NUMERICAL METHODS FOR THE VALUATION OF FINANCIAL DERIVATIVES

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A thesis submitted in partial fulfilment of the requirements for the degree of Master of Science, in the Department of Mathematics and Applied Mathematics, University of Western Cape, South Africa.

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November 2005

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KEYWORDS

Black Scholes.

Financial Derivatives.

Exotic Options.

Futures.

American and European Options.

Stochastic Differential Equations.

Multi Period Binomial Model.

Finite Difference Method.

Monte Carlo Simulation.

Differential Equations.

ABSTRACT

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MSc thesis, Department of Mathematics and Applied Mathematics,

University of Western Cape.

Numerical methods form an important part of the pricing of financial derivatives and especially in cases where there is no closed form analytical formula.

We begin our work with an introduction of the mathematical tools needed in the pricing of financial derivatives. Then, we discuss the assumption of the log-normal returns on stock prices and the stochastic differential equations. These lay the foundation for the derivation of the Black Scholes differential equation, and various Black Scholes formulas are thus obtained. Then, the model is modified to cater for dividend paying stock and for the pricing of options on futures.

Multi-period binomial model is very flexible even for the valuation of options that do not have a closed form analytical formula. We consider the pricing of vanilla options both on non dividend and dividend paying stocks. Then show that the model converges to the Black-Scholes value as we increase the number of steps.

We discuss the Finite difference methods quite extensively with a focus on the Implicit and Crank-Nicolson methods, and apply these numerical techniques to the pricing of vanilla options. Finally, we compare the convergence of the multi-period binomial model, the Implicit and Crank Nicolson methods to the analytical Black Scholes price of the option.

We conclude with the pricing of exotic options with special emphasis on path dependent options. Monte Carlo simulation technique is applied as this method is very versatile in cases where there is no closed form analytical formula. The method is slow and time consuming but very flexible even for multi dimensional problems.

The data analysis is presented by means of tables and graphs. The sample programs used in generating the tables are based on Matlab programming language and are listed in the appendix.

DECLARATION

I declare that *Numerical Methods for the Valuation of Financial Derivatives* is my own work, that it has not been submitted for any degree or examination in any other University, and that all the sources I have or quoted have been indicated and acknowledged by complete references.

Davis Bundi Ntwiga November 2005

Signed: ....

ACKNOWLEDGMENTS

First and foremost, my special gratitude goes to the University of the Western Cape through the Dean, Faculty of Science, and the African Institute for Mathematical Sciences (AIMS) for their financial support toward my masters' studies. Sincere thanks to Prof. Neil Turok, the founder of AIMS.

I am also grateful to my supervisor Prof. Jean Claude Ndogmo for his guidance, suggestions and assistance during the course of this work.

I thank Prof. Peter Witbooi for his gratifying support during my stay and in particular my arrival at the University of Western Cape, the effort and sacrifice shown by Justin, Elmerie, Paul and Ian for assisting me in proof reading the document, and not forgetting the staff and students in the Department for creating a warm and friendly environment that enabled me to undertake my research.

Finally, I am highly indebted to my family and relatives back at home, and friends both at home and here in South Africa. The love, patience, encouragements and their support through constant communication have been an invaluable asset for me. I wish to mention a few names, Doris, Martin, Michael, Maina, Benta, Timothy, Edwin, James, Kendi and my Dad, Evasius as the list is endless.

To my late mum,

Hellen Kagendo.

LIST OF ACRONYMS

NSE	Nairobi Stock Exchange.
MTM	Marking To Market.
GBM	Geometric Brownian Motion.
SDE	Stochastic Differential Equation.
CRR	Cox, Ross and Rubinstein.
PDE	Partial Differential Equation.
FDE	Finite Difference Equation.
SOR	Successive Over Relaxation.
JSE	Johannesburg Stock Exchange.
ATS	Automated Trading System.
SSF	Single Stock Futures.
SAFEX	South African Futures Exchange.
MCS	Monte Carlo Simulation.

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Chapter 1

Introduction

1.1 Background

In 1973, the Chicago Board of Trade (CBOT) became the first organized exchange to start trading so-called *options*, as well as *futures* and other *financial derivatives*. The question that arose was what price was the buyer willing to pay for those financial instruments?

Black, Scholes and Merton approached the problem of pricing an option in a physicist's way by assuming a reasonable model for the price of a risky asset. This search had not started at this point but can be traced back to the time of a botanist Robert Brown (1827), who first described the unusual motion exhibited by a small particle that is totally immersed in a liquid or gas. Then, Albert Einstein (1905) expounded more on this motion. Further, Norbert Wiener gave in a mathematically concise definition of the theory on Brownian Motion in a series of papers originating in 1918. Before Einstein, a young French PhD student, called Louis Bachelier proposed in his 1900 thesis, Brownian motion as a model for speculative prices. In 1960's the economist Samuelson propagated to his students at Massachusetts Institute of Technology (MIT) the exponential of Brownian motion (*Geometric Brownian motion*) for modeling prices which are subject to uncertainty.

Brownian motion is a natural analogue of a random walk in continuous time. A random walk is defined as discrete time equidistant instants of time. In finance, we use it to model prices at every instant of time and the notion of continuous time becomes important. The use of Brownian motion has a major flaw when used to model stock prices since it assumes that the price of a stock follows a normal random variable. This leads amongst others, to negative stock values which is not realistic.

The geometric Brownian motion is the basic mathematical model for price movements and Black, Scholes and Merton used this principle. However, they realized that Brownian motion is closely related to *stochastic* or *Itô calculus,* named after the Japanese mathematician Kiyosi Itô who developed this theory in the 1940's. They further used the notion that the derivative security can be exactly replicated by a dynamic trading strategy utilizing the stocks and the risk-less bond. A trading strategy which *replicates* the value of the option at maturity is called a *hedge.* When they developed the Black-Scholes formula, they argued that if the option was sold at a price other than the Black-Scholes price, a rational investor could make a profit without any accompanying risk (called *arbitrage).* In 1973, Black and Scholes published their analysis of the European call option in a paper titled 'The Pricing of Options and Corporate liabilities' [5].

This laid the foundation for the rapid growth of markets for derivatives and the starting point for the pricing of other kinds of financial derivatives. The modification of the Black Scholes formula to cater for a large number of derivatives has taken place while new pricing techniques and models have been developed. This has enhanced the theoretical understanding of financial markets.

1.1.1 Aims of Study

The aims of this research can be summarized as:

(a). To introduce the concept of financial derivatives, definitions and mathematical tools vital in the valuation of financial derivatives.

(b). To discuss and apply some of the numerical methods used in the valuation of financial derivatives. The methods to be discussed are: (1) Black Scholes model. (2) Binomial model. (3) Finite difference methods. (4) Monte Carlo simulation method.

In recent years, a large variety of financial products have been created by financial institutions. The pricing of these new products became a challenge. The advent of financial mathematics has led to exploitation of advanced tools like martingale theory, functional analysis, stochastic control and partial differential equations in the pricing of these new products. We concentrate on the pricing of options, forwards and futures using these advanced mathematical tools.

1.2 Financial Derivatives

A financial derivative is a contract whose value is determined by the value of one or more underlying assets.

This term is very broad due to the introduction of complex and varying derivatives in the market. Thus, a financial derivative can have a large number of properties and so it's value can depend on one or more characteristics exhibited by the underlying asset(s).

The contract specifies the rights and obligations between the writer of the security and the holder to receive or deliver future cash flows based on some future event. We say that the future value of a derivative is a stochastic process due to its uncertainty.

The main groups of underlying assets are stocks, foreign currencies, interest rates, stock indices and commodities. Figure 1.1 shows a sample path of the stochastic process followed by a stock index. We have four main types of derivatives namely futures, forwards, swaps and options. We intend to use the stock as our main underlier and the principles applied can be generalized in pricing derivatives with other underlying assets.

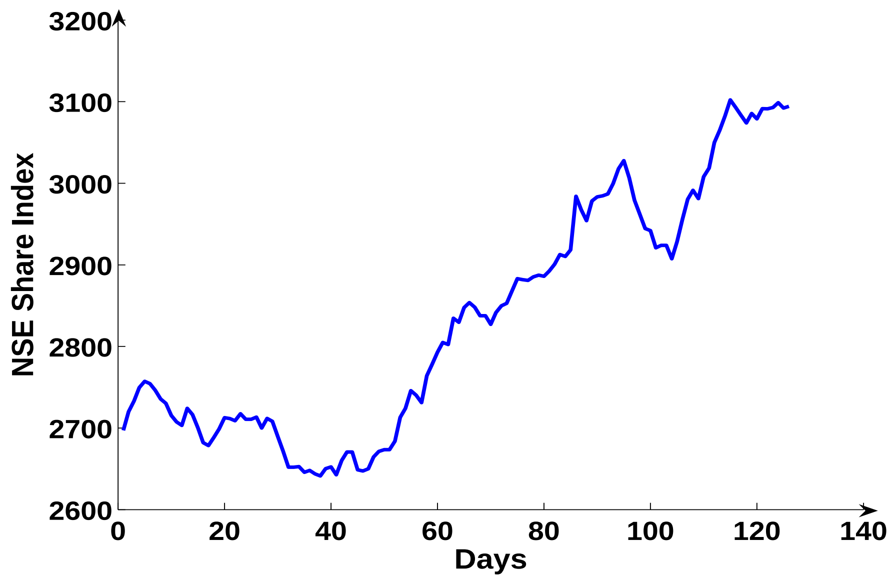

Figure 1.1: A sample path of the NSE 20 Share Index. The data are from August 2004 to January 2005.

1.2.1 Futures

A futures contract is an agreement that one places in advance to buy or sell an asset or commodity at a certain date for a certain price called the delivery price.

As time passes, the forward price is liable to change but the delivery price remains the same. The price is fixed when the order is placed but the payment is not made until the delivery date. The contract can be reversed before expiration by taking an equal and opposite position in the same futures contract. The holder of the long (short) position gains if the futures price at which the position is reversed is above (below) the initial futures price. The following specifications and conditions characterize futures contracts:

- They have standardized contract terms. Futures contracts are highly uniform and with a well specified commitments describing the good to be delivered at a certain time and in certain manner. The contract also stipulates the minimum price fluctuation or *tick* size and the *daily price limit.* The *tick* is the minimum permissible price fluctuation.

- At the start of a futures contract, the prospective trader must deposit funds referred to as an *initial margin.* It is the minimum amount of money that must be in an investment account on the day of transaction. This amount acts as a financial safeguard to ensure that traders will fulfill their contract obligations. Traders know their daily price changes through a process referred to as *marking-to-market (MTM). MTM* is a process in which the daily price changes are paid by the parties incurring losses to the parties making profits.

  A trader receives a *margin call* when the value of the initial margin falls below the *maintenance margin.* A *maintenance margin* is the minimum amount of money that must be kept in a margin account on any day other than the day of a transaction. The *margin call* requests the trader to replenish the account to bring it back to its initial level. This additional amount brought by the trader is called the *variation margin.* The *variation margin* is the money added to or subtracted from a futures account that reflects profits or losses accruing from the daily settlement or the marking-to-market.

- The parties in the contract have an obligation to the clearing house which in turn ensures that the parties concerned honor the contract. The clearing house takes no active position in the market but interposes itself between parties to every transaction [21].

1.2.2 Forwards

A forward contract is an agreement between two parties, a buyer and a seller, to purchase or sell an asset at a later date at a price agreed upon today.
A forward contract is characterized by the following:

- Forward markets have no formal corporate body organized as the market. They trade in an over-the-counter market among major financial institutions. An *over-the-counter* is trading in financial instruments off organized exchanges with the risk that performance by the counter parties is not guaranteed by an exchange.
- The parties in a forward contract incur the obligation to ultimately buy or sell the asset.
- Forward contracts are tailor made to meet the specific needs of the parties involved.

The futures and forward contracts have differences and similarities, in that:

- They differ in the institutional setting in which they trade but the principles of pricing are the same.
- While the profit or loss from a futures contract is evaluated everyday, that of the forward contract is only realized at the expiry date.
- It costs nothing to enter into a forward contract but there is an initial margin for the futures contract.
- Both the futures and forward contracts are *linear* instruments. This implies that the contract price changes has a direct relationship with the price changes of the underlying asset.
- The forward and the futures contracts do not contain the element of choice, the parties concerned are obligated to honor the contract ([21], [10], [11]).

1.2.3 Options

An option gives the holder the *right* but not the *obligation* to buy or sell an asset in the future at a price that is agreed upon today. Every option has the exercise date or expiration date, exercise price or strike price and command a premium, also called the price of the option.

The option is said to be exercised when the holder chooses to buy or sell the underlying stock. The writer of the option is the other party to the contract. The holder (writer) is said to be in the long (short) position of the option contract [17].

The standard derivatives or 'plain vanilla options' are the European and American options. Other options are called Exotic or non standard derivatives. Examples include; Asian, Lookback, Barrier options among others.

European Options

A European call (put) option gives the holder the right but not the obligation to buy (sell) the underlying asset with an initial price S, at a given maturity date T and for a fixed price K, called the strike price. Let the price of European call (put) option be denoted by $c(p)$. These notations will be used throughout our work to denote the European call and put option. The payoff of a European call at maturity time T is

$$c = \max(\mathrm{S}_T - K, 0). \tag{1.1}$$

If $S_T < K$, the call will be worthless and the holder will not exercise the right. The payoff of a European put is

$$p = \max(K - \mathrm{S}_T, 0). \tag{1.2}$$

If $S_T > K$, the put will be worthless and the holder will not exercise the right. The call - put parity is the relationship between a European call and put, given by

$$c + K\mathrm{e}^{-rt} = p + S, \tag{1.3}$$

where r denotes the risk free interest rate and S the initial stock price.

American Options

American call (put) option gives to its holder the right but not the obligation to buy (sell) the underlying asset at any time t $(0 < t < T)$, up to maturity date T, for a strike price K. Let the price of the American call (put) option be denoted by $C(P)$. These notations will be used throughout our work to denote the American call and put option. The payoff of an American call at maturity time T is

$$C = \max(\mathrm{S}_T - K, 0). \tag{1.4}$$

The payoff of an American put is

$$P = \max(K - \mathrm{S}_T, 0). \tag{1.5}$$

The price boundary and put-call parity for the American option is given by

$$S - K \leq C - P \leq S - K\mathrm{e}^{-rt}. \tag{1.6}$$

1.2.4 Value of an Option

In option pricing, the value of the option is a function of both the underlying asset and time, $c_t = f(S_t, t)$. The calculation of the price of an option (premium) is our prime concern. The premium is the fair value of an option contract determined in the competitive market, which the option buyer pays to the option writer.

The intrinsic value of a call option is $\max(S_t - K, 0)$ and that of the put option is $\max(K - S_t, 0)$ for $0 \leq t \leq T$. This value represents the profit an investor can make by immediately exercising the option.

The time value of a call is the difference between the price of the call and its intrinsic value. The time value of an option decreases as the time remaining to expiration decreases. The European options do not have time value because they can only be exercised at maturity time and at this time T, the time value of the option is zero.

The sum of an option intrinsic and time values is the total value of an option. The intrinsic value is the possible profit resulting from selling the option at the present time and does not take a negative value. If the present value of the underlying asset is lower than the strike price the intrinsic value of the call vanishes.

Example, if the call premium is \$$7.20$ and the price of the underlying stock is \$$40$ with a strike price of \$$35$, the intrinsic value is \$5 and the time value is \$$2.20$.

When an option is *in-the-money* (ITM), the intrinsic value is non-zero. When the strike price is the same as the spot price, we say the option is *at-the- money* (ATM) and the intrinsic value is zero. Otherwise the option is *out-of-the money* (OTM).

Why Trade Options?

- Investors prefer to trade options rather than stocks in order to save transaction costs and avoid market restrictions. A trader can use options to take a particular risk position and pay lower transaction costs than stocks would require.

- The stock and option markets have their own institutional rules. The differences in these rules may stimulate option trading. Example, selling stock short is highly restricted but by trading in the option market, it is possible to replicate a short sale of stock to avoid some stock market restrictions.

- They are attractive to *speculators*, who have a view on how the asset price will evolve and wish to gamble. A *speculator* is an investor who is taking a position in the market with a view that the price of an asset will go up or down. The aim of the bet is to gain from the price movement. The option price is more volatile than the price of the underlying stock, so investors can get more price action per dollar of investment by investing in options instead of investing in the stock itself.

- They are attractive to individuals and institutions wishing to mitigate their exposure to risk. Options may be regarded as insurance policies against unfavorable movements in the market. By trading options in conjuction with their stock portfolios, investors can carefully adjust the risk and return of their investments.

- There is a logical, systematic theory for working out how much an option should cost ([19], [21], [10]).

The process followed by derivative prices due to uncertainty of their future value is vital in enabling us to fully understand and model the behavior of the financial derivatives. These processes are discussed in the next section.

1.3 The Dynamics of Derivative Prices

We turn our attention to the continuous time models of financial asset prices. We assume that assets are traded and prices evolve on a continuous basis, that is, they change continuously and can be expressed in arbitrarily fine fractions.

We treat some discrete time models in our definitions and highlight how they converge in a limiting case to the continuous time models.

1.3.1 Stochastic Process

Definition 1.

A stochastic Process $X = \{X(t), t \in I\}$ is a collection of random variables with index set I, where t is time. A realization of X is called a sample path. A continuous time stochastic process $\{X(t)\}$ is said to have independent increments if for all $t_0 < t_1 < \ldots < t_n$, the random variables

$$X(t_1) - X(t_0), X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_n) - X(t_{n-1})$$

are independent. It is said to possess stationary increments if $X(t+s) - X(t)$ has the same distribution for all t and the distribution depends only on s [23].

1.3.2 Markov Process

Definition 2.

A Markov process is a stochastic process for which everything that we know about its future is summarized by its current value. A continuous time stochastic process $X = \{X(t), t \geq 0\}$ is Markovian if

$$\text{Prob}[X(t) \leq x | X(u), 0 \leq u \leq s] = \text{Prob}[X(t) \leq x | X(s)] \quad \text{for } s < t.$$

1.3.3 Martingale

Suppose we observe a family of random variables and let the observed process be denoted by $\{S_t, t \in [0, T]\}$. Let us assume that time is continuous and that over an interval $[0, T]$, we can represent the various time periods as $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = T$. Let $\{I_t, t \in [0, T]\}$ represent a family of information sets that become continuously available to the investor as time passes. Given $s < t < T$, this family of information sets will satisfy

$$I_s \subseteq I_t \subseteq I_T \ldots.$$

This set, $\{I_t, t \in [0,T]\}$ is called a *filtration.* At some particular time t, if the value of the price process is S_t and if it is included in the information set I_t for $t \geq 0$, then it is said that $\{S_t, t \in [0,\infty)\}$ is *adapted* to $\{I_t, t \in [0,\infty)\}$. This implies that the value of S_t will be known given the information set I_t [24].

Definition 3.

A stochastic process $M_t,\ t \geq 0$ is a martingale with respect to the family of information sets I_t and with respect to the probability Q, if for all $t \geq 0$,

(i). $E_Q[|M_t|] < \infty$.

(ii). Whenever $0 \leq s < t$, then $E_Q[M_t|I_s] = M_s$.

A martingale, (1) makes the expected future value conditional on its present value or on the set of information that is known. (2) is not expected to drift upwards or downwards and thus it is a notion of a fair game. (3) is always defined with respect to some information set, and with respect to some probability measure [24].

In the discrete time setting, a martingale means that $E[X_{n+1}|X_1, X_2, \ldots, X_n] = X_n$. A financial pricing model describes the dynamics of price changes.

1.3.4 Brownian Motion

Definition 4.

A random process $B_t,\ t \in [0,\infty)$ is a Brownian motion if

(i). B_t has both stationary and independent increments.

(ii). B_t is a continuous function of time, with $B_o = 0$, unless otherwise stated.

(iii). For $0 \leq s \leq t,\ B_t - B_s$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2|t-s|$. That is, $(B_t - B_s) \sim N\left[\mu(t-s), \sigma^2|t-s|\right]$, where μ and $\sigma \neq 0$ are real numbers.

Such a process is called a (μ,σ) Brownian motion with drift μ and variance σ^2 [23].

Definition 5.

(i). The $(0,1)$ Brownian motion is called the normalized Brownian motion, or again the Wiener process.

(ii). A (μ,σ) Brownian motion is also called a generalized Wiener process or the Wiener - Bachelier process.

1.3.5 Brownian Motion as the Limit of a Random Walk

A (μ,σ) Brownian motion is a limiting case of a random walk. Suppose that a particle moves κ either to the left with probability $1-q$ or to the right with probability q, where κ is the size of the step in the ith position. The successive steps are independent.

Let X_n denote the position of the random walk after n steps. The stochastic process $\{X_n, n \geq 0\}$ is called a random walk process. Now, assume that $n = t/\Delta t$ is an integer. The particle's position at time t is

$$Y(t) = k[X_1 + X_2 + \ldots + X_n] \tag{1.7}$$

where

$$X_i = \begin{cases} +1, & \text{if the } i \text{ th move is to the right} \\ -1, & \text{if the } i \text{ th move is to the left} \end{cases}$$

and X_i are independent with Prob$[X_i = 1] = q$ and Prob$[X_i = -1] = 1-q$. Then,

$$\begin{aligned} E[X_i] &= (2q-1), \\ Var[X_i] &= 1-(2q-1)^2. \end{aligned} \tag{1.8}$$

Therefore the respective mean and variance of $Y(t)$ are

$$\begin{aligned} E[Y(t)] &= n\kappa(2q-1), \\ Var[Y(t)] &= n\kappa^2[1-(2q-1)^2]. \end{aligned} \tag{1.9}$$

Let $\kappa = \sigma\sqrt{\Delta t}$ and $q = \frac{1}{2}[1+\mu/\sigma\sqrt{\Delta t}]$. Then, (1.9) is expressed as

$$\begin{aligned} E[Y(t)] &= n\sigma\sqrt{\Delta t}(\frac{\mu}{\sigma})\sqrt{\Delta t} = \mu t, \\ Var[Y(t)] &= n\sigma^2\Delta t\left[1-(\frac{\mu}{\sigma})^2\Delta t\right] \longrightarrow \sigma^2 t \quad \text{as} \quad \Delta t \to 0. \end{aligned} \tag{1.10}$$

This shows that by the central limit theorem $\{Y(t), t \geq 0\}$ converges to a (μ, σ) Brownian motion. When $\mu = 0$ is chosen, then it becomes a Brownian motion with a zero drift which is a limiting case of a symmetric random walk [23].

1.3.6 Geometric Brownian Motion

Definition 6.

If X(t) is a Brownian motion with drift rate μ and variance rate σ^2, the process $\{Y(t) = e^{X(t)}, t \geq 0\}$ is called a *geometric Brownian motion,* or the exponential Brownian motion, or again the lognormal diffusion. The mean and variance are given respectively by

$$\begin{aligned} E[Y(t)] &= e^{(\mu+\sigma^2/2)t}, \\ Var[Y(t)] &= e^{(2\mu+\sigma^2)t}[e^{\sigma^2 t} - 1]. \end{aligned} \tag{1.11}$$

We consider Arbitrage which is the basic principle for the pricing of financial instruments in that the market prices are at equilibrium and thus no risk free profits are available..

1.4 Arbitrage

Arbitrage is a trading strategy that involves two or more securities being mispriced relative to each other to realise a profit without taking a risk.

In practice, arbitrage opportunities are normally rare, short-lived and therefore immaterial with respect to the volume of transactions. Thus, the market does not allow risk-free profits, (that is '*there is no free lunch*'). The main tools used to determine the fair price of a security or a derivative asset rely on the no-arbitrage principle. It is a fundamental assumption about the market. Further:

- The no-arbitrage principle is that a portfolio yielding a zero return in every possible scenario must have a zero present value. Any other value would imply arbitrage opportunities, which one can realize by shorting the portfolio if its value is positive and buying it if its value is negative [24].

- If one makes risk free profit in the market, then arbitrage opportunities exist and it implies that the economy is in an *economic disequilibrium.* An *economic disequilibrium* is a situation in which there is mispricing in the market and investors trade. Their trading causes prices to change, moving them to new economic equilibrium. The mispricing is corrected by trading and arbitrage opportunities no longer exist [24].

The lemmas in the next part proves some of the arbitrage-free conditions that options must satisfy.

Lemma 1.

A European call option with a higher strike price cannot be worth more than an otherwise identical call with a lower strike price.

Proof: Let K_1 and K_2 be the strike prices with $K_1 < K_2$. Suppose $c_{K_1} < c_{K_2}$. Then, buy the low-priced c_{K_1}, and write the high priced c_{K_2}, generating a positive return.

Lemma 2.

An American call with a longer time to expiration cannot be worth less than an otherwise identical call with a shorter time to expiration.

Proof: Suppose that $C_{t_1} > C_{t_2}$, where $t_1 < t_2$. We buy C_{t_2} and sell C_{t_1} to generate a net cash flow of $C_{t_1} - C_{t_2}$ at time zero. At time t_1, let $t_1 = \tau$, the short call either expires or is exercised, and the position is worth $C_\tau - \max(S_\tau - K, 0)$. If this value is positive, close out the position with a profit by selling the remaining call. Otherwise, $\max(S_\tau - K, 0) > C_\tau \geq 0$ and the short call is exercised. In this case, we exercise the remaining call and have a net cash flow of zero.

In both cases, the total payoff is positive without any initial investment. To avoid locking in a risk-free profit, the arbitrage free relation must hold [23].

1.4.1 Arbitrage-Free Market

A market is Arbitrage-free if it satisfies any of the following conditions

(a). Market Efficiency

Market efficiency is the characteristic of a market in which the prices of the instruments trading therein reflect the true economic values to investors. In an efficient market, prices fluctuate randomly and investors cannot consistently earn returns above those that would compensate them for the level of risk they assume. Thus, the efficient market hypothesis states that “prices of securities fully reflect available information.”

If the securities market is efficient, then information is widely and cheaply available to investors and all relevant and ascertainable information is already reflected in security prices. The efficient market hypothesis comes in three different forms.

- The *weak form* asserts that stock prices already reflect all information contained in the history of past prices. It is impossible to earn superior returns simply by looking for patterns in stock prices as price changes are random.
- The *semi strong form* asserts that stock prices already reflect all publicly available information. It is impossible to make consistently superior returns just by reading the newspaper, looking at the company's annual accounts and so on.
- The *strong form* states that stock prices reflect all relevant information including *insider information.* The *insider information* is the material information about a company's activities that has not been disclosed to the public. It is hard to get insider information because you are competing with thousands, perhaps millions of active, intelligent and greedy investors [19].

Efficient market hypothesis is a fair game model or a martingale and can be expressed as $\bar{\mu}_{i,t} = E[\bar{\mu}_{i,t+1}|I_t]$. Where $\bar{\mu}_{i,t}$ is the actual return on security i in period t and $E[\bar{\mu}_{i,t+1}|I_t]$ is the expected return on security i in period $t+1$ conditional on I_t the set of information available in period t.

(b). Self Financing Strategy

It is a trading strategy in which the value change in a portfolio is as a result of a change in the value of the underlying asset and not because of change in the portfolio structure.

If we have ϕ_t units of a stock S_t and ψ_t units of a bond B_t, then the portfolio's value is $V_t = \phi_t S_t + \psi_t B_t$.

The strategy is self-financing if $\phi_{t-1} S_t = \phi_t S_t$ and $\psi_{t-1} B_t = \psi_t B_t$. That is, we have re-adjusted the portfolio while the prices have remained the same, and the total value has not changed [4]. We illustrate with an example.

Example 1.4.1.

If $\Delta X = X_t - X_{t-1}$ for any random variable, then

$$\begin{aligned} V_t &= \phi_t S_t + (\phi_{t-1} S_{t-1} - \phi_t S_{t-1}) + \psi_t B_t + (\psi_{t-1} B_{t-1} - \psi_t B_{t-1}), \\ &= V_{t-1} + \phi_t . \Delta S_t + \psi_t . \Delta B_t. \end{aligned} \tag{1.12}$$

Thus, the change in portfolio is expressed as

$$\Delta V_t = \phi_t . \Delta S_t + \psi_t . \Delta B_t. \tag{1.13}$$

which clearly shows how changes in the prices determine changes in the portfolio value. An arbitrage opportunity is a self-financing trading strategy with the property that $V_o = 0$ and $V_t \geq 0$ for all $t > 0$ but $V_t > 0$ with positive probability for some $t > 0$.

1.4.2 Risk Neutral Valuation

It is the valuation of a derivative assuming the world is risk neutral. A *risk neutral world* is a world where assets are valued solely in terms of their expected return. The return on all securities is the risk-free interest rate and all individuals are indifferent to risk.

Thus the risk neutral valuation principle is important in option pricing. Indeed it implies that all expected returns must be zero. As a consequence, derivative prices are determined by the expected present value payoff. We assume that the world is risk neutral and the price obtained is correct not just in a risk-neutral world but also in the real world.

Risk

We can define the risk in a portfolio as the variance of the return. This definition does not take into account the distribution of the return. Example, a bank savings account or a government bond has a guaranteed return with no variance, and is thus termed as risk-less (or risk-free). A highly volatile stock with a very uncertain return has a large variance and is a risky asset. We assume the existence of risk-free investments that give a guaranteed return with no chance of default.

We have two types of risk: specific and non-specific, called market or systematic risk. Specific risk is the component of risk associated with a single asset or a sector of the market.

Example, an unstable management would affect an individual company but not the market, or may be a highly volatile share. Non-specific risk is associated with factors affecting the whole market. Example, a change in interest rates would affect the market as a whole.

Diversifying away specific risk can be achieved by having a portfolio with a large number of assets from different sectors of the market. It is not possible to diversify away non-specific risk. Market risk can be eliminated from a portfolio by taking similar positions in the assets which are highly negatively correlated; as one decreases in value, the other increases [32].

We have explained some of the tools needed in the pricing of financial derivatives. These tools have laid the foundation and will be applied in the building of numerical models for pricing derivatives in the subsequent chapters.

Chapter 2

Black - Scholes Model

2.1 Stochastic Differential Equation

A stochastic differential equation (or SDE) is a differential equation in which one or more of the terms is a stochastic process, thus resulting in a solution which is itself a stochastic process.

The SDE's can model the randomness of the underlying asset in financial derivatives. They are utilized in pricing derivative assets because they give a formal model of how an underlying asset's price changes over time. In pricing derivative assets, the randomness of the underlying instrument is essential. After all, it is the desire to eliminate or take risk that leads to the existence of derivative assets.

A trader continuously tries to forecast the price of an asset at any time interval, $\delta t.$ These 'new events' recorded as time passes contain some parts that are unpredictable. After that, they become known and become part of the new information set $\{I_t\}$ the trader possesses. The formal derivation of SDE's is compatible with the way dealers behave in financial markets.

Let S_t be the price of a security. A trader will be interested in knowing $dS_t,$ the next instant's incremental change in the security price. The dynamic behaviour of the asset price in a time interval dt can then be represented by the SDE given by

$$dS_t = \alpha(S_t, t)dt + \sigma(S_t, t)dW_t \qquad \text{for } \ t \in [0, \infty) \tag{2.1}$$

where dW_t is an innovation term representing unpredictable events that occur during the infinitesimal interval $dt,$ $\alpha(S_t, t)$ is the drift parameter and $\sigma(S_t, t)$ the diffusion parameter which depend on the level of observed asset price S_t and on time $t.$ The drift and diffusion parameters are assumed to satisfy the conditions

$$P\left[\int_0^t |\alpha(S_u,u)|du < \infty\right] = 1$$
$$P\left[\int_0^t \sigma(S_u,u)^2 du < \infty\right] = 1. \tag{2.2}$$

These conditions require that the drift and diffusion parameters do not vary 'too much' over time. They are functions of bounded variation with probability one [24].

We have seen that SDE's are vital in our stochastic environment as the evolution of the asset price S at time t contains uncertainty. In the next part, we consider It$\hat{o}$'s process and It$\hat{o}$'s lemma which are important tools of stochastic calculus.

2.1.1 It$\hat{o}$ Process

The stochastic process $X = \{X_t, t \geq 0\}$ that solves

$$X_t = X_o + \int_0^t a(X_s,t)ds + \int_0^t b(X_t,t)dW_s \tag{2.3}$$

is an It$\hat{o}$ process. The corresponding stochastic differential equation is given by

$$dX_t = a(X_t,t)dt + b(X_t,t)dW_t, \tag{2.4}$$

where $a(X_t,t)$ is the drift form, $b(X_t,t)$ is the diffusion form and W_s is a standard Wiener process.

2.1.2 It$\hat{o}$'s Lemma

Definition 7.

Let $F(S,t)$ be a twice differentiable function of t and of the random process S_t, and suppose that S_t follows the It$\hat{o}$ process

$$dS_t = a_t dt + \sigma_t dW_t, \;\; t \geq 0 \tag{2.5}$$

with well behaved drift and diffusion parameters a_t and σ_t. Then,

$$dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}\sigma_t^2 dt. \tag{2.6}$$

We substitute (2.5) into (2.6) for dS_t and by using the relevant stochastic differential equation we have

$$dF_t = \left[\frac{\partial F}{\partial S_t} a_t + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}\sigma_t^2\right] dt + \frac{\partial F}{\partial S_t}\sigma_t dW_t \tag{2.7}$$

which is known as It$\hat{o}$'s lemma and it has proved to be very vital in mathematical modeling of derivative prices. The F_t follows an It$\hat{o}$ process with the drift rate

$$\left[\frac{\partial F}{\partial S_t} a_t + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}\sigma_t^2\right]$$

and the variance rate

$$\left[\frac{\partial F}{\partial S_t}\right]^2 \sigma_t^2.$$

In general, with the It$\hat{o}$ formula we can determine the stochastic differential equation for the financial derivative given the SDE of the underlying asset [4]. It$\hat{o}$'s lemma is also useful in evaluating It$\hat{o}$ integral.

If a variable $S(t)$ follows a geometric Brownian motion, then it obeys a stochastic differential equation of the form

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.8}$$

and It$\hat{o}'s$ lemma is given for any function $F(S,t)$ as

$$dF = \left[\frac{\partial F}{\partial S}\mu S + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right] dt + \frac{\partial F}{\partial S}\sigma S dW, \tag{2.9}$$

where μ and σ are constants.

Example 2.1.1.

A stock price process S follows the random process in (2.8). We are interested in the process followed by Log$S$. Let $F(S,t) = $ Log$S$. Then

$$\begin{aligned} \frac{\partial F}{\partial S} &= \frac{1}{S}, \\ \frac{\partial F}{\partial t} &= 0, \\ \frac{\partial^2 F}{\partial S^2} &= \frac{-1}{S^2}. \end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.9), we get

$$d(\log S) = (\mu - \sigma^2/2)dt + \sigma dW.$$

This shows that Log$S$ is a Brownian motion with drift parameter $(\mu - \sigma^2/2)$ and variance parameter σ^2. Integrating the above expression between 0 and T, we derive an explicit formulation for the evolution of the stock price

$$\begin{aligned} \int_0^T d(\log S) &= (\mu - \sigma^2/2)\int_0^T dt + \sigma \int_0^T dW \\ \log(S_T) - \log(S_0) &= (\mu - \sigma^2/2)T + \sigma(W(T) - W(0)) \\ \text{and} \qquad S_T &= S_0 \exp\left[(\mu - \sigma^2/2)T + \sigma Z\sqrt{T}\right], \end{aligned} \tag{2.11}$$

where $Z \sim N(0,1)$. Therefore, stock dynamics follows a log-normal distribution [4].

Figure 2.1 shows the evolution of a stock price in a geometric Brownian motion path using (2.11) and this graph of simulated data can be compared to Figure 1.1 which is based on real life data. This enhances the understanding of the stochastic behaviour of the underlying assets and the assumption that stock returns are log normally distributed.

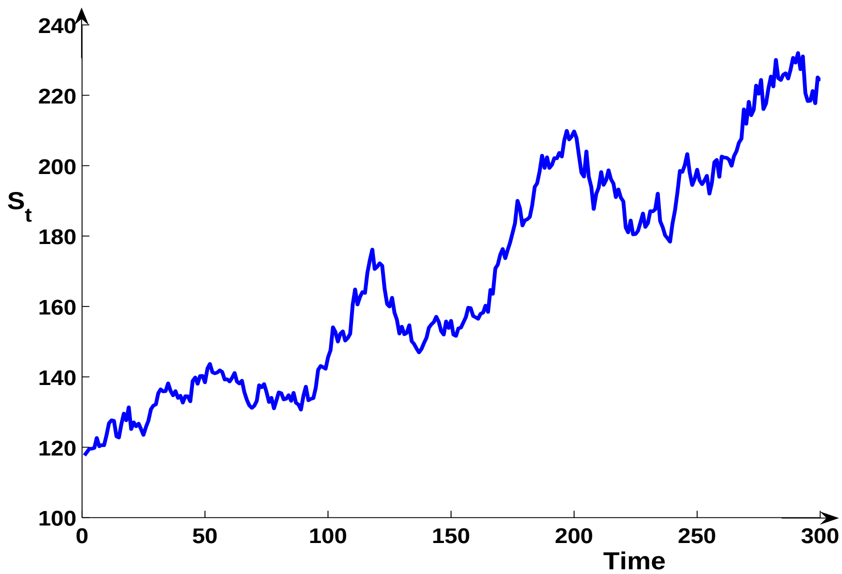

Figure 2.1: Simulation of a geometric Brownian motion path with, $S_0 = 120, \sigma = 0.30, \mu = 0.15$, $T = 1$, and $N = 300$ as samples drawn from the standard normal distribution.

Solution of an SDE

Now, consider the finite difference approximation in small discrete intervals of equal length h,

$$S_\kappa - S_{\kappa-1} = \alpha(S_{\kappa-1}, \kappa)h + \sigma(S_{\kappa-1}, \kappa)\Delta W_\kappa \qquad \text{for } \kappa = 1, 2, \ldots, n. \tag{2.12}$$

The solution to this equation is a random process S_t. We need a solution when h goes to zero as $n \to \infty$ for the partition of $[0, T]$. If a continuous time process S_t satisfies the equation

$$\int_0^t dS_u = \int_0^t \alpha(S_u, u)du + \int_0^t \sigma(S_u, u)dW_u \qquad \text{for } t > 0, \tag{2.13}$$

then we say that S_t is the solution of (2.1). We determine a process S_t such that

$$S_t = S_0 + \int_0^t \alpha(S_u, u)du + \int_0^t \sigma(S_u, u)dW_u \qquad \text{for } t \in [0, \infty). \tag{2.14}$$

The solution, $S_t = f(\alpha, \sigma, S_0, t, W_t)$ is a stochastic process and to solve the SDE, we will consider a candidate solution and apply Ito's lemma to check if it satisfies the SDE.

Example 2.1.2.

Consider the standard SDE called the *geometric SDE* given in (2.1) which we can write as

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\alpha(S_t, t) = \mu S_t$ is the drift parameter and $\sigma(S_t, t) = \sigma S_t$ is the diffusion parameter. We calculate the implied integral equation

$$\int_0^t \frac{dS_u}{S_u} = \int_0^t \mu du + \int_0^t \sigma dW_u. \tag{2.15}$$

The first integral at the right of (2.15) does not contain any random terms and the second integral contains a random term, but the coefficient dW_u is a time invariant constant. We write them as

$$\begin{aligned} \int_0^t \mu du &= \mu t \\ \text{and} \qquad \int_0^t \sigma dW_u &= \sigma[W_t - W_0], \end{aligned}$$

where $W_0 = 0$. We consider the candidate solution

$$S_t = S_0 e^{(\alpha - \sigma^2/2)t + \sigma W_t}.$$

By Ito's lemma, we have

$$\begin{aligned} dS_t &= S_0 e^{(\alpha-\sigma^2/2)t+\sigma W_t}\left[(\alpha-\frac{1}{2}\sigma^2)dt+\sigma dW_t+\frac{1}{2}\sigma^2 dt\right] \\ &= S_t[\alpha dt+\sigma dW_t] \end{aligned} \tag{2.16}$$

which is the original SDE with α equal to μ [24].

Earlier, we considered the Brownian motion. The next section illustrates the lognormal dynamics of derivative prices and the advantages of lognormal over the normal distribution.

2.2 Lognormal Dynamics

The rate of return of a stock can be expressed as

$$\frac{S_{t+\delta t}-S_t}{S_t}=\mu\delta t+\sigma Z\sqrt{\delta t}, \tag{2.17}$$

where $Z \sim N(0,1)$. Then, (2.17) tells us that as time passes by an amount of δt, the rate of return changes by $\mu\delta t$, and also jumps up or down by a random amount $\sigma Z\sqrt{\delta t}$.

Since there is a random change every δt time, then there are several random variables over any given time period and these sequences of random variables are called random processes. We noted that a trader tries to forecast the price of an asset in this time interval, δt.

Let us make the time intervals smaller and smaller. In the limit as $t \rightarrow 0$, the random process becomes a continuous process. Let us write $W_t = Z\sqrt{\delta t}$, then (2.17) can be expressed as

$$\frac{dS_t}{S_t}=\mu dt+\sigma dW_t. \tag{2.18}$$

Let us denote the right hand side of (2.18) by

$$dX_t=\mu dt+\sigma dW_t. \tag{2.19}$$

The variable μ is called the drift rate. We now use the fact that $dS_t/S_t = d(\text{Log}S_t)$, and write S_t as

$$S_t=S_0e^{X_t}. \tag{2.20}$$

This means that the logarithm of S_t is normally distributed. Hence, we say that the distribution of S_t is lognormal [32]. The lognormal distribution has the following advantages:

- A lognormally distributed variable can only take on positive values (between zero and infinity) unlike a normal distribution which allows variables to take both positive and negative values.

- It is mathematically tractable, and so we can obtain solutions for the value of the options if stock returns are log-normally distributed. The value of the option prices that we compute are very good approximations of actual market prices.

- It differs from the symmetric normal distribution in that it exhibits a skew with its mean and median all differing from that in a normal distribution. The stock dynamics will be treated as log-normally distributed with a specified mean and variance [4].

We consider the various factors affecting the value of an option and then move on to the derivation of the Black Scholes model.

2.2.1 Factors Affecting Option Value

The fundamental direct determinants of option value are, the current stock price S, the interest rate r, the strike price K, the expiration date T, the stock price volatility σ, and the dividend D expected during the life of the option. It is also important to consider whether the option is an American or European style option. These factors affecting an option value are summarised in table 2.1 for both the call and put option. Then we note that:

- An increase in S, means a higher intrinsic value if the call was in-the-money and hence the higher the premium. If it was out-of-the-money, then the higher the chance of being in-the-money. The reverse applies to put options.

- Increasing the strike price increases the intrinsic value of a put while lowering the intrinsic value of a call.

- The longer an option has to run, the greater the probability that it will be possible to exercise the option profitably, hence the greater the time value of the option (this argument is more intuitive for American option and can be proved mathematically for European option).

- The greater the expected movement in the price of the underlying instrument due to high volatility, the greater the probability that the option can be exercised for a profit and hence the more valuable the option is.

- Dividend payment lowers the current stock price and this increases the chance for a call option to be out-of-money and this in turn decreases the value of the option. For the put option, the decrease in stock price increases the chance of the put to be in-the-money.

- The higher the interest rate, the lower the present value of the exercise price the call buyer has contracted to pay in the event of exercise. A call option is the right to buy the underlying asset at the discounted value of the exercise price and thus the higher the degree of discount the more valuable is the right. Similarly, a put option is the right to sell the underlying asset at the discounted value of the exercise price and the higher the interest rate, the lower the value of the right [12].

Table 2.1: A summary of the general effect of each of the six variables.

Factor	**Call Option**	**Put Option**
Strike Price, K	Decrease	Increase
Spot price, S_0	Increase	Decrease
Interest rate, r	Increase	Decrease
Time to maturity, T	Increase	Increase
Volatility, σ	Increase	Increase
Dividend, D	Decrease	Increase

In 1973, Black and Scholes formulated and solved the partial differential equation governing the behaviour of contingent claims and this changed the general view of pricing derivatives as financial instruments.

2.3 Black - Scholes Equation

It was under the assumption of the lognormal dynamics of derivatives that Fischer Black, Myron Scholes and Merton developed their European option pricing model. They further made the following assumptions [5]:

- The probability of the rate of return for a stock is lognormally distributed with the mean same as the risk-free rate of return.

- There are no transaction costs or taxes.

- No risk-free arbitrage opportunities exist.

- There are no dividends during the life of the options.

- The risk-free interest rate r is known and constant over time.

- The variance of the return is constant over the life of the option.

- The underlying asset trading is continuous and the change of its price is continuous.

Let a stock price follow

$$dS = \mu S dt + \sigma S dW, \tag{2.21}$$

where μ is the trend, σ is the volatility and W follows a Wiener process. Now, suppose that f is the price of a call option or other derivative contingent on S. The variable f must be some function of S and t. Hence, by It$\hat{o}$'s lemma

$$df = \left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}\right]dt + \frac{\partial f}{\partial S}\sigma S dW. \tag{2.22}$$

The discrete versions of (2.21) and (2.22) are

$$\begin{aligned} \delta S &= \mu S \delta t + \sigma S \delta W \\ \text{and} \qquad \delta f &= \left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}\right]\delta t + \frac{\partial f}{\partial S}\sigma S \delta W. \end{aligned} \tag{2.23}$$

The Wiener process underlying f and S are the same and can be eliminated by choosing an appropriate portfolio of the stock and the derivative. We choose a portfolio of

$$\begin{aligned} -1 &: \text{ derivative} \\ +\frac{\partial f}{\partial S} &: \text{ shares.} \end{aligned}$$

The holder is short one derivative and long an amount $\partial f/\partial S$ of shares. We define Θ as the value of the portfolio and we have

$$\Theta = -f + \frac{\partial f}{\partial S}S. \tag{2.24}$$

The change $\delta\Theta$ in the value of the portfolio in the time interval δt is given by

$$\delta\Theta = -\delta f + \frac{\partial f}{\partial S}\delta S. \tag{2.25}$$

Substituting (2.23) into (2.25), we get

$$\delta\Theta = \left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right]\delta t. \tag{2.26}$$

The portfolio is now risk-less due to elimination of the δW term. It must then earn a return similar to other short term risk-free securities. Therefore

$$\delta\Theta = r\Theta\delta t, \tag{2.27}$$

where r is the risk-free interest rate. Substituting (2.24) and (2.26) into (2.27), we obtain

$$\left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right]\delta t = r\left[f - \frac{\partial f}{\partial S}S\right]\delta t. \tag{2.28}$$

Thus, we have

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf \tag{2.29}$$

which is the Black-Scholes-Merton differential equation.

Solving the partial differential equation in (2.29) gives an analytical formula for pricing the European style options. These options can only be exercised at the maturity date. The American style options are exercised any time up to the maturity date. Thus, the analytical formula we will derive is not appropriate for pricing them due to this early exercise privilege [17].

In the next section we examine the upper and lower boundary conditions for the American and European style options. Then, the boundary conditions for the European options will be applied to solve (2.29).

2.3.1 American and European Options

American options are just like European options, except that the American option allows the early exercise privilege. If we know the price of a European option, we can price the parallel American option by determining the impact of the early exercise privilege. The value of the right to exercise before expiration is the *early exercise premium*. Thus the American option must be worth at least as much as the European option. Therefore,

$$\begin{aligned} C(S,t,K) &\geq c(S,t,K) \\ P(S,t,K) &\geq p(S,t,K), \end{aligned} \tag{2.30}$$

where the American (European) call and put options are denoted by $C(c)$ and $P(p)$ respectively.

American and European Puts

The respective American and European lower boundary conditions that are determined by the arbitrage-free option prices are

$$\begin{aligned} P(S,t,K) &\geq K - S_t \\ p(S,t,K) &\geq Ke^{-r(T-t)} - S_t. \end{aligned} \tag{2.31}$$

The upper boundary conditions are

$$\begin{aligned} P(S,t,K) &\leq K \\ p(S,t,K) &\leq Ke^{-r(T-t)}. \end{aligned} \tag{2.32}$$

The price difference between American and European options depends largely on the extent to which the option is in-the-money, the interest rate and the amount of time remaining. The early exercise of an American put discards the value of waiting to see how stock prices evolve.

For an American put on a dividend paying stock, the optimal time to exercise is generally immediately after a dividend payment. The dividend payment reduces the stock price and this pushes the put further into-the-money. A put option when held in conjuction with the stock, insures the holder against the stock price falling below a certain level. However, it may be optimal for an investor to forgo this insurance and exercise early in order to realize the strike price immediately. It is optimal to exercise a put before the maturity date on a non-dividend paying stock [10].

American and European Calls

For a non-dividend paying stock, early exercise is never optimal, and the price of an American call carries the same value as its European counterpart. The respective lower boundary and upper boundary conditions are given by

$$\begin{aligned} & C(S,t,K),\ \ c(S,t,K) \ \ \geq \ \ S_t - Ke^{-r(T-t)} \\ \text{and} \qquad & C(S,t,K),\ \ c(S,t,K) \ \ \leq \ \ S_t. \end{aligned} \tag{2.33}$$

If the underlying stock pays a dividend, it can be rational to exercise early, and an American call can be worth more than the European call. The early exercise should occur immediately before a dividend payment as a dividend payout lowers the current stock price and this in turn lowers the call intrinsic value.

The American call on a non-dividend paying stock should not be exercised early as the call option acts like an insurance to the holder against the stock price falling below the exercise price. This insurance vanishes when the option is exercised. The latter the strike price is paid out, the better for the option holder [21].

We have considered the boundary conditions for both the American and European options. The boundary conditions for the European call option will be applied in solving the Black Scholes PDE.

2.3.2 Solution of the Black-Scholes Equation

The Payoff condition is $f(S, t = T) = \max(S - K, 0)$. The lower and upper boundary conditions are given by (2.33). These are the conditions that should be satisfied by the PDE.

Let $\tau = T - t$, where T is the expiration time and t the present time. Then, (2.29) can be written as

$$\frac{\partial f}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf. \tag{2.34}$$

Taking $y = \ln S$

$$\begin{aligned} \frac{\partial f}{\partial S} &= \frac{1}{S}\frac{\partial f}{\partial y} \\ \frac{\partial^2 f}{\partial S^2} &= -\frac{1}{S^2}\frac{\partial f}{\partial y} + \frac{1}{S^2}\frac{\partial^2 f}{\partial y^2}. \end{aligned} \tag{2.35}$$

We now introduce a new notation $w(y, \tau) = e^{r\tau} f(y, \tau)$. Using (2.35), the Black-Scholes PDE becomes a diffusion equation

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial y^2} + \left[r - \frac{\sigma^2}{2}\right]\frac{\partial w}{\partial y} \tag{2.36}$$

and has a fundamental solution as a normal function

$$\phi(y,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}}\mathrm{exp}\Bigg[-\frac{[y+(r-\sigma^2/2)\tau]^2}{2\sigma^2\tau}\Bigg]. \tag{2.37}$$

The solution to (2.36) is

$$w(y,\tau) = \int_{-\infty}^{\infty} w(\xi,0)\phi(y-\xi,\tau)d\xi. \tag{2.38}$$

We use the payoff condition and the fundamental solution of (2.37) to obtain

$$\begin{aligned} w(y,\tau) &= \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\infty}\max(\mathrm{e}^{\xi}-K,0)\mathrm{exp}\Bigg[-\frac{[y-\xi+(r-\sigma^2/2)\tau]^2}{2\sigma^2\tau}\Bigg]d\xi \\ &= \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty}(\mathrm{e}^{\xi}-K)\mathrm{exp}\Bigg[-\frac{[y-\xi+(r-\sigma^2/2)\tau]^2}{2\sigma^2\tau}\Bigg]d\xi. \end{aligned} \tag{2.39}$$

We denote the distribution function for a normal variable by $N(x)$:

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{-\mathrm{u}^2/2}du. \tag{2.40}$$

We can express (2.39) as

$$\begin{aligned} w(y,\tau) = &\ \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty}\mathrm{e}^{\xi}\mathrm{exp}\Bigg[-\frac{(-\xi+A)^2}{2\sigma^2\tau}\Bigg]d\xi \\ - &\ \frac{K}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{\infty}\mathrm{exp}\Bigg[-\frac{(-\xi+A)^2}{2\sigma^2\tau}\Bigg]d\xi, \end{aligned} \tag{2.41}$$

where $A = y+(r-\sigma^2/2)\tau = \ln S+(r-\sigma^2/2)\tau$. We consider the second term in the right-hand side of (2.41). Let

$$Z = (-\xi+A)/\sigma\sqrt{\tau}. \tag{2.42}$$

Then using (2.42), the $d\xi$ becomes

$$d\xi = -\sigma\sqrt{\tau}dZ. \tag{2.43}$$

and the limits of (2.41) using (2.42) are given as

$$\begin{aligned} Z &= -\infty \text{ when } \xi = \infty \\ Z &= \frac{-\ln K + A}{\sigma\sqrt{\tau}} = \frac{-\ln K + \ln S + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \equiv d_2 \text{ when } \xi = \ln K. \end{aligned} \tag{2.44}$$

Changing the variable from ξ to Z, the second term of (2.41) becomes

$$\begin{aligned} \frac{K}{\sqrt{2\pi}}\int_{d_2}^{-\infty}\mathrm{e}^{-Z^2/2}dZ &= -\frac{K}{\sqrt{2\pi}}\int_{-\infty}^{d_2}\mathrm{e}^{-Z^2/2}dZ \\ &= -KN(d_2). \end{aligned} \tag{2.45}$$

The integrand of the first term in (2.41) is expressed as

$$\begin{aligned}
& \mathrm{e}^{\xi}\exp\left[-\frac{(-\xi+A)^2}{2\sigma^2\tau}\right] \\
= \;& \exp\left[-\frac{\xi^2-2(A+\sigma^2\tau)\xi+A^2}{2\sigma^2\tau}\right] \\
= \;& \exp\left[-\frac{\xi^2-2(A+\sigma^2\tau)\xi+(A+\sigma^2\tau)^2-(A+\sigma^2\tau)^2+A^2}{2\sigma^2\tau}\right] \\
= \;& \exp\left[-\frac{[\xi-(A+\sigma^2\tau)]^2}{2\sigma^2\tau}+\tfrac{1}{2}\sigma^2\tau+A\right] \\
= \;& \mathrm{e}^{\frac{1}{2}\sigma^2\tau+A}\exp\left[-\frac{[\xi-(A+\sigma^2\tau)]^2}{2\sigma^2\tau}\right]. \qquad (2.46)
\end{aligned}$$

We use the definition of A to have

$$\mathrm{e}^{\frac{1}{2}\sigma^2\tau+A}=\mathrm{e}^{y+r\tau}=S\mathrm{e}^{r\tau}. \qquad (2.47)$$

Inserting (2.46) and (2.47) into the first term of (2.41), that first term becomes

$$\frac{1}{\sigma\sqrt{2\pi\tau}}S\mathrm{e}^{r\tau}\int_{\ln K}^{\infty}\exp\left[-\frac{[\xi-(A+\sigma^2\tau)]^2}{2\sigma^2\tau}\right]d\xi. \qquad (2.48)$$

By changing the variables as we did in the previous case, we get

$$\frac{1}{\sqrt{2\pi}}S\mathrm{e}^{r\tau}\int_{-\infty}^{d1}\mathrm{e}^{-Z^2/2}dZ=S\mathrm{e}^{r\tau}N(d_1). \qquad (2.49)$$

The last line of (2.39) can be written as

$$w(y,\tau)=\mathrm{e}^{r\tau}SN\left[\frac{\ln(S/K)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right]-KN\left[\frac{\ln(S/K)+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right] \qquad (2.50)$$

and it implies that

$$c=SN(d_1)-Ke^{-r\tau}N(d_2) \qquad (2.51)$$

where

$$d_1=\frac{\ln(S/K)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \qquad \text{and} \qquad d_2=d_1-\sigma\sqrt{\tau}. \qquad (2.52)$$

This is the Black-Scholes formula for the price at time zero of a European call option on a non dividend paying stock [5]. We can derive the corresponding European put option formula for a non-dividend paying stock by using the call - put parity given by $p=c+Ke^{-rT}-S$. The European put analytical formula is

$$p=Ke^{-r\tau}N(-d_2)-SN(-d_1). \qquad (2.53)$$

The European call and put analytical formulas have gained popularity in the world of finance due to the ease with which one can use the formula to value the European options. When calculating the value of options, the other parameters apart from the volatility can easily be observed from the market. Thus it becomes necessary to find appropriate methods of estimating the volatility.

2.3.3 Volatility

Volatility is the standard deviation measure of an asset's potential deviating from its current price. This is the simple definition we gave for risk. The price volatility creates greater value for a given option, for the greater the volatility of the underlying, the greater the value of the option. For options, volatility is 'good' while for other financial assets, volatility is 'bad'. This is due to the fact that the purchaser of options enjoy only the upside potential, not downside risk. Other financial assets have both risks.

Investors are usually assumed to be risk averse and they place a lower value on highly priced volatile assets. Volatility gives uncertain values and therefore risk of loss.

The price volatility in asset markets is caused mainly by information release, the process of trading, and market-making for financial instruments. Information release fall into two categories, the anticipated and unanticipated information.

Anticipated information includes economic statistics, political and social information. The impact of the information is often driven by market expectations and can be analysed with reference to past releases as we can develop probabilistic expectations of anticipated asset price volatility from the historical reaction of the market.

Unanticipated information include wars, natural disasters, etc. The information can have substantial unpredictable impact on asset price volatility. It is difficult to predict this type of information release.

The volatility estimate is a measure of the uncertainty about the returns on the asset. When pricing options, the volatility is assumed to be: (1) Time homogenous, that is, the same over the life of the asset. (2) Constant between the pricing date and option expiry [12].

The two major approaches for the estimation of volatility are the historical and implied volatility methods.

Historical/Empirical Method

This method estimates the volatility by calculating the standard deviation of the logs of the price changes of a sample time series of historical data for the asset price. The daily return is given as, $X_t = \ln(S_t/S_{t-1})$.

The variance is estimated by the sample variance, which is normalised by $(n-1)$ to make it an unbiased statistic

$$\text{Var} = \sigma^2_{\text{day}} = \frac{1}{n-1}\left[\sum_{t=1}^{n}(X_t^2 - \bar{X}^2)\right].$$

The standard deviation computed equates to the daily volatility if daily data is used. We get the annual volatility by

$$\sigma_{\text{yr}} = \sigma_{\text{day}} \times \sqrt{252}$$

where σ_{yr} is the annual volatility, σ_{day} is the daily volatility and $\bar{X}$ is the mean value of the daily returns. We normally take 252 days as the number of trading days in a year. The assumed uncertainty about the asset does not increase linearly.

If the asset pays dividends, then the asset price sequence must be adjusted to reflect the non-homogenous nature of the data series. The transition from cum-dividend to ex-dividend will affect the price of the asset. A dividend payment increases the return to be paid to the buyer. If the buyer has an asset that pays a dividend $D,$ then the daily price return is restated as $\ln[(S_t + D)/S_{t-1}]$.

Implied Volatility

The implied volatility is the volatility of the underlying which when substituted into the Black Scholes formula gives a theoretical price equal to the market price.

The implied volatility is deficient in that: (1) The options currently trading volatility is treated as being the true constant asset price volatility parameter. (2) Options with different strike prices and same maturity often demonstrate different implied volatilities (so called 'volatility smile') [12].

The major value of implied volatility as a volatility estimation is that it provides an observable measure of the relevant option market expectations as to volatility. The Newton-Raphson method or any suitable numerical method is used in the derivation of the volatility with respect to the option price. We have to solve numerically for σ the equation $V_{BS}(S_o, T, \sigma, r, K) =$ known call/put value, where V_{BS} is the Black Scholes value.

2.3.4 Dividend Paying Stock

We relax the assumption that no dividends are paid during the life of the option and examine the effect of dividend on the value of European options by modifying the Black Scholes PDE to cater for these dividends payments.

Continuous Dividend Yield Model

Let λ denote the constant continuous dividend yield which is known. This means that the holder receives a dividend $\lambda S \delta t$ within the time interval δt. The share value is lowered after the payout of the dividend and so the expected rate of return μ of a share becomes $(\mu - \lambda)$. The geometric Brownian motion model in (2.21) becomes

$$\frac{dS}{S} = (\mu - \lambda)dt + \sigma dZ \tag{2.54}$$

and the modified PDE in (2.29) is given by

$$\frac{\partial c}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + (r - \lambda) S \frac{\partial c}{\partial S} - rc = 0. \tag{2.55}$$

Let $\tau = T - t$. Solving (2.55) by applying the same method in section 2.3.2, the European call option for a dividend paying stock is given by

$$c = Se^{-\lambda\tau} N(\hat{d}_1) - Ke^{-r\tau} N(\hat{d}_2) \tag{2.56}$$

and the European put option is

$$p = Ke^{-r\tau} N(-\hat{d}_2) - Se^{-\lambda\tau} N(-\hat{d}_1) \tag{2.57}$$

where

$$\hat{d}_1 = \frac{\ln(S/K) + (r - \lambda + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \qquad \text{and} \qquad \hat{d}_2 = \hat{d}_1 - \sigma\sqrt{\tau}.$$

The results in (2.56) and (2.57) can similarly be achieved by considering the non-dividend paying formulas in (2.51), (2.52) and (2.53). The dividend payment lowers the stock price from S to $Se^{-\lambda\tau}$ and the risk- free interest rate which is the rate of return from r to $(r - \lambda)$ [17].

Discrete Dividend

Suppose that the underlying asset pays N discrete dividends at known payments dates $t_1, t_2, \ldots, t_N$ of amounts $D_1, D_2, \ldots, D_N$, respectively. Since the actual amounts of dividends and ex-dividend dates are known then we can assume that the asset price is composed of two components:

- The risk-free component that will be used to pay the known dividends during the life of the option. This risk-free component is taken to be the present value of the future dividends discounted at the risk free interest rate. By the time the option matures, the dividends will have been paid and the risk-less component will no longer exist.

- Risky component which follows a stochastic process. The value of the risky component denoted as $\tilde{S}_t$ is

$$\tilde{S}_t = S_t - D_i e^{-rt_i} \qquad \text{for } \ i = 1, 2, \ldots, N. \tag{2.58}$$

The new asset price $\tilde{S}_t$ is then used to compute the value of the option. Example 2.3.1 shows how to price an asset on a stock with discrete dividend payments.

Example 2.3.1.

Consider a European call option on a stock when there are ex-dividend dates in three and five months time. The dividend on each ex-dividend date is expected to be \$ 0.60. The current share price is \$ 42, the exercise price is \$ 42, the stock volatility is 25% per annum, the risk-free rate of interest is 10% per annum, and the time to maturity is six months.

The present value of the dividend is calculated as, $\$0.6e^{-0.25*0.1} + \$0.6e^{-0.4167*0.1} = \$1.161$.

Therefore, $S_0 = \$40.839, K = \$42, r = 0.1, T = 0.5$ and $\sigma = 0.25$.

We apply the European call and put analytical formula in (2.56) and (2.57).

The call price $= \$3.31$ and the put price $= \$2.42$.

2.4 Options on Futures

The options on futures or futures option is a contract that grants the holder the right, but not the obligation, to buy or sell a futures contract at a fixed price called the strike price, up to a specified expiration date.

An option to buy (sell) a futures is a call (put). A futures option can either be an American or European style option.

The underlying asset of the options on futures is a futures contract which normally matures shortly after the expiration of the option. When the call (put) is exercised, the holder acquires long (short) position in the underlying futures contract plus a cash amount equal to the excess of futures (strike) price over the strike (futures) price.

An option on a futures contract differs from an option on the spot instrument in that upon exercise, the option holder establishes a position in the underlying futures contract which expires after the options on futures. The value of a call option on a futures contract should be lower than the value of a call option on the physical asset. The futures price should already impound the carrying costs associated with the physical commodity [10].

2.4.1 American Options on Futures

The minimum value of an American call and put on a futures is its intrinsic value. The respective call and put intrinsic values are

$$\begin{aligned} C &\geq \max(F-K,0) \\ P &\geq \max(K-F,0), \end{aligned} \tag{2.59}$$

where F is the futures price and K the strike price. We recall that in the absence of dividend on a stock, a call option on the stock would not be optimal to exercise early, however, a put option might be optimal. For the option on a futures contract, either a call or a put might be exercised early.

We consider a deep-in-the-money American call. A call on a futures will move nearly one for one with the futures price. Thus, the call on the futures will act almost exactly like a long position in a futures contract. By exercising the call and replacing it with a long position in the futures, the investor frees up funds tied up in the call and has the same opportunity to profit.

The futures price $f+$ is the price at which the American call will equal its intrinsic value. When $\max(F-K,0) \geq f+$, then the American call will be exercised early as the long position in a futures contract and in a call option will offer the same return to the investor. This is illustrated in figure 2.2.

Similarly, for put options on futures, a deep-in-the-money American put tend to be exercised early. The price of the American put option on futures approaches its intrinsic value of $(K - F)$. The futures price $f*$ is the price at which the American put equals its intrinsic value. When $\max(K - F, 0) \leq f*$, the American put will be exercised early as the long position in a futures contract and a put option will offer the same return to the investor [10].

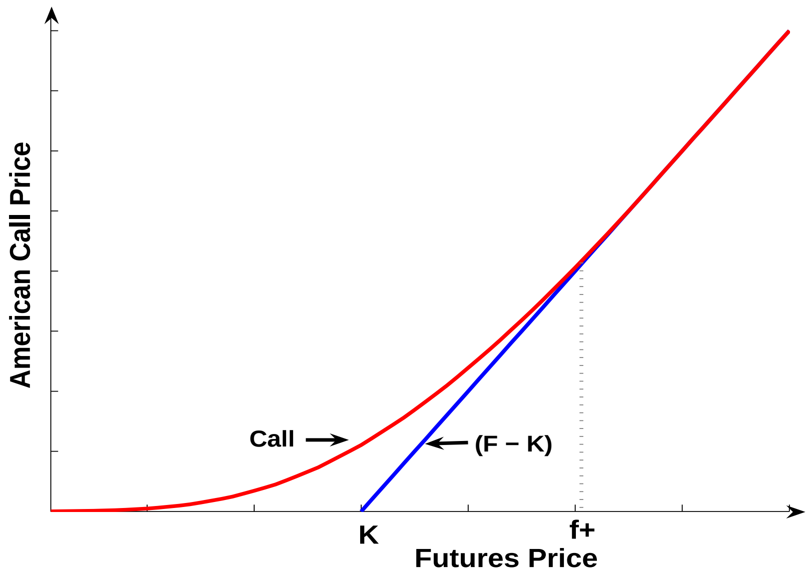

Figure 2.2: At a price $f+$, the American call will behave almost identically to the futures contract and the call will be exercised early.

2.4.2 European Options on Futures

The lower bound of a European call and put are respectively given by

$$\begin{aligned} c &\geq \max[(F - K)\mathrm{e}^{-r(T-t)}, 0] \\ p &\geq \max[(K - F)\mathrm{e}^{-r(T-t)}, 0]. \end{aligned} \tag{2.60}$$

The European options on futures do not have the early exercise privilege like the American counterparts. This implies that the time value is zero as they can only be exercised at maturity date and at this time T, the time value of the option is zero.

If we compare the lower bound of the European options on futures contract with that of the spot price, we have made a substitution of $F\mathrm{e}^{-r(T-t)} = S$.

Put-Call Parity

We construct two portfolios, $\mathbf{A}$ and $\mathbf{B}$. Portfolio $\mathbf{A}$ will consist of a long futures and a long put on the futures. This can be thought of as a *protective put.* A *protective put* is an investment strategy involving the use of a long position in a put and a stock to provide a minimum selling price for the stock.

Portfolio $\mathbf{B}$ will consist of a long call and a long bond with a face value equal to the exercise price of the futures contract minus the futures price.

The current value of portfolio $\mathbf{A}$ long futures (long put) position will equal the portfolio B long call (long bond) position.

The current value of portfolio $\mathbf{A}$ is p which is a protective put and that of portfolio $\mathbf{B}$ is $c + (K - F)\mathrm{e}^{-r(T-t)}$. Since portfolio $\mathbf{B}$ is also like a protective put, then its current value is equal to the current value of portfolio $\mathbf{A}$. We conclude that

$$p = c + (K - F)\mathrm{e}^{-r(T-t)} = c + K\mathrm{e}^{-r(T-t)} - F\mathrm{e}^{-r(T-t)}. \tag{2.61}$$

We can replace S by $F\mathrm{e}^{-r(T-t)}$ in (1.3) to derive the put-call parity for the futures options [10].

2.4.3 Black's Model

The options on futures and the underlying futures expire on different dates. In $1976,$ Fischer Black developed a variation of his own Black Scholes model for pricing European options on futures under the assumption that: (1) The option and the futures expires simultaneously. (2) The futures price equals the forward price. For the futures and forward prices to be equal, we have to assume that interest rates are non stochastic.

The modification of the Black-Scholes analytical formula for the European call option on a spot instrument gives us the European call option on futures as

$$c = e^{-r\tau}\Bigg[FN(d_1) - KN(d_2)\Bigg]. \tag{2.62}$$

The futures price F takes the place of the stock price S. We use the put-call parity relationship in (2.61) to obtain the European put option on futures

$$p = e^{-r\tau}\Bigg[KN(-d_2) - FN(-d_1)\Bigg] \tag{2.63}$$

where

$$d_1 = \frac{\ln(F/K) + (\sigma^2\tau/2)}{\sigma\sqrt{\tau}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{\tau}.$$

d_1 does not contain the risk-free rate r as it does in Black-Scholes model in (2.52). The risk free rate captures the opportunity cost of funds tied up in the stock. If the option is on futures contract, no funds are invested in the futures, thus no opportunity cost as the *cost to carry* is zero. The *cost to carry* is the cost involved in holding or storing an asset that consists of storage costs and interest lost on funds tied up. The futures price reflects the cost to carry on the underlying spot asset, but the futures itself does not have a cost to carry because there are no funds tied up and no storage costs.

The dividends do not show up in the Black's model unlike the Black Scholes model. The reason is that even though dividends do affect the call price, it is in an indirect manner as the futures price captures all the effects of the dividends [11].

Futures and Forward Price Difference

The futures and forward prices are not identical due to varying interest rate patterns in the two prices and this has effect on the daily settlement cash flows on the futures. By assuming that interest rates are non stochastic, then the futures and forward prices are equal.

When both the futures prices and interest rates rise, the holder of the long position will invest the excess funds at this rising interest rate. When both the futures prices and interest rates fall, the holder of the long position will borrow to cover the daily marking to market at the falling interest rate.

Thus there will be a preference for the futures contracts when the interest rates and futures prices move in the same direction.

However, if the futures prices and interest rates move in opposite direction, then the preference will be for forward contracts.

When the futures prices rise and interest rates fall, the holder of the long position will invest the excess funds from the daily marking to market at this falling interest rate. When the futures prices fall and interest rates rise, the holder of the long position will borrow to cover the daily marking to market at the rising interest rate.

Thus, the long futures contracts are preferred over the long forward contracts when interest rates rise (fall) and the futures prices rise (fall). The forward contracts are preferred over the long futures contracts when interest rates rise (fall) and the futures prices fall (rise) [10].

We conclude that futures price is a good proxy for the forward price. The model works well for any forward price. Because of its relative mathematical simplicity, Black's model is widely used in industry to price options on interest rate futures as well as other interest rate options.

In this chapter, we have discussed and derived the Black Scholes analytical formula for the European options. The modified version of the Black Scholes for the dividend paying stock and the Black's model for the options on futures contracts were also discussed.

In the subsequent chapters, we discuss some numerical methods used in the valuation of options. The convergence of these methods to the price of the option will be compared to the exact Black Scholes and Black's value of the spot and futures options respectively.

Chapter 3

Binomial Model

The binomial models were first suggested by Cox, Ross and Rubinstein (CRR) in 1979 [17], and assumes that stock price movements are composed of a large number of small binomial movements. Binomial models come in handy particularly when the holder has early exercise decisions to make prior to maturity or when exact formulas are not available. These models can accommodate complex option pricing problems.

First, we divide the life time $[0, T]$ of the option into N time subintervals of length δt, where $\delta t = T/N$. Suppose that S is the stock price at the beginning of a given time period. Then the binomial model of price movements assumes that at the end of each time period, the price will either go up to uS with probability q or down to dS with probability $(1-q)$, where u and d are the up and down factors with $d < 1 < u$.

We recall that by the principle of risk neutral valuation, the expected return from all traded options is the risk-free interest rate. We can value future cash flows by discounting their expected values at the risk free interest rate. The parameters u, d and q satisfy the conditions for the risk-neutral valuation and the lognormal distribution of the stock price and we have

$$\begin{aligned} S\mathrm{e}^{r\delta t} &= Squ + S(1-q)d \\ \mathrm{e}^{r\delta t} &= qu + (1-q)d. \end{aligned} \tag{3.1}$$

Since S follows a lognormal distribution, its variance is given by

$$\mathrm{Var}[S] = S^2\mathrm{e}^{2r\delta t}[\mathrm{e}^{\sigma^2\delta t} - 1] \tag{3.2}$$

where $\mathrm{Var}(S) = E(S^2) - [E(S)]^2$. This can be expressed as

$$S^2\mathrm{e}^{2r\delta t}[\mathrm{e}^{\sigma^2\delta t} - 1] = qu^2S^2 + (1-q)d^2S^2 - [quS + (1-q)dS]^2, \tag{3.3}$$

and this can be simplified to yield

$$\mathrm{e}^{2r\delta t+\sigma^2\delta t} = qu^2 + (1-q)d^2. \tag{3.4}$$

If we assume that $u = 1/d$, then it follows from (3.1) and (3.4) that

$$\begin{aligned} u &= \mathrm{e}^{\sigma\sqrt{\delta t}}, \\ d &= \mathrm{e}^{-\sigma\sqrt{\delta t}}, \\ q &= \frac{\mathrm{e}^{r\delta t} - d}{u - d}. \end{aligned} \tag{3.5}$$

The probability q obtained in (3.5) is called the risk neutral probability. It is the probability of an upward movement of the stock price that ensures that all bets are fair, that is, it ensures that there is no arbitrage.

The expectation of the share price can be written as

$$E[S_1] = quS + d(1-q)S, \tag{3.6}$$

where S_1 is the share price after one period, and using the value of q in (3.5), we find that $E[S_1] = S\mathrm{e}^{r\delta t}$ which naturally follows from our assumption of the risk-neutral valuation.

In the next part, we discuss the CRR model.

3.1 CRR Model

The Cox, Ross and Rubinstein model contains the Black-Scholes analytical formula as the limiting case as the number of steps tends to infinity.

We know that after one time period, the stock price can move up to uS with probability q or down to dS with probability $(1-q)$. Therefore the corresponding value of the call option at the first time movement δt is given by

$$\begin{aligned} c_u &= \max(uS-K, 0) : \text{after upward movement}, \\ c_d &= \max(dS-K, 0) : \text{after downward movement}. \end{aligned} \tag{3.7}$$

We need to derive a formula to calculate the fair value of the option. The risk neutral call option price at the present time is

$$c = [qc_u + (1-q)c_d]e^{-r\delta t}. \tag{3.8}$$

We extend the binomial model to two periods. Let c_{uu} denote the call value at time $2\delta t$ for two consecutive upward stock movement, c_{ud} for one downward and one upward movement and c_{dd} for two consecutive downward movement of the stock price. Then we have

$$\begin{aligned} c_{uu} &= \max(u^2S - K, 0) \\ c_{ud} &= \max(udS - K, 0) \\ c_{dd} &= \max(d^2S - K, 0), \end{aligned} \tag{3.9}$$

which are illustrated in figure 3.1 for the three different states of the asset and call prices in the two period binomial model. Since q is the risk neutral probability, the values of call options at time, δt are

$$\begin{aligned} c_u &= \mathrm{e}^{-r\delta t}[qc_{uu} + (1-q)c_{ud}] \\ c_d &= \mathrm{e}^{-r\delta t}[qc_{ud} + (1-q)c_{dd}]. \end{aligned} \tag{3.10}$$

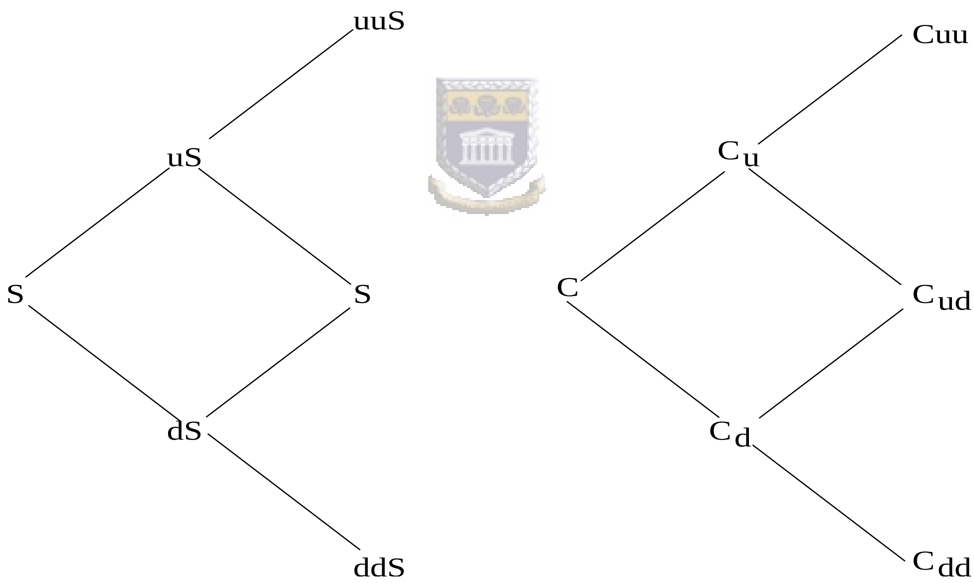

Figure 3.1: Binomial tree for the respective asset and call price in a two-period model.

We substitute (3.10) into (3.8) and this gives us the current call value using time $2\delta t$ as

$$c = \mathrm{e}^{-2r\delta t}[q^2c_{uu} + 2q(1-q)c_{ud} + (1-q)^2c_{dd}]. \tag{3.11}$$

We generalize the result in (3.11) to value an option which expires at $T = N\delta t$ as

$$\begin{aligned} c &= \mathrm{e}^{-Nr\delta t}\sum_{j=0}^{N}\binom{N}{j}q^j(1-q)^{N-j}c_{u^jd^{N-j}} \\ &= \mathrm{e}^{-Nr\delta t}\sum_{j=0}^{N}\binom{N}{j}q^j(1-q)^{N-j}\max(u^jd^{N-j}S-K,0), \end{aligned} \tag{3.12}$$

where $\binom{N}{j} = N!/j!(N-j)!$ is the binomial coefficient. We assume that m is the smallest integer for which the option's intrinsic value in (3.12) is greater than zero. This implies that $u^m d^{N-m}S \geq K$. Then, (3.12) is written as

$$c = S\mathrm{e}^{-Nr\delta t}\sum_{j=m}^{N}\binom{N}{j}q^j(1-q)^{N-j}u^jd^{N-j} - K\mathrm{e}^{-Nr\delta t}\sum_{j=m}^{N}\binom{N}{j}q^j(1-q)^{N-j} \tag{3.13}$$

which gives us the present value of the call option.

The term $e^{-Nr\delta t}$ is the discounting factor that reduces c to its present value. The first term $\binom{N}{j}q^j(1-q)^{N-j}$ is the binomial probability of j upward movements to occur after the first N trading periods and $u^jd^{N-j}S$ is the corresponding value of the asset after j upward move of the stock price. The second term is the present value of the option's strike price. Let $R = \mathrm{e}^{r\delta t}$. We substitute R in the first term in (3.13) to yield

$$\begin{aligned} c &= SR^{-N}\sum_{j=m}^{N}\binom{N}{j}q^j(1-q)^{N-j}u^jd^{N-j} - K\mathrm{e}^{-Nr\delta t}\sum_{j=m}^{N}\binom{N}{j}q^j(1-q)^{N-j} \\ &= S\sum_{j=m}^{N}\binom{N}{j}\Big[R^{-1}qu\Big]^j\Big[R^{-1}(1-q)d\Big]^{N-j} - K\mathrm{e}^{-Nr\delta t}\sum_{j=m}^{N}\binom{N}{j}q^j(1-q)^{N-j} \end{aligned} \tag{3.14}$$

Now, let $\Phi(m; N, q)$ be the binomial distribution function. That is

$$\Phi(m; N, q) = \sum_{j=m}^{N}\binom{N}{j}q^j(1-q)^{N-j} \tag{3.15}$$

is the probability of at least m success in N independent trials, each resulting in a sucess with probability q and in a failure with probability $1-q$. Then, letting $q' = R^{-1}qu$, we easily see that $R^{-1}(1-q)d = 1-q'$.

Consequently it follows from the second equality in (3.14) that

$$c = S\Phi(m; N, q') - K\mathrm{e}^{-rT}\Phi(m; N, q) \tag{3.16}$$

where $T = N\delta t$.

The model in (3.16) was developed by Cox, Ross and Rubinstein and we will refer to it as the CRR model. The corresponding value of the European put option can be obtained using the call-put parity relationship in (1.3).

3.2 Numerical Implementation

When stock price movements are governed by a multi-step binomial tree, we can treat each binomial step separately. The multi-step binomial tree can be used for the American and European style options.

Like the Black Scholes model, the CRR formula in (3.16) can only be used in the valuation of European style options and can easily be implemented in Matlab. To overcome this problem, we use a different multi-period binomial model for the American style options on both the dividend and non dividend paying stocks.

The no-arbitrage arguments are used and no assumptions are required about the probabilities of up and down movements in the stock price at each node. We now explain the procedure for the implementation of the multi-period binomial model.

At time zero, the stock price S is known. At time $\delta t,$ there are two possible stock prices uS and $dS.$ At time $2\delta t,$ there are three possible stock prices $u^2S,\ udS,\ d^2S,$ and so on. In general, at time $i\delta t,$ where $0 \leq i \leq N,\ (i+1)$ stock prices are considered, given by

$$Su^j d^{N-j} \qquad \text{for } j = 0, 1, \ldots, N \tag{3.17}$$

where N is the total number of movements and j is the total number of up movements. The multi-period binomial model can reflect numerous stock price outcomes if there are numerous periods. Fortunately, the binomial option pricing model is based on recombining trees, otherwise the computational burden would quickly become overwhelming as the number of moves in the tree is increased.

Options are evaluated by starting at the end of the tree at time T and working backward. We know the worth of a call and a put at time T is $\max(S_T - K, 0)$ and $\max(K - S_T, 0)$ respectively. Because we are assuming the risk-neutral world, the value at each node at time $(T - \delta t)$ can be calculated as the expected value at time T discounted at rate r for a time period $\delta t.$ Similarly, the value at each node at time $(T - 2\delta t)$ can be calculated as the expected value at time $(T - \delta t)$ discounted for a time period δt at rate $r,$ and so on. By working back through all the nodes, we are able to obtain the value of the option at time zero.

Suppose that the life of an European option on a non-dividend paying stock is divided into N subintervals of length $\delta t.$ Denote the jth node at time $i\delta t$ as the (i, j) node, where $0 \leq i \leq N$ and $0 \leq j \leq i.$ Define $f_{i,j}$ as the value of the option at the (i, j) node. The stock price at the

(i, j) node is $Su^j d^{i-j}$. Then, the respective European call and put can be expressed as

$$f_{N,j} = \max(Su^j d^{N-j} - K, 0) \qquad \text{for } j = 0, 1, \ldots, N,$$
$$f_{N,j} = \max(K - Su^j d^{N-j}, 0) \qquad \text{for } j = 0, 1, \ldots, N. \tag{3.18}$$

There is a probability q of moving from the (i, j) node at time $i\delta t$ to the $(i+1, j+1)$ node at time $(i+1)\delta t$, and a probability $(1-q)$ of moving from the (i, j) node at time $i\delta t$ to the $(i+1, j)$ node at time $(i+1)\delta t$. The risk neutral valuation is

$$f_{i,j} = \mathrm{e}^{-r\delta t}[qf_{i+1,j+1} + (1-q)f_{i+1,j}] \text{ and } 0 \le i \le N-1, \; 0 \le j \le i. \tag{3.19}$$

For an American option, we check at each node to see whether early exercise is preferable to holding the option for a further time period δt. When early exercise is taken into account, this value of $f_{i,j}$ must be compared with the option's intrinsic value [17] and we have

$$f_{i,j} = \max\left[K - Su^j d^{i-j}, \; e^{-r\delta t}(qf_{i+1,j+1} + (1-q)f_{i+1,j})\right]. \tag{3.20}$$

We compute the values of both European and American style options. See appendix A.1 for the Matlab code.

The results in table 3.1 for the American and European options using the multi-period binomial model are compared to those obtained using the the Black Scholes model. The convergence of the multi-period model to the Black Scholes value of the option is also made more intuitive by the graph in figure 3.2. Table 3.1 and figure 3.2 uses the parameters $S = 45, K = 40, T = 0.5, r = 0.1$ and $\sigma = 0.25$ in computing the options prices, and as we increase the number of steps denoted by N.

Table 3.1: Comparison of the Multi-step binomial and CRR analytical formula to Black Scholes value of the option as we increase N.

Option	**10**	**30**	**70**	**120**	**200**	**270**	**BS value**
European Call	7.5849	7.6222	7.6219	7.6229	7.6213	7.6215	7.6200
American Call	7.5849	7.6222	7.6219	7.6229	7.6213	7.6215	
European Put	0.6341	0.6714	0.6711	0.6721	0.6705	0.6707	0.6692
American Put	0.6910	0.7258	0.7238	0.7238	0.7224	0.7223	

Therefore, Black Scholes formula for the European call option can be used to value American call option for it is never optimal to exercise an American call option before maturity. The value of the American put option is higher than the corresponding European put option due to what we called *early exercise premium.* Sometimes the early exercise of the American put option can be optimal.

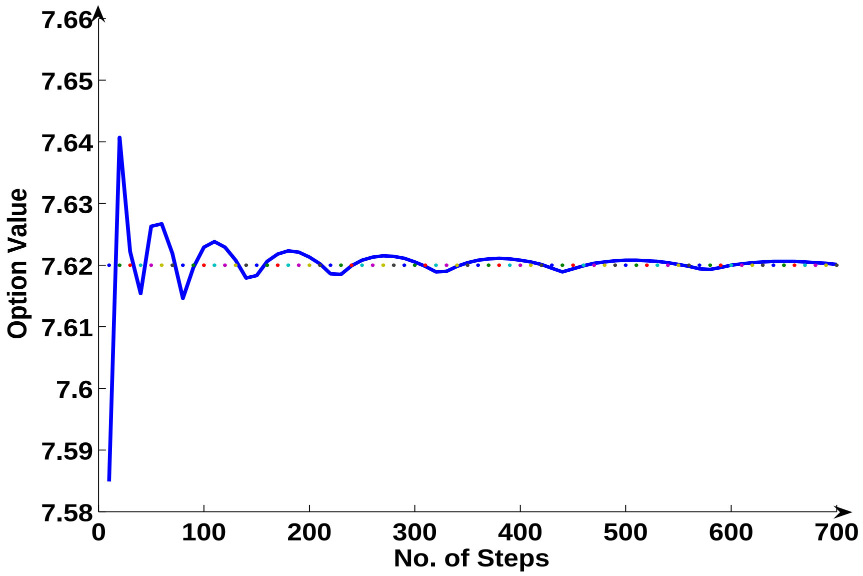

Figure 3.2: Convergence of the European call price for a non-dividend paying stock using the multi-period binomial model to the Black Scholes value of 7.62.

3.2.1 Dividend Paying Stock

Continuous Dividend Yield

We explored Merton's model, the adjustment for the Black-Scholes model to cater for European options on stocks that pay continuous dividend. Referring to (2.56) and (2.57), we saw that the risk free interest rate is modified from r to $(r - \lambda)$, where λ is the continuous dividend yield. We apply the same principle in our binomial model for the valuation of the options. The risk neutral probability in (3.5) is modified but the other parameters remains the same

$$\begin{aligned} u &= e^{\sigma\sqrt{\delta t}}, \\ d &= e^{-\sigma\sqrt{\delta t}}, \\ q &= \frac{\mathrm{e}^{(r-\lambda)\delta t} - d}{u - d}. \end{aligned} \tag{3.21}$$

These parameters apply when generating the binomial tree of stock prices for both the American and European options on stocks paying a continuous dividend and the tree will be identical in both cases. The probability of a stock price increase varies inversely with the level of the continuous dividend rate, λ.

Known Dividend Yield

It is assumed that there is a single dividend on a particular date τ and the dividend yield is a percentage of the stock price which is known. If the time $i\delta t$ is prior to the stock going ex-dividend, the nodes on the tree correspond to stock prices

$$Su^i d^{N-j} \qquad \text{for } j = 0, 1, \ldots, N. \tag{3.22}$$

If the time $i\delta t$ is after the stock goes ex-dividend, the nodes correspond to stock prices

$$S(1-\lambda)u^j d^{N-j} \qquad \text{for } j = 0, 1, \ldots, N. \tag{3.23}$$

Matlab Implementation

For the European call and put options, the Matlab code takes into consideration only the prices at the maturity date T and the stock prices will be as in (3.23).

For the American call and put options, the Matlab code will incorporate the early exercise privilege and the date τ, when the dividend will be paid. Then, it implies that the stock prices will exhibit (3.22) and (3.23). See appendix A.3 for the Matlab code.

Table 3.2 shows that the American option on the dividend paying stock is always worth more than its European counterpart. A very deep-in-the-money American option has a high early exercise premium. The premium of both the put and call option decreases as the option goes out-of-money.

The American and European call options are not worth the same as it is optimal to exercise the American call early on a dividend paying stock. A deep-out-of-the-money American and European call and put options are worth the same. This is due to the fact that they might not be exercised early as they are worthless.

Table 3.2: Out-of-money, at-the-money and in-the-money vanilla options on a stock paying a known dividend yield with $S = 50$, $r = 0.1$, $T = 0.5$, $\sigma = 0.25$, $\tau = 1/6$, and $\lambda = 5\%$.

K	European Call	American Call	Early Exercise Premium	European Put	American Put	Early Exercise Premium
30	18.97	20.50	1.53	0.004	0.004	0.00
45	6.06	6.47	0.41	1.37	1.49	0.12
50	3.32	3.42	0.10	3.38	3.78	0.40
55	1.62	1.63	0.01	6.40	7.31	0.91
70	0.11	0.11	0.00	19.19	21.35	2.16

3.3 Single Stock Futures Contracts

A Single Stock Futures (SSF) contract is a futures contract where the underlying security is an equity listed on the Johannesburg Stock Exchange (JSE). The term single is used to refer to the fact that the stock in the futures contract is from one company. It is a legally binding commitment made through a futures exchange to buy or sell a single equity in the future.

The SSF's were introduced with shares of only four of the leading JSE Securities Exchange listed companies in 1999. The number has gradually expanded to more than 60 listed companies. Each SSF contract is standardized with regard to size, expiration, and *tick* movement. A *tick* is the smallest price change of a contract. For the SSF contract, the tick move is R1 per contract. The value of an SSF contract is equal to 100 times the particular share's futures price and the price is negotiated through an order matching platform called the *Automated Trading System (ATS)*. The *ATS* is an auction based system where members in remote locations enter into and buy/sell orders, which are then matched automatically on the basis of price and time. The trading is conducted through South African Futures Exchange (SAFEX) members [14].

The SSF's follows the same procedure as a futures contract for margining as explained in section 1.2.1. This process ensures that both the buyer and seller are constantly up-to-date in their profits and losses, and are not subjected to potentially massive settlement of their losses upon expiry of the contract.

3.3.1 Options on SSFs

An SSF option is an instrument that conveys to its holder the right, but not the obligation, to buy or sell an SSF future at a fixed price K, called the strike price. Options on SSFs are American or European style options.

The risk in options differs between buyers and sellers. Buying of options involves limited risk which is the premium paid and is known at the outset. The writing of options is a high-risk strategy requiring intimate product knowledge.

The maturity months for the options are exactly the same as those of the underlying futures contracts, with quarterly expirations in March, June, September, and December. Strike prices are established at R5.00 intervals above or below the current futures level.

An example how the options are quoted on the Safex trading system is: *Month of expiry, year of expiry, three letter code of stock followed by letter Q, strike price and option type*. For a company called Smart Data, a call with a $R50.00$ strike price expiring in June 2005 is quoted as JUN05 SDTQ 50.00 c, where Q stands for quote and c for a call option.

As with other options, one can write SSF options as part of investment strategy or to hedge and simultaneously enhance investment returns. Safex options are margined options and this means that the buyer and the seller put up initial margin at the beginning of the contract. The initial margin requirement for individual equity options varies amongst stocks and is set by the Risk Management Committee. The seller (buyer) does not receive (pay) the full premium at inception of the contract. The premium is paid to the seller over the life of the option through the daily process of marking-to-market. The marking-to-market is accomplished at the Safex by taking the average midpoint of a selection of *bid* and *offer* prices on a traded SSF and using this as the market price for the SSF. The *Bid (offer)* price is the quoted price at which a particular market dealer is willing to buy (sell).

To exit a SSF option position, the holder has three alternatives:

- Exercise in the future and this means the holder assumes a futures position at the strike price of the option paid by the writer.
- Offset the option by entering into an equal and opposite trade.
- Let the option expire, if the option is in-the-money, it will be exercised with delivery

of the physical shares. The writer delivers the shares to the holder of the contract. If out-of-money, then it will be worthless.

Why trade SSFs and Options on SSFs:

- **Easy stock Exposure.** SSFs provide a quick and simple mechanism for gaining exposure to a specific stock. They enable investors to create a very simple long or short position in a share with cost effective purchase or sale of a single stock futures contract.

- **Hedging Stock Positions.** If a shareholder anticipates a short term fall in price, the holder can sell a future (or buy a put option) to avoid making a loss, without having to sell the share.

- **Shorting.** An investor can take advantage of a predicted fall in price by selling a futures contract. You do not need to own the underlying shares to be able to short. As the underlying share falls, the seller of the futures makes a positive return, because they can be able to buy back the futures at a lower price. The same result is obtained by purchasing a put option on the SSF.

- **Increased Gearing.** When buying the actual stock, the buyer has to pay the seller the full value. When buying a futures or an option, no money changes hands between the buyer and the seller. Only an initial margin for the futures contract and a premium for the options.

- **Pairs Trading.** It involves the buying of one share and selling of another. The objective is to take a position on the relative performance of two stocks, usually from the same sector. This is possible if one stock has a better outlook than the other and the overall gain or loss depends on the performance of the two stocks.

- **Index Composition.** SSFs allows fund managers to hedge against the arrival or demotion of a share on a particular index. When a particular stock is added to indices, index tracking fund managers rush to accumulate that stock to maintain their portfolio weightings and there is a significant skew in prices due to the purchase in this limited environment. Thus SSFs allow fund managers to gradually ease their way into these desired stocks [14].

3.3.2 Valuation of Options on Futures

We define the delta of an option, Δ, as the rate of change of the option price with respect to the price of the underlying asset. It is the number of units of the futures contract that should be held for each option contracts shorted in order to create a risk less hedge. We can also refer to it as delta hedging denoted as, Δ hedging.

We set up a risk-less hedge in a portfolio consisting of a short position in one option contract and a long position in one futures contract.

We said that a binomial model of price movements assumes that at the end of each time period, the price will either go up with a factor u with probability q or down with a factor d with probability $(1-q)$.

The futures price starts at F_o and is anticipated to rise to F_ou or move down to F_od over the time period δt. The option contract maturing at the end of the time period δt will have a payoff of fu if the price moves up and fd if the price moves down.

For Δ hedging, we have

$$\Delta F_ou - fu = \Delta F_od - fd \tag{3.24}$$

and this implies

$$\Delta = \frac{fu - fd}{F_ou - F_od} \tag{3.25}$$

where $fd = \max(F_od - K, 0)$ and $fu = \max(F_ou - K, 0)$. The value of the portfolio today is

$$-f = [(F_ou - F_o)\Delta - fu]e^{-r\delta t}. \tag{3.26}$$

We substitute Δ in (3.25) into (3.26) to get

$$f = e^{-r\delta t}\left[fu\frac{(1-d)}{u-d} + fd\frac{(u-1)}{u-d}\right]. \tag{3.27}$$

In a risk neutral world we have

$$f = e^{-r\delta t}\left[qfu + (1-q)fd\right]. \tag{3.28}$$

Comparing (3.27) and (3.28) yields

$$q = \frac{1-d}{u-d}. \tag{3.29}$$

The probability q does not contain the term $e^{r\delta t}$ as a futures price is analogous to a stock providing a dividend yield. The dividend yield is equal to the domestic risk-free interest rate, that is, $r = \lambda$.

We implement the multi-period binomial model for the options on futures with our new probability q, but the other parameters remains the same as in (3.5) [17]. See appendix A.2 for a Matlab code.

The American call option on a non dividend paying stock can be priced using the Black Scholes formula for it is never optimal to exercise it before the maturity date. However, section 2.4.1 explains why it is optimal to exercise the American call option on futures contract. Table 3.3 shows that the American call option on a futures contract is worth more than the European counterpart due to the early exercise privilege.

Table 3.3: Convergence of the multi-period binomial model to the Black's value as the number of steps N increase. $F = 514.80, K = 500, T = 1.0, r = 0.07$ and $\sigma = 0.2$.

Option Type	**10**	**40**	**80**	**140**	**250**	**Black's Value**
European call	45.3451	45.2105	45.0775	44.9794	44.9841	44.9832
American call	46.3143	46.0705	45.9331	45.8393	45.8438	
European put	31.5457	31.4111	31.2781	31.1800	31.1847	31.1838
American put	31.9904	31.8905	31.7606	31.6642	31.6639	

The multi-period binomial model is very flexible in pricing options. This was evident in pricing American put options for which the Black Scholes model is not suited to price.

The next chapter considers the finite difference methods. Then, a comparison of the convergence of the multi-period binomial model and the finite difference methods to the Black Scholes value of the option will be considered.

Chapter 4

Finite Difference Methods

The finite difference methods attempt to solve Black Scholes Partial differential equation by approximating the differential equation over the area of integration by a system of algebraic equations. They are a means of obtaining numerical solutions to Partial differential equations. They also constitute a very powerful and therefore flexible technique that is capable of generating accurate numerical solutions to PDE's arising in financial and other physical sciences.

The most common finite difference methods for solving the Black Scholes Partial differential equation are the Explicit method, the fully Implicit method and the Crank-Nicolson method. These are closely related but differ in stability, accuracy and execution speed.

In the formulation of a partial differential equation problem there are three components to consider: (1) The partial differential equation. (2) The region of space-time on which the partial differential equation is required to be satisfied. (3) The auxiliary boundary and initial conditions to be met.

4.1 Discretization of the Equation

The finite difference method consists of discretizing the partial differential pricing equation and the boundary conditions using a forward or a backward difference approximation. The Black Scholes PDE given by (2.29) can we written as

$$\frac{\partial f(S_t,t)}{\partial S_t} r S_t + \frac{\partial f(S_t,t)}{\partial t} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f(S_t,t)}{\partial S_t^2} = r f(S_t,t). \tag{4.1}$$

We discretize the equation with respect to time and to the underlying asset price. Divide the (S, t) plane into a sufficiently dense grid or mesh, and approximate the infinitesimal steps ΔS

and Δt by some small fixed finite steps. Further, define an array of $N+1$ equally spaced grid points $t_0, t_1, \dots, t_N$ to discretize the time derivative with $t_{n+1} - t_n = \Delta t$ and $\Delta t = T/N$.

We know that the stock price cannot go below 0 and we have assumed that $S_{\text{max}} = 2S_0$. We have $M+1$ equally spaced grid points $S_o, S_1, \dots, S_M$ to discretize the stock price derivative with $S_{m+1} - S_m = \Delta S$ and $\Delta S = S_{\text{max}}/M$

This gives us a rectangular region on the (S,t) plane with sides $(0, S_{\text{max}})$ and $(0,T)$. The grid coordinates (n,m) enables us to compute the solution at discrete points.

The time and stock price points define a grid consisting of a total of $(M+1) \times (N+1)$ points. The (n,m) point on the grid is the point that corresponds to time $n\Delta t$ for $n = 0, 1, \dots, N$, and stock price $m\Delta S$ for $m = 0, 1, \dots, M$. Figure 4.1 illustrates the discretized stock price and time derivatives into $(M+1)$ and $(N+1)$ grid points respectively. We will denote the value of the derivative at time step t_n when the underlying asset has value S_m as

$$f_{n,m} = f(n\Delta t, m\Delta S) = f(t_n, S_m) = f(t, S) \tag{4.2}$$

where n and m are the number of discrete increments in the time to maturity and stock price respectively. The discrete increments in the time to maturity and the stock price are given by Δt and ΔS, respectively.

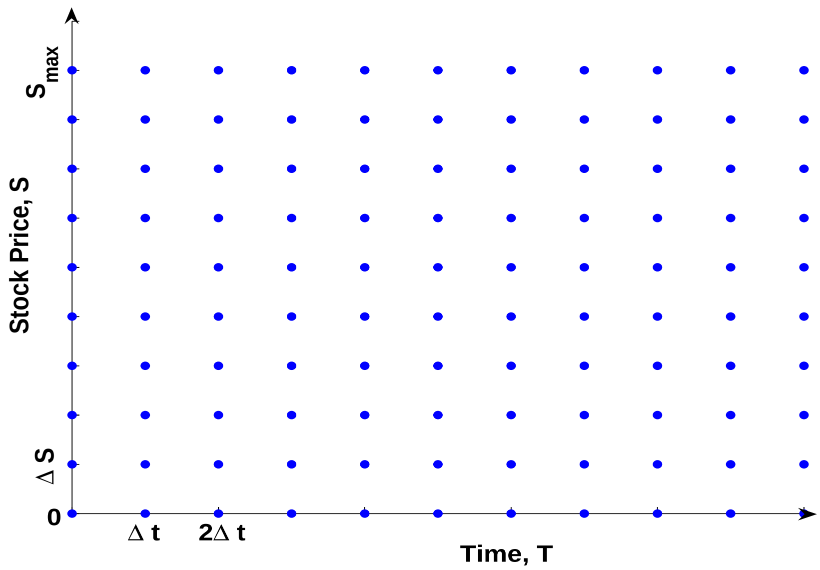

Figure 4.1: The mesh points for the fi nite difference approximation.

Let $f_n = f_{n,0}, f_{n,1}, \ldots, f_{n,M}$ for $n = 0, 1, \ldots, N$. Then, the quantities $f_{0,m}$ and $f_{N,m}$ for $m = 0, 1, \ldots, M$ are referred to as the boundary values which may or may not be known ahead of time but in our PDE they are known. The quantities $f_{n,m}$ for $n = 1, 2, \ldots, N-1$ and $m = 0, 1, \ldots, M$ are referred to as interior points or values.

We classify partial differential equations as: (1) Boundary value problems, where we need to specify the full set of boundary conditions. (2) Initial value problems, where only the value of the function at one particular time needs to be specified. The majority of derivative security pricing problems, including most of the options valuation problems, are initial value problems.

4.1.1 Finite Difference Approximations

The idea underlying finite difference methods is to replace the partial derivatives occurring in the PDE's by approximations based on Taylor series expansions of functions near the point or points of interest. The derivative we seek is expressed with any desired order of accuracy.

Assuming that $f(t, S)$ is represented in the grid by $f(n, m)$, the respective expansions of $f(t, S + \Delta S)$ and $f(t, S - \Delta S)$ in Taylors series are

$$f(t, S + \Delta S) = f(t, S) + \frac{\partial f}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\Delta S^2 + \frac{1}{6}\frac{\partial^3 f}{\partial S^3}\Delta S^3 + O(\Delta S^4), \quad (4.3)$$

$$f(t, S - \Delta S) = f(t, S) - \frac{\partial f}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\Delta S^2 - \frac{1}{6}\frac{\partial^3 f}{\partial S^3}\Delta S^3 + O(\Delta S^4). \quad (4.4)$$

Using (4.3), the forward difference is given by

$$\begin{aligned} \frac{\partial f}{\partial S}(t, S) &= \frac{f(t, S + \Delta S) - f(t, S)}{\Delta S} + O(\Delta S) \\ &\approx \frac{f_{n,m+1} - f_{n,m}}{\Delta S}, \end{aligned} \quad (4.5)$$

and (4.4) gives the corresponding backward difference as

$$\begin{aligned} \frac{\partial f}{\partial S}(t, S) &= \frac{f(t, S) - f(t, S - \Delta S)}{\Delta S} + O(\Delta S) \\ &\approx \frac{f_{n,m} - f_{n,m-1}}{\Delta S}. \end{aligned} \quad (4.6)$$

Subtracting (4.4) from (4.3) and taking the first order partial derivative results in the central difference given by

$$\begin{aligned} \frac{\partial f}{\partial S}(t, S) &= \frac{f(t, S + \Delta S) - f(t, S - \Delta S)}{2\Delta S} + O(\Delta S^2) \\ &\approx \frac{f_{n,m+1} - f_{n,m-1}}{2\Delta S}. \end{aligned} \quad (4.7)$$

The second order partial derivatives can be estimated by the symmetric central difference approximation. We sum (4.4) and (4.3) and take the second order partial derivative to have

$$\begin{aligned}\frac{\partial^2 f}{\partial S^2}(t,S) &= \frac{f(t,S+\Delta S)-2f(t,S)+f(t,S-\Delta S)}{\Delta S^2}+O(\Delta S^2)\\ &\approx \frac{f_{n,m+1}-2f_{n,m}+f_{n,m-1}}{\Delta S^2}.\end{aligned} \tag{4.8}$$

Although there are other approximations, this approximation to $\partial^2 f/\partial S^2$ is preferred, as its symmetry preserves the reflectional symmetry of the second order partial derivative. It is also invariant and more accurate than other similar approximations.

We expand $f(t+\Delta t,S)$ in Taylors series

$$f(t+\Delta t,S)=f(t,S)+\frac{\partial f}{\partial t}\Delta t+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}\Delta t^2+\frac{1}{6}\frac{\partial^3 f}{\partial t^3}\Delta t^3+O(\Delta t^4). \tag{4.9}$$

The forward difference for the time is given by

$$\begin{aligned}\frac{\partial f}{\partial t}(t,S) &= \frac{f(t+\Delta t,S)-f(t,S)}{\Delta t}+O(\Delta t)\\ &\approx \frac{f_{n+1,m}-f_{n,m}}{\Delta t}.\end{aligned} \tag{4.10}$$

Replacing the first and second derivatives in the Black Scholes PDE will result in a difference equation which gives an equation that we use to approximate the solution $f(S,t)$ [17].

4.1.2 Boundary and Initial Conditions

A partial differential equation without the auxiliary boundary or initial conditions will either have an infinity of solutions, or have no solution. We need specify the boundary and initial conditions for the European put option whose payoff is given by $\max(K-S_T,0)$. When the stock is worth nothing, a put is worth its strike price K. That is,

$$f_{n,0}=K \qquad \text{for } n=0,1,\ldots,N. \tag{4.11}$$

As the price of the underlying asset price increases, the value of the put option approaches zero. Accordingly, we choose $S_{\max}=S_{\mathrm{M}}$ and from this we get

$$f_{n,M}=0 \qquad \text{for } n=0,1,\ldots,N. \tag{4.12}$$

We know the value of the put option at time T and can impose the initial condition

$$f_{N,m}=\max(K-m\Delta S,0) \qquad \text{for } m=0,1,\ldots,M. \tag{4.13}$$

The initial condition gives us the values of f at the end of the time period and not at the beginning. This means that we move backward from the maturity date to time zero. The price of the put option is given by $f_{0,\frac{M+1}{2}}$ when M is odd and by $f_{0,\frac{M}{2}}$ when M is even. This method is suited for European put options where early exercise is not permitted. The call-put parity in (1.3) is used to obtain the corresponding value of the European call option.

To value an American put option, where early exercise is permitted, we need make only one simple modification ([32], [30]). After each linear system solution, we need to consider whether early exercise is optimal or not. We compare $f_{n,m}$ with the intrinsic value of the option, $(K - m\Delta S)$. If the intrinsic value is greater, then set $f_{n,m}$ to the intrinsic value.

The American call options are handled in almost exactly the same way. For a call option, (4.13) becomes

$$f_{N,m} = \max(m\Delta S - K, 0) \qquad \text{for } m = 0, 1, \ldots, M. \tag{4.14}$$

It is computationally more efficient to use finite difference methods with ln$S$ rather than S as the underlying variable. We consider the log transform or the change of variable of the Black Scholes PDE.

4.1.3 Log Transform of the Black Scholes Equation

The log transform method was suggested in [8] by Brennan and Schwartz. When S is a stock price, it is efficient to use ln$S$ rather than S as the underlying variable when the finite difference methods are applied. This is because as indicated in [18] by Hull and White, when σ is constant, the instantaneous standard deviation of ln$S$ is constant. The standard deviation of changes in ln$S$ in a time interval Δt is independent of S and t.

We define $y = \ln S$ and $f(t, S) = g(t, y)$ as the price of the call at time t. This is the price of the call in terms of the transformed asset price and time. We price the call in terms of the log of the asset price and time t.

$$\begin{aligned} \frac{\partial f}{\partial S} &= \frac{\partial g}{\partial y}\mathrm{e}^{-y}, \\ \frac{\partial^2 f}{\partial S^2} &= \left[\frac{\partial^2 g}{\partial y^2} - \frac{\partial g}{\partial y}\right]\mathrm{e}^{-2y}, \\ \frac{\partial f}{\partial t} &= \frac{\partial g}{\partial t}. \end{aligned} \tag{4.15}$$

The transformed equations are similar to those in (2.35). We drop the y and t notations and substitute (4.15) into (4.1) to obtain

$$\frac{\partial g}{\partial t} + (r - \sigma^2/2)\frac{\partial g}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial y^2} - rg = 0. \tag{4.16}$$

We partition a reasonable range of the log of the asset price into finite intervals with $\{y_0, y_1, \dots, y_M\}$ equally spaced $M+1$ grid points and $N+1$ equally spaced grid points $\{t_0, t_1, \dots, t_N\}$ of time. The stock price is assumed to be log-normally distributed and thus can be at a minimum of zero and a maximum of infinity. Since $\ln S \to -\infty$ as $S \to 0$, we must choose a small ε such that $\ln S = \varepsilon$ for $S < 1$, to avoid negative stock prices.

Boundary and Initial Conditions

We define the boundary conditions for our transformed PDE in (4.16). If the asset price is zero, the put is worth its strike price K regardless of the time to expiration,

$$f(t, 0) = f_{n,0} = K \qquad \text{for all } \ t, n.$$

For the change of variable technique, we have $\ln S = \varepsilon$ with ε very close to zero. This condition can be specified as,

$$g(t, \varepsilon) = g(t, \ln S) = 0 \qquad \text{for } \ S < 1.$$

As the price of the underlying asset prices, the value of the put option approaches zero

$$f_{n,M} = 0 \qquad \text{for } \ n = 0, 1, \dots, N.$$

For the change of variable technique, when $S \to \infty$, then the put option is zero as $\ln S \to \infty$

$$g(t, y) = g(t, \ln S) = g_{n,M} = 0 \qquad \text{for } \ n = 0, 1, \dots, N.$$

When $S \to \infty$, the first derivative of the call price with respect to the asset price is 1

$$\lim_{S \to \infty} \frac{\partial f}{\partial S} = 1 \qquad \text{for all } \ t.$$

This shows that for sufficiently high values of the underlying asset, the option behaves like the underlying asset. Since $(\partial f(t, S)/\partial S) = (\partial g(t, S)/\partial y)\mathrm{e}^{-y}$, we have

$$\frac{\partial f(t, S)}{\partial y} = \mathrm{e}^y = S \qquad \text{for all } \ t \ \text{ when } \ \ln S \to \infty.$$

The intrinsic value at expiration which gives the initial condition is given as

$$f(T, S) = \max(K - S_T, 0) \qquad \text{for all } \; S.$$

In terms of y, for the change of variable technique gives

$$g(T, y) = \max(K - \mathrm{e}^y, 0) \qquad \text{for all } \; y.$$

This last equation representing the initial condition helps us to fill the entire rightmost column with the stock prices at time T.

4.2 The Explicit Finite Difference Method

Given that we know the value of an option at the maturity time, it is possible to give an expression that gives us the next value $f_{m,n}$ explicitly in terms of the given values $f_{m-1,n+1}$, $f_{m,n+1}$ and $f_{m+1,n+1}$.

We discretize the Black Scholes PDE in (4.1) by taking the forward-difference for time discretization and the central difference for the stock price discretization. This yields

$$\begin{aligned} &\frac{f_{n+1,m} - f_{n,m}}{\Delta t} + \frac{rm\Delta S}{2\Delta S}\Big[f_{n+1,m+1} - f_{n+1,m-1}\Big] \\ &+ \frac{\sigma^2 m^2 \Delta S^2}{2\Delta S^2}\Big[f_{n+1,m-1} - 2f_{n+1,m} + f_{n+1,m+1}\Big] = rf_{n,m}, \end{aligned} \tag{4.17}$$

and re-arranging we have

$$\begin{aligned} f_{n,m} = \frac{1}{1 + r\Delta t}&\Big[\beta_{1m} f_{n+1,m-1} + \beta_{2m} f_{n+1,m} + \beta_{3m} f_{n+1,m+1}\Big] \\ &\text{for} \quad n = 0, 1, \ldots, N-1 \text{ and } m = 1, 2, \ldots, M-1. \end{aligned} \tag{4.18}$$

The forward difference for time discretization is accurate to $O(\Delta t)$ and the central difference for stock discretization to $O(\Delta S^2)$. Therefore the finite difference method is accurate to $O(\Delta t, \Delta S^2)$. The weights in (4.18) are given by

$$\begin{aligned} \beta_{1m} &= \frac{1}{2}\sigma^2 m^2 \Delta t - \frac{1}{2} rm\Delta t, \\ \beta_{2m} &= 1 - \sigma^2 m^2 \Delta t, \\ \beta_{3m} &= \frac{1}{2} rm\Delta t + \frac{1}{2}\sigma^2 m^2 \Delta t. \end{aligned} \tag{4.19}$$

These weights sum to unity. They are the risk neutral probabilities of the three asset prices $S - \Delta S, S$ and $S + \Delta S$ at $t + \Delta t$. We are assuming that the expected returns on the asset is also true in a risk neutral world. For the explicit version of the finite difference to work well, the three "probabilities" should be positive. The problem associated with the explicit method is that some probabilities are negative. This produces results that do not converge to the solution of the differential equation. The condition to have non-negative probabilities is that $\sigma^2 m^2 \Delta t < 1$ and $r < \sigma^2 m$ [17].

The stock price and time in the system of equations in (4.18) gives rise to a tridiagonal system written as $\mathbf{Au} + \epsilon = \mathbf{b}$. The vector ϵ arises as a result of the boundary conditions at $m = 0$ and M for all $n > 0$. The system is represented as

$$\begin{bmatrix} \beta_{20} & \beta_{30} & 0 & \dots & 0 & 0 & 0 \\ \beta_{11} & \beta_{21} & \beta_{31} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{1M-1} & \beta_{2M-1} & \beta_{3M-1} \\ 0 & 0 & 0 & \dots & 0 & \beta_{1M} & \beta_{2M} \end{bmatrix} \begin{bmatrix} f_{n+1,0} \\ f_{n+1,1} \\ \vdots \\ f_{n+1,M-1} \\ f_{n+1,M} \end{bmatrix} = \begin{bmatrix} f_{n,0} \\ f_{n,1} \\ \vdots \\ f_{n,M-1} \\ f_{n,M} \end{bmatrix} \tag{4.20}$$

This system of equations can be written in the form $\mathbf{Af_{n+1,m}} = \mathbf{f_{n,m}}$, for $m = 0, 1, \dots, M$ and we ignore the error terms as the boundary conditions will take care of them.

The vector of asset prices $\mathbf{f_{n+1,m}}$ is known at time T from our initial condition. We can work backward by solving for $\mathbf{f_{n,m}}$ $(m = 0, 1, \dots, M)$ using the matrix $\mathbf{A}$ which comprises of the probabilities, $\beta_{\kappa m}$ $(\kappa = 1, 2, 3)$ that are known. These backward iterations leads us to the value of the option obtained at time zero.

The iterations in finding the solution leads to rounding errors as the difference equation is solved to give the numerical solution. If these rounding errors are not magnified at each iteration, the system is stable, otherwise it is unstable. When using finite difference grids, we encounter two kinds of problems, the stability and accuracy of the method. Our concern is to obtain an accurate solution with as few computations as possible and that's why stability and accuracy are of importance.

4.2.1 Stability Analysis

The two fundamental sources of error are, the truncation error in the stock price discretization and in the time discretization. The implication of truncation error is that the numerical scheme solves a problem that is not exactly the same as the problem we are trying to solve.

The three fundamental factors that characterize a numerical scheme are:

- Consistency - A finite difference representation of a partial differential equation is consistent if the difference between the PDE and finite difference equation (FDE) vanishes as the grid interval and time step size approach zero. That is, the truncation error vanishes so that

  $$\lim_{\Delta t \to 0} (PDE - FDE) = 0.$$

  Consistency deals with how well the FDE approximates the PDE and it is the necessary condition for convergence.

- Stability - For a stable numerical scheme, the errors from any source will not grow un boundedly with time.

- Convergence - It means that the solution to a FDE approaches the true solution to the PDE as both grid interval and time step sizes are reduced.

These three factors that characterize a numerical scheme are linked together by

- **Lax Equivalence Theorem** - It states that, *given a properly posed linear initial value problem and a consistent finite difference scheme, stability is the necessary and sufficient condition for convergence* [28].

In general, a problem is properly posed if:

- A solution to the problem exists.

- The solution is unique when it exists.

- The solution depends continuously on the problem data.

A Necessary and Sufficient Condition for Stability

Let $\mathbf{f_{n+1}} = \mathbf{A}\mathbf{f_n}$ be a system of equations. Matrix $\mathbf{A}$ and the column vectors $\mathbf{f_{n+1}}$ and $\mathbf{f_n}$ are as represented in (4.20). We have

$$\begin{aligned}
\mathbf{f_n} &= \mathbf{A}\mathbf{f_{n-1}} \\
&= \mathbf{A^2}\mathbf{f_{n-2}} \\
&\vdots \\
&= \mathbf{A^n}\mathbf{f_0} \qquad \text{for } n = 1, 2, \ldots, N
\end{aligned} \tag{4.21}$$

where $\mathbf{f_0}$ is the vector of initial values. We are concerned with stability and we investigate the propagation of a perturbation. Perturb the vector of initial values $\mathbf{f_0}$ to $\mathbf{f_0^*}$. The exact solution at the n^{th} time-row will then be

$$\mathbf{f_n^*} = \mathbf{A^n}\mathbf{f_0^*}. \tag{4.22}$$

Let the perturbation or 'error' vector $\mathbf{e}$ be defined by

$$\mathbf{e} = \mathbf{f^*} - \mathbf{f},$$

and using the perturbation vector, (4.21) and (4.22) we have

$$\begin{aligned}
\mathbf{e_n} &= \mathbf{f_n^*} - \mathbf{f_n} \\
&= \mathbf{A^n}(\mathbf{f_0^*} - \mathbf{f_0}) \\
&= \mathbf{A^n}\mathbf{e_0} \qquad \text{for } n = 1, 2, \ldots, N.
\end{aligned} \tag{4.23}$$

Hence, for compatible matrix and vector norms [28]

$$||\mathbf{e_n}|| \leq ||\mathbf{A^n}|| \;\; ||\mathbf{e_0}||.$$

Lax and Richtmyer defined the difference scheme to be stable when there exists a positive number L, independent of $n, \Delta t$ and ΔS such that

$$||\mathbf{A^n}|| \leq L, \qquad \text{for } n = 1, 2, \ldots, N.$$

This limits the amplification of any initial perturbation and therefore of any arbitrary initial rounding errors because it implies that

$$||\mathbf{e_n}|| \leq L||\mathbf{e_0}||.$$

Since

$$||\mathbf{A^n}|| = ||\mathbf{AA^{n-1}}|| \leq ||\mathbf{A}||\ \ ||\mathbf{A^{n-1}}|| \leq ... \leq ||\mathbf{A}||^n$$

then the Lax-Richtmyer definition of stability is satisfied when

$$||\mathbf{A}|| \leq 1. \tag{4.24}$$

Condition (4.24) is the necessary and sufficient condition for the difference equations to be stable [28]. Since the spectral radius $\rho(\mathbf{A})$ satisfies

$$\rho(\mathbf{A}) \leq ||\mathbf{A}||$$

it follows automatically from (4.24) that

$$\rho(\mathbf{A}) \leq 1.$$

We note that if matrix $\mathbf{A}$ is real and symmetric, then by definition [28], we have

$$\begin{aligned} ||\mathbf{A}||_\infty &= \text{moduli of the maximum row of matrix}\mathbf{A} \\ ||\mathbf{A}||_2 &= \rho(\mathbf{A}) = \max_i |\lambda_i| \end{aligned} \tag{4.25}$$

where λ_i is an eigenvalue of matrix $\mathbf{A}$.

The Eigenvalues of a Common Tridiagonal Matrix

The other method used in the analysis of stability is the use of eigenvalues of the tridiagonal system. The eigenvalues of the $N \times N$ matrix

$$\begin{bmatrix} y & z & & & \\ x & y & z & & \\ & \ddots & \ddots & \ddots & \\ & & x & y & z \\ & & & x & y \end{bmatrix}$$

are $\lambda_n = y + 2[\sqrt{xz}]\cos\frac{n\pi}{N+1},$ for $n = 1, 2, \ldots, N$, where x, y and z may be real or complex [28].

The Stability Issue of Explicit Method

We use the matrix $\mathbf{A}$ in (4.20) to analyze the stability of the explicit finite difference method, where the $\beta_{\kappa m}$, for $\kappa = 1, 2, 3$ are given by (4.19). Matrix $\mathbf{A}$ is real and symmetric. If υ_n is the nth eigenvalue of $\mathbf{A}$, then we have [28]

$$||\mathbf{A}||_2 = \rho(\mathbf{A}) = \max_n |\upsilon_n|.$$

The eigenvalues λ_n are given by

$$\lambda_n = \beta_{2m} + 2[\beta_{1m}\beta_{3m}]^{1/2}\cos\frac{n\pi}{N} \qquad \text{for } n = 1, 2, \ldots, N-1. \tag{4.26}$$

Substituting the values of β's, we have

$$\lambda_n = 1 - \sigma^2 m^2 \Delta t + \sigma^2 m^2 \Delta t \left[1 - \frac{r^2}{\sigma^4 m^2}\right]^{1/2} \left[1 - 2\sin^2\frac{n\pi}{2N}\right] \tag{4.27}$$

for $n = 1, 2, \ldots, N-1$. Further, we apply the binomial expansion on the square root part and ignore some terms. Re-arranging we get

$$\lambda_n \approx 1 - 2\sigma^2 m^2 \Delta t \sin^2\frac{n\pi}{2N}.$$

Therefore the equations are stable when

$$||\mathbf{A}||_2 = \max\left|1 - 2\sigma^2 m^2 \Delta t \sin^2\frac{n\pi}{2N}\right| \leq 1,$$

that is,

$$-1 \leq 1 - 2\sigma^2 m^2 \Delta t \sin^2\frac{n\pi}{2N} \leq 1 \qquad \text{for } n = 1, 2, \ldots, N-1 \tag{4.28}$$

as $\Delta t \to 0,\ N \to \infty$ and $\sin^2\frac{(N-1)\pi}{2N} \to 1$. Hence

$$0 \leq \sigma^2 m^2 \Delta t \leq 1. \tag{4.29}$$

Alternatively, when $1 - \sigma^2 m^2 \Delta t \geq 0$, then $\sigma^2 m^2 \Delta t \leq 1$, and

$$||\mathbf{A}||_\infty = \beta_{1m} + \beta_{2m} + \beta_{3m} = 1.$$

When $1 - \sigma^2 m^2 \Delta t < 0,\ \ \sigma^2 m^2 \Delta t > 1$, then $|1 - \sigma^2 m^2 \Delta t| = \sigma^2 m^2 \Delta t - 1$, and

$$||\mathbf{A}||_\infty = 2\sigma^2 m^2 \Delta t - 1 > 1.$$

Therefore by Lax's equivalence theorem, the scheme is stable, convergent and consistent for $0 \leq \sigma^2 m^2 \Delta t \leq 1$.

In (4.19), the other condition is that $r < \sigma^2 m$. These conditions are necessary for the weights $\beta_{\kappa m}$ $(\kappa = 1, 2, 3)$ to be positive, otherwise, they will be negative. These weights are 'probabilities' and should always be non negative. We said that the main disadvantage of the Explicit method is that some weights are negative and thus the scheme does not converge to the solution of the differential equation.

4.2.2 Change of Variable - The Explicit Method

The boundary conditions considered and the differential equation in (4.16) will be applied in deriving the explicit finite difference method for change of variable method.

We discretize the stock price with the central difference scheme and time by forward difference and substitute it into (4.16) to get

$$\frac{g(t+\Delta t, y) - g(t, y)}{\Delta t} + \frac{(r - \sigma^2/2)}{2\Delta y}\Big[g(t+\Delta t, y+\Delta y) - g(t+\Delta t, y-\Delta y)\Big]$$
$$+\frac{\sigma^2}{2\Delta y^2}\Big[g(t+\Delta t, y-\Delta y) - 2g(t+\Delta t, y) + g(t+\Delta t, y+\Delta y)\Big] = rg(t, y). \qquad (4.30)$$

Re-arranging we get

$$g_{n,m} = \frac{1}{1+r\Delta t}\Big[\beta_1^* g_{n+1,m-1} + \beta_2^* g_{n+1,m} + \beta_3^* g_{n+1,m+1}\Big] \qquad (4.31)$$

where our new weights in (4.31) are given by

$$\begin{aligned}
\beta_1^* &= \frac{1}{2}\left[\frac{\sigma}{\Delta y}\right]^2 \Delta t - \frac{1}{2}\frac{(r-\sigma^2/2)}{\Delta y}\Delta t, \\
\beta_2^* &= 1 - \left[\frac{\sigma}{\Delta y}\right]^2 \Delta t, \\
\beta_3^* &= \frac{1}{2}\left[\frac{\sigma}{\Delta y}\right]^2 \Delta t + \frac{1}{2}\frac{(r-\sigma^2/2)}{\Delta y}\Delta t. \qquad (4.32)
\end{aligned}$$

The log transform allows the weights β_κ^*, for $\kappa = 1, 2, 3$ sum to unity. The weights can be made non-negative which is important since they are probabilities by choosing $\Delta t \leq \Delta y^2/\sigma^2$ and $\Delta y \leq \sigma^2/(r - \sigma^2/2)$ [17].

The Stability Issue of the Change of Variable

We use the matrix method to analyse the stability of the explicit FDM. Consider the weights β_κ^*, for $\kappa = 1, 2, 3$ that make up the matrix under consideration. The parameters in (4.32) will enable us to carry out the analysis. When

$$1-\left[\frac{\sigma^2}{\Delta y}\right]^2\Delta t \geq 0, \qquad \text{then } \left[\frac{\sigma^2}{\Delta y}\right]^2\Delta t \leq 1,$$

and

$$||\mathbf{A}||_\infty = \beta_1^* + \beta_2^* + \beta_3^* = 1.$$

When

$$1-\left[\frac{\sigma^2}{\Delta y}\right]^2\Delta t < 0,\ \left[\frac{\sigma^2}{\Delta y}\right]^2\Delta t > 1, \qquad \text{then } \left|1-\left[\frac{\sigma^2}{\Delta y}\right]^2\Delta t\right| = \left[\frac{\sigma^2}{\Delta y}\right]^2\Delta t - 1,$$

and

$$||\mathbf{A}||_\infty = 2\left[\frac{\sigma^2}{\Delta y}\right]^2\Delta t - 1 > 1.$$

Therefore by Lax's equivalence theorem, the scheme is stable, convergent and consistent for $0 \leq [\sigma^2/\Delta y]^2\Delta t \leq 1$.

4.3 The Implicit Finite Difference Method

We express $f_{n+1,m}$ implicitly in-terms of the unknowns $f_{n,m-1}$, $f_{n,m}$ and $f_{n,m+1}$. We discretize the Black Scholes PDE in (4.1) using the forward difference for time and central difference for the stock price to have

$$\begin{aligned}&\frac{f_{n+1,m}-f_{n,m}}{\Delta t} + rm\Delta S\left[\frac{f_{n,m+1}-f_{n,m-1}}{2\Delta S}\right]\\ &+\frac{1}{2}\sigma^2 m^2\Delta S^2\left[\frac{f_{n,m+1}-2f_{n,m}+f_{n,m-1}}{\Delta S^2}\right] = rf_{n+1,m}.\end{aligned} \tag{4.33}$$

Rearranging, we get

$$\begin{aligned}&f_{n+1,m} = \frac{1}{1-r\Delta t}\left[\alpha_{1m}f_{n,m-1} + \alpha_{2m}f_{n,m} + \alpha_{3m}f_{n,m+1}\right]\\ &\text{for } n = 0,1,\ldots,N-1 \text{ and } m = 1,2,\ldots,M-1.\end{aligned} \tag{4.34}$$

Similar to the explicit method, the implicit method is accurate to $O(\Delta t, \Delta S^2)$. The parameters $\alpha_{\kappa m}$'s for $\kappa = 1,2,3$ are given as

$$\begin{aligned}\alpha_{1m} &= \frac{1}{2}rm\Delta t - \frac{1}{2}\sigma^2 m^2\Delta t,\\ \alpha_{2m} &= 1+\sigma^2 m^2\Delta t,\\ \alpha_{3m} &= -\frac{1}{2}rm\Delta t - \frac{1}{2}\sigma^2 m^2\Delta t.\end{aligned} \tag{4.35}$$

The system of equations can be expressed as a tridiagonal system

$$\begin{bmatrix} f_{n+1,0} \\ f_{n+1,1} \\ \vdots \\ f_{n+1,M-1} \\ f_{n+1,M} \end{bmatrix} = \begin{bmatrix} \alpha_{20} & \alpha_{30} & 0 & \dots & 0 & 0 & 0 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{1M-1} & \alpha_{2M-1} & \alpha_{3M-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_{1M} & \alpha_{2M} \end{bmatrix} \begin{bmatrix} f_{n,0} \\ f_{n,1} \\ \vdots \\ f_{n,M-1} \\ f_{n,M} \end{bmatrix}$$

which can be written as $\mathbf{Af_{n,m}} = \mathbf{f_{n+1,m}}$. for $m = 0, 1, \dots, M$. Let $\mathbf{f_{n,m}} = \mathbf{f_n}$. We need to solve for $\mathbf{f_n}$ given matrix $\mathbf{A}$ and column vector $\mathbf{f_{n+1}}$ and this implies that $\mathbf{f_n} = \mathbf{A^{-1}f_{n+1}}$. The matrix A has $\alpha_{2m} = 1 + \sigma^2 m^2 \Delta t$ in the diagonal which is positive. The product of the diagonal elements are non zero and therefore the matrix is nonsingular. We can solve the system by finding the inverse matrix $\mathbf{A}^{-1}$.

When we apply the boundary conditions together with (4.34), this gives rise to some changes in the elements of matrix $\mathbf{A}$ with $\alpha_{20}, \alpha_{2M} = 1$ and $\alpha_{30}, \alpha_{1M} = 0$.

Our initial condition give values for the N^{th} time step, and we solve for f_n at t_n in terms of f_{n+1} at t_{n+1}. We set the right hand side of the system to our initial condition and solve the system to produce a solution to the equation for time step $N-1$. By repeatedly iterating in such a manner, we can obtain the value of f at any time step $0, 1, \dots, N-1$.

The implicit method allows us to use a large number of S-mesh points without having to take ridiculously small time-steps. We can solve our system of linear equations using either the LU decomposition method or the SOR method. The use of these techniques makes implicit method as almost as efficient as the explicit method in terms of arithmetical operations per time-step. As fewer time-steps need to be taken, the implicit finite difference method, which is unconditionally stable, is more efficient over-all than the explicit method.

The Stability Issue of Implicit Method

We analyzed the stability of the explicit method. We apply the same principle to test for the stability of the implicit finite difference method.

The eigenvalues λ_n are given by

$$\lambda_n = \alpha_{2m} + 2[\alpha_{1m}\alpha_{3m}]^{1/2}\cos\frac{n\pi}{N} \qquad \text{for } n = 1, \dots, N-1. \tag{4.36}$$

Substituting the values of α's in (4.35), we have

$$\lambda_n = 1 + \sigma^2 m^2 \Delta t + \sigma^2 m^2 \Delta t \left[1 - \frac{r^2}{\sigma^4 m^2}\right]^{1/2} \left[1 - 2\sin^2 \frac{n\pi}{2N}\right] \tag{4.37}$$

for $n = 1, 2, \ldots, N-1$. Furthermore, applying the binomial expansion on the square root part and re-arranging we have

$$\lambda_n \approx 1 + 2\sigma^2 m^2 \Delta t - 2\sigma^2 m^2 \Delta t \sin^2 \frac{n\pi}{2N}$$

where there is change of sign due to the truncation of the binomial expansion. Therefore the equations are stable when

$$||\mathbf{A}||_2 = \max \left| 1 + 2\sigma^2 m^2 \Delta t - 2\sigma^2 m^2 \Delta t \sin^2 \frac{n\pi}{2N} \right| \leq 1$$

that is,

$$-1 \leq 1 + 2\sigma^2 m^2 \Delta t - 2\sigma^2 m^2 \Delta t \sin^2 \frac{n\pi}{2N} \leq 1 \qquad \text{for } n = 1, 2, \ldots, N-1. \tag{4.38}$$

As $\Delta t \to 0,\ N \to \infty$ and $\sin^2 \frac{(N-1)\pi}{2N} \to 1$, (4.38) reduces to $|1| \leq 1$.

Alternatively,

$$1 + \sigma^2 m^2 \Delta t \geq 0 \quad \text{and} \quad ||\mathbf{A}||_\infty = 1.$$

Therefore by Lax's equivalence theorem, the scheme is unconditionally stable, convergent and consistent.

4.3.1 Solving Systems of Linear Equations

We can apply the direct solvers or iterative solvers in solving our system of linear equations. A direct solver is one that achieves the solution within a finite number of steps. The accuracy of the solution is not a controllable parameter and it depends on the particulars of the implementation and the characteristics of the algorithm itself. The popular direct solver is the tridiagonal solver which is the Gaussian elimination method applied to tridiagonal equations.

An iterative solver achieves a solution on the basis of satisfying an accuracy criterion. This use of accuracy as a termination criterion gives iterative solvers a dimension of flexibility and efficiency. The two main types of iterative solvers are stationary and non stationary methods. Stationary methods use iteration schemes with parameters that remain fixed during the iterations. Examples are Jacobi, Gauss-Seidel, and Successive over-relaxation (SOR) methods [30].

In practice, we have far more efficient solution techniques than matrix inversion. The matrix $\mathbf{A}$ in the implicit method is tridiagonal and has the property that, only the diagonal, super-diagonal and sub-diagonal elements are non-zero.

This has the following advantages: (1) It means that we do not have to store all the zeros but just the non zero elements [32]. The inverse of $\mathbf{A}, \mathbf{A^{-1}}$, is not tridiagonal and requires a high storage space. If N is the dimension of the system, storing $\mathbf{A}^{-1}$ requires N^2 real numbers, whereas storing the non-zero elements of $\mathbf{A}$ requires $3N-2$ storage space. (2) The tridiagonal structure of $\mathbf{A}$ means that there are highly efficient algorithms for solving $\mathbf{Af_n} = \mathbf{f_{n+1}}$ in $O(N)$ arithmetic operations per solution. We turn our attention to two of these algorithms, *LU decomposition* and *SOR*.

The LU Method

This is a direct method for solving systems of equations and it aims to find the unknowns exactly in one pass. In this method, we are concerned about the decomposition of the matrix $\mathbf{A}$ into a product of a lower triangular matrix **L** and an upper triangular matrix **U**, namely **A**=**LU**, of the form

$$\begin{bmatrix} \alpha_{20} & \alpha_{30} & 0 & \dots & 0 & 0 & 0 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{1M-1} & \alpha_{2M-1} & \alpha_{3M-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_{1M} & \alpha_{2M} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ l_1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & l_{M-1} & 1 & 0 \\ 0 & 0 & \dots & 0 & l_M & 1 \end{bmatrix} \begin{bmatrix} y_0 & z_0 & 0 & \dots & 0 & 0 \\ 0 & y_1 & z_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & y_{M-1} & z_{M-1} \\ 0 & 0 & 0 & \dots & 0 & y_M \end{bmatrix} \tag{4.39}$$

The parameters α_{1m}, α_{2m} and α_{3m} are as given in (4.35).We need to determine the quantities l_m, y_m and z_m which can only be calculated at once. We simply multiply together the two matrices on the right hand side of (4.39) and equate the result to the left hand side. After some operations we find that

$$\begin{aligned} y_0 &= \alpha_{20}, \\ y_m &= \alpha_{2m} - \frac{\alpha_{1m}\alpha_{3m}}{y_{m-1}} \qquad \text{for } m = 1,2,\ldots,M, \\ z_m &= \alpha_{3m} \qquad \text{for } m = 0,1,\ldots,M, \\ l_m &= \frac{\alpha_{1m}}{y_{m-1}} \qquad \text{for } m = 1,2,\ldots,M. \end{aligned} \tag{4.40}$$

Then, the only quantities we need to calculate and save are the $y_m, m = 0,1,\ldots,M$. The original problem $\mathbf{Af_n} = \mathbf{f_{n+1}}$ can be written as $\mathbf{L}(\mathbf{Uf_n}) = \mathbf{f_{n+1}}$, which can be broken down as

$$\mathbf{Uf_n} = \mathbf{x_n} \text{ and } \mathbf{Lx_n} = \mathbf{f_{n+1}}$$

where $\mathbf{x_n}$ is an intermediate vector. We have eliminated the l_m in the lower triangular matrix and the z_m from the upper triangular matrix using (4.40). The solution procedure is to solve the two subproblems

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ \alpha_{11}/y_0 & 1 & \ldots & 0 & 0 \\ 0 & \alpha_{12}/y_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \alpha_{1M}/y_M & 1 \end{bmatrix} \begin{bmatrix} x_{n,0} \\ x_{n,1} \\ \vdots \\ x_{n,M-1} \\ x_{n,M} \end{bmatrix} = \begin{bmatrix} f_{n+1,0} \\ f_{n+1,1} \\ \vdots \\ f_{n+1,M-1} \\ f_{n+1,M} \end{bmatrix} \tag{4.41}$$

and

$$\begin{bmatrix} y_0 & \alpha_{30} & 0 & \ldots & 0 & 0 \\ 0 & y_1 & \alpha_{31} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & y_{M-1} & \alpha_{3M-1} \\ 0 & 0 & 0 & \ldots & 0 & y_M \end{bmatrix} \begin{bmatrix} f_{n,0} \\ f_{n,1} \\ \vdots \\ f_{n,M-1} \\ f_{n,M} \end{bmatrix} = \begin{bmatrix} x_{n,0} \\ x_{n,1} \\ \vdots \\ x_{n,M-1} \\ x_{n,M} \end{bmatrix} \tag{4.42}$$

For the Implicit method, the vector $\mathbf{f_{n+1}}$ is known and the intermediate vector quantities $\mathbf{x_n}$ are easily found by forward substitution. We can read off the value $x_{n,0}$ directly, while any other equation in the system relates only to $x_{n,m}$ and $x_{n,m-1}$. If we solve the system in increasing $m-$ indicial order, we have $x_{n,m-1}$ available at the time we have to solve for $x_{n,m}$. We can see that this generalizes to

$$\begin{aligned} x_{n,0} &= f_{n+1,0}, \\ x_{n,m} &= f_{n+1,m} - \frac{\alpha_{1m}x_{n,m-1}}{y_{m-1}} \qquad \text{for } m = 1,2,...,M. \end{aligned} \tag{4.43}$$

Similarly, solving (4.42) for the $\mathbf{f_{n,m}}$ is easily achieved by backward substitution. Indeed, $f_{n,M}$ can be read off directly. If we solve in decreasing $m-$ indicial order we can find all of the $f_{n,m}$ in the same manner. As $\mathbf{x_{n,m}}$, for $m = 1, 2, \ldots, M$ is known, we express $f_{n,m}$ in the form

$$\begin{aligned} f_{n,M} &= \frac{x_{n,M}}{y_M}, \\ f_{n,m-1} &= \frac{x_{n,m-1} - \alpha_{3m-1} f_{n,m}}{y_{m-1}} \qquad \text{for } m = 1, 2, \ldots, M. \end{aligned} \tag{4.44}$$

Our aim was to find the vector $\mathbf{f_n}$ which gives us the solution to the system of linear equations in (4.34) using the LU method [32].

The SOR Method

SOR stands for Successive Over-Relaxation. It is an example of an iterative method. In the iterative method, one starts with a guess for the solution and successively improves it until it converges to the exact solution. They have an advantage over the direct method in that they are easier to program and they generalize in straightforward ways to American option problems. The SOR is a refinement of Gauss-Seidel iterative method, which in turn is a development of the Jacobi method [30].

4.3.2 Change of Variable - The Implicit Method

We discretize the asset price with the central difference, and time with forward difference. Substituting the finite difference approximations for the asset price and time into (4.16), we obtain

$$\begin{aligned} &\frac{g(t+\Delta t, y) - g(t, y)}{\Delta t} + \frac{(r - \sigma^2/2)}{2\Delta y}\Big[g(t, y+\Delta y) - g(t, y-\Delta y)\Big] \\ &+ \frac{\sigma^2}{2\Delta y^2}\Big[g(t, y-\Delta y) - 2g(t, y) + g(t, y+\Delta y)\Big] = rg(t+\Delta t, y) \end{aligned}$$

and re-arranging we get

$$g_{n+1,m} = \frac{1}{1 - r\Delta t}\Big[\alpha_1^* g_{n,m-1} + \alpha_2^* g_{n,m} + \alpha_3^* g_{n,m+1}\Big] \tag{4.45}$$

where the α_κ^* for $\kappa = 1, 2, 3$ are given by

$$\begin{aligned}
\alpha_1^* &= \frac{1}{2}\frac{(r-\sigma^2/2)}{\Delta y}\Delta t - \frac{1}{2}\left[\frac{\sigma}{\Delta y}\right]^2 \Delta t, \\
\alpha_2^* &= 1 + \left[\frac{\sigma}{\Delta y}\right]^2 \Delta t, \\
\alpha_3^* &= -\frac{1}{2}\frac{(r-\sigma^2/2)}{\Delta y}\Delta t - \frac{1}{2}\left[\frac{\sigma}{\Delta y}\right]^2 \Delta t.
\end{aligned} \tag{4.46}$$

This method is generally better but a bit more difficult to implement than the Explicit finite difference method. It requires solving simultaneous equations. The methods for solving linear equations that we have discussed can be applied in solving the system of linear equations in this scheme.

We can show by Lax's equivalence theorem that the change of variable scheme is unconditionally stable, convergent and consistent.

4.4 The Crank Nicolson Method

The Crank Nicolson implicit finite difference method is the average of the implicit and explicit methods. The explicit scheme is given by (4.18) and the implicit by (4.34). We take the average of the two equations to get

$$\begin{aligned}
&\frac{f_{n+1,m} - f_{n,m}}{\Delta t} + \frac{rm\Delta S}{4\Delta S}\left[f_{n+1,m+1} - f_{n+1,m-1} + f_{n,m+1} - f_{n,m-1}\right] \\
&+ \frac{\sigma^2 m^2 \Delta S^2}{4\Delta S^2}\left[f_{n,m-1} - 2f_{n,m} + f_{n,m+1} + f_{n+1,m-1} - 2f_{n+1,m} + f_{n+1,m+1}\right] \\
&= \frac{1}{2}\left[rf_{n,m} + rf_{n+1,m}\right].
\end{aligned} \tag{4.47}$$

Re-arranging we get

$$\begin{aligned}
&\left[\frac{1}{4}rm\Delta t - \frac{1}{4}\sigma^2 m^2 \Delta t\right] f_{n,m-1} + \left[1 + \frac{1}{2}r\Delta t + \frac{1}{2}\sigma^2 m^2 \Delta t\right] f_{n,m} \\
&+ \left[-\frac{1}{4}\sigma^2 m^2 \Delta t - \frac{1}{4}rm\Delta t\right] f_{n,m+1} = \left[\frac{1}{4}\sigma^2 m^2 \Delta t - \frac{1}{4}rm\Delta t\right] f_{n+1,m-1} \\
&+ \left[1 - \frac{1}{2}r\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t\right] f_{n+1,m} + \left[\frac{1}{4}rm\Delta t + \frac{1}{4}\sigma^2 m^2 \Delta t\right] f_{n+1,m+1}
\end{aligned} \tag{4.48}$$

and we simplify to get

$$\rho_{1m} f_{n,m-1} + \rho_{2m} f_{n,m} + \rho_{3m} f_{n,m+1} = \chi_{1m} f_{n+1,m-1} + \chi_{2m} f_{n+1,m} + \chi_{3m} f_{n+1,m+1} \tag{4.49}$$

for $n = 0, 1, \ldots, N-1$ and $m = 1, 2, \ldots, M-1$. Then, the parameters $\rho_{\kappa m}$ and $\chi_{\kappa m}$ for $\kappa = 1, 2, 3$ are given as

$$\begin{aligned}
\rho_{1m} &= \frac{1}{4}rm\Delta t - \frac{1}{4}\sigma^2 m^2 \Delta t, \\
\rho_{2m} &= 1 + \frac{1}{2}r\Delta t + \frac{1}{2}\sigma^2 m^2 \Delta t, \\
\rho_{3m} &= -\frac{1}{4}\sigma^2 m^2 \Delta t - \frac{1}{4}rm\Delta t, \\
\chi_{1m} &= \frac{1}{4}\sigma^2 m^2 \Delta t - \frac{1}{4}rm\Delta t, \\
\chi_{2m} &= 1 - \frac{1}{2}r\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t, \\
\chi_{3m} &= \frac{1}{4}rm\Delta t + \frac{1}{4}\sigma^2 m^2 \Delta t.
\end{aligned} \tag{4.50}$$

We express the system of equations in (4.49) as $\mathbf{Cf_n} = \mathbf{Df_{n+1}}$. This results into a tridiagonal system given by

$$\begin{bmatrix}
\rho_{20} & \rho_{30} & 0 & \ldots & 0 & 0 & 0 \\
\rho_{11} & \rho_{21} & \rho_{31} & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \rho_{1M-1} & \rho_{2M-1} & \rho_{3M-1} \\
0 & 0 & 0 & \ldots & 0 & \rho_{1M} & \rho_{2M}
\end{bmatrix}
\begin{bmatrix}
f_{n,0} \\ f_{n,1} \\ \vdots \\ f_{n,M-1} \\ f_{n,M}
\end{bmatrix}$$

$$= \begin{bmatrix}
\chi_{20} & \chi_{30} & 0 & \ldots & 0 & 0 & 0 \\
\chi_{11} & \chi_{21} & \chi_{31} & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \chi_{1M-1} & \chi_{2M-1} & \chi_{3M-1} \\
0 & 0 & 0 & \ldots & 0 & \chi_{1M} & \chi_{2M}
\end{bmatrix}
\begin{bmatrix}
f_{n+1,0} \\ f_{n+1,1} \\ \vdots \\ f_{n+1,M-1} \\ f_{n+1,M}
\end{bmatrix} \tag{4.51}$$

The elements of vector $\mathbf{f_{n+1}}$ are known at maturity time T, and we express the system as $\mathbf{f_n} = \mathbf{C^{-1}Df_{n+1}}$. By repeatedly iterating from time T to time zero, we obtain the value of f as the price of the option. The diagonal entries of matrix $\mathbf{C}$ is $\rho_{2m} = 1 + r\Delta t/2 + \sigma^2 m^2 \Delta t/2$ are always positive and thus the diagonal elements are non zero. Therefore the matrix is non singular as the diagonal entries are non zero.

The boundary conditions and (4.49) results in some entry changes in the tridiagonal matrices $\mathbf{C}$ and $\mathbf{D}$. For the matrix $\mathbf{C}$, $\rho_{20}, \rho_{2M} = 1$ and $\rho_{30}, \rho_{1M} = 0$. For the matrix $\mathbf{D}$, $\chi_{20}, \chi_{2M} = 1$ and $\chi_{30}, \chi_{1M} = 0$.

Accuracy - Crank Nicolson Method

The finite difference approximations from the Taylors series expansion leads to truncation errors and this affects the accuracy of the scheme. The Crank Nicolson method is more accurate than the Explicit and Implicit methods with an accuracy of up to O$(\Delta t^2, \Delta S^2)$. We show this accuracy by equating the central difference and the symmetric central difference at $f_{n+\frac{1}{2},m} \equiv f(t+\Delta t/2, S)$. We expand $f_{n+1,m}$ in Taylor series at $f_{n+\frac{1}{2},m}$ to yield

$$f_{n+1,m} = f_{n+\frac{1}{2},m} + \frac{1}{2}\frac{\partial f}{\partial t}\Delta t + O(\Delta t^2) \tag{4.52}$$

and expanding $f_{n,m}$ at $f_{n+\frac{1}{2},m}$ gives

$$f_{n,m} = f_{n+\frac{1}{2},m} - \frac{1}{2}\frac{\partial f}{\partial t}\Delta t + O(\Delta t^2). \tag{4.53}$$

Taking the average of these two equations yields

$$\frac{1}{2}\left[f_{n,m} + f_{n+1,m}\right] = f_{n+\frac{1}{2},m} + O(\Delta t^2).$$

The subscript m was arbitrary and we can write this for subscripts $m-1, m$ and $m+1$ as follows

$$\begin{aligned} & f_{n+\frac{1}{2},m-1} - 2f_{n+\frac{1}{2},m} + f_{n+\frac{1}{2},m+1} \\ &= \frac{1}{2}\left[f_{n,m-1} - 2f_{n,m} + f_{n,m+1}\right] + \frac{1}{2}\left[f_{n+1,m-1} - 2f_{n+1,m} + f_{n+1,m+1}\right] + O(\Delta t^2). \end{aligned} \tag{4.54}$$

The right hand side of (4.54) is an average of two symmetric central differences centered at grid points n and $n+1$. Dividing by ΔS^2 we obtain the equality

$$\frac{\partial^2 f(t+\frac{1}{2}\Delta t, S)}{\partial S^2} = \frac{1}{2}\left[\frac{\partial^2 f(t,S)}{\partial S^2} + \frac{\partial^2 f(t+\Delta t, S)}{\partial S^2}\right] + O(\Delta t^2, \Delta S^2) \tag{4.55}$$

which is the second order partial derivative defined by the symmetric central difference approximation. The subscript m is arbitrary and we derive the central difference approximation as follows

$$\begin{aligned} & f_{n+\frac{1}{2},m+1} - f_{n+\frac{1}{2},m-1} \\ &= \frac{1}{2}\left[f_{n,m+1} - f_{n,m-1}\right] + \frac{1}{2}\left[f_{n+1,m+1} - f_{n+1,m-1}\right] + O(\Delta t^2). \end{aligned} \tag{4.56}$$

We divide the equation by $2\Delta S$ to get the equality

$$\frac{\partial f(t+\frac{1}{2}\Delta t, S)}{\partial S} = \frac{1}{2}\left[\frac{\partial f(t,S)}{\partial S} + \frac{\partial f(t+\Delta t, S)}{\partial S}\right] + O(\Delta t^2, \Delta S^2) \tag{4.57}$$

which is the first order partial derivative defined by the symmetric central difference approximation. Now, subtract (4.53) from (4.52) to obtain the approximation of $\partial f/\partial t$ centered at $(t+\frac{1}{2}\Delta t, S)$

$$\frac{\partial f(t+\frac{1}{2}\Delta t, S)}{\partial t} = \frac{f_{n+1,m} - f_{n,m}}{\Delta t} + O(\Delta t^2). \tag{4.58}$$

Hence the Black Scholes PDE centered at $(t+\frac{1}{2}\Delta t, S)$ has a finite difference approximation

$$\begin{aligned} &\frac{f_{n+1,m} - f_{n,m}}{\Delta t} + \frac{rm\Delta S}{4\Delta S}\Big[f_{n,m+1} - f_{n,m-1} + f_{n+1,m+1} - f_{n+1,m-1}\Big] \\ &+ \frac{\sigma^2 m^2 \Delta S^2}{4\Delta S^2}\Big[f_{n,m-1} - 2f_{n,m} + f_{n,+m+1} + f_{n+1,m-1} - 2f_{n+1,m} + f_{n+1,m+1}\Big] \\ &= rf_{n,m} \end{aligned} \tag{4.59}$$

and re-arranging, we get an equation of the form (4.49) which is the exact Crank Nicolson scheme. Therefore, the scheme has a leading error of order O$(\Delta t^2, \Delta S^2)$ [20].

4.4.1 Options on Futures

We applied the Black's model (2.62) and the binomial model (3.28) in pricing of the options on futures. We now apply the Crank Nicolson finite difference method. This method is more accurate and converges faster than the other two finite difference methods.

The PDE underlying the Black's model is given by

$$\frac{\partial c(F_t, t)}{\partial t} + \frac{\sigma^2}{2}F^2\frac{\partial^2 c(F_t, t)}{\partial F_t^2} = rc(F_t, t). \tag{4.60}$$

We apply the same discretization procedure and finite difference approximations as in our previous work in the Black Scholes PDE. The only major change is to replace the underlying asset S with the futures price F. For the Crank Nicolson method, we average the implicit and the explicit finite difference methods for (4.60) to get

$$\begin{aligned} &\frac{\sigma^2 m^2 \Delta F^2}{4\Delta F^2}\Big[(f_{n,m-1} - 2f_{n,m} + f_{n,m+1}) + (f_{n+1,m-1} - 2f_{n+1,m} + f_{n+1,m+1})\Big] \\ &+ \frac{f_{n+1,m} - f_{n,m}}{\Delta t} = \frac{1}{2}\Big[rf_{n,m} + rf_{n+1,m}\Big]. \end{aligned} \tag{4.61}$$

Re-arranging we have

$$\omega_{1m} f_{n,m-1} + \omega_{2m} f_{n,m} + \omega_{1m} f_{n,m+1} = \gamma_{1m} f_{n+1,m-1} + \gamma_{2m} f_{n+1,m} + \gamma_{1m} f_{n+1,m+1} \tag{4.62}$$

where $\omega_{\kappa m}$ and $\gamma_{\kappa m}$ for $\kappa = 1, 2$ are given as

$$\begin{aligned}
\omega_{1m} &= -\frac{1}{4}\sigma^2 m^2 \Delta t, \\
\omega_{2m} &= 1 + \frac{1}{2} r \Delta t + \frac{1}{2}\sigma^2 m^2 \Delta t, \\
\gamma_{1m} &= \frac{1}{4}\sigma^2 m^2 \Delta t, \\
\gamma_{2m} &= 1 - \frac{1}{2}\sigma^2 m^2 \Delta t - \frac{1}{2} r \Delta t.
\end{aligned} \tag{4.63}$$

The tridiagonal system is expressed as

$$\begin{bmatrix} \omega_{20} & \omega_{10} & 0 & \dots & 0 & 0 & 0 \\ \omega_{11} & \omega_{21} & \omega_{11} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \omega_{1M-1} & \omega_{2M-1} & \omega_{1M-1} \\ 0 & 0 & 0 & \dots & 0 & \omega_{1M} & \omega_{2M} \end{bmatrix} \begin{bmatrix} f_{n,0} \\ f_{n,1} \\ \vdots \\ f_{n,M-1} \\ f_{n,M} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_{20} & \gamma_{10} & 0 & \dots & 0 & 0 & 0 \\ \gamma_{11} & \gamma_{21} & \gamma_{11} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_{1M-1} & \gamma_{2M-1} & \gamma_{1M-1} \\ 0 & 0 & 0 & \dots & 0 & \gamma_{1M} & \gamma_{2M} \end{bmatrix} \begin{bmatrix} f_{n+1,0} \\ f_{n+1,1} \\ \vdots \\ f_{n+1,M-1} \\ f_{n+1,M} \end{bmatrix} \tag{4.64}$$

and can be written as $\mathbf{Af_n} = \mathbf{Bf_{n+1}}$, where $\mathbf{f_{n+1}}$ is known at the maturity time T. Then we have $\mathbf{f_n} = \mathbf{A^{-}Bf_{n+1}}$. The same analysis for matrix $\mathbf{A}$ applies as in our previous work. We apply the same boundary condition principles that are given by (4.11) and (4.12) for the options on spot price. The payoff of a European put futures option is given by max$(K - F_T, 0)$. We impose the initial condition as the value of the put option is known at time T

$$f_{N,m} = \max(K - m\Delta F, 0) \qquad \text{for } \; m = 0, 1, \dots, M \tag{4.65}$$

where K is the strike price and F_T is the futures price at the maturity date T. We apply the boundary conditions to the system of equations in (4.62) and this results in entry changes of matrix $\mathbf{A}$ and $\mathbf{B}$. For matrix $\mathbf{A}$, $\omega_{10}, \omega_{1M} = 0$ and $\omega_{20}, \omega_{2M} = 1$. For the matrix $\mathbf{B}$, $\gamma_{10}, \gamma_{1M} = 0$ and $\gamma_{20}, \gamma_{2M} = 1$.

4.4.2 Matlab Implementation

We are dealing with tridiagonal matrices which will generally be large. Very large matrices occupy huge amounts of memory, and processing them can take up a lot of computer time. For example, a system of n simultaneous linear equations requires n^2 matrix entries, and the computing time to solve them is proportional to n^3. In our case, the matrices have very few non-zero entries. Such matrices are called sparse as opposed to full. Matlab has facilities for exploiting the sparsity of matrices, and has the potential of saving huge amounts of memory and processing time.

We said that the implicit finite difference method can be expressed as $\mathbf{f_n} = \mathbf{A^{-1}f_{n+1}}$. Matlab has an inbuilt function to cater for the inverse of a matrix. It is accurate and efficient as it uses the Gauss elimination method [16]. This inbuilt function will ease our implementation of the Implicit and Crank Nicolson methods in Matlab.

Options on Spot Price

Earlier in chapter 3, we examined the multi period binomial model. We consider the convergence of the fully implicit, the Crank Nicolson method and the multi period model with relation to the Black Scholes value of the option. We price the American put option on a non dividend paying stock with $S = 20, K = 22, r = 0.1, T = 0.5$ and $\sigma = 0.25$. See appendix A.1, A.4 and A.5 for Matlab codes.

Table 4.1 shows that the Crank Nicolson finite scheme in (4.49) converges faster than the fully implicit finite scheme in (4.34) as $N \to \infty, \Delta t \to 0$ and as $M \to \infty, \Delta S \to 0$. The multi-period binomial model is closer to the solution for small values of N than the two finite difference methods. When N and M are different, the finite difference methods converges faster than when N and M are the same.

Options on Futures

To implement the options on futures in Matlab, the procedure is the same as in the option on spot assets. We apply the matrix in (4.64). The option prices in table 4.2 are as a result of using the Crank Nicolson method to approximate (4.60) for options on futures, when the option is an American put with $F = 514.80, K = 500, r = 0.07, T = 1.0$ and $\sigma = 0.2$. See appendix A.6

Table 4.1: The Comparison of the convergence of the Implicit method, the Crank Nicolson method and the multi-period binomial model as we increase N and M.

N = M	**Multi-period Binomial**	**Fully Implicit**	**Crank Nicolson**	**N**	**M**	**Fully Implicit**	**Crank Nicolson**
10	2.2344	2.0574	2.0637	10	20	2.1326	2.1596
20	2.2483	2.1546	2.1694	20	40	2.2091	2.2209
30	2.2477	2.2204	2.2302	30	60	2.2234	2.2340
40	2.2457	2.2177	2.2238	40	80	2.2287	2.2369
50	2.2436	2.2286	2.2354	50	100	2.2328	2.2388
60	2.2439	2.2317	2.2369	60	120	2.2352	2.2405
70	2.2459	2.2342	2.2385	70	140	2.2366	2.2413
80	2.2466	2.2352	2.2395	80	160	2.2377	2.2418
90	2.2463	2.2379	2.2413	90	180	2.2387	2.2422
100	2.2453	2.2374	2.2407	100	200	2.2393	2.2426

for a Matlab code.

Table 4.2 shows that the finite difference methods are suited for pricing American put options on futures contracts. The price of the American put option with the same parameters is also displayed in table 3.3 using the multi-period binomial model.

Table 4.2: Crank Nicolson finite difference method for the American put option on futures.

N	10	20	40	60	100	160	210
M	40	50	70	90	130	190	240
American Put	31.6035	31.5903	31.5488	31.6377	31.6384	31.6537	31.6576

We conclude that the finite difference methods, just like the binomial model are very powerful in pricing of vanilla options. The Crank Nicolson method has a higher accuracy than the implicit method and this means that it converges faster, though the two methods are unconditionally stable. The option prices in table 4.1 highlighted this fact.

Our last chapter is on Monte Carlo simulation method. This technique is very flexible and emerging in popularity as an alternative method for the pricing of exotic options.

Chapter 5

Monte Carlo Simulation

In the previous three chapters, we considered the analytical techniques for pricing plain vanilla products. We now discuss the use of Monte Carlo simulation method for pricing exotic or non-standard options. Sometimes, these options do not have a convenient analytical formula available for pricing them. Even in cases where there is an analytical formula available, Monte Carlo simulation can be applied to give an estimate of the option's price.

5.1 Simulation

Simulation is a numerical technique for conducting experiments by imitating a situation using mathematical and logical models in order to estimate the likelihood of various possible outcomes over a period of time.

There are a number of situations where simulation can be used successfully:

- When it is extremely expensive or impossible to obtain data from certain processes in the real world. For example, the effect of an advertising campaign on the total sales, the effect of proposed tax cuts on the economy. The simulated data is then necessary to formulate hypothesis about the system.

- It may be either impossible or very costly to perform validating experiments on the mathematical models describing the system. We say that the simulated data can be used to test alternative hypothesis.

- The observed system may be very complex that it cannot be described in terms of a set of mathematical equations for which analytical solutions are obtainable. For example, it is difficult to describe the operation of a business firm or an industry in terms of a few simple equations.

Computer simulation enables us to *replicate* an experiment. Replication means re-running an experiment with selected changes in parameters involved, but without changing the outcomes. This permits a considerable degree of freedom so that a model can bear a close correspondence to the system being studied.

Even though simulation is an invaluable and versatile tool in those problems where analytical techniques are inadequate, it is not an ideal tool. Simulation is an imprecise technique, providing only statistical estimates rather than exact results. It is a slow and costly way to study a problem, and requires a large amount of time and great expense for analysis and programming.

We have defined simulation in a broader sense. Then, *stochastic simulation* is experimenting with the model over time and it involves sampling stochastic variates from a probability distribution, that is, it is a statistical sampling experiment [27]. The sampling from a particular distribution involves the use of random numbers. Thus, stochastic simulation is sometimes called *Monte Carlo Simulation* (MCS).

5.1.1 Monte Carlo Method

Monte Carlo method is an analytical technique for solving a problem by performing a large number of trial runs, called simulations, and inferring a solution from the collective results of the trial runs.

The term "Monte Carlo" was introduced by Von Neumann and Ulam during World War II, as a code for the secret work at Los Alamos. The standard Monte Carlo technique uses random or pseudorandom numbers which are independent random variables uniformly distributed over the unit interval $[0, 1)$ [27].

Monte Carlo simulation has been applied in many fields, including the pricing of financial derivatives. This method can be used in estimating option prices for derivatives that do or do not have a convenient analytical formula. It uses the risk-neutral valuation in which the expected payoff in a risk neutral world is calculated using a sampling procedure, and discounted at the risk-free interest rate. In an efficient market, the pricing of an option is equivalent to evaluating the expectation of its discounted payoff under a specified measure.

The use of Monte Carlo simulation in pricing options was first published by Boyle (1977) in [6]. Twenty years later, Boyle, Broadie and Glasserman (1997) described in [7] research advances that had improved efficiency and broadened the types of problem where simulation can be applied. The research undertaken has proved that simulation is a valuable tool for pricing options. What makes Monte Carlo simulation method popular?

- It is easy to apply to many problems even for complicated or high dimensional financial models.

- Its good performance on high-dimensional problems. The rate of convergence of a MCS estimate does not depend on the dimension of the problem. The high dimension is due to models of markets that have derivative securities depending in a non-trivial way on prices at many times. MCS is becoming increasingly important as securities markets and financial risk management become more sophisticated.

- The confidence interval provided for by the MCS estimate makes it possible to assess the quality of the estimate. We can then deduce how much more computational effort is needed to achieve the desired results or acceptable quality.

The increased availability of powerful computers and easy to use software has enhanced the appeal of simulation to price derivatives. There are some disadvantages of Monte Carlo simulation but in recent years progress has been to overcome these problems. For a very complex problem, a large number of replications may be required to obtain precise results. Different variance reduction techniques have been developed to enhance precision. We consider two of these techniques later in our work, the control variate and antithetic variate method [7].

In our analysis, we make the usual assumptions underlying the Black-Scholes-Merton model, in that: (1) The price of the underlying asset of an option follows a log-normal random walk. (2) There are no arbitrage opportunities. (3) The price of the underlying asset is expected to appreciate at the risk-free rate of interest.

When using Monte Carlo simulation, the main steps followed are [7]:

- Simulate a path of the underlying asset under the risk neutral condition within the desired time horizon.

- Discount the payoff corresponding to the path at the risk-free interest rate. The structure of the security in question should be adhered to.

- Repeat the procedure for a high number of simulated sample paths.

- Average the discounted cash flows over sample paths to obtain the option's value.

A Monte Carlo simulation can be used as a procedure for sampling random outcomes of a process followed by the stock price

$$dS = \mu S dt + \sigma S dW_t, \tag{5.1}$$

where dW_t is a Wiener process and S is the stock price. If δS is the increase in the stock price in the next small interval of time δt then

$$\frac{\delta S}{S} = \mu\delta t + \sigma Z\sqrt{\delta t}, \tag{5.2}$$

where $Z \sim N(0,1)$, σ is the volatility of the stock price and μ is its expected return in a risk-neutral world. (5.2) is expressed as

$$S(t+\delta t) - S(t) = \mu S(t)\delta t + \sigma S(t) Z\sqrt{\delta t}. \tag{5.3}$$

We can calculate the value of S at time $t+\delta t$ from the initial value of S, then the value of S at time $t+2\delta t$ from the value at time $t+\delta t$, and so on. We use N random samples from a normal distribution to simulate a trial for a complete path followed by S. It is more accurate to simulate ln$S$ than S, we transform the asset price process using Itô's lemma

$$\begin{aligned} d\text{ln}S &= (\mu - \sigma^2/2)dt + \sigma dW_t \\ \text{so that} \qquad \text{ln}S(t+\delta t) - \text{ln}S(t) &= (\mu - \sigma^2/2)\delta t + \sigma Z\sqrt{\delta t} \\ \text{or} \qquad S(t+\delta t) &= S(t)\text{exp}[(\mu - \sigma^2/2)\delta t + \sigma Z\sqrt{\delta t}]. \end{aligned} \tag{5.4}$$

MCS is particularly relevant when the financial derivative's payoff depends on the path followed by the underlying asset during the life of the option, that is, for path dependent options. The method can also be applied when the value of the financial derivative depends only on the final value of the underlying asset. An example is the European style option whose payoff depends on the value of S at maturity time T [17]. The stock price process for a European option can be expressed as

$$S_T^i = S\text{exp}[(\mu - \sigma^2/2)T + \sigma z\sqrt{T}], \tag{5.5}$$

where $i = 1, 2, \ldots, M$ and M denotes the number of trials or the different states of the world. These M simulations are the possible paths that a stock price can have at maturity date T. The estimated European call option value is

$$c = \frac{1}{M}\sum_{i=1}^{M} \mathrm{e}^{-rT}\max[S_T^i - K, 0]. \tag{5.6}$$

This is an unbiased estimate of the derivative's price. When the number of trials M is large, the central limit theorem provides a confidence interval for the estimate, based on the sample variance of the discounted payoff. The M independent trials carried out depends on the accuracy required. If ω is the standard deviation and $\bar{\mu}$ is the mean of the discounted payoffs given by (5.6), then the standard error is estimated by $\omega/\sqrt{M}$. A 95% confidence interval for the price f of the derivative is therefore, given by

$$\bar{\mu} - \frac{1.96\omega}{\sqrt{M}} < f < \bar{\mu} + \frac{1.96\omega}{\sqrt{M}}, \tag{5.7}$$

under the assumption that f is normally distributed [17].

5.2 Variance Reduction Procedures

The uncertainty about the value of the derivative is inversely proportional to the square root of the number of trials. Then, if the simulation is to give accurate results, very large number of simulated sample paths is usually necessary. This is very expensive in terms of computational time. The variance reduction technique refines and improves the efficiency of the simulation.

5.2.1 Antithetic Variable Technique

In this technique, a simulation trial involves calculating two values of the derivative. The first value f_1 is calculated in the usual way. The second value f_2 is calculated by changing the sign of all the random samples from the standard normal distribution. If Z is a sample used to calculate f_1, then $-Z$ is the corresponding sample used to calculate f_2. For example, if we use (5.5), then we have two equations of the form

$$\begin{aligned} S_T &= S\text{exp}[(\mu - \sigma^2/2)T + Z\sigma\sqrt{T}] \\ S_T &= S\text{exp}[(\mu - \sigma^2/2)T - Z\sigma\sqrt{T}]. \end{aligned} \tag{5.8}$$

We prefer to use the random inputs obtained from the collection of antithetic pairs $(Z, -Z)$ as they are more regularly distributed than a collection of $2N$ independent samples. The pair is called antithetic because they exhibit negative independence. The sample mean of the antithetic pairs always equals the population mean of zero. The mean over finitely many independent samples is almost surely different from zero. We denote $\bar{f}$ as the average of f_1 and f_2

$$\bar{f} = \frac{f_1 + f_2}{2}.$$

Then

$$\text{Var}(\bar{f}) = \text{Var}[\frac{1}{2}(f_1 + f_2)] = \frac{1}{4}\text{Var}[f_1] + \frac{1}{4}\text{Var}[f_2] + \frac{1}{2}\text{Cov}[f_1, f_2].$$

If the covariance, Cov$[f_1, f_2]$, between f_1 and f_2 is negative this will yield a smaller estimate of the variance than an independent estimate.

The confidence interval is computed by estimating the standard error using the sample standard deviation of the N averaged pairs $(f_1 + f_2)/2$ and not the $2N$ individual observations [7]. Thus the antithetic variate exploits the existence of the negative correlation between two estimates.

5.2.2 Control Variate Technique

In this technique, we replace the evaluation of an unknown expectation with the evaluation of the difference between the unknown quantity and a related quantity, whose expectation is known.

The control variate uses a second estimate with a high positive correlation with the estimate of interest. We carry out two simulations using the same number streams and the same δt. Let f_A and f_B be the respective values of A and B. Then we can write $f_A = \text{E}[f_A^*]$ and $f_B = \text{E}[f_B^*]$, where f_A^* and f_B^* are estimate values of A and B respectively.

Derivative A whose value is f_A is the security under consideration. Derivative B whose value is f_B, is similar to derivative A and has an analytical solution available. A random variate f_B is a control variate for f_A if it is correlated with f_A. Then

$$\hat{f}_A = f_A^* + (f_B - f_B^*),$$

where f_B is the known value of B. The known error $(f_B - f_B^*)$ is used as a control in the estimation of f_A. The value $\hat{f}_A$ adjusts the estimator f_A according to the difference between the known value f_B and the observed value f_B^*. We aim to reduce the variance and comparing the values of derivative A and B, we have

$$\mathrm{Var}[\hat{f}_A] = \mathrm{Var}[f_A^*] + \mathrm{Var}[f_B] + \mathrm{Var}[f_B^*] - 2\mathrm{Cov}[f_A^*, f_B^*] \tag{5.9}$$

and $\mathrm{Var}[f_B] = 0$ since f_B is the known value of B and thus not a random variable. This control variate technique is effective if the covariance between f_A^* and f_B^* is large, that is, if $2\mathrm{Cov}[f_A^*, f_B^*] > \mathrm{Var}[f_A^*] + \mathrm{Var}[f_B^*]$, then the variance is reduced [7].

5.3 Exotic Options

Financial engineers have created various exotic products to meet the different market needs. These products are: (1) Designed to meet a genuine hedging need in the market, that is, they are 'tailor-made.' (2) Sometimes designed to reflect a corporate treasurer's view on potential future movements in particular market variables. (3) Sometimes attractive due to tax, legal or regulatory reasons in the market [17].

We can classify exotic options as: (1) Path dependent options, like for example, Asian, Barrier and Lookback. (2) Correlation options, like for example, Basket, Exchange, Foreign-Equity, Quanto and Spread. (3) Other exotic options, like for example, Digital, Chooser and Contingent premium.

Even though some exotic options have an exact pricing formula, we approximate the options price using Monte Carlo simulation. We consider two path dependent options, the Asian and lookback options.

5.3.1 Path Dependent Option

A Path dependent option is an option whose value depends on the sequence of prices of the underlying asset during the whole or part of the option's life rather than just the final price of the asset.

Asian Options

Asian or Average options are options whose payoff depends on the average price of the underlying asset during at least some part of the life of the option.

Let N denote the number of trading days of the option, T the maturity date of the option, and $S(t_j)$ the security's price at the end of the day j, where $j = 1, 2, \ldots, N$, and $t_N = T$. Then, the average of the underlying asset price can be calculated using two methods, namely the arithmetic and geometric average.

- **Arithmetic Average:** Let $S_A(t)$ be the arithmetic average value of the underlying asset calculated over the life of the option. The arithmetic average is calculated using

$$\begin{aligned} S_A(t) &= \frac{S(t_1) + S(t_2) + \ldots + S(t_N)}{N} \\ &= \frac{1}{N} \sum_{j=1}^{N} S(t_j). \end{aligned} \tag{5.10}$$

- **Geometric Average:** Let $S_G(t)$ be the geometric average value of the underlying asset calculated over the life of the option. Then the geometric average is given in [7] as

$$\begin{aligned} S_G(t) &= \left[\prod_{j=1}^{N} S(t_j) \right]^{1/N} \\ &= [S(t_1)S(t_2) \ldots S(t_N)]^{1/N}. \end{aligned} \tag{5.11}$$

The two types of Standard Asian options obtained using the arithmetic or geometric average of the underlying asset are:

(i) Average Price Option

- An average price call payoff is $\max(\bar{S}(t) - K, 0)$.
- An average price put payoff is $\max(K - \bar{S}(t), 0)$.

(ii) Average Strike Price Option

- An average strike call payoff is $\max(S_T - \bar{S}(t), 0)$.
- An average put payoff is $\max(\bar{S}(t) - K, 0)$,

where $\bar{S}(t)$ is either given by the arithmetic average in (5.10) or geometric average in (5.11).

Average price options are more appropriate to meet some needs of corporate treasurer and they are less expensive. For example, a South African corporate treasurer expects to receive a cash flow of 120 million U.S dollars spread evenly over the next year from the company's

U.S subsidiary. Then the treasurer is likely to be interested in an option that guarantees that the average exchange rate realized during the year is above some level. The Asian put option can easily achieve this than a regular put option.

Asian options have gained popularity in the foreign currency market, interest rate and commodity markets. They are attractive to traders for the following reasons: (1) There is a minimal chance of the underlying asset price manipulation as the final payoff depends on the average price during the life of the option. The manipulation is easy for options whose payoff depends only on the final asset price. (2) They sell at a lower premium than the vanilla options. The volatility in the average asset price tends to be lower than the volatility of the underlying asset in the vanilla options. Note that we are primarily concerned with European style options [9].

The product of log-normal prices is itself log-normal. Thus the geometric average has a closed form analytical formula while the arithmetic average do not because they lack an analytically tractable properties.

The other type of Asian options are the Flexible Asian options which are an extension of standard Asian options. The pricing differs in that the weighting is equal for the Standard Asian options. For the Flexible Asian options, the weights are different and are assigned depending on the needs of the investor. These options will not be considered in our work.

We can express the standard Average strike price Asian call option payoff as

$$f_c(S,T) = \max\left[S(T) - \frac{1}{T}\int_0^T S(\tau)d\tau, 0\right], \tag{5.12}$$

where its value depends on the history of the asset price, not simply its final value [32]. The Asian put is expressed as

$$f_p(S,T) = \max\left[\frac{1}{T}\int_0^T S(\tau)d\tau - S(T), 0\right]. \tag{5.13}$$

One of the fundamental concerns is the frequency with which the price will be observed over the averaging period. To price (5.12) by Monte Carlo, we choose a positive integer N and subdivide the time interval $[0,T]$ into N equal subintervals and $\Delta t = T/N$. We simulate the asset price

$$S[(\kappa+1)\Delta t] = S(\kappa\Delta t)\text{exp}\left[(r - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}Z_\kappa\right] \tag{5.14}$$

where $Z_\kappa \sim N(0,1)$ for $\kappa = 0, 1, \ldots, N-1$. Set $S_\kappa = S(\kappa\Delta t)$. Then (5.14) implies

$$\begin{aligned} \ln\left[\frac{S_{\kappa+1}}{S_\kappa}\right] = X_\kappa &= \left[(r - \frac{\sigma^2}{2})\Delta t + \sigma Z_\kappa \sqrt{\Delta t}\right] \\ &= \mu\Delta t + \sigma Z_\kappa \sqrt{\Delta t} \end{aligned} \tag{5.15}$$

where $\mu = (r - \sigma^2/2)$ is the drift parameter of a risk-neutral GBM, and $X_\kappa \sim N(\mu\Delta t, \sigma^2\Delta t)$. Since

$$\ln\left[\frac{S_{\kappa+1}}{S_\kappa}\right] = X_\kappa$$

then it implies that

$$\begin{aligned} S_{\kappa+1} &= S_\kappa \mathrm{e}^{X_\kappa} \\ &= S_{\kappa-1}\mathrm{e}^{X_{\kappa-1}}\mathrm{e}^{X_\kappa} \\ &= S_o \mathrm{e}^{X_o + \ldots + X_\kappa}. \end{aligned} \tag{5.16}$$

Equation (5.16) gives an explicit formula, while (5.14) gives a recurrence relation for S_κ. We can approximate the time average integral by the trapezium rule

$$\int_0^T S(\tau)d\tau \approx \frac{1}{N}\left[\frac{1}{2}S(0) + \frac{1}{2}S(T) + \sum_{\kappa=1}^{N-1} S(\kappa\Delta t)\right] \tag{5.17}$$

and this gives a discrete approximation $\bar{S}_t$. The discretely monitored Asian call option has the estimated value in the ith path given by

$$c^i = \mathrm{e}^{-rT}\max[S_T - \bar{S}_t, 0]. \tag{5.18}$$

This is repeated for $i = 1, 2, \ldots, M$ and the final estimated option value is

$$c = \frac{1}{M}\sum_{i=1}^{M} c^i. \tag{5.19}$$

Table 5.1 shows the results of Monte Carlo simulation and we have assumed there are 252 trading days in a year. We have taken $N = 126$ days which corresponds to $T = 0.5$ years, and $M = 10000$ as the number of simulation, each of which corresponds to possible path that can be taken by the asset price during the life of the option. See appendix A.7 for the Matlab code.

The initials *'Conf. Interval'* in Table 5.1 stands for confidence interval. The simulation results have a confidence interval for which the geometric analytical formula values lies in. We apply the principle in (5.7) to help us calculate the confidence interval. These calculations are part of the Matlab code in appendix A.7.

The values obtained using the geometric averaging method are more accurate than those of the arithmetic averaging. The vanilla option with the same parameters as a standard Asian option is more expensive. This is due to the fact that the average asset price tends to have a lower volatility than that of the underlying asset in the vanilla options.

The geometric averaging analytical formula used was formulated by Kemna and Vorst in 1990 [9]. They altered the volatility and in formulating the formula they had the advantage that the geometric average of the underlying prices follows a log normal distribution.

Table 5.1: MCS results and the geometric formula pricing of the Asian average price options compared to the Black Scholes model for vanilla options. $K = 25, r = 0.12, T = 0.5, \sigma = 0.4$.

	Stock Price, S	**20**	**25**	**30**
Arithmetic Average Method	Call	0.214	1.952	5.771
	Conf. Interval	(0.147, 0.282)	(1.868, 2.035)	(5.670, 5.873)
	Put	4.368	1.207	0.189
	Conf. Interval	(4.301, 4.496)	(1.124, 1.291)	(0.087, 0.291)
Geometric Average Method	Call	0.180	1.848	5.602
	Conf. Interval	(0.112, 0.247)	(1.765, 1.931)	(5.500, 5.704)
	Put	4.467	1.265	0.218
	Conf. Interval	(4.400, 4.534)	(1.182, 1.349)	(0.1159, 0.3194)
Geo. Average Analytical Formula	Call	0.186	1.844	5.587
	Put	4.449	1.290	0.2118
Black Scholes for Vanilla options	Call	1.069	3.518	7.268
	Put	4.613	2.063	0.812

Lookback Options

A lookback call (put) is an option whose strike price corresponds to the minimum (maximum) price recorded by the underlying asset during the option's life. The lookback call (put) involves the right to buy (sell) at the lowest (highest) price in the life of the option.

There are two types of lookback options:

- **The Floating Strike Price Lookbacks**

If the exercise time is at the end of N trading days, then the payoff of a floating strike lookback call option is the difference between the minimum underlying asset price achieved during the life of the option and the final asset price. Thus the payoff is

$$c_{\text{ft}} = \max[S_T - \min_{1 \leq j \leq N} S(t_j), 0].$$

Similarly, the payoff of a floating strike lookback put option is the difference between the maximum underlying asset price achieved during the life of the option and the final asset price. Hence the payoff is

$$p_{\text{ft}} = \max[\max_{1 \leq j \leq N} S(t_j) - S_T, 0].$$

- **The Fixed Strike Price Lookbacks**

The payoff of a fixed strike lookback option has similar payoff to that of a standard option, with the strike price K, except that the final underlying asset price S_T is replaced by the maximum (minimum) asset price reached during the life of the option for a call (put). Thus, the respective payoffs of the fixed strike lookback call and put options are

$$c_{\text{fd}} = \max[M - K_f, 0]$$

$$p_{\text{fd}} = \max[K_f - m, 0],$$

where $M = \max S(t_j), m = \min S(t_j)$ for $j = 1, 2, \ldots, N$, and K_f is the fixed strike price known at the onset of the contract. When the final underlying asset price is the maximum value recorded during the option's life, the fixed strike lookback call's payoff is equal to that of a standard call.

The floating strike price lookback options allow investors with special information on the range of the asset price to take advantage of such information [9]. The lookback options are more expensive than standard options and this has hindered their popularity in actual markets. This has led to the creation of partial lookback options.

The principles of pricing the floating strike price lookback options are similar to those applied in pricing the standard Asian options. We illustrate the use of antithetic variance reduction technique in pricing lookback options.

We simulate the asset price for N days using the equation in (5.14). For the antithetic technique we write the equations as

$$\begin{aligned} S^i_+[(j+1)\Delta t] &= S(j\Delta t)\text{exp}[(r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}Z_j] \\ S^i_-[(j+1)\Delta t] &= S(j\Delta t)\text{exp}[(r-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}Z_j], \end{aligned} \tag{5.20}$$

for $j = 0, 1, \ldots, N-1$, $i = 1, 2, \ldots, M$, and where S_+ and S_- are positive and negative antithetic values of the stock price respectively. This gives us a collection of antithetic pairs of stock prices. We obtain the maximum (minimum) asset price for the put (call) reached during the life of the option for each of the equations in (5.20). Then the estimates of the ith simulation for the call option are given by ,

$$\begin{aligned} c^i_{\text{+ft}} &= \max[S_+(t_N) - m_+, 0] \\ c^i_{\text{-ft}} &= \max[S_-(t_N) - m_-, 0], \end{aligned} \tag{5.21}$$

where $m_+ = \min S_+(t_j)$ and $m_- = \min S_-(t_j)$ for $j = 1, 2, \ldots, N$. The values $S_+(t_N)$ and $S_-(t_N)$ are the stock prices at maturity time T. The estimates of the put option are given as

$$\begin{aligned} p^i_{\text{+ft}} &= \max[M_+ - S_+(t_N), 0] \\ p^i_{\text{-ft}} &= \max[M_- - S_-(t_N), 0], \end{aligned} \tag{5.22}$$

where $M_+ = \max S_+(t_j)$ and $M_- = \max S_-(t_j)$ for $j = 1, 2, \ldots, N$. We repeat the procedure for M simulated sample paths. The respective estimated call and put option prices are

$$\begin{aligned} c_{\text{ft}} &= \frac{\text{e}^{-rT}}{2M}\left[\sum_{i=1}^{M} c^i_{\text{+ft}} + \sum_{i=1}^{M} c^i_{\text{-ft}}\right], \\ p_{\text{ft}} &= \frac{\text{e}^{-rT}}{2M}\left[\sum_{i=1}^{M} p^i_{\text{+ft}} + \sum_{i=1}^{M} p^i_{\text{-ft}}\right]. \end{aligned} \tag{5.23}$$

The algorithm can easily be implemented in Matlab to estimate the prices of the floating strike price lookback options.

We conclude that Monte Carlo simulation is a versatile tool in option pricing where analytical formulas do or do not exist. It is flexible in handling varying and even high-dimensional financial problems. However, it is costly in terms of time and computing resources. The advances in research has led to faster computers and in variance reduction techniques to save on time. Thus, Monte Carlo simulation is becoming more appealing and gaining popularity in derivative pricing due to the increasing number of more and more sophisticated derivative and other products in the financial markets.

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Appendix A

Appendix

We have listed the various Matlab codes used in our work to generate the tabulated results. Further, we have listed mainly the European and American put options but the same principle applies to the European and American call options.

A.1 Binomial Model for a Non-Dividend Paying Stock

function $[c, p] =$ Europut_nondiv_binomial(S, K, T, r, σ, N)

$\delta t = T/N;$

$A =$ zeros$(N + 1);$

$a =$ exp$(r * \delta t);$

$u =$ exp$(\sigma * \sqrt{\delta t});$

$q = (a * u - 1)/(u^2 - 1);$

$A(N + 1, :) =$ max$[K - S * u.\hat{}(2 * (0 : N) - N), 0];$

for $i = N : -1 : 1$

$A(i, 1 : i) = [q * A(i + 1, (1 : i) + 1) + (1 - q) * A(i + 1, 1 : i)]/a;$

end

$p = A(1, 1);$

$c = p + S - K *$ exp$(-r * T);$

Similarly, to get the price of an American put, we replace the body of the **for** loop by

$A(i, 1:i) = \max\Big[(K - S * u.\hat{}\{2*(1:i)-i-1)\}, \{q * A(i+1,(1:i)+1) + (1-q) * A(i+1, 1:$
$i)\}/a\Big];$

A.2 Binomial Model for Options on Futures.

function $[c, p] =$ Europut_futures_binomial(F, K, T, r, σ, N)

$\delta t = T/N;$

$A =$ zeros$(N + 1);$

$a =$ exp$(r * \delta t);$

$u =$ exp$(\sigma * \sqrt{\delta t});$

$q = (u - 1)/(u^2 - 1);$

$A(N + 1, :) = \max[K - F * u.\hat{}(2 * (0 : N) - N), 0];$

for $i = N : -1 : 1$

$A(i, 1 : i) = [q * A(i + 1,(1 : i) + 1) + (1 - q) * A(i + 1, 1 : i)]/a;$

end

$p = A(1, 1);$

$c = p - (F - K) *$ exp$(-r * T);$

Similarly, to get the price of an American put, we replace the body of the **for** loop by

$A(i, 1:i) = \max\Big[\{K - F * u.\hat{}(2*(1:i)-i-1)\}, \{q * A(i+1,(1:i)+1) + (1-q) * A(i+1, 1:$
$i)\}/a\Big];$

A.3 Binomial Model for a Known Dividend Paying Stock.

function $[c, p] =$ Amerput_div_binomial$(S, K, T, r, \sigma, N, \lambda, \tau)$

% The parameter τ is the dividend payment date and λ is the known percentage (%) of the dividend paid.

$\delta t = T/N;$

$A =$ zeros$(N + 1);$

$a =$ exp$(r * \delta t);$

$u =$ exp$(\sigma * \sqrt{\delta t});$

$n = \tau/\delta t;$

$q = (a * u - 1)/(u^2 - 1);$

$A(N + 1, :) = \max(K - S * (1 - \lambda) * u.\hat{}(2 * (0 : N) - N), 0);$

for $i = N : -1 : n$

$A(i, 1 : i) = \max\Big[\{K - S * (1 - \lambda) * u.\hat{}(2 * (1 : i) - i - 1)\}, \{q * A(i + 1, (1 : i) + 1) + (1 - q) * A(i + 1, 1 : i)\}/a\Big];$

% This loop values the stock prices after the dividend pay out.

end

for $i = n : -1 : 1$

$A(i, 1 : i) = \max\Big[\{K - S * u.\hat{}(2 * (1 : i) - i - 1)\}, \{q * A(i + 1, (1 : i) + 1) + (1 - q * A(i + 1, 1 : i)\}/a\Big];$

% This second loop values the stock prices prior to the dividend day.

end

$P = A(1, 1);$

Similarly, to get the price of an European put, we replace the two bodies of the **for** loop by one **for** loop given by,

$A(i, 1 : i) = [q * A(i + 1, (1 : i) + 1) + (1 - q) * A(i + 1, 1 : i)]/a;$

A.4 Fully Implicit Finite Difference Method for Options on Spot

function$[p] =$ Amerput_implicit$(S, K, r, \sigma, T, N, M)$;

$dt = T/N;$

$ds = 2 * S/M;$

$A =$ sparse$(M + 1, M + 1)$;

$f = \max[K - (0 : M) * ds, 0];$

for $m = 1 : M - 1$

$x = 1/(1 - r * dt);$

$A(m + 1, m) = x * (r * m * dt - \sigma^2 * m^2 * dt)/2;$

$A(m + 1, m + 1) = x * (1 + \sigma^2 * m^2 * dt);$

$A(m + 1, m + 2) = x * (-r * m * dt - \sigma^2 * m^2 * dt)/2;$

end

$A(1, 1) = 1;$

$A(M + 1, M + 1) = 1;$

for $i = N : -1 : 1$

$f = A\backslash f;$

$f = \max[f, (K - (0 : M) * ds)'];$

end

$P = f$[round$((M + 1)/2)$];

A.5 Crank Nicolson Finite Difference Method for Options on Spot

function$[p] =$ Amerput_CrankN_Spot$(S, K, r, \sigma, T, N, M)$;

$dt = T/N;$

$ds = 2 * S/M;$

$A =$ sparse$(M + 1, M + 1)$;

$f = \max[K - (0 : M) * ds, 0];$

for $m = 1 : M - 1$

$A(m + 1, m) = (r * m * dt - \sigma^2 * m^2 * dt)/4;$

$A(m + 1, m + 1) = 1 + 0.5 * r * dt + 0.5 * \sigma^2 * m^2 * dt;$

$A(m + 1, m + 2) = (-r * m * dt - \sigma^2 * m^2 * dt)/4;$

end

$A(1, 1) = 1;$

$A(M + 1, M + 1) = 1;$

for $m = 1 : M - 1$

$B(m + 1, m) = (-r * m * dt + \sigma^2 * m^2 * dt)/4;$

$B(m + 1, m + 1) = 1 - 0.5 * r * dt - 0.5 * \sigma^2 * m^2 * dt;$

$B(m + 1, m + 2) = (r * m * dt + \sigma^2 * m^2 * dt)/4;$

end

$B(1, 1) = 1;$

$B(M + 1, M + 1) = 1;$

for $i = N : -1 : 1$

$f = A\backslash(B * f);$

$f = \max(f, (K - (0 : M) * ds)');$

end

$P = f[\text{round}((M+1)/2)]$;

The European call and put options code can be obtained for both the implicit and Crank Nicolson finite difference method by excluding the early exercise privilege check in the program, that is, delete $f = \max[f, (K-(0:M)*ds)']$.

A.6 Crank Nicolson Finite Difference Method for Options on Futures

function$[p] =$ Amerput_CrankN_ Futures$(S, K, r, \sigma, T, N, M)$;
$dt = T/N$;
$ds = 2*S/M$;
$A =$ sparse$(M+1, M+1)$;
$f = \max[K-(0:M)*ds, 0]$;

for $m = 1 : M-1$
$A(m+1, m) = -\sigma^2 * m^2 * dt/4$;
$A(m+1, m+1) = 1 + 0.5*r*dt + 0.5*\sigma^2*m^2*dt$;
$A(m+1, m+2) = -\sigma^2 * m^2 * dt/4$;
end

$A(1,1) = 1$;
$A(M+1, M+1) = 1$;

for $m = 1 : M-1$
$B(m+1, m) = \sigma^2 * m^2 * dt/4$;
$B(m+1, m+1) = 1 - 0.5*r*dt - 0.5*\sigma^2*m^2*dt$;
$B(m+1, m+2) = \sigma^2 * m^2 * dt/4$;
end

$B(1,1) = 1;$

$B(M+1, M+1) = 1;$

for$i = N : -1 : 1$

$f = A\backslash(B * f);$

$f = \max[f, (K - (0:M) * ds)'];$

end

$P = f[\text{round}((M+1)/2)];$

A.7 Standard Asian Options

$\text{function}[c, p] = \text{Euro\_Standard\_Asian}(S_0, K, r, \sigma, T, N, M);$

$dt = T/N;$

for $i = 1 : M$

$S(1) = S_0 * \exp[(r - 0.5 * \sigma^2) * dt + \sigma * \sqrt{dt} * \text{randn}];$

$S_{\text{value}}(1) = [S(1)];$

for $j = 1 : N - 1$

$S(j+1) = S(j) * \exp[(r - 0.5 * \sigma^2) * dt + \sigma * \sqrt{dt} * \text{randn}];$

$S_{\text{value}} = S[1 : j+1];$

end

$S_{\text{mean}}(i) = \text{mean}(S_{\text{value}});$

% The geometric and arithmetic mean for the stock price after the first simulated sample path of N days.

$S_G(t) = [\text{prod}(S_{\text{value}})]^{1/N};$

$S_A(t) = \text{sum}(S_{\text{value}})/N;$

% The Geometric call and put options estimated price after the first trial of N days.

Gc$(i) = \max(S_G(t) - K, 0);$

Gp$(i) = \max(K - S_G(t), 0);$

%The Arithmetic call and put options estimated price after the first trial of N days.

Ac$(i) = \max(S_A(t) - K, 0);$

Ap$(i) = \max(K - S_A(t), 0);$

% Assign the stock prices after the first simulated path to a null vector.

$S(1 : j + 1) = [];$

end

Geocall $=$ exp$(-r * T) *$ mean$(Gc);$

Geoput $=$ exp$(-r * T) *$ mean$(Gp);$

Arithcall $=$ exp$(-r * T) *$ mean$(Ac);$

Arithput $=$ exp$(-r * T) *$ mean$(Ap);$

width $= 1.96 *$ std$(S_{\text{mean}})/\sqrt{M};$

Confidence Interval $=$ [Geocall $-$ width, Geocall $+$ width];

% The same can be applied to obtain the confidence interval of the other three option prices. That is, the Geometric put, Arithmetic call and Arithmetic put.