

Methods of optimizing investment portfolios



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A thesis submitted in fulfilment of the requirements for the degree Magister Scientiae in the Department of Mathematics and Applied Mathematics, University of the Western Cape.

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October 2013

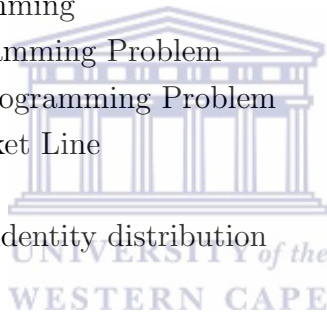
KEYWORDS

Optimisation
Convex Optimisation
Modern Portfolio Theory
Portfolio Management
Quadratic Programming
Markowitz Mean Variance Optimisation
Capital Asset Pricing Models
Value at Risk



Abbreviations

MPT - Modern Portfolio Theory
QP - Quadratic Programming
CAPM - Capital Asset Pricing Model
VaR - Value at Risk
MVO - Mean-Variance Optimisation
CVaR - Conditional Value at Risk
HS - Historical Simulation
MC - Monte-Carlo Simulation
FHS - Filtered Historical Simulation
KKT - Karush Kuhn Tucker
CAL - Capital Allocation Line
LP - Linear Programming
LPP - Linear Programming Problem
QPP - Quadratic Programming Problem
CML - Capital Market Line
EXP - Exponential
i.i.d. - independent identity distribution



Abstract

In this thesis, we discuss methods for optimising the expected rate of return of a portfolio with minimal risk. As part of the work we look at the Modern Portfolio Theory which tries to maximise the portfolio's expected rate of return for a certain amount of risk. We also use Quadratic Programming to optimise portfolios. Generally it is recognised that portfolios with a high expected return, carry higher risk. The Modern Portfolio Theory assists when choosing portfolios with the lowest possible risk. There is a finite number of assets in a portfolio and we therefore want to allocate them in such a way that we're able to optimise the expected rate of return with minimal risk. We also use the Markowian approach to allocate these assets. The Capital Asset Pricing Model is also used, which will help us to reduce our efficient portfolio to a single portfolio. Furthermore we use the Black-Litterman model to try and optimise our portfolio with a view to understanding the current market conditions, as well as considering how the market will perform in the future. An additional tool we'll use is Value at Risk. This enables us to manage the market risk. To this end, we follow the three basic approaches from Jorion [Value at Risk. USA: McGraw-Hills, 2001]. The Value at Risk tool has become essential in calculating a portfolio's risk over the last decade. It works by monitoring algorithms in order to find the worst possible scenarios within the portfolio. We perform several numerical experiments in MATLAB and Microsoft Excel and these are presented in the thesis with the relevant descriptions.

October 2013.

Declaration

I declare that “**Methods of optimising investment portfolios**” is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Thoriso P.J. Seepi



October 2013

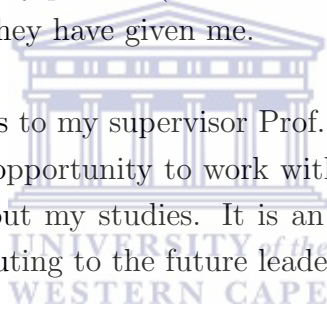
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Acknowledgements

I would like to thank the Almighty for blessing me and for the opportunities that He has given me. My parents are still alive to witness this important milestone in my life and I would like to thank God for that.

I would also like to thank my parents (Michael and Joy Seepi) and my entire family for the love and support they have given me.

My greatest gratitude goes to my supervisor Prof. Kailash C. Patidar for believing in me and giving me the opportunity to work with him. He has provided me with helpful guidance throughout my studies. It is an honour to work with him and I hope he continues contributing to the future leaders of our beautiful country.



DEDICATION

This thesis is dedicated to my late grandparents (both grandfathers and grandmothers), my uncle and my cousins. I would also like to dedicate this thesis to the entire Seepi and Rangoaga family as well as my church.



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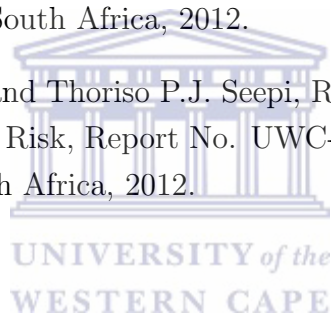
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List of Publications

The following technical reports have been generated from this thesis. Their revised forms will be submitted as research papers for publication:

- Kailash C. Patidar and Thoriso P.J. Seepi, Portfolio optimization by estimating the input with GARCH, Report No. UWC-MRR 2012/16, University of the Western Cape, South Africa, 2012.
- Kailash C. Patidar and Thoriso P.J. Seepi, Reverse optimization by minimizing conditional Value at Risk, Report No. UWC-MRR 2012/17, University of the Western Cape, South Africa, 2012.



Chapter 1

General Introduction

1.1 A quick review of portfolio theory

It has been said that the dinosaur known as the ‘Stock Broker’, is now being replaced by the ‘Financial Consultant’. The primary advice that financial consultants should give to their clients, is to have a portfolio. A portfolio is a collection of financial assets, also known as securities. This collection may consist of stocks, bonds and futures amongst others. It is essential for an investor to have a portfolio consisting of stocks, bonds or futures as this allows the risk to be spread, as opposed to having all of their eggs in one basket. Every finance book out there will give the same advice. The big challenge comes when selecting these assets to suit the investor.

As a new investor, the logical thing to do in selecting assets for a portfolio is to select a number of assets, review their historical returns, rank them in descending order with regards to their returns and then choose the most profitable stocks to construct the portfolio with. In fact, before the 1950’s, investors were only interested in the rate of return, as they were not able to quantify risk. This goes against the Modern Portfolio Theory (a fundamental theory in investment) founded by Nobel Prize winner, economist, Harry Markowitz ([51]).

The fundamental principle in the Modern Portfolio Theory (MPT) is based on the random walk hypothesis, which states that the future price of an asset is unable to be calculated with any certainty ([13]). The asset’s future price is unknown, i.e. the path of the asset’s price is usually unpredictable. It is important to note, that since there is uncertainty regarding the future price of an asset or security the ‘expected

return' is estimated. This is where another phenomenon known as 'risk' comes into this study. Risk is not possible if there is certainty regarding a future outcome. This also implies that two movements are possible; either upward or downward movements. This indicates that the security prices are random, as stated above, so it is assumed that the rate of return will follow a normal distribution. Other distributions may be followed, but as the sample size increases, so the distribution becomes virtually normal, due to the central limit theorem. The expected return is the expected value (mean) of this probability distribution. There is also standard deviation which provides a range of possible outcomes, as well as a description of the width and shape of the probability distribution.

Due to the fact that expected returns can be different from actual returns, risk can now be defined as the probability between the differences of the actual and expected returns. Since standard deviation is the range of possible outcomes, it can be concluded that having a small standard deviation will mean that the actual return won't be significantly different from the expected return. This means that little risk has been taken. The larger the standard deviation ranges the higher the risk. However, as there are two potential movements, big profits can also be made. In simple terms, the more uncertainty (a wide range of standard deviation) there is about what the actual return will be, the more risk the asset carries. So risk is directly proportional to the expected return and therefore the return of an asset is dependent on the risk. Harry Markowitz regarded the standard deviation as the conventional way of measuring risk. So, if standard deviation is regarded as risk, the main goal of investors is to try and minimise the standard deviation and maximise the expected return of a portfolio. This will form the basis of Quadratic Programming which we explore later. It should be noted that variance measures uncertainty and this is not the same thing as risk.

1.2 Portfolio management or optimisation

Optimisation is a branch of applied mathematics ([18]). Simply put, it is about minimising or maximising the given objective function, with several decision variables, which will satisfy functional constraints. The fundamental components of any optimisation problems are as follows:

- (i) a objective function;

- (ii) a decision variable; and
- (iii) a set of constraints.

In a general optimisation problem, we are given a function and a set, the problem solves.

$$\begin{aligned} & \min_x f(x) \\ & \text{subject to } x \in S \end{aligned}$$

There are generally two kinds of optimisation problems: constrained and unconstrained ([19]). For the purpose of this thesis we only be looking at the constrained optimisation problems. We deal with this category of problems due to the classical Markowitz ([50]) Mean-Variance Optimisation Model. In instances regarding constrained optimisation problems, a portfolio manager will give their clients advice on how they should allocate their wealth. The foundation of this advice is built on optimisation.

The Markowitz ([51]) model of portfolio selection only deals with convex and concave problems. This will be dealt with in the ‘efficient frontier’ discussion.

Portfolios are constructed and held as a part of an investment strategy for the purpose of diversification ([13]). Within the available portfolios, the primary interest area is regarding optimal portfolios. Optimal portfolios are constructed by using optimisation and the Markowitz framework. This model deals with two main measures including the variance and the return. So if this model is being used, a portfolio with a low variance and some level of return is preferred. It is important to note that variance is not a measure of risk, but rather a measure of uncertainty. A lower variance means a higher diversification level ([59]). Variance measures the fluctuations of the variable about r_i its mean, so that the larger the values of σ_i the riskier the investments.

To manage such risk the concept of diversification is introduced. This method was regarded by Markowitz ([51]) as being a practical way of managing risk. The Modern Portfolio Theory further suggests that the formulation of portfolio management risk should be primarily based on the entire portfolio and not individual or specific risk. In a portfolio, uncorrelated assets need to be put together. Consider this example: suppose there is one asset in a portfolio, namely Nokia. If Nokia’s share price goes

down, the investor will lose. However, if a second asset, namely Sasol, is added to the portfolio, the wealth is evenly distributed. Now, if the fuel price goes down, Sasol's share price will be harmed, however the Nokia share price will go up as a result of the reduction in the fuel price. These two assets have offset each other and this shows the usefulness of the diversification strategy. Adding more securities tends to improve the effect of diversification. However, diversification merely reduces the portfolio risk and does not eliminate it. Certain types of risk cannot be diversified and this is called Market Risk ([13]). Market Risk, also known as non-diversifiable risk (including war, inflation etc.), is part of the securities risk and investors are not rewarded for incurring this risk. They are rewarded for the stand-alone (see Figure 1.1) or systematic risk that can be eliminated by diversification.

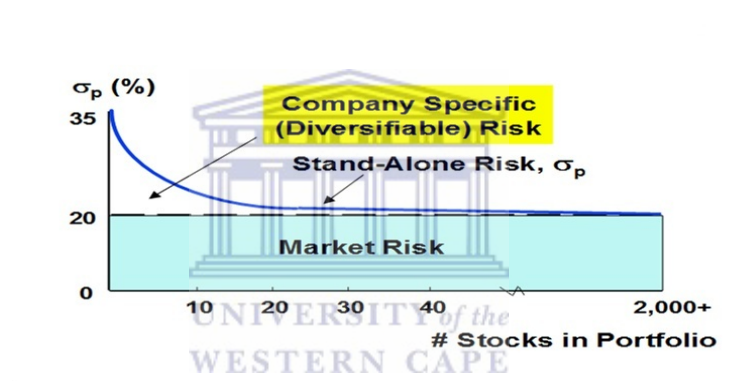


Figure 1.1: Portfolio's stand alone risk vs market risk ([16]) P76

Diversification is known as 'the rule of thumb', however it wasn't well understood. Previous authors presumed that diversification could eliminate all risk through the law of large numbers; to their thinking, as long as there were a large number of stocks in a portfolio they presumed that there was no risk. The advice that followed was to invest in those assets which had the highest expected return.

The Markowitz Model is based on the assumption that investors are risk averse and they are only concerned with the portfolio's mean risk (see Figure 1.2). This will lead to optimal portfolios being chosen. The feasible set (the shaded grey part on the diagram) of portfolios represents all portfolios that can be constructed from a given set of stocks. An efficient portfolio is one that offers the most return for a given amount of risk, or the least risk for a given amount of return. The collection of efficient portfolios is called the efficient set or efficient frontier, which represents

the asset mix with the highest expected returns for each given level of risk. This can be seen in the diagram. These are the sets of portfolios that were constructed after the returns, standard deviation and correlation coefficients were determined.

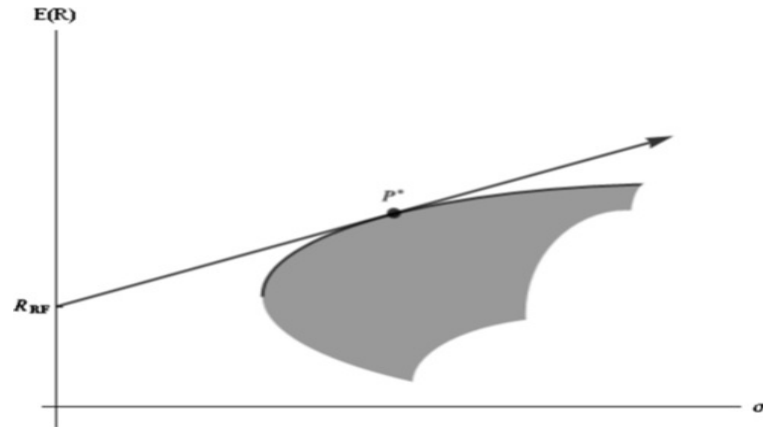


Figure 1.2: Markowitz mean-variance efficient frontier illustration diagram ([65] P437)

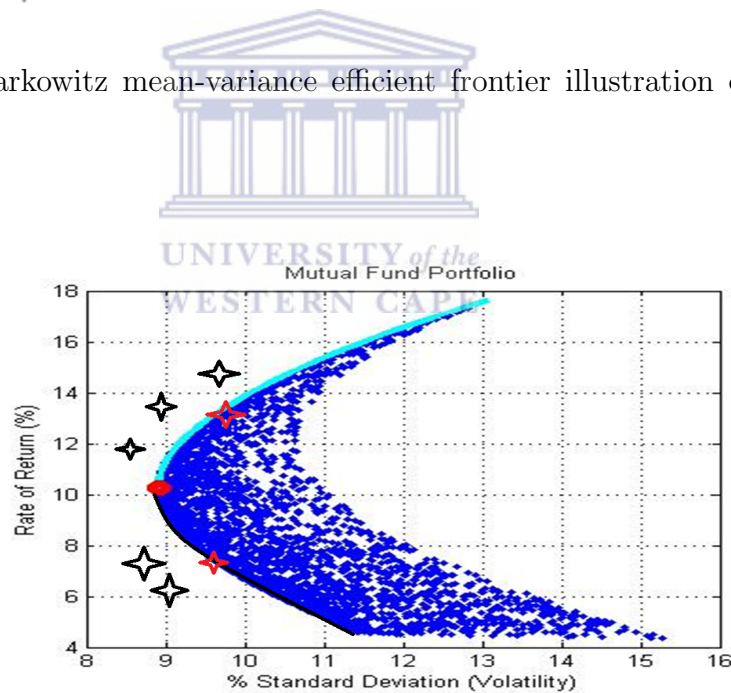


Figure 1.3: Portfolio envelope diagram of the mean-variance optimisation

Figure 1.3 presents an envelope of portfolios. The stars are the non-efficient portfolios, as they are outside the feasible region. The black curve is part of the envelope but is not efficient, as the portfolio on the black line will have a smaller return than the one exactly above it with the same standard deviation. The blue blocks are the efficient portfolios. The turquoise curve is called the *efficient frontier*. Portfolios on

the efficient frontier are superior to those efficient portfolios inside the curve. This is due to the fact that they have a high return. The red circle is called the *Global Minimum Variance Portfolio* and this is the lowest *efficient portfolio* in terms of risk and return. It is difficult to imagine how the stock returns depend on each other by looking directly at the covariance matrix. Every investor knows that there is a trade-off between risk and return. In order to increase the expected return on investment, an investor must be willing to tolerate greater risks. Portfolio theory, studies how to model this trade-off given a collection of possible investments with returns ([57]).

The correlation measures the tendency of the return on investments to move in the same direction. Two investments whose returns tend to rise and fall together, have a positive correlation; the nearer the two investments are the more closely they track each other. Investments whose returns tend to move in opposite directions have a negative correlation.

We are interested in portfolios for which the expected return is large, while the variance is small. In the model proposed by Markowitz, we combine these two aims into a single objective function, with the aid of a risk tolerance parameter denoted by A , and solve the following problem to find the optimal portfolio:

$$\max x^T \mu - Ax^T V x$$

subject to

$$\sum_i^n x_i = 1, x \geq 0.$$

In portfolio theory we weigh up the securities, in order to understand the proportion of each asset. This is known as asset allocation. This is described below.

Asset allocation

Asset allocation forms part of portfolio management. When managing portfolios it assists in making decisions regarding which assets to include in any given portfolio and when to include them. This proportion is obtained by dividing the total number of assets in the portfolio. The sum of those proportions must be equal to one. In performing asset allocation we are trying to diversify the risk. As mentioned earlier,

the fundamental advice that any investor will receive is to own a portfolio ([52]). With this in mind we now look at various strategies to help when constructing efficient portfolios. Each client has different expectations and these assets need to be allocated in such a way that their expectations will be met. For example, if a long term retirement plan is being considered, a younger investor should consider investing in higher risk assets, such as stocks and perhaps later investing in bonds. An older investor should consider investing in less risky assets such as bonds. The goal is to combine these assets to get the optimal return with minimal risk. Risk tolerance, age and goals all need to be considered in finding the right combination.

We now look into the general approach (see Figure 1.4) of asset allocation. The overall approach to asset allocation is called Integrated Asset Allocation ([68]). We also have other types of asset allocation approaches including strategic, tactical and insured. The integrated strategy is the most widely used, as unlike other strategies, it is able to account for investment risk tolerance. This is due to the fact that the net worth of the investor at the end of the period is the current investment tolerance. In this strategy there are two streams being followed; the first stream deals with capital markets and the second stream is specific to the investor. When considering these two streams, it is clear that this strategy comprises aspects of the other strategies. Not only does it account for expectations, but it also tackles changes in the market.

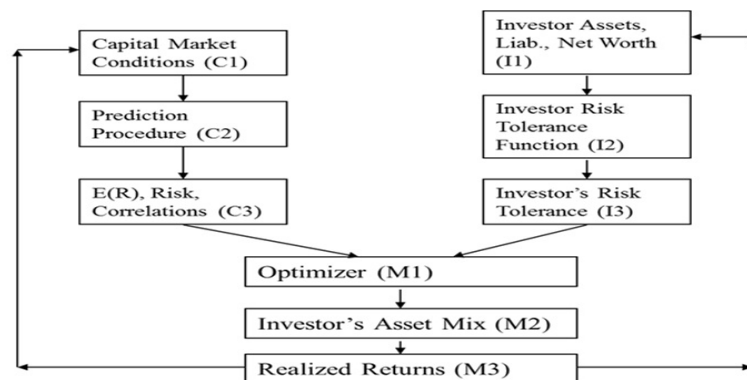


Figure 1.4: Integrated asset allocation: A diagrammatic overview ([68]) P26

Strategic asset allocation

Strategic asset allocation is based on the proportions which are the expected rates and rates of return. It is used for each asset and each asset class. These proportions

need to be calculated periodically, perhaps once every couple of years. This is important as there are few assets in a portfolio. This method follows the base policy mix. For example, if stocks have historically returned 20% per year and bonds have returned 10% per year, a mix of 50% stocks and 50% bonds would be expected to return 15% per year. For more information on this type of asset allocation strategy, the reader is referred to ([48]).

Tactical asset allocation

The tactical asset allocation strategy differs from strategic asset allocation. In tactical asset allocation there is a short time frame within which to try and exploit market changes. It should be assumed that asset class performance is mean-reverting ([48]). If stocks have performed above average in relation to bonds, it would be recommended that stocks should be under weighted and bonds should be over weighted for the next period ([59]).

Insured asset allocation

Insured asset allocation also uses short time frames to exploit changes in an investor's objectives and constraints. This type of asset allocation is based on the assumption that an investors wealth changes and that they become more risk tolerant as wealth increases ([59]).

Passive management

The passive management technique, also known as the buy-and-hold strategy, is a basic concept in portfolio management with regards to the period of buying and selling assets. The more complex alternative technique, which is active management, is based on the same principles as the passive technique. Passive management is simpler and less expensive than active management. In passive management one can construct a highly diversified portfolio and wait to see to if the investor reaches an expected target ([13]).

Simple portfolio calculation

Calculating parameters is often needed in order to perform a portfolio calculation. For each asset (stocks, bonds, etc.) statistical parameters are required, such as

the expected return, standard deviation, correlations and covariances. Firstly, the returns are calculated in order to get the means which are either arithmetic or geometric. We assume these are compound rates i.e. the investor's returns are reinvested. Let's suppose there are n assets in a portfolio ($n \geq 2$). The rate of return r_i , the expected return μ_i , the standard deviation σ and w_i the weight or probability of the asset is represented where $\sum_{i=1}^n w_i = 1$. We also have the $\rho_{i,j}$ as the correlation coefficient, where $i \neq j$.

For simplicity we start by looking at two assets. The return on the portfolio is

$$E(r_p) = w_1 r_1 + w_2 r_2$$

with

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_1 \sigma_2 \rho$$

and

$$cov(1, 2) = \frac{1}{M} \sum_{t=1}^T [r_{1,t} - E(r_1)][r_{2,t} - E(r_2)]$$

where

$$\rho = \frac{cov(1, 2)}{\sigma_1 \sigma_2}$$

From the above equations, the return r_p on the portfolio has been calculated by adding the product of the returns of the respective assets and their weights. We have also calculated the portfolio variance σ^2 which gives us the standard deviation σ . For a general case the following calculation is used:

$$E(r_p) = \sum_{i=1}^n w_i r_i,$$

$$Var_p = \sum_{i=1}^n w_i^2 Var(r_i) + 2 \sum_{i=1}^n \sum_{j \neq i}^n w_i w_j Cov(r_i, r_j).$$

The risk-free asset

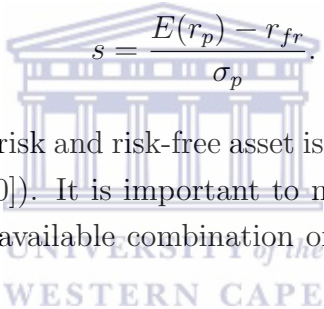
After calculating the risk with regards to the rate of return, it can now be combined with the risk-free rate (r_{fr}).

$$E(r_p) = r_{fr} + \left[\frac{r_p - r_{fr}}{\sigma_R} \right] \sigma_p$$

The following equation is then used to calculate risk aversion.

$$A = \frac{E(r_p) - r_f}{\frac{1}{2}\sigma_p^2}.$$

The alternative to risk aversion is risk tolerance. The Capital Allocation Line (CAL) is the gradient of this allocation. The reward to variability ratio is given by.


$$s = \frac{E(r_p) - r_{fr}}{\sigma_p}.$$

The combination of a high risk and risk-free asset is one of the steps required in order to diversify a portfolio ([20]). It is important to note that only government bonds are risk-free. Finding the available combination of risk and return is the technical part of asset allocation.

1.3 Literature review

Optimisation is an abstract tool, in the sense that it can be applied to various disciplines such as engineering, biology, finance, etc. There is uncertainty regarding where optimisation originated and who created this concept, but it can be traced back to the work of Newton and Leibnitz in calculus, who laid the foundations of optimisation ([58]). These foundations paved the way for Bernoulli, Euler, Lagrange and Weirstrass to devise minimising functions. Lagrange then came up with the Constrained Optimisation Problem ([58]). The development of the simplex method used in Linear Programming was introduced by Dantzig ([19]). Numerical methods for constrained optimisation became popular in the 1960's.

Modern Portfolio Theory (MPT), which is based on Quadratic Programming, was formulated by the Nobel Prize winner Markowitz ([50]). This was the result of an investor not being able to quantify the risks involved when holding stocks. Markowitz

then came up with a method of selecting portfolios by minimising the portfolio risk for a given level of return. The suitable portfolio would therefore be dependent on the risk aversion of the investor. Markowitz presented this as a form of diversification. The algorithm which is used to select assets within portfolios is well documented in his monograph ([51]). Tobin ([69]) introduced the concept of dominance. He envisioned an investor who was free to select his portfolio from a set of high risk assets and one risk-free asset i.e. cash. He showed that one set of relative proportions of the high risk assets would dominate all other possible combinations, in the sense that, for any given level of risk, it gave the investor ‘the highest possible expectation of return available to him at that level of risk’. Therefore, in an optimal portfolio, ‘the proportionate composition of the non-cash assets is independent on their aggregate share of the investment balance’. An investor’s attitude toward risk would be reflected in the percentage of the value of cash assets in the portfolio, rather than the proportionate composition of the non-cash assets.

Capital Asset Pricing Model (CAPM) is an extension of Markowitz’s model. It was proposed by Nobel Prize winner Sharpe ([65]) in 1964, Treynor ([72]) (who is recognised by French ([26])), Linter ([45]) and Mossin (cite Mossin-1966). They all worked independently on the foundation of the Markowitz model. This is not the process of direct portfolio selection, but is rather the pricing of securities in relation to market conditions. Sharpe suggested that the market portfolio was in equilibrium and that investors should buy into this portfolio. This is known as the tangency portfolio in which a line can be drawn from the portfolio. This line is called the Capital Market Line (CML). So, taking various investors’ risk tolerance into consideration, the variation should fall along that particular line. Sharpe argues that when in equilibrium, there is a simple linear relationship between the anticipated return and standard deviation. The main function of this is to give an indication of whether the assets are over-priced or underpriced. CAPM mainly assesses the asset’s risk against the return. CAPM also claims that the company’s events have very little influence towards the asset’s anticipated return. This assumes that an investor will be holding a diversified portfolio as a result of the Markowitz efficient frontier. CAPM has two basic fundamentals which compare the market portfolio and the market risk.

The CAPM test was performed by Jensen ([38]) and Fama ([23]). CAPM therefore states that in equilibrium, only the systematic (market) risk is priced, and not the total risk; investors do not need to be compensated for unique risk. The equation

of the CAPM reads as

$$R_i = r_f + \beta_i[(E(R_M) - r_f)],$$

where

R_i =anticipated return of the asset i

r_f =risk-free rate

β_i =asset beta

$E(R_M)$ =markets' anticipated rate of return. Beta is the covariance of the asset's i returns and the market returns, divided by the markets standard deviation i.e.,

$$\beta_i = \frac{Cov(i, M)}{\sigma_M}.$$

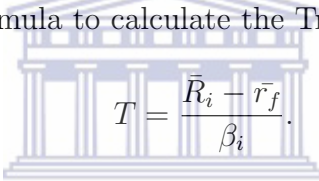
CAPM was criticised by Roll ([62]). He argued that the entire CAPM theory was based on the market portfolio and that it is almost impossible to calculate this, as it must contain every high risk asset in the universe

Black and Litterman ([11]) developed an extension, which was considered a breakthrough with regards to the Markowitz model. This was their first paper and it was documented under Goldman Sachs. It was first published in 1991 ([11]) and the extended version was then published in 1992 ([12]). The model is highly sophisticated and enables a far more intelligent portfolio choice, as opposed to what the classic Markowitz mean-variance optimisation model produces. The first step starts with the reverse optimisation of Black's zero-beta model ([9]). This assists in obtaining the implied returns of an equilibrium portfolio. The combination of reverse optimisation, CAPM ([65]), mixed estimation ([70]), the universal hedge ratio, Black's global CAPM ([10]), ([7]), ([32]), and Markowitz's ([51]) mean-variance optimisation model enabled Black and Litterman to combine the current market conditions with the anticipated rate of returns. The returns need to be amended by following a thorough process. Best ([8]) proved that a small change in the anticipated return can result in a huge shift from the original portfolio. Best and Grauer's ([8]) paper investigates the sensitivity of the mean-variance (MV) efficient portfolios which can change the means of individual assets. For example, if only a budget constraint is imposed on an investment problem, the analytical results indicate that an MV-efficient portfolio's weights, means and variance can be extremely sensitive to changes in asset means. When non-negativity constraints are then also imposed on the problem, the computational results confirm that a positively weighted MV-efficient portfolio's

weights are extremely sensitive to changes in asset means; however the portfolio's returns are not. A surprisingly small increase in the mean of just one asset could drive half the securities from the portfolio. Even with this change, the portfolio's expected return and standard deviation are virtually unchanged.

Portfolio performance measure

The Composite Portfolio Performance Measurement was introduced by Treynor ([72]). He focused on a method of quantifying the performance of a portfolio, regardless of the different policies or the risk tolerance of the investor ([59]). He came up with a characteristic line that identifies a portfolio's return volatility against the returns of the market or benchmark. From that relationship he then suggested that an investor should always consider portfolios that are close to the line. The composite quantitative formula to calculate the Treynor measure is:


$$T = \frac{\bar{R}_i - \bar{r}_f}{\beta_i}.$$

Another commonly used composite portfolio measurement, which is similar to Treynor's, is called Sharpe's Ratio. This measurement, which is an extension to Treynor's measure, was introduced by Sharpe ([66]). He was attempting to extend Treynor's ratio by subjecting the proposed measure to an empirical test, in order to evaluate its predictive ability. Furthermore, he was trying to make the relationships between the development in capital theory and the alternative models of mutual fund performance more explicit. He also aimed to subject these alternative models to empirical tests. The difference between Treynor and Sharpe's measurement is that Sharpe looks at not only the systematic risk but the whole portfolio's risk, as we can see in the following equation:

$$S = \frac{\bar{R}_i - \bar{r}_f}{\sigma_i}.$$

Another well-known statistical portfolio performance measure is the Information Ratio. It is also known as the Appraisal Ratio. This ratio quantifies the ability of a portfolio to outperform the benchmark. It divides the excess return of a portfolio and benchmark by the standard deviation ([59]). Due to the fact that the standard deviation is always positive, this gives an indication as to whether the portfolio will

perform better than the benchmark. The information ratio is calculated with the following formula:

$$\mathbb{R}_i = \frac{\bar{R}_i - \bar{R}_B}{\sigma_{ER}}.$$

Value at Risk

Value at Risk (VaR) is a risk measure which provides an alternative to the Markowitz ([50]) standard deviation. It is a commonly used methodology for estimating market risks. However, there is very little consensus as to which method is superior ([15]). Alexandra ([1]) claims that VaR was developed by Baumol in 1963, Markowitz ([50]) had the notion which clearly stated that investors should concentrate on far more than just the expected return. There is also evidence that Roy ([63]) came up with the confidence based risk measures. Regardless of the above, a significant need came about in the late 1980's due to worldwide financial disasters. Understandably, banks wanted to protect themselves against credit risk and this is where confidence based risk measures were needed. VaR became popular in the 1990's and the theoretical background was formally laid down by Jorion in 1994 ([40]). In 1994 Morgan ([54]) also launched RiskMetrics. Regulators and the financial industry advisory committees, such as the Basel Committee, recommended VaR as a method to measuring risk.

VaR summarises the expected maximum loss over a specified time horizon with a specific confidence level ([40]). The level of confidence is chosen based on specific needs. For example, the Basel Committee requires a 99% confidence level for their official reports ([4]). The most commonly used levels of confidence are 90%, 95% and 99%. VaR assumes the portfolio is frozen for the time horizon, or more generally, assumes the risk profile remains constant. There are three common approaches which estimate VaR, namely; the parametric approach, the non-parametric approach and the Monte-Carlo methods. These common approaches are formally laid down by Jorion ([40]) and by Morgan's RiskMetrics technical document ([54]).

It is also important for portfolio managers to study the assets contribution of risk to the whole portfolio. Hallerbach ([31]) shows a general analysis of decomposition by looking at (i) the marginal contribution of the individual portfolio components to the diversified portfolio VaR, (ii) the proportion of the diversified portfolio VaR that

can be attributed to each of the individual components constituting the portfolio, and *(iii)* the incremental effect on VaR of adding a new instrument to the existing portfolio.

VaR gives us the total risk of a portfolio. It is an intuitive approach for investors who are not too concerned about the finer details within a portfolio. It is essential for portfolio managers to study the assets' contribution of risk to the entire portfolio. Hallerbach ([31]) demonstrates a general analysis of decomposition, by looking at *(i)* the marginal contribution of individual portfolio components to the diversified portfolio VaR. *ii)* He looks at the proportion of the diversified portfolio VaR that can be attributed to each of the individual components constituting the portfolio, as well as the *(iii)* incremental effect on VaR when adding a new instrument to the existing portfolio. Expressions for these marginal and component VaR metrics were derived by Garman ([28]) ([29]) under the assumption that returns are drawn from a multivariate normal distribution. Whenever these deviations from normality are expected to cause serious distortions in VaR calculations, one has to resort to either alternative distribution specifications or historical and Monte-Carlo simulation methods. Although these approaches to overall VaR estimation have received substantial interest in the literature, there were no substantial procedures for estimating marginal VaR and component VaR in a non-normal analytical setting or a Monte-Carlo or historical simulation context. He tried to fill this gap by investigating these VaR concepts in a general distribution-free setting. He derived a general expression for the marginal contribution of an instrument to the diversified portfolio VaR, whether this instrument was already included in the portfolio or not. In the general sense, he showed how a total portfolio VaR can be decomposed in partial VaR which can then be attributed to individual instruments comprised in the portfolio. This component of VaR has an appealing property that aggregates linearly into the diversified portfolio VaR. He not only shows how standard results under normality can be generalised to non-normal analytical VaR approaches, but he also presents an explicit procedure for estimating marginal VaR in a simulation framework. The marginal VaR estimate and the component VaR readily follow.

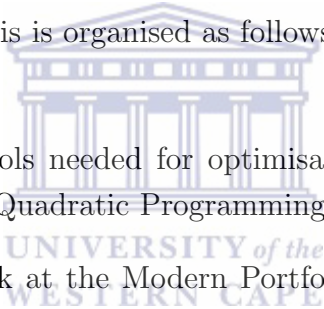
Portfolio optimisation using VaR and CVaR

This approach was introduced by Rockafellar and Uryasev ([60]) to optimise or hedge a portfolio of financial instruments to reduce the portfolio's risk. They presented and

tested various applications. They also focused on minimising Conditional Value-at-Risk (CVaR) rather than minimising Value-at-Risk (VaR). However, portfolios with low CVaR don't necessarily have low VaR as well. CVaR, also called Mean Excess Loss, Mean Shortfall or Tail VaR, is considered a more consistent measure of risk than VaR. Central to this new approach is a technique for portfolio optimisation which calculates VaR and optimises CVaR simultaneously. This technique is suitable for use by investment companies, brokerage firms, mutual funds and any business that evaluates risks. It can be combined with analytical or scenario-based methods to optimise portfolios with large numbers of instruments, in which case the calculations often come down to Linear Programming or non-smooth programming.

1.4 Outline of the thesis

The remainder of this thesis is organised as follows:

- 
- In Chapter 2 the tools needed for optimisation are explored. We focus on Convex, Linear and Quadratic Programming.
 - In Chapter 3 we look at the Modern Portfolio Theory which focuses on the passive management of Markowitz's portfolio selection using historical returns and computing returns. We also follow the Capital Asset Pricing Model and the Black-Litterman model in order to incorporate our view.
 - In Chapter 4 we look at the Value at Risk (VaR), an alternative approach to the risk measure.
 - In Chapter 5 we explore some applications of the findings through various simulation results.
 - In Chapter 6 we present our concluding remarks and scope for future research.

Chapter 2

Optimisation Techniques for Modern Portfolio Theory

2.1 Introduction

In this chapter we study the basic concepts surrounding the problems associated with optimisation. We do this by exploring the underlying terminologies surrounding optimisation. The Markowitz Portfolio Selection Model is both a concave and convex set. We start with Convex Programming and its properties. We also deal with the sufficient and necessary conditions called Karush Kuhn Tucker (KKT). These conditions enable us to compute the Quadratic Programming Problem (QPP), which we deal with later. Before we deal with QPP, we cover the Linear Programming Problem (LPP), as this needs to be understood in order to understand QPP.

2.2 Constrained optimisation techniques

Convex Programming

Mean-Variance Optimisation deals with convex sets. We now look at the properties of Convex Programming and we start by defining a convex set as:

Definition 2.2.1. ([44]) *If we have a set of points belonging to domain D , and we then join a line between any two points, and the line is in Domain D , then the set is called a convex set.*

Mathematically speaking a convex set is the set of all points x_1, x_2 in D such that

$$\alpha x_1 + (1 - \alpha)x_2 \in D.$$

This is reported in Figure 2.1.

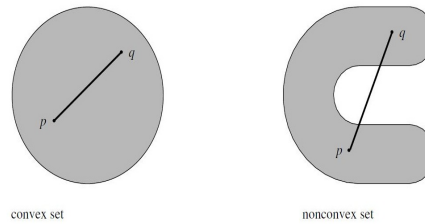


Figure 2.1: Convex and non Convex Set: An illustrative example ([33]) P57

Convex functions are easy to optimise. It is interesting to note that the convexity property gives the assurance that diversification is possible.

In general optimisation problems we are trying to solve what we call the Global Minimum Point of the feasible set; see Figure 2.2. This is done by searching for that point using optimisation methods, such as Linear Programming (which uses numerical methods for instance Simplex and Interior Point Methods) or Quadratic Programming (which uses numerical methods for example the Active Set and Interior Point Methods).

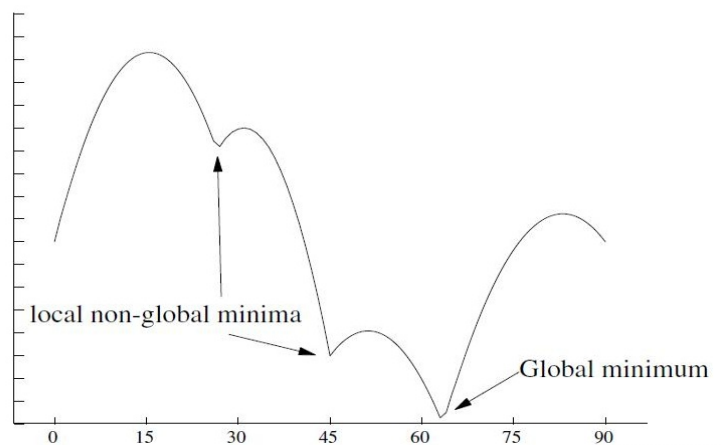


Figure 2.2: Local minima and global minimum ([33]) P2

Properties of convex functions

- Not all convex functions are differentiable. If a convex function is twice continuously differentiable, then the corresponding Hessian is a positive semi-definite matrix.
- All convex functions are continuous if they are considered within an open set.
- The local minima of a convex function is global. If a convex function f is twice continuously differentiable, then the global minimum is obtained in the points solving the first-order condition, $\nabla f(x) = 0$.
- A sum of convex functions is a convex function.

Optimality Constraints

These constraints are usually described through the following theorem:

Theorem 2.2.1. ([41], [57]) (Karush Kuhn Tucker (KKT) Conditions for QPP)
A feasible solution $x^ \in \mathbb{R}^n$ to the Quadratic Programming Problem (QPP)*

$$\text{minimise } a^T x + \frac{1}{2} x^T B x,$$

$$\text{subject to } Cx = d,$$

$$x \geq 0,$$

is optimal if, and only if, an m -vector exists λ and an n -vector $\mu \geq 0$ such that

$$a^T + x^{*T} B = -\lambda^T C + \mu^T,$$

$$\mu^T x^* = 0.$$

Theorem 2.2.2. ([41], [57]) (Karush Kuhn Tucker (KKT) Conditions for Unconstrained Quadratic Programming Problem (UQPP))

A feasible solution $x^* \in \mathbb{R}^n$ to the Unconstrained Quadratic Programming Problem

$$\text{minimise } a^T x + \frac{1}{2} x^T B x,$$

$$\text{subject to } Cx = d,$$

$$x \geq 0,$$

is optimal if, and only if, an m -vector exists such that

$$a^T + x^T B = -\lambda^T C.$$

Corollary. ([6]) Any vector $x \geq 0$ satisfying $Cx = d$ is an optimal solution to QPP if, and only if, there is an optimal solution to UQPP.

One of the most general frameworks in which the KKT conditions are necessary and sufficient is that of Convex Programming. Adding the assumption of convexity makes the KKT condition necessary and sufficient ([58]).

2.3 Linear Programming

Linear Programming is one the popular optimisation problems. The popularity of this method lies in its simplicity. Non-linear Programming is built upon this optimisation technique. So in simple terms, this means that in order for us to complete the computation of Quadratic Programming it is useful to have some background of Linear Programming. A typical Linear Programming Problem reads as follows:

$$\min_x f(x)$$

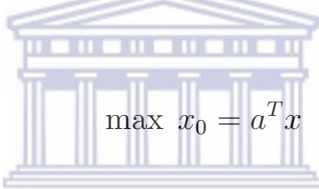
subject to

$$Cx = d$$

and

$$x'_i s \geq 0.$$

In the above system we have $f(x)$ as the objective function, x is the decision variable, $C \in \mathbb{R}^{m \times n}$ and d is a vector of certain coefficients. There are three linear outcomes for a Linear Programme, namely; it is not feasible, it has an unbound optimum or it has an optimal solution. It is important to note that a problem may not always be given as a standard problem. We may have a maximisation problem and the inequalities may be given as greater or less than. Necessary adjustments may need to be made in order to obtain the standard form of the problem. We illustrate this adjustment by illustrating an example using the Simplex Method. In the system of equations above, it is seen that all the equations have to be linear, and this marks the difference in the optimisation problems. Without loss of generality we assume the following:



$$\begin{aligned} \max \quad & x_0 = a^T x \\ \text{subject to} \quad & Cx \leq d, \quad d \geq 0, \\ & x \geq 0. \end{aligned}$$

Table 2.1: Simplex Algorithm

| | x_1 | x_2 | \dots | x_s | \dots | x_n | x_{n+1} | \dots | x_{n+r} | \dots | x_{n+m} | b |
|-----------|----------|----------|----------|----------|----------|----------|-----------|----------|-----------|----------|-----------|----------|
| x_{n+1} | c_{11} | c_{12} | \dots | c_{1s} | \dots | c_{1n} | 1 | \dots | 0 | \dots | 0 | b_1 |
| x_{n+2} | c_{21} | c_{22} | \dots | c_{2s} | \dots | c_{2n} | 0 | \dots | 0 | \dots | 0 | b_2 |
| \vdots | \vdots | \vdots | \ddots | \vdots | \vdots | \vdots | 0 | \ddots | \vdots | \ddots | \vdots | \vdots |
| x_{n+r} | c_{r1} | c_{r2} | \dots | c_{rs} | \dots | c_{rn} | 0 | \dots | 1 | \dots | 0 | b_r |
| \vdots | \vdots | \vdots | \ddots | \vdots | \ddots | \vdots | 0 | \ddots | \vdots | \ddots | \vdots | \vdots |
| x_{n+m} | c_{m1} | c_{m2} | \dots | c_{ms} | \dots | c_{mn} | 0 | \dots | 0 | \dots | 1 | b_r |
| x_0 | $-a_1$ | $-a_2$ | \dots | \dots | \dots | a_n | 0 | \dots | 0 | \dots | 0 | 0 |

In Table 2.1 we have $x_{n+i}; i = 1, \dots, m$ as the slack variables. The variables that were present in the original equation $x_i; i = 1 \dots n$ are called the decision variables.

The slack variable, or the basic variable, is obtained from the system of equations by setting the decision variables to zero. Some relevant results regarding LPP are

Lemma 2.3.1. ([19]) *Any solution to a Linear Programming Problem that is a local minimum solution, is also a global minimum solution.*

Theorem 2.3.2. ([19]) *A basic feasible solution to a LPP $Cx = d$, $x \geq 0$, $a^T x = \min$ corresponds to an extreme point in the convex set of feasible solutions to the LPP.*

Theorem 2.3.3. ([58]) *The feasible region of a Linear Programming Problem is a convex set.*

Theorem 2.3.4. ([44]) *Every basic feasible solution is an extreme point of the convex set of feasible solutions.*

Theorem 2.3.5. ([58]) *A function $f(X)$ is convex if the Hessian matrix $H(X) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$ is positive semi-definite.*

Theorem 2.3.6. [33] *Any local minimum of a convex function $f(X)$ is a global minimum.*

If the Hessian matrix H is positive semi-definite, it is said that the Quadratic Programming Problem's (QPP) standard form is a convex QPP. In this case, the problem is not that much more difficult to solve than a Linear Programming Problem (LPP). Non-convex QPP, in which H is an indefinite matrix, can be more challenging, since they can have several stationary points and local minima, see ([57]) for further clarifications.

According to the Markowitz Model, we can see that we have a QPP. QPP is more complex than LPP. If Q is not positive semi-definite, then our problem will be non-convex and therefore within a non-convex problem. There may be several local minimisers that are not global. The importance of these are seen when optimisation, using the covariance matrices, need to be performed.

Simplex Method

The Simplex Method was developed by Dantzig ([19]). This method obtains the basic feasible solution by searching for the optimal solution. If the optimal solution is not achieved, it will continue to search until the lower or equal value is reached.

This is the simplest and most common method to use in linear optimisation when looking for the optimal solution. It is also known as the Tableau Method. We explore this method through an illustrative example. The problem is to

$$\begin{aligned}
 & \text{maximise } p = \frac{1}{2}x_1 + 3x_2 + x_3 \\
 & \text{subject to } x_1 + x_2 \leq 4 \\
 & \quad 2x_1 + x_2 - x_3 \geq 3 \\
 & \quad x_1 + 3x_3 \leq 5 \\
 & \quad x_1, x_2, x_3 \geq 0
 \end{aligned} \tag{2.3.1}$$

Now the inequalities are removed to obtain a standard form, by introducing the slack variables. There are three inequalities and three decision variables conditions, so we have s_1, s_2, \dots, s_6 to obtain the following

$$\begin{aligned}
 & \text{maximise } p = \frac{1}{2}x_1 + 3x_2 + x_3 \\
 & \text{subject to } x_1 + x_2 + s_1 \leq 4 \\
 & \quad 2x_1 + x_2 - x_3 + s_2 \geq 3 \\
 & \quad x_1 + 3x_3 + s_3 \leq 5 \\
 & \quad x_1, x_2, x_3, s_1, s_2, \dots, s_6 \geq 0
 \end{aligned} \tag{2.3.2}$$

and

$$\begin{aligned}
 & p - \frac{1}{2}x_1 - 3x_2 - x_3 \quad \text{row1} \\
 & \quad x_1 + x_2 + s_1 \leq 4 \\
 & \quad 2x_1 + x_2 - x_3 + s_2 \geq 3 \\
 & \quad x_1 + 3x_3 + s_3 \leq 5
 \end{aligned} \tag{2.3.3}$$

The above equation is solved in terms of the decision variables x_1, x_2 and x_3 . Suppose we set $x_1 = x_2 = x_3 = 0$, we then get $s_1 = 4, s_2 = 3$ and $s_3 = 5$ with $p = 0$ and these are the basic solutions. Since we want to increase p as much as possible, we can ask whether we are still able to increase this. Since x_1, x_2 and $x_3 \leq 0$ we can see that we are still able to increase p due to the negative decision variables. The following is then implemented to obtain a solution:

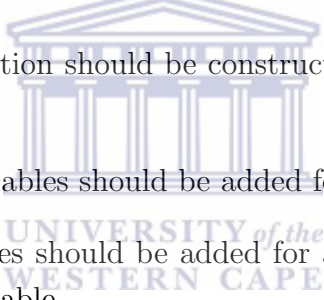
- (i) If all variables in *row1* are non-negative, we can then conclude that the basic solution is optimal. If so, any variable can be chosen that has a negative coefficient.
- (ii) For the non-*row1* rows, the computed ratio of the entered variable and pivot row should be the one with the minimum ratio.

We then use these rules to obtain results as in Table 2.2. The optimal solution is $p = 49/4$; $x_1 = 1/2$, $x_2 = 7/2$, $x_3 = 3/2$.

Wolfe's Method

Wolfe's ([73]) Method is an extension of the Simplex Method, which can be applied to QPPs in which all the problem variables are non-negative. To solve a QPP we can re-write the original constraints with slack variables and use these complementary slack conditions to ensure that the second and third K-T conditions are satisfied. For a QPP, the main K-T condition is obviously linear. We are therefore able to calculate the main K-T condition (1st condition) for the given QPP and use an extended Simplex Method (Wolfe's Method) that ensures the complementary slackness conditions are satisfied.

Wolfe's Simplex Method can further be extended as follows:

- 
- (i) The main K-T condition should be constructed for all the non-negative variables.
 - (ii) Slack and excess variables should be added for the inequalities.
 - (iii) The artificial variables should be added for any equation that does not have an obvious basic variable.
 - (iv) The Simplex Method is applied with the modification of never pivoting the excess e_i from the i^{th} constraint and x_i such that they both become basic.

Our focus is on Quadratic Programming, and we use the Wolfe Method in the following section to solve a QPP through the application of Linear Programming. In order to understand the principles of Quadratic Programming (QP) we briefly start with Non-linear Programming.

2.4 Non-linear programming

Non-linear programming background

These are constrained optimisation problems with inequalities that are in the form of:

$$\text{maximise } Z = f(x),$$

CHAPTER 2. OPTIMISATION TECHNIQUES FOR MODERN PORTFOLIO THEORY

Table 2.2: Results obtained by using the Simplex Method

| Table 1 | | | | | | | | | | |
|------------------|------------|-------------|-------------|-------|-------------|-------|-------|--------|-----|-------|
| x_1 | x_2 | x_3 | s_1 | s_2 | s_3 | s_4 | s_5 | s_6 | p | |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 4 |
| 2 | 1 | -1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 3 |
| 1 | 0 | 3 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 5 |
| 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 |
| -0.5 | -3 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Table 2 | | | | | | | | | | |
| x_1 | x_2 | x_3 | s_1 | s_2 | s_3 | s_4 | s_5 | s_6 | p | |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 4 |
| 0 | 1 | -1 | 0 | -1 | 0 | 2 | 0 | 0 | 0 | 3 |
| 0 | 0 | 3 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 5 |
| 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 |
| 0 | -3 | -1 | 0 | 0 | 0 | -0.5 | 0 | 0 | 1 | 0 |
| Table 3 | | | | | | | | | | |
| x_1 | x_2 | x_3 | s_1 | s_2 | s_3 | s_4 | s_5 | s_6 | p | |
| 0 | 1/2 | 1/2 | 1 | 1/2 | 0 | 0 | 0 | 0 | 0 | 2.5 |
| 0 | 1/2 | -1/2 | 0 | -1/2 | 0 | 1 | 0 | 0 | 0 | 1.5 |
| 0 | -1/2 | 3/2 | 0 | 1/2 | -1 | 0 | 0 | 0 | 0 | 3.5 |
| 1 | 1/2 | -1/2 | 0 | -1/2 | 0 | 0 | 0 | 0 | 0 | 1.5 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 |
| 0 | -2 3/4 | -1 1/4 | 0 | -1/4 | 0 | 0 | 0 | 0 | 1 | 0.75 |
| Table 4 | | | | | | | | | | |
| x_1 | x_2 | x_3 | s_1 | s_2 | s_3 | s_4 | s_5 | s_6 | p | |
| 0 | 1/2 | 0 | 1 | 1/2 | 0 | 0 | 0 | 1/2 | 0 | 2 1/2 |
| 0 | 1/2 | 0 | 0 | -1/2 | 0 | 1 | 0 | -1/2 | 0 | 1 1/2 |
| 0 | -1/2 | 0 | 0 | 1/2 | -1 | 0 | 0 | 3/2 | 0 | 3 1/2 |
| 1 | 1/2 | 0 | 0 | -1/2 | 0 | 0 | 0 | -1/2 | 0 | 1 1/2 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 |
| 0 | -2 3/4 | 0 | 0 | -1/4 | 0 | 0 | 0 | -1 1/4 | 1 | 3/4 |
| Table 5 | | | | | | | | | | |
| x_1 | x_2 | x_3 | s_1 | s_2 | s_3 | s_4 | s_5 | s_6 | p | |
| 0 | 4/7 | 0 | 1 | 3/7 | 1/7 | 0 | 0 | 0 | 0 | 2 |
| 0 | 3/7 | 0 | 0 | -3/7 | -1/7 | 1 | 0 | 0 | 0 | 2 |
| 0 | -1/7 | 0 | 0 | 1/7 | -2/7 | 0 | 0 | 1 | 0 | 1 |
| 1 | 3/7 | 0 | 0 | -3/7 | -1/7 | 0 | 0 | 0 | 0 | 2 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 0 | 0 |
| 0 | -1/7 | 1 | 0 | 1/7 | -2/7 | 0 | 0 | 0 | 0 | 1 |
| 0 | -3 | 0 | 0 | -0 | -1/3 | 0 | 0 | 0 | 1 | 2 |
| Table 6 | | | | | | | | | | |
| x_1 | x_2 | x_3 | s_1 | s_2 | s_3 | s_4 | s_5 | s_6 | p | |
| 0 | 4/7 | 0 | 1 | 3/7 | 1/7 | 0 | 0 | 0 | 0 | 2 |
| 0 | 3/7 | 0 | 0 | -3/7 | -1/7 | 1 | 0 | 0 | 0 | 2 |
| 0 | -1/7 | 0 | 0 | 1/7 | -2/7 | 0 | 0 | 1 | 0 | 1 |
| 1 | 3/7 | 0 | 0 | -3/7 | -1/7 | 0 | 0 | 0 | 0 | 2 |
| 0 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | -1/7 | 1 | 0 | 1/7 | -2/7 | 0 | 0 | 0 | 0 | 1 |
| 0 | -3 | 0 | 0 | -0 | -1/3 | 0 | 0 | 0 | 1 | 2 |
| Table 7 | | | | | | | | | | |
| x_1 | x_2 | x_3 | s_1 | s_2 | s_3 | s_4 | s_5 | s_6 | p | |
| 0 | 1 | 0 | 1 3/4 | 3/4 | 1/4 | 0 | 0 | 0 | 0 | 3 1/2 |
| 0 | 0 | 0 | -3/4 | -3/4 | -1/4 | 1 | 0 | 0 | 0 | 1/2 |
| 0 | 0 | 0 | 1/4 | 1/4 | -1/4 | 0 | 0 | 1 | 0 | 1 1/2 |
| 1 | 0 | 0 | -3/4 | -3/4 | -1/4 | 0 | 0 | 0 | 0 | 1/2 |
| 0 | 0 | 0 | 1 3/4 | 3/4 | 1/4 | 0 | 1 | 0 | 0 | 3 1/2 |
| 0 | 0 | 1 | 1/4 | 1/4 | -1/4 | 0 | 0 | 0 | 0 | 1 1/2 |
| 0 | 0 | 0 | 41/8 | 17/8 | 3/8 | 0 | 0 | 0 | 1 | 49/4 |
| Optimal solution | $P = 49/4$ | $x_1 = 1/2$ | $x_2 = 7/2$ | and | $x_3 = 3/2$ | | | | | |

subject to $g(x) \leq 0$.

In non-linear optimisation there is no explicit condition that $x \geq 0$, so if there is a restriction that $x \geq 0$, then it has to be included as a constraint within the problem. It is important to note that every minimisation problem can be written as a maximisation problem, with some changes within the objective function.

In order to solve a Non-linear Optimisation Programming (NOP) problem, we must first convert the inequality $g(x) \leq 0$ into an equation. This gives:

$$g(x) + S^2 = 0.$$

The term S^2 comes from the fact that there is no explicit restriction that S is greater or equal to zero. Remember that $g(x) + S^2 = 0$ is a set of constraints. Similarly, the function $Z = f(x)$ involves more than one variable, and we generally have m variables and n constraints for $g(x) \leq 0$.

Let's now take $g(x) + S^2 = 0$ into the objective function by introducing the Lagrange Multiplier. We define Lagrangian by

$$L = f(x) - \lambda(g(x) + S^2).$$

Partial differentiation of L with respect to S will give us the conditions to find the optimum. This leads to

$$\frac{\partial L}{\partial x} = \nabla f(x) - \lambda \nabla g(x), \tag{2.4.4}$$

$$\frac{\partial L}{\partial S} = 2\lambda S, \tag{2.4.5}$$

$$\frac{\partial L}{\partial \lambda} = -(g(x) + S^2). \tag{2.4.6}$$

We are able to modify these three conditions. The constraints are relaxed and therefore restrict the problem with an additional constraint which will form the original problem. Maximising this gives

$$\lambda \geq 0, \tag{2.4.7}$$

$$\nabla f(x) - \lambda \nabla g(x) = 0, \tag{2.4.8}$$

$$\lambda S = 0, \tag{2.4.9}$$

$$g(x) \leq 0. \tag{2.4.10}$$

The above equations (2.4.7) to (2.4.10) give us the Kuhn Tucker (KT) or Karush Kuhn Tucker (KKT) conditions.

Quadratic Programming

Having gone through the above, we can now understand Quadratic Programming. Quadratic Programming is a special type of Non-linear Programming. This is due to the fact that it has a quadratic objective function, but it has linear constraints. We write the general form of a Quadratic Programming problem as

$$\max Z = ax + \frac{1}{2}x^T Bx, \tag{2.4.11}$$

subject to

$$Cx \leq d, \tag{2.4.12}$$

with

$$x \geq 0. \tag{2.4.13}$$

We note that x_1, x_2, \dots, x_n is a set of decision variables. The objective function is quadratic due to the term $x^T Bx$. We also have the linear term in the objective function, namely ax . Constraints are linear: $Cx \leq d$. Here x is explicitly stated as well as $x \geq 0$.

Definition 2.4.1. ([33]) *Let B be a symmetric $n \times n$ matrix. A is called positive definite if $x^T Bx > 0$ for all $x \in \mathbb{R}, x \neq 0$. Matrix B is called positive semi-definite if $x^T Bx \geq 0$ for all $x \in \mathbb{R}$. The notion of negative (semi) definite is defined analogously.*

gously. Matrix B is called indefinite if vectors x_1 and x_2 exist such that $x_1^T B x_1 > 0$ and $x_2^T B x_2 < 0$.

The above is applied in the portfolio theory, as we look at the allocation of wealth to each part of the portfolio such that we

minimise risk

subject to

$$E(\text{return}) \geq R$$

and

$$\sum_{i=1}^n x_i = 1.$$

This problem is modelled as a QPP because risk is seen as a variance/covariance of returns. If we have a QPP like this, we then write the KKT conditions. This QPP has an explicit condition that $x \geq 0$. The KKTs are derived from the general Non-linear Programming. So, in order to write down the KKT conditions they are actually written as (2.4.11), (2.4.12) and (2.4.13) as constraints and then written as the KKT conditions.

In order to do this, we write the problem as follows:

$$\max ax + x^T Bx,$$

$$G(x) = C - Ix - d \leq 0.$$

We introduce the Lagrangian multiplier λ for $Cx \leq d$ and μ for $x \geq 0$. So now we have as many λ 's as the number of constraints and as many μ 's as the number of variables. We also have as many x 's as the number of variables. This leads to

$$\lambda, \mu \geq 0,$$

$$a + 2X^T B - (\lambda\mu)C - I = 0,$$

$$\lambda g(x) = 0,$$

$$Cx \leq d,$$

$$x \geq 0.$$

A further modification of the above equations gives the following:



$$\lambda, \mu \geq 0, \tag{2.4.14}$$

$$-X^T B + \lambda^T C - \mu^T = a, \tag{2.4.15}$$

$$\lambda_i(C_i x_i - d) = 0, \tag{2.4.16}$$

$$\mu_j x_j \leq d, \tag{2.4.17}$$

$$x \geq 0, \tag{2.4.18}$$

$$Cx + S = d. \tag{2.4.19}$$

The term $\lambda_i g_i(x) = 0$ has been written from the KKT form. It has to be written for both sets of constraints $Cx \leq d$ and $x \geq 0$. Note that λ_i actually represents λ and μ , where λ is for $Cx \leq d$ set and μ is for $x \geq 0$ set. This gives us 2.4.16 and 2.4.17. Since we have $Cx \leq d$, we can introduce a slack variable S so that we have 2.4.19. We therefore have

$$\lambda_i S_i = 0, \text{ and}$$

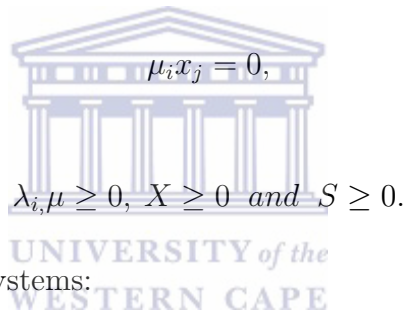
$$\mu_i x_j = 0.$$

Now the KKT conditions are reduced to

$$-2X^T B + \lambda^T C - \mu^T = a,$$

$$CX + S = d,$$

$$\lambda_i S_i = 0,$$



We therefore have three systems:

$$\left. \begin{aligned} -2X^T B + \lambda^T C - \mu^T &= a, \\ CX + S &= d. \end{aligned} \right\} \quad (2.4.20)$$

$$\left. \begin{aligned} \lambda_i S_i &= 0, \\ \mu_j x_j &= 0. \end{aligned} \right\} \quad (2.4.21)$$

$$\left. \begin{aligned} \lambda, \mu &\geq 0, \\ x &\geq 0, \\ S &\geq 0. \end{aligned} \right\} \quad (2.4.22)$$

We can write the above as

$$\begin{pmatrix} -2B & C & -I & 0 \\ C & 0 & 0 & I \end{pmatrix} \begin{pmatrix} x \\ \lambda \\ \mu \\ s \end{pmatrix} = \begin{pmatrix} a \\ d \end{pmatrix}. \quad (2.4.23)$$

This is a set of linear equations. We can now see that we initially had a QPP and we reduced the optimality conditions after applying KKT conditions to a linear equation. This is a set of non-negative variables and a set of product relationships among the variables. If we temporarily relax (2.4.21), we then get a set of linear equations (2.4.20) subject to (2.4.22). We can then model this as a LPP. We are able to do this, however it must be subject to the fact that (2.4.21) is not violated. This is done by ensuring that every iteration we come across, when solving the LPP, (2.4.21) must not be violated.

It is important to note that we can also solve the QPP by using numerical methods in order to solve it directly without employing LPP. To do this we have three algorithms, namely: the Interior Point Algorithm; the Active Set Method and the Gradient-Projection Method ([57]). We only employ the Interior Point Algorithm and the Active Set Method to verify the results.

2.5 Summary

In this chapter we have investigated the important requirements needed in order to perform optimisation. We have also looked at Linear Programming and Quadratic Programming. In the portfolio optimisation following the mean-variance optimisation, we performed only quadratic optimisation, but we have seen that we are able to transform a QPP to a LPP following Wolfe's method. The Modern Portfolio Theory builds on the Markowitz model portfolio selection after applying the mathematical concepts to perform a QPP. As we've performed the algorithms of QPP we now apply this to calculate Markowitz's portfolio selection in the next chapter.

Chapter 3

Modern Portfolio Theory

3.1 Introduction

In this chapter we deal with one of the main aims in the Modern Portfolio Theory, which is to diversify a portfolio in order to eliminate non-systematic risk through the use of optimisation. The main step towards diversification is to hold only the risk-free asset and the tangent portfolio. The pricing model is obtained by recognising the tangent portfolio. We do this through use of the Modern Portfolio Theory. Our first step is to look at the mean-variance optimisation (MVO) theory, which allows us to allocate assets within a portfolio, in order to maximise the expected return, while minimising the risk. Following that we use Capital Asset Pricing, which extends on MVO, by introducing a risk-free asset. This leads us to choosing the tangent portfolio. We then consider the Black-Litterman model, which further extends on the MVO and CAPM by allowing users to include their views about the prediction of the expected returns. Finally, we look at the composite measures of portfolio performance by studying the ratios of these portfolios.

3.2 Portfolio selection using Mean-Variance Optimisation

Mean-Variance Optimisation marked the birth of the Modern Portfolio Theory (MPT). This was pioneered by Markowitz ([50]). He developed the framework of MVO. In the investment world, it is widely accepted that there is trade-off between risk and return. In order to balance this trade-off, investors understand that the

more risk averse they are, the less compensation they will receive. The higher the risk, the higher the potential is to receive greater compensation. Markowitz made a huge contribution to MPT by highlighting the trade-off between risk and return as the problem with regards to the selection of portfolios or securities (or asset classes), in a manner that trades off the expected returns and the perceived risk of potential portfolios ([30]).

The Markowitz model is built on a number of assumptions in relation to market conditions and the conduct of investors. These include,

- (i) Investors want to achieve the maximum possible expected return from their wealth.
- (ii) Investors are maximising a single period for the expected utility.
- (iii) Investors are risk adverse and prefer a higher return on the same level of risk and vice versa.
- (iv) Decisions are based only on the risk and expected return plane.
- (v) There are no taxes and transaction costs.

To gain some more insight into MPT, let us consider assets $S_1, S_2, \dots; S_n (n \geq 2)$ with random returns. Let μ_i and σ_i denote the return and the standard deviation of return on an asset i . For $i \neq j$; $\rho_{i,j}$ denotes the correlation coefficient of the returns on assets S_i and S_j . Let $\mu = [\mu_1, \dots, \mu_n]^T$ and $V = \sigma_{ij}$ be the $n \times n$ symmetric covariance matrix with $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ for $i \neq j$. Denoting x_i by the proportion of the total funds invested in security i , one can represent the expected return and the variance of the resulting portfolio $x = [w_1; \dots; w_n]$ as follows:

$$E[w] = w_1\mu_1 + \dots + w_n\mu_n = \mu^T w,$$

and

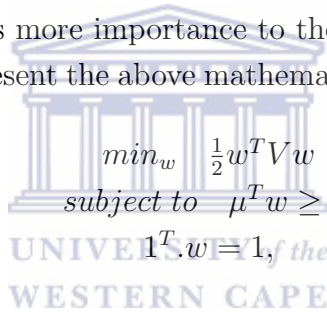
$$Var[w] = \sum_{i,j}^n \rho_{ij}\sigma_i\sigma_j w_i w_j = w^T V w; \text{ where } \rho_{ii} = 1.$$

Since variance is always non-negative, it follows that $w^T V w \geq 0$ for any w , i.e. V is positive semi-definite. We assume that it is in fact positive-definite, which is

essentially equivalent to assuming that there are no redundant assets in the collection $S_1; S_2; \dots; S_n$ ([18]). Investors are more concerned with portfolios that will be in-line with their risk tolerance. The Markowitz model is a combination of the expected return and the variance, which also bears in mind the risk tolerance of an investor. Markowitz combines risk with the expected return to obtain the following objective function:

$$\begin{aligned} \max f(w) : & \mu^T w - A w^T V w \\ \text{subject to} & \sum_i^n w_i = 1, \\ & w \geq 0, \end{aligned} \tag{3.2.1}$$

where $\mu^T w$ is the linear part, $w^T V w$ is the quadratic part and A is the risk tolerance. Note that A is a non-negative factor which represents a trade-off between risk and return. If A is large, this implies its risk is considered more important, while if A is small, the model assigns more importance to the return. If $V = 0$ we get a LPP. Alternatively, we can represent the above mathematically, in the following two ways:



$$\begin{aligned} \min_w & \frac{1}{2} w^T V w \\ \text{subject to} & \mu^T w \geq R \\ & 1^T . w = 1, \end{aligned} \tag{3.2.2}$$

or

$$\begin{aligned} \max_w & \mu^T w \\ \text{subject to} & w^T V w \leq \sigma^2 \\ & 1^T . w = 1. \end{aligned} \tag{3.2.3}$$

KKT conditions for the Markowitz Model

Looking at equation (3.2.2) we obtain the following Lagrangian

$$L(w, \lambda, \zeta) = \frac{1}{2} w^T V w - \lambda (\mu^T w - \mu) - \zeta (1^T . w - 1),$$

and the KKT conditions are

$$V w - \lambda \mu - \zeta . 1,$$

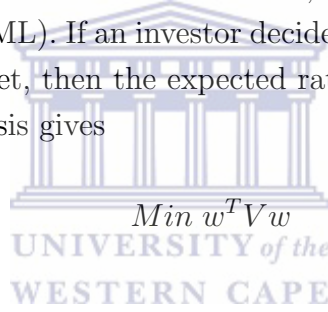
$$\mu^T w \geq R, \quad 1^T . w = 1, \quad \lambda \geq 0,$$

$$\lambda^T(\mu w - R) = 0,$$

for $\lambda, \zeta \in \mathbb{R}$.

Adding the Risk-Free Rate

If we add a risk-free asset to an investment universe, the efficient frontier changes. In fact, Tobin ([69]), Sharpe ([65]) and Linter ([45]) showed that efficient portfolios, with a risk-free asset added to the investment universe, are more superior than the portfolios available to investors without the risk-free asset. When having a number of efficient portfolios on the mean-standard deviation plane, there is one particular unique portfolio available to each risk-free level, called the market portfolio (this portfolio constructs the CML). If an investor decides to allocate a proportion of their wealth to the risk-free asset, then the expected rate of return is $Er_p = w_r\mu + w_fr_f$ and the mean-value analysis gives



$$\begin{aligned} & \text{Min } w^T V w \\ & \text{UNIVERSITY of the} \\ & \text{WESTERN CAPE} \\ & \text{subject to } w^T \mu + w_f = 1 \end{aligned}$$

$$w^T \mu + w_fr_f \geq R$$

$$w \geq 0, w_f \leq 1.$$

Theorem 3.2.1. (Two fund theorem) ([52]) *Given any two efficient portfolios, any two portfolios can be matched (same mean-variance) as a linear combination of the other two.*

The above theorem fails if short selling is not allowed. When short selling is allowed, the optimisation chooses more assets on the efficient portfolios as opposed to fewer assets on the efficient portfolios. We use the Huang and Litzenburg ([34]) approach to calculate the envelope of portfolios. We now investigate the extension of the MVO.

3.3 Asset pricing using the Capital Asset Pricing Model

The Capital Asset Pricing Model (CAPM) was developed by Sharpe ([65]) and Linter ([45]). This model was formulated on the basis of the Mean-Variance Optimisation problem of Markowitz ([50]). This method attempts to modify the rate of return of an asset that we want to include in a diversified portfolio if we are given its non-diversifiable risk. The method recognises the beta, i.e. the asset's sensitivity towards the market risk or simply the relationship between the asset's risk and the market risk. The goal with this method is to obtain the tangent portfolio or the market portfolio.

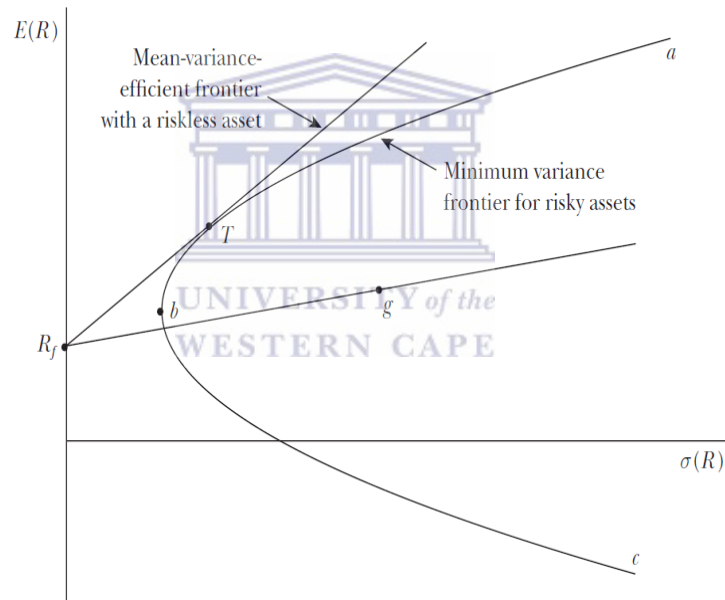


Figure 3.1: Mean-Variance efficient frontier and a portfolio envelope diagram ([25]) P27

When we look at Figure 3.1 it is clear that instead of just having the envelope portfolio, we also have two lines. We need to remember that the model also has the assumption that the investor needs to put their wealth in two baskets, namely the risk-free and the high risk assets. We can see from the above picture that the two lines represent the risk-free asset. We are maximising these assets by shifting the line $R_f g$ until it touches the efficient frontier once. We can also see that the point where the assets touch the efficient frontier is determined by the risk-free rate. We

apply the very same notion that Markowitz had, in that we should ignore line G, as there are inferior portfolios on this line. Instead we need to look at line T which is called the Capital Allocation Line (CAL). The different portfolios on the lines are dependent on the risk aversion of the investor. We can see that we obtain a different portfolio when we shift R_f up and down.

Mathematically, we know from the previous chapter, that we have a concave and a convex set. Based on this deduction, we are able to search for an optimal point by shifting the line crossing the y-axis at point R_f until it touches the efficient frontier once.

Markowitz came up with a technical approach to obtaining the frontier of high risk assets. His model showed a general solution to asset allocation. Tobin ([69]) showed that under the same conditions, the Markowitz model could be broken down into two phases of investment choices; high risk and risk-free assets. Accordingly, a decision can be made in two stages. Firstly, the efficient sets of investment plans are found, and secondly the investment plan is chosen from among this set. A plan is said to be efficient if, and only if, there is no alternative with either:

1. The same expected return and a lower standard deviation;
2. The same standard deviation and a higher expected return, or;
3. A higher expected return and a lower standard deviation.

The CAPM is an equilibrium model that specifies the relationship between risk and the required rate of return for assets held in well-diversified portfolios.

Assumptions of the CAPM:

- Investors all think in terms of a an identical time horizon.
- All investors are anticipating the same goal (in terms of returns and risk).
- Investors can borrow or lend unlimited amounts at the risk-free rate.
- All assets are perfectly divisible.
- There are no taxes and no transactions costs.
- Trading does not affect the assets' prices.

- Quantities of all assets are given and fixed.

The beta can be calculated using the following formula:

$$\text{Market Beta } \beta_{i,M} = \frac{\text{cov}(R_i, R_M)}{\sigma_M^2}$$

and the formula to calculate the expected stock return is as follows:

$$E(R_i) = R_f + [E(R_M) - R_f]\beta_{i,M}, \quad i = 1 \dots N.$$

Zero Beta Model

This model was introduced by Black in 1972 ([9]). Assuming that the risk-free asset exists, the market or tangent portfolio can be computed. In simple terms, this means that as the risk-free asset has zero risk, then its beta is zero. From this Black deduced the following formula to calculate the market portfolio:

$$w = \frac{V^{-1}\{E(R_p) - c\}}{\text{Total}([V^{-1}\{E(R_p) - c\}])}$$

In the above formula c is any constant and $E(R_p)$ is the expected return of the portfolio. This model fails when short selling is not allowed.

CAPM TEST(1970's)

After the computation of a portfolio, the portfolio can be tested using the CAPM test. This was introduced in the 1970's by Jensen ([38]). The steps to testing a portfolio is as follows:

- (i) Pick a set of high risk assets (e.g. stocks, mutual funds, etc.);
- (ii) Choose a candidate for a market portfolio. A popular candidate is the S&P500;
- (iii) First pass regression: determine β_i for each asset i

$$R_i = \alpha_i + \beta_i R_{MT};$$

- (iv) Second pass regression: Regress asset average returns on their β 's $\bar{R} = \varphi_0 + \varphi_i \beta_i$.

3.4 Estimation of the Covariance Matrix

This topic is often neglected in the finance world, as most people are more interested in the estimation of the expected returns. Remembering that the mean-variance, studies the trade-off between risk and return, if the variances are ignored we are able to ask if the estimation towards diversifying a portfolio is complete? This is why we have the implied returns and not the implied Covariances.

The Markowitz equation

$$\min w'Vw$$

$$\text{subject to } w'1 = 1,$$

$$w'\mu = q,$$

can be solved analytically and the solution to the equation above is as follows:

$$w = \frac{C - qB}{AC - B^2} V^{-1}1 + \frac{qA - B}{AC - B^2} V^{-1}\mu,$$

where $A = 1'V^{-1}1$, $B = 1'V^{-1}\mu$ and $C = \mu'V^{-1}\mu$. If the Markowitz set of equations has inequality, then it cannot be solved analytically, and the KKT conditions have to be employed. From the above computation of the weights, we can see the importance of the covariance matrix. In portfolio optimisation, it is recommended that users should utilise historical data as much as possible to achieve better results.

Variance Covariance estimation using Shrinkage Methods

The sample covariance matrix is not recommended for portfolio optimisation ([43]) and for this reason it is preferable to move away from this method. The covariance matrix can be calculated through alternative methods such as the single index, constant correlation or shrinkage amongst others. For purposes of this thesis we only investigate the shrinkage method. The reader can refer to ([21]) for other calculation methods. When a substantial amount of data is used, this increases the estimation error from the covariance matrix and hence can lead to undesired portfolio optimi-

sation ([21]).

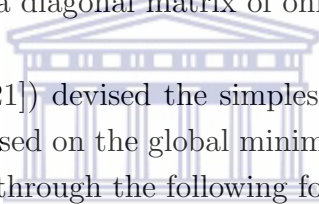
The concept of shrinkage has been around for some time. It was first introduced by Charles Stein in 1955; however it is not related to covariance estimation ([21]). In relation to portfolio selection, the shrinkage estimation was first applied by Frost and Savarino ([27] and Jorion ([39]). We follow the theory of Ledoit and Wolf ([42]) as this is the more recent method of performing shrinkage.

The ‘so called’ shrinkage method assumes that the variance matrix is a convex combination of the sample covariance matrix and another matrix. It is given by the following:

$$\lambda \times \text{sample covariance} + (1 - \lambda) \times \text{other matrix} \quad 0 < \lambda < 1,$$

where the other matrix is a diagonal matrix of only variances.

Disatnik and Benninga ([21]) devised the simplest shrinkage estimation approach. The approach is mostly based on the global minimum variance portfolio, which can be analytically calculated through the following formulation:


$$GMVP = \frac{1 \cdot V^{-1}}{1 \cdot V^{-1} \cdot 1^T},$$

where V^{-1} is the inverse of the sample covariance matrix and 1 is a vector of ones.

They recommended that the best shrinkage operator λ , is obtained by varying the operator (λ). A good operator is given by having a wholly positive GMVP. Regardless of the simplicity of this method, we used the approach of Ledoit ([43]). In order to minimise the error from the true covariance matrix, we chose all optional shrinkage operators. The reader is referred to ([43]) for the derivation of the estimate of the optimal shrinkage operator.

3.5 Estimation of anticipated returns using the Black-Litterman Model

There is no doubt that Markowitz ([50]) pioneered the Modern Portfolio Theory. It was also developed by several other people including Sharpe ([65]) and Mossin

([56]). Even so, this was insufficient due to the market being so unpredictable. The main problem facing the MPT is that it uses historical data, which can be a false predictor of future expectations. In simple terms, this means that if a company has been performing well for the past thirty years, then this is not necessarily an indicator that they will continue to perform well for the next two years. There are also potentially high performing investments that may not have enough or any historical data.

Market data and the MVO can produce unrealistic or concentrated portfolios. So Black and Litterman ([12]) came up with a model that moves away from unrealistic portfolios. Best and Grauer ([8]) demonstrated that a small increase in the expected return of a portfolio's assets, could force half of the assets out of the portfolio. Therefore, the most important inputs in MVO are the expected returns. Unfortunately, mean returns are also the most difficult input to estimate. This model also extends by allowing investors to incorporate their own view in the optimisation of their portfolios.

Instead of computing portfolios following the normal optimisation approach, the Black-Litterman model starts by assuming the chosen portfolio is optimal, allowing the investor to express their views on the expected future return. This approach combines both the investors' and the markets' views regarding future expected returns. This approach is followed by most professional financial advisors, which demonstrates the effectiveness of the model when there are benchmarks, risk, a beta or any other target ([11]). As mentioned above, the model achieves good results if there is a benchmark. We define this benchmark as

Definition 3.5.1. ([11]) *The standard used to define the risk of other portfolios. If a benchmark is defined, the risk of a portfolio is measured as the volatility of the tracking error - the difference between the portfolio's return and those of the benchmark.*

This is the reverse engineering of normal portfolio optimisation. In normal circumstances, the variance-covariance matrix is used and the expected return is then computed to get the optimal portfolio. However, in the Black-Litterman model the optimal portfolio is taken and the returns are then computed. The weights are obtained by applying the current price and multiplying it by the outstanding shares to get the market value of a company. The market values are added and then each

divided by the total. This creates a complete market portfolio as it is assumed that the chosen stocks are available within the universe.

Steps towards Black-Litterman optimisation

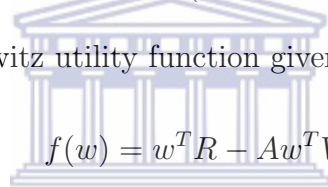
The market portfolio weights are obtained through the use of market capitalisation. This is done by taking the outstanding shares and the current stock price. So we have

$$\text{Market Cap} = \text{Outstanding share} \times \text{Current stock price},$$

and then the weight of asset i is obtained from

$$W_i = \frac{\text{Market Cap}_i}{\text{Sum}(\text{Market cap})}.$$

Going back to the Markowitz utility function given in (3.2.1)



$$f(w) = w^T R - A w^T V w,$$

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such that $w^T \mathbf{1} = 1$.

Now

$$\frac{df}{dw} = R - AVw, \tag{3.5.4}$$

where A is the risk aversion. We maximise by allowing

$$R - AVw = 0, \tag{3.5.5}$$

which implies that

$$\mathbb{\Pi} = AVw_{mkt}, \tag{3.5.6}$$

where

$\mathbb{\Pi}$ is the Implied Excess Equilibrium Return Vector (an $N \times 1$ column vector),

A is the risk aversion coefficient. A can also be seen as the normalising factor (A is directly proportional to the return). This A can be computed by using the following

formula:

$$A = \frac{E(r_m) - r_{fr}}{\sigma_M},$$

where $E(r_m)$ and σ_m are the components of the market portfolio.

V is the covariance matrix of excess returns (an $N \times N$ matrix), and

w_{mkt} are the market capitalisation weights (an $N \times 1$ column vector) of the assets. By re-arranging the above equation (3.5.6), we obtain the following:

$$w = (AV)^{-1}\mu,$$

where μ is any vector of excess returns. The equation (3.5.6) lets us compute the vector of returns of the assumed market weights.

Estimating Returns

Here we follow the Black-Litterman model to estimate the inputs. The new vector of returns is given by the following:

$$\widehat{\Pi} = \Pi + \tau V P^T [(P \tau V P^T) + \Omega^{-1}] [Q - P \Pi]$$

where

$\widehat{\Pi}$ is the new estimated Combined Return Vector (an $N \times 1$ column vector)

Π is the Implied Equilibrium Return Vector ($N \times 1$ column vector). This vector of returns is computed by use of reverse optimisation which uses the weights obtained by Market Capitalisation and the returns are obtained using equation (3.5.6).

τ is a scalar which is used in the Black-Litterman formula. They set the value to 0.025. There is very little literature on this value, but most authors use the value of one, and here we assume this value.

V is the covariance matrix of excess returns ($N \times N$ matrix). We are able to get the covariance from the historical returns or from the shrinkage.

Ω is a diagonal covariance matrix of error in terms of the expressed views representing the uncertainty in each view ($K \times K$ matrix). This is a diagonal matrix as the views are uncorrelated. This can be calculated in several ways and in this example we deploy $\partial^T \tau \sum p$.

Q is the view vector ($K \times 1$ column vector). These are the views we know about the market deviation. There are two classes of views and they are the absolute and relative views. Relative views are commonly used in practice and they sum to zero in the matrix. Absolute views are less commonly used and they sum up to one.

P is a matrix that identifies the assets involved in the views ($N \times N$ matrix or $1 \times N$ row vector in the special case of one view). This is also known as the link matrix which defines views.

This model assumes that the expected returns are distributed normally. So we have $E(r) \sim N(\mu, V)$ and we also define μ as a random mean return and $\mu \sim N(\pi, V_\pi)$ where π is the estimate. With these two uncorrelated variances we have the following formula of V_r as

$$V_r = V_\pi + V.$$

This is essentially the product of two multivariate normal distributions. The model uses the first distribution as the neutral starting point. This is the equilibrium return which uses market data. We also need to compute the posterior variance. By defining

$$M = ((\tau V)^{-1} + P^T \Omega^{-1} P)^{-1},$$

we have the posterior variance as

$$V_p = V + M.$$

The second distribution is the investors view. Here we incorporate this view. We

follow the method of Idzorek ([36]) to incorporate the confidence level. This method is the simplest, and is also different to what He and Litterman ([32]) used. The other alternatives can be found in Meucci ([53]) and Litterman ([47]). Idzorek represents the views as a percentage from 0% to 100%. We now use $\Omega = \alpha PVP^T$. When we are 100% confident, alpha goes to zero and vice versa when confidence decreases. From this fact we can see that

$$\alpha = \frac{1 - \text{confidence}}{\text{confidence}}.$$

Summary of the Black Litterman Methodology

- (i) We start by calculating the market capitalisation in order to compute the markets weights. We also compute the covariance matrix and the investor decides on the risk benchmark (their risk aversion).
- (ii) We then perform reverse optimisation in order to obtain the excess returns.
- (iii) We clearly state our views together with our confidence levels towards those views.
- (iv) We incorporate our views to obtain our new vector of returns.
- (v) We then go back to basics to run the mean-variance optimisation.

3.6 Sharpe's style analysis to track benchmarks

This style analysis was modelled by Sharpe ([67]). Asset allocation is generally defined as the allocation of an investor's portfolio amongst a number of asset classes. Clearly such a generalisation cannot be made operational without defining these classes ([67]). Here we basically allocate the securities in such a way that the portfolio's return tracks the index ([37]). This is achieved by solving the following Quadratic Programming Problem:

$$\min \text{Var}(e_i) = \text{Var}(R_S - R_M),$$

$$\sum_{i=1}^n w_i = 0,$$

$$w_i \geq 0,$$

where e_i is the error between the style returns and the index return (S&P500). We track this by minimising the variance of the error between the style returns and the index.

3.7 Use of Portfolio Performance Measure to back-test chosen portfolios

High betas are required for rising markets and low betas for declining markets. Previously, MPT investors were only interested in the rate of return, as they were unable to quantify and measure risk in terms of the variability of returns (adjusting returns for the level of risk).

Treynor Portfolio Performance Measure

Treynor ([71]) observed a new way to rate the performance of a fund's investment management. He provided a basis for reviewing the performance of fund management by using beta as a measure for risk. We calculate this measure using the following:

$$T = \frac{\bar{R}_i - \bar{r}_f}{\beta_i},$$

Where

\bar{R}_i = the average rate of return for portfolio i during a specified time period,

\bar{r}_f = the average risk-free rate for the same period, and

β_i = Portfolio's beta.

A larger T value indicates a larger slope and an improved portfolio for all investors (regardless of their risk preferences). As the numerator of this ratio $\bar{R}_i - \bar{r}_f$ is the risk premium, and the denominator is a measure of risk, the total expression indicates the portfolio's risk premium return per unit of risk. All risk-averse investors would prefer to maximise this value. Note that the risk variable beta measures systematic risk and provides no indication regarding the diversification of the portfolio. It

implicitly assumes a completely diversified portfolio, which means that systematic risk is the relevant risk measure.

Sharpe Portfolio Performance Measure

On the other hand the Sharpe Portfolio Performance Measure is given by the following:

$$S = \frac{\bar{R}_i - \bar{r}_f}{\sigma_i},$$

Where

\bar{R}_i = the average rate of return for portfolio i during a specified time period,

\bar{r}_f = the average risk-free rate for the same period, and

σ_i = the standard deviation of portfolio i .

Sharpe's ratio is similar to Treynor's but looks at the total risk of the portfolio. It seeks to measure this total risk, by including the standard deviation of returns rather than considering only the systematic risk summarised by beta. A completely diversified portfolio would give similar results and a poorly diversified portfolio would give a high Treynor and a low Sharpe measure. Sharpe's ratio can be used to compute a set of efficient portfolios by maximising them in order to obtain two portfolios. It will then use the two fund theorem to construct the whole efficient frontier.

Information Ratio Portfolio Performance Measure

The Information Ratio is one of the most famous statistical portfolio performance measures. It is also known as the Appraisal Ratio. It quantifies the ability to outperform the benchmark, by dividing the excess return of the chosen portfolio and the benchmark, by their standard deviation ([59]). Due to the fact that the standard deviation is always positive, this gives an indication that, on average, a portfolio performs better than the benchmark. We calculate the Information Ratio with the following formula:

$$\mathbb{R}_i = \frac{\bar{R}_i - \bar{R}_B}{\sigma_{ER}},$$

where

\mathbb{R}_i is the information ratio of portfolio i ,

\bar{R} is the average return of portfolio i for the specified time period,

\bar{R}_B is the average return of the benchmark portfolio for the specified time, and

σ_{ER} is the standard deviation of the excess returns between the benchmark portfolio and portfolio i .

3.8 Summary

In this chapter we have looked at the theory or the heart of the Modern Portfolio Theory. We started by constructing an efficient frontier following the Mean-Variance Optimisation theory which was modelled by Markowitz. We then investigated the extension of the MPT by looking at the Capital Asset Pricing Model. This gave us a more refined portfolio selection, by reducing the set of portfolios from the efficient frontier to one portfolio choice. We then observed the Black-Litterman model which enabled us to incorporate our returns from the benchmark or market portfolio. By using Sharpe's Style Analysis, we were able to consider alternative ways to track the benchmark. By looking at the measurements of the selected portfolios, we were able to conclude how the portfolios performed in comparison to the benchmark.

Chapter 4

Portfolio Risk Assessment using Value at Risk (VaR)

4.1 Introduction

Investors won't always get the expected return on investment from their portfolios. There is always the possibility that they will lose on an investment, and this is where Value at Risk (VaR) comes in. VaR is an estimate of what an investor could lose, trying to show how bad things could get. It tries to answer the question of how much they could lose in their current position, as well as taking the current market conditions into account ([5]). According to Hull ([35]), a financial manager will need to make a statement based on the form: We are x percent certain that we will not lose more than V dollars in the next N days.

It is important to note that VaR is an estimate and not an exact number. VaR doesn't estimate the worst case scenario as rare conditions are not taken into account. VaR will take the given stock price and estimate a drop in that price or value which could occur in a day or a week ($N=1$ or $N=5$ respectively) with a probability of perhaps 1% or 5% (given the confidence levels of 99% and 95% respectively). Let's consider the following example: suppose you are told that your VaR is \$100 to a 95% confidence level. Essentially, this means that if the market is normal, then there is a 5% chance that you could lose more than \$100 in a day or a week depending on your specified period. This period is normally within a short space of time.

VaR is a measure which includes every sub-section of risk, in order to give an overall market risk of a particular portfolio. This has made VaR a popular measure in practice, especially by banks in order to protect themselves. Central bank regulators also use VaR to determine the capital a bank requires in order to reflect the market risks that they are bearing ([49]).

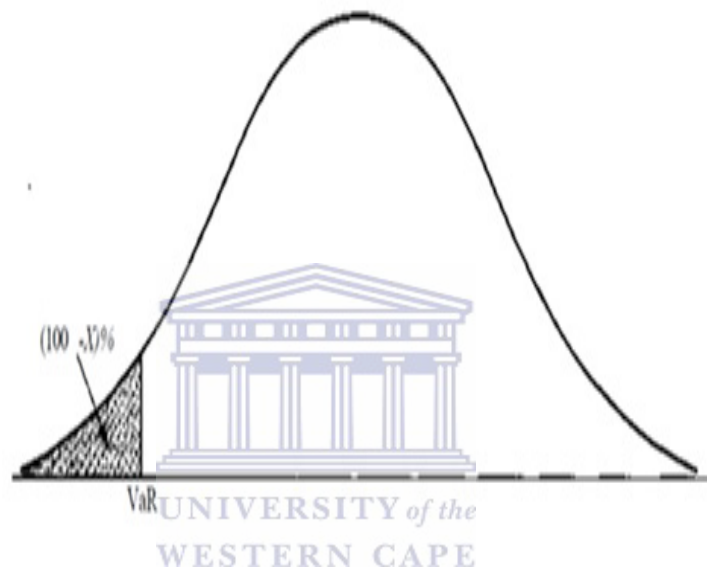


Figure 4.1: VaR at 100-X% confidence ([35]) P444

There are three popular approaches in calculating VaR namely:

- Parametric VaR/ Variance-Covariance method;
- Historical Simulation;
- Monte-Carlo Simulation or Stochastic Simulation.

4.2 VaR Valuation using the Parametric Method

This is also known as the Variance-Covariance VaR, or Delta normal amongst others. This is the foundation of VaR as it is the easiest and most widely used method ([30]). It was introduced by J.P. Morgan's RiskMetrics system ([55]). This is called a parametric method, as it assumes the probability distribution as normal. In order to compute the parametric method, the statistical measures such as variance ([49]) are

needed first and therefore the historical information is also required. This method is also the fastest of the three; however it only deals with linear models. It doesn't have the ability to capture non-linear and non-normal models. If the underlying market value of the portfolio has a linear function of the parameters, then that distribution of the profit and loss is normal. We calculate VaR by obtaining the profit and loss figures and from there calculate the variances. VaR is calculated as follows:

$$VaR = \alpha \sqrt{wV_p w},$$

where w are the weights of the assets in the portfolio, and V is the Variance-Covariance matrix. The parameter α can be obtained from the normal distribution tables, depending on the level on confidence (99% \rightarrow 2.33 and 95% \rightarrow 1.65). It is also expressed as the probability or the quantile of losses.

4.3 Numerical Simulation to estimate VaR

VaR Valuation using Historical Simulation

As the title suggests, in this instance, we only need returns from the past time horizon. In order to calculate VaR the historical return data is used. This moves away from the assumptions of linearity and normality (constant correlation and deltas). As this data deals with actuals, although historical, this method is able to capture those rare events and they are included in the model. Correlation is also captured. The main draw-back with this method is that historical data is not an accurate reflection of what could happen in the future. Following the Marrison ([49]) algorithm, we refer back to the rate of return and tomorrow's scenario is calculated as follows:

$$r_{tomorrow} = r_{today}(1 + \Delta_t).$$

After simulating, we then observe the lowest quantile (depending on the alpha) from our simulated data. The returns are then sorted in ascending order and VaR is chosen according to the confidence level required.

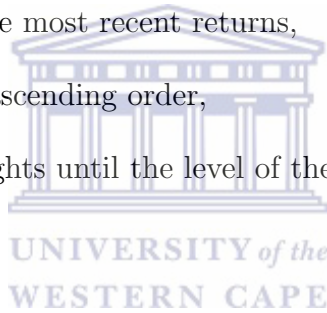
Weighted Historical Simulation

This method of VaR is based on traditional historical simulation; however it actually assigns lower weights to the return volatility. It uses the exponential moving average:

$$(1 - \lambda) \times \lambda^{r-1}.$$

So we essentially have a combination of approaches. The Exponential (EXP) approach, proposed by ([15]), makes the point that estimating condition volatilities and correlations, which is precisely the task at hand, could benefit from giving greater weight to recent returns. In Historical Simulation (HS), there is no assumption of any distribution and consequently parameterisation is avoided. This allows the data to assign the shape of the distribution. We obtain the VaR as follows:

1. Assign weights to the most recent returns,
2. Sort the returns in ascending order,
3. Accumulate the weights until the level of the VaR is reached.



Bootstrapping

Monte-Carlo simulation uses an algorithm to generate possible future paths of a portfolio where else the bootstrapping¹ uses the historical return instead of the algorithm. Firstly we index the daily returns. We then randomly select a cross-section vector of returns; a particular day from the historical window is chosen and the returns from this data are used for each asset in the portfolio. Possible paths are simulated and the Historical Simulation Methodology is utilised. The procedure is as follows:

- We start by indexing the returns.
- We randomly select one index return from our available set of indexes.
- We then take the randomly selected index to simulate forward for m (depending on time horizon) number of times.
- Sort these and look for the worst case (path) depending on our alpha.

¹This method is employed from Willmot ([75]) under simulation

With this approach we did not need to specify the distributional assumption, such as the normal distribution etc. We also have the cross-section of returns ([24]) which automatically incorporates the correlation of the returns.

VaR calculation using Bootstrapping and Filtered Historical Simulation

This is an improvement on the traditional Historical Simulation Method (Bootstrapping). This approach takes the change in past and current volatilities of historical returns into account, and makes the least number of assumptions about the statistical properties of future prices ([3]).

Barone-Adesi *et al.* ([2]) proposed filtering historical simulation through the GARCH process in order to model the future distribution of asset correlations. This method implicitly takes into account an asset's correlations, without restricting their values over time or compacting them explicitly. Historical Simulation assigns equal probability to past returns, neglecting current market conditions.

This model does not assume any theoretical distribution. It is a combination of parametric and non-parametric. This is due to the fact that GARCH modelling is parametric and historical simulation is non-parametric. The main extension is adoption of the historical residual returns to the current market conditions; this is achieved through scaling the current, over the past conditional volatilities, by the applicable ratios ([2]). This is done by dividing historical residual returns by this volatility. The standardised residuals are then scaled by volatility forecasts that reflect the current market conditions. The estimated returns are based on these residuals.

A pathway is simulated by taking each asset's returns and simulating the price path of a portfolio through randomly picking an 'indexed' set of returns. Each index corresponds to the price of the asset which occurred in that day. The residual return is then computed. The next step is to calculate volatilities for each asset, using a GARCH model; however any model can be applied. The sample residuals are changed according to the scale, by using the modelled volatility over the historical volatility.

Simulation of a single path This approach does not impose any theoretical distribution. In order for the returns to have independent identity distribution (i.i.d), it is required to remove serial correlations and volatility clusters. This is done by adding a Moving Average term into the Conditional Mean Equation. The volatility cluster is captured by modelling returns through the GARCH process ([14]). The serial dependence is removed by adding the moving average term. ARMA-GARCH(1,1) model can be written as follows:

$$r_t = \mu r_{t-1} + \theta \epsilon_{t-1} + \epsilon_t \quad N(0, \sigma_t),$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^q \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^p \beta_i \sigma_{t-1}^2,$$

where

μ is the AR(1) term,

θ is the MA term,

α_0 is a constant and,

ϵ_t is the random residual.



Returns are standardised by dividing the estimated residual $\hat{\epsilon}_k$.

VaR Valuation using Monte-Carlo Simulation

The Monte-Carlo method is more powerful and flexible than the other two methods as it has the ability to capture all non-linear scenarios of a portfolio's value with respect of the risk factor ([49]). It also generates an infinite number of scenarios and therefore tests numerous future outcomes. Due to this, Monte-Carlo Simulation is fast becoming recognised as the optimal quantitative methodology for measuring VaR.

The overview of Monte-Carlo (MC) is that it computes an infinite number of tests or paths. Simulation is then used to compute these tests. Simulation is needed as it's impractical to compute an infinite number of tests manually. To calculate the VaR using MC simulation, the portfolio prices are simulated first. It makes more sense to incorporate the correlation into the simulation, i.e. the asset prices are simulated in conjunction with their proportion. Having the asset's returns increase

in line with the proportion, we can therefore assume that the volatility is a constant stochastic process. The series of returns are calculated from the following equation:

$$\frac{ds}{s} = \mu dt + \sigma dz = \mu dt + \sigma \epsilon \sqrt{dt},$$

where

s is the stock price,

σ is the volatility of the asset price,

ϵ is the random number drawn from the standardised normal distribution.

For MC simulation, daily stock prices are used. The logarithmic returns are computed in order to get the daily standard deviations and correlations. At this point, in order to get VaR, simulated prices are sorted in ascending order to get the fifth percent number of the simulated price. The covariance matrix is not directly computed but is instead calculated using the standard deviations and correlations. To compute VaR for a monthly period, the asset prices are simulated for 21 days as we assume there are 252 trading days in a calendar year. We compute the following algorithm:

1. Choose the time horizon: 1 for daily VaR, 5 for weekly VaR, 21 for monthly VaR and 252 for yearly VaR. The length of the analysis is determined by the estimates of the historical returns.
2. Increment of the stock prices will be based on the random number drawn from the generator. The stochastic process is used here.
3. Increase or decrease the stock price until the end of the stock horizon.
4. Repeat the paths for N a number of times.
5. Sort the numbers from lowest to highest. VaR is then obtained from the final stock price, according to the level of confidence required.

4.4 Marginal and Component VaR

Now that we've looked at computing VaR, it is important to note that we calculated the total VaR and not the individual assets. In this section we consider the effects

as well as the importance of diversification with the Markowitz model ([51]. When the total risk of a portfolio is calculated, the total value of the risk is different to the added individual risk of the stocks. The algorithm laid out by Jorion ([40]) is followed here.

Marginal VaR

Here marginal VaR is defined as the partial derivative with respect to the component weight. It measures the change in a portfolio's VaR resulting from adding additional value to a component, Jorion ([40]).

This is the algorithm which shows the impact of a stock on a portfolio, if the value of the VaR of a particular stock is changed. We now look at the diagram obtained from Jorion ([40]) to gain a better understanding.

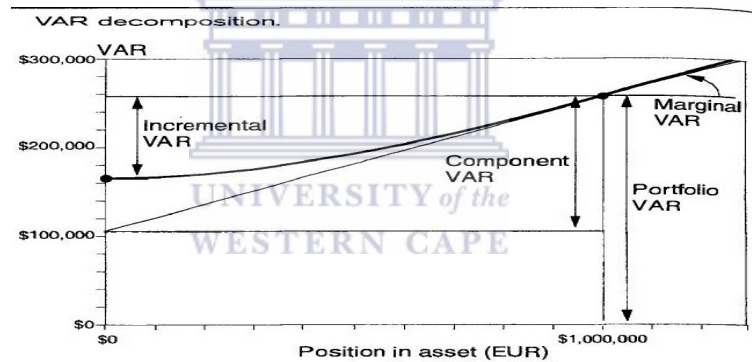


Figure 4.2: Marginal and Conditional VaR ([40]) P162

If we focus on the concave line, the line is the plot of a portfolio's VaR. The x-axis is the position of one of a portfolio's assets or components. Due to the convexity of the line, a point can be picked and a tangent line can be drawn; this is the marginal VaR. This shows that the first derivative is the key sensitivity. The marginal VaR of the i^{th} component is then calculated with the following formula:

$$\Delta VaR_i = \frac{VaR}{W} * \beta_i.$$

Essentially what we are aiming to understand is how much a portfolio's risk will change due to the removal of a particular stock, because that particular risk value

does not equal the individual risk due to diversification. It's important to note that the formula can be adjusted due to beta.

Component VaR

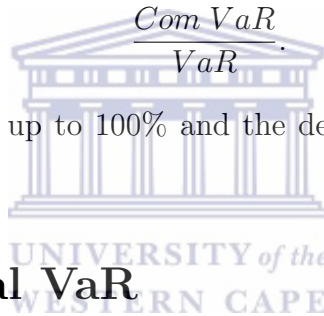
Component VaR is defined as a partition of the portfolio VaR which indicates a change in VaR if a given component is deleted ([40]). This is based on the Marginal VaR, and in order to calculate the component VaR, we first calculate the component VaR of the i^{th} asset as:

$$Com VaR = VaR.w_i.\beta_i$$

and the contribution percentage of the i^{th} asset as

$$\frac{Com VaR}{VaR}$$

The percentage must add up to 100% and the deviations or differences should be 0%.



4.5 Conditional VaR

This is a further development of VaR. It was first introduced by Rockafellar ([60]). He extended the model by asking questions regarding how much could be lost if things got really bad. CVaR is always greater than VaR, as is evident from Figures 4.1 and 4.3 as they have the same VaR but Figure 4.3 has a riskier portfolio. CVaR seems like a desirable measure, but VaR however, remains the most prevalent measure in practice ([35]). This can be optimised and this is done by following the Rockafellar and Uryasev models ([60]). By definition, a specified probability level α (normally 0.9, 0.95 or 0.99), the α VaR of a portfolio is the lowest amount θ such that, with probability α , the loss will not exceed θ , whereas the α CVaR is the conditional expectation of losses above that amount. CVaR is also known as Mean Excess, Mean Short, or Tail and it's considered more consistent than VaR ([35]).

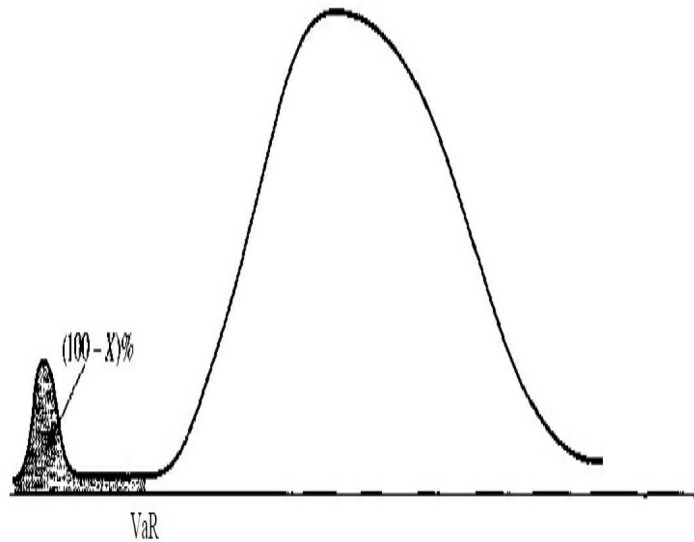


Figure 4.3: CVaR lowest percentile ([35]) P445

Application to portfolio

Following the previous notion of Quadratic Programming we have the following form:

$$\min \sigma^2(w),$$

$$\text{subject to } -w^T q \leq R,$$

$$w_i \geq 0,$$

where $g(w, q) = -w^T q$.

We solve for

$$\theta_\alpha(w) = \mu(w) + s_1(\alpha)\sigma(w)$$

and

$$\Omega_\alpha(w) = \mu(w) + s_2(\alpha)\sigma(w),$$

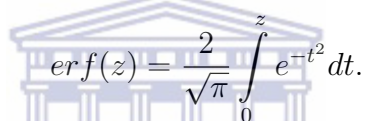
where

$$s_1 = \sqrt{2} \operatorname{erf}^{-1}(2\alpha - 1)$$

and

$$s_2 = (\sqrt{2\pi} \exp(\operatorname{erf}^{-1}(2\alpha - 1))^2 (1 - \alpha))^{-1}.$$

The term $\operatorname{erf}(z)$ is an error function also known as the Gauss error function and it is of the formula



$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

Asset Assessment using Hotspots and the Best Hedge report

We've looked at marginal VaR, which deals with the individual contribution of risk to a portfolio. We now extend upon that by following the process of Goldman Sachs by Litterman ([46]). He shows how to identify the primary source of risk "Hotspot" and how to reduce the risk hedge. This is an important process for portfolio managers to understand. It provides an algorithm to highlight the biggest risk contributors in a portfolio, so that managers are able to reduce a portfolio's risk. This is reported in The Best Hedge Report. This is broken down in two phases. Firstly, the hotspot needs to be identified. In order to do this we look at the highest to lowest contributors of risk in a portfolio. This highlights the asset's impact on a portfolio's risk for a risk manager. The next step is to try and hedge the portfolio risk using the potential assets after they've been identified. This is done by looking at the Trade Risk profile' ([46]). In fact we do this through portfolio optimisation to complete the process ([46]). We go back to the VaR and CVaR calculation by Rockafellar ([60]). The portfolio risk is then minimised by changing the portion to obtain the best hedge for each asset through optimisation.

4.6 Summary

In this chapter we have looked at the basic tools to estimate VaR including some common approaches namely Parametric, Historical Simulation and Monte-Carlo methods. It is clear that each method has its advantages and disadvantages. We also looked at an indirect approach to the Historical Simulation method by bootstrapping in order to estimate the VaR. We have covered the tools of VaR through Marginal and Component VaR. These play an important role as they provide an indication of an asset's risk and not just the portfolio. Finally we concluded this chapter by looking at CVaR, which is based on VaR but has a different notion towards the estimation of loss. In the next chapter, we present numerical simulation for certain portfolio investment models.



Chapter 5

Numerical Simulation for Investment Portfolio Models

5.1 Introduction

In this chapter we chose twenty nine stocks, S&P500 index and the one year T-bill rate. We used historical data from <http://www.finance.yahoo.com> dating from 01/01/1996 to 31/12/2010. We then applied the various methods that we've covered in order to compute efficient portfolios assuming our position is at 01/01/2010 and we have a lump sum of \$1 000 000. From there we measured the portfolio's performance in 2011 and we also show how we have measured their performance. As mentioned above, our stock price data was obtained from <http://www.finance.yahoo.com> and the risk-free rate is from the United States Department of the Treasury.

Here we are going to conduct experiments based on the reported literature.

- First, we compute our portfolios using Quadratic Programming to obtain portfolios based on Markowitz's principles.
- Constrains are then added to move away from concentrated or abnormal portfolios that can sometimes be seen with the Markowitz principles.
- The CAPM theory is then applied in order to move us even further away from some of the abnormal Markowitz portfolios. We also try to diversify more by adding a risk-free asset.
- The problem still exists even after applying CAPM.

- We then follow the Black-Litterman model to reduce the problem further.
- VaR is now considered in order to try and quantify risk as a value for future disasters. We calculate VaR with a 95% confidence level in all the calculations.

Before we present our numerical results, we would just like to highlight that as we are dealing with a huge dimension of assets, we are only able to show some rows and columns as all of the data cannot be presented in this thesis due to space limitations. We also removed the asset allocation of allowing short selling as well as groups without short selling.

5.2 Evaluation of Markowitz Portfolios

We obtained twenty nine historical prices of stocks of the S&P500 and the S&P500 from <http://www.finance.yahoo.com>. All the assets were adjusted for dividends except the S&P500. Using the historical prices, we were able to compute the continuous returns of each stock to obtain the basic statistical parameters, which are the means, variances and the correlations as follows on Table 5.1 and Table 5.2. For the purpose of this discussion we use the geometric mean as opposed to the arithmetic mean. The following formula is used in order to obtain the mean:

$$\mu_t = \left(\prod_{t=1}^T (1 - r_{it}) \right)^{\frac{1}{T}} - 1.$$

Table 5.1: Computation of the asset statistical measures

| Stock Name | Stock Abbreviation | Arithmetic Mean | Geometric Mean | Variance | Sigma | Beta | Alpha |
|------------------------------|--------------------|-----------------|----------------|----------|-------|--------|---------|
| Hewlett | HPQ | 1.208% | 0.613% | 0.012 | 11% | 1.3777 | 0.0053 |
| Whirlpool | WHR | 1.099% | 0.482% | 0.013 | 12% | 1.3268 | 0.0070 |
| AT&T | T | 0.642% | 0.352% | 0.006 | 8% | 0.7601 | 0.0027 |
| Autozone | AZO | 1.780% | 1.364% | 0.008 | 9% | 0.5405 | 0.0151 |
| JP Morgan | JPM-PW | 1.109% | 0.600% | 0.011 | 11% | 1.4172 | 0.0041 |
| Chevron Group | CVX | 1.163% | 0.980% | 0.004 | 6% | 0.6509 | 0.0084 |
| Chubb Group | CB | 0.967% | 0.714% | 0.005 | 7% | 0.6359 | 0.0065 |
| Coca Cola | KO | 0.707% | 0.476% | 0.005 | 7% | 0.6005 | 0.0041 |
| Noble Energy | NBL | 1.578% | 1.049% | 0.011 | 10% | 0.7735 | 0.0120 |
| Dell | Dell | 2.452% | 1.552% | 0.018 | 13% | 1.7087 | 0.0161 |
| United Health | UNH | 1.316% | 0.868% | 0.011 | 11% | 0.8498 | 0.0090 |
| Exxon Mobil | XOM | 1.055% | 0.925% | 0.003 | 5% | 0.4858 | 0.0081 |
| Kellogg | K | 0.578% | 0.392% | 0.004 | 6% | 0.2635 | 0.0045 |
| McDonalds | MCD | 0.998% | 0.767% | 0.005 | 7% | 0.7126 | 0.0065 |
| Mc Graw Hill | MHP | 1.139% | 0.837% | 0.006 | 8% | 0.9024 | 0.0069 |
| Microsoft | MSFT | 1.541% | 1.009% | 0.011 | 10% | 1.3765 | 0.0086 |
| Good Year | GT | 0.408% | -0.656% | 0.023 | 15% | 1.7621 | -0.0046 |
| Nike | NIKE | 1.456% | 1.016% | 0.01 | 10% | 0.8612 | 0.0103 |
| Pfizer | PFE | 0.697% | 0.457% | 0.005 | 7% | 0.6835 | 0.0036 |
| General Electric | GE | 0.761% | 0.416% | 0.007 | 9% | 1.2936 | 0.0012 |
| Walmart | WMT | 1.259% | 1.028% | 0.005 | 7% | 0.6270 | 0.0095 |
| American International Group | AIG | 1.295% | -1.111% | 0.206 | 45% | 2.0764 | 0.0027 |
| Alcoa AA | AA | 0.898% | 0.217% | 0.017 | 13% | 1.6639 | 0.0007 |
| Caterpillar CAT | CAT | 1.674% | 1.195% | 0.01 | 10% | 1.2666 | 0.0105 |
| General Mills | GIS | 0.933% | 0.801% | 0.003 | 5% | 0.1974 | 0.0084 |
| Ace Ltd | ACE | 1.297% | 0.967% | 0.007 | 8% | 0.7759 | 0.0091 |
| American Express AXP | AXP | 1.252% | 0.755% | 0.009 | 10% | 1.5017 | 0.0051 |
| IBM | Ibm | 1.396% | 1.035% | 0.007 | 8% | 1.1379 | 0.0083 |
| Ford | Ford | 1.538% | 0.460% | 0.025 | 16% | 1.6973 | 0.0070 |
| Standard and Poor | S&P 500 | 0.495% | 0.382% | | | 1.0000 | 0.0000 |

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Table 5.2: Computation of the asset covariance and correlation matrices

| covariance Matrix | | | | | | | | | |
|-------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| | HPQ | WHR | T | AZO | GIS | ACE | AXP | IBM | Ford |
| HPQ | 0.012 | 0.005 | 0.002 | 0 | 0 | 0.002 | 0.004 | 0.005 | 0.005 |
| WHR | 0.005 | 0.013 | 0.002 | 0.002 | 0.001 | 0.003 | 0.007 | 0.003 | 0.01 |
| T | 0.002 | 0.002 | 0.006 | 0.001 | 0.001 | 0.001 | 0.003 | 0.002 | 0.003 |
| AZO | 0 | 0.002 | 0.001 | 0.008 | 0 | 0.001 | 0.002 | 0.001 | 0.002 |
| WMT | 0 | 0.002 | 0.001 | 0.002 | 0 | 0.001 | 0.002 | 0.001 | 0.003 |
| AIG | 0.005 | 0.012 | 0.008 | 0.005 | 0 | 0.005 | 0.015 | 0.004 | 0.005 |
| AA | 0.007 | 0.009 | 0.003 | 0.002 | 0 | 0.002 | 0.006 | 0.005 | 0.013 |
| CAT | 0.003 | 0.008 | 0.003 | 0.002 | 0 | 0.003 | 0.006 | 0.003 | 0.009 |
| GIS | 0 | 0.001 | 0.001 | 0 | 0.003 | 0.001 | 0.001 | 0 | 0 |
| ACE | 0.002 | 0.003 | 0.001 | 0.001 | 0.001 | 0.007 | 0.004 | 0.001 | 0.002 |
| AXP | 0.004 | 0.007 | 0.003 | 0.002 | 0.001 | 0.004 | 0.009 | 0.003 | 0.008 |
| IBM | 0.005 | 0.003 | 0.002 | 0.001 | 0 | 0.001 | 0.003 | 0.007 | 0.005 |
| Ford | 0.005 | 0.01 | 0.003 | 0.002 | 0 | 0.002 | 0.008 | 0.005 | 0.025 |

| Correlations | | | | | | | | | |
|--------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| | HPQ | WHR | T | AZO | GIS | ACE | AXP | IBM | Ford |
| HPQ | 1 | 0.3215 | 0.2011 | 0.0363 | 0.0378 | 0.197 | 0.3397 | 0.5353 | 0.2588 |
| WHR | 0.3215 | 1 | 0.2134 | 0.1556 | 0.1942 | 0.3434 | 0.5958 | 0.2982 | 0.5133 |
| T | 0.2011 | 0.2134 | 1 | 0.1775 | 0.2469 | 0.2141 | 0.3019 | 0.2795 | 0.2737 |
| AZO | 0.0363 | 0.1556 | 0.1775 | 1 | 0.0675 | 0.1661 | 0.1937 | 0.1214 | 0.1534 |
| GIS | 0.0378 | 0.1942 | 0.2469 | 0.0675 | 1 | 0.1776 | 0.1116 | -0.026 | 0.0548 |
| ACE | 0.197 | 0.3434 | 0.2141 | 0.1661 | 0.1776 | 1 | 0.4243 | 0.1877 | 0.2597 |
| AXP | 0.3397 | 0.5958 | 0.3019 | 0.1937 | 0.1116 | 0.4243 | 1 | 0.3328 | 0.6636 |
| IBM | 0.5353 | 0.2982 | 0.2795 | 0.1214 | -0.026 | 0.1877 | 0.3328 | 1 | 0.2577 |
| Ford | 0.2588 | 0.5133 | 0.2737 | 0.1534 | 0.0548 | 0.2597 | 0.6636 | 0.2577 | 1 |

| | HPQ | WHR | T | AZO | GIS | ACE | AXP | IBM | Ford | S&P500 |
|--------|--------|--------|--------|--------|--------|--------|---------|---------|--------|--------|
| Alpha | 0.0053 | 0.007 | 0.0027 | 0.0151 | 0.0084 | 0.0091 | 0.0051 | 0.0083 | 0.007 | 0 |
| SE | 0.0067 | 0.6321 | 0.0051 | 0.0067 | 0.0038 | 0.0055 | 0.0059 | 0.0051 | 0.0102 | 0 |
| T-Stat | 0.7903 | | 0.5207 | 2.2727 | 2.1805 | 1.6622 | 0.863 | 1.6388 | 0.6839 | |
| Beta | 1.3777 | 1.3268 | 0.7601 | 0.5405 | 0.1974 | 0.7759 | 1.5017 | 1.1379 | 1.6973 | 1 |
| SE | 0.1402 | 0.1475 | 0.1075 | 0.1403 | 0.0808 | 0.1157 | 0.1242 | 0.1071 | 0.2152 | 0.0482 |
| T-Stat | 9.8286 | 8.9957 | 7.0719 | 3.8525 | 2.4433 | 6.7056 | 12.0873 | 10.6257 | 7.8867 | |

Huang and Litzenburg Approach for Asset Allocation

First we start by computing the efficient frontier of the Markowitz model by using the Huang and Litzenburg approach. This is a simple algebraic computation. This approach is suitable when there are no constraints on our assets. By using this approach we solve the following model:

$$\min w'Vw$$

$$\text{subject to } w'1 = 1,$$

$$w'\mu = q,$$

$$w = \frac{C - qB}{AC - B^2}V^{-1}1 + \frac{qA - B}{AC - B^2}V^{-1}\mu,$$

where

$$A = 1'V^{-1}1, B = 1'V^{-1}\mu \text{ and } C = \mu'V^{-1}\mu.$$

We also have one as the vector of ones and V^{-1} as the inverse of the covariance matrix. This computes two envelope portfolios. We then use the Two Fund Theorem to obtain the rest of the envelope.

We constructed ten portfolios with annual returns of 6% to 33% in Table 5.3. The table shows that portfolios 1 and 2 carry higher risk and a lower expected return than portfolios 3 and 4. From the shape of the graph in Figure 5.1(a), we can see that we obtained an envelope portfolio. We cannot put constraints on this approach, and this provides limitations, as the efficient frontier cannot be constructed. This is crucial due to the assumption that we should be risk averse and we are therefore only interested in the efficient frontier. This could also be a problem for investors who don't want to go short as this approach only sums up the weights of 100% regardless of the sign. From the values at the end of the period in Figure 5.1(b), we can see that every portfolio value has increased and every portfolio is greater than \$1 million. It is interesting to note that the profit is increasing in-line with the increment of the expected return and the standard deviation. We also note

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that portfolio's 1 and 2, which are not efficient, have actually made less profit than 3 and 4. This shows that the portfolio's which are under the minimum variance portfolio or in a lower expected return with the same standard deviation are not good investment options.

Table 5.3: Asset allocation using Huang and Litzenburg Approach

| Portfolios | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---------------|--------------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| Sigma | 11.10% | 10.00% | 9.92% | 10.88% | 12.65% | 14.95% | 17.56% | 20.37% | 23.31% | 26.33% |
| Return | 6.00% | 9.00% | 12.00% | 15.00% | 18.00% | 21.00% | 24.00% | 27.00% | 30.00% | 33.00% |
| | PORTFOLIO WEIGHTS | | | | | | | | | |
| HPQ | 7.33% | 5.26% | 3.19% | 1.12% | -0.95% | -3.02% | -5.09% | -7.16% | -9.23% | -11.30% |
| WHR | 5.67% | 5.19% | 4.72% | 4.24% | 3.76% | 3.29% | 2.81% | 2.33% | 1.86% | 1.38% |
| T | 1.21% | -1.77% | -4.75% | -7.73% | -10.71% | -13.68% | -16.66% | -19.64% | -22.62% | -25.60% |
| AZO | 3.10% | 5.34% | 7.59% | 9.84% | 12.09% | 14.33% | 16.58% | 18.83% | 21.08% | 23.32% |
| JPM | 0.21% | -0.74% | -1.70% | -2.65% | -3.61% | -4.57% | -5.52% | -6.48% | -7.44% | -8.39% |
| CVX | -4.42% | -4.38% | -4.35% | -4.32% | -4.28% | -4.25% | -4.22% | -4.18% | -4.15% | -4.12% |
| CB | 8.87% | 6.64% | 4.42% | 2.20% | -0.02% | -2.24% | -4.46% | -6.68% | -8.91% | -11.13% |
| KO | 9.92% | 6.72% | 3.51% | 0.31% | -2.90% | -6.10% | -9.31% | -12.51% | -15.71% | -18.92% |
| NBL | 8.11% | 7.14% | 6.17% | 5.20% | 4.23% | 3.26% | 2.29% | 1.32% | 0.35% | -0.62% |
| Dell | -10.23% | -5.84% | -1.46% | 2.93% | 7.32% | 11.70% | 16.09% | 20.48% | 24.86% | 29.25% |
| UNH | -6.56% | -4.70% | -2.84% | -0.98% | 0.88% | 2.74% | 4.60% | 6.46% | 8.32% | 10.18% |
| XOM | 24.64% | 27.08% | 29.53% | 31.98% | 34.42% | 36.87% | 39.31% | 41.76% | 44.21% | 46.65% |
| K | 23.06% | 17.11% | 11.16% | 5.21% | -0.74% | -6.69% | -12.64% | -18.59% | -24.54% | -30.49% |
| MCD | -1.46% | -0.06% | 1.35% | 2.75% | 4.16% | 5.56% | 6.96% | 8.37% | 9.77% | 11.18% |
| MHP | 9.40% | 10.79% | 12.17% | 13.55% | 14.93% | 16.31% | 17.69% | 19.07% | 20.46% | 21.84% |
| MSFT | -1.30% | -1.99% | -2.68% | -3.37% | -4.06% | -4.76% | -5.45% | -6.14% | -6.83% | -7.52% |
| GT | -0.34% | -4.18% | -8.02% | -11.86% | -15.70% | -19.54% | -23.38% | -27.22% | -31.06% | -34.90% |
| NKE | -0.46% | -0.80% | -1.14% | -1.48% | -1.82% | -2.15% | -2.49% | -2.83% | -3.17% | -3.51% |
| PFE | 14.15% | 9.95% | 5.75% | 1.54% | -2.66% | -6.86% | -11.06% | -15.27% | -19.47% | -23.67% |
| GE | 3.58% | 2.17% | 0.75% | -0.66% | -2.07% | -3.49% | -4.90% | -6.31% | -7.73% | -9.14% |
| WMT | 12.77% | 11.54% | 10.32% | 9.09% | 7.87% | 6.64% | 5.42% | 4.19% | 2.97% | 1.74% |
| AIG | 0.60% | 0.06% | -0.48% | -1.02% | -1.56% | -2.10% | -2.64% | -3.18% | -3.72% | -4.26% |
| AA | 3.26% | -0.35% | -3.96% | -7.57% | -11.18% | -14.79% | -18.40% | -22.01% | -25.62% | -29.23% |
| CAT | -17.35% | -8.62% | 0.12% | 8.85% | 17.58% | 26.31% | 35.05% | 43.78% | 52.51% | 61.25% |
| GIS | 13.91% | 19.84% | 25.78% | 31.71% | 37.65% | 43.58% | 49.51% | 55.45% | 61.38% | 67.32% |
| ACE | -5.94% | -3.10% | -0.27% | 2.57% | 5.40% | 8.24% | 11.07% | 13.91% | 16.75% | 19.58% |
| AXP | -9.58% | -8.43% | -7.29% | -6.14% | -5.00% | -3.85% | -2.71% | -1.56% | -0.42% | 0.73% |
| Ibm | 8.93% | 11.01% | 13.09% | 15.17% | 17.25% | 19.33% | 21.41% | 23.49% | 25.56% | 27.64% |
| Ford | -1.10% | -0.90% | -0.69% | -0.49% | -0.28% | -0.08% | 0.12% | 0.33% | 0.53% | 0.73% |
| Sum | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% |

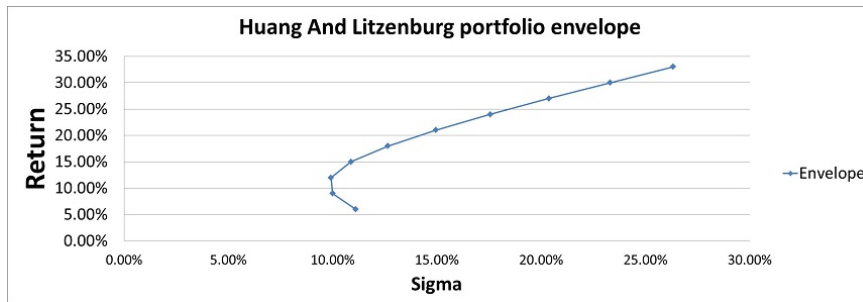
Asset allocation with no short selling allowed

We now solve the weights by using Matlab. We also add an additional constrain of $w_i \geq 0$, i.e. where no short selling is allowed. So we solve this for

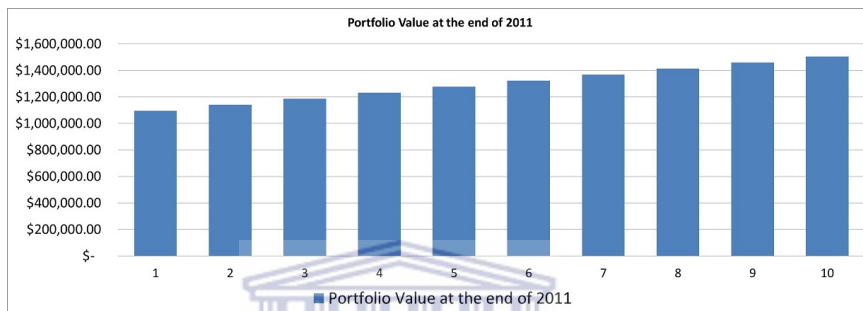
$$\min w'Vw,$$

$$\text{subject to } w'1 = 1,$$

$$w'\mu = q,$$



(a) Huang and Litzenburg Portfolio Envelope



(b) Huang and Litzenburg Portfolio Value at the end of 2011

Figure 5.1: Huang and Litzenburg envelope and value.

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$$w_i \geq 0.$$

Where w_i 's are the weights of the assets, V is the covariance matrix and, μ is the vector of geometric means. When we allow short selling we remove the last equation if we have no bounds, but here we limit the short selling by up to 50% and the value of each asset by 150%.

Results: In Table 5.4 we have ten portfolios obtained by not allowing short selling and also allowing short selling. Most of the assets were left out by performing optimisation. Some portfolios are concentrated with particular assets such as the Dell stock. This creates a problem in that the portfolio is highly volatile. From the ten portfolios, the rate of return goes from an annual return of 10.2% to 18.63% and an annual standard deviation of 10.97% to 45.96%. The numbers are not unreasonable; however, this disregards our intention of diversification, i.e. having as many assets as possible through the law of large numbers. This optimisation selected an average of six assets out of the possible twenty nine assets. Following this we try and limit

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our assets by including an additional constrain of $0.1 \leq w_i \leq 0.5$ however, this failed due to the fact that there is no solution which satisfies this model.

When we allow short selling, the optimisation chooses several assets and even so these are concentrated within certain assets. In this scenario, the expected return goes from 10.73% to 98.51% while the standard deviation goes from 9.82% to 171.65%. It is interesting to note that the expected returns of portfolios 1 to 6 are less than their standard deviation. These are the benefits of short selling, however with that it also brings high volatility to some portfolios. We can also see from the comparison in Figure 5.2(c) how the no short selling efficient frontier, is enclosed under the short selling efficient frontier. This profit tends to decrease while the standard deviations increase.

Regardless of the concentration of assets, when short selling is not allowed, we see from Figure 5.2(d) there is profit at the end of the period. Also when allowing for short selling, there is profit, and in fact, some portfolios have doubled and even tripled due to short selling.

Table 5.4: Markowitz Portfolio Selection with no short selling allowed and with short selling

| Portfolio Weights when short selling is not allowed | | | | | | | | |
|---|--------------|---------------|---------|---------|---------|---------|---------|---------|
| Ports | Annual Sigma | Annual Return | HPQ | WHR | T | AXP | Ibm | Ford |
| 1 | 11.0% | 10.2% | 0.0% | 0.0% | 0.0% | 0.0% | 4.8% | 0.0% |
| 2 | 11.2% | 11.1% | 0.0% | 0.0% | 0.0% | 0.0% | 5.5% | 0.0% |
| 3 | 12.1% | 12.1% | 0.0% | 0.0% | 0.0% | 0.0% | 3.6% | 0.0% |
| 4 | 13.9% | 13.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.4% | 0.0% |
| 5 | 16.4% | 13.9% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| 6 | 19.3% | 14.9% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| 7 | 22.4% | 15.8% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| 8 | 26.0% | 16.8% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| 9 | 31.6% | 17.7% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| 10 | 46.0% | 18.6% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| Portfolio Weights when short selling is allowed | | | | | | | | |
| Ports | Annual Sigma | Annual Return | HPQ | WHR | T | AXP | Ibm | Ford |
| 1 | 9.82% | 10.73% | 4.07% | 4.92% | -3.48% | -7.77% | 12.21% | -0.78% |
| 2 | 14.52% | 20.48% | -2.66% | 3.37% | -13.17% | -4.05% | 18.97% | -0.12% |
| 3 | 23.54% | 30.23% | -9.39% | 1.82% | -22.86% | -0.32% | 25.73% | 0.55% |
| 4 | 33.56% | 39.99% | -16.13% | 0.28% | -32.55% | 3.40% | 32.49% | 1.21% |
| 5 | 44.09% | 49.74% | -26.51% | -5.53% | -45.00% | 8.27% | 40.08% | -0.63% |
| 6 | 55.84% | 59.50% | -44.89% | -17.52% | -50.00% | 21.67% | 48.03% | -10.46% |
| 7 | 69.98% | 69.25% | -50.00% | -29.54% | -50.00% | 39.55% | 52.02% | -23.75% |
| 8 | 87.68% | 79.01% | -50.00% | -50.00% | -50.00% | 49.71% | 58.06% | -40.16% |
| 9 | 110.86% | 88.76% | -50.00% | -50.00% | -50.00% | 17.58% | 79.39% | -50.00% |
| 10 | 171.65% | 98.51% | -50.00% | -50.00% | -50.00% | -50.00% | 150.00% | -50.00% |

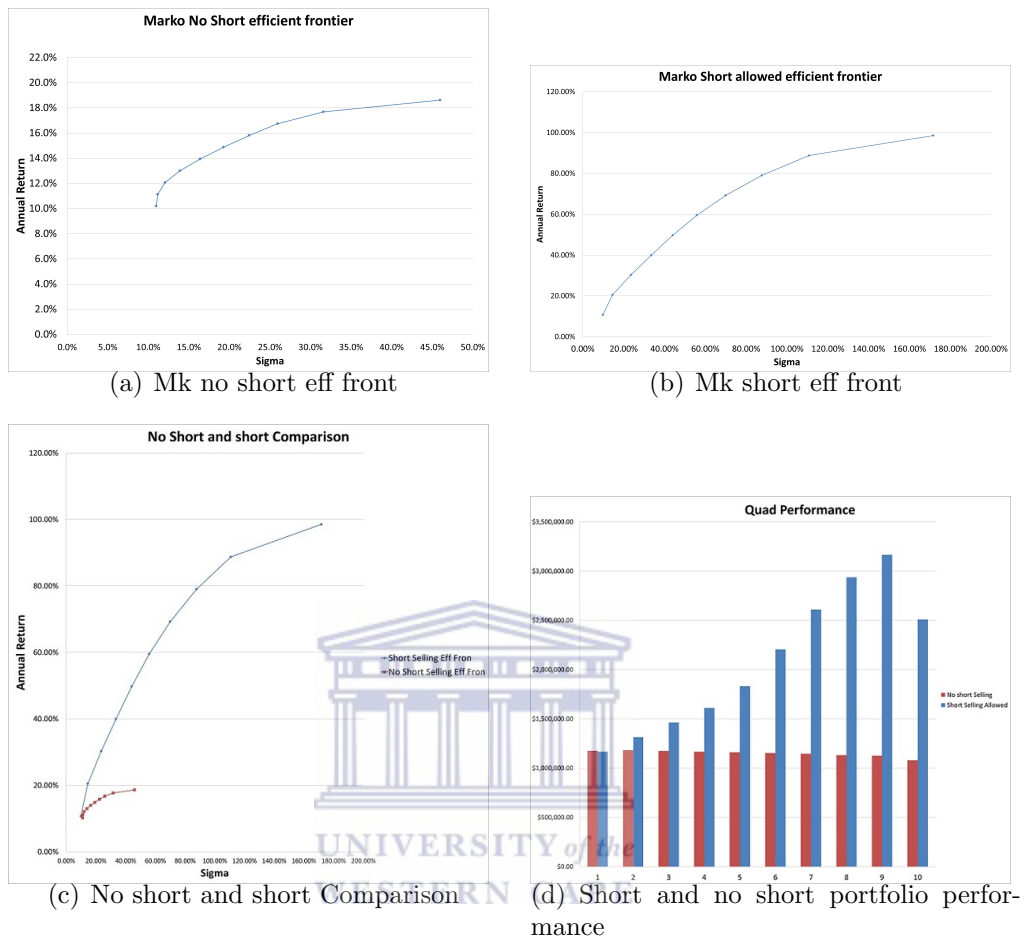


Figure 5.2: Markowitz efficient frontier and performance.

Asset allocation through Grouping

We have seen above, that the model fails when we impose restrictive bounds to the assets. In this section we are going to allow short selling of -0.5 and will limit the assets to 1.5. These groups will then be divided into their respective sectors: Consumer Discretionary (WHR, AZO, MCO, GT, NKE and FORD) from 0 up to 0.7; Consumer Staples (KO, K, WMT and GIS) from 0 up to 0.55; Energy (NBL, CVX and XOM) from 0 up to 0.25; Financials (JPM, CB, AIG, ACE and AXP) from 0 up to 0.6; Health Care (UNH and PFE) from 0 up to 0.2; Industrial (GE and CAT) from 0 up to 0.2; Information technology (HPQ, DELL, MSFT and IBM) from 0 up to 0.55; Materials (AA) from 0 up to 0.1; Telecommunication Services (T) from 0 up to 0.1. This makes the assets flexible but restricts their sector.

Results: Setting group bounds tends to improve our results in Table 5.5. There is an increase in the number of assets allocated i.e. concentration is reduced. We can see that the assets have an average allocation to a possible eleven out of twenty nine. When short sales are introduced we obtain a good return but the volatility still persists, and the problem of having assets holding very high short positions also still exists. When short selling is not allowed, there is hardly any improvement in the asset allocation spread. In fact, in the consumer discretion sector, we have one dominant asset (AZO), which is still allocated in every portfolio. We have a similar situation for all the other sectors. This tells us that when short selling is not allowed, there is little improvement in terms of distribution of the wealth. We can also see that the expected return of the minimum variance portfolio through grouping is less than that with no grouping. It also has a higher standard deviation (return from 10.2 \rightarrow 9.99% and STD 11 \rightarrow 11.02). When short selling is allowed, there is no improvement as the assets, like AA and T, are left out of the allocation. In some assets, the excessive position is reduced but with minor significance. We also see that the expected return of the minimum variance portfolio of the grouped approach, has actually decreased from 10.73% to 10.60% and the standard deviation increased from 9.82% to 9.94% when comparing it with the no group approach.

We then reduced the margin of the group, and we bound consumer discretionary from 0.3 to 0.7, consumer staples from 0.2 to 0.55 financials from 0.3 to 0.6 and information technology from 0.25 to 0.55. When we reduced these margins optimisation failed. Following this outcome, we widened the margin of information technology and it moved from 0.1 to 0.55. The expected return of the minimum variance portfolio of the grouping approach is higher than the no grouping approach. When we look at the groups in Figure 5.3 under the no short selling approach performance, we can see that portfolios 3 to 8 are all in the negative position, compared to their initial position. When short selling is allowed, portfolios 3 and 4 made a loss. Under the no short selling approach portfolios 5 to 8 have made significant losses. This is surprising as the efficient frontier from Figure 5.2 of the ungrouped data, is close to the grouped data. There were no losses in the ungrouped data.

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Table 5.5: Portfolio Selection by grouping assets and imposing group bounds

| Portfolio Weights when short selling is not allowed with group constrains | | | | | | | | |
|---|---------------|-------|-------|-------|-------|-------|-------|--|
| Annual Sigma | Annual Return | HPQ | WHR | T | AXP | IBM | Ford | |
| 11.02% | 9.99% | 0.04% | 0.00% | 0.00% | 0.00% | 6.09% | 0.00% | |
| 11.18% | 10.84% | 0.00% | 0.00% | 0.00% | 0.00% | 7.53% | 0.00% | |
| 11.74% | 11.68% | 0.00% | 0.00% | 0.00% | 0.00% | 7.82% | 0.00% | |
| 13.00% | 12.53% | 0.00% | 0.00% | 0.00% | 0.00% | 4.95% | 0.00% | |
| 14.90% | 13.38% | 0.00% | 0.00% | 0.00% | 0.00% | 1.34% | 0.00% | |
| 17.23% | 14.22% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | |
| 19.88% | 15.07% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | |
| 22.78% | 15.92% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | |
| 25.99% | 16.76% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | |
| 30.72% | 17.61% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | |

| Portfolio Weights when short selling is allowed with group constrains | | | | | | | | |
|---|---------------|---------|---------|-------|--------|--------|---------|--|
| Annual Sigma | Annual Return | HPQ | WHR | T | AXP | IBM | Ford | |
| 9.94% | 10.60% | 3.00% | 3.19% | 0.00% | -6.21% | 12.18% | -1.59% | |
| 13.76% | 18.67% | -5.89% | 2.51% | 0.00% | -0.38% | 17.94% | -3.15% | |
| 21.39% | 26.74% | -15.03% | 1.64% | 0.00% | 6.66% | 23.62% | -4.62% | |
| 30.16% | 34.80% | -24.70% | 2.55% | 0.00% | 15.78% | 29.94% | -5.72% | |
| 39.44% | 42.87% | -36.90% | -0.10% | 0.00% | 25.79% | 36.37% | -8.67% | |
| 50.39% | 50.94% | -50.00% | -7.20% | 0.00% | 39.03% | 43.76% | -15.55% | |
| 64.26% | 59.00% | -50.00% | -19.96% | 0.00% | 48.73% | 33.18% | -26.56% | |
| 81.03% | 67.07% | -50.00% | -34.05% | 0.00% | 52.82% | 6.41% | -39.60% | |
| 100.61% | 75.14% | -50.00% | -50.00% | 0.00% | 55.45% | 5.00% | -50.00% | |
| 135.00% | 83.20% | -50.00% | -50.00% | 0.00% | 0.00% | 5.00% | -50.00% | |

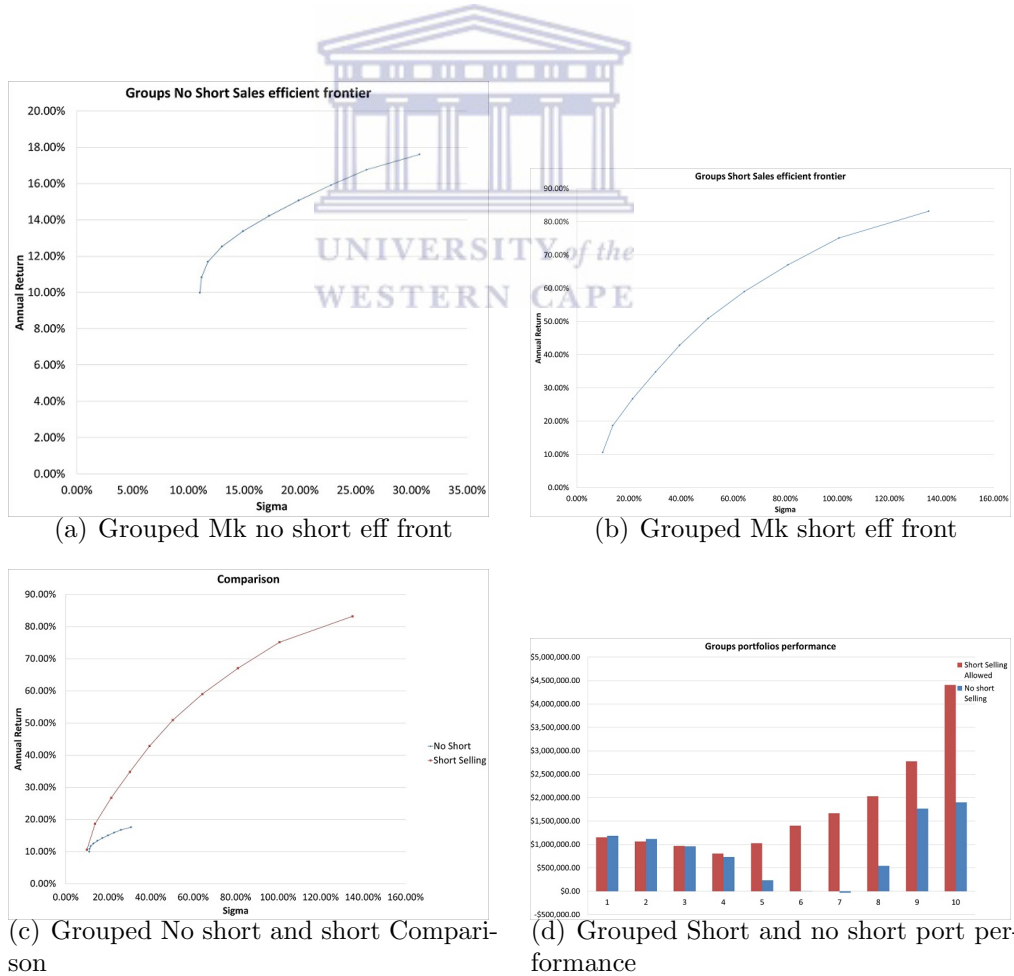


Figure 5.3: Grouped Markowitz efficient frontier and performance.



Figure 5.4: Portfolio efficient frontier group bounds comparison

5.3 Evaluation of portfolios using the CAPM anticipated returns

We now change the expected return in order to see if the problem can be resolved. We estimate the expected returns by using the CAPM. We also introduce the risk-free asset. The risk-free rate is 6% p.a. which implies that it is 0.5% monthly. We start by estimating betas of every asset by taking the S&P500 to be the market portfolio. We calculate the expected returns by using the CAPM formula

$$E(R_i) = r_f + \beta_i[E(R_M) - r_f],$$

where,

r_f is the risk-free rate,

β is the covariance of the asset i against the S&P500 and $E(R_M)$ is the S&P500 expected return.

The process of calculating the efficient frontier is then repeated using Quadratic Programming through setting the constraints, i.e. we restrict them by not allowing short sales, allowing short sales, grouping and not allowing short sales as well as allowing grouped short sales. We calculate the market or benchmark portfolio using

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Blacks zero beta to see whether the efficient frontier is moving above or below the market portfolio.

Results: In all our calculations in Table 5.6, we can see the efficient frontier is on the market portfolio. It is interesting to note that by using the zero beta model we obtained the same expected return as the S&P500 geometric mean of 0.38% or the annual mean of 4.6%. Another interesting finding is that even in the restricted groups i.e. no short sales, the efficient frontier is above the market portfolio. When short selling is allowed the market portfolio is very close to the efficient frontier. The portfolios that are close to the market portfolio are evenly distributed, with the wealth spread across various assets. When all the efficient frontiers are on the same set of axes, we limit the optimisation to provide good returns for the level of risk.

When we look at Figure 5.3 under CAPM, by allowing short selling at the output of $t+1$ we see that when holding a portfolio with a high standard deviation an investor can go from wealthy to bankrupt. The portfolio with a standard deviation of 135% made almost 100% return, however a portfolio with a standard deviation of 161% lost almost 10%. The portfolio with a 210% standard deviation (portfolio number 10 under the short selling approach) has created debt of 60% of the original value. This is due to the high position assumed in the portfolio. When we group these assets, we see that we have avoided losses that were in the short selling approach.

Table 5.6: Portfolio selection by using CAPM returns

| Risk free | Expected Market Return | Market sigma | | | | | | |
|------------|------------------------|---|-----|----|-----|-----|------|--|
| 0.00025 | 0.045791879 | 0.170979 | | | | | | |
| Port Sigma | Port Return | Portfolio Weights when short selling is not allowed | | | | | | |
| | | HPQ | WHR | T | AXP | IBM | Ford | |
| 11% | 2% | 0% | 0% | 0% | 0% | 5% | 0% | |
| 12% | 3% | 4% | 0% | 0% | 0% | 10% | 0% | |
| 14% | 4% | 6% | 0% | 2% | 3% | 11% | 0% | |
| 17% | 5% | 7% | 0% | 4% | 7% | 12% | 0% | |
| 21% | 5% | 7% | 0% | 5% | 14% | 11% | 0% | |
| 24% | 6% | 7% | 0% | 4% | 23% | 9% | 0% | |
| 29% | 7% | 3% | 0% | 0% | 34% | 0% | 0% | |
| 38% | 8% | 0% | 0% | 0% | 7% | 0% | 11% | |
| 81% | 8% | 0% | 0% | 0% | 0% | 0% | 0% | |
| 157% | 9% | 0% | 0% | 0% | 0% | 0% | 0% | |

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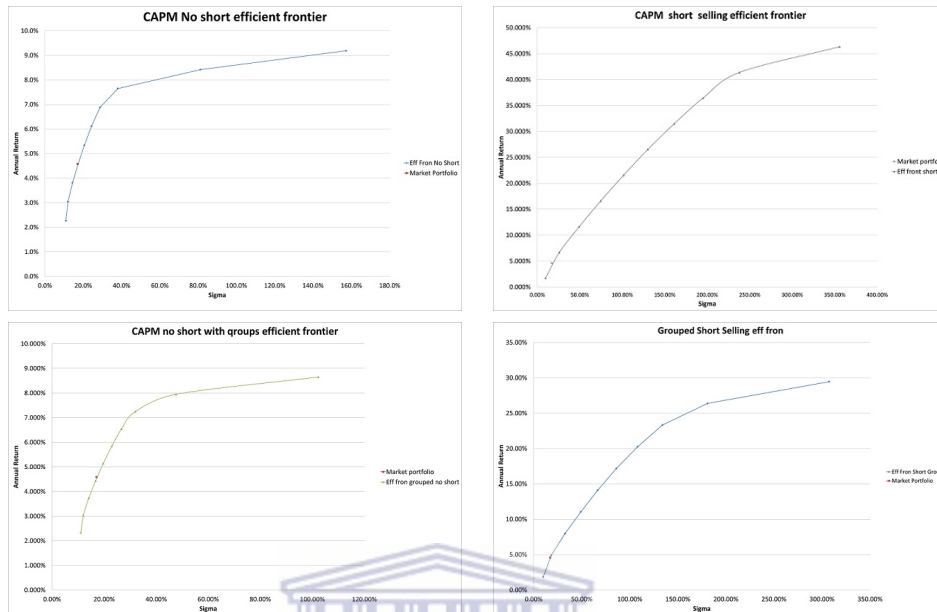


Figure 5.5: CAPM efficient frontiers

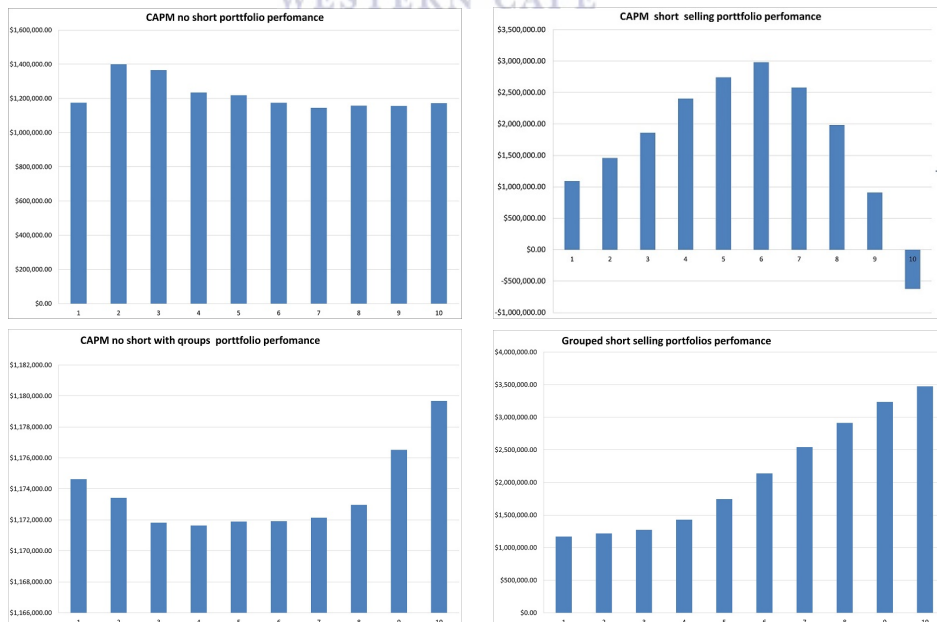
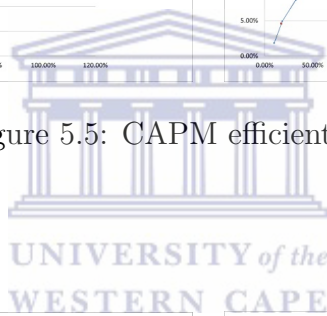


Figure 5.6: CAPM portfolio performances

5.4 Tracking the benchmark portfolio using the Sharpe Style Analysis

In this section we take a look at the historical returns of the assets. Having the benchmark portfolio, we then obtain the style by multiplying the weights of the portfolio with the returns of period t . We then compute the error by subtracting the S&P500 rate of return at time t , from the style at time t . We then use Quadratic Programming to achieve the weights of the desired portfolio which will track the index.

Results: Both for long and short positions we see from Table 5.7 that the rate of return is just over 5% p.a. This is close to the geometric mean of the S&P500 which is 4.56% p.a. Using the weights of no short selling and where short selling is allowed, the values at the end of the period are \$1,051,882.84 and \$1 057 917 respectively. If we invested our lump sum in the S&P500, the value at the end of the period would be \$967 103 with a negative position. This is in fact less than the two portfolios; however, we have tracked the expected return of the S&P500. So using this, we can see that the S&P500 is over performed.

Table 5.7: Asset allocation by tracking the benchmark using Sharpe style error

| Date | Style | Error | | Style Port No Short Selling | Style Portfolio Short Selling |
|-----------|---------|---------|---------------|-----------------------------|-------------------------------|
| 12/1/2010 | 0.0706 | -0.0053 | HPQ | 7.64% | 7.76% |
| 11/1/2010 | -0.0084 | 0.0737 | WHR | 0.00% | -0.24% |
| 10/1/2010 | 0.0396 | 0.0257 | T | 2.59% | 2.62% |
| 9/1/2010 | 0.0886 | -0.0233 | AZO | 3.33% | 3.46% |
| 8/2/2010 | -0.0394 | 0.1047 | JPM | 5.68% | 5.89% |
| 7/1/2010 | 0.0829 | -0.0176 | CVX | 1.51% | 1.85% |
| 6/1/2010 | -0.0461 | 0.1114 | CB | 2.73% | 3.59% |
| 5/3/2010 | -0.0932 | 0.1585 | KO | 7.43% | 7.85% |
| 4/1/2010 | 0.0051 | 0.0602 | NBL | 4.59% | 4.93% |
| 3/1/2010 | 0.0465 | 0.0188 | Dell | 0.64% | 0.67% |
| 2/1/2010 | 0.0150 | 0.0503 | UNH | 1.13% | 1.04% |
| 1/4/2010 | -0.0236 | 0.0889 | XOM | 5.40% | 5.00% |
| 12/1/2009 | 0.0225 | 0.0428 | K | 1.28% | 1.01% |
| 11/2/2009 | 0.0630 | 0.0023 | MCD | 2.98% | 2.64% |
| 10/1/2009 | 0.0002 | 0.0651 | MHP | 7.58% | 7.77% |
| 9/1/2009 | 0.0315 | 0.0338 | MSFT | 5.55% | 5.40% |
| 8/3/2009 | 0.0416 | 0.0237 | GT | 0.07% | 0.36% |
| 7/1/2009 | 0.0807 | -0.0154 | NKE | 0.00% | -0.58% |
| 6/1/2009 | -0.0005 | 0.0658 | PFE | 5.61% | 5.88% |
| 5/1/2009 | 0.0450 | 0.0203 | GE | 12.13% | 12.27% |
| 4/1/2009 | 0.1157 | -0.0504 | WMT | 3.57% | 3.90% |
| 3/2/2009 | 0.1138 | -0.0485 | AIG | 0.35% | 0.30% |
| 2/2/2009 | -0.1120 | 0.1773 | AA | 3.81% | 3.64% |
| 1/2/2009 | -0.1009 | 0.1662 | CAT | 1.94% | 2.09% |
| 12/1/2008 | 0.0008 | 0.0645 | GIS | 3.88% | 4.04% |
| 11/3/2008 | -0.0555 | 0.1208 | ACE | 0.00% | -1.90% |
| 10/1/2008 | -0.1451 | 0.2104 | AXP | 1.35% | 2.05% |
| 9/2/2008 | -0.0605 | 0.1258 | IBM | 7.23% | 7.18% |
| 8/1/2008 | 0.0119 | 0.0534 | Ford | 0.00% | -0.50% |
| 7/1/2008 | 0.0134 | 0.0519 | | | |
| 6/2/2008 | -0.0771 | 0.1424 | Monthly Mean | 0.73% | 0.72% |
| 5/1/2008 | 0.0085 | 0.0568 | Monthly Sigma | 4.95% | 4.94% |
| 4/1/2008 | 0.0274 | 0.0379 | | | |
| 3/3/2008 | 0.0119 | 0.0534 | Annual Mean | 8.72% | 8.66% |
| 2/1/2008 | -0.0137 | 0.0790 | Annual Sigma | 17.13% | 17.12% |

5.5 Numerical asset allocation using the Black-Litterman model

Benchmark

As we stated earlier, the Black-Litterman model is the reverse engineering of optimization. First we start by getting the benchmark portfolio. The benchmark portfolio is obtained by estimating the outstanding shares to compute the market value of each asset. We then obtain the proportion by dividing each market value by the total market value of the portfolio. Now that we have the benchmark portfolio we are now able to compute the rate of returns required in order to obtain that benchmark. This is the complete opposite of what we've been doing all along by calculating the expected returns in order to allocate the wealth. We now use these returns to calculate the efficient frontier.

Results: With the Black-Litterman model we can see from Table 5.8 that there are portfolios which are evenly distributed even when we don't allow short selling. The portfolio with the standard deviation of 4.85% and the return of 0.94% only excluded two assets. This is not seen when assets are grouped. From the portfolios at $t + 1$ in Figure 5.5, we see that on the efficient frontier of Figure 5.5, the portfolios shown before the benchmark portfolio are the ones that have a positive rate of return. When we allow short selling and limit the groups, the portfolio didn't perform particularly well, as half of them had over -100% rate of return.

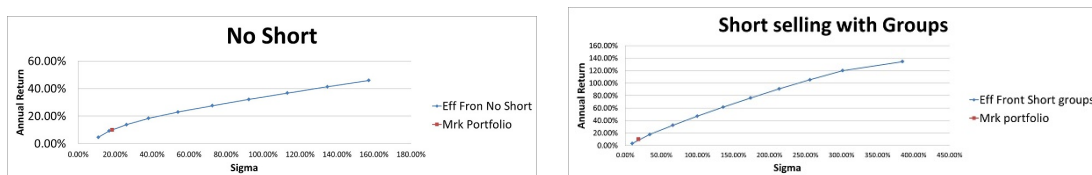


Figure 5.7: Efficient frontiers using market returns



Figure 5.8: Benchmark portfolio performances

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Table 5.8: Market portfolio computation and asset allocation using market portfolio as the benchmark

| Computation of the benchmark or market portfolio returns and its asset allocation | | | | | | | |
|---|--------------------|--|------------|------------|------------|-------------|-------------|
| Benchmark Portfolio | HPQ | WHR | T | AXP | IBM | Ford | |
| | 3.36% | 0.27% | 6.53% | 1.99% | 6.81% | 0.01% | |
| Risk-free | 0.025% | | | | | | |
| Expected Return | 0.87% | 1.02% | 0.65% | 1.11% | 0.79% | 1.24% | |
| Port Sigma | Port Return | Portfolio Weights when short selling is not allowed | | | | | |
| | | HPQ | WHR | ACE | AXP | IBM | Ford |
| 10.97% | 4.60% | 0.00% | 0.00% | 0.00% | 0.00% | 4.81% | 0.00% |
| 16.81% | 9.19% | 3.53% | 0.50% | 0.53% | 0.84% | 7.29% | 0.00% |
| 26.26% | 13.77% | 1.48% | 0.00% | 0.00% | 7.90% | 5.36% | 2.09% |
| 38.11% | 18.36% | 0.00% | 0.00% | 0.00% | 9.49% | 0.00% | 7.28% |
| 54.10% | 22.95% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 15.20% |
| 72.67% | 27.53% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 21.72% |
| 92.41% | 32.12% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 27.07% |
| 113.20% | 36.71% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 29.54% |
| 134.88% | 41.29% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 14.77% |
| 157.24% | 45.88% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |
| Port Sigma | Port Return | Portfolio Weights when short selling is allowed | | | | | |
| | | HPQ | WHR | ACE | AXP | IBM | Ford |
| 9.82% | 3.09% | 4.07% | 4.92% | -1.50% | -7.77% | 12.21% | -0.78% |
| 34.10% | 17.68% | 2.57% | -4.90% | 3.47% | 12.85% | 0.79% | 0.88% |
| 66.11% | 32.28% | 1.06% | -18.14% | 8.06% | 36.85% | -10.96% | 3.73% |
| 100.02% | 46.87% | -10.05% | -37.81% | 16.09% | 58.59% | -20.20% | 12.70% |
| 135.99% | 61.47% | -18.81% | -50.00% | 11.59% | 85.62% | -33.19% | 21.01% |
| 174.01% | 76.06% | -25.40% | -50.00% | -14.17% | 118.50% | -44.41% | 24.81% |
| 213.81% | 90.65% | -31.01% | -50.00% | -50.00% | 150.00% | -50.00% | 33.49% |
| 256.25% | 105.25% | -43.85% | -50.00% | -50.00% | 150.00% | -50.00% | 52.55% |
| 301.66% | 119.84% | -50.00% | -50.00% | -50.00% | 150.00% | -50.00% | 75.59% |
| 384.81% | 134.44% | -50.00% | 100.00% | -50.00% | 150.00% | -50.00% | 150.00% |

Adjusting rates of returns

Alternatively, after estimating the returns from the above, the Black-Litterman approach allows us to add our thoughts regarding what we feel the mispriced returns are. In essence, this means that we are able to change the returns based on what we think they are for the various assets. Due to a level of uncertainty, we only use the 10% confidence levels on the assets we are changing. We estimate the HPQ rate of return to be 1.1% instead of 1% and that the NBL rate will be 0.6% instead of 0.85%. With our estimations, we can see that we are able to obtain a portfolio that performs very close to the benchmark (S&P500). However, it's important to note that this can cause major problems as these returns are correlated. If we are wrong then we are also mispricing the other assets.

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Table 5.9: Portfolio selection following Black-Litterman with one opinion of one asset

| Computation of the benchmark or market portfolio and its return using the Black-Litterman Approach | | | | | | | |
|--|-------------|-------|-------|-------|--------|-------|--------|
| Port Sigma | Port Return | HPQ | WHR | T | AXP | IBM | Ford |
| 10.97% | 4.82% | 0.00% | 0.00% | 0.00% | 0.00% | 4.81% | 0.00% |
| 16.45% | 9.40% | 6.07% | 0.27% | 4.93% | 0.33% | 8.02% | 0.00% |
| 25.52% | 13.99% | 7.12% | 0.00% | 6.43% | 7.57% | 7.39% | 1.56% |
| 37.09% | 18.57% | 2.80% | 0.00% | 0.00% | 10.17% | 0.00% | 5.07% |
| 53.01% | 23.16% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 11.18% |
| 71.72% | 27.74% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 16.06% |
| 91.70% | 32.33% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 19.29% |
| 112.85% | 36.92% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 19.55% |
| 134.74% | 41.50% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 14.86% |
| 157.24% | 46.09% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% |

| Portfolio Weights when short selling is allowed with group constrains | | | | | | | |
|---|-------------|---------|---------|--------|---------|---------|---------|
| Port Sigma | Port Return | HPQ | WHR | T | AXP | IBM | Ford |
| 9.94% | 3.52% | 3.00% | 3.19% | 0.00% | -6.21% | 12.18% | -1.59% |
| 26.18% | 14.49% | 8.59% | -2.37% | 10.00% | 7.42% | 5.61% | 0.15% |
| 50.53% | 25.46% | 17.68% | -3.96% | 10.00% | 20.85% | -7.63% | 4.08% |
| 76.58% | 36.43% | 26.29% | -5.24% | 4.31% | 36.49% | -26.49% | 10.73% |
| 104.45% | 47.40% | 26.59% | -7.53% | 0.00% | 41.24% | -50.00% | 26.27% |
| 136.45% | 58.36% | 11.28% | -10.21% | 0.00% | 9.57% | -50.00% | 54.55% |
| 171.92% | 69.33% | -11.57% | -26.35% | 0.00% | -13.66% | -50.00% | 82.71% |
| 209.94% | 80.30% | -30.64% | -50.00% | 0.00% | -43.95% | -50.00% | 106.13% |
| 251.63% | 91.27% | -50.00% | -50.00% | 0.00% | -50.00% | -50.00% | 109.26% |
| 313.10% | 102.24% | -50.00% | -50.00% | 0.00% | 40.00% | -50.00% | 150.00% |

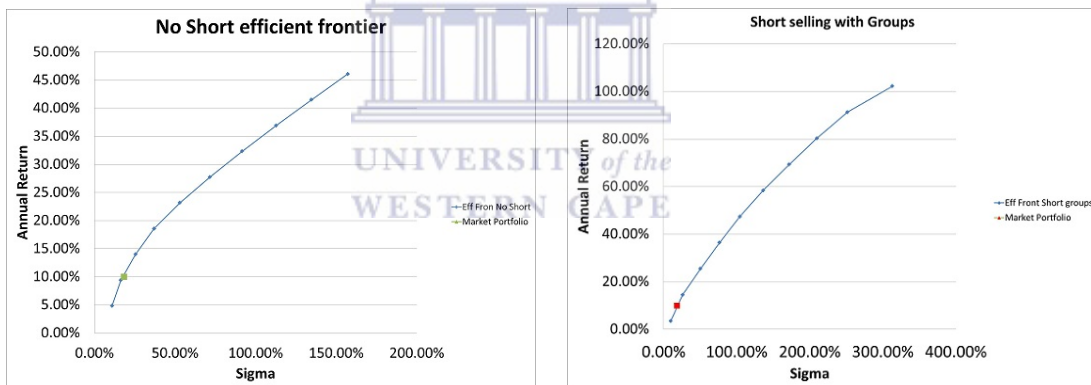


Figure 5.9: Black-Litterman efficient frontiers using one opinionated market return

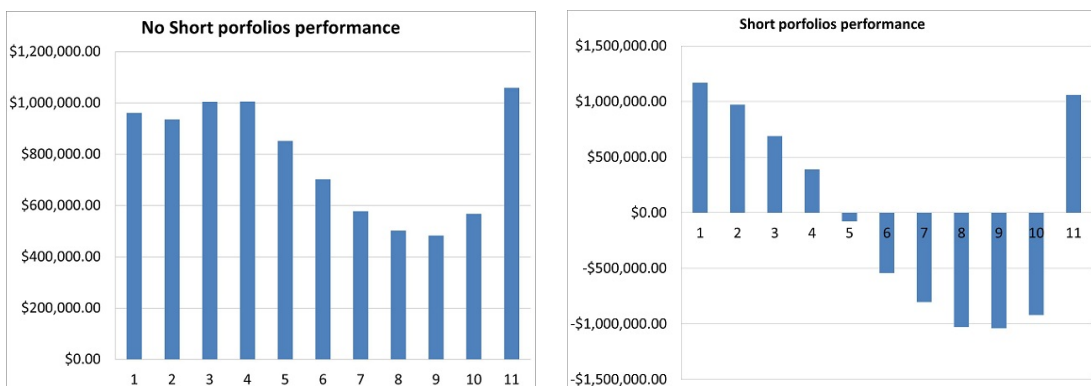


Figure 5.10: Black-Litterman portfolio performances using one opinionated market return

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Table 5.10: Portfolio selection following Black-Litterman with two opinions
 Computation of the benchmark or market portfolio and its return using the Black-Litterman Approach

| Port Sigma | Port Return | Portfolio Weights when short selling is not allowed | | | | | | |
|------------|-------------|---|-------|-------|-------|-------|--------|--|
| | | HPQ | WHR | T | AXP | IBM | Ford | |
| 10.97% | 4.79% | 0.00% | 0.00% | 0.00% | 0.00% | 4.81% | 0.00% | |
| 16.46% | 9.37% | 6.13% | 0.29% | 4.91% | 0.35% | 8.11% | 0.00% | |
| 25.52% | 13.96% | 7.28% | 0.00% | 6.59% | 7.45% | 7.59% | 1.57% | |
| 37.08% | 18.55% | 3.20% | 0.00% | 0.00% | 9.77% | 0.00% | 5.33% | |
| 53.02% | 23.13% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 11.46% | |
| 71.74% | 27.72% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 16.48% | |
| 91.73% | 32.30% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 20.00% | |
| 112.86% | 36.89% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 20.41% | |
| 134.74% | 41.48% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 14.86% | |
| 157.24% | 46.06% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | |

| Port Sigma | Port Return | Portfolio Weights when short selling is allowed with group constrains | | | | | | |
|------------|-------------|---|---------|--------|---------|---------|---------|--|
| | | HPQ | WHR | T | AXP | IBM | Ford | |
| 9.94% | 3.50% | 3.00% | 3.19% | 0.00% | -6.21% | 12.18% | -1.59% | |
| 26.17% | 14.45% | 8.70% | -2.35% | 10.00% | 7.40% | 5.82% | 0.17% | |
| 50.47% | 25.40% | 17.93% | -3.62% | 10.00% | 20.75% | -7.65% | 4.20% | |
| 76.49% | 36.35% | 26.62% | -4.72% | 4.55% | 36.37% | -26.71% | 10.83% | |
| 104.27% | 47.30% | 27.43% | -7.27% | 0.00% | 40.81% | -50.00% | 26.47% | |
| 136.14% | 58.25% | 12.71% | -9.77% | 0.00% | 8.70% | -50.00% | 54.85% | |
| 171.49% | 69.21% | -9.61% | -25.75% | 0.00% | -14.86% | -50.00% | 83.14% | |
| 209.40% | 80.16% | -28.40% | -50.00% | 0.00% | -45.51% | -50.00% | 107.43% | |
| 251.02% | 91.11% | -50.00% | -50.00% | 0.00% | -50.00% | -50.00% | 110.39% | |
| 313.10% | 102.06% | -50.00% | -50.00% | 0.00% | 40.00% | -50.00% | 150.00% | |

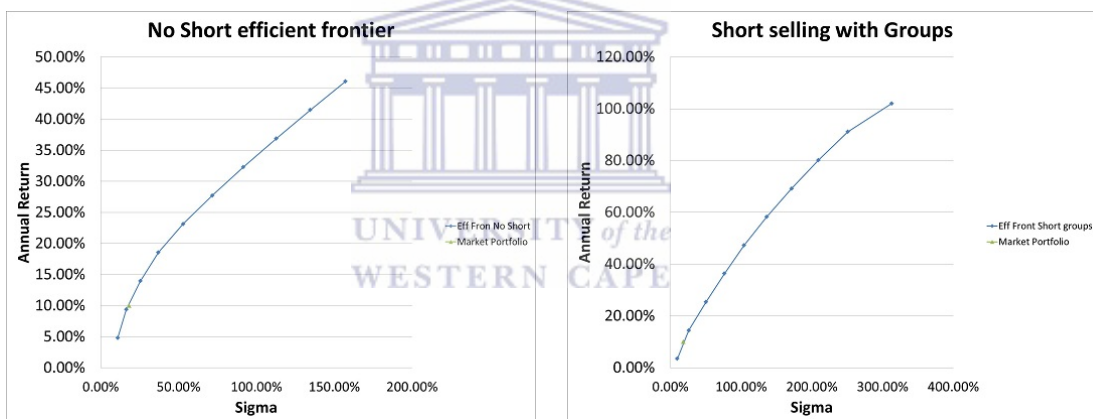


Figure 5.11: Black-Litterman efficient frontiers using two opinionated market returns

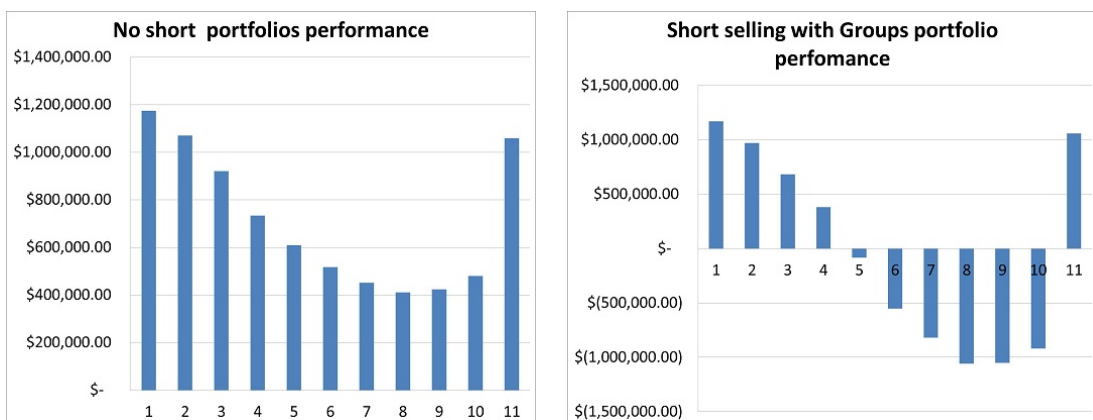


Figure 5.12: Black-Litterman portfolio performances using two opinionated market returns

Black-Litterman using relative views and comparison with other models

Here we use the Black-Litterman formula to incorporate our views. We start by computing the market capitalisation. Looking at the equilibrium, the expected returns hold the following views: HPQ will outperform CVX by 50 basis points, PFE will outperform MCD by 75 basis points and AZO will outperform XOM by 5 basis points. We are 10% confident in our views. The expected returns we obtain from the Black-Litterman approach give us the weights with 100% confidence. We then compute the W_{hat} from the 100% confidence weights. W_{100} , W_{hat} , W_{mkt} , W_{capm} , W_{hist} are portfolios which are obtained from Black-Litterman returns with 100% confidence of views, 10% confidence, market capitalisation, CAPM returns and plain historical returns respectively.

Results: Looking at the estimation of the expected returns by the Black-Litterman approach, equilibrium returns, geometric historical and CAPM, we see that the returns of the Black-Litterman and equilibrium returns are close to one another. Some of the geometric and CAPM returns are also close to each other, however most of the Black-Litterman and equilibrium returns are not close to geometric and CAPM returns. More than half of the stocks' expected returns of Black-Litterman and market are more than double the geometric and CAPM expected returns. The biggest attraction is on AIG and GT. This is due to the fact that the Black-Litterman expected returns are positive and the geometric returns are negative with a difference of 7.33% and 2.7% respectively. In observing our views we can see that the HPQ is greater than CVX. This means that HPQ has outperformed CVX and we see the same result for XOM against AZO and PFE and MCD respectively. To be precise, the difference between HPQ and CVX is 0.48%, XOM and AZO is 0.06% and PFE and MCD is 0.33%. The percentages are not exactly the same as our views, but are very close.

Portfolio Statistics results

From Table 5.12 W_{hist} has the highest expected return of 1.53%. The W_{100} , W_{hat} and W_{mkt} are close to each other at 1.34%, 1.3% and 1.3% respectively. The W_{capm} has the lowest expected return of 0.38%. The W_{hist} has the highest expected return and the lowest standard deviation. The W_{mkt} has a beta of one as expected, followed by W_{hat} with a beta of 1.013 which is very close to one. The W_{100} , W_{capm} and W_{hist}

have a beta of 1.135. For the Treynor (see Figure 5.13(a)) and Sharpe measures (see Figure 5.13(b)) the W_{capm} is above the SML and CML lines respectively, showing us that its output performed S&P500 index, W_{100} , W_{hist} , W_{hat} , W_{mkt} are all under the CML and SML. This indicates that they did not perform as well as the S&P500 index did. The information ratio of W_{mkt} and W_{hat} has a zero indication which shows that they are very close to the S&P500 index, which is as expected. W_{100} and W_{capm} are negative, indicating that they underperformed against the S&P500. W_{hist} has a ratio of 0.4, which shows that it performed above the aggregate index.



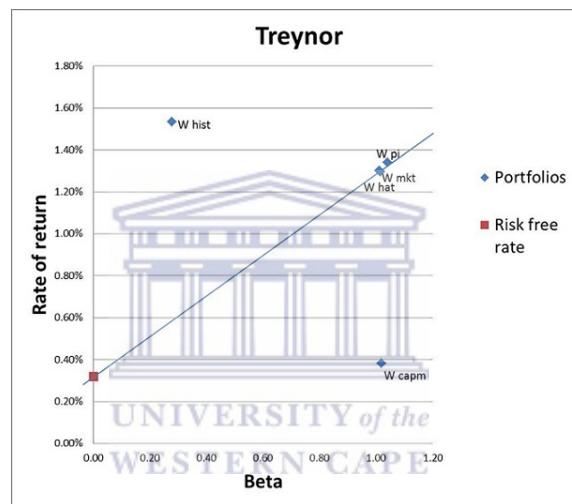
Table 5.11: Black-Litterman anticipated returns and weights of portfolios through relative views

| | Opinion Based Returns Vector ($\hat{\Pi}$) | Equilibrium Return Vector (Π) | Geometric Historical Return Vector | CAPM Returns Vector | Opinion Based Weights (W_{100}) | Equilibrium Weights (W_{mkt}) | Geometric Historical Weights (W_{hsit}) | CAPM Weights (W_{capm}) | Opinion Based with confidence (\hat{W}) |
|------------|--|---|--|---------------------------|---|---|--|-----------------------------------|---|
| HPQ | 1.32% | 1.36% | 0.613% | 0.516% | 3.74% | 3.36% | -1.23% | 6.51% | 3.40% |
| WHR | 1.61% | 1.61% | 0.482% | 0.498% | 0.27% | 0.27% | 3.70% | -0.38% | 0.27% |
| T | 0.98% | 1.01% | 0.352% | 0.296% | 6.53% | 6.53% | -11.13% | 5.35% | 6.53% |
| AZO | 0.60% | 0.63% | 1.364% | 0.218% | 2.60% | 0.42% | 12.40% | 3.78% | 0.64% |
| JPM | 1.60% | 1.59% | 0.600% | 0.530% | 6.45% | 6.45% | -3.74% | 0.63% | 6.45% |
| CVX | 0.84% | 0.90% | 0.980% | 0.257% | 6.70% | 7.08% | -4.27% | 3.07% | 7.05% |
| CB | 0.68% | 0.69% | 0.714% | 0.252% | 0.65% | 0.65% | -0.30% | 5.04% | 0.65% |
| KO | 0.75% | 0.78% | 0.476% | 0.239% | 5.93% | 5.93% | -3.33% | 9.42% | 5.93% |
| NBL | 0.92% | 0.95% | 1.049% | 0.301% | 0.62% | 0.62% | 4.11% | 7.78% | 0.62% |
| Dell | 1.93% | 1.90% | 1.552% | 0.634% | 0.98% | 0.98% | 7.93% | 3.82% | 0.98% |
| UNH | 1.23% | 1.17% | 0.868% | 0.328% | 1.53% | 1.53% | 1.14% | -2.37% | 1.53% |
| XOM | 0.65% | 0.71% | 0.925% | 0.198% | 11.47% | 13.65% | 34.76% | -2.20% | 13.43% |
| K | 0.30% | 0.30% | 0.392% | 0.119% | 0.72% | 0.72% | -1.58% | -1.65% | 0.72% |
| MCD | 0.75% | 0.95% | 0.767% | 0.279% | -11.42% | 3.12% | 4.34% | -0.56% | 1.67% |
| MHP | 1.07% | 1.03% | 0.837% | 0.347% | 0.41% | 0.41% | 15.13% | 13.17% | 0.41% |
| MSFT | 1.63% | 1.60% | 1.009% | 0.516% | 9.32% | 9.32% | -4.16% | 6.60% | 9.32% |
| GT | 2.04% | 2.14% | -0.656% | 0.653% | 0.12% | 0.12% | -16.24% | 1.26% | 0.12% |
| NKE | 1.05% | 1.08% | 1.016% | 0.332% | 1.59% | 1.59% | -1.86% | -3.39% | 1.59% |
| PFE | 1.08% | 0.88% | 0.457% | 0.269% | 19.64% | 5.10% | -3.24% | 7.41% | 6.55% |
| GE | 1.60% | 1.57% | 0.416% | 0.486% | 7.67% | 7.67% | -2.27% | 10.82% | 7.67% |
| WMT | 0.74% | 0.76% | 1.028% | 0.249% | 7.24% | 7.24% | 7.70% | 6.06% | 7.24% |
| AIG | 6.21% | 6.12% | -1.111% | 0.765% | 3.57% | 3.57% | -1.64% | -1.98% | 3.57% |
| AA | 2.14% | 2.20% | 0.217% | 0.618% | 0.67% | 0.67% | -11.69% | -1.48% | 0.67% |
| CAT | 1.60% | 1.64% | 1.195% | 0.477% | 2.46% | 2.46% | 18.80% | 5.49% | 2.46% |
| GIS | 0.22% | 0.22% | 0.801% | 0.095% | 0.91% | 0.91% | 38.48% | 1.72% | 0.91% |
| ACE | 0.94% | 0.90% | 0.967% | 0.302% | 0.85% | 0.85% | 5.74% | -2.65% | 0.85% |
| AXP | 1.72% | 1.75% | 0.755% | 0.561% | 1.99% | 1.99% | -4.83% | 10.99% | 1.99% |
| IBM | 1.18% | 1.23% | 1.035% | 0.431% | 6.81% | 6.81% | 17.54% | 11.14% | 6.81% |
| Ford | 1.92% | 1.95% | 0.460% | 0.630% | 0.01% | 0.01% | -0.26% | -3.41% | 0.01% |
| SUM | | | | | 100.000% | 100.000% | 100.000% | 100.000% | 100.000% |

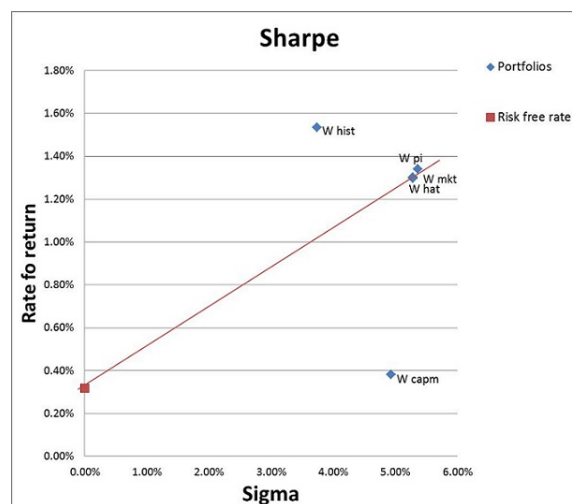
CHAPTER 5. NUMERICAL SIMULATION FOR INVESTMENT PORTFOLIO MODELS

Table 5.12: Black-Litterman portfolio statistics and measures

| <i>Portfolio Statistics</i> | | | | | |
|-----------------------------|--------------|--------------|--------------|---------------|---------------|
| | <i>W 100</i> | <i>W mkt</i> | <i>W hat</i> | <i>W Hist</i> | <i>W CAPM</i> |
| Risk-Free | 0.03% | 0.03% | 0.03% | 0.03% | 0.03% |
| Expected Returns | 1.34% | 1.30% | 1.30% | 1.53% | 0.38% |
| Excess Returns | 1.31% | 1.28% | 1.27% | 1.51% | 0.36% |
| Variance | 0.29% | 0.28% | 0.28% | 0.14% | 0.24% |
| Standard Deviation | 5.37% | 5.29% | 5.28% | 3.74% | 4.94% |
| Beta | 1.04 | 1.01 | 1.01 | 0.28 | 1.02 |
| Alpha | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| Sharpe Ratio | 0.08 | 0.13 | 0.12 | 0.63 | 0.08 |
| Treynor Ratio | 0.00 | 0.01 | 0.01 | 0.08 | 0.01 |
| Information Ratio | -0.17 | 0.00 | 0.00 | 0.41 | -0.04 |



(a) Treynor's portfolio measurement plane



(b) Sharpe's portfolio measurement plane

Figure 5.13: Performance measurements of Black-Litterman portfolios

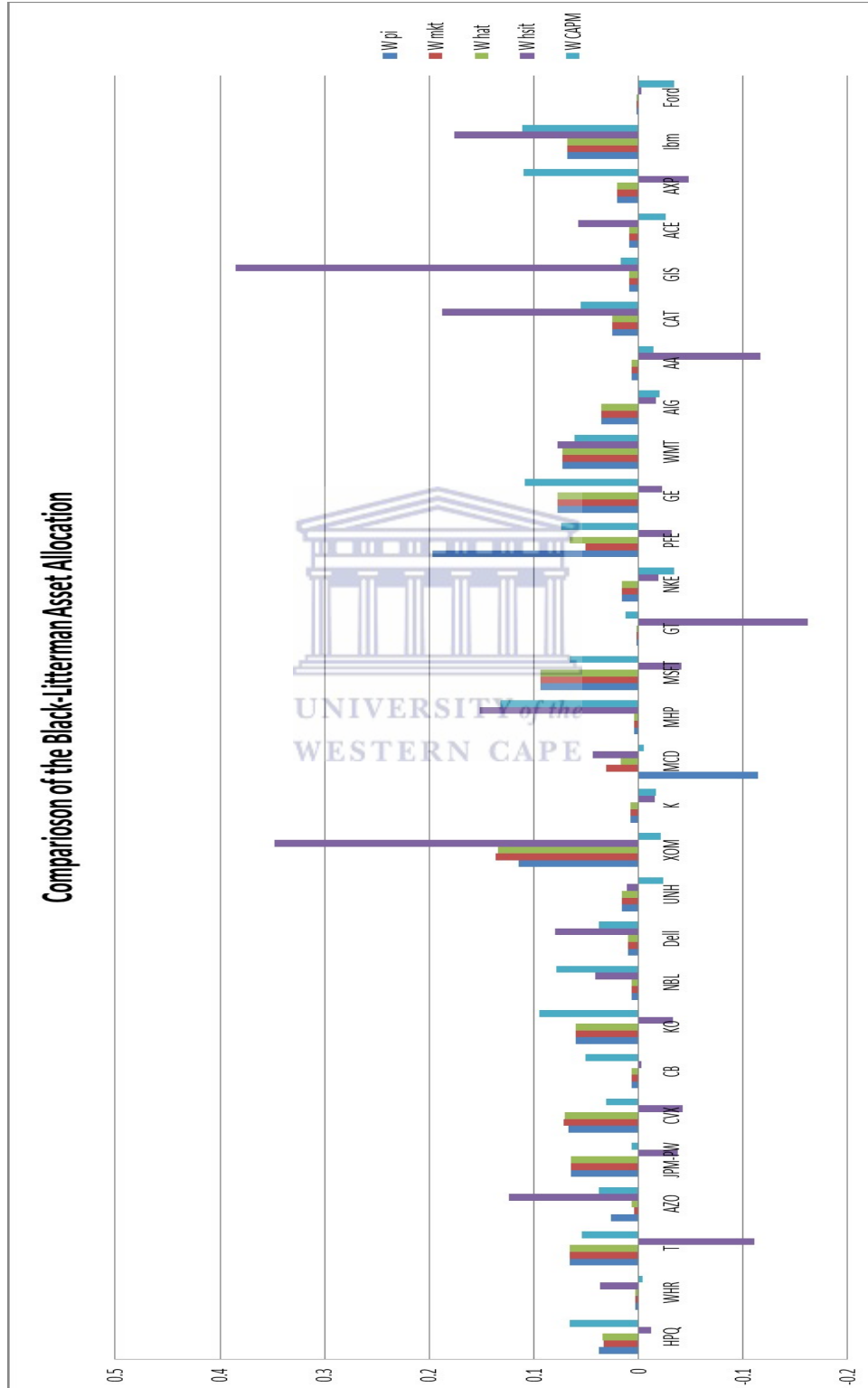


Figure 5.14: Comparison of Black-Litterman's Asset Allocation

5.6 Portfolio measurement using alternative approaches to back-test various portfolio performances

Here we look at the three portfolio performance measurement approaches, following the literature. Basically, a back-test for a period of one year is used. We take monthly returns, and for the period of one year, calculate the standard deviation, betas and returns of the portfolios. The average risk-free rate for 2010 is 0.3%.

Results: When looking at the Treynor ratio in Table 5.13, the highest ratio is with portfolio 4 under the Markowitz grouped short category. This portfolio has a monthly expected return of 1.3 and a monthly volatility of 3.048%. Out of the top ten portfolios, five portfolios are in the Markowitz short category, three under Markowitz short category and two are under Huang and Litzenburg. High volatile portfolios under Huang and Litzenburg are in the top ten. Neither portfolios 1 nor 2 are in the top ten, nor is the lowest ratio, portfolio 10 with a very high volatility.

Now looking at the Sharpe ratio in Table 5.13, the highest is portfolio 10 of Huang and Litzenburg. The top ten portfolios are only under the Huang and Litzenburg and Markowitz Short categories. Five portfolios under Huang and Litzenburg approach are highly volatile. Under Markowitz short category, portfolios 3-6 are the common ones in the top ten under Sharpe's and Treynor's ratios.

Portfolios 4-10 of CAPM with no short and in the group's category, have Treynor ratios greater than Sharpe ratios. Black-Litterman 1 short, Black-Litterman 1 short with groups each has four portfolios with a higher Treynor than Sharpe. This is an indication of the diversification degree as these portfolios or methods are not well diversified. We have no such portfolios under Huang & Litzenburg, Markowitz no short with groups, Black-Litterman no short, Black-Litterman no short with groups, Black-Litterman 1 no short, Black-Litterman 1 no short with groups, Black-Litterman 2 no short, Black-Litterman 2 no short with groups, style and benchmark.

Information Ratio: From Table 5.14, four portfolios from Markowitz no short with groups are in the top ten, with all of them over a ratio of 0.5. Three of them are from Markowitz no short, two of them are from Huang and Litzenburg and

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one from Black-Litterman no short. The minimum ratio of the top ten is at 0.49 (Black-Litterman) of which the results show that these portfolios performed above average than the index within all the categories of the top ten portfolios. The top ten portfolios are all over the top five of the optimal portfolios i.e. they all had a higher return and volatility.



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Table 5.13: Treynor and Sharpe ratio calculation results

| Treynor | | | | | | | | | | | |
|----------------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|----|
| rf bar | 3.44% | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Marko No short | 0.1469 | 0.1666 | 0.1656 | 0.1585 | 0.1495 | 0.142 | 0.1371 | 0.1325 | 0.1163 | 0.0889 | |
| Marko Short | 0.2203 | 0.5569 | 0.9523 | 1.4234 | 2.0498 | 2.6477 | 3.0506 | 3.4968 | 3.227 | 1.0199 | |
| Marko No short with Groups | 0.1836 | 0.2157 | 0.2494 | 0.2941 | 0.3738 | 0.4309 | 0.632 | 1.0021 | 455.79 | -2.554 | |
| Marko Short with Groups | 0.151 | 0.3168 | 0.4584 | 0.5462 | 0.6044 | 0.6437 | 0.6713 | 0.6973 | 0.7131 | 0.6742 | |
| CAPM No short | 0.0414 | 0.0451 | 0.0494 | 0.0544 | 0.0603 | 0.0672 | 0.0755 | 0.0857 | 0.0986 | 0.1153 | |
| CAPM Short | 0.0631 | -0.0638 | -0.033 | -0.0258 | -0.0226 | -0.0208 | -0.0196 | -0.0188 | -0.0182 | -0.0177 | |
| CAPM No short with Groups | 0.0397 | 0.0415 | 0.0434 | 0.0454 | 0.0476 | 0.0499 | 0.0524 | 0.0552 | 0.0582 | 0.0614 | |
| CAPM Short with Groups | 0.0575 | -0.1863 | -0.0533 | -0.0363 | -0.0296 | -0.026 | -0.0238 | -0.0222 | -0.0211 | -0.0203 | |
| Benchmark | 0.0532 | | | | | | | | | | |
| BL No short | 0.1109 | 0.0879 | 0.0809 | 0.0833 | 0.0938 | 0.1029 | 0.1122 | 0.1226 | 0.1327 | 0.1421 | |
| BL Short | 0.1273 | 0.077 | 0.0717 | 0.0708 | 0.0704 | 0.0703 | 0.071 | 0.073 | 0.0757 | 0.0789 | |
| BL No short with Groups | -0.0617 | 0.4797 | 0.2827 | 0.068 | -0.022 | -0.0209 | -0.0199 | -0.019 | -0.0182 | -0.0177 | |
| BL Short with Groups | 0.1211 | 0.0802 | 0.0759 | 0.0751 | 0.0761 | 0.0795 | 0.0837 | 0.0885 | 0.0961 | 0.0929 | |
| BL1 No short | 0.1145 | 0.0909 | 0.0832 | 0.0852 | 0.095 | 0.1039 | 0.1139 | 0.1241 | 0.1335 | 0.1429 | |
| BL1 Short | 0.1328 | 0.0791 | 0.0734 | 0.0724 | 0.0718 | 0.0714 | 0.0719 | 0.0739 | 0.0772 | 0.0796 | |
| BL1 No short with Groups | 0.1117 | 0.0942 | 0.0869 | 0.0826 | 0.084 | 0.0911 | 0.0979 | 0.1041 | 0.1103 | 0.1175 | |
| BL1 Short with Groups | 0.1263 | 0.0826 | 0.0777 | 0.0764 | 0.0775 | 0.0806 | 0.0844 | 0.0892 | 0.096 | 0.0935 | |
| BL2 No short | 0.1038 | 0.0872 | 0.0812 | 0.0834 | 0.0942 | 0.1039 | 0.1123 | 0.1212 | 0.1312 | 0.1407 | |
| BL2 Short | 0.1193 | 0.0818 | 0.0774 | 0.0758 | 0.075 | 0.0744 | 0.0739 | 0.0752 | 0.0787 | 0.0782 | |
| BL2 No short with Groups | 0.1027 | 0.0897 | 0.0836 | 0.0807 | 0.082 | 0.0898 | 0.0971 | 0.1038 | 0.1097 | 0.1143 | |
| BL2 Short with Groups | 0.1146 | 0.0842 | 0.0811 | 0.081 | 0.0822 | 0.0858 | 0.0899 | 0.0946 | 0.1011 | 0.0938 | |
| Huang & Litz | 0.0747 | 0.1659 | 0.2614 | 0.3614 | 0.4663 | 0.5763 | 0.692 | 0.8137 | 0.9418 | 1.0771 | |
| Style | 0.0543 | | 0.0536 | | | | | | | | |

| Sharpe | | | | | | | | | | | |
|----------------------------|---------|--------|--------|--------|--------|---------|---------|---------|---------|---------|----|
| rf bar | 3.44% | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Marko No short | 0.6159 | 0.6894 | 0.7153 | 0.6881 | 0.6409 | 0.5935 | 0.5516 | 0.5128 | 0.4512 | 0.3303 | |
| Marko Short | 0.7413 | 1.173 | 1.1379 | 1.0888 | 1.0501 | 1.0037 | 0.9404 | 0.8618 | 0.7696 | 0.5539 | |
| Marko No short with Groups | 0.5938 | 0.661 | 0.7017 | 0.6986 | 0.6665 | 0.6254 | 0.5847 | 0.5474 | 0.5125 | 0.461 | |
| Marko Short with Groups | 0.7202 | 1.1062 | 1.0891 | 1.0398 | 0.9996 | 0.9424 | 0.8646 | 0.7852 | 0.7126 | 0.5908 | |
| CAPM No short | 0.1735 | 0.1767 | 0.1788 | 0.1794 | 0.1771 | 0.1715 | 0.1631 | 0.1523 | 0.1404 | 0.1279 | |
| CAPM Short | 0.2124 | 0.1628 | 0.1225 | 0.1052 | 0.0956 | 0.0887 | 0.0828 | 0.0775 | 0.0724 | 0.0615 | |
| CAPM No short with Groups | 0.1708 | 0.1726 | 0.174 | 0.1751 | 0.1756 | 0.1756 | 0.175 | 0.1736 | 0.1713 | 0.1621 | |
| CAPM Short with Groups | 0.2061 | 0.1771 | 0.1312 | 0.1084 | 0.0949 | 0.0855 | 0.0779 | 0.0714 | 0.0639 | 0.0545 | |
| Benchmark | 0.2798 | | | | | | | | | | |
| BL No short | 0.4648 | 0.4647 | 0.4007 | 0.3474 | 0.2948 | 0.2568 | 0.2313 | 0.2128 | 0.1987 | 0.1877 | |
| BL Short | 0.4282 | 0.3765 | 0.3247 | 0.3009 | 0.2848 | 0.2722 | 0.2619 | 0.2522 | 0.2429 | 0.2128 | |
| BL No short with Groups | -0.2656 | 2.5423 | 1.4662 | 0.3316 | -0.095 | -0.0735 | -0.0581 | -0.0474 | -0.0398 | -0.0337 | |
| BL Short with Groups | 0.4344 | 0.4026 | 0.334 | 0.3036 | 0.284 | 0.2656 | 0.2493 | 0.2355 | 0.2227 | 0.1998 | |
| BL1 No short | 0.4801 | 0.4882 | 0.423 | 0.3648 | 0.306 | 0.2637 | 0.2355 | 0.2152 | 0.2002 | 0.1887 | |
| BL1 Short | 0.4467 | 0.3942 | 0.3393 | 0.3143 | 0.2972 | 0.2835 | 0.2723 | 0.2615 | 0.2506 | 0.2299 | |
| BL1 No short with Groups | 0.4805 | 0.5056 | 0.4587 | 0.4133 | 0.3719 | 0.3275 | 0.29 | 0.2618 | 0.2411 | 0.2234 | |
| BL1 Short with Groups | 0.4531 | 0.4233 | 0.3489 | 0.3154 | 0.2937 | 0.2724 | 0.2537 | 0.2384 | 0.2244 | 0.201 | |
| BL2 No short | 0.4354 | 0.4621 | 0.4067 | 0.3531 | 0.2965 | 0.2559 | 0.2293 | 0.2109 | 0.1968 | 0.1859 | |
| BL2 Short | 0.4013 | 0.386 | 0.338 | 0.3183 | 0.3022 | 0.2889 | 0.2769 | 0.2658 | 0.2549 | 0.2381 | |
| BL2 No short with Groups | 0.4417 | 0.4747 | 0.4371 | 0.3977 | 0.3605 | 0.3193 | 0.2833 | 0.2562 | 0.2361 | 0.2173 | |
| BL2 Short with Groups | 0.411 | 0.4108 | 0.3461 | 0.3161 | 0.2975 | 0.2789 | 0.2608 | 0.2444 | 0.2291 | 0.2041 | |
| Huang & Litz | 0.2302 | 0.5555 | 0.8624 | 1.0617 | 1.1502 | 1.1744 | 1.1705 | 1.1563 | 1.1393 | 1.1226 | |
| Style | 0.3077 | | 0.3044 | | | | | | | | |

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Table 5.14: Information Ratio calculation results

| Information Ratio | | | | | | | | | | |
|--------------------------------|---------------|---------------|--------------|--------------|---------------|---------------|---------------|---------------|---------------|---------------|
| Marko No short | -0.0921 9 | -0.0355 8 | 0.0536 7 | 0.1832 6 | 0.4160 4 | 0.6918 3 | 0.8395 2 | 0.8874 1 | 0.2106 5 | -0.2471 10 |
| Marko Short | 0.3061 10 | 0.3412 8 | 0.3519 5 | 0.3570 3 | 0.3578 1 | 0.3570 2 | 0.3530 4 | 0.3499 6 | 0.3489 7 | 0.3361 9 |
| Marko No short with Gr | -0.0874 10 | -0.0416 9 | 0.0241 8 | 0.1091 7 | 0.2754 6 | 0.5184 4 | 0.7445 3 | 0.8471 2 | 0.8874 1 | 0.2976 5 |
| Marko Short with Group | -0.0385 10 | 0.2230 9 | 0.3389 5 | 0.3615 1 | 0.3538 2 | 0.3438 4 | 0.3380 6 | 0.3228 8 | 0.3359 7 | 0.3463 3 |
| CAPM No short | -0.0921 3 | -0.1188 4 | -0.1553 6 | -0.1545 5 | -0.1844 7 | -0.1993 8 | -0.3376 10 | -0.2349 9 | 0.0622 2 | 0.3662 1 |
| CAPM Short | -0.0445 1 | -0.0636 3 | -0.0534 2 | -0.1062 5 | -0.1335 6 | -0.1352 7 | -0.1497 8 | -0.1682 9 | -0.2335 10 | -0.0827 4 |
| CAPM No short with Gr | -0.0874 2 | -0.1187 3 | -0.1506 4 | -0.1638 6 | -0.1769 7 | -0.1771 8 | -0.2644 9 | -0.3019 10 | -0.1518 5 | 0.1872 1 |
| CAPM Short with Group | -0.0385 2 | -0.1436 4 | -0.2053 5 | -0.2370 6 | -0.2769 8 | -0.3403 10 | -0.3112 9 | -0.2674 7 | -0.0737 3 | 0.0984 1 |
| Benchmark | 0.0101 | | | | | | | | | |
| BL No short | -0.0921 10 | -0.0395 9 | 0.0922 7 | -0.0173 8 | 0.1798 6 | 0.3103 5 | 0.4068 3 | 0.4903 1 | 0.4207 2 | 0.3662 4 |
| BL Short | -0.0445 10 | 0.0444 3 | 0.0448 2 | 0.0254 4 | 0.0204 5 | -0.0141 6 | -0.0254 8 | -0.0193 7 | -0.0323 9 | 0.1095 1 |
| BL No short with Groups | -0.0874 9 | -0.1011 10 | 0.0769 6 | 0.0646 7 | -0.0275 8 | 0.1211 5 | 0.2459 3 | 0.3397 2 | 0.4004 1 | 0.1995 4 |
| BL Short with Groups | -0.0385 6 | 0.0233 4 | -0.0430 7 | -0.0700 9 | -0.1075 10 | -0.0432 8 | 0.0066 5 | 0.0420 3 | 0.0985 2 | 0.2318 1 |
| BL1 No short | -0.0921 9 | -0.1385 10 | 0.0433 7 | -0.0640 8 | 0.1254 6 | 0.2436 5 | 0.3351 4 | 0.4067 2 | 0.4211 1 | 0.3662 3 |
| BL1 Short | -0.0445 6 | 0.0176 3 | -0.0250 2 | 0.0036 4 | -0.0073 5 | -0.0451 7 | -0.0612 8 | -0.0650 9 | -0.0752 10 | 0.0765 1 |
| BL1 No short with Group | -0.0874 9 | -0.1483 10 | -0.0189 7 | 0.0117 6 | -0.0792 8 | 0.0933 5 | 0.2174 3 | 0.3096 2 | 0.3760 1 | 0.1995 4 |
| BL1 Short with Groups | -0.0385 6 | -0.0213 5 | -0.0777 8 | -0.1051 9 | -0.1302 10 | -0.0664 7 | -0.0153 4 | 0.0311 3 | 0.0882 2 | 0.2318 1 |
| BL2 No short | -0.0921 9 | -0.1499 10 | 0.0430 7 | -0.0615 8 | 0.1302 6 | 0.2502 5 | 0.3414 4 | 0.4138 2 | 0.4211 1 | 0.3662 3 |
| BL2 Short | -0.0445 6 | 0.0136 3 | 0.0225 2 | 0.0009 4 | -0.0068 5 | -0.0463 7 | -0.0624 8 | -0.0659 9 | -0.0774 10 | 0.0765 1 |
| BL2 No short with Group | -0.0874 9 | -0.1530 10 | -0.0299 7 | 0.0128 6 | -0.0798 8 | 0.0951 5 | 0.2199 3 | 0.3127 2 | 0.3798 1 | 0.1995 4 |
| BL2 Short with Groups | -0.0385 6 | -0.0239 5 | -0.0816 8 | -0.1086 9 | -0.1305 10 | -0.0672 7 | -0.0164 4 | 0.0296 3 | 0.0867 2 | 0.2318 1 |
| Huang & Litz | -0.2583 10 | -0.1252 9 | 0.0149 8 | 0.1458 7 | 0.2566 6 | 0.3445 5 | 0.4117 4 | 0.4627 3 | 0.5014 2 | 0.5311 1 |
| Style | -0.113769 | | -0.122816 | | | | | | | |

| Overall Ranking | | | | | | | | | | |
|--------------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Marko No short | 166 | 126 | 89 | 69 | 15 | 6 | 4 | 2 | 64 | 203 |
| Marko Short | 51 | 40 | 33 | 30 | 28 | 29 | 32 | 34 | 35 | 44 |
| Marko No short with Gr | 161 | 133 | 99 | 76 | 53 | 8 | 5 | 3 | 1 | 52 |
| Marko Short with Group | 127 | 61 | 42 | 27 | 31 | 38 | 43 | 47 | 45 | 36 |
| CAPM No short | 166 | 177 | 192 | 191 | 197 | 198 | 210 | 201 | 88 | 23 |
| CAPM Short | 136 | 146 | 142 | 173 | 181 | 182 | 186 | 194 | 200 | 160 |
| CAPM No short with Gr | 161 | 176 | 188 | 193 | 195 | 196 | 205 | 208 | 189 | 68 |
| CAPM Short with Group | 127 | 184 | 199 | 202 | 207 | 211 | 209 | 206 | 153 | 78 |
| Benchmark | 108 | | | | | | | | | |
| BL No short | 166 | 132 | 81 | 117 | 70 | 49 | 18 | 10 | 14 | 23 |
| BL Short | 136 | 91 | 90 | 97 | 102 | 114 | 122 | 119 | 125 | 75 |
| BL No short with Groups | 161 | 171 | 84 | 87 | 123 | 74 | 56 | 41 | 20 | 65 |
| BL Short with Groups | 127 | 100 | 134 | 152 | 174 | 135 | 109 | 94 | 77 | 60 |
| BL1 No short | 166 | 183 | 92 | 147 | 73 | 57 | 46 | 19 | 13 | 23 |
| BL1 Short | 136 | 103 | 98 | 110 | 113 | 140 | 143 | 148 | 154 | 85 |
| BL1 No short with Group | 161 | 185 | 118 | 107 | 157 | 80 | 63 | 50 | 22 | 65 |
| BL1 Short with Groups | 127 | 120 | 156 | 172 | 179 | 150 | 115 | 95 | 82 | 58 |
| BL2 No short | 166 | 187 | 93 | 144 | 72 | 55 | 39 | 16 | 12 | 23 |
| BL2 Short | 136 | 105 | 101 | 111 | 112 | 141 | 145 | 149 | 155 | 85 |
| BL2 No short with Group | 161 | 190 | 124 | 106 | 158 | 79 | 62 | 48 | 21 | 65 |
| BL2 Short with Groups | 127 | 121 | 159 | 175 | 180 | 151 | 116 | 96 | 83 | 58 |
| Huang & Litz | 204 | 178 | 104 | 71 | 54 | 37 | 17 | 11 | 9 | 7 |

5.7 Numerical results for VaR using various approaches

Following the Capital Asset Pricing Model, it advises us to only hold two assets which are the market or equilibrium portfolios as well as the risk-free assets. It also states that investors are only compensated for the market risk which is associated with beta. This portfolio also has the maximum Sharpe ratio. From this point on we only concentrate of the five equilibrium portfolios which are the W_{100} (Black-Litterman with 100% views), W_{hat} (Black-Litterman with 10% confidence level in our view), W_{mkt} (equilibrium or market portfolio), W_{capm} (weights obtained from the CAPM returns) and W_{hist} (weights obtained from the geometric historical returns).

Calculation of VaR using the Parametric Method and Historical Simulation

Parametric VaR

We follow our literature to estimate the Parametric VaR. We calculate the monthly VaR at 5% using the expected volatility and return.

Results: From Table 5.15 and Figure 5.15, we can see that W_{hist} has the lowest VaR, whereas W_{hat} and W_{mkt} have almost the same VaR. They are then followed by W_{100} and W_{capm} with the highest VaR.

We now use historical simulation. We calculate one month of VaR at 5% using historical returns from 01/01/1996 to 31/12/2010 (fifteen years).

Results: From Table 5.16 (page 90) we see that W_{hist} has the lowest VaR at 3.78%. It is then followed by W_{capm} which is almost double the VaR of W_{capm} . W_{100}, W_{mkt} and W_{hat} are very close to each other with W_{mkt} having the highest VaR at 8.86%. Since we are looking at historical data, it is no surprise that W_{hist} has the lowest VaR. We see from Figure 5.7 (page 91) that W_{hist} has the steepest upward daily closings and fewer dips than the other portfolios.

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Table 5.15: Calculation of VaR using the Parametric Method

| Parametric Value at a 5% risk level | | | | | |
|-------------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| | W 100 | W hat | W mkt | W capm | W hist |
| Mean | 1.34% | 1.30% | 1.30% | 0.38% | 1.53% |
| Sigma | 5.37% | 5.28% | 5.29% | 4.94% | 3.74% |
| Initial Investment | \$ 1,000,000.00 | \$ 1,000,000.00 | \$ 1,000,000.00 | \$ 1,000,000.00 | \$ 1,000,000.00 |
| Mean Investment Value | \$ 1,013,399.11 | \$ 1,012,958.62 | \$ 1,013,014.27 | \$ 1,003,821.25 | \$ 1,015,348.16 |
| Sigma of Investment Value | \$ 53,651.37 | \$ 52,844.85 | \$ 52,875.28 | \$ 49,357.39 | \$ 37,378.44 |
| Cut-off | \$ 925,150.46 | \$ 926,036.58 | \$ 926,042.17 | \$ 922,635.57 | \$ 953,866.09 |
| Cumulative PDF | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| VaR at 1% risk level | \$74,850 | \$73,963 | \$73,958 | \$77,364 | \$46,134 |

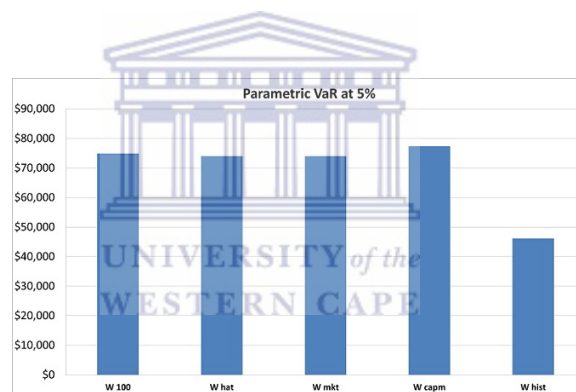


Figure 5.15: Parametric VaR in values

Table 5.16: Historical Simulation for VaR calculations

| VaR based Historical Simulation | | | | | |
|---------------------------------|--------------|--------------|--------------|--------------|--------------|
| | W100 | W hat | W mkt | W capm | W hist |
| VaR @95% | -8.46% | -8.80% | -8.86% | -7.59% | -3.87% |
| VaR @95% | \$ 84,622.05 | \$ 87,956.84 | \$ 88,587.02 | \$ 75,902.32 | \$ 38,739.38 |

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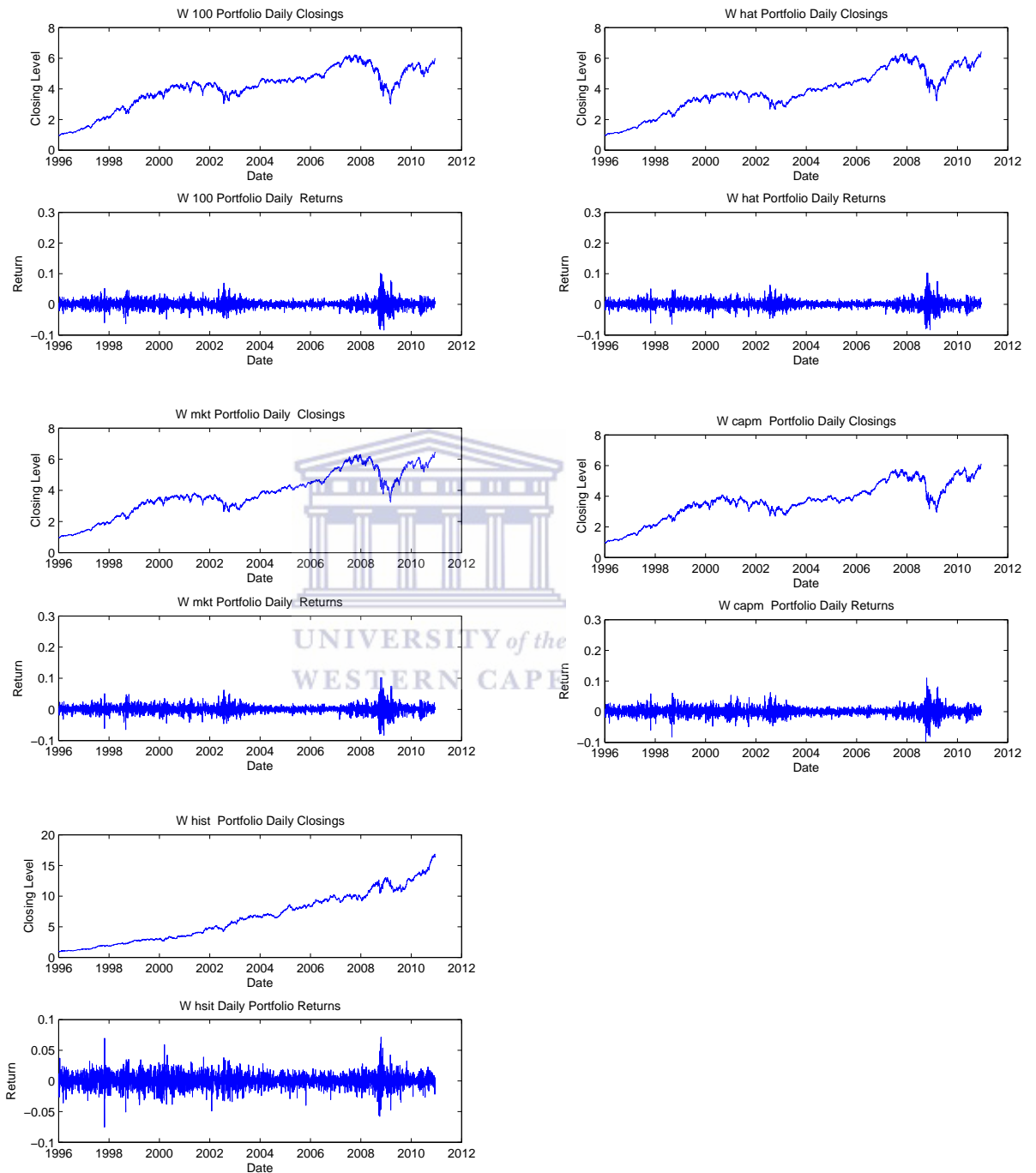


Figure 5.16: Portfolios daily closings and returns

Calculation of VaR using Bootstrapping and filtered Historical Simulation

Here we run a simulation following the algorithm from the previous chapter. We use geometric daily returns from the period 01/01/1996 to 31/12/2010, and multiply them with the weights of the portfolios. We then test if the series of returns are independent and identically distributed. We do this by checking if there is autocorrelation and heteroskedasticity. If the returns are not *i.i.d* then we produce them by applying an autoregressive model to the conditional mean of the portfolio returns

$$r_\tau = c + \theta r(\tau - 1) + \epsilon(\tau), \quad N(0, \sigma_t)$$

and the GARCH model to the conditional variance

$$\sigma_t^2(\tau) = \alpha_0 + \sum_{i=1}^q \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^p \beta_i \sigma_{t-1}^2(\tau - 1).$$

We then standardise the residuals by following

$$\text{standardised residuals} = \frac{\text{residual}}{\sigma}.$$

We then compare the autocorrelation of the standardised residuals and squared standardised residuals. Once the standardised residuals are *i.i.d*, we can then bootstrap historical simulation to obtain VaR. We now calculate one month's VaR by simulating 10, 000 paths with a 21 day time horizon.

Results: From Figure 5.17 we can see that the portfolio returns have some serial correlation. We also see that the squared returns have some degree of persistence. From Figure 5.18 we observe that the standardised residuals are now almost independently identically distributed when compared to the raw returns. Note that we have only shown figures of W_{100} and we have similar results for other portfolios. From Table 5.17 the W_{100} has the highest VaR followed by W_{capm} , W_{hat} , W_{mkt} and W_{hist} respectively. Another important point to note is that W_{hist} not only has the lowest VaR but also the maximum profit is higher than the maximum loss.

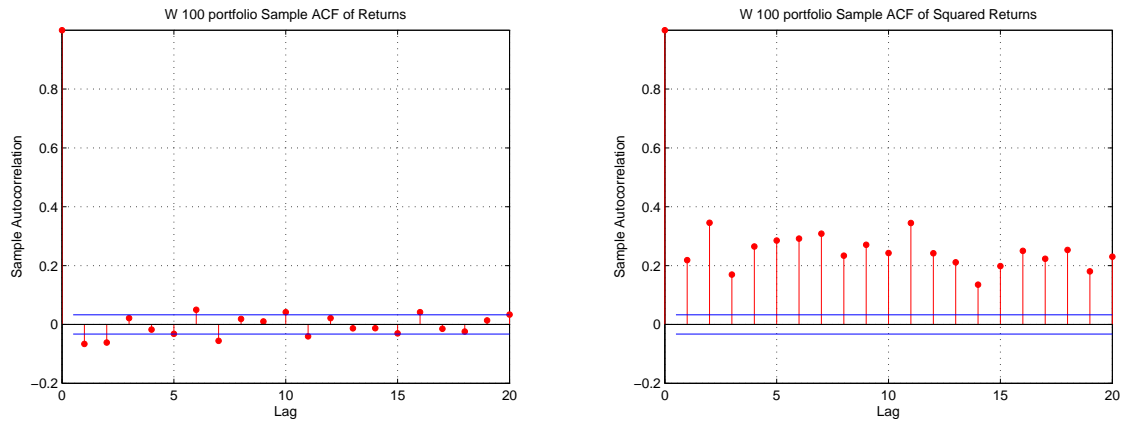


Figure 5.17: Valuation of the W_{100} portfolio's sample autocorrelation

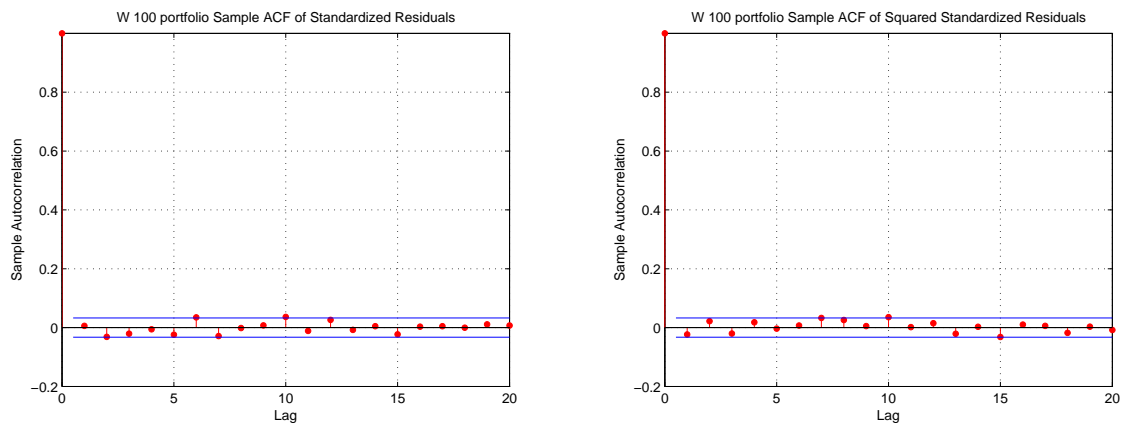


Figure 5.18: Valuation of the W_{100} portfolio's standardised residual autocorrelation

Table 5.17: VaR using FHS

| VaR using FHS | | | | | |
|------------------------------------|--------------------|--------------------|--------------------|---------------------|---------------------|
| Parameter | W_{100} Value | W_{hat} Value | W_{mkt} Value | W_{capm} Value | W_{hist} Value |
| C | 0.000724 | 0.000821 | 0.000829 | 0.000762 | 0.000814 |
| AR)(1) | -0.04103 | -0.04565 | -0.04561 | -0.05117 | -0.0354 |
| alpha | 7.71E-07 | 7.79E-07 | 7.96E-07 | 9.57E-07 | 1.19E-06 |
| GARCH(1) | 0.92928 | 0.92401 | 0.92346 | 0.9251 | 0.94397 |
| ARCH(1) | 0.068071 | 0.072784 | 0.073114 | 0.070922 | 0.046577 |
| Maximum loss at: | 22.82% | 23.54% | 23.39% | 24.93% | 16.05% |
| Maximum gain at: | 17.48% | 15.86% | 15.90% | 17.08% | 19.58% |
| VaR at 95% confidence level | -5.53% | -5.01% | -4.95% | -5.33% | -4.92% |

VaR using Monte-Carlo

Here we get the daily return based on fifteen years of data. We then calculate the correlation for the twenty nine stocks. We also convert the prices to logarithmic returns. We calculate the covariance using correlation and standard deviations. In this exercise we assume that our portfolio value is \$1000000. To calculate one month's VaR, we also assume our time horizon to be 21 days. We then simulate 10,000 paths. We then find the Monte-Carlo by looking for the fifth lowest portfolio price. Note: we only show the figures from one of the portfolios

Results: With the simulated paths shown in Figure 5.19, W_{hist} simulation has the lowest VaR at 6.86% from Table 5.18. This means that there is a 5% chance that our monthly VaR will not be greater than \$68 616.04. It is then followed by the W_{hat} and surprisingly the W_{mkt} and W_{100} are close to each other at 8.47% and 8.49% respectively. This also means that the monthly VaR will not be greater than \$84 658.81 and \$84 867.77 respectively. The W_{capm} has the highest VaR at 9.91% and there is a 5% chance that we lose more than \$99 079.80.

Table 5.18: Monte-Carlo Simulation for VaR calculation

| Monte-Carlo VaR at a 5% confidence level | | | | | |
|--|-----------------|-----------------|-----------------|-----------------|-----------------|
| | W100 | W hat | W mkt | W capm | W hist |
| Mean Value | \$ 1,044,057.69 | \$ 1,036,133.45 | \$ 1,011,543.55 | \$ 1,028,881.30 | \$ 1,033,753.26 |
| Sigma Value | \$ 139,832.81 | \$ 140,975.08 | \$ 145,943.53 | \$ 137,678.40 | \$ 122,582.29 |
| Top Sigma Value | \$ 1,183,890.50 | \$ 1,177,108.54 | \$ 1,157,487.09 | \$ 1,166,559.71 | \$ 1,156,335.55 |
| Bottom Sigma Value | \$ 904,224.88 | \$ 895,158.37 | \$ 865,600.02 | \$ 891,202.90 | \$ 911,170.97 |
| Cut-off at 5% | \$ 915,132.23 | \$ 917,012.90 | \$ 915,341.19 | \$ 900,920.20 | \$ 931,383.96 |
| MC VaR at 5% | \$ 84,867.77 | \$ 82,987.10 | \$ 84,658.81 | \$ 99,079.80 | \$ 68,616.04 |

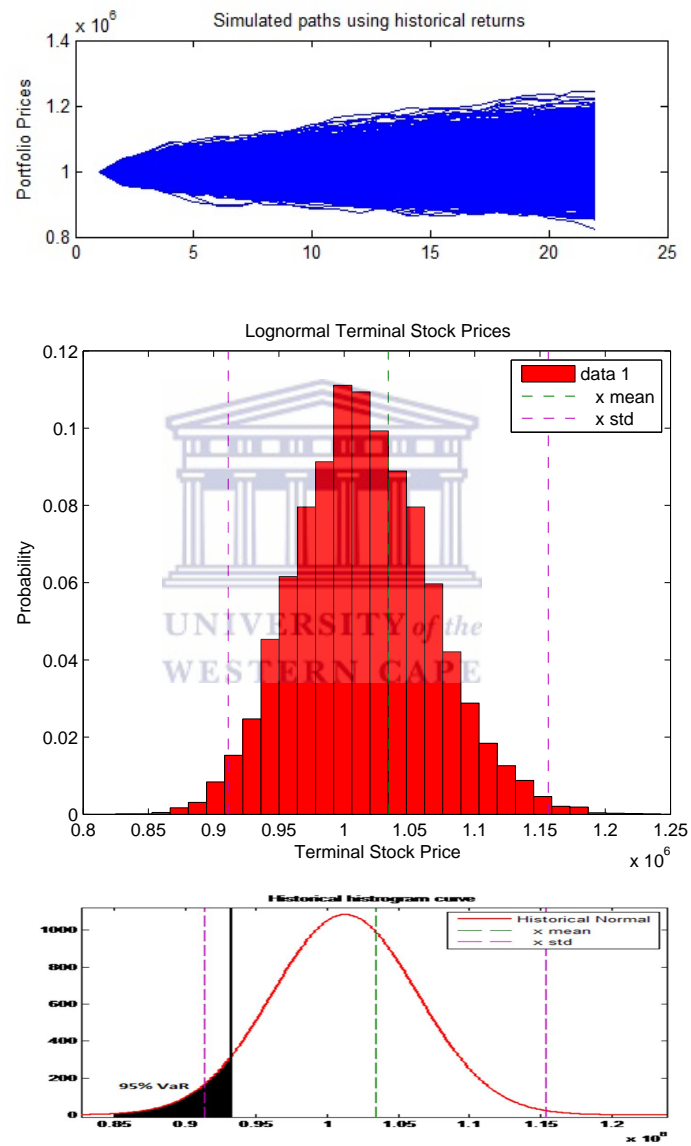


Figure 5.19: Monte-Carlo simulated paths and histogram with basic stats

VaR Back-testing

We now run the back-testing procedure on five of our portfolios. For each portfolio we calculate their VaR on a rolling basis. We then plot the preceding VaR against the current actual return (profit and/or loss). We then observe whether the profit and/or loss is greater than the estimated VaR of the previous month. VaR is calculated using an alpha at 5% confidence level. Our goal is to look for a volatility method which provides a more effective estimation of which portfolio has the highest number of violations in all the methods. We use conventional historical simulation, weighted historical simulation, riskmetrics, GARCH and t-GARCH. When the profit and/or loss exceeds the VaR we then see this as a violation and the ideal method produces a low number of violations. While the first 179 observations are used for in-sample forecasting of volatility, the last 89 and (50%) observations are used for an out-sample forecast of volatility, as well as the VaR calculation and the back-testing procedure. A total of 89 observations are used for the out-sample volatility forecast.

W_{capm}

Parameters

Alpha = 0.05

In- Sample size = 90

Out- Sample size = 89

Bandwidth size (for rolling estimates) = 89

Risk Metrics Parameter = 0.94

Weighted Historical Simulation parameter = 0.98

GARCH parameters: CST = 0.0092 GARCH = 0.2042 GARCH = 0.7859

GARCH parameters: CST = 0.0092 GARCH = 0.2042 GARCH = 0.7851 Degree of Freedom = 200.0000

Using the above set of parameters, the results are plotted and presented in Figure 5.23 (page 103).

Results: Using the CAPM weights we found that by using the historical method we had four violations giving us the frequency of 4.5%. We also found six violations for the weighted historical simulation, giving us the empirical frequency of violations as 6.74%. The riskmetric method was violated five times, giving an empirical frequency

of 5.62%. The GARCH and t-GARCH methods produced only one violation each. This gives us a total of seventeen violations.

W_{100}

Parameters

Alpha = 0.05

In- Sample size = 90

Out- Sample size = 89

Bandwidth size (for rolling estimates) = 89

Risk Metrics Parameter = 0.94

Weighted Historical Simulation Parameter = 0.98

GARCH parameters: CST = 0.0058 GARCH = 0.3018 GARCH = 0.6077

GARCH parameters: CST = 0.0057 GARCH = 0.3021 GARCH = 0.6084 Degree of Freedom = 200.0000

Using the above set of parameters, the results are plotted and presented in Figure 5.20 (page 100).

Results: Using the weight of 100% confidence level from Black-Litterman, the risk-metrics method had the highest number of violations at six, and an empirical frequency violation of 6.74%. The second was the weighted historical simulation which gave us five violations with an empirical frequency of 5.62%. The third was the historical with a total number of three violations and an empirical frequency of 3.37%. The GARCH and the t-GARCH had the least at one and zero with an empirical frequency of 1.12% and 0% respectively. These weights of the W_{100} produced a total number of fifteen violations.

W_{hat}

Parameters

Alpha = 0.05

In- Sample size = 90

Out- Sample size = 89

Bandwidth size (for rolling estimates) = 89

Risk Metrics Parameter = 0.94

Weighted Historical Simulation Parameter = 0.98

GARCH parameters: CST = 0.0091 GARCH = 0.3375 GARCH = 0.5993

GARCH parameters: CST = 0.0091 GARCH = 0.3383 GARCH = 0.5999 Degree of Freedom = 200.0

Using the above set of parameters, the results are plotted and presented in Figure 5.21 (page 101).

Results: Using the Black-Litterman weights at a 10% confidence level, the weighted historical and the risk-metrics had the highest number of violations at five and the empirical frequency violations were 5.62%. These were followed by the historical method with three violations and the empirical frequency violations were 3.3%. The GARCH and the t-GARCH didn't have any violations. This produced a total of thirteen violations for the Black-Litterman weights with a 10% confidence level.

W_{hist}

Parameters

Alpha = 0.05

In- Sample size = 90

Out- Sample size = 89

Bandwidth size (for rolling estimates) = 89

Risk Metrics Parameter = 0.94

Weighted Historical Simulation Parameter = 0.98

GARCH parameters: CST = 0.0113 GARCH = 0.0840 GARCH = 0.0000

GARCH parameters: CST = 0.0113 GARCH = 0.0849 GARCH = 0.0000 Degree of Freedom = 200.0000

Using the above set of parameters, the results are plotted and presented in Figure 5.24 (page 104).

Results: For the last 89 observations using the historical weights, the weighted historical produced the highest number of violations at four, and the empirical frequency violations were 4.49%. The historical, GARCH and t-GARCH had an equal number of violations at three and empirical frequency violations were 3.37%. The

risk-metrics had two violations, which were the least, and the empirical frequency violations were 2.25%. The total number of violations from the historical weights, out of the five methods, is fifteen.

W_{mtk}

Parameters

Alpha = 0.05

In- Sample size = 90

Out- Sample size = 89

Bandwidth size (for rolling estimates) = 89

Risk Metrics Parameter = 0.94

Weighted Historical Simulation Parameter = 0.98

GARCH parameters: CST = 0.0095 GARCH = 0.3366

GARCH = 0.6023 GARCH parameters: CST = 0.0095 GARCH = 0.3374 GARCH = 0.6029 Degree of Freedom = 200.0000

Using the above set of parameters, the results are plotted and presented in Figure 5.22 (page 102).

Results: Using the market weights, the weighted historical and the risk-metrics, these produced the highest number of violations at five and the empirical frequency violations were 5.62%. These were followed by the historical method with three violations and the empirical frequency violations were 3.3%. The GARCH and the t-GARCH had no violations. This brought a total of thirteen violations for the market weights.

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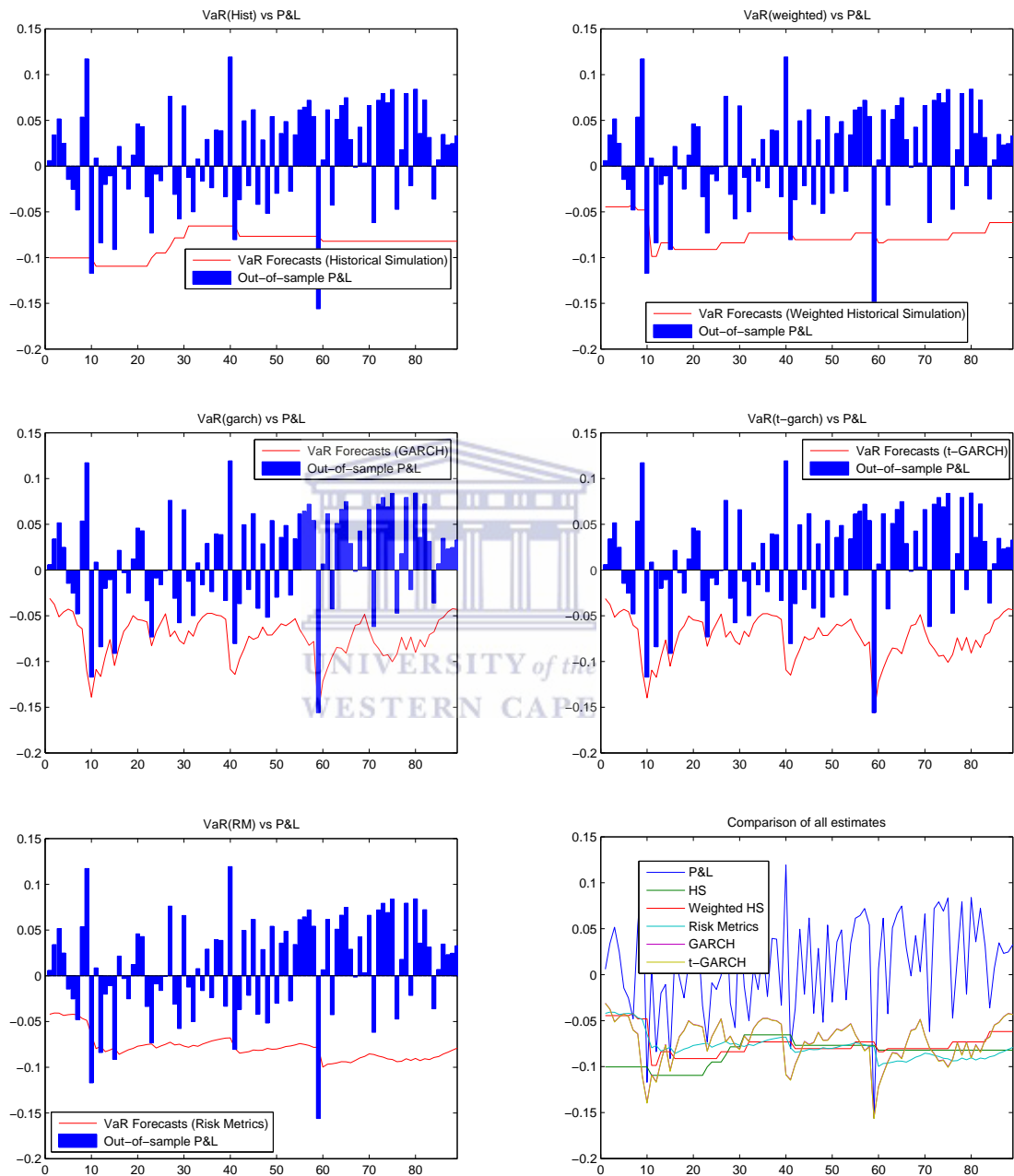


Figure 5.20: W_{100} backtesting

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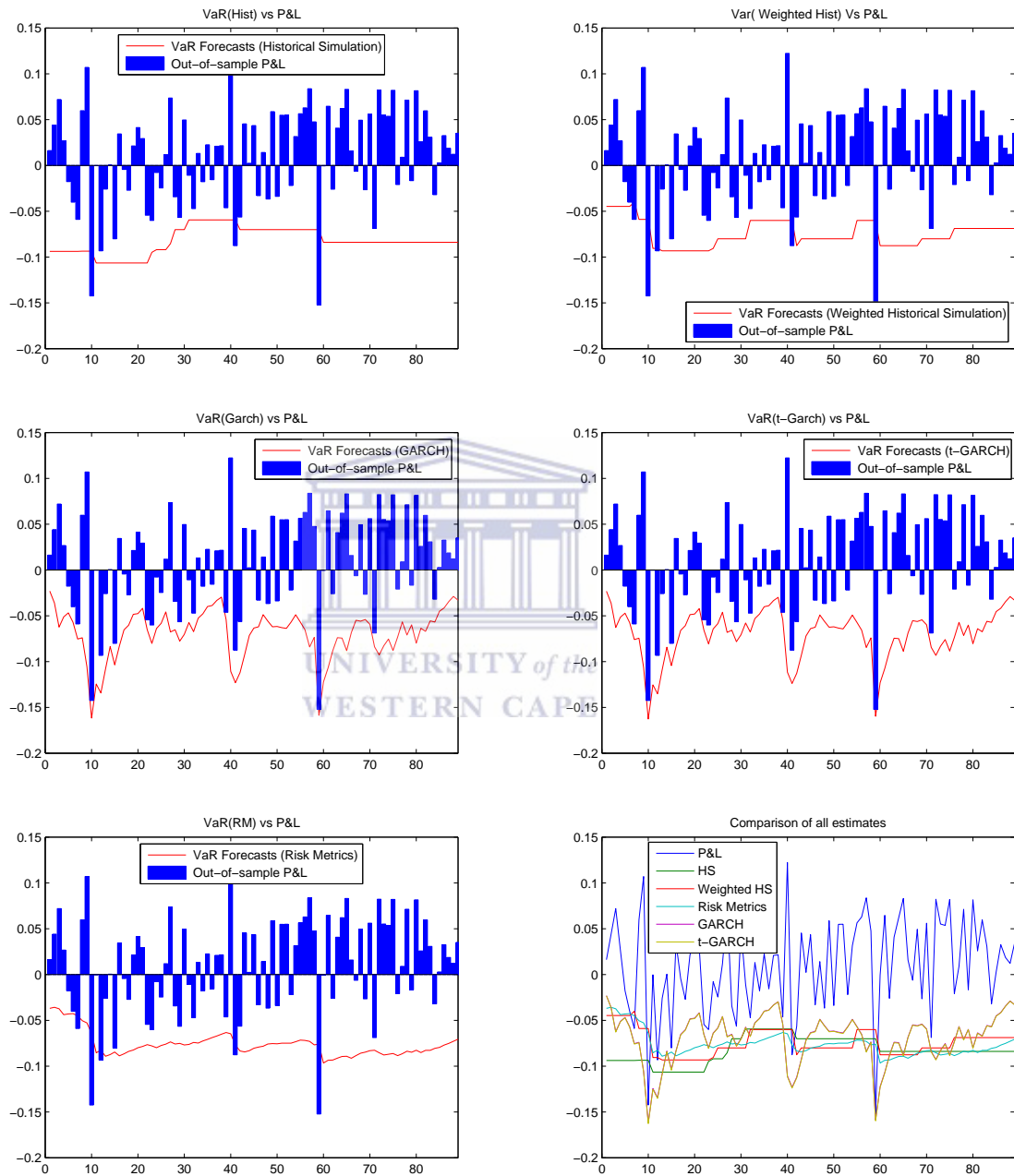


Figure 5.21: W_{hat} backtesting

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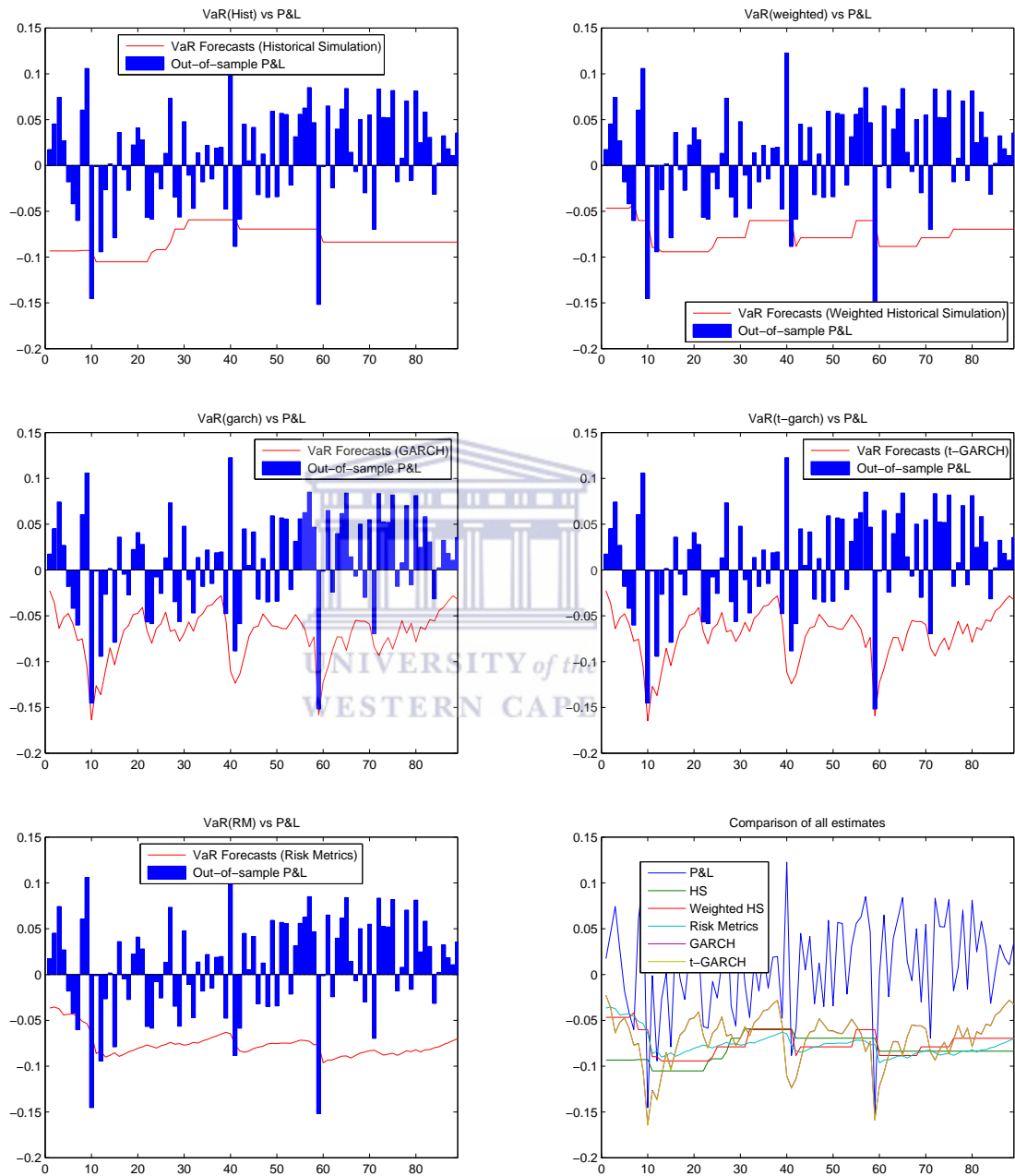


Figure 5.22: W_{mkt} backtesting

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Figure 5.23: W_{capm} backtesting

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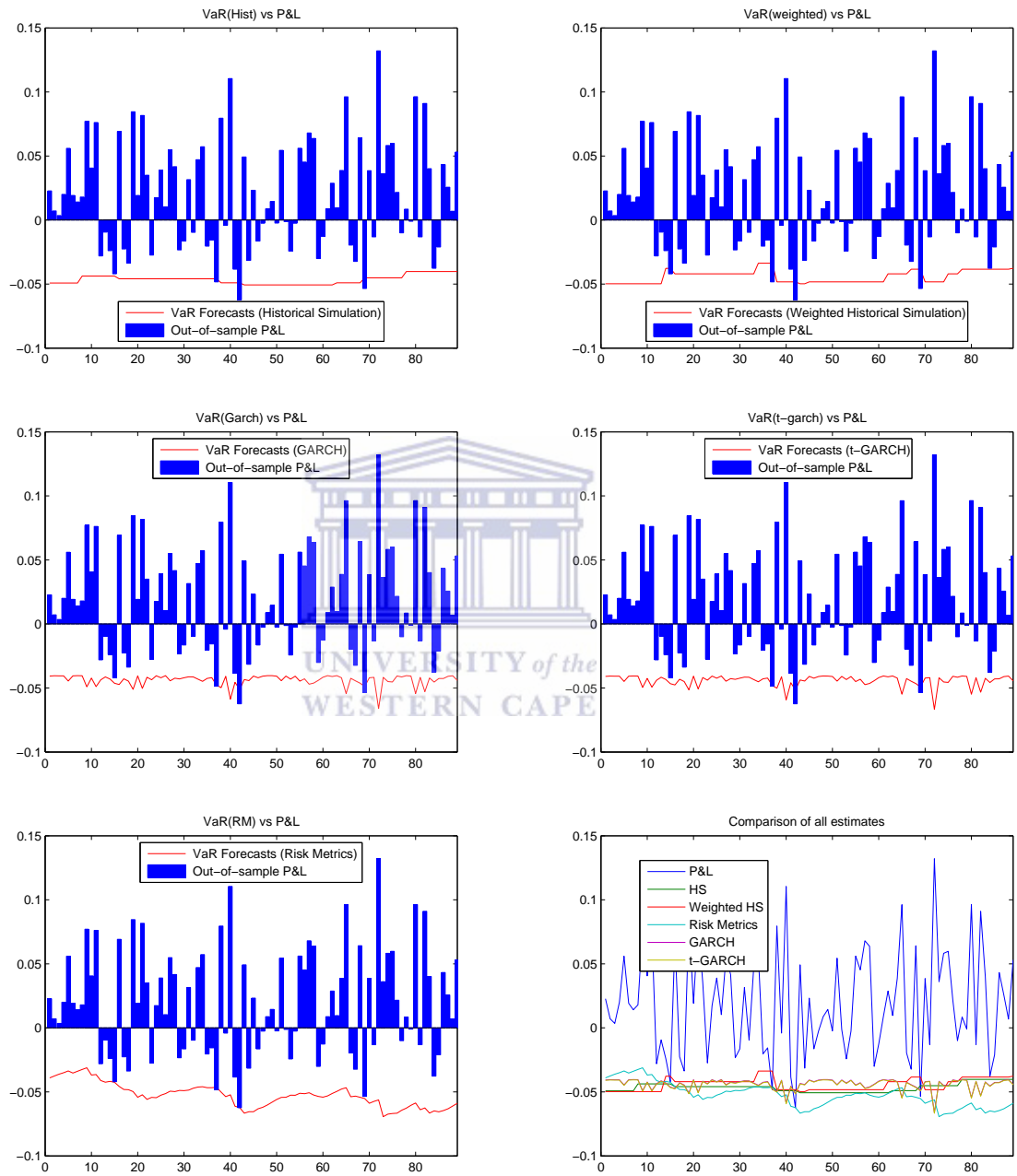


Figure 5.24: W_{hist} backtesting

Conditional VaR

We now follow Rockafellar ([61]) to calculate VaR and CVaR at a 5% confidence level for one month. The inverse of the error function is 1.1630 giving us S1 and S2 f 1.6448 and 2.06271. We want to calculate our losses for a monthly period should disaster strike.

Results

In this context of CVaR, we are unable to say which portfolio will suffer the highest losses, because CVaR calculates the minimum loss. Under non-normal conditions from Table 5.19 and Figure 5.7, W_{hist} will have the lowest minimum loss at 6.18% (\$61 752,84) which is 1.56% more than the maximum loss under normal conditions. W_{hat} is the second lowest minimum loss at 9.6% (\$96 000) if disaster should strike. W_{100} is the third and W_{mkt} has the highest at 10.07% (\$100 700). The CVaR and VaR of W_{capm} , W_{hat} , W_{capm} and W_{100} are very close together at 2.1% to 2.2% respectively. W_{hist} has the highest return and lowest volatility, but also has the lowest VaR and CVaR. The ratio gives an indication of what has happened in the past. We then move from Ex Post to Ex Ante by calculating VaR. This qualifies the future risk we can expect under normal circumstances. We can therefore calculate our minimum loss should the market behave abnormally.

Table 5.19: CVaR Parameters and Results

| Conditional Value at Risk (CVaR) | | | | | |
|----------------------------------|--------------|---------------|--------------|---------------|---------------|
| Beta = | 0.95 | | | | |
| inverse(erf) | 1.163087154 | | | | |
| c1 | 1.644853627 | | | | |
| c2 | 2.062712808 | | | | |
| | W100 | W mkt | W hat | W hist | W capm |
| Port Return | 1.34% | 0.84% | 1.30% | 1.53% | 0.38% |
| Port Risk | 5.37% | 5.29% | 5.28% | 3.74% | 4.94% |
| VaR | 7.48% | 7.86% | 7.40% | 4.61% | 7.74% |
| CVaR | 9.73% | 10.07% | 9.60% | 6.18% | 9.80% |
| VaR(In \$) | \$ 74,849.54 | \$ 78,584.72 | \$ 73,963.42 | \$ 46,133.91 | \$ 77,364.43 |
| CVaR(In \$) | \$ 97,268.26 | \$ 100,679.14 | \$ 96,045.13 | \$ 61,752.84 | \$ 97,988.87 |

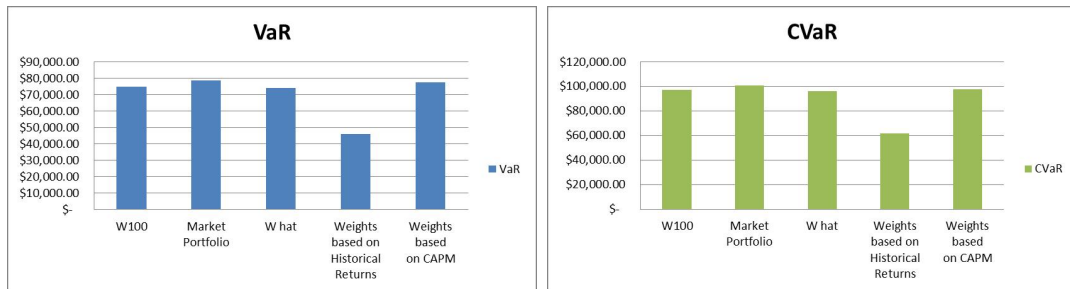


Figure 5.25: CVaR and VaR

5.8 Assets Risk Assessment using Marginal and Component VaR

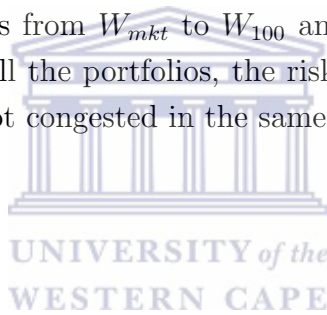
In this section we use the literature from our previous chapter in order to calculate the assets marginal and component VaR. We also calculate the assets VaR in order to compare the reduction of the portfolios VaR due to diversification.

Results: From Table 5.20, the portfolio with the weight from historical returns (W_{hist}) has produced the highest gains at almost 27% through diversification. Not only does it have the highest percentage in diversification, but it also has the lowest VaR amongst other portfolios. W_{capm} has the second lowest gains from the VaR. W_{mkt} and W_{hat} have close gains at 5.90% and 5.82% respectively. This gives us an indication that, W_{mkt} and W_{hat} had expected returns that gave a more diversified portfolio than other portfolios. It is also important to note that W_{100} portfolio had the lowest expected return with the highest standard deviation.

Looking at W_{mkt} , W_{100} and W_{hat} portfolios, AIG, GT and AA are the top contributors to risk, with AIG being the highest risk contributor. As we started with the computation of the market portfolio, it implies that the contribution percentage of AIG dropped to 0.218% for both W_{100} and W_{hat} portfolios. This also highlights the hotspots. It also shows that AIG will have the most hedging effects, followed by GT and AA. This is as we expected as the three assets in all these portfolios have the highest expected returns. It is surprising that portfolio W_{100} and W_{hat} in Table 5.21 MCD and PFT have the highest deviation from the W_{mkt} portfolio, but they are one of the lowest risk contributors. However in Table 5.20, both of them combined contribute about 5% to the total marginal risk of the portfolio.

The total sum ¹ of marginal VaR's of the W_{hist} is less than all the other portfolios. Dell, GT and AZO are the top risk contributors to the portfolios respectively. GT, which is ranked second within the other portfolios, is actually negative in this portfolio W_{hist} . This is an indication that if we remove GT from the portfolio, we actually increase the portfolio risk i.e. GT is fully hedged.

For the W_{capm} portfolio we have similar results obtained from the W_{mkt} , W_{100} and W_{hat} . Looking at the asset we changed in the Black-Litterman model, we see on the table that the HPQ marginal VaR decreased in the W_{100} portfolio and increased for W_{100} from 0.085% to 0.083% and 0.086% respectively. We have a similar case for CVX, and these are the assets we changed against each other for the AZO and XOM for MCD. The marginal risk decreases from the W_{mkt} to W_{100} and W_{hat} for PFE, the marginal risk increases from W_{mkt} to W_{100} and W_{hat} . Due to the law of large numbers, we see that in all the portfolios, the risk is gradually evenly distributed, i.e. the marginal risk is not congested in the same assets.



¹The sum of the assets doesn't have to be equal to the VaR of the portfolio, in fact it is always less than the portfolio VaR

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Table 5.20: Marginal VaR calculation results

| Gains from Diversification | | | | | | |
|-------------------------------------|------------------|------------|--------|-------------------------------------|-----------------------|--------|
| | Port return | Variance | Sigma | Port Var | Undiversified VaR | Gains |
| Market Portfolio | 0.71% | 0.00245385 | 4.95% | 7.44% | 13.34% | 5.90% |
| W100 | 0.67% | 0.00250118 | 5.00% | 7.55% | 15.79% | 8.24% |
| W hat | 0.71% | 0.00240761 | 4.91% | 7.36% | 13.19% | 5.82% |
| Weights based on Historical Returns | 1.53% | 0.00140257 | 3.75% | 4.63% | 31.57% | 26.93% |
| Weights based on CAPM | 0.84% | 0.0024625 | 4.96% | 7.32% | 19.63% | 12.30% |
| Marginal VaR | | | | | | |
| | Market Portfolio | W100 | W hat | Weights based on Historical Returns | Weights based on CAPM | |
| HPQ | 0.085% | 0.083% | 0.086% | 0.022% | 0.101% | |
| WHR | 0.097% | 0.097% | 0.098% | 0.019% | 0.098% | |
| T | 0.065% | 0.061% | 0.064% | 0.015% | 0.059% | |
| AZO | 0.026% | 0.024% | 0.027% | 0.042% | 0.030% | |
| JPM-PW | 0.103% | 0.103% | 0.105% | 0.019% | 0.100% | |
| CVX | 0.051% | 0.046% | 0.050% | 0.032% | 0.043% | |
| CB | 0.044% | 0.042% | 0.044% | 0.021% | 0.042% | |
| KO | 0.049% | 0.045% | 0.048% | 0.016% | 0.042% | |
| NBL | 0.051% | 0.046% | 0.049% | 0.035% | 0.054% | |
| Dell | 0.110% | 0.112% | 0.112% | 0.060% | 0.125% | |
| UNH | 0.061% | 0.062% | 0.060% | 0.030% | 0.057% | |
| XOM | 0.039% | 0.034% | 0.038% | 0.030% | 0.030% | |
| K | 0.016% | 0.016% | 0.016% | 0.012% | 0.016% | |
| MCD | 0.056% | 0.041% | 0.055% | 0.026% | 0.047% | |
| MHP | 0.056% | 0.057% | 0.056% | 0.028% | 0.068% | |
| MSFT | 0.101% | 0.102% | 0.103% | 0.034% | 0.101% | |
| GT | 0.137% | 0.129% | 0.137% | -0.009% | 0.143% | |
| NKE | 0.064% | 0.060% | 0.063% | 0.029% | 0.055% | |
| PFE | 0.055% | 0.068% | 0.057% | 0.016% | 0.050% | |
| GE | 0.099% | 0.099% | 0.099% | 0.016% | 0.098% | |
| WMT | 0.043% | 0.042% | 0.044% | 0.032% | 0.040% | |
| AIG | 0.223% | 0.218% | 0.218% | 0.028% | 0.169% | |
| AA | 0.127% | 0.121% | 0.127% | 0.022% | 0.125% | |
| CAT | 0.092% | 0.087% | 0.092% | 0.043% | 0.089% | |
| GIS | 0.007% | 0.007% | 0.007% | 0.024% | 0.008% | |
| ACE | 0.054% | 0.055% | 0.054% | 0.029% | 0.052% | |
| AXP | 0.108% | 0.104% | 0.108% | 0.025% | 0.114% | |
| IBM | 0.075% | 0.071% | 0.075% | 0.039% | 0.083% | |
| Ford | 0.001% | 0.118% | 0.122% | 0.018% | 0.122% | |

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Table 5.21: Component VaR calculation results

| Component VaR | | | | | | |
|---------------|------------------|---------|---------|-------------------------------------|-----------------------|--|
| | Market Portfolio | W100 | W hat | Weights based on Historical Returns | Weights based on CAPM | |
| HPQ | 3.42% | 4.09% | 3.98% | -0.57% | 9.02% | |
| WHR | 0.00% | 0.35% | 0.36% | 1.49% | -0.51% | |
| T | 6.08% | 5.30% | 5.66% | -3.64% | 4.29% | |
| AZO | 0.00% | 0.83% | 0.23% | 11.34% | 1.53% | |
| JPM-PW | 8.29% | 8.80% | 9.19% | -1.51% | 0.86% | |
| CVX | 4.83% | 4.09% | 4.83% | -2.98% | 1.79% | |
| CB | 0.59% | 0.36% | 0.39% | -0.14% | 2.88% | |
| KO | 3.93% | 3.55% | 3.88% | -1.16% | 5.35% | |
| NBL | 0.69% | 0.38% | 0.41% | 3.12% | 5.71% | |
| Dell | 1.48% | 1.45% | 1.48% | 10.29% | 6.53% | |
| UNH | 1.65% | 1.26% | 1.26% | 0.73% | -1.85% | |
| XOM | 7.33% | 5.15% | 6.95% | 22.84% | -0.89% | |
| K | 0.22% | 0.15% | 0.16% | -0.41% | -0.37% | |
| MCD | 2.25% | -6.15% | 1.24% | 2.46% | -0.36% | |
| MHP | 0.00% | 0.31% | 0.31% | 9.17% | 12.24% | |
| MSFT | 12.19% | 12.65% | 13.00% | -3.09% | 9.11% | |
| GT | 0.00% | 0.21% | 0.22% | 3.17% | 2.46% | |
| NKE | 1.72% | 1.27% | 1.37% | -1.16% | -2.57% | |
| PFE | 3.73% | 17.79% | 5.04% | -1.08% | 5.08% | |
| GE | 10.61% | 10.01% | 10.29% | -0.79% | 14.54% | |
| WMT | 4.06% | 4.04% | 4.33% | 5.29% | 3.33% | |
| AIG | 12.01% | 10.31% | 10.58% | -1.00% | -4.58% | |
| AA | 1.71% | 1.08% | 1.16% | -5.60% | -2.53% | |
| CAT | 2.47% | 2.83% | 3.07% | 17.61% | 6.63% | |
| GIS | 0.10% | 0.09% | 0.09% | 20.17% | 0.18% | |
| ACE | 0.72% | 0.62% | 0.62% | 3.55% | -1.88% | |
| AXP | 2.89% | 2.74% | 2.91% | -2.62% | 17.07% | |
| Ibm | 7.04% | 6.45% | 6.98% | 14.64% | 12.60% | |
| Ford | 0.00% | 0.02% | 0.02% | -0.10% | -5.67% | |
| Sum | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | |

Best Hedge Report

Here we want to control risk by assessing the Best Hedge of every asset. We do this through the use of optimisation. We want to calculate best hedge and we do this by minimising VaR and CVaR, varying one asset at a time and keeping the position of the other assets constant. We then sort the reduction of VaR and CVaR from highest to lowest and the highest reduction gives the best hedge.

Results: From the portfolio of W_{hist} in Table 5.23 (page 112) , XOM, GIS and AZO are the top three assets that have the biggest opportunity for risk reduction in selling. We also see that we almost have the same rank of assets in the VaR and CVaR. GE, Ford and WHR have the lowest ability to reduce the risk. Even so, we see that in all the assets, even those in the top three, there is not that much significance in the risk reduction of the portfolio when selling the assets, as the highest reduction is 7.5% for VaR and 8.06% CVaR which is low compared to each portfolio. This could be an indication of how well diversified this portfolio is. In this portfolio we can see that some of the assets need to reach the best hedge which is positive. This is due to the fact we are already short in those assets. In Table 5.23 (page 112), under W_{capm} portfolio, AXP, GE and MSFT are the best hedging assets while AIG, GIS and AZO are the lowest respectively. Surprisingly, Ford is not one of the lowest best hedging assets and GIS is one of the lowest even though it is the second in the W_{hist} . In Table 5.22 (page 111), under the W_{mkt} portfolio, GE, AXP and AA are the top portfolio risk reducers, while AZO, KG and GIS are the lowest. Here Ford has a VaR reduction of -14.46% and this is an indication that the asset was fully hedged at the original position. We increase the portfolio risk by selling the asset. In W_{hat} AXP, AA and CAT are the top three VaR reducers and AA, MSFT and GE are the top three CVaR reducers. In W_{100} GE, AXP and AA are the top three risk reducers while GIG, AZO and K are the lowest risk reducers. In this portfolio the ranking of the asset is exactly the same in VaR and CVaR. This is due to the fact that there is the same degree of difference in the asset percentage reduction.

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Table 5.22: Best Hedge Report for W_{100} , W_{hat} and W_{mkt}

| Best Hedge Report for the W100 portfolio | | | | | | | | |
|--|-------------|-------------------|-----------------------------|--|--|--------------------|-----------------------------|--|
| Current Value at Risk at 7.48% | | | | | Current Conditional Value at Risk at 9.73% | | | |
| Rank | Hedge Asset | VaR at best Hedge | Percentage reduction in VaR | Trade required to reach the best hedge | Hedge Asset | CVaR at best Hedge | Percentage Reduction in VaR | Trade required to reach the best hedge |
| 1 | GE | 5.27% | 29.53% | \$ (31,804.82) | GE | 6.78% | 30.24% | \$ (32,546.67) |
| 2 | AXP | 5.53% | 26.06% | \$ (6,940.85) | AXP | 7.13% | 26.73% | \$ (7,112.49) |
| 3 | AA | 5.88% | 21.46% | \$ (1,586.91) | AA | 7.58% | 22.05% | \$ (1,629.18) |
| 26 | NBL | 7.07% | 5.57% | \$ (975.06) | NBL | 9.17% | 5.76% | \$ (1,007.86) |
| 27 | AZO | 7.27% | 2.92% | \$ (3,354.19) | AZO | 9.43% | 3.02% | \$ (3,471.19) |
| 28 | K | 7.36% | 1.63% | \$ (1,039.66) | K | 9.56% | 1.69% | \$ (1,076.55) |
| 29 | GIS | 7.39% | 1.28% | \$ (1,393.99) | GIS | 9.60% | 1.32% | \$ (1,443.69) |

| Best Hedge Report for the W hat portfolio | | | | | | | | |
|---|-------------|-------------------|-----------------------------|--|--|--------------------|-----------------------------|--|
| Current Value at Risk at 7.40% | | | | | Current Conditional Value at Risk at 9.60% | | | |
| Rank | Hedge Asset | VaR at best Hedge | Percentage reduction in VaR | Trade required to reach the best hedge | Hedge Asset | CVaR at best Hedge | Percentage Reduction in VaR | Trade required to reach the best hedge |
| 1 | AXP | 5.30% | 28.35% | -7100.80 | AA | 5.64% | 41.32% | -1680.25 |
| 2 | AA | 5.63% | 23.82% | -1639.90 | MSFT | 5.95% | 38.07% | -27104.47 |
| 3 | CAT | 5.76% | 22.07% | -7526.91 | GE | 6.71% | 30.17% | -32066.09 |
| 27 | NBL | 6.94% | 6.18% | -1012.04 | AZO | 9.27% | 3.46% | -900.51 |
| 28 | AZO | 7.15% | 3.35% | -872.28 | K | 9.44% | 1.71% | -1069.34 |
| 29 | K | 7.27% | 1.65% | -1033.01 | GIS | 9.48% | 1.31% | -1414.57 |

| Best Hedge Report for the Wmkt portfolio | | | | | | | | |
|--|-------------|-------------------|-----------------------------|--|--|--------------------|-----------------------------|--|
| Current Value at Risk at 7.41% | | | | | Current Conditional Value at Risk at 9.62% | | | |
| Rank | Hedge Asset | VaR at best Hedge | Percentage reduction in VaR | Trade required to reach the best hedge | Hedge Asset | CVaR at best Hedge | Percentage Reduction in VaR | Trade required to reach the best hedge |
| 1 | GE | 5.24% | 29.24% | \$ (31,265.40) | GE | 6.73% | 29.98% | \$ (31,998.32) |
| 2 | AXP | 5.30% | 28.42% | \$ (7,114.78) | AXP | 6.82% | 29.09% | \$ (7,278.30) |
| 3 | AA | 5.63% | 23.94% | \$ (1,644.98) | AA | 7.26% | 24.52% | \$ (1,685.28) |
| 27 | K | 7.28% | 1.65% | \$ (1,031.85) | AZO | 9.28% | 3.49% | \$ (597.26) |
| 28 | GIS | 7.31% | 1.25% | \$ (1,361.89) | K | 9.45% | 1.71% | \$ (1,068.20) |
| 29 | Ford | 8.48% | -14.46% | \$ (7.99) | GIS | 9.49% | 1.30% | \$ (1,410.89) |

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Table 5.23: Best Hedge report for W_{capm} and W_{hist}

| Best Hedge Report for the Wcapm portfolio | | | | | | | | | |
|---|-------------|-------------------|-----------------------------|--|--|--------------------|-----------------------------|--|--|
| Current Value at Risk at 7.03% | | | | | Current Conditional Value at Risk at 9.09% | | | | |
| Rank | Hedge Asset | VaR at best Hedge | Percentage reduction in VaR | Trade required to reach the best hedge | Hedge Asset | CVaR at best Hedge | Percentage Reduction in VaR | Trade required to reach the best hedge | |
| 1 | AXP | 4.81% | 31.62% | \$ (38,597.67) | AXP | 6.15% | 32.36% | \$ (39,395.88) | |
| 2 | GE | 5.01% | 28.65% | \$ (41,114.09) | GE | 6.41% | 29.50% | \$ (42,080.89) | |
| 3 | MSFT | 5.13% | 26.94% | \$ (17,090.45) | IBM | 6.97% | 23.36% | \$ (39,874.38) | |
| 27 | AZO | 6.77% | 3.72% | \$ (5,111.59) | AZO | 8.74% | 3.84% | \$ (5,266.67) | |
| 28 | GIS | 6.92% | 1.56% | \$ (2,711.80) | GIS | 8.94% | 1.61% | \$ (2,798.30) | |
| 29 | AIG | 6.94% | 1.20% | \$ 310.37 | AIG | 8.96% | 1.45% | \$ 346.05 | |
| Best Hedge Report for the W hist portfolio | | | | | | | | | |
| Current Value at Risk at 5.87% | | | | | Current Conditional Value at Risk at 7.43% | | | | |
| Rank | Hedge Asset | VaR at best Hedge | Percentage reduction in VaR | Trade required to reach the best hedge | Hedge Asset | CVaR at best Hedge | Percentage Reduction in VaR | Trade required to reach the best hedge | |
| 1 | XOM | 5.43% | 7.50% | \$ (96,338.08) | XOM | 6.83% | 8.06% | \$ (100,068.20) | |
| 2 | GIS | 5.48% | 6.62% | \$ (97,968.91) | GIS | 6.93% | 6.75% | \$ (99,349.59) | |
| 3 | AZO | 5.54% | 5.64% | \$ (16,327.79) | AZO | 6.99% | 5.85% | \$ (16,703.65) | |
| 4 | CVX | 5.56% | 5.27% | \$ 8,346.70 | CVX | 7.00% | 5.80% | \$ 8,771.34 | |
| 5 | WMT | 5.58% | 4.96% | \$ (12,959.03) | WMT | 7.03% | 5.35% | \$ (13,498.18) | |
| 6 | ACE | 5.72% | 2.48% | \$ (5,659.18) | IBM | 7.22% | 2.79% | \$ (17,909.79) | |
| 7 | IBM | 5.73% | 2.42% | \$ (16,622.42) | ACE | 7.22% | 2.78% | \$ (6,013.21) | |
| 8 | Dell | 5.74% | 2.10% | \$ (4,467.57) | Dell | 7.25% | 2.47% | \$ (4,854.16) | |
| 9 | WICD | 5.75% | 2.05% | \$ (4,648.87) | WICD | 7.25% | 2.37% | \$ (4,944.71) | |
| 20 | KO | 5.84% | 0.49% | \$ 1,730.08 | KO | 7.38% | 0.62% | \$ 1,965.93 | |
| 21 | AXP | 5.85% | 0.33% | \$ 1,483.72 | AXP | 7.39% | 0.53% | \$ 1,886.42 | |
| 22 | AA | 5.86% | 0.21% | \$ (11,754.19) | PFE | 7.41% | 0.32% | \$ 1,373.71 | |
| 23 | PFE | 5.86% | 0.19% | \$ 1,043.63 | HPQ | 7.41% | 0.28% | \$ 303.38 | |
| 24 | HPQ | 5.86% | 0.18% | \$ 243.51 | JPM-PW | 7.41% | 0.20% | \$ 827.14 | |
| 25 | JPM-PW | 5.86% | 0.10% | \$ 578.07 | T | 7.42% | 0.10% | \$ 2,384.51 | |
| 26 | T | 5.86% | 0.04% | \$ 1,524.85 | AA | 7.42% | 0.10% | \$ (11,754.19) | |
| 27 | WHR | 5.87% | 0.01% | \$ (156.25) | WHR | 7.43% | 0.05% | \$ (360.33) | |
| 28 | Ford | 5.87% | 0.00% | \$ (3.37) | GE | 7.43% | 0.03% | \$ 221.36 | |
| 29 | GE | 5.87% | 0.00% | \$ (5.00) | Ford | 7.43% | 0.00% | \$ 5.60 | |

Applying the Best Hedge Report

Looking at the Best Hedge Report for the W_{mkt} portfolio, we now experiment to try and reduce the VaR of a new portfolio. We have called this portfolio W_{newBL} . We reduce GE from 1.57% to 1.2% with a 70% confidence level. GIS will outperform AXP by 30 basis points with a 50% confidence level. DELL will outperform AA by 60 basis points with 35% confidence level. We apply these views in the Black-Litterman model. We then compare W_{100} and W_{newBL} to see if we can achieve a lower VaR. In the computation of the Black-Litterman model, we use the reduced covariance matrix instead of the sample covariance matrix.

Results: W_{newBL} has a monthly expected return of 1.107% and a standard deviation of 4.81%. These are both less than the W_{100} . Using the various approaches, we find that W_{newBL} has a lower VaR than W_{100} except for the Historical Simulation approach. We were able to reduce VaR, but the expected return was lower.

5.9 Summary

In this chapter we applied our literature by choosing twenty-nine stocks, S&P500 index and the one year T-bill rate. We took the data and began by computing the envelope portfolio following the Huang and Litzenburg approach. We have seen how the portfolios performed and also observed that all of our ten portfolios brought about a profit at the end of the year. We then looked at the performance of the Markowitz model throughout the period of 2011, and we also saw profits until we grouped the stocks. We saw how volatile the mean-variance optimisation was. We then looked at the CAPM in 2011 and saw that it performed well only when following the Markowitz mean-variance optimisation without allowing short positions. We saw huge losses when we allowed short selling. The CAPM advises investors to only invest in the market portfolio. We have seen that the market portfolios made a profit in all scenarios. Using the CAPM our model was still sensitive, and we looked at Black and Litterman. We have seen that we can also get very poor results if our views are wrong. We then tracked the S&P500 index by use of the Sharpe's style analysis, and we observed little difference (both by allowing short selling and not allowing short selling) between the market portfolio (obtained by market capitalisation) in terms of profits. We then shifted focus to five portfolios. We measured their performance by using the Sharpe's, Treynor's and

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Information Ratios. We have seen that the portfolio with the highest ratios was a strong performing portfolio. We also looked at measuring the risk using VaR and CVaR. We have seen that W_{hist} had the highest expected return ratio and even the lowest VaR in all our calculations. We have also seen that a portfolio with a low VaR also has a low CVaR and vice versa.



Chapter 6

Concluding Remarks and Scope for Future Research

We have looked at the basic components of optimisation which we need for portfolio optimisation. We have seen how Quadratic Programming builds from Linear Programming. In future we will like to look at QPP with more complex constraints, such as orthogonal constraints, and also the more complex QPP such as Quadratic Fractional Programming. We will also look at Non-Convex Programming as there are cases that are non-convex in the extension of the QPP.

There is no doubt that the Modern Portfolio Theory is the foundation of finance. Since this is the foundation, any investor regardless of their preferences, should base their investment goals on the MPT. Studying various assets or securities individually is just an extra step towards portfolio optimisation. There are cases where traders and investors have criticised the MPT, and therefore don't use it. Professor Merton stated that "some people didn't fully understand the assumptions behind the Black-Scholes model and they applied the model incorrectly" ([17]). Understanding these basics will allow us to relax some of the assumptions, in order to apply them in the real world. We've investigated the MPT thoroughly, and now fully understand the basic assumptions underlying it. We have seen the power of MPT, starting from the Mean-Variance Optimisation (MVO). We have also seen that the MVO produces highly efficient and highly risky portfolios at the same time; however, an investor can choose their portfolios according to their risk tolerance, which at the end of the day is critical to making decisions. We've also identified its weaknesses and we were able to deal with these. It's important not to neglect this model, and rather use the

modifications that we've seen with the Capital Asset Pricing Model (CAPM). This builds and extends on the MVO. Even the CAPM was criticised, however we've observed that it brings about an important factor of being able to hold a single market or equilibrium portfolio in line with the risk preference of the investor. The critics of the CAPM developed an alternative method called the Asset Pricing Theory (APT), which has some assumptions that are related to the CAPM. We have also seen the enormous contribution that Black and Litterman have made to the MPT, by allowing investors to include their views on the market MVO or CAPM returns. These returns are no longer sensitive to small changes as in mean-variance optimization.

We've also observed a strong connection between risk and return. The importance of risk management has been identified as well as VaR, which also starts by looking at the portfolio's risk measure. Due to diversification, the importance of risk management is within the portfolio and not within the distinctive assets of the portfolio. Another observation has been that, we can change the expected return of the asset in order to expand any hedging opportunities, and this is done after carefully studying the overall portfolio.

In future we would like to consider adding transaction costs, options and foreign stocks and bonds to our portfolio. We would also like to take robust portfolio optimisation into account, as we've seen that optimisation is strongly dependent on the estimation of the parameters. This will ultimately give us a more accurate estimation. We will also look at the active management to review whether it is worth the effort when compared to Markowitz.

Appendix

Variance estimation approaches

ARCH

The clustering of volatilities changes over time so this plays a very important role in risk management as we need to try and predict the volatilities. If volatility increases so will the Value at Risk [40]). So in general we normally assume a constant variance towards our model. This problem of getting away from the constant variance assumption was introduced by Engle [22]. From the term ARCH, the change of variance is stipulated by heteroskedasticity. The formula follows a time series and it is computed as follows

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^q \alpha_i \epsilon_{t-i}^2$$

where $\epsilon_t = \sigma_t z_t$ and z_t is a random variable with a white noise process, $\alpha_0 > 0$ and $\alpha_i \geq 0, i > 0$.

GARCH

The ARCH laid the foundation of estimating non-constant variance and the generalised model was introduced by Bollerslev ([14]). The formula is as follows

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^q \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^p \beta_i \sigma_{t-i}^2,$$

where α_0 is a constant p lags of squared error q lags of conditional variance.

We compute and plot the autocorrelation of ϵ^2 by

$$\rho = \frac{\sum_{t=i+1}^T (\hat{\epsilon}_t^2 - \hat{\sigma}_t^2)(\hat{\epsilon}_{t-1}^2 - \hat{\sigma}_{t-1}^2)}{\sum_{t=1}^T (\hat{\epsilon}_t^2 - \hat{\sigma}_t^2)^2}.$$



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