Modelling of asset allocation in banking using the mean-variance approach

by

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Abstract

Bank asset management mainly involves profit maximization through investment in loans giving high returns on loans, investment in securities for reducing risk and providing liquidity needs. In particular, commercial banks grant loans to creditors who pay high interest rates and are not likely to default on their loans. Furthermore, the banks purchase securities with high returns and low risk. In addition, the banks attempt to lower risk by diversifying their asset portfolio. The main categories of assets held by banks are loans, treasuries (bonds issued by the national treasury), reserves and intangible assets. In this mini-thesis, we solve an optimal asset allocation problem in banking under the mean-variance frame work. The dynamics of the different assets are modelled as geometric Brownian motions, and our optimization problem is of the mean-variance type. We assume the Basel II regulations on banking supervision. In this contribution, the bank funds are invested into loans and treasuries with the main objective being to obtain an optimal return on the bank asset portfolio given a certain risk level. There are two main approaches to portfolio optimization, which are the so called martingale method and Hamilton Jacobi Bellman method. We shall follow the latter. As is common in portfolio optimization problems, we obtain an explicit solution for the value function in the Hamilton Jacobi Bellman equation. Our approach to the portfolio problem is similar to the presentation in the paper [Hojgaard, B., Vigna, E., 2007. Mean-variance portfolio selection and efficient frontier for defined contribution pension schemes. ISSN 1399-2503. On-line version ISSN 1601-7811]. We provide much more detail and we make the application to banking. We illustrate our findings by way of numerical simulations.

Declaration

I declare that, *Modelling of asset allocation in banking using the mean-variance approach* is my work, it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

B.C. Kaibe   May 2012

Signed.....................................

Acknowledgements

I take this opportunity to start first by thanking God the Almighty for being with me through this tough trying times. He guided and gave me the strength to hold on until the last moment. To my parents, wife and kids, find in this work the expression of my sincere love to you.

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Index of Abbreviations

a.s., almost surely;
SDE, Stochastic Differential Equation;
HJB, Hamilton-Jacobi-Bellman;
CAR, Capital Adequacy Ratio;
TRWA, Total Risk Weighted Asset;
LQ, Linear Quadratic;
PDE, Partial Differential Equation;
BCBS, Basel Committee on Banking Supervision;
BRC, Bank Regulatory Capital;
FDIC, Federal Deposit Insurance Corporation;
DIF, Deposit Insurance Fund.

List of Notations

Treasury securities at time $t : y_0(t)$;
Bank Loans at time $t : L(t)$;
Bank Deposits at time $t : D(t)$;
Bank Borrowing at time $t : B(t)$
Bank Reserve at time $t : R_v(t)$;
Bank Capital at time $t : C(t)$;
Tier 1 Capital at time $t : C_{T1}(t)$;
Tier 2 Capital at time $t : C_{T2}(t)$;
Tier 3 Capital at time $t : C_{T3}(t)$;
Risk-free rate of interest : r;
Sharpe ratio of the loans : δ;
Real-world probability measure : $\mathbb{P}$;
Natural filtration : $\{\mathcal{F}\}_{t \geq 0}$;
Total risk-weighted assets at time $t : a_w(t)$;
Capital Adequacy Ratio at time $t : R(t)$;
Stochastic process at time $t : \{X_t\}_{t \geq 0}$;
n-dimensional Brownian motion at time $t : dW(t)$.

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Chapter 1

General Introduction

1.1 Introduction

Commercial banks hold substantial amounts of liquid assets. A portion of the investment portfolio of commercial banks is held in short-term securities, especially treasury securities. They hold a certain amount of liquid assets to offset the risk of the large volume of volatile transactions deposits. Commercial banks administers the nation's payments system, and the demand for loans. The main categories of assets held by banks are loans, treasuries (bonds issued by the national treasury), reserves and intangible assets. Bank asset management mainly involves achieving profit maximization via high returns on loans and securities, reducing risk and providing for liquidity needs. More specifically, according to Petersen and Mukuddem-Petersen [13], banks try to manage their assets in the following ways. They endeavour to grant loans to creditors who are likely to pay high interest rates and unlikely to default on their loans. Secondly, banks try to purchase securities with high returns and low risk. Also, in managing their assets, banks attempt to lower risk by diversifying their investment portfolio. The study of the dynamics of these risk minimization strategies has always been an important issue in the management of banks.

Bank capital is a fundamental building block of the banking business, it is essential for survival and growth of the bank. The decision about the amount of capital the bank should hold and how it should be accessed is referred to as capital adequacy management [22]. According to the Basel Committee on Banking Supervision (BCBS), a committee established in the mid-1980s

to set common standards for banking regulations and to improve stability of the banking system, bank capital can be classified into Tier 1, Tier 2 and Tier 3 capital. Tier 1 capital consists of equity and reserves; Tier 2 capital consists of revaluations, undefined profits, soft debts and general provisions; and Tier 3 capital consists of subordinated debt with a term of at least 5 years and redeemable preference shares which may not be redeemed for at least 5 years. The 1988 Basel Accord, also known as Basel I, required that all banks should hold available capital equal to at least 8% of their risk-weighted assets (RWAs). More precisely, it required that the sum of Tier 1, Tier 2 and Tier 3 capital should be equal to 8% of the risk-weighted assets, and that the Tier 1 capital alone must be at least 4% of RWA. Currently the banking regulation is embodied by the Basel II capital Accord which has been implemented globally from the year 2007. Basel II adopts a three-pillared approach with the ratio of the bank capital to risk-weighted assets, also known as Capital Adequacy Ratio (CAR), playing a vital role as an index used to measure the strength of the bank. In our contribution, CAR is expressed thus,

$$CAR = \frac{\text{Bank Capital}}{\text{Total RWAs}} \tag{1.1}$$

where the total RWAs are comprised of risk-weighted loans and treasuries. In this regard, our banking model presents a balance sheet that comprises of assets (loans, treasuries, and reserves), liabilities (deposits), and bank capital (share-holder equity and subordinate debt). As a consequence of this, we are able to formulate a minimization problem that determines the optimal return on the bank asset portfolio given a certain risk level.

The study of financial portfolio selection theory dates back to the 1950s with Markowitz's pioneering work [20] on mean-variance efficient portfolios for a single-period investment. His work was recognized as the foundation for modern financial portfolio theory. Mean-variance portfolio selection in simple terms, means to allocate total wealth among a number of assets (risk-free and risky), with the main objective being to maximize the expected level of return, $E(x(T))$, and to minimize the level of risk on the investment $x(t)$, T being a certain terminal time. In his framework, he used the variance of the final wealth, $Var(x(T))$ as the measure of risk. Then the problem was how to minimize the portfolio's variance subject to the given level of return. The portfolio which achieves the minimum variance, given the expected level of return z, is said to be optimal. The pair $(Var(x(T)), z)$ is called a variance minimizing frontier. If in this portfolio, the maximum expected level of return among the portfolios with the same variance can be achieved, then this portfolio is said to be efficient. The pair of minimum variance and maximum expected level of return

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is then called the efficient frontier. Since Markowitz's marvellous award winning work, this subject has been of great interest to many researchers. There has been significant development from the single-period case to multi-period discrete-time (see: Smith (1967); Chen, Jen and Zionts (1971); etc.) and continuous time cases (see: Merton (1969); Cox and Huang (1989); etc.). However in the multi-period case, instead of using mean-variance model, the expected utility of the terminal wealth, $E(U(x(T)))$, was used and the U represented the utility function. The problem then was to maximize the $E(U(x(T)))$. Mean-variance portfolio selection for multi-period model, as much as it seemed the ideal way to deal with portfolio selection problems, especially when the market is less volatile, has not been studied further and developed very intensively until recently (Li and Ng, 2000). One of the difficulties encountered by researchers before Li and Ng, 2000, was the term $[E(x(T))]^2$ which resulted from $\mathrm{Var}(x(T))$. In solving a stochastic optimal control problem one typically uses the "smoothing" property of the expectation operator, but the variance operator does not satisfied this property. This contributed to the unavailability of analytical and efficient numerical results (see: Samuelson (1986); Hakansson (1971); Grauer and Hakansson (1993); and Pliska (1997)) by the research that came prior to Li and Ng (2000). In 2000, the work done by Li and Ng [18] can be viewed as a breakthrough since they extend Markowitz's single-period analytical result to a multi-period, discrete time portfolio selection. They used a so-called embedding technique to combine $E(x(T))$ and $\mathrm{Var}(x(T))$ to be a single objective $J(U(.)) = -E(x(T)) + \mu \mathrm{Var}(x(T))$, where μ can be any positive number. Thus they came up with an optimal portfolio and efficient frontier. This was then viewed as an extension of Markowitz's work into multi-period. In the case of the continuous time, extension was a bit more complicated. It could not be simply seen as the limit of the multi-period model by easily dividing the investment period again and again in order to make it go to infinitesimal. Nevertheless, research on continuous-time Markowitz's mean-variance model still became more and more active; and results were obtained. Up until now, the only two main methods that have been used to solve these types of problems are the so-called stochastic linear-quadratic control approach and martingale approach. In 2000, Zhou and Li [17] introduced stochastic linear-quadratic control for the first time as a framework for solving continuous time mean-variance problems. The martingale approach is another important method used in solving mean-variance portfolio selection problems. This method was first used to solve the portfolio optimization problem (under the expected utility framework) by Harrison and Kreps (1979) and Pliska (1982, 1986) where the use of risk neutral (equivalent martingale) probability measure was incorporated. Majority of the past researches, in view of mean-variance framework, assumed

that the market is complete. But, Jin (2004), Jin and Zhou (2005) applied the martingale approach to solve the mean-variance problem in an incomplete market. In their work, they studied the following four scenarios respectively: portfolios are unconstrained, shorting is prohibited, bankruptcy is prohibited, and both short-selling and bankruptcy are prohibited. In our work we solve an optimal asset allocation problem in banking under the mean-variance framework by making use of stochastic linear-quadratic control, introduced by Zhou and Li (2000).

In the case of the linear-quadratic method, the previous researches use analytical techniques to solve the non-linear Hamilton Jacobi Bellman (HJB) PDE for special cases. In order to obtain analytic solutions, the authors make assumptions which allow for the possibility of unbounded borrowing and infinite negative wealth (bankruptcy). However, some analytical solution have been developed for handling specific constraints: no stock shorting (Li et al.,2002) (but shorting the bond is still allowed) and the no bankruptcy case (Bielecki et al.,2005) (but again allowing for shorting the bond). In this contribution, like in [14], we solve the problem without putting restrictions on the optimal investment allocation, since there are some difficulties that arise when constraints are introduced in the model.
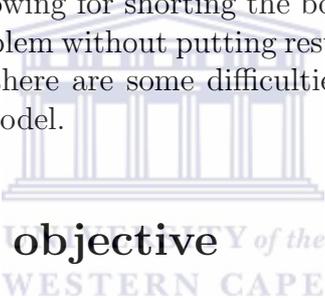
1.2   Research objective

In this work, we solve an optimal asset allocation problem in banking under the mean-variance frame work. We assume that bank funds are invested into loans and treasuries with the main objective being to obtain an optimal return on the bank asset portfolio given a certain risk level. Following the work by Zhou and Li (2000), we define and solve a mean-variance portfolio selection problem in banking and find the optimal policy and the efficient frontier of feasible portfolios in closed form. The solution is then obtained by transforming the mean-variance problem in a linear-quadratic control problem, which has been solved through standard techniques of stochastic optimal control theory.

1.3   Structure of the thesis

The rest of the thesis is organised as follows. In chapter 2, we give mathematical preliminaries, the definitions of the mathematical tools that will be

used throughout our work. In chapter 3, we review a continuous time dynamic model for a commercial bank in which the bank hold assets (uses of funds) and has liabilities (sources of funds) that behave in a stochastic manner. In chapter 4, we formulate and give a solution to the mean-variance optimization problem in banking. In chapter 5, we derive the dynamics for the Basel II capital adequacy ratio of a commercial bank. Furthermore, in chapter 6 we show results by way of simulation and lastly, chapter 7 concludes the mini-thesis.

Chapter 2

Mathematical Preliminaries

In this chapter, we give some definitions of mathematical tools that have been used throughout our dissertation. We define concepts such as Brownian motion, stochastic process, filtration, random variables, stochastic integration, etc., and give some basic results. Our main references on such basics are Etheridge [4], Wilmott, Howison and Dewynne [5], Oksendal [15] and Grimmett and Stirzaker [7].

2.1 Random Variables and Stochastic Processes

In order for one to talk about a random variable in a formal way, the concept of probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ needs to be specified. Ω is a set called the sample space, $\mathcal{F}$ is a collection of subsets of Ω, and $\mathbb{P}$ specifies the probability of each event $A \in \mathcal{F}$. The collection $\mathcal{F}$ is a σ-field, that is, $\Omega \in \mathcal{F}$ and $\mathcal{F}$ closed under the operations of countable union and taking complements. The probability $\mathbb{P}$ must satisfy the following axioms of probability

1. $0 \leq \mathbb{P}[A] \leq 1$, for all $A \in \mathcal{F}$

2. $\mathbb{P}[\Omega] = 1$

3. $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$ for any disjoint $A, B \in \mathcal{F}$,

4. If $A_n \in \mathcal{F}$ for all $n \in N$ and $A_1 \subseteq A_2 \subseteq ...$, then $\mathbb{P}[A_n] \uparrow \mathbb{P}[\bigcup_n A_n]$ as $n \uparrow \infty$.

Definition 2.1.1

Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a σ-algebra $\mathcal{F}_t$. Assume further that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s < t < \infty$ and $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t$.

Then we call the collection $\mathcal{F}_t$ of σ-algebras a filtration and $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ is called a filtered probability space.

We consider $\mathcal{F}_t$ as the set of information available to the observer (e.g. the bank manager) up to time t. More generally, we consider $\{\mathcal{F}_t\}_{t>0}$ as describing the flow of information over time, where we suppose that the bank does not lose information as time passes (hence why we say $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$).

Definition 2.1.2

A real-valued stochastic process is an indexed family of real-valued functions, $\{X_t\}_{t \geq 0}$ on Ω. $\{X_t\}_{t \geq 0}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if X_s is $\mathcal{F}_t$-measurable for each $t \geq s$.

2.2 Brownian Motion

In the year 1827, Robert Brown observed the complex and erratic motion of grains of pollen suspended in a liquid. It was later discovered that such irregular motion comes from extremely large number of collisions of the suspended pollen grains with the molecules of the liquid. Norbert Wiener presented a mathematical model for this motion based on the theory of stochastic processes. The position of a particle at each time $t \geq 0$ is a three dimensional random vector W_t.

Definition 2.2.1

A real-valued stochastic process $\{W_t\}_{t \geq 0}$ is a $\mathbb{P}$-Brownian motion (or a $\mathbb{P}$-Wiener process) if for some real constant σ, under $\mathbb{P}$,
1. for each $s \geq 0$ and $t > 0$ the random variable $W_{t+s} - W_s$ has the normal distribution with mean zero and variance $\sigma^2 t$,
2. for each $n \geq 1$ and any times $0 \leq t_0 \leq t_1 \leq t_2 .... \leq t_n$, the random variables $\{W_{t_r} - W_{t_{r-1}}\}$ are independent,
3. $W_0 = 0$,
4. W_t is continuous in $t \geq 0$.

2.3 Stochastic Integration

The history of stochastic integration and the modelling of risky asset prices both begin with Brownian motion [21]. Wiener and others proved many properties of the paths of Brownian motion. Two key properties relating to stochastic integration are that (1) the paths of Brownian motion have a non-zero finite quadratic variation, such that on an interval (s,t), the quadratic variation is $(t-s)$ and (2) the paths of Brownian motion have infinite variation on compact time intervals, almost surely. Processes used to model stock price are usually functions of one or more Brownian motions. In this regard, suppose that the stock price is of the form $S_t = f(t, W_t)$. Using Taylor's theorem, we can write

$$
\begin{aligned}
f(t+\delta t, W_{t+\delta t}) - f(t, W_t) &= \delta t \, \dot{f}(t, W_t) + O(\delta t^2) + (W_{t+\delta_t} - W_t) f'(t, W_t) \\
&+ \frac{1}{2!}(W_{t+\delta t} - W_t)^2 f''(t, W_t) + ...
\end{aligned}
\tag{2.1}
$$

where the notation $\dot{f}$, f' and f'' must be interpreted as $\dot{f}(t,x) = \frac{\partial f}{\partial t}(t,x)$, $f'(t,x) = \frac{\partial f}{\partial x}(t,x)$ and $f''(t,x) = \frac{\partial^2 f}{\partial x^2}(t,x)$. The dynamics of a stock price is commonly modeled by way of a stochastic differential equation as follows (see for example Etheridge [4, p 75]):

$$
dS_t = \dot{f}(t, W_t)dt + f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.
\tag{2.2}
$$

It is convenient to write the differential equation above in integrated form,

$$
S_t = S_0 + \int_0^t \dot{f}(s, W_s)ds + \int_0^t f'(W_s)dW_s + \int_0^t \frac{1}{2}f''(W_s)ds.
\tag{2.3}
$$

2.3.1 Itô Process

A stochastic process $X = \{X_t, t \geq 0\}$ that solves an equation of the form

$$
X_t = X_0 + \int_0^t a(X_t, t)ds + \int_0^t b(X_t, t)dW_s
\tag{2.4}
$$

is called an Itô process. Another manner of writing equation (2.4) is

$$
dX_t = a(X_t, t)dt + b(X_t, t)dW_t,
\tag{2.5}
$$

where $a(X_t, t)$ is the drift rate, $b(X_t, t)$ is the variance rate or diffusion and W_s is a standard Wiener process.

2.3.2 Itô Formula

Let X_t be an Itô process given by

$$dX_t = udt + vdB_t. \tag{2.6}$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$ (i.e., *g is twice continuously differentiable on* $[0,\infty) \times \mathbb{R}$).
Then

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t).(dX_t)^2, \tag{2.7}$$

where differentials are multiplied according to the rules

$$dt.dt = dt.dB_t = dB_t.dt = 0, \qquad dB_t.dB_t = dt. \tag{2.8}$$

2.4 A Discussion and Brief Literature Review of Modern Portfolio Theory

Research on modern portfolio theory dates back to the 1950s with Markowitz's nobel-prize winning work on mean-variance efficient portfolios for a single-period investment [20]. The most important contribution of Markowitz's work is the introduction of quantitative and scientific approaches to risk management and analysis. When short-selling is not allowed, efficient portfolios are obtained computationally via solving a quadratic programming problem. Merton [23], later derived an analytical solution to the single-period mean-variance problem under the assumption that the covariance matrix is positive definite and short selling is allowed.

Modern portfolio theory is a theory of finance which attempts to maximize portfolio expected return for a given amount of portfolio risk, or equivalently, to minimize risk for a given level of expected return, by carefully choosing the proportions of various assets.

Mean-variance portfolio selection (see [19]) refers to the problem of finding

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an allowable investment policy (i.e., a dynamic portfolio satisfying all the constraints) such that the expected terminal wealth satisfies $E(x(T)) = d$ while the risk measured by the variance of the terminal wealth

$$\text{Var}(x(T)) = E[x(T) - E(x(T))]^2 = E[x(T) - d]^2 \qquad (2.9)$$

is minimized.

After Markowitz's pioneering work, there has been significant development. According to authors in paper [19] extensions have taken somewhat different tack to Markowitz's original formulation. Specifically, instead of treating the $\text{Var}(x(T))$ and $\text{E}(x(T))$ of a portfolio as separate quantities and finding the relationship between them, a single quantity known as the expected utility of terminal wealth $\text{EU}(x(T))$ is considered. The utility function U is commonly a power, log, exponential, or quadratic form. According to authors in papers [17] and [19] the disadvantage of this approach is that the relationship between risk and return is contained only implicitly in the utility function. Hence, it is not clear in general what relationship exists between the risk and the return of the derived policy.

In extending Markowitz's idea to the multi-period or continuous time setting there were some difficulties encountered by several researchers. They found that the variance $\text{Var}(x(T))$ involves a term $E[x(T)^2]$ that is hard to analyze due to its non-separability in the sense of dynamic programming; see [17] for a more detailed discussion on this point. It is only recently that Li and Ng in paper [18] have faithfully extended Markowitz's mean-variance model to the multi-period setting by using an idea of embedding the problem into a tractable auxiliary problem.

In the paper by Zhou and Li [17], the continuous-time mean-variance problem is studied by incorporating the embedding technique used in Li and Ng [18]. However, the main contribution of [17] is not the explicit mean-variance efficient frontier it obtained *per se*; rather it is unifying framework, i.e., that of the stochastic linear-quadratic optimal control, it introduced to solve certain finance problems including the mean-variance portfolio selection.

In this contribution we use stochastic linear quadratic control as the framework to solve a mean-variance portfolio selection problem in banking.

Chapter 3

The Stochastic Banking Model

In this section we describe a banking model, very closely related to those in [8], [9], [10] and [13].

The bank capital is the difference between the values of assets and liabilities. According to [11], it is very difficult to get an accurate measure of the true value of illiquid assets such as loans. Basel II capital Accord encourages banks to view balance sheet items from the view point of the riskiness of assets held and the adequacy of their capital. In this regard, in order to understand the operation and management of banks, we study its balance sheet, which records the bank assets (uses of funds) and bank liabilities (sources of funds). The items on the balance sheet behave in an unpredictable manner, arising from the uncertain behaviour of the activities related to the evolution of treasuries, loan demand, risky and riskless investments, deposits, loan repayments, borrowings and eligible regulatory capital. These components of the balance sheet can be related as mentioned above, by the relation:

$$\text{Total Assets} = \text{Total Liabilities} + \text{Bank Capital}. \qquad (3.1)$$

As in the paper [9] of Mukkudem-Petersen and Petersen, a commercial bank's balance sheet at time t can be represented as

$$y_0(t) + S(t) + L(t) = D(t) + B(t) + C(t) \qquad (3.2)$$

where,
y_0, S, L, D, B and C are treasuries, securities, loans, deposits, borrowings and bank capital, respectively. All these components are regarded as functions from $\Omega \times T$ to $\mathbb{R}_+$.

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3.1   Bank Assets

In this subsection, the bank assets that we discuss are loans, treasuries and reserves. We suppose that we are working with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on a time period $T = [t_0, t_1]$. Here we assume that $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is a complete, right continuous filtration generated by one-dimensional Brownian motion $\{W(t)\}_{t \geq 0}$.

3.1.1   Loans

In our model, the dynamics followed by the loans (risky asset) is assumed to be given by:

$$dL(t) = \lambda L(t)dt + \sigma L(t)dW(t) \tag{3.3}$$

where,
$L : \Omega \times T \to \mathbb{R}_+$ is a stochastic process, λ is the drift rate, σ is the volatility of the loans and $W : \Omega \times T \to \mathbb{R}$ is a Brownian motion whose value at time t is denoted by $W(t)$.

The graph in Figure 3.1 shows the behaviour of the risky asset-loans following the Brownian motion path over a period of 20 years.
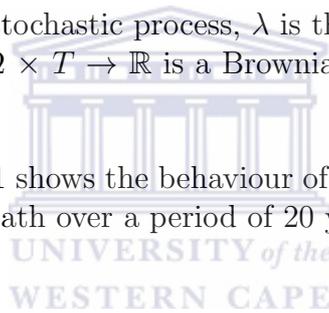
Table 3.1: Parameter values for the loans process $L(t)$.

λ	σ	L	dt	T(yrs)
0.08	0.05	10	0.5	20
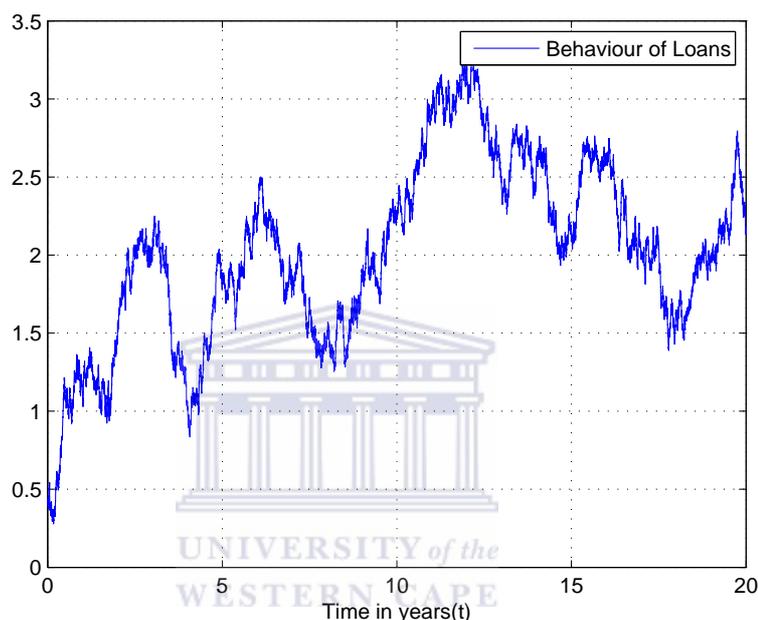
Figure 3.1: A simulation of the behaviour of risky asset-loans.

3.1.2 Treasury securities

Treasury securities include multiple types of securities that are issued by the national treasuries to help raise capital. Monies received from the sale of the treasury securities help to pay for the operation of the government. Treasury securities, in this sense, can be regarded as the debt financing instruments of the government. They are often referred to as "treasuries." Treasury bills, treasury notes, treasury bonds, and treasury inflation protected securities (or savings bonds) are all forms of treasury securities traded on the secondary market. All of the treasury securities besides savings bonds are very liquid. The dynamics followed by the treasuries (risk-free asset) is assumed as being

given by:

$$dy_0(t) = y_0(t)rdt, \qquad y_0(0) = 1, \qquad (3.4)$$

where $y_0 : \Omega \times T \to \mathbb{R}_+$ is the stochastic process followed by the treasuries and r is a deterministic rate of return.

3.1.3 Reserves

Bank reserves are the deposits held in accounts with a national agency (e.g., the federal reserve for banks) plus money that is physically held by banks (vault cash). Such reserves are constituted by money that is not lent out but is earmarked to cater for withdrawals by depositors. Since it is uncommon for depositors to withdraw all of their funds simultaneously, only a portion of total deposits will be needed as reserves. The bank uses the remaining deposits to earn profit either by issuing loans or by investing in assets such as treasuries and stocks.

3.2 Liabilities

Liabilities constitute the sources of the funds for banks. These funds are used to purchase income-earning assets. The dynamics of the bank's liabilities is stochastic because its value has a reliance on, for instance, deposits that have randomness associated with them. In this regard, the bank item that we discuss under liabilities will only be deposits.

The majority of a bank's liabilities consists of retail deposits, which are fully insured by a deposit insurance fund (DIF). In our study the term deposits include both chequeable and nontransaction deposits. Deposits, $D : \Omega \times T \to \mathbb{R}_+$, can be modelled as a stochastic process because there is a great deal of randomness associated with them. The deposits, D, and the reserves, R_v, can be related by (see, page 207 of chapter 9 in Mishkin, 2004): $R_v(t) = \alpha\, D(t)$, where α is a real-valued constant.

3.3 Bank Capital

Bank capital is raised by selling new equity, retaining earnings, issuing debt or building up loan-loss reserves. The dynamics of bank capital is stochastic in

nature because it depends in part on the uncertainty related to debt and shareholder contributions. In theory, the bank can decide on the rate at which debt and equity is raised. The underlying principle governing this decision is that the level of capitalization of the bank has to be taken into account. Roughly speaking, the rate at which debt and equity is raised can be reduced during times when the bank is adequately capitalized and should be increased when the bank is under capitalized. When using the Basel II risk-based approach to assets, RWAs are defined by placing each on- and off-balance item into a risk category with prescribed risk weight. In this regard, the riskier the asset the higher the risk-weight. In our case, on- and off-balance sheet assets are allocated to five categories each with a different weight. The first category carries a 0% weight and includes items that have little default risk, such as reserves and government securities. Category 2 has a 20% weight and includes claims on banks. Category 3 carries a weight of 50% and includes municipal bonds and residential mortgages. Category 4 has the maximum weight of 100% and includes loans to customers and corporations. Off-balance sheet items form the fifth category and are treated in a similar manner by assigning a credit-equivalent percentage that converts them to on-balance sheet items to which appropriate risk weight applies. The main constituents of this category are intangible assets that carry a risk weight of 100% and are used in determining the value of Tier 1. Bank's capital, C, has the form

$$C(t) = C_{T1}(t) + C_{T2}(t) + C_{T3}(t) \tag{3.5}$$

where C_{T1}, C_{T2}, C_{T3} are Tier 1, Tier 2 and Tier 3 capital, respectively.

3.3.1 Tier 1 Capital

Tier 1 capital is the book value of the bank's stock or equity held by shareholders plus retained earnings. It is always available and acts as a buffer against losses without a bank being required to cease trading. Also, the amount of Tier 1 capital affects returns for shareholders in the bank while a minimum amount of such is required by regulatory authorities.

3.3.2 Tier 2 and 3 Capital

Tier 2 and Tier 3 capital, collectively known as supplementary capital, is the sum of loan-loss reserves and subordinate debt held by debt holders. Tier 2 capital includes unaudited retained earnings; revaluations reserves; general

provisions for bad debts; perpetual cumulative preference shares (i.e., preference shares with no maturity date whose dividends accrue for future payment even if the bank's financial condition does not support immediate payment) and perpetual subordinated debt (i.e., debt with no maturity date which ranks in priority behind all creditors except shareholders). Tier 2 can absorb losses in the event of a wind-up and so provides a lesser degree of protection to depositors, e.g., long term subordinated debt. Tier 3 capital consists of subordinated debt with a term of at least 5 years and redeemable preference shares which may not be redeemed for at least 5 years. Tier 3 capital can be used to provide a hedge against losses caused by market risks if Tier 1 and Tier 2 capital are insufficient for this.

3.3.3 Dynamics of total Bank Capital

The dynamics of the bank capital can be represented as a diffusion process (see, [9]) in the form

$$dC(t) = b(t)dt + \sigma dZ(t), \qquad C_{t0} = C(t_0) \qquad (3.6)$$

where b is the bank capital contribution rate and $C(t_0) = C_0$. In reality, b may depend on such factors as profit flow, asset substitution and transaction costs. Subsequently, we assume that b is a measurable adapted process with respect to the filtration $\{\mathcal{F}_t\}$ that satisfies

$$\int_0^\infty |b(s)|ds < \infty, \qquad a.s. \qquad (3.7)$$

Chapter 4

The Mean-Variance Optimization Program

In order for a bank to determine an optimal rate at which additional debt and equity should be raised and a strategy for allocation of equity, it is imperative that a well-defined objective function with appropriate constraints is considered. In the sections that follow, we shall be studying the choice of an optimal investment portfolio based mostly on that in reference [14]. In this regard, we consider a financial market that consists of two assets, namely: treasuries and loans. Suppose the loans (risky asset) follow the dynamics shown in (3.3) and the treasuries (risk-free asset) follow the dynamics shown in (3.4). Suppose the bank also continuously pays some of its capital in-flow from equities into a portfolio at a constant contribution rate c, paid at a unit rate. Our objective, is to maximize the mean terminal wealth, and at the same time to minimize the variance of the terminal wealth.

4.1 Stochastic dynamics of the asset portfolio

Suppose that the bank has an initial wealth $x_0 > 0$ and the total wealth of the bank position at time $t \geq 0$ is $X(t)$. The proportions of portfolio invested in loans and treasuries at time t are denoted by $y(t)$ and $(1 - y(t))$, respectively. By making use of (3.3) and (3.4), we find that the dynamics of the wealth at

time t may be represented by the stochastic differential equation (SDE):

$$
\begin{aligned}
\frac{dX(t)}{X(t)} &= y(t)\frac{dL(t)}{L(t)} + (1 - y(t))\frac{dy_0(t)}{y_0(t)} \\
&= y(t)\left[\lambda dt + \sigma dW(t)\right] + (1 - y(t))\, r dt \\
&= y(t)\lambda dt + y(t)\sigma dW(t) + r dt - y(t) r dt \\
&= \left[y(t)\left(\lambda - r\right) + r\right] dt + y(t)\sigma dW(t)
\end{aligned} \tag{4.1}
$$

that is,
$$
dX(t) = X(t)\left[y(t)(\lambda - r) + r\right] dt + X(t)y(t)\sigma dW(t). \tag{4.2}
$$

Now we assume that the bank pays some of its capital in-flow from equities into a portfolio at a constant rate c. Then

$$
dX(t) = X(t)\left[y(t)(\lambda - r) + r\right] dt + X(t)y(t)\sigma dW(t) + c dt. \tag{4.3}
$$

Rearranging in (4.3), we then obtain that at time t the wealth, $X(t)$, grows according to the following SDE:

$$
\begin{cases}
dX(t) = \left\{X(t)\left[y(t)(\lambda - r) + r\right] + c\right\} dt + X(t)y(t)\sigma dW(t) \\
X(0) = x_0 \geq 0.
\end{cases} \tag{4.4}
$$

Recall that $W(t)$ is a standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, with $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$.

Table 4.1: Parameter values for the Wealth process $X(t)$

r	λ	σ	c	y	X	dt	T(years)
0.02	0.08	0.05	0.01	0.01	10	0.5	20
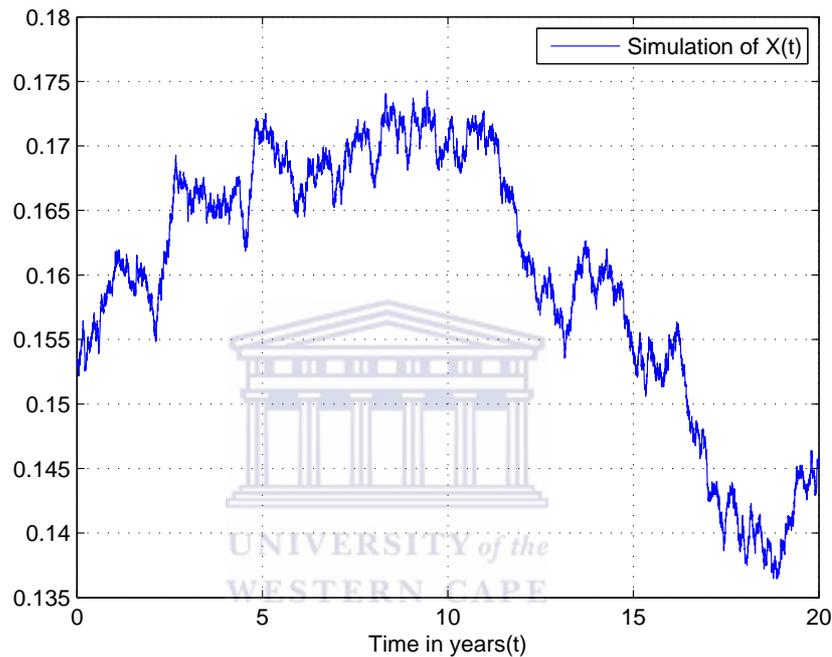
Figure 4.1: A simulation of the behaviour of the wealth process $X(t)$.

Figure 4.1 shows the graph of the wealth process $X(t)$ following the Brownian motion path over a period of 20 years.

4.2 Formulation of Optimal Asset Allocation Problem for Banks

Shareholders of a commercial bank expect a good return on their capital investment while minimizing their risk. In order to maximize return per risk, bank management needs to strategically allocate the shareholder's equity. In order to maximize shareholder return versus risk, the mean terminal

wealth, $E(X(T))$, will be maximized; and the variance of the terminal wealth, $\text{Var}(X(T))$, will be minimized. That is, the commercial bank seeks to, in some sense, minimize the vector $[-E(X(T)), \text{Var}(X(T))]$.

Definition 4.2.1
An investment strategy $u(.)$ is said to be admissible if $u(.) \in L_{\mathcal{F}}^2(0, T; \mathbb{R})$.

Mean-Variance Problem
The mean-variance optimization problem is mathematically defined as, (see [14], [17], [18]),

$$\text{Minimize} \left(J_1(y(.)), J_2(y(.)) \right) \equiv (-E(X(T)), \text{Var}(X(T))) \qquad (4.5)$$

$$\text{subject to} \begin{cases} y(.) \text{ admissible} \\ X(.), y(.) \text{ satisfy (4.4)}. \end{cases}$$

Our objective is to identify efficient portfolios along with the efficient frontier. An efficient portfolio is one where there exists no other portfolio better than it with respect to both the mean and variance criteria ([14], [17], [18]). By standard multi-objective optimization theory, an efficient portfolio can be found by solving a single-objective optimization problem where the objective is a weighted average of the two original criteria under certain convexity condition (see e.g., [28]), which are satisfied in the present case. The efficient frontier can then be generated by varying the weights. Therefore, problem (4.5) can be solved via the following optimal control problem

$$\text{Minimize} \qquad J_1(y(.)) + \alpha J_2(y(.)) \equiv -E(X(T)) + \alpha \text{Var}(X(T))$$

$$\text{subject to} \begin{cases} y(.) \text{ admissible} \\ X(.), y(.) \text{ satisfying (4.4)} \end{cases} \qquad (4.6)$$

where the parameter (representing the weight) $\alpha \geq 0$. Denote problem (4.6) above by $P(\alpha)$.
Define

$$\Pi_{P(\alpha)} = \{ y(.) \mid y(.) \text{ is an optimal control of P}(\alpha) \}. \qquad (4.7)$$

4.3 Construction of Optimal Asset Allocation Problem for Banks

Note that problem $P(\alpha)$ is not a standard stochastic optimal control problem and it is hard to solve due to the term $[EX(T)]^2$ in its objective function,

which is nonseparable in the sense of dynamic programming. As in [17], we propose to embed problem $P(\alpha)$ into a tractable auxiliary problem that turns out to be a stochastic LQ problem. To do this, set the following problem:

$$\text{Minimize} \left(J\left(y(.) \right), \alpha, \beta \right) \equiv E\left[\alpha X^2(T) - \beta X(T) \right] \tag{4.8}$$

$$\text{subject to} \begin{cases} y(.) \text{ admissible} \\ X(.), y(.) \text{ satisfy (4.4)} \end{cases}$$

where $\alpha > 0$ and β is some other parameter. We call the above, problem $A(\alpha, \beta)$.
Define

$$\Pi_{A(\alpha,\beta)} = \left\{ y(.) \mid y(.) \text{ is an optimal control of A}(\alpha, \beta) \right\}. \tag{4.9}$$

The following result shows the relationship between problems $P(\alpha)$ and $A(\alpha, \beta)$.

Theorem 4.3.1[17] *For any $\alpha > 0$, one has*

$$\Pi_{P(\alpha)} \subseteq \bigcup_{-\infty < \beta < +\infty} \Pi_{A(\alpha,\beta)}. \tag{4.10}$$

Moreover, if $\overline{y}(.) \in \Pi_{P(\alpha)}$, then $\overline{y}(.) \in \Pi_{A(\alpha,\overline{\beta})}$ with $\overline{\beta} = 1 + 2\alpha E(\overline{X}(T))$, where $\overline{X}(.)$ is the corresponding wealth trajectory.

The proof is on page 24 of paper [17], and in detail. We therefore omit it.

4.4   Optimal Bank Asset Allocation

In this section, we want to find the solution to problem $A(\alpha, \beta)$ introduced in section (4.3). To do this, like in [17] we set:

$$\gamma = \frac{\beta}{2\alpha} \qquad \text{and} \qquad Z(t) = X(t) - \gamma$$

According to [14] and [17], with these settings, problem $A(\alpha, \beta)$ is equivalent to minimizing

$$E\left[\frac{1}{2}\alpha Z(T)^2 \right] \tag{4.11}$$

subject to

$$\begin{cases} dZ(t) = \left\{ (Z(t) + \gamma) \left[y(t) \left(\lambda - r \right) + r \right] + c \right\} dt + (Z(t) + \gamma) \sigma y(t) dW(t) \\ Z(0) = x_0 - \gamma. \end{cases}$$

$$\tag{4.12}$$

The class of admissible controls associated with (4.11) is the set $\Psi[0,T] = L^2_{\mathcal{F}}(0,T;\mathbb{R})$. L^2 is the set of square-integrable functions; see [34] for a more detailed discussion on this point. Given $y(.) \in \Psi[0,T]$, the pair $(X(.),y(.))$ is referred to as an *admissible pair* if $X(.) \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ is a solution of the stochastic differential equation (4.12) associated with $y(.) \in \Psi[0,T]$. In this regard, our task is to find an optimal $y(.)$ that minimizes the quadratic (terminal) cost function

$$J(y(.);\alpha) = E\left[\frac{1}{2}\alpha Z(T)^2\right]. \tag{4.13}$$

To this end, let us define the value function associated with the LQ problem $(4.12)-(4.13)$ by

$$V(t,z) = \inf_{y(.)} J\left(y(.);\alpha\right) = \inf_{y(.)} E_{t,z}\left[\frac{1}{2}\alpha Z(T)^2\right] \tag{4.14}$$

4.4.1 Value Function and Optimal Control

In order to find an optimal control $y(.)$ that minimizes the quadratic cost function in (4.13), we state and prove the Theorems below.

Theorem 4.4.1
Suppose that (4.12) and (4.14) hold. Then the optimal fraction of portfolio to be invested in loans at time t is given by

$$\overline{y}(t,z) = -\frac{(\lambda - r)}{\sigma^2}\frac{1}{(z+\gamma)}\frac{V_z}{V_{zz}} \tag{4.15}$$

Proof. The value function V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} \inf_{y\in\mathbb{R}}\left\{\frac{\partial V}{\partial t} + [(z+\gamma)(y(\lambda-r)+r)+c]\frac{\partial V}{\partial z} + \frac{1}{2}(z+\gamma)^2\sigma^2 y^2\frac{\partial^2 V}{\partial z^2}\right\} = 0 \\ V(T,z) = \frac{1}{2}\alpha z^2. \end{cases}$$
$$\tag{4.16}$$

Now note that $\frac{\partial V}{\partial t}$ is not a function of y. Consequently we can rewrite the HJB equation (4.16) as follows,

$$\frac{\partial V}{\partial t} + \inf_y H(y) = 0 \tag{4.17}$$

where $H(y)$ is defined by

$$H(y) = [(z+\gamma)(y(\lambda-r)+r)+c]V_z + \frac{1}{2}(z+\gamma)^2\sigma^2 y^2 V_{zz}. \tag{4.18}$$

Let us assume that $\frac{\partial^2 V}{\partial z^2} \geq 0$. Then, our stochastic control problem reduces to the minimization problem

$$\inf_y H(y) = [(z+\gamma)(y(\lambda-r)+r)+c]V_z + \frac{1}{2}(z+\gamma)^2\sigma^2 y^2 V_{zz}. \qquad (4.19)$$

If we differentiate (4.18) with respect to y and set it equal to zero we find an optimal fraction of portfolio to be invested in the loans, $\overline{y}(t,z)$, at time t:

$$(z+\gamma)(\lambda-r)V_z + (z+\gamma)^2\sigma^2 y V_{zz} = 0 \qquad (4.20)$$

implying,

$$\overline{y}(t,z) = -\frac{(\lambda-r)}{\sigma^2}\frac{1}{(z+\gamma)}\frac{V_z}{V_{zz}}. \qquad (4.21)$$

$\square$

Next, we substitute the optimal control (4.21) in the HJB equation (4.16). This results in the following non-linear PDE for the value function V:

$$V_t + [(z+\gamma)r + c]\,V_z - \frac{1}{2}\delta^2\frac{V_z^2}{V_{zz}} = 0 \qquad (4.22)$$

where we set δ to be

$$\delta = \frac{(\lambda-r)}{\sigma}. \qquad (4.23)$$

We get the solution of (4.22) via the introduction of the theorem below.

Theorem 4.4.2
If $A(t)$, $B(t)$ and $C(t)$ are functions for which the following conditions are satisfied, then

$$V(t,z) = A(t)z^2 + B(t)z + C(t) \qquad (4.24)$$

is a solution of (4.22). The said conditions are:

$$\begin{cases} A'(t) = (\delta^2 - 2r)\,A(t), \\ B'(t) = (\delta^2 - r)\,B(t) - 2\,(\gamma r + c)\,A(t) \\ C'(t) = \frac{\delta^2 B(t)^2}{4A(t)} - (\gamma r + c)\,B(t) \end{cases} \qquad (4.25)$$

with boundary conditions

$$A(T) = \frac{1}{2}\alpha \qquad B(T) = 0 \qquad C(T) = 0. \qquad (4.26)$$

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We prove Theorem 4.4.2 via proving the Propositions 4.4.1 to 4.4.3 below.

Proposition 4.4.1
The function

$$A(t) = \frac{1}{2}\alpha e^{-(\delta^2 - 2r)(T-t)} \tag{4.27}$$

is a solution of equation

$$A'(t) = \left(\delta^2 - 2r\right) A(t). \tag{4.28}$$

Proof. We can write (4.28) in the form

$$\frac{A'(t)}{A(t)} = (\delta^2 - 2r). \tag{4.29}$$

Integrating from T to t in (4.29) we obtain:

$$\ln\frac{A(t)}{A(T)} = (\delta^2 - 2r)(t - T) = -(\delta^2 - 2r)(T - t) \tag{4.30}$$

or

$$A(t) = A(T)e^{-(\delta^2 - 2r)(T-t)}. \tag{4.31}$$

Now substituting $A(T)$ from (4.26) in (4.31), yields:

$$A(t) = \frac{1}{2}\alpha e^{-(\delta^2 - 2r)(T-t)}. \tag{4.32}$$

$\square$

Proposition 4.4.2
The function

$$B(t) = \frac{\alpha\left(\gamma r + c\right)}{r} e^{-\left(\delta^2 - 2r\right)(T-t)} \left[1 - e^{-r(T-t)}\right] \tag{4.33}$$

is a solution of equation

$$B'(t) = \left(\delta^2 - r\right) B(t) - 2\left(\gamma r + c\right) A(t). \tag{4.34}$$

Proof. Equation (4.34) is a first order linear differential equation. If we substitute $A(t)$ from (4.26) in (4.34), we can then write equation (4.34) in standard form as follows:

$$B'(t) - \left(\delta^2 - r\right) B(t) = -\alpha\left(\gamma r + c\right) e^{-(\delta^2 - 2r)(T-t)}. \tag{4.35}$$

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Multiplying both sides of (4.35) by the integrating factor

$$e^{-(\delta^2 - r)\int_T^t dt} = e^{-(\delta^2 - r)(t-T)} = e^{(\delta^2 - r)(T-t)}, \tag{4.36}$$

we get

$$B'(t)e^{(\delta^2 - r)(T-t)} - \left(\delta^2 - r\right) B(t)e^{(\delta^2 - r)(T-t)} = -\alpha \left(\gamma r + c\right) e^{-(\delta^2 - 2r)(T-t)}e^{(\delta^2 - r)(T-t)} \tag{4.37}$$

which can also be written as

$$\frac{d[B(t)e^{(\delta^2 - r)(T-t)}]}{dt} = -\alpha \left(\gamma r + c\right) e^{r(T-t)}. \tag{4.38}$$

Integrating from T to t and substituting $B(T)$ from (4.26), we get

$$\frac{B(t)}{e^{-(\delta^2 - r)(T-t)}} = -\alpha \left(\gamma r + c\right) \int_T^t e^{r(T-t)}dt \tag{4.39}$$

which implies that,

$$
\begin{aligned}
B(t) &= -\alpha \left(\gamma r + c\right) e^{-(\delta^2 - r)(T-t)} \left[-\frac{1}{r}e^{r(T-t)}\right]_T^t \\
&= \frac{\alpha \left(\gamma r + c\right)}{r}e^{-\left(\delta^2 - r\right)(T-t)} \left[e^{r(T-t)} - 1\right] \\
&= \frac{\alpha \left(\gamma r + c\right)}{r}e^{-\left(\delta^2 - r\right)(T-t)}e^{r(T-t)} \left[1 - \frac{1}{e^{r(T-t)}}\right] \\
&= \frac{\alpha \left(\gamma r + c\right)}{r}e^{-\left(\delta^2 - 2r\right)(T-t)} \left[1 - e^{-r(T-t)}\right].
\end{aligned}
\tag{4.40}
$$

$\square$

Proposition 4.4.3
The function
$$C(t) = \int_T^t \left[\frac{\delta^2 B(s)^2}{4A(s)} - \left(\gamma r + c\right) B(s)\right] ds \tag{4.41}$$
is a solution of equation
$$C'(t) = \frac{\delta^2 B(t)^2}{4A(t)} - \left(\gamma r + c\right) B(t). \tag{4.42}$$

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Proof. If we integrate (4.42) from T to t, we get:

$$C(t) - C(T) = \int_T^t \frac{\delta^2 B(s)^2}{4A(s)} ds - (\gamma r + c) \int_T^t B(s) ds \qquad (4.43)$$

Substituting $C(T)$ from (4.26), we result with:

$$C(t) = \int_T^t \left[\frac{\delta^2 B(s)^2}{4A(s)} - (\gamma r + c) B(s) \right] ds. \qquad (4.44)$$

$\square$

We are now in a position to present the optimal investment strategy for problem $A(\alpha, \beta)$. Similar to the presentation of Hojgaard and Vigna in [14] we deduce the following Theorem.

Theorem 4.4.3
An optimal investment strategy of problem $A(\alpha, \beta)$ is given by

$$\overline{y}(t, x) = -\frac{\lambda - r}{\sigma^2 x} \left[x - \gamma e^{-r(T-t)} + \frac{c}{r} \left(1 - e^{-r(T-t)} \right) \right]. \qquad (4.45)$$

Proof. Making use of the results established in Theorems 4.4.1, 4.4.2 and replacing the partial derivatives of V in (4.15) and also $(z + \gamma)$ with x we get an optimal investment strategy for problem $A(\alpha, \beta)$ as

$$\overline{y}(t, x) = -\frac{\lambda - r}{\sigma^2 x} \left[x - \gamma e^{-r(T-t)} + \frac{c}{r} \left(1 - e^{-r(T-t)} \right) \right]. \qquad (4.46)$$

$\square$

4.5 Efficient Strategy and Efficient Frontier

In this section we derive the efficient frontier for the portfolio selection problem (4.5). That is, we specify the relationship between the variance and the expected value of the terminal wealth. The following observations (Proposition 4.5.1 and Proposition 4.5.2) are crystallized from the paper of Hojgaard and Vigna [14].

Proposition 4.5.1
The expected terminal wealth, $\mathrm{E}(\overline{X}(T))$, is given by:

$$E(\overline{X}(T)) = x_0 e^{rT} + c \frac{e^{rT} - 1}{r} + \frac{e^{\delta^2 T} - 1}{2\alpha}. \qquad (4.47)$$

Proof. Under the optimal control (4.45), the wealth equation (4.4) evolves according to the following SDE:

$$
\begin{aligned}
d\overline{X}(t) &= \left[(r - \delta^2)\overline{X}(t) + e^{-r(T-t)} \left(\delta^2 \gamma + \frac{\delta^2 c}{r} \right) + \left(c - \frac{\delta^2 c}{r} \right) \right] dt \\
&\quad + \left[-\delta \overline{X}(t) + e^{-r(T-t)} \left(\delta \gamma + \frac{\delta c}{r} \right) - \frac{\delta c}{r} \right] dW(t).
\end{aligned}
\tag{4.48}
$$

If we take the expectation on both sides of (4.48), then $E(\overline{X}(t))$ satisfy the following non-homogeneous linear ordinary differential equation:

$$
\begin{cases}
dE(\overline{X}(t)) = \left[(r - \delta^2)E(\overline{X}(t)) + e^{-r(T-t)}\delta^2(\gamma + \frac{c}{r}) + (c - \frac{\delta^2 c}{r}) \right] dt \\
E\left(\overline{X}(0)\right) = x_0.
\end{cases}
\tag{4.49}
$$

When we solve (4.49), we obtain the expected value of the wealth under optimal control at time t as

$$
E(\overline{X}(t)) = \left(x_0 + \frac{c}{r} \right) e^{-(\delta^2-r)t} + \left(\gamma + \frac{c}{r} \right) e^{-r(T-t)} - \left(\gamma + \frac{c}{r} \right) e^{-r(T-t)-\delta^2 t} - \frac{c}{r}.
\tag{4.50}
$$

At terminal time T, (4.50) reduces to:

$$
E(\overline{X}(T)) = (x_0 + \frac{c}{r})e^{-(\delta^2-r)T} + \gamma(1 - e^{-\delta^2 T}) - \frac{c}{r}e^{-\delta^2 T}.
\tag{4.51}
$$

By Theorem 4.3.1, an optimal solution of problem $P(\alpha)$, if it exists, can be found by selecting $\overline{\beta}$ so that

$$
\begin{aligned}
\overline{\beta} &= 1 + 2\alpha E(\overline{X}(T)) \qquad \text{(using (4.51))} \\
&= 1 + 2\alpha \left((x_0 + \frac{c}{r})e^{-(\delta^2-r)T} + \gamma(1 - e^{-\delta^2 T}) - \frac{c}{r}e^{-\delta^2 T} \right).
\end{aligned}
\tag{4.52}
$$

Rearranging the terms in (4.52), and using the fact that $\gamma = \frac{\overline{\beta}}{2\alpha}$, we obtain γ as a decreasing function of α:

$$
\gamma = \frac{e^{\delta^2 T}}{2\alpha} + x_0 e^{rT} + \frac{c}{r}(e^{rT} - 1).
\tag{4.53}
$$

Using (4.53) we then write the expected optimal final wealth in (4.51) in terms of α as follows:

$$
E(\overline{X}(T)) = x_0 e^{rT} + c\frac{e^{rT} - 1}{r} + \frac{e^{\delta^2 T} - 1}{2\alpha}.
\tag{4.54}
$$

$\square$

Remark 4.5.1

1. *The Sharpe ratio of the loans, δ, is directly proportional to the expected optimal final wealth, $E(\overline{X}(T))$. That is, if the Sharpe ratio of the loans, δ, is high then the expected optimal final wealth will also be high; but if it is low then the expected optimal final wealth will also be low.*

2. *The minimization of the variance of the terminal wealth, α, is inversely proportional to the expected optimal final wealth, $E(\overline{X}(T))$. That is, if the importance given to the minimization of the variance of the final wealth, α, is high then the mean will be low; but if it is low then the mean will be high.*

Proposition 4.5.2
The expected square of the terminal wealth, $\mathrm{E}(\overline{X}^2(T))$, is given by:

$$
\begin{aligned}
E(\overline{X}^2(T)) &= (x_0 + \frac{c}{r})^2 e^{-(\delta^2 - 2r)T} + \gamma^2(1 - e^{-\delta^2 T}) - \frac{2c}{r}(x_0 + \frac{c}{r})e^{-(\delta^2 - r)T} \\
&+ \frac{c^2}{r^2}e^{-\delta^2 T}.
\end{aligned}
\tag{4.55}
$$

Proof. Applying Ito's lemma to (4.48), we obtain that $\overline{X}^2(t)$ evolves according to the following SDE:

$$
\begin{aligned}
d\overline{X}^2(t) &= \left[(2r - \delta^2)\overline{X}^2(t) + 2c\overline{X}(t) + \delta^2\left((\gamma + \frac{c}{r})e^{-r(T-t)} - \frac{c}{r}\right)^2\right] dt \\
&- 2\delta\left\{\overline{X}^2(t) - \left[(\gamma + \frac{c}{r})e^{-r(T-t)} - \frac{c}{r}\right]\overline{X}(t) + \frac{c}{r}\right\} dW(t). \quad (4.56)
\end{aligned}
$$

Taking the expectation on both sides of (4.56), we then obtain that $E(\overline{X}^2(t))$ satisfy the following non-homogeneous linear ordinary differential equation:

$$
\begin{cases}
dE(\overline{X}^2(t)) = \left[(2r - \delta^2)E(\overline{X}^2(t)) + 2cE(\overline{X}(t)) + \delta^2\left((\gamma + \frac{c}{r})e^{-r(T-t)} - \frac{c}{r}\right)^2\right] dt \\
E(\overline{X}^2(0)) = x_0^2.
\end{cases}
\tag{4.57}
$$

When we solve (4.57), we obtain the expected value of the square of the wealth under optimal control at time t as

$$
\begin{aligned}
E(\overline{X}^2(t)) &= \left(x_0 + \frac{c}{r}\right)^2 e^{-(\delta^2 - 2r)t} - (\gamma + \frac{c}{r})^2 e^{-2r(T-t) - \delta^2 t} - \frac{2c}{r}(\gamma + \frac{c}{r})e^{-r(T-t)} \\
&+ \frac{2c}{r}(\gamma + \frac{c}{r})e^{-r(T-t) - \delta^2 t} - \frac{2c}{r}(x_0 + \frac{c}{r})e^{-(\delta^2 - r)t} + (\gamma + \frac{c}{r})^2 e^{-2r(T-t)} \\
&+ \frac{c^2}{r^2}.
\end{aligned}
\tag{4.58}
$$

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At terminal time T, (4.58) reduces to:

$$E(\overline{X}^2(T)) = (x_0 + \frac{c}{r})^2 e^{-(\delta^2 - 2r)T} + \gamma^2(1 - e^{-\delta^2 T}) - \frac{2c}{r}(x_0 + \frac{c}{r})e^{-(\delta^2 - r)T}$$
$$+ \frac{c^2}{r^2}e^{-\delta^2 T}. \tag{4.59}$$

$\square$

We are now in a position to present the optimal investment strategy for problem $P(\alpha)$. Similar to the presentation of Hojgaard and Vigna in [14] we deduce the following Theorem.

Theorem 4.5.1

An optimal investment strategy of problem $P(\alpha)$ is given by:

$$\overline{y}(t,x) = -\frac{\lambda - r}{\sigma^2 x}\left[x - \left(E[\overline{X}(T)]e^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)})\right) - \frac{e^{-r(T-t)}}{2\alpha}\right]. \tag{4.60}$$

Proof. By Theorem 4.4.3 and Proposition 4.5.1, we obtain (4.60) immediately.

$\square$

The following result is a version of [14]. The proof that we include contains much more detail than that in [14].

Proposition 4.5.3

The variance of the terminal wealth, $\mathrm{Var}(\overline{X}(T))$, is given by:

$$\mathrm{Var}(\overline{X}(T)) = \frac{e^{\delta^2 T} - 1}{4\alpha^2}. \tag{4.61}$$

Proof. Let us start by introducing the following notation:

$$\begin{cases} w_0 \equiv x_0 + c \\ \theta \equiv 1 - e^{-\delta^2 T} \\ \rho \equiv e^{-(\delta^2 - r)T} \\ \phi \equiv e^{-(\delta^2 - 2r)T}. \end{cases} \tag{4.62}$$

With this notation we can express $E(\overline{X}(T))$ in (4.51) and $E(\overline{X}^2(T))$ in (4.55) as explicit functions of γ as follows:

$$E(\overline{X}(T)) = w_0\rho - \frac{c}{r}(1 - \theta) + \gamma\theta \tag{4.63}$$

29

and

$$E(\overline{X}^2(T)) = w_0^2\phi - 2\frac{c}{r}w_0\rho + \frac{c^2}{r^2}(1-\theta) + \gamma^2\theta. \tag{4.64}$$

Then we have

$$
\begin{aligned}
\mathrm{Var}(\overline{X}(T)) &= E(\overline{X}^2(T)) - E\left(\overline{X}(T)\right)^2 \\
&= (w_0^2\phi - 2\frac{c}{r}w_0\rho + \frac{c^2}{r^2}(1-\theta) + \gamma^2\theta) - (w_0\rho - \frac{c}{r}(1-\theta) + \gamma\theta)^2 \\
&= (w_0^2\phi - 2\frac{c}{r}w_0\rho + \frac{c^2}{r^2}(1-\theta) + \gamma^2\theta) - ((w_0\rho + \gamma\theta) - \frac{c}{r}(1-\theta))^2 \\
&= (w_0^2\phi - 2\frac{c}{r}w_0\rho + \frac{c^2}{r^2}(1-\theta) + \gamma^2\theta) - ((w_0\rho + \gamma\theta)^2 \\
&\quad + 2((w_0\rho + \gamma\theta)(-\frac{c}{r}(1-\theta)) + \frac{c^2}{r^2}(1-\theta)^2) \tag{4.65} \\
&= (w_0^2\phi - 2\frac{c}{r}w_0\rho + \frac{c^2}{r^2}(1-\theta) + \gamma^2\theta) - (w_0^2\rho^2 + 2w_0\rho\gamma\theta + \gamma^2\theta^2 \\
&\quad + 2\frac{c}{r}\theta w_0\rho - 2\frac{c}{r}w_0\rho + 2\frac{c}{r}\gamma\theta^2 - 2\frac{c}{r}\gamma\theta + \frac{c^2}{r^2} - 2\frac{c^2}{r^2}\theta + \frac{c^2\theta^2}{r^2}) \\
&= w_0^2\phi - 2\frac{c}{r}w_0\rho + \frac{c^2}{r^2}(1-\theta) + \gamma^2\theta - w_0^2\rho^2 - \frac{c^2}{r^2}(1-\theta^2) \\
&\quad - \gamma^2\theta^2 + 2w_0\rho\frac{c}{r}(1-\theta) - 2w_0\rho\gamma\theta + 2\frac{c}{r}(1-\theta)\gamma\theta.
\end{aligned}
$$

As time passes, using the fact that $\theta - \rho^2 = \phi\theta$, we have

$$\mathrm{Var}(\overline{X}(T)) = w_0^2\theta\phi + \theta(1-\theta)\left(\gamma + \frac{c}{r}\right)^2 - 2w_0\rho\theta\left(\gamma + \frac{c}{r}\right). \tag{4.66}$$

In (4.63), we can rearrange the terms in this form:

$$\theta\left(\gamma + \frac{c}{r}\right) = E(\overline{X}(T)) - w_0\rho + \frac{c}{r}. \tag{4.67}$$

Substituting (4.67) in (4.66), we get:

$$
\begin{aligned}
\mathrm{Var}(\overline{X}(T)) &= w_0^2\theta\phi + \theta(1-\theta)\frac{(E(\overline{X}(T)) - w_0\rho + \frac{c}{r})^2}{\theta^2} - 2w_0\rho(E(\overline{X}(T)) - w_0\rho + \frac{c}{r}) \\
&= \frac{1-\theta}{\theta}\left[\frac{w_0^2\phi\theta^2}{1-\theta} + (E(\overline{X}(T)) - w_0\rho + \frac{c}{r})^2 - \frac{2w_0\rho\theta}{1-\theta}(E(\overline{X}(T)) - w_0\rho + \frac{c}{r})\right] \\
&= \frac{1-\theta}{\theta}[\frac{\phi\theta^2 + \rho^2 + \rho^2\theta}{1-\theta}w_0^2 + 2E(\overline{X}(T))\frac{c}{r} + \frac{c^2}{r^2} + E(\overline{X}(T))^2 \\
&\quad - 2w_0\frac{\rho}{1-\theta}(E(\overline{X}(T)) + \frac{c}{r})]. \tag{4.68}
\end{aligned}
$$

Note that the term $\frac{\phi\theta^2+\rho^2+\rho^2\theta}{1-\theta}$ in (4.68) is equal to e^{2rT} and $\frac{\rho}{1-\theta}$ is equal to e^{rT}. Substituting these terms in (4.68), we obtain:

$$
\begin{aligned}
\operatorname{Var}(\overline{X}(T)) &= \frac{1-\theta}{\theta}\left[w_0^2 e^{2rT} + (E(\overline{X}(T)) + \frac{c}{r})^2 - 2w_0 e^{rT}(E(\overline{X}(T)) + \frac{c}{r})\right] \\
&= \frac{1-\theta}{\theta}\left[(E(\overline{X}(T)) + \frac{c}{r}) - w_0 e^{rT}\right]^2 \qquad (4.69) \\
&= \frac{e^{-\delta^2 T}}{1-e^{-\delta^2 T}}\left[E(\overline{X}(T)) - \left(x_0 e^{rT} + c\frac{e^{rT}-1}{r}\right)\right]^2
\end{aligned}
$$

where in the last equality we have used (4.62). Applying (4.47) in the expression above results in an expression that consists of the variance of the final wealth being written in terms of α and δ as follows

$$
\operatorname{Var}(\overline{X}(T)) = \frac{e^{-\delta^2 T}}{1-e^{-\delta^2 T}}\left(\frac{e^{\delta^2 T}-1}{2\alpha}\right)^2 = \frac{e^{\delta^2 T}-1}{4\alpha^2}. \qquad (4.70)
$$

$\square$

The relation (4.69) reveals explicitly the tradeoff between the mean and variance. For example, if the bank has set an expected return level, then (4.69) tells us the risk that the bank has to take; and vice versa. In particular, if the bank does not want to take any risk, namely, $\operatorname{Var}(\overline{X}(T)) = 0$, then we see from (4.69) that $E(\overline{X}(T))$ has to be

$$
x_0 e^{rT} + c\frac{e^{rT}-1}{r} \qquad (4.71)
$$

meaning that the bank can only invest in treasuries. If we denote the standard deviation of the terminal wealth by $\sigma(\overline{X}(T))$, then (4.69) gives

$$
E(\overline{X}(T)) = x_0 e^{rT} + c\frac{e^{rT}-1}{r} + \sqrt{\frac{1-e^{-\delta^2 T}}{e^{-\delta^2 T}}}\sigma(\overline{X}(T)). \qquad (4.72)
$$

Remark 4.5.2
The efficient frontier in the mean-standard-deviation diagram is a straight line, which is also termed the capital market line (see [31], [32], Figure 4.2 and Figure 4.3 below). The slope of the line (4.72) is called the price of risk.

This is simply telling us that if the volatility of the final wealth increases by one unit then the mean of the final wealth will increase by a certain amount as well.

Table 4.2: Parameter values for an efficient frontier 1

| r	λ	σ	c	x_0	$T(\text{yrs})$
0.03	0.08	0.5	0.1	1.0	20
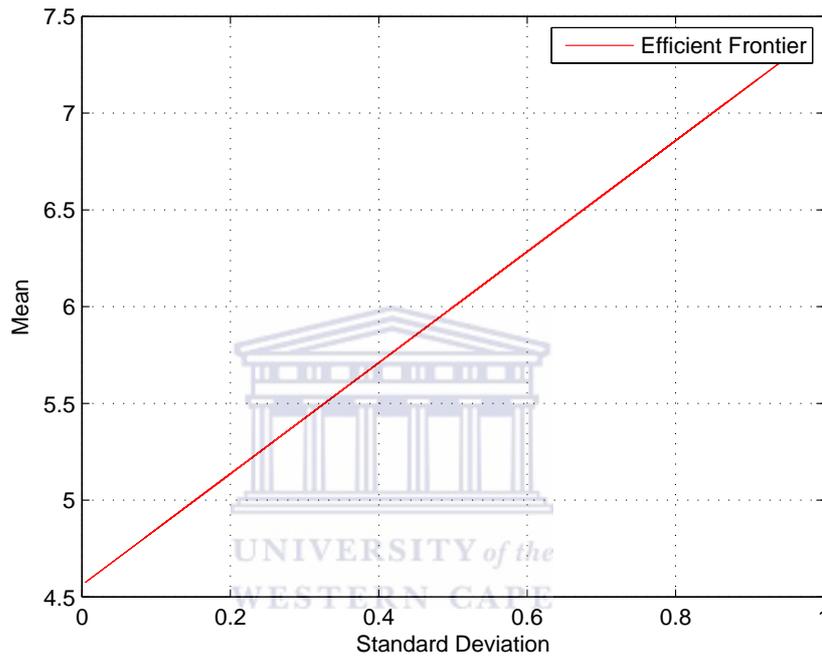
Figure 4.2: An Efficient Frontier 1

The price of risk using the parameters and Figure 4.2 above is obtained as 2.8684. That is, if the volatility of the final wealth increases by one unit then the mean of the final wealth will increase by 2.8684.

If we use different values for our parameters as shown in Table 4.3 below, we see from Figure 4.3 that the efficient frontier is still a straight line.

Table 4.3: Parameter values for an efficient frontier 2

| r	λ	σ	c	x_0	$T(\text{yrs})$
0.04	0.06	0.05	0.2	2.0	20
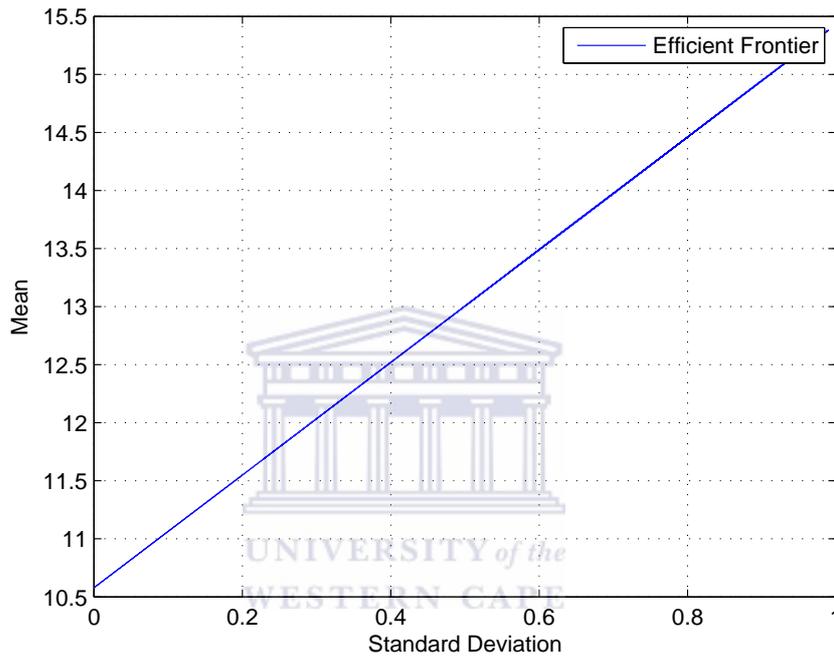
Figure 4.3: An Efficient Frontier 2

The price of risk using the parameters and Figure 4.3 above is obtained as 4.8510. That is, if the volatility of the final wealth increases by one unit then the mean of the final wealth will increase by 4.8510.

Chapter 5

The Explicit Formula for the Capital Adequacy Ratio

In this section, we state and prove the theorem for obtaining an explicit stochastic differential equation (SDE) for the Basel II Capital adequacy ratio (CAR) of a bank. But before we do that, we give a review of what the literature has to offer about the Basel Accords and the Capital Adequacy Ratio.

5.1 Capital Adequacy Ratio

The Basel Committee on Banking Supervision (BCBS) is a committee established in the mid-1980s in Basel, Switzerland to create common standards for banking regulations and to improve the stability of the banking system. The committee have publications which can be classified into 3 categories: research papers, consultative documents, and Accords. The purpose of the Accords is to lay out the rules to be followed by the national regulators in such matters as setting the minimum capital requirements.

In 1988, the Basel Committee on Banking Supervision published their first Accord, known as the 1988 Basel Accord or Basel I. The main aim of the Accord was to set common standards as to how banks should manage and regulate their capital requirements. The 1988 Basel Accord regarded capital requirements as the cornerstone of bank regulation. The Accord require that all banks should hold available capital equal to at least 8% of their risk-weighted asset (see chapter 23 Marrison [11], [24], [30]). However, the 1988 Basel Accord was

later criticised as being too crude and out of line with the evolving standards for managing and assessing bank performance. According to [29], most critics noted that the 1988 Accord treated all corporate credits alike and thereby were inviting regulatory arbitrage. In reacting to the criticisms, the BCBS made several adjustments to the 1988 Basel Accord and this led to the introduction of the new accord (known as Basel II) in the year 2004. Currently the banking regulation is embodied by the Basel II capital Accord which has been implemented globally from the year 2007. Basel II adopts a three-pillared approach with the ratio of the bank capital to risk-weighted assets, also known as capital adequacy ratio (CAR), playing a vital role as an index used to measure the strength of the bank. The capital adequacy ratio is a measure of the amount of a bank's capital relative to the amount of its credit exposures. In this study, we concentrate our efforts on the Basel II risk-based capital adequacy ratio (Basel II CAR) given by

$$CAR(R) = \frac{\text{Bank Capital(C)}}{\text{Total RWAs}(a_w)}. \tag{5.1}$$

In December 2010, the Basel Committee on Banking Supervision released a near final version of its new bank capital and liquidity standards, referred to as "Basel III". Basel III is a series of amendments to the existing Basel II framework. According to [36] and [37], the key elements of the Basel III framework include among others the following

A. Capital Ratios

1. Core solvency ratio retained at 8% of risk weighted assets.

2. Minimum "common equity" component will be 4.5% instead of the current 2% minimum.

3. Overall Tier 1 element of the capital base (including common equity) will be 6% instead of the current 4% minimum.

B. Constituents of Capital

1. The common equity component of Tier 1 will be comprised of the ordinary share capital and retained profits.

2. Tier 2 capital will no longer be divided into lower Tier 2 (principally, dated term preference shares and subordinated debt) and upper Tier

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2 (including certain perpetual preferred instruments and subordinated debt). Instead, all Tier 2 instruments will be required to be either convertible into common equity or written down in the event of the institution becoming non-viable without a bail-out.

3. Tier 3 capital will be abolished.

C. Leverage Ratio

1. A backstop 3% ratio of Tier 1 capital as against all of a bank's assets and certain off-balance sheet exposures will be introduced. The assets will be treated on a non-risk adjusted basis with limited or no recognition of collateralization or credit risk mitigation associated with assets. Effectively, this would amount to a leverage ratio of 33 : 1.

The diagrammatic overview of amendments of Basel II to Basel III by Shearman and Sterling$_{\text{LLP}}$, a financial institution advisory and financial regulatory, is shown below (see [37])
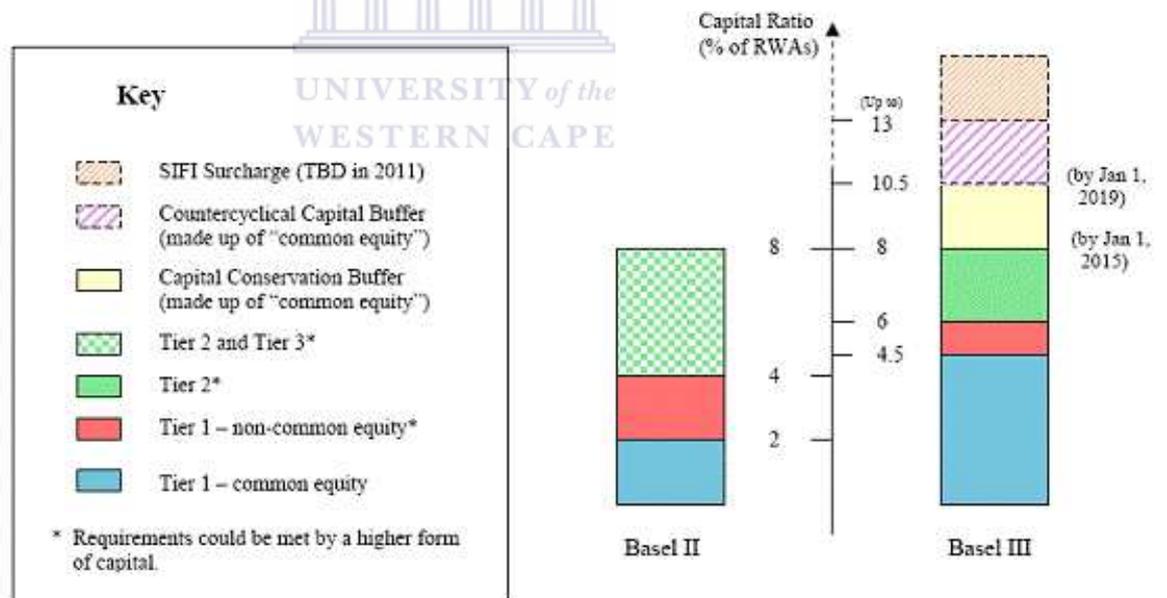

Figure 5.1: Diagrammatic overview of amendments of Basel II to Basel III.

5.2 Dynamics of Capital Adequacy Ratio

In this study we derive an explicit formula for the total risk-based capital adequacy ratio. To this end we first derive the dynamics for the total risk-weighted assets (TRWAs).

Proposition 5.2
Suppose that the dynamics of the treasuries and loans are as described in (3.3) and (3.4), respectively, and that we invest only in loans and treasuries according to (4.2). Then the dynamics for the TRWAs at time t, $a_w(t)$, is given by:

$$da_w(t) = a_w(t)[0.5\lambda y(t)dt + 0.5\sigma y(t)dW(t)]. \tag{5.2}$$

Proof. Using the risk weights in section 3.3 we have

$$
\begin{aligned}
\frac{da_w(t)}{a_w(t)} &= 0 \times (1 - y(t))\frac{dy_0(t)}{y_0(t)} + 0.5y(t)\frac{dL(t)}{L(t)} \\
&= 0.5y(t)\left[\lambda dt + \sigma dW(t)\right] \\
&= 0.5\lambda y(t)dt + 0.5\sigma y(t)dW(t)
\end{aligned}
$$

This implies,

$$da_w(t) = a_w(t)[0.5\lambda y(t)dt + 0.5\sigma y(t)dW(t)]. \tag{5.3}$$

$\square$

Theorem 5.2 (Explicit SDE for the Capital Adequacy Ratio of a Bank)
Suppose that the dynamics of bank capital $C(t)$ and total risk-weighted assets $a_w(t)$ are described by (3.6) and (5.3), respectively. Then the dynamics of the total risk-based capital adequacy ratio $R(t)$ of a bank may be represented by

$$dR(t) = \{R(t)(\alpha_1 - \beta_1) + \alpha_2\}\,dt - R(t)[\beta_2 dW(t) - \beta_3 dZ(t)] \tag{5.4}$$

where

$$\alpha_1 = 0.25\sigma^2 y^2(t) \qquad \alpha_2 = \frac{b(t)}{a_w(t)}$$

$$\beta_1 = 0.5\lambda y(t) \qquad \beta_2 = 0.5\sigma y(t) \qquad \beta_3 = \frac{\sigma}{C(t)}.$$

Proof. In this proof we derive (5.4) by mainly using the general Itô formula. Let $f(a_w(t)) = \frac{1}{a_w(t)}$.
Then,

$$
\begin{aligned}
df(a_w(t)) &= \frac{\partial f(t)}{\partial t}dt + \frac{\partial f(t)}{\partial a_w(t)}da_w(t) + \frac{1}{2}\frac{\partial^2 f(t)}{\partial a_w^2(t)}[da_w(t)]^2 \\
&= 0dt - \frac{da_w(t)}{a_w^2(t)} + \frac{[da_w(t)]^2}{a_w^3(t)} \\
&= -\frac{1}{a_w(t)}[0.5\lambda y(t)dt + 0.5\sigma y(t)dW(t)] + \frac{1}{a_w(t)}[0.5\lambda y(t)dt \\
&+ 0.5\sigma y(t)dW(t)]^2 \\
&= -\frac{1}{a_w(t)}[0.5\lambda y(t)dt + 0.5\sigma y(t)dW(t)] + \frac{1}{a_w(t)}[0.25\sigma^2 y^2(t)dt] \\
&= -\frac{1}{a_w(t)}\left\{0.5y(t)(\lambda - 0.5\sigma^2 y(t))dt + 0.5\sigma y(t)dW(t)\right\} \qquad (5.5)
\end{aligned}
$$

The CAR is expressed as:

$$
R(t) = \frac{C(t)}{a_w(t)} = C(t)f(a_w(t)). \qquad (5.6)
$$

Applying the product rule to $R(t)$, we have:

$$
dR(t) = f(a_w(t))dC(t) + C(t)df(a_w(t)) + dC(t)df(a_w(t)) \qquad (5.7)
$$

But the Brownian motion Z in $dC(t)$ and W in $df(a_w(t))$ are independent,

therefore the term $dC(t)df(a_w(t))$ in (5.7) is equal to zero. Then we obtain

$$
\begin{aligned}
dR(t) &= f(a_w(t))dC(t) + C(t)df(a_w(t)) \\
&= f(a_w(t))[b(t)dt + \sigma dZ(t)] - \frac{C(t)}{a_w(t)}[0.5y(t)(\lambda - 0.5\sigma^2 y(t))dt \\
&+ 0.5\sigma y(t)dW(t)] \\
&= \frac{1}{a_w(t)}[b(t)dt + \sigma dZ(t)] - R(t)[(0.5\lambda y(t) - 0.25\sigma^2 y^2(t))dt \\
&+ 0.5\sigma y(t)dW(t)] \\
&= \left[R(t)(0.25\sigma^2 y^2(t) - 0.5\lambda y(t)) + \frac{b(t)}{a_w(t)}\right]dt + \frac{\sigma}{a_w(t)}dZ(t) \\
&- 0.5R(t)\sigma y(t)dW(t) \\
&= \left[R(t)(0.25\sigma^2 y^2(t) - 0.5\lambda y(t)) + \frac{b(t)}{a_w(t)}\right]dt \\
&- R(t)\left[0.5\sigma y(t)dW(t) - \frac{\sigma}{C(t)}dZ(t)\right] \\
&= \{R(t)[\alpha_1 - \beta_1] + \alpha_2\}dt - R(t)[\beta_2 dW(t) - \beta_3 dZ(t)] \qquad (5.8)
\end{aligned}
$$

where α_1, α_2, β_1, β_2 and β_3 are as defined in Theorem 5.2. $\qquad\square$
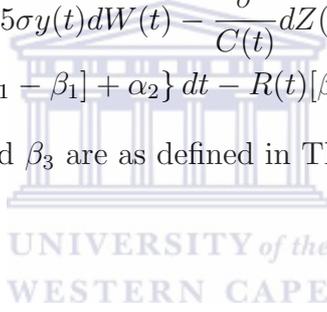
Chapter 6

Simulation Results

In this section, we provide the numerical simulation of the optimal investment strategy, the Basel II CAR and the optimized wealth process $\overline{X}(t)$ that were derived in the previous sections. We choose the following values for our parameters: $r = 0.03$, $\lambda = 0.08$, $\sigma = 0.5$, $c = 0.1$, $x_0 = 1.0$, $\alpha = 0.5$, $Z(1) = 1.0$, $a_w(t) = 0.05$, $b(t) = 0.01$, $R = 0.01$, $C(t) = 0.1$ and $T = 20$.
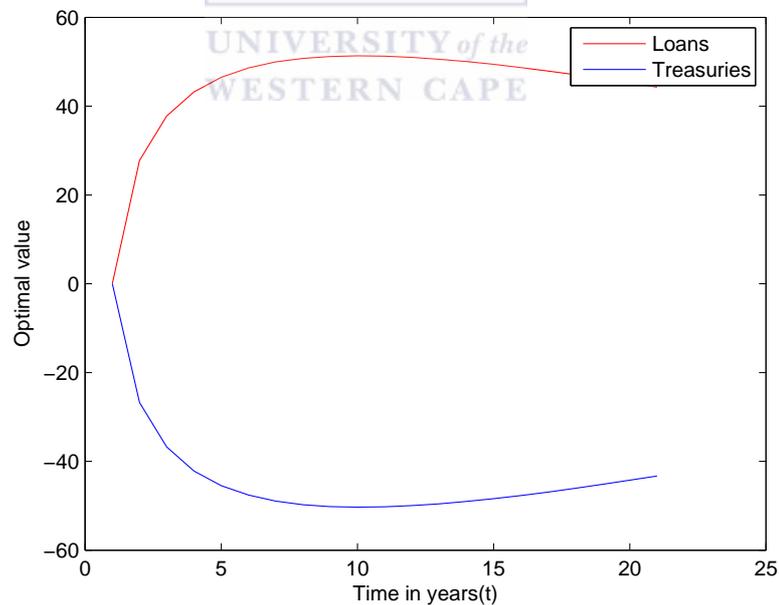

Figure 6.1: A simulation of the optimal investment allocation of a commercial bank invested in treasuries and loans.

The optimal investment strategy in Figure 6.1 above indicates the optimal proportion invested in the loans to be more heavier. This is resembled by an increasing red curve. According to [14], the reason for this is the choice given to α, the minimization of variance. The higher the weight given to the minimization of the variance, the lower the amount invested in the loans, and vice versa. In our case we chose $\alpha = 0.5$, which is too low, hence the reason why the proportion invested in the loans is heavier. The proportion invested in the treasuries is resembled by the decreasing blue curve in Figure 6.1. Its curve is the reverse of that of the loans. That is, if the minimization of variance is high then the investment in treasuries will be high. In this regard, in order for the bank to invest the wealth entirely in the treasuries, then $\alpha = +\infty$. That is, the strategy to invest the whole portfolio in the treasuries will be optimal if and only if zero importance is allocated to the maximization of the final wealth. The results obtained here are consistent with those in paper [14]. The weight given to the minimization of variance, α, is inversely proportional to the proportion invested in the risky asset-loans. The investment strategy illustrated in Figure 6.1 consequently leads to the Basel II CAR in Figure 6.2 below.
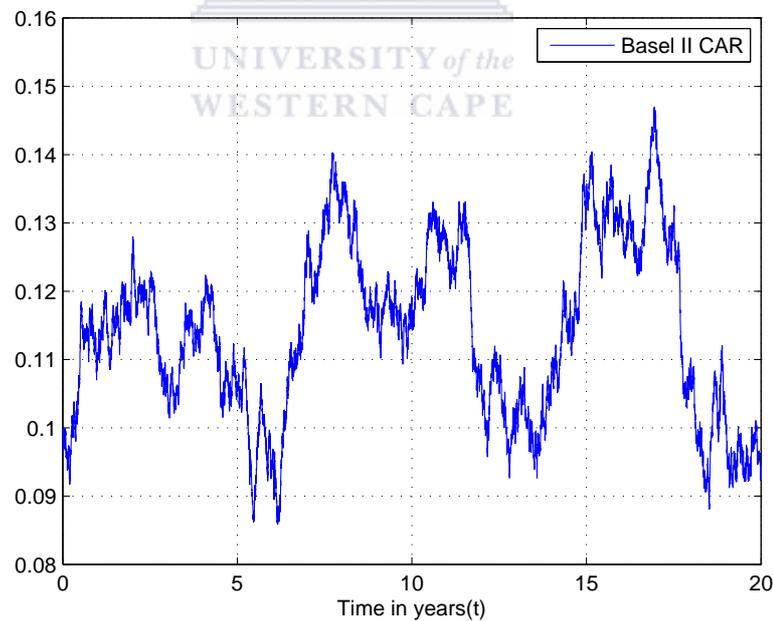

Figure 6.2: A simulation of the behaviour of the Basel II CAR of a commercial bank subject to the optimal investment allocation strategy.

The Basel II Capital Accord recommends a minimum Basel II CAR value of 0.08 in order to ensure that banks can absorb a reasonable level of losses before going insolvent. The authors in [8] and [13] recommend that banks set the control objective to keep its Basel II CAR in the range of [12%, 20%]. Our simulation findings of Basel II CAR indicate that our Basel II CAR is above 8% and below 20% which is consistent with the recommendations made by the authors in papers [8] and [13]. Again we notice that the nature of our Basel II CAR in Figure 6.2 is consistent with the stochastic models in (3.3), (3.4), (3.6) and (5.3) suggested in this work.
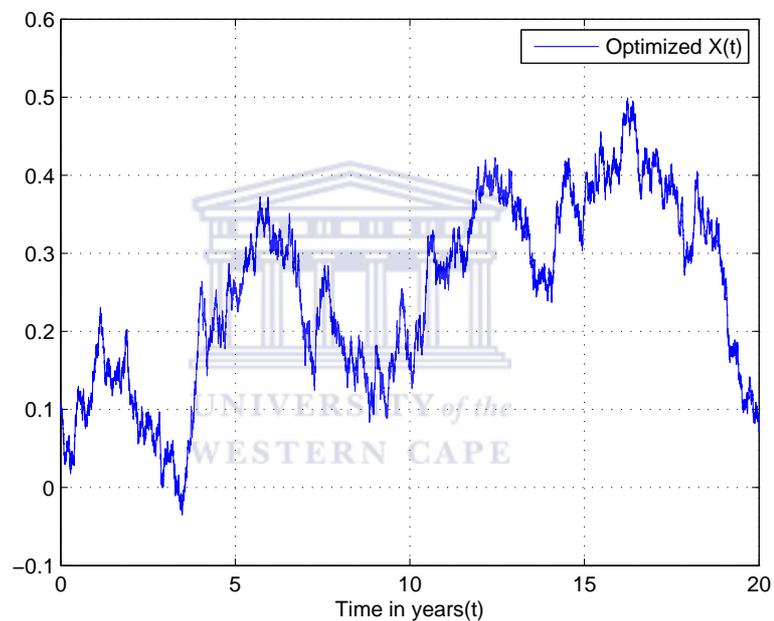

Figure 6.3: A simulation of the behaviour of the optimized wealth process $\overline{X}(t)$.

Figure 6.3 above shows the optimized wealth process $\overline{X}(t)$ following the Brownian motion path over a period of 20 years.

Chapter 7

Conclusion

In this contribution, we applied a mean-variance approach to solve an optimal asset allocation problem in banking. The problem we addressed involved obtaining an optimal investment allocation strategy that optimizes the bank's asset portfolio consisting of two assets, namely: treasuries and loans. To achieve this we first obtained the stochastic differential equation satisfied by the dynamics followed by our asset portfolio. Using the embedding technique proposed by Zhou and Li [17], we transformed the mean-variance selection problem into an optimal control problem; which we then solved via the minimization of the Hamilton Jacobi Bellman equation to obtain an optimal bank asset allocation. Furthermore, we derived an efficient frontier for our portfolio selection problem. We obtained an efficient frontier to be a straight line known as the capital market line (CML), and the tradeoff between the mean (return) and the variance (risk) is given by equation (4.69).

Next we derived the dynamics for the risk weighted assets (or asset portfolio of the bank) and Basel II CAR. Basel II CAR, a ratio of the bank capital to risk-weighted assets, measures the strength of the bank and according to Basel II capital Accord a healthy bank should have CAR value greater than 0.08 or 8%. We observe in Figure 6.2 that the trajectories of the capital adequacy ratio always remain above the stipulated minimum requirement of 8% as suggeted by Basel II capital Accord. From Theorem 4.5.1 we observed that the optimal investment strategy is inversely proportional to the minimization of variance, α. Our simulation results in Figure 6.1 confirms this fact; the investment in the loans proves to be more heavier in the case where α is chosen as small as 0.5. But if instead we choose α as big as $+\infty$, then this will mean the whole portfolio is invested entirely in the treasuries.

In the year 2010, the Basel Committee on Banking Supervision released a close to final version of its new bank capital and liquidity standards, known as "Basel III". Basel III is a series of amendments to the existing Basel II. In the year 2015 when Basel III is fully phased, major changes that it is going to introduce include among others the following: overall Tier 1 element of the capital base being 6% instead of the curent 4%; Tier 2 capital being no longer divided into lower Tier 2 and upper Tier 2; and Tier 3 capital being completely abolished.

In this contribution, we have solved the mean-variance selection problem in banking without putting restrictions on the optimal investment allocation. This is because of the difficulties that arises when constraints are introduced in our model. Therefore, for future research, it may be of interest for one to tackle this problem in the case where restrictions are put on the optimal investment allocation.

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