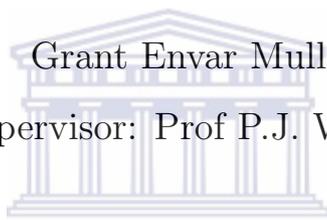


Discrete and continuous time methods of optimization  
in pension fund management.



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A thesis submitted in partial fulfilment of the requirements  
for a Master of Science degree,  
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# Declaration

I declare that

*Discrete and continuous time methods of optimization in pension fund management* is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.



Grant Envar Muller

February, 2010

Signed: .....

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# Abstract

Pensions are essentially the only source of income for many retired workers. It is thus critical that the pension fund manager chooses the right type of plan for his/her workers. Every pension scheme follows its own set of rules when calculating the benefits of the fund's members at retirement. Whichever plan the manager chooses for the members, he/she will have to invest their contributions in the financial market. The manager is therefore faced with the daunting task of selecting the most appropriate investment strategy as to maximize the returns from the financial assets. Due to the volatile nature of stock markets, some pension companies have attached minimum guarantees to pension contracts. These guarantees come at a price, but ensure that the member does not suffer a loss due to poorly performing equities.

In this thesis we study four types of mathematical problems in pension fund management, of which three are essentially optimization problems. Firstly, following Blake [5], we show in a discrete time setting how to decompose a pension benefit into a combination of European options. We also model the pension plan preferences of workers, sponsors and fund managers. We make a number of contributions additional to the paper by Blake [5]. In particular, we contribute graphic illustrations of the expected values of the pension fund assets, liabilities and the actuarial surplus processes. In more detail than in the original source, we derive the variance of the assets of a defined benefit pension plan. Secondly, we dedicate Chapter 6 to the problem of minimizing the cost of a minimum guarantee included in defined contribution (DC) pension contracts. Here we work in discrete time and consider multi-period guarantees similar to those in Hipp [25]. This entire chapter

is original work. Using a standard optimization method, we propose a strategy that calculates an optimal sequence of guarantees that minimizes the sum of the squares of the present value of the total price of the guarantee. Graphic illustrations are included to indicate the minimum value and corresponding optimal sequence of guarantees. Thirdly, we derive an optimal investment strategy for a defined contribution fund with three financial assets in the presence of a minimum guarantee. We work in a continuous time setting and in particular contribute simulations of the dynamics of the short interest rate process and the assets in the financial market of Deelstra et al. [19]. We also derive an optimal investment strategy of the surplus process introduced in Deelstra et al. [19]. The results regarding the surplus are then converted to consider the actual investment portfolio pertaining to the wealth of the fund. We note that the aforementioned paper does not use optimal control theory. In order to illustrate the method of stochastic optimal control, we study a fourth problem by including a discussion of the paper by Devolder et al. [21] in Chapter 3. We enhance the work in the latter paper by including some simulations. The specific portfolio management strategies are applicable to banking as well (and is being pursued independently).

**JEL Classification:** G11; G23; C61

**2000 AMS Subject Classification:** 91B28

**Keywords:** Pension fund; Defined benefit; Defined contribution; Targeted money purchase; Minimum guarantee; Short-rate process; European options; Lagrangian; Risky asset; Optimal investment strategy

# List of Acronyms

Defined Benefit (DB)

Defined Contribution (DC)

Targeted Money Purchase (TMP)

Cox-Ingersoll-Ross (CIR)

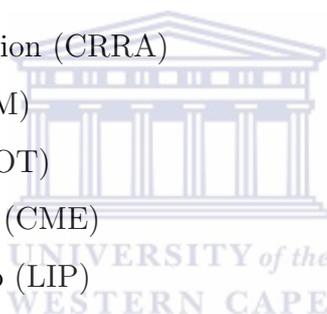
Constant Relevant Risk Aversion (CRRA)

Asset-Liability Modeling (ALM)

Chicago Board Of Trade (CBOT)

Chicago Mercantile Exchange (CME)

Liability Immunizing Portfolio (LIP)



# List of Notations

## Chapter 5

$t$  : Time

$P(t)$  : Price of a European put option at time  $t$ ;

$C(t)$  : Price of a European call option at time  $t$ ;

$L$  : Exercise/strike price of a European option;

$F(t)$  : Expected value of the accumulated financial assets at time;

$X(t)$  : Expected discounted value of the remaining contributions until retirement;

$Y(0)$  : Starting income of the pension fund member;

$\gamma$  : Contribution rate into the scheme;

$g_Y$  : Expected growth rate in income;

$r_{F(t)}$  : Expected yields of the assets bought with the contributions at time  $t$ ;

$\tau$  : Rate of tax relief on the contributions;

$T$  : Number of years of pensionable service;

$Z$  : Expected pension at retirement;

$g_Z$  : Expected growth rate in the pension;

$p(t)$  : One-year survival probability at time  $t$ ;

$A(t)$  : Expected member's pension assets at time  $t$ ;

$L(t)$  : Expected liabilities at time  $t$ ;

$D_{F(t)}$  : Duration of the financial assets;

$D_{X(t)}$  : Duration of the remaining contributions;

$\sigma_{A(t)}^2$  : Variance of the pension assets;

$\sigma_r$  : Standard deviation of the rates of change in the yields on financial assets;  
 $\sigma_g$  : Standard deviation of the rates of change in the earnings;  
 $\eta$  : Specific risk on the return on the pension assets;  
 $g_I$  : Inflation;  
 $g_E$  : Dividend growth;  
 $\sigma_{AL(t)}$  : Covariance of the return processes of the assets and liabilities;  
 $\eta_{AL}$  : Covariance between the specific risks on assets and liabilities returns;

## Chapter 6

$c(t)$  : Periodic contributions to the fund;  
 $g(t)$  : Guarantee associated with periodic premiums;  
 $\rho(t)$  : Premium of the associated guarantee;  
 $V$  : Sum of the squares of the present value of the total price of the guarantee;  
 $S(t)$  : Value of an investment at time  $t$ ;  
 $G(t)$  : Partial sums of the guarantee increments at time  $t$ ;  
 $C(t)$  : Expected value of the member's portfolio at time  $t$ ;  
 $P(t)$  : Price of the guarantee at time  $t$ ;

## Chapters 7 and 8

$Z(t)$  and  $Z_r(t)$  : One dimensional Brownian motions;  
 $S_0(t)$  : Price of a riskless asset at time  $t$ ;  
 $r(t)$  : Short-rate interest process at time  $t$ ;  
 $S(t)$  : Price of a risky asset at time  $t$ ;  
 $B(t, T)$  : Price of a zero-coupon bond with maturity  $T$  at time  $t$ ;  
 $H(t)$  : Optimal growth portfolio;  
 $G(T)$  : Minimum guarantee;

$W(t)$  : Wealth of the pension fund at time  $t$ ;

$v(y)$  : CRRA utility function;

$\mathcal{A}$  : Set of admissible controls;

$Y(t)$  : Surplus process of the fund;

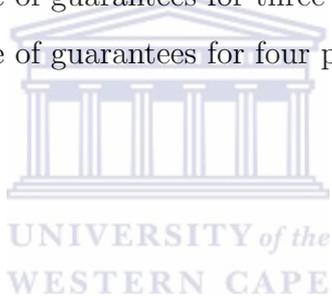
$y^B(t)$  : Proportion of the surplus process invested in the zero-coupon bond;

$y^S(t)$  : Proportion of the surplus process invested in the risky asset



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# Chapter 1

## Introduction and scope of the thesis

The first decision the manager or employer has to make when setting up a pension scheme is which type of plan will be offered to the employees. There exist two main types of pension schemes. They are defined benefit (DB) and defined contribution (DC) pension schemes, but targeted money purchase (TMP) schemes have also become popular as of late. The aforementioned pension schemes adhere to different sets of rules in calculating the benefits for the workers at retirement. We briefly describe each of the three schemes.

The benefits of an employee holding a DB contract are fixed in advance by the manager or sponsor [1]. A DB scheme calculates the benefit in relation to factors such as final salary, length of pensionable service and the age of the member [5]. Contributions made by the DB scheme member are initially set but adjusted in order to maintain the fund's balance [1]. According to Deelstra et al. [20] only the contributions are considered random variables since the benefit at retirement is fixed. Risk associated with a DB scheme is assumed by the sponsor agent [30]. In contrast to DB schemes, DC plans calculate its member's benefits by using the full value of the fund's assets to determine the amount of pension the member will receive, which might be high or low depending on the success of the fund manager [5]. Contributions for the aforementioned pension plan are fixed and the member is exposed to market risk since the benefit depends on the performance of the asset portfolio (see [1] and [30]). Blake [5] described TMP plans as pension schemes

that employ DC plans in order to target a particular pension at retirement. The pension at retirement might be the same as that resulting from a final salary scheme, but also benefits from any upside potential in the value of the fund assets above that required to deliver this target level. In other words, the TMP scheme aims to provide a minimum pension but not a maximum pension, see [5] for instance.

A class of problems that have been studied frequently and have become popular among researchers of pension funds is the problem of optimal pension fund management. The attention received by these problems is justified by the fact that pension funds have over the years become one of the most important institutions in financial markets. This is a result of huge amounts of funds invested by the pension fund managers on behalf of their clients and the fact that pension funds assist the government by serving as a form of income for retired workers. Some of the contributions made by researchers on the optimal management of pension funds we can mention, are on the optimal investment strategies in pension schemes by Vigna and Haberman [49], Deelstra et al. [19], managing risk in pension funds by Josa-Fombellida and Rincon-Zapatero [30], the optimal form of the guarantee for a pension scheme by Deelstra et al. [20] and the optimal management of a pension to maintain solvency in the fund by Petersen et al. [44]. We can also mention Deelstra et al. [18] who derived a more general optimal investment strategy that can be applied in pension fund management.

The scope of this thesis is as follows. There are two preliminary chapters and four main chapters. In the preliminaries, all relevant concepts from finance and probability and stochastic theory are covered. In Section 3 of Chapter 3 (the first of the preliminary chapters) we introduce and apply stochastic optimal control theory to the asset allocation problem in pension funds. We particularly derive an optimal strategy for investing in the risky assets by presenting the power utility case from the paper by Devolder et al. [21]. We contribute graphic illustrations of the evolution equation of the fund assets. The fourth chapter (second preliminary chapter) is not very mathematical, but serves as

a brief introduction to financial derivatives.

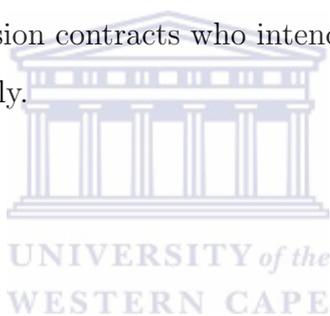
The first of the main chapters (Chapter 5) focus on the composition of pension schemes in a discrete time setting. In the second of these chapters (Chapter 6) we apply optimization theory to a DC pension scheme with embedded minimum guarantees in a discrete time setting, with the aim of minimizing the cost of the guarantees. This is followed by two chapters (Chapters 7 and 8) which address the investment strategy of a pension fund manager responsible for investing the member's contributions into a market of three assets in continuous time.

In particular Chapter 5 presents the main ideas of the paper by Blake [5], who showed that different pension schemes are related through a combination of put and call options. This chapter also includes a brief description how the author of [5] modeled the preferences of the pension scheme members and managers.

In Chapter 6 we present new work on minimizing the premiums paid by the member of a defined contribution pension plan on the guarantees included in the contract. This is achieved by pricing the premiums as the difference between two European put options at times  $t$  and  $t - 1$  respectively. We apply the method of Lagrange multipliers in order to find an optimal sequence of guarantees that minimizes the sum of the squares of the present value of the total price of the guarantee for multi-period contracts. Graphic illustrations and numeric computations are presented as part of our solution to the problem at hand.

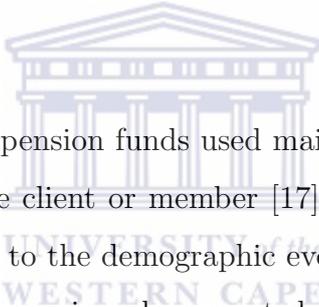
Chapter 7 is an introduction of the model of Deelstra et al. [19] in which the financial market of the pension scheme is discussed. In this chapter we introduce the dynamics of the assets of the pension fund. In Chapter 8 we introduce the optimization problem which is then transformed to a relatively easily solvable problem. This ultimately leads to the optimal strategy for investing in the three assets.

We wish to note the explicit contributions of the author to the development presented in this thesis. The entire Chapter 6 is original. In Chapter 5 we contribute graphic illustrations of the expected values of the fund assets and liabilities as well as the actuarial surplus in [5] over fifteen and twenty years respectively. We also show in more detail how to derive the variance of the returns of the pension assets. The contributions we make in Chapter 7 include finding explicit solutions for the stochastic differential equations that describe the short-rate process, the risky asset and the zero-coupon bond in [19]. We include graphic illustrations for each of the processes over a period of ten years. Chapter 8 includes detailed proofs for the relevant lemmas and theorems in [19] necessary for deriving the optimal investment strategy. We finally simulate the optimal investment strategies for members of pension contracts who intend to retire after twenty and twenty five years of service respectively.



# Chapter 2

## Historical background and literature review



From a historical perspective, pension funds used mainly the DB pension method which is the preferred method by the client or member [17]. New systems have however been introduced in recent years due to the demographic evolution and development of the equity markets [20]. Most of the pension plans created in recent years have been based on DC schemes [7]. The implication is that the equity market risk is transferred to the clients [20]. This is an inconvenience and can be moderated by introducing a minimum guarantee on the future benefit that will be paid out to the pension fund members. The minimum guarantee can be very complex and the question is to find the optimal form that it should take in order to maximize the utility of the client [20]. In addition to the aforementioned optimization problem, there is also the problem of how to manage the pension fund asset portfolio. This entails investing the member's contributions in a combination of assets in order to maximize their benefits at retirement. Numerous authors have applied optimization theory in order to find optimal investment strategies.

One of these authors is Blake [5], who showed that the three principle types of pension schemes (DB, DC and TMP schemes) are related through a set of options on the un-

derlying assets in the fund. In the aforementioned paper, the author also shows how these options are valued and modeled the preferences of the scheme members, sponsors and the pension fund managers. This is very important, since from the member's point of view each of the schemes have different costs, different expected returns and different risks. The author of [5] finally concentrated on a DB pension scheme and derived an optimal strategy to manage the pension fund assets. This strategy was derived on the assumption that the surplus risk has to be minimized subject to the constraint that the surplus is zero on the date of maturity and never below zero prior to the maturity date.

In their paper, Vigna and Haberman [49] considered DC pension schemes where the member's contributions are invested into only two assets with different levels of risk. Of the two assets, one is a high-risk asset and the other a low-risk asset, for which the amount invested into the assets change every year. The authors of [49] studied the two components of risk in the fund, namely the investment risk borne by the member during the accumulation phase and the annuity risk for the annuity bought at retirement. The results from their study suggest that the lifestyle strategy is appropriate for reducing investment risk. In their paper the authors found an optimal investment strategy for the members of such a DC pension scheme by using dynamic programming techniques that have been used in the literature.

Boulier et al. [7] focused on the pension fund management issue in a frictionless and continuously open market with no arbitrage. The authors of [7] considered DC pension schemes in a continuous time framework in which the contributions are invested into three assets. These assets are stocks, equities and bonds, but for simplicity only one equity asset was considered which represents the index of the stock market. For the aforementioned DC pension scheme, the authors found an optimal strategy for investing in the assets. Their results are modified to the strategic process of the fund. Not only does the wealth investment in the assets change with time, but also with the short interest rate and the value of the contributor's savings account. This allows for a practical tool to

be implemented in order to aid a pension fund manager in selecting the composition of his portfolio. An important feature of the model is that with deterministic interest rates the cash and bonds assets are theoretically equivalent. This means that it is senseless to find an asset allocation between the two assets. When the interest rate is stochastic, the interest rate risk management is easier. This is due to the fact that with every movement of the rate the manager knows exactly how to react and balance his portfolio.

Deelstra et al. [18] studied an optimal investment problem for a financial market containing three assets which can be bought and sold without incurring restrictions such as short sales or trading costs. The three assets are a riskless asset, a zero-coupon bond and a risky asset, and the short-rate interest rates in the market follows the Cox-Ingersoll-Ross (CIR) [15] dynamics. The optimal investment strategy was found by transforming the optimization problem into a less complicated and easily solvable problem, using the Cox-Huang methodology [14]. In the aforementioned paper, the authors also analyzed the behavior of the solution near the time of maturity. They found that when the risk aversion is high, their solution suggests that the proportion of wealth of the fund invested in the zero-coupon bond should be invested in the cash asset as time elapses, while the investment in the risky asset remains unchanged. For the case of a lower risk aversion the proportion invested in the zero-coupon bond should increase since the cash asset is considered risky in the stochastic interest rates framework. In this case the risk premium of the bond should be preferred in order to improve the performance of the cash asset.

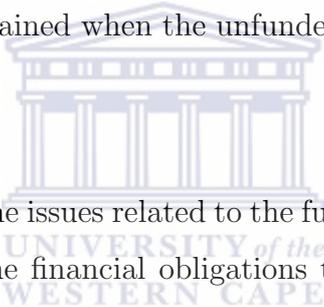
In Deelstra et al. [19] an explicit optimal strategy is obtained for a DC pension scheme with a minimum guarantee. The authors worked in a continuous time framework and assumed that the interest rate follows the dynamics of Duffie and Khan [22] in its one-dimensional version. This includes as special cases the CIR and Vasiček models (see [15] and [48]). The financial market considered in Deelstra et al. [19] consists of three assets, which are a riskless asset, a stock and a zero-coupon bond. The assets can be bought or sold continuously without incurring any restriction as short sales constraints or trading

costs. The optimization problem was to find an optimal strategy of investing the initial wealth and contribution flow into the three assets in the market in order to maximize the expected utility of the terminal wealth and include a minimum guarantee at expiration. The authors reduced the original optimization problem to a simple investment problem and solved it by introducing an auxiliary process (surplus process) on the assumption that the utility function is a Constant Relevant Risk Aversion (CRRA) function. This was followed by solving the original problem by specifying the contribution process and the guarantee.

In a discrete time setting, Huang and Cairns [26] studied a DB pension plan. In [26] the authors propose a model that makes it possible to determine the appropriate contribution rate for a DB pension plan where the interest rates are stochastic and rate of returns random. This is done by extending previous research by introducing a Vasiček model for short-term interest rates that helps to control the contribution rate volatility. The authors of [26] assumed that the pension fund has three assets in which the pension fund member's contributions are invested. The pension fund assets consist of a one-year bond (cash), a long dated bond and an equity-asset. By modeling three assets instead of one, the authors were able to compare different investment strategies for the assets. Following this approach, the authors provided illustrative examples that indicate that the adjustment to the contribution rate given in the aforementioned paper and taking into account different interest rates actually improve stability. This is particularly the case when there is a strong degree of persistence in interest rates. The aforementioned authors learned from this approach that the standard approach to liability valuation using an artificial valuation interest rate can be improved by making an appropriate adjustment for market conditions.

In Josa-Fombellida and Rincon-Zapatero [30] a DB pension plan similar to the one in Haberman and Sung [24], Boulier et al. [8], Cairns [11] and Josa-Fombellida and Rincon-Zapatero [28] and Josa-Fombellida and Rincon-Zapatero [29] is studied. In the latter

papers the pension scheme manager is faced with the responsibility of keeping the fund assets close to the actuarial liability as possible. In Josa-Fombellida and Rincon-Zapatero [30] the authors looked at three different objectives for the sponsor. First the sponsor wishes to maximize the probability that the fund assets achieve a fixed target value before a predetermined insolvency value. The second aim of the sponsor is to minimize the expected discounted penalization cost associated with an high value of the unfunded actuarial liability. The third aim is to maximize some utility function along an infinite horizon when the date of termination of the funding process is random and has an exponential distribution. The main result of [30] is that the optimal investment decisions in all three cases are proportional to the difference between the liabilities and the fund assets, known as the unfunded actuarial liability. This means that the optimal policies require more risk, which is obtained when the unfunded actuarial liability assumes larger values as it approaches zero.



Petersen et al. [44] attend to the issues related to the funding of a continuous time pension that accumulate capital for the financial obligations to be paid in future. Under these circumstances there exists a pension fund that holds assets and possibly has contractual or semi-contractual liabilities. The management of the pension fund under consideration involves minimizing the contributions supporting the fund and to maintain a reasonable level of solvency. The authors of the above mentioned paper applied stochastic optimal control theory in order to determine how the pension fund solvency is influenced by the contributions paid by the pension fund members and different asset allocation strategies. The authors of [44] accomplished this by deriving a stochastic model for the dynamics of the associated asset-liability ratio that could be used to assess the solvency level of the fund. From this a nonlinear optimal stochastic control problem originated, which was solved analytically using methods linked to dynamic programming and asset-liability modeling (ALM). The above mentioned authors also introduced an asset-liability reference process with respect to which the optimal sequence of member contributions and investment policy are characterized.

In Devolder et al. [21] it is shown how stochastic optimal control theory can be applied to find an optimal investment policy before and after retirement for a defined contribution pension plan. The benefits of the plan are paid under the form of annuities which are guaranteed during a certain fixed period of time. The financial market in [21] consists of two assets. The authors of [21] particularly investigated the problem of finding the optimal investment strategy for the assets backing the pension liabilities during the whole life of the participant in the plan. During the activity period of the contract, the contributions of the member are invested in either the risky or riskless asset. The reserve obtained at retirement is the amount accumulated without any special guarantee given by the insurer. The insurer uses the guarantee to buy a paid-up annuity at retirement. He/she is responsible for paying the annuity and faced with the decision of how much of the mathematical reserve should be invested in the two assets. Devolder et al. [21] split the problem into two periods due to the presence of the liability only at retirement. In the first period, i.e., the period before retirement without liability, the authors optimize the utility of the final wealth at retirement. In the second period (period after retirement) the problem was to maximize the expected utility of the final surplus after payment of pension during  $T$  periods. In both periods two forms of utility functions (power law utility and exponential utility) were used. The main results of this paper can be summarized as follow. In the power law case the optimal policy after retirement the surplus instead of the total assets should be invested, which confirms that the portion invested in the risky asset should decrease. In the exponential utility case after retirement there is no such effect, but in the case of periodic contributions before retirement the same kind of effects were established: no effect for the exponential utility case and the opposite effect for power law utility leading to a natural "lifestyle investment strategy" (see [21]).

# Chapter 3

## Preliminaries

In this chapter some basic concepts from finance, probability and measure theory are introduced. We provide relevant definitions and theorems that are used in the chapters that follow. Our main references on such basics and in particular are Tuckman [47], Nielsen [42], Musiela and Rutkowski [41], Etheridge [23] and Baz and Chacko [2]. The final section of this chapter introduces an optimal control problem from the paper by Devolder et al. [21].

### 3.1 Concepts from finance

**Definition 3.1.1:** (*Duration of a financial security*) (see [47, p98]) The duration of a financial security, which is a measure of its interest rate sensitivity, measures the percentage change in the value of the security for a unit change in interest rates. If we let  $D$  denote duration, then mathematically duration is written as

$$D \equiv -\frac{1}{P} \frac{\Delta P}{\Delta y},$$

where  $\Delta P$  and  $\Delta y$  denote the change in price of the security and change in interest respectively. If an explicit formula for the price-rate function of a financial security is available, the derivative of the price-rate function may be used for the change in price

divided by the change in rate. We then write the duration as

$$D = -\frac{1}{P} \frac{dP}{dy}.$$

Next we define the concept of elasticity. It is a very basic concept and widely applied in economics. We nevertheless provide its formal definition.

**Definition 3.1.2:** (*Elasticity*) An elasticity measures a specific form of responsiveness: the percentage change in one variable that accompanies a one percent change in another variable. The elasticity between two variables is thus the ratio of percentage changes in the variables (see Klein [32, p183]). Mathematically elasticity can be defined as follows. If  $f$  is a differentiable function at  $x$  and  $f(x) \neq 0$ , then according to Sydsaeter and Hammond [45, p239] the elasticity of  $f$  with respect to  $x$  is

$$E_x f = \frac{\frac{df(x)}{f(x)}}{\frac{dx}{x}} = \frac{x}{f(x)} f'(x).$$

**Definition 3.1.3:** (*Zero-coupon bond*) (see [41, p265]) Let  $T > 0$  be a fixed maturity date for all market activities. A zero-coupon bond (or discount bond) of maturity  $T$  is a financial security paying one unit of cash to its holder at a specified date  $T$  in the future. By convention this means that the bond's principal (face or nominal) value at time  $T$  is one unit of currency. It will be assumed throughout that the bonds are default free, i.e. the possibility of default by the bond's issuer is excluded.

The price of a zero-coupon bond with maturity  $T$  at any time  $t \leq T$ , denoted  $B(t, T)$  has the property  $B(T, T) = 1$  for any maturity date  $T^*$ . Since there are no other payments to the holder, in practice a discount bond sells for less than the principal before maturity—i.e., at a discount. This is because one could carry cash at virtually no cost, and thus would have no incentive to invest in a discount bond costing more than its face value. It will be assumed that for any fixed maturity  $T \leq T^*$ , the bond price follows a strictly positive and adapted process on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .

**Remark 3.1.3:** ( see Levy [34, p8]) We now explain the difference between *riskless* and *risky* assets. The future value of a riskless asset can only assume one value, say  $x$ , with probability  $\mathbb{P}[x] = 1$ . The future value of a risky asset may assume more than one value, say  $x_i$ , with probability  $0 < \mathbb{P}[x_i] < 1$ .

## 3.2 Concepts from probability and measure theory

**Definition 3.2.1:** ( *$\sigma$ -algebra*) (see [42, p317]) Let  $\Omega$  be any non-empty set. A  $\sigma$ -algebra or  $\sigma$ -field on  $\Omega$  is a class  $\mathcal{F}$  of subsets of  $\Omega$  with the following three properties:

1.  $\Omega \in \mathcal{F}$ .
2. If  $\{A(t)\}$  is a finite or infinite sequence of sets in  $\mathcal{F}$ , then  $\bigcup A(t) \in \mathcal{F}$ .
3. If  $A \in \mathcal{F}$  then  $A^c \in \mathcal{F}$ .

**Definition 3.2.2:** (*Filtration*) (see [42, p14]) A filtration is a family  $\{\mathcal{F}(t)\}_{t \in J}$  of  $\sigma$ -algebras  $\mathcal{F}(t) \subset \mathcal{F}$  which is increasing in the sense that whenever  $s, t \in J$  and  $s \leq t$   $\mathcal{F}(s) \subset \mathcal{F}(t)$  .

**Definition 3.2.3:** (*Probability triple*) (see [23, p29]) A probability triple  $(\Omega, \mathcal{F}, \mathbb{P})$ , consists of a set  $\Omega$  (sample space), a collection of subsets  $\mathcal{F}$  of  $\Omega$  (events) and a probability measure  $\mathbb{P}$ , which specifies the probability of each event  $A \in \mathcal{F}$ . The collection  $\mathcal{F}$  is assumed closed under the operations of countable union and taking complements ( $\sigma$ -field).  $\mathbb{P}$  must of course satisfy the following axioms:

1.  $0 \leq \mathbb{P}(A) \leq 1$  for all  $A \in \mathcal{F}$ ;
2.  $\mathbb{P}[\Omega] = 1$ ;
3.  $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$  for any disjoint  $A$  and  $B$  in  $\mathcal{F}$ ;
4. If  $A(n) \in \mathcal{F}$  for all  $n \in \mathbb{N}$  and  $A(1) \subseteq A(2) \subseteq \dots$  then  $\mathbb{P}[A(n)] \uparrow \mathbb{P}[\bigcup_n A(n)]$  as  $n \uparrow \infty$ .

**Definition 3.2.4:** (*Stochastic process*) (see [42, p2]) Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and let  $J$  be a time interval. Specifically, assume that  $J = [0, \infty)$  or  $J = [0, T]$  for some  $T$ . A  $k$ -dimensional stochastic process is a mapping  $X : \Omega \times J \rightarrow \mathbb{R}^k$  such that for each fixed  $t \in J$ , the mapping

$$X(t) : \omega \mapsto X(\omega, t) = X(t)(\omega) : \Omega \rightarrow \mathbb{R}^k$$

is measurable. A stochastic process is said to be *adapted* to a filtration  $\{\mathcal{F}(t)\}_{t \in J}$  if for each  $t \in J$ , the random variable or vector resulting from the latter mapping is measurable with respect to  $\mathcal{F}(t)$ . This means that the value  $X(t)$  of  $X$  at  $t$  depends only on information available at time  $t$ .

**Definition 3.2.5:** (*Standard Brownian motion*) (see [42, p5]) A  $k$ -dimensional standard Brownian motion is a  $k$ -dimensional process  $\{W(t)\}_{t \geq 0}$  such that

1.  $W(0) = 0$  with probability one.
2.  $W$  is continuous.
3. if  $0 \leq t_0 \leq \dots \leq t_n$ , then the increments  $W(t_1) - W(t_0), \dots, W(t_n) - W(t_{n-1})$  are independent.
4. if  $0 \leq s \leq t$ , then the increment  $W(t) - W(s)$  is normally distributed with mean zero and covariance matrix  $(t - s)I$ , where  $I$  is the  $k \times k$  identity matrix.

If  $W$  is a one-dimensional standard Brownian motion, and if  $0 \leq s \leq t$ , then the increment  $W(t) - W(s)$  is normally distributed with mean zero and variance  $t - s$ . A one-dimensional process is called a geometric Brownian motion if it has the form  $e^Z$ , where  $Z$  is a one-dimensional generalized Brownian motion with constant initial value  $Z(0)$ .

**Definition 3.2.6:** (*Martingale*) (see [42, p16]) Let  $\{\mathcal{F}(t)\}_{t \geq 0}$  be a filtration. A process  $X$  is a martingale if it is integrable and adapted and whenever  $s, t \in J$  and  $0 \leq s \leq t$

$$\mathbb{E}[X(t) | \mathcal{F}(s)] = X(s).$$

**Remark 3.2.7:** (see [23, p79]) Suppose that  $f$  is a simple function; then

1. the process  $\int_0^t f(s, \omega) dW(s)$  is a continuous  $(\mathbb{P}, \{\mathcal{F}(t)\}_{t \geq 0})$ -martingale.

2. 
$$\mathbb{E} \left[ \left( \int_0^t f(s, \omega) dW(s) \right)^2 \right] = \int_0^t \mathbb{E}[f(s, \omega)^2] ds.$$

The process  $\int_0^t f(s, \omega) dW(s)$  in Remark 3.2.7. is called an Itô integral and in addition to being continuous is also adapted to the filtration  $\mathcal{F}(t)$  (see [42]).

**Remark 3.2.8:** (see [23, p93]) Consider stochastic processes  $Y(t)$  and  $Z(t)$  satisfying the SDE's:

$$dY(t) = \mu(t, Y(t))dt + \sigma(t, Y(t))dW(t)$$

and

$$dZ(t) = \mu(t, Z(t))dt + \sigma(t, Z(t))dW(t).$$

Then we obtain two  $(\mathbb{P}, \{\mathcal{F}(t)\}_{t \geq 0})$ -martingales  $\{M^Y(t)\}_{t \geq 0}$  and  $\{M^Z(t)\}_{t \geq 0}$  defined by

$$M^Y(t) = \int_0^t \sigma_1(s, Y(s)) dW(s)$$

and

$$M^Z(t) = \int_0^t \sigma_2(s, Z(s)) dW(s)$$

with associated quadratic variation processes

$$[M^Y(t)] = \int_0^t \sigma_1^2(s, Y(s)) ds$$

and

$$[M^Z(t)] = \int_0^t \sigma_2^2(s, Z(s)) ds.$$

Then

$$Cov_t(M^Y, M^Z) = \int_0^t \sigma_1(s, Y(s)) \sigma_2(s, Z(s)) ds.$$

**Remark 3.2.9:** (*Itô's formula*) (see [23, p85]) Suppose that  $H(T)$  represents the set of functions for which  $f: \mathbb{R}_+ \rightarrow \mathbb{R}$  for which  $f(t, \phi)$  is  $\{\mathcal{F}(t)\}_{t \geq 0}$ -predictable for  $0 \leq t \leq T$  and

$$\int_0^T \mathbb{E} \left[ f(s, \phi)^2 \right] ds < \infty.$$

Then for  $f$  such that the partial derivatives  $\frac{\partial f}{\partial t}$ ,  $\frac{\partial f}{\partial x}$  and  $\frac{\partial^2 f}{\partial x^2}$  exist and are continuous, as well as  $\frac{\partial f}{\partial x} \in H(T)$  for each  $t$ ,

$$f(t, W(t)) - f(0, W(0)) = \int_0^t \frac{\partial f}{\partial x}(s, W(s)) dW(s) + \int_0^t \frac{\partial f}{\partial s}(s, W(s)) ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, W(s)) ds.$$

Sometimes the following more general version of Itô's formula is required:

$$\begin{aligned} f(t, S(t)) - f(0, S(0)) &= \int_0^t \frac{\partial f}{\partial u}(u, S(u)) du + \int_0^t \frac{\partial f}{\partial x}(u, S(u)) dS(u) \\ &+ \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(u, S(u)) \sigma^2 S^2(u) du \\ &= \int_0^t \frac{\partial f}{\partial u}(u, S(u)) du + \int_0^t \frac{\partial f}{\partial x}(u, S(u)) \sigma S(u) dW(u) \\ &+ \int_0^t \frac{\partial f}{\partial x}(u, S(u)) \mu S(u) du + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(u, S(u)) \sigma^2 S^2(u) du. \end{aligned}$$

In differential notation Itô's formula is given by

$$df(t, W(t)) = f_s(t, W(t)) dW(t) + f_t(t, W(t)) dt + \frac{1}{2} f_{ss}(t, W(t)) dt.$$

**Remark 3.2.10:** ( see [42, p107]) Let  $a$ ,  $b$ , and  $\sigma$  be constants, and let  $r_0$  be a random variable which is measurable with respect to  $\mathcal{F}(0)$ . The process that solves the equation

$$dr(t) = a(b - r(t))dt + \sigma dW(t), \text{ with initial value } r(0),$$

is given by

$$r(t) = e^{-at} r(0) + (1 - e^{-at})b + \sigma e^{-at} \int_0^t e^{au} dW(u).$$

The above mentioned process, with  $a > 0$ , is called an Ornstein-Uhlenbeck process. It is often used in finance to model the dynamics of interest rates. Under the assumption

that  $a > 0$ , the drift term  $a(b - r)$  is positive whenever  $b < r$ , and more so if  $a$  is large. The implication is that the process has the tendency to drift toward  $b$ , known as its *mean reversion level*. The parameter  $a$  is interpreted as the speed of adjustment.

### 3.3 Stochastic control and asset allocation

In this section we present an example of an optimal control problem related to pension funds. We first introduce the following theorem from the book by Baz and Chacko [2] that will be used in the optimal control example that follows.

**Theorem 3.3.1:** *The Hamilton-Jacobi-Bellman equation (hereafter the HJB-equation) of optimal control for Itô processes for the optimization problem*

$$J(X, 0) = \max_y \mathbb{E} \left[ \int_0^T f(X, t, y) dt + B(X(T), T) \middle| \mathcal{F}(0) \right]$$

subject to the constraints

$$dX = \mu(X, t, y)dt + \sigma(X, t, y)dW$$

and with  $X(0)$  fixed, is given by

$$-\frac{\partial J(X, t)}{\partial t} = \max_y \left\{ f(X, t, y) + \frac{\partial J(X, t)}{\partial(X, t, y)} \mu(X, t, y) + \frac{1}{2} \frac{\partial^2 J(X, t)}{\partial X^2(t)} \sigma^2(X, t, y) \right\}.$$

This is a partial differential equation with boundary condition

$$J(X(T), T) = B(X(T), T) \text{ (see [2, p247]).}$$

The variable  $y$  is generally called the control variable. □

In the example that follows we employ Theorem 3.3.1 to derive an optimal investment strategy for particularly investing in the risky assets of pension funds. When making investments into such assets, the investment manager must select the best portfolio. This

means that given the various possible investment choices with different risk levels and expected returns, the best combination must be chosen. The paper of Merton [38] presents a very useful method of portfolio selection or asset allocation. The stochastic control method of Merton [38] has been applied extensively in this regard. We present the asset allocation problem from the paper of Devolder et al. [21]. The problem and its solution is presented very briefly. The aim is to show the main features of a very fundamental method in financial modeling, and to run a simulation. We shall not include a long description of the model.

**Example 3.3.2:** We present the power utility case of the problem addressed in the paper of Devolder et al. [21]. Let  $F(t)$  be the total value of the investment at time  $t$ , and let  $u(t)$  be the fraction of  $F(t)$  which is invested in the risky asset at time  $t$ . The particular expected utility we consider is, for some  $\gamma < 0$ :

$$J(u, F) = \mathbb{E}\left[-\frac{1}{\gamma} F(T)^\gamma | \mathcal{F}(0)\right].$$

We need to find the value  $u = u^*$  which maximizes  $J$ , so that

$$J(u^*, F) = \max_{\{u\}} \mathbb{E}\left[-\frac{1}{\gamma} F(T)^\gamma | \mathcal{F}(0)\right].$$

We introduce the value function

$$W(t, F(\cdot)) = \max_{\{u\}} \left[-\frac{1}{\gamma} F(T)^\gamma | \mathcal{F}(t)\right].$$

Now solving this problem entails maximizing an expression in the HJB-equation arising from the original problem in [21]:

$$0 = \max_{\{u\}} \left[ \frac{\partial W}{\partial t} + [u(t)(\alpha - r) + r]F \frac{\partial W}{\partial F} + \frac{1}{2} u^2(t) \sigma^2 F^2 \frac{\partial^2 W}{\partial F^2} \right].$$

We compute the relevant derivative with respect to  $u$  and determine the critical values.

The derivative is

$$0 = (\alpha - r)F \frac{\partial W}{\partial F} + u^*(t)F^2 \sigma^2 \frac{\partial^2 W}{\partial F^2}.$$

Then

$$u^*(t) = -\frac{\partial W/\partial F}{F(\partial^2 W/\partial F^2)} \frac{\alpha - r}{\sigma^2}.$$

An explicit expression for  $W$  is obtained by inspection as in Baz and Chacko [2]. We note that the usual way of solving for such a  $W$  is via the Feynman-Kac stochastic representation theorem (see [23, p103]). Eventually we find the optimal portfolio fraction  $u^*(t)$  :

$$u^*(t) = -\frac{b(t)F^{\gamma-1}}{Fb(t)(\gamma-1)F^{\gamma-2}} \frac{\alpha - r}{\sigma^2}$$

which simplifies to

$$u^*(t) = \frac{\alpha - r}{\sigma^2} \frac{1}{1 - \gamma}.$$

Therefore the amount of the portfolio which is invested in the risky asset at time  $t$  is

$$u^*(t)F(t) = \frac{\alpha - r}{\sigma^2} \frac{1}{1 - \gamma} F(t),$$

where

$$F(t) = F(0) \exp \left[ \left( u(t)\alpha + (1 - u(t))r - \frac{1}{2}u^2(t)\sigma^2 \right) t + u(t)\sigma w(t) \right] \text{ and}$$

$w(t)$  is a standard Brownian motion.

We now simulate the latter equation, i.e. the evolution equation  $F(t)$  of the fund assets, for constant volatility  $\sigma$  over periods of twenty and thirty years respectively. The short-rate process,  $r$ , is assumed to be an Ornstein-Uhlenbeck process known as the Vasiček short rate model (see Remark 3.2.10).

The parameters used in the simulations are as follows. The parameters for the short-rate process in Fig.3.1. are  $a = 0.01$ ,  $b = 0.03$ , which appear in the drift term, the volatility is  $\eta = 0.01$  and the initial value of the short rate process is  $r(0) = 0.03$ . The parameters that make up the drift term and volatility of the fund assets are  $\alpha = 0.07$ ,  $\sigma = 0.3$  respectively,

$\gamma = -1$  and the initial value of the fund assets is assumed  $F(0) = 1$ .

For Fig.3.2. these parameters are  $a = 0.01$ ,  $b = 0.03$ ,  $\eta = 0.01$ ,  $r(0) = 0.07$ ,  $\alpha = 0.08$ ,  $\sigma = 0.6$ ,  $\gamma = -1$ . For this simulation we also consider an initial value  $F(0) = 1$  for the pension fund assets.

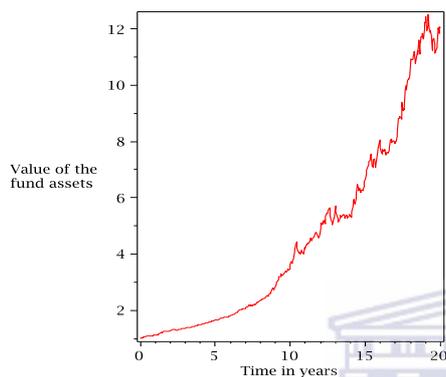


Figure 3.1: A simulation of the fund as-  
sets over  $T = 20$  years.

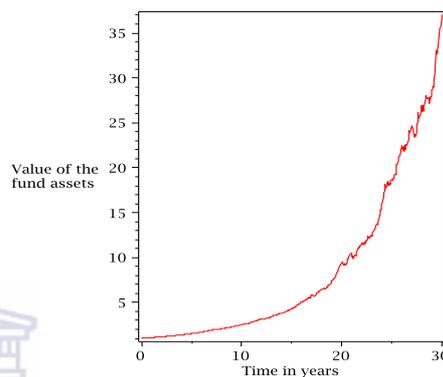
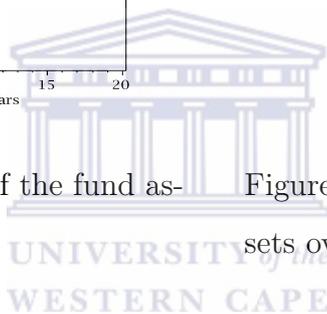
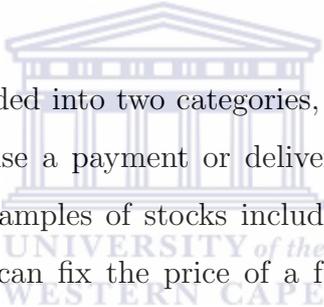


Figure 3.2: A simulation of the fund as-  
sets over  $T = 30$  years.



# Chapter 4

## Preliminaries on forward, futures and option contracts



Financial markets can be divided into two categories, namely the underlying stocks and their derivatives, which promise a payment or delivery at a future date, based on the performance of the stock. Examples of stocks include foreign currencies, commodities, equities and bonds. Traders can fix the price of a future transaction by using one of the derivatives, resulting in either an increase in the risk of a loss on the transaction, or reducing the risk [23]. Examples of such derivatives are forward contracts, futures contracts and options contracts. We briefly discuss the first two types of derivatives, which will be followed by a more detailed discussion of European options contracts, since they are the most applicable form of financial security to the relevant parts of this thesis.

### 4.1 Forward and futures contracts

A *forward contract* is an agreement to buy or sell an asset for a specified price, say  $K$  and at a certain date in the future known as the maturity date  $T$ . The buyer is said to be the holder of the long position, while the seller holds the short position (see Etheridge [23]). Forward contracts are free (no cost) to enter and the payoff for the holder with a long position in a futures contract is given by  $B = S(T) - K$ . Here  $S(t)$  denotes the time  $t$

price of the underlying commodity. The payoff for the short position, on the other hand, is given by  $B = K - S(T)$ .

A *futures contract* is an agreement to buy or sell an asset at a certain time in the future for a certain price known as the futures price. The investor who has agreed to sell holds the short futures position, while the investor who has agreed to buy holds the long futures position (see Hull [27]). The only difference between futures and forward contracts is that futures are traded on exchanges like the Chicago Board of Trade (CBOT) and Chicago Mercantile Exchange (CME) (see Hull [27]). These exchanges specify certain features of the contract and particular form of settlement (see Etheridge [23]).

## 4.2 European options

Options contracts have become popular with investors, even though they have not been traded quite as long as futures contracts. An *option contract* gives its holder either the right to purchase or the right to sell an asset. In particular, a *European call* option gives its holder the right, but not the obligation to buy an asset for a specified price,  $K$ , at a specified date,  $T$ . The European put option is the right, but not the obligation to sell an asset at time  $T$  for the price  $K$ . *European* options can only be exercised on the date of expiration, whereas *American* options can be exercised at any time on the interval  $[0, T]$ , i.e. the time from the purchase of the option to its maturity (see [23].)

The payoff of European options depends on what it is worth at expiration. This feature can be explained as follows. Suppose that an investor holds a European call option on a stock. If at expiration of the option the actual price of the underlying stock is  $S(T)$  and  $S(T)$  is greater than the strike price  $K$ , the option will be exercised and the stock will be purchased for the strike price  $K$ . If on the other hand,  $S(T)$  is less than  $K$ , the investor will choose not to exercise the option and purchase the stock on the open market,

since the option is worthless. Hence the payoff of a European call option is

$$(S(T) - K)^+ = \max(S(T) - K, 0).$$

Now suppose that a trader holds a European put option to sell an underlying stock. He will exercise the option if at expiration the stock price  $S(T)$  is less than  $K$  to sell the underlying stock for  $K$ . In the case of  $K$  being greater than  $S(T)$  at maturity, the trader will not exercise the option as it the option will be worthless. The payoff of a European put option is

$$(K - S(T))^+ = \max(K - S(T), 0).$$

An option is said to be at the money if  $S(T) = K$ . (The intrinsic value of an option is defined as the maximum of zero and the value it would have if it were exercised immediately. For a European call this is  $\max(S(T) - K, 0)$  and for the European put  $\max(K - S(T), 0)$ .)

#### 4.2.1 The Black-Scholes model

We will now briefly look at the pricing formulas proposed by Black and Scholes [3] for pricing European call and put options. Their options pricing model is based on the assumption that stock prices follow the phenomenon known as random walk. This means that proportional changes that occur in the stock price have a normal distribution. This implies that the price of the stock at any time in the future follows a lognormal distribution. Unlike normally distributed variables that can take positive or negative values, only positive values can be assumed by lognormal variables. Where normal variables are symmetrical, lognormal variables have skew distributions and their mean, modes and medians are different from each other (see [27]).

The two most important factors influencing the behavior of a stock is its expected rate of return and its volatility. The expected rate of return of a stock, denoted  $\mu$ , is the annual average return earned by investors and depends on the riskiness of the stock as well as the level of interest rate in the economy. Risky assets have high expected rate of

returns and high levels of interest rate in the economy will result in a high expected rate of returns. The volatility,  $\sigma$ , measures the uncertainty in stock price movements and is usually expressed as a percentage. Since the underlying stock in the Black-Scholes model has a lognormal distribution, it follows that its natural logarithm is normally distributed, which implies that  $\ln S(T)$  is normally distributed and has mean  $\ln S(T) + (\mu - \frac{\sigma^2}{2})$  and standard deviation  $\sigma\sqrt{T}$ .

In order to derive the prices for European call and put options on an underlying asset with a lognormal distribution, Black and Scholes [3] had to assume the following:

1. The behavior of the price of the stock corresponds to the lognormal model with constant drift terms and volatility.
2. There are no transaction costs or taxes and that all securities are perfectly divisible.
3. There are no dividends on the stock during the life of the option.
4. No riskless arbitrage opportunities exist.
5. Security trading is continuous.
6. Investors can borrow or lend at the same risk-free interest rate.
7. The short-term risk-free rate of interest  $r$  is constant.

They derived the formulas for the prices of the European call and put options on non-dividend paying stocks respectively as

$$C(t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_1) \quad (4.1)$$

and

$$P(t) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_2) \quad (4.2)$$

where

$$d_1 = \frac{\ln(\frac{S(t)}{K}) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}$$

and

$$\begin{aligned}d_2 &= \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}} \\ &= d_1 - \sigma\sqrt{T - t}.\end{aligned}$$

The function  $N(x)$  is the cumulative probability function for a standardized normal variable,  $C$  and  $P$  are the European call and put prices. The strike price of the options are given by  $K$ , the stock price by  $S(t)$  and  $T$  represents the maturity date of the option [27].

## 4.2.2 Factors influencing option prices

In this section, we calculate the prices of European call and put options using Eqs. (4.1) and (4.2). We consider the scenarios where the underlying stock has stock price 48 units currency and assume that the two options have maturity  $T = 1$  year. We also discuss the factors that influence the prices of these options.

In the table that follows, the prices of European call and put options are calculated for parameters  $r = 0.07$ ,  $\sigma = 0.06$ , a stock price of 48 units currency and different values of the strike price  $K$ . Notice how the price of a European call option becomes cheaper (the put option becomes more expensive) as the strike price  $K$  increases, whereas the put option becomes cheaper (the call option becomes more expensive) as  $K$  decreases.

The primary determining factors of option values are the current stock price  $S(t)$ , the strike price  $K$ , the date of expiration  $T$ , the stock price volatility  $\sigma$ , and the dividend expected during the life of the option. These factors are summarized for European call options in Table 4.2 from which the following can be noted: (see Das [16])

1. A higher premium value goes hand in hand with an increase in  $S(t)$ , which means a higher intrinsic value if the call option was in the money. If it was out-of-the-money, the higher the chance of it being out-of-the-money. For put options the reverse holds.

Table 4.1: The effect of the strike price on European options.

$r$	$\sigma$	$S(T)$	$K$	$C$	$P$
0.07	0.06	48	50	1.95	0.57
			49	2.64	0.32
			48	3.41	0.17
			47	4.25	0.08
			46	5.14	0.03

2. The intrinsic value of a call option decreases with an increase in the strike price  $K$  while the intrinsic value of a put option increases.
3. The greater the time until expiration of an option, the greater the probability that it will be exercised profitably, and hence the time value of the option will be greater.
4. The option becomes more valuable if the volatility of the stock price is high, since the probability that the option will be exercised profitably is high.
5. Dividend payments lead to lower current stock prices which increase the chance of call options to be out-of-the-money. This in turn decreases the value of the option. For the put option, an increase in the stock price increases the chance of the put to be in-the-money.
6. The higher the interest rate, the lower the present value of the exercise price the call buyer has agreed to pay in the event of the option being exercised. Since the call option is the right to buy the underlying asset at a discounted value of the exercise price, the right becomes more valuable with higher degrees of discount. The value of the put option decreases with an increase in the interest rate.

Table 4.2: Factors influencing call option prices.

Factor	Call option
Strike price, $K$	Decrease
Spot price, $S(0)$	Increase
Interest rate, $r$	Increase
Time to maturity, $T$	Increase
Volatility, $\sigma$	Increase
Dividend, $D$	Decrease



# Chapter 5

## Pension schemes as options

In this chapter we show how Blake [5] explained the relationship between the three pension scheme types. As our contribution, we derive in more detail, using the concept of elasticity, the variance of the pension fund's assets and simulate for a DB pension fund the expected value of a member's pension assets, the expected value of the fund's liabilities and the expected actuarial surplus process.

Blake [5] described a funded pension scheme as the composition of a pension fund plus a pension annuity. Defined contribution plans use the full value of the fund's assets to determine the amount of pension which depends on the pension fund manager's success, whereas the benefits from defined benefit plans are calculated considering the age, salary and length of service of the member. According to Blake, TMP schemes use DC schemes to target a particular pension at retirement, but take advantage of any upside potential in the value of the fund assets above that required target level. TMP schemes thus provide minimum pension and not a maximum pension. He examined the relationship between three different schemes, namely DC, DB and TMP plans, in terms of the differing sets of options implicit in their structure. We examine the structures of DC and DB pension schemes in terms of the different put and call combinations used in their structures, as well as review the pricing methods of the options as proposed in the aforementioned paper.

## 5.1 Blake's pension schemes decomposition

In the aforementioned paper it was found that the present value of a DC plan at retirement depends only on the value of the fund assets on that particular date, but contrary to DC plans, the present value of a DB plan is independent of the value of the fund's assets. According to Blake [5], DB pension plans can be replicated using a long put ( $P$ ) and a short call ( $-C$ ), both with exercise price ( $L$ ) on the underlying assets of the fund, say  $A$ . The put option is held by the scheme member and written by the sponsor. The call option, on the other hand, is regarded as written by the member and held by the sponsor. The expiry date of the options is the same as the date of retirement of the member, on which one of the options are certain to be exercised.

The decision of which option will be exercised depends on the stock price at that particular moment. If the value of the assets drops below the exercise price of the option, the scheme shows an actuarial deficit, upon which the scheme member will surely exercise the put option against the sponsor, who will then make a deficiency payment, say  $(L - A)$ . On the other hand, the rise of the assets value above the exercise price of the options will result in an actuarial surplus so the sponsor will exercise the call option against the member to recover the surplus  $(A - L)$ . From the two aforementioned scenarios it is clear that a DB member is not exposed to asset market risk and Blake recognized that DB and DC plans are somehow related. The relationship is due to the fact that a DC plan is invested only in the underlying assets, whereas a DB scheme is invested in a portfolio containing the underlying assets plus a put option minus a call option on the underlying assets. This relationship can be formalized as follows:

$$\begin{aligned} DB &= L \\ &= A + P - C \\ &= DC + P - C. \end{aligned} \tag{5.1}$$

Blake [5] also showed that the TMP pension scheme can be replicated using a long put

option, also called a protective put option,  $P$  on the underlying assets of the fund,  $A$ , with exercise price  $L$ . The put option is held by the scheme member and written by the scheme sponsor and at retirement of the scheme member, the option will be exercised if the value of the assets is less than the exercise price. The effect of the option is to place a floor on the value of the pension received by the member. The present value of the TMP was calculated to be the larger of the two present values provided by the DC and DB schemes, and Blake expressed them as:

$$\begin{aligned}
 TMP &= A + P \\
 &= \max(A, L) \\
 &= \max(DC, DB) \\
 &= C + L.
 \end{aligned} \tag{5.2}$$

From Eq.(5.2) the aforementioned author concluded that a TMP scheme is equivalent to the member holding a call option (or floor) on the underlying pension fund assets with an exercise price  $L$  and a riskless pure discount bond with a maturity value of  $L$ , where the call option will be exercised at maturity if  $A$  exceeds  $L$ .

## 5.2 Pricing the underlying options of the pension schemes

At first Blake [5] concentrated on DB plans and made the following assumptions:

1. The conditioning date is the start-up date of the scheme ( $t = 0$ ), such that values dated ahead of  $t = 0$  are expected values conditioned on information available at  $t = 0$ .
2. The expected value of a scheme member's pension assets at any date  $t$  will equal the expected value of the accumulated financial assets,  $F(t)$ , plus the expected discounted value of the remaining contributions until the retirement date,  $X(t)$ .

3. The aforementioned quantities depend on starting income of the member,  $Y(0)$ , the contribution rate as a proportion,  $\gamma$  of income into the scheme, the expected future growth rate in income, say  $g_Y$ , assumed to be consistent over the whole period, the expected yields on the investments in the financial assets purchased with the contributions,  $r_{F(t)}$ , the rate of tax relief on the contributions,  $\tau$ , the number of years of pensionable service ( $T$ ) and the one-year survival probabilities from date  $t = 0$ , say  $p(t)$ .
4. The appropriate discount rates used to discount the remaining contributions are the expected returns on financial assets held during the relevant period. The expected value of the member's pension assets at any date  $t \in \mathbb{N}$  is given by:

$$\begin{aligned}
A(t) &= F(t) + X(t) \\
&= \sum_{k=1}^t \frac{p(k)\gamma Y(0)(1+g_Y)^{k-1}}{1-\tau} \prod_{j=k+1}^t (1+r_{F(j)}) \\
&\quad + \sum_{k=t+1}^T \frac{p(k)\gamma Y(0)(1+g_Y)^{k-1}}{(1-\tau) \prod_{j=t+1}^k (1+r_{F(j)})}, \quad \text{for } t = 1, \dots, T. \quad (5.3)
\end{aligned}$$

Eq.(5.3) is consistent with the notes of Tepper [46]. However according to [5], other economists have used the after-tax rate of return on corporate bonds which is more in line with the accountancy profession in which the use of the yield on long-term government bonds are recommended. In this equation, terms associated with  $j > t$  are set to unity since it is assumed that all cash flows arise at the end of the relevant period. Blake [5] considered DB schemes, where the liabilities at retirement depend on the expected pension at retirement ( $Z$ ), the expected growth rate in the pension,  $g_Z$ , and the one-year survival probabilities in retirement ( $p(T+t)$ ). This can be considered as the probability that the fund member survives a period of  $t$  years after retirement. Suppose that the retirement pension,  $Z$  is equal to some proportion, say  $\theta$ , of the expected income at retirement ( $Y_0, (1+g_Y)^{T-1}$ ). The expected value of the liabilities at any date  $t$  is expressed as

$$L(t) = \sum_{k=1}^{\infty} p(T+k) Z \left[ \frac{1+g_Z}{1+r_B} \right]^k \left[ \frac{1}{\prod_{j=t+1}^T (1+r_{F(j)})} \right], \quad \text{for } t = 1, \dots, T. \quad (5.4)$$

The discount rates from the retirement date onwards are the discount rates,  $r_B$  on government bonds with maturity of approximately fifteen years. This is based on the assumption that these type of bonds are used to finance annuities. The discount rates from the retirement date back to date  $t$  and are the same as those rates used to discount projected contributions. In other words the rates are the same as the expected returns on the financial assets in the pension fund (see [5]).

The difference between (5.3) and (5.4) represents the actuarial surplus with a DB scheme:

$$S(t) = A(t) - L(t), \quad \text{for } t = 1, \dots, T. \quad (5.5)$$

We proceed to draw two graphs for Eqs. (5.3), (5.4), and (5.5). In Fig.5.1. we simulate the dynamics of the three processes for a pension fund member with length of service  $T = 15$  years, whereas Fig.5.2. represents the dynamics of the processes for a member with length of service  $T = 20$  years.

The simulations are based on the assumption that a proportion ( $\gamma$ ) of the member's starting salary is invested into the pension fund assets at fixed times  $t = 0 \dots T - 1$ . The value of assets are assumed to follow a geometric Brownian motion process, which we denote by  $S(t)$ . Each investment is expected to pay a dividend of  $\delta S(t)$  one year after the investment has been made. In order to simplify the simulation, we assume that the dividend is not invested into the fund assets. We particularly assume that the values of the assets follow

$$S(t) = S(0)(1 - \delta)^{n(t)} \exp(\mu t + \sigma W(t)),$$

where  $n(t)$  is the number of dividend payments made by time  $t$ ,  $\mu$  is the drift term and  $\sigma$  the volatility of the asset (see [23, p130]). We calculate the expected yield ( $r_{F(t)}$ ) of the investments by taking the sum of the individual yields of the investments at time  $t$ .

In both figures we have chosen the starting wealth of the member as  $Y(0) = 1$  and the proportion of the starting salary invested in the assets remain constant at  $\gamma = 0.1$ . The

top line represents the expected value of the member's assets ( $A(t)$ ), the middle line the expected liabilities ( $L(t)$ ) and the bottom line the actuarial surplus ( $S(t)$ ).

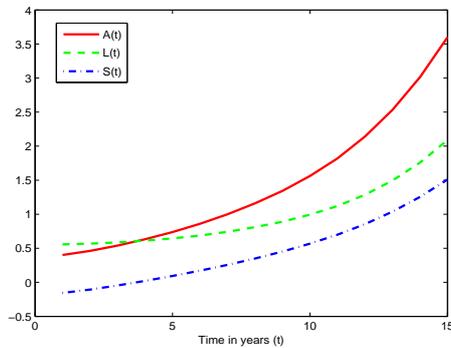


Figure 5.1: A simulation of the expected values of the assets, liabilities and the surplus processes of a DB scheme with retirement date  $T = 15$  years.

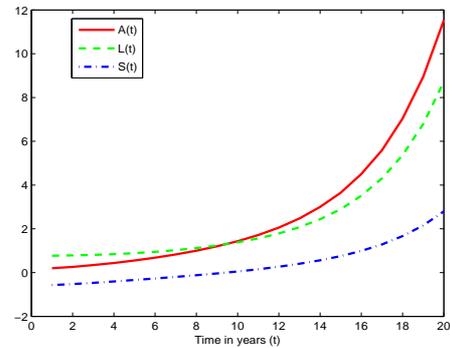


Figure 5.2: A simulation of the expected values of the assets, liabilities and the surplus processes of a DB scheme with retirement date  $T = 20$  years.

According to Blake [5], there is no actuarial surplus in the case of a DC scheme, since in the case of a DC scheme (5.3) represents the present value of both the assets and liabilities, but in the case of a TMP, the liabilities are the larger of Eqs.(5.3) and (5.4). As is the case with a DC scheme there is no surplus for a TMP scheme.

The options embedded in the DB and TMP schemes have the following properties:

1. Since the options can only be exercised at maturity they are European.
2. The underlying asset does not make payouts prior to the expiry date of the option.
3. The exercise price varies unlike in the Black-Scholes model and equals the value of the liability.

The appropriate model for pricing the options is based on the modification of the Black-Scholes [3] framework which recognizes that the options in (5.1) and (5.3) are exchange

options that are options to exchange risky assets at an exercise price that is indexed to the uncertainty in the value of the liabilities. The call option in Eq.(5.1) is given by

$$C(t) = N(h(t))A(t) - N(k(t))L(t), \quad (5.6)$$

where

$$h(t) = \frac{\ln\left(\frac{A(t)}{L(t)}\right) + \frac{1}{2}\sigma_{S(t)}^2(T-t)}{\sigma_{S(t)}\sqrt{T-t}}, \quad (5.7)$$

$$k(t) = h(t) - \sigma_{S(t)}\sqrt{T-t}, \quad (5.8)$$

and  $N$  is the cumulative normal distribution function

$$N(\alpha) = \int_{-\infty}^{\alpha} (\sqrt{2\pi})^{-1} e^{-z^2/2} dz. \quad (5.9)$$

The variance of the surplus (or the square of the surplus risk) (Eq.(5.5)) depends on the standard deviations of the rate of returns of the assets and liabilities as well as the covariance between the two and is

$$\sigma_{S(t)}^2 = \sigma_{A(t)}^2 + \sigma_{L(t)}^2 - 2\sigma_{AL(t)}. \quad (5.10)$$

Blake [5] considered the most appropriate way of modeling the components of Eq.(5.10), which appeared to involve identifying sources of variability common to both assets and liabilities, involving substantial simplifying assumptions. The assumptions were not realistic and needed a more realistic and consequently more complex framework. This was overcome by taking the assumption that the key sources of volatility facing the assets and liabilities are the volatilities attached to the interest and growth rates. Furthermore the volatilities are assumed to be scaled by the different durations of the assets and liabilities based on the work of Macaulay [36]. Blake assumed that there would be specific components to the asset and liability volatilities.

From Eq.(5.3) Blake [5] recognized that the volatility of the rate of change in the value of the pension assets depends on their duration. The duration in turn, equals the weighted sum of the durations of the existing financial assets ( $D_{F(t)}$ ) and of the remaining contributions ( $D_{X(t)}$ )

$$\begin{aligned} D_{A(t)} &= \alpha(t)D_{F(t)} + (1 - \alpha(t))D_{X(t)} \\ &= \alpha(t)D_{F(t)} + (1 - \alpha(t)) \left[ \sum_{k=t+1}^T \frac{(k-t)p(k)\gamma Y_0(1+g_Y)^{k-1}}{(1-\tau)X(t) \prod_{j=t+1}^k (1+r_{F(j)})} \right], \end{aligned} \quad (5.11)$$

for  $t = 1, \dots, T$ .

Here

$$\alpha(t) = \frac{F(t)}{F(t) + X(t)}$$

is the weight of the existing financial assets in total pension assets at time  $t$ . The standard deviations of the rates of change in the yields on financial assets and the rate of change in the earnings are respectively given by  $\sigma_r$  and  $\sigma_g$ . As a first-order linear approximation, Blake found that the variance of the pension asset returns is

$$\sigma_{A(t)}^2 = D_{A(t)}^2(\sigma_r^2 + \sigma_g^2) + \eta_A^2, \quad \text{for } t = 1, \dots, T. \quad (5.12)$$

Here  $\eta_A$  represents the specific risk on the return on the pension assets. For simplicity he took the following assumptions that are unlikely to hold in the real world:

1. Financial asset returns and growth rates are uncorrelated.
2. The standard deviation of the rate of change in financial assets remains constant over time.
3. The standard deviations of earnings growth ( $g_Y$ ), pension growth ( $g_Z$ ), and later inflation ( $g_I$ ) as well as the dividend growth ( $g_E$ ) all remain constant over time and are equal. This implies that when the four growth rates differ, they differ by constant amounts.

The duration of the financial assets, which is a measure of an asset's sensitivity to interest rates, equals the value-weighted sum of the durations of the individual assets in the portfolio (see Blake [4]). According to Blake [5] the duration measure is highly sensitive to the underlying model of the term structure of interest rates used. In fact, this was shown by Boyle [9]. The author of [9] also stated that models with parallel yield curve shifts result in long-term assets having substantially greater durations than models with mean reversions.

We now show, using the concept of elasticity, how Blake [5] derived Eq.(5.12).

**Theorem 5.1:** *Suppose that in Eq.(5.3) the  $r_{F(t)}$  are expected to be constant over time and that the  $p(k)$  are fixed at  $p$ . If we define*

$$d = \frac{p\gamma Y(0)}{1 - \tau}$$

and write  $A(0)$  at time  $t = 0$  as :

$$A(0) = X(0) = \sum_{k=1}^T \frac{d(1 + g_Y)^k}{(1 + r_F)^k},$$

then the volatility of the rate of return process is given by

$$\sigma_{A(0)} = (\sigma_r^2 + \sigma_g^2) + \eta_A^2.$$

*Proof.* The elasticity of  $A(0)$  with respect to  $(1 + g_Y)$  is given by

$$E_1 A(0) = E_1 X(0) = \frac{(1 + g_Y)X(0)'}{X(0)} = \frac{(1 + g_Y)}{X(0)} \frac{\partial X(0)}{\partial (1 + g_Y)}.$$

But

$$\begin{aligned} A(0) &= X(0) \\ &= \frac{d(1 + g_Y)}{(1 + r_F)} + \frac{d(1 + g_Y)^2}{(1 + r_F)^2} + \frac{d(1 + g_Y)^3}{(1 + r_F)^3} + \dots + \frac{d(1 + g_Y)^T}{(1 + r_F)^T}. \end{aligned}$$

Consequently we have the following explicit form for  $E_1A(0)$ :

$$\begin{aligned}
E_1A(0) &= E_1X(0) \\
&= \frac{(1+g_Y)}{A(0)} \frac{\partial}{\partial(1+g_Y)} \left[ \frac{d(1+g_Y)}{(1+r_F)} + \frac{d(1+g_Y)^2}{(1+r_F)^2} + \frac{d(1+g_Y)^3}{(1+r_F)^3} + \dots + \frac{d(1+g_Y)^T}{(1+r_F)^T} \right] \\
&= \frac{(1+g_Y)}{A(0)} \left[ \frac{d}{(1+r_F)} + \frac{2d(1+g_Y)}{(1+r_F)^2} + \frac{3d(1+g_Y)^2}{(1+r_F)^3} + \dots + \frac{Td(1+g_Y)^{T-1}}{(1+r_F)^T} \right] \\
&= \frac{(1+g_Y)}{A(0)} \sum_{k=1}^T \frac{kd(1+g_Y)^{k-1}}{(1+r_F)^k} \\
&= \frac{1}{A(0)} \sum_{k=1}^T \frac{kd(1+g_Y)^k}{(1+r_F)^k} \\
&\equiv D_{A(0)}.
\end{aligned}$$

Similarly, the elasticity of  $A(0)$  with respect to  $(1+r_F)$  is given by

$$E_2A(0) = E_2X(0) = \frac{(1+r_F)X(0)'}{X_0} = \frac{(1+r_F)}{X(0)} \frac{\partial X(0)}{\partial(1+r_F)}.$$

From the expansion of  $A(0) = X(0)$  above, it is clear that

$$\begin{aligned}
E_2A(0) &= E_2X(0) \\
&= \frac{(1+r_F)}{A(0)} \frac{\partial}{\partial(1+r_F)} \left[ \frac{d(1+g_Y)}{(1+r_F)} + \frac{d(1+g_Y)^2}{(1+r_F)^2} + \frac{d(1+g_Y)^3}{(1+r_F)^3} + \dots + \frac{d(1+g_Y)^T}{(1+r_F)^T} \right] \\
&= \frac{(1+r_F)}{A(0)} \left[ -\frac{d(1+g_Y)}{(1+r_F)^2} - \frac{2d(1+g_Y)^2}{(1+r_F)^3} - \frac{3d(1+g_Y)^3}{(1+r_F)^4} - \dots - \frac{Td(1+g_Y)^T}{(1+r_F)^{T+1}} \right] \\
&= -\frac{(1+r_F)}{A(0)} \sum_{k=1}^T \frac{kd(1+g_Y)^k}{(1+r_F)^{k+1}} \\
&= -\frac{1}{A(0)} \sum_{k=1}^T \frac{kd(1+g_Y)^k}{(1+r_F)^k} \\
&= -D_{A(0)}.
\end{aligned}$$

The total differential of  $A(0)$  satisfies the equation:

$$\begin{aligned}
\frac{dA(0)}{A(0)} &= \frac{\partial A(0)}{\partial(1+g_Y)} \frac{(1+g_Y)}{A(0)} \frac{dg_Y}{(1+g_Y)} + \frac{\partial A(0)}{\partial(1+r_F)} \frac{(1+r_F)}{A(0)} \frac{dr_F}{(1+r_F)} + \epsilon_A \\
&= -D_{A(0)} \left[ -\frac{dg_Y}{(1+g_Y)} + \frac{dr_F}{(1+r_F)} \right] + \epsilon_A.
\end{aligned}$$

Here  $\epsilon_A$  is a serially and contemporaneously uncorrelated specific risk component to the rate of return on the pension assets. It follows that the volatility of the rate of return on pension assets is

$$\sigma_{A(0)}^2 = D_{A(0)}^2(\sigma_r^2 + \sigma_g^2) + \eta_A^2.$$

The variables  $\sigma_r^2$  and  $\sigma_g^2$  are the volatilities of the rates of change in interest rates and growth rates respectively.

□

Using the work of Langetieg et al. [33], Blake [5] found that the volatility of the rate of change in the value of the pension liabilities also depend on their duration, where the duration is given by:

$$D_{L(t)} = \sum_{k=1}^{\infty} \frac{kp(T+k)}{L(T)} \left[ \frac{1+gZ}{1+r_f} \right] + (T-t), \quad \text{for } t = 1, \dots, T. \quad (5.13)$$

Blake [5] again used a first-order linear approximation and expressed the variance of the liability returns as

$$\sigma_{L(t)}^2 = D_{L(t)}^2(\sigma_r^2 + \sigma_g^2) + \eta_L^2, \quad \text{for } t = 1, \dots, T. \quad (5.14)$$

The standard deviation of the growth rate in pension is  $\sigma_g$  and the specific risk on liability returns is given by  $\eta_L$ . The covariance between the return processes of the assets and liabilities was also derived. Blake found it to be

$$\sigma_{AL(t)} = D_{A(t)}D_{L(t)}(\sigma_r^2 + \sigma_g^2) + \eta_{AL}, \quad \text{for } t = 1, \dots, T, \quad (5.15)$$

where  $\eta_{AL}$  is the covariance between the specific risks on assets and liability returns.

Blake [5] derived the value of the put options in Eq.(5.1) from the put-call parity by using Eqs.(5.5) and (5.6) as follows.

$$P(t) = L(t) - A(t) + C(t).$$

Thus

$$\begin{aligned}
P(t) &= C(t) - (A(t) - L(t)) \\
&= [A(t)N(h(t)) - L(t)N(k(t))] - (A(t) - L(t)) \\
&= A(t)N(h(t)) - A(t) + L(t) - L(t)N(k(t)) \\
&= A(t)[N(h(t)) - 1] + L(t)[1 - N(k(t))] \\
&= L(t)[1 - N(k(t))] - A(t)[1 - N(h(t))]. \tag{5.16}
\end{aligned}$$

Blake [5] stated that the call and put option values in Eqs.(5.6) and (5.16) do not depend explicitly on the riskless rate of interest as in the standard Black-Scholes model. This is because pension liabilities provide a natural hedge for the pension assets against both interest rate and growth rate risks. He explained that the appropriate manner to define the risk is not the risk given by Eq.(5.12), attached to the pension assets (Eq.5.3), but the risk given by Eq.(5.10) attached to the pension surplus.

The reason for the first feature comes from the Black-Scholes innovation of constructing a riskless hedge portfolio. This can be achieved by hedging against changes in both the value of the underlying assets and the exercise price. By holding the assets, the changes in asset values can be hedged. Blake [5] found that the cost of the hedge equals the rate of return on the assets, but since the assets form part of the portfolio, the return from the portfolio counteracts the cost of the hedge against changes in the asset values. For this reason the rate of return of the assets is not included in the option pricing theory. Only the riskless rate of interest appears in the standard Black-Scholes formula, since the hedge portfolio is riskless and generates no risky rate of return. The author of [5] explained that changes in the exercise price are hedged by holding in the portfolio assets whose returns are perfectly correlated with changes in the exercise price (i.e. changes in the value of the liabilities), which is achieved by holding a portfolio of assets that exactly tracks any changes in the value of the liabilities embedded in the main portfolio. This portfolio was denoted the liability immunizing portfolio (LIP) and took the assumption that it was indeed possible to construct such a portfolio. The LIP would be a risky component of the portfolio, but

would be riskless relative to the liabilities that it would be immunizing and so it would also generate a riskless rate of return. The rate of return on the hedge portfolio is 0 due to the fact that the return on the liability hedge counteracts the return on the asset hedge.

The second feature is based on the fact that pension asset and liabilities respond to abrupt changes in the same way. The author of [5] considered changes in Eqs.(5.3) and (5.4) and assumed unexpected changes in the yields and growth rates. As an example he found that an unexpected increase in yields reduces present value of both the assets and the liabilities, while an unexpected increase in growth rates will have the opposite effect. The volatilities are common to both assets and liabilities and the differential effects of the volatilities on the variances of asset and liability returns come from the different duration of assets and liabilities, seen in Eqs.(5.12), (5.14) and (5.15). Blake found that the following is obtained if these equations are substituted into Eq.(5.10):

$$\sigma_{S(t)}^2 = (\sigma_r^2 + \sigma_g^2)(D_{A(t)} - D_{L(t)})^2 + \eta_A^2 + \eta_L^2 - 2\eta_{AL}. \quad (5.17)$$

This, according to Blake [5], shows that the volatility of the surplus depends on the squared duration gap between assets and liabilities, the variances of the rate of change in yields and growth rates as well as the relationship between the specific risks on assets and liabilities. In the case of a financial asset portfolio constructed to have returns that are perfectly correlated with changes in the value of the liabilities,

$$\eta_A^2 = \eta_L^2 = \eta_{AL}$$

will hold and the terms involving  $\eta$  in Eq.(5.17) will disappear and the portfolio would be classified as LIP. In addition to the fact that the portfolio is LIP, the surplus risk can be eliminated if the duration of the assets is kept equal to that of the liabilities. A more complex version of Eq.(5.17) would be needed if Blake's simplifying assumptions are not valid, but he reckoned that it would be unlikely that the surplus risk could be completely eliminated in his general framework (see [5]).

### 5.3 Preferences of the members, sponsors and managers

Associated with pension schemes are different costs, different expected returns as well as different risks. Blake [5] used the framework of Merton [38] and took the assumption that a typical member's reward-risk preferences can be represented by an isoelastic utility function with a constant relative risk aversion parameter  $\beta$ , which suggests that the risk-reward indifference curves for the three types of schemes are given as :

$$U_{DB} = g_Y - \frac{1}{2}\beta\sigma_g^2, \quad (5.18)$$

$$U_{DC} = r_{F(t)} - \frac{1}{2}\beta[D_{A(t)}^2(\sigma_r^2 + \sigma_g^2) + \eta_A^2] \quad (5.19)$$

and

$$U_{TMP} = r_{F(t)} - P(t)/L(T-t) - \frac{1}{2}\beta[(D_{A(t)} - D_{L(t)})^2(\sigma_r^2 + \sigma_g^2) + \eta_A^2 + \eta_L^2 - 2\eta_{AL}]. \quad (5.20)$$

Here Eqs. (5.18), (5.19) and (5.20) represent the risk-reward indifference curves of the DB scheme, DC scheme and TMP schemes respectively. Blake [5] found that the lowest expected return was associated with a DB scheme, which was equal to the expected growth rate in the member's earnings, with the volatility of the risk being a measure of the risk. Another result from Blake's analysis is that a DC scheme offers the highest expected return, which equals the expected return of the financial assets, but coupled to this is the highest risk of all three of the pension schemes. The TMP scheme on the other hand has a lower expected return than the DC scheme due to the cost of buying the protective put option, leading to a lower risk.

The author of [5] found that the ranking of the preference depends on the degree of risk aversion as follows:

$$-\infty < \beta \leq \beta_1(t) \Rightarrow U_{DC} > U_{TMP} > U_{DB},$$

$$\beta_1(t) < \beta \leq \beta_2(t) \Rightarrow U_{TMP} > U_{DC} > U_{DB},$$

$$\beta_2(t) < \beta \leq \beta_3(t) \Rightarrow U_{TMP} > U_{DB} > U_{DC},$$

$$\beta_3(t) < \beta \leq \infty \Rightarrow U_{DB} > U_{TMP} > U_{DC}.$$

Here

$$\beta_1(t) = \frac{2P(t)/L(T-t)}{(2D_{A(t)}D_{L(t)} - D_{L(t)}^2)(\sigma_r^2 + \sigma_g^2) + 2\eta_{AL} - \eta_L^2}, \quad (5.21)$$

$$\beta_2(t) = \frac{2(r_{F(t)} - g_Y)}{D_{A(t)}(\sigma_r^2 + \sigma_g^2) + \eta_A^2 - \sigma_g^2} \quad (5.22)$$

and

$$\beta_3(t) = \frac{2(r_{F(t)} - P(t)/L(T-t) - g_Y)}{(D_{A(t)} - D_{L(t)})^2(\sigma_r^2 + \sigma_g^2) + (\eta_A^2 + \eta_L^2 - 2\eta_{AL}) - \sigma_g^2} \quad (5.23)$$

for  $t = 1, \dots, T$ .

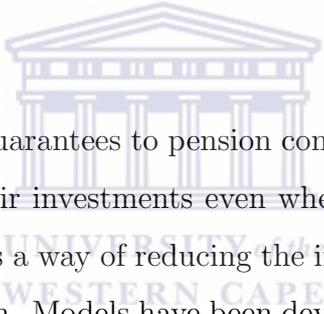
Blake [5] found that  $\beta_2(t)$  and  $\beta_3(t)$  will be positive and that  $\beta_1(t)$  will only assume negative values if the duration of the assets is less than half the value of the liabilities. He also found that DB schemes would be desirable to individuals who are high risk averse, and that substantial risk takers will prefer DC schemes. TMP schemes, on the other hand will be preferred by those individuals who are moderately risk averse and moderately risk takers. If the durations of assets and liabilities are continuously equalized, then Eq.(5.23) shows that the TMP scheme will always be preferred to the DB scheme. Blake [5] described the risk shared between members, sponsors and fund managers. In the case of the DC scheme, all the risk attached to the pension fund assets is borne by the member and none by the sponsor or pension fund manager, but in the long run the sponsor or fund manager could go out of business if the portfolio performance is poor. The DB scheme

member bears no financial risk and the member's benefits are calculated according to a pre-set formula, which is not influenced by the value of the financial assets at retirement. All the risk is borne by the sponsor, but in the event of the portfolio performing better than expected, the sponsor will retain all the upside potential. In the case of a TMP scheme, all the risk is borne by the sponsor, while any upside potential is retained by the scheme member (see [5]).



# Chapter 6

## Minimizing the cost of the guarantee in discrete time



Companies attach minimum guarantees to pension contracts in order to ensure that their members earn a return on their investments even when the investment performs poorly in the financial market. This is a way of reducing the investment risk that the member or contributor faces to a minimum. Models have been developed where the guarantee is only applicable at the maturity of the pension contract, for instance Brennan and Scwhartz [10]. Hipp [25] recognized that the guarantees included in many life insurance contracts are not maturity guarantees. They are in fact multi-period guarantees, which secure a minimum rate of return at the end of each period. Authors such as Persson and Aase [43] and Miltersen and Persson [39] studied two period guarantees, whereas Lindset [35] extended the work of these authors to a case of multi-period guarantees in a setting of stochastic interest rates.

In this chapter we propose an optimal strategy to minimize the premium that the holder of a DC pension contract has to pay for each guarantee associated with regular contributions to the fund. We use a Lagrangian optimization method as can be found in the book by Conrad and Clarke [13], in a discrete time setting. The optimization is done in terms

of utility maximization. We now proceed to the very description of the model and the optimization problem.

## 6.1 Formulating the problem

The member holding the defined contribution pension contract is assumed to pay periodically at times  $t(0), t(1), t(2), \dots, t(n-1)$  a sequence of contributions  $c(0), c(1), c(2), \dots, c(n-1)$ . If we denote the contribution to be made at time  $t(\iota)$  by  $c(\iota)$ , where  $0 \leq \iota \leq n-1$ , then with every contribution there is an associated guarantee increment denoted by  $g(\iota)$ . At time  $t(\iota)$ , the premium  $\rho(\iota)$  for this guarantee increment must be paid by the contributor, and the member thus contributes  $c(\iota) + \rho(\iota)$ , but only  $c(\iota)$  is invested in the portfolio.

We wish to minimize the sum of the squares of the present value of the total price of the guarantee, say  $V$ , by finding an optimal sequence  $g(0), g(1), g(2), \dots, g(n-1)$  of incremental guarantees for a given value of  $\Gamma$ . We assume that the wealth of the pension fund follows the geometric Brownian motion. More precisely if at time  $t(0)$  an amount  $S$  is invested into the fund, then over a period  $t(0) < t < t(1)$  the value  $S(t)$  of the investment evolves as a geometric Brownian motion with constant drift  $\mu$  and constant volatility  $\sigma$ , i.e.,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where  $W(t)$  is the standard Brownian motion. We shall denote the riskless interest rate by  $r$  and assume it to be constant.

Let the partial sums of the guarantee increments be denoted by

$$G(j) = \sum_{\iota=0}^j g(\iota).$$

For  $j = 0, 1, 2, \dots, n-1$ , let  $C(j)$  be the expected value of the member's portfolio at time

$t(j)$ . Then

$$C(j) = \sum_{\iota=0}^j c(\iota) e^{\mu t(\iota)}.$$

We propose a strategy in which the guarantee increment is priced, using the Black-Scholes formula, as the difference between the prices of two European put options, i.e at time  $t(j)$  we sell the option bought at time  $t(j-1)$  with strike price  $G(j-1)$  and buy a new option with strike price  $G(j)$ . The calculation is done at time zero and the expected value of the member's portfolio,  $C(j)$ , is used as the stock price of the options.

The price of the guarantee at time  $t(0)$  is priced, as a European put option as in Brennan and Schwartz [10], as:

$$P(0) = g(0)e^{-rt(n)}N(-k(0)) - c(0)N(-h(0))$$

where

$$h(0) = \frac{\ln\left(\frac{c(0)}{g(0)}\right) + \left(r + \frac{1}{2}\sigma^2\right)t(n)}{\sigma\sqrt{t(n)}}$$

and

$$k(0) = h(0) - \sigma\sqrt{t(n)}.$$

Also,

$$N(-\alpha) = \int_{\infty}^{-\alpha} (\sqrt{2\pi})^{-1} e^{-z^2/2} dz.$$

The incremental premium at time  $t(j)$  ( $j > 0$ ) is priced as the difference between the prices of the options purchased at times  $t(j)$  and  $t(j-1)$  as follows:

$$\begin{aligned} P(j) &= G(j)e^{-r(t(n)-t(j))}N(-k(j)) - C(j)N(-h(j)) \\ &\quad - G(j-1)e^{-r(t(n)-t(j))}N(-\kappa(j-1)) + C(j-1)e^{\mu(t(j)-t(j-1))}N(-\eta(j-1)), \end{aligned}$$

where

$$h(j) = \frac{\ln\left(\frac{C(j)}{G(j)}\right) + \left(r + \frac{1}{2}\sigma^2\right)(t(n) - t(j))}{\sigma\sqrt{t(n) - t(j)}},$$

$$k(j) = h(j) - \sigma\sqrt{t(n) - t(j)},$$

$$\eta(j-1) = \frac{\ln\left(\frac{C(j-1)e^{\mu(t(j)-t(j-1))}}{G(j-1)}\right) + (r + \frac{1}{2}\sigma^2)(t(n) - t(j))}{\sigma\sqrt{t(n) - t(j)}},$$

and

$$\kappa(j-1) = \eta(j-1) - \sigma\sqrt{t(n) - t(j)}.$$

For a start we consider, for argument sake, two period contracts and calculate the prices for different sequences of guarantee increments, which add up to  $\Gamma$ . For simplicity we set the contributions  $c(0) = c(1) = 1.00$  units of currency in the top half of the table and  $c(0) = c(1) = 1.50$  in the bottom half. We use different values for the parameters  $r$  and  $\sigma$  in both cases.



Table 6.1: The optimal sequence of guarantees for two period contracts.

$r$	$\sigma$	$g(0)$	$g(1)$	$\Gamma$	$V$
0.05	0.08	1.00	1.00	2.00	0.04
		1.26	0.74		
0.03	0.06	1.00	1.00	2.00	0.01
		1.16	0.84		
0.05	0.08	1.50	1.50	3.00	0.09
		1.89	1.11		
0.03	0.06	1.50	1.50	3.00	0.0003
		1.33	1.67		

We now proceed to draw the graph of  $V$  as a function of  $g(0)$  by assuming  $r = 0.05$  and  $\sigma = 0.07$ . In Fig.6.1.  $\mu = 0.05$  and  $\Gamma = 215$ . In Fig.6.2. we have chosen  $\mu = 0.04$  and  $\Gamma = 210$ . Note that  $g(0)$  is regarded as an independent variable and that  $g(1)$  is determined by the condition  $g(1) = V - g(0)$ .

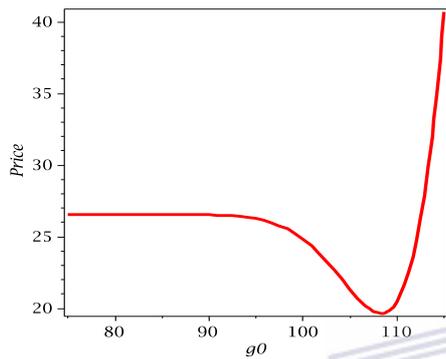


Figure 6.1: A simulation of the price of the guarantee for a two period contract with  $\mu = 0.05$ .

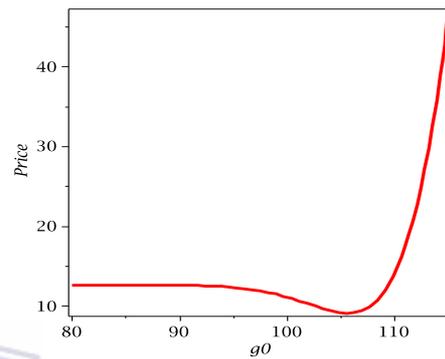


Figure 6.2: A simulation of the price of the guarantee for a two period contract with  $\mu = 0.04$ .

From Fig.6.1. and Fig.6.2. it can be noted that there is a minimum value for  $V$ . The minimum value  $V^*$  of  $V$  and the associated guarantee values  $g(0)^*$  and  $g(1)^*$  for the two figures can be read off the graphs. These values are approximately  $g(0)^* = 108$  and  $V^* = 19.5$  in Fig.6.1. In Fig.6.2. the values are approximately  $g(0)^* = 106$  and  $V^* = 9.5$ . However, more accurate solutions can be calculated by using optimization methods.

If we extend the results to a 3-period contract we deduce from the graph in Fig.6.3. below that a minimum can be obtained if the graph of the surface is concave upward. For the three period scenario we have 3 variables, where  $g(0)$  and  $g(1)$  are independent and  $g(2)$  is determined by the condition  $g(2) = \Gamma - g(0) - g(1)$ . In Fig.6.3. the parameters are  $r = 0.05$ ,  $\sigma = 0.08$ ,  $\mu = 0.05$  and  $\Gamma = 330$ .

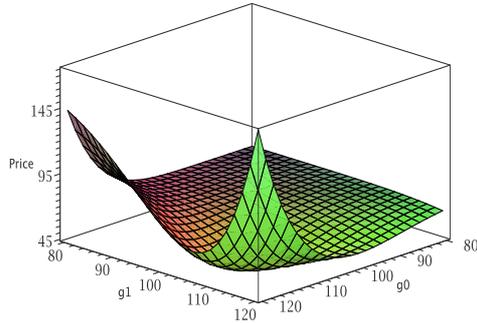


Figure 6.3: A simulation of the price of the guarantee for a three period contract with  $\mu = 0.05$ .

By rotating the graph in Fig.6.3. on the computer monitor we obtain Fig.6.4. and Fig. 6.5., from which we can get a fairly accurate solution. The minimum value  $V^*$  of  $V$  and the associated guarantee values  $g(0)^*$  and  $g(1)^*$  are read from the graph as approximately  $V^* = 47$ ,  $g(0)^* = 113$  and  $g(1)^* = 109$ .

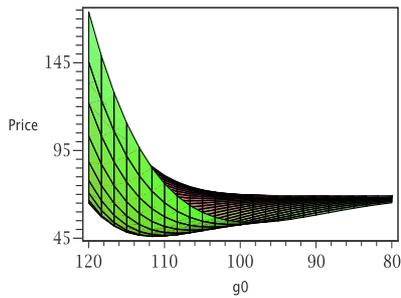
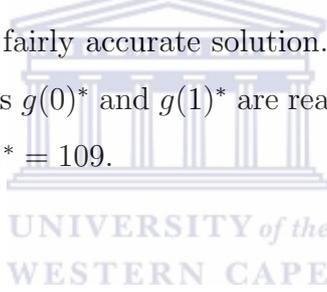


Figure 6.4: A rotation of the graph of the price of the guarantee for a three period contract with  $\mu = 0.05$ .

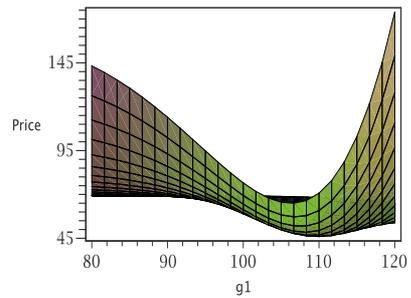


Figure 6.5: A rotation of the graph of the price of the guarantee for a three period contract with  $\mu = 0.05$ .

For multi-period contracts with maturities exceeding three years such graphing methods

are unfortunately not available. In such cases the method of Lagrange multipliers is very useful for obtaining the solution.

In the optimization problem we consider the sum  $V$  of the squares of the discounted payments on the guarantee increments. Motivated by the observations above, we seek a sequence of incremental guarantees that minimizes  $V$ . The problem we propose to solve is as follows:

**Optimization problem:** For a time horizon  $[0, t(n)]$  and a given sequence of member contributions,  $c(0), c(1), \dots, c(n-1)$  with a fixed  $\Gamma = \sum_{\iota=0}^{n-1} g(\iota)$ , what is the corresponding sequence of guarantees  $g(0), g(1), \dots, g(n-1)$  for which  $V = \sum_{j=0}^{n-1} [e^{-rt(j)} P(j)]^2$  is at its minimum? Writing the problem mathematically gives the optimization problem as :

$$\begin{aligned} \text{Minimize: } V &= \sum_{j=0}^{n-1} \left[ e^{-rt(j)} P(j) \right]^2 \\ \text{subject to: } \sum_{j=0}^{n-1} g(j) &= \Gamma. \end{aligned} \tag{6.1}$$

## 6.2 The solution

We solve the constrained optimization problem by using the method of Lagrange undetermined multipliers. We require only one Lagrange multiplier since we have only one constraint, and it is an equality constraint. The Lagrangian is given below:

$$L(G, \lambda) = V(G) - \lambda \left[ \sum_{j=0}^{n-1} g(j) - \Gamma \right],$$

where  $G$  denotes the  $T$ -tuple variable  $G = (G(0), G(1), \dots, G(n-1))$ ,  $V$  is the objective function that we seek to minimize,  $\lambda$  is a Lagrangian multiplier and  $\Gamma$  is the minimum guarantee of the benefit at maturity.

In the proof of Proposition 6.1 below we require certain derivatives which are relatively simple conceptually, but clumsy in writing. We give the detail of such toward the end of

the chapter.

**Proposition 6.1:** *The following conditions are necessary for a sequence  $g(0), g(1), \dots, g(n-1)$  together with  $\lambda$  that solves Problem (6.1):*

(a) 
$$\sum_{\iota=0}^{n-1} g(\iota) = \Gamma.$$

(b) *At time  $t(j)$ , we have the following situation:*

(i) *If  $i = j = 0$ , then*

$$2P(0) \frac{\partial P(0)}{\partial g(0)} = 2P(0) \left[ e^{-rt(n)} N(-k(0)) + \frac{e^{-\frac{1}{2}k(0)^2} e^{-rt(n)}}{\sqrt{2\pi\sigma}\sqrt{t(n)}} - \frac{c(0)}{g(0)} \frac{e^{-\frac{1}{2}h(0)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)}} \right].$$

(ii) *If  $j = i > 0$ , then*

$$2 \left[ e^{-rt(j)} \right]^2 P(j) \frac{\partial P(j)}{\partial g(j)} = 2 \left[ e^{-rt(j)} \right]^2 P(j) \left\{ e^{-r(t(n)-t(j))} N(-k(j)) + \frac{e^{-r(t(n)-t(j))} e^{-\frac{1}{2}k(j)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} - \frac{C(j)}{G(j)} \frac{e^{-\frac{1}{2}h(j)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} \right\}.$$

(iii) *If  $j > i$ , then*

$$2 \left[ e^{-rt(j)} \right]^2 P(j) \frac{\partial P(j)}{\partial g(j)} = 2 \left[ e^{-rt(j)} \right]^2 P(j) \left\{ e^{-r(t(n)-t(j))} N(-k(j)) + \frac{e^{-r(t(n)-t(j))} e^{-\frac{1}{2}k(j)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} - \frac{C(j)}{G(i)} \frac{e^{-\frac{1}{2}h(i)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} - \frac{e^{-r(t(n)-t(j))} e^{-\frac{1}{2}\kappa(j-1)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} - e^{-r(t(n)-t(j))} N(-\kappa(j-1)) + \frac{C(j-1)e^{\mu(t(j)-t(j-1))}}{G(j-1)} \frac{e^{-\frac{1}{2}\eta(j-1)}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} \right\}.$$

(c) *At time  $t(j)$ ,*

(i) *If  $i = 0$ , then*

$$2 \sum_{j=0}^{n-1} \left[ e^{-rt(j)} \right]^2 P(j) \frac{\partial P(j)}{\partial g(i)} = \lambda.$$

(ii) If  $0 < i < n - 1$ , then

$$2 \sum_{j=i}^{n-1} \left[ e^{-rt(j)} \right]^2 P(j) \frac{\partial P(j)}{\partial g(i)} = \lambda.$$

(iii) If  $i = n - 1$ , then

$$2 \left[ e^{-rt(j)} \right]^2 P(j) \frac{\partial P(j)}{\partial g(i)} = \lambda.$$

*Proof.* We apply the first order conditions of Lagrange's method. The condition **(a)** obviously holds, following from the condition  $\frac{\partial L}{\partial \lambda} = 0$ . We prove **(b)** by computing the partial derivatives of  $\sum_{j=0}^{n-1} \left[ e^{-rt(j)} P(j) \right]^2$  and equating it to zero. The assertion **(c)** can be deduced by expanding and differentiating the Lagrangian,  $L$ , with respect to  $g(i)$  (for  $0 \leq i \leq n - 1$ ) and equating it to zero.

We prove **(c)**. For each  $t(j)$  we have:

$$\frac{\partial L}{\partial g(i)} = 0, \text{ i.e., } \frac{\partial V}{\partial g(i)} - \lambda = 0. \quad (6.2)$$

By considering different intervals of  $i$  in the proposition, **(c)** is deduced from the derivative

$$\frac{\partial V}{\partial g(i)} = \lambda$$

Now we prove part **(b)**.

Let  $A(j, i)$  denote the derivative of  $P(j)$  with respect to  $g(i)$ .

Then

$$A(j, i) = \frac{\partial P(j)}{\partial g(i)}.$$

Recall that the guarantee premium  $P(0)$  at time  $t(0)$  is given by

$$P(0) = g(0)e^{-rt(n)}N(-k(0)) - c(0)N(-h(0)).$$

Therefore at time  $t(0)$ ,

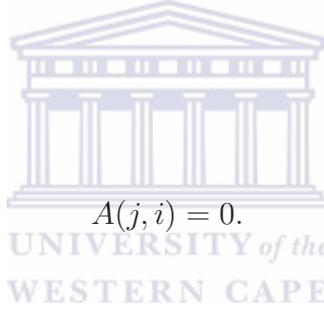
$$\begin{aligned} A(0,0) &= g(0)e^{-rt(n)}\frac{\partial N(-k(0))}{\partial g(0)} + e^{-rt(n)}N(-k(0)) - c(0)\frac{\partial N(-h(0))}{\partial g(0)} \\ &= e^{-rt(n)}N(-k(0)) + \frac{e^{-\frac{1}{2}k(0)^2}e^{-rt(n)}}{\sqrt{2\pi\sigma}\sqrt{t(n)}} - \frac{c(0)}{g(0)}\frac{e^{-\frac{1}{2}h(0)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)}}. \end{aligned}$$

Recall that the guarantee premium  $P(j)$  at time  $t(j)$  is given by

$$\begin{aligned} P(j) &= G(j)e^{-r(t(n)-t(j))}N(-k(j)) - C(j)N(-h(j)) \\ &\quad - G(j-1)e^{-r(t(n)-t(j))}N(-\kappa(j-1)) + C(j-1)e^{\mu(t(j)-t(j-1))}N(-\eta(j-1)). \end{aligned}$$

When differentiating the guarantee premium at time  $t(j)$  with respect to  $g(i)$ , we consider the three cases:

If  $0 < j < i$ , then



$$A(j,i) = 0.$$

If  $j = i$ , then

$$\begin{aligned} A(j,i) &= e^{-r(t(n)-t(j))}N(-k(j)) + G(j)e^{-r(t(n)-t(j))}\frac{\partial N(-k(j))}{\partial g(j)} - C(j)\frac{\partial N(-h(j))}{\partial g(j)} \\ &= e^{-r(t(n)-t(j))}N(-k(j)) + \frac{e^{-r(t(n)-t(j))}e^{-\frac{1}{2}k(j)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} - \frac{C(j)}{G(j)}\frac{e^{-\frac{1}{2}h(j)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}}. \end{aligned}$$

If  $j > i$ , then

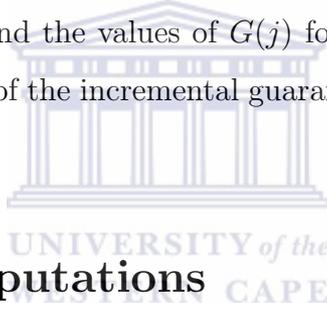
$$\begin{aligned} A(j,i) &= e^{-r(t(n)-t(j))}N(-k(j)) + G(j)e^{-r(t(n)-t(j))}\frac{\partial N(-k(j))}{\partial g(i)} - C(j)\frac{\partial N(-h(j))}{\partial g(i)} \\ &\quad - e^{-r(t(n)-t(j))}N(-\kappa(j-1)) - G(j-1)e^{-r(t(n)-t(j))}\frac{\partial N(-\kappa(j-1))}{\partial g(i)} \\ &\quad + C(i-1)\frac{\partial N(-\eta(j-1))}{\partial g(i)}. \end{aligned}$$

The latter expression simplifies to

$$\begin{aligned}
A(j, i) = & e^{-r(t(n)-t(j))} N(-k(j)) + \frac{e^{-r(t(n)-t(j))} e^{-\frac{1}{2}k(j)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} - \frac{C(j)}{G(j)} \frac{e^{-\frac{1}{2}k(j)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} \\
& - \frac{e^{-r(t(n)-t(j))} e^{-\frac{1}{2}\kappa(j-1)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}} - e^{-r(t(n)-t(j))} N(-\kappa(j-1)) \\
& + \frac{C(j-1)e^{\mu(t(j)-t(j-1))}}{G(j-1)} \frac{e^{-\frac{1}{2}\eta(j-1)^2}}{\sqrt{2\pi\sigma}\sqrt{t(n)-t(j)}}.
\end{aligned}$$

The required result is obtained by applying a left multiplication on  $A(j, i)$  by a factor of  $2 \left[ e^{-rt(j)} \right]^2 P(j)$  for each of the time scenarios above. □

The proposition provides us with a system of  $n + 1$  simultaneous equations to be solved for the unknowns, that is  $\lambda$  and the values of  $G(j)$  for  $j = 0, 1, \dots, n - 1$ . These values constitute the optimal design of the incremental guarantee sequence required to minimize the objective function  $V$ .



## 6.3 Sample Computations

In this section we employ Proposition 6.1 to solve Problem 6.1. We use the MAPLE 10 programming language to generate these results. In the calculation we used the *fsolve* command in MAPLE, together with approximations to the exponential and normal distributions. The following approximations to the exponential and the normal distribution functions in our simulations were used respectively.

Let

$$\Lambda(x) \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!}.$$

For better accuracy in our calculations we approximate the exponential distribution function by

$$\exp(x) = \frac{\Lambda(x/2)}{\Lambda(-x/2)}.$$

The normal distribution function is approximated by

$$N(x) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \left[ x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} - \frac{x^{11}}{42240} + \frac{x^{13}}{599040} - \frac{x^{15}}{9676800} + \frac{x^{17}}{175472640} \right].$$

We now present the optimal sequences of contributions that were calculated with the approximations above for three and four periods respectively.

For three period pension contracts, we have set the contributions  $c(i) = 1.00$  in the top half of the table and  $c(i) = 1.50$  in the bottom half. The riskless interest rate  $r$  and the volatilities  $\sigma$  are given as percentages. We present a constant sequence of incremental contributions with its optimal solution of incremental guarantees below it for each pair  $(r, \sigma)$ . We have taken  $\mu = 0.05$  and rounded the values of the guarantees to 2 decimal places after the calculation.



Table 6.2: The optimal sequence of guarantees for three period contracts.

$r$	$\sigma$	$g(0)$	$g(1)$	$g(2)$	$\Gamma$	$V$
0.05	0.08	1.00	1.00	1.00	3.00	0.0001
		0.88	0.88	1.24		
0.03	0.06	1.00	1.00	1.00	3.00	0.06
		1.32	1.04	0.64		
0.05	0.08	1.50	1.50	1.50	4.50	0.32
		2.23	1.61	0.66		
0.03	0.08	1.50	1.50	1.50	4.50	0.39
		2.19	1.57	0.73		

In calculating the optimal sequence of contributions of four period pension contracts, we have used the same parameters of  $r$  and  $\sigma$  as in the case of three period pension plans. The constant sequences of contributions and corresponding optimal sequences of guarantees can be seen in the table that follows. The contributions for the four period contact

are also chosen as  $c(i) = 1.00$  in the top half of the table and  $c(i) = 1.50$  in the bottom half.

Table 6.3: The optimal sequence of guarantees for four period contracts.

$r$	$\sigma$	$g(0)$	$g(1)$	$g(2)$	$g(3)$	$\Gamma$	$V$
0.05	0.08	1.00	1.00	1.00	1.00	4.00	0.0005
		0.82	0.79	1.48	0.91		
0.03	0.06	1.00	1.00	1.00	1.00	4.00	0.02
		1.52	1.10	0.47	0.90		
0.05	0.08	1.50	1.50	1.50	1.50	6.00	0.63
		2.52	1.65	1.73	0.09		
0.03	0.08	1.50	1.50	1.50	1.50	6.00	1.11
		1.11	3.24	1.44	0.21		

Computations for values of  $n > 4$  are quite attainable, but will drift the discussion more into numerical analysis. Thus we shall for now be content with what is demonstrated in the three and four period cases. Let us also observe that the paper of Lindset [35] reported far more complicated numerical work.

## 6.4 Derivatives of Lagrangian

In the process of calculating the Lagrangian of Problem 6.1 the following derivatives were obtained and used in the MAPLE program to solve the three and four period versions of Problem 6.1:

$$N(-\alpha) = \int_{\infty}^{-\alpha} (\sqrt{2\pi})^{-1} e^{-z^2/2} dz \quad \text{thus} \quad \partial N(-\alpha)/\partial \alpha = -(2\pi)^{-\frac{1}{2}} e^{-\alpha^2/2}.$$

At time  $t(0)$ ,

$$h(0) = \frac{\ln\left(\frac{c(0)}{g(0)}\right) + \left(r + \frac{1}{2}\sigma^2\right)t(n)}{\sigma\sqrt{t(n)}}.$$

$$\frac{\partial h(0)}{\partial g(0)} = -\frac{1}{g(0)\sigma\sqrt{t(n)}}.$$

Therefore

$$\frac{\partial k(0)}{\partial g(0)} = \frac{\partial h(0)}{\partial g(0)}$$

since  $k(0) = h(0) - \sigma\sqrt{t(n)}$ .

Furthermore using the chain rule of differentiation we obtain

$$\begin{aligned} \frac{\partial N(-h(0))}{\partial g(0)} &= \frac{\partial N(-h(0))}{\partial h(0)} \frac{\partial h(0)}{\partial g(0)} \\ &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}h^2(0)} \left( -\frac{1}{g(0)\sigma\sqrt{t(n)}} \right) \\ &= \frac{e^{-\frac{1}{2}h^2(0)}}{\sqrt{2\pi}g(0)\sigma\sqrt{t(n)}}. \end{aligned}$$

In a similar manner, we obtain

$$\begin{aligned} \frac{\partial N(-k(0))}{\partial g(0)} &= \frac{\partial N(-k(0))}{\partial k(0)} \frac{\partial k(0)}{\partial g(0)} \\ &= \frac{e^{-\frac{1}{2}k^2(0)}}{\sqrt{2\pi}g(0)\sigma\sqrt{t(n)}}. \end{aligned}$$

Furthermore, at time  $t(j)$  for  $j > 0$ , we express the partial derivatives as piecewise defined functions below:

$$\frac{\partial h(j)}{\partial g(i)} = \frac{e^{-\frac{1}{2}h(j)^2}}{\sqrt{2\pi}G(j)\sigma\sqrt{t(n)-t(j)}}, \text{ if } j \geq i,$$

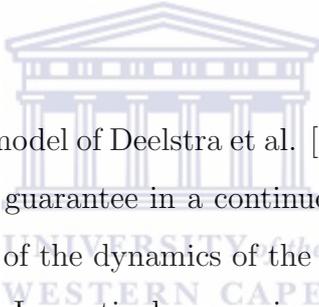
$$\frac{\partial k(j)}{\partial g(i)} = \frac{e^{-\frac{1}{2}k(j)^2}}{\sqrt{2\pi}G(j)\sigma\sqrt{t(n)-t(j)}}, \text{ if } j \geq i,$$

$$\frac{\partial \eta(j-1)}{\partial g(i)} = \frac{e^{-\frac{1}{2}\eta(j-1)^2}}{\sqrt{2\pi}G(j-1)\sigma\sqrt{t(n)-t(j)}}, \text{ if } j > i,$$

$$\frac{\partial \kappa(j-1)}{\partial g(i)} = \frac{e^{-\frac{1}{2}\kappa(j-1)^2}}{\sqrt{2\pi}G(j-1)\sigma\sqrt{t(n)-t(j)}}, \text{ if } j > i.$$

# Chapter 7

## The continuous time model of Deelstra et al. (2003)



In this chapter we present the model of Deelstra et al. [19] on defined contribution schemes in the presence of a minimum guarantee in a continuous-time framework. We add some detail and present simulations of the dynamics of the three underlying assets of the pension fund as our contribution. In particular, we give more detailed proofs of Lemmas 2 and 5 in [19], but confine ourselves to the special case where  $\eta_1 = 0$  and replace  $\eta_2$  by  $\eta$ .

We consider a pension fund manager who is responsible for maximizing the wealth of the fund. The manager has to maximize the wealth under the constraint that the terminal wealth of the fund must exceed the minimum guarantee, which is accomplished by investing the initial wealth of the fund as well as the regular contribution flow into the financial market. Interest rates are assumed to be stochastic and the optimal investment strategies are obtained based on the assumption of a (CRRA) utility function and a complete financial market. In Section 7.1 we describe the financial market and in Section 7.2 we describe and give graphical presentations of the dynamics short-rate process and the riskless asset. In Section 7.3 we present the dynamics of the risky asset and in Section 7.4 the dynamics of the zero-coupon bond.

## 7.1 The financial market

We describe the market model of Deelstra et al. [19]. The randomness in the financial market is driven by a two-dimensional standard Brownian motion given by

$$\underline{Z}(t) = (Z(t), Z_r(t))'; t \in [0, \infty).$$

The Brownian motion is defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\mathbb{P}$  is the real world probability. The information generated by the Brownian motion is assumed to satisfy the usual conditions, represented by  $\mathcal{F} = \{\mathcal{F}(t)\}_{t \geq 0}$ . We denote the conditional expectation under the real world probability by  $\mathbb{E}(\cdot | \mathcal{F}(t))$ . The conditional variance and covariance under the real world probability are represented by  $\mathbb{V}(\cdot | \mathcal{F}(t))$  and  $\text{COV}(\cdot | \mathcal{F}(t))$  respectively.

We take the assumption that the market is composed of three financial assets, to be bought and sold continuously without incurring any restrictions as short sales and trading cost. The assets under consideration are a riskless asset (cash), a risky asset (stock) and a zero-coupon bond with maturity  $T$ . We assume that the cash asset yields an instantaneous risk-free interest rate of the type of an Ornstein-Uhlenbeck process, known as the Vasiček model of the term structure of interest rates as in Boulier et al. [7].

We now introduce the short rate interest rate and each of the financial assets and provide graphs of their dynamics.

## 7.2 The riskless asset

We continue with the model of Deelstra et al. [19]. The first asset in the market is a riskless asset, with price  $S_0(t)$ , for  $t \geq 0$ , and evolves according to the differential equation

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \text{ where } S_0(0) = 1.$$

The short rate process,  $r(t)$ , follows the stochastic differential equation

$$dr(t) = (a - br(t))dt - \sqrt{\eta}dZ_r(t) \quad (7.1)$$

where  $a$ ,  $b$ ,  $\eta$  are positive constants and  $r(0)$  is a positive random number. In all the simulations hereafter we assign a positive value to  $r(0)$ .

**Remark 7.1:** If the price of an asset follows

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt,$$

then it can be shown that the price at time  $\tau$  of the asset is  $S_0(\tau) = S_0(0) \exp(\int_0^\tau r(t)dt)$ , but from the initial condition  $S_0(0) = 1$ , the solution of the differential equation is  $S_0(\tau) = \exp(\int_0^\tau r(t)dt)$ .

We proceed to draw the graph of the riskless asset. The parameters for the process in Fig.7.1. are chosen as  $a = 0.01$ ,  $b = 0.07$ ,  $\sqrt{\eta} = 0.03$ ,  $r(0) = 0.08$  and  $S_0(0) = 1$ . The parameters  $a$  and  $b$  appear in the drift term of the short rate process and the volatility of the short rate process is parameter  $\sqrt{\eta}$ . The parameters  $r(0)$  and  $S_0(0)$  are respectively the initial values of the short rate process and the riskless asset.

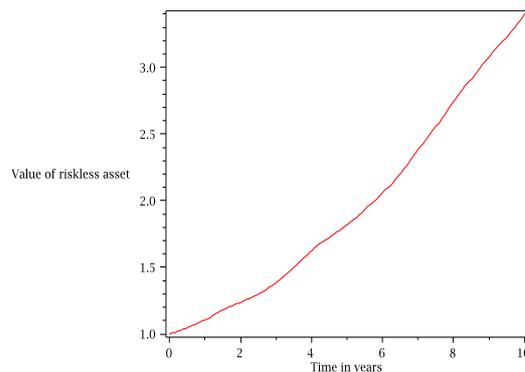


Figure 7.1: A simulation of the riskless asset over  $T = 10$  years.

**Proposition 7.2:** The short term interest rate that solves Eq.(7.1) is

$$r(t) = e^{-bt}r(0) + \alpha_1(1 - e^{-bt}) - \sqrt{\eta} \int_0^t e^{-b(t-s)} dZ_r(s)$$

*Proof.* If we write Eq.(7.1) as

$$dr(t) = b(\alpha_1 - r(t))dt - \sqrt{\eta}dZ_r(t)$$

and apply Itô's lemma to  $f(r(t), t) = r(t)e^{bt}$ , then we obtain the following:

$$\begin{aligned} df(r(t), t) &= br(t)e^{bt}dt + e^{bt}dr(t) \\ &= br(t)e^{bt}dt + e^{bt}[(b\alpha_1 - br(t))dt - \sqrt{\eta}dZ_r(t)] \\ &= e^{bt}b\alpha_1dt - e^{bt}\sqrt{\eta}dZ_r(t). \end{aligned}$$

Integrating both sides yields

$$\int_0^t df(r(s), s) = \int_0^t (e^{bs}b\alpha_1ds - e^{bs}\sqrt{\eta}dZ_r(s)).$$

Thus

$$\begin{aligned} r(t) &= e^{-bt}r(0) + e^{-bt} \int_0^t b\alpha_1e^{bs}ds - e^{-bt}\sqrt{\eta} \int_0^t e^{bs}dZ_r(s) \\ &= e^{-bt}r(0) + \alpha_1e^{-bt}(e^{bt} - e^0) - \sqrt{\eta} \int_0^t e^{-bt}e^{bs}dZ_r(s) \\ &= e^{-bt}r(0) + \alpha_1e^{-bt}(e^{bt} - 1) - \sqrt{\eta} \int_0^t e^{-b(t-s)}dZ_r(s) \\ &= e^{-bt}r(0) + \alpha_1(1 - e^{-bt}) - \sqrt{\eta} \int_0^t e^{-b(t-s)}dZ_r(s), \end{aligned}$$

with  $\alpha_1 = \frac{a}{b}$ .

□

We now proceed to draw the graph of the short-rate process. It is simulated over a period of ten years in Fig.7.2. The parameters for this process seen in the figure are chosen as  $a = 0.01$  and  $b = 0.07$  for the drift term of the process,  $\sqrt{\eta} = 0.03$ , which is the volatility term and  $r(0) = 0.08$  is the initial value of the short rate process.

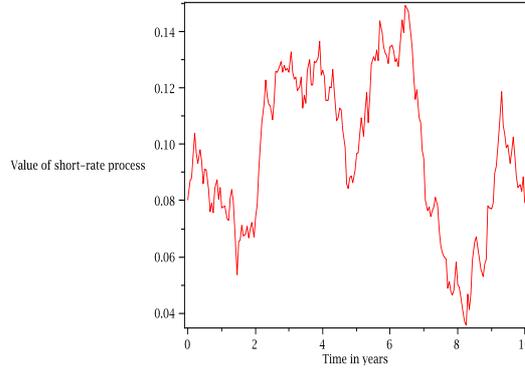


Figure 7.2: A simulation of the short interest rate process over  $T = 10$  years.

### 7.3 The risky asset

The second asset in the financial market of Deelstra et al. [19] is a stock whose price, denoted by  $S(t)$ , for  $t \geq 0$ , and has dynamics

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma_1(dZ(t) + \lambda_1 dt) + \sigma_2\sqrt{\eta}(dZ_r(t) + \lambda_2\sqrt{\eta}dt), \quad (7.2)$$

with  $S(0) = 1$  and  $\lambda_1, \lambda_2, \sigma_1$  and  $\sigma_2$  positive constants. In the proposition that follows, the price for the risky asset in Eq.(7.2) is obtained.

**Proposition 7.3:** *If we re-write Eq.(7.2) as*

$$\frac{dS(t)}{S(t)} = (r(t) + \sigma_1\lambda_1 + \eta\sigma_2\lambda_2)dt + \sigma_1dZ(t) + \sigma_2\sqrt{\eta}dZ_r(t),$$

*then the following process will solve Eq.(7.2):*

$$S(t) = \exp \left[ (r(t) + \sigma_1\lambda_1 + \eta\sigma_2\lambda_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2\eta))t + \sigma_1Z(t) + \sigma_2\sqrt{\eta}Z_r(t) \right].$$

*Proof.* We let  $F(t, S(t)) = \log S(t)$  and use Itô's lemma to get

$$dF(t, S(t)) = \frac{\partial F(t, S(t))}{\partial S(t)}dS(t) + \frac{\partial F(t, S(t))}{\partial t}dt + \frac{1}{2} \frac{\partial^2 F(t, S(t))}{\partial S^2(t)}(dS(t))^2$$

$$\begin{aligned}
&= \frac{dS(t)}{S(t)} - \frac{1}{2} \frac{(dS(t))^2}{S^2(t)} \\
&= (r(t) + \sigma_1\lambda_1 + \eta\sigma_2\lambda_2)dt + \sigma_1dZ(t) + \sigma_2\sqrt{\eta}dZ_r(t) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2\eta)dt \\
&= (r(t) + \sigma_1\lambda_1 + \eta\sigma_2\lambda_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2\eta))dt + \sigma_1dZ(t) + \sigma_2\sqrt{\eta}dZ_r(t).
\end{aligned}$$

Integrating both sides we obtain

$$\begin{aligned}
\log S(t) &= \log S(0) + \int_0^t dF(u, S(u)) \\
&= \log S(0) + \int_0^t (r(u) + \sigma_1\lambda_1 + \eta\sigma_2\lambda_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2\eta))du + \int_0^t \sigma_1dZ(u) \\
&\quad + \int_0^t \sigma_2\sqrt{\eta}dZ_r(u) \\
&= \log S(0) + (r(t) + \sigma_1\lambda_1 + \eta\sigma_2\lambda_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2\eta))t + \sigma_1Z(t) \\
&\quad + \sigma_2\sqrt{\eta}Z_r(t).
\end{aligned}$$

Therefore

$$S(t) = S(0) \exp \left[ (r(t) + \sigma_1\lambda_1 + \eta\sigma_2\lambda_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2\eta))t + \sigma_1dZ(t) + \sigma_2\sqrt{\eta}Z_r(t) \right].$$

Substituting  $S(0) = 1$  leads to the required result

$$S(t) = \exp \left[ (r(t) + \sigma_1\lambda_1 + \eta\sigma_2\lambda_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2\eta))t + \sigma_1Z(t) + \sigma_2\sqrt{\eta}Z_r(t) \right].$$

□

We proceed to graph the dynamics of the risky asset of the market over a period of ten years. The parameters for the risky asset in Fig.7.3. are chosen as  $a = 0.01$ ,  $b = 0.07$ ,  $\sqrt{\eta} = 0.03$ ,  $\sigma_1 = 0.07$ ,  $\sigma_2 = 0.08$ ,  $\lambda_1 = 0.05$ ,  $\lambda_2 = 0.05$ . The parameters  $r(0) = 0.08$  and  $S(0) = 1$  are the initial values of the short rate process and risky asset respectively.

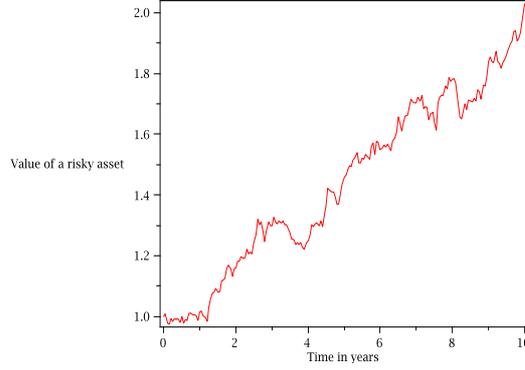


Figure 7.3: A simulation of the risky asset over  $T = 10$  years.

## 7.4 The zero-coupon bond

The last asset under consideration in Deelstra et al. [19] is a zero-coupon bond with maturity  $T$  and price  $B(t, T)$ , where  $0 \leq t \leq T$ . In order to calculate the price of the zero-coupon bond, we formulate a lemma which will enable us to fix the dynamics of the zero-coupon bond as in Deelstra et al. [19]. The proof of the following lemma is along the lines of the relevant one in Deelstra et al. [19], but we provide more detail.

**Lemma 7.4:** *If the interest rates follow Eq.(7.1), then there exists two deterministic functions  $K(\Phi, \Psi, T - t)$  and  $L(\Phi, \Psi, T - t)$  such that for  $\Phi \in \mathbb{R}$  and  $\Psi > 0$*

$$\mathbb{E} \left[ \exp(-\Phi r(T) - \Psi \int_t^T r(t^*) dt^*) \middle| \mathcal{F}(t) \right] = K(\Phi, \Psi, T - t) \exp[-r(t)L(\Phi, \Psi, T - t)], \quad (7.3)$$

where

$$\begin{aligned} K(\Phi, \Psi, T - t) &= \exp \left\{ \alpha_1 \left[ \left( \frac{\Psi}{b} - \Phi \right) (1 - e^{-b(T-t)}) - \Psi(T-t) \right] \right. \\ &+ \frac{\eta \Phi^2}{2} \int_t^T \left( e^{-2b(T-s)} \right) ds + \frac{\eta \Psi^2}{2b^2} \int_t^T \left( 1 - e^{-b(T-s)} \right)^2 ds \\ &\left. + \frac{\eta \Phi \Psi}{b} \int_t^T e^{-b(T-s)} \left( 1 - e^{-b(T-s)} \right) ds \right\}, \end{aligned}$$

and

$$L(\Phi, \Psi, T - t) = \Phi e^{-b(T-t)} + \frac{\Psi}{b}(1 - e^{-b(T-t)}).$$

*Proof.* Recall that the interest rate in Eq.(7.1) can be written as

$$dr(t) = b(\alpha_1 - r(t))dt - \sqrt{\eta}dZ_r(t)$$

with solution

$$r(t) = e^{-bt}r(0) + \alpha_1(1 - e^{-bt}) + \sqrt{\eta} \int_t^0 e^{bs}dZ_r(s).$$

If we define

$$Q(\Phi, \Psi, T - t) := -\Phi r(T) - \Psi \int_t^T r(t^*)dt^*,$$

as in Deelstra et al. [19], it follows that since  $Q(\Phi, \Psi, T - t)$  is Gaussian,

$$\begin{aligned} & \mathbb{E} \left[ \exp(-\Phi r(T) - \Psi \int_t^T r(t^*)dt^*) \middle| \mathcal{F}(t) \right] \\ &= \exp \left\{ \mathbb{E} \left[ Q(\Phi, \Psi, T - t) \middle| \mathcal{F}(t) \right] + \frac{1}{2} \mathbb{V} \left[ Q(\Phi, \Psi, T - t) \middle| \mathcal{F}(t) \right] \right\}. \end{aligned}$$

We use Proposition 7.2 to define  $r(t^*)$  for any  $t^* \geq t$  as

$$r(t^*) = e^{-b(t^*-t)}r(t) + \alpha_1(1 - e^{-b(t^*-t)}) - \sqrt{\eta} \int_t^{t^*} e^{-b(t^*-s)}dZ_r(s).$$

We now calculate the expected value of  $r(T)$ , i.e. the expected value of the interest rate at maturity  $T$ :

$$\begin{aligned} \mathbb{E}[r(T)|\mathcal{F}(t)] &= \mathbb{E} \left[ e^{-b(T-t)}r(t) + \alpha_1(1 - e^{-b(T-t)}) - \sqrt{\eta} \int_t^T e^{-b(T-s)}dZ_r(s) \middle| \mathcal{F}(t) \right] \\ &= e^{-b(T-t)}r(t) + \alpha_1(1 - e^{-b(T-t)}). \end{aligned}$$

The variance of the underlying interest rate at maturity  $T$  is calculated as follows:

$$\begin{aligned} \mathbb{V}[r(T)|\mathcal{F}(t)] &= \mathbb{E} \left[ (r(T) - \mathbb{E}[r(T)|\mathcal{F}(t)])^2 \middle| \mathcal{F}(t) \right] \\ &= \mathbb{E} \left[ \left( -\sqrt{\eta} \int_t^T e^{-b(T-s)}dZ_r(s) \right)^2 \middle| \mathcal{F}(t) \right] \\ &= \int_t^T \mathbb{E} \left[ (\eta e^{-2b(T-s)}) \middle| \mathcal{F}(t) \right] ds \\ &= \eta \int_t^T (e^{-2b(T-s)}) ds. \end{aligned}$$

Integrating  $r(t^*)$  over the interval  $[t, T]$ , where  $t \leq t^* \leq T$ , yields

$$\begin{aligned}
\int_t^T r(t^*) dt^* &= \int_t^T \left( e^{-b(t^*-t)} r(t) + \alpha_1(1 - e^{-b(t^*-t)}) - \sqrt{\eta} \int_t^{t^*} e^{-b(t^*-s)} dZ_r(s) \right) dt^* \\
&= \frac{e^{-b(T-t)}}{-b} r(t) + \frac{e^{-b(t-t)}}{b} r(t) + \alpha_1 T - \alpha_1 t + \frac{\alpha_1 e^{-b(T-t)}}{b} - \frac{\alpha_1 e^{-b(T-t)}}{b} \\
&\quad - \sqrt{\eta} \int_t^T \left( \frac{e^{-b(T-s)}}{-b} + \frac{e^{-b(t-t)}}{b} \right) dZ_r(s) \\
&= \frac{1 - e^{-b(T-t)}}{b} r(t) + \alpha_1(T - t) - \alpha_1 \frac{1 - e^{-b(T-t)}}{b} \\
&\quad - \sqrt{\eta} \int_t^T \frac{1 - e^{-b(T-s)}}{b} dZ_r(s).
\end{aligned}$$

The expected value of  $\int_t^T r(t^*) dt^*$  is

$$\begin{aligned}
\mathbb{E} \left[ \int_t^T r(t^*) dt^* \middle| \mathcal{F}(t) \right] &= \mathbb{E} \left[ \frac{1 - e^{-b(T-t)}}{b} r(t) + \alpha_1(T - t) - \alpha_1 \frac{1 - e^{-b(T-t)}}{b} \right. \\
&\quad \left. - \sqrt{\eta} \int_t^T \frac{1 - e^{-b(T-s)}}{b} dZ_r(s) \middle| \mathcal{F}(t) \right] \\
&= \frac{1 - e^{-b(T-t)}}{b} r(t) + \alpha_1(T - t) - \alpha_1 \frac{1 - e^{-b(T-t)}}{b},
\end{aligned}$$

The variance of  $\int_t^T r(t^*) dt^*$  is

$$\begin{aligned}
\mathbb{V} \left[ \int_t^T r(t^*) dt^* \middle| \mathcal{F}(t) \right] &= \mathbb{E} \left[ \left( \int_t^T r(t^*) dt^* - \mathbb{E} \left[ \int_t^T r(t^*) dt^* \middle| \mathcal{F}(t) \right] \right)^2 \middle| \mathcal{F}(t) \right] \\
&= \mathbb{E} \left[ \left( -\sqrt{\eta} \int_t^T \left( \frac{1 - e^{-b(T-s)}}{b} \right) dZ_r(s) \right)^2 \middle| \mathcal{F}(t) \right] \\
&= \int_t^T \mathbb{E} \left[ \eta \left( \frac{1 - e^{-b(T-s)}}{b} \right)^2 \middle| \mathcal{F}(t) \right] ds \\
&= \frac{\eta}{b^2} \int_t^T \left( 1 - e^{-b(T-s)} \right)^2 ds.
\end{aligned}$$

The covariance of  $r(T)$  and  $\int_t^T r(t^*) dt^*$  is calculated as follows:

$$\begin{aligned}
\text{Cov} \left[ r(T), \int_t^T r(t^*) dt^* \middle| \mathcal{F}(t) \right] &= \mathbb{E} \left[ \left( r(T) - \mathbb{E} \left[ r(T) \middle| \mathcal{F}(t) \right] \right) \right. \\
&\quad \left. \left( \int_t^T r(t^*) dt^* - \mathbb{E} \left[ \int_t^T r(t^*) dt^* \middle| \mathcal{F}(t) \right] \right) \right] \\
&= \mathbb{E} \left[ \left( -\sqrt{\eta} \int_t^T e^{-b(T-s)} dZ_r(s) \right) \right]
\end{aligned}$$

$$\begin{aligned}
& \left( -\sqrt{\eta} \int_t^T \left( \frac{1 - e^{-b(T-s)}}{b} \right) dZ_r(s) \right) \Big| \mathcal{F}(t) \Big] \\
&= \mathbb{E} \left[ \eta \int_t^T \left( e^{-b(T-s)} \right) \left( \frac{1 - e^{-b(T-s)}}{b} \right) ds \Big| \mathcal{F}(t) \right] \\
&= \frac{\eta}{b} \int_t^T \left[ e^{-b(T-s)} \left( 1 - e^{-b(T-s)} \right) \right] ds.
\end{aligned}$$

We now derive the two functions  $K(\Phi, \Psi, T-t)$  and  $L(\Phi, \Psi, T-t)$ . According to Deelstra et al. [19],

$$\begin{aligned}
& \mathbb{E} \left[ \exp \left( -\Phi r(T) - \Psi \int_t^T r(t^*) dt^* \right) \Big| \mathcal{F}(t) \right] \\
&= \exp \left[ -\Phi \mathbb{E} \left[ r(T) \Big| \mathcal{F}(t) \right] - \Psi \mathbb{E} \left[ \int_t^T r(t^*) dt^* \Big| \mathcal{F}(t) \right] + \frac{\Phi^2}{2} \mathbb{V} \left[ r(T) \Big| \mathcal{F}(t) \right] \right. \\
& \quad \left. + \frac{\Psi^2}{2} \mathbb{V} \left[ \int_t^T r(t^*) dt^* \Big| \mathcal{F}(t) \right] + \Phi \Psi \text{Cov} \left[ \left( r(T), \int_t^T r(t^*) dt^* \right) \Big| \mathcal{F}(t) \right] \right],
\end{aligned}$$

which can be written as

$$K(\Phi, \Psi, T-t) \exp[-r(t)L(\Phi, \Psi, T-t)].$$

Here

$$\begin{aligned}
K(\Phi, \Psi, T-t) &= \exp \left\{ \alpha_1 \left[ \left( \frac{\Psi}{b} - \Phi \right) (1 - e^{-b(T-t)}) - \Psi(T-t) \right] \right. \\
& \quad + \frac{\eta \Phi^2}{2} \int_t^T \left( e^{-2b(T-s)} \right) ds + \frac{\eta \Psi^2}{2b^2} \int_t^T \left( 1 - e^{-b(T-s)} \right)^2 ds \\
& \quad \left. + \frac{\eta \Phi \Psi}{b} \int_t^T e^{-b(T-s)} \left( 1 - e^{-b(T-s)} \right) ds \right\},
\end{aligned}$$

and

$$L(\Phi, \Psi, T-t) = \Phi e^{-b(T-t)} + \frac{\Psi}{b} (1 - e^{-b(T-t)}).$$

□

**Remark 7.5:** For the remainder of this thesis, we assume that the dynamics of the zero-coupon bond follows (as in [19])

$$\frac{dB(t, T)}{B(t, T)} = r(t)dt + \sigma_B(T-t, r(t))(dZ_r(t) + \lambda_2 \sqrt{\eta} dt) \quad (7.4)$$

with  $B(T, T) = 1$  and where  $\sigma_B(T - t, r(t)) = h(T - t)\sqrt{\eta}$ , with

$$h(t) = \frac{(e^{bt} - 1)}{be^{bt}}, \text{ for } t \geq 0. \quad (7.5)$$

Also

$$H(t) = \exp \left\{ - \int_0^t r(t^*) dt^* - \int_0^t \underline{\lambda}' dZ(t^*) - \frac{1}{2} \int_0^t |\underline{\lambda}(t^*)|^2 dt^* \right\} \quad (7.6)$$

where  $\underline{\lambda}'(t^*) = (\lambda_1, \lambda_2\sqrt{\eta})$ .

The process  $H(t)$  is also called the growth-optimum portfolio, while the process  $H^{-1}(t)$  represents the shadow state process (see Merton [38] and Deelstra et al [18]) respectively.

In order to simulate the dynamics of the zero-coupon bond we have resorted to choosing very small parameters for Eq.(7.4). This ensures that the simulation adheres to the definition of the zero-coupon bond.

The parameters used in Fig.7.4. are  $a = 0.00001$ ,  $b = 0.0025$ ,  $\eta = 0.0000001$ ,  $\sigma_1 = 0.00001$ ,  $\sigma_2 = 0.00001$ ,  $\lambda_1 = 0.001$ ,  $\lambda_2 = 0.001$  and  $r(0) = 0.06$ .

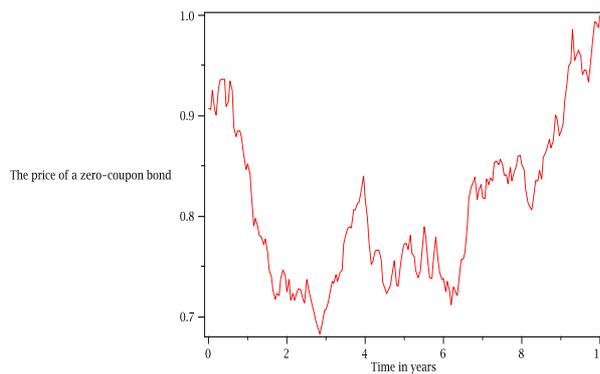
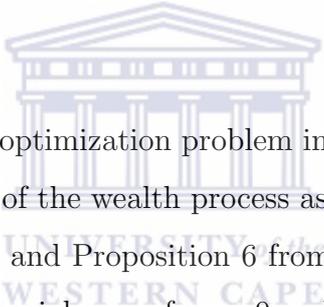


Figure 7.4: A simulation of the price of the zero-coupon bond over  $T = 10$  years.

# Chapter 8

## The optimization problem of Deelstra et al. (2003)



In this chapter we present the optimization problem in Deelstra et al. [19]. We add some detail and present simulations of the wealth process as our contribution. We particularly prove Proposition 4, Lemma 5 and Proposition 6 from the paper by Deelstra et al. [19], but restrict the study to the special case of  $\eta_1 = 0$  and  $\eta_2 = \eta$ .

According to the aforementioned authors, it turns out that a pension fund is built by handing over the asset allocation to a professional investing manager. This is beneficial since the client may not have the time or the competency to participate in the stock market. It also reduces transaction or informational costs. The manager promises a minimum amount to the client. As long as this commitment has not been met, commissions due to the manager are in practice not definitively acquired and stocked on a separate account. Therefore we follow their method of modeling the manager's remuneration as being paid at maturity only. The total commission will be positive only if the wealth of the pension fund exceeds the minimum amount at the retirement of the employee. This emphasizes the importance of the wealth process.

These remarks were formalized by Deelstra et al. [19] as follows. On one hand, the pension fund has an initial wealth  $W(0)$ , which is strictly positive, and receives a non-negative, progressively measurable and square-integrable process at a rate denoted by  $c(t)$  which represents the contributions made to the fund at any time  $t \geq 0$  by participants. On the other hand, the pension fund must in any case (and thus ignoring death) provide at date  $T$ , at least the minimum guarantee  $G(T)$ , a strictly positive square-integrable  $\mathcal{F}(T)$ -measurable random variable; it can be the value at time  $T$  of a benchmark portfolio or an annuity.

Let  $W(t)$  denote the wealth of the fund at date  $t \in [0, T]$  and assume that the process  $H(t)$  is defined as in chapter 7. Furthermore assume that

$$\mathbb{E}[H(T)G(T)] < W(0) + \mathbb{E} \left[ \int_0^T H(t)c(t)dt \right], \quad (8.1)$$

which means that the manager can always choose a strategy such that, at date  $T$ , the surplus  $W(T) - G(T)$  is strictly positive. Furthermore, it is assumed that the manager is controlled by a regulator and will select a strategy in which  $W(T) - G(T) \geq 0$  almost surely. The manager's remuneration is modeled by a function  $\omega(\cdot)$  of the surplus only. From the manager's rationality, it follows that the function  $\omega(\cdot)$  has to be increasing and the assumption is made that this function is concave. The surplus is shared according to the rule below.

1. the manager takes  $\omega(W(T) - G(T))$ , where  $\omega(\cdot)$  is a strictly increasing concave function such that  $\omega(0) = 0$ ;
2. the contributors receive  $W(T) - G(T) - \omega(W(T) - G(T))$ .

The manager's preferences are described by a CRRA utility function defined by

$$v(y) = \frac{y^\gamma}{\gamma}, \quad (8.2)$$

with  $\gamma \in (-\infty, 0) \cup (0, 1)$ .

The program of the manager is then to maximize the expected utility of his terminal wealth under feasibility constraints, namely

$$\max_{\underline{u}(t) \in \mathcal{A}} \mathbb{E} \left[ U(W(t) - G(t)) \right], \quad (8.3)$$

where  $t \in [0, T]$  and

1.  $U := v \circ \omega$  is strictly increasing, strictly concave and satisfies the conditions  $U'(+\infty) = 0$  and  $U'(0) = +\infty$ ,
2. the wealth process  $\{W(t)\}_{t \geq 0}$  is defined by the dynamics

$$\begin{aligned} dW(t) &= W(t) \underline{u}'(t) \text{diag}[\underline{S}(t)]^{-1} d\underline{S}(t) + c(t) dt \\ &= W(t) \left( 1 - u^B(t) - u^S(t) \right) \frac{dS_0(t)}{S_0(t)} + W(t) u^S(t) \frac{dB(t, T)}{B(t, T)} \\ &\quad + W(t) u^S(t) \frac{dS(t)}{S(t)} + c(t) dt, \end{aligned} \quad (8.4)$$

with  $W(0) > 0$ ,  $\underline{u}(t) = ((1 - u^B(t) - u^S(t)), u^B(t), u^S(t))'$  and  $\underline{S}(t) = (S_0(t), B(t, T), S(t))'$ ,

3.  $\mathcal{A}$  is the set of admissible controls, i.e.

$$\mathcal{A} = \{ \underline{u}(t) : \underline{u}(t) \in \mathcal{F}(t), W(t) \underline{u}(t) \text{ is square integrable, and } W(T) - G(t) \geq 0 \} \quad (8.5)$$

From Eq.(8.1) it can be seen that  $\mathcal{A}$  is non-empty.

The quantities  $1 - u^B(t) - u^S(t)$ ,  $u^B(t)$  and  $u^S(t)$  represent the proportions of wealth invested into the riskless, the bond and the risky asset respectively.

## 8.1 Transforming the initial problem

In this section, we include the method of how in Deelstra et al. [19] an auxiliary process was introduced in order to obtain an equivalent simplified program. They achieved this

by defining the surplus process along the lines of Boulier et al. [7], and proved that this surplus process is self-financing. Proposition 8.1 is taken from the paper by Deelstra et al. [19] and proves that the two optimization problems are equivalent, but we first need to define the following processes as done by these authors.

We note that the surplus process  $Y(t)$ , for  $t \geq 0$  is defined by

$$Y(t) = W(t) + D(t) - G(t), \quad (8.6)$$

and that as in Deelstra et al. [19]

$$D(t) = \mathbb{E} \left[ \int_t^T \frac{H(s)}{H(t)} c(s) ds \middle| \mathcal{F}(t) \right] \text{ and } G(t) = \mathbb{E} \left[ \frac{H(T)}{H(t)} G(T) \middle| \mathcal{F}(t) \right].$$

Then this surplus process, at time  $t$ , is equal to the sum of the following terms:

1. the value of the portfolio  $W(t)$ ,
2. plus the discounted value of the future engagements coming from the contributor  $D(t)$ ,
3. minus the discounted value of the pension fund future engagement (that is the guarantee)  $G(t)$ .

The value of the process at time  $T$  is equal to the surplus  $W(T) - G(T)$ , while the initial wealth is given by

$$Y(0) = W(0) + \mathbb{E} \int_0^T H(s) c(s) ds - \mathbb{E}[H(T)G(T)] = Y(0) > 0. \quad (8.7)$$

**Proposition 8.1:** (i) *The surplus process is self-financing, i.e., there exists a progressive measurable random process  $(\underline{y}(t)) = \left( (1 - y^B(t) - y^S(t)), y^B(t), y^S(t) \right)$ , where  $t \in [0, T]$ , denoting the proportions of  $Y(t)$  invested into  $S_0(t)$ ,  $B(t, T)$  and  $S(t)$  respectively, such that*

$$dY(t) = Y(t) \underline{y}'(t) \text{diag}[\underline{S}(t)]^{-1} d\underline{S}(t). \quad (8.8)$$

(ii) Let  $\mathcal{A}^Y = \{(\underline{y}(t)) : \underline{y}(t) \in \mathcal{F}(t), Y(t)\underline{y}(t) \text{ is square integrable and (8.8) holds almost surely}\}$  denote the set of admissible controls of the problem

$$\max_{\underline{y}(t) \in \mathcal{A}^Y} \mathbb{E}U(Y(T)), \quad (8.9)$$

where  $t \in [0, T]$ . Then problem (8.3) is equivalent to problem (8.9).

*Proof.* (i) For the process  $L(t)$ , let  $\hat{L}(t)$  denote the process  $\hat{L}(t) := H(t)L(t)$ . Then

$$d\hat{Y}(t) = d\hat{W}(t) + d\hat{D}(t) - d\hat{G}(t).$$

From Eqs.(7.2), (7.6) and (8.4)

$$d(\hat{W}(t)) = \hat{W}(t) \left( \hat{u}(t)\sigma(t, r(t)) - \underline{\lambda}'(t) \right) d\underline{Z}(t) + \hat{c}(t)dt,$$

where  $\hat{u} = (u^B(t), u^S(t))$  and

$$\sigma(t, r(t)) = \begin{pmatrix} 0 & \sigma_B(T-t, r(t)) \\ \sigma_1 & \sigma_2\sqrt{\eta} \end{pmatrix}.$$

Using the martingale representation theorem for the Brownian motion as in Karatzas and Shreve [31], it turns out that there exists a unique square integrable process  $(\underline{\chi}(t))$  with  $\underline{\chi}(t) = (\chi(t), \chi^r(t))'$ , that satisfies

$$\int_0^T |\underline{\chi}(t)|^2 < +\infty \text{ P-a.e.} \quad (8.10)$$

such that

$$d(\hat{D}(t)) = -\hat{c}(t)dt + \underline{\chi}'(t)d\underline{Z}(t), \quad (8.11)$$

where  $t \in [0, T]$ .

Analogously, there is a unique square integrable process  $\underline{\rho}(t)$ , with  $t \in [0, T]$  and  $\underline{\rho}(t) = (\rho(t), \rho^r(t))'$ , satisfying

$$\int_0^T |\underline{\rho}(t)|^2 dt < +\infty \text{ P-a.e.} \quad (8.12)$$

such that

$$d(\hat{G}(t)) := d(\mathbb{E}[\hat{G}(T)|\mathcal{F}(t)]) = \underline{\rho}'(t)d\underline{Z}(t).$$

Finally,

$$d\hat{Y}(t) = \left( \hat{W}(t)(\hat{\underline{u}}'(t)\sigma(t, r(t))) - \underline{\lambda}'(t) + \underline{\chi}'(t) - \underline{\rho}'(t) \right)$$

and therefore the process  $Y(t)$  is self-financing. In order to prove Eq.(8.8), it is sufficient to define  $\hat{\underline{y}}'(t) = \left( y^B(t), y^S(t) \right)$  as

$$\hat{\underline{y}}(t) = \frac{1}{Y(t)} W(t) \hat{\underline{u}}'(t) + (D(t) - G(t))[\sigma(t, r(t))']^{-1} \underline{\lambda}(t) + H^{-1}(t)[\sigma(t, r(t))']^{-1}(\underline{\chi}(t) - \underline{\rho}(t)), \quad (8.13)$$

which proves (i).

(ii) The strategy  $(\underline{y}(t))$  in (8.13) (and respectively  $(\underline{u}(t))$ ), for  $t \in [0, T]$ , is admissible for (8.9) (and respectively 8.3), since all the terms in the right-hand side of Eq.(8.13) are square integrable. This in turn implies that the optimal values of Eqs.(8.3) and (8.9) are equal.

□

The explicit form of Eq.(8.8) is

$$\begin{aligned} \frac{dY(t)}{Y(t)} &= r(t)dt + y^B(t)\sigma_B(T-t, r(t))(dZ_r(t) + \lambda_2\sqrt{\eta}dt) + y^S(t)[\sigma_1(dZ(t) + \lambda_1dt) \\ &+ \sigma_2\sqrt{\eta}(dZ_r(t) + \lambda_2\sqrt{\eta}dt)], \end{aligned} \quad (8.14)$$

where  $Y(0) > 0$ , and where  $(\underline{y}(t)) = \left( (1 - y^B(t) - y^S(t)), y^B(t), y^S(t) \right)$  is linked by (8.13) with  $(\underline{u}(t))$  for  $t \in [0, T]$ .

## 8.2 The explicit solution of the optimization problem in the power utility case

Deelstra et al. [19] noticed in Eq.(8.13) that the optimal strategies  $(\underline{u}(t))$  for  $t \in [0, T]$  of the initial optimization program 8.3 are linked with the controls  $(\underline{y}(t))$  of program 8.9. In this section the explicit expressions of  $(\underline{y}(t))$  of program 8.9 with a CRRA utility function  $U$  as defined in Eq.(8.2). In this section we show the method proposed in Deelstra et al. [19] can be used to derive the explicit expressions of  $(\underline{y}(t))$  for different choices of contributions and guarantees.

In order to determine the solution  $(\underline{y}(t))$  of program (8.9), the explicit expression of the following quantity is needed, and is determined in Lemma 8.2:

$$\mathbb{E} \left[ \left( \frac{H(t)}{H(T)} \right)^{\frac{\gamma}{1-\gamma}} \right].$$

**Lemma 8.2:** *Suppose that  $c$  is a real number such that  $c(1 - \lambda_2 b) \leq 0$ . Then there exist two deterministic functions  $k(t, c)$  and  $l(t, c)$  such that*

$$\mathbb{E} \left[ \left( \frac{H(t)}{H(T)} \right)^c \middle| \mathcal{F}(t) \right] = k(T - t, c) \exp \left\{ -r(t)l(T - t, c) \right\}. \quad (8.15)$$

*Proof.* We prove the statement for  $c \neq 0$  by following the same reasoning as Deelstra et al. [18]. Recall that

$$H(t) = \exp \left\{ - \int_0^t r(t^*) dt^* - \int_0^t \underline{\lambda}'(t^*) d\underline{Z}(t^*) - \frac{1}{2} \int_0^t |\underline{\lambda}(t^*)|^2 dt^* \right\}$$

with  $\underline{\lambda}'(t^*) = (\lambda_1, \lambda_2 \sqrt{\eta})$ .

We calculate the quantity  $\frac{H(t)}{H(T)}$  as

$$\begin{aligned}
\frac{H(t)}{H(T)} &= \exp \left\{ - \int_0^t r(t^*) dt^* - \int_0^t \underline{\lambda}' d\underline{Z}(t^*) - \frac{1}{2} \int_0^t |\underline{\lambda}(t^*)|^2 dt^* \right. \\
&\quad \left. - \left( - \int_0^T r(t^*) dt^* - \int_0^T \underline{\lambda}' d\underline{Z}(t^*) - \frac{1}{2} \int_0^T |\underline{\lambda}(t^*)|^2 dt^* \right) \right\} \\
&= \exp \left\{ - \int_0^t r(t^*) dt^* - \int_0^t \underline{\lambda}' d\underline{Z}(t^*) - \frac{1}{2} \int_0^t |\underline{\lambda}(t^*)|^2 dt^* + \int_0^T r(t^*) dt^* \right. \\
&\quad \left. + \int_0^T \underline{\lambda}' d\underline{Z}(t^*) + \frac{1}{2} \int_0^T |\underline{\lambda}(t^*)|^2 dt^* \right\} \\
&= \exp \left\{ - \int_0^t r(t^*) dt^* + \int_0^T r(t^*) dt^* - \int_0^t \underline{\lambda}' d\underline{Z}(t^*) + \int_0^T \underline{\lambda}' d\underline{Z}(t^*) \right. \\
&\quad \left. - \frac{1}{2} \int_0^t |\underline{\lambda}(t^*)|^2 dt^* + \frac{1}{2} \int_0^T |\underline{\lambda}(t^*)|^2 dt^* \right\} \\
&= \exp \left\{ \int_t^T r(t^*) dt^* + \int_t^T \underline{\lambda}' d\underline{Z}(t^*) + \frac{1}{2} \int_t^T |\underline{\lambda}(t^*)|^2 dt^* \right\} \\
&= \exp \left\{ \int_t^T r(t^*) dt^* + \int_t^T (\lambda_1, \lambda_2 \sqrt{\eta}) d\underline{Z}(t^*) + \frac{1}{2} \int_t^T (\lambda_1^2 + \lambda_2^2 \eta) dt^* \right\} \\
&= \exp \left\{ \int_t^T r(t^*) dt^* + \frac{1}{2} \int_t^T (\lambda_1^2 + \lambda_2^2 \eta) dt^* + \int_t^T \lambda_1 dZ(t^*) \right. \\
&\quad \left. + \int_t^T \lambda_2 \sqrt{\eta} dZ_r(t^*) \right\} \\
&= \exp \left\{ \int_t^T (r(t^*) + \frac{1}{2} (\lambda_1^2 + \lambda_2^2 \eta)) dt^* + \int_t^T \lambda_1 dZ(t^*) \right. \\
&\quad \left. + \int_t^T \lambda_2 \sqrt{\eta} dZ_r(t^*) \right\}. \tag{8.16}
\end{aligned}$$

By rewriting Eq.(7.1) and integrating (see Deelstra et al. [18]) we find that

$$\begin{aligned}
\int_t^T dr(t^*) - \int_t^T b\alpha_1 dt^* + \int_t^T br(t^*) dt^* &= - \int_t^T \sqrt{\eta} dZ_r(t^*) \\
\lambda_2 [r(T) - r(t) - b\alpha_1(T-t) + \int_t^T br(t^*) dt^*] &= - \int_t^T \lambda_2 \sqrt{\eta} dZ_r(t^*).
\end{aligned}$$

Substituting back and taking the expectations in Eq.(8.16) we calculate

$$\begin{aligned}
\mathbb{E} \left[ \left( \frac{H(t)}{H(T)} \right)^c \middle| \mathcal{F}(t) \right] &= \mathbb{E} \left[ \exp \left\{ c \left[ \int_t^T (r(t^*) + \frac{1}{2} (\lambda_1^2 + \lambda_2^2 \eta)) dt^* + \int_t^T \lambda_1 dZ(t^*) \right. \right. \right. \\
&\quad \left. \left. - \lambda_2 \left[ r(T) - r(t) - b\alpha_1(T-t) + \int_t^T br(t^*) dt^* \right] \right] \right\} \middle| \mathcal{F}(t) \right] \\
&= f(T-t, r(t), c) \times B, \tag{8.17}
\end{aligned}$$

where  $B = \mathbb{E} \left[ \exp \left\{ -\Phi r(T) - \Psi \int_t^T r(t^*) dt^* \right\} \middle| \mathcal{F}(t) \right]$ ,

$$f(T-t, r(t), c) = \exp \left\{ c \left[ \frac{1}{2} (\lambda_1^2 + \lambda_2^2 \eta) (T-t) + \lambda_2 b \alpha_1 (T-t) + \lambda_2 r(t) \right] \right\}, \quad (8.18)$$

$$\Phi = c \lambda_2 \quad (8.19)$$

and

$$\Psi = -c(1 - b \lambda_2). \quad (8.20)$$

By applying Lemma 7.4 to Eq.(8.17) we obtain

$$\mathbb{E} \left[ \left( \frac{H(t)}{H(T)} \right)^c \middle| \mathcal{F}(t) \right] = f(T-t, r(t), c) K(\Phi, \Psi, T-t) \exp\{-r(t)L(\Phi, \Psi, T-t)\}. \quad (8.21)$$

From Eqs. (8.21) and (8.18) the result is obtained with

$$k(t, c) = K(\Phi, \Psi, t) \exp\{c[(\lambda_1^2 + \lambda_2^2 \eta)t + \lambda_2 b \alpha_1 t]\} \quad (8.22)$$

and

$$l(t, c) = -c \lambda_2 + L(\Phi, \Psi, t). \quad (8.23)$$

Following the approach of Deelstra et al. [19] we obtain the price of the bond with  $c = -1$ . □

**Proposition 8.3:** *For the optimal surplus process,  $Y(t)$ , we have the following decomposition (where the specific value of  $*$  is irrelevant):*

$$\frac{dY(t)}{Y(t)} = \frac{1}{1-\gamma} \frac{dH^{-1}(t)}{H^{-1}(t)} + \frac{l(T-t, \frac{\gamma}{1-\gamma})}{h(T-t)} \frac{dB(t, T)}{B(t, T)} + [*]dt.$$

*Proof.* The optimal surplus, according to Deelstra et al. [18], is given by

$$Y(t) = \mathbb{E} \left[ \frac{H(T)}{H(t)} I(\theta H(T)) \middle| \mathcal{F}(t) \right],$$

where  $\theta$  is determined by  $Y(0) = \mathbb{E}[H(T)I(\theta H(T))]$  and  $I(x) = (U')^{-1}(x) = x^{\frac{1}{\gamma-1}}$ .

If we apply Lemma 7.6 we obtain

$$\begin{aligned}
Y(t) &= \theta^{\frac{1}{\gamma-1}} H^{-1}(t) \mathbb{E} \left[ H(T)^{\frac{\gamma}{\gamma-1}} \middle| \mathcal{F}(t) \right] \\
&= \theta^{\frac{1}{\gamma-1}} H(t)^{\frac{1}{\gamma-1}} \mathbb{E} \left[ \left( \frac{H(t)}{H(T)} \right)^{\frac{\gamma}{\gamma-1}} \middle| \mathcal{F}(t) \right] \\
&= \left( \theta H(t) \right)^{\frac{-1}{1-\gamma}} k \left( T-t, \frac{\gamma}{1-\gamma} \right) \exp \left\{ -r(t) l \left( T-t, \frac{\gamma}{1-\gamma} \right) \right\} \\
&= \theta^{\frac{-1}{1-\gamma}} [H^{-1}(t)]^{\frac{1}{1-\gamma}} k \left( T-t, \frac{\gamma}{1-\gamma} \right) \exp \left\{ -r(t) l \left( T-t, \frac{\gamma}{1-\gamma} \right) \right\}.
\end{aligned}$$

If we let  $x_1 = H^{-1}(t)$ ,  $x_2 = r(t)$  and  $x_3 = t$ , then

$$dY(t) = \frac{\partial Y(t)}{\partial x_1} dx_1 + \frac{\partial Y(t)}{\partial x_2} dx_2 + \frac{\partial Y(t)}{\partial x_3} dx_3$$

where

$$\begin{aligned}
\frac{\partial Y(t)}{\partial x_1} &= \frac{1}{1-\gamma} \theta^{\frac{-1}{1-\gamma}} [H^{-1}(t)]^{\left(\frac{1}{1-\gamma}-1\right)} k \left( T-t, \frac{\gamma}{1-\gamma} \right) \exp \left\{ -r(t) l \left( T-t, \frac{\gamma}{1-\gamma} \right) \right\} \\
&= \frac{1}{1-\gamma} \frac{1}{H^{-1}(t)} Y(t),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial Y(t)}{\partial x_2} &= \theta^{\frac{-1}{1-\gamma}} [H^{-1}(t)]^{\frac{1}{1-\gamma}} k \left( T-t, \frac{\gamma}{1-\gamma} \right) \exp \left\{ -r(t) l \left( T-t, \frac{\gamma}{1-\gamma} \right) \right\} \\
&= -l \left( T-t, \frac{\gamma}{1-\gamma} \right) Y(t)
\end{aligned}$$

and we denote  $\frac{\partial Y(t)}{\partial x_3}$  by  $[*] \cdot Y(t)$ , where  $[*]$  is the locally deterministic factors.

Then

$$dY(t) = \frac{1}{1-\gamma} \frac{1}{H^{-1}(t)} Y(t) dH^{-1}(t) - l \left( T-t, \frac{\gamma}{1-\gamma} \right) Y(t) dr(t) + [*] Y(t) dt.$$

Thus

$$\frac{dY(t)}{Y(t)} = \frac{1}{1-\gamma} \frac{dH^{-1}(t)}{H^{-1}(t)} - l \left( T-t, \frac{\gamma}{1-\gamma} \right) dr(t) + [*] dt$$

$$\begin{aligned}
&= \frac{1}{1-\gamma} \frac{dH^{-1}(t)}{H^{-1}(t)} - l(T-t, \frac{\gamma}{1-\gamma}) \left( (a - br(t))dt - \sqrt{\eta} dZ_r(t) \right) + [*]dt \\
&= \frac{1}{1-\gamma} \frac{dH^{-1}(t)}{H^{-1}(t)} + l(T-t, \frac{\gamma}{1-\gamma}) \sqrt{\eta} dZ_r(t) + [*]dt \\
&= \frac{1}{1-\gamma} \frac{dH^{-1}(t)}{H^{-1}(t)} + l(T-t, \frac{\gamma}{1-\gamma}) \frac{\sigma_B(T-t, r(t))}{h(T-t)} dZ_r(t) + [*]dt \\
&= \frac{1}{1-\gamma} \frac{dH^{-1}(t)}{H^{-1}(t)} + \frac{l(T-t, \frac{\gamma}{1-\gamma})}{h(T-t)} \left[ \frac{dB(T, t)}{B(T, t)} - r(t)dt - \sigma_B(T-t, r(t)) \lambda_2 \sqrt{\eta} dt \right] \\
&+ [*]dt \\
&= \frac{1}{1-\gamma} \frac{dH^{-1}(t)}{H^{-1}(t)} + \frac{l(T-t, \frac{\gamma}{1-\gamma})}{h(T-t)} \frac{dB(t, T)}{B(t, T)} + [*]dt, \tag{8.24}
\end{aligned}$$

□

The optimal investment strategy that follows from Proposition 8.3 is summarized in the remark below and will be used in simulating the wealth function of the DC pension fund.

**Remark 8.4:**

1. Eq.(8.24) implies that the surplus process  $Y(t)$  can be replicated as a combination of the three processes  $H^{-1}(t)$ ,  $B(t, T)$  and  $S_0(t)$  with weights  $\frac{1}{1-\gamma}$ ,  $\frac{l(T-t, \frac{\gamma}{1-\gamma})}{h(T-t)}$  and  $1 - \frac{1}{1-\gamma} - \frac{l(T-t, \frac{\gamma}{1-\gamma})}{h(T-t)}$  respectively. Since the diffusion terms are equal in Eq.(8.24), it follows that the drift terms are also equal due to arbitrage arguments.

2. Following Deelstra et al. [19], we assume that the strategy that replicates the process  $H^{-1}(t)$  is given by

$$\left( \left( 1 - \frac{\lambda_1}{\sigma_1} - \frac{\sigma_1 \lambda_2 - \sigma_2 \lambda_1}{\sigma_1 h(T-t)} \right), \frac{\lambda_1}{\sigma_1}, \frac{\sigma_1 \lambda_2 - \sigma_2 \lambda_1}{\sigma_1 h(T-t)} \right).$$

3. We further assume under the hypothesis of Lemma 8.2 and following Proposition 8.3 that the trading strategy that solves Eq.(8.9) is given by

$$y^{S_0}(t) = 1 - y^B(t) - y^S(t),$$

$$y^B(t) = \frac{l(T-t, \frac{\gamma}{1-\gamma})}{h(T-t)} + \frac{1}{1-\gamma} \frac{\sigma_1 \lambda_2 - \sigma_2 \lambda_1}{\sigma_1 h(T-t)},$$

$$y^S(t) = \frac{1}{1-\gamma} \frac{\lambda_1}{\sigma_1}.$$

We proceed to draw the strategy that replicates the process  $H^{-1}(t)$  in Remark 8.4. The graphs in the two figures that follow can be explained as follow. The quantity  $1 - \frac{\lambda_1}{\sigma_1} - \frac{\sigma_1 \lambda_2 - \sigma_2 \lambda_1}{\sigma_1 h(T-t)}$  is represented by  $x$  (initially higher curve) in the figures, the quantity  $\frac{\lambda_1}{\sigma_1}$  is represented by  $y$  (initially middle curve) and finally the quantity  $\frac{\sigma_1 \lambda_2 - \sigma_2 \lambda_1}{\sigma_1 h(T-t)}$  is represented by  $z$  (initially lower curve).

The two figures below represent two different replicating strategies of  $H^{-1}(t)$ . In Fig.8.1. we have set  $\lambda_1 = 0.02$ ,  $\sigma_1 = 0.2$ ,  $\lambda_2 = 0.7$ ,  $\sigma_2 = 0.01$  and  $b = 0.09$ . In Fig.8.2. we have chosen  $\lambda_1 = 0.03$ ,  $\sigma_1 = 0.2$ ,  $\lambda_2 = 0.8$ ,  $\sigma_2 = 0.03$  and  $b = 0.05$ . The replicating strategies are graphed over periods of  $T = 20$  and  $T = 30$  years respectively.

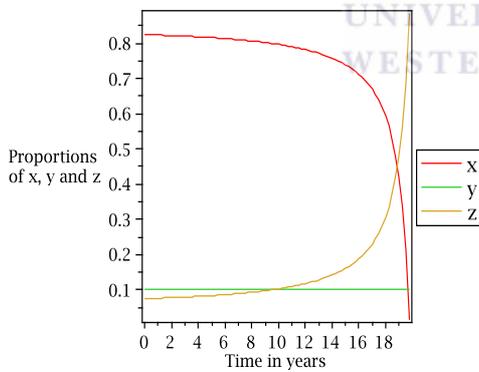


Figure 8.1: A simulation of the strategy that replicates the process  $H^{-1}(t)$  over  $T = 20$  years.

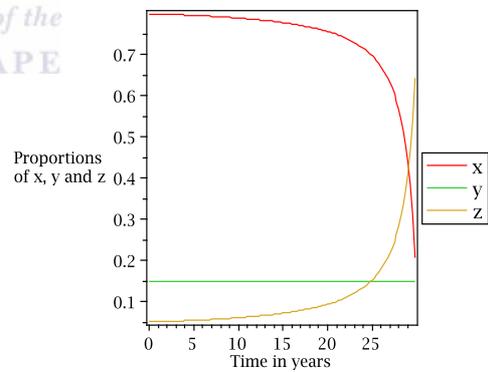


Figure 8.2: A simulation of the strategy that replicates the process  $H^{-1}(t)$  over  $T = 30$  years.

We now convert the results regarding the surplus to consider the actual investment portfolio pertaining to the wealth  $W$ .

We note that  $W(t) = Y(t) + G(t) + D(t)$ ,

and

$$\frac{dG(t)}{dt} = 0.$$

Therefore

$$\begin{aligned} dW(t) &= dY(t) + dD(t) \\ &= dY(t) + c(t)dt. \end{aligned}$$

Thus

$$\frac{dW(t)}{W(t)} = \frac{Y(t)}{W(t)} \left[ \frac{dY(t)}{Y(t)} + \frac{c(t)dt}{Y(t)} \right].$$

We note that given the  $\frac{dY(t)}{Y(t)}$  decomposition in the actual investment strategy, the cash component is all that increases. Therefore we have discovered:

**Proposition 8.5:**

$$\frac{dW(t)}{W(t)} = \frac{Y(t)}{W(t)} \left[ y^B(t) \frac{dB(t)}{B(t)} + y^S(t) \frac{dS(t)}{S(t)} + \left\{ y^{S_0}(t) + \frac{c(t)}{Y(t)} \right\} dt \right]$$

□

Further we note that

$$\frac{Y(t)}{W(t)} y^B(t) + \frac{Y(t)}{W(t)} y^S(t) + \frac{Y(t)}{W(t)} y^{S_0}(t) + \frac{c(t)}{W(t)} = 1.$$

Therefore

$$\frac{Y(t)}{W(t)} \left( y^B(t) + y^S(t) + y^{S_0}(t) \right) + \frac{c(t)}{W(t)} = 1.$$

Since  $\left( y^B(t) + y^S(t) + y^{S_0}(t) \right) = 1$ , it follows that

$$\frac{Y(t)}{W(t)} + \frac{c(t)}{W(t)} = 1.$$

Thus

$$W(t) = c(t) + Y(t).$$

We now proceed to plot the actual proportions of wealth invested in the assets. The member is assumed to pay contributions on a quarterly basis. If we let  $c(0)$  denote the very first contribution made by the member, then the expected contribution at time  $t$  is given by  $c(0)e^{\nu t}$ . We simulate the investment of wealth for twenty and twenty five year pension contracts assuming  $\nu = 0.1$ .

Parameters to simulate the proportions of wealth invested used in Fig.8.3. are  $a = 0.01$ ,  $b = 0.1$ ,  $\sqrt{\eta} = 0.15$ ,  $\lambda_1 = 0.14$ ,  $\lambda_2 = 0.21$ ,  $\sigma_1 = 0.02$ ,  $\sigma_2 = 0.14$  and  $\gamma = -1$ . The initial value of the short rate process is  $r(0) = 0.08$ , initial value of the optimal surplus  $Y(0) = 0.095$  and the contribution 0.1. The parameters in Fig.8.4. are  $a = 0.01$ ,  $b = 0.01$ ,  $\sqrt{\eta} = 0.014$ ,  $\lambda_1 = 0.13$ ,  $\lambda_2 = 0.21$ ,  $\sigma_1 = 0.02$ ,  $\sigma_2 = 0.13$ ,  $\gamma = -1$ ,  $r(0) = 0.08$ ,  $Y(0) = 0.095$  for a contribution of 0.1. The time until retirement in the two figures are respectively  $T = 20$  and  $T = 25$  years.

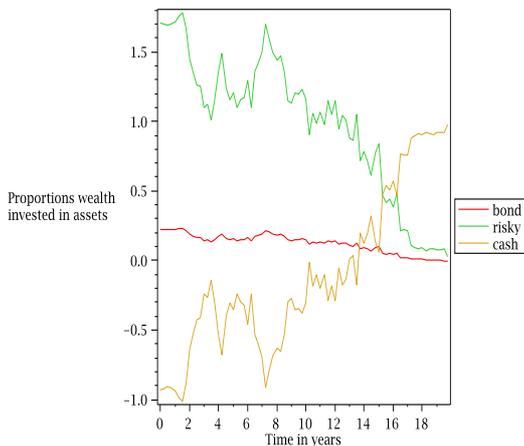


Figure 8.3: A simulation of the optimal proportions of wealth invested over  $T = 20$  years.

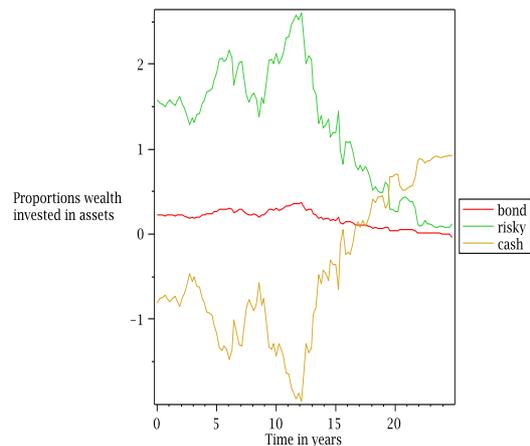


Figure 8.4: A simulation of the optimal proportions of wealth invested over  $T = 25$  years.

From the graphs of the optimal wealth investment proportions it can be noted that the largest proportion of wealth should be invested in the risky asset (initially higher curve) and the second largest proportion in the bond asset (initially middle curve). As the retirement of the employee approaches, the cash asset becomes the dominant asset as the largest proportion of wealth is invested in it, and replaces the risky asset. It is also interesting to note that the proportion invested in the cash asset starts in the negative. This phenomenon is explained in [19] and [7] as the fact that the guarantee can be seen as a put option on the value of the pension fund sold by the manager to the client. Dynamical hedging of this put requires a short position in cash and a long position in risky assets.



# Chapter 9

## Conclusion

We now discuss the main results of this investigation. In Chapter 5, we have showed, using the work of Blake [5] how the three principal types of pension funds are related through a set of options on the underlying assets in the pension scheme with exercise prices related to the value of the liabilities. The relationships between the pension schemes can be summarized as follows. While the contributions of a DC pension scheme member is invested in the underlying fund assets, a DB scheme's contributions are invested into a portfolio containing the underlying assets plus a European put option minus a European call option. This means that a DB scheme is in some degree a DC scheme. The put and call options are written by the scheme member and sponsor respectively. The present value of the the TMP plan on the retirement date is the larger of the two present values provided by the DC and DB plans. A TMP is thus equivalent to a call option held by the member on the asset plus a riskless discount bond. Whether the call option is exercised depends on whether the value of the assets exceeds the maturity value of the bond. We also found that the reward-risk preferences of pension fund members, sponsors and managers can be represented by an isoelastic utility function with a constant relative risk aversion parameter.

In Chapters 7 and 8 we have discussed the work of Deelstra et al. [19] who studied the problem of deriving an optimal strategy of investment for a DC plan consisting of three

assets, namely a riskless asset such as cash, a risky asset such as a stock and a zero-coupon bond. In Chapter 7 we introduced the dynamics of the short-rate process and the assets of the DC plan in the financial market. In Chapter 8 we transformed the optimization problem to an easily solvable problem and found, using the method proposed by Deelstra et al. [19], optimal proportions of the surplus process that should be invested in the fund assets in order to maximize the member's benefits at maturity. We then converted the results regarding the surplus to an investment portfolio pertaining to the wealth of the fund. From the simulations run for the optimal proportions of wealth invested in the assets, it turned out that the the largest proportion of wealth should be invested in the risky asset, while the proportion invested in the cash asset starts in the negative. However, as the retirement date of the employee approaches, the proportion invested in the cash asset should exceed that of the bond and the risky assets. The proportion of wealth invested in the risky asset decreases accordingly.

Chapter 6 is a piece of original work in which we addressed the problem of minimizing the cost of the embedded minimum guarantees associated with the contributions made by the member of the pension fund. By pricing the premiums of the guarantees as the difference between two put options, we found that by applying the method of Lagrange multipliers we can obtain an optimal sequence of guarantees that minimizes the sum of the squares of the present value of the total price of the guarantees. Although the minimum value of the sum of the squares and the optimal sequence of guarantees can be fairly accurately obtained from graphs in the cases of two and three period contracts, it is not quite possible to do so for multi-period contracts exceeding three periods. This is where the method of Lagrange multipliers is particularly useful. In our investigation we were content with simulating the Lagrange multipliers method for three and four period contracts in order to avoid complicated numerical calculations.

There are several directions of future research concerning the original piece of work in Chapter 6. The approach can for instance be used to solve the optimization problem

for pension plans with maturities exceeding twenty and even thirty years. This would involve more numeric calculations which can be done by implementing complicated iterative computer algorithms. It is also worth mentioning that in order to solve the optimization problem in chapter 6, the premiums of the incremental guarantees could possibly be priced as a combination of European call options. Lastly it would also be interesting to derive an optimal sequence of guarantees for a pension contract where the sum of the incremental guarantees is not known in advance.

On a completely different note, the investment strategies presented here, may well have relevance in a different kind of financial practice. In particular, methods such as these are applicable to the banking industry. The Basel II conditions on internationally active banks (see Mukuddem-Petersen and Petersen [40] for instance) require bank management to become more and more sophisticated. For example in [40] the authors examine a problem related to the optimal risk management of banks in a stochastic dynamic setting. Mukuddem-Petersen and Petersen [40] particularly studied the problem of minimizing the market and capital adequacy risk that involves the safety of the securities held and the stability of sources of funds by applying stochastic optimal control theory. Collaborative work, jointly with G.J. Van Schalkwyk and P.J. Witbooi is in progress on the application of the asset allocation strategy of Chapter 8 in the context of banking operations as mentioned in the abstract.

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