Block Toeplitz Operators with Rational Symbols and Discrete Singular Systems

by

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Chapter 0

Introduction

This thesis concerns block Toeplitz operators (equations). Consider the block Toeplitz operator $T = [\Phi_{k-j}]_{k,j=0}^{\infty}$, where the Φ_k are complex $m \times m$ matrices such that

$$(0.1) \qquad \sum_{\nu=-\infty}^{\infty} \|\Phi_\nu\| < \infty.$$

The norm in (0.1) is the usual operator norm on an $m \times m$ matrix. The condition (0.1) means that the *symbol*

$$(0.2) \qquad \Phi(\lambda) = \sum_{\nu=-\infty}^{\infty} \lambda^\nu \Phi_\nu, \qquad |\lambda| = 1,$$

belongs to the Wiener class $\mathcal{W}^{m\times m}$ of all absolutely convergent sequences of complex $m \times m$ matrices. Let $1 \leq p \leq \infty$ be fixed. The block Toeplitz operator T induces a bounded linear operator (also denoted by T) on l_p^m, namely,

$$(0.3) \qquad y_k = (Tx)_k = \sum_{\nu=0}^{\infty} \Phi_{k-\nu} x_\nu, \qquad k = 0, 1, 2, \ldots,$$

where $x = (x_0, x_1, x_2, \ldots) \in l_p^m$.

Here Φ_k, $k = 0, \pm 1, \pm 2, \ldots$, are the Fourier coefficients of a rational $m \times m$ matrix function Φ given by (0.2). In [BGK1, BGK2], equation (0.3) was analyzed and solved explicitly for the case when the symbol Φ is both analytic at infinity and $\Phi(\infty)$ is invertible. Recently, the general rational matrix case (i.e., without any restriction on the behaviour at infinity) was analyzed and solved in [GK1]. In [GK1] the analysis is based on the following representation of the symbol

$$(0.4) \qquad \Phi(\lambda) = I + C(\lambda G - A)^{-1} B, \qquad |\lambda| = 1.$$

Here A and G are square matrices of which the order n may be larger than m, the pencil $\lambda G - A$ is regular on the unit circle $|\lambda| = 1$, and the matrices B and C have sizes $n \times m$ and $m \times n$, respectively. The results in [GK1] are expressed in terms of A, G, B, C and matrices derived from A, G, B and C.

In this thesis we carry out a similar program as in [GK1], but with a different representation, namely,

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad |\lambda| = 1. \tag{0.5}$$

Here A, G, B and C are as in (0.4) and D is an invertible $m \times m$ matrix. Choose $\alpha \neq 0$ such that α is neither a pole nor a zero of Φ. Then any rational $m \times m$ matrix function Φ without poles on $|\lambda| = 1$ admits a representation of the form (0.5). The representation (0.5) has the advantage that the matrices A, G are in general of smaller size than the matrices A, G which appear in the representation (0.4), and hence (0.5) leads to formulas of lower numerical complexity than those arising from (0.4). The representations (0.4) and (0.5) derive from mathematical systems theory and are called *realizations*. The main ideas from [GK1] are extended to the case considered here. The exposition is based on a separation of spectra argument for linear operator pencils (the so-called spectral decomposition of pencils), which may be found in F. Stummel [S].

Furthermore, the method of [BGK2, BGK3], which is based on an equivalence of linear systems with boundary conditions is reviewed and extended here. The systems which correspond to (0.5) are singular systems (cf. [VLK] and [C]) and have the following form:

$$\begin{cases} A\rho_{k+1} &= G\rho_k + Bx_k, \qquad k = 0,1,2,\ldots, \\ y_k &= C(\alpha\rho_{k+1} - \rho_k) + Dx_k, \quad k = 0,1,2,\ldots, \\ (I - Q)\rho_0 &= 0. \end{cases} \tag{0.6}$$

The matrices A, G, B, C and D are the same as in (0.5) and Q is the projection

$$Q = \frac{1}{2\pi i}\int_{|\lambda|=1} (\lambda G - A)^{-1} G d\lambda.$$

The equivalence between (0.3) and (0.6) provides a method to invert (0.3) and enables one to compute the Fredholm properties of a block Toeplitz operator T with rational symbol Φ. Also, this method is applied to invert finite block Toeplitz matrices. Moreover, the inversion formulas are obtained in a form which is similar to the formula for the general solution of a system of ordinary differential equations with constant coefficients. In addition, we construct a generalized inverse directly.

The thesis consists of three chapters (not counting the present introduction).

Chapter 1 contains preliminaries, the spectral decomposition of operator pencils and the power representation of the Fourier coefficients of Φ corresponding to the realization (0.5).

Chapter 2 explores the inversion of Toeplitz operators with rational symbols. We calculate the inverse of double infinite block Toeplitz operators with rational symbols. The inversion of semi-infinite block Toeplitz operators is calculated via equivalence to singular systems with boundary conditions. Inversion of finite block Toepltz matrices is also treated in this chapter.

In chapter 3 we compute Fredholm properties of block Toeplitz operators with rational symbols. Fredholm characteristics are derived and a generalized inverse for a block Toeplitz operator with rational symbol is constructed directly. A Riemann-Hilbert problem is solved as an application. Finally, we illustrate the theory with an example.

Chapter 1

Preliminaries and Spectral Decomposition

1.1 Preliminaries

We first give some preliminaries on notation. The unit circle in the complex plane $\mathbb{C}$ will be denoted by $\mathbb{T}$. We write $\mathbb{D}_+$ for the open unit disc and $\mathbb{D}_-$ for the complement on the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ of the set $\mathbb{D}_+ \cup \mathbb{T}$. By a Cauchy contour Γ we mean the positively oriented boundary of a bounded Cauchy domain in $\mathbb{C}$. Such a contour consists of a finite number of nonintersecting closed rectifiable Jordan curves. The set of points inside Γ is called the inner domain of Γ and will be denoted by Δ_+. The outer domain of Γ is the set $\Delta_- = \mathbb{C}_\infty \setminus \overline{\Delta_+}$. We shall always assume that 0 belongs to Δ_+. By definition $\infty \in \Delta_-$.

We denote by $L_2(\mathbb{T})$ the space of all functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$t \to f(e^{it}),$$

is Lebesgue measurable and square integrable on the interval $[-\pi, \pi]$. The space $L_2(\mathbb{T})$ is a Hilbert space. Its inner product and norm are given by

$$< f, g >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \overline{g(e^{it})} dt,$$

$$\|f\| = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^2 dt \right)^{\frac{1}{2}}.$$

An orthonormal basis for $L_2(\mathbb{T})$ are the functions $\zeta^n, \zeta = e^{it}, n \in \mathbb{Z}$. The numbers

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int} dt =< f(e^{it}), e^{int} >, \ n = 0, \pm 1, \pm 2, \ldots$$

are called the *Fourier coefficients* of f. The subspace of $L_2(\mathbb{T})$ consisting of all functions $f \in L_2(\mathbb{T})$ for which the Fourier coefficients $c_{-1}, c_{-2}, \ldots$ are zero will be denoted by $H_2(\mathbb{T})$. That is,

$$H_2(\mathbb{T}) = \left\{ f \in L_2(\mathbb{T}) : < f, e^{int} >= 0, \ n = -1, -2, \ldots \right\}.$$

The space $H_2(\mathbb{T})$ is called the *Hardy space* of square integrable functions on the unit circle.

We denote by $l_2(\mathbb{Z})$ the Hilbert space of all square summable double infinite sequences of complex numbers. The symbol l_2 shall stand for the usual Hilbert space of all square summable infinite sequences of complex numbers. We shall identify l_2 with its canonical image in $l_2(\mathbb{Z})$, that is,

$$l_2 = \left\{ (u_j)_{j=-\infty}^{\infty} \in l_2(\mathbb{Z}) : u_j = 0 \text{ for } j < 0 \right\}.$$

The map U which assigns to a function $f \in L_2(\mathbb{T})$ its sequence of Fourier coefficients

$$Uf = (c_n)_{n=-\infty}^{\infty}, \qquad c_n = < f, e^{int} >, \tag{1.1}$$

is a unitary operator from $L_2(\mathbb{T})$ onto $l_2(\mathbb{Z})$, which carries $H_2(\mathbb{T})$ over into l_2.

Given a Hilbert space H, we denote by H^m the Cartesian product of m copies of H. An element $x = col(x_i)_{i=1}^m$ of H^m is an m-tuple of elements from H written as a column with $x_1, \ldots, x_m$ in H. The space H^m is a Hilbert space. Its inner product and norm are given by

$$< x, y >= \sum_{j=1}^{m} < x_j, y_j >,$$

$$\|x\| = \left(\sum_{j=1}^{m} \|x_j\|^2 \right)^{\frac{1}{2}}.$$

The unitary map $U : L_2(\mathbb{T}) \to l_2(\mathbb{Z})$ defined by (1.1) extends in a natural way to a unitary operator, also denoted by U, from $L_2^m(\mathbb{T}) = L_2(\mathbb{T})^m$ onto $l_2^m(\mathbb{Z}) = l_2(\mathbb{Z})^m$, namely

$$Uf = U col(f_i)_{i=1}^m = col(Uf_i)_{i=1}^m \in l_2^m(\mathbb{Z}).$$

The map U is called the *Fourier transformation* on $L_2^m(\mathbb{T})$ and Uf is called the *Fourier transform* of f. If $Uf = (c_n)_{n=-\infty}^{\infty}$ then f has a complex *Fourier series* representation of the form

$$f = \sum_{n=-\infty}^{\infty} e^{int} c_n. \tag{1.2}$$

The series in the right hand side of (1.2) converges in the norm of $L_2^m(\mathbb{T})$. From (1.2) we can see that the elements of the Hardy space $H_2^m(\mathbb{T})$ may be identified as those functions $f \in L_2^m(\mathbb{T})$ that have an extension to an analytic $\mathbb{C}^m$-valued function inside the unit circle.

We shall denote the set of all $m \times m$ matrices with entries in $L_2(\mathbb{T})$ by $L_2^{m\times m}(\mathbb{T})$. If $\Phi \in L_2^{m\times m}(\mathbb{T})$ then a complex Fourier series representation of Φ is given by

$$\Phi(\zeta) = \sum_{\nu=-\infty}^{\infty} \zeta^{\nu} \Phi_{\nu}, \qquad \zeta = e^{it}, \tag{1.3}$$

where

$$\Phi_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{it}) e^{-ikt} dt \tag{1.4}$$

is called the k-th *Fourier coefficient* of Φ.

For $1 \leq p \leq \infty$ we denote by l_p^m the Banach space of all sequences $(x_0, x_1, x_2, \ldots)$ of vectors in $\mathbb{C}^m$ such that the corresponding sequence of norms, $(\|x_k\|)_{k=1}^{\infty}$, belongs to l_p, the space of all p summable infinite sequences of complex numbers. The space of all double infinite sequences of this type is denoted by $l_p^m(\mathbb{Z})$, where

$$l_p^m(\mathbb{Z}) = \{x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) : x_i \in \mathbb{C}^m,$$

$$i \in \mathbb{Z} : (\|x_k\|)_{k=-\infty}^{\infty} \in l_p(\mathbb{Z})\}.$$

1.2 Spectral Decomposition of operator pencils

In this section we recall (from [GK1]) a spectral decomposition theorem which summarizes the extension to operator pencils of the classical Riesz theory about separation of spectra. Let X be a complex Banach space, and let G and A be bounded linear operators on X. The expression $\lambda G - A$, where λ is a complex parameter, will be called a *(linear) pencil* of operators on X. Given a non-empty subset Δ of the Riemann sphere $\mathbb{C}_\infty$, we say that $\lambda G - A$ is Δ-regular if $\lambda G - A$ (or just G if $\lambda = \infty$) is invertible for each λ in Δ. Assume that 0 is inside Γ, where Γ is a *Cauchy contour* in $\mathbb{C}$.

We now recall the spectral decomposition theorem.

Theorem 1.1 ([GK1], Theorem 2.1). *Let Γ be a Cauchy contour with Δ_+ and Δ_- as inner and outer domain, respectively, and let $\lambda G - A$ be a Γ-regular pencil of operators on the Banach space X. Then there exists a projection P and an invertible operator E, both acting on X, such that relative to the decomposition $X = \ker P \oplus \mathrm{im} P$, the following partitioning holds:*

$$(1.5) \qquad (\lambda G - A)E = \begin{bmatrix} \lambda\Omega_1 - I_1 & 0 \\ 0 & \lambda I_2 - \Omega_2 \end{bmatrix} : \ker P \oplus \mathrm{im}P \to \ker P \oplus \mathrm{im}P,$$

where I_1 (resp. I_2) denotes the identity operator on $\ker P$ (resp. $\mathrm{im}P$), the pencil $\lambda\Omega_1 - I_1$ is $\overline{\Delta}_+$-regular and $\lambda I_2 - \Omega_2$ is $\overline{\Delta}_-$-regular. Furthermore, P and E (and hence also the operators Ω_1 and Ω_2) are uniquely determined. In fact,

$$(1.6) \qquad P = \frac{1}{2\pi i}\int_\Gamma G(\lambda G - A)^{-1} d\lambda,$$

$$(1.7) \qquad E = \frac{1}{2\pi i}\int_\Gamma (1 - \lambda^{-1})(\lambda G - A)^{-1} d\lambda,$$

$$(1.8) \qquad \Omega = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} = \frac{1}{2\pi i}\int_\Gamma (\lambda - \lambda^{-1}) G(\lambda G - A)^{-1} d\lambda.$$

Proof. We have to modify the arguments which are used to derive the properties of the Riesz projections. Only the main differences will be explained. Let P be defined by (1.6). We also need the following operator

$$(1.9) \qquad Q = \frac{1}{2\pi i}\int_\Gamma (\lambda G - A)^{-1} G d\lambda.$$

We shall see that P and Q are projections. For a pencil, a generalized resolvent identity holds, namely

$$(1.10) \qquad (\lambda G - A)^{-1} - (\mu G - A)^{-1} = (\mu - \lambda)(\lambda G - A)^{-1} G (\mu G - A)^{-1},$$

where λ and μ are points where the pencil is invertible. Introduce the following auxiliary operator

$$K = \frac{1}{2\pi i}\int_\Gamma (\lambda G - A)^{-1} d\lambda.$$

Note that

$$KG = Q, \qquad GK = P. \tag{1.11}$$

Using the generalized resolvent equation (1.10) and the usual contour integration arguments we show that $KGK = K$. Indeed, let Γ_1 be a Cauchy contour in the inner domain of Γ. Then

$$\begin{aligned}
KGK &= \left(\frac{1}{2\pi i}\int_{\Gamma_1}(\lambda G - A)^{-1} d\lambda\right) G \left(\frac{1}{2\pi i}\int_\Gamma (\mu G - A)^{-1} d\mu\right)\\
&= (\frac{1}{2\pi i})^2 \int_{\Gamma_1}\int_\Gamma (\lambda G - A)^{-1} G (\mu G - A)^{-1} d\mu d\lambda\\
&= (\frac{1}{2\pi i})^2 \int_{\Gamma_1}\int_\Gamma \left\{\frac{(\lambda G - A)^{-1} - (\mu G - A)^{-1}}{\mu - \lambda}\right\} d\mu d\lambda\\
&= (\frac{1}{2\pi i})^2 \int_{\Gamma_1}\int_\Gamma \frac{(\lambda G - A)^{-1}}{\mu - \lambda} d\mu d\lambda - (\frac{1}{2\pi i})^2 \int_{\Gamma_1}\int_\Gamma \frac{(\mu G - A)^{-1}}{\mu - \lambda} d\mu d\lambda\\
&= \frac{1}{2\pi i}\int_{\Gamma_1}(\lambda G - A)^{-1}\left(\frac{1}{2\pi i}\int_\Gamma \frac{1}{\mu - \lambda} I d\mu\right) d\lambda\\
&\quad -(\frac{1}{2\pi i})^2 \int_\Gamma\int_{\Gamma_1} \frac{(\mu G - A)^{-1}}{\mu - \lambda} d\lambda d\mu\\
&= \frac{1}{2\pi i}\int_{\Gamma_1}(\lambda G - A)^{-1} d\lambda - \frac{1}{2\pi i}\int_\Gamma (\mu G - A)^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_1}\frac{1}{\mu - \lambda} I d\lambda\right) d\mu\\
&= K - 0 = K,
\end{aligned}$$

since

$$\int_\Gamma \frac{d\mu}{\mu - \lambda} = 2\pi i \ \ (\lambda \in \Gamma_1), \qquad \int_{\Gamma_1} \frac{d\lambda}{\mu - \lambda} = 0 \ \ (\mu \in \Gamma).$$

Note these identities hold, because Γ_1 is in the inner domain of Γ. Furthermore, in the computation of the second integral, the interchange of integrals are justified because the integrand is a continuous operator function on $\Gamma_1 \times \Gamma$, or, alternatively by an application of Fubini's theorem. Thus the identities in (1.11) imply that P and Q are projections. We also have

$$GQ = PG, \qquad AQ = PA, \qquad K = KP = QK. \tag{1.12}$$

The first identity in (1.12) follows from (1.6) and (1.9), the third is a corollary of (1.11) and the fact that $K = KGK$, and the second identity in (1.12) is a consequence of the following formula:

$$A(\lambda G - A)^{-1}G = G(\lambda G - A)^{-1}A, \qquad \lambda \in \rho(G, A). \tag{1.13}$$

Formula (1.12) allows us to partition the operators G, A and K in the following way:

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} : \ker Q \oplus \mathrm{im} Q \to \ker P \oplus \mathrm{im} P, \tag{1.14}$$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \ker Q \oplus \mathrm{im} Q \to \ker P \oplus \mathrm{im} P, \tag{1.15}$$

$$K = \begin{bmatrix} 0 & 0 \\ 0 & L \end{bmatrix} : \ker P \oplus \mathrm{im} P \to \ker Q \oplus \mathrm{im} Q. \tag{1.16}$$

The identities in (1.11) imply that G_2 is invertible and $G_2^{-1} = L$. Next, consider

$$T(\lambda) = \frac{1}{2\pi i}\int_\Gamma (\lambda - \mu)^{-1}(\mu G - A)^{-1} d\mu, \qquad \lambda \notin \Gamma.$$

Using the generalized resolvent identity and both Cauchy's integral formula and integral theorem one checks that

$$\begin{aligned} T(\lambda)(\lambda G - A) &= \frac{1}{2\pi i}\int_\Gamma (\lambda - \mu)^{-1}(\mu G - A)^{-1}(\lambda G - A)\, d\mu \\ &= \frac{1}{2\pi i}\int_\Gamma \left[\frac{(\lambda G - A)^{-1}}{\lambda - \mu} + (\lambda G - A)^{-1} G(\mu G - A)^{-1}\right] \cdot \\ &\quad (\lambda G - A) d\mu. \end{aligned}$$

From the generalized resolvent equation we deduce that

$$\begin{aligned} (\mu G - A)^{-1}G(\lambda G - A)^{-1} &= \frac{(\mu G - A)^{-1} - (\lambda G - A)^{-1}}{\lambda - \mu} \\ &= \frac{(\lambda G - A)^{-1} - (\mu G - A)^{-1}}{\mu - \lambda} \\ &= (\lambda G - A)^{-1}G(\mu G - A)^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} T(\lambda)(\lambda G - A) &= \frac{1}{2\pi i}\int_\Gamma \frac{I}{\lambda-\mu}d\mu + \frac{1}{2\pi i}\int_\Gamma (\mu G - A)^{-1}G d\mu \\ &= \begin{cases} Q - I & \text{for } \lambda \text{ inside } \Gamma, \\ Q & \text{for } \lambda \text{ outside } \Gamma. \end{cases} \end{aligned}$$

Thus

$$T(\lambda)(\lambda G - A) = \begin{cases} Q - I \text{ for } \lambda \in \Delta_+, \\ Q \qquad \text{for } \lambda \in \Delta_-. \end{cases} \tag{1.17}$$

Similarly, we find that

$$\begin{aligned} (\lambda G - A)T(\lambda) &= \frac{1}{2\pi i}\int_\Gamma (\lambda G - A)\cdot \\ &\quad \left\{\frac{(\lambda G - A)^{-1}}{\lambda - \mu} + (\mu G - A)^{-1}G(\lambda G - A)^{-1}\right\} d\mu \\ &= \frac{1}{2\pi i}\int_\Gamma \left\{\frac{I}{\lambda-\mu} + (\lambda G - A)(\mu G - A)^{-1}G(\lambda G - A)^{-1}\right\} d\mu. \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{-1}{\mu - \lambda} I d\mu + \frac{1}{2\pi i}\int_\Gamma G(\mu G - A)^{-1}d\mu. \end{aligned}$$

Now, using Cauchy's integral formula, we get

$$(\lambda G - A)T(\lambda) = \begin{cases} P - I \text{ for } \lambda \in \Delta_+, \\ P \qquad \text{for } \lambda \in \Delta_-. \end{cases} \tag{1.18}$$

Here I is the identity operator on X. From the generalized identity (1.10) it follows that

$$\begin{aligned} T(\lambda)P &= \left(\frac{1}{2\pi i}\int_\Gamma (\lambda-\mu)^{-1}(\mu G - A)^{-1}d\mu\right)\left(\frac{1}{2\pi i}\int_{\Gamma_1} G(sG - A)^{-1}ds\right) \\ &= (\frac{1}{2\pi i})^2 \int_\Gamma\int_{\Gamma_1} (\lambda-\mu)^{-1}(\mu G - A)^{-1}G(sG - A)^{-1}dsd\mu \\ &= (\frac{1}{2\pi i})^2 \int_\Gamma\int_{\Gamma_1} (\lambda-\mu)^{-1}(sG - A)^{-1}G(\mu G - A)^{-1}dsd\mu \\ &= \frac{1}{2\pi i}\int_\Gamma \left[\frac{1}{2\pi i}\int_{\Gamma_1} (sG - A)^{-1}G ds\right](\lambda - \mu)^{-1}(\mu G - A)^{-1}d\mu \end{aligned}$$

Then

$$
\begin{aligned}
A\rho_{k+1} &= AE\Omega^{k+1}x + AE\Omega^{N-k}y + \sum_{\nu=0}^{k} AE\Omega^{k-\nu}(I-P)\varphi_\nu \\
&\quad - \sum_{\nu=k+1}^{N} AE\Omega^{\nu-k-1}P\varphi_\nu \\
&= \Omega^{k+1}x + \Omega^{N+1-k}y + \sum_{\nu=0}^{k} \Omega^{k-\nu}(I-P)\varphi_\nu \\
&\quad - \sum_{\nu=k+1}^{N} \Omega^{\nu-k}P\varphi_\nu \\
&= GE\Omega^{k}x + GE\Omega^{N+1-k}y + \sum_{\nu=0}^{k-1} GE\Omega^{k-1-\nu}(I-P)\varphi_\nu \\
&\quad +(I-P)\varphi_k - \sum_{\nu=k}^{N} GE\Omega^{\nu-k}P\varphi_\nu + P\varphi_k \\
&= G\Bigg(E\Omega^{k}x + E\Omega^{N+1-k}y + \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)\varphi_\nu \\
&\quad - \sum_{\nu=k}^{N} E\Omega^{\nu-k}P\varphi_\nu\Bigg) + \varphi_k \\
&= G\rho_k + \varphi_k.
\end{aligned}
$$

The converse statement is proved as follows. Decompose (1.24) as

$$
\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix}, \tag{1.27}
$$

where

$$
A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \ker Q \oplus \mathrm{im} Q \to \ker P \oplus \mathrm{im} P,
$$

$$
G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} : \ker Q \oplus \mathrm{im} Q \to \ker P \oplus \mathrm{im} P,
$$

$$
\rho_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix} \text{ and } \varphi_k = \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix}.
$$

Equation (1.27) can now be written as two separate difference equations, one going forwards and the other going backwards. They are

$$A_1 x_{k+1} = G_1 x_k + \alpha_k \tag{1.28a}$$

and

$$A_2 y_{k+1} = G_2 y_k + \beta_k. \tag{1.28b}$$

From (1.28a) we have

$$x_{k+1} = A_1^{-1} G_1 x_k + A_1^{-1}\alpha_k,$$

since A_1 is invertible. Put $x_0 = A_1^{-1}x$, where x is an arbitrary vector in $\ker P$. Now establish a general formula for x_k by solving (1.28a) forward in time as follows.

$$\begin{aligned} x_1 &= A_1^{-1}\left(G_1 A_1^{-1}\right) x + A_1^{-1}\alpha_0, \\ x_2 &= A_1^{-1} G_1 x_1 + A_1^{-1}\alpha_1 \\ &= A_1^{-1}\left(G_1 A_1^{-1}\right)^2 x + A_1^{-1}\left(G_1 A_1^{-1}\right)\alpha_0 + A_1^{-1}\alpha_1, \\ x_3 &= A_1^{-1} G_1 x_2 + A_1^{-1}\alpha_2 \\ &= A_1^{-1}\left(G_1 A_1^{-1}\right)^3 x + A_1^{-1}\left(G_1 A_1^{-1}\right)^2 \alpha_0 + A_1^{-1}\left(G_1 A_1^{-1}\right)\alpha_1 + A_1^{-1}\alpha_2. \end{aligned}$$

Continuing in this way, we obtain

$$\begin{aligned} x_k &= A_1^{-1}\left(G_1 A_1^{-1}\right)^k x + \sum_{\nu=0}^{k-1} A_1^{-1}\left(G_1 A_1^{-1}\right)^{k-1-\nu}\alpha_\nu \\ &= E\Omega^k x + \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)\varphi_\nu. \end{aligned} \tag{1.29a}$$

Making y_k the subject of the formula, we deduce from (1.28b) that

$$y_k = G_2^{-1} A_2 y_{k+1} - G_2^{-1}\beta_k,$$

since G_2 is invertible. Put $y_{N+1} = G_2^{-1}y$, where y is an arbitrary vector in $\mathrm{im}P$. A general formula for y_k can be found similarly by solving (1.28b) backward in time. Now

$$y_N = G_2^{-1} A_2 y_{N+1} - G_2^{-1}\beta_N$$

$$
\begin{aligned}
&= G_2^{-1}\left(A_2G_2^{-1}\right)y - G_2^{-1}\beta_N,\\
y_{N-1} &= G_2^{-1}A_2y_N - G_2^{-1}\beta_{N-1}\\
&= G_2^{-1}\left(A_2G_2^{-1}\right)^2 y - G_2^{-1}\left(A_2G_2^{-1}\right)\beta_N - G_2^{-1}\beta_{N-1},\\
y_{N-2} &= G_2^{-1}A_2y_{N-1} - G_2^{-1}\beta_{N-2}\\
&= G_2^{-1}\left(A_2G_2^{-1}\right)^3 y - G_2^{-1}\left(A_2G_2^{-1}\right)^2\beta_N - G_2^{-1}\left(A_2G_2^{-1}\right)\beta_{N-1} - G_2^{-1}\beta_{N-2}.
\end{aligned}
$$

Continuing in this manner, we get

$$y_{N-k} = G_2^{-1}\left(A_2G_2^{-1}\right)^{k+1} y - \sum_{\nu=0}^{k} G_2^{-1}\left(A_2G_2^{-1}\right)^{\nu}\beta_{N+\nu-k}.$$

If we make a change of variable $N-k \leftrightarrow k$ we get

$$y_k = G_2^{-1}\left(A_2G_2^{-1}\right)^{N+1-k} y - \sum_{\nu=0}^{N-k} G_2^{-1}\left(A_2G_2^{-1}\right)^{\nu}\beta_{k+\nu}.$$

And another change of variable $\nu \leftrightarrow \nu - k$ yields

$$y_k = G_2^{-1}\left(A_2G_2^{-1}\right)^{N+1-k} y - \sum_{\nu=k}^{N} G_2^{-1}\left(A_2G_2^{-1}\right)^{\nu-k}\beta_{\nu},$$

i.e.,

$$y_k = E\Omega^{N+1-k}y - \sum_{\nu=k}^{N} E\Omega^{\nu-k}P\varphi_\nu. \qquad (1.29b)$$

Combining (1.29a) and (1.29b) we get

$$
\begin{aligned}
\rho_k &= \begin{bmatrix} x_k \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_k \end{bmatrix} = E\Omega^k x + E\Omega^{N+1-k}y + \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)\varphi_\nu\\
&\quad - \sum_{\nu=k}^{N} E\Omega^{\nu-k}P\varphi_\nu, \qquad k = 0,\ldots,N+1. \natural
\end{aligned}
$$

In what follows Γ will often be taken to be the unit circle $\mathbb{T}$. In this case the regularity conditions on the pencils $\lambda\Omega_1 - I_1$ and $\lambda I_2 - \Omega_2$ in (1.5) are just equivalent to the requirement that Ω_1 and Ω_2 have their spectra in the open unit disc.

Corollary 1.3 ([GK1], Corollary 2.3). *Let $\lambda G - A$ be a $\mathbb{T}$-regular pencil of operators. Then the corresponding associated operator Ω has its spectrum in the open unit disc.*

Proof. Use that Ω is given by the first identity in (1.8) and apply the remark preceding the present corollary. ♮

1.3 Realization and Power Representation

Let Φ be an $m \times m$ rational matrix function, and choose $\alpha \neq 0$ such that α is neither a pole nor a zero of Φ (see [GK2] and [G1]). Then Φ admits a representation

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B. \tag{1.30}$$

The representation (1.30) is derived from classical realization results by applying the Möbius transformation

$$\phi(\lambda) = \alpha\frac{2\lambda - 1}{2\lambda + 1}, \qquad \phi^{-1}(z) = -\frac{1}{2}\frac{z + \alpha}{z - \alpha} \tag{1.31}$$

to the realization

$$\Phi(\lambda) = D + C(\lambda - A)^{-1}B. \tag{1.32}$$

Indeed, a rational matrix function $\hat{\Phi}(\lambda)$ which is analytic and invertible at infinity can be represented as (see, e.g., [BGK1])

$$\hat{\Phi}(\lambda) = \hat{D} + \hat{C}(\lambda - \hat{A})^{-1}\hat{B},$$

where $\hat{D} = \hat{\Phi}(\infty)$ and $\hat{A}$, $\hat{B}$ and $\hat{C}$ are matrices of appropriate sizes. Now put

$$\begin{aligned}
\Phi(\lambda) &= \hat{\Phi}(\phi^{-1}(\lambda)) \\
&= \hat{D} + \hat{C}\left[-\frac{1}{2}\frac{\lambda + \alpha}{\lambda - \alpha} - \hat{A}\right]^{-1}\hat{B} \\
&= \hat{D} + (\lambda - \alpha)\hat{C}\left[-\frac{1}{2}(\lambda + \alpha) - \hat{A}(\lambda - \alpha)\right]^{-1}\hat{B} \\
&= \hat{D} + (\lambda - \alpha)\hat{C}\left[\lambda(-\frac{1}{2} - \hat{A}) - \alpha(\frac{1}{2} - \hat{A})\right]^{-1}\hat{B}.
\end{aligned}$$

If we define $A = \alpha(\frac{1}{2} - \hat{A})$, $G = -\frac{1}{2} - \hat{A}$, $B = \hat{B}$, $C = \hat{C}$, $D = \hat{D}$, then we get

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B$$

(see Theorem 1.9, [BGK1]).

We refer to the right hand side of (1.30) as a *realization* of Φ. The realization (1.30) is said to be *minimal* if the order of A and G is as small as possible among all possible realizations. Here G and A are square matrices of order, say $n \times n$. The matrices B and C are of size $n \times m$ and $m \times n$ respectively, while D is a square matrix of order $m \times m$.

Assume $\Phi(\lambda)$ has no poles on the Cauchy contour Γ. Then the pencil $\lambda G - A$ in (1.30) can always be chosen to be Γ-regular. Indeed, if the realization is minimal, then Γ-regularity is assured.

The next two lemmas will be useful later. They are the natural analogues of Theorem 4.2 and Lemma 4.3 in [GK1].

Lemma 1.4 ([G1], Lemma 2.1). *Let*

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \Gamma, \tag{1.33}$$

where $\lambda G - A$ is Γ-regular, be a given realization. Put $G^{\times} = G + BD^{-1}C$ and $A^{\times} = A + \alpha BD^{-1}C$. Then det $\Phi(\lambda) \neq 0$ *for each $\lambda \in \Gamma$ if and only if the pencil $\lambda G^{\times} - A^{\times}$ is Γ-regular, and in this case*

$$\Phi(\lambda)^{-1} = D^{-1} - (\lambda - \alpha)D^{-1}C(\lambda G^{\times} - A^{\times})^{-1}BD^{-1}, \qquad \lambda \in \Gamma. \tag{1.34}$$

Proof. We prove a stronger (pointwise) version of the theorem. Take a fixed $\zeta \in \Gamma$. Since $\det(I - TS) = \det(I - ST)$, we have

$$\begin{aligned}
\det\Phi(\zeta) &= \det[D + (\zeta - \alpha)C(\zeta G - A)^{-1}B] \\
&= \det D[I + (\zeta - \alpha)D^{-1}C(\zeta G - A)^{-1}B] \\
&= \det D \det[(\zeta G - A)^{-1}\{(\zeta G - A) + (\zeta - \alpha)BD^{-1}C\}] \\
&= \det D \det[(\zeta G - A)^{-1}(\zeta G^{\times} - A^{\times})] \\
&= \det D \frac{\det(\zeta G^{\times} - A^{\times})}{\det(\zeta G - A)}.
\end{aligned}$$

–

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It follows that $\det\Phi(\zeta) \neq 0$ if and only if $\det(\zeta G^\times - A^\times) \neq 0$. In particular, $\det\Phi(\lambda) \neq 0$ for each $\lambda \in \Gamma$ if and only if the pencil $\lambda G^\times - A^\times$ is Γ-regular.

Next, assume that $\det(\lambda G^\times - A^\times) \neq 0$, and let us solve the equation $\Phi(\lambda)x = y$. Introduce a new unknown by setting $z = (\lambda G - A)^{-1}Bx$. Then given y we have to compute x from

$$\begin{cases} \lambda G z &= Az + Bx, \\ y &= (\lambda - \alpha)Cz + Dx. \end{cases} \tag{1.35}$$

Apply BD^{-1} to the second equation in (1.35) and subtract the result from the first equation in (1.35). This yields the following equivalent system

$$\begin{cases} \lambda G^\times z &= A^\times z + BD^{-1}y, \\ y &= (\lambda - \alpha)Cz + Dx. \end{cases} \tag{1.36}$$

Hence $z = (\lambda G^\times - A^\times)^{-1}BD^{-1}y$ and

$$\Phi(\lambda)^{-1}y = D^{-1}y - (\lambda - \alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}BD^{-1}y,$$

which proves (1.34). ♮

Lemma 1.5 ([G1], Lemma 2.2). *Let Φ be as in (1.33), where $\lambda G - A$ is Γ-regular. Assume that $\det\Phi(\lambda) \neq 0$ for each $\lambda \in \Gamma$, and set $G^\times = G + BD^{-1}C$ and $A^\times = A + \alpha BD^{-1}C$. Then for $\lambda \in \Gamma$,*

$$\begin{aligned} \Phi(\lambda)^{-1}C(\lambda G - A)^{-1} &= D^{-1}C(\lambda G^\times - A^\times)^{-1}, \\ (\lambda G - A)^{-1}B\Phi(\lambda)^{-1} &= (\lambda G^\times - A^\times)^{-1}BD^{-1}, \\ (\lambda G^\times - A^\times)^{-1} &= (\lambda G - A)^{-1} - (\lambda - \alpha) \\ &\quad \cdot(\lambda G - A)^{-1}B\Phi(\lambda)^{-1}C(\lambda G - A)^{-1}. \end{aligned}$$

Proof. First note that

$$(\lambda - \alpha)BD^{-1}C = \lambda(G + BD^{-1}C) - (A + \alpha BD^{-1}C) + (A - \lambda G),$$

i.e.,

$$(\lambda - \alpha)BD^{-1}C = (\lambda G^\times - A^\times) - (\lambda G - A). \tag{1.37}$$

We also know, from Lemma 1.4, that $\lambda G^\times - A^\times$ is invertible for each $\lambda \in \Gamma$. For the first identity

$$
\begin{aligned}
&\Phi(\lambda)^{-1}C(\lambda G - A)^{-1}\\
&= \{D^{-1}C - (\lambda - \alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}BD^{-1}C\}(\lambda G - A)^{-1}\\
&= \{D^{-1}C - D^{-1}C(\lambda G^\times - A^\times)^{-1}[(\lambda G^\times - A^\times) - (\lambda G - A)]\}(\lambda G - A)^{-1}\\
&= D^{-1}C(\lambda G^\times - A^\times)^{-1}.
\end{aligned}
$$

For the second identity

$$
\begin{aligned}
&(\lambda G - A)^{-1}B\Phi(\lambda)^{-1}\\
&= (\lambda G - A)^{-1}\{BD^{-1} - (\lambda - \alpha)BD^{-1}C(\lambda G^\times - A^\times)^{-1}BD^{-1}\}\\
&= (\lambda G - A)^{-1}\{BD^{-1} - [(\lambda G^\times - A^\times) - (\lambda G - A)](\lambda G^\times - A^\times)^{-1}BD^{-1}\}\\
&= (\lambda G^\times - A^\times)^{-1}BD^{-1}.
\end{aligned}
$$

And for the third identity

$$
\begin{aligned}
&(\lambda G - A)^{-1} - (\lambda - \alpha)(\lambda G - A)^{-1}B\Phi(\lambda)^{-1}C(\lambda G - A)^{-1}\\
&= (\lambda G - A)^{-1} - (\lambda - \alpha)(\lambda G^\times - A^\times)^{-1}BD^{-1}C(\lambda G - A)^{-1}\\
&= (\lambda G - A)^{-1} - (\lambda G^\times - A^\times)^{-1}\{(\lambda G^\times - A^\times) - (\lambda G - A)\}(\lambda G - A)^{-1}\\
&= (\lambda G^\times - A^\times)^{-1}. \natural
\end{aligned}
$$

If Γ is identified with the unit circle $\mathbb{T}$ in (1.33), then the realization (1.33) can be used to compute the Fourier coefficients Φ_k of Φ. This leads to the following corollary, which is the natural analogue of Corollary 3.2 in [GK1].

Corollary 1.6 ([JJ]). *Let Φ be a rational $m \times m$ matrix function without poles on the unit circle $\mathbb{T}$, and let*

$$
\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \mathbb{T}, \tag{1.38}
$$

be a realization of Φ. Then the k-th Fourier coefficient Φ_k of Φ admits the following representation:

$$
\Phi_k = \begin{cases} -CE(\Omega^{k-1} - \alpha\Omega^k)(I-P)B, & k > 0,\\ D + \alpha CE(I-P)B + CEPB, & k = 0,\\ CE(\Omega^{-k} - \alpha\Omega^{-k-1})PB, & k < 0. \end{cases} \tag{1.39}
$$

Here P, E and Ω are, respectively, the separating projection, the right equivalence operator and the associated operator corresponding to $\lambda G - A$ and $\mathbb{T}$, that is, P, E

and Ω are given by (1.6)-(1.8). In particular, Ω has all its eigenvalues in the open unit disc and Ω commutes with P.

Proof. Let Ω be as in (1.8). Since $\lambda\Omega_1 - I_1$ is regular on $\mathbb{D}_+ \cup \mathbb{T}$ and $\lambda I_2 - \Omega_2$ is regular on $\mathbb{D}_- \cup \mathbb{T}$, the matrices Ω_1 and Ω_2 have all their eigenvalues in $\mathbb{D}_+$. Hence the eigenvalues of the matrix Ω have the required location. According to Theorem 1.1,

$$\begin{aligned}
\Phi(\lambda) &= D + (\lambda - \alpha)CE \begin{bmatrix} (\lambda\Omega_1 - I_1)^{-1} & 0 \\ 0 & (\lambda I_2 - \Omega_2)^{-1} \end{bmatrix} B, \qquad \lambda \in \mathbb{T} \\
&= D + (\lambda - \alpha)CE \begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^{\nu}\Omega_1^{\nu} & 0 \\ 0 & \sum_{\nu=0}^{\infty} \lambda^{-\nu-1}\Omega_2^{\nu} \end{bmatrix} B \\
&= D + (\lambda - \alpha)CE \begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^{\nu}\Omega_1^{\nu} & 0 \\ 0 & \sum_{\nu=-1}^{-\infty} \lambda^{\nu}\Omega_2^{-\nu-1} \end{bmatrix} B \\
&= D + CE \begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^{\nu+1}\Omega_1^{\nu} & 0 \\ 0 & \sum_{\nu=-1}^{-\infty} \lambda^{\nu+1}\Omega_2^{-\nu-1} \end{bmatrix} B \\
&\quad -\alpha CE \begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^{\nu}\Omega_1^{\nu} & 0 \\ 0 & \sum_{\nu=-1}^{-\infty} \lambda^{\nu}\Omega_2^{-\nu-1} \end{bmatrix} B \\
&= D + CE \begin{bmatrix} \sum_{\nu=1}^{\infty} -\lambda^{\nu}\Omega_1^{\nu-1} & 0 \\ 0 & \sum_{\nu=0}^{-\infty} \lambda^{\nu}\Omega_2^{-\nu} \end{bmatrix} B \\
&\quad -\alpha CE \begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^{\nu}\Omega_1^{\nu} & 0 \\ 0 & \sum_{\nu=-1}^{-\infty} \lambda^{\nu}\Omega_2^{-\nu-1} \end{bmatrix} B,
\end{aligned}$$

since

$$(\lambda\Omega_1 - I_1)^{-1} = \sum_{\nu=0}^{\infty} -\lambda^{\nu}\Omega_1^{\nu}, \qquad \lambda \in \mathbb{T},$$

$$(\lambda I_2 - \Omega_2)^{-1} = \sum_{\nu=0}^{\infty} \lambda^{-\nu-1}\Omega_2^{\nu}, \qquad \lambda \in \mathbb{T}.$$

It follows that

$$\begin{aligned}
\Phi_k &= CE \begin{bmatrix} -\Omega_1^{k-1} & 0 \\ 0 & 0 \end{bmatrix} B - \alpha CE \begin{bmatrix} -\Omega_1^{k} & 0 \\ 0 & 0 \end{bmatrix} B \\
&= -CE(\Omega^{k-1} - \alpha\Omega^{k})(I - P)B, \qquad k > 0,
\end{aligned}$$

$$\begin{aligned}\Phi_0 &= D + CE\begin{bmatrix}0 & 0\\ 0 & I_2\end{bmatrix}B - \alpha CE\begin{bmatrix}-I_1 & 0\\ 0 & 0\end{bmatrix}B\\ &= D + \alpha CE(I-P)B + CEPB,\end{aligned}$$

$$\begin{aligned}\Phi_k &= CE\begin{bmatrix}0 & 0\\ 0 & \Omega_2^{-k}\end{bmatrix}B - \alpha CE\begin{bmatrix}0 & 0\\ 0 & \Omega_2^{-k-1}\end{bmatrix}B\\ &= CE(\Omega^{-k} - \alpha\Omega^{-k-1})PB, \qquad k < 0,\end{aligned}$$

and the corollary is proved. ♮

We refer to (1.39) as the *power representation* of the Fourier coefficients of Φ corresponding to the realization (1.38).

Chapter 2

Inversion of Toeplitz operators with rational symbols

2.1 Inversion of Double Infinite Block Toeplitz Operators with rational symbols

We review and modify Section 4 from [GK1]. See also Section 3, [G2]. In this section $L = [\Phi_{i-j}]_{i,j=-\infty}^{\infty}$ is a double infinite block Toeplitz operator on $l_p^m(\mathbb{Z})$. We assume that the symbol

$$\Phi(\lambda) = \sum_{\nu=-\infty}^{\infty} \lambda^{\nu} \Phi_{\nu}, \qquad \lambda \in \mathbb{T},$$

is a rational matrix function. Since Φ has no poles on $\mathbb{T}$, it admits a realization. The next theorem describes the inversion of L in terms of the data appearing in the realization of its symbol.

Theorem 2.1. *Let L be a double infinite block Toeplitz operator on $l_p^m(\mathbb{Z})$ with a rational symbol*

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \mathbb{T}, \tag{2.1}$$

given in realized form, where $\alpha \neq 0$ is neither a pole nor a zero of Φ. Put $G^{\times} = G + BD^{-1}C$ and $A^{\times} = A + \alpha BD^{-1}C$. Then L is invertible if and only if the pencil $\lambda G^{\times} - A^{\times}$ is $\mathbb{T}$-regular, and in this case $L^{-1} = [\Phi_{i-j}^{\times}]_{i,j=-\infty}^{\infty}$, with

$$\Phi_k^{\times} = \begin{cases} D^{-1}CE^{\times}[(\Omega^{\times})^{k-1} - \alpha(\Omega^{\times})^{k}](I - P^{\times})BD^{-1}, & k > 0, \\ D^{-1} - D^{-1}CE^{\times}[P^{\times} + \alpha(I - P^{\times})]BD^{-1}, & k = 0, \\ D^{-1}CE^{\times}[\alpha(\Omega^{\times})^{-k-1} - (\Omega^{\times})^{-k}]P^{\times}BD^{-1}, & k < 0. \end{cases} \tag{2.2}$$

Here $P^\times$, $E^\times$ and $\Omega^\times$ are, respectively, the separating projection, the right equivalence operator, and the associated operator corresponding to the pencil $\lambda G^\times - A^\times$ and Γ, with $\Gamma = \mathbb{T}$, i.e.,

$$P^\times = \frac{1}{2\pi i}\int_\Gamma G^\times(\lambda G^\times - A^\times)^{-1}d\lambda, \tag{2.3}$$

$$E^\times = \frac{1}{2\pi i}\int_\Gamma (1-\lambda^{-1})(\lambda G^\times - A^\times)^{-1}d\lambda, \tag{2.4}$$

$$\Omega^\times = \begin{bmatrix} \Omega_1^\times & 0 \\ 0 & \Omega_2^\times \end{bmatrix} = \frac{1}{2\pi i}\int_\Gamma (\lambda-\lambda^{-1})G^\times(\lambda G^\times - A^\times)^{-1}d\lambda. \tag{2.5}$$

Proof. The symbol Φ is continuous on $\mathbb{T}$. It is known (see [GKr]) that L is invertible if and only if $\det\Phi(\lambda) \neq 0$ for each $\lambda \in \mathbb{T}$, and in this case $L^{-1} = [\Phi^\times_{i-j}]_{i,j=-\infty}^{\infty}$, where $\Phi_k^\times$ is the k-th Fourier coefficient of $\Phi(\cdot)^{-1}$. Now apply Lemma 1.4 with $\Gamma = \mathbb{T}$. Then L is invertible if and only if $\lambda G^\times - A^\times$ is $\mathbb{T}$-regular.

Next, assume that L is invertible. Lemma 1.4 implies that

$$\Phi(\lambda)^{-1} = D^{-1} - (\lambda-\alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}BD^{-1}, \qquad \lambda \in \mathbb{T}. \tag{2.6}$$

Apply Corollary 1.6 and compute the power representation of the Fourier coefficients of $\Phi(\cdot)^{-1}$ corresponding to the realization (2.6). This gives the formula (2.2). Indeed,

$$\begin{aligned}
\Phi(\lambda)^{-1} &= D^{-1} - (\lambda-\alpha)D^{-1}CE^\times \begin{bmatrix} (\lambda\Omega_1^\times - I_1^\times)^{-1} & 0 \\ 0 & (\lambda I_2^\times - \Omega_2^\times)^{-1} \end{bmatrix} BD^{-1} \\
&= D^{-1} - (\lambda-\alpha)D^{-1}CE^\times \\
&\quad \cdot \begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^\nu(\Omega_1^\times)^\nu & 0 \\ 0 & \sum_{\nu=0}^{\infty} \lambda^{-\nu-1}(\Omega_2^\times)^\nu \end{bmatrix} BD^{-1} \\
&= D^{-1} - (\lambda-\alpha)D^{-1}CE^\times \\
&\quad \cdot \begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^\nu(\Omega_1^\times)^\nu & 0 \\ 0 & \sum_{\nu=-1}^{-\infty} \lambda^{\nu}(\Omega_2^\times)^{-\nu-1} \end{bmatrix} BD^{-1} \\
&= D^{-1} - D^{-1}CE^\times \\
&\quad \cdot \begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^{\nu+1}(\Omega_1^\times)^\nu & 0 \\ 0 & \sum_{\nu=-1}^{-\infty} \lambda^{\nu+1}(\Omega_2^\times)^{-\nu-1} \end{bmatrix} BD^{-1}
\end{aligned}$$

$$
\begin{aligned}
&+\alpha D^{-1}CE^{\times}\begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^{\nu}(\Omega_1^{\times})^{\nu} & 0 \\ 0 & \sum_{\nu=-1}^{-\infty} \lambda^{\nu}(\Omega_2^{\times})^{-\nu-1} \end{bmatrix} BD^{-1} \\
= \; & D^{-1} - D^{-1}CE^{\times}\begin{bmatrix} \sum_{\nu=1}^{\infty} -\lambda^{\nu}(\Omega_1^{\times})^{\nu-1} & 0 \\ 0 & \sum_{\nu=0}^{-\infty} \lambda^{\nu}(\Omega_2^{\times})^{-\nu} \end{bmatrix} BD^{-1} \\
&+\alpha D^{-1}CE^{\times}\begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^{\nu}(\Omega_1^{\times})^{\nu} & 0 \\ 0 & \sum_{\nu=-1}^{-\infty} \lambda^{\nu}(\Omega_2^{\times})^{-\nu-1} \end{bmatrix} BD^{-1},
\end{aligned}
$$

since

$$
(\lambda\Omega_1^{\times} - I_1^{\times})^{-1} = \sum_{\nu=0}^{\infty} -\lambda^{\nu}(\Omega_1^{\times})^{\nu}, \qquad \lambda \in \mathbb{T},
$$

$$
(\lambda I_2^{\times} - \Omega_2^{\times})^{-1} = \sum_{\nu=0}^{\infty} \lambda^{-\nu-1}(\Omega_2^{\times})^{\nu}, \qquad \lambda \in \mathbb{T}.
$$

We deduce that

$$
\begin{aligned}
\Phi_k^{\times} \;=\; & -D^{-1}CE^{\times}\begin{bmatrix} -(\Omega_1^{\times})^{k-1} & 0 \\ 0 & 0 \end{bmatrix} BD^{-1} \\
&+\alpha D^{-1}CE^{\times}\begin{bmatrix} -(\Omega_1^{\times})^{k} & 0 \\ 0 & 0 \end{bmatrix} BD^{-1} \\
=\; & D^{-1}CE^{\times}((\Omega^{\times})^{k-1} - \alpha(\Omega^{\times})^{k})(I - P^{\times})BD^{-1}, \qquad k > 0,
\end{aligned}
$$

$$
\begin{aligned}
\Phi_0^{\times} \;=\; & D^{-1} - D^{-1}CE^{\times}\begin{bmatrix} 0 & 0 \\ 0 & I_2^{\times} \end{bmatrix} BD^{-1} \\
&+\alpha D^{-1}CE^{\times}\begin{bmatrix} -I_1^{\times} & 0 \\ 0 & 0 \end{bmatrix} BD^{-1} \\
=\; & D^{-1} - D^{-1}CE^{\times}[P^{\times} + \alpha(I - P^{\times})]BD^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\Phi_k^{\times} \;=\; & -D^{-1}CE^{\times}\begin{bmatrix} 0 & 0 \\ 0 & (\Omega_2^{\times})^{-k} \end{bmatrix} BD^{-1} \\
&+\alpha D^{-1}CE^{\times}\begin{bmatrix} 0 & 0 \\ 0 & (\Omega_2^{\times})^{-k-1} \end{bmatrix} BD^{-1} \\
=\; & -D^{-1}CE^{\times}((\Omega^{\times})^{-k} - \alpha(\Omega^{\times})^{-k-1})P^{\times}BD^{-1}, \qquad k < 0. \natural
\end{aligned}
$$

2.2 Inversion of Semi Infinite Block Toeplitz Operators via equivalence to singular systems

Here we review and modify Section 7 of [GK1]. In this section, we develop an approach for inverting block Toeplitz operators with rational symbols, which is based on connections between Toeplitz operators and discrete singular systems with boundary conditions. Theorems 2.2 and 2.4 are, respectively, the natural analogues of Theorems 7.1 and 7.3 in [GK1], whereas Lemma 7.2 remains unchanged as Lemma 2.3.

Theorem 2.2. *Let* $1 \leq p \leq \infty$, *and let* $T = [\Phi_{j-k}]_{j,k=0}^{\infty}$ *be a block Toeplitz operator on* l_p^m *with symbol*

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \mathbb{T}, \tag{2.7}$$

given in realized form, where $\alpha \neq 0$ *is neither a pole nor a zero of* Φ. *Then the Toeplitz equation*

$$Tx = z, \qquad z \in l_p^m \tag{2.8)}$$

is equivalent to the following discrete boundary value system:

$$\begin{cases} A\rho_{k+1} &= G\rho_k + Bu_k, \quad k = 0, 1, 2, \ldots, \\ y_k &= C(\alpha\rho_{k+1} - \rho_k) + Du_k, \quad k = 0, 1, 2, \ldots, \\ (I - Q)\rho_0 &= 0. \end{cases} \tag{2.9}$$

Here Q *is the projection given by (1.9) with* $\Gamma = \mathbb{T}$ *and the equivalence between (2.8) and (2.9) has to be understood in the following sense: If* $x = (x_k)_{k=0}^{\infty}$ *in* l_p^m *is a solution of (2.8), then the system (2.9) with input* $u_k = x_k$ $(k = 0, 1, 2, \ldots)$ *has output* $y_k = z_k$ $(k = 0, 1, 2, \ldots)$, *and, conversely, if the system (2.9) with input* $u = (u_k)_{k=0}^{\infty}$ *from* l_p^m *has output* $y_k = z_k$ $(k = 0, 1, 2, \ldots)$, *then* $x = u$ *is a solution of (2.8).*

The statement of the theorem is made precise by noting that the system (2.9) with input $u = (u_k)_{k=0}^{\infty}$ from l_p^m is said to have output $y = (y_k)_{k=0}^{\infty}$ if and only if there exists $\rho = (\rho_k)_{k=0}^{\infty}$ in l_p^n, where n is the order of the matrices A and G, such that $(I-Q)\rho_0 = 0$ and the sequence $\rho_0, \rho_1, \ldots$ satisfies the two equations in (2.9). In the proof of Theorem 2.2 we shall see that in this case ρ is uniquely determined by the input u. Theorem 2.2 therefore states that the system (2.9) has a well-defined input/output map which is equal to the block Toeplitz operator T. We need the following lemma in the proof of Theorem 2.2.

Lemma 2.3 ([GK1], Lemma 7.2). *Let $\lambda G - A$ be a $\mathbb{T}$-regular pencil of $n \times n$ matrices. Fix $1 \leq p \leq \infty$, and let $(\varphi_k)_{k=0}^{\infty}$ be in l_p^n. Then the general solution in l_p^n of the equation*

$$A\rho_{k+1} = G\rho_k + \varphi_k, \qquad k = 0, 1, 2, \ldots, \tag{2.10}$$

is given by

$$\begin{aligned} \rho_k \quad = \quad & E\Omega^k \eta + \textstyle\sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)\varphi_\nu \\ & - \textstyle\sum_{\nu=k}^{\infty} E\Omega^{\nu-k} P\varphi_\nu, \qquad k = 0, 1, 2, \ldots . \end{aligned} \tag{2.11}$$

Here P, E and Ω are given by (1.6)-(1.8) with $\Gamma = \mathbb{T}$ and η is an arbitrary vector in $\ker P$.

Proof. Let η be an arbitrary vector in $\mathbb{C}^n$, and let $\rho = (\rho_k)_{k=0}^{\infty}$ be given by (2.11). We first prove that $\rho \in l_p^n$. Put

$$\rho = g + S\varphi$$

where

$$g = (E\Omega^k \eta)_{k=0}^{\infty}, \qquad S : l_p^n \to l_p^n.$$

The operator $S : l_p^n \to l_p^n$ is defined by

$$(Su)_k = \sum_{\nu=0}^{\infty} M_{k-\nu} u_\nu, \qquad k = 0, 1, 2, \ldots,$$

where

$$M_k = \begin{cases} E\Omega^{k-1}(I-P), & k = 1, 2, \ldots, \\ -E\Omega^{-k}P, & k = 0, -1, -2, \ldots . \end{cases}$$

If we can show that $g \in l_p^n$ and S is a well-defined block Toeplitz operator, then $\rho = g + S\varphi \in l_p^n$, since l_p^n is a vector space. Since Ω has all its eigenvalues in the open unit disc (see Corollary 1.3), $\|\Omega\| < 1$, and

$$\|g\|_p^p = \sum_{k=0}^{\infty} \|E\Omega^k \eta\|^p \leq \|E\|^p \|\eta\|^p \sum_{k=0}^{\infty} \|\Omega\|^p < \infty.$$

Thus $g \in l_p^n$. To show that S is a block Toeplitz operator on l_p^n, we only need to show that the entries in M_k are bounded. The defining function of S is $\Phi(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n M^n$. Since $\|\Omega\| < 1$ we have that $\|\Omega\|^k < 1$ for each $k \in \mathbb{N}$, and so $\|M_k\| \leq \|E\|$. Therefore the entries in M_k are bounded. Thus $S : l_p^n \to l_p^n$ is a block Toeplitz operator.

Next, we show that $\rho = (\rho_k)_{k=0}^{\infty}$ given by (2.11) is a solution of the difference equation (2.10). Take $N \geq 0$, and note that the first $N+1$ elements in ρ may be rewritten as

$$\begin{aligned}
\rho_k &= E\Omega^k \eta + \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)\varphi_\nu - \sum_{\nu=k}^{\infty} E\Omega^{\nu-k} P\varphi_\nu \\
&= E\Omega^k \eta + \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)\varphi_\nu - \sum_{\nu=k}^{N} E\Omega^{\nu-k} P\varphi_\nu - \sum_{\nu=N+1}^{\infty} E\Omega^{\nu-k} P\varphi_\nu \\
&= E\Omega^k \eta + \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)\varphi_\nu - \sum_{\nu=k}^{N} E\Omega^{\nu-k} P\varphi_\nu \\
&\quad + E\Omega^{N+1-k}\left((-)\sum_{\nu=0}^{\infty} \Omega^\nu P\varphi_{\nu+N+1}\right) \\
&= E\Omega^k \eta + E\Omega^{N+1-k} y_{N+1} + \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)\varphi_\nu - \sum_{\nu=k}^{N} E\Omega^{\nu-k} P\varphi_\nu,
\end{aligned}$$

where

$$y_{N+1} = -\sum_{\nu=0}^{\infty} \Omega^\nu P\varphi_{\nu+N+1}.$$

Since

$$\Omega = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} : \ker P \oplus \mathrm{im} P \to \ker P \oplus \mathrm{im} P,$$

we have $y_{N+1} \in \mathrm{im} P$. But then we can apply Lemma 1.2 to show that $\rho_0, \ldots, \rho_{N+1}$ is a solution to the finite difference equation

$$A\rho_{k+1} = G\rho_k + \varphi_k, \qquad k = 0, \ldots, N. \tag{2.12}$$

Since N is arbitrary, this implies that ρ is a solution of (2.10).

To prove the converse, let $\rho = (\rho_k)_{k=0}^{\infty}$ in l_p^n be a solution of (2.10). Take $N \geq 0$. Then $\rho_0, \ldots, \rho_{N+1}$ is a solution of (2.12). So, using Lemma 1.2 we get the form

$$\begin{aligned} \rho_k &= E\Omega^k x_{N+1} + E\Omega^{N+1-k} y_{N+1} + \textstyle\sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)\varphi_\nu \\ &\quad - \textstyle\sum_{\nu=k}^{N} E\Omega^{\nu-k} P\varphi_\nu, \qquad k = 0, \ldots, N+1, \end{aligned} \tag{2.13}$$

where $x_{N+1} \in \ker P$ and $y_{N+1} \in \mathrm{im} P$. Then

$$\rho_0 = Ex_{N+1} + E\Omega^{N+1} y_{N+1} - \sum_{\nu=0}^{N} E\Omega^{\nu} P\varphi_\nu$$

and

$$\rho_{N+1} = E\Omega^{N+1} x_{N+1} + Ey_{N+1} + \sum_{\nu=0}^{N} E\Omega^{N-\nu}(I-P)\varphi_\nu.$$

Recall that $Q = EPE^{-1}$ and $\Omega P = P\Omega$, where Q is given by (1.9) with $\Gamma = \mathbb{T}$. Since $x_{N+1} \in \ker P$, $y_{N+1} \in \mathrm{im} P$, $P\varphi_\nu \in \mathrm{im} P$ we have

$$\begin{aligned} (I-Q)\rho_0 &= (I-Q)Ex_{N+1} + (I-Q)E\Omega^{N+1} y_{N+1} - (I-Q)\sum_{\nu=0}^{N} E\Omega^{\nu} P\varphi_\nu \\ &= Ex_{N+1} + 0 + 0 \end{aligned}$$

and

$$\begin{aligned} Q\rho_{N+1} &= QE\Omega^{N+1} x_{N+1} + QEy_{N+1} + Q\sum_{\nu=0}^{N} E\Omega^{N-\nu}(I-P)\varphi_\nu \\ &= 0 + Ey_{N+1} + 0. \end{aligned}$$

Thus

$$(I-Q)\rho_0 = Ex_{N+1}, \qquad Q\rho_{N+1} = Ey_{N+1} \tag{2.14}$$

where Q is given by (1.9). The first identity in (2.14) implies that x_{N+1} is independent of N. Put $\eta = E^{-1}(I-Q)\rho_0$. Then $\eta = x_{N+1}$ for each N and $\eta \in \ker P$. Since $\rho \in l_p^n$, the sequence $\rho_0, \rho_1, \ldots$ is a bounded sequence in $\mathbb{C}^n$. Thus $y_k = E^{-1}Q\rho_k$ is also a bounded sequence in $\mathbb{C}^n$. Therefore $E\Omega^{N+1-k}y_{N+1} \longrightarrow 0$ as $N \longrightarrow \infty$ since $\|\Omega\| < 1$. Furthermore, since $(\varphi_k)_{k=0}^{\infty} \in l_p^n$ and $\|\Omega\| < 1$ we have that

$$\left\| \sum_{\nu=k}^{\infty} E\Omega^{\nu-k} P\varphi_\nu \right\| \leq \|E\|\|P\|\|\varphi\|_\infty \sum_{\nu=0}^{\infty} \|\Omega\|^\nu < \infty,$$

where $\|\varphi\|_\infty = \sup\{\|\varphi\|_p : 1 \leq p < \infty\} < \infty$ as $\varphi \in l_p^n$. So $\sum_{\nu=k}^{\infty} E\Omega^{\nu-k}P\varphi_\nu$ is absolutely convergent. Since $\mathbb{C}^n$ is a Banach space, the series is also convergent. Thus (2.13) becomes (2.11) as $N \longrightarrow \infty$. ♮

Proof of Theorem 2.2. Since the symbol Φ is given by (2.7), the entries of T admit the following power representation:

$$\Phi_k = \begin{cases} -CE(\Omega^{k-1} - \alpha\Omega^k)(I-P)B, & k > 0, \\ D + \alpha CE(I-P)B + CEPB, & k = 0, \\ CE(\Omega^{-k} - \alpha\Omega^{-k-1})PB, & k < 0, \end{cases} \tag{2.15}$$

where P, E and Ω are given by (1.6)-(1.8) with $\Gamma = \mathbb{T}$. Assume $x = (x_k)_{k=0}^{\infty} \in l_p^m$ is a solution of (2.8). Put

$$\rho_k = \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)Bx_\nu - \sum_{\nu=k}^{\infty} E\Omega^{\nu-k}PBx_\nu, \quad k = 0, 1, 2, \ldots . \tag{2.16}$$

Note that ρ_k $(k = 0, 1, 2, \ldots)$ is the same as in (2.11) provided that in (2.11) we take $\eta = 0$ and $\varphi_k = Bx_k$, $k = 0, 1, 2, \ldots$. So Lemma 2.3 implies that $\rho = (\rho_k)_{k=0}^{\infty}$ is in l_p^n and the sequence $\rho_0, \rho_1, \ldots$ satisfies the first equation in (2.9) with $u_k = x_k$, $k = 0, 1, 2, \ldots$. The power representation (2.15) implies that $(\rho_k)_{k=0}^{\infty}$ satisfies the second equation in (2.9) with $y_k = z_k$ and $u_k = x_k$, $k = 0, 1, 2, \ldots$. This can be seen as follows. Note that we can write

$$(Tx)_k = \sum_{\nu=0}^{\infty} \Phi_{k-\nu} x_\nu,$$

where Φ_k is given by (2.15). So

$$\begin{aligned} z_k &= (Tx)_k \\ &= \sum_{\nu=0}^{\infty} \Phi_{k-\nu} x_\nu = \sum_{\nu=0}^{k-1} \Phi_{k-\nu} x_\nu + \Phi_0 x_k + \sum_{\nu=k+1}^{\infty} \Phi_{k-\nu} x_\nu \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\nu=0}^{k-1} -CE(\Omega^{k-\nu-1} - \alpha\Omega^{k-\nu})(I-P)Bx_\nu \\
&\quad +(D + \alpha CE(I-P)B + CEPB)x_k + \sum_{\nu=k+1}^{\infty} CE(\Omega^{\nu-k} - \alpha\Omega^{\nu-(k+1)} PBx_\nu \\
&= \alpha C \left\{ \sum_{\nu=0}^{k-1} E\Omega^{k-\nu}(I-P)Bx_\nu + E(I-P)Bx_k \right. \\
&\quad \left. - \sum_{\nu=k+1}^{\infty} E\Omega^{\nu-(k+1)})PBx_\nu \right\} \\
&\quad -C \left\{ \sum_{\nu=0}^{k-1} E\Omega^{k-\nu-1}(I-P)Bx_\nu - EPBx_k \right. \\
&\quad \left. - \sum_{\nu=k+1}^{\infty} E\Omega^{\nu-k} PBx_\nu \right\} + Dx_k \\
&= \alpha C \left\{ \sum_{\nu=0}^{k} E\Omega^{k-\nu}(I-P)Bx_\nu - \sum_{\nu=k+1}^{\infty} E\Omega^{\nu-(k+1)} PBx_\nu \right\} \\
&\quad -C \left\{ \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)Bx_\nu - \sum_{\nu=k}^{\infty} E\Omega^{\nu-k} PBx_\nu \right\} + Dx_k.
\end{aligned}
$$

Thus

$$z_k = C(\alpha\rho_{k+1} - \rho_k) + Dx_k, \qquad k = 0, 1, 2, \ldots .$$

Furthermore, $\Omega P = P\Omega$ and $EP = QE$. This follows from

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} : \ker P \oplus \mathrm{im} P \to \ker Q \oplus \mathrm{im} Q$$

and

$$\Omega = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} : \ker P \oplus \mathrm{im} P \to \ker P \oplus \mathrm{im} P.$$

Now, $\rho_0 = -\sum_{\nu=0}^{\infty} E\Omega^\nu PBx_\nu$. Thus

$$
\begin{aligned}
(I-Q)\rho_0 &= -(I-Q)E\sum_{\nu=0}^{\infty} \Omega^\nu PBx_\nu = -E(I-P)\sum_{\nu=0}^{\infty} P\Omega^\nu Bx_\nu \\
&= -E(I-P)P\sum_{\nu=0}^{\infty} \Omega^\nu Bx_\nu = 0,
\end{aligned}
$$

since P is a projection. Thus $(\rho_k)_{k=0}^{\infty} \in l_p^n$ is a solution of (2.9) with $u_k = x_k$ and $y_k = z_k$, $k = 0, 1, 2, \dots$.

To prove the converse, suppose $\rho = (\rho_k)_{k=0}^{\infty}$ is a solution in l_p^n of the singular system (2.9) with $u = (u_k)_{k=0}^{\infty}$ from l_p^n. Put $x_k = u_k$ and $z_k = y_k$, $k = 0, 1, 2, \dots$. Then x and z are in l_p^m. We want to show that $Tx = z$. Observe that Lemma 2.3 implies that

$$\rho_k = E\Omega^k \eta + \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)Bx_\nu - \sum_{\nu=k}^{\infty} E\Omega^{\nu-k}PBx_\nu, \; k = 0, 1, 2, \dots,$$

where η is some vector in $\ker P$. Since $\rho_0 = -\sum_{\nu=0}^{\infty} E\Omega^\nu PBx_\nu + E\eta$, and

$$\begin{aligned} 0 = (I-Q)\rho_0 &= -(I-Q)\sum_{\nu=0}^{\infty} E\Omega^\nu PBx_\nu + (I-Q)E\eta \\ &= 0 + E(I-P)\eta \\ &= E\eta, \end{aligned}$$

(using the boundary condition in (2.9) and the fact that $\eta \in \ker P$) we have $\eta = 0$. So the sequence $\rho_0, \rho_1, \dots$ is uniquely determined and given by (2.16). Using the second equation in (2.9) and the power representation (2.15), we obtain

$$\begin{aligned} y_k &= C(\alpha\rho_{k+1} - \rho_k) + Dx_k \\ &= \alpha C\left\{\sum_{\nu=0}^{k} E\Omega^{k-\nu}(I-P)Bx_\nu - \sum_{\nu=k+1}^{\infty} E\Omega^{\nu-(k+1)}PBx_\nu\right\} \\ &\quad -C\left\{\sum_{\nu=0}^{k-1} E\Omega^{k-\nu-1}(I-P)Bx_\nu - \sum_{\nu=k}^{\infty} E\Omega^{\nu-k}PBx_\nu\right\} + Dx_k \\ &= \sum_{\nu=0}^{k-1} -CE\left(\Omega^{k-\nu-1} - \alpha\Omega^{k-\nu}\right)(I-P)Bx_\nu \\ &\quad + [D + \alpha CE(I-P)B + CEPB]\,x_k \\ &\quad + \sum_{\nu=k+1}^{\infty} CE\left(\Omega^{\nu-k} - \alpha\Omega^{\nu-k-1}\right)PBx_\nu \\ &= \sum_{\nu=0}^{k-1} \Phi_{k-\nu}x_\nu + \Phi_0 x_k + \sum_{\nu=k+1}^{\infty} \Phi_{k-\nu}x_\nu \end{aligned}$$

$$= \sum_{\nu=0}^{\infty} \Phi_{k-\nu} x_\nu = (Tx)_k = z_k$$

Hence $x = (x_k)_{k=0}^{\infty} \in l_p^m$ solves $Tx = z$. ♮

Note that the last part of the proof of Theorem 2.2 shows that for given input and output in l_p^m the solution $\rho = (\rho_k)_{k=0}^{\infty}$ of (2.9) in l_p^n is unique (assuming it exists).

The equivalence in Theorem 2.2 implies that we may get solutions of equation (2.8) by inverting the system (2.9). This is done as follows. First interchange in (2.9) the roles of input and output. Apply BD^{-1} to the second equation to give

$$BD^{-1}y_k = BD^{-1}C(\alpha\rho_{k+1} - \rho_k) + Bu_k.$$

Now subtract this equation from the first equation. This yields

$$A\rho_{k+1} = G\rho_k + BD^{-1}y_k - BD^{-1}C(\alpha\rho_{k+1} - \rho_k).$$

Thus

$$(A + \alpha BD^{-1}C)\rho_{k+1} = (G + BD^{-1}C)\rho_k + BD^{-1}y_k.$$

Therefore the inverse system is

$$(2.17) \qquad \begin{cases} A^\times \rho_{k+1} &= G^\times \rho_k + BD^{-1}y_k, & k = 0, 1, 2, \dots, \\ u_k &= -D^{-1}C(\alpha\rho_{k+1} - \rho_k) + D^{-1}y_k, & k = 0, 1, 2, \dots, \\ (I - Q)\rho_0 &= 0, \end{cases}$$

where $A^\times = A + \alpha BD^{-1}C$ and $G^\times = G + BD^{-1}C$. We may assume that $y = (y_k)_{k=0}^{\infty}$ is a given element in l_p^m. The problem is now to find $(\rho_k)_{k=0}^{\infty}$ in l_p^n satisfying the first equation in (2.17) and the boundary condition $(I - Q)\rho_0 = 0$. Note that the projection Q comes from the pencil $\lambda G - A$ and is not directly related to $\lambda G^\times - A^\times$, and hence it is not straightforward to find a sequence $(\rho_k)_{k=0}^{\infty}$ with the desired properties. In fact, the problem may not be solvable or if it is solvable it may have many solutions. However, if such a sequence $(\rho_k)_{k=0}^{\infty}$ has been found, then a solution of the equation $Tu = y$ is obtained by taking $u_k = -D^{-1}C(\alpha\rho_{k+1} - \rho_k) + D^{-1}y_k$, $k = 0, 1, 2, \dots$. In this manner we are led to the following theorem.

Theorem 2.4. *Let* $1 \leq p \leq \infty$, *and let* $y = (y_k)_{k=0}^{\infty}$ *be in* l_p^m. *Consider the block Toeplitz equation*

$$\sum_{\nu=0}^{\infty} \Phi_{k-\nu} u_\nu = y_k, \qquad k = 0, 1, 2, \ldots, \tag{2.18}$$

where Φ_k *are the Fourier coefficients of a rational matrix function*

$$\Phi(\lambda) = D + (\lambda - \alpha) C (\lambda G - A)^{-1} B, \qquad \lambda \in \mathbb{T}, \tag{2.19}$$

given in realized form, where $\alpha \neq 0$ *is neither a pole nor a zero of* Φ. *Put* $A^\times = A + \alpha B D^{-1} C$ *and* $G^\times = G + B D^{-1} C$, *and assume that the pencil* $\lambda G^\times - A^\times$ *is* $\mathbb{T}$-*regular. Then the equation (2.18) is solvable in* l_p^m *if and only if*

$$\sum_{\nu=0}^{\infty} (\Omega^\times)^\nu P^\times B D^{-1} y_\nu \in \operatorname{im} P + \ker P^\times, \tag{2.20}$$

and in this case the general solution in l_p^m *of (2.18) is given by*

$$u_k = D^{-1} C E^\times \left[(\Omega^\times)^{k-1} - \alpha (\Omega^\times)^k \right] \eta + \sum_{\nu=0}^{\infty} \Phi_{k-\nu}^\times y_\nu, \; k = 1, 2, \ldots . \tag{2.21}$$

Here P *is the separating projection corresponding to* $\lambda G - A$ *and* $\mathbb{T}$, *and the operators* $P^\times$, $E^\times$ *and* $\Omega^\times$ *are, respectively, the separating projection, the right equivalence operator and the associate operator corresponding to* $\lambda G^\times - A^\times$ *and* $\mathbb{T}$,

$$\Phi_k^\times = \begin{cases} D^{-1} C E^\times \left[(\Omega^\times)^{k-1} - \alpha (\Omega^\times)^k \right] (I - P^\times) B D^{-1}, & k > 0, \\ D^{-1} - D^{-1} C E^\times \left[P^\times + \alpha (I - P^\times) \right] B D^{-1}, & k = 0, \\ D^{-1} C E^\times \left[\alpha (\Omega^\times)^{-k-1} - (\Omega^\times)^{-k} \right] P^\times B D^{-1}, & k < 0, \end{cases} \tag{2.22}$$

and η *is an arbitrary vector in* $\ker P^\times$ *such that*

$$\eta - \sum_{\nu=0}^{\infty} (\Omega^\times)^\nu P^\times B D^{-1} y_\nu \in \operatorname{im} P. \tag{2.23}$$

In particular, the general solution in l_p^m *of the homogeneous equation*

$$\sum_{\nu=0}^{\infty} \Phi_{k-\nu} u_\nu = 0, \qquad k = 0, 1, 2, \ldots, \tag{2.24}$$

is given by

$$u_k = D^{-1}CE^\times\left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right]\eta, \qquad k = 1, 2, \ldots, \tag{2.25}$$

where η *is an arbitrary vector in* $\ker P^\times \cap \mathrm{im}P$.

Proof. Let Q be the projection defined by (1.9) with $\Gamma = \mathbb{T}$, and let $Q^\times$ be the corresponding projection for $\lambda G^\times - A^\times$ and $\mathbb{T}$, that is,

$$Q^\times = \frac{1}{2\pi i}\int_\Gamma (\lambda G^\times - A^\times)^{-1}G^\times d\lambda, \qquad \Gamma = \mathbb{T}. \tag{2.26}$$

From Theorem 2.2, and the statements made in the discussion preceding this theorem, it follows that (2.18) is solvable in l_p^m if and only if there exists $(\rho_k)_{k=0}^\infty$ in l_p^n satisfying the first equation in (2.17) and the boundary condition $(I - Q)\rho_0 = 0$. According to Lemma 2.3 the general solution in l_p^n of the first equation in (2.17) is given by

$$\begin{aligned}\rho_k \;=\; & E^\times(\Omega^\times)^k\gamma + \textstyle\sum_{\nu=0}^{k-1} E^\times(\Omega^\times)^{k-1-\nu}(I - P^\times)BD^{-1}y_\nu \\ & - \textstyle\sum_{\nu=k}^{\infty} E^\times(\Omega^\times)^{\nu-k}P^\times BD^{-1}y_\nu, \qquad k = 0, 1, 2, \ldots,\end{aligned} \tag{2.27}$$

where γ is an arbitrary vector in $\ker P^\times$. Note that

$$\rho_0 = E^\times\gamma - \sum_{\nu=0}^{\infty} E^\times(\Omega^\times)^\nu P^\times BD^{-1}y_\nu.$$

Since $E^\times\gamma \in \ker Q^\times$, the first equation in (2.17) has a solution $(\rho_k)_{k=0}^\infty$ in l_p^n satisfying the boundary condition $(I - Q)\rho_0 = 0$ if and only if

$$\sum_{\nu=0}^{\infty} E^\times(\Omega^\times)^\nu P^\times BD^{-1}y_\nu \in \ker Q^\times + \mathrm{im}Q. \tag{2.28}$$

This can be seen as follows. Note that $(I - Q)\rho_0 = 0$ implies that $Q\rho_0 = \rho_0$, i.e., $\rho_0 \in \mathrm{im}Q$. Thus

$$E^\times\gamma - \sum_{\nu=0}^{\infty} E^\times(\Omega^\times)^\nu P^\times BD^{-1}y_\nu \in \mathrm{im}Q.$$

Thus

$$\sum_{\nu=0}^{\infty} E^\times(\Omega^\times)^\nu P^\times BD^{-1}y_\nu = E^\times\gamma - \rho_0 \in \ker Q^\times + \mathrm{im}Q.$$

In this case the output $u = (u_k)_{k=0}^{\infty}$ of (2.17) is given by

$$
\begin{aligned}
u_k &= -D^{-1}C(\alpha\rho_{k+1} - \rho_k) + D^{-1}y_k \\
&= -D^{-1}\{\alpha C\rho_{k+1} - C\rho_k - y_k\} \\
&= -D^{-1}\Bigg\{\alpha C\Bigg[E^\times(\Omega^\times)^{k+1}\gamma + \sum_{\nu=0}^{k} E^\times(\Omega^\times)^{k-\nu}(I-P^\times)BD^{-1}y_\nu \\
&\qquad - \sum_{\nu=k+1}^{\infty} E^\times(\Omega^\times)^{\nu-(k+1)}P^\times BD^{-1}y_\nu\Bigg] \\
&\qquad -C\Bigg[E^\times(\Omega^\times)^k\gamma + \sum_{\nu=0}^{k-1} E^\times(\Omega^\times)^{k-1-\nu}(I-P^\times)BD^{-1}y_\nu \\
&\qquad - \sum_{\nu=k}^{\infty} E^\times(\Omega^\times)^{\nu-k}P^\times BD^{-1}y_\nu\Bigg] - y_k\Bigg\} \\
&= D^{-1}CE^\times\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right]\gamma \\
&\qquad + \sum_{\nu=0}^{k-1} D^{-1}CE^\times\left[(\Omega^\times)^{k-1-\nu} - \alpha(\Omega^\times)^{k-\nu}\right](I-P^\times)BD^{-1}y_\nu \\
&\qquad -\alpha D^{-1}CE^\times(\Omega^\times)^0(I-P^\times)BD^{-1}y_k - D^{-1}CE^\times(\Omega^\times)^0P^\times BD^{-1}y_k \\
&\qquad +D^{-1}y_k + \sum_{\nu=k+1}^{\infty} D^{-1}CE^\times\left[\alpha(\Omega^\times)^{\nu-k-1} - (\Omega^\times)^{\nu-k}\right]P^\times BD^{-1}y_\nu \\
&= D^{-1}CE^\times\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right]\gamma \\
&\qquad + \sum_{\nu=0}^{k-1} D^{-1}CE^\times\left[(\Omega^\times)^{(k-\nu)-1} - \alpha(\Omega^\times)^{k-\nu}\right](I-P^\times)BD^{-1}y_\nu \\
&\qquad + \left\{D^{-1} - D^{-1}CE^\times\left[P^\times + \alpha(I-P^\times)\right]BD^{-1}\right\}y_k \\
&\qquad + \sum_{\nu=k+1}^{\infty} D^{-1}CE^\times\left[\alpha(\Omega^\times)^{-(k-\nu)-1} - (\Omega^\times)^{-(k-\nu)}\right]P^\times BD^{-1}y_\nu \\
&= D^{-1}CE^\times\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right]\gamma + \sum_{\nu=0}^{k-1}\Phi_{k-\nu}^\times y_\nu + \Phi_0^\times y_k + \sum_{\nu=k+1}^{\infty}\Phi_{k-\nu}^\times y_\nu .
\end{aligned}
$$

Thus

$$
u_k = D^{-1}CE^\times\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right]\gamma + \sum_{\nu=0}^{\infty}\Phi_{k-\nu}^\times y_\nu,\ k = 0,1,2,\ldots, \tag{2.29}
$$

where the $\Phi_k^\times$ are defined by (2.22) and γ is an arbitrary vector in $\ker P^\times$ such that

$$E^\times\gamma - \sum_{\nu=0}^{\infty} E^\times(\Omega^\times)^\nu P^\times BD^{-1}y_\nu \in \mathrm{im}Q. \tag{2.30}$$

Therefore (2.28) is a necessary and sufficient condition for (2.18) to have a solution in l_p^m. If this condition is satisfied, then the general solution $(u_k)_{k=0}^\infty$ in l_p^m of (2.18) is given by (2.29).

For the remainder of the proof we must show that (2.20) is equivalent to (2.28) and (2.21) gives the same set of sequences as (2.29). Denote the left hand side of (2.20) by x_0. Then $x_0 = \sum_{\nu=0}^\infty (\Omega^\times)^\nu P^\times BD^{-1}y_\nu = P^\times \sum_{\nu=0}^\infty (\Omega^\times)^\nu BD^{-1}y_\nu$ implies that $x_0 \in \mathrm{im}P^\times$. So $G^\times E^\times x_0 = G^\times E^\times P^\times x_0 = P^\times x_0 = x_0$. Next, note that the operators

$$Q^\times|\mathrm{im}Q : \mathrm{im}Q \longrightarrow \mathrm{im}Q^\times \text{ and } P^\times|\mathrm{im}P : \mathrm{im}P \longrightarrow \mathrm{im}P^\times$$

are equivalent. Indeed, we know that

$$G^\times Q^\times = P^\times G^\times, \qquad G^\times Q = PG^\times.$$

Moreover, $G^\times$ maps $\mathrm{im}Q^\times$ (resp. $\mathrm{im}Q$) in a one-one manner onto $\mathrm{im}P^\times$ (resp. $\mathrm{im}P$). Therefore the operators

$$G^\times|\mathrm{im}Q : \mathrm{im}Q \longrightarrow \mathrm{im}P, \qquad G^\times|\mathrm{im}Q^\times : \mathrm{im}Q^\times \longrightarrow \mathrm{im}P^\times$$

are invertible and

$$(G^\times|\mathrm{im}Q^\times)(Q^\times|\mathrm{im}Q) = (P^\times|\mathrm{im}P)(G^\times|\mathrm{im}Q). \tag{2.31}$$

It follows that

$$\begin{aligned} E^\times x_0 \in \ker Q^\times + \mathrm{im}Q &\iff E^\times x_0 \in \mathrm{im}(Q^\times|\mathrm{im}Q) \\ &\iff G^\times E^\times x_0 \in \mathrm{im}(P^\times|\mathrm{im}P) \\ &\iff x_0 \in \mathrm{im}(P^\times|\mathrm{im}P) \\ &\iff x_0 \in \ker P^\times + \mathrm{im}P, \end{aligned}$$

which proves the equivalence of (2.20) and (2.28).

Also, note that $\Omega^\times(I-P^\times) = G^\times E^\times(I-P^\times)$ (see the first identity in (2.22b), [GK1]). Thus

$$\Omega^\times\gamma = G^\times E^\times\gamma, \qquad \gamma\in\ker P^\times.$$

Let L_1 be the set of all $\gamma\in\ker P^\times$ satisfying (2.30), i.e., $E^\times\gamma - E^\times x_0 \in \mathrm{im}Q$. Let L_2 be the set of all $\eta\in\ker P^\times$ such that (2.23) holds, i.e., $\eta - x_0\in\mathrm{im}P$. To prove that (2.21) and (2.29) define the same set of sequences, it suffices to show that $G^\times E^\times(L_1) = L_2$. Take $\gamma\in L_1$. Thus $E^\times\gamma - E^\times x_0\in\mathrm{im}Q$. Since $G^\times$ maps $\mathrm{im}Q$ into $\mathrm{im}P$, this implies that

$$G^\times E^\times\gamma - x_0 = G^\times E^\times\gamma - G^\times E^\times x_0 \in \mathrm{im}P.$$

Also, $G^\times E^\times(\ker P^\times)\subseteq\ker P^\times$. So $G^\times E^\times\gamma\in L_2 \Longrightarrow G^\times E^\times(L_1)\subseteq L_2$. Conversely, take $\eta\in L_2$. Then there exists $u\in\mathrm{im}Q$ such that $\eta - x_0 = G^\times u$. Thus

$$-G^\times E^\times x_0 = -x_0 = -P^\times x_0 = P^\times(\eta - x_0) = P^\times G^\times u = G^\times Q^\times u.$$

Thus $-E^\times x_0 = Q^\times u$ ($\in \mathrm{im}(Q^\times|\mathrm{im}Q)$) since $G^\times$ is one-one on $\mathrm{im}Q^\times$. But then there exists $\gamma\in\ker P^\times$ such that $E^\times\gamma - E^\times x_0 = u$. So $\gamma\in L_1$ and

$$G^\times E^\times\gamma - x_0 = G^\times E^\times\gamma - G^\times E^\times x_0 = G^\times u = \eta - x_0.$$

Thus $\eta = G^\times E^\times\gamma\in G^\times E^\times(L_1) \Longrightarrow L_2\subseteq G^\times E^\times(L_1)$ and the theorem is proved. ♮

2.3 Inversion of Finite Block Toeplitz Matrices

In this section the inversion method based on equivalence to linear systems, which was used in the previous section, is developed further for finite block Toeplitz matrices.

Theorem 2.5. *Consider the finite block Toeplitz equation*

$$\sum_{\nu=0}^{N}\Phi_{k-\nu}x_\nu = z_k, \qquad k=0,\dots,N, \tag{2.32}$$

where $\Phi_{-N}, \ldots, \Phi_N$ are the $-N$ to N Fourier coefficients of a rational matrix function

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \mathbb{T} \tag{2.33}$$

given in realized form, where $\alpha \neq 0$ is neither a pole nor a zero of Φ. Then Equation (2.32) is equivalent to the following discrete boundary value system:

$$\begin{cases} A\rho_{k+1} &= G\rho_k + Bu_k, \qquad k = 0, 1, \ldots, N, \\ y_k &= C(\alpha\rho_{k+1} - \rho_k) + Du_k, \quad k = 0, 1, \ldots, N, \\ (I - Q)\rho_0 &= 0, \quad Q\rho_{N+1} = 0, \end{cases} \tag{2.34}$$

where Q is the projection given by (1.9) with $\Gamma = \mathbb{T}$. The equivalence between (2.32) and (2.34) has to be understood in the following sense: If $x = (x_k)_{k=0}^N$ is a solution of (2.32), then the system (2.34) with input $u_k = x_k$ $(k = 0, 1, \ldots, N)$ has output $y_k = z_k$ $(k = 0, 1, \ldots, N)$, and, conversely, if the system (2.34) with input $u = (u_k)_{k=0}^N$ has output $y_k = z_k$ $(k = 0, 1, \ldots, N)$, then $x = u$ is a solution of (2.32).

Proof. Since the symbol Φ is given by (2.33), the matrix coefficients $\Phi_{-N}, \ldots, \Phi_N$ in (2.32) are given by

$$\Phi_k = \begin{cases} -CE(\Omega^{k-1} - \alpha\Omega^k)(I - P)B, & k = 1, 2, \ldots, N, \\ D + \alpha CE(I - P)B + CEPB, & k = 0, \\ CE(\Omega^{-k} - \alpha\Omega^{-k-1})PB, & k = -1, -2, \ldots, -N, \end{cases} \tag{2.35}$$

where P, E and Ω are given by (1.6)-(1.8) with $\Gamma = \mathbb{T}$. Assume $x_0, \ldots, x_N$ is a solution of (2.32). Put

$$\begin{aligned} \rho_k &= \textstyle\sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I - P)Bx_\nu \\ &\quad - \textstyle\sum_{\nu=k}^{N} E\Omega^{\nu-k}PBx_\nu, \ k = 0, 1, \ldots, N+1. \end{aligned} \tag{2.36}$$

Using the identities $AE(I - P) = I - P$, $AEP = \Omega P$, $GE(I - P) = \Omega(I - P)$ and $GEP = P$ we get

$$\begin{aligned} A\rho_{k+1} &= \sum_{\nu=0}^{k} AE\Omega^{k-\nu}(I - P)Bx_\nu - \sum_{\nu=k+1}^{N} AE\Omega^{\nu-k-1}PBx_\nu \\ &= \sum_{\nu=0}^{k} \Omega^{k-\nu}(I - P)Bx_\nu - \sum_{\nu=k+1}^{N} \Omega^{\nu-k}PBx_\nu \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\nu=0}^{k-1} \Omega^{k-\nu}(I-P)Bx_\nu - \sum_{\nu=k}^{N} \Omega^{\nu-k} PBx_\nu \\
&\quad + \Omega^0 (I-P)Bx_k + \Omega^0 PBx_k \\
&= \sum_{\nu=0}^{k-1} \Omega(I-P)\Omega^{k-1-\nu} Bx_\nu - \sum_{\nu=k}^{N} P\Omega^{\nu-k} Bx_\nu + Bx_k \\
&= G\sum_{\nu=0}^{k-1} E(I-P)\Omega^{k-1-\nu} Bx_\nu \\
&\quad - G\sum_{\nu=k}^{N} EP\Omega^{\nu-k} Bx_\nu + Bx_k \\
&= G\left\{\sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)Bx_\nu - \sum_{\nu=k}^{N} E\Omega^{\nu-k} PBx_\nu\right\} + Bx_k \\
&= G\rho_k + Bx_k.
\end{aligned}
$$

Thus $\rho_0, \ldots, \rho_{N+1}$ is a solution of the first equation in (2.34) with $u_k = x_k,\ k = 0, 1, \ldots, N$. Also,

$$
\begin{aligned}
z_k &= \sum_{\nu=0}^{N} \Phi_{k-\nu} x_\nu = \sum_{\nu=0}^{k-1} \Phi_{k-\nu} x_\nu + \Phi_0 x_k + \sum_{\nu=k+1}^{N} \Phi_{k-\nu} x_\nu \\
&= \sum_{\nu=0}^{k-1} -CE(\Omega^{k-1-\nu} - \alpha\Omega^{k-\nu})(I-P)Bx_\nu \\
&\quad + [D + \alpha CE(I-P)B + CEPB]x_k \\
&\quad + \sum_{\nu=k+1}^{N} CE(\Omega^{\nu-k} - \alpha\Omega^{\nu-k-1})PBx_\nu \\
&= C\left\{\sum_{\nu=0}^{k-1} -E\left(\Omega^{k-1-\nu} - \alpha\Omega^{k-\nu}\right)(I-P)Bx_\nu \right. \\
&\quad \left. + \left(\alpha E(I-P)B + EPB\right)x_k + \sum_{\nu=k+1}^{N} E\left(\Omega^{\nu-k} - \alpha\Omega^{\nu-k-1}\right)PBx_\nu \right\} + Dx_k \\
&= C\left\{\alpha\left[\sum_{\nu=0}^{k-1} E\Omega^{k-\nu}(I-P)Bx_\nu + E(I-P)Bx_k - \sum_{\nu=k+1}^{N} E\Omega^{\nu-k-1}PBx_\nu\right] \right. \\
&\quad \left. - \left[\sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)Bx_\nu - EPBx_k - \sum_{\nu=k+1}^{N} E\Omega^{\nu-k}PBx_\nu\right]\right\} + Dx_k
\end{aligned}
$$

$$
\begin{aligned}
&= C\left\{\alpha\left[\sum_{\nu=0}^{k} E\Omega^{k-\nu}(I-P)Bx_\nu - \sum_{\nu=k+1}^{N} E\Omega^{\nu-k-1}PBx_\nu\right]\right. \\
&\quad \left. - \left[\sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)Bx_\nu - \sum_{\nu=k}^{N} E\Omega^{\nu-k}PBx_\nu\right]\right\} + Dx_k \\
&= C(\alpha\rho_{k+1} - \rho_k) + Dx_k \\
&= y_k.
\end{aligned}
$$

Therefore $\rho_0, \ldots, \rho_{N+1}$ satisfy the second equation in (2.34) with $u_k = x_k$ and $y_k = z_k$, $k = 0, 1, \ldots, N$. Furthermore, since $EP = QE$ and $\Omega P = P\Omega$, $\rho_0 = -\sum_{\nu=0}^{N} E\Omega^\nu PBx_\nu$ implies that $(I-Q)\rho_0 = -(I-Q)\sum_{\nu=0}^{N} E\Omega^\nu PBx_\nu = -(E-EP)P\sum_{\nu=0}^{N} \Omega^\nu Bx_\nu = -(EP-EP^2)\sum_{\nu=0}^{N} \Omega^\nu Bx_\nu = -(EP-EP)\sum_{\nu=0}^{N} \Omega^\nu Bx_\nu = 0$. Also, $\rho_{N+1} = \sum_{\nu=0}^{N} E\Omega^{N-\nu}(I-P)Bx_\nu$ implies that $Q\rho_{N+1} = \sum_{\nu=0}^{N} QE\Omega^{N-\nu}(I-P)Bx_\nu = \sum_{\nu=0}^{N} EP(I-P)\Omega^{N-\nu}Bx_\nu = \sum_{\nu=0}^{N} E(P-P^2)\Omega^{N-\nu}Bx_\nu = \sum_{\nu=0}^{N} E(P-P)\Omega^{N-\nu}Bx_\nu = 0$. Thus with input $u_k = x_k$ $(k = 0, 1, \ldots, N)$ the system (2.34) has output $y_k = z_k$ $(k = 0, 1, \ldots, N)$.

To prove the converse statement, let $\rho_0, \ldots, \rho_N$ be a solution of (2.34) with $y_k = z_k$, $k = 0, 1, \ldots, N$. We need to show that if u_k is the input with output y_k, then $T_\Phi x = z$, where $x = u$ and $y = z$. From Lemma 1.2 we know that ρ_k is given by

$$
\begin{aligned}
\rho_k &= E\Omega^k\eta + E\Omega^{N+1-k}\xi + \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I-P)Bx_\nu \\
&\quad - \sum_{\nu=k}^{N} E\Omega^{\nu-k}PBx_\nu, \qquad k = 0, 1, \ldots, N+1,
\end{aligned}
$$

where $\eta \in \ker P$ and $\xi \in \mathrm{im} P$. Thus $\rho_0 = E\eta + E\Omega^{N+1}\xi - \sum_{\nu=0}^{N} E\Omega^\nu PBx_\nu$ and $\rho_{N+1} = E\Omega^{N+1}\eta + E\xi + \sum_{\nu=0}^{N} E\Omega^{N-\nu}(I-P)Bx_\nu$. From the first boundary condition we get

$$
0 = (I-Q)\rho_0 = (I-Q)E\eta + (I-Q)E\Omega^{N+1}\xi - (I-Q)\sum_{\nu=0}^{N} E\Omega^\nu PBx_\nu,
$$

i.e.,

$$0 = (I - Q)E\eta + (I - Q)E\Omega^{N+1}\xi.$$

Similarly, from the second boundary condition we get

$$0 = Q\rho_{N+1} = QE\Omega^{N+1}\eta + QE\xi + \sum_{\nu=0}^{N} QE(I - P)\Omega^{N-\nu}Bx_\nu,$$

i.e.,

$$0 = QE\Omega^{N+1}\eta + QE\xi.$$

Here we have two equations in ξ and η. The second equation yields $EP\Omega^{N+1}\eta + EP\xi = E\Omega^{N+1}P\eta + EP\xi = E\xi = 0$. Thus $\xi = 0$. From the first equation we get $(I - Q)E\eta = E\eta - QE\eta = E\eta - EP\eta = E\eta = 0$. Thus $\eta = 0$. Therefore ρ_k can be expressed as

$$\rho_k = \sum_{\nu=0}^{k-1} E\Omega^{k-1-\nu}(I - P)Bx_\nu - \sum_{\nu=k}^{N} E\Omega^{\nu-k}PBx_\nu,\ k = 0, 1, \ldots, N + 1,$$

which is the same as (2.36). Using the second equation in (2.34) we find that

$$z_k = y_k = C(\alpha\rho_{k+1} - \rho_k) + Du_k,\ k = 0, 1, \ldots, N.$$

Thus

$$z_k = \sum_{\nu=0}^{N} \Phi_{k-\nu}u_\nu,\ k = 0, 1, \ldots, N.$$

This is obtained from a previous calculation in the proof of Theorem 2.2. Thus $x_k = u_k,\ k = 0, 1, \ldots, N$ is a solution of (2.32). $\natural$

Using the equivalence in Theorem 2.5 one may solve Equation (2.32). The final result is the following theorem.

Theorem 2.6. *Let $y_0, y_1, \ldots, y_N$ be given vectors in $\mathbb{C}^m$, and consider the equation*

$$\sum_{\nu=0}^{N} \Phi_{k-\nu}u_\nu = y_k, \qquad k = 0, \ldots, N, \tag{2.37}$$

where $\Phi_{-N}, \ldots, \Phi_N$ are the $-N$ to N Fourier coefficients of a rational matrix function

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \mathbb{T},$$

given in realized form, where $\alpha \neq 0$ *is neither a pole nor a zero of* Φ. *Put* $A^\times = A + \alpha BD^{-1}C$ *and* $G^\times = G + BD^{-1}C$, *and assume that the pencil* $\lambda G^\times - A^\times$ *is* $\mathbb{T}$-*regular. Introduce*

$$
(2.38) \qquad \begin{aligned} V_N &= (I-Q)E^\times(I-P^\times) + (I-Q)E^\times(\Omega^\times)^{N+1}P^\times \\ &\quad + QE^\times(\Omega^\times)^{N+1}(I-P^\times) + QE^\times P^\times, \end{aligned}
$$

where Q *is the projection given by (1.9) with* $\Gamma = \mathbb{T}$ *and* $P^\times$, $E^\times$ *and* $\Omega^\times$ *are, respectively, the separating projection, the right equivalence operator and the associated operator corresponding to* $\lambda G^\times - A^\times$ *and* $\mathbb{T}$. *Then Equation (2.37) is solvable if and only if*

$$
(2.39) \qquad \begin{aligned} &\textstyle\sum_{\nu=0}^{N} \big[(I-Q)E^\times(\Omega^\times)^{\nu}P^\times \\ &-QE^\times(\Omega^\times)^{N-\nu}(I-P^\times)\big] BD^{-1}y_\nu \in \operatorname{im} V_N, \end{aligned}
$$

and in this case the general solution of (2.37) is given by

$$
(2.40) \qquad \begin{aligned} u_k &= D^{-1}CE^\times\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right](I-P^\times)\eta \\ &\quad + D^{-1}CE^\times\left[(\Omega^\times)^{N+1-k} - \alpha(\Omega^\times)^{N-k}\right]P^\times\eta \\ &\quad + \textstyle\sum_{\nu=0}^{N} \Phi^\times_{k-\nu} y_\nu, \ k = 0, 1, \ldots, N, \end{aligned}
$$

where η *is an arbitrary vector in* $\mathbb{C}^n$ *(with* n *the order of the matrices* G *and* A*) such that* $V_N\eta$ *is equal to the left side of (2.39) and*

$$
(2.41) \qquad \Phi^\times_k = \begin{cases} D^{-1}CE^\times\left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right](I-P^\times)BD^{-1}, & k > 0, \\ D^{-1} - D^{-1}CE^\times\left[P^\times + \alpha(I-P^\times)\right]BD^{-1}, & k = 0, \\ D^{-1}CE^\times\left[\alpha(\Omega^\times)^{-k-1} - (\Omega^\times)^{-k}\right]P^\times BD^{-1}, & k < 0. \end{cases}
$$

In particular, the general solution of the homogeneous equation

$$
\sum_{\nu=0}^{N} \Phi_{k-\nu}u_\nu = 0, \qquad k = 0, \ldots, N,
$$

is given by

$$
(2.42) \qquad \begin{aligned} u_k &= D^{-1}CE^\times\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right](I-P^\times)\eta + D^{-1}CE^\times \cdot \\ &\quad \left[(\Omega^\times)^{N+1-k} - \alpha(\Omega^\times)^{N-k}\right]P^\times\eta, \ k = 0, \ldots, N, \end{aligned}
$$

where η *is an arbitrary vector in* $\ker V_N$. *Furthermore, the block Toeplitz matrix*

$$
T_N = [\Phi_{k-j}]_{k,j=0}^{N}
$$

is invertible if and only if $\det V_N \neq 0$, *and in this case the entries of the inverse* $T_N^{-1} = [\Gamma_{kj}^N]_{k,j=0}^N$ *admits the following representation:*

$$\Gamma_{kj}^N = \Phi_{k-j}^\times + K_{kj}^N, \qquad k, j = 0, \ldots, N, \tag{2.43}$$

where $\Phi_{-N}^\times, \ldots, \Phi_N^\times$ *are as in (2.41) and*

$$\begin{aligned} K_{kj}^N &= D^{-1}CE^\times \left\{ \left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1} \right] (I - P^\times) \right. \\ &\quad \left. + \left[(\Omega^\times)^{N+1-k} - \alpha(\Omega^\times)^{N-k} \right] P^\times \right\} V_N^{-1} \cdot \\ &\quad \left\{ (I-Q)E^\times(\Omega^\times)^j P^\times - QE^\times(\Omega^\times)^{N-j}(I-P^\times) \right\} BD^{-1} \end{aligned}$$

Proof. By Theorem 2.5 the sequence $u = (u_k)_{k=0}^N$ is a solution of (2.37) if and only if there exist $\rho_0, \rho_1, \ldots, \rho_{N+1}$ satisfying (2.34). The inverse system of (2.34) is

$$\begin{cases} A^\times \rho_{k+1} &= G^\times \rho_k + BD^{-1}y_k, \qquad k = 0, \ldots, N, \\ u_k &= -D^{-1}C(\alpha\rho_{k+1} - \rho_k) \\ &\quad + D^{-1}y_k, \qquad k = 0, \ldots, N, \\ (I-Q)\rho_0 &= 0, \; Q\rho_{N+1} = 0. \end{cases} \tag{2.44}$$

From Lemma 1.2 we know that the general solution of the first equation in (2.44) is given by

$$\begin{aligned} \rho_k &= E^\times(\Omega^\times)^k(I-P^\times)\eta + E^\times(\Omega^\times)^{N+1-k}P^\times\eta \\ &\quad + \textstyle\sum_{\nu=0}^{k-1} E^\times(\Omega^\times)^{k-1-\nu}(I-P^\times)BD^{-1}y_\nu \\ &\quad - \textstyle\sum_{\nu=k}^{N} E^\times(\Omega^\times)^{\nu-k}P^\times BD^{-1}y_\nu, \; k = 0, 1, \ldots, N+1, \end{aligned} \tag{2.45}$$

where η is an arbitrary vector in $\mathbb{C}^n$. Thus
$\rho_0 = E^\times(I-P^\times)\eta + E^\times(\Omega^\times)^{N+1}P^\times\eta - \sum_{\nu=0}^N E^\times(\Omega^\times)^\nu P^\times BD^{-1}y_\nu$ and
$\rho_{N+1} = E^\times(\Omega^\times)^{N+1}(I-P^\times)\eta + E^\times P^\times\eta + \sum_{\nu=0}^N E^\times(\Omega^\times)^{N-\nu}(I-P^\times)BD^{-1}y_\nu$. The first boundary condition is

$$\begin{aligned} 0 &= (I-Q)\rho_0 \\ &= (I-Q)E^\times(I-P^\times)\eta + (I-Q)E^\times(\Omega^\times)^{N+1}P^\times\eta \\ &\quad -(I-Q)\sum_{\nu=0}^N E^\times(\Omega^\times)^\nu P^\times BD^{-1}y_\nu, \end{aligned}$$

while the second boundary condition is

$$\begin{aligned} 0 &= Q\rho_{N+1} \\ &= QE^{\times}(\Omega^{\times})^{N+1}(I-P^{\times})\eta + QE^{\times}P^{\times}\eta \\ &\quad +Q\sum_{\nu=0}^{N} E^{\times}(\Omega^{\times})^{N-\nu}(I-P^{\times})BD^{-1}y_{\nu}. \end{aligned}$$

From these two equations for the boundary conditions it follows that

$$\begin{aligned} &\left[(I-Q)E^{\times}(I-P^{\times}) + (I-Q)E^{\times}(\Omega^{\times})^{N+1}P^{\times} \right. \\ &\left. +QE^{\times}(\Omega^{\times})^{N+1}(I-P^{\times}) + QE^{\times}P^{\times}\right]\eta \\ &= \textstyle\sum_{\nu=0}^{N}\left[(I-Q)E^{\times}(\Omega^{\times})^{\nu}P^{\times} - QE^{\times}(\Omega^{\times})^{N-\nu}(I-P^{\times})\right]BD^{-1}y_{\nu}. \end{aligned}$$

Thus the vectors $\rho_0, \ldots, \rho_{N+1}$ in (2.45) satisfy the boundary conditions in (2.44) if and only if

$$V_N\eta = \sum_{\nu=0}^{N}\left[(I-Q)E^{\times}(\Omega^{\times})^{\nu}P^{\times} - QE^{\times}(\Omega^{\times})^{N-\nu}(I-P^{\times})\right]BD^{-1}y_{\nu}.$$

From (2.44) we have that

$$\begin{aligned} u_k &= -D^{-1}C(\alpha\rho_{k+1} - \rho_k) + D^{-1}y_k \\ &= -D^{-1}\{\alpha C\rho_{k+1} - C\rho_k - y_k\} \\ &= -D^{-1}\left\{\alpha C\left[E^{\times}(\Omega^{\times})^{k+1}(I-P^{\times})\eta + E^{\times}(\Omega^{\times})^{N-k}P^{\times}\eta \right.\right. \\ &\quad + \sum_{\nu=0}^{k} E^{\times}(\Omega^{\times})^{k-\nu}(I-P^{\times})BD^{-1}y_{\nu} \\ &\quad \left. - \sum_{\nu=k+1}^{N} E^{\times}(\Omega^{\times})^{\nu-k-1}P^{\times}BD^{-1}y_{\nu}\right] \\ &\quad -C\left[E^{\times}(\Omega^{\times})^{k}(I-P^{\times})\eta + E^{\times}(\Omega^{\times})^{N+1-k}P^{\times}\eta \right. \\ &\quad + \sum_{\nu=0}^{k-1} E^{\times}(\Omega^{\times})^{k-1-\nu}(I-P^{\times})BD^{-1}y_{\nu} \\ &\quad \left.\left. - \sum_{\nu=k}^{N} E^{\times}(\Omega^{\times})^{\nu-k}P^{\times}BD^{-1}y_{\nu}\right] - y_k\right\} \\ &= D^{-1}CE^{\times}\left[(\Omega^{\times})^{k} - \alpha(\Omega^{\times})^{k+1}\right](I-P^{\times})\eta \end{aligned}$$

$$
\begin{aligned}
&+D^{-1}CE^{\times}\left[(\Omega^{\times})^{N+1-k}-\alpha(\Omega^{\times})^{N-k}\right]P^{\times}\eta \\
&+\sum_{\nu=0}^{k-1}D^{-1}CE^{\times}\left[(\Omega^{\times})^{k-1-\nu}-\alpha(\Omega^{\times})^{k-\nu}\right](I-P^{\times})BD^{-1}y_{\nu} \\
&-\alpha D^{-1}CE^{\times}(\Omega^{\times})^{0}(I-P^{\times})BD^{-1}y_k-D^{-1}CE^{\times}(\Omega^{\times})^{0}P^{\times}BD^{-1}y_k \\
&+D^{-1}y_k+\sum_{\nu=k+1}^{N}D^{-1}CE^{\times}\left[\alpha(\Omega^{\times})^{\nu-k-1}-(\Omega^{\times})^{\nu-k}\right]P^{\times}BD^{-1}y_{\nu} \\
= \;& D^{-1}CE^{\times}\left[(\Omega^{\times})^{k}-\alpha(\Omega^{\times})^{k+1}\right](I-P^{\times})\eta \\
&+D^{-1}CE^{\times}\left[(\Omega^{\times})^{N+1-k}-\alpha(\Omega^{\times})^{N-k}\right]P^{\times}\eta \\
&+\sum_{\nu=0}^{k-1}D^{-1}CE^{\times}\left[(\Omega^{\times})^{k-1-\nu}-\alpha(\Omega^{\times})^{k-\nu}\right](I-P^{\times})BD^{-1}y_{\nu} \\
&+\left\{D^{-1}-D^{-1}CE^{\times}\left[P^{\times}+\alpha(I-P^{\times})\right]BD^{-1}\right\}y_k \\
&+\sum_{\nu=k+1}^{N}D^{-1}CE^{\times}\left[\alpha(\Omega^{\times})^{-k-1+\nu}-(\Omega^{\times})^{-k+\nu}\right]P^{\times}BD^{-1}y_{\nu} \\
= \;& D^{-1}CE^{\times}\left[(\Omega^{\times})^{k}-\alpha(\Omega^{\times})^{k+1}\right](I-P^{\times})\eta \\
&+D^{-1}CE^{\times}\left[(\Omega^{\times})^{N+1-k}-\alpha(\Omega^{\times})^{N-k}\right]P^{\times}\eta \\
&+\sum_{\nu=0}^{k-1}\Phi^{\times}_{k-\nu}y_{\nu}+\Phi^{\times}_{0}y_k+\sum_{\nu=k+1}^{N}\Phi^{\times}_{k-\nu}y_{\nu}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
u_k \;=\; & D^{-1}CE^{\times}\left[(\Omega^{\times})^{k}-\alpha(\Omega^{\times})^{k+1}\right](I-P^{\times})\eta \\
&+D^{-1}CE^{\times}\left[(\Omega^{\times})^{N+1-k}-\alpha(\Omega^{\times})^{N-k}\right]P^{\times}\eta \\
&+\textstyle\sum_{\nu=0}^{N}\Phi^{\times}_{k-\nu}y_{\nu},\; k=0,1,\ldots,N.
\end{aligned}
$$

In particular, the homogeneous equation

$$
\sum_{\nu=0}^{N}\Phi_{k-\nu}u_{\nu}=0, \qquad k=0,1,\ldots,N
$$

has solution as given by (2.42), where $V_N\eta=0$ since $y_k=0,\ k=0,1,\ldots,N$. Thus $\eta\in\ker V_N$. In the nonhomogeneous case the solution for u_k is given by (2.40) with η an arbitrary element in $\mathbb{C}^n$. Suppose V_N invertible $\Longleftrightarrow$ η unique $\Longleftrightarrow$ u_k unique $\Longleftrightarrow$ T_N invertible.

Let us return to the nonhomogeneous case. We have that

$$V_N\eta = \sum_{\nu=0}^{N} \left[(I-Q)E^\times(\Omega^\times)^\nu P^\times - QE^\times(\Omega^\times)^{N-\nu}(I-P^\times)\right] BD^{-1}y_\nu.$$

Thus

$$\eta = \sum_{\nu=0}^{N} \left\{V_N^{-1}(I-Q)E^\times(\Omega^\times)^\nu P^\times BD^{-1} - V_N^{-1}QE^\times(\Omega^\times)^{N-\nu}(I-P^\times)BD^{-1}\right\} y_\nu,$$

since the inverse V_N^{-1} exists. But then for $k=0,1,\ldots,N$ we have that

$$\begin{aligned} u_k &= D^{-1}CE^\times\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right](I-P^\times)\eta \\ &\quad + D^{-1}CE^\times\left[(\Omega^\times)^{N+1-k} - \alpha(\Omega^\times)^{N-k}\right]P^\times\eta \\ &\quad + \sum_{\nu=0}^{N}\Phi_{k-\nu}^\times y_\nu,\ k=0,1,\ldots,N, \\ &= \sum_{\nu=0}^{N} D^{-1}CE^\times\left\{\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right](I-P^\times)\right. \\ &\quad \left. + \left[(\Omega^\times)^{N+1-k} - \alpha(\Omega^\times)^{N-k}\right]P^\times\right\}V_N^{-1}\cdot \\ &\quad \left\{(I-Q)E^\times(\Omega^\times)^\nu P^\times - QE^\times(\Omega^\times)^{N-\nu}(I-P^\times)\right\}BD^{-1}y_\nu \\ &\quad + \sum_{\nu=0}^{N}\Phi_{k-\nu}^\times y_\nu,\ k=0,1,\ldots,N. \end{aligned}$$

Thus

$$u_k = \sum_{j=0}^{N}\Gamma_{kj}^N y_j = \sum_{j=0}^{N}\left[\Phi_{k-j}^\times + K_{kj}^N\right]y_j,\ k,j=0,1,\ldots,N,$$

where

$$\begin{aligned} K_{kj}^N &= D^{-1}CE^\times\left\{\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right](I-P^\times)\right. \\ &\quad \left. + \left[(\Omega^\times)^{N+1-k} - \alpha(\Omega^\times)^{N-k}\right]P^\times\right\}V_N^{-1}\cdot \\ &\quad \left\{(I-Q)E^\times(\Omega^\times)^j P^\times - QE^\times(\Omega^\times)^{N-j}(I-P^\times)\right\}BD^{-1}. \ \natural \end{aligned}$$

Chapter 3

Fredholm properties of Block Toeplitz Operators with rational symbols

3.1 Fredholm characteristics and generalized inverse

In this section we derive the Fredholm properties and generalized inverse for a block Toeplitz operator with rational symbol. The symbol is given in realized form and all results are expressed explicitly in terms of the data appearing in the realization. In what follows the term generalized inverse is used in a weak sense, i.e., an operator S is said to have a *generalized inverse* S^+ whenever $S = SS^+S$. Recall that a bounded linear operator $A : X \to Y$, acting between complex Banach spaces X and Y, is called a Fredholm operator if its range im$A$ is closed and the numbers

(3.1) $$n(A) = \dim \ker A,\ d(A) = \dim(Y/\mathrm{im}A)$$

is finite. In this case the ind $A = n(A) - d(A)$ is said to be the index of A. Note that $\dim(Y/\mathrm{im}A)$ is also written as codim (im$A$).

Lemma 3.1. *Suppose T is a block Toeplitz operator with rational symbol*

(3.2) $$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \mathbb{T},$$

given in realized form, where $\alpha \neq 0$ is neither a pole nor a zero of Φ. Assume that $\lambda G^\times - A^\times$ is $\mathbb{T}$- regular. If $\phi \in \ker T$ then

(3.3) $$\phi_k = D^{-1}CE^\times \left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right]\eta$$

where $\eta \in \ker P^\times \cap \mathrm{im}P$.

Proof. Let $\phi \in \ker T$. Then there exists ρ (see (2.9)) such that

$$\begin{cases} A\rho_{k+1} & = & G\rho_k + B\phi_k, & k = 0,1,2,\dots, \\ 0 & = & C(\alpha\rho_{k+1} - \rho_k) + D\phi_k, & k = 0,1,2,\dots, \\ (I-Q)\rho_0 & = & 0. \end{cases}$$

By the first equation of the inverse system (2.17) we obtain

$$A^\times \rho_{k+1} = G^\times \rho_k, \qquad k = 0,1,2,\dots .$$

But then from (2.27) we obtain

$$\rho_k = E^\times(\Omega^\times)^k\gamma, \qquad k = 0,1,2,\dots,$$

and so (see(2.29))

$$\begin{aligned} \phi_k &= D^{-1}CE^\times\left[(\Omega^\times)^k - \alpha(\Omega^\times)^{k+1}\right]\gamma, \qquad k = 0,1,2,\dots, \\ &= D^{-1}CE^\times\left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right]\Omega^\times\gamma, \qquad k = 1,2,\dots . \end{aligned}$$

Here $\gamma \in \ker P^\times$. Thus $G^\times E^\times\gamma = \Omega^\times\gamma \in \ker P^\times$. Also, $(I-Q)\rho_0 = 0$ implies that $E^\times\gamma \in \mathrm{im}Q$ (see (2.30)). Thus $G^\times E^\times\gamma \in \mathrm{im}P$. Put $\eta = \Omega^\times\gamma = G^\times E^\times\gamma$. Then

$$\phi_k = D^{-1}CE^\times\left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right]\eta,$$

where $\eta \in \ker P^\times \cap \mathrm{im}P$. ♮

Lemma 3.2. *Suppose T is a block Toeplitz operator with rational symbol*

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \mathbb{T},$$

given in realized form, where $\alpha \neq 0$ is neither a pole nor a zero of Φ. Assume that $\lambda G^\times - A^\times$ is $\mathbb{T}$- regular. If $\phi \in \mathrm{im}T$ then

$$\sum_{\nu=0}^{\infty}(\Omega^\times)^\nu P^\times BD^{-1}\phi_\nu \in \mathrm{im}P + \ker P^\times. \tag{3.4}$$

Furthermore, codim $(\mathrm{im}T)$ = codim $(\mathrm{im}P + \ker P^\times)$.

Proof. Let $\phi \in \text{im}T$. Put $Tf = \phi$, $f \in l_p^m$. Then there exists ρ such that

$$\begin{cases} A\rho_{k+1} &= G\rho_k + Bf_k, \qquad k = 0, 1, 2, \dots, \\ \phi_k &= C(\alpha\rho_{k+1} - \rho_k) + Df_k, \quad k = 0, 1, 2, \dots, \\ (I - Q)\rho_0 &= 0. \end{cases}$$

But then, for the inverse system, we get

$$\begin{cases} A^\times \rho_{k+1} &= G^\times \rho_k + BD^{-1}\phi_k, \qquad k = 0, 1, 2, \dots, \\ f_k &= -D^{-1}C(\alpha\rho_{k+1} - \rho_k) + D^{-1}\phi_k, \quad k = 0, 1, 2, \dots, \\ (I - Q)\rho_0 &= 0. \end{cases}$$

By Lemma 2.3 we have that

$$\begin{aligned} \rho_k &= E^\times(\Omega^\times)^k\gamma + \sum_{\nu=0}^{k-1} E^\times(\Omega^\times)^{k-1-\nu}(I - P^\times)BD^{-1}\phi_\nu \\ &\quad - \sum_{\nu=k}^{\infty} E^\times(\Omega^\times)^{\nu-k}P^\times BD^{-1}\phi_\nu, \; k = 0, 1, 2, \dots, \end{aligned}$$

where $\gamma \in \ker P^\times$. Therefore

$$\rho_0 = E^\times\gamma - \sum_{\nu=0}^{\infty} E^\times(\Omega^\times)^\nu P^\times BD^{-1}\phi_\nu \in \text{im}Q.$$

We know that $G^\times E^\times \gamma = \Omega^\times\gamma \in \ker P^\times$ and

$$\begin{aligned} G^\times\rho_0 &= G^\times E^\times\gamma - \sum_{\nu=0}^{\infty} G^\times E^\times(\Omega^\times)^\nu P^\times BD^{-1}\phi_\nu \\ &= \Omega^\times\gamma - \sum_{\nu=0}^{\infty} P^\times(\Omega^\times)^\nu BD^{-1}\phi_\nu \;\; (G^\times E^\times P^\times = P^\times) \\ &= \Omega^\times\gamma - \sum_{\nu=0}^{\infty}(\Omega^\times)^\nu P^\times BD^{-1}\phi_\nu \in \text{im}P. \end{aligned}$$

Thus

$$\sum_{\nu=0}^{\infty}(\Omega^\times)^\nu P^\times BD^{-1}\phi_\nu \in \text{im}P + \ker P^\times.$$

Define a mapping $\theta : l_p^m/\text{im}T \to \mathbb{C}^n/(\text{im}P + \ker P^\times)$ by $[\phi] \mapsto [R(\phi)]$, where $R(\phi) = \sum_{\nu=0}^{\infty} (\Omega^\times)^\nu P^\times BD^{-1}\phi_\nu$, or equivalently, $\theta([\phi]) = [R(\phi)]$ or $\phi + \text{im}T \mapsto R(\phi) + \text{im}P + \ker P^\times$.

To show θ is injective, suppose $\theta([\phi]) = [0]$, i.e., $[\phi] \in \ker\theta$. Then $[R(\phi)] = [0]$ or $R(\phi) \in \mathrm{im}P + \ker P^{\times}$. But then $\phi \in \mathrm{im}T$, showing that $[\phi] = [0]$. Hence θ is injective.

To show that θ is surjective, it would suffice to show that $\mathbb{C}^n = \mathrm{im}R + \mathrm{im}P + \ker P^{\times}$. To this end, let $w \in \ker P$ and let f be the function with

$$f_k = \begin{cases} -\alpha CEw, & k = 0, \\ -CE\left(\alpha\Omega^k - \Omega^{k-1}\right)w, & k \geq 1. \end{cases}$$

Then

$$\begin{aligned} R(f) &= -\alpha P^{\times}BD^{-1}CEw - \sum_{\nu=1}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}BD^{-1}CE\left(\alpha\Omega^{\nu} - \Omega^{\nu-1}\right)w \\ &= -\alpha P^{\times}BD^{-1}CEw - \alpha\sum_{\nu=1}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}BD^{-1}CE\Omega^{\nu}w \\ &\quad + \sum_{\nu=1}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}BD^{-1}CE\Omega^{\nu-1}w \\ &= \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}(-\alpha BD^{-1}C)E\Omega^{\nu}w \\ &\quad + \sum_{\nu=0}^{\infty}P^{\times}(\Omega^{\times})\left(\Omega^{\times}\right)^{\nu}(+BD^{-1}C)E\Omega^{\nu}w \\ &= \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}(A - A^{\times})E\Omega^{\nu}w \\ &\quad - \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}\Omega^{\times}(G - G^{\times})E\Omega^{\nu}w \\ &= \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}AE\Omega^{\nu}w - \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}A^{\times}E\Omega^{\nu}w \\ &\quad - \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}\Omega^{\times}GE\Omega^{\nu}w + \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}\Omega^{\times}G^{\times}E\Omega^{\nu}w \\ &= \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}\Omega^{\nu}w - \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}\Omega^{\times}G^{\times}E\Omega^{\nu}w \\ &\quad - \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}\Omega^{\times}\Omega\Omega^{\nu}w + \sum_{\nu=0}^{\infty}\left(\Omega^{\times}\right)^{\nu}P^{\times}\Omega^{\times}G^{\times}E\Omega^{\nu}w \\ &= \sum_{\nu=0}^{\infty}P^{\times}\left(\Omega^{\times}\right)^{\nu}\Omega^{\nu}w - \sum_{\nu=1}^{\infty}P^{\times}\left(\Omega^{\times}\right)^{\nu}\Omega^{\nu}w \end{aligned}$$

$$= P^\times w,$$

since $A - A^\times = -\alpha BD^{-1}C$, $G - G^\times = -BD^{-1}C$, $AE(I-P) = I - P$, $P^\times A^\times = A^\times Q^\times = A^\times E^\times (E^\times)^{-1} Q^\times = A^\times E^\times P^\times (E^\times)^{-1} = \Omega^\times P^\times (E^\times)^{-1} = \Omega^\times P^\times G^\times$ and $GE(I-P) = \Omega(I-P)$. Thus $w = P^\times w + (I - P^\times)w = R(f) + (I - P^\times)w$ from which it follows that $z \in \mathbb{C}^n$, $z = Pz + (I-P)z = Pz + w = Pz + R(f) + (I - P^\times)w$. Hence θ is an invertible linear operator. Thus

$$\text{codim}\,(\text{im}T) = \dim(l_p^m/\text{im}T) = \dim\frac{\mathbb{C}^n}{\text{im}P + \ker P^\times} = \text{codim}\,(\text{im}P + \ker P^\times). \natural$$

Theorem 3.3. *Let T be a block Toeplitz operator on l_p^m with rational symbol*

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \mathbb{T},$$

given in realized form, where $\alpha \neq 0$ is neither a pole nor a zero of Φ. Put $A^\times = A + \alpha BD^{-1}C$ and $G^\times = G + BD^{-1}C$. Then T is a Fredholm operator if and only if $\lambda G^\times - A^\times$ is a $\mathbb{T}$-regular pencil. Assume that the latter condition holds. Then

$$(3.5)\qquad \ker T = \left\{\left(D^{-1}CE^\times\left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right]\eta\right)_{k=0}^\infty \middle| \eta \in \ker P^\times \cap \text{im}P\right\},$$

$$(3.6)\qquad \text{im}T = \left\{(\phi_k)_{k=0}^\infty \in l_p^m \middle| \sum_{\nu=0}^\infty (\Omega^\times)^\nu P^\times BD^{-1}\phi_\nu \in \text{im}P + \ker P^\times\right\},$$

$$(3.7)\qquad n(T) = \dim(\ker P^\times \cap \text{im}P),\ d(T) = \dim\frac{\mathbb{C}^n}{\text{im}P + \ker P^\times},$$

$$(3.8)\qquad \text{ind}\,(T) = \text{rank}\ P - \text{rank}\ P^\times,$$

and a generalized inverse of T is given by $T^+ = [\Gamma_{ij}^+]_{i,j=0}^\infty$ with

$$(3.9)\qquad \Gamma_{ij}^+ = \Phi_{i-j}^\times + K_{ij}^+, \qquad i,j = 0,1,2,\ldots,$$

$$(3.10)\qquad \Phi_k^\times = \begin{cases} D^{-1}CE^\times\left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right](I - P^\times)BD^{-1}, & k > 0, \\ D^{-1} - D^{-1}CE^\times\left[P^\times + \alpha(I - P^\times)\right]BD^{-1}, & k = 0, \\ D^{-1}CE^\times\left[\alpha(\Omega^\times)^{-k-1} - (\Omega^\times)^{-k}\right]P^\times BD^{-1}, & k < 0, \end{cases}$$

$$(3.11)\qquad K_{ij}^{+} = -D^{-1}CE^{\times}\left[(\Omega^{\times})^{i-1} - \alpha(\Omega^{\times})^{i}\right](I-P^{\times})(J^{\times})^{+}(\Omega^{\times})^{j}P^{\times}BD^{-1},$$

where $(J^{\times})^{+}$ *is a generalized inverse of the operator*

$$(3.12)\qquad J^{\times} = P^{\times}|\text{im}P : \text{im}P \to \text{im}P^{\times}.$$

Here P *is the separating projection corresponding to* $\lambda G - A$ *and* $\mathbb{T}$*, and the operators* $P^{\times}$, $E^{\times}$ *and* $\Omega^{\times}$ *are, respectively, the separating projection, the right equivalence operator and the associated operator corresponding to* $\lambda G^{\times} - A^{\times}$ *and* $\mathbb{T}$.

Proof. Gohberg and Feldman (1974) proved that T is Fredholm if and only if $\det\Phi(\lambda) \neq 0,\ \lambda \in \mathbb{T}$. By Lemma 2.1 ([G1]) the latter condition is equivalent to the requirement that $\lambda G^{\times} - A^{\times}$ is $\mathbb{T}$-regular.

Suppose T is Fredholm, i.e., $\lambda G^{\times} - A^{\times}$ is $\mathbb{T}$-regular. From Lemma 3.1 and Lemma 3.2 the formulas for $\ker T$ and $\text{im}T$ follows. From Lemma 3.1 it follows that

$$n(T) = \dim\ker T = \dim(\text{im}P \cap \ker P^{\times});$$

and from Lemma 3.2 it follows that

$$d(T) = \dim \frac{\mathbb{C}^n}{\text{im}P + \ker P^{\times}}.$$

Therefore

$$\begin{aligned}
\text{ind}\,(T) &= n(T) - d(T) \\
&= \dim(\text{im}P \cap \ker P^{\times}) - \dim \frac{\mathbb{C}^n}{\text{im}P + \ker P^{\times}} \\
&= \left\{\dim \text{im}P + \dim\ker P^{\times} - \dim(\text{im}P + \ker P^{\times})\right\} \\
&\quad - \left\{\dim \mathbb{C}^n - \dim(\text{im}P + \ker P^{\times})\right\} \\
&= \dim(\text{im}P) + \dim\ker P^{\times} - n \\
&= \dim(\text{im}P) - (n - \dim\ker P^{\times}) \\
&= \dim(\text{im}P) - \dim \text{im}(P^{\times}) \\
&= \text{rank } P - \text{rank } P^{\times}.
\end{aligned}$$

What remains to be checked, is the formula for a generalized inverse. If $Tf = \phi$, we know from Theorem 2.4 that

$$f_k = D^{-1}CE^\times \left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right]\eta + \sum_{\nu=0}^{\infty} \Phi_{k-\nu}^\times \phi_\nu,\ k = 1, 2, \ldots,$$

where $\eta \in \ker P^\times$. Put $T^+ = [\Gamma_{ij}^+]_{i,j=0}^\infty$, so

$$\begin{aligned}(T^+f)_k &= \sum_{j=0}^{\infty} -D^{-1}CE^\times \left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right] \cdot \\ &\quad (I - P^\times)(J^\times)^+ P^\times(\Omega^\times)^j BD^{-1} f_j + \sum_{\nu=0}^{\infty} \Phi_{k-\nu}^\times f_\nu \\ &= -D^{-1}CE^\times \left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right] \cdot \\ &\quad (I - P^\times)(J^\times)^+ \sum_{\nu=0}^{\infty} P^\times(\Omega^\times)^\nu BD^{-1} f_\nu + \sum_{\nu=0}^{\infty} \Phi_{k-\nu}^\times f_\nu.\end{aligned}$$

To show that T^+ is a generalized inverse for T we need to show that $TT^+Tf = Tf$ for every $f \in D(T)$. Suppose that $Tf = \phi$. Then $\sum_{\nu=0}^{\infty}(\Omega^\times)^\nu P^\times BD^{-1}\phi_\nu = R(\phi)$ $\in \text{im}P + \ker P^\times$. Then $\chi_\phi = -(I - P^\times)(J^\times)^+ R(\phi) \in \ker P^\times$. Note from the expression for f_k above that

$$\begin{aligned}(T^+\phi)_k &= -D^{-1}CE^\times \left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right] \cdot \\ &\quad (I - P^\times)(J^\times)^+ \sum_{\nu=0}^{\infty} P^\times(\Omega^\times)^\nu BD^{-1}\phi_\nu + \sum_{\nu=0}^{\infty} \Phi_{k-\nu}^\times \phi_\nu \\ &= D^{-1}CE^\times \left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right] \chi_\phi + \sum_{\nu=0}^{\infty} \Phi_{k-\nu}^\times \phi_\nu \\ &= f_k\end{aligned}$$

(from Theorem 2.4). Thus $T(T^+\phi) = T(T^+Tf) = Tf$, showing that Tf coincides with TT^+Tf, or equivalently, $T = TT^+T$. ♮

3.2 Riemann-Hilbert problem

Let Φ be an $m \times m$ rational matrix function without poles on the unit circle $\mathbb{T}$. Recall that a pair of $\mathbb{C}^m$-valued functions (ψ_+, ψ_-) is said to be a *solution of the* *homogeneous Riemann-Hilbert boundary problem* (see, e.g., [CG]) for $\Phi(\lambda)$ relative to $\mathbb{T}$ if ψ_+ is analytic in $\mathbb{D}_+$, continuous on $\overline{\mathbb{D}}_+$, the function

$$\psi_-(\lambda) = \Phi(\lambda)\psi_+(\lambda), \qquad \lambda \in \mathbb{T}, \tag{3.13}$$

extends to an analytic function in $\mathbb{D}_-$, is continuous on $\overline{\mathbb{D}}_-$ and $\psi_-(\lambda)$ has the value zero at infinity.

Theorem 3.4. *Let*

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \qquad \lambda \in \mathbb{T}, \tag{3.14}$$

be given in realized form, where $\alpha \neq 0$ is neither a pole nor a zero of Φ. Put $A^\times = A + \alpha BD^{-1}C$ and $G^\times = G + BD^{-1}C$. Let P and $P^\times$ be the projections given by

$$P = \frac{1}{2\pi i}\int_\Gamma G(\lambda G - A)^{-1}d\lambda, \; P^\times = \frac{1}{2\pi i}\int_\Gamma G^\times(\lambda G^\times - A^\times)^{-1}d\lambda, \; \Gamma = \mathbb{T}.$$

Then the general solution of the Riemann-Hilbert boundary value problem (3.13) is given by

$$\begin{aligned} \psi_+(\lambda) &= -(\lambda - \alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}x, \\ \psi_-(\lambda) &= -(\lambda - \alpha)C(\lambda G - A)^{-1}x, \end{aligned} \tag{3.15}$$

where x is an arbitrary vector in $\operatorname{im}P \cap \ker P^\times$. Moreover, the vector x is uniquely determined by the solution (ψ_+, ψ_-).

Proof. From the $\mathbb{T}$-spectral decomposition of the pencil $\lambda G - A$ we know, since $x \in \operatorname{im}P$, that $\psi_-(\lambda) = -(\lambda - \alpha)C(\lambda G - A)^{-1}x = -(\lambda - \alpha)C(\lambda G - A)^{-1}Px$ has an analytic continuation to $\mathbb{D}_-$, also denoted by ψ_- and $\psi_-(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow \infty$. Similarly, since $x \in \ker P^\times$, the $\mathbb{T}$-spectral decomposition of the pencil $\lambda G^\times - A^\times$ allows us to conclude that $\psi_+(\lambda) = -(\lambda - \alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}x$ has an analytic continuation to $\mathbb{D}_+$, which we also denote by ψ_+. Also, recall from Lemma 1.5 that

$$\Phi(\lambda)^{-1}C(\lambda G - A)^{-1} = D^{-1}C(\lambda G^\times - A^\times)^{-1}.$$

So, any pair (ψ_+, ψ_-) of the form (3.15) is a solution of the Riemann-Hilbert problem for $\Phi(\lambda)$ relative to $\mathbb{T}$.

Conversely, let (ψ_+, ψ_-) be a solution of the Riemann-Hilbert problem for $\Phi(\lambda)$ relative to $\mathbb{T}$. We know that a block Toeplitz operator T_Φ with defining function Φ is unitarily equivalent to the compression to the Hardy space $H_2^m(\mathbb{T})$ of the operator of multiplication by Φ on $L_2^m(\mathbb{T})$. That is,

$$U^{-1} T_\Phi U f = \mathbb{P} M_\Phi f, \qquad f \in H_2^m(\mathbb{T}),$$

where U is the Fourier transformation on $H_2^m(\mathbb{T})$, the operator M_Φ is the operator of multiplication by Φ on $L_2^m(\mathbb{T})$ and $\mathbb{P}$ is the orthogonal projection of $L_2^m(\mathbb{T})$ on $H_2^m(\mathbb{T})$ (see Corollary 3.3, [GGK2]). Then

$$U^{-1} T_\Phi U \psi_+(\lambda) = \mathbb{P} \Phi(\lambda) \psi_+(\lambda) = \mathbb{P} \psi_-(\lambda) = 0.$$

Therefore $T_\Phi U \psi_+(\lambda) = 0$. So, clearly $U\psi_+ \in \ker T_\Phi$. From Theorem 3.3, and the fact that $\psi_+ \in H_2^m(\mathbb{T})$, $U\psi_+$ is of the form

$$U\psi_+ = (c_k)_{k=0}^\infty = \left(D^{-1} C E^\times \left[(\Omega^\times)^{k-1} - \alpha (\Omega^\times)^k \right] x \right)_{k=0}^\infty, \quad x \in \operatorname{im} P \cap \ker P^\times,$$

where $(c_k)_{k=0}^\infty$ are the Fourier coefficients of $\psi_+(\lambda) = -(\lambda - \alpha) D^{-1} C (\lambda G^\times - A^\times)^{-1} x$, $x \in \operatorname{im} P \cap \ker P^\times$, with $c_n = 0$ for $n = -1, -2, -3, \ldots$. This can be seen as follows:

$$\begin{aligned}
\psi_+(\lambda) &= -(\lambda - \alpha) D^{-1} C (\lambda G^\times - A^\times)^{-1} x \\
&= -(\lambda - \alpha) D^{-1} C E^\times \begin{bmatrix} (\lambda \Omega_1^\times - I_1^\times)^{-1} & 0 \\ 0 & (\lambda I_2^\times - \Omega_2^\times)^{-1} \end{bmatrix} x \\
&= -(\lambda - \alpha) D^{-1} C E^\times \begin{bmatrix} \sum_{\nu=0}^\infty -\lambda^\nu (\Omega_1^\times)^\nu & 0 \\ 0 & \sum_{\nu=0}^\infty \lambda^{-\nu-1} (\Omega_2^\times)^\nu \end{bmatrix} x \\
&= -(\lambda - \alpha) D^{-1} C E^\times \begin{bmatrix} \sum_{\nu=0}^\infty -\lambda^\nu (\Omega_1^\times)^\nu & 0 \\ 0 & \sum_{\nu=-\infty}^{-1} \lambda^{\nu} (\Omega_2^\times)^{-\nu-1} \end{bmatrix} x \\
&= -D^{-1} C E^\times \begin{bmatrix} \sum_{\nu=0}^\infty -\lambda^{\nu+1} (\Omega_1^\times)^\nu & 0 \\ 0 & \sum_{\nu=-\infty}^{-1} \lambda^{\nu+1} (\Omega_2^\times)^{-\nu-1} \end{bmatrix} x \\
&\quad + \alpha D^{-1} C E^\times \begin{bmatrix} \sum_{\nu=0}^\infty -\lambda^{\nu} (\Omega_1^\times)^\nu & 0 \\ 0 & \sum_{\nu=-\infty}^{-1} \lambda^{\nu} (\Omega_2^\times)^{-\nu-1} \end{bmatrix} x \\
&= -D^{-1} C E^\times \begin{bmatrix} \sum_{\nu=1}^\infty -\lambda^{\nu} (\Omega_1^\times)^{\nu-1} & 0 \\ 0 & \sum_{\nu=-\infty}^{0} \lambda^{\nu} (\Omega_2^\times)^{-\nu} \end{bmatrix} x
\end{aligned}$$

$$+\alpha D^{-1}CE^\times \begin{bmatrix} \sum_{\nu=0}^{\infty} -\lambda^\nu (\Omega_1^\times)^\nu & 0 \\ 0 & \sum_{\nu=-\infty}^{-1} \lambda^\nu (\Omega_2^\times)^{-\nu-1} \end{bmatrix} x$$

Thus

$$\begin{aligned} c_0 &= -D^{-1}CE^\times \begin{bmatrix} 0 & 0 \\ 0 & I_2^\times \end{bmatrix} x + \alpha D^{-1}CE^\times \begin{bmatrix} -I_1^\times & 0 \\ 0 & 0 \end{bmatrix} x \\ &= -D^{-1}CE^\times [P^\times + \alpha(I - P^\times)]x, \\ &= -D^{-1}CE^\times [P^\times + \alpha I - \alpha P^\times]x \\ &= -\alpha D^{-1}CE^\times x \quad (P^\times x = 0), \end{aligned}$$

$$\begin{aligned} c_k &= -D^{-1}CE^\times \begin{bmatrix} -(\Omega_1^\times)^{k-1} & 0 \\ 0 & 0 \end{bmatrix} x + \alpha D^{-1}CE^\times \begin{bmatrix} -(\Omega_1^\times)^{k} & 0 \\ 0 & 0 \end{bmatrix} x,\ k > 0, \\ &= D^{-1}CE^\times \left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right](I - P^\times)x,\ k > 0, \\ &= D^{-1}CE^\times \left[(\Omega^\times)^{k-1} - \alpha(\Omega^\times)^k\right]x,\ k > 0, \end{aligned}$$

$$\begin{aligned} c_k &= -D^{-1}CE^\times \begin{bmatrix} 0 & 0 \\ 0 & (\Omega_2^\times)^{-k} \end{bmatrix} x + \alpha D^{-1}CE^\times \begin{bmatrix} 0 & 0 \\ 0 & (\Omega_2^\times)^{-k-1} \end{bmatrix} x,\ k < 0, \\ &= -D^{-1}CE^\times \left[(\Omega^\times)^{-k} - \alpha(\Omega^\times)^{-k-1}\right] P^\times x,\ k < 0, \\ &= 0,\ k < 0. \end{aligned}$$

Now it is plain that

$$\begin{aligned} \psi_-(\lambda) &= \Phi(\lambda)\psi_+(\lambda) \\ &= -(\lambda - \alpha)\Phi(\lambda)D^{-1}C(\lambda G^\times - A^\times)^{-1}x \\ &= -(\lambda - \alpha)C(\lambda G - A)^{-1}x,\ x \in \operatorname{im} P \cap \ker P^\times, \end{aligned}$$

(see first identity, Lemma 1.5).

It remains to show that x in (3.15) is uniquely determined. If $\psi_+(\lambda) = -(\lambda-\alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}x_1 = -(\lambda-\alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}x_2$, with $x_1, x_2 \in \text{im}P \cap \ker P^\times$, then

$$-(\lambda-\alpha)D^{-1}C(\lambda G^\times - A^\times)^{-1}(x_1 - x_2) = 0,$$

$$-(\lambda-\alpha)C(\lambda G - A)^{-1}(x_1 - x_2) = 0.$$

Thus

$$(\lambda-\alpha)BD^{-1}C(\lambda G^\times - A^\times)^{-1}(x_1 - x_2) = 0,$$

$$(\lambda-\alpha)BD^{-1}C(\lambda G - A)^{-1}(x_1 - x_2) = 0.$$

Using (2.8) we see that

$$\left[(\lambda G^\times - A^\times) - (\lambda G - A)\right](\lambda G^\times - A^\times)^{-1}(x_1 - x_2) = 0,$$

$$\left[(\lambda G^\times - A^\times) - (\lambda G - A)\right](\lambda G - A)^{-1}(x_1 - x_2) = 0.$$

Thus

$$x_1 - x_2 = (\lambda G - A)(\lambda G^\times - A^\times)^{-1}(x_1 - x_2),$$

$$(\lambda G^\times - A^\times)(\lambda G - A)^{-1}(x_1 - x_2) = x_1 - x_2,$$

i.e.,

$$(\lambda G - A)^{-1}(x_1 - x_2) = (\lambda G^\times - A^\times)^{-1}(x_1 - x_2), \ \lambda \in \mathbb{T}.$$

Therefore, for $\Gamma = \mathbb{T}$

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma G^\times(\lambda G - A)^{-1}d\lambda(x_1 - x_2) &= \frac{1}{2\pi i}\int_\Gamma G^\times(\lambda G^\times - A^\times)^{-1}d\lambda(x_1 - x_2)\\ &= P^\times(x_1 - x_2)\\ &= 0.\end{aligned}$$

That is,

$$\frac{1}{2\pi i}\int_\Gamma (G + BD^{-1}C)(\lambda G - A)^{-1}d\lambda(x_1 - x_2) = 0$$

implies that

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma G(\lambda G - A)^{-1} d\lambda (x_1 - x_2) &= -\frac{1}{2\pi i}\int_\Gamma BD^{-1}C(\lambda G - A)^{-1} d\lambda (x_1 - x_2) \\ &= 0. \end{aligned}$$

(from Cauchy's theorem). Thus

$$0 = P(x_1 - x_2) = (x_1 - x_2)$$

since P is a projection, i.e., $x_1 - x_2 = 0$. That is, $x_1 = x_2$. ♮

3.3 Example

In this section we calculate the inverse of a block Toeplitz operator with rational symbol using discrete singular systems with boundary conditions. Note that all calculations were done with the aid of MAPLE. Let T be a block Toeplitz operator on l_p^m with rational symbol

(3.16) $$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \ \lambda \in \mathbb{T},$$

given in realized form. Consider the finite block Toeplitz equation

(3.17) $$\sum_{\nu=0}^{N} \Phi_{k-\nu} x_\nu = z_k, \ k = 0, \ldots, N.$$

We use the results of Section 2.3 to invert a finite block Toeplitz matrix that corresponds to Equation (3.17). Finally we calculate the formula for the inverse as given in Section 3.1.

Let T be the block Toeplitz operator with symbol

(3.18) $$\Phi(\lambda) = \begin{bmatrix} 0 & -1 \\ (3\lambda^2 + 13\lambda + 4)/3\lambda & 2(3\lambda + 4)(\lambda - 1)/3\lambda \end{bmatrix}, \ \lambda \in \mathbb{T}.$$

We first write Φ as a transfer function of a system. Introduce

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}, \ \alpha = -4/3,$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1/2 & -1 \end{bmatrix}, \ C = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \ D = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}.$$

Clearly $\Phi(\alpha) = D$, which is well-defined with inverse

$$\Phi(\alpha)^{-1} = D^{-1} = \begin{bmatrix} 0 & 1/2 \\ -1 & 0 \end{bmatrix}.$$

The pencil $\lambda G - A$ is $\mathbb{T}$-regular and one finds that

$$(\lambda G - A)^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2/(2\lambda - 1) & 0 \\ 0 & 0 & -2/\lambda \end{bmatrix}, \ \lambda \in \mathbb{T}.$$

Thus

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \ \lambda \in \mathbb{T}.$$

Calculation of the projections P and Q, the right equivalence operator E and associated operator Ω yields

$$P = Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \ \Omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

One checks that the identities

$$PG = GQ, \ PA = AQ, \ \Omega P = P\Omega,$$

$$(\lambda G - A)E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \lambda - 1/2 & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda\Omega_1 - I_1 & 0 \\ 0 & \lambda I_2 - \Omega_2 \end{bmatrix}$$

are satisfied.

Taking $N = 2$ we compute the Fourier coefficients $\Phi_{-2}, \Phi_{-1}, \Phi_0, \Phi_1$ and Φ_2 as

$$\Phi_{-2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; \Phi_{-1} = \begin{bmatrix} 0 & 0 \\ 4/3 & -8/3 \end{bmatrix},$$

$$\Phi_0 = \begin{bmatrix} 0 & -1 \\ 13/3 & 2/3 \end{bmatrix},$$

$$\Phi_1 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \; \Phi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The block Toeplitz matrix (with $N = 2$) is

$$T_N = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 13/3 & 2/3 & 4/3 & -8/3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 2 & 13/3 & 2/3 & 4/3 & -8/3 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 13/3 & 2/3 \end{bmatrix}. \tag{3.19}$$

The next step is to analyze the pencil $\lambda G^\times - A^\times$. One finds that

$$A^\times = \begin{bmatrix} 5/3 & -8/3 & 2/3 \\ 0 & -1/2 & 0 \\ 1/3 & 4/3 & 1/3 \end{bmatrix}, \; G^\times = \begin{bmatrix} -1/2 & 2 & -1/2 \\ 0 & -1 & 0 \\ -1/4 & -1 & -3/4 \end{bmatrix}.$$

The determinant of $\lambda G^\times - A^\times$ is $(-1/4)\lambda^3 - (23/24)\lambda^2 + (5/24)\lambda + 1/6$ with roots $\lambda = 1/2, -1/3, -4$. Thus the pencil $\lambda G^\times - A^\times$ is $\mathbb{T}$-regular, i.e., no roots lie on the unit circle. Before we can calculate the Fourier coefficients $\Phi_k^\times$ of $\Phi(\cdot)^{-1}$ we first have to compute the separating projection $P^\times$, the right equivalence operator $E^\times$ and the associate operator $\Omega^\times$ corresponding to the pencil $\lambda G^\times - A^\times$ and $\mathbb{T}$. We note that

$$\Phi(\lambda)^{-1} = \begin{bmatrix} 2(3\lambda + 4)(\lambda - 1)/(3\lambda^2 + 13\lambda + 4) & 3\lambda/(3\lambda^2 + 13\lambda + 4) \\ -1 & 0 \end{bmatrix}.$$

The projections and operators are

$$P^\times = \begin{bmatrix} -1/11 & -160/99 & 6/11 \\ 0 & 1 & 0 \\ -2/11 & -80/297 & 12/11 \end{bmatrix}, \; Q^\times = \begin{bmatrix} -1/11 & 80/27 & -4/11 \\ 0 & 1 & 0 \\ 3/11 & -20/27 & 12/11 \end{bmatrix},$$

$$E^\times = \begin{bmatrix} 7/11 & -776/297 & 2/11 \\ 0 & -1 & 0 \\ 1/11 & 788/297 & -17/11 \end{bmatrix}, \ \Omega^\times = \begin{bmatrix} -8/33 & -64/33 & -1/22 \\ 0 & 1/2 & 0 \\ 1/66 & -164/99 & -15/44 \end{bmatrix}.$$

Once more the identities

$$P^\times G^\times = G^\times Q^\times, P^\times A^\times = A^\times Q^\times, \Omega^\times P^\times = P^\times \Omega^\times$$

are satisfied. The eigenvalues of $\Omega^\times$ are $1/2, -1/3, -1/4$ and one computes that

$$(\Omega^\times)^k_{[1,1]} = -\frac{1}{11}(-\frac{1}{3})^k + \frac{12}{11}(-\frac{1}{4})^k, \ (\Omega^\times)^k_{[1,2]} = -\frac{112}{45}(\frac{1}{2})^k + \frac{48}{55}(-\frac{1}{3})^k + \frac{160}{99}(-\frac{1}{4})^k,$$

$$(\Omega^\times)^k_{[1,3]} = \frac{6}{11}(-\frac{1}{3})^k - \frac{6}{11}(-\frac{1}{4})^k, \ (\Omega^\times)^k_{[2,1]} = 0, \ (\Omega^\times)^k_{[2,2]} = (\frac{1}{2})^k, \ (\Omega^\times)^k_{[2,3]} = 0,$$

$$(\Omega^\times)^k_{[3,1]} = -\frac{2}{11}(-\frac{1}{3})^k + \frac{2}{11}(-\frac{1}{4})^k, \ (\Omega^\times)^k_{[3,2]} = -\frac{272}{135}(\frac{1}{2})^k + \frac{96}{55}(-\frac{1}{3})^k + \frac{80}{297}(-\frac{1}{4})^k,$$

$$(\Omega^\times)^k_{[3,3]} = \frac{12}{11}(-\frac{1}{3})^k - \frac{1}{11}(-\frac{1}{4})^k.$$

Thus

$$\Phi^\times_k = \begin{cases} (-\frac{1}{4})^k \begin{bmatrix} -20/11 & 3/11 \\ 0 & 0 \end{bmatrix}, & k > 0, \\ \begin{bmatrix} 2/11 & 3/11 \\ -1 & 0 \end{bmatrix}, & k = 0, \\ (-\frac{1}{3})^{-k} \begin{bmatrix} 24/11 & 3/11 \\ 0 & 0 \end{bmatrix}, & k < 0. \end{cases} \tag{3.20}$$

One finds that

$$V_{N_{[1,1]}} = \frac{8}{11} - \frac{1}{11}(-\frac{1}{3})^{N+1}, \ V_{N_{[1,2]}} = \frac{320}{297} + \frac{48}{55}(-\frac{1}{3})^{N+1} - \frac{616}{135}(\frac{1}{2})^{N+1},$$

$$V_{N_{[1,3]}} = -\frac{4}{11} + \frac{6}{11}(-\frac{1}{3})^{N+1}, \ V_{N_{[2,1]}} = 0, \ V_{N_{[2,2]}} = -1, \ V_{N_{[2,3]}} = 0,$$

$$V_{N_{[3,1]}} = -\frac{2}{11}(-\frac{1}{4})^{N+1} + \frac{3}{11}, \ V_{N_{[3,2]}} = -\frac{80}{297}(-\frac{1}{4})^{N+1} + \frac{868}{297},$$

$$V_{N_{[3,3]}} = \frac{1}{11}(-\frac{1}{4})^{N+1} - \frac{18}{11}.$$

The determinant of V_N is equal to

$$\det(V_N) = 12/11 - 1/11(-1/3)^{N+1}(-1/4)^{N+1},$$

which has no zeroes for positive integer values N. Thus

$$K_{kj}^{N} = \frac{1}{11}(-\frac{1}{3})^{j}(-\frac{1}{4})^{k}\begin{bmatrix} -2 & -1/4 \\ 0 & 0 \end{bmatrix} + \frac{1}{11}(-\frac{1}{4})^{N-j}(-\frac{1}{3})^{N-k}\begin{bmatrix} 5/3 & -1/4 \\ 0 & 0 \end{bmatrix}. \tag{3.21}$$

So $T_N^{-1} = [\Phi_{k-j}^{\times} + K_{kj}^{N}]_{k,j=0}^{N}$, where $\Phi_k^{\times}$ is given by (3.20) and K_{kj}^{N} by (3.21). Putting $N = 2$, we find

$$T_2^{-1} = \begin{bmatrix} \frac{2}{1885} & \frac{471}{1885} & -\frac{1264}{1885} & -\frac{12}{145} & \frac{448}{1885} & \frac{48}{1885} \\ -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{72}{145} & -\frac{9}{145} & \frac{26}{145} & \frac{39}{145} & -\frac{112}{145} & -\frac{12}{145} \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -\frac{216}{1885} & \frac{27}{1885} & \frac{792}{1885} & -\frac{9}{145} & \frac{626}{1885} & \frac{471}{1885} \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}. \tag{3.22}$$

If we multiply (3.19) with (3.22) we get the required 6×6 identity matrix.

Finally, we find the inverse T^{-1} of a semi-infinite Toeplitz operator. We use Theorem 3.3 to show that T is invertible. Since $\mathrm{im}P = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\ker P^{\times} = \mathrm{span}\left\{ \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} \right\}$, we have that $\ker P^{\times} \cap \mathrm{im}P = \{0\}$. Thus $n(T) = \dim(\ker P^{\times} \cap \mathrm{im}P) = 0$, so T is injective. And

$$d(T) = \dim \frac{\mathbb{C}^n}{\mathrm{im}P + \ker P^{\times}} = \dim \mathbb{C}^n - \dim(\mathrm{im}P + \ker P^{\times}) = 3 - 3 = 0$$

implies that T is surjective. Thus T is invertible. We calculate $J^{\times}$ from

$$J^{\times} = P^{\times}|\mathrm{im}P = \begin{bmatrix} P_1 & -160/99 & 6/11 \\ P_2 & 1 & 0 \\ P_3 & -80/297 & 12/11 \end{bmatrix},$$

where P_1, P_2 are arbitrary, and $P_3 = -7/2P_1 + 388/27P_2 - 25/6$. One possible $J^{\times}$ is

$$J^{\times} = \begin{bmatrix} -25/33 & -160/99 & 6/11 \\ 0 & 1 & 0 \\ -50/33 & -80/297 & 12/11 \end{bmatrix}.$$

A generalized inverse of $J^\times$ is

$$(J^\times)^+ = \begin{bmatrix} -33/25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 80/27 & 0 \end{bmatrix}.$$

One easily verifies that $J^\times = P^\times|\text{im}P : \text{im}P \to \text{im}P^\times$ and that $J^\times = J^\times(J^\times)^+J^\times$. The inverse of T is given by

$$\Gamma_{ij}^+ = \Phi_{i-j}^\times + K_{ij}^+, \qquad i, j = 0, 1, 2, \ldots,$$

where $\Phi_k^\times$ is given by (3.20) and K_{ij}^+ by

$$K_{kj}^+ = \frac{1}{11}(-\frac{1}{3})^j(-\frac{1}{4})^k \begin{bmatrix} -2 & -1/4 \\ 0 & 0 \end{bmatrix}.$$

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List of Symbols

$\subset$ subset

$\mathbb{R}$ set of real numbers

$\mathbb{Z}$ set of integers

$\mathbb{C}$ set of complex numbers

$\mathbb{C}_\infty$ Riemann sphere $\mathbb{C} \cup \infty$

$\mathbb{T}$ unit circle in the complex plane

$\mathbb{D}_+$ open unit disc in $\mathbb{C}$

$\mathbb{D}_-$ complement of $\mathbb{D}_+ \cup \mathbb{T}$

Γ Cauchy contour in $\mathbb{C}$

Δ_+ inner domain of Γ

Δ_- outer domain of Γ

$L_2(\mathbb{T})$ set of all Lebesgue measurable and square integrable functions on the interval $[-\pi, \pi]$

$H_2(\mathbb{T})$ space of all square integrable functions on the unit circle

$L_2(\mathbb{Z})$ Hilbert space of all square summable infinite sequences of complex numbers

$\ker T$ kernel (nullspace) of the operator T

$\operatorname{im} T$ image (range) of the operator T

Summary

In this dissertation we studied the modern state space method for inverting semi-infinite block Toeplitz operators with rational matrix symbols explicitly from the representation of its symbol in realization form. A rational matrix function Φ which is analytic and invertible at infinity, may be represented in the form

$$\Phi(\lambda) = D + C(\lambda I - A)^{-1}B, \tag{1}$$

where A is an $n \times n$ square matrix, say, B and C are $n \times m$ and $m \times n$ matrices, respectively, and D is an invertible $m \times m$ matrix. The method for constructing explicit formulas for the inverse of a semi-infinite block Toeplitz operator with rational symbol is well-known for rational matrix functions in the form (1). However, in our work, we have emphasized the case where Φ does not have these properties at infinity and has a realization of the form

$$\Phi(\lambda) = D + (\lambda - \alpha)C(\lambda G - A)^{-1}B, \tag{2}$$

where A, B, C and D are as above and G is of the same order as A. In the main results in Chapter 2, we give necessary and sufficient conditions for the equivalence between block Toeplitz operators with rational symbol and discrete singular systems with boundary conditions. In addition, this equivalence implies that the explicit formulas (in realized form (2)) for the inverse may be written in terms of the matrices A, G, B, C and D and various other matrices derived from them. We also deal with the special case of finite block Toeplitz matrices. Different Fredholm characteristics are computed and a Riemann-Hilbert problem is solved as an application. The exposition is based on extensive use of a separation of spectra argument for linear operator pencils, the so-called spectral decomposition of the pencil $\lambda G - A$.

T^{-1} inverse of the operator T

T^{+} generalized inverse of the operator T, i.e., $T = TT^{+}T$

$T|_X$ restriction of the operator T to the set X

ind T index of the operator T

$I_X,\ I_m$ identity operator on X, $m \times m$ identity matrix

$X \oplus Y$ direct sum of the linear spaces X and Y

$\mathbb{C}^n$ Unitary space of dimension n over the field $\mathbb{C}$

$< x,\ y >$ inner product of x and y

σ non-empty subset of the complex plane

$\sigma(G, A)$ spectrum of the operator pencil $\lambda G - A$

$\rho(G, A)$ resolvent set of the operator pencil $\lambda G - A$

diag $(\lambda_j)_{j=1}^m$ $m \times m$ diagonal matrix with diagonal entries λ_1 up to λ_m

$l_p^m(\Gamma)$ space of $\mathbb{C}^m$-valued p-summable sequences on Γ

$\mathcal{L}(X)$ class of all bounded linear operators on X

$L_p^m(\Gamma)$ space of $\mathbb{C}^m$-valued p-integrable functions on Γ

$\mathcal{W}^{m \times m}$ $m \times m$ matrix Wiener algebra