A mathematical model for managing equity-linked pensions

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Declaration

I declare that *A mathematical model for managing equity-linked pensions* is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Elmerie Julie December 2006

Signed: ....

Acknowledgement

I dedicate this mini-thesis to everybody that assisted me in the completion of my mini-thesis and to my family and friends who offered me encouragement and support. I wish to express my thanks and appreciation to my supervisor, Professor P. Witbooi, for the guidance and contributions he gave me. I would also like to acknowledge my fellow students for their support.

Abstract

Pension fund companies manage and invest large amounts of money on behalf of their members. In return for their contributions, members expect a benefit at termination of their contract. Due to the volatile nature of returns that pension funds attain, pension companies started attaching a minimum guaranteed amount to member's benefits. In this mini-thesis we look at the pioneering work of Brennan and Schwartz [10] for pricing these minimum guarantees. The model they developed prices these minimum guarantees using option pricing theory. We also look at the model proposed by Deelstra *et al.* [13] which prices minimum guarantees in a stochastic financial setting. We conclude this mini-thesis with new contributions where we look at simple alternative ways of pricing minimum guarantees. We conclude this mini-thesis with an approach, related to the work of Brennan and Schwartz [10], whereby the member's benefit is maximised for a given minimum guaranteed amount, which comprises of multi-period guarantees. We formulate a method to find the optimal stream of these multi-period guarantees.

Key words: pension fund, defined benefit, defined contribution, minimum guarantee, maximum benefit, return on investment, sharing rule in pension funds, call option, put option, Lagrangian.

Chapter 1

Introduction

Mathematical modelling is widely applied in pension fund management. Pension fund management involves the investment of huge amounts of funds and thus pension funds are subject to general investment practices. On the other hand there are also specific needs and specialised functions that distinguish them from other funding systems. Pension funds need to ensure that they receive enough contributions and make wise investment decisions over the life of the fund to cover the benefits the fund must pay out in the future. Pension funds also have legal requirements to adhere to and are only allowed to invest in certain asset types and must maintain adequate asset-liability ratios. Thus there are methods especially developed for pension funds. One class of problems that occur frequently in pension funds are the optimisation problems. Here we can mention contributions such as on optimal risk management practices by Josa-Fombellida and Rincón-Zapatero [24], the optimal form of the minimum benefit guarantee such as Deelstra, Grasselli and Koehl [13], optimal investment choices during the accumulation phase such as Huang and Cairns [21] and Deelstra *et al.* [14], investment of the benefit subsequent to retirement of a member from the fund by Gerrard, Haberman and Vigna [18] and the optimal management with regard to solvency level of the fund, for example Petersen, Raubenheimer and van Schuppen [31].

Scope of this mini-thesis

This mini-thesis is divided into three distinct parts. Each section focuses on pricing minimum guarantees under different scenarios and using different mathematical models.

In particular in the first two parts we shall look in detail at two important instances, which are presented in a paper by Brennan and Schwartz [10] and another approach to minimum guarantees as presented in the paper by Deelstra *et al.* [13]. These two papers are not unrelated but they differ in the following sense. In Deelstra *et al.* [13] the presentation is in terms of stochastic calculus and stochastic optimisation but in the older Brennan and Schwartz paper [10] the methods are deterministic, with stochasticity captured in the Black-Scholes formula, which is key to the paper. Thus presentation of two such papers allow for a good display of the relevant methods.

In the final part we present new work on the pricing of minimum guarantees. We end off this mini-thesis by finding a sequence of guarantees that maximises the expected benefit the contributor receives when he/she leaves the fund.

PART I

The Brennan and Schwartz model for pricing guaranteed minimum benefit

The first part of this mini-thesis focuses on the model for pricing minimum guarantees in equity-linked insurance policies proposed by Brennan and Schwartz [10]. We give an overview of the pioneering work of Brennan and Schwartz [10] on the pricing of minimum guarantees on the benefit. The papers by Brennan and Schwartz [10] and Boyle and Schwartz [9] are considered to be the ground-breaking papers on equity-linked funds with a minimum guarantee. Numerous research papers on equity-linked funds are based on these two papers. In both papers the authors show that a portion of the benefit payable under equity-linked contracts can be modelled as either call or put options. Hence, the benefits can be priced using well-known option pricing theory.

In chapter 2 we give a brief introduction to equity-linked funds and option pricing techniques since this forms the basis for the work in this mini-thesis. Chapter 3 is a literature review on equity-linked funds. All the papers researched for this mini-thesis consider the basic model proposed by Brennan and Schwartz [10] but modify certain assumptions which Brennan and Schwartz [10] make in their model. In chapter 4 we discuss the Brennan and Schwartz model in some detail and we conclude this section with computations based on the Brennan and Schwartz model. We observe how the results produced in chapter 4 of this part is consistent with the properties known about option prices.

Chapter 2

Equity-linked funds

Two extreme types of pension contracts exist: defined benefit and defined contribution. Defined benefit pension contracts ensures that the contributor knows at the beginning of the contract what his or her benefit will be at the end of the contract. However, the contributions the member makes to the pension fund, to pay for the benefit, will vary over the life of the contract. This variability of the contributions is due to the variability of external factors such as interest rates, mortality rates, etc..

This is in contrast to defined contribution funds where the contributor makes fixed contributions to the fund. The benefit that the contributor receives at the end of the contract is stochastic and depends on the investment return achieved by the pension fund over the life of the contract. The drawback of defined contribution funds is that the contributor bears all the risk of the investment decisions made by the pension fund manager. To reduce some of the investment risk faced by the contributor, the pension fund ensures a minimum guaranteed amount as part of the benefit to the contributor.

A defining feature of equity-linked contracts is that the benefit payable at expiration of the contract is linked to a financial asset or a reference portfolio. A portion of the contributions paid by a member of an equity-linked pension fund is used to purchase stock, which then makes up the reference portfolio. The benefit that the contributor then receives depends on the market value of this portfolio at expiration of the contract. This random benefit is a distinguishing feature of equity-linked contracts. The minimum guarantee then offers protection against the policyholder's benefit being too low. The minimum guarantee is attached to the benefit to minimise the investment risk the policyholder is exposed to. The contributor has to pay a premium above the normal contribution to cover the cost of this minimum guarantee. Hardy [20] gives a comprehensive discussion on equity-linked funds and investment guarantees.

Since the Brennan and Schwartz [10] model uses option pricing techniques to price the benefits in equity-linked funds we end this chapter with a definition of basic options. For a more comprehensive discussion on options and other derivatives and how they are priced the reader is referred to Hull [22].

A derivative security or contract is one that derives its value from the price of an underlying asset at maturity of the derivative contract. A call option on an asset gives the contract holder the right, but not the obligation, to purchase the asset at a predetermined price, called the exercise price. At a specified time or over a specified time period, put options give the holder the right, but not the obligation, to sell the underlying asset at the exercise price. European options can only be exercised on the expiration date as opposed to American options that can be exercised on any date prior to expiration.

Chapter 3

Literature review of equity-linked funds

Numerous authors have gone on to modify the model proposed by Brennan and Schwartz [10] by changing some of the assumptions that Brennan and Schwartz made. Despite the fact that the model proposed by Brennan and Schwartz [10] is considered to be the poineering work on equity-linked funds, a major disadvantage to the model is that their model assumes constant interest rates. This is unacceptable and unrealistic due to the long term nature of pension funds. The main problems that authors have subsequently considered are: the type of guarantee paid out, whether a deterministic or stochastic interest rate is considered and whether the contributor pays a single premium or makes periodic payments to the fund.

In their paper, Nielsen and Sandmann [29] consider stochastic interest rates and periodic premiums. The benefit they consider is a function of the history of the spot price of the stock throughout the life of the contract. Therefore, the life-insurance contract contains an embedded Asian option and is priced accordingly. In another paper [30] they derive a model where the contributor receives a bonus in addition to the guaranteed amount depending on the investment returns over the period of the contract. This bonus is priced as an Asian option.

In their paper, Bacinello and Ortu [1] first build a model that extends the original work of Brennan and Schwartz [10] to include interest rate risk. Here the reference portfolio is an all-equity fund. They then go on to consider the case where the reference portfolio is made up of fixed income assets. They use stochastic interest rates with a single premium policy of the endowment type. In another paper, Bacinello and Ortu [2] consider a model where the reference portfolio comprises of interest rate sensitive assets. They distinguish between the case where the payoffs received from the reference portfolio are reinvested and where they are not reinvested. They conclude by providing numerical results to assess the effect of various parameters in their model.

Bacinello and Persson [3] propose a model for a periodic premium endowment policy under stochastic interest rates. The guarantee they consider is expressed as a number of units in the reference portfolio. This implies that the benefit is a number of units of the reference portfolio and not a fixed monetary amount. The authors consider such a contract because they feel it reflects real world contracts. Also, a distinguishing feature of this contract is the closed form solutions that is found for the periodic premium contract. They compare the model they developed with the one originally proposed by Brennan and Schwartz [10]. They conclude with a numerical analysis discussing how the various parameters affect their model and how their model compares with Brennan and Schwartz's [10] model.

All the papers considered, in this literature review, concentrates on the financial side of the contract and mortality is usually considered to be diversified away. In his paper, Jacques [23], considers a single contract so that mortality cannot be diversified. He tries to close the gap between pricing using financial theory and actuarial methods.

Since Brennan and Schwartz [10] is considered the pioneering paper on equity-linked funds with an asset value guarantee, in the next chapter we discuss their model.

Chapter 4

The Brennan and Schwartz model

In their paper, Brennan and Schwartz [10] considered an equity-linked policy with an asset value guarantee that expires at a known time t. The contributor pays regular premiums to the insurance company to receive a benefit at expiration of the contract. A portion of the premiums is then invested in equity (stock), which then makes up the reference portfolio. The benefit that the contributor will receive, at maturity of the contract, will then be the greater of the reference portfolio and a minimum guaranteed amount.

Define the following variables:
$X(t)$: value of the reference portfolio at time t,
$G(t)$: minimum guarantee at time t,
$B(t)$: the benefit payable at time t.

Then the benefit paid out at time t is

$$\max[G(t), X(t)] \tag{4.1}$$

This decomposes to

$$\begin{aligned} B(t) &= G(t) + \max[X(t) - G(t), 0] && (4.2) \\ &= X(t) + \max[G(t) - X(t), 0] && (4.3) \end{aligned}$$

If we look at the second term on the right side of equation (4.2), $\max[X(t) - G(t), 0]$, is the payoff the holder of a European call option receives, where $X(t)$ is the value of the underlying asset and $G(t)$ is the exercise price at time t. The underlying asset is therefore the reference portfolio and the strike price is the minimum guarantee. Similarly for equation (4.3), $\max[G(t) - X(t), 0]$ is the payoff the holder of the put option receives. The initial cost of purchasing the option is ignored.

Equation (4.2) defines the benefit the contributor receives as he/she receiving the minimum guarantee and the payoff from holding a call option on the reference portfolio. Equation (4.3) describes the benefit as receiving the payoff from receiving the value of the reference portfolio and holding a put option on the reference portfolio.

Define further:
$C(t, G(t))$: value of a call option at time 0, that expires at time t, with strike price $G(t)$,
$P(t, G(t))$: value of a put option at time 0, that expires at time t, with strike price $G(t)$,
$V_0X(t)$: discounted time 0 value of the reference portfolio,
$V_0B(t)$: discounted time 0 value of benefit that is payable at time t,
r: risk-free interest rate.

Corresponding to equations (4.2) and (4.3) the discounted values are as follows respectively:

$$V_0B(t) = G(t)\exp(-rt) + C(t, G(t)) \quad (4.4)$$

$$= V_0X(t) + P(t, G(t)) \quad (4.5)$$

This is the price (and hence the premium) the contributor must pay to receive the benefit defined by equations (4.2) and (4.3) respectively. Looking at equation (4.5), $V_0X(t)$ represents the part of the premium the contributor must pay for receiving the value of the reference fund at maturity of the contract. Therefore, $P(t, G(t))$ represents the extra premium the contributor must pay for having the security of the minimum guarantee attached to his/her policy.

Equating equations (4.4) and (4.5) leads to

$$P(t, G(t)) = G(t)\text{exp}(-rt) + C(t, G(t)) - V_0 X(t) \tag{4.6}$$

Therefore to derive the extra premium the insurance company must charge for providing the minimum guarantee, one just needs to derive the value of the call option and then solve equation (4.6).

Assume the dynamics of the reference portfolio can be described by the following stochastic differential equation

$$\frac{dX(t)}{X(t)} = \mu dt + \sigma dW_t$$

where μ is the expected return on the portfolio, σ is the volatility of the portfolio, and dW_t is the Standard Brownian Motion. Assume further that we are working with a single premium contract, then deriving the value of the call option is equivalent to the well-known Black-Scholes problem of pricing an option on a non-dividend paying stock.

Based on these assumptions the well-known solution for the call option in equation (4.6) is

$$C(t, G(t)) = X(0)N(d_1) - G(t)\text{exp}(-rt)N(d_2) \tag{4.7}$$

where

$$d_1 = \frac{\text{ln}X(0) - \text{ln}G(t) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \text{exp}(-\frac{1}{2}u^2)du \qquad \text{is the cumulative normal distribution.}$$

For the derivation of the Black-Scholes formula see [6], [22], [34].

Now that we have the formula for the price of a call option in equation (4.7), we can use equation (4.6) to solve for the extra premium the contributor must pay for receiving the minimum guarantee.

Alternatively, one can use the following put price formula to calculate the extra premium that is charged for the guarantee:

$$P(t, G(t)) = \Big(G(t)\exp(-rt)\Big)\Big(1 - N(d_2)\Big) - X(0)\Big(1 - N(d_1)\Big).$$

The scenario described above assumes that the contributor makes a single payment. No unique solution can be found in the case of periodic premiums and numerical methods are used to find the solution.

Chapter 5

Calculations based on the Brennan and Schwartz model

In the calculations that follows we only consider single premium contracts and will not consider the effects of mortality. The computations in this chapter are done using Matlab. For more details on how the programme works and an introduction to Matlab see Hahn [19]. To do computations on equity-linked contracts using the Black-Scholes model we must specify a risk-free interest rate and the volatility or variance rate of the portfolio. For each example these are specified in the tables. As mentioned previously, the assumption of constant interest rates is unrealistic. However, we assume constant interest rates for ease of computations.

In table (5.1) various results for intermediate time periods are displayed for what we will consider our base case in comparisons. The first column shows the time to maturity of the contract and therefore each line represents the values that would be applicable to contract of that duration. The second column shows the premium that the contributor pays to the fund. For our base case we assume that the premium is 100 units irrespective of the time to maturity of the contract. This is the amount that is invested in the reference portfolio. In the base case we assume that the guaranteed amount, shown in column 3, on

$T-t$	Premium	Guarantee	Call	Put	Contribution
1	100	100	7.50	3.58	103.58
5	100	100	22.31	4.19	104.19
10	100	100	36.26	3.29	103.29
15	100	100	47.53	2.41	102.41
20	100	100	56.80	1.73	101.73

Table 5.1: Put prices charged for guarantees with $\sigma = 0.01846$ and $r = 4\%$

the contract is 100% of the premium, in the second column. The third and fourth columns shows the call and put prices for the contracts, and is calculated using equations (4.7) and (4.6) respectively. The put column is the values that is of most interest to us since this is the extra premium the fund must charge the member to have the guaranteed amount attached to his/her contract. The final column is the total contribution the member pays to the fund and is the sum of columns 2 and 5.

Figure (5.1) shows the effect that changes in the interest rate and the time to maturity has on the put values (the extra premium charged). We use a term to maturity of 40 years, so that the effect can be seen more clearly, and a volatility rate of 0.01846. We consider interest rates of 8%, 4% and 1% with a premium and guarantee of 100 units. From the graph we see that as the term of the contract increases the value of the put increases for a while and then starts to decrease. It however reaches a limiting value and only for high interest rates and longer time periods does the put value decrease to zero. We also observe that the smaller the interest rate the longer it takes before the put price starts decreasing. If we look at one point in time the graph shows us that as the risk-free interest rate increases the value of the put decreases.

From graph (5.2), which shows the sensitivity of the model to changes in variance, we see that as the volatility increases the put prices increase. For this graph we assumed a term

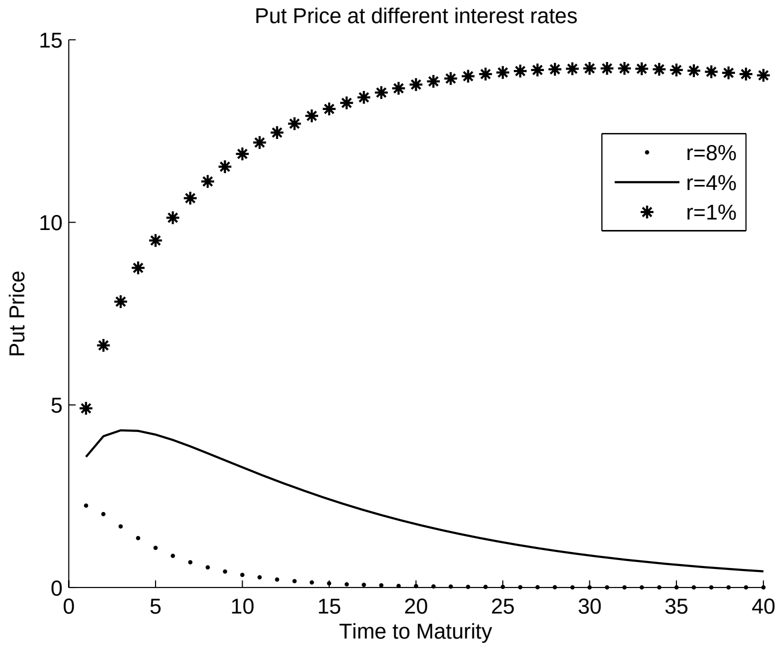

Figure 5.1: Put Prices at different interest rates

to maturity of 1 year, a risk-free interest rate of 4% and a guarantee and premium of 100 units.

The observations drawn from the table and graphs above are in line with well-known properties about changes in option prices due to changes in the various parameters.

Variance	Put Price
0.01	2.26
0.02	3.79
0.03	4.98
0.04	6.00
0.05	6.91

Table 5.2: Put prices for a one period contract at different volatility rates

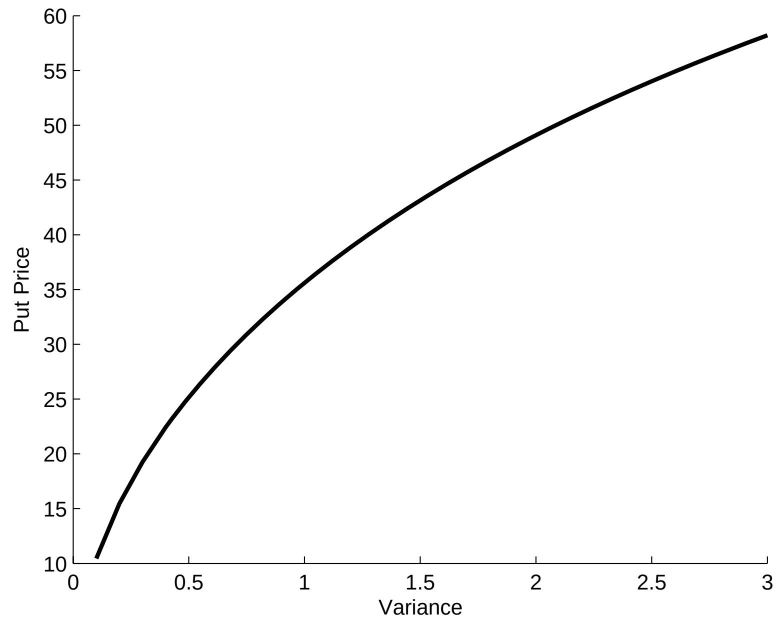

Figure 5.2: Put Price at different variance rates

Table (5.2) shows the value of the put option for a one period contract for increasing variance rates. We see that the put price increases at a decreasing rate and it increases at a smaller rate than the increases in the volatility. For table (5.2) we use the same parameters as in graph (5.2).

In the above calculations we assumed a constant premium and guarantee irrespective of the term to maturity of the contract. It would be more realistic to consider premiums and guarantees that increases as time to maturity increases. This is what we consider in the following section. We first consider a premium that increases linearly with time and then we consider a premium that increases by a factor of time.

Table (5.3) shows the put prices for a linearly increasing premium and guarantee. The guarantee is still 100% of the premium. Table (5.3) shows that the premium one pays for the guarantee still increases with time at first, then starts to decrease as in table (5.1)

$T-t$	Premium	Guarantee	Call	Put	Contribution
1	100	100	7.50	3.58	103.58
5	500	500	111.56	20.93	520.93
10	1000	1000	362.57	32.89	1032.89
15	1500	1500	712.98	36.20	1536.20
20	2000	2000	1136.01	34.67	2034.67

Table 5.3: Results with guarantee that increases linearly with time and $\sigma = 0.01846$ and $r = 4\%$

which shows the base case. However, the amount of the put price is much higher when the premium increases linearly.

Next we assume the premium increases linearly with time but that the guaranteed amount increases exponentially with time. The results are displayed in table (5.4).

$T-t$	Premium	Guarantee	Call	Put	Contribution
1	100	195.12	0.00	87.47	187.47
5	500	889.40	9.45	237.62	737.62
10	1000	1606.53	140.77	217.66	1217.66
15	1500	2208.55	447.53	159.61	1659.61
20	2000	2735.76	881.99	111.25	2111.25

Table 5.4: Premium increasing linearly but guarantee increasing exponentially and $\sigma = 0.01846$ and $r = 4\%$

The guarantee is calculated as: guarantee=premium(t)+premium(t)*(exp(alpha*(t))), where alpha is a function of time. We observe that the put price still increases but decreases as a proportion of the total contribution as time increases. This is in contrast to what is observed in the base case in table (5.1).

$T-t$	Base case	Linear	Exponential
1	3.45	3.45	46.66
5	4.02	4.02	32.21
10	3.19	3.19	17.88
15	2.35	2.35	9.62
20	1.70	1.70	5.27

Table 5.5: Put price as a percentage of the contribution

Table (5.5) shows the values for the put price as a percentage of the contribution, for the different guarantees considered, namely a constant guarantee (base case), a guarantee increasing linearly and a guarantee increasing exponentially. We observe that for the constant guarantee and the guarantee that increases linearly the percentages are exactly the same. The case where the guarantees increase exponentially is always significantly higher than in the other two cases.

PART II

The work of Deelstra *et al.* on minimum guarantees

This part looks at the model for the optimal form of the minimum guarantee proposed by Deelstra *et al.* [13]. The analysis is fully stochastic in nature and looks at stochastic optimisation and utility maximisation. Their model incorporates a sharing rule, that splits the surplus of the pension fund between the contributor and pension fund manager. The surplus of the pension fund is the difference between the value of the fund at the end of the contract, and the guarantee. The contributor receives a portion as part of his/her benefit and the pension fund manager takes the remainder as compensation for managing the fund.

We begin this part with a description of the financial model of [13] and assumptions that are used in this part of the mini-thesis. We also discuss the dynamics of the pension fund and the surplus process of the fund. In the last chapter of this part we look at a utility maximisation problem as in [13]. We look to find the minimum guarantee such that the utility the contributor derives from his/her final benefit is maximised.

Chapter 6

Financial setting assumptions

Let us introduce the model for pension fund management as of Deelstra *et al.* [13]. We assume there are $n + 1$ assets in the financial market we are working in: n risky assets and 1 risk-free asset. The risky assets considered will be non-dividend paying stock, and the risk-free asset will be similar to a bank account. We assume that the market is arbitrage-free to ensure that no investor can make a profit without taking some form of risk. We also assume that the market is complete which means that every contingent claim is attainable and can be priced. We work in a continuous time setting with $t \in [0, T]$ for $0 < T < \infty$.

The fact that the stock prices change randomly over very short periods of time is captured by the n-dimensional Brownian Motion $W(t) = (W_1(t); W_2(t); \ldots; W_n(t))'$. All the information generated by the Brownian Motion (and hence the stock prices) until time t is captured in the filtration $\mathcal{F}(t)$. This filtration satisfies the usual conditions of being right continuous and complete. For a detailed discussion on filtrations see [27].

The Brownian Motion is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the probability space and $\mathcal{F}$ is a σ-algebra. $\mathbb{P}$ denotes the real world probability of an event, say $A \in \mathcal{F}$ occurring. See Bass [4] for more on probability spaces.

6.1 The risk-free asset

The price of the bank account at time t is $P_0(t)$. If one unit of money is invested in the bank account at time 0, then $P_0(t)$ evolves according to

$$\frac{dP_0(t)}{P_0(t)} = r(t)dt \qquad P_0(0) = 1, \tag{6.1}$$

where $r(t)$ is the instantaneous interest rate. This is the continuous interest rate earned if you deposit money in the bank account now and withdraw the money an instant later.

Solving equation (6.1) leads to the value of the bank account at time t as follows:

$$P_0(t) = \exp(\int_0^t r(s)ds) \tag{6.2}$$

In equation (6.2) if $r(t)$ is assumed non-negative then equation (6.2) is an increasing function. Therefore, the bank account's value can only increase and it is considered risk-free. Since $r(t)$ is $\mathcal{F}_t$ measurable the interest rate is always known at time t [26].

6.2 The risky assets

The price process of the stock is modelled as an Itô process:

$$dP_i(t) = P_i(t)\Big[b_i(t)dt + \sum_{i=1}^{n} \sigma_{ij}(t)dW_j(t)\Big], \qquad P_i(0) > 0 \qquad i\ ,\ j = 1, \ldots, n. \tag{6.3}$$

In equation (6.3) the randomness is captured by the Brownian Motion $W(t)$. Volatility measures how uncertain we are about the price movements of the stock, therefore, $(\sigma_{ij}(t))$ indicates how the j-th source of uncertainty affects stock i at time t. The volatility matrix of the stock is represented by $\sigma = (\sigma_{ij}(t))$ and the returns (drift process) of the various stock is represented by $b(t) = (b_1(t), \ldots, b_n(t))'$.

Equation (6.3) states that the change in the price of the stock depends on that stock's return over that period and the random volatility of that stock. The assumption is made that $r(t)$, $b(t)$ and $\sigma(t)$ are progressively measurable with respect to $\mathcal{F}_t$ and $\sigma(t)$ is invertible.

Using Itô's lemma, one finds the solution to equation (6.3) to be

$$P_i(t) = P_i(0)\text{exp}\Big[\sum_{j=1}^{n}\int_0^t \sigma_{ij}(s)dW_s^j + \int_0^t \Big(\mu_s^i - \frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2(s)\Big)ds\Big]$$

6.3 The exponential process

Since the market is arbitrage-free and complete there is a unique process

$$\theta(t) = \sigma^{-1}(t)[b(t) - r(t)\mathbf{1}_n] \qquad \mathbf{1_n} = (\mathbf{1}, ..., \mathbf{1})^{\mathbf{n}} \in \mathbb{R}^{\mathbf{n}} \qquad [25].$$

This represents the market price of risk or is referred to as the relative risk.

Theorem 6.1

Define the exponential process

$$Z(t) = \text{exp}\Big[-\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\Big].$$

Then Z(t) is a local martingale.

Proof

Let $L(t) = -\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t ||\theta'(s)||^2 ds$

Then $dL(t) = -\theta'(t)dW(t) - \frac{1}{2}||\theta'(t)||^2 dt$

Now $Z(t) = \text{exp}(L(t))$

By the famous Itô's formula

$$\begin{aligned} dZ(t) &= \text{exp}(L(t))dL(t) + 0dt + \frac{1}{2}\text{exp}(L(t))(-\theta'(t))^2 dt \\ &= Z(t)dLt + \frac{1}{2}Z(t)\theta'(t)^2 dt \end{aligned}$$

$$
\begin{aligned}
&= Z(t)[-\theta(t)dW(t) - \frac{1}{2}\theta^2 dt] + \frac{1}{2}Z(t)\theta^2(t)dt \\
&= -Z(t)\theta(t)dW(t) - \frac{1}{2}Z(t)\theta^2 dt + \frac{1}{2}\theta^2 Z(t)dt \\
&= -\theta(t)Z(t)dW(t).
\end{aligned}
$$

Therefore $Z(t)$ is an Itô integral and hence a local martingale under the real world probability, $\mathbb{P}$. $\square$

This implies that

$$\mathbb{Q}(A) = \mathbb{E}[Z(T)1_A] = \int_A Z(t)d\mathbb{P} \qquad A \in \mathcal{F}(t)$$

is the risk-neutral equivalent martingale measure.

To ensure there is no arbitrage in the market the state-price density process $h(t)$ is introduced

$$h(t) = \frac{Z(t)}{P_0(t)} = \exp\Big[-\int_0^t r(s)ds - \int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\Big]$$

For a more detailed and technical explanation of the various topics covered in this chapter the reader is referred to [5], [16], [25], [26], [27] .

Chapter 7

Dynamics of the pension fund

A member makes an initial contribution of X_0 to the fund at time 0 when the member has a defined contribution contract and then pays periodic contributions at a continuous rate *c(t)* over the life of the contract. This contributions process is a square integrable, non-negative progressively measurable process, i.e., $\int_0^T c^2(t)dt < \infty, a.s.$

Based on this the discounted value of all contributions that the contributor makes is the initial contribution he/she makes plus the discounted value of all the contributions still to be paid by the contributor.

$$X_0' = X_0 + \mathbb{E}\Big[\int_0^T h(s)c(s)ds\Big] \tag{7.1}$$

The pension fund manager receives a portion of the surplus as compensation for managing the pension fund and therefore wants to maximise the portion of the fund he/she receives. The manner in which the surplus of the fund is divided is determined by the sharing rule. We assume that the fund manager receives the proportion β of the surplus. The spectrum of sharing rules provides a continuum between defined benefit funds and defined contribution funds and allows transfer of risk between the contributor and the pension fund manager.

With $\beta = 0$, the fund manager receives no compensation and with $\beta = 1$, the contributor receives no portion of the surplus. These are trivial cases so we only consider $\beta \in (0,1)$.

Further, for the pension fund manager to make a profit we assume that the discounted value of the contributions is greater than the discounted value of the guarantee at termination of the contract.

$$X_0' > \mathbb{E}[h(T)G(T)] a.s.$$

The pension fund manager has control over the portion he/she receives by the investment decisions he/she makes. If we consider that the pension fund manager invests the proportion $\pi_i(t)$ of the funds he/she has available in the risky asset i such that $\sum_{i=1}^{n} \pi_i(t) = \pi(t)$ is the total proportion invested in the risky assets, then $1-\pi(t)$ is the proportion invested in the riskless asset.

Changes in the value of the fund, over an infinitesimal time period, are described by the following dynamics:

- the proportion invested in the riskless asset will change by the interest rate $X(t)(1-\pi(t))r(t)dt$
- the proportion invested in the risky asset will change due to the stock return and the volatility of the stock $X(t)\pi(t)b(t)dt + X(t)\pi(t)\sigma(t)dW(t)$
- the new premium is held for a short period of time, after which it gets absorbed into the fund. Then it is subject to the dynamics described above.

The fund therefore follows the following stochastic differential equation:

$$\begin{aligned} dX(t) &= X(t)(1-\pi(t))r(t)dt + X(t)\pi(t)b(t)dt + X(t)\pi(t)\sigma(t)dW(t) + c(t)dt \\ &= \Big(X(t)r(t) + c(t) + X(t)\pi(t)\Big(b(t) - r(t)\Big)\Big)dt + X(t)\pi(t)\sigma(t)dW(t). \end{aligned}$$

Chapter 8

The surplus process

In Deelstra *et al.* [13] the *surplus process* of the pension fund at time t for $t \geq 0$ is defined as consisting of the value that the reference portfolio has accumulated to at time t, plus the discounted value of contributions still to be received from time t till the end of the contract less the value of the minimum guarantee that would be paid out at that time t. Steffensen [33] looks at the dynamics of the surplus process in the life insurance context.

Define the following variables:
$X(t)$: the value of the reference portfolio,
$D(t)$: the expected discounted value of the remaining contributions,
$G(t)$: the expected discounted value of the guarantee,
$Y(t)$: the value of the surplus.

Then using the above definition

$$Y(t) = X(t) + D(t) - G(t),$$

where

$$D(t) = \mathbb{E}_t \int_t^T \frac{h(s)}{h(t)} c(s) ds$$

$$G(t) \;=\; \mathbb{E}_t\Big[\frac{h(T)}{h(t)}G(T)\Big]$$

The value of the surplus at time 0 is the value of the initial contribution plus the expected discounted value of all the contributions that will be paid by the contributor less the expected discounted value of the guarantee, i.e.,

$$\begin{aligned} Y(0) &= X(0) + D(0) - G(0) \\ &= X_0 + \mathbb{E}_0 \int_0^T \frac{h(s)}{h(0)} c(s)ds - \mathbb{E}_0\Big[\frac{h(T)}{h(0)}G(T)\Big] \\ &= X_0 + \mathbb{E}\Big[\int_0^T h(s)c(s)ds\Big] - \mathbb{E}\Big[\frac{h(T)}{h(0)}G(T)\Big] \\ &= X_0' - \mathbb{E}[h(T)G(T)]. \end{aligned} \tag{8.1}$$

In the second line we are taking the conditional expectations under the real-world probability. This expectation is conditional upon the information generated till the indicated time point. Since we are taking the expectation at time 0, no information has been generated and therefore the normal expectation is taken. The last line uses X_0' which is defined in equation 7.1 as the discounted value of all contributions the contributor will make.

Theorem 8.1

There exists a random Y(t) process such that

$$\frac{dY(t)}{Y(t)} = \Big[r(t) + y'(t)\Big(b(t) - r(t)\Big)\Big]dt + y'(t)\sigma(t)dW(t)$$

Proof

Define $\tilde{K}(t) = h(t)K(t)$, then $d\tilde{Y}(t) = d\tilde{X}(t) + d\tilde{D}(t) - d\tilde{G}(t)$.

From the price process of the stock, the state price density process and the process of the reference portfolio, the change in the discounted value of the fund is defined as follows

$$d\tilde{X}(t) = \Big(\pi'(t)\sigma(t) - \theta'(t)\Big)dW(t) + \tilde{c}(t)dt \tag{8.2}$$

Let $D_1(t) = D(t) + \int_0^t \frac{h(u)c(u)}{h(t)}du$, then

$$
\begin{aligned}
\mathbb{E}_s[h(t)D_1(t)] &= \mathbb{E}_s\Big[h(t)\mathbb{E}_t\Big(\int_t^T \frac{h(u)}{h(t)}c(u)du\Big) + h(t)\int_0^t \frac{h(u)c(u)}{h(t)}du\Big] \\
&= \mathbb{E}_s\Big[\mathbb{E}_t\Big(\int_t^T h(u)c(u)du\Big) + \int_0^t h(u)c(u)d(u)\Big] \\
&= \mathbb{E}_s\Big[\mathbb{E}_t\Big(\int_t^T h(u)c(u)du\Big) + \mathbb{E}_s\Big(\int_0^t h(u)c(u)du\Big)\Big] \\
&= \mathbb{E}_s\Big(\int_t^T h(u)c(u)du\Big) + \mathbb{E}_s\Big(\int_0^t h(u)c(u)du\Big) \\
&= \mathbb{E}_s\Big(\int_0^T h(u)c(u)du\Big) \\
&= \mathbb{E}_s\Big[\int_0^s h(u)c(u)du + \int_s^T h(u)c(u)du\Big] \\
&= \int_0^s h(u)c(u)du + \mathbb{E}_s\Big[\int_s^T h(u)c(u)du\Big] \\
&= h(s)\int_0^s \frac{h(u)}{h(s)}c(u)du + h(s)\mathbb{E}_s\Big[\int_s^T \frac{h(u)c(u)}{h(s)}du\Big] \\
&= h(s)\int_0^s \frac{h(u)}{h(s)}c(u)du + h(s)D(s) \\
&= h(s)\Big[\int_0^s \frac{h(u)}{h(s)}c(u)du + D(s)\Big] \\
&= h(s)D_1(s).
\end{aligned}
$$

Therefore $h(t)D_1(t)$ is a $\mathbb{P}$ martingale. By the Brownian motion martingale representation theorem, there exists a unique process $\zeta(t)$ such that

$$\tilde{D}_1(t) = D_0 + \int_0^t \zeta'(t)dW(t).$$

Then,

$$
\begin{aligned}
d\tilde{D}_1(t) &= \zeta'(t)dW(t) \\
d\tilde{D}(t) &= -\tilde{c}(t)dt + \zeta'(t)dW(t) \qquad (8.3)
\end{aligned}
$$

Similarly,

$$\mathbb{E}_s[h(t)G(t)] = \mathbb{E}_s\Big[h(t)\mathbb{E}_t\Big(\frac{h(T)}{h(t)}G(T)\Big)\Big]$$

$$
\begin{aligned}
&= \mathbb{E}_s\Big[\mathbb{E}_t\Big(h(T)G(T)\Big)\Big] \\
&= \mathbb{E}_s\Big[h(T)G(T)\Big] \\
&= h(s)\mathbb{E}_s\Big[\frac{h(T)}{h(s)}G(T)\Big] \\
&= h(s)G(s).
\end{aligned}
$$

Therefore, $h(t)G(t)$ is a $\mathbb{P}$ martingale and applying the Brownian motion martingale representation theorem, there exists a unique process $\rho(t)$ such that

$$\tilde{G}(t) = G(0) + \int_0^t \rho'(t)dW(t).$$

Then

$$d\tilde{G}(t) = \rho'(t)dW(t) \tag{8.4}$$

Combining equations 8.2, 8.3 and 8.4 we get

$$
\begin{aligned}
d\tilde{Y}(t) &= d\tilde{X}(t) + d\tilde{D}(t) - d\tilde{G}(t) \\
&= \tilde{X}(t)\Big(\pi'(t)\sigma(t) - \theta'(t)\Big)dW(t) + \tilde{c}(t)dt \\
&+ \Big(-\tilde{c}(t)dt + \zeta'(t)dW(t)\Big) - \rho'(t)dW(t) \\
&= \Big[\tilde{X}(t)\Big(\pi'(t)\sigma(t) - \theta'(t)\Big) + \zeta'(t) - \rho'(t)\Big]dW(t).
\end{aligned} \tag{8.5}
$$

The following equation solves the stochastic differential equation defined in equation 8.5:

$$Y(t) = Y(0)\exp\Big\{\int_0^t \Big[r(s) + y'(s)\Big(b(s) - r(s)\Big) - \frac{1}{2}||y'(s)\sigma(s)||^2\Big]ds + \int_0^t y'(s)\sigma(s)dW(s)\Big\}$$

This completes the proof of the proposition. □

Therefore,

$$Y(T) = Y(0)\exp\Big\{\int_0^T \Big[r(s) + y'(s)\Big(b(s) - r(s)\Big) - \frac{1}{2}||y'(s)\sigma(t)||^2\Big]ds + \int_0^T y'(s)\sigma(s)dW(s)\Big\},$$

with Y(0) defined in equation 8.1.

Define

$$\exp\Big\{\int_0^T\Big[r(s)+y'(s)\Big(b(s)-r(s)\Big)-\frac{1}{2}||y'(s)\sigma(t)||^2\Big]ds+\int_0^T y'(s)\sigma(s)dW(s)\Big\}=\varphi,$$

then $Y(T)=Y(0)\varphi=\Big(X_0'-\mathbb{E}[h(T)G(T)]\Big)\varphi.$

Now,

$$\mathbb{E}[h(T)Y(T)]=Y(0) \tag{8.6}$$

and

$$\mathbb{E}[h(T)\varphi]=1. \tag{8.7}$$

Chapter 9

Utility maximisation

The contributor receives only a portion of the surplus Y_T, as part of his/her benefit, because the pension fund manager takes the remainder as compensation for managing the fund. The surplus at time T is the difference between the reference fund and minimum guarantee. The assumption is made that the fund manager only receives his compensation at the end of the contract.

The contributor's benefit is then made up of the minimum guarantee and the portion he/she receives from the surplus. If β is the proportion of the surplus that the fund manager takes, then

$$\begin{aligned} B_T &= G_T + (1-\beta)(X_T - G_T) \\ &= G_T + (1-\beta)Y(T) \end{aligned}$$

is the total benefit the contributor receives at the end of the contract.

This implies that

$$G_T = B_T - (1-\beta)Y(T). \tag{9.1}$$

When choosing a contract, the contributor only has control over the type of guarantee he/she chooses. This is because the proportion the pension fund manager takes as compensation is fixed and the value of the reference portfolio is unknown at the time he/she chooses a contract. Therefore, when choosing a contract the contributor will choose the contract that offers him/her the best guarantee.

9.1 Utility Maximisation Problem

On deciding what contract offers the optimal guarantee, the contributor will choose a contract such that the expected utility he/she receives from the final benefit is maximised. Utility is a measure of the satisfaction that the contributor will receive from the benefit. For the guarantee to add value to the pension contract, the minimum guarantee has to be greater than zero. The expected value of the benefit is calculated by discounting the benefit in the risk-neutral probability measure. The optimal guarantee therefore depends on the expected value of the benefit. The optimal guarantee will therefore lie in a set such that there is a solution for the expected value of the benefit.

The discounted expected value of the benefit the contributor receives will be the time 0 value of the expected contributions less the time 0 value of the manager's portion of the surplus.

$$k = X_0' - B_0^M,$$

where k is the time 0 value of the benefit and B_0^M is the time 0 value of the manager's portion.

From the above discussion the contributor's maximisation programme is defined in problem 9.1.

Problem 9.1

The contributor's maximisation programme is

$$\text{Maximise}_{\{G_T\}}\mathbb{E}\Big[u\Big(G_T + (1-\beta)(X_T - G_T)\Big)\Big] \tag{9.2}$$

subject to

$$\begin{aligned} k &= \mathbb{E}\Big[h(T)\Big(G_T + (1-\beta)(X_T - G_T)\Big)\Big] \\ \mathbb{G} &= \{G_T : \exists k \in [0, X_0'[\text{s.t. } G_T \text{ is a solution of k}\} \\ G_T &\geq 0 \end{aligned}$$

□

The set $\mathbb{G}$ is used in solving for k. This problem cannot be solved so we need to transform the problem so that the problem can be solved using calculus of variations. To transform the problem we use features about the manner in which the pension fund is managed and features about the surplus process. This was covered in chapters 7 and 8 respectively. Using the properties from those two chapters we transform the contributor's maximisation programme next.

Expression (9.2) can also be written as $\mathbb{E}\Big[u\Big(G_T + (1-\beta)(Y_T)\Big)\Big]$ and the first constraint as $k = \mathbb{E}\Big[h(T)\Big(G_T + (1-\beta)(Y_T)\Big)\Big]$

Looking at the constraint this can be simplified as follows:

$$\begin{aligned} k &= \mathbb{E}\Big[h(T)\Big(G_T + (1-\beta)(Y_T)\Big)\Big] \\ k &= \mathbb{E}\Big[h(T)\Big(G_T + (1-\beta)\Big(\Big(X_0' - \mathbb{E}[h(T)G(T)]\Big)\varphi\Big)\Big)\Big] \\ k &= \mathbb{E}[h(T)(1-\beta)X_0'\varphi] - \mathbb{E}[h(T)(1-\beta)\varphi\mathbb{E}[h(T)G_T] \\ &+ \mathbb{E}[h(T)G_T] \\ k - \mathbb{E}[h(T)(1-\beta)X_0'\varphi] &= \mathbb{E}[h(T)G_T][1 - \mathbb{E}[h(T)(1-\beta)\varphi] \\ \mathbb{E}[h(T)G_T] &= \frac{k - \mathbb{E}[h(T)(1-\beta)X_0'\varphi]}{1 - \mathbb{E}[h(T)(1-\beta)\varphi]} \end{aligned}$$

$$\mathbb{E}[h(T)G_T] = \frac{k-(1-\beta)X_0'}{\beta}$$

The last equality holds due to (8.7). Now,

$$\begin{aligned} Y(T) &= \Big[X_0' - \mathbb{E}[h(T)G(T)]\Big]\varphi \\ &= \varphi\Big[X_0' - \Big[\frac{k-(1-\beta)X_0'}{\beta}\Big]\Big] \\ &= \varphi\Big[X_0' - \frac{k}{\beta} + \frac{X_0'}{\beta} - \frac{\beta X_0'}{\beta}\Big] \\ &= \varphi\Big[\frac{X_0'-k}{\beta}\Big] \end{aligned}$$

This changes the benefit defined in equation (9.1) to

$$G(T) + \varphi\Big[\frac{X_0'-k}{\beta}(1-\beta)\Big]$$

and the guarantee to

$$B(T) - (1-\beta)\varphi\Big[\frac{X_0'-k}{\beta}\Big]$$

This changes the maximisation programme defined in the problem to

$$\text{Maximise}_{\{G_T\}}\mathbb{E}\Big[u\Big(G_T + (1-\beta)\varphi\Big(\frac{X_0'-k}{\beta}\Big)\Big)\Big]$$

subject to

$$\begin{aligned} k &= \mathbb{E}\Big[h(T)(G_T + (1-\beta)\varphi\Big(\frac{X_0'-k}{\beta}\Big)\Big] \\ \mathbb{G} &= \{G_T : \exists k \in [0, X_0'[\text{s.t. } G_T \text{ is a solution of k}\} \\ G_T &\geq 0 \end{aligned}$$

This is further simplified to

$$\text{Maximise}_{\{B_T\}}\mathbb{E}[u(B(T))]$$

subject to:

$$\begin{aligned} k &= \mathbb{E}[h(T)(B(T)] \\ B(T) &\geq (1-\beta)\varphi\Big[\frac{X_0'-k}{\beta}\Big] \end{aligned}$$

This problem is solved in Deelstra *et al.* [13] using calculus of variations.

PART III

New contributions

10. Maximising the expected benefit for a given minimum guarantee on the benefit

In the final section of this mini-thesis we include some original contributions.

In conclusion, as a novelty we present a particular discrete optimisation problem, which maximises the expected benefit for a given sequence of member contributions and a given minimum guarantee on the benefit. We provide illustrative computational examples on this particular problem which shows that the member can find a sequence of guarantees that will result in him/her receiving a maximum benefit.

Chapter 10

Maximising the expected benefit for a given minimum guarantee on the benefit

In the final chapter of this mini-thesis we try to find a strategy that will maximise the benefit a pension fund member will receive given that he/she makes regular defined contributions to the fund and has a given minimum guarantee attached to his/her pension contract.

Pension fund companies started attaching minimum guarantees to contracts to ensure that members still earned a return on their investments, even when the pension fund achieved poor investment returns, and to minimise the investment risk that contributors are exposed to. In this mini-thesis we looked at the pricing of these guarantees and we saw that these guarantees add value to the contract. In the first section of this mini-thesis we concentrated on the model proposed by Brennan and Schwartz [10] where these minimum guarantees are priced as put options. These put prices are the additional premiums that contributors pay to the fund for the minimum guarantee being attached to their contract.

In their paper Brennan and Schwartz [10] considered maturity guarantees whereby the minimum guarantee that is attached to the contract is only applicable at the termination of the contract. Other authors, for instance, Lindset [28] have developed models that price multi-period guarantees since they argue that minimum guarantees are applicable at the end of each period and not only at the end of the contract. In this chapter of this mini-thesis we try to find the optimal sequence of incremental guarantees that will maximise the expected benefit the contributor will receive.

10.1 The optimisation problem

We work in the time period $0 = t_0, t_1, t_2, ..., t_{n-1}, t_n = T$, and the multi-period incremental guarantees $g_0, g_1, g_2, ..., g_{n-1}$ we consider add up to the minimum guarantee G. We consider the problem whereby we want to maximise the expected benefit $\bar{B}$ the contributor receives, with the minimum guarantee G attached to the contributor's contract, comprised of multi-period guarantee increments $g_0, g_1, g_2, ..., g_{n-1}$. We assume that the contributor makes fixed contributions $c_0, c_1, c_2, ..., c_{n-1}$ to the fund at times $t_0, t_1, t_2, ..., t_{n-1}$. These contributions c_i splits as $c_i = p_i + x_i$ for $i = 1, ..., n-1$. The amount x_i is the effective contribution, and it is only this amount that is invested in the member's portfolio. The remainder p_i is the premium that the contributor pays for the minimum guarantee. Under the assumption that the dynamics of the pension fund follows a geometric Brownian Motion, the p_i's are the prices of European put options as explained in chapter 4 of this mini-thesis. The strike price of the put option over the period $[t_i, T]$ is g_i, and the (time t_i)-value of the risky asset is x_i. We price the option using the Black-Scholes formula, therefore $p_i = BSP(T - t_i, g_i)$. The expected benefit $\bar{B}$ that we want to maximise is then derived from these effective contributions that are invested in the fund. The expected benefit $\bar{B}$ is therefore the accumulated value of the effective contributions, where the effective contributions accumulate at rate μ.

Thus the optimisation problem is:

$$\text{Maximise}_{\{g_i\}}\ B = \Sigma_{i=0}^{n-1} e^{(T-t_i)\mu} x_i \tag{10.1}$$

$$\text{subject to } G = \Sigma_{i=0}^{n-1} g_i\ ,\ c_i = x_i + p_i \text{ (for all } i). \tag{10.2}$$

It should be noted the the sum G of the incremental guarantee is actually not the final minimum benefit. Every call option can be considered individually. Thus it may happen that the total portfolio of the member may have a maturity value $\bar{B}$ already slightly bigger than G, but the member is entitled to claim even more than $\bar{B}$. The reason for the latter is that some of the incremental guarantees may still be claimable (i.e. in the cases where the relevant incremental of the portfolio ended up below the strike value, g_i).

Now let us express p_i as a function $p_i = f_i(x_i, g_i)$, which is of course the Black-Scholes formula for a put option striking at g_i with the initial value of the underlying asset at x_i. The second part of expression (10.2) can be expressed as $x_i = c_i - p_i$. Since the c_i are fixed and $p_i = f_i(x_i, g_i)$, we obtain the identity

$$x_i = c_i - f_i(x_i, g_i). \tag{10.3}$$

This says that x_i is implicitly a function of g_i. This means that x_i is a fixed point of the function $F_i(u)$, where

$$F_i(u) = c_i - f_i(u, g_i).$$

In order to calculate the contributor's expected benefit we need to calculate x_i. In the following two sections we describe two iterative methods of calculating x_i.

We first prove an important observation for implementing the method discussed in this chapter.

Proposition 11.1. (a) *The assumption* $c_i = p_i + x_i$ *implies* $g_i < c_i e^{r(T-t_i)}$ *for each* i.

(b) *In particular, for interest rate* $r = 0$, *the assumption* $c_i = p_i + x_i$ *implies* $g_i < c_i$ *for each* i.

Proof. (a) For time $0 < t < T$ the following relationship holds for European put and call options:

$$C_t - P_t = S_t - Ke^{-r(T-t)}$$

This relationship is known as the put-call parity (see Hull [22] for instance). Noticing that $C_t > 0$, the put-call parity gives the following:

$$-P_t < S_t - Ke^{-r(T-t)}, \text{ i.e., } S_t + P_t > Ke^{-r(T-t)}.$$

If we consider the special case where $K = g_i,\ t = t_i,\ S_t = x_i$ and $P(t) = p_i$, then

$$x_i + p_i > g_i e^{-r(T-t_i)}, \text{ i.e., } g_i < c_i e^{r(T-t_i)}.$$

(b) This follows immediately from (a). □

10.2 Newton's method for finding roots

The Newton-Raphson method is a technique used to find the roots of an equation using the derivative. If $\varphi(x)$ is the function we want the root of using the Newton-Raphson formula, we approximate the root as

$$x_{n+1} = x_n - \frac{\varphi(x)}{\varphi'(x)}. \tag{10.4}$$

We guess a value x_0 in order that the sequence of iterations will converge to the value we are looking for. The iterations of the Newton-Raphson will converge as long as $\varphi, \varphi', \varphi''$ are continuous near the root, φ' does not equal zero at the root and the initial value chosen is sufficiently close to the answer [17].

We therefore use the Newton-Raphson method to calculate the root of equation (10.3).

Now,

$$\varphi(x_i) = c_i - p_i - x_i$$

so that

$$\varphi'(x_i) = -\frac{\partial p_i}{\partial x_i} - 1 \tag{10.5}$$

where

$$\frac{\partial p_i}{\partial x_i} = g_i e^{-rt}\Big(-\frac{e^{-\frac{k_i^2}{2}}}{x_i\sigma\sqrt{2\pi\tau}}\Big) - N(-h_i) + \frac{e^{-\frac{k_i^2}{2}}}{\sigma\sqrt{2\pi\tau}} \tag{10.6}$$

$$h_i = \frac{\Big[\ln\frac{x_i}{g_i} + (r + \frac{1}{2}\sigma^2)(\tau)\Big]}{(\sigma\sqrt{\tau})} \tag{10.7}$$

$$k_i = \frac{\Big[\ln\frac{x_i}{g_i} + (r - \frac{1}{2}\sigma^2)(\tau)\Big]}{(\sigma\sqrt{\tau})} \tag{10.8}$$

$\tau = T_i - t$ and $N(\cdot)$ is the cumulative normal distribution.

We calculate values for x_i using Maple 9.5. For an introduction to the programme Maple see [12]. We found that in certain instances the iterations did stabilise but for certain parameters no stabilisation of the iterations occurred. For the examples in this section we assume a volatility rate of 0.03 and that there are only two time periods. We also assume the total contribution c_i is 1.

We find that the method gives no results for guarantee values g_i greater than 1, if we assume $c_i = x_i + p_i(x, g) = 1$. The iterations are shown in appendix A. The following proposition explains why no solution is found for guarantees greater than 1.

Suppose we have an asset with value that has the dynamics $\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_t$ and let us denote by $p_i(x, g)$ the Black-Scholes price of a European put option with strike g and $S_0 = x$ while we assume that T, r, σ are fixed. Since we assume that the contribution is 1 we let $F(u) = 1 - p_i(u, g)$.

We can prove the following.

Proposition 11.2

$F(u)$ has no fixed point if $g > 1$.

Proof

For time $0 < t < T$ the following relationship holds for European put and call options:

$$c_t - p_t = S_t - Ke^{-r(T-t)}$$

This relationship is known as the put-call parity. For details on how the put-call parity relationship arises see Hull [22]. Noticing that $c_t > 0$, the put-call parity gives the following:

$$\begin{aligned} -p_t &< S_t - Ke^{-r(T-t)} \\ S_t + p_t &> Ke^{-r(T-t)} \end{aligned}$$

If we consider the special case where $K = g,\ r = 0,\ t = 0,\ S_t = x$, then

$$\begin{aligned} x + p_t &> g \\ 1 &> g \end{aligned}$$

$\square$

Since the incremental guarantees g_i are less than one, $\Sigma_{i=0}^{n-1} g_i$ will always be less than n. This means that if we take $c_i = 1$ for all i, then the total minimum guaranteed amount G always has to be less than the time to maturity of the contract.

In the following section we discuss another method that can be used to find the effective contributions, namely the fixed point iteration method.

10.3 Fixed point iteration method

The fixed point iteration method is an iterative technique that aims to find a fixed point of a given function. A fixed point is that point of the function where the $y = x$ line crosses that graph. We therefore use the fixed point method since x_i is implicit in p_i. If we assume that the constant contribution that is paid is one unit then the effective contribution is $1 - p_i = x_i$. The aim is to find the fixed point of this equation. If $g(x)$ is the function we want to find the fixed point of then $x_{n+1} = g(x)$. For an initial value this will converge towards the fixed point which is the solution we are looking for. The question now is whether a fixed point exist for our problem and will the iterations converge.

Appendix B shows an example where the fixed point iterations stabilises after 10 iterations.

For fixed strike=g and variable stock=x the graph of $p(x)$ has slope $\frac{\partial p}{\partial x}$, which is defined by equation (10.6). We observe that

$$-1 < \frac{\partial p}{\partial x} < 0$$

This means that $x_i = 1 - p_i$ has slope between 0 and 1. We also observe that $\frac{\partial^2 p}{\partial x^2} < 0$ over the whole interval. Consequently, the iteration method will always produce a fixed point.

For a more in-depth discussion of the Newton-Raphson and the fixed point method the reader is referred to [17].

Now having shown that we can determine the contributor's effective contribution in the following section we show some numerical examples where we calculate the member's expected benefit. In particular we can now calculate the expected benfit for a given sequence of incremental guarantees.We use the fixed point iteration method to calculate the member's effective contribution in the rest of this chapter.

We note in table 10.1 that we obtain an increase on the expected benefit if instead of the "level" sequence, case (a), we take an increasing sequence of g_i's, case (b).

Table 10.1: $r = 0.04$, $\mu = 0.06$, $\sigma = 0.08$, $G = 95828$ (and in fact G was decided to be $\Sigma_{j=1}^{8} c_j$), (c_i) is a geometric sequence with factor 1.04.

	c_i :	10400	10816	11249	11699	12167	12653	13159	13686	
(a)	g_i :	10400	10816	11249	11699	12167	12653	13159	13686	
	x_i :	10324	10723	11134	11556	11991	12435	12890	13360	$\bar{B}_a = 123412$
(b)	g_i	11087	11295	11547	11765	12025	12326	12667	13117	
	x_i :	10250	10664	11091	11546	12018	12512	13034	13588	$\bar{B}_b = 123659$

10.4 The 2 and 3 period cases

Let us suppose that there are only two periods in the contract. We show graphically the existence of a maximum in the expected benefit for the member. In the 2-period case there are only two contributions, c_0 and c_1 and consequently we consider only g_0 and g_1. Since the minimum guarantee G is fixed, we can consider g_0 to be an independent variable but then g_1 is automatically determined by $g_1 = G - g_0$. We plot $\bar{B}$ versus g_0.

The graph (10.1) shows the expected benefit as a function of g_0 given that the minimum guarantee is 1.95 units, for contributions $c_0 = 1 = c_1$ while we work with volatility value $\sigma = 0.08$.

We now turn to the 3-period case where there are 3 g_i's. Again the minimum guarantee G is fixed, the expected benefit $\bar{B}$ is a function of two independent variables g_0 and g_1 while $g_2 = G - g_0 - g_1$. We observe that the graph in figure (10.2) is concave down, which indicates that once again a maximum of the expected benefit is attained. For the graph we consider the case: μ=0.01, a volatility rate of 0.08, a minimum guarantee of 2.8 units and constant contributions of 1 unit.

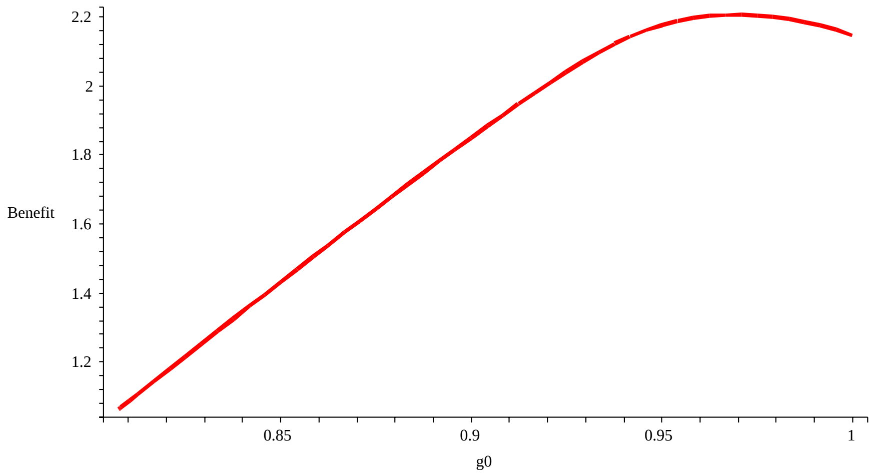

Figure 10.1: Benefit with G=1.95, $\sigma = 0.08$, $\mu = 0.1$

Figure 10.2: Expected benefit for a three period contract

From the sketch (note that on the computer we can rotate the graph and make more accurate readings from the screen) we read off the approximate values for g_0^* and g_1^* where the maximum benefit will occur:

$$g_0^* \cong 0.920 \ , \ \ g_1^* \cong 0.928.$$

With a total $G = 2.8$ we thus have $g_2^* \cong 2.8 - 1.848 = 0.952$. The approximate maximum value of $\bar{B}$ is $\bar{B}^* \cong 2.98$.

The graphs for the 2 period case and the 3 period case show that a maximum benefit amount can be attained for a suitable choice of the numbers g_i. This motivates the formulation of a solution method to find the optimal stream of g_i's, which is done in the next section.

10.5 Solving the optimisation problem

To solve the optimisation problem mentioned in section (11.1) we use the Lagrangian method of optimisation. The Lagrangian method of optimisation is used when we want to determine the optimal value of a function that has more than two variables, that are possibly inter-related. A detailed explanation of the Lagrangian method of optimisation is found in [11].

Let g denote the T-tuple variable $g = (g_0, g_1, ..., g_{n-1})$, $\bar{B}$ is the objective function that we seek to minimise, λ is a Lagrangian multiplier. The T-tuple $(x_0, x_1, ..., x_{n-1})$ is denoted by x. The Lagrangian is:

$$L(x, g, \lambda, \lambda_0, \lambda_1, ..., \lambda_{n-1}) = \bar{B}(x) + \lambda\Big[\sum_{i=0}^{n-1} g_i - G\Big] + \sum_{i=0}^{n-1} \lambda_i\Big(c_i - p_i(x_i, g_i) - x_i\Big). \quad (10.9)$$

Proposition 11.3

The following conditions (together with the constraints $p_i = c_i - x_i and \sum_{1=0}^{n-1} g_i = G)arenecessaryforaseque$
that solves the optimisation problem. For each $i = 0, 1, 2, ..., n-1$*:*

$$e^{\mu i}/\lambda_i = 1 - N(-h_i) + (e^{-h_i^2/2} - \frac{g_i}{x_i}e^{-r\tau_i - k_i^2/2})/\sigma\sqrt{2\pi\tau_i} \quad (10.10)$$

$$\lambda/\lambda_i = e^{-r\tau_i}N(-k_i) + (e^{-r\tau-\frac{1}{2}k_i^2} - \frac{x_i}{g_i}e^{-\frac{1}{2}h_i^2})/(\sigma\sqrt{2\pi\tau_i}), \quad (10.11)$$

where for each i,

$$\tau_i = T - t_i \quad (10.12)$$

$$h_i = \frac{\ln\frac{x_i}{g_i} + (r + \frac{1}{2}\sigma^2)\tau_i}{\sigma\sqrt{\tau_i}} \quad (10.13)$$

$$k_i = \frac{\ln\frac{x_i}{g_i} + (r - \frac{1}{2}\sigma^2)\tau_i}{\sigma\sqrt{\tau_i}} \quad (10.14)$$

Proof

The first order conditions that need to be satisfied are that the first order partial derivatives of L with respect to the variables x_i and g_i and with respect to the costate variables $\lambda, \lambda_0, \lambda_1, ...\lambda_{n-1}$ must vanish. Considering the costate variables we obtain the constraints. Furthermore we have

$0 = \frac{\partial L}{\partial x_i} = e^{\mu\tau_i} + \lambda_i(-\frac{\partial p_i}{\partial x_i} - 1)$
$0 = \frac{\partial L}{\partial g_i} = \lambda - \lambda_i \frac{\partial x_i}{\partial g_i}$.

Now we calculate these partial derivatives. We note that

$$\frac{\partial N(-h_i)}{\partial x_i} = \frac{dN(-h_i)}{dh_i}\frac{\partial h_i}{\partial x_i} = -\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}h_i^2} \cdot \frac{1}{x_i\sigma\sqrt{\tau_i}}.$$

We also note that

$$\frac{\partial k_i}{\partial x_i} = \frac{\partial h_i}{\partial x_i} = \frac{1}{x_i\sigma\sqrt{\tau_i}}.$$

This gives

$$\frac{\partial p_i}{\partial x_i} = e^{r\tau_i}g_i\frac{\partial N(-k_i)}{\partial x_i} - N(-h_i) - x_i\frac{\partial N(-h_i)}{\partial x_i}$$

$$= -N(-h_i) + (e^{-h_i^2/2} - \frac{g_i}{x_i}e^{-r\tau_i - k_i^2/2})/\sigma\sqrt{2\pi\tau_i}.$$

Similarly we can calculate $\frac{\partial p_i}{\partial g_i}$. These values when substituted into $\frac{\partial L}{\partial x_i}$ and $\frac{\partial L}{\partial g_i}$ above gives us the stated conditions. □

Now the necessary conditions constitute a system simultaneous equations which we can solve for the unknown $g_0, g_1, g_2, ..., g_{n-1}$ together with λ and all the λ_i. In the next section we give sample numerical solutions.

10.6 Solving for the incremental guarantees and the expected benefit

To solve for the incremental guarantees in higher time periods we solve the system of equations in proposition 11.2. We use the powerful *f-solve* function in the computing programme Maple to solve the system of equations.
When calculating these values we use approximations for the exponential and normal function since the computer code could not solve the problem with the true values. We use the following approximations: $e^x \cong \Sigma_{n=0}^{10}\frac{x^n}{n!}$ and

$$N(x) \cong x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} - \frac{x^{11}}{42240} + \frac{x^{13}}{599040} - \frac{x^{15}}{9676800}$$

It is sufficient to use these orders for the approximations since our x values are small. However, using an approximation is a significant drawback of the method. Yet, for 8 period problems it ran very fast (a few seconds). Given a solution, the test (calculation of $\bar{B}$ and comparison with other g_i sequences) can be made without making approximations, and we are happy that we obtain sufficiently precise solutions. The case (b) of the table (10.1) above is in fact an optimal solution obtained via this method. In what follows we include some further computations.

We consider an 8-period problem with a constant sequence of contributions and with the parameters $r = 0.02$, $\mu = 0.06$, $\sigma = 0.08$, and G is just the sum of the contributions $G = 48690$. The sequence of incremental guarantees in (a) is the one for which $g_i = c_i$, and the optimal sequence of incremental guarantees is given in (b).

Table 10.2: $r = 0.02$, $\mu = 0.06$, $\sigma = 0.08$, $G = 48690$ (and in fact G was decided to be $\Sigma_{j=1}^{8} 1.02^j$), (c_i) is a geometric sequence with factor 1.04).

	c_i :	5000	5200	5400	5600	6100	6530	6860	8000	
(a)	g_i :	5000	5200	5400	5600	6100	6530	6860	8000	
	x_i :	4809	4992	5174	5355	5832	6224	6535	7638	$\bar{B} = 60345$
	g_i^* :	5138	5319	5441	5628	6084	6480	6759	7840	
	x_i^* :	4748	4933	5152	5338	5833	6262	6625	7810	$\bar{B}^* = 60448$

Table 10.3: $r = 0.05$, $\mu = 0.08$, $\sigma = 0.115$, $G = 50000$

c_i	5000	5200	5400	5600	6100	6530	6860	8000	
g_i^*	5532	5724	5787	5841	6223	6504	6666	7723	
x_i^*	4841	5010	5208	5411	5901	6338	6698	7846	$\bar{B}^* = 66984$

Conclusion

In this mini-thesis we calculate and discuss various results regarding minimum guarantees that are present in investment contracts. We concentrated on the models proposed by Brennan and Schwartz [10] and Deelstra *et al.* [13]. We found that the inclusion of these minimum guarantees to a pension fund member's contract adds value to his/her contract.

In the final chapter of this mini-thesis we formulate a method to find an optimal stream of incremental multi-period guarantees that would maximise the benefit the contributor receives from his/her contract. We conclude that the member can attain the highest benefit when his/her guarantees are increasing. We include numerical examples for a two, three and four period contract. We did not include major multi-period cases since that may include numerical work taking us beyond the scope of a mini-thesis. Instead we used the very powerful *f-solve* function of Maple.

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Appendix A

Iteration	Initial	x_i
1	1	0.99130718
2	0.99130718	0.99009306
3	0.99009306	0.988997875
4	0.988997875	0.98996891
5	0.98996891	0.98996806
6	0.98996806	0.98996799
7	0.98996799	0.98996800
8	0.98996800	0.98996800

Table 4: Newton-Raphson iterated values with gaurantee = 0.95

Table (4) shows an instance where the sequence stabilises. We observe that with a guarantee of 0.95 the sequence stabilises after 8 iterations.

Table (5) is produced using code that produces a value for the effective contribution using the Newton-Raphson method after 10 iterations. We iterate 5 times to get a value after 50 iterations. From table (5) we see that the series with a guarantee of 1 still does not stabilise after 50 iterations.

From table (6) we see that the iterations also does not stabilise after 50 iterations when the guarantee is 1.01.

Iteration	Initial	x_i
1	1	1.0237582
2	1.0237582	0.97035619
3	0.97035619	1.0382054
4	1.0382054	0.98033454
5	0.98033454	1.0188994

Table 5: Newton-Raphson iterated values with gaurantee = 1

Iteration	Initial	x_i
1	1	0.95998052
2	0.95998052	1.2243362
3	1.2243362	0.99999884
4	0.99999884	0.95980852
5	0.95980852	1.2243362

Table 6: Newton-Raphson iterated values with gaurantee = 1.01

Iteration	Initial	x_i
1	1	0.99927183
2	0.99927183	0.99927183

Table 7: Newton-Raphson iterated values with gaurantee = 0.9

In table (7) we see that the series stabilises after twenty iterations.

From this we conclude that the Newton-Raphson method only converges when the guarantee is less than 1.

Appendix B

Iteration	Initial	x_i
1	1	0.99225739
2	0.99225739	0.99049877
3	0.99049877	0.99009114
4	0.99009114	0.98999656
5	0.98999656	0.98997462
6	0.98997462	0.98996952
7	0.98996952	0.98996835
8	0.98996835	0.98996806
9	0.98996806	0.98996800
10	0.98996800	0.98996800

Table 8: Fixed-Point Iterated values with guarantee = 0.95

Table (8) shows the iterated values using the fixed point method with guarantee equal to 0.95. We see that this method converges after 10 iterations. We again assume a volatility of 0.03 and two time periods.