ON ERRORS IN ELEMENTARY DIFFERENTIAL CALCULUS: A CASE STUDY AT A TECHNIKON

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Acknowledgements

I wish to dedicate this mini-thesis to my daughter Kirshia Naidoo who spent most parts of 1994 and 1995 patiently waiting for my attention while I was engaged with my mini-thesis work.

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Abstract

The thesis sought to investigate the errors made in elementary differential calculus by students studying engineering at technikons. A sample of 45 first year students from a technikon's engineering faculty were interviewed and questioned on their understanding of ideas considered to be important in elementary differentiation. Differentiation tasks were used to determine the kind of errors first year technikon students make in elementary differential calculus. Subsections of the tasks were regrouped to form twelve items, each item relating to one aspect of differentiation. These aspects were grouped into four sections: elementary algebra, rate of change, limits and infinity, and differentiation. The errors in the four sections were analyzed according to a classification of errors. This classification of errors was linked to concepts in cognitive theory. Analysis of the data reveals that there were more structural errors than executive or arbitrary errors in the sections on elementary algebra, rate of change and differentiation. There were more executive errors than structural errors in the section on limits and infinity. The structural errors were due to the students not applying the correct group of principles to the tasks while the executive errors were due to the students either omitting or replacing one substage in a correct rule by an inappropriate or incorrect operation. It is recommended that the errors can be alleviated by the use of appropriate computer technology such as spreadsheet and differential calculus software.

Declaration

I, the undersigned, Richard Naidoo, declare that "On errors in elementary differential calculus: A case study at a technikon" is my own work and that all the sources I have used or quoted have been indicated and acknowledged by means of complete references.

Signed : ....

Date : ....

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CHAPTER 1

INTRODUCTION AND STATEMENT OF THE PROBLEM

Mathematics is a compulsory subject for students following engineering courses at technikons in South Africa. The first year course in mathematics consists mainly of basic mathematics and calculus. A major component in the course is calculus which in turn places emphasis on differentiation.

In my experience as a mathematics lecturer at ML Sultan Technikon (in Durban, South Africa) I have found that students perform consistently poor in solving problems on differentiation. For example, in the examination at the end of the first semester in 1994 only 32% passed the section on differentiation. Similar views are expressed at annual meetings of mathematics lecturers of technikons in South Africa. For example, in one meeting it was mentioned that students have a mechanistic perception of the derivative. This phenomenon was also observed by Morgan (1990) in England. He investigated engineering students' understanding of differentiation. The students were studying at undergraduate and higher national diploma level at Middlesex Polytechnic in England. He identified some of the problems students experienced with differentiation. His findings were as follows:

(i) If the principle $dy/dx=0$ for a maximum or minimum is being used, where the function y is given

> as an algebraic fraction or as a sum of two functions, many students appear to think that the two parts may be considered separately.
>
> (ii) Students appear to have difficulty with the chain rule of differentiation. They do not appear to realize what variable they are differentiating with respect to. Sometimes they ignore the chain rule altogether and differentiate with respect to the wrong variable.
>
> (iii) Partial differentiation seems to cause much difficulty. Students do not seem to realize which parameters are constant and which are variable.
>
> (iv) Very often, when considering a maximum and minimum problem, students do not recall that they have a mathematics procedure for dealing with such problems and often resort to guessing.
>
> (Morgan,1990:979)

Orton (1983) also investigated the understanding of differentiation. He, however, concentrated on high school students and training college students. He concluded that both groups found the same items difficult and the same items easy.

Another dimension to the problem of understanding differentiation is the classroom practice of the lecturer which has an influence on how the students are taught (Ernest,1991).

Mathematics is a service course at South African Technikons. I suspect that this situation could possibly be a contributing factor to the problems technikon students are experiencing with differentiation. This issue was discussed at an international conference on 'mathematics as a service course' held in Japan in 1988. At this conference Hodgson and Muller (1989) indicated that many faculty members consider the mathematics service course as one or more of the following:

(i) large classes with a majority of uninterested students with rather a weak mathematics preparation;

(ii) restricted and overloaded syllabuses, too difficult for the students and with topics remote from research interests, with emphasis on techniques;

(iii) a task for which little technikon credit is given.

At South African technikons students do not specialise in mathematics. It is offered as a non-specialist subject. Subsequently students tend to focus less on the mathematics than on the other subjects which they must pass to be allowed into the next semester. This constitutes a problem for motivating the students.

Another issue is the time afforded for the study of mathematics. Restricted time mitigates against the teaching for understanding in differentiation. Indeed the time available does not allow for distinction to be made between processes in differentiation and the ideas or concepts which underlie these processes.

In the above we have identified some of the problems impacting on the understanding of differentiation at technikons. Since differential calculus is fundamentally important for engineering studies at technikons, it is important that we know the difficulties students experience and that we consider ways of minimizing these difficulties. This study sets out to investigate how students at a technikon understand introductory concepts of differentiation.

In order to get an indication of whether any previous study of technikon students' understanding of differentiation in South Africa was done a data base search was executed. The research data bases at the ML Sultan Technikon Library and Natal University Library were used. A search by Human Research Council (HSRC) was also performed. These searches indicated no previous research has been attempted in South Africa.

In this chapter we have discussed aspects of students' poor performance in understanding of differentiation in South Africa and England. We alluded to other factors, such as the fact that mathematics is a service course, students are not specialising in mathematics and students having little time to study mathematics, which may contribute to the problem students experience with differentiation. The primary objective of this study is to investigate some of the concepts and processes associated with differentiation which, to my knowledge and experience, could be considered as the basis for students finding it difficult to handle differentiation.

Chapter 2 reviews theoretical issues in the literature which explore some of the concepts and processes associated with differentiation. The theoretical framework underpinning these concepts and processes is discussed in chapter 3. A brief exposition of the research methodology and design is performed in chapter 4.
Chapter 5 consists of the data analysis which leads to the quantification of the results. The concluding remarks and recommendations on ways for improving the understanding of elementary differentiation at technikon level are discussed in chapter 6.

CHAPTER 2

THE CONCEPTS AND PROCESSES CONSIDERED TO BE CONTRIBUTING TO STUDENTS' DIFFICULTIES

In chapter 1 we alluded to the notion that students at technikons are experiencing a degree of difficulty in understanding differentiation. Chapter 2 reviews theoretical issues in the literature which explore some of the concepts and processes associated with differentiation. These concepts and processes could be considered as areas where students experience difficulty in understanding differentiation.

2.1 DIFFERENTIATION AS AN ANALYSIS COMPONENT

In his research of the problems experienced by engineering students with mathematics in the Netherlands, van Streun (1991), showed that analysis constitutes the biggest conceptual stumbling block for such students. Differentiation is categorised as a component of analysis. Differentiation techniques can be applied in the sketching of the graphs of algebraic polynomial and rational functions. A particular application of differentiation in this regard is the determination of relative extrema, points of inflection, intervals on which the function is increasing or decreasing, and the concavity of the function. The students at the M L Sultan Technikon usually fail to appreciate the role which inequalities play in curve sketching. This comes to the fore especially where students have to determine the interval where a function increases or decreases.

Students often confuse relative maxima with relative minima when they use the first derivative test. For a relative minimum the derivative changes from positive to negative. This may lead students to believe that the function is decreasing and therefore attains a relative minima.

The second derivative test which determines whether the relative extrema is maximum or minimum is often easier to use than the first derivative test. However, it can happen that $f''(x)=0$ at a maximum or minimum point, and in such cases it is necessary that we use the first derivative test. Students normally do not go back to the first derivative test. In using the second derivative test students find it difficult to grasp that $f''(x)$ is negative at a maximum point and positive at a minimum point.

The points at which the curve changes from concave up to concave down, or from concave down to concave up, are known as points of inflection (stationary points). Students find it difficult to sketch the point of inflection. They interpret the second derivative test as the rate of change of $f(x)$ to be equal to zero at the point of inflection. Due to this misconception students sketch the inflection point (x,y) as a horizontal line.$(y=c)$.

2.2 DIFFERENTIATION AS A COMPLEX CONCEPT

A derivative can be seen as a concept which is built up from other concepts. The derivative can be seen as a function, a number if it is evaluated at a point or limit of the sequence of secant slopes. There do not seem to be clear-cut characteristics that set advanced mathematical concepts (algebra) apart from those in elementary mathematics (arithmetic). Each advanced concept is based on elementary concepts and cannot be grasped without a solid and sometimes very specific understanding of these elementary concepts. Thus the concepts of advanced mathematics carry an intrinsic complexity. Students cannot grasp what is meant by a differential equation or interpret its solution unless they have understood the concepts and not just the techniques of differentiation. Differentiation assumes an understanding of the function concept. The function concept assumes an understanding of the notion of a variable which in turn presupposes the number concept. This network or sequence leads to interrelated ideas, each idea integrating some of the more elementary ones into an added structure. For example, a function is not only a variable but two variables that stand in a relationship that must obey certain rules. Differentiation generates a new function, the derived function, from a given one. For example, if $f(x)=ax^2+bx+c$ then $f'(x)=2ax+b$. However, $f'(x)$ can also be interpreted as a dependent variable, say $y=2ax+b$. At $x=2$, $f'(2)=4a+b$, is a real number. It is precisely this complexity of the concepts that tends to make differentiation difficult for students to grasp them as entities.

2.3 GRAPHS AND DIFFERENTIATION

One of the ways of interpreting the meaning of a derivative is to consider the concept of the 'gradient of the graph of a function'. This interpretation, basic to the understanding of calculus, deals with the slope of the line tangent at a point on a curve. Consider the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in figure 1 below. The slope of the line through these points is given by $m= \Delta y/\Delta x = y_2-y_1/x_2-x_1$ which according to Skemp (1970) is the ratio of a pair of corresponding changes. This, however, represents the slope of the line through P and Q and no other line. If we now allow

Q_1 to be a point closer to P, the slope of PQ_1 will more closely approximate the slope of a line drawn tangent to the curve at P in figure 2 below. In fact, the closer Q is to P, the better this approximation becomes. It is not possible to allow Q to coincide with P, for then it would not be possible to define the slope of PQ in terms of the two points. The slope of the tangent line, often referred to as the slope of the curve, is the limiting value of the slope of PQ as Q approaches P.

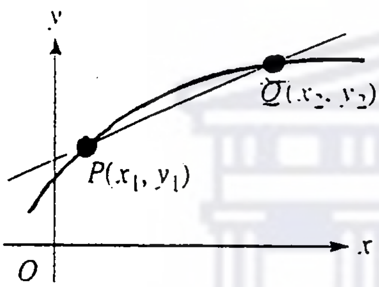


Figure 1

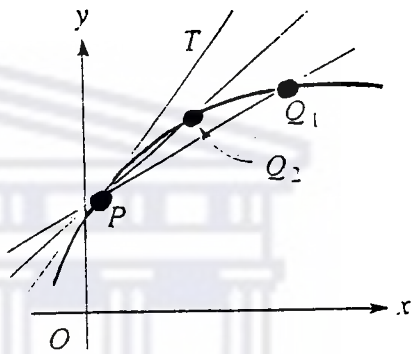


Figure 2

The derivative is defined as the limit of the ratio $\Delta y/\Delta x$ as Δx approaches zero. Therefore the derivative is the gradient of the line tangent to the curve. The average rate of change is given by $\Delta y/\Delta x$. The derivative is then a measure of the rate of change of y with respect to x at a point P. The notation dy/dx is used for the derivative. Kerslake (1977) found that students have very little understanding of gradients of lines and therefore of the derivative.

Hughes-Hallet (1989) reported that students should be able to estimate derivatives graphically as well as to work with functions given graphically. She states that there is an

> ...alarming number of students who are expert at finding derivatives analytically, but who have no idea what they are doing graphically... For example students find derivatives of $1/(\cos^3(x^4))$ quite easily but who (sic) are completely unable to estimate the derivative at a point on the graph. In fact they have an attitude towards graphs which is reminiscent of the attitude many people express towards mathematics. Such student say 'Oh I just can't do graphs' and laugh as though this were rather quaint and quite unimportant. After all, in their view, 'real mathematics' is manipulating x's.
>
> (Hughes-Hallet,1989:32)

I found similar responses from my students at ML Sultan Technikon. Hughes-Hallet goes on to state the 'rule of three' which is based on the belief that in order to understand an idea, students need to see it from several points of view, and to build a web of connections between the different viewpoints. In calculus most ideas should be presented three ways: graphically,numerically and algebraically. This implies that for teaching the derived function of x^2, it should be dealt as a limit of the sequence of ratios, numerical values of the difference quotient and a sequence of secants fixed at one point on the graph approaching a tangent to that fixed point.

Dick (1989) proposes that the derivative should be introduced via magnification. He states this as

> The epsilon-delta's of scaling...using graphing technology in calculus...the definition of derivative is often illustrated with secant

lines approaching a tangent line. The limiting process is visualized using a fixed graph on which a sequence of secants are drawn with one of the points determining the secant approaching the other point...nice software versions of this visual limiting process exist, but their specialized nature would make many instructors think twice about programming them on a general graphing package or calculator... unlike the fixed graph illustrations above, we can formulate visual meanings for these definitions in terms of the effects of changes of scaling of graphs of functions.

He then proposes the following scaling definition.

The function f is said to be differentiable at $x=a$ if and only if for a screen centred at the pixel $(a,f(a))$, there exists a common horizontal and vertical scaling factor such that the graph of $y=f(x)$ is indistinguishable from the line whose graph is $y=m(x-a)+f(a)$. In other words, if we 'zoom in' on the graph of a differentiable function close enough, its graph should appear to be a straight line whose slope is the derivative of the function at that point.

(Dick, 1989:145-147)

Strang (1989) agrees with this approach. He combines two approaches. Initially the study of the actual machine-produced graph is done and then the understanding of the mathematics of the graphs including concepts such as slope, concavity, scaling and inflection.

Both Strang and Dick make understanding of the derivative much simpler provided students have an access to computers and the required software.

2.4 THE COGNITIVE ASPECTS IN DIFFERENTIATION

The cognitive structure existing in the mind of every individual yields a variety of personal mental images when a concept is evoked. For the concept differentiation it could evoke mental images of tangents, rate of change, secants and limits.

Accomplished mathematicians thought about researching mathematics. They took the mathematical content and its structure as a basis for their thoughts. However, sometimes they do not sufficiently take account of the students involved in learning the mathematics or of the details of their understanding and how it is acquired. There was no evidence that they investigated students' thought processes in their written work. For example, the first account of the differential calculus was published by Leibniz in 1684 under the title 'Nova methodus pro maximis ... ' Here Leibniz gave the formulas $dxy=xdy+ydx$, $d(x/y)=(ydx-xdy)/y^2$...These formulas were derived by neglecting infinitesimals of higher order (Boyer,1989:450). No mention was made by Leibniz about the other terms in the expression or how small the infinitesimals were. Lecturers tend to use Leibniz's techniques in their lessons.

Some serious crises have arisen in the lecturing of college mathematics. One such example, the crisis in teaching calculus as given by Steen (1987). One of the reasons for such crises is precisely that in most college or technikon mathematics teaching there is no consideration of cognitive processes but only to mathematical manipulations.

2.5 DIFFERENTIATION AND FIRST YEAR STUDENTS

First year students arrive in their calculus classes with far less knowledge, skill and understanding than their instructors assume. In France, for example, a large scale study has shown that beginning students are reasonably competent in algebra. They have difficulties in logic (interpretation and manipulation of statements that include quantifiers) and graphing (both in producing and interpreting) (Robert and Boschet,1984).

Research by Artigue and Viennot (1988) has led to the following conclusions about first year calculus students: they can compute derivatives but cannot work with linear approximations (nor do they conceive of a derivative as an approximation), their geometric images are normally poor and their functional thinking is relatively weak.

Students have a strong tendency to reduce differentiation to a collection of algebraic algorithms, while avoiding graphics as well as geometric images. As a consequence they lack the ability to grasp the role of approximations, which is fundamental for understanding the concept of the derivative.

Another startling deduction was made by Beckman (1989) who reported that one difficulty in calculus is that students will phrase responses in numbers as natural numbers. This concept of a number is incompatible with the study of continuous phenomena (Confrey, 1980). It will certainly not help the student to understand differentiation from a method involving the 'slope of secant to slope of tangent of a curve' where a sequence of secant slopes is generated.

J.and A. Selden (1989) observed that the calculus courses based on textbooks produced before the introduction of electronic technology are not producing the desired results. Not only do apparently successful students fail to appreciate the relationship between theory and problems, but some even have a poor grasp of the fundamental concepts. In addition, students were rarely able to solve unseen problems, an important ability for anyone interested in applications. Furthermore, traditional calculus books contain perhaps too many sample solutions and detailed algorithms for solving problems. Consequently students stress procedural aspects over conceptual ones. For example, students who were asked to discuss differentiability of $f(x,y)=2x+4y+y^3((1-\cos x)^{1/2}+x^2)$ immediately started computing the partial derivatives of f rather than studying the structure of the expression (Alibert,1988). Students were found to look at differentials as purely fictional elements and they do not see differentials as approximations, functions nor a single variable.

2.6 THE CONCEPT IMAGE VERSUS FORMAL CONCEPT DEFINITION

There is a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived. The term concept image describes the total cognitive structure that is associated with the concept. This includes all mental pictures and associated properties and processes. Vinner (1982) stated that students in a service level calculus course were given the graph $y=x^3$. Most of them stated correctly that it is possible to draw exactly one tangent to the curve at the origin, but fewer than 20% were able to draw the tangent correctly (Vinner,1982). Their concept image did not include a horizontal tangent at a point other than at a maximum or a minimum point.

In coming to understand mathematical concepts at school, students evolve mental pictures at a concrete level. For example, to understand rate of change students may evoke pictures of a moving car. The mental pictures which served the students well at school level may now become an impediment. Bruner (1986) suggested that iconic processing limited ideas and urged a movement onto the symbolic level. The student with an inadequate concept image may find such a development difficult to achieve.

2.7 CONCLUSION

In differentiation as an analysis component students find inequalities, first derivative and second derivative tests and sketching the derivative of the curve at the inflection point difficult. Differentiation is built up of a network of other concepts which makes it a complex concept.

Using the graphical nature of the differentiation opens the possibility for determining gradients of secants at a sequence of points and then finally arrive at a tangent for the curve at a point as the limit of a sequence. Due to the cognitive structure of the student, technikon mathematics teaching must take into account cognitive processes. For some of the first year students differentiation will be a problem because their fundamental mathematical knowledge and understanding are limited. Students with an inadequate concept image may find developing other concepts difficult to achieve. In this chapter some of the concepts and processes that may explain the errors made by students in elementary differential calculus at technikons, have been expounded. Chapter 3 is a discussion on the theoretical framework underpinning this study.

CHAPTER 3

THEORETICAL FRAMEWORK

In this chapter I will discuss the theoretical framework underpinning this study. Errors in students' understanding of differentiation are categorised as structural, executive and arbitrary by Donaldson (1963). This categorisation will be integrated with the theoretical framework of Davis (1984).

3.1 DONALDSON'S ERROR CLASSIFICATION FORMAT

From my experience I observed that students make the kind of errors expounded by Donaldson (1963). For example, to find the derivative of $y=1/x$ at $x=0$ most students will write and make the structural error by either giving 0 or infinity as the answer. Structural errors are described as those which arise from some failure to appreciate the relationships involved in a problem or group of principles essential to the solution of the problem. There is a distinction to be drawn between principles of wide generality, applicable to all problem solving behaviour, and principles relevant to the form or structure of a particular problem. As the name 'structural' implies, it is failure to grasp a principle of the latter kind. (Donaldson,1963).

Failure to handle relationships in a problem is seen as arising from a false expectation of the problem. Structural errors may arise in connection with variable interaction. Successful solution demands, in the first place, a grasp of the notion that perfect correlation makes possible inference from the possession of one property to the possession of another. Structural errors arise when the subject is acting on the basis of the given in so far s/he is able to comprehend it and grasp what it implies i.e., it occurs within the deductive mode: they occur whenever the subject reasons deductively but fallaciously. One may expect that failure to perceive inconsistency or consistency would be a common source of structural error. (Donaldson,1963).

Another example of a common error occurs when students are required to find the derivative of $y=x^n$. If they experience difficulty they recall the logarithm law of $\log x^n = n \log x$ and then equate this with the derivative. This type of error is an executive error which involves failure to carry out manipulations, although the principles may have been understood. Some defect of concentration, attention or immediate memory lies at their origin. The most prevalent of this class of errors is loss of hold on reasoning. (Donaldson,1963).

The third type of error is when a student differentiates $y=3x^2$ to get $dy/dx=2x$ and ignores the constant 3. This type of error is classified as an arbitrary error. Arbitrary errors are those in which the subject behaves arbitrarily and fails to take account of the constraints laid down in what was given. These are errors which have as their outstanding common feature a lack of loyalty to the given. Sometimes the subject appears to be constrained by her/his knowledge of what is 'true' by some considerations drawn from 'real-life' experience. Sometimes there is no constraint of any kind. The subject simply decided 'it is so'. (Donaldson,1963:184).

3.2 THE COGNITIVIST THEORY OF DAVIS

Davis (1984) proposed a cognitivist theory as a language to describe mathematical behaviour. This theory regards mathematical thought processes as fundamental. The theory relates observations to a postulated theory of 'metaphoric' processes with information of how the individual thinks about some mathematical problem. This theory borrows its basic concepts from the field of artificial intelligence.
In what follows I will comment on four of the basic concepts of Davis's theory and link them to Donaldson's classification of errors. In addition to the four basic concepts of Davis's theory there are also pointers, descriptors, metaphor and isomorphism, flexibility, planning space, planning language and meta-language.

3.2.1 SEQUENTIAL PROCESSES

Sequential processes are devices for guiding mathematical problem solving activity. It makes use of procedures which are algorithmic and step-by-step activities. There are at least two kinds of procedures: visually moderated sequences and integrated sequences. The input in the visually moderated sequences cues the retrieval of a procedure. The execution of the procedure modifies the visual input. The modified visual input cues the retrieval of a new procedure. The cycle continues until some process triggers termination. (Davis,1984). For example, to differentiate the polynomial function $f(x)=4x^3$ we use the procedure $f'(x)=4d(x^3)/dx$ i.e., the procedure $d(cf(x))/dx=cd(f(x)/dx$. The modified visual input $d(x^3)/dx$ cues the retrieval of the procedure $d(x^n)/dx=nx^{n-1}$ when $n=3$ which yields $d(x^3)/dx=3x^2$. The two new visually modified inputs viz., the constant 4 and the differentiated component $3x^2$ triggers the termination $f'(x)=4.3x^2=12x^2$. Sequences which through sufficient practice, have become independent of visual cues for program guidance are called integrated sequences. The power of a 'procedure' is that once a procedure has been synthesized in a student's mind, that procedure can be given a name and new procedures can be synthesized which use this name as if it were a command in the student's internal cognitive 'programming language'. (Davis,1984). For example, the integrated procedure for $d(4x^3)/dx$ is $4.nx^{n-1}$ or $d(cx^n)/dx=c.nx^{n-1}$ which is the synthesized procedure and a command name can be 'polynomial times a constant'.

There are relations among procedures. One procedure A may call upon or transfer control to a second procedure B. When B has completed its assigned task, it returns control to procedure A. Procedure B is said to be a subprocedure A. Procedure A is called the superprocedure (Davis,1984). For example, to differentiate polynomials the superprocedure cnx^{n-1} is used for differentiation of subprocedure x^3. The errors caused by the use of superprocedures and subprocedures can be classified by Donaldson's error classification format. If a wrong superprocedure has been selected the error will be a structural one. If the wrong subprocedure has been selected the error is an executive error. The student may just lose track of the algorithm s/he is trying to use. S/he has to keep track of all the sub-assemblies. Keeping track of all the sub-assemblies is an example of a control task. The human-mind have internal mechanisms for keeping track of where we are, while working through a task. (Minsky,1980). These executive errors are mainly made by beginners. These behaviours are mainly due to distractibility within the context of the problem i.e., s/he may take a different route in solving a problem although s/he may be familiar with the solution route.

3.2.2 CRITIC

A critic is an information-processing operator that is capable of detecting certain kinds of errors. The critic may detect arbitrary or executive errors. For example, to differentiate the function $x^2+y^2=9$ the student may do the following $2xdx+2y=0$. The critic will look for the dy in the equation and the student will be aware that his solution is incorrect. The critic operator gives the student the motivation and the apparatus for stepping back and critiquing her/his own thinking as well as saying something about her/his errors (Brown,1978).

3.2.3 FRAMES

Information in one's mind must typically be organized into quite large chunks (Davis and McKnight,1979; Minsky,1975). A frame is an abstract formal structure, stored in memory, that somehow encodes and represents a sizeable amount of knowledge. Minsky (1975:212) states that " when one encounters a new situation...one selects from memory a substantial structure called a frame. This is a remembered framework to be adapted to fit reality by changing details as necessary". A frame differs from a procedure in that it is not sequential. It allows multiple points of entry and provides some flexibility in its use. A frame can be retrieved when needed. The retrieval occurs almost instantly.

3.2.3.1 FRAME SELECTION PROCEDURES

Davis (1984) lists six possible frame selection procedures:

(i) Bootstrapping

This involves what one sees in the given.

(ii) Not knowing too much

The students learn and apply the concept of differentiation when confronted with a minimum or maximum problem.

(iii) Focus on some key cue

Students may focus on some cues whose presence would be taken as evidence for the retrieval of some specific frame. For example, the rules for differentiation of polynomials.

(iv) Using context

Students use the context to influence their choice.

For example, in curve sketching, a section which comes under the differentiation in the Technikon syllabus, the students may use differentiation techniques.

(v) Using systematic search

The student may develop systematic procedures for searching his/her memory. In curve sketching the student first finds the roots, differentiate once to find the minimum or maximum points. Differentiate one more time to determine the exact minimum position or maximum position.

(vi) Parameter-adjusting or spreading activation

The student sees an array of 'assimilation' candidates. Whenever one of these is satisfied, its 'expectation value' is increased. For example differentiating a term with x to some non zero positive power indicates that it is a polynomial. It increases the expectation value that derivative is of the form nx^{n-1}. The frames become active and assume control. If it is an incorrect frame it will contribute to a structural error.

3.2.3.2 INTERNAL ORGANISATION OF THE FRAME

Frames possess considerable internal organisation. The frame variables or slots seek specific values from input data. When the input does not provide enough information to permit certain slots to be filled, the frame may insert some tentative 'guess', based on past experience. When slots are filled in this way, it is called default evaluations. If the tentative guess is incorrect, then it is an arbitrary error.

3.2.3.3 PRE-DIFFERENTIATION FRAMES

Everyone possesses a large and powerful repertoire of frames for dealing with operations. Differentiation is a collage of these operations. The successful mathematician has built on these pre-differentiation frames and synthesized the abstract frames appropriate to what we recognize as mathematical thought. Not all students have done this. All Technikon students are supposed to have pre-differentiation frames and if we can bring them to bear on a differentiation problem, they can probably solve the problem. The retrieval of appropriate mathematical frames, and not the following of natural language sentences,is essential if one is to succeed in mathematics.

Papert (1980) stated that anything is easy if one can assimilate it into one's collection of mental models. He found that working with gears made him relate new concepts to his earlier ideas of gears. This is an expert testimony on behalf of the importance of pre-differentiation frames.

It can be postulated that one mechanism for dealing with a novel situation involves the following:

(i) Some initial interpretive frame is retrieved, based on some cues in the input data.

(ii) A possession of a collection of basic information processing frames, such as the differentiation frame.

(iii) The initial frame recognizes that the input data deal with differentiation so it causes the retrieval of the differentiation frame or some other appropriate frame.

(iv) The retrieval of the basic differentiation frame automatically lists the previous uses of this frame. This allows one to consider whether any of these are similar to the present task.

(v) If so, selection of pieces of this previous 'solution' is executed.

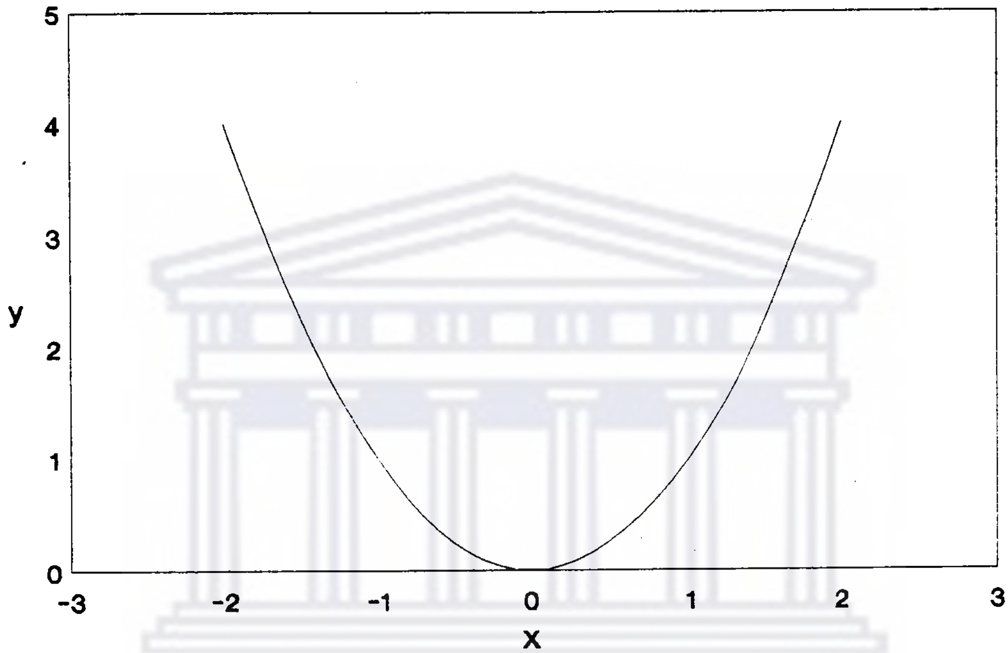
The above is also a recipe for building a new frame. The entire sequence we have just gone through can be 'welded together' into a single cognitive entity (Davis,1984). In general, a problem may be quite easy if you have an effective representation for the problem itself, and the effective representations for the relevant areas of knowledge.

The definition of differentiation does not induce the creation of adequate frames. Maybe an appropriate frame can only be synthesized from experience with some suitable collection of examples and counter-examples. The mental representation of the concept derivative depends in a central way on:

(i) the ability to recall or to invent candidate exemplars. These may be examples or counter-examples. For example, the differentiability of a function at the point (0,0) graphed below.

Graph 1:

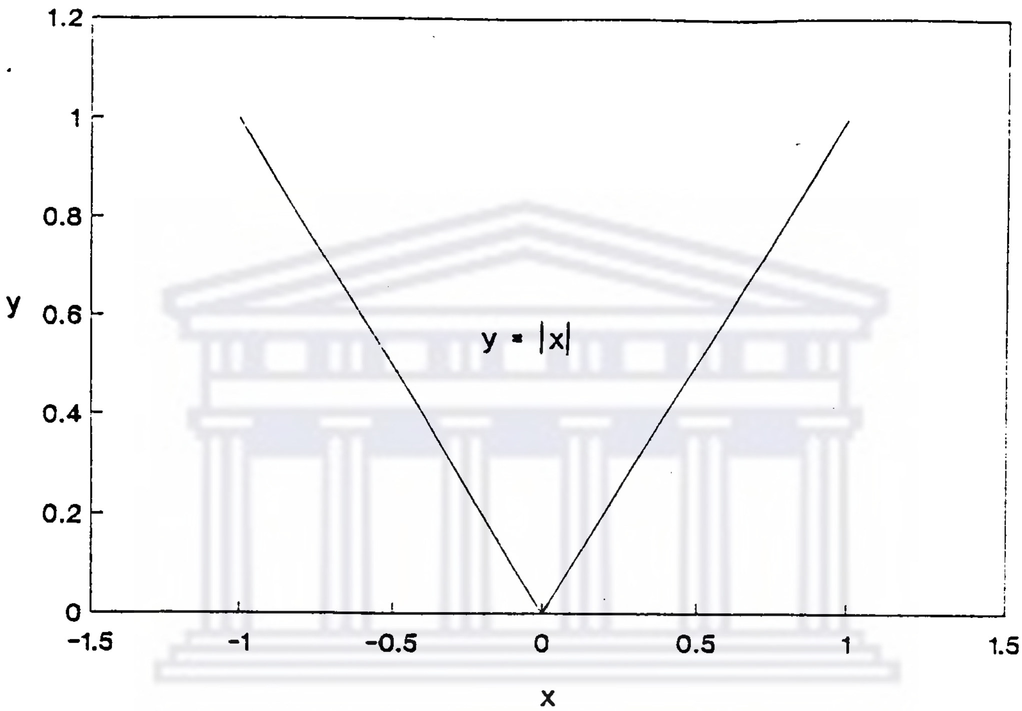
$y=x^2$



The above function is differentiable for all x.

Graph 2:

$y = |x|$



The above function is not differentiable at $x = o$.

(ii) the capability of making judgements on exemplar candidates. For example, polynomials are differentiable everywhere.

3.2.3.4 DIFFICULTIES WITH DIFFERENTIATION FRAMES

With respect to the ideas expressed above one can expect that students will be faced with three difficulties when dealing with differentiation, viz.,

(1) finding the correct pieces in the mind,

(2) determining the units in the input data, and

(3) finding the correct mapping of input units into frame slots after the frame has been selected.

For example, a problem on differentiation of polynomials will elicit a frame with 'differentiation of polynomials with rules of differentiation'. This results in 'top-down' and 'bottom-up' processing. In the top-down processing the frame helps as a guide to the input data. In the bottom-up processing it may turn out that the transformation did not facilitate the application of the main final solution. If the units in the input data are incorrect then one will not be able to slot it in the frames. A correct mapping must be selected. While the 'building' is in progress the partially completed representation will be subjected to meta-analysis to guide the search for a solution. In 'recognition problems' the student knows the usual facts and techniques. For example, the process of finding a derivative of a polynomial. S/he may run into difficulty because none of the familiar tools may suffice to work through a problem. The student must retrieve a correct 'assimilation paradigm' and devise a correct paradigm for it.

A visually-similar cue may elicit a well-established frame which was learned earlier. Since eliciting cues are not sharply defined, confusion may result. Consequently an established frame is preferred to a tentative new one. For example, to find a slope of a tangent at a point in a parabolic graph the student may determine the slope to a secant cutting two points on the graph. Such an error can be classified as structural error.

When a student has learned differentiation s/he would have built in her/his mind a knowledge representation structure or frame. The frame will seek inputs for certain key frame variables. The frame will seek the type of function we want to differentiate and the rules that are applicable. If the student cannot apply the rule then we can say the student had not synthesized an adequate knowledge representation structure of differentiation. The student's frame is sketchy and incomplete.

When the student uses the matching frame without the required input then this will constitute an error in the mathematical solution. This error is the same as Donaldsons' arbitrary error.

3.2.3.5 REAL-TIME CONSTRUCTION

What we need may not be retrievable from memory. Therefore we must be able to create new knowledge representation structures at the moment when we need them. This is called real-time constructions. We see a two-part process viz., 'top-down' and 'bottom-up' being performed simultaneously. Suppose a student was asked from a set of data to calculate the rate of change (derivative). The frame may ask whether the graph obtained from the set of data is a straight line or whether the gradient of the tangent is the same as the rate of change. The student may possess a frame labelled 'how to solve maths problems'. (Davis,1984:263). This frame embodies a number of heuristic principles. It has variables that allow one to input a description of the problem to be solved. As one of its outputs it will supervise the creation of an ad hoc representation structure for this particular problem. It will make use of representations retrieved from memory. One of its tasks is to guide the process of locating these items in memory. The frame directs a search in the problem statement in order to find what was being asked. The frame has the responsibility for assembling a procedure for solving the problem. The frame may resort to 'backward-chaining' to show that rate of change is the same as the derivative. (Larkin,1980). We have the two processes converging: first the heuristic problem analysis sequence and, secondly, the building up of a cognitive representation of the solution.

3.2.4 DEEPER-LEVEL PROCEDURES

Matz (1980) postulates two levels of procedures which are stateable as rules. The surface level rules are ordinary rules of algebra. The deeper level rules serve the purpose of creating superficial-level rules, modifying superficial rules or changing the control structure. The deeper level procedures are by no means infallible. They produce many superficial level procedures that are erroneous. The deeper level procedures operate in a systematic and consistent fashion. For example, $d(\sin 2x)/dx = 2d(\sin x)/dx$. These symtematic errors are similar to Donaldson's structural errors. Apparently some aspects of 'familiarity' weighs the outcome in favour of the more familiar frame. Students making the above error are not doing a careful enough job of verifying that frame selection and instantiation have proceeded correctly (Davis,1984).

Students retrieve incorrect frames which function in certain situations but should not be used in a context under consideration. One way of dealing with this is that higher level goals must be achieved. The constraints can then be relaxed in order to satisfy these higher level goals. For example, implicit differentiation must be used to find the derivative of $y=(\sin x)^2$. If the higher level goals cannot be achieved then the error made will be structural in nature.

3.3 CONCLUSION

The three errors of Donaldson were linked to the sequential processes, critic, frames and the deeper level procedures. The learning of differentiation does not require verbatim repeating of verbal statements but the development of appropriate mental frames to represent the concepts and procedures of differentiation. Structural errors are caused by incorrect frame retrieval, sketchy or incomplete frames, deep-level procedures and superprocedures.

Executive errors are caused by incorrect subprocedures and control structure of sub-assemblies.

Arbitrary errors are caused by mapping incorrect inputs to the retrieved frame.

The theoretical framework refers to the ways students are thinking with respect to the mathematical tasks. This necessitates that one has to get information from students whilst they are engaged in specific mathematical tasks. The mathematical tasks and the research methodology are discussed in the next chapter.

CHAPTER 4

METHODOLOGY AND DESIGN

In this chapter I will give an overview of the subjects, methodology and the design of my research experiment.

4.1 THE SUBJECTS

The aim of the present study was to investigate the understanding of elementary differentiation by first year technikon engineering students. The subjects of the study were from the engineering faculty of the ML Sultan Technikon, Durban, South Africa. All students studied the elementary notions of differentiation at school. They had also completed the first semester mathematics course at the Technikon. Students matriculated from the, now discontinued, Department of Education and Training and Departments of Education and Culture (House of Representatives and House of Delegates). There were 45 students in the sample. In selecting the sample to be interviewed there were problems in terms of balance between males and females. This can be attributed to the fact that presently more males than females pursue engineering courses.

4.2 METHODOLOGY: DATA COLLECTION

Orton's (1983) battery of tests was used as the testing instrument. The battery contains 37 tasks of which 12 dealt with elementary differentiation here for the sake of convenience renumbered and combined into 10 tasks. These 10 tasks were divided into 12 items described on page 44. These 12 items were used in this study.

Interviews were conducted on Technikon premises and lasted from forty five minutes to sixty minutes. Responses to the differentiation tasks were given orally or written on paper. A tape recording was taken of the students' oral responses. I used the clinical method which stresses the importance of the subjective experience of individuals.

4.3 THE PILOT STUDY

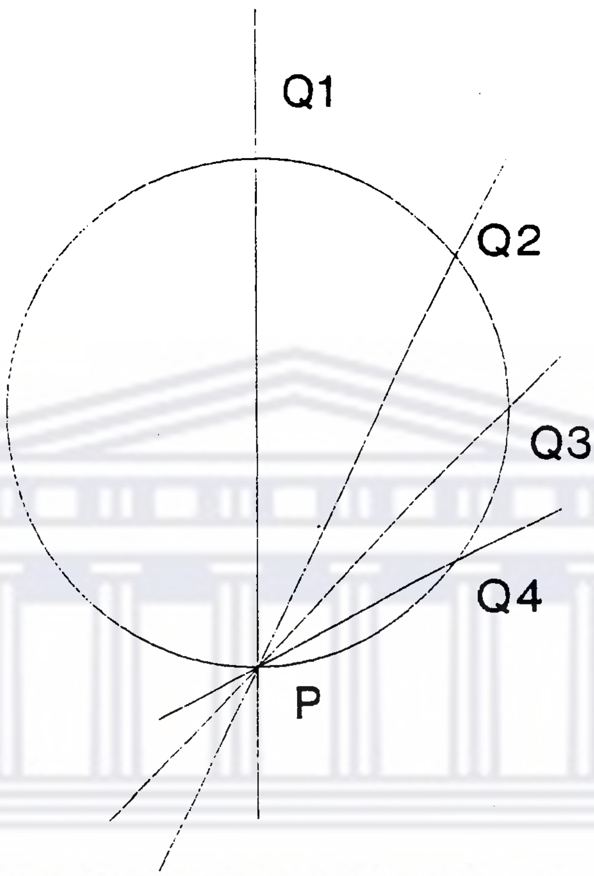
The Orton tasks were piloted with three students and later used in the main study. The students were selected to reflect the attainments on which the main study was to be based. From the nature of the responses it became clear that some tasks were not properly understood or, perhaps, not well formulated. For example, Task 6 required clarification as to whether it is a general curve or an exponential curve.

4.4 THE TASKS

These tasks come from Orton's instruments for the understanding of differentiation.(Orton, 1983).

TASK 1

The diagram shows a circle and a fixed point P on the circle. Secant lines PQ are drawn from P to points Q on the circle and are extended in both directions.



(1.1) How many different secants could be drawn in addition to the ones already in the diagram?

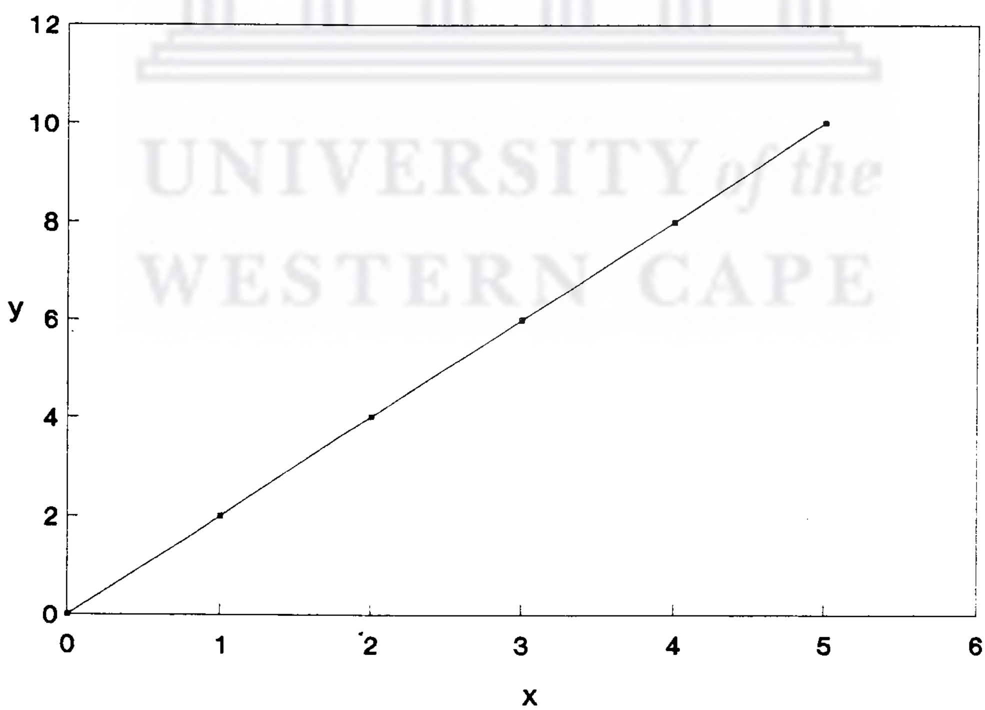
(1.2) As Q gets closer and closer to P starting from Q_1 what happens to the secant?

DISCUSSION

It was intended to mirror the following situation in differentiation. The moving point, Q, approaches the fixed point, P. The secant approaches the tangent to the circle at the fixed point. The slope of the tangent at a fixed point can be considered as the limit of the sequence of slopes of secants through the same fixed point. The frame the student should retrieve incorporates the following: secant cutting two points on a curve, sequence of secants through Q and P with Q approaching P and the secant approaching the tangent to the curve at the fixed point, P. The student could also synthesize the above frames welded into a single frame or could construct each frame from assemblies. The student could also represent the problem with the curve being cut by a secant and approaching a point (x,y).

TASK 2

Water is flowing into a tank at a constant rate, such that for each unit increase in time the depth of the water increases by two units. The graph illustrates this situation.



What is the rate of increase in the depth when $x=2\frac{1}{2}$? when

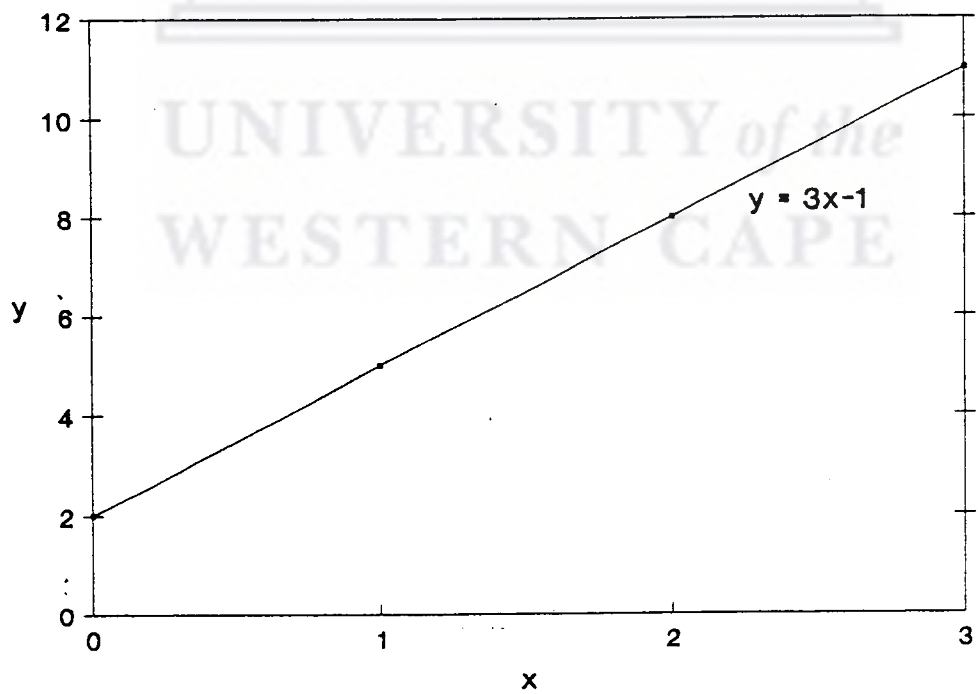
$x = T$?

DISCUSSION

Both questions related to the general theme of rate of change, are based on the same graphical situation. The student should retrieve the frame 'a tank being filled with water', 'a straight line graph with gradient 2', and 'rate of change equal to gradient'. The "tank being filled" can be taken as pre-mathematical frames or collages for the synthesized frames. The procedure of the frame (top-down processing) is to recognise that constant rate relates to straight line graphs and that every point on the x-axis yields the same rate of change.

TASK 3

The graph below represents $y = 3x - 1$



(3.1) What is the value of y when $x = a$? [a is any real number]

(3.2) What is the value of y when x = a + h? [h is any increment]

(3.3) What is the increase in y as x increases from a to a + h?

(3.4) What is the rate of increase of y as x increases from a to a + h?

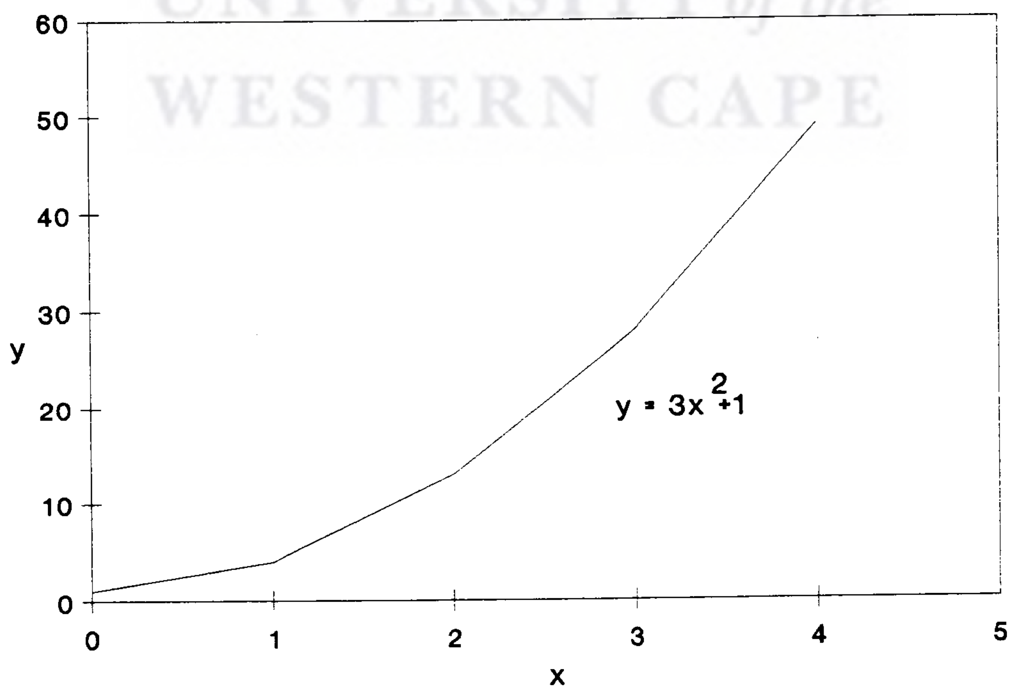
(3.5) What is the rate of increase of y at x = 2½? at x = X?

DISCUSSION

Task 3 was complementary to Task 2. Both included a variety of questions following the theme of rate of change. Task 3 was based on a linear function whereas Task 2 was based on a real world problem. This type of task is usually found in engineering courses. The frame to be retrieved in this task is similar to the previous one. The frame will require inputs for a function y=f(x), change of y, change of x, rate of change = $\Delta y/\Delta x$ and this is constant throughout the x-axis.

TASK 4

The graph below represents $y = 3x^2 + 1$, from x = 0 to x = 4.



(4.1) What is the value of y when $x = a$? [a is any real number]

(4.2) What is the value of y when $x = a + h$? [h is any increment on the x-axis]

(4.3) What is the change in y as x increases from a to $a + h$?

(4.4) What is the average rate of change in y in the x-interval a to $a + h$?

(4.5) Can you use the result of (4.4) to obtain the rate of change of y at $x = 2\frac{1}{2}$? at $x = T$? If so, how?

DISCUSSION

This task complemented the previous one by considering the same kind of questions but with a different type of function. The task is aimed at extracting information concerning students' capabilities and understanding relating to rate of change based on graphs. The retrieved frame will be similar to the above except that the input function is a quadratic and that average rate is now $\Delta y/\Delta x$. Recall that in the linear graph the rate of change is the same as the average rate of change. Using a superprocedure within the frame, $\lim_{h\to 0} \Delta y/\Delta x$ = the rate of change, the student will be able to determine rate of change at $x=2\frac{1}{2}$ and at $x=T$. The subprocedures are the determination of Δy, Δx and the limit. These subprocedures can also be taken as assemblies.

TASK 5

(5.1) What is the formula for the rate of change for the equation $y = x^n$? [n is an element of the natural numbers]

(5.2) What is the rate of change formula for each of the following equations:-

$y = 3x^3$?

$y = 4$?

$y = 2/x^2$?

DISCUSSION

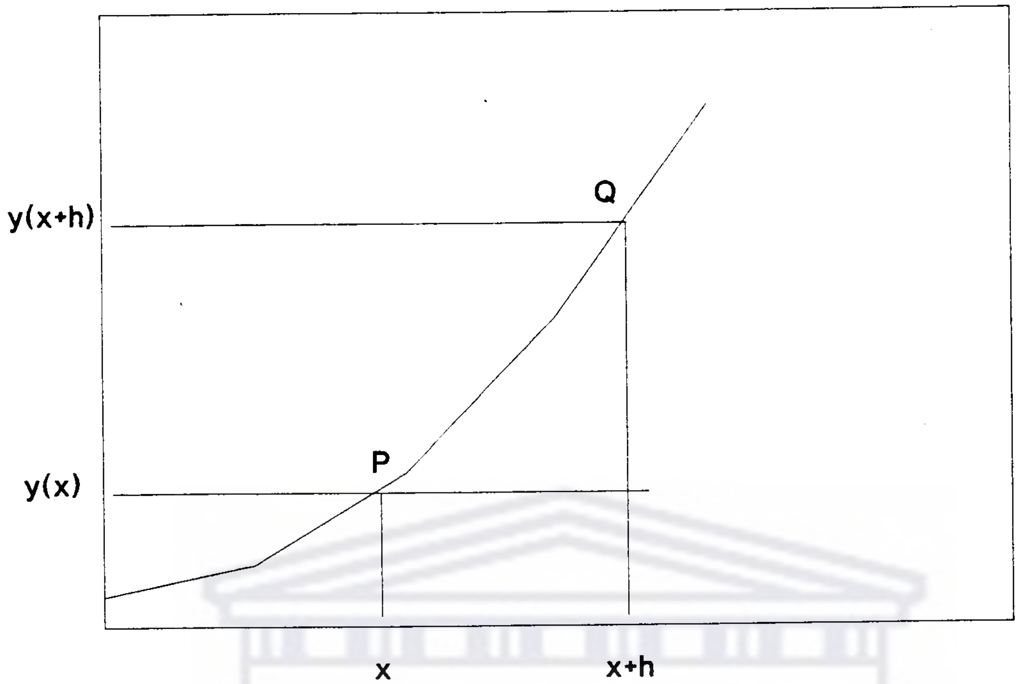
Tasks (5.1) and (5.2) are typical problems found in mathematics at first year level at technikons. The student retrieves the rules of differentiation frame. The higher order rules could be production rules which are the 'if-then' rules. For example, if the student is confronted with a polynomial mathematical expression then he should first identify it as a polynomial. This is a higher order rule. The lower order rule will elicit the polynomial rule for differentiation.

TASK 6

The diagram below is used to introduce the definition of the derivative, viz., $dy/dx = \lim_{h \to 0} [y(x+h)-y(x)]/h$ in engineering mathematics, where y is any function and h is an increment in x.

(6.1) At which point or points of the graph does the formula measure the rate of change?

(6.2) Explain why the formula defines this rate of change?

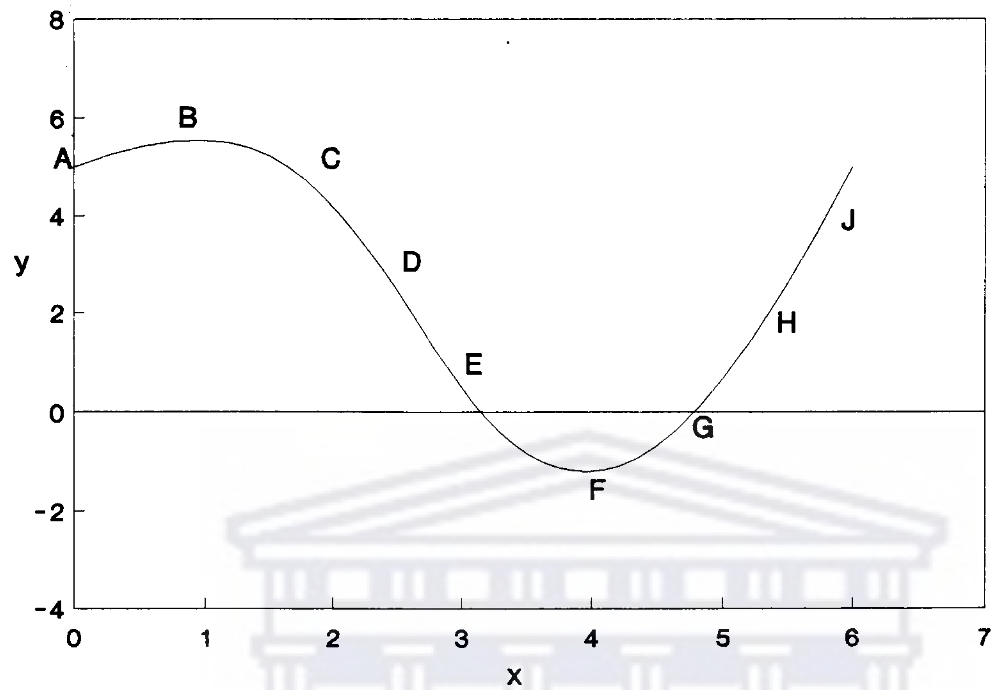


DISCUSSION

In the pilot study I discovered that students thought that the above curve was an exponential curve. This led me to rephrase the problem statement as 'the diagram below represents a general curve'. The frame to be retrieved could be the sequential secant tending towards a tangent to the curve at a point and the slope of the tangent is a representation of a rate of change at that point. Furthermore the rate of change represents differentiation.

TASK 7

The graph of y for a certain equation, for $x = 0$ to $x = 6$, is shown.



What is the average rate of change of y with respect to x,

(7.1) From A to B?

(7.2) From B to E?

(7.3) From A to J?

DISCUSSION

The main aim of this task is that the idea of rate of change can be introduced in the sinusoidal wave which is often encountered by engineering students. The frame to be retrieved is that average rate of change can be calculated from any two points irrespective of the curve.

TASK 8

Explain the meaning of each of the following symbols:-

(8.1) δx,

(8.2) δy,

(8.3) $\delta y/\delta x$,

(8.4) dx

(8.5) dy

(8.6) dy/dx

(8.7) What is the relationship between $\delta y/\delta x$ and dy/dx ?

DISCUSSION

The understanding of various symbols used in connection with differentiation was tested. The frame retrieved gives a meaning to each symbol and the relationship between the symbols and related concepts such as differentiation or limits or average rate of change. The symbols can be seen as pointers to other concepts which are descriptors of another concept. For example, the symbol $\delta y/\delta x$ is a pointer for rate of change which is the descriptor for the derivative.

TASK 9

In each of the following, calculate the rate change at the point indicated, and explain the significance of your answer:-

(9.1) $y = x^2-4x+1$ at $x=1$,

(9.2) $y= x^2-4x+1$ at $x=2$,

(9.3) $y = 1/x$ at $x=0$.

DISCUSSION

This task tested understanding of zero and negative derivatives in relation to the graph. It also tested whether the students could obtain the derivatives. Part (9.3) is also encountered in engineering. For example, in electronics the student is required to calculate the rate of change of current with respect to the resistance of the alternating current resonance circuit. The equation is $i=E/R$ where E is the constant voltage, i is the current and R is the resistance of the circuit. The circuit experiences zero resistance at resonance. Therefore $i=E/R$ in this situation is similar to $y=1/x$ at $x=0$, where $y=i$, $1=E$ and $x=R$.

The retrieved frame incorporated 'rules for differentiation', calculate a value for the derivative by substituting it into the derivative equation, using rules to interpret the values for the derivative at a certain point.

TASK 10

Find the coordinates of the point or points on the curve

$$y = x^3 - 3x^2 + 4$$

at which there is a turning point or stationary point. Determine also what kind of point you have found.

DISCUSSION

This is a typical mathematics problem which are used to test applications of differentiation at first year technikon level. The frame retrieved must be able to draw curve, use first differentiation and second differentiation higher order rules to calculate turning points and stationary points.

4.5 THE ITEMS AND THE TASKS

In several tasks similar types of skills and concepts were required. Responses to appropriate subdivisions of the tasks were re-grouped to form items (Orton,1983). Each item related to just one aspect of elementary differential calculus. The result of the re-grouping was twelve items. The following is the summary of the tasks and items.

ITEMS AND RELATED TASKS

Item no.	Item Description	Related Tasks
1	Infinite geometric sequences	1.1
2	Limits of geometric sequences	1.2
3	Substitution and increases from equations	3.1;3.2;3.3 4.1;4.2;4.3
4	Rate of change from straight line graph	2 3.5
5	Rate, average rate and instantaneous	3.4 4.4;4.5
6	Average rate of change from curve	7.1;7.2;7.3
7	Carrying out differentiation	5.1;5.2
8	Differentiation as a limit	6.1;6.2
9	Use of the δ-symbolism	8.1;8.2;8.3;8.4;8.5 8.6;8.7
10	Significance of rates of change from differentiation	9.1;9.2;9.3
11	Gradient of tangent to curve by differentiation	10
12	Stationary points on a graph	10

4.6 CONCLUSION

Forty five subjects were chosen for the study. Using the clinical method, responses to a selection of Orton's tasks were elicited. Orton's tasks on differentiation were listed and discussed as to relevance and type of the frame retrieved. The tasks were then itemised according to required skills and concepts. There were 12 items listed in the table above.

In the following chapter the responses of the students to these tasks will be analyzed according to Donaldson's classification of errors.

CHAPTER 5

ANALYSIS OF DATA

In chapter 4 the methodology and design of this study were discussed. The tasks for the students were also discussed. The present chapter deals with the analysis of the data of the study. At the M L Sultan Technikon mathematical concepts in differential calculus are conveniently divided into four sections: (i) elementary algebra, (ii) rate of change, (iii) limits and infinity, and (iv) differentiation. Errors demonstrated by the students in these sections will be classified in terms of structural, executive and arbitrary errors. Examples of students' work are presented in APPENDIX.

5.1 ELEMENTARY ALGEBRA

A considerable proportion of the algebraic content of the tasks was concerned with graphs of functions. Item 3 concerns finding function values for small increments in the independent variable. Item 3 consists of tasks 3.1, 3.2, 3.3, 4.1, 4.2 and 4.3 (see chapter 4, p35-37). Item 12 consists of task 10 (see chapter 4, p42). The table below indicates the items and the number of errors made by the students in terms of structural, executive and arbitrary errors.

CLASSIFICATION OF ERRORS IN ELEMENTARY ALGEBRA

ERRORS

Item	Structural	Executive	Arbitrary
3	10	6	3
12	13	5	3

In what follows tasks from items 3 and 12 will be used as exemplars for the various types of errors made by students.

STRUCTURAL: TASK 4.3

$y= 3x^2+1=3((a+h)-a)+1$	[incorrect representation for
$= 3(a+h-a)+1$	Δy by not calculating
$= 3(h)+1$	function at a+h and a...
$=3h+1$	structural error]

Another exemplar is the 'zero product principle error' which illustrates Davis' 'rules creating rules' (Davis,1984).

$x^3-3x^2+4=0$

$x^3- 3x^2=-4$

$x^2(x-3)=-4$

$x^2=-4$

or $x=-4+3$

$x=\sqrt{-4}$

or $x=-1$...

EXECUTIVE: TASK 3.3

$y = 3a-1-(3a+3h-1)$	[Δy and not y ...
$= 3a-1-3a-3h+1$	and change in y taken from f(a) to
$= -3h$	f(a+h)...executive error]

ARBITRARY: TASK 4.3

$\Delta x=(3a^2+3h^2+1)-(3a^2+1)$	[Δy and not Δx...
$=3a^2+3h^2+1-1-3a^2-1$	arbitrary error]
$=3h^2$	

For the tasks under item 12 see chapter 4 task 10 p42-43.

STRUCTURAL: TASK 10

$y = x^3-3x^2+4$

$y'=3x^2-6x$

$y'=0$

$3x^2-6x=0$

$x^2-2x=0$

$x(x-2)=0$

$x=0$

or $x=2$

$y''=6x-6$

[Incorrect use of the second derivative test... structural error]

$6x-6=0$

$6x=6$

$x=1$

$y(1)=x^3-3x^2+4$

$=1-3+4$

$=2$

(1;2)

for $x=0$ $y=4$ (0;4)

maximum for $x=2$

$y=8-12+4=0$ (2;0)

minimum for $x=0$

$y''=6x-6=-6$

$y''(2)=6(12)-6=66.$

EXECUTIVE: TASK 10

$3x^2-6x=0$

$3x^2=6x$

$x^2=2x$

$x=2$

[...finds minimum and maximum point but cannot determine which is maximum or minimum...omits one root viz., 0...lack of control...executive error]

ARBITRARY: TASK 10

$3x^2-6x=0$

$x(3x-3)=0$

$x=0$

$3x-3=0$

$3x=3$

$x=1.$

[...finds maximum or minimum points but cannot determine which is maximum and minimum point, writes 3 instead of 6...arbitrary error]

5.2 RATE OF CHANGE

Three tasks on graphs, two linear relationships (tasks 2 and 3) and one quadratic relationship (task 4) (see chapter 4, p34-37), were considered to be important. Graphs form an important basis for the study of rate of change in differentiation. Task 7 (see chapter 4, p39-40) provided further questions on the rate of change. Task 2 is an elementary real life rate of change problem. The table below indicates the items and the corresponding number of structural, executive and arbitrary errors made by the students.

CLASSIFICATION OF ERRORS ON RATE OF CHANGE

ERRORS

Item	Structural	Executive	Arbitrary
4	32	3	2
5	24	6	-
6	10	1	-

In the item 4 (see chapter 4, p34-35) the following are the exemplars of the structural, executive and arbitrary errors.

STRUCTURAL: TASK 2

$m=y_2-y_1/x_2-x_1$

$=4-2/2-1$

$=2/1$

$m=2$

$y=mx+c$

$4=2(2)+c$

$c=0$

$y=2*\frac{1}{2}$

$y=2.$

$\therefore$the rate of increase is

two times when $x=2\frac{1}{2}$ ...

When x=T

y=2T

∴ rate will be 2T. [takes y as rate of change...structural error]

EXECUTIVE: TASK 2

rate =5,3/T [divides estimated y-value at 2½ with T...but cannot give an exact answer...executive error]

ARBITRARY: TASK 2

... =T

y=2T

At x=2½

x=T the graph is a straight line so that the rate of change is [no numerical answer was given... omits the constant from equation...arbitrary error]

In item 5 (see chapter 4, p36-37) the following are examples of the structural and executive errors.

STRUCTURAL: TASK 4

Average=

$(3a^2+6ah+3h^2+1+3a^2+1)/2$

[incorrect formula for average...structural error]

EXECUTIVE: TASK 4

$f'(x)=\lim_{h\to 0} 3a^2+6h+3h^2+1-3a^2-1/h$

$=\lim_{h\to 0} 6h+3h^2/h$

$=\lim_{h\to 0} 6+3h\ 6+3(0)\ =6$

[incorrect squaring of a+h...executive error]

In item 6 (see chapter 4, p39-40) the following are exemplars of structural and executive errors.

STRUCTURAL: TASK 7

0-6=-6

[change in y incorrect ...omitted change in x...structural error]

EXECUTIVE: TASK 7

...to use the average

rate of change in y

in the x-interval a

to a+h, we may use

the differentiation

$\Delta y = \lim_{h \to 0} f(a+h)-f(a)/h$

$= \lim_{h \to 0} 3(a+h)^2+1-(3a+1)/h$

$= \lim_{h \to 0} 3a^2+6ah+3h^2+1-3a^2-1/h$

$= 6a$ [...change in y taken as differentiation...omits change in x...executive error]

5.3 LIMITS AND INFINITY

Limits are important for a conceptual understanding of differentiation. Items 1 and 2 were based on the limits of the infinite sequences of Task 1 (see chapter 4, p32-33). The idea of a sequence of secants through a fixed point resulting in a sequence of non-fixed points associated with the sequence of secants approaching the fixed point was intended to relate to the definition of a derivative of a function. It was considered to be an important task in giving further evidence concerning the level of understanding of the tangent as a limit. The table below indicates the items and the corresponding number of structural and executive errors made by the students.

CLASSIFICATION OF ERRORS ON LIMITS AND INFINITY

	ERRORS	
Item	Structural	Executive
1	2	19
2	2	3

In the item 1 (see chapter 4, p32-33) the following are examples of the structural and executive errors.

STRUCTURAL: TASK 1

$2\pi r$ circumference -can
be drawn to all parts. [takes the circumference to mean secant...structural error]

EXECUTIVE: TASK 1

179 [takes secant as being angle in the semi circle...executive error]

In the item 2 (see chapter 4, p32-33) the following are exemplars of the structural and executive errors.

STRUCTURAL: TASK 1

the length of the

secant increases [structural error]

EXECUTIVE: TASK 1

The secant disappears

and a perpendicular is

formed. [executive error]

5.4 DIFFERENTIATION

The concept of "limit" and "the rate of change" both appeared in the tasks on differentiation. Items 7, 8, 9, 10, 11, and 12 (see chapter 4 p37-43 for tasks and p44 for grouping into items) were used to examine and explain the errors. The table below gives the number of structural, executive or arbitrary errors in each of the above items made by students.

CLASSIFICATION OF ERRORS ON DIFFERENTIATION

Items	Structural	Executive
7	8	1
8	39	3
9	124	2
10	13	-
11	9	4
12	3	-

In item 7 the following are exemplars of structural and executive errors.

STRUCTURAL: TASK 5.1

$y=2/x^2$...

$dy/dx = 2/2x$. [differentiated the denominator...incorrect rule...structural error]

Further exemplars of the structural error are given below:

TASK 5.1: (1) $y=x^n$...$dx/dt=x^n$[associates rate of change with the variable time... structural error]

(2) $y=x^n$...nlogx... [interprets n as a variable rather

than an integer… structural error]

TASK 5.2: (1) $y=3x^3$…

$dx/dt=3x^3$…$y=4$…$y=2/x^2$…

$y=2/x^2$…$dx/dt=2/x^2$

[associates rate of change variable time… structural error]

(2) $y=3x^3=\ln 3+3\ln x$…

$y=4$…$y=0$

…$y=2/x^2=2x^{-2}=-2\log 2x$.

[whenever a variable is raised to a power it is operated on by logarithms …structural error]

(3) $3=y=3x^3$…$y=4$…

$2=y=2/x^2$

[takes the rate of change as the constants…structural error]

EXECUTIVE: TASK 5.2

$dy/dx=-2/x^3$

[omits the constant 2 …executive error]

A further exemplar is

TASK 5.1: $y=x^n$…$\log y=\log x^n$…

$\log y=n\log x$…$y=3x^3$…

$\log y=3\log x^3$

[uses logarithms on all functions... seem to be stuck on a rule... executive error]

In item 8 the following are exemplars of structural and executive errors.

STRUCTURAL: TASK 6.2

It is because point :Y+K depends on Y and x+h depends on x ∴rate of y upon x is y+k-y/x+h-x.

[takes average rate of change as rate of change... structural error]

EXECUTIVE: TASK 6.1

(1) measured from point x→(x+h)

[omits sequence of secant...structural error]

(2) The formula compares the change in x and the change in y as the x-variable tends to zero ie. gets smaller.

[omitted the h tends to zero ... executive error]

In item 9 the following are examples of structural and executive errors.

STRUCTURAL

TASK 8.3

derivative of

function/equation wrt x

[takes average rate of change as the derivative which is the rate of change... structural error]

TASK 8.4:

derivative in

respect to x

[takes part of the symbol for the derivative as being the derivative of x...structural error]

TASK 8.5:

derivative in

respect to y

[takes part of the symbol for the derivative as being the derivative of y...structural error]

TASK 8.7:

dy/dx is derivative

of $\delta y/\delta x$

[derivative is taken as the average

...structural error]

EXECUTIVE

TASK 8.1:

function of equation

in respect to x [takes change in x for function of x ...executive error]

TASK 8.2:

function of equation

in respect to y [takes change in y as function... executive error]

In item 10 the following are example of structural errors.

STRUCTURAL:

TASK 9.3

At $x=0$ $y'=0$ [1/0 is taken 0...structural error]

Further exemplars are given below:

TASK 9.3: (1) rate of change $=0$. [1/0 is taken as 0... structural error]

(2) $dy/dx=\ln x=\ln 0$ [takes the antiderivative as the derivative ...structural error]

(3) $y=x^{-1}=-1x^{-2}$...

$-1(0)^{-2}=0$. [dividing by 0 is taken as

	0...structural error]
(4) $y'(0) = 0$.	[dividing by zero is taken as zero...structural error]
(5) $dy/dx=0$	[dividing by zero is taken zero...structural error]
(6) $f'(0) = -(0)^{-1} = 0$.	[dividing by zero is taken as zero...structural error]
(7) $dy/dx = \ln x = \ln 0$	[takes the antiderivative as the derivative...structural error]

In item 11 the following are examples of structural and executive errors.

STRUCTURAL: TASK 10

$d^2y/d^2x = 6x-6 = 6(3)-6$

$m=12$. [takes the second derivative as being the tangent at $x=3$...structural error]

Another exemplar is:

TASK 10: Gradient of the tangent

$=y=x^3-3x^2+4=(3)^3-3(3)^2+4$

$=9-27+4=-14$ [y value taken as rate of change...structural error]

EXECUTIVE: TASK 10

$dy/dx=3x^2+6x$ ...at $x=3$....

$dy/dx = 3(9)+18=27+18=45$. [omitted the negative sign...executive error]

In item 12 the following is the structural error

STRUCTURAL: TASK 10

Turning points

$d^2y/dx^2=0$ $2x-3$

$=0$ $x=3/2$ $\therefore$

where $x=3/2$:

$y=(3/2)^3-3(3/2)^2+4$

$=3,375-6,75+4$

$=0,625$. [uses the second derivative for the turning points instead of the first derivative...structural error]

5.5 CONCLUSION

The error types were more structural than executive or arbitrary with the exception of the classification of errors on limits where there were more executive errors than structural ones. It was in the structural error that students show an obvious misunderstanding in the concept of the sequence of secants fixed at one point approaching the tangent at the fixed position on the circle. Structural errors were particularly noticeable in the section on rate of change where the differentiation symbols caused the students some difficulty.

A number of students divided by zero and gave a result of zero. Few students retrieved the Davis's classical error of the 'zero product principle'.

In the following chapter I will conclude the thesis by connecting the analysis with the relevant literature and make recommendations for teaching differentiation at technikons.

CHAPTER 6

CONCLUDING REMARKS AND RECOMMENDATIONS

Some of the general factors that may contribute to the students' improper understanding of differentiation at the technikon could be the following: (i) lack of importance attached to mathematics as a service course, (ii) students have little time per semester to study mathematics and little time to study differential calculus as an analysis component, and (iii) classroom practices of lecturers.

The tasks performed by students for this thesis formed a suitable base from which research into students' understanding of elementary differential calculus has been carried out. A study of errors was performed.

Many of the errors were structural and executive. A small proportion was arbitrary. In the set of tasks used for this study, students made more structural errors than executive errors. This might be attributed to students tendency to rote learning the elementary differential calculus. These structural errors indicate that students do not understand differentiation principles.

Students did experience a number of serious difficulties. A core of three items was found to be difficult in the sense that even "good" students i.e., students getting less than two items wrong, could not cope with them. These are items 4, 8 and 9. With these three items we investigated an understanding of differentiation based on rate of change, differentiation as a limit and the use of δ symbolism, all of which constitute a real stumbling block in learning elementary differentiation. Over 60% of the students could not grasp at least one of these important steps in learning differentiation. This finding indicates that differentiation is a complex concept. Differentiation becomes a complex concept in that it relies upon other concepts for its understanding. For example, differentiation relies upon rate of change, limits and δ symbolism for its definition. Therefore cognitive processes should be taken into consideration.

In chapter 5 it was clear that many students experienced many difficulties with algebra such as indices, logarithms, squaring of a binomial. In item 3 many students experienced great difficulties with substitution and increases in y and x values in equations. Arithmetic, which is the foundation for algebra, seems to be a problem for the students. For example, many students divided by zero. Many of the frames retrieved by the students were sketchy or incomplete. Some of the frames retrieved were the Davis's classical error 'rules creating rules'. It is doubtful that the introduction to differentiation can be completely meaningful in conventional lectures if the algebra within studies of gradient, rate of change of functions and sketching of graphs for maximum and minimum points, and inflection points still causes problems. Some students had conceptual difficulties with algebra. They could not carry out the procedures they had in mind without error. There are mathematical technology software packages such as Maple, Derive and Mathematica which automatically treat the algebraic procedures and allow the student to concentrate on the notion of the slopes of secant lines 'becoming' the slope of the tangent line.

Rate of change needs to be studied intensely. Many students experienced great difficulties with the items 4, 5 and 6 involving rate of change. For example, students could not distinguish between rate of change, average rate of change and Δy. It appeared in many responses by students that difficulties which might at first seem to be associated with understanding differentiation, might be attributed to insufficient attention given to the study of rate of change.

Students made more executive errors than structural errors in items 1 and 2 which concern the concept limit. This implied that students encountered difficulties with the control of the sub-assemblies associated with the concept of the limit and infinity. The limit is used in the definition of the derivative at a point which is often interpreted as the limit of a sequence of slopes of secant lines approaching a tangent at the same point. Perhaps using Dick's (1989) 'epsilon-delta of scaling ...using graphing technology in calculus' and Strang's (1989) procedure 'centering and zoom...graphing technology' may alleviate the pressure of understanding the limit concept in the context of differentiation.

There were twenty times more structural errors than executive errors in item 9. This suggests that great care needs to be taken when introducing the notation of differential calculus. Students found the notations $\delta y/\delta x$, dy and dx and the relationship between $\delta y/\delta x$ and dy/dx difficult.

It is in the graphs of tasks 3, 4, 6, and 7 and the diagram of task 1 that students experienced difficulties. In particular at the technikon level the graphical approach could be used more frequently. Hughes-Hallet (1989) mentioned the fact that students can differentiate complicated functions analytically but cannot interpret differentiation graphically. She further advocates the rule of three for the learning of differentiation viz., algebraic, numerical and graphical methods. Eventhough the technikon syllabus encourages the use of graphical approach it is usually neglected. In items 10, 11 and 12 a number of structural errors were noted. The results from items 10, 11 and 12 support Hughes-Hallet (1989) assertion. It indicated that a number of students could not interpret the derivative graphically although they could differentiate the functions analytically. This finding is also consistent with Artique and Viennot (1988), Alibert (1988), Vinner (1982) and Robert and Boschet (1984). The behaviour of the curve should grow out of studying the graph of the curve in the interval. This could prevent the use of algorithmic procedures.

As stated earlier a major concern that flows from this study is students' difficulties with basic algebra and its impact on understanding differentiation. Errors in algebra were structural, executive and arbitrary. Booth (1984) captures the nature of the students' errors in her study in a different form when she reports that students have difficulty with algebra because they don't understand: (i) what letters mean, (ii) the various notations in algebra, (iii) the basic properties of real numbers which are the foundations of algebraic manipulation, (iv) that a response does not have to be a number and (v) that you can express the same value or relationship in more than one way. These errors were also observed in this study. Lins (1992) considers that thinking arithmetically is a component of thinking algebraically. Some students divided by zero and got a value of zero in item 10. These errors will further exacerbate the problems in algebra and consequently differential calculus.

Further research is required to determine whether technikon students would understand differentiation to a greater degree if they use the computer techniques as expounded by Strang (1989) and Dick (1989). Computer tasks could be designed to foster student construction of the differentiation concept. It permits students to learn a host of techniques for calculation by performing them quickly and easily on the computer. Visual representations of functions are constructed automatically. This may motivate students to acquire a deeper meaning of the mathematical principles underlying differential calculus. Consequently structural errors may be minimised.

While this study concentrated on identifying and analyzing the errors students made in the tasks performed, it may be useful to analyze all items the students performed correctly. Such an analysis may indicate some unique thought processes acquired and demonstrated by the students. It may also give the lecturer some insight into future structures of the lesson plan.

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APPENDIX

ELEMENTARY ALGEBRA

Student 12 Item 3 **Structural**

$y = 3x^2 + 1$ his x increase from a to $a+h$

$\therefore a+h-a$

$= h$

$= 3((a+h)-a)+1$

$= 3(a+h-a)+1$

$= 3(h)+1$

$= 3h+1$

Student 16 Item 3 **Executive**

$y = 3x - 1 - (3x + 3h - 1)$

$= 3x - 1 - 3x - 3h + 1$

$= -3h$

Student 22 Item 3 **Arbitrary**

$\Delta x = (3a^2 + 3h^2 + 1) - (3a^2 + 1)$

$= 3a^2 + 3h^2 + 1 - 3a^2 - 1$

$= 3h^2$

Student 13 Item 12 **Structural**

Test 17

$y = x^3 - 3x^2 + 4$

$y' = 3x^2 - 6x$

$y' = 0$

$3x^2 - 6x = 0$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ or $x = 2$

$y'' = 6x - 6$

$6x - 6 = 0$

$6x = 6$

$x = 1$

$y(1) = x^3 - 3x^2 + 4$

$= 1 - 3 + 4$

$= 2$

$(1; 2)$

for $x = 0$

$y' = 0$

$0 = 0$

$y = 4$

$(0; 4)$

~~Minimum T.P~~

Maximum T.P

for $x = 0$

$y'' = 6x - 6 = -6$

$y''(2) = 6(12) - 6 = 66$

gradient of the tangent when $x = 3$

$y' = 3(3)^2 - 6(3)$

$= 27 - 18$

$= 9$

Student 17 Item 12 **Executive**

turning point $\frac{dy}{dx} = 3x^2 - 6x$

$\therefore$ solve for

$\therefore 3x^2 - 6x = 0$

$3x^2 = 6x$

$x^2 = 2x$

$x = 2$

Student 43 Item 12 **Arbitrary**

$y' = x^3 - 3x^2 + 4$

$= 3x^2 - 6x = 0$

$= x(3x-3) = 0$

$= \underline{x = 0} \quad 3x - 3 = 0$

$3x = 3$

$x = 1$

Student 17 Item 5 **Structural**

$$\text{average} = \frac{3a^2 + 6ah + 3h^2 + 1 + 3a^2 + 1}{2}$$

$$= \frac{6a^2 + 6ah + 3h^2 + 2}{2}$$

$$= 3a^2 + 2ah + \frac{3}{2}h^2 + 1 \quad = ?$$

Student 16 Item 5 **Executive**

$$f'(a) = \lim_{h \to 0} \frac{3a^2 + 6h^2 + 3h^2 + 1 - 3a^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{6h^2 + 3h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(6 + 3h)}{h}$$

$$\lim_{h \to 0} 6 + 3h$$

$$6 + 3(0)$$

$$= 6$$

Student 5 Item 6 **Structural**

vii $\quad 0 - 6 = -6$

Student 24 Item 6 **Executive**

$$\Delta Y = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Average rate of change $= \frac{\Delta Y}{\Delta x}$

$$= \lim_{h \to 0} \frac{3(a+h)^2 + 1 - (3a+1)}{h}$$

$$= \lim_{h \to 0} \frac{3a^2 + 6ah + 3h^2 + 1 - 3a^2 - 1}{h}$$

$$= \underline{\quad 6a \quad}$$

LIMITS AND INFINITY

Student 22 Item 1 **Structural**

2πr circumference – can be drawn to all
(i)

Student 27 Item 1 **Executive**

$\boxed{(a) = 179}$

Student 7 Item 2 **Structural**

(b) the length of the secant increases.

Student 6 Item 2 **Executive**

b) THE SECANT DISAPPEARS AND A PERPENDICULAR IS FORMED

DIFFERENTIATION

Student 21 Item 7 **Structural**

$$y = 2/x^2$$

$$\frac{dy}{dx} = \frac{2}{2x}$$

Student 33 Item 7 **Executive**

(v) $dy/dx = \frac{-2}{x^3}$

Student 21 Item 8 **Structural**

Point "y+k" depends on "y" and "x+h

rate of "y" upon "x" is $\frac{y+k-y}{x+h-x}$

Student 34 Item 8 **Executive**

Measured from point x → (x+h)

The formula compares the change in x

Student 5 Item 9 **Structural**

Derivative of function/equation with respect to x.

Student 5 Item 9 **Executive**

function of equation in respect to u

Student 35 Item 10 **Structural**

$$y = x^{-1}$$
$$= -1x^{-2}$$
$$-1(0)^{-2}$$
$$= 0.$$

Student 29 Item 11 **Structural**

$$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 6x - 6$$
$$= 6(3) - 6$$
$$m = 12$$

Student 30 Item 11 **Executive**

$\frac{dy}{dx} = 3x^2 + 6x$ plug in $x = 3$

$\frac{dy}{dx} = 3(9) + 18) = 27 + 18$ at $x = 3$

$= 45$

Student 14 Item 12 **Structural**

Turning pts. $\frac{d^2y}{dx^2} = 0$

$$2x - 3 = 0$$
$$x = 3/2$$

where $x = 3/2$: $y = (3/2)^3 - 3(3/2)^2 + 4$

$= 3{,}375 - 6{,}75 + 4$

$= 0{,}625$