $\hat{U}^{m+1}$  satisfies

$$(I + \Delta\tau L_x)\hat{U}^{m+1}(x) = U(x, \tau^m),$$

and as the solution of (5.3.8) is smooth enough, we have

$$\begin{aligned} U(\tau^m) &= U(\tau^{m+1}) + \Delta\tau L_x U(\tau^{m+1}) + \int_{\tau^m}^{\tau^{m+1}} (\tau^m - s) \frac{\partial^2 U}{\partial \tau^2}(s) ds \\ &= (I + \Delta\tau L_x)U^{m+1}(x) + \mathcal{O}(\Delta\tau^2). \end{aligned} \quad (5.3.34)$$

Then  $e_{m+1}$  is the solution of a boundary value problem of type

$$(I + \Delta\tau L_x)e_{m+1} = \mathcal{O}(\Delta\tau^2), \quad e_{m+1}(x_{-\infty}) = e_{m+1}(x_\infty) = 0. \quad (5.3.35)$$

Thus (5.3.33) follows when applying the stability result (5.3.30).

**Theorem 5.3.1** (Global error estimate). *Under the hypotheses of Lemma 5.3.1, we have*

$$\|E_m\|_\infty \leq C_0\Delta\tau, \quad \forall m \leq \frac{T}{\Delta\tau}. \quad (5.3.36)$$

**Proof.** Using the local error estimate up to the  $m^{\text{th}}$  time level given by Lemma (5.3.1), we get the following global error estimate at  $(m + 1)^{\text{th}}$  time level

$$\begin{aligned}
 \|E_{m+1}\|_{\infty} &= \left\| \sum_{l=1}^m e_l \right\|, \quad m \leq \frac{T}{\Delta\tau} \\
 &\leq \|e_1\|_{\infty} + \|e_2\|_{\infty} + \dots + \|e_m\|_{\infty} \\
 &\leq C_1(m\Delta\tau)\Delta\tau \quad \text{using (5.3.33)} \\
 &\leq C_1(T)\Delta\tau \quad \text{since } m\Delta\tau \leq T \\
 &= C_0\Delta\tau.
 \end{aligned} \tag{5.3.37}$$

Therefore the time semi-discretization process converge with order one.

□

Now we prove that the B-spline collocation method converge with order two in the spatial direction. To proceed with, we first prove the following lemma:

**Lemma 5.3.2** *The B-splines  $B_{-1}, B_0, B_1, \dots, B_{n+1}$  defined in equation (5.3.14), satisfy the inequality*

$$\sum_{i=-1}^{n+1} |B_i(x)| \leq 10, \quad x_{-\infty} \leq x \leq x_{\infty}.$$

**Proof.** We know that

$$\left| \sum_{i=-1}^{n+1} B_i(x) \right| \leq \sum_{i=-1}^{n+1} |B_i(x)|.$$

At any node  $x_i$ , we have

$$\sum_{i=-1}^{n+1} |B_i| = |B_{i-1}| + |B_i| + |B_{i+1}| = 6 < 10.$$

Also

$$|B_i(x)| \leq 4 \quad \text{and} \quad |B_{i-1}(x)| \leq 4, \quad x_{i-1} \leq x \leq x_i.$$

Similarly

$$|B_{i-2}(x)| \leq 1 \quad \text{and} \quad |B_{i+1}(x)| \leq 1, \quad x_{i-1} \leq x \leq x_i.$$

Therefore, for any point  $x_{i-1} \leq x \leq x_i$ , we have

$$\sum_{i=-1}^{n+1} |B_i(x)| = |B_{i-2}| + |B_{i-1}| + |B_i| + |B_{i+1}| \leq 10.$$

□

**Theorem 5.3.2** *Let  $S(x)$  be the approximation from the space of cubic splines  $S^3(\omega)$  to the solution  $\hat{U}^{m+1}(x)$  of the semi-discrete boundary value problem (5.3.31) at the  $(m+1)^{\text{th}}$  time level. If  $f(x) \in C^2[x_{-\infty}, x_{\infty}]$ , then the uniform error estimate is given by*

$$\|\hat{U}^{m+1}(x) - S(x)\|_{\infty} \leq Mh^2,$$

where  $M$  is a positive constant independent of  $h$ .

**Proof.** To estimate the error  $\|\hat{u}^{m+1} - S(x)\|_{\infty}$ , let us assume that  $Y_n$  be the unique spline interpolant from  $S^3(\omega)$  to the solution  $\hat{u}^{m+1}(x)$  of our semi-discrete boundary value problem (5.3.31). If  $f(x) \in C^2[x_{-\infty}, x_{\infty}]$ , then  $\hat{u}^{m+1}(x) \in C^4[x_{-\infty}, x_{\infty}]$ , and it follows from the de Boor-Hall error estimates [11] that

$$\|D^j(\hat{u}^{m+1}(x) - Y_n)\|_{\infty} \leq \zeta_j h^{4-j}, \quad j = 0, 1, 2, \quad (5.3.38)$$

where  $\zeta_j$ 's are constants independent of  $h$  and  $m$ .

Let

$$Y_n(x) = \sum_{i=-1}^{n+1} b_i B_i(x). \quad (5.3.39)$$

It is clear from the estimates (5.3.38) that

$$\begin{aligned} |LS(x_i) - LY_n(x_i)| &= |f(x_i) - LY_n(x_i) + L\hat{u}^{m+1}(x_i) - L\hat{u}^{m+1}(x_i)| \\ &\leq \lambda h^2, \end{aligned} \quad (5.3.40)$$

where

$$\lambda = \left[ \frac{1}{2}\sigma^2\zeta_2 + \left(r - \frac{1}{2}\sigma^2\right)\zeta_1 h + r\zeta_0 h^2 \right]. \quad (5.3.41)$$

Also

$$LS(x_i) = L\hat{u}^{m+1}(x_i) = f(x_i).$$

Let

$$LY_n(x_i) = \hat{f}_n(x_i), \quad \forall i$$

and

$$\hat{f}^n = (\hat{f}_n(x_0), \hat{f}_n(x_1), \dots, \hat{f}_n(x_n))^T.$$

From system (5.3.26) and (5.3.40), it is clear that the  $i^{\text{th}}$  component of  $A(x^N - y^N)$ , where  $y^N = (b_0, b_1, \dots, b_N)^T$ , satisfies the inequality

$$|[A(x^N - y^N)]_i| = h^2 |f_i - \hat{f}_i| \leq \lambda h^4. \quad (5.3.42)$$

Now

$$(Ax^N)_i = h^2 f(x_i)$$

and

$$(Ay^N)_i = h^2 \hat{f}(x_i), \quad \forall i = 1, 2, 3, \dots, n.$$

Also

$$(Ax^N)_0 = h^2 f(x_0) - (6a - 3bh - gh^2)u_{-\infty}(\tau)$$

and

$$(Ay^N)_0 = h^2 \hat{f}_n(x_0) - (6a - 3bh - gh^2)u_{-\infty}(\tau).$$

However, the  $i^{th}$  component of  $[A(x^N - y^N)]$  is the  $i^{th}$  equation

$$(6a - 3bh - gh^2)\eta_{i-1} - (12a + 4gh^2)\eta_i + (6a + 3bh - gh^2)\eta_{i+1} = \xi_i, \quad 1 \leq i \leq n-1, \quad (5.3.43)$$

where

$$\eta_i = b_i - c_i, \quad -1 \leq i \leq n+1,$$

and

$$\xi_i = h^2[f(x_i) - \hat{f}_n(x_i)], \quad 1 \leq i \leq n-1.$$

Obviously

$$|\xi_i| \leq \lambda h^4.$$

Let

$$\xi = \max_{1 \leq i \leq n-1} |\xi_i|,$$

and consider

$$\eta = (\eta_{-1}, \eta_0, \dots, \eta_{n+1})^T.$$

Then define

$$\varrho_i = |\eta_i| \quad \text{and} \quad \tilde{\varrho} = \max_{1 \leq i \leq n} |\varrho_i|.$$

equation (5.3.43) then becomes

$$-(12a + 4gh^2)\eta_i = \xi_i + (gh^2 - 6a)(\eta_{i-1} + \eta_{i+1}) + 3bh(\eta_{i-1} - \eta_{i+1}), \quad 1 \leq i \leq n-1. \quad (5.3.44)$$

Taking absolute value and simplifying, we have

$$(12a + 4gh^2)\varrho_i \leq \xi + 2\tilde{\varrho}(gh^2 + 3bh - 6a). \quad (5.3.45)$$

Therefore,

$$(12a + 4gh^2)\varrho_i \leq \xi + 2\tilde{\varrho}(gh^2 + 3bh - 6a) \leq \xi + 2\tilde{\varrho}(gh^2 + 3bh + 6a).$$

In particular,

$$(12a + 4gh^2)\tilde{\varrho} \leq \xi + 2\tilde{\varrho}(gh^2 + 3bh + 6a). \quad (5.3.46)$$

Solving for  $\tilde{\varrho}$ , we obtain

$$(2gh^2 - 6bh)\tilde{\varrho} \leq \xi \leq \lambda h^4,$$

which gives

$$\tilde{\varrho} \leq \frac{\lambda h^3}{2gh - 6b}. \quad (5.3.47)$$

Now to estimate  $\varrho_{-1}$ ,  $\varrho_0$ ,  $\varrho_n$  and  $\varrho_{n+1}$ , we observe that the first equation of the system  $A(x^N - y^N) = h^2(f^n - \hat{f}^n)$  where  $f^n = (f_0, f_1, \dots, f_n)$  yields

$$(-36a + 12bh)\eta_0 + 6bh\eta_1 = h^2(f_0 - \hat{f}_0).$$

Taking absolute value and simplifying, we have

$$\varrho_0 \leq \frac{2\lambda gh^5}{(-36a + 12bh)(2gh - 6b)}. \quad (5.3.48)$$

Similarly, we obtain

$$\varrho_n \leq \frac{2\lambda gh^5}{(36a + 12bh)(2gh - 6b)}. \quad (5.3.49)$$

Now  $\varrho_{-1}$  and  $\varrho_{n+1}$  can be evaluated using the boundary conditions given by Eqs. (5.3.21) and (5.3.22) (note that  $\eta_{-1} = (0 - 4\eta_0 - \eta_1)$  and  $\eta_{n+1} = (-4\eta_n - \eta_{n-1})$ ) as

$$\varrho_{-1} \leq \frac{\lambda h^3}{2gh - 6b} \left[ \frac{2gh^2 - 9a + 3bh}{-9a + 3bh} \right] \quad (5.3.50)$$

and

$$\varrho_{n+1} \leq \frac{\lambda h^3}{2gh - 6b} \left[ \frac{2gh^2 + 9a + 3bh}{9a + 3bh} \right]. \quad (5.3.51)$$

Using the value from 5.3.41, it is easy to see that there exists a constant  $\tilde{C}$  such that

$$\varrho = \max_{-1 \leq i \leq n+1} \{\varrho_i\} \leq \tilde{C}h^2. \quad (5.3.52)$$

The above inequality enables us to estimate  $\|S(x) - Y_n(x)\|_\infty$ , and hence  $\|\hat{U}^{m+1}(x) - S(x)\|_\infty$ . In particular, we will have

$$S(x) - Y_n(x) = \sum_{i=-1}^{n+1} (c_i - b_i)B_i(x). \quad (5.3.53)$$

Thus,

$$|S(x) - Y_n(x)| = \max |c_i - b_i| \sum_{i=-1}^{n+1} |B_i(x)|. \quad (5.3.54)$$

Since

$$\sum_{i=-1}^{n+1} |B_i(x)| \leq 10, \quad x_{-\infty} \leq x \leq x_\infty, \quad (\text{using Lemma 5.3.2}). \quad (5.3.55)$$

Combining Eqs. (5.3.52), (5.3.54) and (5.3.55), we see that

$$\|S - Y_n\|_\infty \leq 10\tilde{C}h^2. \quad (5.3.56)$$

Moreover,

$$\|\hat{U}^{m+1} - Y_n\|_\infty \leq \zeta_0 h^4$$

and

$$\|\hat{U}^{m+1}(x) - S(x)\|_\infty \leq \|\hat{U}^{m+1}(x) - Y_n\|_\infty + \|Y_n - S(x)\|_\infty.$$

This implies that

$$\|\hat{U}^{m+1}(x) - S(x)\|_\infty \leq Mh^2, \quad (5.3.57)$$

where  $M = 10\tilde{C} + \zeta_0 h^2$ .

□

We have therefore proved the following main result.

**Theorem 5.3.3** *Let  $U(x, \tau)$  be the solution of problem (5.3.8) and  $S(x, \tau^m)$  be the B-spline collocation approximation from the space  $S^3(\omega)$  to the solution  $U(x, \tau^m)$ . If  $f(x, \tau^m) \in C^2[x_{-\infty}, x_{\infty}]$ , then under the hypotheses of Theorems 5.3.1 and 5.3.2, the error estimate is given by*

$$\|U(x, \tau^m) - S(x)\|_{\infty} \leq \widetilde{M}(\Delta\tau + h^2), \quad (5.3.58)$$

where  $\widetilde{M}$  is independent of mesh parameters.

**Proof.** The proof is accomplished by using the results from theorems 5.3.1 and 5.3.2. □

**Remark 5.3.1** To determine the functional relationship between the two step-sizes used in the numerical simulation, we apply the conventional von-Neumann stability analysis for the system (5.3.20). Using

$$c_j^m = \varepsilon^m \exp(i\beta jh), \quad i = \sqrt{-1}, \quad (5.3.59)$$

along with (5.3.20), where  $\varepsilon$  is the growth factor and  $\beta$  is the mode number, we obtain at  $m$ th time level

$$\alpha^- c_{j-1}^{m+1} + \tilde{\alpha} c_j^{m+1} + \alpha^+ c_{j+1}^{m+1} = c_{j-1}^m + 4c_j^m + c_{j+1}^m, \quad \forall j = 0, 1, \dots, n, \quad (5.3.60)$$

where

$$\alpha^- = -r_1 + r_2 + r_3, \quad \tilde{\alpha} = 2r_1 + 4r_3, \quad \alpha^+ = -r_1 - r_2 + r_3,$$

$$r_1 = 6a \frac{\Delta\tau}{h^2}, \quad r_2 = 3b \frac{\Delta\tau}{h}, \quad r_3 = 1 + r\Delta\tau.$$

Using equation (5.3.59) and the recurrence relation (5.3.60), we get

$$\varepsilon^{m+1} [\alpha^- \exp(-i\beta h) + \tilde{\alpha} + \alpha^+ \exp(i\beta h)] = \varepsilon^m [\exp(-i\beta h) + 4 + \exp(i\beta h)]. \quad (5.3.61)$$

Equation (5.3.61) can be rewritten in a simple form as

$$\varepsilon = \frac{X_1}{X - iY}, \quad (5.3.62)$$

where  $X_1$ ,  $X$  and  $Y$  are expressed as

$$\begin{aligned} X_1 &= 2 \left[ 1 + 2 \cos^2 \left( \frac{\beta h}{2} \right) \right], \\ X &= 2r_3 \left[ 1 + 2 \cos^2 \left( \frac{\beta h}{2} \right) \right] + 4r_1 \left[ 1 - \cos^2 \left( \frac{\beta h}{2} \right) \right], \\ Y &= 2r_2 \sin(\beta h). \end{aligned}$$

We note that  $X_1 < X$ , and therefore

$$|\varepsilon| = \sqrt{\frac{X_1^2}{X^2 + Y^2}} < 1.$$

Hence the proposed numerical scheme is unconditionally stable.

### 5.3.3 Numerical computation of the American put option

It is well known that the American options valuation can be treated as a free boundary value problem and until recently no analytical formula is available. The American options allow early exercise at any time  $\tau \in [0, T]$  with optimal exercise stock value  $S = S_f(\tau)$ . The difficulty for most numerical methods to an accurate solution for the American options is due to the unknown free boundary  $S_f(\tau)$ . To satisfy this early optimal exercise, the Black-Scholes equation for the American put options valuation

(5.2.1) and (5.2.2) is imposed by Wilmott *et al.* [155] as

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial U}{\partial x} - rU, \quad x > x_{\text{opt}}(\tau)$$

$$U(x, \tau) = \max\{U(x, \tau), U(x, 0)\}, \quad x \leq x_{\text{opt}}(\tau)$$
(5.3.63)

where  $x_{\text{opt}}(\tau) = \log(S_f(\tau))$  is the corresponding optimal exercise point due to the transformations  $\tau = T - t$  and  $S = e^x$  and  $U(x, 0) = E - e^x$  is the transformed payoff value given by the equation  $V(S, T) = \max(E - S, 0)$ . The region  $x \leq x_{\text{opt}}(\tau)$  corresponds to where the American options should be exercised early to attain the optimal value  $U(x, \tau)$ .

The difficulty to solve equation (5.3.63) is due to the unknown optimal exercise point  $x_{\text{opt}}(\tau)$ . The valuation of the American put options can easily be performed by modifying the boundary update procedure for the European put options. To satisfy this early optimal exercise for the valuation of the American put options, we used the update procedure as mentioned by Hon [67], we update at each time level in the valuation of the European put option, the elements of  $U^m$  by

$$U^m(j) = \max\{E - e^{x_j}, U^m(j)\}.$$

This makes the valuation of the American options relatively simple. Note that in the physical point of view, the difference between pricing the American options and the European options is the propagation process as an effect of the moving of the unknown free boundary  $x_{\text{opt}}(\tau)$ . This places an additional restriction at any time  $\tau$  on the solution that its value must be at least  $U(x, 0)$  (see [157] for further details).

We now go back to equation (5.3.63). Following lemma will be useful in obtaining the total error when the update procedure is used.

**Lemma 5.3.3** [157] *Let  $\nu, w, z$  be any functions or vectors. Then*

$$|\max\{\nu, w\} - \max\{\nu, z\}| \leq |w - z|,$$

where the inequality holds for functions and every entry of vectors point-wise.

**Proof.** Lemma 5.3.3 follows simply from the following facts that when  $w > z$ ,

$$\max\{\nu, w\} = \begin{cases} \nu, & \nu > w > z, \\ w, & w > \nu > z, \\ w, & w > z > \nu, \end{cases}$$

and

$$\max\{\nu, z\} = \begin{cases} \nu, & \nu > w > z, \\ \nu, & w > \nu > z, \\ z, & w > z > \nu. \end{cases}$$

Similar situation occur when  $w < z$ .

Hence,

$$\max\{\nu, w\} - \max\{\nu, z\} = \begin{cases} 0, & \nu > w > z, \\ w - \nu < w - z, & w > \nu > z, \\ w - z, & w > z > \nu. \end{cases}$$

□

Comparing Eqs. (5.3.8) and (5.3.63), we can see that both Eqs. (5.3.8) and (5.3.63) satisfy a diffusion process but equation (5.3.63) has a restriction that its solution values must be greater than or equal to  $U(x, 0)$ . This turns out to be an easy task in our proposed method by making a simple updating of data at every time step  $\tau + \Delta\tau$  as follows. The difference between solving Eqs. (5.3.8) and (5.3.63) is that solution (5.3.63) needs an extra updating of solution values at every time step  $\tau + \Delta\tau$ . The total

error is in fact can be obtained by using Lemma 5.3.3. From Theorem 5.3.2 we then obtain an estimate of the same convergence order of solution for equation (5.3.63). In conclusion, we have

**Theorem 5.3.4** *Let  $S(x)$  be the approximation from the space of cubic splines  $S^3(\omega)$  to the solution  $U(x)$  of the problem (5.3.63). Then the error can be estimated as*

$$\|U(x) - S(x)\|_{\infty} \leq \mathcal{O}(h^2).$$

From Lemma 5.3.3, we finally have

**Theorem 5.3.5** *The error estimate given in Theorem 5.3.3 also holds for the free boundary-value problem.*

## 5.4 Numerical results

In this section, we present some numerical results for the solution of the Black-Scholes equation describing European and American put options. Note that algorithmically first we solve a European option pricing problem and then we use an update procedure, therefore, it is important to give comparative numerical results obtained for European option pricing problem too so that the update procedure is pre-justified.

In Table 5.4.1, we have tabulated the comparative results. It consists the exact, Quasi-RBFs and B-spline solutions for the European put option for  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ . The numerical result shown in Table 5.4.1 indicate that the B-spline approach provide a highly accurate approximation to the solution of the European option. The actual error is controlled via the numbers  $n$  and  $m$ .

To demonstrate the accuracy of this B-spline method for the American put options, we have tabulated the binomial, Quasi-RBFs and B-spline solutions (see Table 5.4.2). Numerical simulations are done with  $E = 100$ ,  $r = 0.1$ ,  $T = 1$ , and  $\sigma = 0.30$ . In

Table 5.4.1: Comparison between the exact, Quasi-RBFs [67] and B-spline solutions for the European put option.  $E = 10$ ,  $r = 0.05$ ,  $T = 0.5$ , and  $\sigma = 0.20$ , with different time and space step-sizes.

$S$	Exact solution	Quasi-RBF solution [67]	B-spline solutions	
			$\Delta x = 0.006$ $\Delta t = 0.0004$	$\Delta x = 0.003$ $\Delta t = 0.0002$
2.00	7.7531	7.7531	7.7531	7.7531
4.00	5.7531	5.7531	5.7531	5.7531
6.00	3.7532	3.7532	3.7532	3.7532
7.00	2.7568	2.7568	2.7568	2.7568
8.00	1.7987	1.7988	1.7987	1.7987
9.00	0.9880	0.9881	0.9881	0.9880
10.00	0.4420	0.4420	0.4420	0.4420
11.00	0.1606	0.1606	0.1606	0.1606
12.00	0.0483	0.0483	0.0484	0.0483
13.00	0.0124	0.0124	0.0124	0.0124
14.00	0.0028	0.0028	0.0028	0.0028
15.00	0.0006	0.0006	0.0006	0.0006
16.00	0.0001	0.0001	0.0001	0.0001

Table 5.4.2: Comparison between the Binomial, Quasi-RBFs [67] and B-spline solutions for the American put option.  $E = 100$ ,  $r = 0.1$ ,  $T = 1$ , and  $\sigma = 0.30$ , with different time and space step-sizes.

$S$	Binomial solution [67]	Quasi-RBF solution [67]	B-spline solutions	
			$\Delta x = 0.01$ $\Delta t = 0.001$	$\Delta x = 0.001$ $\Delta t = 0.0001$
80	20.2689	20.2655	20.2665	20.2684
85	16.3467	16.3427	16.3438	16.3450
90	13.1228	13.1185	13.1173	13.1202
95	10.4847	10.4813	10.4818	10.4826
100	8.3348	8.3363	8.3367	8.3373
105	6.6071	6.6020	6.6021	6.6027
110	5.2091	5.2079	5.2066	5.2084
115	4.0976	4.0935	4.0935	4.0938
120	3.2059	3.2072	3.2069	3.2074

this computation, we take  $x \in [-5, 5.5]$  so that  $S \in [e^{-5}, e^{5.5}]$ . The numerical result shown in Table 5.4.2 also indicate that the B-spline approach provide a highly accurate approximation to the solution of the American option.

## 5.5 Summary and discussions

A numerical method is developed to solve the nonlinear Black-Scholes partial differential equation for modelling European and American options pricing on a single asset. This method is based on the implicit Euler method for the temporal discretization and the B-spline collocation method in the spatial direction on a uniform mesh. Numerical results show that the B-spline collocation method, offers a very high accuracy in the computations of both European and American options.

The free boundary condition in the valuation of American options usually places a great difficulty to most existing numerical methods for obtaining an accurate approximation. This, however, does not apply to this proposed method because we first evaluate an analogous European option and then use an update procedure to evaluate the actual American option. As can be seen from the tabular results, the proposed approach gave the results which are comparable with those obtained by quasi-radial basis functions [67].

In next chapter, we will explore the use of B-splines to solve a class of exotic options.

## Chapter 6

# B-spline approximation method for pricing the barrier options

Barrier options are financial derivative contracts that are activated or extinguished when the price of the underlying asset crosses a certain level. Most models for pricing barrier options assume continuous monitoring of the barrier. However in practice many (if not all) barrier options traded in markets are discretely monitored.

There are two main types of difficulties in solving problems for discrete barrier options: I. When the barrier is discretely monitored, a numerical method may be used to value the option. However this method will increase calculation time exponentially with the numbers of barrier. II. For problems pricing discrete barrier options, the trinomial method is useful, but it is less effective when the barrier is very close to the current asset price.

In order to resolve these two problems, we construct a new class of numerical method. This method is based on the spline approximations for the solution of the nonlinear Black-Scholes partial differential equation modeling barrier options. We use the classical Euler implicit method for the time discretization and the B-spline collocation

tion method for the spatial discretization. The method is shown to be unconditionally stable and accurate of order  $\mathcal{O}((\Delta x)^2 + \Delta\tau)$ . The computational performance of the proposed method is compared with other methods seen in the literature.

## 6.1 Introduction

There is a very large over-the-counter market in which financial institutions sell a variety of exotic options tailored to meet particular client demands. Examples of these options include Asian, Lookback and Barrier options. Asian options feature payoffs which depend on the average price of the underlying asset during the contract. Lookback options depend on the highest or lowest price reached by the underlying asset. Barrier options are financial derivatives that are activated or extinguished when the price of the underlying asset crosses a certain level. Barrier options can take either American or European forms, and despite their seemingly complex payoffs, they are widely used in the markets and are generally cheaper than plain vanilla options.

Options with the barrier feature are considered to be the simplest types of path dependent options. Barrier option's distinctive feature is that the payoff depends not only on the final price of the underlying asset, but also on whether the asset price has breached (one-touch) some barrier level during the life of the option. Barrier options can be classified into knock-out and knock-in options. Assuming that the barrier price is  $X$ , the knock-out option can be exercised unless the asset price  $S$  reaches the barrier  $X$  during the day of purchase and expiration day. The knock-in option can be exercised if the asset price  $S$  overtakes the barrier  $X$ . The knock-out options can be classified into "up-and-out" and "down-and-out". The up-and-out option can be exercised unless the asset price  $S$  reaches the barrier  $X$  from below the barrier and the down-and-out option can be done unless the asset price reaches the barrier from above the barrier. The knock-in options can be classified into "up-and-in" and "down-and-in". The up-and-in option can be exercised if the asset reaches the barrier from below the barrier

whereas the down-and-in option can be exercised if the asset price reaches the barrier from above the barrier.

Barrier options have become widespread, particularly for foreign currency contracts. There are also a variety of other instruments with similar kinds of contingent payoffs, including capped options, ladder options, and interest rate corridors. The problem can be readily generalized to incorporate early exercise, although we must then find solutions numerically. In principle, barrier features may be applied to any options. The valuation algorithms of the options are almost similar and therefore we discuss only the down-and-out option.

As far as the relevant research in this direction is concerned, we list some of them below.

Arciniega and Allen [5] analyzed the fully implicit and Crank–Nicolson difference schemes for solving option prices. They proved that the error expansions for the difference methods have the correct form for applying Richardson extrapolation to increase the order of accuracy of the approximations. They applied the difference methods to European, American, and down-and-out knock-out call options. Their computational results indicated that Richardson extrapolation significantly decreases the amount of computational work in estimation of option prices.

Boyle and Tian [15] considered an explicit finite difference approach to solve problem of pricing barrier options. They discuss the issue of aligning grid points with barriers by constructing a grid which lies right on the barrier and mentioned that if necessary, interpolation can be used to find the option value corresponding to the initial stock price.

In [48], Figlewski and Gao illustrated the application of an adaptive mesh technique to solve a barrier option problem. Their basic idea is to use a fine mesh in regions where

it is required (e.g. close to a barrier) and to graft the computed results from this onto a coarser mesh which is used in other regions. This is an interesting approach and would appear to be both quite efficient and flexible, though in their chapter, they only examined a simple case of a down-and-out European call option with a flat, continuously monitored barrier.

Goto *et al.* [60] described the valuation scheme of European, barrier, and Asian options of single asset by using radial basis function approximation. They discretized the equation with Crank–Nicolson scheme and then the option price was approximated with the radial basis functions. They showed that for European and barrier options, the prices are governed by the Black–Scholes equation, but the governing equation for Asian options is different from them. To solve the latter, they adopted another radial basis functions than that for the original Black–Scholes equation.

Zvan *et al.* [166], proposed an implicit method which has superior convergence (when the barrier is close to the region of interest) and stability properties as well as offering additional flexibility in terms of constructing the spatial grid. Their method also allows to place grid points either near or exactly on barriers. They in fact presented an implicit method which can be used for PDE models with general algebraic constraints on the solution. Examples of constraints can include early exercise features as well as barriers. Also in their method, barrier options with or without American constraints can be handled.

For some further reading on barrier options, the reader may refer to [20, 21, 69, 71, 106, 111, 133, 147, 153, 154].

The rest of the chapter is organized as follows. In Section 6.2, we describe the differential equation model that price a down-and-out barrier option. Construction of the numerical method is given in Section 6.3. In Section 6.4, we analyze the complete

method for convergence. Comparative numerical results are presented in Section 6.5. Finally, we summarize the main findings in Section 6.6.

## 6.2 Problem description

Consider the portfolio of one European in-option and one European out-option; both having the same barrier, strike price and date of expiration. The sum of their values is simply the same as that of a corresponding European option with the same strike price and date of expiration. This is obvious since only one of the two barrier options survives at expiry and either payoff is the same as that of the European option. Hence, provided there is no rebate payment upon knock-out, we have [105]

$$C_{\text{ordinary}} = C_{\text{down-and-out}} + C_{\text{down-and-in}}, \quad (6.2.1)$$

$$P_{\text{ordinary}} = P_{\text{down-and-out}} + P_{\text{down-and-in}}, \quad (6.2.2)$$

where  $C$  and  $P$  denote call and put values, respectively, and  $C_{\text{ordinary}}$  and  $P_{\text{ordinary}}$  are the Black-Scholes ordinary options cost. Above relations imply that the value of an in-option can be found easily once the value of the corresponding out-option is available, and vice versa.

In this section, we shall consider the down-and-out option with the exercise price  $E$  and the barrier  $X$ . The option becomes invalid if the asset price  $S$  reaches the barrier  $E$  from above the barrier during the day of purchase and the expiration date. Unless the asset price  $S$  reaches the barrier  $E$ , i.e.,  $S > X$ , the option is a European call option.

The value of the down-and-out option, denoted by  $V = V(S, t)$ , is governed by the

equations

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (S > X), \quad (6.2.3)$$

$$V = 0 \quad (S \leq X), \quad (6.2.4)$$

where  $S$  is the current value of the underlying asset at time  $t$ ,  $\sigma$  is the annual volatility of the underlying asset,  $T$  is the expiry time and  $r$  is the interest rate.

The strike condition on the expiration day is given by

$$V(S, T) = \max(S(T) - E, 0). \quad (6.2.5)$$

As  $S$  becomes large, the likelihood of the barrier being activated becomes negligible and so assuming no dividends are paid, we have

$$V(S, t) \sim S \quad \text{as } S \rightarrow \infty. \quad (6.2.6)$$

The problem looks identical to that for a vanilla call (see [155]). However, it differs in that the second boundary condition is applied at  $S = X$  rather than  $S = 0$ . If  $S$  reaches  $X$ , the option is invalid; thus on the line  $S = X$  the value of the option is zero, i.e.,

$$V(X, t) = 0 \quad \text{on } S = X. \quad (6.2.7)$$

Therefore, the payoff  $K$  is

$$K = \begin{cases} \max(S - E, 0) & (S > X), \\ 0 & (S \leq X). \end{cases} \quad (6.2.8)$$

To keep this chapter self-contained, we again discuss the reduction of the above problem to a standard diffusion equation [155].

We use a log transformation to transform the equation (6.2.3) as a standard diffusion equation. With the transformations

$$S = Ee^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V(S, t) = E \exp \left[ -\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau \right] u(x, t), \quad (6.2.9)$$

and setting  $k = 2r/\sigma^2$  which transforms the barrier to

$$x_0 = \log(X/E),$$

we see that the barrier option problem (6.2.3) becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \quad (6.2.10)$$

The payoff (6.2.8) is transformed to

$$u(x, 0) = u^0(x) = \max \left( \exp \left[ \frac{1}{2}(k-1)x \right] - \exp \left[ \frac{1}{2}(k+1)x \right], 0 \right), \quad x \geq x_0, \quad (6.2.11)$$

whereas the boundary conditions become

$$u(x_0, \tau) = 0, \quad (6.2.12)$$

and

$$u(x, \tau) \sim \exp \left[ \frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau \right] = u_\infty(\tau) \quad \text{as } x \rightarrow \infty. \quad (6.2.13)$$

We solve problem (6.2.10)-(6.2.13) using a B-spline described in the next section and then recover the solution of the original problem (6.2.3)-(6.2.8) using (6.2.9).

### 6.3 Construction of the numerical method

Our numerical method is based on B-spline for the discretization in space and the finite difference techniques for the temporal one. We consider a two-dimensional grid as follows:

Let  $\Delta\tau$  and  $\Delta x$ , be the mesh step-sizes in the  $\tau$  and  $x$ -directions. The step-size in  $\tau$ -direction is given by  $\Delta\tau = \tau_{\max}/m$  with  $\tau_{\max} = \sigma^2 T/2$  where  $m$  is an integer.

The calculation of the step-size for the  $x$ -discretization is little complicated. The infinite interval  $-\infty < x < \infty$  must be replaced by a finite interval  $x_{-\infty} \leq x \leq x_{\infty}$ . The end values  $x_{-\infty} = x_{\min} < 0$  and  $x_{\infty} = x_{\max} > 0$  should be chosen in such a way that for  $S_{\min} = Ee^{x_{-\infty}}$ ,  $S_{\max} = Ee^{x_{\infty}}$  and the interval  $S_{\min} \leq S \leq S_{\max}$ , a sufficient smooth approximation is obtained. Then for a suitable integer  $n$  the step length in  $x$  is defined by  $\Delta x = (x_{\infty} - x_{-\infty})/n$ . This defines a two-dimensional uniform grid.

Note that the equidistant grid is defined in terms of  $x$  and  $\tau$ , and not for  $S$  and  $t$ . Transforming the  $(x, \tau)$ -grid via the transformation in (6.2.9) back to the  $(S, t)$ -plane, leads to a nonuniform grid with unequal distances of the grid lines  $S = S_i = Ee^{x_i}$ . The actual error is then controlled via the numbers  $n$  and  $m$  of grid lines.

To discretize (6.2.10) in the temporal discretization, we use an implicit finite difference technique with uniform step-size  $\Delta\tau$  and obtain the following system of linear ordinary differential equations:

$$u^0 = u^0(x), \quad x_0 < x < x_{\infty}, \quad (6.3.1a)$$

$$\frac{u^{m+1} - u^m}{\Delta\tau} = u_{xx}^{m+1}, \quad x_0 < x < x_{\infty}, \quad \tau > 0 \quad (6.3.1b)$$

with the boundary conditions,

$$u^{m+1}(x_0) = u_0(\tau^{m+1}), \quad u^{m+1}(x_\infty) = u_\infty(\tau^{m+1}), \quad (6.3.1c)$$

where  $u^{m+1}$  is the solution of Eq.(6.2.10) at  $(m + 1)^{th}$  time level. Here  $u^m = u(x, \tau^m)$ , where the superscript  $m$  denotes  $m^{th}$  time level, i.e.,  $\tau^m = m\Delta\tau$ .

We rewrite Eq.(6.3.1b) as

$$-\delta u_{xx}^{m+1} + u^{m+1} = u^m, \quad (6.3.2)$$

where  $\delta = \Delta\tau$ .

The solution of (6.3.2) is sought in form of

$$S(x) = \sum_{i=-1}^{n+1} c_i B_i(x), \quad (6.3.3)$$

where  $c_i$  are unknown real coefficients and  $B_i(x)$  are cubic B-spline functions defined as

$$B_i(x) = \begin{cases} \left(\frac{x-x_{i-2}}{h}\right)^3, & \text{if } x \in [x_{i-2}, x_{i-1}], \\ 1 + 3\left(\frac{x-x_{i-1}}{h}\right) + 3\left(\frac{x-x_{i-1}}{h}\right)^2 - 3\left(\frac{x-x_{i-1}}{h}\right)^3, & \text{if } x \in [x_{i-1}, x_i], \\ 1 + 3\left(\frac{x_{i+1}-x}{h}\right) + 3\left(\frac{x_{i+1}-x}{h}\right)^2 - 3\left(\frac{x_{i+1}-x}{h}\right)^3, & \text{if } x \in [x_i, x_{i+1}], \\ \left(\frac{x_{i+2}-x}{h}\right)^3, & \text{if } x \in [x_{i+1}, x_{i+2}], \\ 0, & \text{otherwise.} \end{cases} \quad (6.3.4)$$

It is required that the approximate solution  $S(x)$  satisfies the given problem (6.3.2) at mesh points  $\omega$  as well as boundary conditions at  $x = x_0$  and  $x = x_n$ . Therefore, we have introduced two extra splines  $B_{-1}$  and  $B_{n+1}$  to force  $S(x)$  to satisfy the boundary conditions.

Let  $S(x)$  satisfy the equation (6.3.2), then we have

$$LS(x_j) = f(x_j), \quad 0 \leq x_j \leq n, \quad f(x_j) = u^m(x_j), \quad (6.3.5)$$

where  $Lu^{m+1} \equiv -\delta u_{xx}^{m+1} + u^{m+1}$ , and therefore

$$-\delta \sum_{i=-1}^{n+1} c_i B_i''(x_j) + \sum_{i=-1}^{n+1} c_i B_i(x_j) = f_j, \quad f_j = f(x_j).$$

By solving this equation and noting that the support of the function  $B_i(x)$  is the segment  $[x_{i-2}, x_{i+2}]$ , we have

$$\begin{aligned} c_{j-1}(-\delta B_{j-1}''(x_j) + B_{j-1}(x_j)) + c_j(-\delta B_j''(x_j) + B_j(x_j)) \\ + c_{j+1}(-\delta B_{j+1}''(x_j) + B_{j+1}(x_j)) = f_j, \quad \forall j = 0, 1, \dots, n, \end{aligned} \quad (6.3.6)$$

where we recall that

$$B_i(x_j) = \begin{cases} 4, & \text{if } i = j, \\ 1, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2, \end{cases} \quad (6.3.7)$$

and that  $B_i(x) = 0$  for  $x \geq x_{i+2}$  and  $x \leq x_{i-2}$ .

Furthermore, we can show that

$$B'_i(x_j) = \begin{cases} 0, & \text{if } i = j, \\ \pm \frac{3}{h}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2, \end{cases} \quad (6.3.8)$$

and

$$B''_i(x_j) = \begin{cases} -\frac{12}{h^2}, & \text{if } i = j, \\ \frac{6}{h^2}, & \text{if } i - j = \pm 1, \\ 0, & \text{if } i - j = \pm 2. \end{cases} \quad (6.3.9)$$

By using equations (6.3.7) and (6.3.9) we get

$$(h^2 - 6\delta)c_{j-1} + (4h^2 + 12\delta)c_j + (h^2 - 6\delta)c_{j+1} = h^2 f_j, \quad \forall j = 0, 1, \dots, n. \quad (6.3.10)$$

The boundary conditions (6.2.12) and (6.2.13) becomes

$$c_{-1} + 4c_0 + c_1 = 0, \quad (6.3.11)$$

and

$$c_{n-1} + 4c_n + c_{n+1} = u_\infty(\tau). \quad (6.3.12)$$

The equations (6.3.10), (6.3.11) and (6.3.12) lead to a  $(n + 3) \times (n + 3)$  system of equations with  $(n + 3)$  unknowns  $c_{-1}, c_0, \dots, c_{n+1}$ . By eliminating  $c_{-1}$  from the first



## 6.4 Analysis of the numerical method

As we have pointed out previously, we discretize only the time variable by means of the Euler implicit rule with uniform step-size  $\Delta\tau$ . Using  $L_1 \equiv -\frac{\partial^2}{\partial x^2}$ , we rewrite (6.3.1) in the form

$$u^0 = u^0(x), \quad x_0 < x < x_\infty, \quad (6.4.1a)$$

$$(I + \Delta\tau L_1)u^{m+1} = u^m, \quad x_0 < x < x_\infty, \quad \tau > 0, \quad (6.4.1b)$$

$$u^{m+1}(x_0) = u_0(\tau^{m+1}), \quad u^{m+1}(x_\infty) = u_\infty(\tau^{m+1}), \quad (6.4.1c)$$

which gives semi-discrete approximations  $u^m(x)$  to the exact solution  $u(x, \tau)$  of (6.2.10) at the  $m^{\text{th}}$  time level  $\tau_m$ .

The stability of (6.4.1) follows from the maximum principle for the operator  $I + \Delta\tau L_1$ , because

$$\|(I + \Delta\tau L_1)^{-1}\|_\infty \leq \frac{1}{1 + \tilde{b}\Delta\tau}. \quad (6.4.2)$$

The local truncation error of the time semi-discretization method (6.4.1) is given by

$$e_{m+1} = u(\tau^{m+1}) - \hat{u}^{m+1}, \quad (6.4.3)$$

where  $\hat{u}^{m+1}$  is the solution of

$$(I + \Delta\tau L_1)\hat{u}^{m+1}(x) = u(x, \tau^m), \quad x_0 < x < x_\infty, \quad \tau > 0, \quad (6.4.4a)$$

$$\hat{u}^{m+1}(x_0) = u_0(\tau^{m+1}), \quad \hat{u}^{m+1}(x_\infty) = u_\infty(\tau^{m+1}). \quad (6.4.4b)$$

This error measures the contribution of each time step to the global error of the time semi-discretization which is defined as

$$E_m \equiv u(x, \tau^m) - u^m(x). \quad (6.4.5)$$

Then the following accuracy result follows.

**Lemma 6.4.1** (Local error estimate). *If*

$$\left| \frac{\partial^i}{\partial t^i} u(x, \tau) \right| \leq C_0, \quad x_0 < x < x_\infty, \quad 0 < \tau < \frac{1}{2} \sigma^2 T, \quad 0 \leq i \leq 2, \quad (6.4.6)$$

then the local error satisfies

$$\|e_{m+1}\|_\infty \leq C_0 (\Delta\tau)^2, \quad (6.4.7)$$

where  $C_0$  is a positive constant independent of  $\Delta\tau$ .

**Proof.** Since the function  $\hat{u}^{m+1}$  satisfies

$$(I + \Delta\tau L_1) \hat{u}^{m+1}(x) = u(x, \tau^m),$$

and as the solution of (6.2.10) is smooth enough, we have

$$\begin{aligned} u(\tau^m) &= u(\tau^{m+1}) + \Delta\tau L_1 u(\tau^{m+1}) + \int_{\tau^m}^{\tau^{m+1}} (\tau^m - s) \frac{\partial^2 u}{\partial \tau^2}(s) ds \\ &= (I + \Delta\tau L_1) u^{m+1}(x) + \mathcal{O}(\Delta\tau^2). \end{aligned} \quad (6.4.8)$$

Then  $e_{m+1}$  is the solution of a boundary value problem of type

$$(I + \Delta\tau L_1) e_{m+1} = \mathcal{O}(\Delta\tau^2), \quad e_{m+1}(x_0) = e_{m+1}(x_\infty) = 0. \quad (6.4.9)$$

Thus (6.4.7) follows when applying the stability result (6.4.2).

**Theorem 6.4.1** (Global error estimate). *Under the hypotheses of Lemma 6.4.1, we have*

$$\|E_m\|_\infty \leq C_0 \Delta\tau, \quad \forall m \leq \frac{\sigma^2 T}{2\Delta\tau}. \quad (6.4.10)$$

**Proof.** Using the local error estimate up to the  $m^{\text{th}}$  time level given by Lemma (6.4.1), we get the following global error estimate at  $(m + 1)^{\text{th}}$  time level

$$\begin{aligned}
 \|E_{m+1}\|_{\infty} &= \left\| \sum_{l=1}^m e_l \right\|, \quad m \leq \frac{\sigma^2 T}{2\Delta\tau} \\
 &\leq \|e_1\|_{\infty} + \|e_2\|_{\infty} + \dots + \|e_m\|_{\infty} \\
 &\leq C_1(m\Delta\tau)\Delta\tau \quad \text{using (6.4.7)} \\
 &\leq C_1 \left( \frac{1}{2}\sigma^2 T \right) \Delta\tau \quad \text{since } m\Delta\tau \leq \frac{1}{2}\sigma^2 T \\
 &= C_0\Delta\tau.
 \end{aligned} \tag{6.4.11}$$

Therefore the time semi-discretization process converge with order one.

□

Now we prove that the B-spline collocation method converge with order two in the spatial direction. To proceed with, we first prove the following lemma:

**Lemma 6.4.2** *The B-splines  $B_{-1}, B_0, B_1, \dots, B_{n+1}$  defined in Eq.(6.3.4), satisfy the inequality*

$$\sum_{i=-1}^{n+1} |B_i(x)| \leq 10, \quad x_0 \leq x \leq x_{\infty}.$$

**Proof.** We know that

$$\left| \sum_{i=-1}^{n+1} B_i(x) \right| \leq \sum_{i=-1}^{n+1} |B_i(x)|.$$

At any node  $x_i$ , we have

$$\sum_{i=-1}^{n+1} |B_i| = |B_{i-1}| + |B_i| + |B_{i+1}| = 6 < 10.$$

Also

$$|B_i(x)| \leq 4 \quad \text{and} \quad |B_{i-1}(x)| \leq 4, \quad x_{i-1} \leq x \leq x_i.$$

Similarly

$$|B_{i-2}(x)| \leq 1 \quad \text{and} \quad |B_{i+1}(x)| \leq 1, \quad x_{i-1} \leq x \leq x_i.$$

Therefore for any point  $x_{i-1} \leq x \leq x_i$ , we have

$$\sum_{i=-1}^{n+1} |B_i(x)| = |B_{i-2}| + |B_{i-1}| + |B_i| + |B_{i+1}| \leq 10.$$

□

**Theorem 6.4.2** *Let  $S(x)$  be the approximation from the space of cubic splines  $S^3(\omega)$  to the solution  $\hat{u}^{m+1}(x)$  of the semi-discrete boundary value problem (6.4.4) at the  $(m+1)^{\text{th}}$  time level. If  $f(x) \in C^2[x_0, x_\infty]$ , then the error estimate is given by*

$$\|\hat{u}^{m+1}(x) - S(x)\|_\infty \leq Mh^2,$$

where  $M$  is a positive constant independent of  $h$ .

**Proof.** To estimate the error  $\|\hat{u}^{m+1} - S(x)\|_\infty$ , let us assume that  $Y_n$  be the unique spline interpolant from  $S^3(\omega)$  to the solution  $\hat{u}^{m+1}(x)$  of our semi-discrete boundary value problem (6.4.4). If  $f(x) \in C^2[x_0, x_\infty]$ , then  $\hat{u}^{m+1}(x) \in C^4[x_0, x_\infty]$ , and it follows from the de Boor-Hall error estimates [11] that

$$\|D^j(\hat{u}^{m+1}(x) - Y_n)\|_\infty \leq \zeta_j h^{4-j}, \quad j = 0, 1, 2, \quad (6.4.12)$$

where  $\zeta_j$ 's are constants independent of  $h$  and  $m$ .

Let

$$Y_n(x) = \sum_{i=-1}^{n+1} b_i B_i(x). \quad (6.4.13)$$

It is clear from the estimates (6.4.12) that

$$|LS(x_i) - LY_n(x_i)| = |f(x_i) - LY_n(x_i) + L\hat{u}^{m+1}(x_i) - L\hat{u}^{m+1}(x_i)| \leq \lambda h^2, \quad (6.4.14)$$

where

$$\lambda = [\delta\zeta_2 + \zeta_0 h^2]. \quad (6.4.15)$$

Also

$$LS(x_i) = L\hat{u}^{m+1}(x_i) = f(x_i).$$

Let

$$LY_n(x_i) = \hat{f}_n(x_i), \quad \forall i$$

and

$$\hat{f}^n = (\hat{f}_n(x_0), \hat{f}_n(x_1), \dots, \hat{f}_n(x_n))^T.$$

From system (6.3.15) and (6.4.14), it is clear that the  $i^{\text{th}}$  component of  $A(x^N - y^N)$ , where  $y^N = (b_0, b_1, \dots, b_N)^T$ , satisfies the inequality

$$|[A(x^N - y^N)]_i| = h^2 |f_i - \hat{f}_i| \leq \lambda h^4. \quad (6.4.16)$$

Now

$$(Ax^N)_i = h^2 f(x_i)$$

and

$$(Ay^N)_i = h^2 \hat{f}_i(x_i), \quad \forall i = 0, 1, 2, \dots, n-1.$$

Also

$$(Ax^N)_n = h^2 f(x_\infty) - (h^2 - 6\delta)u_\infty(\tau)$$

and

$$(Ay^N)_n = h^2 \hat{f}_n(x_\infty) - (h^2 - 6\delta)u_\infty(\tau).$$

However, the  $i^{\text{th}}$  component of  $[A(x^N - y^N)]$  is the  $i^{\text{th}}$  equation

$$(h^2 - 6\delta)\eta_{i-1} + (4h^2 + 12\delta)\eta_i + (h^2 - 6\delta)\eta_{i+1} = \xi_i, \quad 1 \leq i \leq n - 1, \quad (6.4.17)$$

where

$$\eta_i = b_i - c_i, \quad -1 \leq i \leq n + 1$$

and

$$\xi_i = h^2[f(x_i) - \hat{f}_n(x_i)], \quad 1 \leq i \leq n - 1.$$

Obviously

$$|\xi_i| \leq \lambda h^4.$$

Let

$$\xi = \max_{1 \leq i \leq n-1} |\xi_i|,$$

and consider

$$\eta = (\eta_{-1}, \eta_0, \dots, \eta_{n+1})^T.$$

Then define

$$\varrho_i = |\eta_i| \quad \text{and} \quad \tilde{\varrho}_i = \max_{1 \leq i \leq n} |\varrho_i|.$$

Eq. (6.4.17) then becomes

$$(4h^2 + 12\delta)\eta_i = \xi_i + (6\delta - h^2)(\eta_{i-1} + \eta_{i+1}), \quad 1 \leq i \leq n - 1. \quad (6.4.18)$$

Taking absolute value and simplifying, we have

$$(4h^2 + 12\delta)\varrho_i \leq \xi + 2\tilde{\varrho}_i(6\delta - h^2). \quad (6.4.19)$$

Therefore,

$$(4h^2 + 12\delta)\varrho_i \leq \xi + 2\tilde{\varrho}_i(6\delta - h^2) \leq \xi + 2\tilde{\varrho}_i(6\delta + h^2).$$

In particular,

$$(4h^2 + 12\delta)\tilde{\varrho} \leq \xi + 2\tilde{\varrho}(6\delta + h^2). \quad (6.4.20)$$

Solving for  $\tilde{\varrho}$ , we obtain

$$2h^2\tilde{\varrho} \leq \xi \leq \lambda h^4,$$

which gives

$$\tilde{\varrho} \leq \frac{1}{2}\lambda h^2. \quad (6.4.21)$$

Now to estimate  $\varrho_{-1}$ ,  $\varrho_0$ ,  $\varrho_n$  and  $\varrho_{n+1}$ , we observe that the first equation of the system  $A(x^N - y^N) = h^2(f^n - \hat{f}^n)$ , where  $f^n = (f_0, f_1, \dots, f_n)$  yields

$$36\delta\eta_0 = h^2(f_0 - \hat{f}_0),$$

which gives

$$\varrho_0 \leq \frac{\lambda h^4}{36\delta}. \quad (6.4.22)$$

Similarly, we obtain

$$\varrho_n \leq \frac{\lambda h^4}{36\delta}. \quad (6.4.23)$$

Now  $\varrho_{-1}$  and  $\varrho_{n+1}$  can be evaluated using the boundary conditions given by Eqs. (6.3.11) and (6.3.12) (note that  $\eta_{-1} = (0 - 4\eta_0 - \eta_1)$  and  $\eta_{n+1} = (-4\eta_n - \eta_{n-1})$ ) as

$$\varrho_{-1} \leq \frac{\lambda h^4}{9\delta} + \frac{1}{2}\lambda h^2 \quad (6.4.24)$$

and

$$\varrho_{n+1} \leq \frac{\lambda h^4}{9\delta} + \frac{1}{2}\lambda h^2. \quad (6.4.25)$$

Using 6.4.15, it is easy to see that there exists a constant  $\tilde{C}_0$  such that

$$\varrho = \max_{-1 \leq i \leq n+1} \{\varrho_i\} \leq \tilde{C}_0 h^2. \quad (6.4.26)$$

The above inequality enables us to estimate  $\|S(x) - Y_n(x)\|_\infty$ , and hence  $\|\hat{u}^{m+1}(x) - S(x)\|_\infty$ . In particular, we will have

$$S(x) - Y_n(x) = \sum_{i=-1}^{n+1} (c_i - b_i) B_i(x). \quad (6.4.27)$$

Thus

$$|S(x) - Y_n(x)| = \max |c_i - b_i| \sum_{i=-1}^{n+1} |B_i(x)|. \quad (6.4.28)$$

Since

$$\sum_{i=-1}^{n+1} |B_i(x)| \leq 10, \quad x_0 \leq x \leq x_\infty, \quad (\text{using Lemma 6.4.2}). \quad (6.4.29)$$

Combining (6.4.26), (6.4.28) and (6.4.29), we see that

$$\|S - Y_n\|_\infty \leq 10\tilde{C}_0 h^2. \quad (6.4.30)$$

Moreover,

$$\|\hat{u}^{m+1} - Y_n\|_\infty \leq \zeta_0 h^4$$

and

$$\|\hat{u}^{m+1}(x) - S(x)\|_\infty \leq \|\hat{u}^{m+1}(x) - Y_n\|_\infty + \|Y_n - S(x)\|_\infty.$$

This implies that

$$\|\hat{u}^{m+1}(x) - S(x)\|_\infty \leq M h^2, \quad (6.4.31)$$

where  $M = 10\tilde{C}_0 + \zeta_0 h^2$ .

□

We have therefore proved the following main result.

**Theorem 6.4.3** *Let  $u(x, \tau)$  be the solution of problem (6.2.10) and  $S(x, \tau^m)$  be the collocation approximation from the space  $S^3(\omega)$  to the solution  $u(x, \tau^m)$ . If  $f(x, \tau^m) \in C^2[x_0, x_\infty]$ , then under the hypotheses of Theorems 6.4.1 and 6.4.2, the error estimate*

is given by

$$\|u(x, \tau^m) - S(x)\|_\infty \leq \widetilde{M}(\Delta\tau + h^2), \quad (6.4.32)$$

where  $\widetilde{M}$  is independent of mesh parameters.

**Proof.** The proof is accomplished by using the results from theorems 6.4.1 and 6.4.2. □

**Remark 6.4.1** As in the previous cases, to determine the functional relationship between the two step-sizes used in the numerical simulation, we apply the conventional von-Neumann stability analysis for the system (6.3.10). Using

$$c_j^m = \varepsilon^m \exp(i\beta jh), \quad i = \sqrt{-1}, \quad (6.4.33)$$

along with (6.3.10), where  $\varepsilon$  is the growth factor and  $\beta$  is the mode number, we obtain at  $m^{\text{th}}$  time level

$$\alpha c_{j-1}^{m+1} + \tilde{\alpha} c_j^{m+1} + \alpha c_{j+1}^{m+1} = c_{j-1}^m + 4c_j^m + c_{j+1}^m, \quad \forall j = 0, 1, \dots, n, \quad (6.4.34)$$

where

$$\alpha = 1 - r_1, \quad \tilde{\alpha} = 4 + r_2, \quad r_1 = 6 \frac{\Delta\tau}{h^2}, \quad r_2 = 2r_1.$$

Using Eq. (6.4.33) and the recurrence relation (6.4.34), we get

$$\varepsilon^{m+1} [\alpha \exp(-i\beta h) + \tilde{\alpha} + \alpha \exp(i\beta h)] = \varepsilon^m [\exp(-i\beta h) + 4 + \exp(i\beta h)] \quad (6.4.35)$$

which implies that

$$\varepsilon = \frac{3 - 2 \sin^2(\frac{\beta h}{2})}{3 - 2 \sin^2(\frac{\beta h}{2}) + 2r_1 \sin^2(\frac{\beta h}{2})}. \quad (6.4.36)$$

Clearly,  $0 < \varepsilon \leq 1$  for all  $r_1 > 0$  and all  $\beta$ . Therefore, the proposed numerical method is unconditionally stable.

## 6.5 Numerical results

In this section, we present some numerical experiments to find the numerical solutions of Black–Scholes equation which describe the European down-and-out call options. The parameters used in the numerical simulations are

Expiration date	$T = 0.5$ (year)
Exercise price	$E = 10.0$
Risk free interest rate	$r = 0.05$
Volatility	$\sigma = 0.2$
Barrier value	9.0
Transformed time-step size	$\Delta(\tau) = 0.00001$
Transformed space-step size	$\Delta(x) = 0.005$

In Table 6.5.1 we have tabulated the comparative results. It contains the exact, radial basis function approximations as in [60] and B-spline solution obtained by our method.

Table 6.5.1: Comparative numerical results for a European down-and-out call option

Stock $S$	Exact	$V_{MQ}$	$V_{RMQ}$	B-spline
1.00	0.0000	0.0000	0.0000	0.0000
3.00	0.0000	0.0000	0.0000	0.0000
5.00	0.0000	0.0000	0.0000	0.0000
7.00	0.0000	0.0000	0.0000	0.0000
9.00	0.0000	0.0000	0.0000	0.0000
11.00	1.3998	1.3985	1.3985	1.3997
13.00	3.2591	3.2589	3.2589	3.2592
15.00	5.2475	5.2474	5.2474	5.2475
17.00	7.2469	7.2469	7.2465	7.2469
19.00	9.2469	9.2466	9.2483	9.2469

$V_{MQ}$  : Approximate solution using Multi-Quadratic radial basis function [60].

$V_{RMQ}$ : Approximate solution using Reciprocal Multi-Quadratic radial basis function [60].

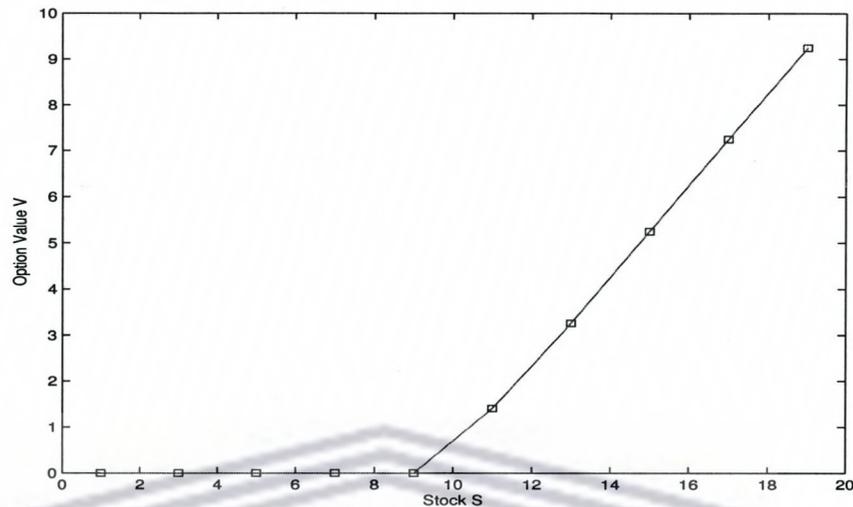


Figure 6.5.1: Numerical values of European down-and-out call option obtained via B-spline approach. Here we use  $T = 6/12$ ,  $E = 10$ ,  $\sigma = 0.20$ ,  $\Delta\tau = 0.0001$ ,  $\Delta x = 0.05$  and  $x \in (x_0, 10)$

## 6.6 Summary and discussions

Exotic options are now widely used in global financial markets such as barrier options. Their popularity calls for the development of faster and more stable numerical methods. In general, a closed-form valuation equation exists only in European options with a continuous barrier. For discrete barrier options, some difficulties arise in the pricing process. The majority of valuation methods are based on a lattice or other correction methods, which are limited to handle this feature. In this chapter, we develop an alternative evaluation model using B-spline to solve the problem. This method is based on the implicit Euler method for the temporal discretization and the B-spline collocation method in the spatial direction on a uniform mesh. The method is shown to be uniformly convergent. As is seen from the tabular results, the proposed approach gave the results which are comparable with those obtained by the radial basis function approximation [60].

## Chapter 7

# Concluding remarks and scope for future research

This thesis deals with the application of robust numerical methods to solve option pricing problems.

A basic theory and some properties of spline functions are described in Chapter 2, these properties are very useful in proving some of the theoretical results.

In Chapter 3, we have made a thorough comparison of various numerical methods to solve a typical option pricing problems. This includes, the application of method of lines and cubic spline interpolation to discretize the problem in the spatial direction. For the time integration of the system obtained via method of lines, we have used a number of MATLAB ode solvers whereas for the one obtained by using cubic spline, we use an implicit Euler method. We also presented results obtained via B-spline in this chapter. After comparing, we found that the results obtained by B-spline are more suitable for a number of reasons which were indicated in subsequent chapters. Also it is noteworthy that B-splines have the smallest support size among all splines and therefore, we decided to use them further to solve other option pricing problems.

Chapter 4 deals with a thorough derivation of B-spline for solving problem that price a European option. The method is analyzed for stability and convergence.

This method presented in Chapter 4 was extended in Chapter 5 to solve American option problems where a derivation of the method is discussed by reducing the problem to a constant coefficient problem. Then using an update procedure, we solve the American option problem.

In Chapter 6 we solve some exotic options.

Regarding the scope for future research, we indicate the following:

- The applicability of Spline approximation schemes for the solution of multi-asset American option problems. In this case we intend to use multivariate splines in the spatial direction.
- We also intend to investigate the applicability of B-splines to solve American options with stochastic volatility.
- In Chapter 5 we used an update procedure to solve the American option problem. Currently we are investigating the use of B-splines techniques to approximate the spatial derivatives with the penalty approach to handle the Black-Scholes partial differential equation for both single and two-asset American options.
- The proposed approach can also be extended to solve some Jump-diffusion models in option pricing theory.
- The cases when volatility is very small, we may further design some exponentially fitted methods (those originally designed to solve singularly perturbed problems).
- We can also extend the B-splines techniques to the Heston partial differential equation ([63]) which plays an important role in financial option pricing theory.

A feature of this time-dependent, two-dimensional convection-diffusion-reaction equation is the presence of a mixed spatial-derivative term, which stems from the correlation between the two underlying stochastic processes for the asset price and its variance.

- Since the invention of finite element methods in 1950s, there have been demands in construction of smooth finite element shape functions over discretizations of an arbitrary domain of multiple dimensions. This is because in many engineering applications the related Galerkin weak formulations may be involved with higher order derivatives of unknown functions. It is known that the B-splines form a basis of spline spaces and it form a partition of unity, we may further design some numerical methods based on B-splines basis functions.

The particular approach that we intend to investigate is the Reproducing Kernel Element Method (RKEM). This RKEM is constructed by combining the virtues of finite element approximations and reproducing kernel particle approximations. In RKEM, the global partition polynomials are patched together by associating them with compactly supported functions defined through a kernel to satisfy the required reproducing conditions [110]. Such RKEM are advantageous to the usual FEM in the sense that (i) The smoothness of the global basis functions is solely determined by that of the kernel function, and is not limited by the smoothness of the finite elements, and (ii) The global basis functions of RKEM have the Kronecker delta property at the associated nodes, provided that some conditions on the support size of the kernel function are met.

- Toward an attempt to improve accuracy in space, one may also think of using high order splines, for example, Quintic B-splines, Sextic B-splines, or Septic B-splines. The only challenge would be to adopt the initial and boundary conditions appropriately. Furthermore, in order to achieve higher order accuracy, some extrapolation may be used.

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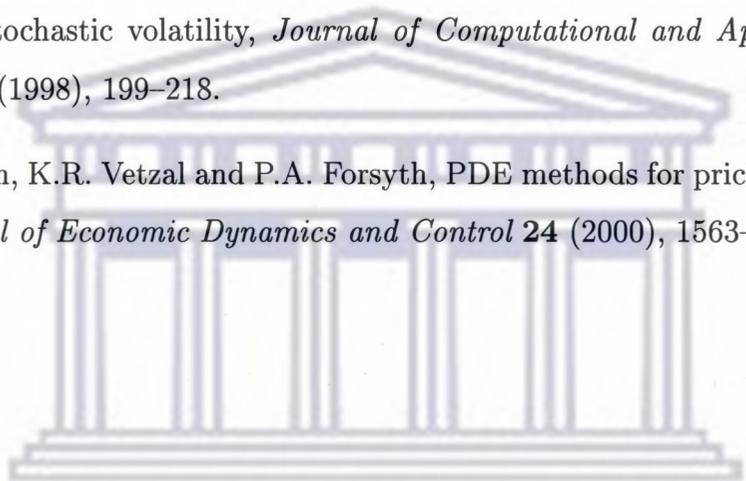
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