

**Mathematical modeling in the sustainable use of
natural resources.**

by

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Thesis submitted in partial fulfillment of the requirements
for the degree of
Master of Science
in the

Department of Mathematics and Applied Mathematics
University of the Western Cape

Supervisor: Prof P.J. Witbooi
February 2015

Keywords

Sustainability
Natural resource management
Utility
Stability
Deterministic
Stochastic differential equation
Savanna
Eutrofication
Pollution
Fisher information
Renewable resources
Optimality
Biological pest control
Predator-prey



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Abstract

The sustainable use of natural resources is of utmost importance for every community. In particular, it is important for every given generation to plan in such a way that proper provision is made for future generations. The scientific understanding of resources use and appreciation for its life-supporting capacity is therefore essential. Mathematical modeling has proved useful to inform the planning and management of strategies for sustainable use of natural resources. Some specific topics in resource management has been studied intensively through many decades. In particular, mining, fisheries, forestry and water resources are among these. Instead of presenting a study of the latter topics, this dissertation presents a variety of cases of mathematical modeling in resource management. The aim is to improve the general understanding of the relevant problems. We expand on existing literature, papers of other authors, and add to such studies by focussing on specific items in the work, illuminating it with further explanations and graphs, or by modifying the models through the introduction of stochastic perturbations. In particular this dissertation makes contributions by giving more explanation, on the so-called *environmental Fisher information* or EFI for brevity (Section 2.4 and Chapter 6), and by introducing stochasticity into a pest control model (Chapter 4) and into a savanna vegetation model (Chapter 5). In Chapter 3 we present a model from the literature pertaining to the problem of shifting cultivation, i.e, the use of forest land when used for subsistence level agricultural purposes, until the land is so degraded that the occupants abandon it and move on to a new stand. The model used to study the shifting period is similar to the forest rotation problem.

A model, already in the literature, for biological control of a pest is studied in Chapter 4. Onto the deterministic model we impose a stochastic pertur-

bation, so that we obtain a stochastic differential equation model. We prove stochastic stability of the disease-free state, when the basic reproduction number of the pest is below unity. We have performed simulations of solutions of the stochastic system.

In Chapter 5 we review an existing ordinary differential equation model for the competition between trees and grass in savanna environment. The competition between them is for soil water, fed by annual rainfall. On the other hand, trees and grass are perturbed by fire, and some other environmental forcings such as herbivores. For this ODE model, we introduce stochastic perturbations. The stochastic perturbations are in the form of three mutually independent Brownian motions. Simulations to illustrate the effect of the stochasticity are shown.

We present a three-tiered predator-prey model and consider its stability in terms of Fisher information. This appears as Chapter 6. The Fisher information is defined on the basis of the so-called sustainable measures hypotheses. The model is already in the literature and in the dissertation we present several computations to show the influence of carrying capacity of prey and of mortality rate on EFI.

Another problem that we consider, in Chapter 7, is that of lake eutrophication caused by excessive phosphorus inflow. The computations illustrates the management of the runoff nutrients into or out of the lake. Necessary and the sufficient conditions for an optimal utility management are obtained using standard optimal control theory.

The results of this dissertation demonstrate the modeling techniques in the sustainable use of natural resources. Sustainability is the quest for equal opportunities over all generations. The manner in which this sustainability is quantified in models is being debated and improved all the time. The discourse on sustainability is especially important in view of a growing world population, and with forcings such as climate change. The most important original contribution in this dissertation is the stochastic analysis on the pest control model and the savanna model.

Declaration

I declare that *Mathematical modeling in the sustainable use of natural resources* is my work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.



L. Mthombeni

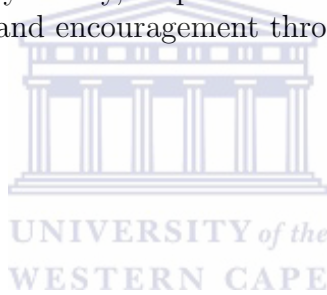
February 2015

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Acknowledgments

To Almighty God be honour and glory for giving me the strength, wisdom and perseverance to see this study through. I thank the NRF (National Research Foundation) for financial support. To my supervisor, a great thanks to Professor Peter Witbooi for guidance throughout the entire study. I also wish to thank all mathematics PhD students for moral support.

Special thanks go to my family, in particular to my sisters Agnes, Rhandzu and Pearl, for support and encouragement throughout the study.



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Chapter 1

Introduction and literature review

A natural resource is a material source that occurs in a natural state and also has economic value. This project entails a study of the use of mathematical models in the sustainable use of natural resources. Thus we shall mostly be concerned with the sustainable growth, development of renewable resources and control of pollution. We explore the efficient use of natural resources and we work mathematically towards ways and means of identifying practices and strategies for environmentally sound management. The mathematical methods are mainly around optimality conditions and stability analysis. We look at resource consumption that would not jeopardise the earth's life-support systems in the long-term.

The scientific and technical understanding of resources use and its life-supporting capacity is important, to inform the planning and management of strategies for sustainable use of natural resources. Sustainable use in ecology is also essential due to the life-support capacity of ecosystem services. Some specific topics in resource management has been studied intensively through many decades. In particular, mining, fisheries and forestry are among these, obviously due to their very crucial economic value. For more on these topics the reader is referred to the book [25] of Hanley, Shogren and White or [71] of Tietenberg and Lewis, [51] Pearce and Turner, and [34] De Lara and Doyen. Another extremely important natural resource that is being studied with the aid of mathematical models is water. Especially in countries with a general water scarcity such as South Africa, the management of the water resource management becomes more important every year.

Instead of presenting a study of the latter topics, in this dissertation we present a variety of cases of mathematical modeling in resource management as can be seen from the index page. We work on papers of other authors, and add to such studies by focussing on specific items in the work, illuminating it with further explanations and graphs, or by modifying the models through the introduction of stochastic perturbations.

1.1 Historical background

In 1961, the meteorologist E. L. Lorenz used greatly simplified forecast equations to show that tiny errors in initial conditions could make forecasts outside of a certain time period impossible (deterministic chaos). This chaotic behaviour was also found in models of basic growth processes. For example, the equation developed by Verhulst in 1845 (the discrete version of which is today known as logistic growth, or growth with limited food), also produces chaotic behaviour. An interesting offshoot from that discovery was the development of wonderful two-dimensional computer creativity. It was and is still often desirable in ecology to analyse distributed growth processes in food chains (A. J. Lotka, 1888-1949, E. P. Odum, 1983). Lotka and Volterra developed their famous predator-prey model along these lines. Models including only three species could display chaotic behaviour, depending on the non-linearities of the coupled species. The basic tools include models for logistic growth, delayed logistic growth (M. Smith 1968), exponential growth, etc. Depending on the ecosystems to be analysed, transporting diffusions and other processes has to be modelled. An enormous problem was to determine the initial conditions for model runs, in the case of forecasting, climate research, ecosystems research etc. Extensive data assimilation via different methods, including coastal measuring was needed. It became important to plan and control the manner in which people and the natural environment interact. It brings together land use planning, water management, biodiversity conservation, and the future sustainability of industries like agriculture, mining, tourism, fisheries and forestry [35]. The historical emphasis on sustainability traced back an early attempt to understand the ecological nature of rangelands in the late 19th century. Sustainability awareness has emerged very prominently in a series of meetings and reports during the 1970's, and the resource conservation movement originated from these events. Globally we find that the 20th century was marked by devastating world wars, [19]. The result thereof were the vast colonial empires. The rapid advances in science and technology from the first airplane

flight to the first moon landing also indicates a sharp rise in living standards. Increased concerns about the environment includes the loss of natural forests, shortages of energy and water, decline in biological diversity, air pollution, the onset of epidemics, and the ultimate emergence of world superpowers. The planet's population continues to explode from 1 billion in 1820 to 2 billion in 1930, 3 billion in 1960, 4 billion in 1974, 5 billion in 1987, and 6 billion in 1999. As for the 21st century, the continued exponential growth in science and technology raises both hopes and fears. Development depended on natural resources even more, and so on the sustainable use of natural resources. The use of resources was considered within the framework of process and practice of how these have shaped access and control of natural resources, see [41]. In South Africa the land reform process and restitution in particular, has been a means toward greater equity and more inclusive broad-based forms of natural resource management, [52]. The harmful consequences that result from bad environmental usages have a wide variety of causes and can be attributed to both natural and human reasons. In Africa, natural causes include drought or floods caused by a shortage or excess of precipitation, volcanoes, hurricanes, and earthquakes. The human causes include over-logging, dam construction, biological warfare, and environmental pollution. South Africa has a growing market economy, [19] with a plethora of natural resources. Gold, platinum and precious stones such as diamonds account for nearly half of South Africa's exports. Automobile assembly, textiles, iron, steel, chemicals and commercial ship repair also play a role in the country's economy. Nature preservations (or nature reserves) and its exports were and still is significant in the economy of South Africa. Jeopardizing and ignoring the need for sustainable use is seriously affecting the quality of life. Ignorance on proper management strategies and policies led to disruption or extinction of traditional habitat, temporarily even permanently, because of a marked environmental degradation, [19].

A variety of environmental applications carry with them a number of special demands. The demands have led to a variety of current research themes [39]. The European Research Consortium for Informatics and Mathematics (ERCIM), News No. 61, April 2005, reports on environmental modeling advances through distributed grid computing, knowledge from data, decision support, intelligent/adaptive user interfaces and visualization, standard of system interfacing and automatic access within the information distributes resources.

1.2 Literature survey

The environmental issues including air pollution, over exploitation of ecosystems, and exhaustion of resources has been and still is attracting the attention of the public. It is now clearly noted that human activities impacts seriously on the environments. In particular, human impact includes ecological and environmental stresses, as well as irreversible loss of species. Moreover, the destruction of habitat in some instances are the most dramatic examples of their effects. The activities caused by habitat loss or substantial change in the environment, not only alter the states and dynamics of natural resources and ecosystem. These do also alter human health, well-being, welfare and economic wealth since these resources are the essential support features for human life.

As the need for sustainable growth, renewable resources and pollution has been identified by authors (Lekama et al., [1], and Wirl [78]) for instance, most work present a growth model with an environmental resource which has its own regeneration process. In such research, a negative externality accounting for pollution flow is introduced. The pollution flow is assumed to be proportional to production. In a growth model, the optimal growth path is shown to converge to the so-called *Green Golden Rule* configuration. Similar results were obtained in the work of Heal et al., [13], on a sustainable growth model with environmental assets as a source of utility. These were cases where the object is the maximization of long-run or limiting utility rather than long-run consumption. Pasqual et al. [27] studies the exploitation of natural resources in growth models with sustainable natural resource management. The study defined environmental problems as externalities, and most of these occurred between different problems on sustaining both renewable resource and non renewables. The reasons why environmental solutions were not resolved in the intergenerational context was that it needed improved methods for evaluating the costs and benefits affecting different generations. An intergenerational redistribution of natural resource property rights was essential to guarantee equality of opportunities for all generations to ensure sustainability. Adequate redistribution of property rights depend on the type of resource, as well as on other factors such as the rate of return on its exploitation and the current generation's propensity to consume. In all cases, sustainability was sufficiently guaranteed by conceding property rights, as a whole, to the future.

In a review of sustainable management of natural resources, Balsdon et al., [4] and Brown [9], identified common approaches that shifting cultivation may be applied to dynamic natural resources. In the study, shifting cultivation has been developed in a dynamic theory. Special attention was paid to an environ-

mental impact variable: the length of time that a given field is cultivated before shifting to the next patch. The model produced indicated that poverty reduction will lead in some ways to accelerated extraction of a natural resource, but also to a longer extraction period. The results produced were therefore providing support for claims of an indirect environmental benefit from the primary goal of alleviating rural poverty. Brown [9] examined a spatiotemporal shifting cultivation and forest cover dynamics. The study described a spatiotemporal model of resource extraction adapted to the use of forest resources by shifting cultivators. In contrast to other models of spatial resource exploitation, decision criteria depended on a nonseparable agricultural household model extended to accommodate both the temporal and spatial dimensions. The paper focuses on the theoretical issues related to modeling of shifting cultivation. The discussion informs the development and implementation of a simulation model based on the approaches described.

Fath et al. [20] and Pawlowski et al. [50] presents a sustainability index using information theory. In the study, Fisher information has been explored as the basis for an index of sustainability. Ecosystems under perturbations of varying regularity and intensity is being considered to either remain within the current regime or to make transition. That has also included a regime with different characteristics. The Fisher information index developed is based on the probability of finding the system in a particular state. Fisher information was shown to be sensitive to transients in model generated data. The early detection of transitions to undesirable regimes permitted management intervention. In addition to that, the study in [20] examined an information theory approach towards regime changes in ecological systems. In the study the ecological system index were effectively developed using information theories. The Fisher information index is a measure of system order, and captures the characteristic variation in speed and acceleration along the system's periodic steady-state trajectories. Concepts were illustrated using ecosystems models. When calculated repeatedly over the system period, this index tracks steady states and transient behavior. According to Fath et al., the index could be useful in detecting system flips associated with a regime change, i.e., determining when systems are in a transient between one steady state and another.

According to Pawlowski et al., [31], the results of Fisher information must detect dynamic regime shifts in ecosystems. It must also assess the quality of the shift in terms of intensity and pervasiveness. An information index define the degree of order and sustain the hypothetical system in the different scenarios, Zellner et al., [81]. Karunathi et al., examined the detection and

assessment of ecosystem regime shifts from Fisher information. In the study they have developed a form of Fisher information that measures dynamic order in complex systems. They proposed the use of Fisher information as a means of: (1) detecting dynamic regime shifts in ecosystems, and (2) assessing the quality of the shift in terms of intensity and pervasiveness. Their *intensity* were reflected as the degree of change in dynamic order, and also determined by Fisher information. The *pervasiveness* reflected how many observable variables are affected by the change. They introduced a new robust methodology to calculate Fisher information from time series field data. Demonstrations detected regime shifts on a Fisher information model. Mayer et al., [45] pursue a study in dynamic regime changes in ecological systems and Fisher information. In the study information theory has significantly advanced in the ability to quantify the organizational complexity inherent in systems despite imperfect observations or response from the source system. Fisher information has been described in three ways: (1) A measure of the degree to which a parameter (or state of a system) can be estimated; (2) A measure of the relative amount of information that exists between different states of a system; (3) A measure of the disorder or chaos of a system. Mayer et al., applied a simple, two species Lotka- Volterra predator-prey model and dynamic systems in a periodic steady state for development of the Fisher information.

Findings on the sustainable management of ecosystem showed that changes in the carrying capacity of the system, give rise to different stable steady states [4, 65]. By repeatedly calculating Fisher information over time, transitions or flips between steady states identify changes in Fisher information. If transition phases between regimes can be detected early enough, human activity suspected of contributing to regime changes can be altered (or continued if the resultant steady state is desirable, such as in ecosystem restoration efforts). Diwekar et al., [65] attempts an idea that incorporates sustainability in ecosystem management. Natural regulation examined philosophies which were made on generic food chain models using the objectives derived from the sustainability hypotheses. Their results indicated a strong relationship between the hypotheses and the dynamic behavior of the models, supporting the use of Fisher information as a measure.

Pollution and the negligent degradation of natural resources also presents a huge challenge to sustainable living. This problem has also attracted the attention of researchers so for instance, Le Kama, [1], has introduced a negative externality caused by a pollution flow which is assumed to be proportional to production. The resource consumption requires higher investment in order not to jeopardize the environmental management. The paper [13] of Heal et al.,

describes the utility on a growth model within the environment, and assets as a source of utility in consumption. Thus a production asset is valued in terms of the long-run utility.



Chapter 2

Mathematical preliminaries and technical terminology

In this chapter we define the basic concepts and theorems such as of ordinary differential equations that will be used in our project to illustrate models. The first subsections cover the necessary stability notions and theorems and the second subsection is on optimization and control. An important item that we feature here, in the third subsection, is a general presentation of Fisher information (FI). FI was introduced for the purpose of optimization measures. It has since been used in other applications if the relevant variables can be appropriately translated. One such instance can be observed in the work [11] of Heriberto and Fath, who applied the idea to ecology. See also the follow up paper [20] of Fath et al. In this chapter we present an interpretation of FI in ecology by using measure theoretic notions. This measure theoretic presentation is an independent (maybe even original) piece of contribution. Finally, in this chapter we also touch on the necessary material from the theory of stochastic analysis, for use in two of the applications chapters.

2.1 Stability in ODEs

Notation: An *autonomous* system of differential equation has the form

$$\frac{dX}{dt} = F(X), \quad X(t_0) = X_0 \tag{2.1}$$

where

$$X = (x_1, \dots, x_n)^{\text{tr}},$$

$$F(X) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))^{\text{tr}}$$

and F does not depend explicitly on t . The notation “matrix^{tr}” means the *transpose* of “matrix”.

The following terminology is standard and can be found in books such as for instance those of Allen [3], Jordan and Smith [28], and other.

Definition 2.1.1. An *equilibrium solution* of the differential system (2.1) is a constant solution \bar{X} satisfying

$$F(\bar{X}) = 0. \tag{2.2}$$

Definition 2.1.2. An equilibrium solution \bar{X} of (2.1) is said to be *locally stable* if for each $\epsilon > 0$ there exist a $\delta > 0$ with the property that for every solution $X(t)$ of (2.1) with initial condition $X(t_0) = X_0$ and with

$$\|X_0 - \bar{X}(t_0)\|_2 < \delta, \tag{2.3}$$

we have

$$\|X_0 - X(t)\|_2 < \epsilon \text{ for all } t \geq t_0. \tag{2.4}$$

If the equilibrium solution is not locally stable it is said to be *unstable*.

Definition 2.1.3. An equilibrium solution \bar{X} is said to be *locally asymptotically stable* if it is locally stable and if there exists $\gamma > 0$ such that

$$\|X_0 - \bar{X}(0)\|_2 < \gamma \tag{2.5}$$

implies that

$$\lim_{t \rightarrow \infty} \|X_0 - X(t)\|_2 = 0. \tag{2.6}$$

Definition 2.1.4. A positive definite function V in an open neighborhood of the origin is said to be a *Liapunov function* for the autonomous differential system,

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y),$$

if $\frac{dV(x,y)}{dt} \leq 0$ for all $(x, y) \in U - \{0, 0\}$. If $\frac{dV(x,y)}{dt} < 0$ for all $(x, y) \in U - \{0, 0\}$, then the function V is called a *strict Liapunov function*.

The following theorem is fundamental in stability analysis.

Theorem 2.1.5 (Liapunov's Stability Theorem) [47]. *Let $(0,0)$ be an equilibrium of the autonomous system*

$$\frac{dx}{dt} = f(x, y); \quad \frac{dy}{dt} = g(x, y), \quad (2.7)$$

and let V be a positive defined C^1 function in an neighborhood U of the origin.

- (i) If $dV(x, y)/dt \leq 0$ for $(0, 0) \in U - \{0, 0\}$, then $(0,0)$ is stable.
- (ii) If $dV(x, y)/dt < 0$ for $(0, 0) \in U - \{0, 0\}$, then $(0,0)$ is asymptotically stable.
- (iii) If $dV(x, y)/dt > 0$ for $(0, 0) \in U - \{0, 0\}$, then $(0,0)$ is unstable.

Proof. In Case (i) the function V is a Liapunov function and in Case (ii) V is a strict Liapunov function.

Case(i) Let $\epsilon > 0$ be sufficiently small so that the neighborhood of the origin of the points $\|(x, y)\| \leq \epsilon$ is contained in U ($\|\cdot\|$ denotes the Euclidean norm). Let m be the minimum value of V on the boundary of the neighborhood, i.e for $\|(x, y)\| = \epsilon$.

Since V is a positive definite and the set $\|(x, y)\| = \epsilon$ is closed and bounded, it follows that $m > 0$. Now, choose a $\delta > 0$ with $0 < \delta \leq \epsilon$ such that $V(x, y) < m$ for a $\|(x, y)\| = \delta$. Such a δ always exist because V is continuous with $V(0, 0) = 0$. If $\|(x_0, y_0)\| \leq \delta$, then the solution with initial condition (x_0, y_0) satisfies $\|(x, y)\| = \epsilon$ for $t \geq 0$ since $dV/dt \geq 0$ implies that $V(x_0, y_0) \leq V(x(t), y(t)) < m$ for $t \geq 0$. Therefore the origin is stable.

Case(ii) The function $V(x(t), y(t))$ decreases along solutions that lie in U . Thus as $t \rightarrow \infty$, $V(x(t), y(t))$ approaches a limit. Suppose $V \rightarrow l$, then it

follows from the uniform continuity of $dV(x(t), y(t))/dt$ solutions are bounded and f and g are C_1 that $dV(x(t), y(t))/dt \rightarrow 0$ in an annular region excluding the origin. This is impossible, since $-dV/dt$ is positive definite, $dV/dt = 0$ only at the origin, and $(x(t), y(t))$ does not tend to the origin when $V \rightarrow l$. It follows that $V(x(t), y(t))$ approaches 0, which implies $(x(t), y(t))$ approaches to $(0,0)$. Therefore the origin is *asymptotically stable*. \square

2.2 Pareto optimality

Pareto optimal solutions are means for desirable allocation of resources such as in Chapter 7 in matter of a value judgment. The term Pareto optimal is usually used in economics, and also has applications in engineering. However, in this section we will show the Pareto optimal allocation of resources with some basic definition. We will also describe the variability approach for multi objective optimization including the principles. The efficiency as in [63, 5] can also be applied to the selection of alternatives in environmental resources such as in chapter (5, 7). Moreover this section shows the minimal and uncontroversial test of optimality of the outcome in multi objective problems. For instance Whinston et al., [44] describe some problems better without making some others worse off through Pareto optimal.

Considering the classical multi-objective problem, defined as follows:

$$\begin{aligned} \min F(x) \\ \text{such that } x \in \Psi \end{aligned} \tag{2.8}$$

where $F = (f_1, f_2, \dots, f_m)$, ($m \geq 2$) is a vector of objectives and $\Psi \subset \mathbb{R}_n$ is the set of feasible solutions.

Therefore the following terminology are basic and can be found in lectures such as for instance those of Selod [63], Silva and Yamakami [67], and other.

Definition 2.2.1. A solution $x^* \in \Psi$ is said to be a Pareto optimal solution of (2.8) if there exists no other feasible $x \in \Psi$ such that $f_i(x) \leq f_i(x^*)$, for every i equals $1, \dots, m$, with strict inequality for at least one i .

Definition 2.2.2. A solution $x^* \in \Psi$ is said to be a locally optimal solution of (2.8) if and only if \exists a real number $\sigma > 0$ such that x^* is Pareto optimal in $\Psi \cap \Pi(x^*, \sigma)$, for instance that \nexists another feasible $x \in \Psi \cap \Pi(x^*, \sigma)$

such that $f_i(x) \leq f_i(x^*)$, for every i equals $1, \dots, m$ with strict inequality for at least one i , where $\Pi(x^*, \sigma)$ represents σ neighbourhood of x^* defined by $\{x \in \mathbb{R} \mid \|x - x^*\| < \sigma\}$.

Definition 2.2.3. A vector that gives the distribution of all *goods* among all resources

$$x = (x_1, x_2, \dots, x_N) \in \prod_{i=1}^N x_i$$

which are individuals or firms is called an *allocation*.

An allocation is *feasible* if

$$\sum_{i \in I} x_g^i \leq W_g$$

for every $g \in G$. $W = (W_1, \dots, W_g, \dots, W_M)$ is the vector of total initial endowments of goods in the economy, where W_g is the total initial endowment of good g . $I = 1, \dots, i, \dots, N$ is a set of N firm or individuals, and is assumed that each resources faces a consumption set X_i and has a utility function

$$u_i : X_i \rightarrow \mathbb{R}.$$

$G = 1, \dots, g, \dots, M$ is a set of M perfectly divisible goods, and hence $X^i = \mathbb{R}_+^M$.

The function

$$x_i = (x_1^i, x_2^i, \dots, x_M^i)$$

is a resource i 's consumption of goods, with x_g^i the quantity of goods g allocated to i .

From definition 2.2.1 to 2.2.3, it follows that an allocation $x^* = (x^{1*}, \dots, x^{i*}, \dots, x^{N*})$ is a Pareto optimum if for every $g \in M$, we have:

$$\sum x_g^i \leq W_g$$

and there does not exist x feasible such that

$$\begin{cases} u_i(x_i) \geq u_i(x^{i*}) & \text{for every } i \in I, \\ \exists j \in I, & u_j(x_j) \geq u_j(x^{j*}). \end{cases}$$

Proposition 2.2.4. *Suppose that the utility functions are quasi-concave and twice continuously differentiable. Then any allocation x^* solution of the following*

$$\begin{cases} \text{Max} & u_1(x_1^1, x_2^1, \dots, x_M^1), \\ \text{such} & \text{that } (x^1, x^2, \dots, x^M) \text{ is feasible,} \\ \text{and} & u_i(x_1^i, x_2^i, \dots, x_M^i) \geq \bar{u}_i \text{ for } i = 2, \dots, N \end{cases} \quad (2.9)$$

is a Pareto optimum. Reciprocally, for any Pareto optimal is a solution of the (2.9) (written with the adequate vector of utility-level parameters $(\bar{u}_2, \dots, \bar{u}_N)$).

2.3 Optimization and Control

Several generations of scientists have considered optimal control as well developed into established research areas. Its applications are found in many fields ranging from mathematics, engineering to biomedical, and management sciences. For an introduction to optimal control we can refer to the books Seierstad and Sydsaeter [62] or the book [18, 61] by Kirk and Schwartz.

In this section we consider from the equation (2.1), to rewrite an *autonomous* system of differential equation to be of the form

$$\dot{x}(t) = \frac{dx}{dt} = f(x(t)), \quad x(t_0) = x_0, \quad t > 0. \quad (2.10)$$

Definition 2.3.1. A state variable $x = [x_1, \dots, x_n]^T$, $n \geq 1$ describing the internal behaviour of a system, together with a so-called control variable $\alpha = [\alpha_1, \dots, \alpha_{n_\alpha}]^T$, $n_\alpha \geq 1$ is called a *control system*.

Here it is understood that f (f as in equation (2.10)) depends on α and also upon some control parameters belonging to some set $S \subset \mathbb{R}^m$, and that makes f into function $f : \mathbb{R}^n \times S \rightarrow \mathbb{R}^n$. If we now take some value of s in S and also consider the corresponding dynamics from (2.8) to be:

$$\dot{x}(t) = f(x(t), s), \quad (t > 0) \quad (2.11)$$

$$x(0) = x_0,$$

then that gives the evolution of our system according to the parameter s which is constantly being reset by the function α . As the value of the parameter

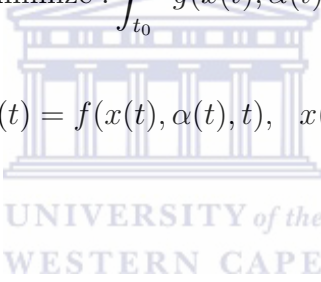
changes, so is the system as well. We assume $\alpha : [0, \infty) \rightarrow S$ to be the function which we regard as the control on the system. Then it is understood that the state of the system also depend on α . Thus:

$$\begin{aligned}\dot{x}(t) &= f(x(t), \alpha(t)), \quad (t > 0) \\ x(0) &= x_0,\end{aligned}\tag{2.12}$$

We therefore define the dynamic system by the equation:

$$\dot{x}(t) = f(x(t), \alpha(t), t), \quad x(t_0) = x_0,\tag{2.13}$$

where $x(t)$ is the state variable, $\alpha(t)$ control variable. The prototypical control problem consists of choosing a continuous differentiable function $\alpha(t)$, $0 \leq t \leq t_1$ at an initial condition, to

$$\begin{aligned}\text{minimize : } & \int_{t_0}^{t_1} g(x(t), \alpha(t), t) dt \\ \text{subject to } & \dot{x}(t) = f(x(t), \alpha(t), t), \quad x(t_0) = x_0.\end{aligned}\tag{2.14}$$


Definition 2.3.2. A real-valued function $\alpha(t)$, $t_0 \leq t \leq t_1$ is said to be *piecewise continuous*, denoted $\alpha \in C[t_0, t_1]$, if there is a finite partition say $t_0 = \beta_0 < \beta_1 \dots \beta_n < \beta_{n+1} = t_1$ such that α may be regarded as a continuous function in $C[\beta_k, \beta_{k+1}]$, for each $k = 0, 1, \dots, n$.

Definition 2.3.3. A piecewise continuous control $\alpha(\cdot)$, defined on some time interval $t_0 \leq t \leq t_1$, with range in the control region A ,

$$\alpha(t) \in A, \forall t \in [t_0, t_1],$$

is said to be an *admissible control*.

The Pontryagin's Maximum Principle converts the maximization or minimization of the objective functional J , coupled with the state variable into pointwise maximizing or minimizing of the Hamiltonian with respect to the control. For this problem the Hamiltonian $H(t, x, \alpha, \lambda)$ is a function of four variables. Time t is the underlying variable for each of x , α , λ and H itself. The function $\lambda(t)$

is called the adjoint variable.

Theorem 2.3.4 *If $\alpha^*(t)$ and $x^*(t)$ are optimal for problem for problem (2.14), then there exists a piecewise differential adjoint variable $\lambda(t)$ such that*

$$H(t, x^*(t), \alpha(t), \lambda(t)) \leq H(t, x^*(t), \alpha(t), \lambda(t))$$

for all controls α at each time t , where the Hamiltonian H is

$$H = g(t, x(t), \alpha(t)) + \lambda(t)f(t, x(t), \alpha(t))$$

and

$$\frac{d\lambda(t)}{dt} = - \frac{\partial H(t, x^*(t), \alpha^*(t), \lambda(t))}{\partial x},$$

$\lambda(t_1) = 0.$

Necessary conditions: If $\alpha^*(t)$ and $x^*(t)$ are optimal, then the following conditions hold:

$$\frac{d\lambda(t)}{dt} = - \frac{\partial H(t, x^*(t), \alpha^*(t), \lambda(t))}{\partial x},$$

$$\lambda(t_1) = 0,$$

$$\frac{\partial H(t, x^*(t), \alpha^*(t), \lambda(t))}{\partial \alpha} = 0.$$

2.4 Fisher information

The notion of Fisher information (FI) was introduced in 1921 [21] in the context of mathematical foundations of theoretical statistics. The FI is a means of optimization measures, and it is defined as follows

$$\text{FI} = \int \frac{1}{p(\epsilon)} \left(\frac{dp(\epsilon)}{d\epsilon} \right)^2 d\epsilon, \quad \text{and}$$

$p(\cdot)$ is the density at the variable ϵ . FI has since been used in other applications where the relevant variables can be appropriately translated. One such appears in the work [11] of Heriberto and Fath, who applied the notion to ecology. In what follows we present an interpretation of FI in ecology by using measure theoretic notions. This measure theoretic approach is independent work in this thesis (and maybe even original).

Consider any measure space (X, \mathcal{A}, μ) with \mathcal{A} a σ -algebra on the set X , and we assume that μ is a measure on X with $\mu(X) < \infty$. Then for any bijective function $g : X \rightarrow Y$, we can define an associated probability space $(Y, \mathcal{A}_g, \mu_g)$ by taking $\mathcal{A}_g = \{g(B) : B \in \mathcal{A}\}$ and for each $D \in \mathcal{A}_g$,

$$\mu_g(D) = \frac{\mu(g^{-1}(D))}{\mu(X)}.$$

Now we consider the special case of X being an interval $[a, b]$, with \mathcal{A} being the Borel σ - algebra on X , and $g : X \rightarrow Y \subseteq \mathbb{R}^n$ being a differentiable function. Let μ be the Lebesgue measure on X . Now we note that in this case, \mathcal{A}_g coincides with the Borel σ - algebra on Y (Y being regarded as a subspace of \mathbb{R}^n). Now we also note that there is a measure λ on Y , determined by curve length on the parametrized curve $\{(t, g(t)) : a \leq t \leq b\}$.

In what follows we present an analysis of the relationship between λ and μ_g . For brevity we write $\mu_g = \Pi$. By $\dot{g}(t)$, we mean the coordinate wise derivative of g with respect to t . Also we assume $\dot{g}(t) \neq 0$.

For any $t \in (a, b)$ and any h such that $t+h \in (a, b)$, let $A_{t,h}$ be the interval:

$$A_{t,h} = \begin{cases} [t, t+h] & \text{if } h > 0, \\ [t+h, t] & \text{if } h < 0. \end{cases}$$

We define the following function $p : (a, b) \rightarrow \mathbb{R}$.

$$p(g(t)) = \lim_{h \rightarrow 0} \frac{\Pi(g(A_{t,h}))}{\|g(t+h) - g(t)\|}.$$

Then p is the probability density function of Π on the parameterized curve Y (or more precisely on the curve $Y^0 = g(a, b)$).

Now we define *ecological Fisher information* (i.e., EFI) as in [65] for instance, by the following equation:

$$\text{EFI} = \int_0^T \frac{1}{pg(t)} \left(\frac{dpg(t)}{d\bar{g}(t)} \right)^2 d\bar{g}(t)$$

In what follows we analyse this formula, transforming it into a directly usable form. The detail we present in the sequel seems not to be in existing literature.

Proposition 2.4.1. *For any $t \in (a, b)$, we have $p(g(t)) = (\mu(t)\|\dot{g}(t)\|)^{-1}$.*

Proof. Note that

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\|g(t+h) - g(t)\|}{\Pi(g(A_{t,h}))} \\ &= \lim_{x \rightarrow 0} \frac{\|g(t+h) - g(t)\|}{\mu(A_{t,h})/\mu(X)} \\ &= \mu(X) \left\| \frac{g(t+h) - g(t)}{h} \right\| \\ &= \mu(X) \|\dot{g}(t)\|. \end{aligned}$$

Since $\|\dot{g}(t)\| \neq 0$, the proposition follows. □

Let us write $\bar{g}(t)$ as denoting the curve length of the curve $(t, g(t))$, with $\bar{g}(0) = 0$. Then we note that

$$\frac{d\bar{g}(t)}{dt} = \sqrt{\dot{g} \cdot \dot{g}} = \|\dot{g}(t)\|$$

By the notation

$$\frac{dp(g(t))}{d\bar{g}(t)} \text{ for } t \in (a, b)$$

we mean

$$\lim_{x \rightarrow 0^+} \frac{pg(t+h) - pg(t)}{\|\bar{g}(t+h) - \bar{g}(t)\|}.$$

This means derivative with respect to curve length. We note that by the chain rule of differentiation we have:

$$\frac{dpg(t)}{d\bar{g}(t)} \cdot \frac{d\bar{g}(t)}{dt} = \frac{dpg(t)}{dt}.$$

From Proposition 2.4.1 we have

$$\frac{dpg(t)}{dt} = \frac{d}{dt} (\mu(t) \|\dot{g}(t)\|)^{-1}.$$

Thus we have the following proposition.

Proposition 2.4.2.

$$\begin{aligned} \frac{dpg(t)}{d\bar{g}(t)} &= \frac{dpg(t)}{d(t)} / \frac{d\bar{g}(t)}{dt} \\ &= \frac{\frac{d}{dt} (\mu(t) \|\dot{g}(t)\|)^{-1}}{\frac{d\bar{g}(t)}{dt}} \\ &= \frac{1}{\mu(X) \|\dot{g}(t)\|} \frac{d}{dt} \|\dot{g}\|^{-1}. \end{aligned} \quad \square$$

We can rewrite

$$\frac{d}{dt} \|\dot{g}(t)\|^{-1} = \frac{d}{dt} (\|\dot{g}(t)\|^2)^{-\frac{1}{2}} = \frac{d}{dt} (\dot{g} \cdot \dot{g})^{-\frac{1}{2}}$$

which is

$$\begin{aligned} &= -\frac{1}{2} (\dot{g} \cdot \dot{g})^{-\frac{3}{2}} \frac{d}{dt} (\dot{g} \cdot \dot{g}) \\ &= -\frac{1}{2} \|\dot{g}\|^{-3} \cdot \left(\sum 2\dot{g}_i \ddot{g}_i \right) \\ &= -\frac{\sum \dot{g}_i \ddot{g}_i}{\|\dot{g}\|^3}. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{p(g(t))} \frac{dpg(t)}{\bar{g}(t)} \frac{dp(g(t))}{dt} \\ &= \mu(X) \|\dot{g}\| \left(\frac{dp(g(t))}{dt} \right) \left(\frac{d\bar{g}(t)}{dt} \right)^{-1} \left(\frac{dp(g(t))}{dt} \right) \\ &= \mu(X) \|\dot{g}\| \|\dot{g}\| \left(\frac{dp(g(t))}{dt} \right)^2 \\ &= \mu(X) \|\dot{g}\|^2 \cdot \frac{1}{\mu(X)^2} \cdot \left(\frac{d}{dt} \frac{1}{\|\bar{g}\|} \right)^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{\|\dot{g}\|^2}{\mu(X)} \cdot \frac{\sum \dot{g}_i \ddot{g}_i}{\|\dot{g}\|^6} \\
&= \frac{(\sum \dot{g}_i \ddot{g}_i)^2}{\mu(X) \|\dot{g}\|^4}
\end{aligned}$$

Now we have a convenient expression for EFI:

Proposition 2.4.3.

$$\text{EFI} = \frac{1}{T} \int_0^T \frac{\sum \dot{g}_i \ddot{g}_i}{\|\dot{g}\|^4} dt.$$

□

2.5 Eigenvalues.

Rational management of this work requires the study of quadratic forms and differential equations information that is both relevant and timely. Our aim is to develop a method on calculating the dynamical ecological system for environmental resources. We describe the characteristics equations for these work as a basis. The author added much more details from Murray, [47, pg.501].

From the autonomous differential system (2.7) we find its trajectories phase to be solutions of

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}. \quad (2.15)$$

Suppose that for any point (x_0, y_0) there is a unique curve besides in the singular points (x_s, y_s) where both functions at its singular points is equal to *zero*. Then, if we let x reflect to $x - x_s$ and y reflect to $y - y_s$ we will have $(0, 0)$ for a singular point of the transformed equation

$$f(x_s, y_s) = g(x_s, y_s) = 0.$$

To have a singular point at the origin we consider (2.8) which gives $f(x_s, y_s) = g(x_s, y_s) = 0$ so that we have $x = 0, y = 0$.

In fact by considering f and g to be analytic near $(0, 0)$ then we can expand f and g in a Taylor series and retain only the linear terms such that we

obtain

$$\frac{dy}{dx} = \frac{ax + by}{cx + dy};$$

and we get

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}. \end{aligned} \tag{2.16}$$

The matrix A is defined from the system and a , b , c and d are the constants. The equivalence of the linear form to the system is

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy. \tag{2.17}$$

Solutions of (2.10) give the parametric forms of the phase curves, where t is the parameters. Let λ_1 and λ_2 be the eigenvalues of A such that

$$0 = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}.$$

Then λ_1 and λ_2 are found to be equal to:

$$1/2(a + d \pm [(a + d)^2 - 4\det A]^{1/2}). \tag{2.18}$$

Then solutions of (2.10) are then

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v_1 e[\lambda_1 t] + c_2 v_2 e[\lambda_2 t],$$

where c_1 and c_2 are arbitrary constants and v_1 , v_2 are the eigenvectors of A corresponding to λ_1 and λ_2 . We therefore present the following cases for the correspondence in the eigenvectors.

Case I: *Real eigenvalues*

In the case of real eigenvalues λ_1 and λ_2 we then find the corresponding eigenvectors V_1 and V_2 in a direction along the solutions that travel toward or away

from the origin. For instance, if λ_1 is positive, then solutions will travel along V_1 and away from the origin. Also if λ_2 is negative, then solutions will travel along V_2 and towards the origin. The generality solutions travel in a direction which is a linear combination of V_1 and V_2 . Therefore the origin real eigenvalues is classified as of either a node or a saddle.

1. *Node*: Both eigenvalues have the same sign and may be distinct or equal, $\lambda_1 \leq \lambda_2 < 0$ and $0 < \lambda_1 \leq \lambda_2$. They can be classified as proper or improper [3]. A node is called *proper* when the eigenvalues are equal and there are two linearly independent eigenvectors; otherwise it is called *improper*. A proper node is also referred to as star point or proper solutions.

2. *Saddle*: Eigenvalues λ_1 and λ_2 have opposite signs, $\lambda_1, \lambda_2 < 0$ (eg $\lambda_1 < 0$ and $\lambda_2 > 0$).

Case II: Complex eigenvalues

In the case of complex eigenvalues, $\lambda_{1,2} = a \pm ib$; a, b parameters and $b \neq 0$. Because solutions to the linear system $dX/dt = AX$ include factors with $\cos(bt)$ and $\sin(bt)$, solutions spiral around the equilibrium. So, if the $\text{Re}(A) < 0$, then the solutions with $e^{at} \cos(bt)$ or $e^{at} \sin(bt)$ spiral inwards, toward the origin. But if $\text{Re}(A) > 0$, then solutions spiral outwards, away from the origin. Finally if $\text{Re}(a) = 0$, the solutions are closed curves encircling the origin.

A center is obtained if eigenvalues are purely imaginary ($a = 0$), $\lambda_{1,2} = \pm ib$.

WESTERN CAPE

2.6 Brownian Motion

Brownian motion has become one of the fundamental building blocks of modern quantitative finance. Indeed, the basic continuous time model for financial asset prices assumes that log-return of a given financial asset follow a Brownian motion with drift. Brownian motion is the irregular random motion of small particles immersed in a liquid or gas, as observed by R. Brown in 1827. There is a formal stochastic process, modelled on the Brownian motion and carrying the same name, called the Brownian process or the Wiener process.

In real life, many phenomena has stochasticity within them or experience randomness from without. Mathematical modellers have realised the importance of the introduction of stochasticity into mathematical models. This can be achieved via different methods, depending on the type of model. One partic-

ular means of incorporating stochasticity into a model is by way of stochastic differential equations (*SDE's*). For the models in this dissertation we shall modify *ODE* models by adding stochastic perturbations, thereby obtaining *SDE* models.

In this section we briefly introduce some of the essential concepts required for the stochastic analysis in two of the applications chapters. Most of these concepts are explained in the textbook [42] of Mao.

Definition 2.6.1. Let (Ω, \mathcal{F}, P) be a probability space with filtration $\{\mathcal{F}_t\}_{t \geq t_0}$. A one-dimensional Brownian motion is a real-valued continuous $\{\mathcal{F}_t\}$ -adapted process $\{B_t\}_{t \geq t_0}$ with the following properties:

- (i) $B_0 = 0$ a.s.;
- (ii) for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$;
- (iii) for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is independent of $\{\mathcal{F}_s\}$
- (vi) B_t is continuous in $t \geq 0$.

Stochastic stability 2.6.2.. Consider the general n -dimensional stochastic system

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t) \quad (2.19)$$

on $t \geq 0$ with initial value $x(0) = x_0$, the solution is denoted by $x(t, x_0)$. Assume that $f(t, 0) = g(t, 0) = 0$ for all $t \geq 0$, so the origin gives an equilibrium point.

Definition 2.6.3. The equilibrium $x = 0$ of the system (2.19) is said to be almost surely exponentially stable if for all $x_0 \in \mathbb{R}^n$

$$\limsup_{x_0 \rightarrow 0} \frac{1}{t} \ln |x(t, x_0)| < 0 \quad a.s.$$

Definition 2.6.4. The equilibrium solution of equation (2.19) is said to be stochastically stable for every pair of $\epsilon \in (0, 1)$ and $r > 0$, \exists a $\delta = \delta(\epsilon, r, t(0)) > 0$ such that

$$P \{|x(t; t(0), x(0))| < r \text{ for all } t \geq t(0)\} \geq 1 - \epsilon$$

whenever $|x(0)| < \delta$.

Definition 2.6.5. The trivial solution of equation (2.19) is said to be stochastically asymptotically stable if it is stochastically stable and, also for $\forall \epsilon \in (0, 1)$, \exists a $\delta_0 = \delta_0(\epsilon, t(0)) > 0$ such that

$$P \left\{ \lim_{t \rightarrow \infty} x(t; t(0), x(0)) = 0 \right\} \geq 1 - \epsilon$$

whenever $|x(0)| < \delta_0$.

The generator differential 2.6.6. The differential operator L which we define below and which is associated with the following equation:

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \quad t \geq t_0,$$

is referred to as the generator of this equation (see Oksendal). The operator L acts on a function of $V \in C^{2,1}(S_h \times \mathbb{R}_+; \mathbb{R}_+)$, as follows:

$$LV = V_t(x, t) + V_x(x, t) + \frac{1}{2} \text{trace} [g^T(x, t)V_{xx}(x, t)g(x, t)],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_d} \right)$, $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{d \times d}$.

Strong law of large numbers 2.6.7. Let $M = \{M_t\}_{t \geq 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then

$$\lim_{t \rightarrow \infty} \langle M, M \rangle_t = \infty \quad a.s. \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{\langle M, M \rangle_t} = 0 \quad a.s.$$

and also

$$\limsup_{t \rightarrow \infty} \frac{\langle M, M \rangle_t}{t} < \infty \quad a.s. \Rightarrow \lim_{t \rightarrow \infty} \frac{M_t}{t} = 0. \quad (2.20)$$

Chapter 3

Shifting cultivation

Shifting cultivation is a way of farming in which farmers use an area of land (usually public land or land under nature conservation) until it cannot produce good crops anymore, and then the farmer moves to a new area of land. It is a type of farming in which the land under cultivation is periodically shifted so that fields that were previously cropped are left to fallow. This occurs mostly in impoverished communities. In the paper [4] of Balsdon, we find a model of shifting cultivation with special attention on environmental impact. The cultivation results in long periods of fallow and the need to move the zone of active cultivation from one location to another over time. A related model is discussed in the paper [9]. The purpose of this chapter is to briefly introduce the model and to describe a conceptual approach in the understanding of the modeling of shifting cultivation.

We consider the objectives described in the paper [4] of Balsdon, whereby the resource stock over a fixed period $[0, T]$ as a classical problem of optimal control is given as follows:

$$V = \int_0^T u[c(t)]e^{nt} dt, \quad (3.1)$$

$$c(t) = N + px(t) \quad (3.2)$$

$$\dot{S} = g[S(t)] - x(t), \quad (3.3)$$

$$S(0) = S(0); S(t) \geq 0. \quad (3.4)$$

The variable S is the stock of soil productivity remaining at time t , and x is the extraction, in the form of agricultural output. The units of x and S are defined so that one extracted unit of the resource stock always yields 1 unit of agricultural output. The parameters p and N are the output price and non-farm income, respectively, so that c is household consumption. The function $u[\cdot]$ is utility with the usual properties.

3.1 Optimization problem

The equation (3.1) for the definition of the objective function can be rewritten as:

$$V = \int_0^T u(N + px(t))e^{\eta t} dt. \quad (3.5)$$

The problem is now to maximize V , subject to the given conditions, with respect to the extraction, $x(t)$. This means that $S(t)$ is the state variable and $x(t)$ is the control. We leave the set of admissible controls open for now. Therefore we have the following.

Problem 3.1

Maximize $V(s, x, t)$ with respect to $x(t)$, subject to the conditions (3.3) and (3.4).

Towards the solution, we require the Hamiltonian, which is as given in equation (3.6) below. At times we shall write

$$e^{\eta t} u = w.$$

We note further that the costate variable α is a function of time, i.e., $\lambda = \lambda(t)$.

$$H(s, x, \lambda, t) = w(x, t) + \lambda(g(S) - x). \quad (3.6)$$

Theorem 3.2

The following condition is necessary for optimality:

$$\dot{w}_x = -w_x g'(s). \quad (3.7)$$

Proof. Let us assume for now that the conditions $S(t) \geq 0$ holds. Now we explore the conditions for optimality that must hold:

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial S}. \quad (\text{a})$$

The control $x(t)$ must maximize H . (b)

Condition (a) implies that:

$$\dot{\lambda} = -\lambda g'(S). \quad (\text{c})$$

Condition (b) implies that

$$0 = \frac{\partial H}{\partial x} = \frac{\partial w}{\partial x} - \lambda. \quad (\text{d})$$

The latter identity implies that:

$$\dot{\lambda} = \frac{d}{dt} \left(\frac{\partial w}{\partial x} \right). \quad (\text{e})$$

Now let us write

$$\frac{\partial w}{\partial x} \text{ as } w_x.$$

Using equations (d) and (e) in (c) we eliminate λ and $\dot{\lambda}$ to obtain:

$$\frac{d}{dt} (+w_x) = -w_x g'(S), \text{ i.e., } \dot{w}_x = -w_x g'(S).$$

Thus, we have

$$-\eta e^{-\eta t} u_x + e^{-\eta t} \dot{u}_x = \dot{w}_x = -w_x g'(S) = -e^{-\eta t} u_x g'(S)$$

which simplifies to:

$$u_x(\eta - g'(S)) = \dot{u}_x.$$

□

3.2 Optimal cultivation period

Let us denote by z , the cost of preparing a new stand for cultivation. The optimal occupation time T^* is calculated similarly as the rotation period in forestry according to the Faustmann model [26, 82].

Proposition 3.1 *The optimal rotation time T satisfies the differential equation:*

$$V'(T) = \frac{\eta(V - z)}{e^{\eta T} - 1}.$$

Proof. Suppose that the occupation time T is fixed. Let W be the discounted wealth obtained over an infinite time horizon. Then

$$\begin{aligned} W &= v - z + e^{\eta t}(v - z) + e^{\eta 2T}(v - z) \\ &= (v - z) \sum_{i=0}^{\infty} e^{-\eta T i} \\ &= \frac{v - z}{1 - e^{-\eta T}}. \end{aligned}$$

For optimal W we set $\frac{dW}{dT} = 0$. Therefore, with $(1 - e^{-\eta T})^{-1} = \alpha$, we have:

$$0 = W'(T) = \alpha V'(T) + (-\alpha^2)(-e^{-\eta T})(-\eta)(V - z).$$

Therefore we obtain

$$V'(T) = \frac{\eta(v - z)e^{-\eta T}}{1 - e^{-\eta T}} = \frac{\eta(v - z)}{e^{\eta T} - 1}.$$

□

Now we note that

$$V'(T) = u(1 + px(T))e^{-\eta T}.$$

This yields

$$u(1 + px(T)) = \frac{\eta(v - z)}{1 - e^{-\eta T}}.$$

This can be rewritten as:

$$(1 - e^{\eta T}) \ln(1 + px(T)) = \eta(v - z).$$

Proposition 3.2 If we assume $S(t) \geq 0$ and take $u(x) = \ln(N + px)$, then

$$\dot{x} = \frac{c}{p} (g'(S) - \eta).$$

Proof. Recall that $c = N + px$. Then (*) implies that

$$\frac{p}{c} (\eta - g'(s)) = \frac{d}{dt} \left(\frac{p}{c} \right) = -\frac{p^2}{c^2} \cdot \dot{x}.$$

Therefore

$$\dot{x} = \frac{c}{p} (g'(S) - \eta). \quad (3.8)$$

□

Proposition 3.3 For $u(x) = (N + px)^2$ we have

$$\dot{x} = \frac{2}{p} (N + px)\eta (\eta - g'(S)).$$

Proof. From Theorem 3.2 we have

$$2p(N + px)\eta (\eta - g'(S)) = p^2 \dot{x}, \text{ i.e.,}$$

$$\dot{x} = \frac{2}{p} (N + px)\eta (\eta - g'(S)). \quad (B)$$

□

Thus we conclude, if the period length is sufficiently long to yield marginal utility, the resource stock can be managed in such a way that exhaustion is avoided. A long period of abandonment allows the rain forest to regenerate at least partially, and the fertility of the land to be restored. The nutrient content of both vegetation and soils is replenished and the regeneration protects the soil from erosion.

Chapter 4

A model for biological control of a pest

Crops and livestock that are essential for human sustenance can be damaged or destroyed by pests. It is important that strategies be developed to control or eradicate such pests [24, 10]. In many cases it is not economically possible or biologically viable to completely eradicate a pest. Nevertheless, the threat needs to be understood, monitored and controlled. Progress in science has brought about and advanced two very significant approaches to pest controls. These two major approaches are by chemical methods (poison) or biological methods such as releasing predators onto the pests or releasing pest infected with a communicable disease. There are other methods too.

Very recently, stochastic perturbation of epidemiological models have been studied extensively, and it has been shown how stochasticity can enhance certain equilibrium points. See for instance the papers [79] and [84]. In this chapter we study a stochastic differential equation model of a system comprising a stock on which we have a pest, with the pest population being subjected to a certain disease. The idea is to control the pest in the population by releasing infected pest from the laboratory. Essentially we shall start with the deterministic model of Chen and Tan [70] and impose a stochastic perturbation on it. The general technique is to choose a living organism, which is referred to as a biocontrol of the pest in the stock. So, for instance, insects can be infected by disease-causing organisms such as bacteria, viruses or fungi. In Chapter 6 we consider various aspects of species interactions in an ecological zone. Such methods are relevant to this study as well. There is a vast literature on the application of microbial disease for suppressing pests, see for instance

[10, 16]. It may be that a disease is present in a pest population but is not active. When conditions become favourable, an outbreak of disease may occur spontaneously. These conditions could include high pest abundance, humidity, temperature, etc.

4.1 The model of Chen

A simple model of releasing infected pests to suppress the abundance of the susceptible pest was proposed by Goh [24]. The model was followed up by a model [10] due to Burgess. Xia [80] presented a continuous time model as well as a model with periodic release of infected pest at regular time intervals. In this presentation we shall focus on the model 4.1 due to Tan and Chen [70].

We denote the quantity of susceptible pest at time t by $S(t)$, and the infected pest by $I(t)$. By $y(t)$ we denote the prey population. The symbol ν represents the release rate of pests which are bred and subsequently infected [10] for release onto the susceptibles. This ν will thus be a controllable parameter. We denote by α a parameter for the transmission coefficient, β represents the death rate of the infected pest population. It is assumed that the infected pests cannot recover, they are not capable of reproducing and cannot attack the crops [83]. We assume that the susceptible pest $S(t)$ is capable of reproducing according to a logistic law. The variables r_1 is a death rate and r_2 is a growth rate of susceptible pests.

$$\begin{cases} \dot{S}(t) = S(t)(-r_1 - \alpha I(t) + k\theta y(t)) \\ \dot{I}(t) = \alpha S(t)I(t) - \beta I(t) + \nu \\ \dot{y}(t) = y(t)(r_2 - \theta S(t) - \omega y(t)) \end{cases} \quad (4.1)$$

The term $y(t)(r_2 - \theta S(t) - \omega y(t))$ in the third equation indicates prey in the absence of any predator which is growing in a logistic form. The contribution of prey to the predator's growth rate is given by $k\theta y(t)S(t)$. We assume that if the pests are infected, they will shortly die and cannot consume the prey population.

The system has 3 equilibrium points, $E_0 = (0, \frac{\nu}{\beta}, 0)$, $E_1 = (0, \frac{\nu}{\beta}, \frac{r_2}{\omega})$ and $E^* = (S^*, I^*, y^*)$, with

$$S^* = \frac{r_2 - \omega y^*}{\theta}, \quad y^* = r_1 + \alpha I^* k \theta,$$

$$I^* = \left[2\omega\alpha^2 \right]^{-1} \left[k\theta^2 \left(\frac{\alpha r_2}{\theta} - \frac{\alpha r_1 \omega}{k\theta^2} - \beta \right) + k\theta^2 \left(\left(\frac{\alpha r_2}{\theta} - \frac{\alpha r_1 \omega}{k\theta^2} - \beta \right)^2 + \frac{4\omega\alpha^2\nu}{k\theta^2} \right)^{1/2} \right]$$

In [70] it is shown that E_1 is stable whenever $R_0 < 1$ and is unstable if $R_0 > 1$, where the threshold R_0 is given by

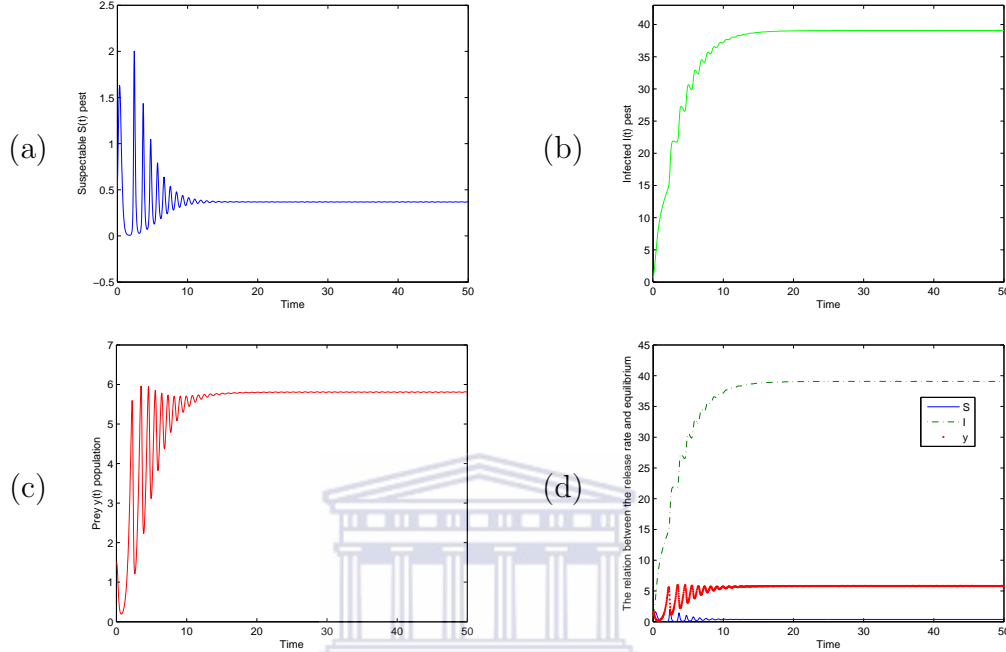
$$R_0 = \frac{k\theta r_2 \beta}{\omega r_1 \beta + \omega \alpha \nu}.$$

In the sequel we form a stochastically perturbed version of the model above, and we prove that the point E_1 is asymptotically exponentially stable. The latter implies global stability of the point E_1 in the deterministic case.

In the paper [70] of Tan and Chen, it is shown how to keep the pest numbers at an economically viable level, by applying the proper intensity on the release rate ν of infected pest. In what follows we give some discussion and graphs on this subject. The insect pest can be successfully controlled by choosing the appropriate value of the parameter ν . Thus we give the graphs to illustrate the relations between the stable equilibrium and release rate ν . Susceptible pests can be eradicated if the release rate ν is too high. The figure (Fig 4.1 (a)) shows a case of control of insect pest such that the number of the susceptible pests is always less than the given economic threshold. Considering the release rate ν as in (b), we see that a good pest control programme should reduce pest populations to a level acceptable to the crop environment. It follows from the equations that in equilibrium we have that I^* is an increasing function of ν and y^* is also an increasing function of ν . This means that the bigger the value of ν , the higher we have y^* . The increase of release rate must be such that the number of susceptible pest will always be less than the economic threshold (ET). Economic threshold, according to Sten, Smith, Vanden Bosch and Hagan, 1959, is the population density of pest population, at which control measures should be started to prevent the population from reaching the economic injury level.

The control of insect pest is observed by the graphs:

Figure 4.1: Simulations showing the control of insect pest in the equilibrium E^* and release rate ν . The economic threshold value $ET = 1$, $r_1 = 0.2$, $\alpha = 0.8$, $k = 0.9$, $\theta = 5$, $\beta = 0.5$, $r_2 = 3$, $\omega = 0.2$.



4.2 The stochastic model and its stability

We now consider a stochastic version of the model system (4.1) by perturbing the contact term in the system above. We use a pair of complementary perturbations. Thus, for some constant parameter σ and a Wiener process $B(t)$, the dynamic behaviour of the system is modeled to be of the form:

$$\begin{cases} dS = (S(t)(-r_1 - \alpha I(t) + k\theta y(t))dt - \sigma I(t)S(t)dB(t) \\ dI = (\alpha S(t)I(t) - \beta I(t) + \nu)dt + \sigma I(t)S(t)dB(t) \\ dy = (y(t)(r_2 - \theta S(t) - \omega y(t)))dt. \end{cases} \quad (4.2)$$

In order for the system of stochastic differential equations to have a solution for any given initial value, the coefficients of the equation have to satisfy certain conditions. We shall omit a discussion of such condition, while only focussing on stability, assuming that we have global positive solutions.

For any $t > 0$, let us define the set:

$$A_t = \{\omega \in \Omega : S(u) > 0, I(u) > 0 \text{ and } y(u) > 0 \text{ for all } 0 \leq u \leq t\}.$$

In other words, we shall assume that every set A_t is of measure 1.

The following observation is popularly used in proving global stability of deterministic systems. We also find it useful in our stochastic analysis.

Proposition 4.1. *Let $g(x) = x - c - c \ln \frac{x}{c}$, for $c > 0$ constant. Then $g(c) = 0$ and $g(x) > 0$ whenever $x \in (0, \infty) \setminus \{c\}$.*

Proof. Differentiating $g(x)$, we have $g'(x) = 1 - c/x$. Thus we have a critical value $x = c$ and we note that $g''(c) = 1/c > 0$, and therefore g has a minimum value at $x = c$. Furthermore we note that $g(c) = 0$. This completes the proof. \square

The following theorem describes stochastic stability of the equilibrium point E_1 . We apply a theorem from the book of Mao, [42, Theorem 2.3 of Chapter 4]

Theorem 4.2. *If $R_0 < 1$ and $\frac{\nu}{\beta} > 1$ then the equilibrium point E_1 is stochastically stable.*

Proof. Let $V_1 = c_1 S + c_2 \left(I - I_1 - I_1 \ln \frac{I}{I_1} \right) + c_3 \left(y - y_1 - y_1 \ln \frac{y}{y_1} \right)$ be as of Chen and Tan [70], and put $U = \ln V_1$. Let L be the operator as in Lahrouz et al., [33]. The operator L is also referred to as the infinitesimal generator of the given system of SDE's (see Oksendal). Now we must calculate LU , which is:

$$\begin{aligned} dU &= \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial I} dI + \frac{\partial U}{\partial y} dy + \frac{1}{2} \left[\frac{\partial^2 U}{\partial S \partial I} dS dI + \frac{\partial^2 U}{\partial S \partial y} dS dy + \frac{\partial^2 U}{\partial I \partial y} dI dy \right] \\ &+ \left[\frac{\partial^2 U}{\partial S \partial S} dS dS + \frac{\partial^2 U}{\partial I \partial I} dI dI + \frac{\partial^2 U}{\partial y \partial y} dy dy \right] \\ &= \frac{c_1}{V_1} (k\theta y S - r_1 S - \alpha I S) + \frac{c_2 (I - I_1)}{V_1 I} (\alpha S I - \beta I + \nu) \\ &+ \frac{c_3 (y - y_1)}{V_1 y} (r_2 y - \theta S y - \omega y^2) + \frac{1}{2} \left[-\frac{C_1^2}{V_1^2} \sigma^2 S^2 I^2 dt \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left[\left(\frac{c_2 I_1}{V_1 I^2} - \frac{c_2^2 (I - I_1)^2}{(V_1 I)^2} \right) \sigma^2 S^2 I^2 dt \right] + \left[\frac{c_1 c_2 I - I_1}{V_1^2 I} \sigma^2 S^2 I^2 dt \right] \\
& = \frac{1}{V_1} \left[c_1 (k\theta y S - r_1 S - \alpha I S) + c_2 \frac{(I - I_1)}{I} (\alpha S I - \beta I + \nu) \right. \\
& \quad \left. + c_3 \frac{(y - y_1)}{y} (r_2 y - \theta S y - \omega y^2) \right] + \frac{(\sigma S I)^2}{2} \left[\left(\frac{c_2 I_1}{V_1 I^2} - \frac{c_2^2 (I - I_1)^2}{(V_1 I)^2} \right) \right. \\
& \quad \left. + 2 \left(\frac{c_1 c_2 I - I_1}{V_1^2 I} \right) - \frac{C_1^2}{V_1^2} \right] \tag{4.3} \\
& = \frac{1}{V_1} \left[-\frac{c_2 \beta (I - I_1)^2}{I} - c_3 \omega (y - y_1)^2 + \frac{\beta}{\omega} (k\theta r_2 \beta - \omega r_1 \beta - \omega \nu \alpha) S \right] \\
& \quad - \frac{(\sigma S I)^2}{2} \left[\frac{c_1^2}{V_1^2} - 2 \frac{c_1 c_2 I - I_1}{V_1^2 I} + \frac{c_2^2}{V_1^2} \left(\frac{I - I_1}{I} \right)^2 - \frac{c_2 I_1}{V_1 I^2} \right] \\
& = -\frac{c_2 \beta (I - I_1)^2}{I} - c_3 \omega (y - y_1)^2 + \frac{\beta}{\omega} (k\theta r_2 \beta - \omega r_1 \beta - \omega \nu \alpha) S \\
& \quad - \frac{1}{2} \left(\frac{\sigma S I}{V_1} \right)^2 \left(c_1 - c_2 \frac{I - I_1}{I} \right)^2 + \frac{1}{2} \frac{(\sigma S)^2}{V_1} c_2 I_1 \\
& = -\frac{1}{2} \left(\frac{\sigma S I}{V_1} \right)^2 \left(c_1 - c_2 + c_2 \frac{I_1}{I} \right)^2 + \frac{1}{2} \frac{(\sigma S)^2}{V_1} c_2 I_1 \\
& \leq \frac{1}{2} \left(\frac{\sigma S}{V_1} \right)^2 [-I^2 - I_1 V_1] \\
& < 0.
\end{aligned}$$

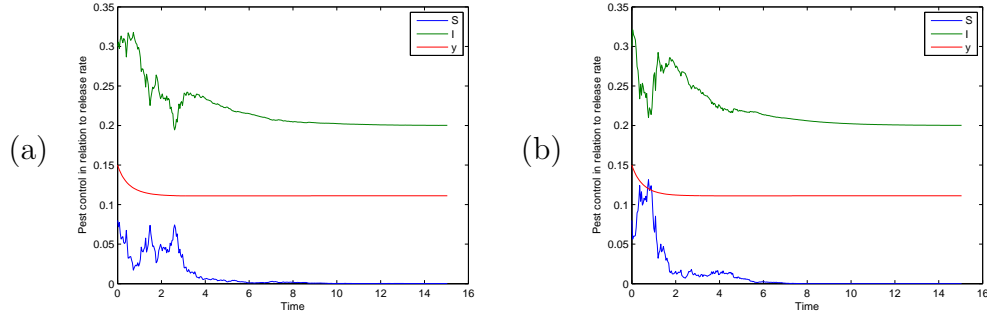
Therefore the point E_1 is stochastically stable. \square

The following are two simulations of solutions of the stochastic system. The parameter values are different in the two simulations. In the simulation displayed in Fig 4.2 (a) we took $\omega = 15$; $\sigma = 10$. This yields $R_0 = 0.83$.

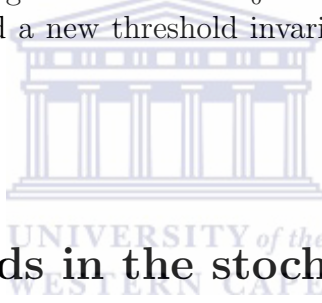
In the second simulation Fig 4.2 (b) we took $\omega = 13.5$; $\sigma = 15$. This yields $R_0 = 0.926$.

In both cases we have $R_0 < 1$ and we also note that the condition $\frac{\nu}{\beta} > 1$ is satisfied. Therefore by Theorem 4.2, we expect the equilibrium point E_1 to be a stable solution. The graphs confirm this.

Figure 4.2: Simulations showing the stochastic trajectories relation in the release rate of $\nu = 0.5$ labeled (a), and $\nu = 0.05$ labeled (b).



Simulations seem to indicate that stochastic stability is obtained even if R_0 is slightly bigger than 1. A more sophisticated stability analysis will most likely yield a better range of values for R_0 to still ensure stochastic stability. We may be able to find a new threshold invariant to indicate stochastic stability of E_1 .



4.3 Thresholds in the stochastic model

In concluding this chapter we consider the problem of retaining, in the long run, a positive level of susceptible prey. As we have mentioned, it may be important to not have the susceptible pest vanish completely. The management of the susceptible pest in the stochastic model can also be explored through simulations. Consider the following parameter values:

$$r_1 = 0.2; \alpha = 0.08; k = 0.04; \theta = 1; \beta = 0.5; r_2 = 1.5; \omega = 0.2; \sigma = 0; a = 0.1,$$

and the initial values

$$S = 1.5; I = 0; y = 5 .$$

Suppose that we seek to have the system to have a stable nontrivial equilibrium with $S^* = 0.5$. The correct ν - level can be determined quite easily by means of graphs through trial and error. We proceed by testing different values of ν , until we get sufficiently close to the desired state. In the sketches below we show the stable curves for the cases $\nu = 0.5$ and for $\nu = 0.05$. The latter value $\nu = 0.05$ seems to be the desired level of release rate of infected prey to keep the numbers of the healthy prey above 0.5 with 0.5 above the equilibrium value.

Having run numerous simulations we can conclude that on the average, a given level of ν give lower S^* values in the stochastic case than in the deterministic case, at least for R_0 relatively small (but above the unity level). We show two such simulations. In fact, the simulations have the same parameters as in the case [70]. However in both cases it shows a clear risk of extinction of S . The control in this case requires more sophisticated analysis, into which we shall not venture now.



Chapter 5

The effects of rain and fire on savanna

Savanna is grassland ecosystem where the trees are sufficiently small or widely spaced such that the canopy does not close. They naturally occur in areas with annual rainfall from 300 to 1800mm. In most cases, competition such as for instance, niche separation with respect to limiting resources such as water, lead to tree-grass coexistence. Factors such as fire, herbivores and rainfall variability promote tree-grass persistence through their dissimilar effects. Classification of savannas can be split into nutrient rich or nutrient poor, fine leafed or broad leafed, semi arid or wet. The transition between moist and dry savanna is expected within a range of 500mm to 700mm annual rainfall, see for instance the papers [57, 60]. This chapter discusses a very simple ordinary differential equation model describing the competition between trees and grass in savanna environment. The competition between them are for soil water, fed by annual rainfall. Ultimately, when tree and grass compete for this soil water, they are also perturbed by fire disturbance and some other environmental forcings such as herbivores. Our main reference to this work is the paper [59] of Scholes et al. The model captures essentially the competition for or availability of water, and the effect of fire. We augment the model in [59] of Scholes et al, by introducing stochastic perturbations. The stochastic perturbations we introduce in the savanna model are in the form of three mutually independent Brownian motions.

5.1 History

Since 1984, savanna studies have been revolutionized and these studies have become increasingly interdisciplinary and international in character. Until the mid-1980s, savanna studies were based on the concepts of ecology which were first developed in Europe and North America between about 1910 and 1940. Plant communities were seen as 'organisms', or ecosystems, in a state of equilibrium. So for instance, in the paper [6] of Provenzale et al., authors regard savannas as organisms responding to changes within ecological determinants. The state can be determined by one or more external ecological or abiotic factors, such as climate, geology, soil or fire. Unfortunately, there was a strong disagreement over which factor controlled savannas. British foresters working in West Africa thought savannas were anthropogenic (human-created) communities, forged out of the forest by cutting and burning trees. In contrast, German scientists working in South West Africa (Namibia) and South America tended to view savannas as a climatically-determined vegetation type, while the French in Indochina (Laos, Cambodia and Vietnam) saw savannas as essentially fire climax. There were also other more idiosyncratic explanations. Certain geographers, for example, argued that savannas were primarily determined by geomorphology (the age of the land surface) and soils. In truth, however, there was very little real debate or discussion between the different schools of thought, and savanna studies soon became somewhat nationalistic, regionally constrained, and ultimately outdated in relation to the more general advances being made in the ecological sciences. It is now thought that all savannas are determined by the complex interplay of ecological determinants. These include moisture, fire, herbivory, frost and wind, as well as major historical anthropogenic events. Historical events, such as the abandonment of human settlement or enhanced global warming, may trigger a savanna into a totally new ecological state. The relationships between the complex of variables are now analysed multivariately, using models or hierarchy theory, and can be used to make basic international, national, and regional savanna comparisons. These new approaches place savanna studies at the centre of ecology. The discussion of global environmental change and ecological determinants link savannas directly with both world-scale and local-scale environmental fluctuations. Fundamentally, savanna studies have at last come of age. Perhaps even more importantly, savannas are no longer seen as being simple equilibrium systems, but rather they are regarded as a vegetation type which is constantly under a regime of change, with their biology driven by both gradual and catastrophic variation in the ecological determinants. Some scholars think of savannas as ever moving between what are called 'multiple equilibrium states', such

as open grassy savannas, wooded savannas, and woodlands.

5.2 The model

We set out to find a model of tree-grass dynamics competing for soil water and perturbed by fire. In the paper [58] of Ratnam et al., the necessity of taking into account both disturbances and competition for resources simultaneously is explained. This has to happen in order to capture their relative importance towards shaping the different types of savannas. The competition of tree-grass leads the co-existence in savannas. Additionally, we include the stochastic fire disturbances significantly which widens the range where coexistence of trees and grasses is found. There is a theory, see Scholes et al [59], that the co-existence of tree and grass in savanna is controlled by fire and rainfall. We assume soil water to be a limiting resource [77], with grasses as the superior competitor but having roots restricted to the topsoil layer, whereas trees root both in the topsoil and the subsoil. Then the range succession model in savannas point the extremes of grassland and forest ref: Figure 5.1.

We now present a description of the model and also discuss its behaviour in the environment. Tilman [72], represent the temporal dynamics of tree and grass as follows

$$\begin{aligned}\frac{dT}{dt} &= c_T T(1 - T) - \xi_T T, \\ \frac{dG}{dt} &= c_G G(1 - G - T) - c_T T G - \xi_G G,\end{aligned}\tag{5.1}$$

The functions T and G represent the fraction of area covered by trees and grass respectively. Of course then, T and G must satisfy the condition $0 \leq T + G \leq 1$. The constants c_T and c_G are the colonization rates, whereas ξ_T and ξ_G are the off-take rates for trees and grasses, respectively. The off-take rates ξ_T , ξ_G and colonization rates c_T , c_G , has to be all positive and are measured in units of $[1/t]$, where t is time. From equation (5.1) our tree T and grass G are dimensionless. If the trees are not present in the area then we have $T = 0$, an area which is fully covered by trees a value $T = 1$, and similarly for grass. The equation above present trees as a superior competitor, while grasses are the inferior competitor. The take off are fire disturbances occurring mainly in the dry season, when the perennial grasses are dormant. We find in the paper [59] of Scholes et al., that fire consumes grass leaves as fuel, but does not generally

kill the dormant grass buds at or below the soil surface. Fire acts differently on grass and tree, can consumes tree leaf litter, but if grass is completely absent it is very difficult for fire to propagate. Now follows the model of tree-grass dynamics competing, and with perturbation by fire.

$$\begin{aligned}\frac{dT}{dt} &= c_T T(1 - T) - \xi_T T - \xi_F fGT, \\ \frac{dG}{dt} &= c_G G(1 - G - T) - c_T TG - \xi_{GO} G - fG\end{aligned}\quad (5.2)$$

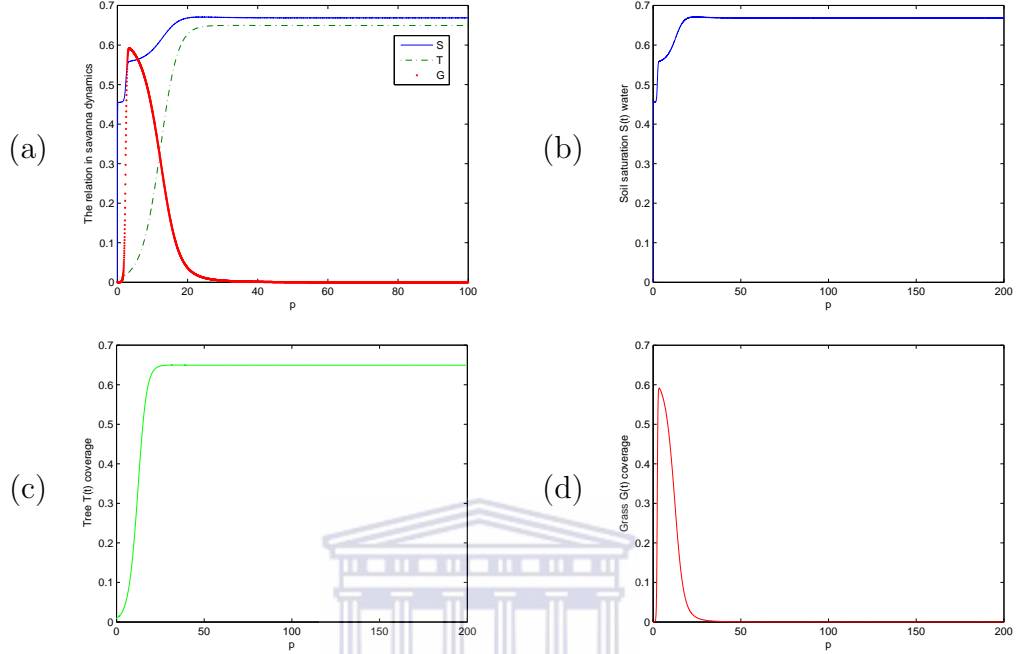
The term f is representing the fire, $-\xi_F fGT$ represents trees' reduction due to fire. Under the grass rate, fG is the amount of grass fuel load. The fire disturbances in equation (5.2) describes the consumption of grass by fire, and is denoted by the term $-fG$. The term $-\xi_{GO} G$ is the grass reduction due to the other causes such as mortality and herbivores. Both of the terms $-\xi_{GO} G$ and $-fG$ are from a removal term $-\xi_G G$ in equation (5.1). The vulnerability of trees to fire ξ_F , and the tree abundance are similar to that in the work of Kumar et al., [30]. Rainfall forcing, soil water, and tree grass dynamics are compared on checking the arid and semi-arid environments. The water in an arid environment is the primary determinant of vegetation composition, structure, and function for water which is present in the rooting zone [6]. Therefore the soil water dynamics is of the form

$$\frac{dS}{dt} = \frac{p}{w_1}(1 - S) - \delta S(1 - T - G) - \tau_T ST - \tau_G SG, \quad (5.3)$$

with S being the degree of saturation in available water capacity. The dynamics of this model describes the water volume present in the root zone. The function S ranges in the interval $[0,1]$; $S = 0$ corresponds to completely dry soil, and $S = 1$ to completely saturated soil [59]. The term p/w_1 , is the rainfall rate over the unit area $p \geq 0$ and normalized by volume w_1 . The volume w_1 is based on the depth of the root zone z with the porosity n . The parameters δ , τ_T , τ_G , normalized by volume w_1 , are the mean evaporation rate from the bare soil, water uptake rate by trees, and water uptake rate by grass. The term $\delta S(1 - T - G)$ is the evaporation from bare soil, and $\tau_T SG$, $\tau_G SG$ are the water uptake by tree and grass respectively. The term $(p/w_1)S$ describes the amount of rainfall exceeding the root zone capacity percolating beyond the root zone depth.

We illustrate the dynamics by way of the following graphs. For these simulations we used the following parameter values:

Figure 5.1: Simulations showing the behaviour of tree and grass in competition for water.



The depth of the root zone z is in the range 0-1 where as the porosity n in 0.55, as used in the paper [60] of Scholes and Walker. The colonization and loss rates are obtained by consideration of the time scales necessary to attain the equilibrium state. The parameter δ is in the range $20yr^{-1}$, $\tau_T = 1yr^{-1}$, $\tau_G = 1yr^{-1}$, $\gamma_T = 2yr^{-1}$, γ_G range in $20 - 200yr^{-1}$, $\xi_T = 0.3yr^{-1}$, $\xi_{GO} = 2.5yr^{-1}$, $\xi_F = 0.02yr^{-1}$, $p = 0mmyr^{-1}$, and $f = 0yr^{-1}$.

From these graphs we observe that, at times where T and G are perturbed, the existence of environmental intrude are found. Trees cannot recover, but only grass at all time. The soil saturation $S(t)$ is capable of reproducing in terms of store rations.

5.3 Savanna dynamics

This section describes a spatially homogeneous model for the competition of trees and grasses in savanna environment. The coupled dynamics emphasizes the competition for soil moisture and the risk of fire. The dynamics illustrated here is due to Scholes et al. [59], and the system of ordinary differential

equations is the following.

$$\begin{aligned}
\frac{dT}{dt} &= \gamma_T ST(1 - T) - \xi_T T - \xi_F fGT, \\
\frac{dG}{dt} &= \gamma_G SG(1 - G - T) - \gamma_T STG - \xi_{GO}G - fG, \\
\frac{dS}{dt} &= \frac{p}{w_1}(1 - S) - \delta S(1 - T - G) - \tau_T ST - \tau_G SG.
\end{aligned} \tag{5.4}$$

This model of Scholes et al. is an extension of the models proposed by De Michele et al., [75] and Luca et al., [49] for the coupled dynamics of soil moisture and one plant functional group in arid and semi-arid ecosystems. From equation (5.2) the colonization of tree and grass is replaced by the term $\gamma_T S$ and $\gamma_G S$ respectively. The parameter γ_T and γ_G are the average colonization rates. In equation (5.4), the functions of the soil moisture in colonizations of tree and grass are in the root zone. Then, we see by equation (5.4) that in equation (5.2) the functions of colonization rates of tree and grass are constant. The dynamic system equation (5.4) is a simple ecohydrologic model of savanna, characterized by parameters w_1 , δ , τ_T , τ_G , γ_T , γ_G , ξ_T , ξ_{GO} , ξ_F . The parameter f is a fire frequency and p is the annual rainfall, and they constitute the main environmental forcings.

The parameter values are of the range:

Symbol	Unit	Range
z	m	0.1 – 1
n	–	0.05 – 0.55
δ	yr^{-1}	20 – 30
τ_T	yr^{-1}	1 – 50
τ_G	yr^{-1}	1 – 50
γ_T	yr^{-1}	0.15 – 2.5
γ_G	yr^{-1}	20 – 200
ξ_T	yr^{-1}	0.03 – 0.3
ξ_{GO}	yr^{-1}	1 – 3
ξ_F	–	0.02 – 0.6
p	$mmyr^{-1}$	0 – 1800
f	yr^{-1}	0 – 2

Table 5.1: The range of parameter values and environmental forcings of the model.

5.4 Stability map and steady state solutions

This section gives the steady state solutions of grass and tree in the savanna environment. As we mentioned in equation (5.2), T , G and S are dimensionless and non-negative, and $T + G \leq 1$. Now the equilibrium points T^* , G^* and S^* are as follows:

$$T(\gamma_T S(1 - T) - \xi_T - \xi_F fGT) = 0;$$

$$T^* = 0 \quad \text{or} \quad T^* = 1 - \frac{(\xi_T + \xi_F fGT)}{\gamma_T S}.$$

$$G[\gamma_G S(1 - G - T) - \gamma_T ST - \xi_{GO} - f] = 0;$$

This implies, $G^* = 0$ or

$$G^* = 1 - \frac{(\gamma_G ST + \gamma_T ST + \xi_{GO} + f)}{\gamma_G S}$$

and finally

$$\frac{p}{w_1}(1 - S) - \delta S(1 - T - G) - \tau_T ST - \tau_G SG = 0;$$

$$\frac{p}{w_1} - S\left(\frac{p}{w_1} + \delta\right) = 0, \quad \text{at } T^* = 0 \text{ and } G^* = 0$$

$$S^* = \left(\frac{p}{w_1}\right) / \left(\frac{p}{w_1} + \delta\right) = 0.$$

The steady state solutions show effective coexistence in response to the forcings. The behaviour of the tree-grass in the forcing are indicated (by calobar), with the different regions of stability. The steady states of the model in equation (5.4), on soil moisture S has $\frac{p}{p - \delta w_1}$ for both T and G equal to zero, and the states shows unvegetated land. Otherwise the grass G has steady state $G^* = 1 - \xi_T / \gamma_T S$ in the absence of trees. At $T = 0$ and $G^* = 1 - (\xi_{GO} + f) / \gamma_G S$, we have the grassland state. The equilibrium states are shown on the map (Figure 5.2). The bare soil is more extensive when the prevalence of fire increases. Permutations such as of saturation limit the availability of fire forcing (Figure 5.3).

5.5 Stability map

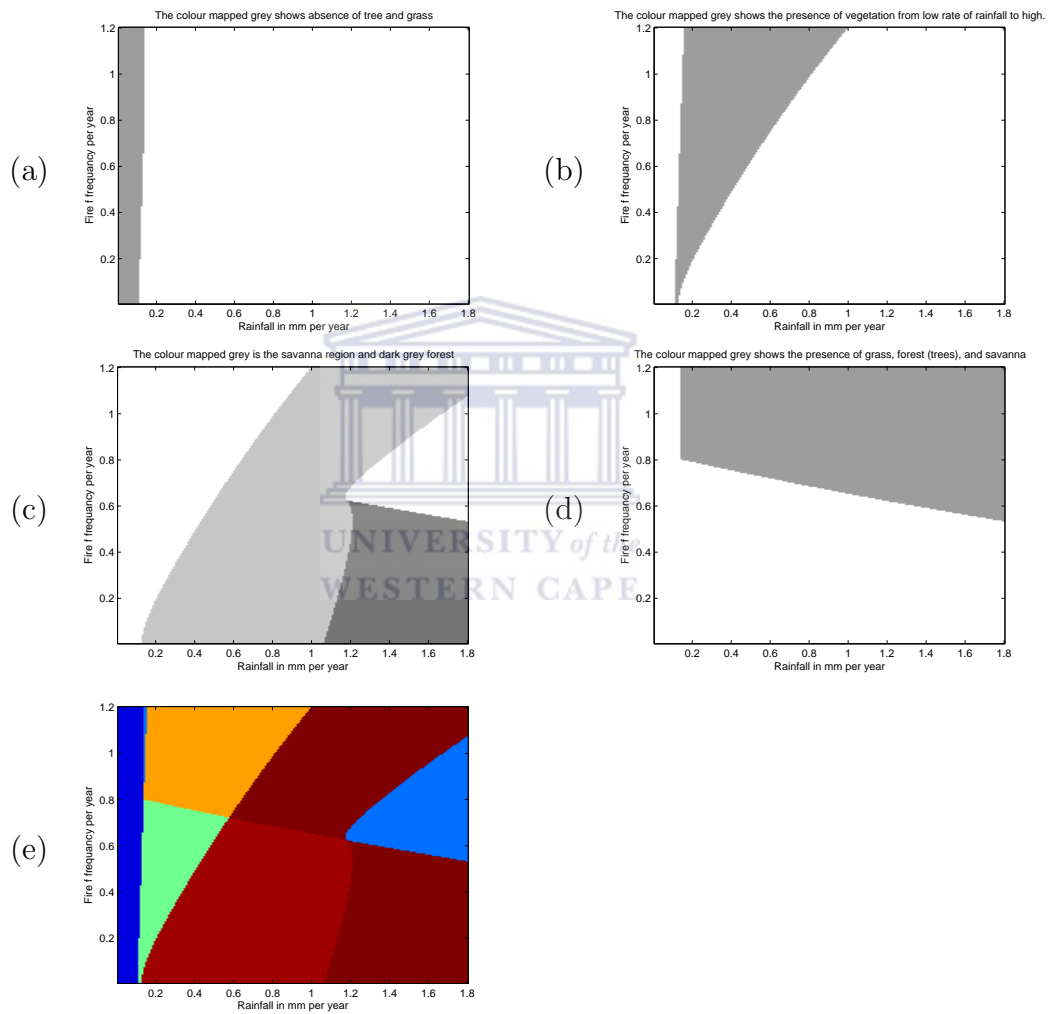
The stability map in Figure 5.2 depicts the environmental space defined by rainfall and fire. Within the environmental space defined by rainfall and fire, the positive values of the environmental forcings and state variables in (5.4) has 7 different regions. These regions are representing the stability and bistability of tree grass coexistence in fire and rainfall. We include the mapping as from the paper [59] of Accatino et al.,.

The parameters assumed are such that domains of environmental forcings and state variables, has 7 different regions of stability and bistability. This predict the changes in vegetation structure along gradients of rainfall and fire disturbances realistically. By also looking at the map result, we have better clarity on the distinction between climate and disturbance dependent ecosystems. The scale used for upper boundary in our map is 1.2 and that is a fire frequency, and is per year. The rainfall is in millimeter per year and that means 1.8 upper boundary in our map, reflects 1800 mm/year.

The region (a) mapped grey shows absence of tree and grass. The result of unvegetated area in (a) is through the high fire frequency and low rate of rainfall. The region (b) mapped grey shows the presence of vegetation from low rate of rainfall to high. The stable equilibrium on (b) is grassland, because rainfall is enough for the existence of grasses. Map named (c) with colour grey is the savanna region and dark grey forest. The rainfall p in (c) allows the occurrence of both trees and grasses, and the stable equilibrium is savanna. The stable equilibria on (d) coloured map grey is showing the rate of grassland and forest, and savanna in continuous state. Therefore this explains that the direct effect of fire on grass is greater than the indirect effect on trees.

Graph (e) in the Figure 5.2 shows the regions on effects result of environmental forcings. The colours mapped blue is unvegetated area, light blue from unvegetated is forest(trees), green is a grass, red is savanna, orange is grass and forest, and dark red from savanna to the right side is forest and to the upper dark red from savanna is forest and savanna. For more references on steady state presentation of these regions, look at Solbriège et al., Sternberg [69].

Figure 5.2: Simulations of the stability map showing different regions of stability and bistability of environmental forcings.



5.6 Results on vegetation changes

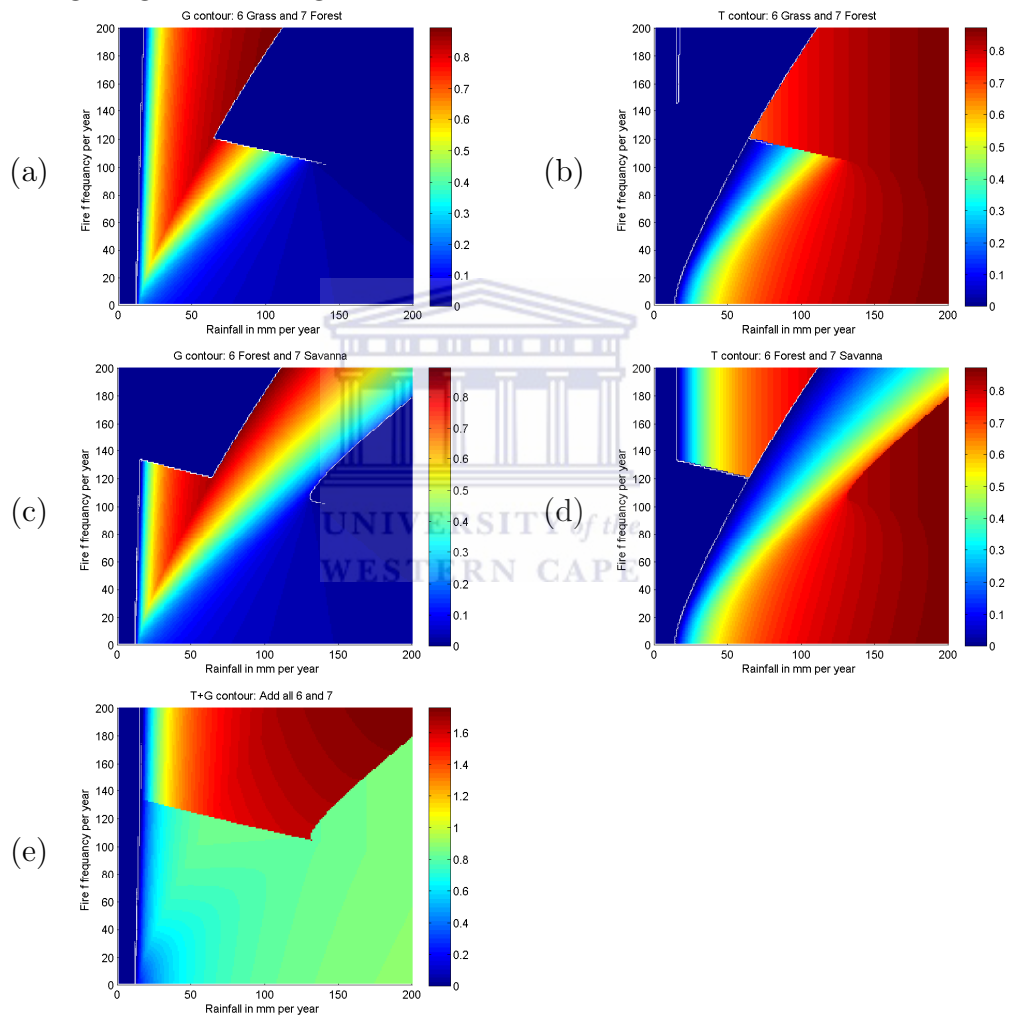
The behaviour of tree-grass coexistence in Figure 5.2 takes the average of different vegetations over a long period. This section is taking heed that the coexistence of vegetation region is more likely than in an undisturbed case without environmental forcings. In the presence of perturbations off-takes rate can be greater. This results in more bare soil area, because fire increases the chance of vegetation death.

Following Scholes et al., [59], the rainfall gradient range is greater than or equal to 0.05 (50 mm/year) and also less than or equal to 1.8 (1800 mm/year). Therefore through the ranges we intend to consider changes in vegetation stable states over which rainfall gradients is low, and high fire conditions. Figure 5.3 is the one showing the pattern on how the gradient range satisfy the states. Then it follows that conditions of the gradient $0 \leq p \leq 200$ as in frequent fire on same gradient, gives the disturbances to be discovered at an earlier stage of the gradient. The vegetation changes along fire gradients $0.2 < f \leq 1$ at the annual rainfall $p = 650$ mm/year is considered. The environment of area covered by trees that survived fire become lower.

However the coexistence as we check (a), (b), (d) on the Figure 5.3, extend in the region where in constant case, only trees are found. If grasses were not damaged too strongly when $f = 0.8$, either forest or grassland could be observed, depending on the initial conditions of tree cover. For these assumptions, check (c), (e), (d) on this figure (5.3).

The fire disturbances indicates that most of the grass and tree covers are destroyed as is shown (b) and (a). Moreover, adult trees which make up forest, are affected less strongly as we observe the calorbar in both (a), (b), (c), (d) at rates from 0.7 till 0.8. Finding the stability, we often notice that fire destroy only part of the adult tree cover, as taller and thicker stems often have higher fire resistance. That is illustrated (e) in this figure. This means fire influences the tree grass ratios illustrated in (d), and (e) as the rainfall availability determines the type of vegetation. The coexistence of tree and grass is permitted by limited amount of water resources at annual rainfall range $25 \leq p \leq 100$. Moreover this range is helping us to clarify the coexistence of tree and grass due to simultaneous influence of rainfall and fire. Therefore our assumptions imply that savanna is the result of the co-occurrence of water limitation and fire existence, check (d), (c), (e).

Figure 5.3: Simulations showing different equilibrium states regions with color bar representing the survived vegetation from low rate of rain fall frequency through high fire forcings.



5.7 Model with stochastic perturbation

Now let us consider a stochastic version of the model (5.4) equation by perturbing the deterministic system. We propose a model of the form below:

$$\begin{aligned}
 dT &= (\gamma_T ST(1 - T) - \xi_T T - \xi_F fGT)dt - \sigma_1 T(t)dB_1(t), \\
 dG &= (\gamma_G SG(1 - G - T) - \gamma_T STG - \xi_{GO}G - fG)dt + \sigma_2 G(t)dB_2(t), \\
 dS &= (p/w_1(1 - S) - \delta S(1 - T - G) - \tau_T ST - \tau_G SG)dt + \sigma_3 S(t)dB_3(t).
 \end{aligned} \tag{5.5}$$

The phrase $\frac{dB_i(t)}{dt}$ (for subscript $\{1, 2, 3\}$) is a noise in the environment with $B_i(t)$ a Brownian motion, and these Brownian motions are considered to be mutually independent. The stochastic perturbations can be interpreted as follows. In the system, the equations associated with T , G and S , the perturbations are (respectively) on the parameters ξ_T , ξ_{GO} and $\delta - p/w_1$. The intensity of the perturbation is determined by the parameters σ_i .

The following graphs compare the deterministic model (5.4) to the model (5.5) with stochastic perturbation:

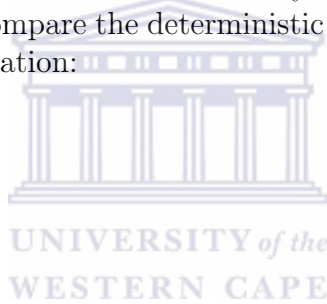
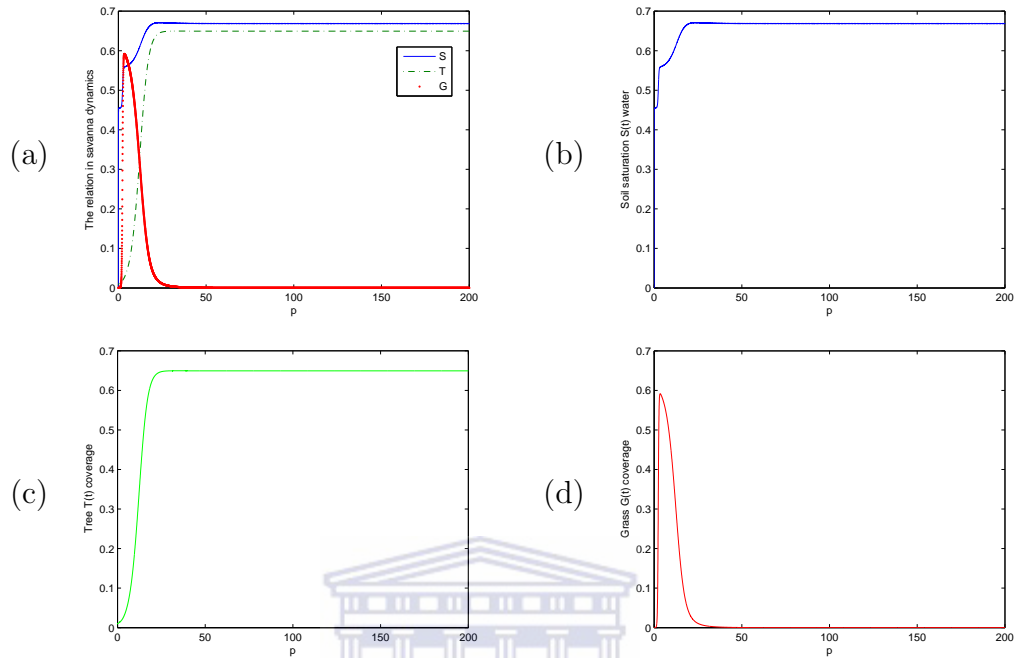
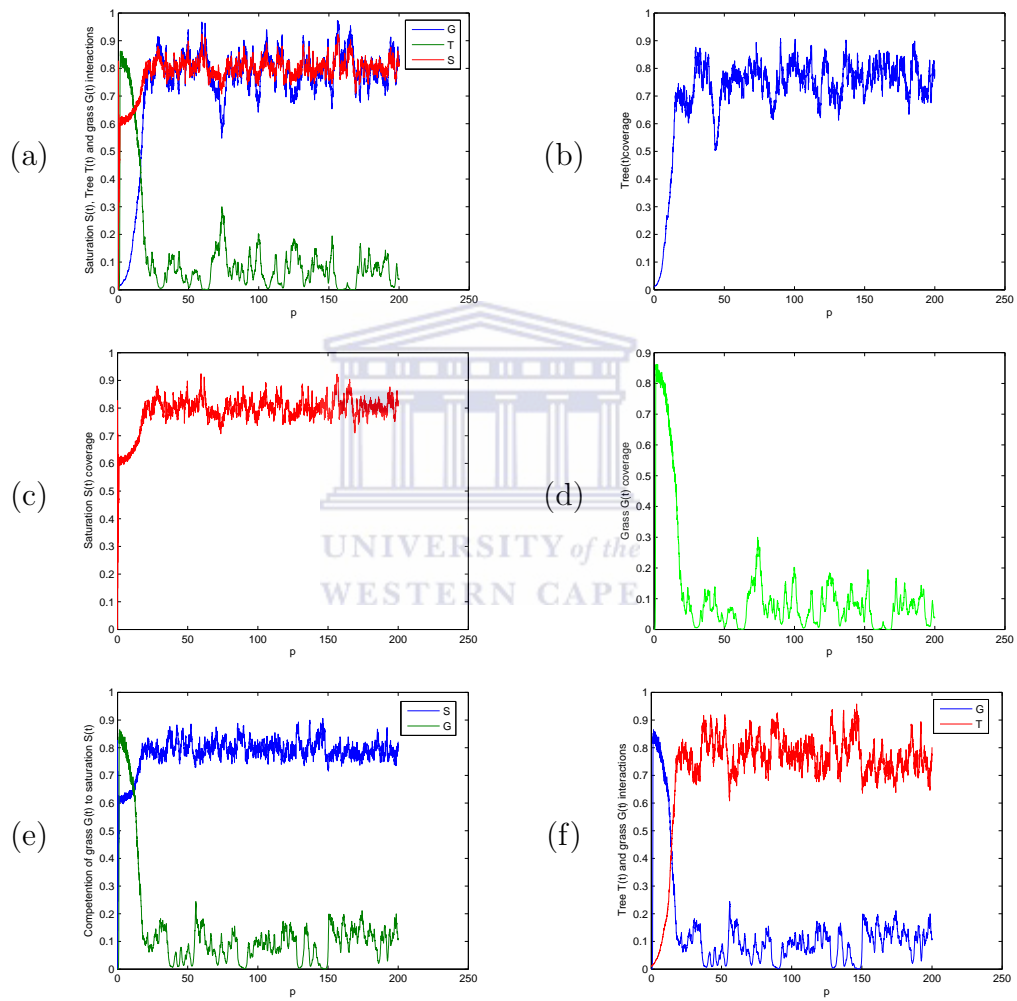


Figure 5.4: Simulations showing savanna dynamics.



The deterministic graphs of 5.4, indicate the systems become stable at the year of rainfall p . At transect of $0 \leq f \leq 1$ and $50 \leq p \leq 100$ for an example the absence of fire disturbance, it is possible to observe the different sequence of tree and grass. In Figure 5.5, the result shows stability disturbed by random intrude. Then the desired results on the comparison of the savannas deterministic to stochastic ones depends on the type of plant species present on the ecosystem. The result of the stochastic disturbance significantly shows the coexistence of trees and grasses. However at the same time, tree and grass coexistence to fire feedback can induce bistability between forest and grassland.

Figure 5.5: Simulations showing stochastic perturbations.



Chapter 6

Stability of a three-tiered predator-prey system

6.1 Introduction

Ecosystems management entails and encourages the ethical use of natural resources. Management strategies depend on the extent to which a natural resource stock responds to it being harvested or exploited, [43]. An important objective of ecosystems management is sustainability of the stock of resources. Sustainable development is a strategy of manipulating and partly altering a natural system or resource (e.g., constructing a building on land that has never been used by humans) for the benefit of human livelihood, and doing it in a manner that does not compromise similar benefits for future generations.

The concept has become a popularly accepted principle, and researchers needed to find ways of quantifying the degree of sustainability of a given development, and in particular, for use in ecosystems management. In this chapter we propose, following [65], a numerical measure of quantifying stability and sustainability of such a resource stock, or rather, of the underlying system. The object of study is a predator-prey interaction in a food chain. In [65] the analysis includes optimal control theory as an important mathematical tool. The authors formulate a control problem based on the so-called Fisher information (explained below, or see Chapter 2) of the ecosystem model. The optimization problem will be that of maximizing the Fisher information, and its solution is a description of the intervention on the system that is most conducive to long term sustainability. The solution of this problem reveals

the best strategy for human interactions with the system in order to maximize the sustainability of the stock. In this dissertation we focus attention on the Fisher information itself and we perform several computations aimed at a better understanding of Fisher information, at least for the given model.

In ecosystem management, preservation of the ecosystem requires consideration of all different interactions and natural fluctuations in the system. Such natural fluctuations may well be important for the long term stability of the system and sustainability of the resource stock. In this chapter we shall introduce the Fisher information concept as a means of quantifying the stability of a system. The Fisher information measure will also encode the effect of resource use by humans. We present some computations, exploring the effect of enrichment of the system on Fisher information, and the effect of harvesting. By enrichment we mean that the system is modified in such a way that the carrying capacity of the prey increases. Regarding the harvesting, we introduce into the system of ODE's, the continuous external removal of prey.

6.2 Information theory and sustainable development

information theory in ecology emerged in the development of methodologies for quantifying the variation in a dynamical ecological system. The use of information theory combines the internal make ups of individual organisms with the overall ecosystems structural organization [20]. An example of such an information theory is the Shannon information (pioneered in [64]) for quantifying diversity in an ecosystem. Considering the long-term effects, benefits and drawbacks in all the decisions relevant to the ecosystem, the use of a suitable information theory is encouraged for the purpose of informing sustainable development strategies. The concept has been recognized to be of paramount importance, and considered essential to have as a quantifying measure in the field of ecosystem management [65]. Sustainable development meets the needs of the present without compromising the ability of the future generations to meet their own needs [68].

6.3 Fisher information

Fisher information quantifies the way in which information on an observable random variable can carry unknown specifications upon which the probability of ecosystem depends. In the paper [11] of Fath and Cabezas, Fisher information is used as the quantifying measure of sustainability and optimal control theory is harnessed to derive the control profiles. Also, Fisher information is the basis of the so-called *sustainable measures hypotheses*, proposed by Fath et al. We present a study of the species interactions in a predator-prey model.

In Chapter 2, the section 2.4 we have observed how the concept can be adjusted to apply to a system of ODE's. For stability of an ecosystem it is important that the system is not losing or gaining species, [65]. As this chapter is a study of the species interactions in a predator-prey model, we consider the sustainable hypothesis. When the state of the system becomes unsustainable, the system begins to either gain or lose Fisher information, see the paper [11] for instance.

6.4 A model of a three level food chain

The basic predator-prey model describes an interaction between two organisms in which one of them acts as predator that captures and feeds on the other organism that serves as the prey. This section review a model that is based on two model with extension to third stream. We now present the model discussed by Diwekar and Shastri, and Ryan et al [65, 56], of a three level food chain. The equations describing the species numbers are illustrated as follows:

$$\begin{cases} \frac{dx}{dt} = x \left[r \left(1 - \frac{x}{K} \right) - \frac{ay}{b+x} \right], \\ \frac{dy}{dt} = y \left[e \frac{ax}{b+x} - \frac{a_1 y_1}{b_1+y} - d \right], \\ \frac{dy_1}{dt} = y_1 \left[e_1 \frac{a_1 y}{b_1+y} - d_1 \right]. \end{cases} \quad (6.1)$$

This model describe x , y , y_1 the population variables of three different species in a food chain. These species are referred to as prey (x), predator (y), and super-predator (y_1). The parameter r is the prey growth rate and the carrying capacity of prey is K . The parameters a , a_1 , b , b_1 , and d are the maximum predation, half saturation constant, efficiency, and death rates of predator(y).

and super-predator (y_1).

The phrase $\frac{a}{b+x}$ is a prey mortality rate, d the predator mortality rate, and $\frac{a_1}{b_1+y}$ is the predator growth rate.

We will use the model parameters for dynamical stability of system as given in the table below

Prey	Predator	Super – predator
$x(0) = 100$	$y(0) = 75$	$y_1(0) = 150$
$r = 1.2$	$a = 2.0$	$a_1 = 0.1$
$K = 710$	$b = 200$	$b_1 = 250$
	$e = 1.0$	$e_1 = 1.12$
	$d = 1.0$	$d_1 = 0.04$

Table 6.1: Stable parameter set of tri-trophic food chain.



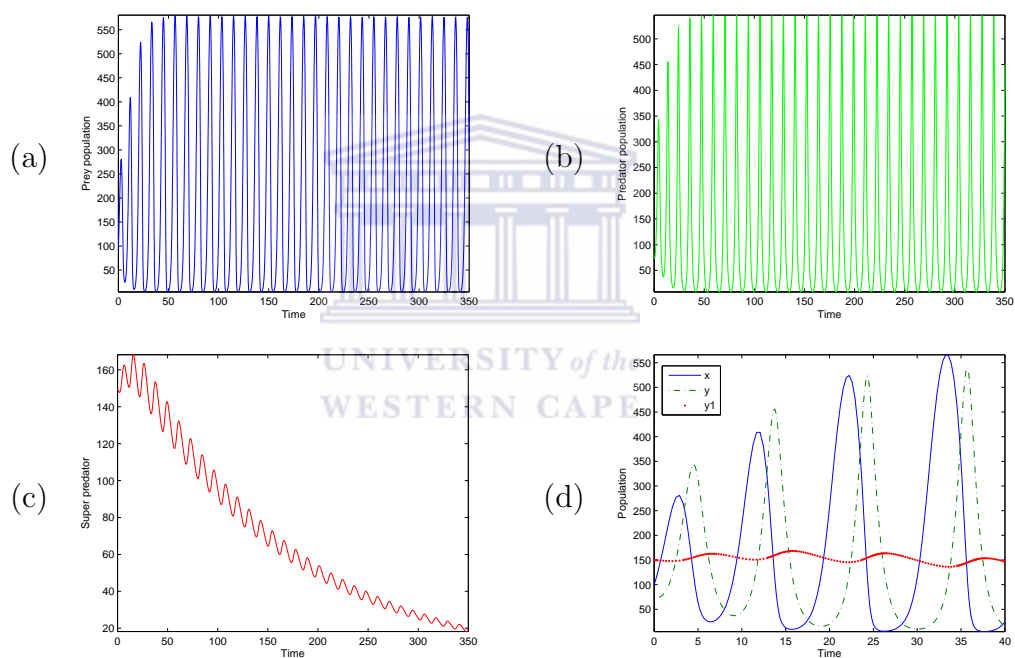
6.5 Simulations

The figure below show the interaction between a prey species x and its predator, super- predator, y , y_1 respectively from equation (6.1). We include the graphs as in the paper of Diwekar and Shastri [65]. The graphs illustrates the recovery of the system from a disturbance in the equilibrium situation. Both the prey and the predator numbers grow in terms of their amplitude, relatively quickly reaching a constant amplitude. We also find that in the super predator there is a significant and a consistent drop in numbers from a value that was initially much too high. This is to be expected by observing b_2 , which is the predator half saturation. Therefore population y_1 which is the super predator, seems to be at risk of going extinct. See Figure 6.1 (c,d).

6.6 Steady states of the system

In Section 6.5 we noticed cyclicity in the system equation 6.1. A phase portrait of the solutions obtained in Section 6.5 is shown in Figure 6.2.

Figure 6.1: Simulations of tri-trophic chain model comparing the maintenance of control options and objectives through growth rate K .



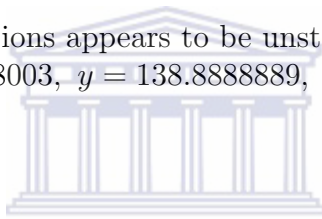
In this section we give the steady state solutions of the predator-prey model, or more particularly, we look at the possibility of equilibrium points. In the simulation of Figure 6.2 we find the equilibrium solutions are periodic. For the same set of parameter values, a MAPLE computation reveals some equilibrium points. Ignoring the negative solutions, we observe the trivial solution (every variable vanish) along with solutions

$$[x = 6000, y = 0, y_1 = 0],$$

$$[x = 200, y = 232, y_1 = 0.],$$

$$[x = 5767.248003, y = 138.8888889, y_1 = 3628.206652].$$

These equilibrium solutions appears to be unstable. In Figure 6.1 we test the solution $([x = 5767.248003, y = 138.8888889, y_1 = 3628.206652])$.



6.7 Fisher information on model refinement

In this section we present some computations, exploring the effect of enrichment of the system on Fisher information, and the effect of harvesting. One can regard the super-predator as being the humans, and then the harvesting is already embodied in the model. As an example, we can consider both the predator and the prey as being two different fish species, the bigger one of which are harvested by humans.

To calculate Fisher information, it is necessary to determine a probability density of (6.1). We assume that the system behaviour can be refined in a continuous dynamic system description. We also assume that the system dynamics be periodic and that must include the equilibria. The refinement of

equation (6.1) is of the following form.

$$\left\{ \begin{array}{l} \frac{dx}{dt} = x(t) \left(r \left(1 - \frac{x}{K} \right) \right) - \tau x(t) - \left(\frac{a}{b+x(t)} \right) x(t)y(t) \\ \equiv \left(r \left(1 - \frac{x}{K} \right) - \tau \right) x(t) - \left(\frac{a}{b+x(t)} \right) x(t)y(t), \\ \\ \frac{dy}{dt} = e \left(\frac{a}{b+x(t)} \right) x(t)y(t) - \left(\frac{a_1}{b_1+y(t)} \right) y_1(t)y(t) - \tau y(t) - dy(t) \\ \equiv e \left(\frac{a}{b+x(t)} \right) x(t)y(t) - (\tau + d)y(t) - \left(\frac{a_1}{b_1+y(t)} \right) y_1(t)y(t), \\ \\ \frac{dy_1}{dt} = e_1 \left(\frac{a_1}{b_1+y(t)} \right) y_1(t)y(t) - \tau y_1(t) - d_1 y(t) \\ \equiv e_1 \left(\frac{a_1}{b_1+y(t)} \right) y_1(t)y(t) - (\tau + d_1)y_1(t). \end{array} \right. \quad (6.2)$$

This model shows the effect of fishing activities. The odd variation in the species catch for an example in this model do with wide variations in fishing activities. The fishing activity decreases the food-fish population at a rate τx where $\tau > 0$ reflects the intensity of fishing activity. Furthermore, fishing activity decreases the population for an example at a rate τy and τy_1 . The paper of Volterra [76] shows the variations and fluctuations in animal living species, whereas the refinements of a model can be observed on Lotka [38].

The effect of enrichment on Fisher information is illustrated in the Figure 6.3 below. The parameter values for this simulation is as given in Table 6.1. The time averaged Fisher information for a system with number n of species is described

$$FI = \int_0^{T_c} \frac{1}{T_c} \left(\frac{s(t)^2}{v(t)^4} \right) d\epsilon, \quad (6.3)$$

where

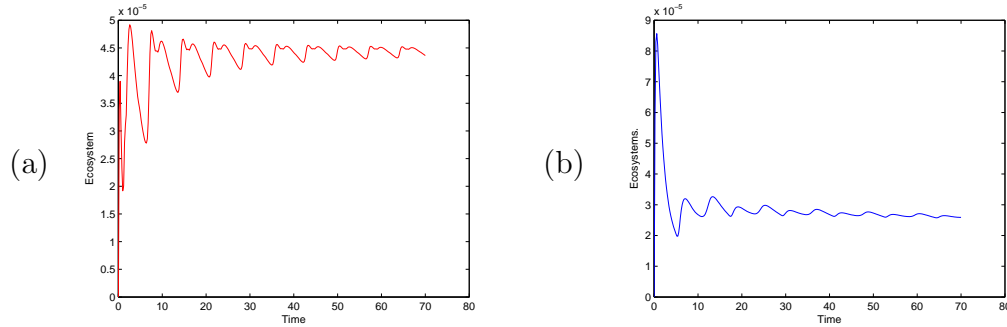
$$v(t) = \sqrt{\sum_{i=1}^n \frac{dx_i}{dt}}, \quad \text{for } x_i \text{ as in equation (6.1)} \quad (6.4)$$

and

$$s(t) = \sqrt{\sum_{i=1}^n \frac{dx_i}{dt} \frac{d^2x_i}{dt^2}}. \quad (6.5)$$

The variables $v(t)$ and $s(t)$ are the velocity and acceleration terms of ecosystem respectively. Note the cyclicity in the system. The period is $T_c = 1$.

Figure 6.3: Graphs showing the Fisher information.



As we can see in our graph, the evolution of the ecosystem over time, seem to be evolving to some stable pattern. The desirable state is to manage in such a way that ecosystem extinction is avoided.

6.8 Further illustrative simulations

Next, we study the effect of K on Fisher information, which is the carrying capacity of prey species. We calculate Fisher information by manipulating the parameter K . For parameters as in Table 6.1, the value of Fisher information is computed using different values K_i of K .

K_i	550	600	650	700	750	800	850	900
FI	0.3187	0.3401	0.3580	0.3731	0.3861	0.3974	0.4073	0.4160

From the table we observe that Fisher information (FI) seems to be increasing with K (FI is an increasing function of K). Increasing FI, means that stability is being compromised. This is of course an interesting phenomenon (which we shall not pursue here), relating to the so-called paradox of enrichment [55], Rosenzweig.

To further determine the response of FI to changes in ecological system, we consider the variation of d_1 which is the predator mortality rate. When we take the values of d_1 at the specified K_i below, then the results of Fisher information in this array tables are of as shown in the tables below:

At $K = 550$,

d_1	0.035	0.04	0.045	0.05
FI	0.3188	0.3187	0.3186	0.3185

at $K = 600$,

d_1	0.035	0.04	0.045	0.05
FI	0.3402	0.3401	0.3400	0.3399

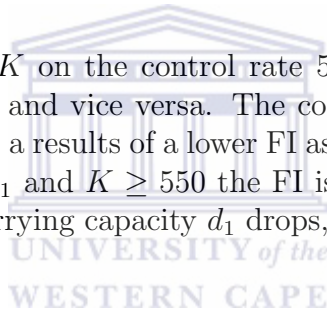
at $K = 650$,

d_1	0.035	0.04	0.045	0.05
FI	0.3580	0.3580	0.3579	0.3578

at $K = 700$,

d_1	0.035	0.04	0.045	0.05
FI	0.3732	0.3731	0.3730	0.3729

Within the respective K on the control rate $500 \leq K \leq 700$, the FI is increasing as d_1 decrease and vice versa. The control of carrying capacity and the growth rate d_1 gives a results of a lower FI as shown at respective $K \leq 550$. At the control higher d_1 and $K \geq 550$ the FI is increasing as well. This conclude that the lower carrying capacity d_1 drops, the higher Fisher information (FI).



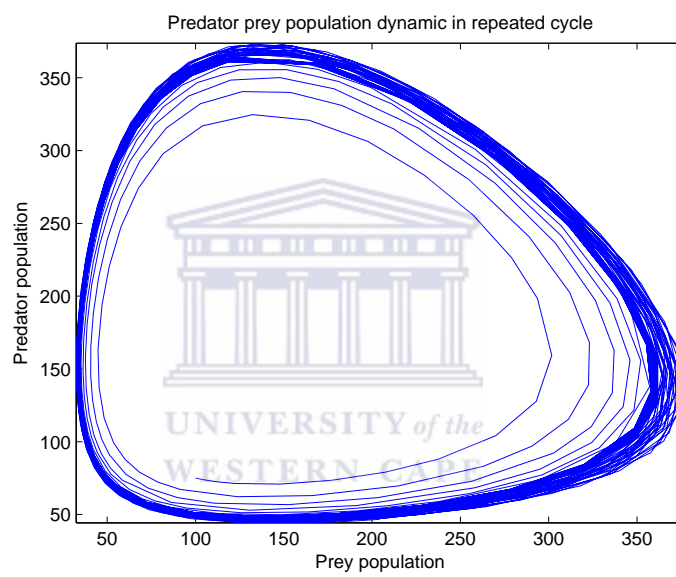


Figure 6.2: Diagram of predator-prey model in growth rate K and a saturation control b

Chapter 7

A model for managing a lake

Human existence depends on services provided by ecological resources. The reliance as in Brauer, Maler [8, 40]) proved more needfull in rural than in industrialised areas. Effective management of these resources is essential. In this chapter we consider one of the problem of eutrophication. The management is subject to alternate states, thresholds, and some states changes of lake behaviours. The problem of a lake eutrophication is considering an excessive phosphorus input. We shall describe a very simple model which illustrates many of the phenomena observed in real lakes. We describe the management of the runoff nutrients into/out of the lake. The view of this chapter is noting the prevention of eutrophication both in urban and rural services. The phrase eutrophication means the enrichment of an ecosystem with chemical nutrients, typically compounds containing nitrogen, phosphorus, or both. The primary cause is usually the excessive inputs of phosphorus, mainly due to runoff from agriculture and urban areas. A lake may be eutrophic with the characteristics of high nutrient input and plant production, dirty water, and toxic molecules. On the other hand, a lake may be oligotrophic with the state characterized by low input of nutrients and plant production, and a fair degree of clear water.

7.1 The model for eutrophication

The section highlight some problems and consider simple models for addressing these questions. The phosphorus concentration in the lake, as explained in

Brock et al., [12] introduced is given by the differential equation

$$\frac{dx}{dt} = W - sx(t) + r \frac{x^q(t)}{n^q + x^q(t)}. \quad (7.1)$$

This model describe the rate W of phosphorus input from the watershed while x is the amount of phosphorus in the water. The phrase $sx(t)$ is the amount of phosphorus present in the water. The rate of phosphorus loss from sedimentation, outflow, and absorption by consumers or plants is assumed to be proportional to the amount of phosphorus present. The parameter s denotes the sedimentation rate at which phosphorus leaves the water column and enters the sediments at the bottom of the lake. The sigmoid function

$$r \frac{x^q(t)}{n^q + x^q(t)} \quad (7.2)$$

is suggested as a recycling rate through the study of approximation. The exponent q describes the steepness of this sigmoid function at its inflection point, and must be greater or equal to 2. The parameter r is the maximum recycling rate of phosphorus and n is the concentration of phosphorus at which recycling is half its maximum rate. The feedback function of (7.2) and constant loading of runoff of phosphorus in water with its rate can be checked in Maler [40, p 654].

Now, a phosphorus loading for when $\frac{dx}{dt} = 0$, is

$$sx(t) = W + r \frac{x^q(t)}{n^q + x^q(t)}. \quad (7.3)$$

In this case the amount of phosphorus present in the water is at equilibrium. The additional methods of intervention to decrease a recycling or increase sedimentation is applied, see Fig 7.2. Whether this understanding of intervention is feasible, depends on the properties of the lake. The health of the lake is accomplished not by considering only the runoff nutrients, but traditional water hyacinths as well.

The growth of water hyacinths or algae in the lakes are at most rare in Africa. Theory and behaviour of eutrophication in terms of ecology study are presented in papers such as of Maler, Brock et al., Brauer [40, 12, 8] and the collective interdisciplinary articles on economics applications.

7.2 Optimal control

We let the instantaneous utility function be $u(x, W)$ and the utility discount rate be α . The social welfare function is defined:

$$K = \int_0^{\infty} e^{-\alpha t} u(x, W) dt.$$

The path of phosphorous loading maximize K subject to the dynamics of the lake.

We assume the utility function to be of the form:

$$u = \ln W - cx(t)^2.$$

The utility is separable in phosphorous loading and the loss in amenities from eutrophication. Therefore necessary and sufficient conditions for an optimum are defined: The Hamiltonian can be written as

$$H = \ln W - cx(t)^2 + \lambda \left(W - sx(t) + r \frac{x^q(t)}{n^q + x^q(t)} \right).$$

The variable λ is the co-state variable. A necessary condition for an optimum is that the partial derivative of H with respect to W is zero:

$$\frac{1}{W} + \lambda = 0$$

\Leftrightarrow

$$W = -\frac{1}{\lambda}.$$

The co-state variable satisfies the differential equation

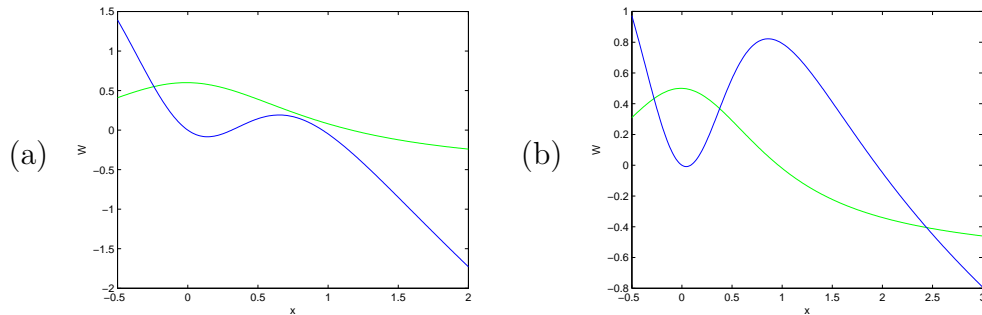
$$\frac{d\lambda}{dt} = \left[\alpha + s - r \frac{2x(t)}{(1 + x^2(t))^2} \right] \lambda + 2cx(t). \quad (7.4)$$

Thus it follows that the loading of phosphorus is

$$\frac{dW}{dt} = - \left[\alpha + s - r \frac{2x(t)}{(1 + x^2(t))^2} \right] W + 2cx(t)W^2. \quad (7.5)$$

In line with Brock et al., [12], the following shows the behaviour of the system. The parameter values that we use are $s = 0.02$, $\alpha = 0.15$, $r = 1$ and $c = 0.01$.

Figure 7.1: Diagram with unique equilibrium labeled (a), and (b) multiple equilibria.



The vertical axes represent the loading of phosphorus W and the horizontal axes represent the stock of phosphorus in the lake.

In Fig 7.1 (a) the two curves corresponds to stationary solutions of the two differential equation (7.4) and (7.5) which intersect at only once. In this case we have it optimal to take the lake back to an oligotrophic state, irrespective of where the original state is. The optimal trajectories converges irrespective of the initial level of eutrophication. Nevertheless, the curves can be found different as is seen in Fig 7.1 (b)

In Fig 7.1 (b) there are three possible equilibria. The middle equilibrium is unstable, but the two others are saddle points and potentially optimal steady states. Following Skiba [66] we analyse the multiple equilibria by looking the concave-convex iterations. There exists a level of initial loading x , such that if the initial stock is less, the optimal path will take the system to convex state, while if the initial stock is great, the optimal path will take the lake to convex function.

If the initial stock happens to be 0.5 following Fig 7.1 (a), then the lake will approach first oligotrophic equilibrium and second eutrophic equilibrium.

7.3 Trophic states

The section reviews the eutrophication process with reference to mineral nutrient status. Following [17] we define trophic category with namely oligotrophic,

mesotrophic, and eutrophic for lake description in eutrophication. The *oligotrophic* lake is a large deep lake with crystal clear waters and a rocky or sandy shoreline, whereas *eutrophic* is a shallow lake rich in organic and mineral nutrient supporting the abundant plant life. Mesotrophic is an intermediate trophic [17] state with the characteristics between the oligotrophic and eutrophic states. Then it follows that we review oligotrophic and eutrophic process in eutrophication. Therefore by considering the eutrophication process in the linear system described in Maler [40], we then note that not only green plants can control the bottom of the lake. Thus it follows that:

$$\frac{dx}{dt} = W - sx(t), \quad (7.6)$$

$$\frac{dy}{dt} = sx(t) - ky(t), \quad s, k > 0, s \neq k, \quad (7.7)$$

The phrase $ky(t)$ is the amount present in water. Here k and s are constants describing the natural removal of phosphorous from the water.

Now, since the system is linear, we know that the solution to this model is

$$x(t) = e^{St} \left[x_0 + \int_0^t e^{Sp} G(p) dp \right],$$

where $G(p) = (d(p), 0)'$. From equations (7.4) and (7.5) we have the following,

$$\mathbf{S} = \begin{pmatrix} -s & 0 \\ s & -k \end{pmatrix}.$$

Since \mathbf{S} has two negative eigenvalues $-s$ and $-k$, $\lim_{t \rightarrow \infty} e^{St} X_0 = 0$. The homogeneous solution represents a transient solution. Therefore,

$$\lim_{t \rightarrow \infty} X(t) = \lim_{t \rightarrow \infty} e^{St} \int_0^t e^{Sp} G(p) dp,$$

where e^{St} is given by

$$\begin{pmatrix} e^{-at} & 0 \\ a \frac{e^{-bt} - e^{-at}}{a-b} & e^{-bt} \end{pmatrix}.$$

Another method to view this system is to approximate the phosphorus amount (x) first, which is a first-order non linear homogeneous equation. Then use x to get y . Suppose there is continuous release of a phosphorus concentration

into lake, i.e., W is constant. Let $W = 1$ and $x(0) = 0 = y(0)$. Then

$$x(t) = \frac{1}{s}(1 - e^{-st}),$$

$$y(t) = \frac{1}{k} + \frac{e^{-st}}{s - k} - \frac{se^{-kt}}{k(s - k)},$$

where their limits as $t \rightarrow \infty$ are:

$$\frac{1}{s}, \frac{1}{k}$$

respectively.

The significant changes in runoffs results of systems (7.4) and (7.5), describes the management expressions in terms of consumer production. Thus it ensures that although eutrophication can increase nutrient input, any activity in the watershed W of a lake that increases nutrient input causes eutrophication.

7.4 The run-off nutrients in the lake

The section describes an estimation of nutrients interactions in the lake. We also describe interaction between the run-off nutrients into the lake and the growth of either water hyacinths or algae.

Definition 7.2: *Run-off nutrients* is discharges into the lake from sources such as fertilizers, sewage, phosphate, etc., which cause the eutrophication. It follows that from the definition (7.2) we are considering a period of time required for the existence of a concentration or amount of phosphorus in the lake. The preference will be on eutrophic lake to oligotrophic as shown on the Fig 7.3. Now, for the estimation nutrients existence in systems (7.4) and (7.5), if in the lake, we have $dx/dy = -sx$, $x(t) = x(0)e^{-st}$, and then the nutrients is the value of t where $x(t) = x(0)/2$, see Fig 7.3.

Therefore the runoff nutrients is $t = \ln(2)/s$. Suppose time is measured in hours and the amount of nutrients of a particular phosphorus in the lake is 1/2 hour and the loading is about 5 hours, then $s = 2\ln(2)$ and $k = \frac{\ln(2)}{5}$.

The solution for the eutrophication reactant is graphed in the figure below. Consider the case where the amount of phosphorus is assumed periodic. This is a reasonable situation because prescription phosphorus are often be put on

specific intervals of time.

Suppose the amount of phosphorus put by a firm or inhabitants is prescribed

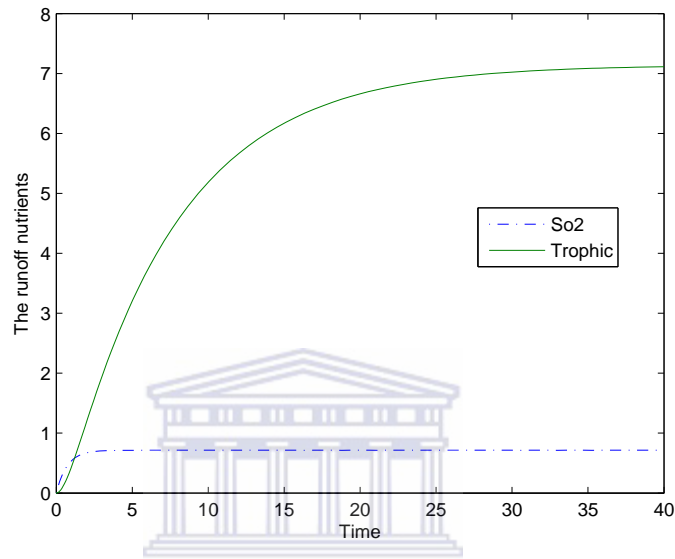


Figure 7.2: Diagram estimating the time existence of runoff nutrients in the lake.

every six hours. Then a reasonable assumption about the concentrations is as follows:

$$d(t) = \begin{cases} 2, & 0 \leq t \leq 1/2, \\ 0, & 1/2 < t < 6, \end{cases}$$

where $d(t+6) = d(t)$. The amount of phosphorus is released into the lake every six hours and can be found in the lake over a half-hour period. Thus it means that the important and interesting questions arise with periodic phosphorus input. For example, if the concentration is changed or the period is changed (for about every 12 hours), how does the maximum or minimum concentration of the phosphorus be known for how long may it be found in the water? So from the figure we see it clearly that in order for the nutrients input to be effective, then a certain minimum concentration needs to be maintained for the phosphorus reluctant not only.

7.5 Management time scale and delays

We find it reasonable that the sediment in the eutrophication can occur with delay, but also having a possibility of no delay depending on the how the lake effects the phosphorus responses. The human activities usually results in an increase in the biological production that occurs in the lake. Then we define the reactants as follows, where for more understanding of time delays of systems, we consider the examples by Loiseau et al., [37]. The following definitions are in terms of nutrients stimulation.

Discrete delay is the reaction to the nutrients that occurs after a fixed period of time.

Continuous delay is the reaction to the concentration that does not occur after a fixed period of time but occurs over a continuous range of times.

Such delays can be incorporated by considering the delay differential equation model with discrete delay of the form:

$$\frac{dx}{dt} = sx(t) + kx(t - T) + f(t). \quad (7.8)$$

The model with continuous delay over an interval $[0, T]$ will give an integro-differential equation of the form

$$\frac{dx}{dt} = sx(t) + k \int_0^T x(t - p) dp + f(t).$$

By letting the variable in the integral be $u = t - p$, leads to the integro differential equation

$$\frac{dx}{dt} = sx(t) + k \int_{t-T}^t x(u) dp + f(t).$$

In a discrete delay differential equation (7.6), it is necessary for the initial condition given on the interval $[-T, 0]$ whereby an interval length is given by the length of delays. Suppose $x(t) = \phi_0$ on $[-T, 0]$. Then the solution of a continuous delay integrodifferential equations is found on the successive interval $[0, T]$ is $[T, 2T]$ and so on.

On the interval $[0, T]$, the differential equation is a non-homogeneous linear differential equation without a delay because $\phi_0(t)$ is known on $[0, T]$.

$$\frac{dx}{dt} = sx(t) + k\phi_0(t - T) + f(t) \quad (7.9)$$

with the initial condition $x(0) = \phi_0$ can be solved to obtain $x(t) = \phi_1$ on $[0, T]$). So, on the interval $[T, 2T]$, the system (7.7) satisfies

$$\begin{aligned}\frac{dx}{dt} &= sx(t) + kx_1(t - T) + f(t) \\ &= sx(t) + k\phi_1(t - T) + f(t)\end{aligned}$$

with the initial condition $x(t) = \phi_1(T)$.

However, when either traditional or industrialized causes increase in intensity and frequency, the results can cause community concern. The results can also cause health problems, and in some cases can be catastrophic to the environment. Thus a time scale and delays on eutrophications can undergo self-purification processes in which the organic materials and waste are broken down by bacterial and other biological activities naturally.



Chapter 8

Conclusion

The scientific and technical understanding of the use of natural resources and its life-supporting capacity is important to inform the planning and management of strategies for sustainable use of natural resources. In this regard, mathematical modeling has an important role. In this thesis we have reviewed a few problems from the existing literature, related to the use of natural resources. The thesis makes a contribution by giving more explanation, especially with respect to EFI (Section 2.4 and Chapter 6), and by considering stochasticity (Chapters 5 and 7). In general we have attempted to improve the general understanding of the relevant problems.

In Chapter 3 we modelled the evolution of shifting cultivation as in Balsdon [4]. This is a complex problem, regarding the degradation of forest land when used for subsistence level agricultural purpose, until the land is so degraded that the occupants abandon it and move on to a new stand. The practise is usually driven by poverty and is difficult to combat. The model used to study the shifting period is similar to the forest rotation problem.

A model for biological control of a pest, as in the model of Chen and Tan [70] is studied in Chapter 4. The ordinary differential equation model assumes that the susceptible $S(t)$ pest is capable of reproducing. We assumed that the infected pests cannot recover, and is not capable of reproducing and attacking the crop in question. Three equilibrium points are shown to be possible, one of them being a disease-free state with non-zero crop. Onto the deterministic model of [70] we impose a stochastic perturbation, so that we obtain a stochastic differential equation model. We prove stochastic stability of the disease-free state mentioned above, when the basic reproduction number of the pest is below unity. We have perform some insightful simulations of

solutions of the stochastic system.

In Chapter 5 an ordinary differential equation model describes the competition between trees and grass in savanna environment, as in Accantino et al., [59]. The competition between them is for soil water, fed by annual rainfall. On the other hand, trees and grass are perturbed by fire, and some other environmental forcings such as herbivores. The model captured essentially the competition for or availability of water, and the effect of fire. The savana dynamics describes a spatially homogeneous model for the competition of trees and grasses in the environment. Steady state solutions of grass and tree in the environment are illustrated, to show effective coexistence in response to the forcings. Stability maps depicts the environmental space defined by rainfall and fire. Into the model of [59] we newly introduce stochastic perturbations. The stochastic perturbations are in the form of three mutually independent brownian motions. We run some simulations to illustrate the effect of the stochasticity.

Chapter 6 presents the predator-prey model of Diwekar and Shastri [65]. We derived new theoretical results on the stability of a three-tiered predator-prey system. We presented some computations, exploring the effect of sustainable development, and even the effect on harvesting. The equilibrium states of the system are shown. The Fisher information is the basis of the so-called sustainable measures hypotheses, as proposed by Fath et al [20]. We present several computations to show the influence of carrying capacity of prey and of mortality rate on FI.

In Chapter 7 following Maler [40] we considered the problem of lake eutrophication caused by excessive phosphorus input. The computations such as in Figure 7.3 illustrate the management of the runoff nutrients time into/out of the lake. Necessary and the sufficient conditions for an optimal utility management is obtained using standard optimal control theory.

The results in this thesis have demonstrated some modeling techniques in the sustainable use of natural resources. The discourse on sustainability is becoming more important, especially in view of a growing world population, and with forcings such as climate change. The key idea in sustainability is the quest for equal opportunities over all generations. The manner in which this sustainability is quantified in models is being debated and improved all the time.

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