Discrete Time Methods of Pricing Asian Options

by

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A mini-thesis submitted in partial fulfillment of the requirements
for the degree of
Masters of Science in Computational Finance
in the
Department of Mathematics and Applied Mathematics
University of the Western Cape

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March 2014

Keywords

Asian option

Basket option

Binary tree

Black-Scholes

Control variate

Cox-Ross-Rubinstein

Levy process

Monte Carlo

Lower bound

Stochastic volatility

Abstract

This dissertation studies the computation methods of pricing of Asian options. Asian options are options in which the underlying variable is the average price over a period of time. Because of this, Asian options have a lower volatility and this render them cheaper relative to their European counterparts. Asian options belong to the so-called path-dependent derivatives; they are among the most difficult to price and hedge both analytically and numerically.

In practice, it is only discrete Asian options that are traded, however continuous Asian options are used for studying purposes. Several approaches have been proposed in the literature, including Monte Carlo simulations, tree-based methods, Taylor's expansion, partial differential equations, and analytical approximations among others. When using partial differential equations for pricing of continuous time Asian options, the high dimensionality is problematic. In this dissertation we focus on the discrete time methods. We start off by explaining the binomial tree method, and our last chapter presents the very exciting and relatively simple method of Tsao and Huang, using Taylor approximations. The main papers that are used in this dissertation are articles by Jan Vecer (2001); LCG Rogers (1995); Eric Benhamou (2001); Gianluca Fusai (2007); Kamizono, Kariya and Nakatsuma (2006) and Tsao and Huang (2007).

The author has provided computations, including graphs and tables dispersed over the different chapters, to demonstrate the utility of the methods. We observe various parameters of influence such as correlation, volatility, strike, etc. A further contribution by the author of this dissertation is, in particular, in Chapter 5, in the presentation of the work of Tsao et al. Here we have provided slightly more detailed explanations and again some further computational tables.

Declaration

I declare that *Discrete Time Methods of Pricing Asian Options* is my work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Neliswa B. Dyakopu March 2014

Signed......................................

Acknowledgements

I want to thank God for being my rock and my source through all my life seasons. I want to acknowledge the University of the Western Cape for the opportunity to study masters program in Computational Finance.

Special thanks to my supervisor Prof P J Witbooi for his in depth comments, patient tutorship and invaluable advice in writing this dissertation. It has been a great blessing and pleasure to study under his supervision. No man is an Island, I want to thank my parents, family and friends for all the patience, understanding and continual support.

Finally I want to thank my mother for being such an inspiration in my life. May I grow to be a woman of wisdom like her, it is my most desire to possess her golden character. Her deeply rooted faith in the Lord, her unquenchable strength and her love motivated me to keep going. Her commitment to a peaceful life makes me want to live long days. She illuminates integrity, her presence means peace and hope. She is my best gift and I love her. God did me well by blessing me with her.

You win if you do not quit, be wise keep going!

List of Acronyms

SDE, Stochastic Differential Equation

PDE, Partial Differential Equation

BSM, Black-Scholes Model

GBM, Geometric Brownian Motion

ARO, Average Rate Options

MCM, Monte-Carlo Method

CEV, Constant Elasticity of Variance

IID, Identically Independent Distributed

List of Symbols

$\{S_t\}_{t\geq 0}$, Value of the underlying stock

$\mathbb{P}$, a real world probability measure

$\mathbb{Q}$, a martingale measure equivalent to the market measure

$(\Omega, \mathcal{F}, \mathbb{Q})$, a probability triple

$\mathcal{F}_{t\geq 0}$, a filtration

$W_{t\geq 0}$ and Z_t, standard Brownian motion

$\mathbb{E}$, expectation operator in the risk neutral

$\mathbb{V}$, variance operator in the risk neutral

$N(x)$, cumulative distribution function of the standard normal variable

$Z \sim N(0, 1)$, the random variable Z has a standard normal distribution

$\mathcal{K}$, The propagator

$\mathcal{L}$, The Lagrangian

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Chapter 1

Introduction and preliminaries

1.1 Introduction and outline of the mini-thesis

In this dissertation we review various computation methods of pricing of Asian options, including some relatively new methods. Asian options are options in which the underlying variable is the average price over a period of time (see Vecer in [37]). Because of this, Asian options have a lower volatility and they are trading cheaper than the European options. Asian options are classified under a type of financial derivatives called exotic, their payoffs depending on the average of a given stock price over a specified period in the future. For an investor who is interested on an average exposure of an asset over a predetermined period in the future, rather than an exposure on a specific date these options are of good use. There is quite a number of reasons that make Asian options popular in the markets. According to Fusai and Meucci (see reference [20]), the exposure to future price movements of the company, is given as the exposure to the average of prices in the future. When the option's life is close to maturity, the average options are less sensitive to the movements of the underlying stock. According to some accounting standards, an average of exchange rates is used for translation of foreign currency assets.

Under a no-arbitrage Black-Scholes framework, this mini-thesis discusses various pricing methods when using these instruments. The thesis is structured as follows. Literature reviewing is interspersed throughout the mini-thesis, rather than included as a chapter on its own. We start off by laying out preliminaries in the remaining part of Chapter 1. In Chapter 2 we explain the Black-Scholes method, along with a discussion of partial differential equations. The various analytical methods for pricing Asian options are discussed in Chapter 3. Chapter 4 is a presentation on Monte-Carlo simulation and we illustrate the method with relevant examples. Chapter 5 elaborates on the very exciting and relatively simple method of Tsao and Huang (see [23]), who uses Taylor approximations. Some concluding remarks ties up the mini-thesis in Chapter 5.

With regard to the more significant contributions we have provided numerous computations, providing graphs and tables dispersed over the different chapters, to demonstrate the utility of the methods. Also some contribution is made by the author of this dissertation in that the presentation based on the idea of Bouaziz et al. (see [5]) and in particular in Chapter 5, the presentation of the work of Tsao and Huang [23]. An approximation formula for European discrete average price Asian options is obtained by employing the Taylor expansion. We enrich this presentation with more detailed explanations and again we illustrate the method with further computational tables. The author provides more detail towards the derivation of the mean and the variance to be used in the approximation formula.

1.2 Preliminaries

This section discusses the construction of the Brownian motion. The nature of Brownian motion automatically leads to the logarithm of the stock being modeled as the stochastic integral which eventually produces a log stock price as an Itó process.

1.2.1 Random walk

Suppose that $\epsilon_1, \epsilon_2, ..., \epsilon_T$ is a sequence of random variables with ϵ_i being a shock (changes in a certain way) only revealed at time $t = i$. Suppose further that the mean for each shock is $\mathbb{E}[\epsilon_{t+1}] = 0$ and the variance $\mathbb{V}[\epsilon_{t+1}] = 1$ (see Kwok [27]).
Such shocks are assumed to have the following:

1. At any of the earlier dates $s < t$, $\mathbb{E}_s[\epsilon_t] = 0$
2. At any earlier date $s < t$, $\mathbb{V}_s[\epsilon_t] = 1$

3. Shocks ϵ_t, ϵ_u are uncorrelated as any of the earlier date $s < u < t$, $\mathrm{Cov}(\epsilon_\mathrm{t}, \epsilon_\mathrm{u}) = 0$.

We therefore assume that the process ϵ_t is identically and independently distributed and we consider the process $B_{1,t}$ that adds up shocks until time t.

$$\begin{aligned} B_{1,0} &= 0, \\ B_{1,1} &= \epsilon_1, \\ B_{1,2} &= \epsilon_1 + \epsilon_2, \\ B_{1,T} &= \epsilon_1 + \epsilon_2 + ... + \epsilon_T. \end{aligned}$$

This process $(B_{1,i})$ is called a discrete-time random walk with time step 1. We consider the properties of each shock and hence conclude that the process $(B_{1,i})$ is a martingale. Arbitrarily we construct a process $(B_{\Delta t,i})$ using δt as the arbitrary time step to yield a so-called random walk $(B_{\delta t,i})_{i \in \mathbb{N}}$ such that

$$\begin{aligned} B_{\Delta t,0} &= 0, \\ B_{\Delta t,\Delta t} &= \epsilon_{\Delta t}, \\ B_{\Delta t,2\Delta t} &= \epsilon_{\Delta t} + \epsilon_{2\Delta t}, \\ B_{1,T} &= \epsilon_{\Delta t} + \epsilon_{2\Delta t} + ... + \epsilon_{T-\Delta t} + \epsilon_T. \end{aligned}$$

Therefore

$$\mathbb{V}(\epsilon_{t+\Delta t}) = \Delta t \text{ and } \mathbb{E}(\epsilon_{t+\Delta t}) = 0.$$

As the number of the time steps increase, the size of the time steps is getting smaller and so is the variance. Thus in the limit as $\Delta t \to 0$, B_t is determined for all $t \in [0, T]$ and hence we obtain continuous-time Brownian motion.

The process (B_t) can be defined as the limit $B_{\Delta t,t}$, as $\Delta t \to 0$, depending on the nature of the underlying stochastic process. With a discrete process, the underlying variable only takes discrete values whereas with a continuous process the underlying variable may take any value within a certain range ([27]).

1.2.2 Brownian Motion

The Brownian Motion is a fundamental stochastic process, studied by the Botanist Robert Brown. It is central to the applications and the theory of stochastic processes. They can be considered as the limiting stochastic process obtained from random walk by letting step size shrink to zero ([27]). More precisely, a Brownian motion is a continuous time stochastic process $(B_t)_{t\geq 0}$ with the following properties:

$B_0 = 0$.

$(B_t)_{t\geq 0}$ has independent increments.

$(B_t)_{t\geq 0}$ has stationary increments.

$B_{t+s} - B_t$ is normally distributed with mean 0 and variance s, i.e. $B_{t+s} - B_t \sim N(0, s)$.

$(B_t)_{t\geq 0}$ has continuous sample paths.

The *geometric* Brownian motion is defined by:

$$S_t = S_0 \exp\left(\nu t + \sigma W_t\right).$$

It is the reference model for stock prices in continuous time where $(W_t)_{t\geq 0}$ is a standard Brownian motion with S_0 as the stock price, ν the mean and σ as the respective volatility.

Quadratic Variation of a Stochastic Process

Suppose that Π is a partition $0 = t_0 < t_1 < ... < t_n = T$ of the time interval $[0, T]$. We let $\delta_{t_{\max}} = \max_k (t_k - t_{k-1})$ and we write $\Delta t_k = t_k - t_{k-1}$. Let Z be the standard Brownian process with corresponding quadratic variation defined by

$$Q_\pi = \sum_{k=1}^{n} \left[Z\left(t_k\right) - Z\left(t_{k-1}\right)\right]^2.$$

Over the time interval $[0, T]$ the quadratic variation of the Brownian process is given by $Q_{[0,T]} = \lim_{\delta t_{\max}\to 0} Q_\pi$. It turns out that $\lim_{\delta t_{\max}\to 0} \mathbb{E}\left[Q_\pi\right] = T$

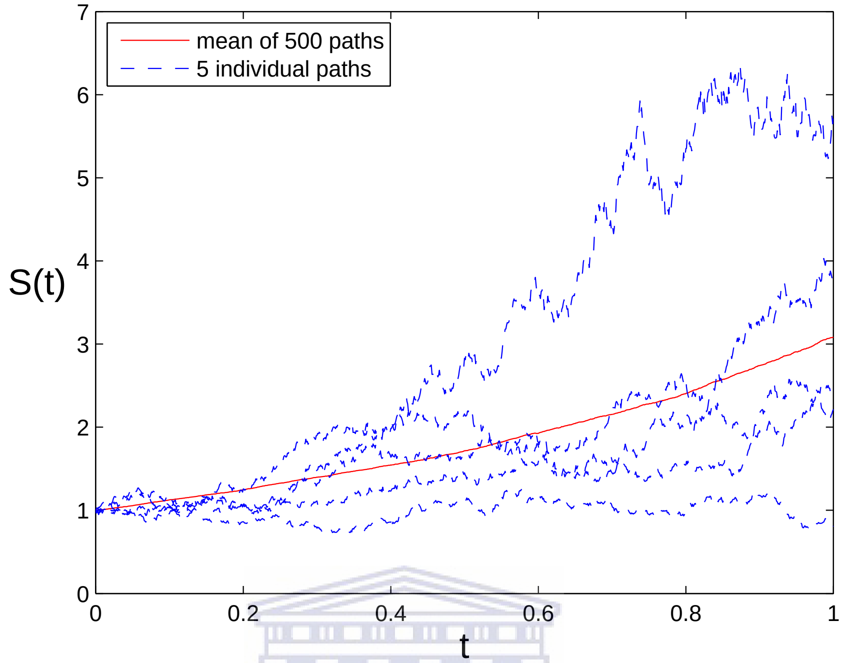

Figure 1.1: We show runs of geometric Brownian motion, more precisely of the process in equation 1.1 with $S_0 = 1$, $\nu = 500$ and $\sigma = 0.5$

and $\lim_{\delta t_{\max}} \mathbb{V}\left[Q_\pi - T\right] = 0$. For the time t, we let $f(t, Z(t))$ be a simple function with the standard Brownian process $Z(t)$. The stochastic integral $\int_0^T f(t, Z(t))dZ(t)$ is defined as follows:

$$\int_0^T f(t, Z(t))dZ(t) = \sum_{k=1}^{n} f(t_{k-1}, Z(t_{k-1}))\left[Z(t_k) - Z(t_{k-1})\right]$$

where the interval $[0, T]$ is partitioned as $0 = t_0 < t_1 < \ldots < t_n = T$. The *Itô stochastic integral* is defined as the limit (in the L^2-norm) of integrals of a sequence of simple functions converging to the given function.

Itô Process

The Itô process defines the general class of continuous stochastic process. We let $\mathcal{F}$ be a filtration generated by a standard Brownian motion $Z(t)$. For all T, we let $\mu(t)$ and $\sigma(t)$ be adapted to $\mathcal{F}_t$ with $\int_0^T \left[\mu(t)\right] dt < \infty$ and $\int_0^T \left[\sigma(t)\right] dt <$

∞ ([27]). The process $X(t)$ is called an Itô process if $X(t) = X(0) + \int_0^t \mu(s)dS + \int_0^t \sigma(s)dZ_s$.

Itô's lemma

Itô's lemma is quite foundational in stochastic analysis since its applications help us with the derivation of the Black Scholes Equation which serves as a tool to price options (see Etheridge in [18]).

Itô Theorem: *Let $f(t,x)$ is twice continuously differentiable function of x and continuously differentiable in t. Then*

$$df(t,W_t) = \int_0^t \frac{\partial f}{\partial x}(s,W_s)\,dW_s + \int_0^t \frac{\partial f}{\partial s}(s,W_s)\,dW_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,W_s)\,dt.$$

The integral formula can also be stated as:

$$df(t,W_t) = f'(t,W_t)\,dW_t + \dot{f}(t,W_t)\,dt + \frac{1}{2}f''(t,W_t)\,dt.$$

Chapter 2

The Black-Scholes Model and Partial Differential Equations

We briefly introduce a generally accepted model for the pricing of a variety of options, and in this section we focus on the European options. In a landmark paper (see [4]) in 1973, Black and Scholes presented the model that has since been so widely used in option pricing. There are several detailed expositions of this so-called Black-Scholes model, such as for instance the books (in reference [10]) by Neil Chriss and (in reference [35]) by Stanley Pliska. We explain the model in the sequel.

2.1 The model

We consider a continuous time trading economy with an infinite horizon. The uncertainty is characterized by a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where Ω is the state space, $\mathcal{F}$ is the σ-algebra representing the measurable events, and $\mathbb{Q}$ is the risk neutral probability measure, assumed to be unique in a complete market with no arbitrage opportunity. The information evolves according to the complete filtration $\{\mathcal{F}_t, t \in \mathbb{R}\}$ generated by a standard one dimensional Brownian motion $\{W_t, t \in \mathbb{R}\}$. We assume that the evolution of the underlying price process $(S_t)_{t \geq 0}$ is described by a stochastic differential equation

$$dS_t = r_t S_t dt + S_t \sigma(t; S_t) dW_t$$

with an initial condition $S_0 = x$. In the stochastic differential equation, r_t

is the deterministic risk free interest rate and $\sigma(t; S_t)$ is called the volatility, which can be either constant (Black Scholes model) or deterministic (like in the Dupire and constant elasticity of variance models).

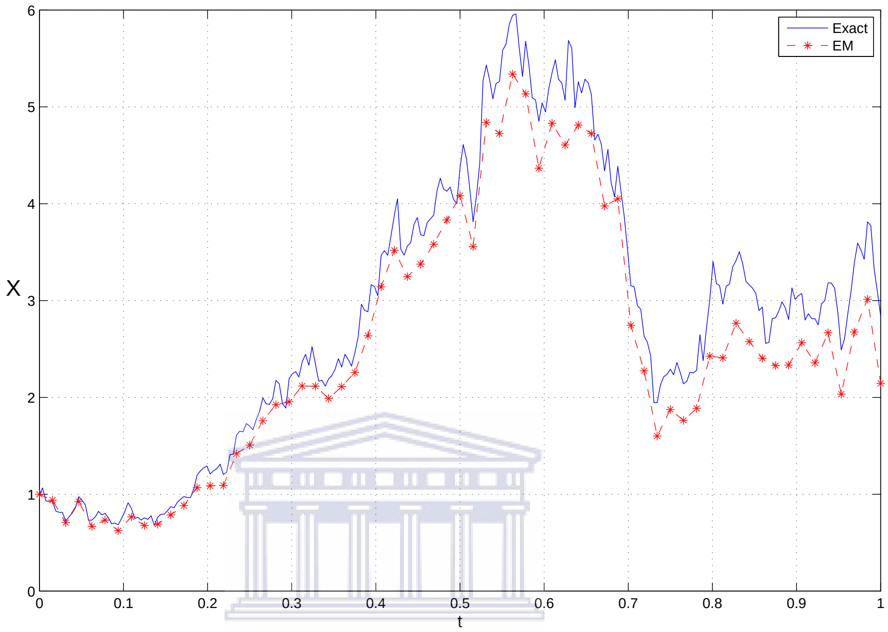

Figure 2.1: Discretized Brownian path with $\mu = 1$ and $\sigma = 2$
.

2.2 Pricing of European Options

Black and Scholes modeled the underlying price processes using a geometric Brownian motion. The model is based on the following assumptions:

1. Stock prices follow a geometric brownian motion with constant drift μ and volatility σ.

2. Risk free arbitrage opportunities do not exist.

3. Any fraction of a share may be bought, all securities are divisible .

4. During the life of an option, there are no dividends.

5. There are no transaction costs when writing or trading options.

6. For all maturities the risk free interest rate is constant.

7. European exercise terms are active. Options may only be exercised at expiration.

Let C be the value of the European call option at time t , S be the value of the stock price, K be the strike price at expiration T and r the continuously compounded risk free rate. Then

$$C = SN\left(\frac{\log\frac{S}{K} + \left(r + \frac{\sigma^2}{2}T - t\right)}{\sigma\sqrt{t}}\right) - e^{rt}KN\left(\frac{\log\frac{S}{K} + \left(r - \frac{\sigma^2}{2}t\right)}{\sigma\sqrt{t}}\right)$$

where N is the cumulative distribution function of a standard normal variable.

For a non-paying dividend stock the Black-Scholes formula for European put option at time = 0 is given by:

$$P = Ke^{-rT}N(-d_2) - S_0N(-d_1),$$

where

$$d_1 = \frac{\ln(S_0) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

2.3 PDEs in Asian option pricing

In the past two decades studies have been conducted on pricing Asian Options, most especially on their path-dependent nature. Using the Black-Scholes frame work, numerous difficulties have been identified by the researchers. In pricing

Asian options, at any future date the price of the underlying asset is modeled by use of a log-normal density function (see Deelstra and Linev [12]). At expiration the payoff measured by the arithmetic average of log-normal random variables is unfortunately not log-normal distributed. This results in a complexity in pricing European Asian Options using arithmetic average since no closed form formula exist. The geometric average, for which there is a closed form solution tends to under-price the value of the Asian call options. Put-call parity is used to value a put option. However the geometric approach tends to overprice the value of the Asian puts.

A more generalized PDE approach was provided by Alziary ,D'ecamps and Koehl (1997) with some applications on an explicit finite difference scheme to approximate numerically the option price. As a means of hedging the instruments, an analysis on the so called delta and gamma was used. Barraquand and Prudet(1976) used finite difference methods to obtain numerical solutions when solving the problems on advection-diffusion PDE. The pricing model was formulated by Dewynne and Wilmot (see Wilmot et al [15]). They employed finite difference method to obtain numerical solutions (see [15]). Moreover, in their article Partial to the Exotic (1993) (see Wilmot and Dewyne [16]), they derived a PDE framework which can be of great use in pricing exotic options. To minimize their derivation of solving a parabolic PDE in two variables, Rogers and Shi (1995) used a property of Brownian motion. They were also able to provide a lower bound for the price of Asian options. The advantage of the PDE approach is that it gives results across the duration of an option, and for all the initial prices and all the running times. A drawback is that the numerical methods are not easy to implement.

2.3.1 The approach of Ingersol et al.

Ingersol (1987) or Forsyth, Vetzal and Zvan (1998) derived the standard PDE for continuous Asian options with non-constant volatility structure as the following expression:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0 \tag{2.1}$$

where $\partial V/\partial t$, $\partial V/\partial S$, $\partial V/\partial I$ and $\partial^2 V/\partial S^2$ denotes the partial derivative with respect to time, underlying, running time and the second order partial derivative function respectively. Barraquand and Pudet (1996) derived the PDE with respect to the running average denoted by A_T and they found the following:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} + \frac{1}{T}(S_T - A_T)\frac{\partial V}{\partial A} - rV \quad = \quad 0. \tag{2.2}$$

The dependence on the underlying of the volatility structure results in the difference with the Black Scholes PDE (see Smith in [34] and Vecer in [36]). However, for discrete Asian options these equations transform to one dimensional PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} - rV = 0,$$

with the condition at the observation date

$$V\left(t_i^-, S, A_{t_i}\right) = V\left(t_i^+, S, A_{t_i} + \alpha_{t_i} S_{t_i}\right)$$

where $V(t, S, A_t)$ denotes the call value at time t with S as the underlying asset and A_t as the average (see Forsyth et al[19]).

2.3.2 The approach of Rogers and Shi

Rogers and Shi (1995), in the case of the Black-Scholes model suggested that to reduce the dimension we can use a change of variable. Benhamou was able to show that the PDE satisfied by an option is a 3 dimensional one (see Benhamou and Duguet [2]) . The call option price is determined as the expected value of the discounted payoff under the risk neutral probability measure.

$$V(t, S_t, A_t) = e^{-r(T-t)}\mathbb{E}\left[g\left(S_T, A_T\right)|F_t\right].$$

It is important to note that the equation (2.1) and (2.2) can be represented by

$$\begin{aligned} V\left(S(t), I(t), t\right) &= e^{-r(T-t)}\mathbb{E}_t\left[g(S(T), I(T), T)\right] \text{ and} \\ V\left(S(t), A(t), t\right) &= e^{-r(T-t)}\mathbb{E}_t\left[g(S(T), A(T), T)\right], \end{aligned} \tag{2.3}$$

respectively. Rogers and Shi (1995) formulated an alternative PDE based on (2.3) and the scaling property of the Brownian Motion by defining a state variable

$$x = \frac{K - \int_0^t S(\tau)\mu(d\tau)}{S_t},$$

where μ is a probability measure with density $\rho(t)$. The density for a fixed strike option is $\rho(t) = \frac{1}{T}$ and for a floating strike option, $K = 0$ and $\rho(t) = \frac{1}{T} - \delta(T - t)$. The value of an Asian option is described by the following PDE according to Rogers and Shi

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial x^2} - (\rho(t) + rx)\frac{\partial W}{\partial x} = 0,$$

with the following conditions for a fixed strike call and a floating strike put as:

$$W(x, T) = \max(0, -x),$$

$$W(x, T) = \max(0, -x - 1),$$

respectively. Therefore the prices of a fixed strike call with a stock price S, strike price K and the price of a floating strike put are defined as $S_0 W(K/S_0, 0)$ and $S_0 W(0, 0)$ respectively. This therefore gives a one dimensional PDE for both fixed and floating strike options. However, the same does not apply with the American style option because of their early exercise nature (see Alziary et al [1] and Mudzimbabwe et al [29]).

Therefore in the Rogers and Shi framework, European style Asian options can be valued using one dimensional PDE for both fixed and floating strike options. However we must solve the two dimensional PDE in equation (2.1) and (2.2) so as to be able to value fixed strike options with early exercise opportunities.

2.3.3 The PDE from Rogers and Shi

From the Black-Scholes formula we have seen that if the risk-free rate and stock volatilities are deterministic, the pricing of contingent claims is straight forward. Let C_t be the contingent claim, the expectation determining the price of the claim is known to be

$$C_t = \exp\left(-\int_t^T r_s dS\right) \mathbb{E}_t^Q \left[g\left(S_T\right)\right].$$

Since we are able to determine the risk-neutral distribution of the stock-prices, this makes it possible to calculate C_t. However if the stock volatility or the risk free rate are stochastic it is not simple to calculate the price of C_t. In such a case we may need to construct a no-arbitrage PDE to compute the solution (see Dubois and Levievre [17]).

Suppose that for a stock price and the risk-free rate, the risk neutral SDE are the following:

$$\begin{aligned} dr_t &= \mu_r\left(r_t, S_t\right)dt + \sigma_r\left(r_t, S_t\right)dB_t^Q, \\ dS_t &= \mu\left(r_t, S_t\right)dt + \sigma\left(r_t, S_t\right)dB_t^Q. \end{aligned} \tag{2.4}$$

According to the Itô formula the processes r_t and S_t are jointly Markov under risk-neutral Q and yields C_t to be a function of time t. Since we know that $\frac{C_t}{\beta_t}$ is a martingale under Q, thus we need

$$\mathbb{E}_t^Q\left[d\left(\frac{C_t}{\beta_t}\right)\right] = 0. \tag{2.5}$$

According to Itô formula

$$d\left[\frac{C_t}{\beta_t}\right] = \frac{1}{\beta_t}\left(dC_t - rC_t dt\right). \tag{2.6}$$

We again apply the Itô formula to find $dC\left(t, t_t, S_t\right)$ given (2.5) and (2.6)

$$\begin{aligned} dC = &\left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial r_t}\mu_r + \frac{\partial C}{\partial S_t}\mu + \left(\frac{1}{2}\frac{\partial^2 C}{\partial r_t^2}\sigma_t^2 + 2\frac{\partial^2 C}{\partial r_t \partial S_t}\sigma_r\sigma + \frac{\partial^2 C\sigma^2}{\partial S_t^2}\right)\right)dt \\ &+ \left(\frac{\partial C}{\partial r_t \sigma_r} + \frac{\partial C}{\partial S_t}\sigma\right)d\beta_t^Q. \end{aligned} \tag{2.7}$$

Together with (2.5) and (2.6), this produces the non-arbitrage PDE

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial r_t}\mu_r + \frac{\partial C}{\partial S_t}\mu + \left(\frac{1}{2}\frac{\partial^2 C}{\partial r_t^2}\sigma_t^2 + 2\frac{\partial^2 C}{\partial r_t \partial S_t}\sigma_r\sigma + \frac{\partial^2 C\sigma^2}{\partial S_t^2}\right) - rC = 0 \tag{2.8}$$

with boundary condition $C\left(T, r, S\right) = g\left(S\right)$.

2.4 The pricing formula of Wilmot

Dewyne et al [15] defines the process such that $X_t := E_t^Q\left[\left(\ln S_T\right)^2\right]$ and by observing

$$d\ln S_t = \left(r - \frac{1}{2}\sigma^2\right)dt + d\beta_t^Q. \tag{2.9}$$

He would conclude that $\ln S_T$ is a Markov process under Q, therefore the value of X_T will be depending on the value of the stock S_T and time T such that $X_t = X(t, \ln S_t)$. According to Wilmot, he would seek to calculate X_0. The martingale proposition states that the process X is a martingale under Q. To find the drift of X_t, Wilmot applies the Itô formula and set $X_t = 0$ such that

$$\frac{\partial X}{\partial t} + \frac{\partial X}{\partial \ln S}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\frac{\partial^2 X \sigma^2}{\partial (\ln S^2)} = 0 \tag{2.10}$$

with

$$X(T, \ln S) = (\ln S)^2 \tag{2.11}$$

as the boundary condition. The PDE in (2.10) yields

$$X(t, \ln S) = a(t)(\ln S)^2 + b(t)\ln S + c(t). \tag{2.12}$$

Wilmott would find from the boundary condition in (2.12) that

$$a(T) = 1, \tag{2.13}$$
$$b(T) = c(T) = 0. \tag{2.14}$$

Substituting (2.12), (2.13) and (2.14) into (2.10) for any S at any t, this would yield

$$a'(t)(\ln S)^2 + (2a(t)\mu + b'(t))\ln S + b(t)\mu + a(t)\sigma^2 + c'(t) = 0. \tag{2.15}$$

However this will only be possible provided the time dependent coefficient are identical and are all equal to zero.

$$\begin{aligned} a'(t) &= 0, \\ 2a(t)\mu + b'(t) &= 0, \\ b(t)\mu + a(t)\sigma^2 + c'(t) &= 0. \end{aligned} \tag{2.16}$$

This would imply that $a(t)$ is a constant, and considering the boundary condition, the constant would have to be 1. Therefore $a(t) = 1$ thus

$$\begin{aligned} b'(t) &= -2\mu, \\ b(t) &= \int_t^T 2\mu ds. \end{aligned}$$

and

$$c\left(t\right)=\mu^{2}\left(T-t\right)^{2}+\sigma^{2}\left(T-t\right). \qquad (2.17)$$

The complete solution would therefore be

$$X\left(t,\ln S_{T}\right)=\left(\ln S_{T}\right)^{2}+2\mu\left(T-t\right)\ln S_{T}+\mu^{2}\left(T-t\right)^{2}+\sigma^{2}\left(T-t\right) \qquad (2.18)$$

with $\mu=r-\frac{1}{2}\sigma^{2}$.

2.5 Pricing as by Filipovic

To find $E_{t}^{Q}\left[\left(\ln S_{T}\right)^{2}\right]$ according to Filipovic (see Černý [9]), he integrates the SDE

$$\ln S_{T}=\ln S_{t}+\left(r-\frac{1}{2}\sigma^{2}\right)\left(T-t\right)+\sigma\left(B_{T}^{Q}-B_{t}^{Q}\right),$$

such that $\ln S_{T}$ is normal and

$$\ln S_{T}\mid\mathcal{F}_{t}\sim^{Q}N\left(\ln S_{t}+\left(r-\frac{1}{2}\sigma^{2}\right)\left(T-t\right),\sigma^{2}\left(T-t\right)\right).$$

From the property $\mathbb{E}\left[X^{2}\right]=\mathbb{V}\left[X\right]+\left(\mathbb{E}\left[X\right]\right)^{2}$ applying $X=\ln S_{T}$ yields

$$\begin{aligned}E_{T}^{Q}\left[\left(\ln S_{T}\right)^{2}\right]&=\mathbb{V}_{t}\left(\ln S_{T}\right)+\left(\mathbb{E}_{t}\left[\ln S_{T}\right]\right)^{2}\\&=\sigma^{2}\left(T-t\right)+\left(S_{t}+\left(r-\frac{1}{2}\sigma^{2}\right)\left(T-t\right)^{2}\right)\end{aligned}$$

which is the same as (2.9).

Table 2.1 gives a comparison of numerical results for pricing the European and Asian call. The techniques employed in pricing the European call is the Black-Scholes and the Maple package based on partial differential equations are used for the pricing of the Asian call. The initial price assumed (S_0) is 80, the risk-free rate (r) is 6%, the dividend yield (d) is 0, the option's time to maturity (T) in months is 4. The considered strike prices (K) are 75, 80 and 85 which represents the in-the-money, at-the-money and out-of-money option respectively.

Table 2.1 computations reveal that the Asian calls are relatively cheaper than their European counterparts across all various volatilities. It is quite evident that there is some level of positive relationship between the volatility of the

Table 2.1: European call versus Asian Call by Black-Scholes and PDE method respectively

σ	K	European Call$_{B-S}$	Asian Call$_{PDE}$
0.05	80	1.9148	1.0109
	85	0.0851	0.0005
0.10	80	2.7245	1.4909
	85	0.6692	0.0787
0.15	80	3.6004	2.0001
	85	1.4664	0.3378
0.20	80	4.4933	2.5168
	85	2.3312	0.7109

stock and the value of the call. The above table reflects a positive relationship between the value of the volatility and the value of the call. The higher the volatility the higher the price of the call. An at-the-money European call with the volatility of 10% and the strike price of 80 is worth 2.72 whereas with the volatility of 20% it is worth 4.49.

Chapter 3

Analytical Methods for Pricing Asian Options

3.1 Introduction

Discrete arithmetic Asian options are path-dependent contingent claims with pay-offs that depend on the average of the underlying asset price over some pre-specified period of time, often a low number of trading days in the discrete averaging case. Such contracts form an attractive specification for thinly traded asset markets where price manipulation on or near a maturity date is possible. There are many different ways of computing prices of Asian options (see Dewyne and Wilmot [14]). We list some of these methods.

1. Monte-Carlo simulation ,
2. Binomial tree ,
3. Convolution method,
4. Direct integration ,
5. Partial differential equation (PDE),
6. Fourier transform (FFT),
7. Approximate analytic method.

All of the above methods involve some trade offs between numerical accuracy and computational efficiency. Options are given in two distinct forms, known as call options and put options. Call options gives the right but not the obligation to buy an asset at a specified time whereas put options gives right to the holder to sell options at a specified time (see Wilmot in [15]). The value of the option depends on the following parameters: S is the underlying asset, T is the expiry period, E is exercise price, σ is the volatility of the underlying asset and r is the interest rate.

The payoff of an Asian option is a function of the average of the asset price over the lifetime of the contract. The following are the pay-offs for geometric and arithmetic Asian options respectively, with strike price X and maturity time T. The number of the trading days are n.

$$\text{Call} : C_T = \max\left(0, \left(\Pi_{i=1}^{n} S_i\right)^{\frac{1}{n}} - X\right),$$

$$\text{Put} : P_T = \max\left(0, X - \left(\Pi_{i=1}^{n} S_i\right)^{\frac{1}{n}}\right).$$

The payoff of arithmetic Asian options are given as:

$$\text{Call} : C_T = \max\left(0, \frac{1}{n}\sum_{i=1}^{n} S_i - X\right),$$

$$\text{Put} : P_T = \max\left(\frac{1}{n}\sum_{i=1}^{n} S_i - X, 0\right).$$

Asians can be exercised in either European style or the American style. These options are popular in the over-the-counter market among institutional investors, and are commonly traded on exchange rates, interest rates and commodity products which have low trading volumes. The common usage of Asian options is to hedge a thinly traded asset over a certain period of time. The hedge is less expensive than a portfolio of regular options. Another advantage of Asian options over the regular options is that they are less affected by price manipulation on the maturity date. However, it is more difficult to value such options than regular options. Closed-form solutions of arithmetic Asian options do not exist when the geometric Brownian motion is assumed to be the underlying asset process (see Etheridge [18]). In the literature, most studies on pricing focus on continuous Asian options using assumptions as in Black and Scholes (1973). We survey some of these methods.

3.2 The Binomial Tree Based Model

In derivative pricing, it is known that pricings are equated to calculating the expected value of its payoff function underneath its probability measure called the risk neutral probability measure. Since Asian options are of path dependent nature, it is quite challenging to find suitable pricing algorithms for these options. The value of these options do not only depend on the final value of the underlying instrument but also upon the path taken to get there. The core challenge in pricing these options is within efficiency, accuracy and convergence. We can make use the binomial tree approach as a pricing algorithm for these options. Hull and White in (1993) proposed the first lattice based model. The binomial model was used by Choo and Lee (1997), they derived an algorithm to price both arithmetic and geometric average options. Binomial models for valuing the price of Asian options was developed by Ritchken, Sankarasubramamian, Vijh (1993), Tan and Vetsal (1995), among other.

The core of the problem when using the pricing algorithm based on a binomial tree, is the underlying asset evolution due to a huge number of arithmetic averages that need to be tracked (see Bennings and Wiener [3]). An increase in time steps that are used to compute option prices makes the number of arithmetic averages grow exponentially. It then becomes difficult to manage these arithmetic averages. Hence Hull and White used a set of representative averages to overcome this problem. They used linear interpolation to compute missing values at each node of the tree. A similar approach based on binomial trees was used by Barraquand and Pudet (1996), Chasalani et al (1998,1999) and Klassen (2001). However they used a different approach when choosing a set of representative averages. Lattice based models play a vital role in pricing Asian options and they can be implemented for both American and European Asians with either continuous or discrete approach. This section reviews the pricing algorithm that is based on a binomial lattice.

An adjusted Binomial Model

This model computes the price of an Asian option from the arithmetic average of the underlying asset. It is based on the Cox-Ross-Rubinstein method, the main difference being in the choosing of representative averages used to compute the option price (see Coastabile et al [11]). This approach chooses a subset of true averages that will still be called representative averages. We make use of node (i, j) with j up steps and $i - j$ down steps. To obtain the set of representative averages we perform the following steps:

1. We compute the maximum average $A_{max}(i, j)$ which originates from $N(i, i)$ and resulted by trajectory $\tau_{max}(i, j)$.

2. The first average computed is denoted by $A(i, j; 1)$ and is the first element in the set of representative averages of $N(i, j)$. Then $A_{min}(i, j)$ is the last element in a set produced by trajectory $\tau_{min}(i, j)$.

3. On the very same $N(i, j)$, the other representative averages are denoted as $A(i, j; k)$, $k = 1, ..., j(1 - j)$ and are computed in the following way: Let $S_{max}(i, j; k)$ be the greatest value of the underlying asset but not from the trajectory $\tau(i, j)$ which produces the average $A(i, j; k)$ then

$$A(i, j; k + 1) = A(i, j; k) - \frac{1}{i + 1}\left[S_{max}(i, j, k) - S_{max}(i, j, k)d^2\right].$$

The meaning of the symbols u and d are as in the Figure 3.1 are defined in terms of risk-neutrality. This means that the $(k+1)$-th representative path is calculated from the k-th representative by substituting $S_{max}(1, j; k)$, the maximum value not from within the trajectory $\tau_{min}(i, j)$ with the value $S_{max}(1, j; k)d^2$. This technique proceeds until the very last trajectory $S_{max}(1, j; k)$ is included. The minimum average is $A_{min}(i, j)$ is produced by the trajectory $\tau_{min}(i, j)$ which is considered from the $N(i, j)$. If we start from $\tau_{max}(i, j)$ we arrive at the $\tau_{min}(i, j)$ after $j(i - j)$ substitutions. At the (i, j)-th node, we consider a set of averages built from $1 + j(i - j)$ elements. Clearly, the minimum and the maximum averages associated to the $N(i, j)$ are found from the first and the last element in the set of representative averages of $N(i, j)$, and are given by:

$$A(i, j; 1) = A_{max}(i, j) = \frac{1}{i + 1}\left(\sum_{h=0}^{j} Su^h + \sum_{h=0}^{i-j-1} Su^{h+2j-1}\right),$$

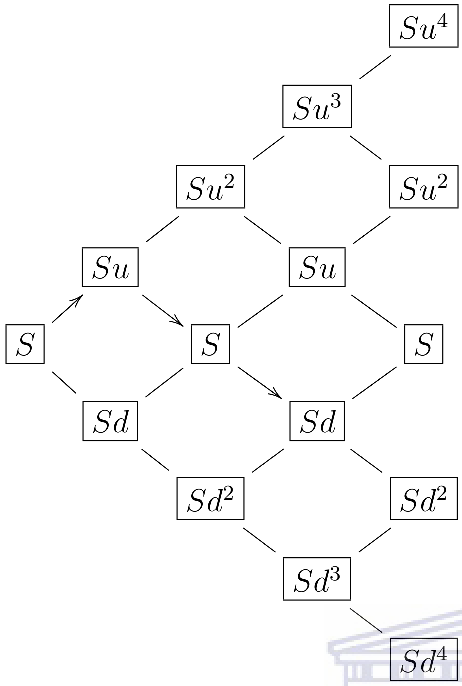

Figure 3.1: A 4 Time-Step CRR Binomial Lattice.

$$A(i,j;1+j(i-j)) = A_{min}(i,j) = \frac{1}{i+1}\left(\sum_{h=0}^{j} Sd^h + \sum_{h=0}^{j-1} Sd^{i-2j+h}\right).$$

This algorithm considers the sets of representative averages as follows:
At $N(4,4)$ there is one trajectory involved and the average associated with the node is computed from the values of $(S, Su, Su^2, Su^3, Su^4)$. However, at $N(4,2)$ the first average set is computed using the values (S, Su, Su^2, Su, S) and the average set is $A(4,2;1) = A_{max}(4,2)$. Therefore $S_{max}(4,2;1) = Su^2$ is used to compute the highest value $A(4,2;1)$. The path obtained through substituting $S_{max}(4,2;1)$ with $S_{max}(4,2;1)d^2 = S$ computes the second average. Therefore the average $A(4,2;2)$ is computed using the vector (S, Su, S, Su, S). The following three averages are computed using the following vectors (S, Sd, S, Su, S), (S, Sd, S, Sd, S) and (S, Sd, Sd^2, Sd, S). Thus from $N(4,2)$ the set of representative averages considered from this node retains all true averages but the one generated by the vector (S, Su, S, Sd, S). However, it is possible to have a case whereby the average $A(i,j;k)$ is produced by the path which reaches $S_{max}(i,j;k)$ more than once.

The representative averages $A(i, j; k+1)$ of such cases are computed by substituting the first value $S_{max}(i, j; k)$ reached by the trajectory with the value $S_{max}(i, j; k)d^2$ see Reisman [30]. In the very same manner, the other nodes of the tree the set are computed. By using the backward induction, the option price can be computed when the sets of representative averages are obtained such that:

$$V(i, j; k) = e^{-r\Delta t}\left[pV(i+1, j+1; k_u) + qV(i+1, j; k_d)\right].$$

Mostly the option values $V(i+1, j+1; k_d)$ and $V(i+1, j+1; k_u)$ are computed by linear interpolation and they are associated with the two averages

$$\frac{(i+1)A(i, j; k) + dS(i, j)}{i+2} \qquad \text{and} \qquad \frac{(i+1)A(i, j; k) + uS(i, j)}{i+2}$$

As for early exercise options, the option price of an American Asian call option is given by:

$$V(i, j, k) = \max\left\{e^{-r\Delta t}\left[pV(i+1, j+1, k_u) + qV(i+1, j, k_d)\right], A(i, j; k) - K\right\}.$$

3.3 Analytical Approximations

A method based on conditioning the geometric mean price was produced by Curran (1992). A closed form approximate expression was derived by Bouaziz, Briys, and Crouhy (1994). This catered for a formal upper bound and the approximation error caused by geometric average. To achieve this they used arithmetic strike options. Levy(1992) assumed that the distribution of the sum of lognormals can be approximated by a lognormal. From this assumption he was able to develop a closed form approximation for pricing European Average rate currency options that worked well only for a limited range of volatilities (see Hansen in the reference [22]). The weakness about this model was when:

$$\text{volatility} \times \sqrt{\text{time till expiration}} > 0.2.$$

To obtain better accuracy, Turnbull and Wakeman (1991) developed a model for volatilities above 0.2 and below 0.3. The reciprocal gamma distribution was used as the state price density function to obtain a closed form expression for the price of an arithmetic cross-currency Asian option. This was also obtained by Milevky and Posner (1998). They extended their work to find the closed form solution for the price of basket options. An Edgeworth series expansion was used by Turnbull and Wakeman (1991) to derive a closed form expansion to price European geometric Asian options. An efficient numerical method for average rate options was derived by Vorst (1992). It provided upper and lower bounds using the geometric average. Zhang (1992) produced a closed form solution for geometric average rate options.

3.4 Geometric Closed Form (Kemna and Vorst 1990)

Geometric averaging options can be priced via a closed form analytic solution, since the geometric average of the underlying prices follows a lognormal distribution as well, whereas with arithmetic average rate options, this condition collapses. Kemna and Vorst (1990) invented a closed form pricing solution to geometric averaging options by altering the volatility and cost of carry term (as stated in Kemna and Vorst [26] and Shreve [32]). The solutions to the geometric averaging Asian call and puts are given as:

$$C_G \approx Se^{-r(b-r)(T-t)}N\left(d_1\right) - Xe^{-r(T-t)}N\left(d_2\right)$$

and

$$P_G \approx Xe^{(T-t)}N(-d_2) - Se^{(b-r)(T-t)}N(-d_1).$$

Here *N(x)* is the cumulative distribution function of the standard normal random variable applied to

$$d_1 = \frac{\ln\frac{S}{X} + (b + 0.5\sigma_A^2)T}{\sigma_A\sqrt{T}}$$

and

$$d_2 = \frac{\ln\frac{S}{X} + (b - 0.5\sigma_A^2)T}{\sigma_A\sqrt{T}}.$$

Thus can be simplified to:

$$d_2 = d_1 - \sigma_A\sqrt{T}.$$

The adjusted volatility and dividend yield are given as:

$$\sigma_A = \frac{\sigma}{\sqrt{3}}$$

and

$$b = \frac{1}{2}(r - D - \frac{\sigma^2}{6})$$

where σ is the observed volatility, r is the risk free rate of interest and D is the dividend yield.

3.5 Arithmetic Average Rate Approximation (Levy 1992)

Levy suggested an analytical approximation that is said to give more accurate approximations:

$$C_{Levy} \approx S_Z N\left(d_1\right) - X_Z e^{rT_2} N\left(d_2\right)$$

where

$$d_1 = \frac{1}{\sqrt{K}}\left[\frac{\ln L}{2} - \ln X_Z\right],$$

$$d_2 = d_1 - \sqrt{K},$$

and

$$X_Z = X - S_{Avg}\frac{T - T_2}{T},$$

$$S_Z = \frac{S}{(r - D)\,T}\left(e^{-DT_2} - e^{-rT_2}\right),$$

$$K = \ln\left(K\right) - 2\left[rT_2 + \ln\left(S_Z\right)\right] \text{ and } L = \frac{M}{T_2},$$

$$M = \frac{2S^2}{r - D + \sigma^2}\left\{\frac{e^{\left[2(r-D)+\sigma^2\right]T_2 - 1}}{2\left(r - D\right) + \sigma^2}\right\} - \frac{e^{(r-D)T_2} - 1}{r - D}.$$

Table 3.1: Geometric Call versus Arithmetic Call by Black-Scholes and PDE method respectively.

σ	K	Geo Call$_{K\&V}$	Arith Call$_{Levy}$
0.1	75	7.0326	7.1089
	80	3.0956	3.1617
	85	0.8383	0.8742
0.2	75	7.7885	8.0285
	80	4.6269	4.8253
	85	2.4431	2.5866
0.3	75	8.9359	9.4172
	80	6.1601	6.5702
	85	4.0544	4.3843
0.4	75	10.1672	10.9736
	80	7.6353	8.3433
	85	5.6087	6.2115
0.5	75	11.3883	12.6101
	80	9.0377	10.1357
	85	7.0900	8.0588

Across all the above stated volatilities, the above computations reflect that the geometric calls are cheaper than the arithmetic calls. An out-of-the money geometric call with a volatility of 10% and a strike price of 85 is worth 0.84 whereas the an out-of-the money geometric call with a volatility of 50% is worth 7.09. This indicates that high volatility has a significant contribution on the price of the call.

3.6 Arithmetic Rate Approximation (Turnbull and Wakeman 1991)

As mentioned before, there are no closed form solutions to arithmetic averages. Due to the popular inexpensive use of the lognormal assumption under this form of averaging, a number of approximations have emerged in literature. The approximation suggested by Turnbull and Wakeman (TW) (1991) makes use of the fact that the distribution under arithmetic averaging is approximately lognormal. They put forward the first and second moments of the average in order to price the option (see Rogers and Shi [31] and Shioura [33]). The analytical approximations for a call and a put under TW are given as:

$$C_{TW} \approx Se^{(b-r)}N(d_1) - Xe^{-rT_2}N(d_2) \tag{3.1}$$

and

$$P_{TW} \approx Xe^{-rT_2}N(d_2) - Se^{(b-r)T_2}N(-d_1) \tag{3.2}$$

and we specify d_1 and d_2 as

$$d_1 = \frac{ln\frac{S}{X} + (b + 0.5\sigma_A^2)T_2}{\sigma_A\sqrt{T_2}} \tag{3.3}$$

and

$$d_2 = d_1 - \sigma_A\sqrt{T_2} \tag{3.4}$$

with T_2 being the time remaining until maturity. For averaging options that already have commenced their averaging period, T_2 is simply T. The adjusted volatility and dividend yield are given as:

$$\sigma_A = \sqrt{\frac{\ln(M_2)}{T} - 2b} \tag{3.5}$$

and

$$b = \frac{\ln(M_1)}{T} \tag{3.6}$$

To generalize the equations, we assume that the averaging period has not yet begun and give the first and second moments as:

$$M_1 = \frac{e^{(r-D)T} - e^{(r-D)\tau}}{(r-D)(T-\tau)} \tag{3.7}$$

$$M_2 = \frac{2e^{2(2(r-D)+\sigma^2)T}}{(\Gamma - D + \sigma^2)(2\Gamma - 2q + \sigma^2)(T-t)^2} + \frac{2e^{(2(r-D)+\sigma^2)r}}{(r-D)(T-t)^2}\left\{\frac{1}{2(r-D)+\sigma^2} + \frac{e^{(r-D)T}}{r-D+\sigma^2}\right\} \tag{3.8}$$

$$X_A = \frac{T}{T_2}X - \frac{(T-T_2)}{T_2}S_{\text{Avg}}. \tag{3.9}$$

Here we reiterate T as the initial time to maturity, T_2 as the remaining time to maturity, X as the original strike price and S_{Avg} is the average asset price. Haug (1998) notes that if $r = D$, the formula will not generate a solution.

3.7 Pricing of an average strike geometric Asian Option

At time T the payoff of an Asian Option can be written as $V_T^A(x_T, \overline{x}_T)$ (see[13]). Then the expected payoff turns out to be:

$$\mathbb{E}\left[V_T^A(x_T, \overline{x}_T)\right] = \int_{-\infty}^{\infty} dx_T \int_{-\infty}^{\infty} d\overline{x}_T V_T^A(x_T, \overline{x}_T)\, \mathcal{K}(x_T, T\,|0\,,0\,|\,\overline{x}_T)\,. \quad (3.10)$$

The price V_T^A of the option is the discounted payoff

$$V_T^A = e^{-rT}\mathbb{E}\left[V_T^A(x_T, \overline{x}_T)\right], \quad (3.11)$$

where r is the risk free interest rate. Expression (3.11) states that any option's price depending on the average of the asset during the lifetime of an option can be calculated. To achieve this, expression (3.11) is evaluated using the payoff:

$$V_T^A(x_T, \overline{x}_T) = \max\left(S_T - \overline{S}_0, 0\right) = S_0 \max\left(e^{x_T} - e^{\overline{x}_T}, 0\right). \quad (3.12)$$

Substituting (3.10) in (3.12) results in

$$V_T^A = S_0 e^{-rT} \int_{-\infty}^{\infty} d\overline{x}_T \int_{-\infty}^{\infty} dx_T \left(e^{x_T} - e^{\overline{x}_T}, 0\right) \mathcal{K}(x_T, T\,|0, 0|\,\overline{x}_T) \quad (3.13)$$

where the lower boundary of the x_T integration depends on $\overline{x}_T$. As for the average call price, its payoff is $\max(S_T - K, 0)$ and this makes the lower boundary of the $\overline{x}_T$ to be $\log \frac{K}{S_0}$. However since the integration of the lower boundary of the present case is complicated, The heaviside function is conveniently used to express this boundary and is expressed as:

$$V_T^A = S_0 e^{-rT} \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} d\overline{x}_T \int_{-\infty}^{\infty} dx_T \int_{-\infty}^{\infty} d\tau \frac{e^{i(x_T - \overline{x}_T)}}{\tau - i\epsilon} \left(e^{x_T} - e^{\overline{x}_T}\right) \times \mathcal{K}(x_T, T\,|0\,,0|\,\overline{x}_T)\,. \quad (3.14)$$

Application of Gaussians have been used to reduce the two original integrals by inserting complex exponential terms. The expression (3.14) can be split into two terms I_1 and I_2 such that:

$$I_1 = S_0 e^{-rT} \frac{\sqrt{3}}{\pi\sigma^2 T} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \frac{1}{\tau - i\epsilon} \int_{-\infty}^{\infty} d\overline{x}_T \int_{-\infty}^{\infty} dx_T \exp\left\{ -\frac{1}{2\sigma^2 T} \left[x_T - \left(\mu - \frac{\sigma^2}{2}\right) T \right]^2 - \frac{\varsigma}{\sigma^2 T} \left(\overline{x}_T - \frac{x_T}{2} \right)^2 + \mathrm{i} \left(x_T - \overline{x}_T \right) \tau + x_T \right\}. \tag{3.15}$$

Here I_2 is of the same form except that the last term of the argument of the exponent of $\overline{x}_T$, x_T is substituted. Then the Gaussian integrals over x_T and $\overline{x}_T$ are calculated to yield:

$$I_1 = S_0 e^{-(r-\mu)T} \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - \mathrm{i}\epsilon} d\tau \tag{3.16}$$

with

$$f(\tau) = \exp\left[-\frac{\sigma^2 T}{\varsigma} \tau^2 + \left(\mu + \frac{\sigma^2}{2} \right) \frac{\mathrm{i}T}{2} \tau \right]. \tag{3.17}$$

By making use of Plemelj's formula (see Gôrski in reference [21]) and also by considering the symmetry, the integral is reduced to

$$\int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - i\epsilon} d\tau = \mathrm{i}\pi \left[\left(\frac{b}{2\sqrt{a}} \right) + 1 \right], \tag{3.18}$$

with

$$\begin{cases} a &= \frac{\sigma^2 T}{\varsigma} \\ b &= \left(\mu + \frac{\sigma^2}{2} \right) \frac{T}{2} \end{cases}. \tag{3.19}$$

Hence the first term becomes

$$I_1 = S_0 e^{-rT} \frac{\sqrt{3}}{\pi\sigma^2 T} \frac{1}{2} \left\{ \left[\sqrt{\frac{3T}{8\sigma^2}} \left(\mu + \frac{\sigma^2}{2} \right) \right] + 1 \right\} \tag{3.20}$$

and similary the second term is evaluated, leading to

$$V_T^A = S_0 e^{-rT} \left(\frac{\sqrt{3}}{\pi\sigma^2 T} \frac{1}{2} \left\{ \left[\sqrt{\frac{3T}{8\sigma^2}} \left(\mu + \frac{\sigma^2}{2} \right) \right] + 1 \right\} - \exp\left[\left(\mu - \frac{\sigma^2}{\varsigma} \right) \frac{T}{2} \right] \left\{ \left[\sqrt{\frac{3T}{8\sigma^2}} \left(\mu - \frac{\sigma^2}{\varsigma} \right) \right] + 1 \right\} \right). \tag{3.21}$$

Using cumulative distribution function of the normal distribution,

$$\phi(x) = \frac{1}{2}\left[1 + \left(\frac{x}{\sqrt{2}}\right)\right],$$

and expression (3.21) can be written in the form:

$$V_0^A = S_0 e^{-rT}\left(e^{\mu T}\phi(d_1) - e^{\left(\mu - \frac{\sigma^2}{\varsigma}\right)\frac{T}{2}}\phi(d_2)\right) \tag{3.22}$$

with $d_1 = \sqrt{\frac{3T}{4\sigma^2}\left(\mu + \frac{4\sigma^2}{2}\right)}$ and $d_2 = \sqrt{\frac{3T}{4\sigma^2}\left(\mu - \frac{4\sigma^2}{2}\right)}$. The analytic pricing formula for an average strike geometric Asian call option is given in expression (3.22) and is stated in the reference Devreese et al [13].

Table 3.2: A comparison of Arithmetic Average Asian Pricing Methods. The Turnbull & Wakeman (TW) with Kemna & Vorst (KV) with the $S_0 = 80$, $T = 252$, $b = 4\%$ $\sigma = 0.20$ and the $r = 0.09$.

m	K	T & W	K & V
63	75	5.9656	6.9202
	80	2.7190	3.3936
	85	0.9260	1.2672
126	75	5.7328	5.6863
	80	2.3416	1.7519
	85	0.6608	0.2057
189	75	5.0361	5.1104
	80	0.7161	0.5738
	85	0.0023	0.0000
252	75	4.9822	4.9887
	80	0.1678	0.1142
	85	0.0000	0.0000

The above table compares the arithmetic average call results of (TW) and (KV) pricing methods within 252 total active trading days across the year. On the first 63 trading days the (TW) pricing method reflects cheaper prices as compared to (KV) pricing method across the three types of calls (in-the-money, at-the-money and out-of-the money). We see a change between the days 126th and 189th. The exact opposite reflects for all call types. At maturity we see a twist, at $K = 75$ (TW) is cheaper than (KV), whereas at $K = 80$ (KV) is cheaper than (TW).

Chapter 4

Monte-Carlo Simulation

4.1 Introduction

A class of computational algorithms that rely on repeated random sampling to compute its solution is named the Monte Carlo Method (MCM). This method is often used in computing on problems in physics and mathematics. Since MCM is a product of repeated computation of random numbers, very powerful computers are required for their calculations and are mostly used when it's impossible to use deterministic algorithms for computing results. The roots of the term Monte Carlo Method were from the physicists in the Los Alamos National Laboratory who were working on nuclear weapon projects during the 1940's (see Kamizono et al [25]). They made applications of the idea originated by Enrico Fermi and Stenislaw Ulam. Often in finance, MCM's are used to evaluate the asset values of companies, investment projects in various business units and financial derivatives. Unlike with the traditional static and deterministic models, Monte Carlo Method permits construction of stochastic or probabilistic financial models and simultaneously enhance the remedy of uncertainty in the evaluations.

4.2 Risk neutral computation

Monte Carlo Simulation employs application of risk neutral valuation when valuing options. To obtain the expected payoff in a risk neutral world, we generate sample paths and we discount the payoff along the risk-free rate. It is assumed that the random variables or the underlying stock prices follow a geometric Brownian motion path such that

$$dS = \mu S dt + \sigma S dW_t \tag{4.1}$$

where W_t is the Wiener process, μ is the expected return and σ is the volatility. The interest rates are assumed to be constant and the underlying market variables or derivatives provides a payoff at time T (see Boyle in reference [6]). The following steps are performed to value the derivatives:

1. Within a desired time horizon, simulate a random path for S in a risk neutral world.
2. According to the structure of the derivative, calculate the payoff.
3. Steps 1 and 2 are to be repeated to get many sample payoffs in the risk neutral world from the derivative.
4. To get the expected payoffs, calculate the mean of the sample payoffs.
5. To get the value of the derivative, the expected payoff should be discounted at the risk free rate .

The path followed by S is simulated by partitioning the life time of the derivative into n units of length δt and approximate the equation (4.1) over the interval dt as:

$$S(t+\delta t) - S(t) = \hat{\mu} S(t)\, \delta t + \sigma S(t)\, \varepsilon \sqrt{\delta t}$$

where

$S(t)$ = the value of S at time t, and ε = random variable which has the standard normal distribution.

This allows for a shift in calculation of S, the value of S from δt is calculated from the value of S at time 0, and the value of S at $2\delta t$ is calculated at δt interval and this goes on up to the very last time interval. Random samples from every trial are constructed. It is more reasonable to simulate $\ln S$ rather than S in practice since $\ln S$ gives better accuracy. An application of Itô's Lemma to the path of $\ln S$, results in:

$$\ln S(t+\delta t) - \ln S = \left(\hat{\mu} - \frac{\sigma^2}{2}\right) \delta t + \sigma \varepsilon \sqrt{\delta t}.$$

If the values of μ and σ are constant, then the above equation can be seen to be equivalent to the (closed form) formula:

$$S(t) = S(0) \exp\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right) t + \sigma\varepsilon\sqrt{t}\right].$$

Black-Scholes formulas can be derived by using this equation, and also the derivatives that provide a non-standard payoff can be valued using the very same equation. Suppose that in the life time of an option the derivate H pays an average of R. Then R sample path ξ will be regarded as being a set $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M\}$ such that

$$R(k, \delta t) = S_0 \exp\left[\sum_{i=1}^{k} \left(\hat{\mu} - \frac{\sigma^2}{2}\right) \delta t + \sigma\varepsilon_i\sqrt{\delta t}\right],$$

where N trials of M normal variables will create N sample paths and thus N values of H (see Hull in [24]). Calculating an average of H will result in a Monte Carlo for this derivative and

$$H(\xi) = \frac{1}{M+1} \sum_{k=0}^{M} R(k\delta t).$$

The error has the order $\epsilon = O\left(N^{-1/2}\right)$ and applying the central limit theorem, it is known that quadrupling the number of the sample paths will halve the errors of the estimated price. The Monte Carlo Method can also be used to generate sample paths of normal correlated variables. It is also effective for the derivatives whose payoff depend on the average value of S followed by the underlying variable. Many payoffs can occur during the lifetime of the derivative, and the method caters for any stochastic process. This method also caters for the derivatives whose payoff depends on a number of underlying market variables. The only disadvantage about Monte Carlo simulation is that it is computationally time consuming and is not most effective for derivatives that expire before maturity.

The crude Monte-Carlo Method

Suppose that for a finite expectation $\mathbb{E}(X)$, we let X be a real valued random variable. Then we approximate $\mathbb{E}(X)$ by the arithmetic mean $\frac{1}{N}\sum_{i=1}^{N}(\omega)$ for some $n \in \mathbb{N}$ (see [24]).

Table 4.1: Crude Monte-Carlo Option Values for Arithmetic Average Asian Call with $S_0 = 80$, $\sigma = 0.2$, $r = 0.06$, $T = 1$ and $N = 10000$

K	m	Geo Asian$_{BS}$	Arith Asian$_{CMC}$	Std Error$_{CMC}$
75	10^1	7.7128	7.7997	0.0770
	10^2	7.7803	7.9351	0.0776
	10^3	7.7897	7.9909	0.0781
80	10^1	4.5339	4.6957	0.0631
	10^2	4.6168	4.8048	0.0647
	10^3	4.6259	4.8311	0.0649
85	10^1	2.3540	2.4363	0.0477
	10^2	2.4334	2.6016	0.0488
	10^3	2.4421	2.5686	0.0500

Table 4.1 is a comparison of geometric Asian calls with arithmetic Asian calls. The methodologies used to price these calls for the geometric Asian calls and crude Monte-Carlo simulations are employed to price the arithmetic Asian calls. The standard error of an arithmetic mean for the at-the-money option for $m = 1$ is 0.0631. This indicates the 95% confidence interval results $[4.5720; 4.8194]$.

Table 4.2: Comparison of European Call, Geometric Asian Call and the Arithmetic Asian Call using Black Scholes and Monte-Carlo with $S_0 = 80$, $\sigma = 0.2$, $r = 0.06$, $T = 1$ and $N = 10000$

K	m	Euro Call (BS)	Euro Call (MCM)	Geo Asian (BS)	Geo Asian (MCM)	Arith Asian (MCM)
75	10^1	11.8046	14.2516	7.7128	8.0814	6.7110
	10^2		11.8701	7.7803	7.9827	7.8846
	10^3		12.0148	7.7877	8.0012	8.0521
80	10^1	8.7916	6.9069	4.5339	4.9501	3.5898
	10^2		9.3685	4.6169	4.7877	4.1048
	10^3		8.7247	4.6259	4.7829	4.5556
85	10^1	6.3495	11.7750	2.3539	2.5658	2.1456
	10^2		4.8519	2.4334	2.5726	1.9926
	10^3		6.9654	2.4421	2.5772	2.6464

The above table compares the prices of European Calls , geometric Asian calls and arithmetic Asian call. The pricing methods employed in constructing the above table are the Black-Scholes and the Monte-Carlo pricing Method. The above results reveal that increasing the number of simulations when using the MCM enhances the accuracy of the European Call. Across all different strike prices we observe that the geometric Asian calls computed with the Black-Scholes method are slightly cheaper than the geometric Asian call computed via the Monte-Carlo method. Arithmetic Asian calls stands out to be the cheapest call across all types of options (in-the-money, at-the-money and out-of-the money).

4.3 Variance reduction

4.3.1 The control variate method

Suppose that an analytical solution is known for derivative Y, while for derivative X the solution is unknown. To get an estimation of this unknown derivative X, two simulation are conducted using the same random number streams and δt. The two estimates f_X^* and f_Y^* are obtained from these simulations (see references Bellalah and Briys [7], Hull [24]). It has been discovered that a better estimate is obtained by using the following formula:

$$\hat{f}_X = f_X^* - f_Y^* + f_Y,$$

where f_Y is a known analytical solution. Since the variance of f_Y is zero and

$$\mathbb{V}\left[\hat{f}_X\right] = \mathbb{V}\left[f_X^*\right] + \mathbb{V}\left[f_Y^*\right] - 2\text{Cov}\left[f_X^*, f_Y^*\right],$$

then the variance is reduced if

$$2\text{Cov}\left[f_X^*, f_Y^*\right] f_Y > \mathbb{V}\left[f_X^*\right] + \mathbb{V}\left[f_Y^*\right].$$

Though there are many possible control variates, we choose two control variates in order to make more efficient approximations. We use the geometric average Asian call option which is given by:

$$\text{CV}_{geo} = e^{-rT}\left[e^{\frac{1}{m+1}\sum_{i=0}^{m}\log S\left(\frac{iT}{m}\right)} - K\right]^+.$$

We also choose a European call option which is given by:

$$\text{CV}_{euro} = e^{-rT}\left(S(T) - K\right)^+.$$

The two schemes are applied in the following two tables.

Table 4.3: Control Variates Option Value for Arithmetic Average Asian Call with $S_0 = 80$, $\sigma = 0.2$, $r = 0.06$, $T = 1$ and $N = 10000$

K	m	ArithAve_{euro}	Std Error_{euro}	ArithAve_{geo}	Std Error_{geo}
75	10^1	7.9610	0.0372	7.9441	0.0020
	10^2	7.9911	0.0407	7.9943	0.0020
	10^3	7.9615	0.0408	8.0038	0.0020
80	10^1	4.7168	0.0317	4.7327	0.0019
	10^2	4.8164	0.0351	4.8013	0.0018
	10^3	4.7855	0.0349	4.8094	0.0019
85	10^1	2.4656	0.0256	2.5215	0.0017
	10^2	2.5723	0.0282	2.5805	0.0016
	10^3	2.5632	0.0276	2.5916	0.0017

Table 4.3 shows us the values we obtained by using control variate methods. Though the option values by the two different control variates are approximately the same, the standard error differ dramatically. The introduction of the geometric average Asian option as the control variate drastically reduces the standard deviation of the estimates. The range of the 95% confidence interval for the 100 time-step Asian option with the strike price of 75 have been reduced from $+0.1605$ to -0.0077.

4.3.2 The Antithetic Variable Technique

The value of the derivatives's uncertainty is inversely proportional to the number of M trials. For more accurate results, a very large number of sample paths are required for a typical problem. The *antithetic variable* technique is called by this name since it uses a variable that displays negative independence [24]. The idea comes from the fact that if W_t is a Wiener process, then also $-W_t$ is a Wiener process. In this technique, two values of the derivatives are calculated for each simulation trial. The first value of the derivative is taken from the generated path $\{-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_N\}$ and then also the path $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M\}$. This reduces the number of sample paths to be taken to construct n paths, and it also improves the accuracy by reducing the variance of the sample paths. The average of the two derivatives is zero and is given by:

$$\bar{f} = \frac{f_1 + f_2}{2}.$$

If the standard deviation of $\bar{f}$ is $\bar{\sigma}$ and M is the number of simulation trials, then $\frac{\bar{\sigma}}{\sqrt{M}}$ is the standard error of the estimate. This is much less than the standard error calculated from $2M$ trials.

Table 4.4: Antithetic Variate Option Value for Arithmetic Average Call with $S_0 = 80$, $\sigma = 0.2$, $r = 0.06$, $T = 1$ and $N = 10000$

K	m	ArithAve	Std Error
75	10^1	7.9336	0.0240
	10^2	8.0147	0.0249
	10^3	8.0381	0.0250
80	10^1	4.7155	0.0301
	10^2	4.7762	0.0309
	10^3	4.7505	0.0306
85	10^1	2.4530	0.0287
	10^2	2.5770	0.0291
	10^3	2.5851	0.0300

The calculations on the method of using antithetic variates are presented in Table 4.3. An appropriate comparison of table 4.1 , 4.2 and 4.3 reveals that it is more efficient to use antithetic variate method than using European option as the control variate. Though paths $-\epsilon_i$ and ϵ_i have perfect negative correlation this does not hold for their respective functions.

4.3.3 Importance sampling

This is best explained as follows: Let K be the strike price of the European call option at the maturity T. We suppose that we are interested in calculating the price of this option when it is deep-out-of the-money. Mostly, a zero payoff is usually obtained when we sample the values of the asset price at time T and this usually waste computation time since it has less contribution in valuing the option. The significant paths selected are paths where stock price is above K at maturity. We let F be the conditional probability distribution for the stock price at time T and $q =$ the probability of the stock price being greater than K at time T. Hence, the probability distribution of the stock price conditional on stock price being greater than K is given by $G = F/q$. The value of the option can be estimated from q multiplied by the average discounted payoff [24].

Importance sampling is a variance reduction technique for Monte Carlo simulation. The principle of this method is based on changing the probability measure from which paths are generated, so that more weight is applied to important outcomes and sampling efficiency is increased.

Example

Suppose that for an out-of-the money European call we have the parameters $S = 80$, $K = 85$, $r = 0.06$, $\sigma = 0.2$ $N = 10000$ and $m = 100$. Since the option is out-of-the money, most of the simulation outcomes done without importance sampling will result in zero payoff. Importance sampling comes to the rescue, by changing the drift so that most of the paths result in a positive payoff, but finally the resulting payoff is adjusted for the change of measure.

Table 4.5: Importance Sampling for an out-of-the money European Call

	Mean	Variance
Normal Sampling	11.825	0.017529
Importance Sampling	11.834	0.043357
Analytical Price = 11.805		

The above table indicates that the mean price of both samples is approximately the same although the variances differ drastically. The analytical price resulting from this technique is 11.805.

Chapter 5

Estimates from Taylor's Expansions

5.1 Pricing of Discrete Asian Options

This chapter is devoted to the approximation method of Tsao and Huang for pricing arithmetic Asian options. The idea of Tsao and Huang is used to obtain the approximation formula for the average strike Asian option through Taylor expansion. The mean and the variance of the approximation formula for pricing discrete arithmetic Asian options is simplified.Whether averages are measured continuously or discretely, option valuation is three dimensional: we must keep track of the asset price, time and a path dependent quantity which for Asian options is the running average (see Lyuu [28], Huang and Tsao [23]). As in the Black Scholes model of option pricing, the price process is assumed to follow the geometric Brownian motion under risk neutral measures and is described by the SDE

$$\frac{dS}{S} = (r - d)dt + \sigma dZ \tag{5.1}$$

where r is the annual risk-free interest rate, d as the dividend yield of the underlying asset and σ is the volatility of the asset. Given a call option at time $t = 0$ and the strike price that is represented by K is fixed for a discrete average price option, its payoff at maturity T is known to be $(\overline{S} - K)^+$. The average price is explicitly described by

$$\overline{S} = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} = \frac{1}{n}\sum_{i=1}^{n} S_0 \exp(\widehat{r}t_i + \sigma Z_{t_i}) \tag{5.2}$$

and n as the number of sampled prices over the time horizon $[0, T]$ to be averaged. We find it convenient to write

$$\hat{r} = (r - d - \frac{\sigma^2}{2}).$$

At time 0, we have the call option price as follows

$$C(S_0, 0, T) = e^{-rT}\mathbb{E}\left[(\overline{S} - K)^+\right] \tag{5.3}$$

where $\mathbb{E}$ is the expectation operator in the risk-neutral probability measure. After substituting equation (5.2) into (5.3), we obtain the following equation:

$$\begin{aligned} C\left(S_0, 0, T\right) &= e^{-rT}\mathbb{E}\left\{\left[\frac{1}{n}\sum_{i=1}^{n} S_0 \exp(\hat{r}t_i + \sigma Z_{t_i}) - K\right]^+\right\} \\ &= S_0 e^{-rT}\mathbb{E}\left\{\left[\frac{1}{n}\sum_{i=1}^{n} \exp(\hat{r}t_i + \sigma Z_{t_i}) - \frac{K}{S_0}\right]^+\right\}. \end{aligned} \tag{5.4}$$

If we let

$$X = \left[\frac{1}{n}\sum_{i=1}^{n} \exp(\hat{r}t_i + \sigma Z_{t_i}) - \frac{K}{S_0}\right]^+ \tag{5.5}$$

then $C(S_0, 0, T) = S_0 e^{-rT}\mathbb{E}\left[X\right]$.

We now seek to find a tractable approximation for X. This will be central to our computation.

5.2 An approximation for X

We apply the second order Taylor's expansion to $\exp\left(\hat{r}t_i + \sigma Z_{t_i}\right)$ with $a = \ln\left(\frac{K}{S_0}\right)$. Using the second order approximation,

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2$$

we obtain:

$$\exp(\hat{r}t_i+\sigma Z_{t_i}) \cong \frac{K}{S_0}+\frac{K}{S_0}\left[\hat{r}t_i+\sigma Z_{t_i}-\ln\left(\frac{K}{S_0}\right)\right]+\frac{K}{2S_0}\left[\hat{r}t_i+\sigma Z_{t_i}-ln\left(\frac{K}{S_0}\right)\right]^2. \tag{5.6}$$

The approximation (5.6) enables us to express X as in the following approximation.

$$\begin{aligned} X \cong{}& \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{K}{S_0}+\frac{K}{S_0}\left[\hat{r}t_i+\sigma Z_{t_i}-\ln\left(\frac{K}{S_0}\right)\right]+\frac{K}{2S_0}\left[\hat{r}t_i+\sigma Z_{t_i}\right.\right. \\ & \left.\left.-\ln\left(\frac{K}{S_0}\right)\right]^2-\frac{K}{S_0}\right\} \\ ={}& \frac{K}{nS_0}\sum_{i=1}^{n}\left\{\left[\hat{r}t_i+\sigma Z_{t_i}-\ln\left(\frac{K}{S_0}\right)\right]+\frac{1}{2}\left[\hat{r}t_i+\sigma Z_{t_i}-\ln\left(\frac{K}{S_0}\right)\right]^2\right\} \\ ={}& \frac{K}{nS_0}\sum_{i=1}^{n}\left\{\left[\hat{r}t_i+\sigma Z_{t_i}-\ln\left(\frac{K}{S_0}\right)\right]+\frac{1}{2}\left[\hat{r}^2t_i^2+\sigma^2Z_{t_i}^2-\ln^2\left(\frac{K}{S_0}\right)\right.\right. \\ & \left.\left.+2\hat{r}t_i\sigma Z_{t_i}-2\hat{r}t_i\ln\left(\frac{K}{S_0}\right)-2\sigma Z_{t_i}\ln\left(\frac{K}{S_0}\right)\right]\right\}. \end{aligned} \tag{5.7}$$

We find the following approximation: If we let Y be such that:

$$\begin{aligned} Y ={}& \frac{K}{nS_0}\sum_{i=1}^{n}\left\{\left[\hat{r}t_i+\sigma Z_{t_i}+\frac{\hat{r}^2t_i^2}{2}+\frac{\sigma^2Z_{t_i}^2}{2}+\hat{r}t_i\sigma Z_{t_i}-\hat{r}t_i\ln\left(\frac{K}{S_0}\right)\right]\right. \\ & \left.-n\ln\left(\frac{K}{S_0}\right)+\frac{n}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\}, \end{aligned} \tag{5.8}$$

then $X \cong Y$. We can rewrite the expression for Y as below.

$$\begin{aligned} Y ={}& \frac{K}{S_0n}\left\{\sum_{i=1}^{n}\left[\hat{r}t_i+\sigma Z_{t_i}+\frac{\hat{r}^2t_i^2}{2}+\hat{r}t_i\sigma Z_{t_i}-\hat{r}t_i\ln\left(\frac{K}{S_0}\right)+\frac{\sigma^2Z_{t_i}^2}{2}\right.\right. \\ & \left.\left.-\sigma Z_{t_i}\ln\left(\frac{K}{S_0}\right)\right]^2-n\ln\left(\frac{K}{S_0}\right)+\frac{1}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2Z_{t_i}\right\}. \end{aligned} \tag{5.9}$$

5.3 The mean of the approximation

We note the following straight forward identities and we apply them to prove the subsequent proposition.

Proposition 5.3.1. *Suppose X and Y are random variables and c is a constant. Then*

$$\begin{aligned}\mathbb{E}[c] &= c \\ \mathbb{E}[cX] &= c\mathbb{E}[X] \\ \mathbb{E}[X+Y] &= \mathbb{E}[X]+\mathbb{E}[Y]\end{aligned}$$

Proposition 5.3.2. *Suppose S_0 denotes the time value of a stock price on which is written a call option at the time 0, and suppose that the fixed strike price for a discrete average price is denoted by K.*

(a) At time 0, the call option price is:

$$C(S_0,0,T) = e^{rT}\mathbb{E}[(\bar{S}-K)^+] = S_0 e^{-rT}\mathbb{E}\left\{\left[\frac{1}{n}\sum_{i=1}^{n}\exp(\hat{r}t_i+\sigma Z t_i)-\frac{K}{S_0}\right]^+\right\}$$

where $\mathbb{E}$ is the expectation operator in the risk-neutral probability measure.

(b) Also,

$$\mathbb{E}[Y] = m_n = \frac{K}{S_0}\left\{\frac{\sigma^2 T(n+1)}{4n}+\frac{\hat{r}T(n+1)}{2n}\left[1-\ln\left(\frac{K}{S_0}\right)\right]+\right.$$
$$\left.\frac{\hat{r}^2T^2(n+1)(2n+1)}{12n^2}-\ln\left(\frac{K}{S_0}\right)+\frac{1}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\}$$

where n is the number of sample prices to be averaged, σ is the instantaneous volatility of the underlying asset, Z is a standard Wiener process, and $\hat{r} = (r-d-\sigma/2)$.

Proof. (a) This is clear.

(b) From Equation (5.9) we have

$$Y = \frac{K}{S_0 n}\left\{\sum_{i=1}^{n}\left[\hat{r}t_i + \sigma Z_{t_i} + \frac{\hat{r}^2 t_i^2}{2} + \hat{r}t_i\sigma Z_{t_i} - \hat{r}t_i\ln\left(\frac{K}{S_0}\right) + \frac{\sigma^2 Z_{t_i}^2}{2}\right.\right.$$
$$\left.\left.-\sigma Z_{t_i}\ln\left(\frac{K}{S_0}\right)\right] - n\ln\left(\frac{K}{S_0}\right) + \frac{n}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\}$$

$$= \frac{K}{S_0}\left\{\sum_{i=1}^{n}\left[\left(1-\ln\left(\frac{K}{S_0}\right)\right)\frac{\hat{r}}{n}t_i + \left(1-\ln\left(\frac{K}{S_0}\right)\right)\frac{\sigma}{n}Z_{t_i} + \frac{\hat{r^2}t_i^2}{2n} + \frac{\hat{r}t_i\sigma Z_{t_i}}{n}\right.\right.$$
$$\left.\left.+\frac{\sigma^2 Z_{t_i}^2}{2n}\right] - \frac{n\ln\left(\frac{K}{S_0}\right)}{n} + \frac{n}{2n}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\}$$

$$= \frac{K}{S_0}\left\{\left(1-\ln\left(\frac{K}{S_0}\right)\right)\frac{\hat{r}}{n}\sum_{i=1}^{n}t_i + \left(1-\ln\left(\frac{K}{S_0}\right)\right)\frac{\sigma}{n}\sum_{i=1}^{n}Z_{t_i} + \frac{\hat{r^2}}{2n}\sum_{i=1}^{n}t_i^2\right.$$
$$\left.+\frac{\hat{r}\sigma}{n}\sum_{i=1}^{n}t_i Z_{t_i} + \frac{\sigma^2}{2n}\sum_{i=1}^{n}Z_{t_i}^2 - \ln\left(\frac{K}{S_0}\right) + \frac{1}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\}. \tag{5.10}$$

Therefore,

$$\mathbb{E}\left[Y\right] = \mathbb{E}\left[\frac{K}{S_0}\left\{\left(1-\ln\left(\frac{K}{S_0}\right)\right)\frac{\hat{r}}{n}\sum_{i=1}^{n}t_i + \left(1-\ln\left(\frac{K}{S_0}\right)\right)\frac{\sigma}{n}\sum_{i=1}^{n}Z_{t_i}\right.\right.$$
$$\left.\left.+\frac{\hat{r^2}}{2n}\sum_{i=1}^{n}t_i^2\frac{\hat{r}\sigma}{n}\sum_{i=1}^{n}t_i Z_{t_i} + \frac{\sigma^2}{2n}\sum_{i=1}^{n}Z_{t_i}^2 - \ln\left(\frac{K}{S_0}\right) + \frac{1}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\}\right]$$

$$= \frac{K}{S_0}\left\{\left(1-\ln\left(\frac{K}{S_0}\right)\right)\frac{\hat{r}}{n}\mathbb{E}\sum_{i=1}^{n}t_i + \left(1-\ln\left(\frac{K}{S_0}\right)\right)\frac{\sigma}{n}\mathbb{E}\sum_{i=1}^{n}Z_{t_i} + \frac{\hat{r^2}}{2n}\mathbb{E}\sum_{i=1}^{n}t_i^2\right.$$
$$\left.+\frac{\hat{r}\sigma}{n}\mathbb{E}\sum_{i=1}^{n}t_i Z_{t_i} + \frac{\sigma^2}{2n}\mathbb{E}\sum_{i=1}^{n}Z_{t_i}^2 - \ln\left(\frac{K}{S_0}\right) + \frac{1}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\}. \tag{5.11}$$

We consider the following elementary, though important identities of summation

$$U_1(n) = 1 + 2 + 3 + ... + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$
$$U_2(n^2) = 1^2 + 2^2 + 3^2 + ... + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We use the preceding properties to simplify the following entities. Here $(z_k)_{k\in\mathbb{N}}$ is a discrete standard Brownian motion.

$$\mathbb{E}\left[\sum_{i=1}^{n} t_i\right] = \frac{T}{n} \times \frac{n(n+1)}{2} = \frac{T(n+1)}{2}.$$
$$\mathbb{E}\left[\sum_{i=1}^{n} t_i^2\right] = \left(\frac{T}{n}\right)^2 \times \frac{T^2 n(n+1)(2n+1)}{6n^2} = \frac{T^2(n+1)(2n+1)}{6n}.$$
$$\mathbb{E}\left[\sum_{i=1}^{n} z_{t_i}\right] = 0.$$
$$\mathbb{E}\left[\sum_{i=1}^{n} z_{t_i}^2\right] = \mathbb{E}\left[\sum_{i=1}^{n} t_i\right] = \frac{T}{n} \times \frac{n(n+1)}{2} = \frac{T(n+1)}{2}.$$

Through applying the preceding identities into Equation (5.11), we then obtain the following

$$\mathbb{E}[Y] = m_n = \frac{K}{S_0}\left\{\frac{\sigma^2 T(n+1)}{4n} + \frac{\hat{r}T(n+1)}{2n}\left[1 - \ln\left(\frac{K}{S_0}\right)\right] + \right.$$
$$\left. \frac{\hat{r}^2 T^2(n+1)(2n+1)}{12n^2} - \ln\left(\frac{K}{S_0}\right) + \frac{1}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\} \tag{5.12}$$

This completes the proof. □

5.4 The variance of the approximation

Towards the variance we note the following very useful formula which we state without (the very elementary) proof.

Proposition 5.4.1. *Let $W = L + U$ for a constant L and a random variable U then*

$$\mathbb{V}[W] = \mathbb{V}[U]$$

.

For the approximation Y of the random variable X we calculate the mean and the variance.

From (5.10)

$$\begin{aligned}
Y &= \frac{K}{S_0 n}\left\{\sum_{i=1}^{n}\left[\hat{r}t_i + \sigma Z_{t_i} + \frac{\hat{r}^2 t_i^2}{2} + \hat{r}t_i\sigma Z_{t_i} - \hat{r}t_i \ln\left(\frac{K}{S_0}\right) + \frac{\sigma^2 Z_{t_i}^2}{2} - \sigma Z_{t_i}\right.\right.\\
&\left.\left.\ln\left(\frac{K}{S_0}\right)\right] - n\ln\left(\frac{K}{S_0}\right) + \frac{n}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\}\\
&= \frac{K}{S_0 n}\left\{\sum_{i=1}^{n}\left[\sigma\left(1 + \hat{r}t_i + \ln\left(\frac{K}{S_0}\right)\right)\right] Z_{t_i} + \frac{\sigma^2 Z_{t_i}^2}{2} + \hat{r}t_i\left[1 + \frac{\hat{r}t_i}{2} - \right.\right.\\
&\left.\left.\ln\left(\frac{K}{S_0}\right)\right] - n\ln\left(\frac{K}{S_0}\right) + \frac{n}{2}\left[\ln\left(\frac{K}{S_0}\right)\right]^2\right\}. \qquad (5.13)
\end{aligned}$$

Therefore in view of the proposition above, we have

$$\mathbb{V}[Y] = \mathbb{V}\left[\frac{K}{S_0 n}\left\{\sum_{i=1}^{n}\left[\sigma\left(1 + \hat{r}t_i + \ln\left(\frac{K}{S_0}\right)\right) Z_{t_i} + \frac{\sigma^2 Z_{t_i}^2}{2}\right]\right\}\right].$$

We let

$$A = \sum_{i=1}^{n}\left[\sigma\left(1 + \hat{r}t_i + \ln\left(\frac{K}{S_0}\right)\right)\right] \quad \text{and} \quad U = \left[AZ_{t_i} + \frac{\sigma^2 Z_{t_i}^2}{2}\right].$$

then

$$\mathbb{V}[Y] = \left(\frac{K^2}{n^2 S_0^2}\right)\mathbb{V}\left[AZ_{t_i} + \frac{\sigma^2 Z_{t_i}^2}{2}\right]. \qquad (5.14)$$

By the variance property

$$\mathbb{V}[U] = \mathbb{E}\left[U^2\right] - \left(\mathbb{E}[U]\right)^2, \qquad (5.15)$$

it follows that

$$\begin{aligned}
\mathbb{E}\left[U^2\right] &= \mathbb{E}\left[\left(AZ_{t_i} + \frac{\sigma^2 Z_{t_i}^2}{2}\right)^2\right]\\
&= \mathbb{E}\left[A^2 Z_{t_i}^2 + \sigma^2 A Z_{t_i}^3 + \frac{\sigma^4 Z_{t_i}^4}{4}\right]. \qquad (5.16)
\end{aligned}$$

Proposition 5.4.2. *Let $X_t = Z_t^n$ where Z_t is a standard Brownian Motion. Then*

$$\mathbb{E}\left[X_t\right] = \mathbb{E}\left[Z_t^n\right] = \frac{n(n-2)}{2}\int_0^t \mathbb{E}\left[Z_s^{n-1}\right] dS$$

In particular, for every $n \in \mathbb{N}, \mathbb{E}\left[\mathbb{Z}^{2n+1}\right] = 0$.

Proof. By the Itô formula,

$$dX_t = \frac{\partial X_t}{\partial t}dt + \frac{\partial X_t}{\partial x}dZ_t + \frac{1}{2}\frac{\partial^2 X_t}{\partial x^2}dt$$

we can write

$$dX_t = 0 + nZ_t^{n-1}dZ_t + \frac{1}{2}n(n-1)Z_t^{n-2}dt.$$

Since $X_0 = 0$, the integrated form of the previously mentioned X is

$$X_t - X_0 = \int_0^t nZ_s^{n-1}dZ_t + \int_0^t \frac{n(n-1)}{2}Z_s^{n-2}dt.$$

By the martingale property the expectation of the first stochastic process on the right hand side vanishes so that:

$$\mathbb{E}\left[X_t\right] = \int_0^t \mathbb{E}\left[\frac{n(n-1)}{2}Z_s^{n-2}\right] dt.$$

This proves our proposition. □

Therefore

$$\mathbb{E}\left[Z_t^n\right] = \frac{n(n-1)}{2}\int_0^t \mathbb{E}\left[Z_s^{n-2}\right] dt.$$

$$\mathbb{E}\left[Z_{t_i}^2\right] = \frac{2(2-1)}{2}\int_0^t \mathbb{E}\left[Z_s^{2-2}\right] dt = t.$$

$$\mathbb{E}\left[Z_{t_i}^3\right] = \frac{3(3-1)}{2}\int_0^t \mathbb{E}\left[Z_s^{3-2}\right] dt = 0.$$

$$\mathbb{E}\left[Z_{t_i}^4\right] = \frac{4(4-1)}{2}\int_0^t \mathbb{E}\left[Z_s^{4-2}\right] dt = 6\int_0^t sdt = 3t^2.$$

However,

$$
\begin{aligned}
(\mathbb{E}[U])^2 &= \left(\mathbb{E}\left[AZ_{t_i} + \frac{\sigma^2 Z_{t_i}^2}{2}\right]\right)^2 \\
&= \left(A\mathbb{E}[Z_{t_i}] + \frac{\sigma^2}{2}\mathbb{E}\left[Z_{t_i}^2\right]\right)^2 \\
&= \left(0 + \frac{\sigma^2}{2} \times t_i\right)^2 = \frac{\sigma^4}{4}t_i^2.
\end{aligned}
$$

From equation (5.15)

$$
\begin{aligned}
\mathbb{V}[U] &= \mathbb{E}\left[U^2\right] - (\mathbb{E}[U])^2 \\
&= A^2\mathbb{E}\left[Z_{t_i}^2\right] + \sigma^2 A\mathbb{E}\left[Z_{t_i}^3\right] + \frac{\sigma^4}{4}\mathbb{E}\left[Z_{t_i}^4\right] - \frac{\sigma^4}{4}t_i^2 \\
&= A^2 t_i + 0 + \frac{\sigma^4}{4} \times 3t_i^2 - \frac{\sigma^4}{4}t_i^2 \\
&= A^2 t_i + \left(\frac{3}{4} - \frac{1}{4}\right)\sigma^4 t_i^2 \\
&= A^2 t_i + \frac{1}{2}\sigma^4 t_i^2. \qquad (5.17)
\end{aligned}
$$

Now we observe that

$$
A = \sum_{i=1}^{n}\left[\sigma\left(1 + \hat{r}t_i + \ln\left(\frac{K}{S_0}\right)\right)\right].
$$

Therefore

$$
\begin{aligned}
A^2 &= \left(\sum_{i=1}^{n} \left[\sigma \left(1 + \hat{r} t_i + \ln \left(\frac{K}{S_0} \right) \right) \right] \right)^2 \\
&= \left(\sigma \sum_{i=1}^{n} 1 + \sigma \hat{r} \sum_{i=1}^{n} t_i + \sigma \ln \left(\frac{K}{S_0} \right) \sum_{i=1}^{n} 1 \right)^2 \\
&= \left[n\sigma + \sigma \hat{r} \frac{Tn(n+1)}{2} + n\sigma \ln \left(\frac{K}{S_0} \right) \right]^2 \\
&= n^2\sigma^2 + \hat{r}^2\sigma^2 T^2 \frac{n^2(n+1)^2}{4} + n^2\sigma^2 \ln^2 \left(\frac{K}{S_0} \right) + 2\hat{r}\sigma^2 T \frac{n^2(n+1)}{2} + \\
&\quad 2n^2\sigma^2 \ln \left(\frac{K}{S_0} \right) + 2\hat{r}\sigma^2 n^2 T \frac{(n+1)}{2} \ln \left(\frac{K}{S_0} \right) \\
&= n^2\sigma^2 \left\{ 1 + \hat{r}^2 T^2 \frac{(n+1)^2}{4} + \ln^2 \left(\frac{K}{S_0} \right) + \hat{r}T(n+1) + 2\ln \left(\frac{K}{S_0} \right) + \right. \\
&\quad \left. \hat{r}T(n+1) \ln \left(\frac{K}{S_0} \right) \right\} \\
&= n^2\sigma^2 \left\{ \left[1 + \ln \left(\frac{K}{S_0} \right) \right]^2 + \hat{r}T(n+1) \times \left[1 + \ln \left(\frac{K}{S_0} \right) \right] + \left[\frac{1}{2} \hat{r} T(n+1) \right]^2 \right\} \\
&= n^2\sigma^2 \left\{ \left[1 + \ln \left(\frac{K}{S_0} \right) \right] + \left[\frac{1}{2} \hat{r} T(n+1) \right] \right\}^2 .
\end{aligned}
$$

From equation (5.17)

$$
\begin{aligned}
\mathbb{V}[U] &= A^2 t_i + \frac{1}{2} \sigma^4 t_i^2 \\
&= \left[n^2\sigma^2 \left\{ \left[1 + \ln \left(\frac{K}{S_0} \right) \right] + \left[\frac{1}{2} \hat{r} T(n+1) \right] \right\}^2 \right] t_i + \frac{1}{2} \sigma^4 t_i^2 .
\end{aligned}
$$

Let $t_i - t_{i-1} = \delta t = \frac{T}{n}$. Then

$$
\begin{aligned}
\mathbb{V}[U] &= \left[n^2\sigma^2 \left\{ \left[1 + \ln \left(\frac{K}{S_0} \right) \right] + \left[\frac{1}{2} \hat{r} T(n+1) \right] \right\}^2 \right] \frac{T}{n} + \frac{1}{2} \sigma^4 \left(\frac{T}{n} \right)^2 \\
&= \left(\frac{T}{n} \right) \left\{ \left[n^2\sigma^2 \left\{ \left[1 + \ln \left(\frac{K}{S_0} \right) \right] + \left[\frac{1}{2} \hat{r} T(n+1) \right] \right\}^2 \right] + \frac{1}{2n} \sigma^4 T \right\} .
\end{aligned}
$$

Substituting the above into equation (5.14) we obtain the following equation

$$\mathbb{V}[Y] = \left(\frac{K^2}{n^2 S_0^2}\right) \times \left(\frac{T}{n}\right) \left\{ \left[n^2\sigma^2 \left\{ \left[1 + \ln\left(\frac{K}{S_0}\right) \right] + \left[\frac{1}{2}\hat{r}T(n+1) \right] \right\}^2 \right] + \frac{1}{2n}\sigma^4 T \right\} \tag{5.18}$$

$$= \left(\frac{K^2 T}{n^3 S_0^2}\right) \left\{ \left[n^2\sigma^2 \left\{ \left[1 + \ln\left(\frac{K}{S_0}\right) \right] + \left[\frac{1}{2}\hat{r}T(n+1) \right] \right\}^2 \right] + \frac{1}{2n}\sigma^4 T \right\}. \tag{5.19}$$

Table 5.1 gives a comparison of numerical results for pricing the discrete Asian call versus arithmetic Asian call through Taylor approximations technique. The initial price assumed (S_0) is 80, the risk-free rate (r) is 0.09, the option's time to maturity (T) in weeks is 4/13 and the number of sample prices to be averaged (n) is 4 and 16. The considered strike prices (K) are 75, 80 and 85 which represents the in-the-money, at-the-money and out-of-money option respectively.

Table 5.1: Comparison of Discrete Asian Call versus Arithmetic Asian Call by Taylor Approximations and PDE respectively

σ	**K**	Asian Call$_{DF}$		Asian Call$_{PDE}$
		$n = 4$	$n = 16$	
0.1	75	5.3689	5.7976	5.9564
	80	1.6011	1.6325	1.6391
	85	0.1041	0.0818	0.0859
0.2	75	5.6638	6.0770	6.2059
	80	2.5091	2.5778	2.5962
	85	0.7323	0.6822	0.7149
0.3	75	6.2751	6.6897	6.7920
	80	3.4410	3.5462	3.5826
	85	1.5578	1.5139	1.5879
0.4	75	7.0168	7.4468	7.5406
	80	4.3709	4.5126	4.5753
	85	2.4390	2.4152	2.5386
0.5	75	7.8108	8.2638	8.3666
	80	5.2920	5.4704	5.5694
	85	3.3379	3.3405	3.5214

The above table shows approximate values for both methodologies. The discrete Asian call values explicitly indicates the impact of the discrete time intervals. The Asian call values of $n = 16$ are slightly higher than those of $n = 4$. Across the various sizes of the volatility, it is evident that high volatility results in higher values of both discrete and arithmetic Asian calls.

Chapter 6

Conclusion

In this dissertation, various method of pricing European discrete Asian Options have been reviewed. We identified three major contributions towards pricing of these options. Firstly, the simplicity of the Black-Scholes pricing formula for pricing European Asian options makes it advantageous for use in private practice. The two main terms that are composed in the formulae are the exponential term and the cumulative distribution function of the standard normal variable (see reference [8]). As with the Black-Scholes formula, they are quite convenient to use. Secondly, the efficiency of control variate techniques is also examined when pricing European Asian Options. The variance reduction technique revealed an improvement of accuracy of the Monte Carlo method. The pricing performance of simulation is considerably enhanced by the judicious choice of the control variate. We have also observed the closed form geometric average rate formula by Kemna and Vorst as the control variate. With all these pricing methods, the trade-off is between accuracy, speed and simplicity. Finally, the method from the idea of Bouaziz et al. (1994) and Tsao et al (2003) in which the Taylor expansion is used to obtain the approximation formula for pricing average strike Asian options is discussed in detail. In order to obtain its distribution, the second-order Taylor expansion is used to average the price under the risk neutral pricing framework.

The literature on pricing of Asian options is vast and yet it is still expanding. With all the different methods available, the practitioner makes selection depending on the need for speed, accuracy or simplicity.

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