On the design and implementation of a hybrid numerical method for singularly perturbed two–point boundary value problems

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A thesis submitted in partial fulfilment of the requirements for the degree of Magister Scientiae in the Department of Mathematics and Applied Mathematics, University of the Western Cape.

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November 2014

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KEYWORDS

Singular perturbation problems

Higher order numerical methods

Fitted finite difference methods

Convergence Analysis

Reaction-diffusion problems

Convection-diffusion problems

Boundary Layer

Layer adapted meshes.

Abstract

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With the development of technology seen in the last few decades, numerous solvers have been developed to provide adequate solutions to the problems that model different aspects of science and engineering. Quite often, these solvers are tailor-made for specific classes of problems. Therefore, more of such must be developed to accompany the growing need for mathematical models that help in the understanding of the contemporary world. This thesis treats two-point boundary value singularly perturbed problems. The solution to this type of problem undergoes steep changes in narrow regions (called boundary or internal layer regions) thus rendering the classical numerical procedures inappropriate. To this end, robust numerical methods such as finite difference methods, in particular fitted mesh and fitted operator methods have extensively been used. While the former consists of transforming the continuous problem into a discrete one on a non-uniform mesh, the latter involves a special discretisation of the problem on a uniform mesh and are known to be more accurate. Both classes of methods are suitably designed to accommodate the rapid change(s) in the solution. Quite often, finite difference methods on piecewise uniform meshes (of Shishkin-type) are adopted. However, methods based on such non-uniform meshes, though layer-resolving, are not easily extendable to higher dimensions. This work aims at investigating the possibility of capitalising on the advantages of both fitted mesh and fitted operator methods. Theoretical results are confirmed by extensive numerical simulations.

November 2014

Declaration

I declare that "On the design and implementation of a hybrid numerical method for singularly perturbed two–point boundary value problems" is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Takura T.A. Nyamayaro November 2014

Signed:...

Dedication

To my parents, Harry and Sibongile.

Acknowledgement

I am grateful to my supervisor, Dr. J.B. Munyakazi, who took me under his wing and introduced me to the study of singular perturbation problems. I would also like to thank him for the patient guidance, encouragement and advice he has provided throughout my time as his student.

I wish to also express my deepest gratitude to my co–supervisor, Prof. K.C. Patidar. His insightful guidance was instrumental to the success of this research.

I am also grateful to the Department of Mathematics and Applied Mathematics for the financial support as well as teaching experience I got through part-time lecturing.

Many thanks to my fellow students in the department, Ayodeji Adebiyi, Lestinah Mtombeni, Jose Itaka, Neliswa Dyakopu, Kelekele Lillo and the late Mr. Felly Illunga. Working in the postgrad lab would have been quite lonely and hebetudinous without them.

I am forever indebted to my good friend, Hasmonia Ziso for his benevolence that inadvertently set me on this path.

Above all, I would like to thank my partner, Zvikomborero for her personal support and great patience at all times. My parents, brothers and sister have given me their unequivocal support throughout this journey, as always, for which my mere expression of gratitude does not suffice.

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Chapter 1

General Introduction

1.1 Introduction

Singular perturbation problems (SPPs) have been a challenge to the research community in science and engineering [6]. These problems arise in a number of disciplines of applied mathematics, for example, geophysical fluid dynamics, oceanic and atmospheric circulation, optimal control, quantum mechanics, plasticity, chemical–reaction theory, aerodynamics, meteorology, modelling of semiconductor devices, diffraction theory and reaction-diffusion processes.

Singular perturbation problems are in general characterised by the highest derivative of a scalar ODE being multiplied by a small parameter, ε, known as the perturbation parameter. There are several different types of SPPs, for the purpose of this research, we consider linear singularly perturbed two point boundary value problems. The general form of such problem is

$$-\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \qquad x \in [x_0, x_f], \tag{1.1}$$

with boundary conditions

$$u(x_0) = u_0, \;\; u(x_f) = u_f, \tag{1.2}$$

where $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions and it is assumed that a unique smooth solution of (1.1) exists. The parameter, ε, is taken to be much smaller than α, the lower bound of the first derivative coefficient, that is, $\varepsilon \ll \alpha \leq a(x)$, else the problem is not singularly perturbed.

The general solution of (1.1) is obtained by finding the roots of the associated polynomial. If the coefficients in the differential equation depend on a parameter ε, as is the case with SPPs, then the roots of this polynomial do so as well.

Ordinarily, analysis of the behaviour of these roots is inadequate to generalize the behaviour of the solution of the differential equation because of the following [51]:

1. A non-homogeneous term in the equation can give rise to a term in the general solution whose behaviour depends only, in part , on the roots of the associated polynomial.

2. As a consequence of imposing boundary conditions, the perturbation parameter, ε, may be present in the constants of the homogeneous terms in the general solution.

3. The set of independent variables under consideration may vary with respect to the parameter ε.

4. The solution of the differential equation will be a function of both ε and the independent variable, say x.

It is well known that classical numerical methods are not appropriate for solving singularly perturbed problems.The chief reason being that the solution profile of such problems exhibits regions of fast variation which are commonly referred to as boundary or interior layers. In these layer regions, the classical methods either fail to capture the behaviour of the solution or are computationally expensive when they do.

For illustration purposes we consider the simple models of SPPs found in the book by Miller *et al.* [40]. We begin by considering an Initial Value Problem (IVP) on the unit interval $\Omega = (0, 1)$.

Example 1.1.1. (Convection–reaction problem)
Given the initial value problem

$$\varepsilon u'(x) + u(x) = 0, \quad u(0) = u_0, \tag{1.3}$$

where $u_0 \in \mathbb{R}$ *is some given constant and* $0 < \varepsilon \leq 1$, *find* $u_\varepsilon \in C^1(\Omega)$, *for all* $x \in \Omega$.

The problem in Example 1.1.1 is elementary and can be solved explicitly to get

$$u(x) = u_0 \exp\left(-\frac{x}{\varepsilon}\right), \qquad \varepsilon > 0. \tag{1.4}$$

If $\varepsilon = 0$, then the order of the equation (1.3) is reduced to a trivial equation $\upsilon_o(x) = 0$ for all $x \in \Omega$. The initial condition at $x = 0$ cannot be imposed as a result of $\upsilon_o(x) \equiv 0$ being completely determined. The differential equation and the reduced equation have the same solution if and only if $u_0 = 0$, otherwise their solutions differ. It therefore follows that there exists a boundary layer near $x = 0$.

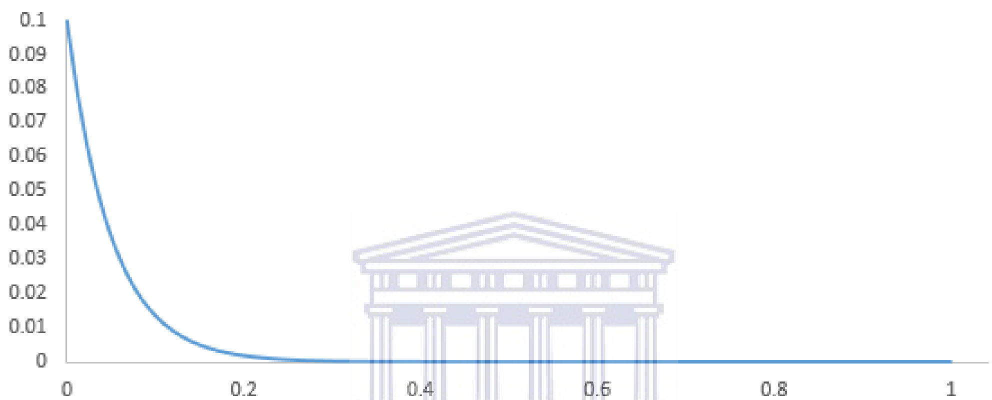

Figure 1.1: Solution profile of Example 1.1.1 with $\varepsilon = 0.05$.

Example 1.1.2. (Reaction–diffusion problem)
Given the equation

$$-\varepsilon u''(x) + u(x) = 0, \tag{1.5}$$

and boundary conditions

$$u(0) = u_0\ ,\ u(1) = u_1, \tag{1.6}$$

where $u_0, u_1 \in \mathbb{R}$ *are given constants and* $0 < \varepsilon \leq 1$, *find* $u \in C^2(\Omega)$, *for all* $x \in \Omega$.

A linear combination of the exponential functions $\{\exp(-x/\sqrt{\varepsilon}), \exp(-(1-x)/\sqrt{\varepsilon})\}$ form the exact solution u. The reduced differential equation is of order zero and no boundary conditions can be imposed on its exact solution, $\upsilon_0 = 0$. There will be a boundary layer at $x = 0$ unless $u_0 = 0$, as well as at $x = 1$ unless $u_1 = 0$.

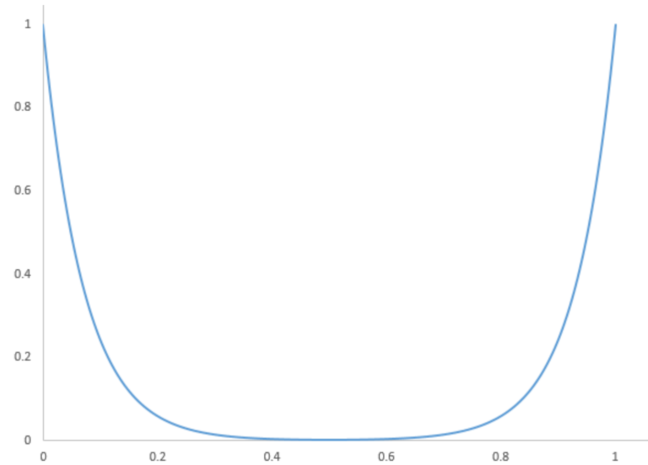

Figure 1.2: Solution profile of Example 1.1.2 with $\varepsilon = 0.005$.

Example 1.1.3. (Convection–diffusion problem)

For a given equation

$$-\varepsilon u''(x) + u'(x) = 0, \tag{1.7}$$

and boundary conditions

$$u(0) = u_0 \ , \ u(1) = u_1, \tag{1.8}$$

where $u_0, u_1 \in \mathbb{R}$ *are given constants and* $0 < \varepsilon \leq 1$, *for all* $x \in \Omega$, *find* $u \in C^2(\Omega)$.

A linear combination of the functions $\{1, \exp(-(1-x)/\sqrt{\varepsilon})\}$ form the exact solution u. The reduced differential equation is of order one, therefore only one boundary condition

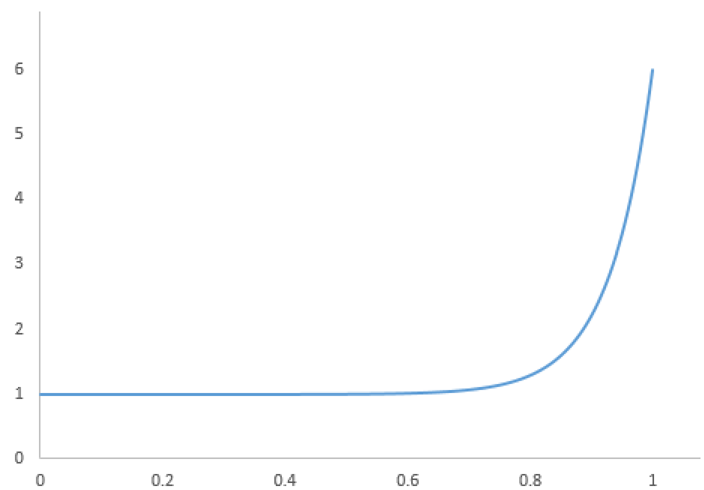

Figure 1.3: Solution profile of Example 1.1.3 with $\varepsilon = 0.005$.

Table 1.1: General solution behaviour for problem (1.1) as given in [6].

$a(x) \neq 0$, $x_0 \leq x \leq x_f$	$a(x) < 0$ $a(x) > 0$	boundary layer at $x = x_0$ boundary layer at $x = x_f$
$a(x) \equiv 0$	$b(x) > 0$ $b(x) < 0$ $b(x)$ changes sign	boundary layer at $x = x_0$ and $x = x_f$ rapidly oscillatory solution classic turning point
$a'(x) \neq b(x)$, $a(x^*) = 0$, $x_0 \leq x^* \leq x_f$	$a'(x^*) < 0$ $a'(x^*) > 0$	no boundary layers, interior layer at $x = x^*$ boundary layers at $x = x_0$ and $x = x_f$, no interior layer at $x = x^*$

can be imposed on its exact solution, $v_0 = 0$. There will be a boundary layer at $x = 0$ unless $u_0 = 0$, as well as at $x = 1$ unless $u_1 = 0$.

We summarize the general behavior we observed in the given examples in Table 1.1 as given by Ascher *et al.* in [5].

1.2 Models depicting singular perturbation problems

We list some models that feature singular perturbation problems.

1. **Slow diffusion with heat production [52]:**

   We consider a one-dimensional bar described by the spatial variable x and length of the bar L; the temperature of the bar is $T(x,t)$, and we assume slow heat diffusion. The endpoints of the bar are kept at a constant temperature. Heat is produced in the bar and also exchanged with its surroundings, so we have that $T = T(x,t)$ is governed by the equation and conditions

   $$\frac{\partial T}{\partial t} = \varepsilon \frac{\partial^2 T}{\partial x^2} - \gamma(x)[T - s(x)] + g(x)$$

   with boundary conditions

   $$T(0,t) = t_0 \quad , \quad T(L,t) = t_1,$$

and initial temperature distribution

$$T(x,0) = \psi(x).$$

The temperature of the neighbourhood of the bar is $s(x)$; $\gamma(x)$ is the exchange coefficient of the heat and $\gamma(x) \geq d$ with d a positive constant so the bar is nowhere isolated.

2. **Relaxation oscillations [44]:**

   Consider the problem of finding the periodic solutions of equations of the form

$$\varepsilon u'' = f(u', u)$$

for small ε when $f(u', u) = 0$ has no periodic solutions. Van der Pol (1927) was the first to treat a problem of this kind in connection with explaining the relaxation oscillations of an electronic circuit governed by the following equation which is named after him [44],

$$u'' + u = \alpha(u' - \frac{1}{3}u'^3). \tag{1.9}$$

Rearranging (1.9), we get

$$u'' = \alpha(u' - \frac{1}{3}u'^3) - u,$$

which leads to

$$\frac{u''}{\alpha} = (u' - \frac{1}{3}u'^3) - \frac{u}{\alpha}.$$

If $\upsilon = u'$ and $x = u/\alpha$, we get

$$\frac{u''}{\alpha} = \upsilon - \frac{\upsilon^3}{3} - x,$$

letting $\varepsilon = \alpha^{-2}$ and using the chain rule, we get

$$\varepsilon \frac{d\upsilon}{dx} = \upsilon^{-1}\left(\upsilon - \frac{\upsilon^3}{3} - x\right).$$

3. **Non-premixed combustion [54]:**

   We consider the model for non–premixed combustion given by the differential equation

$$\varepsilon z'' - z^2 = -t^2, \quad -1 < t < 1,$$

and

$$z(-1) = z(1) = 1.$$

Where ε is a ratio of diffusive effects to the speed of reaction, and t is a distance coordinate, chosen so that $t = 0$ is the location of the flame, where the fuel and the oxidizer meet each other and react. The functions $z - t$ and $z + t$ represent the mass fractions of fuel and oxidizer, respectively.

4. **Motion of a sunflower [26]:**

   The movements of a sunflower can be given by the following model

   $$\varepsilon x''(t) + ax'(t) + b\sin(x(t-\varepsilon)) = 0, \qquad \varepsilon > 0, \;\; t \in [-\varepsilon, 0],$$

   with $x'(0)$ prescribed. Here the function $x(t)$ is the angle of the plant with the vertical at time t, the time lag ε is geotropic reaction, and a and b are positive parameters which can be obtained experimentally.

5. **The Van der Pol oscillator [44]:**

   We consider Van der Pol's oscillator governed by the equation

   $$\frac{d^2u}{dt^2} + u = \varepsilon(1 - u^2)\frac{du}{dt},$$

   where u is the position in space which is a function of the time t, and ε is a scalar parameter showing the nonlinearity and the strength of the damping. When ε is very small, the differential equation is a singular perturbation problem.

6. **Undamped linear spring mass system [26]:**

   Consider a system with a very resistant spring. Let the prescribed specific displacement be at times $t = 0$ and $t = 1$. Then one can obtain the two-point problem

   $$\varepsilon^2 \ddot{x} + x = 0, \quad x(0) = 0, \quad x(1) = 1,$$

   where ε^2 (the ratio of the mass to the spring constant) is small. For nonexceptional small positive values of ε the exact solution oscillates rapidly, so no pointwise limit exists as $\varepsilon \to 0$.

7. **Ground water flow and solute transport [25]:**

Consider the following equation

$$\frac{\partial}{\partial t}C(X,T) = D\frac{\partial^2}{\partial X^2}C(X,T) - \nu\frac{\partial}{\partial X}C(X,T) - \lambda C(X,T), \quad X > 0,\ T > 0, \tag{1.10}$$

where T is the time, X is the horizontal distance taken to be zero at the soil center and measured positive to the right of the soil center; $C(X,T)$ is the solute concentration (mass of solute over volume of solute) at time T; distance X; D is the soil water diffusivity; ν is the average velocity and λ is the decay coefficient (1/time). The contamination in groundwater can be calculated by means of equation (1.10).

The solute transport equation (1.10) represents the mathematical modeling for the unknown concentration $C(X,T)$.We now scale this mathematical problem by selecting the characteristic values for the dependent and independent variables. Consequently, we define dimensionless variables by

$$t = \lambda T,\ x = \frac{\lambda X}{\nu},\ c = \frac{C}{c_0}, \tag{1.11}$$

which lead to the following 3 results

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{\partial}{\partial T}(c_0 c), \\ &= c_0\frac{\partial c}{\partial T}, \\ &= c_0\frac{\partial c}{\partial t}\frac{dt}{dT}, \\ &= c_0\lambda\frac{\partial c}{\partial t}, \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} \frac{\partial C}{\partial X} &= \frac{\partial}{\partial X}(c_0 c), \\ &= c_0\frac{\partial c}{\partial X}, \\ &= c_0\frac{\partial c}{\partial x}\frac{dx}{dX}, \\ &= c_0\frac{\lambda}{\nu}\frac{\partial c}{\partial x}, \end{aligned} \tag{1.13}$$

as well as

$$\begin{aligned}\frac{\partial^2 C}{\partial X^2} &= \frac{\partial}{\partial X}\left(\frac{\partial C}{\partial X}\right), \\ &= \frac{\partial}{\partial X}\left(c_0\frac{\lambda}{\nu}\frac{\partial c}{\partial x}\right), \\ &= c_0\frac{\lambda}{\nu}\frac{\partial^2 c}{\partial x^2}\frac{dx}{dX}, \\ &= c_0\frac{\lambda^2}{\nu^2}\frac{\partial^2 c}{\partial x^2}.\end{aligned} \tag{1.14}$$

Substituting equations (1.12), (1.13) and (1.14) in Equation (1.10), we get

$$\frac{\partial}{\partial t}c(x,t) = \varepsilon\frac{\partial^2}{\partial x^2}c(x,t) - \frac{\partial}{\partial x}c(x,t) - c(x,t), \quad x > 0,\ t > 0,$$

where $\varepsilon = \lambda D/\nu^2 \ll 1$.

1.3 Some general techniques of solving singular perturbation problems

There are various methods available for solving SPPs. These methods can be categorised under analytic methods and numerical methods. We give a brief discussion of both in this section.

1.3.1 Numerical methods

There are several classes of numerical methods used to find numerical approximations to the solutions of differential equations. Some of these classes are the

- Finite Difference Methods (FDMs),
- Finite Element Methods (FEMs),
- Finite Volume Methods (FVMs).

In this section, we consider the class of finite difference methods that are used to solve singular perturbation problems. These are at times referred to as classical discretization methods because they lay the foundation for most numerical schemes. We give the basic steps of FDMs as given by Ascher *et al.* [6].

1. We consider a grid

$$\Omega : x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = x_f \tag{1.15}$$

2. Then we replace the derivatives in the ODE and boundary conditions with difference quotients to form a system of algebraic equations.

3. We then obtain the approximation by solving the system of equations. This gives a set of discrete solution values $U_j \equiv U(x_j)$, at each grid point x_j.

These methods generally give solutions that converge to the exact solution as $N \to \infty$ when solving diffferential equations.

In the case of singularly perturbed differential equations, when the coefficient of the highest derivative, ε, is much smaller than the coefficient of the first derivative, or when $\varepsilon \ll 1$, the solution of the problem may have degrading smoothness [23]. That means that as the value of ε approaches 0, the error in the approximation becomes greater [55]. Therefore, for singularly perturbed problems it is desirable to construct numerical methods for which the accuracy of the approximate solution does not depend on ε, and for which the size of the error depends only on the number of mesh points used [23], that is, methods which converge uniformly with respect to the parameter ε.

We define $\varepsilon-$ convergence as was done by Lins in [36] as follows

Definition 1.3.1. ***(Uniform Convergence)***
Let u be the solution of a singularly perturbed problem and let U be a numerical approximation of u obtained by a numerical method with N grid points. The numerical method is said to be uniformly convergent or robust with respect to the perturbation parameter ε in some norm $\|\ .\ \|$ if there exists positive constants C and N_0 such that

$$\|u - U\| \leq CN^{-p} \ \text{for}\ N \geq N_0,$$

where the $\varepsilon-$uniform error constant, C, is independent of the parameter ε and the $\varepsilon-$uniform rate of convergence, p, is also not dependent of ε.

As far as finite difference methods are concerned, two approaches have ordinarilly been used by numerical analysts to obtain $\varepsilon-$uniform convergent methods for solving SPPs and these are namely fitted operator finite difference methods and fitted mesh finite difference methods.

Fitted Operator Finite Difference Methods (FOFDMs)

These methods are generally categorised into two groups. Exponentially fitted methods constitute one group, while the other group commonly referred to as Non–Standard Finite Difference Methods. The latter group is constructed using a rule first introduced by Mickens [38].The basic idea behind the FOFDMs is to replace the denominator functions of the classical derivatives with positive functions derived in such a way that they capture some notable properties of the governing differential equation [8]. In other words, FOFDMs involve replacing the standard finite difference operator by a different finite difference operator which reflects the singular perturbation nature of the differential operator. For linear problems, such operators may be obtained by choosing the coefficients of the difference operator so that some or all the exponential functions in the null-space of the differential operator, are also in the null space of the finite difference operator. Such fitted operators have been developed by many authors and usually work with uniform meshes. The implementation of these methods is not straightforward and they usually introduce artificial diffusion. We note that these methods can be applied without *a priori* knowledge of the breadth and position of the boundary or interior layers.

Fitted Mesh Finite Difference Methods (FMFDMs)

FMFDMs involve the use of a mesh that is adapted to the singular perturbation. In this group of numerical schemes, meshes are taken such that they are not uniform. There are several approaches to adapting the meshes by way of redistribution of mesh points of which the Bakhavalov and Shishkin meshes are examples.

We employ a basic Shishkin mesh to construct a piecewise-uniform mesh on the interval $\Omega(0,1)$ as described in [40]. The mesh will be in such a way that the boundary layer regions have more mesh points relative to outside these regions. Let us introduce a piecewise-uniform mesh Ω_λ^N which will be generated as follows. Let a point λ satisfy $0 < \lambda \leq 1/2$ and let N be an even number. We divide the interval $[0,1]$ into two subintervals $[0,\lambda]$

and $[\lambda, 1]$

$$x_j - x_{j-1} = \begin{cases} 2\lambda/N, & j = 0, 1, \ldots, N/2 \\ 2(1-\lambda)/N, & j = N/2+1, \ldots, N. \end{cases}$$

The position and definition of λ can be adjusted to fit the particular problem that is being solved ,i.e. λ is adapted according to the position of the layer. We denote the mesh constructed in this manner by Ω_λ^N and example of such a mesh is given in Figure 1.4, where the transition parameter λ is defined as $\lambda = \min\{1/2, \varepsilon \ln N\}$. It should be noted

Figure 1.4: Piecewise uniform mesh Ω_λ^{10}.

that for the case we have just constructed a mesh for, the boundary layer occurs near $x = 0$. If the layer existed near $x = 1$ the mesh would look similar to one in Figure 1.5.

Figure 1.5: Shishkin mesh for a case with a boundary layer near $x = 1$.

λ is still defined as before with

$$x_j - x_{j-1} = \begin{cases} 2(1-\lambda)/N, & j = 0, 1, \ldots, N/2 \\ 2\lambda/N, & j = N/2+1, \ldots, N. \end{cases}$$

We note that in both cases if $\lambda = 1/2$, then all the meshpoints are equidistants and the mesh is uniform.

The third elementary case we anticipate is when there are layers near both $x = 0$ and $x = 1$ boundaries in the domain Ω. In this case the example interval $\Omega(0, 1)$ is divided into 3 subintervals. λ is still chosen satisfying $0 < \lambda \leq 1/2$ and there are two transition points located at $x = \lambda$ and $x = 1 - \lambda$. Intervals $(0, \lambda)$ and $(1 - \lambda, 1)$ are divided into $N/4$ equal subintervals and the interval $(\lambda, 1 - \lambda)$ is divided into $N/2$ subintervals. We define

λ in this case as $\lambda = \min\{1/4, \sqrt{\varepsilon}\ln N\}$. The mesh length is given by

$$x_j - x_{j-1} = \begin{cases} 4\lambda/N, & j = 1, \ldots, N/4, \\ 2(1-2\lambda)/N, & j = N/4+1, \ldots, 3N/4, \\ 4\lambda/N, & j = 3N/4+1, \ldots, N. \end{cases}$$

The grid generated is as illustrated in Figure 1.6. We note that in this case if $\lambda = 1/4$,

Figure 1.6: Shishkin mesh for a case with boundary layers near $x = 0$ and $x = 1$.

then all the meshpoints are equidistants and the mesh is uniform.

Another widely used mesh in FMFDMs is the Bakhvalov mesh. We give the general mesh generating function for these meshes as follows:

$$\phi(x) = \begin{cases} p\sqrt{\varepsilon}\ln\left(\frac{q}{q-x}\right), & x = 0, \ldots, \beta \\ \lambda(\beta) + \lambda'(\beta)(x-\beta), & x = \beta, \ldots, 1/2, \\ 1 - \phi(1-x), & x = 1/2, \ldots, 1, \end{cases}$$

where p and q are constants, independent of ϵ, such that, $q \in (0, 1/2)$ and $p \in (0, q/\sqrt{\epsilon})$. This mesh generating function is for a problem with two layers and it consists of three parts: ϕ_1, ϕ_2, and ϕ_3. The mesh points for the boundary layer regions near $x = 0$ and $x = 1$ are generated by the functions ϕ_1 and ϕ_3 respectively. Function ϕ_2 generates the mesh points outside the layer regions and it is a tangent lone to both ϕ_1 and ϕ_3, and $\phi_2(0.5) = 0.5$

It should be remarked that the construction of FMFDMs requires *a priori* knowledge of the width and position of the boundary or interior layers.

1.3.2 Analytic methods

In the section we briefly discuss another category of generally used methods called analytic methods, specifically those that fall under the sub–group of asymptotic methods. We consider two such methods, namely matched asymptotic expansions and the method of multiple scales.

The Method of Matched Asymptotic Expansions

The initial development of the method of matched asymptotic expansions is credited to Prandtl (1905), who was concerned with the flow of a fluid past a solid body such as an aeroplane wing [24].This method involves finding several different approximate solutions, each valid over different sections of the domain. The different solutions are then matched together and combined to give a single approximate solution that is universally valid.

The Method of Multiple Scales

In essence, this method introduces two scales, a fast and a slow one for the independent variable, and then treats these variables as if they are independent. The secular terms found in each stage can be suppressed by equating the arbitrary functions from one term in the expansion with the next. Thus a single solution is obtained that is valid over the complete domain that can easily be expanded with lower order terms where desired.

1.4 Literature review

The relationship between cause and effect has attracted significant research endeavour in many fields. This becomes more interesting whenever a large effect is a result of a small cause [17]. Such is the case with singular perturbation problems where solutions present so–called layers as the perturbation parameter approaches zero.

Ascher in [5], considers singularly perturbed boundary value problems (BVPs) as given by

$$\varepsilon y' = A(t,\varepsilon)y + f(t,\varepsilon), \qquad 0 \leq t \leq 1,$$

under boundary conditions

$$B_0(\varepsilon)y(0,\varepsilon) + B_1(\varepsilon)y(1,\varepsilon) = \beta(\varepsilon),$$

where the problem defining the reduced solution is singular. Families of symmetric difference schemes, which are equivalent to certain collocation schemes based on Gauss and Lobatto points are used to obtain numerical approximations. They extend convergence results, previously obtained for the ”regular” singularly perturbed case. The grid selection procedure requires a small adjustment for the Lobatto schemes, while the Gauss schemes are extended with no change.

In the paper by Clavero *et al.*[14], two compact monotone finite difference methods to solve singularly perturbed problems of convection–diffusion type are constructed and analyzed. They are defined as HODIE methods of order two and three, i.e., the coefficients are determined by imposing that the local error be null on a polynomial space. For arbitrary meshes, these methods are not adequate for singularly perturbed problems, but using a Shishkin mesh it can be proven that the methods are uniformly convergent of order two and three except for a logarithmic factor.

A centered difference or finite element discretization is applied to a singularly perturbed, one–dimensional boundary value problem in [35]. The discretization uses a piecewise equidistant mesh. It is proved that the pointwise error is (almost) of second order with respect to the number of nodes, uniformly in the perturbation parameter. The proof is based on a monotonicity argument.

A convection–diffusion two–point boundary value problem in conservative form was considered by Kopteva and Stynes [31]. To solve it numerically they applied an upwind conservative finite difference scheme. On an arbitrary mesh they prove bounds, which are weighted by the small diffusion coefficient, on the errors in approximating the derivative of the true solution by divided differences of the computed solution. On a slightly less general mesh Kopteva and Stynes proved unweighted bounds on these errors where the mesh is coarse. These bounds are then made more explicit for the particular cases of Shishkin and Bakhvalov meshes.

Filiz *et al.* [19] propose an $\varepsilon-$uniform finite difference method on an equidistant mesh which requires no exact solution of a differential equation. They begin with a full–fitted operator method reflecting the singular perturbation nature of the problem through a local boundary value problem. However, to solve the local boundary value problem, they employ an upwind method on a Shishkin mesh in local domain, instead of solving it exactly. They further study the convergence properties of the numerical method proposed and prove it nodally converges to the true solution for any ε.

Kanth and Reddy [29] present a numerical method for solving a two point boundary value problem in the interval $[0, 1]$ with regular singularity at $x = 0$. By employing the Chebyshev economizition on $[0, \delta]$, where δ is near the singularity, it is replaced by a regular problem on some interval $[\delta, 1]$. The stable central difference method is then employed to solve the problem over the reduced interval.

In [49], Reddy and Chakravarthy present an exponentially fitted finite difference method for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. A fitting factor is introduced in a tridiagonal finite difference scheme and is obtained from the theory of singular perturbations. Thomas algorithm is used to solve the system. The stability of the algorithm is investigated. Several linear and nonlinear problems are solved to demonstrate the applicability of the method. It is observed that the present method approximates the exact solution very well.

Wazwaz [53] presents an efficient numerical algorithm for approximate solutions of higher–order boundary value problems with two–point boundary conditions. A modified form of the Adomian decomposition method was implemented to construct the solutions. The approach provides the solution in the form of a rapidly convergent series. Kumar [32] presents a three-point finite difference method based on the uniform mesh for the same class of BVPs, in the standard form:

$$(x^{\alpha}y')' = f(x, y)$$

and

$$y(0) = A, \quad y(1) = B, \qquad 0 < \alpha < 1.$$

Here $\alpha \in (0, 1)$ and the constants A, B are finite. This particular method gives approximations that are $O(h^4)$–convergent.

Bellew and O'Riordan [9] examined a coupled system of two singularly perturbed convection–diffusion ordinary differential equations (ODEs). They constructed a numerical method for the system which involves an appropriate piecewise Shishkin mesh. The numerical approximations converge to the continuous solutions uniformly with respect to the singular perturbation parameters.

Cakir and Amiraliyev [11], presented a finite difference method for numerical solutions of singularly perturbed boundary value problems for the second order ODE with nonlocal boundary conditions. By the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder term in the integral form an exponentially fitted difference scheme on a uniform mesh is developed which is shown to be original ε–uniform first order accurate in the discrete maximum norm for original problem. Amiraliyev [1] was concerned with the numerical solution for singular perturbation system of two coupled ordinary differential equations with first and second orders and with initial and boundary conditions, respectively.

Kadalbajoo *et al.* [28] constructed and analysed a FOFDM which is first order ε–uniformly convergent. With the aim of having just one function evaluation at each step, attempts were made to derive a higher order method via Shishkin mesh to which we refer as the FMFDM. This FMFDM is a direct method and ε–uniformly convergent with the nodal error as $O(n^{-2}\ln^2(n))$ which is an improvement over the existing direct methods (i.e. those which do not use any acceleration of convergence techniques, e.g Richardson's extrapolation or defect correction, etc) for such problems on a mesh of Shishkin type that lead the error as $O(n^{-1}\ln(n))$ where n denotes the total number of subintervals of [0,1].

Lubuma and Patidar [37] constructed and analysed non-standard finite difference methods for a class of singularly perturbed differential equations. The class consists of two types of problems: (i) those having solutions with layer behaviour and (ii) those having solutions with oscillatory behaviour. Since no fitted mesh method can be designed for the latter type of problems, other special treatment is necessary, which is one of the aims they attained. The main idea behind the construction of their method is motivated by the modelling rules for non–standard finite difference methods, developed by Mickens. These rules allow one to incorporate the essential physical properties of the differential equations in the numerical schemes so that they provide reliable numerical results. Note that the usual ways of constructing the fitted operator methods need the fitting factor to be incorporated in the standard finite difference scheme and then it is derived by requiring that the scheme be uniformly convergent. The method that they present is fairly simple as compared to the other approaches.

Munyakazi and Patidar [42] investigated the Richardson extrapolation as a convergence acceleration techniques on methods developed by Lubuma and Patidar [37] and Patidar [45], referred to as FOFDM–II and FOFDM–I. The FOFDM–I is fourth and second order accurate for moderate and smaller values of ε, respectively. Unfortunately, Richardson extrapolation does not improve the order of this method. The FOFDM-II is second order uniformly convergent and they show that its order can be improved up to four by using Richardson extrapolation. They conclude in their article that the Richardson extrapolation technique do not perform equally well on all type of methods.

Choo and Schultz [13] developed stable high–order methods, namely the stabilized central difference methods (SCD methods) for solving linear two-point boundary value problems with small coefficients for the second order terms. They developed, in particular, second–, fourth– and sixth–order methods. The methods are proved to be stable and accurate. We have tested these methods for the one-dimensional convection-diffusion equation including problems with and without boundary layers. The results stayed accurate and stable for all values of ε tested, from 1 to 10^{-8}. These methods are significant in the sense that they stabilize the central difference method while improving its accuracy. The authors also prove that the SCD methods are unconditionally stable in the case where both the coefficients of the differential equation are constants. Furthermore, the formulation of these methods is simple and they are applicable to other two-point boundary value problems with either Dirichlet or Neumann boundary conditions.

Richardson extrapolation is also investigated by Natividad and Stynes [43]. The authors consider a convection–diffusion two–point boundary value problem on a piecewise–uniform Shishkin mesh, and show that when simple upwinding is used, a version of Richardson extrapolation improves the accuracy of the computed solution (measured in the discrete L^∞ norm) from $O(N^{-1}\ln N)$ to $O(N^{-2}\ln^2 N)$, where $N+1$ mesh points are used.

Kumar *et al.* [33] developed a scheme in which the original problem is partitioned into inner and outer solutions of differential equations. The method is distinguished by the following fact: the inner region problem is solved as a two–point boundary layer correction problem and the outer region problem of the differential equation is solved as initial-value problem with initial condition at end point.

In 2010, Rao and Kumar [47] presented a high order parameter–robust finite difference method for singularly perturbed reaction–diffusion problem:

$$L_u(x) := -\varepsilon u''(x) + b(x)u(x) = f(x) \quad x \in (0,1),$$

with boundary conditions

$$u(0) = 0 \ , \ u(1) = 0,$$

where $0 < \varepsilon \leq 1$ is a small parameter, $f, b \in C^4(\bar{\Omega})$ with $b(x) \geq \beta > 0$, $x \in \bar{\Omega}$ for some positive β. The problem is discretized using a suitable combination of fourth order compact difference scheme and central difference scheme on generalized Shishkin mesh. The convergence analysis is given and the method is proved to be almost fourth order uniformly convergent in maximum norm with respect to singular perturbation parameter ε.

In their work, Clavero and Gracia [16] were interested in the numerical approximation of one-dimension parabolic singularly perturbed problems of reaction-diffusion type. To approximate the multi–scale solution of this problem they used a numerical scheme combining the classical backward Euler method and central differencing. The scheme is defined on some special meshes which are the tensor product of a uniform mesh in time and a special mesh in space, condensing the mesh points in the boundary layer regions. In this paper three different meshes of Shishkin, Bahkvalov and Vulanovic type are used, proving the uniform convergence with respect to the diffusion parameter. The analysis of the uniform convergence is based on a study of the asymptotic behaviour of the solution of the semi–discrete problems, which are obtained after the time discretization by the Euler method.

In [2],the authors deal with the singularly perturbed boundary value problem for a linear second–order delay differential equation. For the numerical solution of this problem, they use an exponentially fitted difference scheme on a uniform mesh which is accomplished by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder term in integral form. It is shown that one gets first order convergence in the discrete maximum norm, independently of the perturbation parameter.

Kadalbajoo and Patidar [34], a numerical study is made for solving a class of time–

dependent singularly perturbed convection–diffusion problems with retarded terms which often arise in computational neuroscience. A Taylor's series expansion was employed to approximate the retarded terms and the resulting time–dependent SPP is approximated using parameter–uniform numerical methods comprised of a standard implicit finite difference scheme to discretize in the temporal direction on a uniform mesh by means of Rothes method and a B–spline collocation method in the spatial direction on a piecewise–uniform mesh of Shishkin type. The method is shown to be accurate of order $O(M^{-1}+N^{-2}\ln^3 N)$, where M and N are the number of mesh points used in the temporal direction and in the spatial direction respectively.

Bashier and Patidar [8] design a robust fitted operator finite difference method for the numerical solution of a singularly perturbed delay parabolic partial differential equation. Their method is unconditionally stable and is convergent with order $O(k+h^2)$, where k is the time stepsize and h is the space step–size, which is better than the one obtained by Ansari *et al* [4] where a fitted mesh finite difference method was used. Their method was of the order $O(N_t^{-1} + N_x^{-2}\ \ln^2 N_x)$, where N_t and N_x denote the total number of sub-intervals in the time and space directions. Rao and Kumar in 2011 [48], considered a system of $M(\geq 2)$ coupled singularly perturbed equations of reaction–diffusion type.

$$Lu(x) := -Eu''(x) + Au(x) = f(x), \qquad x \in (0,1)$$

and boundary condtions

$$u(0) = a_1, \quad u(1) = a_2,$$

where $E = \text{diag}(\varepsilon, \ldots, \varepsilon)$ is a diagonal matrix with $0 < \varepsilon \ll 1, f = (f_1, \ldots, f_M)^T$. A high order Schwarz domain decomposition method was developed to solve the system numerically. The method splits the original domain into three overlapping subdomains. On two boundary layer subdomains they use a compact fourth order difference scheme on a uniform mesh while on the interior subdomain they use a hybrid scheme on a uniform mesh. They prove that the method is almost fourth order $\varepsilon-$uniformly convergent. Furthermore, they prove that when ε is small, one iteration is sufficient to get almost fourth order $\varepsilon-$uniform convergence.

A comparison of classical methods for singular perturbation problems, such as El–

Mistikawy and Werle scheme and its modifications, to exponential spline collocation schemes is done by Kavčič *et al.* [30]. They discuss subtle differences that exist in applying this method to one dimensional reaction–diffusion problems and advection–diffusion problems. For pure advection–diffusion problems, exponential tension spline collocation is less capable of capturing only one boundary layer, which happens when no reaction term is present. Thus an already existing collocation scheme in which the approximate solution is a projection to the space piecewisely spanned by $\{1, x, \exp(\pm px)\}$ is inferior to the generalization of El–Mistikawy and Werle method proposed by Ramos. The remedy to this situation is obtained by considering projections to spaces locally spanned by $\{1, x, x^2, \exp(px)\}$, where $p > 0$ is a tension parameter. Next, they exploit a unique feature of collocation methods, that is, the existence of special collocation points which yield better global convergence rates and double the convergence order at the knots.

Amodio and Settanni [3] developed a method based on high order finite differences, in particular on the generalized upwind method. Within its simplicity, it uses order variation and continuation for solving any difficult nonlinear scalar problem. Several numerical tests on linear and nonlinear problems are considered. The best performances are reported on problems with perturbation parameters near the machine precision, where most of the existing for two–point BVPs fail.

Rao and Chakravarthy, in their paper [46], present a finite difference method for singularly perturbed linear second order differential–difference equations of convection–diffusion type with a small shift, i.e., where the second order derivative is multiplied by a small parameter and the shift depends on the small parameter. Similar boundary value problems are associated with expected first–exit times of the membrane potential in models of neurons. Here, the study focuses on the effect of shift on the boundary layer behaviour or oscillatory behaviour of the solution via finite difference approach.

In [56], a singularly perturbed semi–linear boundary value problem with two parameters is considered. The problem is solved using exponential spline on a Shishkin mesh. The convergence analysis is derived and the method is convergent independently of the perturbation parameters.

The objective of [22] was to present a numerical method for solving singularly perturbed turning point problems exhibiting an interior layer. The method is based on the asymptotic expansion technique and the reproducing kernel method (RKM). The original problem is reduced to interior layer and regular domain problems. The regular domain problems are solved by using the asymptotic expansion method. The interior layer problem is treated by the method of stretching variable and the RKM.

Brohmer *et al.* [10] give an introductory survey of the defect correction approach. They motivate this approach from its basic idea, that is, for a given mathematical problem and a given approximate solution,

- define the defect as a quantity which indicates how well the problem has been solved,
- use this information in a simplified version of the problem to obtain an appropriate correction quantity,
- apply this correction to the approximate solution to obtain a new improved approximate solution.
- repeat above steps until desired accuracy is obtained.

Cawood *et al.* [12] present a posteriori error estimates for a defect correction method for approximating solutions of convection–diffusion problems. The algorithms and estimators include the possibility of using in the discretization a nonlinear selection mechanism, which they find, improves solution quality in and near layers. Energy norm and L_2 a posteriori error estimates are proven for the full algorithm.

Another defect correction method based on finite difference schemes was considered for a singularly perturbed boundary value problem on a Shishkin mesh by Frohner and Roos [21]. The method combines the stability of the upwind difference scheme and the higher-order convergence of the central difference scheme. The almost second–order convergence of the scheme with respect to the discrete maximum norm, uniformly in the perturbation parameter ε, was proved. Numerical experiments support the theoretical results.

In [7], the well-known method of Iterated Defect Correction (IDeC) based on the following idea: Compute a simple, basic approximation and form its defect with respect to the given ordinary differential equation via a piecewise interpolant. This defect is used todefine an auxiliary, neighbouring problem whose exact solution is known. Solving the neighboring problem with the basic discretization scheme yields a global error estimate. This can be used to construct an improved approximation, and the procedure can be iterated. The fixed point of such an iterative process corresponds to a certain collocation solution. Auzinger *et al.* present a variety of modifications to this algorithm. These modifications are based on techniques like defect quadrature (IQDeC), defect interpolation (IPDeC), and combinations thereof. They investigate the convergence on locally equidistant and nonequidistant grids and show how superconvergent approximations can be obtained. Numerical examples illustrate our considerations.

1.5 Outline of the thesis

The rest of the thesis is organised as follows. In Chapter 2 we consider the fitted operator finite difference methods (FOFDMs) and the fitted mesh finite difference methods (FMFDMs) to solve reaction–diffusion problems. These methods are presented and analysed for convergence. Furthermore, the methods are numerically tested and a brief discussion thereof is presented at the end of the chapter.

Acceleration techniques are introduced and briefly discussed in Chapter 3. Our focus in this chapter is the indirect higher order method known as the defect correction method which consists of the combination of a lower order upwind method with a high order unstable method resulting in a high order and stable method. This method is implemented for the solution of convection-diffusion problems.

Chapter 4 is devoted to the development a new finite difference method to solve two-point boundary value convection–diffusion problems. This method seeks to combine the advantage of accuracy of FOFDMs and the layer–resolving property of FMFDMs. The error analysis of this hybrid method is presented as well as test examples to confirm the theoretical findings.

Finally, in Chapter 5 we provide some concluding remarks of our research and the scope for future work.

Chapter 2

Fitted Operator Vs Fitted Mesh Finite Difference Methods

In the previous chapter, we introduced singular perturbation problems as well as the challenges associated with solving them numerically. We also briefly discussed some of the methods which give $\varepsilon-$uniform approximations when solving these problems. In this chapter, we discuss in further detail some of those methods, namely, the Fitted Operator Finite Difference Method and the Fitted Mesh Finite Difference Method.

2.1 Introduction

In the development of ε–uniform methods for solving SPPs, two approaches have generally been broadly used as far as finite difference methods are concerned. These approaches are namely the Fitted Operator Finite Difference Method, here and after, refered to as FOFDM, and a Fitted Mesh Finite Difference Method, here and after, refered to as FMFDM. In this chapter we compare their perfomance against each other.

The notation $u_j := u(x_j)$ is adopted for the set of meshpoints $\{x_j\}_0^N \in [0, 1]$ and for numerical approximations, capital letters are used with $U(x_j) \approx u(x_j)$ or $U_j \approx u_j$. We also use C to denote a generic positive constant that is independent of both N and the perturbation parameter ε.

2.2 Fitted Mesh Finite Difference Method for reaction diffusion problems

The fitted or adapted mesh finite difference methods require *a priori* knowledge of the width and the location of the regions where the solution displays fast variation. This knowledge is then used to generate an appropriate mesh hence the methods are called fitted or adapted mesh finite difference methods.

We consider the general reaction-diffusion problem given by

$$-\varepsilon u''(x) + b(x)u(x) = f(x), \qquad x \in [0,1], \tag{2.1}$$

with boundary conditions

$$u(0) = u_0, \;\; u(1) = u_f. \tag{2.2}$$

where $b(x)$ and $f(x)$ are sufficiently smooth functions and moreover the coefficient function is assumed to satisfy

$$b(x) \geq \beta > 0, \qquad \text{for all} \quad x \in (0,1).$$

The differential operator for this problem is given by

$$L_\varepsilon \equiv -\varepsilon \frac{d}{dx} + b,$$

and it satisfies the following maximum principle for boundary value problems (BVPs),

Maximum Principle. ([40])
Assume that a function $\psi(x)$ satisfies $\psi(0) \geq 0$ and $\psi(1) \geq 0$. Then $L_\varepsilon\psi(x) \geq 0$, for all $x \in \Omega$, implies that $\psi(x) \geq 0$, for all $x \in \bar{\Omega}$.

Proof. Let x^* be such that $\psi(x^*) = \min_{\bar{\Omega}} \psi(x)$ and suppose that $\psi(x^*) < 0$. Evidently $x^* \notin \{0,1\}$. It follows from elementary calculus that $\psi'(x^*) = 0$ and $\psi''(x^*) \geq 0$. Consequently,

$$\begin{aligned} L_\varepsilon\psi(x^*) &= -\varepsilon\psi''(x^*) + b\psi'(x^*), \\ &< 0, \end{aligned}$$

which is false. Therefore it follows that $\psi(x^*) \geq 0$ and that $\psi(x) \geq 0$, for all $x \in \bar{\Omega}$. $\square$

The following lemma gives a bound on the solution of (2.1).

Lemma 2.2.1. ([40])

Let Ω *be the interval* $[0,1]$ *and* $u_\epsilon \in C^2(\bar{\Omega})$ *be solutions to the problem in (2.1). Then, for* $0 \leq k \leq 4$, *the following bounds hold*

$$||u^{(k)}|| \leq C(1 + \varepsilon^{-k/2}).$$

Proof. We handle first the case when $k = 0$. Consider the functions

$$\psi^{\pm}(x) = \frac{1}{\beta}||f|| + \max\{|u_0|, |u_1|\} \pm u(x).$$

When $x = 0$ we have

$$\begin{aligned}
\psi^{\pm}(0) &= \frac{1}{\beta}||f|| + \max\{|u_0|, |u_1|\} \pm u(0), \\
&\geq \frac{1}{\beta}||f|| \;, \qquad \text{since } \max\{|u_0|, |u_1|\} \geq u(0), \\
&\geq 0.
\end{aligned}$$

When $x = 1$, the proof is analogous to the case when $x = 0$. Now

$$\begin{aligned}
L_\varepsilon \psi^{\pm}(x) &= -\varepsilon(\psi^{\pm}(x))'' + b\psi(x), \\
&= \mp\varepsilon u_\varepsilon''(x) + \frac{b}{\beta}||f|| + b\max\{|u_0|, |u_1|\} \pm u(x), \\
&= \pm f(x) + \frac{b}{\beta}||f|| + b\max\{|u_0|, |u_1|\}, \\
&\geq b\max\{|u_0|, |u_1|\} \;, \qquad \text{since } \frac{b}{\beta}||f|| \geq f(x), \\
&\geq 0.
\end{aligned}$$

Applying the maximum principle, it follows that $\psi^{\pm}(x) \geq 0$, and therefore

$$|u(x)| \leq \frac{1}{\beta}||f|| + \max\{|u_0|, |u_1|\} \;, \quad \text{for all } x \in \bar{\Omega}.$$

We now handle the case when $k = 1$. Let $x \in \Omega$ and construct an associated neighbourhood $N_x = (q, q + \sqrt{\varepsilon})$, such that $x \in N_x$ and $N_x \subset \Omega$. Then, it follows from the

Mean Value Theorem, for some $a \in \bar{N}_x$,

$$\begin{aligned} |u'(a)| &= \frac{u(q+\sqrt{\varepsilon}) - u(q)}{\sqrt{\varepsilon}}, \\ &= \frac{1}{\sqrt{\varepsilon}}|u(q+\sqrt{\varepsilon}) - u(q)|, \\ &\leq \frac{1}{\sqrt{\varepsilon}}\{|u(q+\sqrt{\varepsilon})| + |u(q)|\}, \\ &\leq \frac{1}{\sqrt{\varepsilon}}\{||u|| + ||u||\}, \\ &\leq \frac{2}{\sqrt{\varepsilon}}||u||. \end{aligned}$$

This can be re–written as

$$|u'(a)| \leq 2\varepsilon^{-1/2}||u|| \leq C\varepsilon^{-1/2}||u||.$$

Now

$$\begin{aligned} u'(x) &= u'(a) + u'(x) - u'(a), \\ &= u'(a) + \int_a^x u''(w)dw, \\ &= u'(a) + \int_a^x (bu(x) - f)(w)dw. \end{aligned}$$

Hence

$$|u'| \leq C\varepsilon^{-1/2}.$$

The bounds on u and u' and the differential equation are then used to obtain the bounds on $|u^{(k)}|$ for $k = 2, 3, 4$. □

The difference operator, L_ε^N, approximating the differential operator, L_ε, for the reaction–diffusion problem is

$$L_\varepsilon^N \equiv -\varepsilon\delta^2 + b_j, \tag{2.3}$$

and it satisfies the the following

Discrete Maximum Principle. ([40])
Assume that the mesh function Ψ_j satisfies $\Psi_0 \geq 0$ and $\Psi_N \geq 0$. Then $L_\varepsilon^N \Psi_j \geq 0$, for all $1 \leq j \leq N-1$, implies that $\Psi_j \geq 0$, for all $0 \leq j \leq N$.

Proof. Let k be such that $\Psi_k = \min_j \Psi_j$ and suppose that $\Psi < 0$. Evidently, $k \notin \{0, N\}$, $\Psi_j \leq \Psi_{k+1}$ and $\Psi_j \leq \Psi_{k-1}$. If $\hbar_k = (h_{k+1} + h_k)/2$, it follows that

$$\begin{aligned}
L_\varepsilon^N \Psi_k &= -\varepsilon \delta^2 \Psi_k + b_k \Psi_k, \\
&= -\varepsilon \frac{\Psi_{k+1} - 2\Psi_k + \Psi_{k-1}}{\hbar_k^2} + b_k \Psi_k, \\
&= -\varepsilon \frac{(\Psi_{k+1} - \Psi_k) + (\Psi_{k-1} - \Psi_k)}{\hbar_k^2} + b_k \Psi_k, \\
&< 0,
\end{aligned}$$

which is a contradiction. It follows that $\Psi_k \geq 0$, and thus that $\Psi_j \geq 0$, for all j, $0 \leq j \leq N$. □

Lemma 2.2.2. ([40])

If Φ_j is any mesh function such that $\Phi_0 = \Phi_N = 0$. Then

$$|\Phi_j| \leq \frac{1}{\beta} \max_{1 \leq j \leq N-1} |L_\epsilon^N \Phi_j|, \qquad 0 \leq j \leq N.$$

Proof. We introduce two mesh functions

$$\Psi_j^\pm = \frac{1}{\beta} \max_{1 \leq i \leq N-1} |L_\varepsilon^N \Phi_j| \pm \Phi_i.$$

with $\Phi_j \geq \Phi_{j+1}$ and $\Phi_i \geq \Phi_{j-1}$. Thus,

$$\begin{aligned}
\Psi^\pm(0) &= \frac{1}{\beta} \max_{1 \leq j \leq N-1} |L_\varepsilon^N \Phi_j| \pm \Phi(0), \\
&= \frac{1}{\beta} \max_{1 \leq j \leq N-1} |-\varepsilon \delta^2 \Phi_j + b_j \Phi_j| \pm \Phi(0), \\
&\geq \frac{1}{\beta} \max_{1 \leq j \leq N-1} |-\varepsilon \delta^2 \Phi_j|, \\
&\geq 0.
\end{aligned}$$

It is easy to show analogously that $\Psi^\pm(1) \geq 0$.

Let $M = (1/\beta) \max_{1 \leq j \leq N-1} |L_\varepsilon^N \Phi_j|$, then

$$\begin{aligned}
L_\varepsilon^N \Psi^\pm &= -\varepsilon \delta^2 (M \pm \Phi_i) + b_i (M \pm \Phi_i), \\
&= L_\varepsilon^N M \pm L_\varepsilon^N \Phi_i, \\
&= b_i M \pm L_\varepsilon^N \Phi_i, \\
&\geq 0.
\end{aligned}$$

It then follows from the discrete maximum principle that $\Psi_i^\pm \geq 0$. □

This $\varepsilon-$uniform stability result for the difference operator L_ε^N is a prompt upshot of the discrete maximum principle. In [27] the remark is made that Lemma 2.2 implies that the solution is unique and since the problem under consideration is linear, the existence of the solution is implied by its uniqueness. Further, the boundedness of the solution is implied by Lemma 2.2.2.

In literature, the Shishkin-type meshes are considered to be simpler than, say, Bakhvalov-type meshes when comparing piecewise uniform meshes. Therefore, because of this reason, we use a Shishkin-type mesh for the FMFDM that we are going to develop to solve linear reaction diffusion problems. It is well known that such problems given by (2.1) have layers at both boundaries, hence we utilise the following mesh generating function, $\Omega^N = \{x_j\}_0^N$, such that

$$h_j = \begin{cases} \frac{4\lambda}{N}, & j = 1, \dots, \frac{N}{4} \\ \frac{2(1-\lambda)}{N}, & j = \frac{N}{4}+1, \dots, \frac{3N}{4} \\ \frac{4\lambda}{N}, & j = \frac{3N}{4}+1, \dots, N, \end{cases}$$

where $h_j = x_{j+1} - x_j, N$ is the number of subintervals and λ is defined as

$$\lambda = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\beta}}\ln N\right\}.$$

We develop the scheme by taking the Taylor series expansion for U_ε about x_j, we get

$$U_\varepsilon(x_{j-1}) \approx U_j - h_j U_j' + \frac{h_j^2}{2!}U_j'' - \frac{h_j^3}{3!}U^{(3)} + \frac{h_j^4}{4!}U^{(4)} + \dots \tag{2.4}$$

and

$$U_\varepsilon(x_{j+1}) \approx U_j + h_{j+1} U_j' + \frac{h_{j+1}^2}{2!}U_j'' + \frac{h_{j+1}^3}{3!}U^{(3)} + \frac{h_{j+1}^4}{4!}U^{(4)} + \dots \tag{2.5}$$

We obtain the first order first derivative approximations re-arranging (2.4) and (2.5), which gives us

$$D^-U_j = \frac{U_j - U_{j-1}}{h_j} \qquad \text{and} \qquad D^+U_j = \frac{U_{j+1} - U_j}{h_{j+1}}, \tag{2.6}$$

where $h_j = x_j - x_{j-1}, D^-U_j$ and D^+U_j are commonly referred to as first order forward and backward difference approximation for the first derivative, respectively.

Adding (2.4) and (2.5) and rearranging gives the second order second difference approximation:

$$\delta^2 U_j = \frac{2}{h_j + h_{j+1}} \left(D^+ U_j - D^- U_j \right) = \frac{2}{h_j + h_{j+1}} \left(\frac{U_{j+1} - U_j}{h_{j+1}} - \frac{U_j - U_{j-1}}{h_j} \right). \tag{2.7}$$

Using equations (2.1),(2.2) and (2.7), and simplifying produces a tridiagonal system of equations that can be represented in matrix notation as

$$AU = F, \tag{2.8}$$

where A is the matrix of the system and U and F are corresponding vectors. The various entries of this matrix and the components of the RHS vector are given by

$$\left.\begin{aligned}
(\text{supdiag}(A))_j &= r_j^+, & j &= 1, 2, \ldots, N-2, \\
(\text{maindiag}(A))_j &= r_j^c, & j &= 1, 2, \ldots, N-1, \\
(\text{subdiag}(A))_j &= r_j^-, & j &= 2, 3, \ldots, N-1, \\
F_j &= f_j, & j &= 1, 2, \ldots, N-1, \\
& & F_1 &= f_1 - r_1^- U_0, \\
& & F_{N-1} &= f_{N-1} - r_{N-1}^+ U_N,
\end{aligned}\right\} \tag{2.9}$$

where

$$\left.\begin{aligned}
r_j^+ &= -\varepsilon \tfrac{2}{h_{j+1}(h_j + h_{j+1})}, \\
r_j^c &= \varepsilon \tfrac{2}{(h_j + h_{j+1})} \left(\tfrac{1}{h_{j+1}} + \tfrac{1}{h_j} \right) + b_j, \\
r_j^- &= -\varepsilon \tfrac{2}{h_j (h_j + h_{j+1})}.
\end{aligned}\right\} \tag{2.10}$$

Elementary algebra gives the approximate solutions U_j as a vector in the above set of matrices. We refer, here on, to this numerical scheme as FMFDM.

2.3 Fitted Operator Finite Difference Method for reaction diffusion problems

We now look at the second broadly utilised approach which involves replacing the standard finite difference operator by a fitted finite difference operator which reflects the singularly perturbed nature of the differential operator. Such schemes are, by and large, mentioned as fitted operator finite difference methods [40]. We note that there are two categories of these methods, exponentially fitted methods and Non-Standard Finite Difference Methods developed using rules introduced by Mickens in [38]. The latter methods are the focus of this section, specifically the FOFDM constructed by Lubuma and Patidar [39] using these rules.

The denominator function for the FOFDM from [39] is as follows

$$\varphi_j = \frac{2}{\rho_j} \sinh\left(\frac{\rho_i h}{2}\right), \tag{2.11}$$

with $\rho_j \equiv \rho(x_j) = \sqrt{b_j/\varepsilon}$ and $b_j \equiv b(x_j)$ the coefficients of u_j. The scheme is basically developed by replacing the mesh parameter h_j in the second derivative approximation with the denominator function φ_j leading to $\delta^2 U_j$ in (2.7) being defined by

$$\delta_d^2 U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{\varphi_j^2}. \tag{2.12}$$

Fitted operator finite difference methods are ordinarily implemented on equidistant grid-points. We define the fitted difference operator of this scheme as

$$L_d \equiv -\varepsilon \delta_d^2 + b_j. \tag{2.13}$$

Analogous to the construction of the two FMFDMs we considered, the construction of the FOFDM scheme is completed by the use of equations (2.6) and (2.12), and simplifying produces a system of equations that can be represented in matrix notation as $BU = f$ with

$$\left.\begin{aligned}
(\text{supdiag}(B))_j &= r_j^+, & j &= 1, 2, \ldots, N-2,\\
(\text{maindiag}(B))_j &= r_j^c, & j &= 1, 2, \ldots, N-1,\\
(\text{subdiag}(B))_j &= r_j^-, & j &= 2, 3, \ldots, N-1,\\
f_j &= F_j, & j &= 1, 2, \ldots, N-1,\\
& & f_1 &= F_1 - r_1^- U_0,\\
& & f_{N-1} &= F_{N-1} - r_{N-1}^+ U_N,
\end{aligned}\right\} \tag{2.14}$$

where

$$\left.\begin{aligned}
r_j^+ &= -\tfrac{\varepsilon}{\varphi_j^2},\\
r_j^c &= 2\tfrac{\varepsilon}{\varphi_j^2} + b_j,\\
r_j^- &= -\tfrac{\varepsilon}{\varphi_j^2}.
\end{aligned}\right\} \tag{2.15}$$

Theorem 2.3.1. *Assume that $b(x)$ and $f(x)$ are sufficiently smooth so that $u(x) \in C^4[0,1]$. Then the FOFDM is second-order $\varepsilon-$uniformly convergent in the sense that the numerical solution U of the problem satisfies the error estimate*

$$\max_{1 \leq j \leq N-1} |u(x_j) - U(x_j)| \leq Ch^2.$$

2.4 Numerical simulations

We employ the maximum norm for the measurement of the error as our main objective is to investigate the behaviour of the error in the very small realms in which the boundary or interior layers occur. Norms such as the root mean square, may fail to capture the behaviour of the error in layer regions as rapid changes in the solution may smooth out because by definition they use error averages across the domain. Other examples showing why other norms are not best suited for our objective are given in the book by J. Miller *et al.*[40]. Maximum errors at all the mesh points are evaluated using the formula:

$$e_{N,\varepsilon} := \max_{0 \leq j \leq N} |u(x_j) - U(x_j)|, \tag{2.16}$$

for different values of N. The numerical rates of convergence are computed using the formula:

$$r_{k,\varepsilon} := \log_2 \left(\frac{e_{N_k,\varepsilon}}{e_{2N_k,\varepsilon}} \right), \qquad k = 1, 2, \ldots \tag{2.17}$$

The $\varepsilon-$uniform maximum errors are calculated using

$$E_N := \max_{0 < \varepsilon \leq 1} e_{N,\varepsilon}, \tag{2.18}$$

with the corresponding $\varepsilon-$uniform rates of convergence obtained using

$$r_k := \log_2 \left(\frac{E_{N_k}}{E_{2N_k}} \right), \qquad k = 1, 2, \ldots \tag{2.19}$$

The following linear reaction–diffusion problem considered by Amodio et al in [3], is considered for the comparison of the FMFDM and FOFDM. The test problem is considered over the interval $\Omega = (-1, 1)$.

Example 2.4.1. ([3])

$$\varepsilon u'' - u = -(\varepsilon \pi^2 + 1)\cos(\pi x),$$

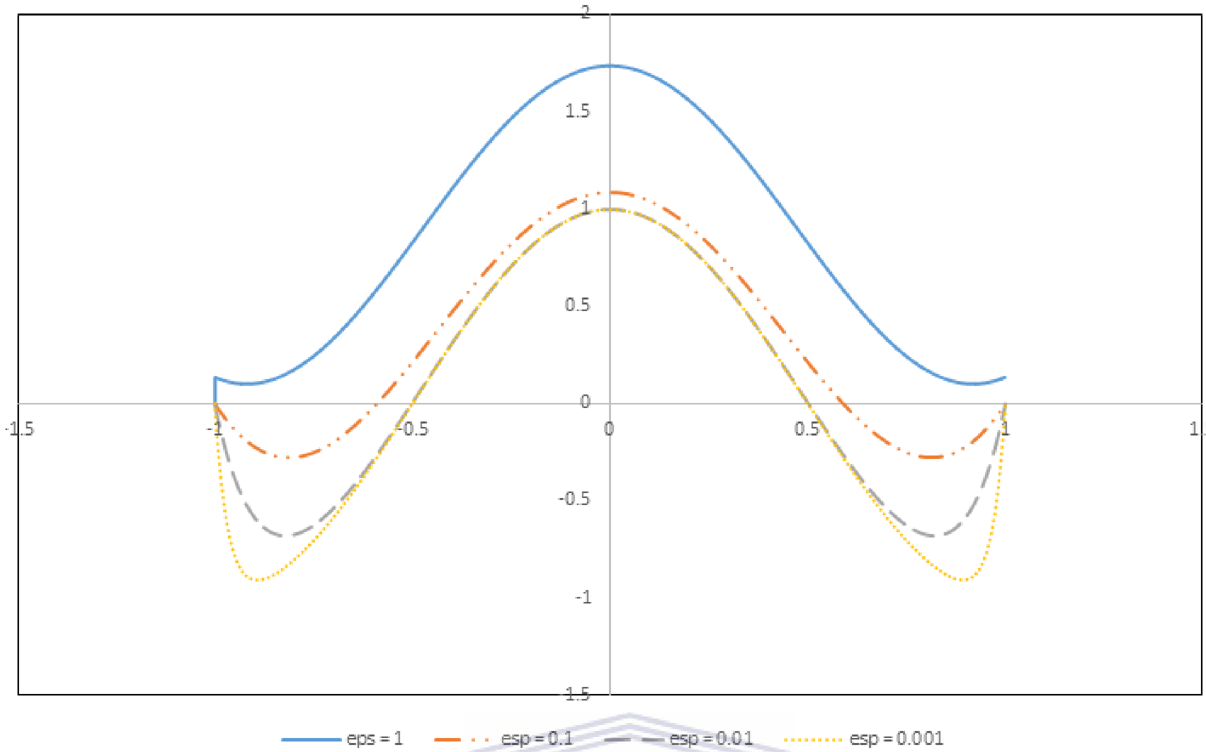

Figure 2.1: Solution profile of Example (2.4.1) for $\varepsilon \in \{1, 10^{-1}, 10^{-2}, 10^{-3}\}$.

with boundary conditions

$$u(-1) = u(1) = \exp(-2/\sqrt{\varepsilon}).$$

The exact solution is given by

$$u(x) = \cos(\pi x) + \exp\left(\frac{x-1}{\sqrt{\varepsilon}}\right) + \exp\left(-\frac{x+1}{\sqrt{\varepsilon}}\right).$$

The problem in Example 2.4.1 has layers of width $O(\sqrt{\varepsilon})$ near $x = -1$ and $x = 1$. Its profile for different values of ε is given in Figure 2.1. Tables 2.1 and 2.2 show that for both methods the maximum error values are bounded as ε becomes very small. However, the FOFDM gives better results than the FMFDM regardless of how small ε gets. Tables 2.3 and 2.4 give the rates for convergence for FMFDM and FOFDM. FMFDM is $\varepsilon-$uniformly convergent of almost 2 and FOFDM is $\varepsilon-$uniformly convergent of exactly 2 for $\varepsilon \ll 1$.

Table 2.1: Maximum errors obtained for Example 2.4.1 using the FMFDM.

ε	$N=20$	$N=40$	$N=80$	$N=160$	$N=320$	$N=640$	$N=1280$	$N=2560$
1	6.65E-02	1.71E-02	4.29E-03	1.07E-03	2.69E-04	6.72E-05	1.68E-05	4.20E-06
10^{-1}	4.59E-02	1.19E-02	3.00E-03	7.49E-04	1.87E-04	4.69E-05	1.17E-05	2.93E-06
10^{-2}	1.89E-01	9.36E-02	3.97E-02	7.58E-04	1.90E-04	4.74E-05	1.19E-05	2.97E-06
10^{-3}	2.24E-01	1.06E-01	5.99E-02	4.40E-02	3.88E-02	3.71E-02	3.63E-02	3.55E-02
10^{-4}	1.94E-01	7.05E-02	2.65E-02	1.21E-02	7.73E-03	6.56E-03	6.44E-03	6.67E-03
10^{-5}	1.80Ee-01	5.62E-02	1.52E-02	3.69E-03	1.83E-03	8.53E-04	4.93E-04	6.18E-04
10^{-6}	1.76E-01	5.13E-02	1.21E-02	3.81E-03	2.38E-03	1.30E-03	6.63E-04	3.15E-04
10^{-7}	1.74E-01	4.97E-02	1.22E-02	4.08E-03	2.54E-03	1.39E-03	7.26E-04	3.69E-04
10^{-8}	1.73E-01	4.92E-02	1.22E-02	4.17E-03	2.59E-03	1.42E-03	7.42E-04	3.79E-04
10^{-9}	1.73E-01	4.91E-02	1.22E-02	4.19E-03	2.60E-03	1.43E-03	7.47E-04	3.81E-04
10^{-10}	1.73E-01	4.90E-02	1.22E-02	4.20E-03	2.61E-03	1.43E-03	7.48E-04	3.82E-04

Table 2.2: Maximum errors obtained for Example 2.4.1 using the FOFDM.

ε	$N=20$	$N=40$	$N=80$	$N=160$	$N=320$	$N=640$	$N=1280$	$N=2560$
1	6.55E-02	1.68E-02	4.23E-03	1.06E-03	2.65E-04	6.62E-05	1.66E-05	4.14E-06
10^{-1}	4.29E-02	1.10E-02	2.78E-03	6.96E-04	1.74E-04	4.36E-05	1.09E-05	2.72E-06
10^{-2}	3.95E-02	1.02E-02	2.56E-03	6.42E-04	1.61E-04	4.016E-05	1.00E-05	2.51E-06
10^{-3}	4.18E-02	1.04E-02	2.58E-03	6.43E-04	1.61E-04	4.016E-05	1.00E-05	2.51E-06
10^{-4}	4.58E-02	1.14E-02	2.69E-03	6.52E-04	1.61E-04	4.02E-05	1.00E-05	2.51E-06
10^{-5}	4.66E-02	1.21E-02	2.98E-03	7.03E-04	1.66E-04	4.05E-05	1.01E-05	2.51E-06
10^{-6}	4.67E-02	1.22E-02	3.06E-03	7.61E-04	1.84E-04	4.30E-05	1.03E-05	2.52E-06
10^{-7}	4.67E-02	1.22E-02	3.07E-03	7.70E-04	1.92E-04	4.72E-05	1.12E-05	2.63E-06
10^{-8}	4.67E-02	1.22E-02	3.07E-03	7.70E-04	1.92E-04	4.81E-05	1.20E-05	2.92E-06
10^{-9}	4.67E-02	1.22E-02	3.07E-03	7.70E-04	1.93E-04	4.82E-05	1.20E-05	3.00E-06
10^{-10}	4.67E-02	1.22E-02	3.07E-03	7.70E-04	1.93E-04	4.82E-05	1.21E-05	3.01E-06

2.5 Discussion

Broad analysis, has been done in literature, of parameter uniform numerical methods for singularly perturbed reaction diffusion problems using fitted meshes of Bakhvalov or Shishkin–type. It now well established that using the pointwise maximum norm, second order (or almost second order in the case of the simpler Shishkin meshes) parameter

Table 2.3: Rates of convergence obtained for Example 2.4.1 using the FOFDM.

ε	r_1	r_2	r_3	r_4	r_5	r_6
1	1.96	1.99	2.00	2.00	2.00	2.00
10^{-1}	1.96	1.99	2.00	2.00	2.00	2.00
10^{-2}	1.96	1.99	2.00	2.00	2.00	2.00
10^{-3}	2.01	2.01	2.00	2.00	2.00	2.00
10^{-4}	2.01	2.08	2.05	2.01	2.00	2.00
10^{-5}	1.95	2.02	2.08	2.08	2.03	2.01
10^{-6}	1.94	1.99	2.01	2.05	2.10	2.07
10^{-7}	1.94	1.99	2.00	2.00	2.02	2.07
10^{-8}	1.94	1.98	2.00	2.00	2.00	2.01
10^{-9}	1.94	1.98	2.00	2.00	2.00	2.00
10^{-10}	1.94	1.98	2.00	2.00	2.00	2.00

Table 2.4: Rates of convergence obtained for Example 2.4.1 using the FMFDM.

ε	r_1	r_2	r_3	r_4	r_5	r_6
1	1.96	1.99	2.00	2.00	2.00	2.00
10^{-1}	1.95	1.99	2.00	2.00	2.00	2.00
10^{-2}	1.94	1.98	2.00	2.00	2.00	2.00
10^{-3}	0.93	0.76	0.57	0.48	0.56	0.93
10^{-4}	1.09	0.96	0.79	0.57	0.32	0.13
10^{-5}	1.47	1.28	1.15	0.99	0.79	0.52
10^{-6}	1.71	1.50	1.46	1.34	1.16	0.97
10^{-7}	1.79	1.60	1.65	1.62	1.50	1.31
10^{-8}	1.80	1.65	1.73	1.77	1.74	1.63
10^{-9}	1.80	1.67	1.75	1.82	1.85	1.82
10^{-10}	1.80	1.68	1.76	1.84	1.89	1.91

uniform convergence can be globally achieved. It is often assumed that the coefficient of the reactive term is strictly positive, i.e. $b_j \geq 0$ throughout the domain [18]. The numerical results obtained for the FMFDM conform to what we have seen in literature. Similarly with the results of the FOFDM as fitted operator finite difference methods generally achieve order 2 convergence.

In the next chapter, we introduce methods of improving the accuracy as well as the rate of convergence of numerical approximations. These methods are generally known as convergence acceleration techniques and we pay particular attention to defect correction methods.

Chapter 3

Defect Correction Methods on Shishkin Meshes

In the previous chapter, we compared two $\varepsilon-$uniform methods, namely, the fitted operator finite difference method and the fitted mesh finite difference method for solving reaction-diffusion singular perturbation problems. In this chapter, we discuss convergence acceleration techniques with particular focus on defect correction methods.

3.1 Introduction

Accuracy and rate of convergence of a system may be improved by direct higher order methods or by indirect higher order methods. Direct higher order methods are constructed directly using, for example, the Taylor series expansions. This is the idea used by Kadalbajoo and Patidar in [28]. Indirect higher order methods include schemes such as the extrapolation methods and defect corrections. The general idea of these methods is to use an easily implemented low-order method and apply some postprocessing technique to the computed solution to improve its accuracy. For extrapolation methods such as the Richardson extrapolation, an initial approximation is found using a coarse mesh with spacing h, then an improved approximation is found using a finer mesh with half the mesh spacing of the coarse mesh,that is, $h/2$. The improved approximations are then extrapolated on to the coarse mesh, with a linear combination of the two approximations then providing the higher order numerical solutions.

Defect correction methods are also indirect higher order methods. They allow low order stabilized methods to combine with higher–order methods that are less stable such as central differences, resulting in a higher–order method with the advantage that only well–conditioned discrete problems have to be solved [20].

We consider this well known method for improving accuracy of finite difference schemes and utilize ideas introduced by Frohner *et al.* in [20] and most of the analysis in this chapter is based on that paper.

We consider the singularly perturbed linear convection–diffusion problem

$$Lu := -\varepsilon u'' - (a(x)u)' + b(x)u = f(x), \qquad u(0) = u(1) = 0, \qquad \text{for } \ x \in (0,1). \quad (3.1)$$

where, as in previous chapters, the perturbation parameter is defined as being $0 < \varepsilon \ll 1$, a is taken to satisfy $a \geq \alpha > 0$ and $b \geq 0$. For f and a sufficiently smooth, the solution of u and its derivatives can be bounded by

$$\left|u^{(k)}(x)\right| \leq C\left(1 + \varepsilon^{-k} \exp\left(-\frac{\alpha x}{\varepsilon}\right)\right), \quad \text{for } \ k = 0,1,2,3 \quad \text{and} \quad x \in [0,1]. \qquad (3.2)$$

3.2 Discretization

We consider other variations of the Shishkin mesh together with the type introduced in Chapter 2. As before N is the discretization parameter taken to be an even positive integer. We denote the mesh transition parameter by λ and let it be defined by

$$\lambda = \min\left(\frac{1}{2}, \frac{\lambda_0 \varepsilon}{\alpha} \ln N\right),$$

where the constant $\lambda_0 > 0$ will be fixed later. In this section we make the mild assumption that $\lambda = \lambda_0 \varepsilon \alpha^{-1} \ln N$, as otherwise N^{-1} is exponentially small compared with ε and the mesh is equidistant. We also assume throughout that $\varepsilon \leq N^{-1}$ as is generally the case in practice. The layer term in (3.2) is ascertained to be smaller than $N^{-\lambda_0}$ on $[\lambda, 1]$ by the choice of the transition point. We consider a mesh $\omega : 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ which is equidistant in $[x_{N/2}, 1]$ but graded in $[0, x_{N/2}]$, where we choose the transition point $[x_{N/2}]$ in the Shishkin's sense, i.e., $x_{N/2} = \lambda$. On $[0, x_{N/2}]$ let our mesh be given by a mesh–generating function φ, with $\varphi(0) = 0$ and $\varphi(1/2) = \ln N$, where φ is continuous, monotonically increasing and piecewise continuously differentiable. Then our mesh is

$$x_j = \begin{cases} \frac{\lambda_0 \varepsilon}{\alpha} \varphi(t_j) & \text{for } t_j = j/N, j = 0, 1, \dots, N/2, \\ 1 - \left(1 - \frac{\lambda_0 \varepsilon}{\alpha} \ln N\right) \frac{2(N-j)}{N} & \text{for } j = N/2 + 1, \dots, N. \end{cases} \tag{3.3}$$

We denote by $h_j = x_j - x_{j-1}$ for $j = 1, \dots, N$ the local mesh sizes and by $h = \max_{j=1,\dots,N} h_j$ the maximal mesh size. For $j = N/2 + 1, \dots, N$ the mesh is uniform and we have $N^{-1} \leq h_j = H \leq 2N^{-1}$. For the maximal mesh size we have $h \leq CN^{-1}$. Examples of the mesh–characterizing function ψ that is closely related to φ by

$$\varphi = -\ln \psi. \tag{3.4}$$

This function, φ, is monotonically decreasing with $\psi(0) = 1$ and $\psi(N/2) = N^{-1}$. We give the followings examples of the mesh–characterizing function ψ as given by Frohner *et al.* in [20] :

- Standard Shishkin mesh:

$$\psi(t) = \exp\left(-2(\ln N)t\right).$$

- Bakhvalov–Shishkin mesh:

$$\psi(t) = 1 - 2(1 - N^{-1})t.$$

- Modified Bakhvalov–Shishkin (in the sense of Vulanovic):

$$\psi(t) = \exp\left(-\frac{t}{q-t}\right) \text{ with } q = \frac{1}{2} + \frac{1}{2\ln N}.$$

- Vulanovic's improved Shishkin mesh with two transition points is generated by

$$\psi(t) = \begin{cases} 4\alpha_N t & \text{if } t \in \left[0, \frac{1}{4}\right], \\ 2\alpha_N - \beta_N + 4(\beta_N - 4\alpha_N)t & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right], \end{cases}$$

with $\alpha_N = \ln(\ln N)$, $\beta_N = \ln N$.

3.3 Defect correction scheme

Fröhner *et al.* in [20], the defect correction scheme is constructed using a first–order accurate upwind scheme given by

$$
\begin{aligned}
[L^u u^u]_j &:= -\varepsilon \left\{ \frac{u^u_{j+1} - u^u_j}{h_{j+1}} - \frac{u^u_j - u^u_{j-1}}{h_j} \right\} - ((au^u)_{j+1} - (au^u)_j) + h_{j+1}(bu^u)_j, \\
&= h_{j+1} f_j, \\
&=: f^u_j,
\end{aligned} \tag{3.5}
$$

together with the unstable second-order central difference scheme

$$
\begin{aligned}
[L^c u^c]_j &:= -\varepsilon \left\{ \frac{u^c_{j+1} - u^c_j}{h_{j+1}} - \frac{u^c_j - u^c_{j-1}}{h_j} \right\} - \frac{(au^c)_{j+1} - (au^c)_{j-1}}{2} + \hbar_j (\widehat{bu^c})_j, \\
&= \hbar_j \widehat{f}_j, \\
&=: f^c_j,
\end{aligned} \tag{3.6}
$$

where $\hbar_j = (h_j + h_{j+1})/2$ and $g_j := (g_{j-1} + 2g_j + g_{j+1})/4$ for any function g. Using these difference operators we formulate our defect–correction method as follows:

1. Compute an initial first–order approximation using the upwind scheme

$$[L^u u^u]_j = f^u_j \qquad \text{for } j = 1, \ldots, N-1, \qquad u^u_0 = u^u_N = 0.$$

2. Estimate the defect τ by using central differences

$$\tau_j = f^c_j - [L^c u^u]_j \qquad \text{for } j = 1, \ldots, N-1. \tag{3.7}$$

3. Find the defect correction δ by solving

$$[L^u \delta]_j = \tau_j \qquad \text{for } j = 1, \ldots, N-1, \qquad \delta_0 = \delta_N = 0.$$

4. Finally, compute the corrected approximation

$$u^{dc} = u^u + \delta.$$

3.4 Error analysis

In this section we carry out analysis of the convergence in our method in the discrete norm defined by

$$||\upsilon||_{\infty,\omega} := \max_{j=1,\dots,N-1} |\upsilon_j|.$$

We also introduce the following norm that we utilize later in this chapter:

$$||\upsilon||_{*,\omega} := \max_{j=1,\dots,N-1} \left| \sum_{i=1}^{j} \upsilon_i \right|.$$

Theorem 3.4.1. Let us assume that the piecewise differentiable mesh–generating function φ satisfies the conditions

$$\max_{t\in[0,1/2]} \varphi'(t) \le CN \tag{3.8}$$

and

$$\int_0^{1/2} \varphi'(t)^2 \le CN. \tag{3.9}$$

Let $\lambda_0 \ge 3$. Then the error of the defect–correction method given by

$$\eta^{dc} := u^{dc} - u,$$

satisfies

$$||\eta^{dc}||_{\infty,\omega} \le C(N^{-1} \max |\psi'|)^2.$$

Proof. The error of the defect–correction method given by

$$\begin{aligned} \eta^{dc} &= u^{dc} - u, \\ &= u^u + \delta - u, \\ &= \delta + \eta^u. \end{aligned} \tag{3.10}$$

Application of the upwind operator then gives

$$\begin{aligned} L^u\eta^{dc} &= L^u\delta + L^u\eta^u, \\ &= f^c - L^c u^u + L^u\eta^u, \\ &= f^c - L^c u^u + L^u\eta^u + L^c u - L^c u, \\ &= L^u\eta^u - (L^c u^u - L^c u) + f^c - L^c u, \\ &= L^u\eta^u - L^c\eta^u + f^c - L^c u, \\ &= (L^u - L^c)\eta^u + f^c - L^c u. \end{aligned} \tag{3.11}$$

We use (3.5) and (3.6) to obtain the following result:

$$
\begin{aligned}
[(L^u - L^c)\upsilon]_j &= -\varepsilon \left\{ \frac{\upsilon_{j+1}-\upsilon_j}{h_{j+1}} - \frac{\upsilon_j - \upsilon_{j-1}}{h_j} \right\} - ((a\upsilon)_{j+1} - (a\upsilon)_j) + h_{j+1}(b\upsilon)_j, \\
&\quad - \left[-\varepsilon \left\{ \frac{\upsilon_{j+1}-\upsilon_j}{h_{j+1}} - \frac{\upsilon_j - \upsilon_{j-1}}{h_j} \right\} - \frac{(a\upsilon)_{j+1} - (a\upsilon)_{j-1}}{2} + \hbar_j(\widehat{b\upsilon})_j \right], \\
&= -\frac{(a\upsilon)_{j+1} - 2(a\upsilon)_j + (a\upsilon)_{j-1}}{2} + h_{j+1}(b\upsilon)_j - \hbar_j(\widehat{b\upsilon})_j,
\end{aligned}
\tag{3.12}
$$

for $j = 1, \ldots, N-1$.

We use the following stability property of the upwind operator without proof.

$$
||\upsilon_j||_{\infty,\omega} \leq C||L^u\upsilon||_{*,\omega} \quad \text{for all } \{\upsilon_j\}_{j=0}^N \text{ with } \upsilon_0 = \upsilon_N = 0. \tag{3.13}
$$

The proof of (3.13) will be given in Section 3.4.2 by the proof of Lemma 3.4.5

We use the fact that

$$
\begin{aligned}
\sum_{i=1}^{j}(h_{i+1}(b\upsilon)_i - \hbar_i(\widehat{b\upsilon})_i) = &-h_1\frac{(b\upsilon)_0 + 2(b\upsilon)_1 + (b\upsilon)_2}{8} \\
&+ h_{j+1}\left((b\upsilon)_j - \frac{(b\upsilon)_{j-1} + 2(b\upsilon)_j + (b\upsilon)_{j+1}}{8}\right) \\
&+ \sum_{i=1}^{j-1} h_{i+1}\left(-\frac{1}{8}(b\upsilon)_{i-1} + \frac{5}{8}(b\upsilon)_i - \frac{3}{8}(b\upsilon)_{i+1} - \frac{1}{8}(b\upsilon)_{i+2}\right)
\end{aligned}
$$

and application of the stabilty property for the upwind operator on (3.12) to get the following result for the upwind scheme error ($\eta^u := u^u - u$) :

$$
\begin{aligned}
||\eta^{dc}||_{\infty,\omega} &\leq C \left\| L^u \eta^{dc} \right\|_{*,\omega}, \\
&\leq C \left\{ \|(L^u - L^c)\eta^u\|_{*,\omega} + \|f^c - L^c u\|_{*,\omega} \right\}, \\
&\leq C \Bigg\{ \left\| - \frac{(av)_{j+1} - 2(av)_j + (bv)_{j-1}}{2} + h_{j+1}(bv)_j - \hbar_j (\widehat{bv})_j \right\|_{*,\omega}, \\
&\qquad + \|f^c - L^c u\|_{*,\omega} \Bigg\} \\
&\leq C \Bigg\{ \left\| \frac{(av)_{j+1} - 2(av)_j + (av)_{j-1}}{2} \right\|_{*,\omega} + \left\| h_{j+1}(bv)_j - \hbar_j (\widehat{bv})_j \right\|_{*,\omega}, \\
&\qquad + \|f^c - L^c u\|_{*,\omega} \Bigg\}, \qquad (3.14) \\
&\leq C \Bigg\{ \frac{1}{2} \left[\max_{j=0,\ldots,N-1} \left| \sum_{i=1}^{j} ((av)_{j+1} - 2(av)_j + (av)_{j-1}) \right| \right], \\
&\qquad\qquad + \max_{j=0,\ldots,N-1} \left| \sum_{i=1}^{j} \Big(h_{j+1}(bv)_j - \hbar_j (\widehat{bv})_j \Big) \right| + \|f^c - L^c u\|_{*,\omega} \Bigg\}, \\
&\leq C \Bigg\{ \frac{1}{2} \left[\max_{j=0,\ldots,N-1} |(av)_0 - (av)_1 - (av)_j + (av)_{j+1}| \right], \\
&\qquad + \max_{j=0,\ldots,N-1} \left| h_{j+1}(bv)_j - \hbar_j (\widehat{bv})_j \right| + \|f^c - L^c u\|_{*,\omega} \Bigg\}, \\
&\leq C \Bigg\{ \max_{j=0,\ldots,N-1} |(a\eta^u)_{j+1} - (a\eta^u)_j| + \max_{j=0,\ldots,N-1} |(b\eta^u)_{j+1} - (b\eta^u)_j|, \\
&\qquad + h||b\eta^u||_{\infty,\omega} + ||f^c - L^c u||_{*,\omega} \Bigg\}, \\
&\leq C \Bigg\{ \max_{j=0,\ldots,N-1} |\eta^u_{j+1} - \eta^u_j| + h||\eta^u||_{\infty,\omega} + ||f^c - L^c u||_{*,\omega} \Bigg\}.
\end{aligned}
$$

Since

$$
C(N^{-1} \max|\psi'|)^2 \geq \begin{cases} \max_{j=0,\ldots,N-1} |\eta^u_{j+1} - \eta^u_j|, \\ h||\eta^u||_{\infty,\omega}, \\ ||f^c - L^c u||_{*,\omega}, \end{cases} \qquad (3.15)
$$

it follows that

$$
||\eta^{dc}||_{\infty,\omega} \leq C(N^{-1} \max|\psi'|)^2.
$$

We will prove the result in (3.15) in Subsection 3.4.3 that will follow. □

Remark 1. From Fröhner *et al.* [20] we note that by theorem 3.4.1, we can analyze the perfomance of the scheme on the Shishkin meshes in a relatively easy way. All mesh–generating functions given earlier, for example, are characterized by a φ that satisfies (3.8) and (3.9). Table 3.1 has the maximum value of $|\psi'|$.

Table 3.1: Maximum values of $|\psi'|$

	$\max \lvert\psi'\rvert$
Standard Shishkin mesh	$C \ln N$
Bakhvalov-Shishkin mesh	C
Modified Bakhvalov-Shishkin mesh	C
Vulanovic improved Shishkin mesh	$C \ln(\ln N)$

3.4.1 Some properties of Shishkin meshes

Lemma 3.4.1. Let (3.8) hold true. Then

$$\int_{x_{j-1}}^{x_j} \left(1 + \varepsilon^{-1} e^{-\alpha\sigma/(k\varepsilon)}\right) d\sigma \le CN^{-1} \max_{t\in[0,1/2]} |\psi'(t)|,$$

for $\lambda_0 \ge k > 0$ and $j = 1, \ldots, N$.

Proof. For $j = N/2 + 1, \ldots, N$ we have

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \left(1 + \varepsilon^{-1} e^{-\alpha\sigma/(k\varepsilon)}\right) d\sigma &= \left[\sigma - \frac{k\varepsilon}{\beta}\left(\varepsilon^{-1} e^{-\alpha\sigma/(k\varepsilon)}\right)\right]_{x_{j-1}}^{x_j}, \\
&= h_j - \frac{k}{\alpha}\left[e^{-\alpha\sigma/(k\varepsilon)}\right]_{x_{j-1}}^{x_j}, \\
&= h_j - \frac{k}{\alpha}\left[e^{-\alpha x_j/(k\varepsilon)} - e^{-\alpha x_{j-1}/(k\varepsilon)}\right], \\
&\le h_j + \frac{k}{\alpha} e^{-\alpha x_{j-1}/(k\varepsilon)}, \\
&\le C\left[N^{-1} + N^{-\lambda_0/\alpha}\right], \quad \left\{x_{N/2} = \frac{\lambda_0 \varepsilon}{\alpha} \ln N \ \text{ at } \ j - 1 = N/2.\right\}, \\
&\le CN^{-1}.
\end{aligned}$$

For $j = 1, \dots, N/2$ we use the transformation $\sigma = \lambda_0 \varepsilon \alpha^{-1} \varphi(t)$ to get

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \left(1 + \varepsilon^{-1} e^{-\alpha\sigma/(k\varepsilon)}\right) d\sigma &= \frac{\lambda_0}{\alpha} \int_{t_{j-1}}^{t_j} \left(\varepsilon + e^{-\lambda_0 \varphi(t)/k}\right) \varphi'(t) dt, \\
&= \frac{\lambda_0}{\alpha} \int_{t_{j-1}}^{t_j} \left(\varepsilon + e^{(1-\lambda_0/k)\varphi(t)}\right) |\psi'(t)| dt, \\
&\leq C \left(\varepsilon + N^{-1} \max_{t \in [t_{j-1}, t_j]} e^{(1-\lambda_0/k)\varphi(t)} \max_{t \in [t_{j-1}, t_j]} |\psi'(t)| \right), \\
&\leq C N^{-1} \max_{t \in [t_{j-1}, t_j]} |\psi'(t)|,
\end{aligned}$$

where we have used the relation between φ and ψ, (3.8) and $0 \leq \varphi(t) \leq \ln N$. Combine the two inequalities and note that $\max |\psi'| \geq C$. □

Lemma 3.4.2. Let (3.8) be satisfied. Then

$$h_j \leq C\varepsilon \quad \text{and} \quad \frac{h_j}{\varepsilon} e^{-\alpha x_j/(\lambda_0 \varepsilon)} \leq C N^{-1} \max_{t \in [0,1/2]} |\psi'(t)| \quad \text{for } j = 1, \dots, N/2.$$

Proof. For $j = 1, \dots, N/2$, we have

$$\begin{aligned}
h_j &= x_j - x_{j-1}, \\
&= \frac{\lambda_0 \varepsilon}{\alpha} \left(\varphi(t_j) - \varphi(t_{j-1})\right), \\
&= \frac{\lambda_0 \varepsilon}{\alpha} \int_{t_{j-1}}^{t_j} \varphi'(t) dt, \\
&\leq \frac{\lambda_0 \varepsilon}{\alpha} N^{-1} \max_{t \in [t_{j-1}, t_j]} |\varphi'(t)|.
\end{aligned} \tag{3.16}$$

From (3.8) we have $\max_{t \in [0,1/2]} \varphi'(t) \leq CN$, therefore it follows that

$$h_j \leq C\varepsilon.$$

Using result for (3.16), we have

$$\begin{aligned}
h_j &= \frac{\lambda_0 \varepsilon}{\alpha} N^{-1} \max_{t \in [t_{j-1}, t_j]} |\varphi'(t)|, \\
&= \frac{\lambda_0 \varepsilon}{\alpha} N^{-1} \frac{\max_{t \in [0,1/2]} |\psi'(t)|}{\min_{t \in [t_{j-1}, t_j]} |\psi(t)|},
\end{aligned}$$

since $\varphi' = -\psi'/\psi$ from definition of ψ. But

$$\begin{aligned}
\min_{t \in [t_{j-1}, t_j]} \psi(t) &= \min_{t \in [t_{j-1}, t_j]} e^{-\alpha x/(\lambda_0 \varepsilon)}, \\
&= e^{-\alpha x_j/(\lambda_0 \varepsilon)},
\end{aligned}$$

hence

$$
\begin{aligned}
h_j &\leq \frac{\lambda_0 \varepsilon}{\alpha} N^{-1} \max_{t\in[0,1/2]} |\psi'(t)| e^{\alpha x_j/(\lambda_0\varepsilon)}, \\
\frac{h_j}{\varepsilon} e^{-\alpha x_j/(\lambda_0\varepsilon)} &\leq \frac{\lambda_0}{\alpha} N^{-1} \max_{t\in[0,1/2]} |\varphi'(t)|, \\
&\leq C N^{-1} \max_{t\in[0,1/2]} |\varphi'(t)|.
\end{aligned}
$$

□

Lemma 3.4.3. Let (3.9) hold true. Then

$$
K_i^j := \prod_{k=i}^{j} \left(1 + \frac{a_k h_k}{\varepsilon}\right)^{-1} \leq e^{-\alpha(x_j - x_{i-1})/\varepsilon}, \quad \text{for } \ 1 \leq i \leq j \leq N/2.
$$

Proof.

$$
\begin{aligned}
\ln\left(\prod_{k=i}^{j}\left(1+\frac{\alpha h_k}{\varepsilon}\right)\right) &\geq \sum_{k=i}^{j}\left[\frac{\alpha h_k}{\varepsilon} - \frac{1}{2}\left(\frac{\alpha h_k}{\varepsilon}\right)^2\right], \text{since } \ \ln(1+t) \geq t - t^2/2, \\
&\geq \frac{\alpha}{\varepsilon}\sum_{k=i}^{j} h_k - \sum_{k=1}^{N/2}\left(\frac{\alpha h_k}{\varepsilon}\right)^2, \\
&\geq \frac{\alpha(x_j - x_{i-1})}{\varepsilon} - \frac{1}{2}\sum_{k=1}^{N/2}\left(\frac{\alpha h_k}{\varepsilon}\right)^2.
\end{aligned}
$$

We multiply terms both sides of the inequality and simplify to get,

$$
\begin{aligned}
\prod_{k=i}^{j}\left(1+\frac{\alpha h_k}{\varepsilon}\right)^{-1} &\leq e^{-\alpha(x_j - x_{i-1})/\varepsilon} \exp\left(\frac{1}{2}\sum_{k=1}^{N/2}\left(\frac{\alpha h_k}{\varepsilon}\right)^2\right), \\
&\leq e^{-\alpha(x_j - x_{i-1})/\varepsilon} \exp\left(\frac{1}{2}\lambda_0^2 \sum_{k=1}^{N/2}(t_k - t_{k-1}) \int_{t_{k-1}}^{t_k} \varphi'(\tau)^2 d\tau\right), \\
&\leq e^{-\alpha(x_j - x_{i-1})/\varepsilon} \exp\left(\frac{1}{2}\lambda_0^2 N^{-1} \int_0^{1/2} \varphi'(\tau)^2 d\tau\right), \\
&\leq e^{-\alpha(x_j - x_{i-1})/\varepsilon} \exp\left(\frac{1}{2}\lambda_0^2 N^{-1}(CN)\right) \quad \text{by (3.9)}, \\
&\leq e^{-\alpha(x_j - x_{i-1})/\varepsilon}.
\end{aligned}
$$

where $(\alpha h_k)/\varepsilon = \lambda_0 \int_{t_{k-1}}^{t_k} \varphi'(\tau) d\tau \qquad \text{for } \ k = 1, \ldots, N/2.$ □

Lemma 3.4.4. Suppose (3.8) and (3.9) are satisfied. Let $\lambda_0 \geq 1$. Then

$$\frac{h_j}{\varepsilon} K_1^j \leq \begin{cases} CN^{-1} \max |\psi'| & \text{for } j = 1, \dots, N/2, \\ CN^{-\lambda_0} & \text{for } j = N/2+1, \dots, N. \end{cases}$$

Proof. For $j = 1, \dots, N/2$ we consider a result from Lemma 3.4.3 as follows:

$$\begin{aligned} K_1^j &\leq e^{-\alpha(x_j - x_0)/\varepsilon}, \\ \frac{h_j}{\varepsilon} K_1^j &\leq \frac{h_j}{\varepsilon} e^{-\alpha(x_j - x_0)/\varepsilon}, \\ &\leq C \frac{h_j}{\varepsilon} e^{-\alpha x_j/(\lambda_0 \varepsilon)}, \\ &\leq CN^{-1} \max |\psi'| \quad \text{using Lemma 3.4.2.} \end{aligned}$$

Since the sequence K_1^j is monotonically decreasing, for $j = N/2+1, \dots, N$ we have

$$\begin{aligned} \frac{h_j}{\varepsilon} K_1^j &\leq \frac{h_j}{\varepsilon} \left(1 + \frac{a_k h_k}{\varepsilon}\right)^{-1} K_1^{N/2}, \\ &\leq \alpha^{-1} K_1^{N/2}, \\ &\leq CN^{-\lambda_0}, \end{aligned}$$

from application of Lemma 3.4.3. □

3.4.2 Analysis of the error of the upwind scheme

In this section we carry out the analysis of the stable first–order method used in the defect–correction method.

Lemma 3.4.5. Let $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary mesh. Let $\{\upsilon_j\}_{i=0}^N$ be an arbitrary mesh function defined on ω with $\upsilon_0 = \upsilon_N = 0$. Then

$$||\upsilon||_{\infty,\omega} \leq 2\alpha^{-1} e^{\gamma^*/\alpha} ||L^u \upsilon||_{*,\omega}$$

where $\gamma^* = \max_{x \in [0,1]} c(x) \geq 0$.

Proof. We rewrite the function υ in the form

$$\upsilon_j = \frac{W_N}{V_N} V_j - W_j, \tag{3.17}$$

where $\{V_j\}_{j=0}^{N}$ is the solution of

$$[A^u V]_j = 1 \qquad \text{for } j = 1, \ldots, N, \qquad V_0 = 0$$

and $\{W_j\}_{j=0}^{N}$ is the solution of

$$[A^u W]_j = \sum_{i-1}^{j-1} [L^u \upsilon]_i \qquad \text{for } j = 1, \ldots, N, \qquad W_0 = 0,$$

with

$$[A^u w]_j := \varepsilon \frac{w_j - w_{j-1}}{h_j} + (aw)_j - \sum_{i-1}^{j-1} h_{i+1}(bw)_i \qquad \text{for any grid function } w. \qquad (3.18)$$

We define a mesh function z by

$$z_j = \prod_{k=1}^{j} \left(1 + \frac{\gamma^*}{\alpha} h_k\right) \qquad \text{for } j = 1, \ldots, N.$$

It follows that

$$1 = z_0 < z_1 < \cdots < z_N,$$

with

$$\begin{aligned}
z_N &= \prod_{k=1}^{N} \left(1 + \frac{\gamma^*}{\alpha} h_k\right), \\
&< \prod_{k=1}^{N} e^{(\gamma^*/\alpha)h_k}, \ \text{since} \left(1 + \frac{\gamma^*}{\alpha} h_k\right) < e^{(\gamma^*/\alpha)h_k}, \\
&< \left(e^{(\gamma^*/\alpha)h_1}\right) \left(e^{(\gamma^*/\alpha)h_2}\right) \ldots \left(e^{(\gamma^*/\alpha)h_N}\right), \\
&< e^{(\gamma^*/\alpha)(h_1 + h_2 + \cdots + N)}, \\
&< e^{(\gamma^*/\alpha)(\sum_{k=1}^{N} h_k)}, \\
&< e^{(\gamma^*/\alpha)}.
\end{aligned}$$

Then again, z satisfies

$$
\begin{aligned}
[A^u z]_j &= \varepsilon \frac{z_j - z_{j-1}}{h_j} + (az)_j - \sum_{i-1}^{j-1} h_{i+1}(bz)_i, \qquad \text{from (3.18)},\\
&= \frac{\varepsilon}{h_j}\left\{\prod_{k=1}^{j}\left(1+\frac{\gamma^*}{\alpha}h_k\right) - \prod_{k=1}^{j-1}\left(1+\frac{\gamma^*}{\alpha}h_k\right)\right\} + a_j\prod_{k=1}^{j}\left(1+\frac{\gamma^*}{\alpha}h_k\right)\\
&\quad - \sum_{i=1}^{j-1} h_{i+1} b_i \prod_{k=1}^{i}\left(1+\frac{\gamma^*}{\alpha}h_k\right),\\
&= \frac{\varepsilon}{h_j}\left\{\left[\left(1+\frac{\gamma^*}{\alpha}h_j\right)-1\right]\prod_{k=1}^{j-1}\left(1+\frac{\gamma^*}{\alpha}h_k\right)\right\} + a_j\prod_{k=1}^{j}\left(1+\frac{\gamma^*}{\alpha}h_k\right)\\
&\quad - \sum_{i=1}^{j-1} h_{i+1} b_i \prod_{k=1}^{i}\left(1+\frac{\gamma^*}{\alpha}h_k\right),\\
&= \varepsilon\frac{\gamma^*}{\alpha}\prod_{k=1}^{j-1}\left(1+\frac{\gamma^*}{\alpha}h_k\right) + a_j\prod_{k=1}^{j}\left(1+\frac{\gamma^*}{\alpha}h_k\right) - \sum_{i=1}^{j-1} h_{i+1} b_i \prod_{k=1}^{i}\left(1+\frac{\gamma^*}{\alpha}h_k\right),\\
&> \alpha\prod_{k=1}^{j}\left(1+\frac{\gamma^*}{\alpha}h_k\right) - \gamma^*\sum_{i=1}^{j-1} h_{i+1}\prod_{k=1}^{i}\left(1+\frac{\gamma^*}{\alpha}h_k\right),\\
&> \alpha\left(1+\frac{\gamma^*}{\alpha}h_1\right),\\
&> \alpha.
\end{aligned}
$$

When we apply the test for the $M-$matrix, we conclude that A^u is a $M-$matrix and moreover, we can prove that

$$
0 < V_j \leq \alpha^{-1} e^{\gamma^*/\alpha} \quad \text{and} \quad |W_j| \leq V_j ||L^u \upsilon||_{*,\omega} \quad \text{for } j = 1, \ldots, N. \tag{3.19}
$$

From (3.17) we get

$$
\begin{aligned}
||\upsilon_j|| &= \left\|\frac{W_N}{V_N}V_j - W_j\right\|,\\
&\leq \left\|\frac{W_N}{V_N}V_j\right\| + \|W_j\|.
\end{aligned}
$$

Using (4.1), we have

$$
\begin{aligned}
||\upsilon_j|| &\leq \alpha^{-1}e^{\gamma^*/\alpha}||L^u\upsilon||_{*,\omega} + \alpha^{-1}e^{\gamma^*/\alpha}||L^u\upsilon||_{*,\omega},\\
&\leq 2\alpha^{-1}e^{\gamma^*/\alpha}||L^u\upsilon||_{*,\omega}.
\end{aligned}
$$

□

Theorem 3.4.2. Let $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary mesh. Then the error $\eta^u = u - u^u$ of the upwind scheme $L^u u^u = f^u$ for (3.1) satisfies

$$||\eta^u||_{\infty,\omega} \leq C \max_{i=1,\dots,N} \int_{x_{i-1}}^{x_i} (1 + |u'(s)|)\, ds.$$

Proof. Integrating (3.1) over $[x_{j+1}, x_j]$ we get

$$-\varepsilon(u'_{j+1} - u'_j) - ((au)_{j+1} - (au)_j) + \int_{x_j}^{x_{j+1}} (bu)(s)ds = \int_{x_j}^{x_{j+1}} f(s)ds,$$

re–arranging gives

$$\varepsilon u'_{j+1} - \varepsilon u'_j - \int_{x_j}^{x_{j+1}} (bu - f)(s)ds = -(au)_{j+1} + (au)_j. \tag{3.20}$$

We define the truncation error of the upwind scheme by

$$\tau^u := L^u u - f^u.$$

Application of (3.20) ot the definition of L^u gives

$$\begin{aligned} \sum_{i=1}^{j-1} \tau^u = &- \varepsilon\left(\frac{u_j - u_{j-1}}{h_j} + u'_j\right) + \varepsilon\left(\frac{u_1 - u_0}{h_1} + u'_1\right) \\ &+ \sum_{i=1}^{j-1}\left\{h_{i+1}(bu - f)_i - \int_{x_i}^{x_{i+1}} (bu - f)(s)ds\right\}. \end{aligned} \tag{3.21}$$

Taking Taylor series expansions of the right–hand side of (3.21) we get

$$\begin{aligned} \varepsilon\left|\frac{u_j - u_{j-1}}{h_j} - u'_j\right| &\leq \varepsilon \int_{x_{j-1}}^{x_j} |u''(s)|ds, \\ &\leq \int_{x_{j-1}}^{x_j} |(f - bu + (au)')(s)|ds \end{aligned}$$

and

$$\left|h_{i+1}(bu - f)_i - \int_{x_i}^{x_{i+1}} (bu - f)(s)ds\right| \leq h_{i+1} \int_{x_i}^{x_{i+1}} |(bu - f)'(s)|\, ds.$$

Employing the boundedness of u, we get

$$\|\eta^u\|_{*,\omega} \leq \max_{i=1,\dots,N} \int_{x_{i-1}}^{x_i} (1 + |u'(s)|ds. \tag{3.22}$$

Use of Lemma 3.4.5 concludes the proof. □

Corollary 3.4.1. Let ω be a Shishkin–type mesh. Suppose that (3.8) holds true. Then the error of η^u of the upwind scheme $L^u u^u = f^u$ satisfies

$$||\eta^u||_{\infty,\omega} \leq CN^{-1} \max |\psi'| \qquad \text{for } \; \lambda_0 \geq 1.$$

Corollary 3.4.1 is a direct result of Theorem 3.4.2, and Lemma 3.4.1 as well as the inequality in (3.2)

Proof. From theorem 3.4.2 we have

$$\begin{aligned}
||\eta^u||_{\infty,\omega} &\leq C \max_{i=1,\dots,N} \int_{x_{i-1}}^{x_i} (1 + |u'(s)|)\, ds, \\
&\leq C \max_{i=1,\dots,N} \left[\int_{x_{i-1}}^{x_i} 1 ds + \int_{x_{i-1}}^{x_i} (|u'(s)|)\, ds \right], \\
&\leq C \max_{i=1,\dots,N} \left[h_i + \int_{x_{i-1}}^{x_i} (|u'(s)|)\, ds \right], \\
&\leq C \left[h + \max_{i=1,\dots,N} \int_{x_{i-1}}^{x_i} (|u'(s)|)\, ds \right], \\
&\leq C \left[h + \max_{i=1,\dots,N} \int_{x_{i-1}}^{x_i} \left(1 + \varepsilon^{-1} e^{-\alpha s/\epsilon}\right) ds \right] \quad \big(\text{from } (3.2)\big), \\
&\leq C \left[h + N^{-1} \max |\psi'| \right] \quad \big(\text{from lemma } 3.4.1\big), \\
&\leq C \left[2N^{-1} \max |\psi'| \right], \\
&\leq CN^{-1} \max |\psi'|.
\end{aligned}$$

□

3.4.3 Approximation of the derivatives

Theorem 3.4.3. ([20]) The error $\eta^u = u - u^u$ of the upwind scheme $L^u u^u = f^u$ for (3.1) on a Shishkin–type mesh satisfies

$$|\eta_j^u - \eta_{j-1}^u| \leq \begin{cases} C(N^{-1} \max |\psi'|)^2 & \text{for } \; j = 1, \dots, N/2 \; \text{ and } \; \lambda_0 \geq 3, \\ CN^{-2} \max |\psi'| & \text{for } \; j = N/2 + 1, \dots, N \; \text{ and } \; \lambda_0 \geq 2. \end{cases}$$

Proof. Recalling the proof of Lemma 3.4.5 with $\upsilon = \eta^u$, we can write the difference as

$$\left|\eta_j^u - \eta_{j-1}^u\right| = \frac{W_N}{V_N}(V_j - V_{j-1}) - (W_j - W_{j-1}), \tag{3.23}$$

where $\{V_j\}_{j=0}^N$ is the solution of

$$[A^u V]_j = 1, \qquad \text{for } j = 1, \dots, N, \qquad V_0 = 0 \tag{3.24}$$

and $\{W_j\}_{i=0}^N$ is the solution of

$$[A^u W]_j = \sum_{i-1}^{j-1} \tau_i^u, \qquad \text{for } j = 1, \dots, N, \qquad W_0 = 0. \tag{3.25}$$

We furthermore, know that

$$|W_N| \le V_N \|\tau^u\|_{*,\omega} . \tag{3.26}$$

We represent the differences $V_j^- := V_j - V_{j-1}$ and $W_j^- := W_j - W_{j-1}$ as the solution of appropriate difference equations involving an $M-$matrix M^u. To derive an upper bound on W^-, we again split

$$W^- = \widetilde{W}^- + \overline{W}_d^- + \overline{W}_l^-,$$

exploiting the specific structure of the matching defining equations. Each addend will be bounded separately.

Let

$$[M^u w]_j := \varepsilon\left(\frac{w_j}{h_j} - \frac{w_{j-1}}{h_{j-1}}\right) + (aw)_j.$$

Taking the difference $[A^u V]_j - [A^u V]_{j-1}$ from (3.24), we see V_j^- is the solution of the first–order difference equation

$$\begin{aligned}
\left[M^u V^-\right]_j &= (a_{j-1} - a_j + h_j b_{j-1}) V_{j-1}, && \text{for } j = 2, \dots, N, \\
\left[M^u V^-\right]_1 &= \varepsilon\left(\frac{V_1^-}{h_1} - \frac{V_0^-}{h_0}\right) + (aV^-)_1, && \text{for } \ j = 1, \\
&= \frac{\varepsilon}{h_1} V_1^- + a_1 V_1^-, && \text{since } \ V_0^- = 0, \\
&= \left(\frac{\varepsilon}{h_1} + a_1\right) V_1^-, \\
&= \frac{h_1}{\varepsilon}\left(1 + \frac{a_1 h_1}{\varepsilon}\right)^{-1} V_1^-,
\end{aligned} \tag{3.27}$$

it follows that

$$\begin{aligned}
V_1^- &= \frac{h_1}{\varepsilon}\left(1 + \frac{a_1 h_1}{\varepsilon}\right)^{-1}, \\
&= \frac{h_j}{\varepsilon} K_1^j,
\end{aligned}$$

using K_i^j from Lemma 3.4.3, where the initial condition is obtained from (3.24) for $j = 1$.

It is evident that M^u is a L^0-matrix. Application of the $M-$matrix criterion with the test function $w_j = h_j$ proves that M^u is an $M-$matrix. From (3.27) we have

$$
\begin{aligned}
\left[M^u V^-\right]_j &= (a' h_j + h_j b_{j-1}) V_{j-1}, \\
&= (a' + b_{j-1}) V_{j-1} h_j, \\
&\leq (a' + \gamma) V_{j-1} h_j, \qquad \text{taking } \ 0 < c(x) \leq \gamma, \\
&\leq (a' + \gamma) \alpha^{-1} e^{\gamma/\alpha} h_j,
\end{aligned}
$$

from the proof of Lemma 3.4.5. Taking absolute values, we get

$$
\begin{aligned}
\left| \left[M^u V^-\right]_j \right| &\leq + \left|(a' + \gamma)\right| \alpha^{-1} e^{\gamma/\alpha} h_j \quad \text{ for } j = 2, \ldots, N, \\
&\leq + \left(\|a'\|_\infty + \gamma\right) \alpha^{-1} e^{\gamma/\alpha} h_j.
\end{aligned}
$$

It follows that

$$
\left|V_j^-\right| \leq \frac{h_j}{\varepsilon} K_1^j + \left(\|a'\|_\infty + \gamma\right) \alpha^{-2} e^{\gamma/\alpha} h_j \quad \text{ for } j = 1, 2, \ldots, N,
$$

by a discrete comparison principle.

We apply Lemma 3.4.4 to the right hand side to obtain

$$
\left|V_j^-\right| \leq \begin{cases} CN^{-1} \max|\psi'| & \text{for } \ j = 1, \ldots, N/2 \\ CN^{-1} & \text{for } \ j = N/2 + 1, \ldots, N. \end{cases} \tag{3.28}
$$

For $W_j^- := W_j - W_{j-1}$ and from (3.25), we have

$$
\left[M^u W^-\right]_j = (a_{j-1} - a_j + h_i b_{j-1}) W_{j-1} + \tau_{j-1}^u, \qquad \text{for } j = 2, \ldots, N, \quad W_0^- = 0.
$$

Splitting of W^- leads to

$$
W^- = \widetilde{W}^- + \overline{W}^-, \tag{3.29}
$$

where $\widetilde{W}^-$ and $\overline{W}^-$ solve

$$
\left[M^u \widetilde{W}^-\right]_j = (a_{j-1} - a_j + h_j b_{j-1}) W_{j-1} \quad \text{ for } j = 2, \ldots, N, \quad \widetilde{W}_0^- = 0
$$

and

$$
\left[M^u \overline{W}^-\right]_j = \tau_{j-1}^u \quad \text{ for } j = 2, \ldots, N, \quad \overline{W}_0^- = 0.
$$

Analogous to the approach used for V^-, we can use the discrete comparison principle to obtain

$$\left|\widetilde{W}_j^-\right| \le (\|a'\|_\infty + \gamma)\,\alpha^{-2}e^{\gamma/\alpha}h_{j+1}\|\tau^u\|_{*,\infty} \qquad \text{for } j=1,2,\ldots,N.$$

Thus on a Shishkin-type mesh, we have

$$\left|\widetilde{W}_j^-\right| \le CN^{-2}\max|\psi'| \qquad \text{for } j=1,2,\ldots,N, \tag{3.30}$$

by (3.22) and Lemma 3.4.1.

Now we bound $\overline{W}^-$. We integrate (3.1) over $[x_{j-1},x_j]$ as we did in the the proof of theorem 3.4.2, we get

$$\begin{aligned}\left[M^u\overline{W}^-\right]_j &= \tau_{j-1}^u,\\ &= -\varepsilon\left(\frac{u_j-u_{j-1}}{h_j}-u_j'\right)+\varepsilon\left(\frac{u_{j-1}-u_{j-2}}{h_{j-1}}-u_{j-1}'\right)\\ &\quad + h_j(bu-f)_{j-1}-\int_{x_{j-1}}^{x_j}(bu-f)(s)ds.\end{aligned}$$

We split $\overline{W}^-$ corresponding to the diffusion terms and lower order terms:

$$\overline{W}^- = \overline{W}_d^- + \overline{W}_l^-, \tag{3.31}$$

where

$$\begin{aligned}\left[M^u\overline{W}_d^-\right]_j &= -\varepsilon\left(\frac{u_j-u_{j-1}}{h_j}-u_j'\right)+\varepsilon\left(\frac{u_{j-1}-u_{j-2}}{h_{j-1}}-u_{j-1}'\right),\\ &= -\varepsilon\left(\chi_j-\chi_{j-1}\right) \qquad \text{with } \chi_j := \frac{u_j-u_{j-1}}{h_j}-u_j', \qquad \overline{W}_{d,1}^- = 0\end{aligned}$$

and

$$\left[M^u\overline{W}_l^-\right]_j = h_j(bu-f)_{j-1}-\int_{x_{j-1}}^{x_j}(bu-f)(s)ds \qquad \overline{W}_{l,1}^- = 0.$$

First we consider $\overline{W}_l^-$ and employing Taylor series expansions, we have

$$\begin{aligned}\left|\left[M^u\overline{W}_l^-\right]_j\right| &\le Ch_j\int_{x_{j-1}}^{x_j}\left(1+\varepsilon^{-1}e^{-\alpha s/\epsilon}\right)ds,\\ &\le Ch_jN^{-1}\max|\psi'|.\end{aligned}$$

Thus

$$\left|\overline{W}_{l,j}^{-}\right| \le Ch_j N^{-1} \max|\psi'|, \\ \le CN^{-2}\max|\psi'| \quad \text{for } j = 1, 2, \ldots, N. \tag{3.32}$$

To study $\overline{W}_{d}^{-}$ we solve the defining difference equation and get

$$\begin{aligned}\overline{W}_{d,j}^{-} &= h_j \sum_{k=2}^{j} K_k^j \left(\chi_{k-1} - \chi_k\right) \\ &= h_j K_2^j \chi_1 + h_j \sum_{k=2}^{j-1} K_k^j \frac{a_k h_k}{\varepsilon}\chi_k - h_j\left(1 + \frac{a_j h_j}{\varepsilon}\right)^{-1}\chi_j,\end{aligned} \tag{3.33}$$

for $j = 2, \ldots, N$. For χ_k, we use Taylor expansions to obtain

$$|\chi_k| \le C\int_{x_{k-1}}^{x_k}\left(1 + \varepsilon^{-1}e^{-\alpha s/\varepsilon}\right) ds$$

and hence derive the two bounds

$$|\chi_k| \le C\left(h_k + \varepsilon^{-1}e^{-\alpha x_{k-1}/\varepsilon}\sinh\frac{\alpha h_k}{2\varepsilon}\right) \tag{3.34}$$

and

$$|\chi_k| \le C\left(h_k + \varepsilon^{-1}e^{-\alpha x_{k-1}/\varepsilon}\right). \tag{3.35}$$

We observe that for $k = 1, \ldots, N/2$ estimate (3.34) is substantially stronger than (3.35) since $\sinh(\alpha h_k/(2\varepsilon)) \le C\alpha h_k/(2\varepsilon) \le CN^{-1}\max|\psi'|$. While on the other hand, for $k > N/2, \sinh(\alpha h_k/(2\varepsilon))$ cannot be bounded uniformly in ε.

We now look for bounds for the three terms on the right hand side of (3.33). For the last term, for $\lambda_0 \ge 2$ and $i = 2, \ldots, N/2$ by (3.34) we have

$$\begin{aligned}h_j\left(1 + \frac{a_j h_j}{\varepsilon}\right)^{-1}|\chi_j| &\le C\left(h_k + \varepsilon^{-1}e^{-\alpha x_{k-1}/\varepsilon}\sinh\frac{\alpha h_k}{2\varepsilon}\right), \\ &\le C\left(h_j^2 + \left(\frac{h_j}{\varepsilon}e^{-\alpha x_j/(\lambda_0\varepsilon)}\right)^2\right), \\ &\le C\left(\varepsilon^2 + \left\{N^{-1}\max|\psi'|\right\}^2\right), \text{ using Lemma 3.4.2} \\ &\le C(N^{-1}\max|\psi'|)^2.\end{aligned}$$

For $j = N/2 + 1, \ldots, N$, we use (3.35) to get

$$\begin{aligned}h_j\left(1 + \frac{a_j h_j}{\varepsilon}\right)^{-1}|\chi_j| &\le C\frac{\varepsilon H}{\varepsilon + a_1 H}\left(H + \varepsilon^{-1}e^{-\alpha x_{j-1}/\varepsilon}\right) \\ &\le C(N^{-2} + e^{-\alpha\lambda/\varepsilon}),\end{aligned}$$

by Lemma 3.4.2. We gather the last two bounds, for $\lambda_0 \geq 2$ and we have

$$h_j \left(1 + \frac{a_j h_j}{\varepsilon}\right)^{-1} |\chi_j| \leq \begin{cases} C(N^{-1}\max|\psi'|)^2 & \text{for } j = 1, \ldots, N/2, \\ CN^{-2} & \text{for } j = N/2+1, \ldots, N. \end{cases} \tag{3.36}$$

We employ (3.34) and Lemma 3.4.4 in order to bound the first term in (3.33) and we obtain

$$\begin{aligned} h_j K_2^j |\chi_1| &\leq C \frac{h_j}{\varepsilon} K_1^j \left(1 + \frac{a_1 h_1}{\varepsilon}\right) N^{-1} \max|\psi'| \\ &\leq \begin{cases} C(N^{-1}\max|\psi'|)^2 & \text{for } j = 2, \ldots, N/2, \\ CN^{-2}\max|\psi'| & \text{for } j = N/2+1, \ldots, N, \end{cases} \end{aligned} \tag{3.37}$$

if $\lambda_0 \geq 1$.

Lastly, we bound the second term in (3.33). For $k < j \leq N/2$ and $\lambda_0 \geq 3$ we have by (3.34) and by lemmas 3.4.1-3.4.3

$$\begin{aligned} h_j K_k^j \frac{a_k h_k}{\varepsilon} |\chi_k| &\leq C h_j e^{-\alpha(x_j - x_{k-1})/\varepsilon} \frac{a_k h_k}{\varepsilon} |\chi_k| \quad \text{by Lemma 3.4.3}, \\ &\leq C h_j e^{-\alpha(x_j - x_{k-1})/\varepsilon} \frac{a_k h_k}{\varepsilon} \left\{ h_k + \varepsilon^{-1} \sinh \frac{\alpha h_k}{2\varepsilon} e^{-\alpha x_{k-1}/\varepsilon} \right\} \text{ using (3.34)}, \\ &\leq C \frac{h_j}{\varepsilon} e^{-\alpha x_j/(\lambda_0 \varepsilon)} \left(\frac{h_k}{\varepsilon} e^{-\alpha x_k/(\lambda_0 \varepsilon)} \right)^2 + C h_j \frac{h_k^2}{\varepsilon}, \\ &\leq C(N^{-1}\max|\psi'|)^2 \int_{t_{k-1}}^{t_k} \frac{-\psi'(\tau)}{\psi(\tau)} d\tau e^{-\alpha x_k/(\lambda_0 \varepsilon)} + C h_j \frac{h_k^2}{\varepsilon}, \\ &\leq C(N^{-1}\max|\psi'|)^2 \int_{t_{k-1}}^{t_k} (-\psi'(\tau)) d\tau + C h_j \frac{h_k^2}{\varepsilon}. \end{aligned}$$

We obtain from the above the following

$$\begin{aligned} \left| h_j \sum_{k=2}^{j-1} K_k^j \frac{a_k h_k}{\varepsilon} \chi_k \right| &\leq C(N^{-1}\max|\psi'|)^2 \int_0^{1/2} (-\psi'(\tau)) d\tau + C \frac{h_j}{\varepsilon} \sum_{k=1}^{N/2} h_k^2, \\ &\leq C(N^{-1}\max|\psi'|)^2 \quad \text{for } j = 2, \ldots, N/2 \text{ and } \lambda_0 \geq 3, \end{aligned} \tag{3.38}$$

by using the result from proof of Lemma 3.4.3 that $\sum_{k=1}^{N/2} h_k^2 \leq C\varepsilon^2$.

Now let $j > N/2$. We consider three distinct cases, namely $k > N/2$, $k = N/2$ and $k < N/2$.

Case 1 ($k > N/2$):

We use (3.35) to get

$$h_j K_k^j \frac{a_k h_k}{\varepsilon} |\chi_k| \leq CH \left(1 + \frac{\alpha H}{\varepsilon}\right)^{k-j-1} \frac{H}{\varepsilon} \left(H + \varepsilon^{-1} e^{-\alpha x_{k-1}/\varepsilon}\right),$$
$$\leq C \frac{H^3 \varepsilon}{\varepsilon + \alpha H} + C \left(\frac{H}{\varepsilon + \alpha H}\right)^2 N^{-\lambda_0} e^{-(k-1-N/2)\alpha H/\varepsilon}.$$

Thus

$$\left| h_j \sum_{k=N/2+1}^{j} K_k^j \frac{a_k h_k}{\varepsilon} \chi_k \right| \leq CN^{-2} \qquad \text{for } \lambda_0 \geq 2. \tag{3.39}$$

Case 2 ($k = N/2$):

Analogous to Case 1, we obtain

$$\begin{aligned} h_j K_k^j \frac{a_k h_k}{\varepsilon} |\chi_k| &\leq CH \left(1 + \frac{\alpha H}{\varepsilon}\right)^{-1} \frac{a_{N/2} h_{N/2}}{\varepsilon} \left(h_{N/2} + \varepsilon^{-1} e^{-\alpha x_{N/2-1}/\varepsilon}\right), \\ &\leq C h_{N/2}^2 + CN^{-\lambda_0}, \\ &\leq CN^{-2} \quad \text{for } \lambda_0 \geq 2, \quad \text{by lemma 3.4.2.} \end{aligned} \tag{3.40}$$

Case 3 ($k < N/2$):

We estimate, similarly to the argument that led to (3.38),

$$h_j K_k^j \frac{a_k h_k}{\varepsilon} |\chi_k| \leq CH \left(1 + \frac{\alpha H}{\varepsilon}\right)^{(j-N/2)} e^{-\alpha(x_{N/2} - x_{k-1})/\varepsilon} \frac{a_k h_k}{\varepsilon} \left(h_k + \frac{h_k}{\varepsilon^2} e^{-\alpha x_{k-1}/\varepsilon}\right),$$
$$\leq CN^{-1} \left(1 + \frac{\alpha H}{\varepsilon}\right)^{-(j-N/2-1)} \left(h_k + \frac{h_k}{\varepsilon} e^{-\alpha x_{k-1}/(\lambda_0 \varepsilon)}\right) \quad \text{if } \lambda_0 \geq 2,$$
$$\leq CN^{-1} \left(1 + \frac{\alpha H}{\varepsilon}\right)^{-(i-N/2-1)} N^{-1} \max |\psi'|,$$

by lemmas 3.4.1 and 3.4.2.

Thus

$$\left| h_i \sum_{k=2}^{N/2-1} K_k^i \frac{a_k h_k}{\varepsilon} \chi_k \right| \leq CN^{-2} \max |\psi'| \qquad \text{for } \lambda_0 \geq 2. \tag{3.41}$$

Using (3.39)-(3.41), we obtain

$$\left| h_j \sum_{k=2}^{j-1} K_k^j \frac{a_k h_k}{\varepsilon} \chi_k \right| \leq CN^{-2} \max |\psi'| \qquad \text{for } j = N/2+1, \ldots, N \text{ and } \lambda_0 \geq 2.$$

This result used with (3.33),(3.36) and (3.37) gives

$$|\overline{W}_{d,j}^{-}| \leq \begin{cases} C(N^{-1}\max|\psi'|)^2 & \text{for } j=1,\ldots,N/2 \text{ and } \lambda_0 \geq 3, \\ CN^{-2}\max|\psi'| & \text{for } j=N/2+1,\ldots,N \text{ and } \lambda_0 \geq 2. \end{cases}$$

The proposition of the theorem follows from the last inequality, (3.22), (3.23), (3.26) and (3.28) □

3.4.4 Consistency error of the central difference method

Theorem 3.4.4. ([20]) Let $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary mesh. Then the consistency error $\tau^c = L^u - f^c$ of the central difference scheme for (3.1) satisfies

$$\|\tau^c\|_{*,\omega} \leq C\left\{\max_{i=1,\ldots,N}\int_{x_{i-1}}^{x_i}\left(1+e^{-\alpha s/(2\varepsilon)}\right)ds\right\}^2.$$

Proof. Integrating (3.1) over $[x_{j-1/2} - x_{j+1/2}]$ we get

$$-\varepsilon\left[u'_{j+1/2} - u'_{j-1/2}\right] - (au_{j+1/2} - au_{j-1/2}) + \int_{j-1/2}^{j+1/2}(bu-f)(s)ds.$$

Thus

$$\begin{aligned}\sum_{i=1}^{j-1}\tau_i^c &= \varepsilon\left(\frac{u_1-u_0}{h_1} - u'_{1/2}\right) - \frac{(au)_1-(au)_0}{2} - (au)_{1/2} \\ &\quad - \varepsilon\left(\frac{u_j-u_{j-1}}{h_j} - u'_{j-1}\right) - \frac{(au)_j-(au)_{j-1/2}}{2} - (au)_{1/2} \\ &\quad + \sum_{i=1}^{j-1}\left\{\hbar_i(\widehat{bu}-\widehat{f})_i\int_{i-1/2}^{i+1/2}(bu-f)(s)ds\right\}.\end{aligned}$$

Taking Taylor series expansions for u, u' and $(au)'$ about the point x_j, we get

$$\begin{aligned}\varepsilon\left|\frac{u_j-u_{j-1}}{h_j} - u'_{j-1/2}\right| &\leq \frac{3\varepsilon}{2}\int_{x_{j-1}}^{x_j}|u'''(s)|(s-x_{j-1})ds \\ &\leq \int_{x_{j-1}}^{x_j}|(f-bu+(au)')'(s)|(s-x_{j-1})ds\end{aligned}$$

and

$$\left|\frac{(au)_j-(au)_{j-1}}{2} - (au)_{1/2}\right| \leq \frac{3}{2}\int_{x_{j-1}}^{x_j}|(au)''(s)|(s-x_{j-1})ds,$$

while a Taylor series expansion for $(bu-f)$ about the point x_{j+1} gives

$$\left|\hbar_j(\widehat{bu}-\widehat{f})_j\int_{j-1/2}^{j+1/2}(bu-f)(s)ds\right| \leq C\hbar_j\int_{x_{j-1}}^{x_{j+1}}|(bu-f)''(s)|(s-x_{j-1})ds.$$

Thus

$$\|\tau^c\|_{*,\omega} \le C \max_{i=1,\ldots,N} \int_{x_{i-1}}^{x_i} \left(1+e^{-\alpha s/\varepsilon}\right)(s-x_{i-1})ds,$$

by (3.2). To bound the right-hand side of this inequality, we use

$$\int_a^b g(x)(x-a)dx \le \frac{1}{2}\left\{\int_a^b g(x)^{1/2}dx\right\}^2,$$

which holds true for any positive monotonically decreasing function g on $[a,b]$. This can be verified by considering the two integrals as functions of the upper integration limit. □

Corollary 3.4.2. ([20]) Let ω be a Shishkin-type mesh. Suppose that (3.8) holds true. Then the truncation error τ^c of the central difference scheme satisfies

$$\|\tau^c\|_{*,\omega} \le C(N^{-1}\max|\psi'|)^2, \text{ for } \lambda_0 \ge 2.$$

3.5 Numerical results

In this section we present some theoretical results for the defect correction methods on convection–diffusion problems. We use a test example previously considered by Clavero *et al.* in [15]. The maximum error solutions and the rates of convergence are obtained using fomulae in (2.16)–(2.19) from Section 2.4.

Example 3.5.1. ([15])

$$\varepsilon u'' + u' = -1, \qquad x\in(0,1), \qquad u(0)=u(1)=1,$$

with the exact solution given by

$$u(x) = x + \frac{\exp(-x/\varepsilon)-\exp(-1/\varepsilon)}{1-\exp(-1/\varepsilon)}.$$

Example 3.5.2. ([28])

$$-\varepsilon u'' + u' = \exp(x), \qquad x\in(0,1), \qquad u(0)=u(1)=0,$$

with the exact solution given by

$$u(x) = \frac{1}{1-\varepsilon}\left[\exp(x) - \frac{1-\exp\{1-(1/\varepsilon\} + \{\exp(1)-1\}\exp\{(x-1)/\varepsilon\}}{1-\exp(-1/\varepsilon)}\right].$$

Table 3.2: Maximum errors obtained for Example 3.5.1 using upwind scheme on Shishkin mesh.

ε	$N = 40$	$N = 80$	$N = 160$	$N = 320$	$N = 640$	$N = 1280$	$N = 2560$
1	1.49E-03	7.49E-004	3.76E-04	1.88E-04	9.43E-05	4.72E-05	2.36E-05
10^{-1}	8.38E-02	5.33E-02	9.22E-02	5.07E-02	2.70E-02	1.39E-02	7.07E-03
10^{-2}	2.54E-01	1.47E-01	8.40E-02	4.68E-02	2.57E-02	1.82E-01	1.07E-01
10^{-3}	8.38E-02	5.33E-02	3.24E-02	1.90E-02	1.09E-02	1.06E-01	6.21E-02
10^{-4}	8.38E-02	5.33E-02	3.24E-02	1.90E-02	1.09E-02	6.08E-03	3.36E-03
10^{-5}	8.38E-02	5.33E-02	3.24E-02	1.90E-02	1.09E-02	6.08E-03	3.36E-03
10^{-6}	8.38E-02	5.33E-02	3.24E-02	1.90E-02	1.09E-02	6.08E-03	3.36E-03
10^{-7}	8.38E-02	5.33E-02	3.24E-02	1.90E-02	1.09E-02	6.08E-03	3.36E-03
10^{-8}	8.38E-02	5.33E-02	3.24E-02	1.90E-02	1.09E-02	6.08E-03	3.36E-03
10^{-9}	8.38E-02	5.33E-02	3.24E-02	1.90E-02	1.09E-02	6.08E-03	3.36E-03
10^{-10}	8.38E-02	5.33E-02	3.24E-02	1.90E-02	1.09E-02	6.08E-03	3.36E-03

Table 3.3: Maximum errors obtained for Example 3.5.1 using defect corrections on Shishkin mesh.

ε	$N = 40$	$N = 80$	$N = 160$	$N = 320$	$N = 640$	$N = 1280$	$N = 2560$
1	6.29E-06	1.57E-06	3.93E-07	9.83E-08	2.46E-08	6.15E-09	1.54E-09
10^{-1}	1.93E-03	4.79E-04	1.20E-04	2.99E-05	7.48E-06	1.87E-06	4.67E-07
10^{-2}	9.65E-03	3.35E-03	1.21E-02	3.02E-03	7.49E-04	1.87E-04	4.68E-05
10^{-3}	9.65E-03	3.35E-03	1.11E-03	3.59E-04	1.12E-04	1.96E-02	4.69E-03
10^{-4}	9.65E-03	3.35E-03	1.11E-03	3.59E-04	1.12E-04	3.45E-05	1.04E-05
10^{-5}	9.65E-03	3.35E-03	1.11E-03	3.59E-04	1.12E-04	3.45E-05	1.04E-05
10^{-6}	9.65E-03	3.35E-03	1.11E-03	3.59E-04	1.12E-04	3.45E-05	1.04E-05
10^{-7}	9.65E-03	3.35E-03	1.11E-03	3.59E-04	1.12E-04	3.45E-05	1.04E-05
10^{-8}	9.65E-03	3.35E-03	1.11E-03	3.59E-04	1.12E-04	3.45E-05	1.04E-05
10^{-9}	9.65E-03	3.35E-03	1.11E-03	3.59E-04	1.12E-04	3.45E-05	1.04E-05
10^{-10}	9.65E-03	3.35E-03	1.11E-03	3.59E-04	1.12E-04	3.45E-05	1.04E-05

Defining the error of the defect correction method as $\eta_N^{dc,\varepsilon} := u - u^{dc}$, we estimate the $\varepsilon-$uniform accuracy by using formulae given in (2.16) to (2.19) from Section 2.4.

ε	r_1	r_2	r_3	r_4	r_5
10^{-1}	0.93	0.96	0.98	0.99	1.00
10^{-2}	0.65	-0.79	0.86	0.91	0.96
10^{-3}	0.65	0.71	0.77	0.81	-3.28
10^{-4}	0.65	0.71	0.77	0.81	0.84
10^{-5}	0.65	0.71	0.77	0.81	0.84
10^{-6}	0.65	0.71	0.77	0.81	0.84
10^{-7}	0.65	0.71	0.77	0.81	0.84
10^{-8}	0.65	0.71	0.77	0.81	0.84
10^{-9}	0.65	0.71	0.77	0.81	0.84
10^{-10}	0.65	0.71	0.77	0.81	0.84

Table 3.4: Rates of convergence obtained for Example 3.5.1 using upwind scheme.

ε	r_1	r_2	r_3	r_4	r_5
10^{-1}	2.01	2.00	2.00	2.00	2.00
10^{-2}	1.53	-1.86	2.01	2.01	2.00
10^{-3}	0.95	0.98	0.99	1.00	-1.98
10^{-4}	1.53	1.59	1.63	1.67	-7.45
10^{-5}	1.53	1.59	1.63	1.67	1.71
10^{-6}	1.53	1.59	1.63	1.67	1.71
10^{-7}	1.53	1.59	1.63	1.67	1.71
10^{-8}	1.53	1.59	1.63	1.67	1.71
10^{-9}	1.53	1.59	1.63	1.67	1.71
10^{-10}	1.53	1.59	1.63	1.67	1.71

Table 3.5: Rates of convergence obtained for Example 3.5.1 using defect corrections.

ε	r_1	r_2	r_3	r_4	r_5
10^{-1}	0.93	0.97	0.98	0.99	1.00
10^{-2}	0.78	-0.32	0.87	0.91	0.96
10^{-3}	0.79	0.81	0.84	0.87	-0.28
10^{-4}	0.79	0.81	0.84	0.87	0.88
10^{-5}	0.79	0.81	0.84	0.87	0.88
10^{-6}	0.79	0.81	0.84	0.87	0.88
10^{-7}	0.79	0.81	0.84	0.87	0.88
10^{-8}	0.79	0.81	0.84	0.87	0.88
10^{-9}	0.79	0.81	0.84	0.87	0.88
10^{-10}	0.79	0.81	0.84	0.87	0.88

Table 3.6: Rates of convergence obtained for Example 3.5.2 using upwind scheme on Shishkin mesh.

ε	r_1	r_2	r_3	r_4	r_5
10^{-1}	0.97	0.99	0.99	1.00	1.00
10^{-2}	0.97	0.98	0.99	1.00	1.00
10^{-3}	0.97	0.98	0.99	1.00	1.00
10^{-4}	0.97	0.98	0.99	1.00	1.00
10^{-5}	0.97	0.98	0.99	1.00	1.00
10^{-6}	0.97	0.98	0.99	1.00	1.00
10^{-7}	0.97	0.98	0.99	1.00	1.00
10^{-8}	0.97	0.98	0.99	1.00	1.00
10^{-9}	0.97	0.98	0.99	1.00	1.00
10^{-10}	0.97	0.98	0.99	1.00	1.00

Table 3.7: Rates of convergence obtained for Example 3.5.2 using defect corrections on Shishkin mesh .

Table 3.8: Maximum errors obtained for Example 3.5.2 using upwind scheme on Shishkin mesh.

ε	$N=40$	$N=80$	$N=160$	$N=320$	$N=640$	$N=1280$	$N=2560$
1	8.88E-02	4.67E-02	2.39E-02	1.21E-02	6.09E-03	3.05E-03	1.53E-03
10^{-1}	2.23E-01	1.30E-01	1.63E-01	8.94E-02	4.77E-02	2.46E-02	1.25E-02
10^{-2}	2.54E-01	1.47E-01	8.40E-02	4.68E-02	2.57E-02	1.82E-01	1.07E-01
10^{-6}	2.58E-01	1.49E-01	8.53E-02	4.75E-02	2.61E-02	1.42E-02	7.62E-03
10^{-7}	2.58E-01	1.49E-01	8.53E-02	4.75E-02	2.61E-02	1.42E-02	7.62E-03
10^{-8}	2.58E-01	1.49E-01	8.53E-02	4.75E-02	2.61E-02	1.42E-02	7.62E-03
10^{-9}	2.58E-01	1.49E-01	8.53E-02	4.75E-02	2.61E-02	1.42E-02	7.62E-03
10^{-10}	2.58E-01	1.49E-01	8.53E-02	4.75E-02	2.61E-02	1.42E-02	7.62E-03

Table 3.9: Maximum errors for Example 3.5.2 using defect corrections on Shishkin mesh.

ε	$N=40$	$N=80$	$N=160$	$N=320$	$N=640$	$N=1280$	$N=2560$
1	2.47E-02	1.31E-02	6.71E-03	3.40E-03	1.71E-03	8.59E-04	4.30E-04
10^{-1}	6.07E-02	2.96E-02	1.49E-02	4.74E-03	2.40E-03	1.22E-03	6.14E-04
10^{-2}	7.87E-02	4.02E-02	2.03E-02	1.02E-02	5.09E-03	3.30E-02	7.60E-03
10^{-6}	8.22E-02	4.20E-02	2.12E-02	1.07E-02	5.35E-03	2.68E-03	1.34E-03
10^{-7}	8.22E-02	4.20E-02	2.12E-02	1.07E-02	5.35E-03	2.68E-03	1.34E-03
10^{-8}	8.22E-02	4.20E-02	2.12E-02	1.07E-02	5.35E-03	2.68E-03	1.34E-03
10^{-9}	8.22E-02	4.20E-02	2.12E-02	1.07E-02	5.35E-03	2.68E-03	1.34E-03
10^{-10}	8.22E-02	4.20E-02	2.12E-02	1.07E-02	5.35E-03	2.68E-03	1.34E-03

For the test example, as was done by Frohner *et al.* in [20], we take $\lambda_0 = 3$ and $\beta = 1$ in the definition of the transition point. Table 3.2 confirm that the maximum errors obtained by the upwind scheme are $\varepsilon-$uniform for Example 3.5.1. The accuracy of the results is seen to be improved by the defect–correction method in table 3.3 as well as maintaining the $\varepsilon-$uniform characteristic of the upwind scheme.

Tables 3.4 and 3.6 show that the the Shishkin-type mesh used in the upwind scheme has an almost order 1 rate of convergence for convection–diffusion problems. This is a confirmation of the theoretical result in Corollary 3.4.1. In tables 3.5 and 3.7 we present the rates of convergence after implementing the defect-correction. Improved rates of almost order 2 are obtained for Example 3.5.1.

In the next chapter, we construct a hybrid method which seeks to incorporate the advantages of fitted mesh finite difference methods and fitted operator finite difference methods.

Chapter 4

A Hybrid Finite Difference Method

In the previous chapter we discussed some higher order methods that improve accuracy of numerical solutions with particular focus on defect corrections. In this chapter, we consider two methods, one that is layer resolving (FMFDM) and the other that is relatively more accurate (FOFDM) [41]. We combine these two methods in Section 4.4 to construct a new hybrid method that has the good properties of both methods and gives better numerical approximations.

4.1 Introduction

We study two-point boundary value problems related with boundary layers such as, for instance, the flows governed by the Navier–Stokes equations [50]. The convection–diffusion equation

$$L_\epsilon y \equiv -\varepsilon y''(x) + a(x)y'(x) = f(x), \quad x \in \Omega = (0,1), \tag{4.1}$$

with boundary conditions

$$y(0) = y_0, \qquad y(1) = y_1,$$

will be used as a model problem. We assume that the singular perturbation parameter can take arbitrary small positive values, $0 < \varepsilon \leq 1$, the functions $a, f \in C^5(0,1)$ and also that $a \geq \alpha > 0, \forall x \in [0,1]$. It is well accepted that the solution of (4.1) has a boundary layer near $x = 1$ and its derivatives satisfy

$$\left|y^{(k)}(x)\right| \leq C\left(1+\varepsilon^{-k}\exp\left(-\frac{\alpha x}{\varepsilon}\right)\right), \quad 0 \leq k \leq 4. \tag{4.2}$$

It is convenient to use an appropriate decomposition of the exact solution y when proving uniform convergence [15]. We choose to use $y = v + w$, where v and w are the regular and singular components of the exact solution respectivley. Clavero *et al.* [15] also note that v and w are solutions of the following boundary value problems

$$L_\varepsilon v = f, \qquad v(0) = v^*(0), \quad v(1) = y(1);$$

$$L_\varepsilon w = 0, \;\; w(0) = y(0) - v^*(0), \;\; w(1) = 0,$$

where $v^*(0)$ is taken so that

$$\left|v^{(j)}(x)\right| \leq C, \quad 0 \leq j \leq 3, \quad \varepsilon\left|v^{(4)}(x)\right| \leq C, \tag{4.3}$$

$$|w(x)| \leq C\exp\left(-\frac{\alpha x}{\varepsilon}\right), \quad \left|w^{(j)}(x)\right| \leq C\varepsilon^{(-j)}, \quad 1 \leq j \leq 4. \tag{4.4}$$

4.2 Fitted Mesh Finite Difference Method for convection–diffusion problems

To approximate the solution of (4.1), we consider a finite difference scheme defined on a Shishkin mesh. Let N be the discretization parameter. We use the transition parameter given by Kadalbajoo and Patidar in [27]

$$\lambda = \min\left\{\frac{1}{2}, 8\varepsilon\ln N\right\}, \tag{4.5}$$

and divide uniformly each one of the subdomains $[0, 1-\lambda], [1-\lambda, 1]$ into $N/2$ intervals. Then the mesh spacing is given by

$$x_j = \begin{cases} 2(1-\lambda)N^{-1}, & j = 1, \ldots, N/2 \\ 2\lambda N^{-1}, & j = N/2+1, \ldots, N-1. \end{cases} \tag{4.6}$$

The fitted mesh scheme is as follows

$$L^N_{up}\bar{u}_j \equiv -\varepsilon\delta^2\bar{u}_j + a_j D^-\bar{u}_j = f_j, \tag{4.7}$$

where

$$\delta^2 \bar{u}_j = \frac{1}{\hbar_j}\left(\frac{\bar{u}_{j+1}-\bar{u}_j}{h_{j+1}} - \frac{\bar{u}_j-\bar{u}_{j-1}}{h_j}\right), \qquad D^-\bar{u}_j = \left(\frac{\bar{u}_j-\bar{u}_{j-1}}{h_j}\right),$$

with $h_j = x_j - x_{j-1}$, and $\hbar_j = (h_{j+1}+h_j)/2$.

4.2.1 Some useful attributes of FMFDMs

We present here some attributes that are facilitatory in the analysis of the fitted mesh finite difference methods.

Discrete Maximum Principle. *Assume that the mesh function* Ψ_j *satisfies* $\Psi_0 \geq 0$ *and* $\Psi_N \geq 0$. *Then* $L^N_{up}\Psi_j \geq 0, \forall\, 1 \leq j \leq N-1$, *implies that* $\Psi_j \geq 0, \forall\, 0 \leq j \leq N$,

Proof. Let k be such that $\Psi_k = \min \Psi_j$ and suppose that $\Psi_k < 0$. Since $\Psi_0 \geq 0$ and $\Psi_N \geq 0$, it follows that $k \neq 0$ and $k \neq N$. Evidently, $\Psi_{k+1} - \Psi_k \geq 0$ and $\Psi_k - \Psi_{k-1} \leq 0$. Therefore,

$$\begin{aligned} L^N_{up}\Psi_k &= -\frac{\varepsilon}{\hbar_k}\left(\frac{\Psi_{k+1}-\Psi_k}{h_{k+1}} - \frac{\Psi_k-\Psi_{k-1}}{h_k}\right) + a_k\frac{\Psi_k-\Psi_{k-1}}{h_k}, \\ &\leq 0. \end{aligned} \tag{4.8}$$

When $\Psi_k - \Psi_{k-1} < 0$ we have $L^N_{up}\Psi_k < 0$. This is clearly not true and therefore $\Psi_k = \Psi_{k-1}$. We repeat this process with $k-2$ instead of $k-1$ and we have $\Psi_k - \Psi_{k-2} \leq 0$,

$$\begin{aligned} L^N_{up}\Psi_k &= -\frac{\varepsilon}{\hbar_k}\left(\frac{\Psi_{k+1}-\Psi_k}{h_{k+1}} - \frac{\Psi_k-\Psi_{k-2}}{h_k}\right) + a_k\frac{\Psi_k-\Psi_{k-2}}{h_k}, \\ &\leq 0, \end{aligned} \tag{4.9}$$

which is also not true, therefore $\Psi_k = \Psi_{k-2}$. We repeat with $k-3, k-4$ and so on, with the following result

$$\Psi_0 = \Psi_1 = \cdots = \Psi_{k-1} = \Psi_k < 0,$$

which is not true as well. It then follows $\Psi_k > 0$ and

$$\Psi_j \geq 0, \quad \text{for all } \; j, 0 \leq j \leq N.$$

□

From the discrete maximum principle we obtain an $\epsilon-$uniform stabilty property for the operator L^N_{up} [40].

Lemma 4.2.1. *If Φ_j is any mesh function such that $\Phi_0 = \Phi_N = 0$, then*

$$|\Phi_j| \leq \frac{1}{\alpha} \max_{1\leq i\leq N-1} \left|L_{up}^N \Phi_i\right|, \quad \forall \;\; 0 \leq j \leq N.$$

Proof. As in [40], we consider two mesh functions Ψ_j^+, Ψ_j^- defined by

$$\Psi_j^{\pm} = \left(\frac{1}{\alpha} \max_{1\leq i\leq N-1} \left|L_{up}^N \Phi_i\right|\right) x_j \pm \Phi_j. \tag{4.10}$$

It follows that

$$\begin{aligned}\Psi_0^{\pm} &= \left(\frac{1}{\alpha} \max_{1\leq i\leq N-1} \left|L_{up}^N \Phi_i\right|\right) x_0 \pm \Phi_0,\\ &= \pm\Phi_0,\\ &= 0,\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\Psi_N^{\pm} &= \left(\frac{1}{\alpha} \max_{1\leq i\leq N-1} \left|L_{up}^N \Phi_i\right|\right) x_N \pm \Phi_N,\\ &= \left(\frac{1}{\alpha} \max_{1\leq i\leq N-1} \left|L_{up}^N \Phi_i\right|\right) \pm \Phi_N,\\ &= \left(\frac{1}{\alpha} \max_{1\leq i\leq N-1} \left|L_{up}^N \Phi_i\right|\right),\\ &\geq 0,\end{aligned} \tag{4.12}$$

and, for all $1 \leq j \leq N-1$,

$$\begin{aligned}L_{up}^N \Psi_i^{\pm} &= \left(\frac{1}{\alpha} \max_{1\leq i\leq N-1} \left|L_{up}^N \Phi_i\right|\right) a_j \pm L_{up}^N \Phi_j,\\ &\geq 0.\end{aligned} \tag{4.13}$$

From the discrete maximum principle, if $\Psi_0 \geq 0$, $\Psi_N \geq 0$ and $L_{up}^N \Psi_j \geq 0$ for all $0 < j < N$ then $\Psi_j^{\pm} \geq 0, \forall\; 0 \leq j \leq N$. □

The reduced problem obtained from (4.1) takes the form

$$a(x)v_0'(x) = f(x)$$

and has the solution

$$v_0(x) = u_0 + \int_0^x \frac{f(t)}{a(t)} dt,$$

and it is evident that, for $0 \le k \le 3$,

$$|v_0^{(k)}(x)| \le C, \qquad \forall x \in \bar{\Omega},$$

from the assumptions on a and f.

The solution y of the problem in (4.1) takes the decomposition

$$y = v_0 + \varepsilon y_1^* + w_0,$$

with v_0 as defined above and y_1^* satisfying

$$L_\varepsilon y_1^* = v_0'', \qquad y_1^*(0) = -\varepsilon^{-1} w_0(0), \qquad y_1^*(1) = 0$$

and w_0 is the solution of the homogeneous problem

$$L_\varepsilon w_0 = 0, \qquad w_0(0) = w_0(1)e^{-\alpha/\varepsilon}, \qquad w_0(1) = u_1 - v_0(1).$$

Evidently

$$|w_0(0)| \le C, \quad |w_0(1)| \le C, \quad |y_1(0)| \le C \quad \text{and} \quad |v_0''(0)| \le C.$$

Therefore, as in [40], we use the argument that y_1^* is the solution of a problem similar to (4.1). This implies that, for $0 \le k \le 3$,

$$|y_1^{*(k)}(x)| \le C(1 + \varepsilon^{-k} e^{-\alpha(1-x)/\varepsilon}).$$

We introduce the functions

$$\Psi^{\pm}(x) = |w_0(1)|e^{-\alpha(1-x)/\varepsilon} \pm w_0(x).$$

Application of the maximum principle gives $\Psi^{\pm}(x) \ge 0$, therefore

$$|w_0(x)| \le Ce^{-\alpha(1-x)/\varepsilon}, \qquad \forall x \in \bar{\Omega}.$$

Rewritting w_0 gives

$$w_0 = w_0(0)\varphi_* + w_0(1)(1 - \varphi_*),$$

with φ defined as

$$\varphi(x)_* = \frac{\int_x^1 e^{-A(t)/\varepsilon} dt}{\int_0^1 e^{-A(t)/\varepsilon} dt}.$$

It follows that

$$w_0' = (w_0(0) - w_0(1))\varphi_*'$$

and hence

$$|w_0'(x)| \le C|\varphi_*'(x)| \le C\varepsilon^{-1}e^{-\alpha(1-x)/\varepsilon}.$$

From the estimates of w_0 and w_0' we obtain

$$|w_0^{(2)}(x)| \le C|\varphi_*^{(2)}(x)| \le C\varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon}$$

and

$$|w_0^{(3)}(x)| \le C|\varphi_*^{(3)}(x)| \le C\varepsilon^{-3}e^{-\alpha(1-x)/\varepsilon}.$$

In general,

$$|w_0^{(k)}(x)| \le C|\varphi_*^{(k)}(x)| \le C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon}.$$

As given earlier,

$$y = \upsilon_0 + \varepsilon y_1 + w_0,$$

so the general derivatives $\forall x \in \bar{\Omega}$ and $0 \le k \le 3$ are given by

$$y^{(k)} = \upsilon_0^{(k)} + \varepsilon y_1^{(k)} + w_0^{(k)},$$

this gives

$$|\upsilon_0^{(k)} + \varepsilon y_1^{(k)}| \le C\varepsilon^{-(k-1)}e^{-\alpha(1-x)/\varepsilon}$$

and

$$|w_0^{(k)}| \le C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon}.$$

We use the idea that y_1 is decomposed analogously to u_ε which leads to the following result

$$u_\varepsilon = \upsilon_\varepsilon + w_\varepsilon,$$

with the following results for $0 \le k \le 3$ and for all $x \in \bar{\Omega}$,

$$|\upsilon_\varepsilon^{(k)}(x)| \le C(1 + \varepsilon^{-(k-2)}e^{-\alpha(1-x)/\varepsilon})$$

and

$$|w_\varepsilon^{(k)}(x)| \le C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon}.$$

We remark that υ_ε and w_ε, satisfy

$$L_\varepsilon \upsilon_\varepsilon = f, \quad L_\varepsilon w_\varepsilon = 0,$$

with boundary conditions

$$\begin{aligned} \upsilon_\varepsilon(0) &= u_0 - w_\varepsilon(0), \quad w_\varepsilon(0) = w_\varepsilon(1)e^{-\alpha/\varepsilon}, \\ \upsilon_\varepsilon(1) &= u_1 - w_\varepsilon(1), \end{aligned}$$

where $w_\varepsilon(1)$ is chosen so that the first and second derivatives of υ_ε are bounded uniformly in ε.

Lemma 4.2.2. *Let $x_j \in \Omega^N$, for a given mesh $\Omega^N = \{x_j\}_0^N$. Then, for any $\vartheta \in C^2(\bar{\Omega})$*

$$\left| \left(D^- - \frac{d}{dx} \right) \vartheta(x_j) \right| \leq \frac{1}{2}(x_j - x_{j-1})|\vartheta|_2,$$

and, for any $\vartheta \in C^3(\bar{\Omega})$

$$\left| \left(\delta^2 - \frac{d^2}{dx^2} \right) \vartheta(x_j) \right| \leq \frac{1}{3}(x_{j+1} - x_{j-1})|\vartheta|_3.$$

Proof.

$$\begin{aligned} \left| \left(D^- - \frac{d}{dx} \right) \vartheta(x_j) \right| &= \left| \frac{\vartheta(x_j) - \vartheta(x_{j-1})}{x_j - x_{j-1}} - \vartheta'(x_j) \right|, \\ &= \frac{1}{x_j - x_{j-1}} \left| \vartheta(x_j) - \vartheta(x_{j-1}) - (x_j - x_{j-1})\vartheta'(x_j) \right|, \\ &= \frac{1}{x_j - x_{j-1}} \Big| \vartheta(x_j) - \vartheta(x_{j-1}) - x_j\vartheta'(x_j) - \dots \\ &\qquad x_{j-1}\vartheta'(x_j) + x_{j-1}\vartheta'(x_{j-1}) - x_{j-1}\vartheta'(x_{j-1}) \Big|, \\ &= \frac{1}{x_j - x_{j-1}} \left| \int_{x_{j-1}}^{x_j} x_{j-1}\vartheta''(s)ds - \int_{x_{j-1}}^{x_j} s\vartheta''(s)ds \right|, \\ &= \frac{1}{x_j - x_{j-1}} \left| \int_{x_{j-1}}^{x_j} (x_{j-1} - s)\vartheta''(s)ds \right|, \\ &\leq \frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} |\vartheta|_2 (s - x_{j-1})ds, \\ &\leq \frac{1}{x_j - x_{j-1}} \left(\frac{(x_j - x_{j-1})^2}{2} |\vartheta|_2 \right), \\ &\leq \frac{1}{2}(x_j - x_{j-1})|\vartheta|_2. \end{aligned}$$

We now prove the second part of the Lemma 4.2.2

$$
\begin{aligned}
\left|\left(\delta^2-\frac{d^2}{dx^2}\right)\vartheta(x_j)\right| &= \left|\frac{1}{\hbar_j}\left(\frac{\vartheta(x_{j+1})-\vartheta(x_j)}{h_{j+1}}-\frac{\vartheta_j-\vartheta_{j-1}}{h_j}\right)-\vartheta''(x_j)\right|,\\
&= \frac{1}{\hbar_j}\left|\left(\frac{\vartheta_{j+1}-\vartheta_j}{h_{j+1}}-\frac{\vartheta_j-\vartheta_{j-1}}{h_j}\right)-\hbar_j\vartheta''(x_j)\right|,\\
&= \frac{1}{\hbar_j}\left|\left(\frac{\vartheta_{j+1}-\vartheta_j}{x_{j+1}-x_j}-\frac{\vartheta_j-\vartheta_{j-1}}{x_j-x_{j-1}}\right)-\frac{x_{j+1}-x_{j-1}}{2}\vartheta''(x_j)\right|,\\
&= \frac{1}{2\hbar_j}\Bigg|-(x_{j+1}-x_j)\vartheta''(x_j)-2\vartheta'(x_j)+2\left(\frac{\vartheta_{j+1}-\vartheta_j}{x_{j+1}-x_j}\right)\\
&\quad -\left\{(x_j-x_{j-1})\vartheta''(x_j)-2\vartheta'(x_j)+2\left(\frac{\vartheta_j-\vartheta_{j-1}}{x_j-x_{j+1}}\right)\right\}\Bigg|,\\
&= \frac{1}{2\hbar_j}\Bigg|\left\{\frac{-(x_{j+1}-x_j)^2\vartheta''(x_j)-2(x_{j+1}-x_j)\vartheta'(x_j)+2\left(\vartheta_{j+1}-\vartheta_j\right)}{x_{j+1}-x_j}\right\}\\
&\quad -\left\{\frac{(x_j-x_{j-1})^2\vartheta''(x_j)-2(x_j-x_{j-1})\vartheta'(x_j)+2\left(\vartheta_j-\vartheta_{j-1}\right)}{x_j-x_{j-1}}\right\}\Bigg|,\\
&= \frac{1}{2\hbar_j}\Bigg|\left\{\frac{-(x_{j+1}-s)^2\vartheta''(s)-2(x_{j+1}-s)\vartheta'(s)+2\vartheta(s)}{x_{j+1}-x_j}\right\}_{x_j}^{x_{j+1}}\\
&\quad -\left\{\frac{(s-x_{j-1})^2\vartheta''(s)-2(s-x_{j-1})\vartheta'(s)+2\vartheta(s)}{x_j-x_{j-1}}\right\}_{x_{j-1}}^{x_j}\Bigg|,\\
&= \frac{1}{2\hbar_j}\Bigg|\frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}}(x_{j+1}-s)^2\vartheta'''(s)ds\\
&\quad -\frac{1}{x_j-x_{j-1}}\int_{x_{j-1}}^{x_j}(s-x_{j-1})^2\vartheta'''(s)ds\Bigg|,\\
&\le \frac{|\vartheta|_3}{2\hbar_j}\Bigg[\frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}}(x_{j+1}-s)^2ds\\
&\quad -\frac{1}{x_j-x_{j-1}}\int_{x_{j-1}}^{x_j}(s-x_{j-1})^2ds\Bigg],\\
&\le \frac{|\vartheta|_3}{x_{j+1}-x_{j-1}}\left[\frac{1}{3}(x_{j+1}-x_j)^2-\frac{1}{3}(x_j-x_{j-1})^2\right],\\
&\le \frac{|\vartheta|_3}{x_{j+1}-x_{j-1}}\left[\frac{1}{3}(x_{j+1}-x_{j-1})^2\right],\\
&\le \frac{1}{3}(x_{j+1}-x_{j-1})|\vartheta|_3.
\end{aligned}
$$

$\square$

Lemma 4.2.3. *Let* $\{Y_j\}_0^N$ *be the solution of (4.1). Given* $Y_0 = e^{-a/\varepsilon}Y_N$*. Then, for all* j*,* $0 \le j \le N/2$*,*

$$0 < Y_j \le CN^{-1}Y_N.$$

Lemma 4.2.4. ([40]) *Let* $\{Y_j\}_0^N$ *be the solution of (4.1) with* $Y_0 = e^{-a/\varepsilon} Y_N$*, and let* Z_j *be the solution of the problem*

$$\begin{cases} -\varepsilon\delta^2 Z_j + b_j D^- Z_j = 0; & 1 \le j \le N-1 \\ Z_0 = e^{-b_0/\varepsilon} Z_N, & Z_N = Y_N, \end{cases}$$

where it is assumed that for all $j, 0 \le j \le N,\ b_j \ge a$*. Then, for all* $j,\ 0 \le j \le N$*,*

$$Z_j = Y_j.$$

4.2.2 Error analysis of the Fitted Mesh Finite Difference Method

Theorem 4.2.1. The error associated with the FMFDM satisfies [15]:

$$\max_j |y(x_j) - \bar{u}_j| \le CN^{-1}(\ln N)^2, \tag{4.14}$$

and therefore it is an almost first order uniformly convergent method.

Proof. Analogous to the decomposition of the solution $y = v + w$, the discrete solution can also be decomposed as

$$\bar{u} = V + W,$$

where V gives the solution of the inhomogeneous problem ([40])

$$L^N_{up} V = f, \qquad V(0) = v(0), \quad V(1) = v(1),$$

and W is the solution of the homogeneous problem

$$L^N_{up} W = 0, \ \ W(0) = w(0), \ \ W(1) = w(1).$$

This enables the error to be written in the following format

$$y - \bar{u} = (v - V) + (w - W), \tag{4.15}$$

which in turn allows for separate estimation of the regular and singular components.

Estimation of the regular component:

$$\begin{aligned} L_{up}^N(v-V) &= L_{up}^N v - L_{up}^N V, \\ &= L_{up}^N v - f, \\ &= L_{up}^N v - L_\epsilon v, \\ &= (L_{up}^N - L_\epsilon) v, \\ &= -\varepsilon\left(\delta^2 - \frac{d^2}{dx^2}\right) v + a\left(D^- - \frac{d}{dx}\right) v. \end{aligned}$$

Applying result from Lemma 4.2.2 we obtain,

$$\begin{aligned} |L_{up}^N(v-V)(x_j)| &\le -\varepsilon\left(\frac{1}{3}(x_{j+1}-x_{j-1})|v|_3\right) + a\left(\frac{1}{2}(x_j - x_{j-1})|v|_2\right), \\ &\le C\left[\varepsilon(x_{j+1}-x_{j-1})|v|_3 + (x_j - x_{j-1})|v|_2\right], \\ &\le C(x_{j+1}-x_{j-1})\left[\varepsilon|v|_3 + |v|_2\right], \\ &\le C(2N^{-1})\left[\varepsilon|v|_3 + |v|_2\right], \\ &\le CN^{-1}\left[\varepsilon|v|_3 + |v|_2\right]. \end{aligned}$$

Using estimates of $|v^{(j)}(x)|$ given earlier we get

$$\left|L_{up}^N(v-V)(x_j)\right| \le CN^{-1}.$$

Using Lemma 4.2.1 to the mesh function $v - V$ yields

$$|(v-V)(x_j)| \le CN^{-1}. \tag{4.16}$$

Estimation of the singular component:

For the singular component, we use the argument from Miller *et al.*[40] that the estimate depends on whether $\lambda = 1/2$ or $\lambda = 8\varepsilon \ln N$.

When $\lambda = 1/2$, we obtain a uniform mesh with $1/2 \le 8\varepsilon \ln N$. Using ideas applied earlier in the estimation of $v - V$, we obtain

$$\begin{aligned} |L_{up}^N(w-W)(x_j)| &\le -\varepsilon\left(\frac{1}{3}(x_{j+1}-x_{j-1})|w|_3\right) + a\left(\frac{1}{2}(x_j - x_{j-1})|w|_2\right), \\ &\le C\left[\varepsilon(x_{j+1}-x_{j-1})|w|_3 + (x_j - x_{j-1})|w|_2\right], \\ &\le C(x_{j+1}-x_{j-1})\left[\varepsilon|w|_3 + |w|_2\right], \\ &\le C(2N^{-1})\left[\varepsilon|w|_3 + |w|_2\right], \\ &\le CN^{-1}\left[\varepsilon|w|_3 + |w|_2\right]. \end{aligned}$$

Employing estimates of $|w^{(j)}(x)|$ given earlier we get

$$\begin{aligned} \left|L_{up}^N(w-W)(x_j)\right| &\le C\varepsilon^{-2}N^{-1}, \\ &\le C(16\ln N)^2N^{-1}, \ \text{ since } \ \varepsilon^{-2} \le 16(\ln N)^2, \\ &\le CN^{-1}(\ln N)^2. \end{aligned}$$

Using Lemma 4.2.1 to the mesh function $w - W$ gives the following estimate

$$|(w-W)(x_j)| \le CN^{-1}(\ln N)^2. \tag{4.17}$$

When $\lambda \neq 1/2$, we have a uniform mesh in the subinterval $[0, 1-\lambda]$ with meshlength $2(1-\lambda)/N$ as well as another uniform mesh in the subinterval $[1-\lambda, 1]$ with meshlength $2\lambda/N$. Different arguments are required to bound $|w - W|$ in the two subintervals.

Since w and W are both small and also from the triangle inequality $|w-W| \le |w|+|W|$, we choose to bound w and W separately in the subinterval without the boundary layer, $[0, 1-\lambda]$.

As in Miller *et al.*[40], we use the fact that

$$\frac{w_0'(x)}{w_0(1)} = -(1 - e^{-\alpha/a})\vartheta(x) > 0,$$

and

$$\frac{w_0(x)}{w_0(1)} = e^{-\alpha/\varepsilon}.$$

Thus $w_0(x)/w_0(1)$ is positive and increasing in the interval (0,1), and as a result, for all x in $[0, 1-\lambda]$,

$$0 \le \frac{w_0(x)}{w_0(1)} \le \frac{w_0(1-\lambda)}{w_0(1)},$$

which leads to

$$|w_0(x)| \le |w_0(1-\lambda)|.$$

The same applies for $w_1(x)$, and since

$$w = w_0 + \varepsilon w_1,$$

it follows that, for all $x \in [0, 1-\lambda]$,

$$|w(x)| \le |w(1-\lambda)|.$$

We employ the relation that $\lambda = 8\varepsilon \ln N$ and the estimate for $|w|$ leading to the following result

$$|w(x)| \leq Ce^{-\alpha\lambda/\varepsilon}$$
$$\leq CN^{-1}.$$

For the bound on W, the auxiliary mesh function W^* is introduced and it is defined analogous to W, with the coeffcient a replaced by its lower bound α. Using result from Lemma 4.2.4,

$$|W(x_j)| \leq |W^*(x_j)|, \qquad \text{for all } \ 0 \leq j \leq N.$$

Application of Lemma 4.2.3 leads to

$$|W(x_j)| \leq CN^{-1}, \qquad \text{for all } \ 0 \leq j \leq N/2.$$

Combining the two estimates w and W, we obtain the following estimate for the interval $[0, 1-\lambda]$,

$$|w(x_j) - W(x_j)| \leq CN^{-1}, \qquad \text{for all } \ 0 \leq j \leq N/2.$$

For the subinterval with the boundary layer, $[1-\lambda, 1]$, it follows from ideas used earlier that for all $N/2+1 \leq j \leq N-1$,

$$\left|L_{up}^N(w-W)(x_j)\right| \leq C\varepsilon^{-2}|x_{j+1} - x_{j-1}|,$$
$$\leq C\varepsilon^{-2}(2\lambda/N),$$

with

$$|w(1) - W(1)| = 0,$$

and using the outcome from the interval without the boundary layer, we have

$$\left|w(x_{N/2}) - W(x_{N/2})\right| \leq \left|w(x_{N/2})\right| - \left|W(x_{N/2})\right|,$$
$$\leq CN^{-1},$$

We introduce the barrier function

$$\Phi_j = (x_j - (1-\lambda))C_1\varepsilon^{-2}\lambda N^{-1} + C_2N^{-1},$$

it follows that for choice of C_1 and C_2, the mesh functions

$$\Psi_j^{\pm} = \Phi_j \pm (w - W)(x_j),$$

the following conditions hold at the boundaries of the interval

$$\Psi^{\pm}_{N/2} \geq 0, \qquad \Psi^{\pm}_{N} = 0,$$

and

$$L^N_{up}\Psi^{\pm}_{N/2} \geq 0, \qquad N/2 + 1 \leq j \leq N - 1.$$

It follows from the discrete maximum principle that

$$\Psi^{\pm}_{j} \geq 0, \qquad N/2 \leq j \leq N$$

and also that

$$\begin{aligned} |(w - W)(x_j)| &\leq \Phi_j, \\ &\leq C_1\varepsilon^{-2}\tau^{-2}N^{-1} + C_2N^{-1}, \\ &\leq CN^{-1}(\ln N)^2, \qquad \text{since } \tau = 8\varepsilon \ln N. \end{aligned}$$

From the two subintervals, we obtain the following estimate for the singular component

$$|(w - W)(x_j)| \leq CN^{-1}(\ln N)^2. \tag{4.18}$$

Taking absolute values of (4.15) and substituting (4.16) and (4.18), we get

$$\begin{aligned} |(y - \bar{u})(x_j)| &\leq |(v - V)(x_j)| + |(w - W)(x_j)|, \\ &\leq CN^{-1} + CN^{-1}(\ln N)^2, \\ &\leq CN^{-1}(\ln N)^2. \end{aligned}$$

□

4.3 Fitted Operator Finite Difference Methods for convection–diffusion problems

In literature there are several Fitted Operator Finite Difference Methods (FOFDMs) developed to solve SPPs based on the rules provided by Mickens. We consider the following denominator function constructed by Lubuma and Patidar in [39] using Mickens rules

$$\phi^2 = \frac{h\varepsilon}{a_j}\left(\exp\left(\frac{a_j h}{\varepsilon}\right) - 1\right), \tag{4.19}$$

with

$$\delta_\phi^2 u_j^* = \frac{u_{j+1}^* - 2u_j^* - u_{j-1}^*}{\phi^2}.$$

the corresponding difference equations given by

$$L_\phi u_j^* \equiv -\varepsilon \frac{u_{j+1}^* - 2u_j^* + u_{j-1}^*}{\phi^2} + a_j \frac{u_j^* - u_{j-1}^*}{h} = f_j. \tag{4.20}$$

The operator in (4.20) satisfies the following condition

Theorem 4.3.1. The error associated with the NSFDM satisfies

$$\max_j \|y(x_j) - u_j^*\| \le CN^{-1} \tag{4.21}$$

and therefore it is a first order uniformly convergent method.

Proof. From (4.19), we replace a_j with its lower bound α and we get

$$\begin{aligned} \frac{1}{\phi^2} &= \frac{1}{\frac{h\varepsilon}{\alpha}\left(\exp\left(\frac{\alpha h}{\varepsilon}\right) - 1\right)}, \\ &= \frac{1}{h^2} - \frac{\alpha}{2h\varepsilon} + \frac{\alpha^2}{12\varepsilon^2} + \mathcal{O}(h^2). \end{aligned}$$

The error for the second derivative estimate is as follows

$$\begin{aligned} \left(\delta_\phi^2 - \frac{d^2}{dx^2}\right)\vartheta(x_j) &= \left(\frac{\vartheta(x_{j+1}) - 2\vartheta(x_j) - \vartheta(x_{j-1})}{\phi^2}\right) - \vartheta''(x_j), \\ &= \frac{1}{\phi^2}\Big(\vartheta(x_{j+1}) - 2\vartheta(x_j) - \vartheta(x_{j-1})\Big) - \vartheta''(x_j), \\ &= \frac{1}{\phi^2}\Bigg(\vartheta(x_j) + h\vartheta'(x_j) + \frac{h^2\vartheta''(x_j)}{2!} + \frac{h^3\vartheta'''(x_j)}{3!} + \frac{h^4\vartheta^{(4)}(\xi_{1,j})}{4!} \\ &\quad - 2\vartheta(x_j) + \vartheta(x_j) - h\vartheta'(x_j) + \frac{h^2\vartheta''(x_j)}{2!} \\ &\quad - \frac{h^3\vartheta'''(x_j)}{3!} + \frac{h^4\vartheta^{(4)}(\xi_{2,j})}{4!}\Bigg) - \vartheta''(x_j), \end{aligned}$$

where $\xi_{1,j} \in (x_j, x_j + h)$ and $\xi_{2,j} \in (x_j - h, x_j)$. Since

$$\frac{h^4\vartheta^{(4)}(\xi_{1,j})}{4!} + \frac{h^4\vartheta^{(4)}(\xi_{2,j})}{4!} = \mathcal{O}(h^4),$$

we get

$$\begin{aligned}\left(\delta_\phi^2 - \frac{d^2}{dx^2}\right)\vartheta(x_j) &= \frac{1}{\phi^2}\left(h^2\vartheta''(x_j) + \mathcal{O}(h^4)\right) - \vartheta''(x_j),\\ &= \left(\frac{1}{h^2} - \frac{\alpha}{2h\varepsilon} + \frac{\alpha^2}{12\varepsilon^2} + \mathcal{O}(h^2)\right)\left(h^2\vartheta''(x_j) + \mathcal{O}(h^4)\right) - \vartheta''(x_j),\\ &= \left(1 - \frac{\alpha h}{2\varepsilon} + \frac{\alpha^2 h^2}{12\varepsilon^2} + \mathcal{O}(1)\right)\left(h^2\vartheta''(x_j) + \mathcal{O}(h^2)\right) - \vartheta''(x_j),\\ &= (1 + \mathcal{O}(h))\vartheta''(x_j) - \vartheta''(x_j),\\ &= \mathcal{O}(h).\end{aligned}$$

It then follows that

$$\left|\left(\delta_\phi^2 - \frac{d^2}{dx^2}\right)\vartheta(x_j)\right| \le C\varepsilon^{-1}N^{-1}. \tag{4.22}$$

Using result from the first part of Lemma 4.2.2 and (4.22), we have

$$\begin{aligned}|L_\phi(y-u^*)(x_j)| &\le \left\{\left|-\varepsilon\left(C\varepsilon^{-1}N^{-1}\right)\right| + (x_j - x_{j-1})|y|_2\right\},\\ &\le C\left\{N^{-1} + N^{-1}|y|_2\right\},\\ &\le CN^{-1}.\end{aligned} \tag{4.23}$$

Finally, using Lemma 4.2.1, we get

$$|(y-u^*)(x_j)| \le CN^{-1}.$$

□

4.4 A Hybrid Finite Difference Method

In this section, we introduce a hybrid method that is constructed using concepts taken from both fitted operator finite difference methods and fitted mesh finite difference methods. For the mesh, we use a simple Shishkin mesh as given in Section 4.2. It is well documented that fitted operator finite difference methods have order of convergence that is superior to that of fitted mesh finite difference methods, therefore our hybrid method also uses the denominator function constructed by Lubuma and Patidar in [39]. However, FOFDMs are traditionally applied on uniform meshes which necessitates the following modifications

$$\psi_1^2 = \frac{\varepsilon}{a_j}\left(\exp\left(\frac{a_j h_j}{\varepsilon}\right) - 1\right), \qquad \psi_2^2 = \frac{\varepsilon}{a_j}\left(\exp\left(\frac{a_j h_{j+1}}{\varepsilon}\right) - 1\right), \tag{4.24}$$

with the second derivative approximation

$$\delta_m^2 u_j^{**} = \frac{1}{\hbar_j}\left(\frac{u_{j+1}^{**} - u_j^{**}}{\psi_2^2} - \frac{u_j^{**} - u_{j-1}^{**}}{\psi_1^2}\right).$$

The numerical scheme is then given by

$$L_\psi u_j^{**} = -\varepsilon\left\{\frac{1}{\hbar_j}\left(\frac{u_{j+1}^{**} - u_j^{**}}{\psi_2^2} - \frac{u_j^{**} + u_{j-1}^{**}}{\psi_1^2}\right)\right\} + a_j\frac{u_j^{**} - u_{j-1}^{**}}{h_j} = f_j. \tag{4.25}$$

As with the FMFDM, we use the transition parameter from (4.5) and the resultant grid (4.6) as the piecewise–uniform mesh.

Theorem 4.4.1. The error associated with the HFDM satisfies

$$\max_j \|y(x_j) - u_j^{**}\| \leq CN^{-1}. \tag{4.26}$$

Proof. As with the proof of Theorem 4.2.1, we look for estimates for the singular and regular components of the solution, where

$$u^{**} = V^{**} + W^{**}.$$

Estimation of the regular component:

$$\begin{aligned} L_\psi(v - V^{**}) &= L_\psi v - L_\psi V^{**}, \\ &= L_\psi v - f, \\ &= L_\psi v - L_\varepsilon v, \\ &= (L_\psi - L_\varepsilon)v, \\ &= -\varepsilon\left(\delta_\psi^2 - \frac{d^2}{dx^2}\right)v + a\left(D^- - \frac{d}{dx}\right)v. \end{aligned}$$

As before, we apply Lemma 4.2.2 to obtain

$$L_\psi(v - V^{**}) \leq CN^{-1}.$$

Using estimates of $|v^{(j)}(x)|$ given earlier we get

$$\left|L_\psi^N(v - V^{**})(x_j)\right| \leq CN^{-1}.$$

Using Lemma 4.2.1 to the mesh function $v - V^{**}$ yields

$$|(v - V^{**})(x_j)| \leq CN^{-1}. \tag{4.27}$$

Estimation of the singular component:

When $\lambda = 1/2$, the mesh is uniform and it follows that

$$\begin{aligned}
\delta_\psi^2 w_j &= \frac{1}{\hbar_j}\left(\frac{w_{j+1} - w_j}{\psi_2^2} - \frac{w_j - w_{j-1}}{\psi_1^2}\right), \\
&= \frac{1}{h}\left(\frac{w_{j+1} - w_j}{\frac{\varepsilon}{a_j}\left(\exp\left(\frac{a_j h}{\varepsilon}\right) - 1\right)} - \frac{w_j - w_{j-1}}{\frac{\varepsilon}{a_j}\left(\exp\left(\frac{a_j h}{\varepsilon}\right) - 1\right)}\right), \\
&= \frac{w_{j+1} - w_j}{\frac{h\varepsilon}{a_j}\left(\exp\left(\frac{a_j h}{\varepsilon}\right) - 1\right)} - \frac{w_j - w_{j-1}}{\frac{h\varepsilon}{a_j}\left(\exp\left(\frac{a_j h}{\varepsilon}\right) - 1\right)}, \\
&= \frac{w_{j+1} - w_j}{\phi^2} - \frac{w_j - w_{j-1}}{\phi^2}, \\
&= \frac{w_{j+1} - 2w_j - w_{j-1}}{\phi^2}, \\
&= \delta_\phi^2 w_j.
\end{aligned}$$

The result above and (4.23) leads to

$$\begin{aligned}
|L_\psi(w - W^{**})(x_j)| &= |L_\phi(w - W^{**})(x_j)|, \\
&\leq CN^{-1}.
\end{aligned}$$

It then follows from Theorem 4.3.1 that

$$|(w - W^{**})(x_j)| \leq CN^{-1}.$$

Given the fact that a Shishkin mesh is piecewise uniform, that is, the meshlength is uniform in the subinterval, $[0, 1-\lambda]$, it is clear that

$$\delta_\psi^2 w_j = \delta_\phi^2 w_j, \qquad 0 \leq j \leq N/2.$$

As above, it follows that

$$\begin{aligned}
|L_\psi(w - W^{**})(x_j)| &= |L_\phi(w - W^{**})(x_j)| \\
&\leq CN^{-1}.
\end{aligned}$$

Consequently, from Theorem 4.3.1 we get

$$|(w - W^{**})(x_j)| \leq CN^{-1}.$$

Using the same arguments for the interval $[1 - \lambda, 1]$, we obtain analogously the result that

$$|(w - W^{**})(x_j)| \leq CN^{-1}.$$

Combining the different estimates for the singular component gives:

$$|(w - W^{**})(x_j)| \leq CN^{-1}.$$

Analogous to the proof of Theorem 4.2.1, we have

$$\begin{aligned} |(y - u^{**})(x_j)| &\leq |(v - V^{**})(x_j)| + |(w - W^{**})(x_j)|, \\ &\leq CN^{-1} + CN^{-1}, \\ &\leq CN^{-1}. \end{aligned}$$

□

4.5 Numerical results

In this section, we present the numerical results of several problems involving the one-dimensional convection–diffusion equation. We shall discuss problems with constant coefficients as well as problems with variable coefficients.

The maximum error solutions and the rates of convergence are obtained using fomulae in (2.16)–(2.19) from Section 2.4.

Example 4.5.1. ([27])

$$-\varepsilon y''(x) + y'(x) = \exp(x), \qquad y(0) = y(1) = 0. \tag{4.28}$$

The exact solution of (4.28) is given by

$$y(x) = \frac{1}{1-\varepsilon}\left[\exp(x) - \frac{1 - \exp\left(1 - \frac{1}{\varepsilon}\right) + \{\exp(1) - 1\}\exp\left(\frac{x-1}{\varepsilon}\right)}{1 - \exp\left(-\frac{1}{\varepsilon}\right)}\right]. \tag{4.29}$$

Table 4.1: Maximum errors obtained for Example 4.5.1 using the FMFDM.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-1}	4.39E-02	2.27E-02	1.15E-02	5.82E-03	2.92E-03
10^{-3}	2.23E-01	1.48E-01	9.28E-02	5.53E-02	3.19E-02
10^{-5}	2.21E-01	1.47E-01	9.23E-02	5.50E-02	3.17E-02
10^{-7}	2.21E-01	1.47E-01	9.23E-02	5.50E-02	3.17E-02
10^{-9}	2.21E-01	1.47E-01	9.23E-02	5.50E-02	3.17E-02
10^{-10}	2.21E-01	1.47E-01	9.23E-02	5.50E-02	3.17E-02

Table 4.2: Maximum errors obtained for Example 4.5.1 using the HFDM.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-1}	2.69E-04	6.74E-05	1.68E-05	4.21E-06	1.05E-06
10^{-3}	2.58E-02	1.42E-02	8.51E-03	5.41E-03	2.98E-03
10^{-5}	2.67E-02	1.34E-02	6.73E-03	3.38E-03	1.71E-03
10^{-7}	2.67E-02	1.34E-02	6.71E-03	3.35E-03	1.68E-03
10^{-9}	2.67E-02	1.34E-02	6.71E-03	3.35E-03	1.68E-03
10^{-10}	2.67E-02	1.34E-02	6.71E-03	3.35E-03	1.68E-03

Example 4.5.2. ([43])

$$-\varepsilon y''(x) + (1 + x(1-x))y'(x) = f(x), \qquad y(0) = y(1) = 0, \tag{4.30}$$

where $f(x)$ is chosen in such a way that the exact solution of (4.30) is given by

$$y(x) = \frac{1 - \exp\left(-\frac{1-x}{\varepsilon}\right)}{1 - \exp\left(-\frac{1}{\varepsilon}\right)} - \cos\left(\frac{\pi x}{2}\right). \tag{4.31}$$

Table 4.3: Rates of convergence obtained for Example 4.5.1 using FMFDM.

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-1}	0.95	0.98	0.99	0.99	1.00	1.00
10^{-3}	0.59	0.67	0.75	0.79	0.83	0.86
10^{-5}	0.59	0.67	0.75	0.79	0.83	0.86
10^{-7}	0.59	0.67	0.75	0.79	0.83	0.86
10^{-9}	0.59	0.67	0.75	0.79	0.83	0.86
10^{-10}	0.59	0.67	0.75	0.79	0.83	0.86

Table 4.4: Rates of convergence obtained for Example 4.5.1 using the HFDM.

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-1}	2.00	2.00	2.00	2.00	2.00	2.00
10^{-3}	0.87	0.73	0.65	8.60	1.10	1.16
10^{-5}	1.00	1.00	0.99	0.98	0.97	0.94
10^{-7}	1.00	1.00	1.00	1.00	1.00	1.00
10^{-9}	1.00	1.00	1.00	1.00	1.00	1.00
10^{-10}	1.00	1.00	1.00	1.00	1.00	1.00

Table 4.5: Maximum errors obtained for Example 4.5.2 using FMFDM.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-1}	2.05E-02	1.07E-02	5.44E-03	2.75E-03	1.38E-03
10^{-3}	1.25E-01	8.30E-02	5.25E-02	3.14E-02	1.82E-02
10^{-5}	1.25E-01	8.27E-02	5.24E-02	3.14E-02	1.81E-02
10^{-7}	1.25E-01	8.27E-02	5.24E-02	3.14E-02	1.81E-02
10^{-9}	1.25E-01	8.27E-02	5.24E-02	3.14E-02	1.81E-02
10^{-10}	1.25E-01	8.27E-02	5.24E-02	3.14E-02	1.81E-02

Table 4.6: Maximum errors obtained for Example 4.5.2 using HFDM.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-1}	2.64E-04	6.60E-05	1.65E-05	4.13E-06	1.03E-06
10^{-3}	2.34E-02	1.24E-02	6.86E-03	3.93E-03	2.02E-03
10^{-5}	2.39E-02	1.21E-02	6.11E-03	3.07E-03	1.55E-03
10^{-7}	2.39E-02	1.21E-02	6.10E-03	3.06E-03	1.53E-03
10^{-9}	2.39E-02	1.21E-02	6.10E-03	3.06E-03	1.53E-03
10^{-10}	2.39E-02	1.21E-02	6.10E-03	3.06E-03	1.53E-03

Table 4.7: Rates of convergence obtained for Example 4.5.2 using FMFDM.

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-1}	0.94	0.97	0.99	0.99	1.00	1.00
10^{-3}	0.56	0.66	0.74	0.79	0.83	0.85
10^{-5}	0.56	0.66	0.74	0.79	0.83	0.85
10^{-7}	0.56	0.66	0.74	0.79	0.83	0.85
10^{-9}	0.56	0.66	0.74	0.79	0.83	0.85
10^{-10}	0.56	0.66	0.74	0.79	0.83	0.85

Table 4.8: Rates of convergence obtained for Example 4.5.2 using HFDM.

ε	r_1	r_2	r_3	r_4	r_5	r_6
10^{-1}	2.00	2.00	2.00	2.00	2.00	2.00
10^{-3}	0.92	0.85	0.80	0.96	1.15	1.19
10^{-5}	0.98	0.99	0.99	0.99	0.98	0.97
10^{-7}	0.98	0.99	1.00	1.00	1.00	1.00
10^{-9}	0.98	0.99	1.00	1.00	1.00	1.00
10^{-10}	0.98	0.99	1.00	1.00	1.00	1.00

Chapter 5

Concluding remarks and scope for future research

This thesis was concerned with two-point boundary value singularly perturbed problems. Two categories of such problems, namely reaction-diffusion and convection-diffusion types, were discussed. Our aim was to consider fitted mesh and fitted operator finite difference methods, understand how they are used and then combine them in the quest to capitalise on their advantages.

In Chapter 2, fitted mesh and fitted operator finite difference methods are used to solve a family of two-point boundary value singular perturbation problems of reaction-diffusion type. The analyses of these methods were presented and comparative results in terms of accuracy and convergence were displayed. The evidence of higher accuracy and faster convergence of FOFDM over FMFDM is indicated theoretically and numerically. It is worth mentioning that, despite this drawback, FMFDMs enjoy the layer-resolving property due to the fact that they are applied on very fine meshes in the layer region(s).

To boost the accuracy of FMFDMs, defect correction methods were discussed in Chapter 3 for convection-diffusion problems. These are post-precessing techniques through which a low order stabilised method is combined with a high-order method that is less stable to obtain a higher-order and stable method. Therefore defect correction methods bear improved results. However, since they are based on piecewise meshes, they are not easily extendable to higher dimensions.

It is well known that FMFDMs are layer-resolving. They are, however, less accurate than FOFDMs which do not enjoy the layer-resolving property. In Chapter 4, we presented the main results of this work. We combined FMFDMs and FOFDMs to design a new finite difference method. This method was analysed for convergence. We found that this hybrid method is more accurate than FMFDMs and is also layer-resolving.

Due to space limitation, we did not investigate the hybridisation above for reaction-diffusion problems. Moreover, like FOFDMs, we believe the proposed hybrid method is extendable to problems in higher dimension. We are currently working in these two directions

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