Numerical techniques for optimal investment consumption models

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A Thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in the Department of Mathematics and Applied Mathematics at the Faculty of Natural Sciences, University of the Western Cape

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November 2014

KEYWORDS

Brownian motion process

Constant Relative Risk Aversion

Consumption

Hyperbolic Absolute Risk Aversion

Log-normal distribution

Markov process

Optimization

ABSTRACT

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The problem of optimal investment has been extensively studied by numerous researchers in order to generalize the original framework. Those generalizations have been made in different directions and using different techniques. For example, Perera [Optimal consumption, investment and insurance with insurable risk for an investor in a Levy market, Insurance: Mathematics and Economics, 46 (3) (2010) 479-484] applied the martingale approach to obtain a closed form solution for the optimal investment, consumption and insurance strategies of an individual in the presence of an insurable risk when the insurable risk and risky asset returns are described by Levy processes and the utility is a constant absolute risk aversion. In another work, Sattinger [The Markov consumption problem, Journal of Mathematical Economics, 47 (4-5) (2011) 409-416] gave a model of consumption behavior under uncertainty as the solution to a continuous-time dynamic control problem in which an individual moves between employment and unemployment according to a Markov process. In this thesis, we will review the consumption models in the above framework and will simulate some of them using an infinite series expansion method − a key focus of this research. Several numerical results obtained by using MATLAB are presented with detailed explanations.

November 2014.

DECLARATION

I declare that *Numerical techniques for optimal investment consumption models* is my own work, that it has not been submitted before for any degree or examination at any other university, and that all sources I have used or quoted have been indicated and acknowledged by complete references.

Bernardin Gael Mvondo   November 2014

Signed . . . .

ACKNOWLEDGEMENT

I would like to acknowledge God Almighty who has been the conductor of this thesis from start to finish, my unbelievable family and my supervisor, his guidance, help and attention were of utmost importance without which this thesis would never be done.
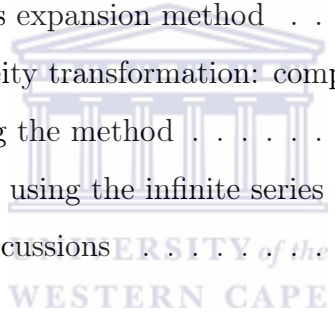
DEDICATION

To My Family

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Chapter 1

General introduction

We start this thesis by giving a background overview on investment consumption problem. In so doing, we will mention what an investment consumption model means and how it is interpreted in the mathematical finance world and finally we briefly mention about the methods used to solve investment consumption models.

1.1 The optimal investment problem

The optimal investment consumption problem explains the optimal strategy an investor can use to spend her money in order to maximize her discounted utility and minimize the risk (loss) when she is confronted with only few investment choices; the money can be saved in the bank account that is the risk free asset (bond), the money can be invested in the risky asset (stock market) and lastly, use the money on consumption.

Such problems of optimal investment have been extensively studied by numerous researchers in order to generalize the original framework proposed by Samuelson [48]. These generalizations have been made in different directions, and using different techniques. Intertemporal investment models in continuous time have been studied by Samuelson [48] who considered a discrete-time investment-consumption model with the objective of maximizing the overall expected utility of consumption. Using the dynamic stochastic programming approach, he succeeded in obtaining the optimal

decision for the investment-consumption model. Merton [38] extended the model of Samuelson [48] to construct an explicit solutions under the assumption that the stock price follows a Geometric Brownian Motion process and the individual preferences are of special type. In particular he showed that under the assumption of log-normal stock returns and hyperbolic absolute risk aversion ($HARA$) utility, the optimal proportion of investment in the risky asset is constant. The investor may also borrow money to finance investment at an interest rate hence in that work, the objective was to choose investments, borrowing, and consumption in order to maximize the total expected discounted utility of consumption, where utility is determined according to a log utility function.

As far as the issue of incompleteness is concerned, Cvitanic and Karatzas [15], Karatzas et al. [26], Harrison and Pliska [21], Pliska [45], Karatzas et al. [27], Cox and Huang [13], Cuoco [14], Brennan et al. [8], Brennen and Schwartz [9], Zariphopoulou [56] and many others have studied incompleteness due to constraints on the portfolio, as they impose that the portfolio must remain in a certain set. Not long ago, Cheung and Yang [55] investigated a dynamic investment-consumption problem in a regime switching environment. In this case, the price process of the risky asset was modeled as a discrete-time regime-switching process, and it was shown that the optimal trading and consumption strategies are consistent with the belief that investors should put a decent amount of their wealth in the risky asset and consume less when the underlying Markov chain is in a better state.

According to some researchers, it may be more realistic to assume that the economic uncertainty is resolved gradually since more and more information is available as time passes. Generally, two approaches are considered for analyzing these problems: stochastic control and martingale analysis. This thesis examines a general investment and consumption problem and discusses a numerical method for solving an optimal consumption problem. Below we give some more specific information regarding necessary mathematics of an investment-consumption model.

The investor consumes wealth X_t at a nonnegative rate C_t and distributes it be-

tween two assets continuously in time. One asset is a bond that is a riskless security with instantaneous rate of return r; the other asset is a stock whose value is driven by a Wiener process. The previous studies in this area have two common features. First one was to provide solutions that relied on the duality approach and/or variational techniques. The second one adopted the assumption of exponential or $HARA$ utility function (especially a logarithmic utility function) in order to obtain explicit solutions. Hence on this investment-consumption model, it has been assumed either that the utility function $U(c)$ is $HARA$ or that R (the interest rate) is equal to r (the risk-free interest rate). When the utility function is $HARA$, there is a simple explicit solution, whereas the Hamilton-Jacobi-Bellman (or dynamic programming) equation can be linearized in case $R = r$ as in [17]. When $U(c)$ is merely asymptotically $HARA$ for small and large c, then the optimal investment and consumption policies are found using asymptotic approximations (see Fleming and Zariphopoulou [18]) for small and large wealth x. For intermediate x, the optimal policies are usually determined numerically. Such numerical studies can be used to confirm or to refute various conjectures about the structure of optimal policies. For instance, the fraction of the investor's wealth in stock is not a monotone function of the wealth x as mentioned in [17].

It would be of great importance to also mention another assumption with optimal investment-consumption problems, that is, the known duration of the planning horizon (such as 10 or 20 years). Usually, when making an investment, the investor knows with certainty the time of eventual exit but in practice, investors may be forced to exit the market before their planned investment horizons due to a variety of reasons such as financial crisis, fatal illness, or death. In these situations, the time of exit is no longer certain. Consequently it is of both practical and theoretical importance to develop a comprehensive theory of optimal investment-consumption decisions under uncertain time horizon as induced by the mortality risk.

The method used in this thesis is the infinite series expansion method which was introduced by Tebaldi and Schwartz [50]. For the logarithmic utility function, there is not much literature or work presented using infinite series expansion.

The purpose of this thesis is to solve an optimal consumption problem in its general framework, derived from Merton's original problem [38]. We will pay close attention in the way the investor should behave when we alternate economics parameters and deduce the optimal policy. It is of utmost importance to underline that we will solely take into consideration the case where we have a logarithmic utility function and an infinite time horizon. It should further be noted that the focus of this thesis is not to provide the underlying theory of the developed method but to provide implementation and simulation results.

In what follows, we provide some mathematical preliminaries that are useful for the smooth reading of the rest of the thesis.

1.2 Mathematical preliminaries

Stochastic processes

A stochastic process is a collection of random variables, which is often used to represent the evolution of some random value, or system, over time and whose development is governed by probability laws. A stochastic process (according to [7]) is defined as

Definition 1.2.1. *Given an index set I, a stochastic process, indexed by I is a collection of random variables $\{X_\lambda : \lambda \in I\}$ on a probability space $(\Omega; F; P)$ taking values in a set S. The set S is called the state space of the process.*

Two important properties of random processes are mean square continuity and mean square differentiation which are defined below (as in [24]).

Definition 1.2.2. *A random process $X(t)$ is said to be mean square (m.s) continuous if*

$$\lim_{\epsilon \to 0} E\left[(X(t+\epsilon) - X(t))^2\right] = 0. \tag{1.2.1}$$

Definition 1.2.3. *The m.s derivative $X^{'}(t)$ of a random process $X(t)$ can be defined as*

$$l.i.m_{\epsilon \to 0} \frac{X(t+\epsilon) - X(t)}{\epsilon} = X^{'}(t), \tag{1.2.2}$$

where l.i.m. denotes limit in the mean (square), provided that

$$\lim_{\epsilon \to 0} E\left[\left(\frac{X(t+\epsilon) - X(t)}{\epsilon} - X^{'}(t) \right)^2 \right] = 0. \tag{1.2.3}$$

Brownian motion process

In the development of our thesis, we will also use a random process called the Brownian motion process also called the Wiener process. The Brownian motion process is continuous stochastic process that is hugely used in physics and finance to model random behaviors that evolve over time. The term Brownian motion can also refer to the mathematical model used to describe such random movements.

In order to clearly state what a Brownian motion process is, the concept of stationary and independent increments are useful. We introduce them as per below.

Definition 1.2.4. *A random process $X(t)$; $t \geq 0$ is said to have independent increments if when ever $0 < t_1 < t_2 < ... < t_n$,*

$$X(0), X(t_1) - X(0), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1}), \tag{1.2.4}$$

are independent. If $X(t)$; $t \geq 0$, has independent increments and $X(t) - X(s)$ has the same distribution as $X(t+h) - X(s+h)$ for all s, t, $h \geq 0$, $s < t$, then the process $X(t)$ is said to have stationary and independent increments.

In the view of definition 1.2.4, the Brownian motion process (denoted again by $X(t)$ to avoid notational complexities) is characterized by the following properties [24]:

1. $X(t)$ has stationary and independent increments,

2. The increment $X(t) - X(s)$; $t > s$, is normally distributed.

3. $E[X(t)] = 0$, and

4. $X(0) = 0$.

Note that the Brownian motion process is the most vital stochastic process. Often, it is utilized to model the behavior of stock prices.

Note also that the investment-consumption model described in this thesis assumes that the market should be perfect, with continuous trading possibilities and no transaction costs and that the investor has a logarithmic utility function, $U(c) = \log(c)$, which fulfill the conditions in the following definition:

Definition 1.2.5. *The utility function $U(c) : S \to \mathbb{R}, S \subseteq \mathbb{R}$ measures the investors risk attitude and preferences. The function has the following properties: $U(c) \in C^2(\mathbb{R}+)$ with $U'(c) > 0$ (non-satiation) and $U''(c) < 0$ (risk aversion).*

Now before we discuss the formulation of the problem along with the associated Hamilton-Jacobi-Bellman equation, we present the economic setting of this problem as follows:

- The evolution equation for the price of the riskless bond with positive interest rate r is described as

$$dB(t) = rB(t)dt, \ t > 0.$$

- The price estimated by the holder of the stock follows the geometric Wiener process, which in differential form can be written as

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW_1(t), \ t > 0,$$

where $\alpha(> r)$ is the rate of return and σ is the volatility, $W_1(t)$ is the Wiener process associated with the stock price.

- The price proposed by the buyer (illiquid asset) with correlation coefficient ρ

satifies

$$\frac{dH(t)}{H(t)} = (\mu - \delta)dt + \eta \left(\rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right), \ H(0) = h, \ t > 0, \ (1.2.5)$$

where μ is the expected rate of return on the risky illiquid asset, ρ is the correlation between the illiquid asset and the stock price, δ is the interest paid by the illiquid asset and η is the continuous standard deviation of the rate of return, $W_2(t)$ is the Wiener process associated with the illiquid asset.

Definition 1.2.6. *If the log-price process* $\ln S(t)$; $t \geq 0$, *is governed by a Brownian motion with a drift,* $S(t) = S(0)e^{\alpha t + \sigma W(t)}$; $t \geq 0$, *where* $\alpha > 0$ *and* $\sigma > 0$, *then the stock price process* $S = (S(t))_{t \geq 0}$ *is called a geometric Brownian motion.*

Now that the processes have been defined and written in differential form, we can now formulate the model analytically approximate its solution.

The continuous model and the Hamilton-Jacobi-Bellman equation

Now we will look at a continuous investment problem and the formulation of a model to solve the latter and as a conclusion, the derivation of the Hamilton-Jacobi-Bellman equation.

The evolution equations for the prices of the bond B_t (the riskless asset) and the stock S_t (the risky asset) are given by

$$dB_t = rB_t dt \ \text{ and}$$
$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

respectively, where $W_t \geq 0$ is a standard Brownian motion. We denote by π_t^0 and π_t the amounts the investor puts at time t in bonds and stocks, respectively, whereas H_t

represents the amount borrowed. The total wealth of the investor is given by $L_t = \pi_t^0 + \pi_t$ - H_t . Taking into account the results in [18], the evolution equation for the investor's wealth is given by

$$dL_t = rL_t dt + [(\alpha - r)\pi_t - (R - r)H_t]dt - c_t dt + \sigma\pi_t dW_t, \qquad (1.2.6)$$

with initial condition $L_t^0 = l$ and $(r - R) = \delta$ (defined in the economic setting). The problem is constrained by the fact that the quantities H_t, π_t, c_t and L_t must all be nonnegative.

We define the value function by

$$V(l) = \sup_A E\left\{ \int_0^\infty e^{-\beta t} U(c_t) dt \right\}, \qquad (1.2.7)$$

where U is the utility function, β is the discount factor and $A(H_t, \pi_t, c_t)$ the set of admissible controls which are measurable with respect to the process W_t. We assume that U satisfies the following assumptions [17]:

Assumption 1.2.7. : *U is a strictly increasing, strictly concave function in* $C[0,\infty) \cap C^3(0,\infty)$. *Furthermore* $\lim_{c \searrow 0} U'(c) = \infty$, $\lim_{c \nearrow \infty} U'(c) = 0$, $U(0) = 0$.

Now a summary of the results from Fleming and Zariphopoulou [18] and Zariphopoulou [56] that will be useful in studying some numerical schemes, are indicated below (see [17] for further details).

Theorem 1.2.8. [17] *The value function* $V(l)$ *is concave and strictly increasing. Furthermore* V *is continuous on* $[0,\infty)$, *with* $V(0) = 0$. *If* $U(c) \nearrow M$ *as* $c \nearrow \infty$, *with* $M < \infty$, *then* $V(l) \nearrow M/\beta$ *as* $l \nearrow \infty$.

Proof. See [17].

To derive the HJB equation we make use of the dynamic programming method in addition to the Ito's formula which is well known from the literature.
We define

$$H(c) = U(c) - cV_l = \log(c) - cV_l, \qquad (1.2.8)$$

and

$$G(\pi) = \frac{1}{2}V_{ll}\pi^2\sigma^2 + V_{lh}\eta\rho\pi\sigma h + \pi(\alpha - r)V_l(l, h). \qquad (1.2.9)$$

The Hamilton-Jacobi-Bellman (HJB) equation for our stochastic control problem is

$$\beta V = G(\pi) + rlV_l + H(c), \; l > 0, \qquad (1.2.10)$$

By rewriting equation (1.2.10) more carefully using the economic setting, we obtain

$$\frac{1}{2}\eta^2 h^2 V_{hh} + (rl + \delta h)V_l + (\mu - \delta)hV_h + \max_{c \geq 0} H(c) + \max_{\pi} G(\pi) = \beta V, \qquad (1.2.11)$$

Next, we reduce equation (1.2.11) to be able to solve it numerically.

Theorem 1.2.9. [17] *The value function is the unique, nondecreasing, concave viscosity solution of equation (1.2.11) on $(0, \infty)$, such that $V(x) \searrow 0$ as $x \searrow 0$.*

Proof. See [17].
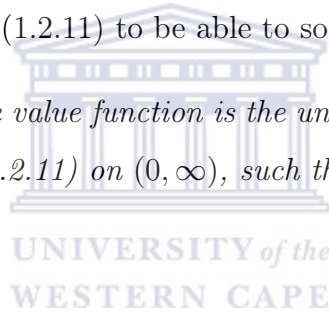
A Reduction of the problem

In order to use the appropriate numerical method, a transformation is used to reduce this problem from a PDE to an ODE. The steps as in ([40]) and ([50]) are as follows:

$$z = \frac{l}{h},$$
$$V(l, h) = K + \frac{\log h}{\beta} + W(z),$$

where K is a constant which we will be set later stage.

Differentiating V, we get

$$V_h = \frac{1}{h\beta} - \frac{l}{h^2}W' \Rightarrow hV_h = \frac{1}{\beta} - zW',$$

$$V_l = \frac{1}{h}W' \Rightarrow \begin{cases} lV_l = zW', \\ hV_l = W', \end{cases}$$

$$V_{hh} = -\frac{1}{h^2\beta} + \frac{2l}{h^3}W' + \frac{l^2}{h^4}W'',$$

$$\Rightarrow h^2 V_{hh} = -\frac{1}{\beta} + zW' + z^2 W'',$$

$$V_{ll} = \frac{1}{h^2}W'' \Rightarrow \begin{cases} h^2 V_{ll} = W'', \\ lhV_{ll} = zW'', \\ l^2 V_{ll} = z^2 W'', \end{cases}$$

$$V_{lh} = -\frac{1}{h^2}W' - \frac{l}{h^3}W'' \Rightarrow h^2 V_{lh} = -W' - zW''.$$

Hence the right hand side in equation (1.2.11) becomes

$$\beta V = \beta K + \log h + \beta W, \tag{1.2.12}$$

whereas the left hand side leads to

$$
\begin{aligned}
\max_{\pi} G(\pi) &= \max_{\pi} \left[\frac{1}{2} V_{ll}\pi^2\sigma^2 + V_{lh}\eta\rho\pi\sigma h + (\alpha - r)V_l\pi \right], \\
&= \max_{\pi} \left[\frac{1}{2} W''\sigma^2\frac{\pi^2}{h^2} - (W' + zW'')\eta\rho\sigma\frac{\pi}{h} + (\alpha - r)W'\frac{\pi}{h} \right], \\
&= \max_{\pi_1} \left[\frac{1}{2} W''\sigma^2\pi_1^2 - (W' + zW'')\eta\rho\sigma\pi_1 + (\alpha - r)W'\pi_1 \right], \\
&= \max_{\pi_1} \left[\frac{1}{2} W''(\sigma^2\pi_1^2 - 2\eta\rho z\pi_1\sigma) - W'\eta\rho\sigma\pi_1 + (\alpha - r)W'\pi_1 \right], \\
&= \max_{\pi_1} \left[\frac{1}{2} W''(\sigma\pi_1 - \eta\rho z)^2 - W'\eta\rho\sigma\pi_1 + (\alpha - r)W'\pi_1 \right] - \frac{\eta^2}{2}\rho^2 z^2 W'', \\
&= \max_{\varphi} \left[\frac{1}{2} W''\sigma^2\varphi^2 + \underbrace{(-\eta\rho\sigma + \alpha - r)}_{k_1}\left(\varphi + \frac{\eta\rho z}{\sigma} \right) W' \right] - \frac{\eta^2}{2}\rho^2 z^2 W'', \\
&= \max_{\varphi} \left[\frac{1}{2} W''\sigma^2\varphi^2 + k_1\varphi W' \right] - \frac{\eta^2}{2}\rho^2 z^2 W'' + \frac{\eta\rho k_1}{\sigma} zW'. \\
\max_{c\geq 0} H(c) &= \max_{c\geq 0} [-cV_t + \log c] = \max_{c\geq 0} \left[-\frac{c}{h}W' + \log c \right] \\
&= \max_{c_1\geq 0} [-c_1 W' + \log c_1] + \log h.
\end{aligned}
$$

In the above, $\pi_1 = \frac{\pi}{h}$, $h > 0$; $\varphi = \pi_1 - \frac{\eta\rho z}{\sigma}$; and $c_1 = \frac{c}{h}$.

The rest of the factor on left hand side is

$$
\begin{aligned}
\frac{1}{2}\eta^2 h^2 V_{hh} + (rl + \delta h)V_l + (\mu - \delta)hV_h &= \frac{\eta^2}{2}\left(-\frac{1}{\beta} + 2zW' + z^2 W'' \right) + (rz + \delta)W', \\
&\quad + (\mu - \delta)\left(\frac{1}{\beta} - zW' \right), \\
&= \frac{\eta^2}{2} z^2 W'' + (\eta^2 + r - (\mu - \delta))zW', \\
&\quad + \delta W' - \frac{\eta^2}{2\beta} + \frac{\mu - \delta}{\beta}.
\end{aligned}
$$

Cancelling the term $\log h$ on both sides of the equation and setting K to be

$$
K = \frac{\mu - \delta}{\beta^2} - \frac{\eta}{\beta^2}, \tag{1.2.13}
$$

equation (1.2.11) can now be written as (see also [3])

$$\frac{\eta^2}{2}(1-\rho^2)z^2W'' + kzW' + \delta W' + \max_{\varphi}\left[\frac{1}{2}W''\sigma^2\varphi^2 + k_1\varphi W'\right]$$
$$+ \max_{c\geq 0}\left[-cW' + \log c\right] = \beta W,$$

where $k = \eta^2 + r - (\mu - \delta) - \frac{\eta\rho k_1}{\sigma}$ and $k_1 = -\eta\rho\sigma + \alpha - r$. We also make the transformation $\varsigma = c - \delta$ to get rid of the term $\delta W'$. By doing so, we end up with the reduced HJB equation ([3])

$$\frac{\eta^2}{2}(1-\rho^2)z^2W'' + kzW' + \max_{\varphi}G_2(\varphi) + \max_{\varsigma\geq-\delta}H_2(\varsigma) = \beta W, \qquad (1.2.14)$$

with

$$G_2(\varphi) = \frac{1}{2}W''\sigma^2\varphi^2 + k_1\varphi W',$$
$$H_2(\varsigma) = -\varsigma W' + \log(\varsigma + \delta).$$
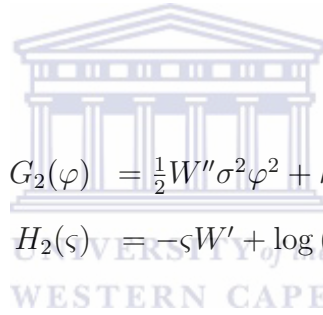
The above equation (1.2.14) will be solved by using the infinite series expansion method in chapter 4.

1.3 Literature review

In this section, we present some works from the literature that deals with optimal investment-consumption choice of an investor who can invest in a risky and safe assets.

Muthuraman [43] considered the presence of fractional transaction costs with the objective of maximizing the discounted utility of consumption where the utility function is a power function. He described an efficient numerical model that transforms the arising free-boundary problem to a sequence of fixed-boundary problems. He proved the convergence of the scheme and also showed that the converged solution is the optimal value function. Finally, he compared and contrasted the results obtained by his procedure with certain well-known results and approximations. His model also

maximizes portfolios with several risky assets and the stock price follows a geometric Brownian motion. The investor has initial choices in both the risky and safe assets. In time, the investor can choose to either spend the money on consumption, spend the money on buying stock or keep the money in the bank by selling stock. Therefore when transacting (buying or selling stock), the investor pays a fraction of the value transacted to a third party that made the transaction possible (transaction costs). Moreover the investor obtains utility by consuming money from the bank.

According to the literature, the investor is allowed to trade in continuous time and in infinitesimal quantities. The optimal strategy suggested by Merton [37] continuously transacts to hold fixed proportions of overall wealth in different stocks and consumes a various but fixed fraction of wealth. Merton's strategy it is required that an infinite number of transactions be made in any define time interval. This implies that Merton's policy can no longer be optimal when transaction costs are present. With the presence of transaction costs, the investor would want to make limited transactions. In particular, the investor would do some transactions only if the fraction of his holdings in stock is sufficiently distant from Merton's optimal fraction to guarantee the transaction. Magill and Constantinides [35] first considered proportional transaction costs and conjectured that the optimal policy would be characterized by an interval of inaction, thus the optimal strategy would not transact as long as the fraction of wealth in stock lies in this interval. When the fraction lies outside the interval the optimal policy would be to buy or sell just enough such that the fraction falls into the interval.

The problem with proportional transaction costs is now understood to be a singular stochastic control problem. Taksar et al. [51] made this observation as they obtained optimal policies for a model without consumption that maximized asymptotic growth rate of portfolio. Davis and Norman [16] solved the Merton problem [38] with proportional transaction costs for the one-stock case. They provided the characteristics of the optimal strategy, and conditions under which the Hamilton-Jacobi-Bellman equation has a smooth solution. They also provided a computational method to derive the optimal policy.

Numerical methods for the one-stock problem can be found in Davis and Norman [16]; Tourin and Zariphopolou [53]. Moreover, the computational complexity of these methods even for the one-stock case are very high. Zariphopolou [57] indicated that usually a computation of these methods on a computer takes hours to obtain satisfactory boundaries, however the method that they presented takes only few seconds to yield boundaries that converge within a tolerance of 10^{-6}.

Muthuraman [43] also gave a description of the model, a brief discussion of the value function and the Hamilton - Jacobi - Bellman (HJB) equation that characterized the value function. Finally, Muthuraman [43] compared and contrasted his results with the results obtained by Davis and Norman [16], asymptotic expansions obtained by Janecek and Shreve [25] and approximations suggested by Constantinides [12].

Now we present some works from the literature that deals with multiperiod optimal investment-consumption strategies with mortality risk and environment uncertainty.

Li et al. [31] derived explicitly the optimal investment consumption strategies for an investor with constant relative risk aversion ($CRRA$) preferences. To do so, they investigated three related investment-consumption problems for a risk-averse investor: (i) an investment-only problem that implies utility from only terminal wealth, (ii) an investment-consumption problem that involves utility from only consumption, and (iii) an extended investment-consumption problem implying utility from both consumption and terminal wealth. Li et al. [31] focused on discrete time rather than the continuous-time frameworks common in the literature of such problem.

As a first contribution, Li et al. [31] introduced a new type of uncertainty: economic environment uncertainty, in addition to the asset return uncertainty. Specifically, they described the economic environment uncertainty by an event tree generated by a finite number of states of nature while the asset return uncertainty will allow for randomness of risky asset returns in any time period and under any given economic state at the beginning of the time period. Thus their contributions was (i) to model these investment-consumption problems using a discrete model that takes into account the environment, mortality and the market risk, (ii) to derive explicit expressions of the op-

timal investment-consumption policies the model problems under study. They tackled the first two model problems by using dynamic programming approach, and the third is solved by using a similar technique in Lakner and Ma-Nygren [30] for a continuous-time investment-consumption problem with known exit time. As a conclusion, they realised that many of their findings are consistent with the well-known findings from the continuous-time models ([5, 6, 37, 38, 47]), even though their models have the extra characteristics of modeling the environment uncertainty and the uncertain exit time.

The common uncertainty considered in the literature is the one due to the economy. Munk [42] used a diffusion model to model the asset price dynamics in which both the drift and diffusion terms are a function of another one-dimensional diffusion process. This one-dimensional diffusion process can be interpreted as a model to the economic uncertainty. In discrete time the economic uncertainty is usually specified by a set of states, each of which is a description of the economic environment for all dates. Moreover, it is commonly assumed that once an economic environment is known, the returns of risky assets in any time period are no longer uncertain. This may be a contradiction to what it commonly observed in practice but in recent years several models have been suggested to tackle this problem. As such, Cheung and Yang [55] suggested the use of the Markovian regime-switching model to capture the economic uncertainty. In their model the underlying economy switches between a finite number of states, and the returns of risky assets during a time period, which depend on the economic state at the beginning of that time period, can still be uncertain (see, Cheung and Yang [55] for further insight).

Now we present some works from the literature that deals with the subject of Taylor series approximations to expected utility and the applicability of the technique to optimal portfolio selection problems by Garlappi and Skoulakis [20].

The use of polynomial approximations for the computation of expected utility is quite broad in the literature (Hlawitschka [22], Loistl [34]). Despite the fact that numerical techniques, such as quadrature and Monte Carlo simulation for approximating integrals have become more sophisticated and powerful since the initial work of Ar-

row [4] and Pratt [46], more recently the subject of polynomial approximations has re-surfaced in the context of dynamic portfolio choice. It turns out that, due to the numerical complexity of these problems, it is often computationally efficient to approximate the utility function by a polynomial obtained via a Taylor expansion. However, the use of Taylor series for the approximation of expected utility is delicate since a few issues must be clarify prior to applying the approximation.

As mentioned in [20], it is important to ensure that a series converges to the exact expected utility as more terms are added. Obviously, convergence depends on the type of utility function: for utility functions such as the exponential, the series converges for all possible level of returns while for other utility functions (power utility), convergence is guaranteed only on a specific range of portfolio returns.

The purpose of Garlappi and Skoulakis [20] paper was to fill the gap observed in the issues that raise from the Taylor's approximation and they did so by establishing a set of conditions under which Taylor series can be used as a sound computational tool for both the evaluation of expected utility of a given portfolio as well as the solution to a portfolio choice problem. The results are found under the assumption of a $HARA$ utility function and a bounded distribution of assets.

Finally, they showed that, when asset returns are skewed, one can improve the precision and efficiency of the Taylor expansion by applying a simple nonlinear transformation to asset returns designed to symmetrize the transformed return distribution and shrink its support.

Now we present some works from the literature that deals with the dynamic consumption and portfolio choice of an investor with habit formation in preferences and access to a complete financial market with time-varying investment opportunities by Munk [41].

Many studies have modeled the dynamics of market prices mostly assuming a power utility of terminal wealth or an additively time separable power utility of consumption. A more plausible representation of preferences is to allow for habit formation in the sense that the utility of a given current consumption level is a decreasing function of the

past consumption level [41]. Munk [41] therefore asked the following questions: what are then the optimal portfolio and consumption strategies for investors with habits for consumption? Do the main qualitative properties of optimal portfolio strategies for investors with standard time-separable preferences carry over to investors with habit formation? Answers to these questions are provided in the same work by Munk [41].

An exact and simple characterization of the optimal behavior under general, possibly non-Markov, dynamics of market prices is derived (see [41]).

With general, possibly non-Markov, dynamics in investment opportunities Munk [41] provided an exact and simple characterization of the optimal consumption and portfolio policies in terms of wealth and habit level and two relatively simple stochastic processes. The optimal portfolio in risky assets is a combination of three portfolios: (i) the standard myopic mean-variance portfolio, (ii) a hedge portfolio providing insurance against adverse movements in investment opportunities as well as variations in future costs of ensuring consumption at the habit level, and (iii) a portfolio ensuring that the agent can consume at least at the habit level in the future. Consequently, the optimal asset mix of an investor with habit formation will differ from that of an investor without habit formation for two reasons: financial assets may differ in their abilities to ensure that future consumption will exceed a certain minimum defined by the habit level and in their abilities to hedge against variations in the habit level.

Finally, Munk [41] studied the optimal consumption and investment strategies with and without habit formation in a model where both the Sharpe ratio of the stock market and the short-term interest rate vary. While this combined model still features a complete market, it does allow for an imperfect correlation between the current price level and the expected return of the stock, in contrast to the Wachter model ([54]) referred to above where interest rates are assumed constant.

1.4   Outline of the thesis

Rest of the thesis is organized as follows:

In Chapter 2, we discuss some applications of the Markov model applied to investment problems.

Chapter 3 deals with a systematic presentation of some of the numerical methods that are used for typical consumption models applied in financial markets.

In Chapter 4, we present the main method, namely, the infinite series expansion method, and then apply it to solve a model problem. We then present and discuss some numerical results and discussion on them in this chapter.

Finally, we conclude this thesis work in Chapter 5, where we also indicate some scope for further research.

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Chapter 2

Some Markov models and their applications

This chapter deals with some consumption models that use the Markov model in particular as a backbone to solve the specific investment problems yielding to interesting conclusions.

2.1 A model of consumption behavior under uncertainty

In this section, we briefly review the problem of consumption behavior under uncertainty developed by Sattinger [49] and an algorithm for its solution. We will not compute the latter nor illustrate it thereof, this will be an overview for the interested reader.

Sattinger [49] argues that, in the Markov consumption problem (hereafter MCP), an individual moves between two states, employment and unemployment, following a continuous-time Markov chain process. The individual earns different incomes in the two states and at each point in time must determine the level of accumulation or decumulation of an asset that yields a constant interest rate [49]. Also a constant

relative risk aversion ($CRRA$) utility function is assumed in this case.

The idea in this model is to develop a method for generating a numerical solution of the differential equations and maximize the expected value, both of which will be discussed in the next section, using consumption levels near the singularity. The approach is formulated by setting the problem in continuous time, applying methods of differential equations instead of dynamic programming.

The next section presents the differential equation model for consumption in the two states and shows some analytic results.

Model setup and solution to the model

Assumption 2.1.1. [49] *An individual moves between two states of a continuous-time Markov process. Let p_1 be the transition rate from state 1 to state 2, and let p_2 be the transition rate from state 2 to state 1. The individual earns income at the rate y_i when in state i. Let $A[t]$ be the individual's assets at time t. The individual earns income from assets at the rate rA, where r is positive and constant over time. Let $C_i[A]$ be the consumption rate chosen if the individual is currently in state i with assets A. If the individual is in state i, the rate of change of assets is*

$$\frac{dA}{dt} = rA + y_i - C_i[A], \; i = 1, 2. \tag{2.1.1}$$

The consumer's instantaneous, time-separable utility takes the Constant Relative Risk Aversion ($CRRA$) form

$$U[C] = \lambda^{-1} C^\lambda, \quad 0 < \lambda < 1. \tag{2.1.2}$$

Future utility is discounted at the rate b. The individual may borrow against future income but may not default on borrowed funds. With $i[t]$ denoting the state the individual is in at time t, the individual seeks to maximize the expected value

$$E_0 \int_0^\infty e^{-bt} \lambda^{-1} C_{i[t]}^\lambda dt,$$

subject to the budget constraint in equation (2.1.1) that determines changes in assets over time, the no-Ponzi condition that the individual may not default on debt, and the stochastic transitions between state 1 (employment) and state 2 (unemployment) [49]. Assume $y_1 > y_2$ and let $V_i[A]$ be the value function for the individual in state i with asset level A in current value form [49]. Then [49]

$$bV_1[A] = Max_{C_1}U[C_1] + p_1(V_2[A] - V_1[A]) + (rA + y_1 - C_1)V_{1A}, \qquad (2.1.3)$$

$$bV_2[A] = Max_{C_2}U[C_2] + p_2(V_1[A] - V_2[A]) + (rA + y_2 - C_2)V_{2A}. \qquad (2.1.4)$$

Under the assumption stated above, with $0 < \lambda < 1$, $y_1 > y_2 \geq 0$ and $p_i \geq 0$, $i = 1, 2$, the optimal consumption levels satisfy the following differential equations:

$$\frac{dC_1}{dA} = \frac{(C_1/(1 - \lambda))(r - b - p_1(1 - (C_1/C_2)^{1-\lambda}))}{rA + y_1 - C_1}, \qquad (2.1.5)$$

$$\frac{dC_2}{dA} = \frac{(C_2/(1 - \lambda))(r - b - p_2(1 - (C_2/C_1)^{1-\lambda}))}{rA + y_2 - C_2}. \qquad (2.1.6)$$

The above assumption is of great importance since it allows us to have a good understanding of what will follow as we will see below.

Note: If $p_1 = p_2 = 0$ in equations (2.1.5) and (2.1.6), the differential equations generate the standard risk-free solution [49]

$$C_i = \frac{b - r\lambda}{1 - \lambda}(rA + y_i).$$

Huggett [23] previously showed that there will be a break-even point in state 1 when a condition equivalent to $b > r$ holds. If we let A_{min} be the lowest allowable asset level, $p_1, p_2 > 0$, with $b > r$ there will be a break-even point at an asset level A_s in the interval (A_{min}, ∞) such that $C_1 = rA_s + y_1$ in state 1, and at A_s, $(r-b)V_{1A} + p_1(V_{2A} - V_{1A}) = 0$,

the differential equation (2.1.5) will have a singularity, and

$$C_2 = C_1 \left(\frac{p_1}{p_1 + b - r} \right)^{1/(1-\lambda)}. \qquad (2.1.7)$$

There will be no singularity or break-even point in state 2 for assets in the interval (A_{min}, ∞).

In what follows, we present the iterative method of deriving a numerical solution to the MCP elaborated by Sattinger [49].

The method shown by Sattinger [49] involves choosing consumption levels in state 1 and state 2 at an asset level near a singularity such that consumption in state 2 approaches zero as assets approach the minimal level. The search for initial conditions then reduces to finding the scalar value of assets at which a singularity occurs.

Given numerical values for all parameters, the steps in the numerical solution as describe in [49] are as follows:

1. Pick an arbitrary $A_s > A_{min}$.

2. Calculate $C_1[A_s]$ as $rAs + y_1$, $C_2[As]$ as $C_1[As](p_1/(p_1+b-r))^{1/(1-\lambda)}$ from equation (2.1.7), dC_1/dA from the positive analytic solution for dC_1/dA , and dC_2/dA from equation (2.1.6).

3. For a small ϵ, calculate $C_1[A_s - \epsilon]$ as $rA_s + y_1 - \epsilon(dC_1/dA)$ and $C_2[A_s - \epsilon]$ as $(rA_s + y_1)(p_1/(p_1 + b - r))^{1/(1-\lambda)} - \epsilon(dC_2/dA)$.

4. Using $C_1[A_s - \epsilon]$ and $C_2[A_s - \epsilon]$ at $A_s - \epsilon$ as initial conditions, solve the differential equations (2.1.5) and (2.1.6) backwards to lower asset levels.

5. Let η be a small increment of assets consistent with accuracy goals. If $A_{min} = -y_2/r$, then from Result 2, C_2 at $A_{min} + \eta$ is approximately $C_{2\eta} = \eta(b + p_2 - r\lambda)/(1 - \lambda)$. If assets in the solution in 4 reach $A_{min} + \eta$ before C_2 reaches $C_{2\eta}$, choose a higher A_s. At a higher A_s, the curves for C_1 and C_2 will be below the

curves at the former A_s, so C_2 at $A_{min} + \eta$ will be lower. If the solution in 4 does not reach $A_{min} + \eta$ before C_2 reaches $C_{2\eta}$, choose a lower A_s. At a lower A_s, the curves for C_1 and C_2 will be higher than the curves at the former A_s, so C_2 reaches $C_{2\eta}$ at a lower asset level.

6. Repeat 2 through 5 until a desired accuracy is reached.

7. With the solution from 6, use $C_1[A_s - \epsilon]$ and $C_2[A_s - \epsilon]$ as initial conditions for the numerical solution of the differential equations below A_s. Use $C_1[A_s + \epsilon] = rA_s + y_1 + \epsilon(dC_1/dA)$ and $C_2[A_s + \epsilon] = (rA_s + y_1)(p_1/(p_1 + b - r))1/(1 - \lambda) + \epsilon(dC_2/dA)$ as initial conditions for the numerical solution above A_s.

This concludes the steps in elaborating an explicit solution to a consumption behavior problem under uncertainty developed by Sattinger [49].

2.2 A model of consumption optimization with transaction costs

Model setup and solution to the model

In this section we consider the work of Li [32], on transaction costs and consumption.

Li [32] assumed that the consumer has an infinite lifespan in a cash-in-advance economy. The consumer can hold either cash or assets. The nominal interest rate for cash is zero, but assets bear a risk free real return, r. The nominal interest rate, i, is equal to $r + \pi$, where π is the inflation rate. The following notations are used in the model ([32]):

- C_t is the consumption in period t,

- θ_t is a binary variable that indicates whether the consumer pays the transaction cost,

- $\theta_t = 1$ if the consumer does and $\theta_t = 0$ if the consumer does not,

- M_t is the real cash holding before receiving the labor income,

- A_t is the real balance of the asset account,

- Δ_t is the real value of labor income,

- ρ is the coefficient of relative risk aversion (CRRA),

- i is the nominal interest rate,

- π is the inflation rate,

- $r = i - \pi$ is the real interest rate, and

- ψ is the transaction cost.

Having described all parameters, Li [32] formulated an optimization problem that the consumer has to solve in order to distribute his cash and the equation is as follows

$$\max_{\theta_t, C_t, M_{t+1}} E_0 \sum_{0}^{\infty} \beta^t U(C_t), \tag{2.2.1}$$

subject to

$$\text{if } \theta_t = 0 \text{ then } \begin{cases} A_{t+1} = (1 + i - \pi) A_t, \\ \\ M_{t+1} = (1 - \pi)(M_t + \Delta_t - C_t), \end{cases} \tag{2.2.2}$$

and

$$\text{if } \theta_t = 1 \text{ then } \begin{cases} A_{t+1} = (1 + i - \pi) \left(A_t M_t + \Delta_t - C_t - \frac{M_{t+1}}{1-\pi} - \psi \right), \\ \\ M_{t+1} \text{ is to be determined by the consumer,} \end{cases} \tag{2.2.3}$$

where

$$C_t \leq M_t + \Delta_t \text{ if } \theta_t = 0, \tag{2.2.4}$$

and

$$\lim_{t \to 0} \frac{A_t}{(1+r)^t} \geq 0. \tag{2.2.5}$$

Because the three control variables are jointly determined at time t, the maximization problem is formulated with respect to the vector $[C_t, \theta_t, M_{t+1}]$ as in [32]. Assuming a $CRRA$ type of utility function, $U(C) = (C^{1-\rho} - 1)/(1 - \rho)$, the optimization is represented as the following Bellman equation as in [32]:

$$V(A_t, M_t, \Delta_t) = \max[V^{NT}(A_t, M_t, \Delta_t), V^{TR}(A_t, M_t, \Delta_t)], \qquad (2.2.6)$$

where

$$V^{NT}(A_t, M_t, \Delta_t) = \max_{C_t} U(C_t) + \beta E_{\Delta_{t+1}} V(A_{t+1}, M_{t+1}, \Delta_{t+1}), \qquad (2.2.7)$$

subject to constraint equations (2.2.2), (2.2.4), (2.2.5) and

$$V^{TR}(A_t, M_t, \Delta_t) = \max_{C_t, M_{t+1}} U(C_t) + \beta E_{\Delta_{t+1}} V(A_{t+1}, M_{t+1}, \Delta_{t+1}), \qquad (2.2.8)$$

subject to constraint equations (2.2.3) and (2.2.5). The system has three value functions V^{NT} (corresponding to $\theta = 0$), V^{TR} (corresponding to $\Theta = 1$), and V, the upper contour of V^{NT} and V^{TR} (see [32] for figure), is the value function of the whole optimization problem. The consumption policy functions for (2.2.7) and (2.2.8) are denoted as $C^{NT}(A_t, M_t, \Delta_t)$ and $C^{TR}(A_t, M_t, \Delta_t)$ respectively [32]. The consumer decides θ_t and C_t according to ([32]):

$$\theta_t = \begin{cases} 1 & \text{if } V^{TR}(A_t, M_t, \Delta_t) > V^{NT}(A_t, M_t, \Delta_t), \\ 0 & otherwise, \end{cases} \qquad (2.2.9)$$

and correspondingly

$$C(A_t, M_t, \Delta_t) = \begin{cases} C^{TR}(A_t, M_t, \Delta_t) & \text{if } \theta = 1, \\ C^{NT}(A_t, M_t, \Delta_t) & \text{if } \theta = 0. \end{cases} \qquad (2.2.10)$$

Furthermore Li [32] argued that the motion of state variables A_t and M_t follows

equations (2.2.2) and (2.2.3). Equation (2.2.2) shows that when $\theta_t = 0$, asset A_t is intact in period t, and accrues real interest, r, or $i - \pi$. A_{t+1} is simply equal to $(1 + i - \pi)A_t$ ([56]). Equation (2.2.3) shows that after making the consumption expenditure, the consumer will carry a real cash balance $(1 - \pi)(M_t + \Delta_t - C_t)$ into the period $t + 1$. When $\theta_t = 1$, consumer will either withdraw cash or save cash [32]. If the consumer chooses to consume the amount C_t and to carry a real cash balance, M_{t+1}, into the next period, $A_t + M_t + \Delta_t - C_t - M_{t+1}/(1 - \pi) - \Psi$ will be left in the asset account. In this case, M_{t+1} becomes a control variable that has to be pinned down by the consumer.

In what follows, we present the results to the model introduced by Li [32] which somehow did not have a closed form solution. Li [32] simulated the model using the converged distribution of the state variables. He chose some baseline parameters on a quarterly basis (see [32] table 2) to simulate the model.

Li [32] then focused his paper on consumption's response to two types of income shocks, unanticipated transitory income shocks and news about one-shot future income changes. Therefore he simply assumed that income shocks are drawn from an independent and identically distributed process. In so doing, a positive probability is assumed, which in the base line parametrization is equal to 1%, in each period that the consumer will receive zero income [32]. If the investor does receive a positive income in a given period, the income is drawn from a log-normal distribution with unit mean ($\mu_\Delta = 1$) and standard deviation (σ_Δ) equal to 0.1 as mentioned in [32].

Following Carroll [10], r is set to be equal to 0.015, yielding a 6% annual rate, and set β at 0.975, which gives an annual discount rate of 10%. More over, following the calibration in Aiyagari and Gertler [1] ψ is assumed to be equal to 0.01.

In his research, Li [32] computed the model using the grid searching method. He iterated the Bellman equations system (2.2.6)-(2.2.8) until the average absolute gap between the policy functions of two consecutive iterations becomes sufficiently small (< 0.0001) as in [32]. Adopting the algorithm introduced in Carroll [11], he discretized the specified log-normal income distribution using 99 grid points. A plot of the con-

sumption function, projected onto the hyperplane of $A = 0.5$ is established (see figure 3 in [32]).

For comparison, he ploted a consumption function of the model with no transaction cost (see the same figure 3 in [32]). The most striking feature of the consumption function with transaction costs is that it is neither continuous nor monotonic.

2.3   A stochastic optimization investment-consumption model

In this section, we consider the work of Alghalith [2] presented in his paper on stochastic investment-consumption model. We present the model and its solution as introduced by Alghalith [2].

In the previous studies in this area, the usual assumption is that the parameters of the model depend on a random external economic factor (stochastic volatility models) in incomplete markets; examples include Focardi and Fabozzi [19], Pham [44] and Liu [33]. In his works, Alghalith [2] derived a general explicit solutions to the investment-consumption model without the restrictive assumption of $HARA$ or exponential utility function and without relying on the existing duality or variational methods.

The model and its solution

The model of Alghalith [2] uses a two-dimensional standard Brownian motion $\{W_{1s}, W_{2s}, F_s\}_{t<s<T}$ based on the probability space (Ω, F_s, P) , where $\{F_s\}_{t<s<T}$ is the augmentation of filtration. Similar to previous models, he considered a risky asset, a risk-free asset and a random external economic factor. The risk-free asset price process is given by $S_0 = e^{\int_t^T r(Y_s)ds}$, where $r(Y_s) \in C_b^2(\mathcal{R})$ is the rate of return and Y_s is the economic factor [2].

The dynamics of the risky asset price are given by [2]

$$dS_s = S_s\{\mu(Y_s)d_s + \sigma(Y_s)dW_{1s}\},$$

where $\mu(Y_s)$ and $\sigma(Y_s)$ are the rate of return and the volatility, respectively.

The economic factor process is given by

$$dY_s = g(Y_s)d_s + \rho dW_{1s} + \sqrt{1-\rho^2}dW_{2s}, \quad Y_t = y,$$

where $|\rho| < 1$ is the correlation factor between the two Brownian motions and $g_{Y_s} \in C^1(\mathcal{R})$ with a bounded derivative [2].

The wealth process is given by

$$X_T^{\pi,c} = x + \int_t^T \{r(Y_s)X_s^{\pi,c} + (\mu(Y_s) - \sigma(Y_s))\pi_s - c_s\}ds + \int_t^T \pi_s\sigma(Y_s)dW_{1s},$$

where x is the initial wealth, $\{\pi_s, F_s\}_{t<s<T}$ is the portfolio process and $\{c_s, F_s\}_{t<s<T}$ is the consumption process, with $\int_t^T \pi_s^2 ds < \infty$, $\int_t^T c_s^2 ds < \infty$, and $c \geq 0$ as in [2]. The trading strategy $(\pi_s, c_s) \in \mathcal{A}(x,y)$ is admissible (that is, $X_s^{\pi,c} \geq 0$). Define $\theta(Y_s) \equiv \sigma^{-1}(Y_s)(\mu(Y_s) - r(Y_s))$.

The investor's objective is to maximize the expected utility of the terminal wealth and consumption

$$V(t, x, \theta(y)) = Sup E_{\pi,c}\left[U^1(X_T^{\pi,c}) + \int_t^T U^2(c_s)ds|F_t\right], \quad (2.3.1)$$

where $V(\cdot)$ is the indirect utility function, $U(\cdot)$ is continuous, bounded and strictly concave utility function [2].

At this point, we now present the results to the model introduced by Alghalith [2].

He started off by rewriting equation (2.3.1) as

$$V(\cdot) = SupE_{\pi,c} \left[U^1 \left(x + \int_t^T \{rX^{\pi,c} + (\mu - r)\pi - ac + b\}ds + \int_t^T \pi\sigma dW_s^1 \right) \right. \quad (2.3.2)$$
$$\left. + \int_t^T U^2(c)ds | F_t \right],$$

where a is a shift parameter with initial value equals one, b is a shift parameter with initial value equals zero [2]. Differentiating both sides of equation (2.3.2) with respect to μ and b, respectively, he obtained

$$V_\mu(\cdot) = (T - t)\pi_t^* E \left[U^{1'}(\cdot) | F_t \right],$$
$$V_b(\cdot) = (T - t)E \left[U^{1'}(\cdot) | F_t \right],$$

where the subscripts denote partial derivatives; thus

$$\pi_t^* = \frac{V_\mu(\cdot)}{V_b(\cdot)}, \quad (2.3.3)$$

where * denotes the optimal value. Similarly he obtained

$$V_a = -(T - t)c_t^* E \left[U^{1'}(\cdot) | F_t \right], \quad (2.3.4)$$

and hence

$$c_t^* = -\frac{V_a(\cdot)}{V_b(\cdot)}. \quad (2.3.5)$$

Consider the following nth-order exact Taylor expansion of $V(\cdot)$

$$V(x, \theta, a, b) = V + V_x x + V_\theta \theta + V_a a + V_b b + ... + \frac{1}{n!} \sum_{n_1,n_2,n_3,n_4} \frac{\delta^n V}{\delta x^{n_1} \delta \theta^{n_2} \delta a^{n_3} \delta b^{n_4}} x^{n_1} \theta^{n_2} a^{n_3} b^{n_4}.$$
$$(2.3.6)$$

Differentiating equation (2.3.6) with respect to μ, a and b (around $b = 0$ and $a = 1$),

respectively, we see that

$$V_\mu(\cdot) = \sigma^{-1}(y)\left(V_\theta + ... + \frac{1}{n!}\sum_{n_1,n_2}\frac{\delta^n V}{\delta x^{n_1}\delta\theta^{n_2}\delta a^{n_3}\delta b^{n_4}}x^{n_1}\theta^{n_2-1}\right), \qquad (2.3.7)$$

$$V_a(\cdot) = V_a + ... + \frac{1}{n!}\sum_{n_1,n_2}\frac{\delta^n V}{\delta x^{n_1}\delta\theta^{n_2}\delta a^{n_3}\delta b^{n_4}}x^{n_1}\theta^{n_2}, \qquad (2.3.8)$$

$$V_b(\cdot) = V_b + ... + \frac{1}{n!}\sum_{n_1,n_2}\frac{\delta^n V}{\delta x^{n_1}\delta\theta^{n_2}\delta a^{n_3}\delta b^{n_4}}x^{n_1}\theta^{n_2}. \qquad (2.3.9)$$

All the derivatives in the right-hand side of equations (2.3.7)-(2.3.9) are constant and therefore for a constant α we have [2]

$$V_\mu(\cdot) = \sigma^{-1}(y)\left(\alpha_0 + ... + \sum_{n_1,n_2}\alpha_{n_1,n_2}x^{n_1}\theta^{n_2-1}\right),$$

$$V_a(\cdot) = \alpha_1 + ... + \sum_{n_1,n_2}\alpha_{n_1,n_2}x^{n_1}\theta^{n_2},$$

$$V_b(\cdot) = \alpha_2 + ... + \sum_{n_1,n_2}\alpha_{n_1,n_2}x^{n_1}\theta^{n_2}.$$

Consequently, using equations (2.3.3) and (2.3.5), he obtained

$$\pi_t^* = \sigma^{-1}(y)\frac{\alpha_0 + ... + \sum_{n_1,n_2}\alpha_{n_1,n_2}x^{n_1}\theta(y)^{n_2-1}}{\alpha_2 + ... + \sum_{n_1,n_2}\alpha_{n_1,n_2}x^{n_1}\theta(y)^{n_2}}, \qquad (2.3.10)$$

$$c_t^* = -\frac{\alpha_1 + ... + \sum_{n_1,n_2}\alpha_{n_1,n_2}x^{n_1}\theta(y)^{n_2}}{\alpha_2 + ... + \sum_{n_1,n_2}\alpha_{n_1,n_2}x^{n_1}\theta(y)^{n_2}}. \qquad (2.3.11)$$

Equations (2.3.10) and (2.3.11) are the optimal strategy an investor will use (c_t^* for the consumption process and π_t^* for the portfolio process) in order to maximize her investment-consumption without the restrictive assumption of $HARA$.

2.4   Summary and discussions

In this chapter we presented a few applications of the Markov model to different investment-consumption problems. The Markov model is indeed a powerful tool in mathematical finance and it is used by many researchers in investment consumption problems to derive explicit solutions to a specific model. We could recommend the Markov model to the interested researchers as cornerstone for future research. In the next chapter, we discuss some methods for solving optimal consumption models.

Chapter 3

Methods for solving consumption models

In previous chapter we presented some Markov models and their solutions. In this chapter we present other methods used to solve optimal consumption models.
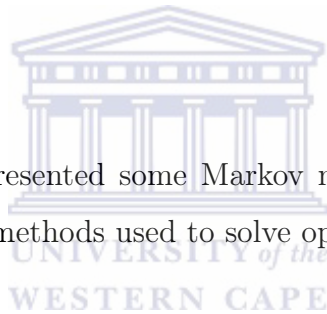
3.1 Markov chain approximation method

Kushner and Dupuis [29] introduced the approximating Markov chain approach and is very well detailed in the literature. In the following subsections we revisit the theory that is used for this method and derive the formulae for the optimal consumption problem and give a description on how the approximate Markov solution is obtained. To help us in our task, the approach of Fitzpatrick and Fleming [17] in developing the approximation is followed.

Given $h \leq 0$ and a positive integer N, the introduction of a grid $S_h^N = \{ih : 0 \leq i \leq N\}$, which shall be the state space for the Markov chain is needed. The

control-dependent transition probabilities are taken to be

$$P_{ii+1}^{\pi,\phi,c} = \left\{ \frac{1}{2}\sigma^2\pi^2 + h[(b-r)\pi + r(hi)] \right\} / Q,$$

$$P_{ii-1}^{\pi,\phi,c} = \left\{ \frac{1}{2}\sigma^2\pi^2 + h[(R-r)\phi + c)] \right\} / Q,$$

$$P_{ii}^{\pi,\phi,c} = 1 - P_{ii+1}^{\pi,\phi,c} - P_{ii-1}^{\pi,\phi,c},$$

for $1 \leq i \leq N-1$ and $(\pi,\phi,c) \in \Gamma_{i,h}^N$ as in [17], where

$$\Gamma_{i,h}^N = \{(\pi,\phi,c) : 0 \leq \pi,\phi,c \leq KNh, \pi - ih \leq \phi\},$$

is the set of admissible control values [17]. The constant K gives an artificial bound on the controls. The boundary probabilities are taken as in [17]

$$P_{00}^{\pi,\phi,c} = 1,$$

$$P_{NN-1}^{\pi,\phi,c} = \left\{ \frac{1}{2}\sigma^2\pi^2 + h[(R-r)\phi + c)] \right\} / Q$$

$$\text{and} P_{NN}^{\pi,\phi,c} = 1 - P_{NN-1}^{\pi,\phi,c}.$$

The normalizing constant Q is taken to be ([17])

$$Q = \max_{\substack{0 \leq \pi,\phi,c \leq KNh \\ 0 \leq i \leq N}} \left\{ \sigma^2\pi^2 + h\left[(b-r)\pi + rih + (R-r)\phi + c\right] \right\},$$

$$= \sigma^2(KNh)^2 + Nh^2\left[(b-r)h + r + (R-r)K + K\right].$$

When dealing with convergence questions, it shall be required that $h \to 0$ and $Nh \to \infty$ as well. This will ensure a total capture of all of the behavior of the continuous problem with the discretized approximation [17]. Kushner [28, 29] showed in his works that, if the time scale is discretized with $\Delta t = h^2/Q$, then the Markov chain with the above transition probabilities will have first and second moments which closely match those of the continuous process. The control problem for the chain is then to determine controls $\{(\pi_k, \phi_k, c_k) : k \geq 0\}$ so that $E_{ih}\left\{\sum_{k=0}^{\infty} \Delta t e^{-\beta t \Delta t} U(c_k)\right\}$ is maximized. The

discrete HJB equation for this problem is therefore

$$V_i = \max_{(\pi,\phi,c)\in\Gamma_{i,h}^N} \left\{ e^{-\beta\Delta t}\sum_j P_{ij}^{\pi,\phi,c}V_j + \Delta t U(c) \right\}, \tag{3.1.1}$$

for $1 \leq i \leq N$ with $V_0 = 0$. Expanding this equation in terms of the prescribed transition probabilities shows that, for $1 \leq i \leq N-1$, hence

$$
\begin{aligned}
V_i ={} & \max_{(\pi,\phi,c)\in\Gamma_{i,h}^N} \left\{ \frac{e^{\beta\Delta t}}{Q}\left[\frac{1}{2}\sigma^2\pi^2(V_{i+1}+V_{i-1}-2V_i)+h(b-r)\pi(V_{i+1}-V_i)\right.\right. \\
& - h(R-r)\phi(V_i-V_{i-1}) - hc(V_i-V_{i-1})] + \Delta t U(c)\} \\
& + rih^2(V_{i+1}-V_i)\frac{e^{-\beta\Delta t}}{Q} + e^{-\beta\Delta t}V_i \\
={} & \max_{\substack{0\leq\pi,\phi,\leq KNh \\ \pi-ih\leq\phi}} \left\{ \frac{e^{\beta\Delta t}}{Q}\left[\frac{1}{2}\sigma^2\pi^2(V_{i+1}+V_i-2V_i)+h(b-r)\pi(V_{i+1}-V_i)\right.\right. \\
& - h(R-r)\phi(V_i-V_{i-1})]\} + e^{-\beta\Delta t}rih^2(V_{i+1}-V_i)/Q \\
& + e^{-\beta\Delta t}V_i + \max_{0\leq c\leq KNh}\left[\Delta t U(c) - \frac{e^{-\beta\Delta t}}{Q}hc(V_i-V_{i-1})\right].
\end{aligned}
$$

As in [17], we note that

$$D^+V_i = (V_i+1-V_i)/h,$$

$$D^-V_i = (V_i-V_{i-1})/h,$$

$$D^2V_i = (V_{i+1}+V_{i-1}-2V_i)/h^2 \text{ and}$$

$$F_{N,h}(y) = \max_{0\leq c\leq KNh}\left[U(c) - e^{-\beta\Delta t}\cdot c\cdot y\right].$$

From Assumption 1.2.7 in chapter 1, we see that $F_{N,h}$ is a strictly decreasing function

of y. Using these definitions, in addition to the fact that $\Delta t = h^2/Q$, yields

$$
\begin{aligned}
V_i \left(\frac{1 - e^{-\beta \Delta t}}{\Delta t} \right) = & \max_{\substack{0 \leq \pi, \phi, \leq KNh \\ \pi - ih \leq \phi}} \left\{ e^{-\beta \Delta t} \left[\frac{1}{2} \sigma^2 \pi^2 D^2 V_i + (b - r)\pi D^+ V_i \right. \right. \\
& \left. \left. + (R - r)\phi D^- V_i \right] \right\}
\end{aligned}
$$

$$+ e^{-\beta \Delta t} r i h D^+ V_i + F_{N,h}(D^- V_i), \quad 1 \leq i \leq N - 1, \quad (3.1.2)$$

an equation which is a finite difference approximation of the continuous HJB equation (1.2.11) [17]. Note that, for $i = N$, we get a slightly different form ([17]):

$$
\begin{aligned}
V_N = & \max_{\substack{0 \leq \pi, \phi, \leq KNh \\ \pi - Nh \leq \phi}} \left[\frac{e^{-\beta \Delta t}}{Q} \left(\frac{1}{2} \sigma^2 \pi^2 (V_{N-1} + V_N) + h(R - r)\pi (V_{N-1} - V_N) \right) \right] \\
& + e^{-\beta \Delta t} V_N + \max_{0 \leq c \leq KNh} \left[\Delta t U(c) - \frac{e^{-\beta \Delta t}}{Q} hc(V_N - V_{N-1}) \right].
\end{aligned}
$$

Later, it shall be shown that $V_{N-1} - V_N \leq 0$, so that this equation reduces to

$$\frac{1 - e^{-\beta \Delta t}}{\Delta t} V_N = F_{N,h}(D^- V_N), \quad (3.1.3)$$

which resembles a boundary condition of $\beta V = F(V_x)$ at the right endpoint. In order to study these equations and their solutions, define the set

$$X^h = \left\{ V_i \in \mathbb{R}^{N+1} : V_0 = 0, \ V_i \geq 0 \ \text{for} \ 1 \leq i \leq N \right\},$$

and the map $X^h \to X^h$ (as in [17]) by

$$
(T_h V)_i = \begin{cases} \max_{\substack{0 \leq \pi, \phi, c \leq NKh \\ \pi - ih \leq \phi}} \left\{ e^{-\beta \Delta t} \sum P_{ij}^{\pi, \phi, c} V_j + \Delta t U(c) \right\} & 1 \leq i \leq N, \\ 0 & i = 0. \end{cases} \quad (3.1.4)
$$

The solution of the discrete HJB equation is, of course, a fixed point of T_h. To establish the existence of such a fixed point, it is important to recall the following result.

Proposition 3.1.1. [17] *The mapping T_h is a contraction on X^h, when X^h is equipped with the supremum norm $\|V\|_h = \sup_{1 \leq i \leq N} |V_i|$.*

Since T_h is a contraction, it has a unique fixed point in X^h. Denote this fixed point by V^h. In the sequel, we shall want to work with V^h as a function on $[0, \infty)$, and therefore we introduce the continuous linear interpolation

$$
v^h(x) = \begin{cases} V_i^h & x = ih, \\ V_i^h + h^-1(V_{i+1}^h - V_i^h)(x - ih) & ih < x < (i+1)h, \\ V_h^N & x \geq Nh. \end{cases}
$$

This is the interpolation that Fleming and Fitzpatrick [17] used to approximate the value function of the continuous problem in their paper. Furthermore, a discussion of some of the properties of the solution V^h of the discrete HJB equation follows.

Property 3.1.2. [17] *The function $v^h(x)$ is nondecreasing on $[0, \infty)$.*

Proof. We show that T_h preserves nondecreasing functions, which implies that the fixed point must be nondecreasing. Let $V \in X^h$ be nondecreasing; that is, $V_i \leq V_{i+1}$ for each i. Suppose that $1 \leq i \leq N - 2$, and choose π^*, ϕ^*, c^* so that they attain the maximum in

$$
T_h V_i = \max_{\substack{0 \leq \pi, \phi, c \leq NKh \\ \pi - ih \leq \phi}} \left[e^{-\beta \Delta t} \sum_j P_{ij}^{\pi, \phi, c} V_j + \Delta t U(c) \right].
$$

We write $P_{ij}^* = P_{ij}^{\pi^*, \phi^*, c^*}$ to simplify the notation.

$$
\begin{aligned}
T_h V_{i+1} - T_h V_i &= \max_{\substack{0 \leq \pi, \phi, c \leq NKh \\ \pi - (i+1)h \leq \phi}} \left[e^{-\beta \Delta t} \sum_j P_{i+1j}^{\pi, \phi, c} V_j + \Delta t U(c) \right] \\
&\quad - \left[e^{-\beta \Delta t} \sum_j P_{ij}^* V_j + \Delta t U(c^*) \right] \\
&\geq e^{-\beta \Delta t} \sum (P_{i+1j}^* - P_{ij}^*) V_j.
\end{aligned}
$$

Examining the definitions of the P_{ij}'s, we see $P_{1+1i+2}^* = P_{ii+1}^* + h^2 r/Q$, $P_{1+1i+2}^* = P_{ii-1}^*$

$+ h^2 r / Q$ and $P^*_{i+1i} = P^*_{ii-1}$, so that

$$
\begin{aligned}
T_h V_{i+1} - T_h V_i \; &\geq \; \Big[P^*_{ii+1}(V_{i+2} - V_{i+1}) + P^*_{ii}(V_{i+1} - V_i) \\
&\quad + P^*_{ii-1}(V_i - V_{i-1}) + \frac{h^2 r}{Q}(V_{i+2} - V_{i+1}) \Big] e^{-\beta \Delta t} \\
&\geq \; 0 \; \text{ if V is increasing.}
\end{aligned}
$$

Of course, $V_i \geq 0$, so that $V_1 \geq V_0 = 0$. We must still check $T_h V_{N-1} \leq T_h V_N$, if $V_{N-1} < V_N$.

$$
\begin{aligned}
T_h V_N - T_h V_{N-1} \; &= \; \max_{\pi,\phi,c} \Big[\sum P^{\pi,\phi,c}_{Nj} V_j e^{-\beta \Delta t} + \Delta t U(c) \Big] \\
&\quad - \max_{\pi,\phi,c} \Big[\sum P^{\pi,\phi,c}_{N-1j} V_j e^{-\beta \Delta t} + \Delta t U(c) \Big] \\
&\geq \; e^{-\beta \Delta t} \Big[P^*_{NN-1} V_{N-1} + (1 - P^*_{NN-1}) V_N - P^*_{N-1N-2} V_{N-2} \\
&\quad - P^*_{N-1N-1} V_{N-1} - P^*_{N-1N} V_N \Big],
\end{aligned}
$$

where π^*, ϕ^*, c^* is optimal for state $N - 1$.

Writing

$$
P_{NN+1} = \left(\frac{1}{2}\sigma^2 \pi^{*2} + h(b-r)\pi^* + hr(Nh) \right) / Q,
$$

$$
P_{NN} = 1 + P_{NN+1} + P_{NN-1},
$$

we have

$$
\begin{aligned}
T_h V_N - T_h V_{N-1} \; &\geq \; e^{-\beta \Delta t} \Big[P^*_{NN-1} V_{N-1} - P^*_{N-1N-2} V_{N-2} + \hat{P}_{NN} V_N - P^*_{N-1N-1} V_{N-1} \\
&\quad + \hat{P}_{NN+1} V_N + P^*_{N-1N} V_N \Big].
\end{aligned}
$$

As before, we see that

$$P_{NN+1} = \frac{h^2 r}{Q} P^*_{N-1N},$$

$$P_{NN+1} = \frac{-h^2 r}{Q} P^*_{N-1N-1},$$

$$P^*_{NN-1} = P^*_{N-1N-2}.$$

Hence, if V is nondecreasing, we have

$$T_h V_N - T_h V_{N-1} \geq e^{-\beta \Delta t} \left[P^*_{NN-1}(V_{N-1}VN-2) + P^*_{N-1N-1}(V_N VN-1) - \frac{h^2 r}{q} V_N \right] \geq 0.$$

Therefore, the unique fixed point of T_h must be nondecreasing.   □

Property 3.1.3. [17] *If $\beta > b$, $v^h(x)$ is concave.*

Proof. We work backwards (in x), showing $D^2 V_i = (V_{i+1} - 2V_i + V_{i-1})/h^2 \leq 0$ for each $i = 1, ..., N-1$. Now

$$V_N = \max_{\pi,\phi,c} \left[\sum P^{\pi,\phi,c}_{Nj} V_j e^{-\beta \Delta t} + \Delta t U(c) \right].$$

Suppose that $D^2 V_{N-1} > 0$. Then from equation (3.1.3), we have

$$
\begin{aligned}
\frac{1 - e^{-\beta \Delta t}}{\Delta t}(V_N - V_{N-1}) = & -\max_{\substack{0 \leq \pi,\phi, \leq NKh \\ \pi-(N-1)h \leq \phi}} \left[e^{-\beta \Delta t} \left(\frac{1}{2}\pi^2 \sigma^2 D^2 V_{N-1} + (b-r)\pi D^+ V_{N-1} \right. \right. \\
& \left. \left. - (R-r)\phi D^- V_{N-1} \right) \right] - e^{-\beta \Delta t} rh(N-1) D^+ V_{N-1} \\
& + F_{N,h}(D^- V_N) - F_{N,h}(D^- V_{N-1}) \\
< & \ 0 + F_{N,h}(D^- V_N) - F_{N,h}(D^- V_{N-1}) \\
< & \ 0,
\end{aligned}
$$

since $D^2 V_{N-1} > 0$ and $F_{N,h}$ is a decreasing function. Hence, $V_N - V_{N-1} < 0$, a contradiction. Thus, we must have $D^2 V_{N-1} \leq 0$. Similarly, suppose $1 < i < N-2$,

and using backward induction on i, assume that $D^2 V_{i+1} < 0$. Then

$$
\begin{aligned}
\frac{1 - e^{-\beta \Delta t}}{\Delta t}(V_{i+1} - V_i) = & \max_{\pi, \phi} \left[e^{-\beta \Delta t} \left(\frac{1}{2} \pi^2 \sigma^2 D^2 V_{i+1} \right. \right. \\
& \left. \left. + (b-r)\pi D^+ V_{i+1} - (R-r)\phi D^- V_{i+1}) \right] \right. \\
& - \max_{\pi, \phi} \left[D^+ V_{i+1} \left(\frac{1}{2} \pi^2 \sigma^2 D^2 V_i \right. \right. \\
& \left. \left. + (b-r)\pi D^+ V_i - (R-r)\phi D^- V_i) \right] \right. \\
& + \left(r(i+1)h D^+ V_{i+1} - r(ih) D^+ V_i \right) e^{-\beta \Delta t} \\
& + F_{N,h}(D^- V_{i+1}) - F_{N,h}(D^- V_i).
\end{aligned}
$$

Let π^*, ϕ^* be optimal for $i + 1$. Choose $\Theta \in [0, 1]$ such that π^* - Θh and ϕ^* are admissible for i.

Put $\hat{\pi} = \pi^* - \Theta h \, (\geq 0)$ then

$$
\begin{aligned}
\frac{1 - e^{-\beta \Delta t}}{\Delta t}(V_{i+1} - V_i) \leq & \; e^{-\beta \Delta t} \left(\frac{1}{2} \pi^{*2} \sigma^2 D^2 V_{i+1} - \frac{1}{2} \sigma^2 \hat{\pi}^2 D^2 V_i \right) \\
& + e^{-\beta \Delta t} \left((b-r)\pi^* D^+ V_{i+1} - (b-r)\hat{\pi} D^+ V_i \right) \\
& - e^{-\beta \Delta t} \left((R-r)\phi^* (D^- V_{i+1} - D^- V_i) \right) \\
& - e^{-\beta \Delta t} \left((r(i+1))h(D^+ V_{i+1} - r(ih)D^+ V_i) \right) \\
& + F_{N,h}(D^- V_{i+1}) - F_{N,h}(D^- V_i), \\
\leq & \; e^{-\beta \Delta t} \left[\frac{1}{2} \pi^{*2} \sigma^2 D^2 V_{i+1} - \frac{1}{2} \sigma^2 \hat{\pi}^2 D^2 V_i \right] \\
& + e^{-\beta \Delta t} \left[(b-r)\pi^* (D^+ V_{i+1} - D^+ V_i) + h(b-r)D^+ V_i \right] \\
& - e^{-\beta \Delta t} \left[(R-r)\phi^* D^2 V_i \cdot h \right] \\
& + e^{-\beta \Delta t} \left[r(i+1))h(D^+ V_{i+1} - D^+ V_i) + r h D^+ V_i \right] \\
& + F_{N,h}(D^- V_{i+1}) - F_{N,h}(D^- V_i) \quad \text{(for } \hat{\pi} \geq \pi^* - h).
\end{aligned}
$$

Collecting terms, we have

$$
\begin{aligned}
\frac{1 - e^{-\beta \Delta t}}{\Delta t}(-e^{-\beta \Delta t}b)(V_{i+1} - V_i) \;\leq\; & e^{-\beta \Delta t}\left(\frac{1}{2}\pi^{*2}\sigma^2 D^2 V_{i+1} - \frac{1}{2}\sigma^2\hat{\pi}^2 D^2 V_i\right) \\
& + e^{-\beta \Delta t}\left[(b - r)\pi^* D^2 V_{i+1} \cdot h\right] \\
& - e^{-\beta \Delta t}\left[(R - r)\phi^* D^2 V_i \cdot h\right] - F_{N,h}(D^- V_{i+1}) \\
& + F_{N,h}(D^- V_i).
\end{aligned}
$$

Now assume that $D^2 V_i > 0$. Then $D^- V_{i+1} > D^- V_i$. Since $D^2 V_{i+1} \leq 0$ and $F_{N,h}$ is strictly decreasing, the right-hand side is < 0.

Also,

$$
\frac{1}{\Delta t}(1 - e^{-\beta \Delta t} - \Delta t e^{-\beta \Delta t}b) = \frac{1}{\Delta t}(1 - e^{-\beta \Delta t}(1 + \beta \Delta t)) > 0 \;\; \text{iff} \;\; \beta > b.
$$

For $\beta > b$, we then have $V_{i+1} - V_i < 0$, contradicting the fact that V_i is nondecreasing in i. Hence, $D^2 V_i \leq 0$ and $V^h(x)$ is concave. $\qquad\square$

The Convergence of the value functions

In this section, it will be shown that $v^h(x) \rightarrow v(x)$ as $h \rightarrow 0$, uniformly on each compact interval in $[0, \infty)$, and that $(v^h)'(x) \rightarrow v'(x)$ as $h \rightarrow 0$ uniformly on compact subintervals of $[0, \infty)$. This information, as we shall see, can be used to prove that the optimal controls for the approximate problem converge to the optimal controls for the continuous problem [17]. The first thing is to deal with the choice of N. It is desirable that $Nh \rightarrow \infty$, so that the approximations cover more and more of the interval $[0, \infty)$, and so that the artificial constraints on the controls disappear. Thus taking $N = N(h)$ to be a function of h such that $N(h) \rightarrow \infty$ and $N(h) \cdot h \rightarrow \infty$, as $h \rightarrow 0$. Furthermore, it is assumed the $\beta > b$ throughout the section.

Lemma 3.1.4. [17] *Suppose that $v^h(x) \to u(x)$ uniformly on [0, B] for each $B > 0$. Then $u(x)$ is a viscosity solution of equation (1.2.11).*

Proof. For $\Psi \in C^2(0, \infty)$ define

$$A\Psi = G(x, \Psi_x, \Psi_{xx}) + rx\Psi + F(\Psi_x) - \beta\Psi. \tag{3.1.5}$$

Then equation (1.2.11) implies that $A\Psi = 0$. We shall show that $u(x)$ must be a viscosity subsolution. Showing that it is a viscosity supersolution is similar. Let us also define $(A_h\Psi)(X_h)$, for $x_h = ih$, $1 \le i \le N - 1$, by

$$
\begin{aligned}
(A_h\Psi)(ih) &= e^{-\beta\Delta t} \max_{\pi,\phi} \left[\frac{1}{2}\pi^2\sigma^2 D^2\Psi(ih) + (b-r)\pi \; D^+\Psi(ih) - (R-r)\phi D^-\Psi(ih) \right] \\
&\quad + e^{-\beta\Delta t} r(ih)D^+\Psi(ih) + F_{Nh}[D^-\Psi(ih)] - \left(\frac{1 - e^{-\beta\Delta t}}{\delta t} \right) \Psi(ih).
\end{aligned}
$$

Note that A_h satisfies

$$T_h\Psi - \Psi = \Delta t A_h\Psi \quad \text{and} \quad A\Psi = \lim_{h \to 0} A_h\Psi, \tag{3.1.6}$$

the convergence being uniform on each compacts subinterval of $(0, \infty)$.

Suppose that $\Psi \in C^2(0, \infty)$, and $x_0 > 0$ are such that $u - \Psi$ has a strict local maximum at x_0. We must show $A\Psi(x_0) \ge 0$.

Since $v^h(x) \to u(x)$ uniformly on each finite interval, there exists x_h such that $x_h \to x_0$ as $h \to 0$ and $v^h(x) - \Psi(x)$ has a local maximum on the grid S_h^N for $x = x_h$.

Now $V_h = T_h V^h$, and $v^h(ih) = V_i^h$, so we must have $\Psi(x_h) \le T_h\Psi(x_h)$, and therefore $A_h\Psi(x_h) \ge 0$. Using equation (3.1.7), we get $A\Psi(x_o) \ge 0$.

This proves lemma 3.1.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The above lemma states that the scheme is consistent; that is, if the approximations converge, they converge to a solution of the continuous problem. We must argue that they do in fact converge and also that they converge to the correct solution. According

to Theorem 1.2.10 in [17], this is the unique, nondecreasing concave viscosity solution which satisfies the boundary condition $v(0) = 0$. To that end, we proceed with more assumptions on U and a sequence of lemmas.

Assumption 3.1.5. [17] *The function $U(c)$ satisfies*

$$\lim_{c \searrow 0} \frac{U(c)}{c^p} = \frac{1}{p} \quad and \quad \lim_{c \nearrow \infty} \frac{U(c)}{c^q} = \frac{1}{q},$$

for some p, q with $0 < p,\ q < 1$.

As established in [18], this assumption states that U acts asymptotically like a $HARA$ function. Furthermore, it is shown in [18] that $v(x)$ satisfies

$$\lim_{x \searrow 0} \frac{v(x)}{x^p} = K \quad \text{and} \quad \lim_{x \nearrow \infty} \frac{v(x)}{x^q} = M,$$

for some $K, M > 0$, so that there exist $x_0 > 0, L > 0$ with $v(x) \leq w(x)$, where $w(x) = Lx^p$ on $[0, x_0]$ and $A_w \leq 0$ on $(0, x_0)$. From the form equation (3.1.6) of the operator A, it is clear that $\hat{L} > L$ implies $\hat{w}(x) = \hat{L}x^p$ satisfies $A\hat{w} \leq 0$ on $(0, x_0)$. The facts lead us to the following lemma.

Lemma 3.1.6. [17] *Suppose that U satisfies Assumption 1.2.8 and Assumption 3.1.5, and that $N_h; 0 < h \leq h_0$ is a collection of positive integers with $N_h \cdot h \to \infty$ as $h \to 0$. Then the functions v^h are equicontinuous at $x = 0$.*

Proof. Choose $\hat{L}$ such that $e^{\beta \Delta t} L < \hat{L}$, and such that

$$\frac{\hat{L}}{L \Delta t}(1 - e^{-\beta \Delta t}) \geq 1,$$

for all $\Delta t = h^2/Q$ with $h \leq h_0$ and N_h as above. Put $\hat{w}(x) = \hat{L}x^p$ as above, and $w(x) = Lx^p$. Fix $0 < h \leq h_0$ and $1 \leq i \leq N - 1$ such that $ih < x_0$. From Taylor's

expansion we see that

$$D^2 w_i = w''(ih) + \frac{h^2}{4!}[w^{(4)}(\alpha_1^h) + w^{(4)}(\alpha_2^h)] \leq w''(ih),$$

$$D^+ w_i = w'(ih) + \frac{h}{2}[w''(\alpha_3^h)] \leq w'(ih), \quad and$$

$$D^- w_i = w(ih) - \frac{h}{2}[w''(\alpha_4^h)] \geq w'(ih),$$

where $(i-1)h \leq \alpha_1^h, \alpha_4^h \leq ih$ and $ih \leq \alpha_2^h, \alpha_3^h \leq (i+1)h$ as in [17].

Then we see that

$$
\begin{aligned}
\frac{1}{\Delta t}[T_h \hat{w}_i - \hat{w}_i] &= \max_{\substack{0 \leq \pi, \phi < KNh \\ \pi - ih \leq ih}} \left(e^{-\beta \Delta t} \left[\frac{1}{2}\pi^2 \sigma^2 D^2 \hat{w}_i + (b-r)\pi D^+ \hat{w}_i \right. \right. \\
&\quad \left. \left. -(R-r)\phi D^- \hat{w}_i \right] \right) \\
&\quad + e^{-\beta \Delta t} rih D^+ \hat{w}_i + F_{N,h}(e^{-\beta \Delta t} D^- \hat{w}_i) - \frac{1-e^{-\beta \Delta t}}{\Delta t}\hat{w}_i \\
&\leq \max_{\substack{0 \leq \pi, \phi \\ \pi - ih \leq phi}} \left(\frac{1}{2}\pi^2 \sigma^2 D^2 w_i + (b-r)\pi D^+ w_i - (R-r)\phi D^- w_i \right) \\
&\quad + rih D^+ w_i + F(D^- w_i) - \frac{1-e^{-\beta \Delta t}}{\Delta t} \cdot \frac{\hat{L}}{L}w_i \\
&= \max_{\substack{0 \leq \pi, \phi \\ \pi - ih \leq phi}} \left(\frac{1}{2}\pi^2 \sigma^2 \left(w''(ih) + \frac{h^2}{4!}[w^{(4)}(\alpha_1^h) + w^{(4)}(\alpha_2^h)] \right) \right), \\
&\quad + (b-r)\pi \left(w'(ih) + \frac{h^2}{2}w''(\alpha_3^h) \right) - (R-r)\phi \left[w'(ih) - \frac{h^2}{2}w''(\alpha_4^h) \right] \\
&\quad + rih \left(w'(ih) + \frac{h^2}{2}w''(\alpha_3^h) \right) - F \left(w'(ih) - \frac{h^2}{2}w''(\alpha_4^h) \right) \\
&\quad - \frac{\hat{L}}{L} \left(\frac{1-e^{-\beta \Delta t}}{\Delta t} \right) w_i.
\end{aligned}
$$

Since $w''(\cdot)$, $w^{(4)}(\cdot)$ are negative, we have

$$
\begin{aligned}
\frac{1}{\Delta t}[T_h \hat{w}_i - \hat{w}_i] &= \max_{\substack{0 \le \pi, \phi \\ \pi - \bar{i}h \le phi}} \left(\frac{1}{2} \pi^2 \sigma^2 w''(ih) + (b-r)\pi w'(ih) \right) \\
&\quad + r(ih)w'(ih) + F(w'(ih)) - \beta w(ih) + \left(\beta - \frac{\hat{L}}{L} \left(\frac{1 - e^{-\beta \Delta t}}{\Delta t} \right) \right) w_i \\
&\le 0 + \beta - \frac{\hat{L}}{L} \left(\frac{1 - e^{-\beta \Delta t}}{\Delta t} \right) w_i,
\end{aligned}
$$

since $Aw(ih) < 0$. Furthermore $\beta - (\hat{L}/L)((1 - e^{-\beta \Delta t})/\Delta t) < 0$, so we have $T_h \hat{w}_i < \hat{w}_i$, which in turn implies $V^h(ih) \le \hat{w}(ih)$. Since $\hat{w}$ is continuous at the origin, and since $v^h \ge 0$ for all h, the proof is complete. $\square$

Next we discuss an extension of equicontinuity to any interval [0,B].

Lemma 3.1.7. [17] *Suppose v satisfies Assumption 1.2.8 and Assumption 3.1.5, and that $Nh : 0 < h \le h_0$ satisfies $N_h \cdot h \nearrow \infty$ as $h \to 0$. Then the functions V^h are equicontinuous on $[0, B]$, for each $B > 0$.*

Proof. From the previous remarks it is clear that there are constants $K, M > 0$ and $0 < \gamma < 1$ such that $v(x) < \bar{w}(x) = Kx^\gamma + M$ for all $x \ge 0$. By an argument analogous to the one of the previous lemma, we have $A_h \bar{w}_i < A\bar{w}(ih) \le 0$, for $1 \le i \le N - 1$. Thus $\bar{w}(x) \ge v^h(x)$ on $[0, (N-1)h]$. From the equation $A_h V_i^h = 0$, we have that

$$
\left(\frac{1 - e^{-\beta \Delta t}}{\Delta t} \right) \bar{w}(ih) \ge \left(\frac{1 - e^{-\beta \Delta t}}{\Delta t} \right) V_i^h \ge r(ih)D^+ V_i e^{-\beta \Delta t},
$$

which yields equicontinuity of the sequence v^h, on any interval $[a, B]$ with $0 < a < B < \infty$. The previous lemma gives equicontinuity at $x = 0$, and the proof is complete. $\square$

These lemmas lead to the following theorem

Theorem 3.1.8. [17] *Assume Assumption 1.2.8, Assumption 3.1.5 and $\beta > b$. Then as $h \to 0$, the functions v^h converge uniformly on compact subsets of $[0, \infty)$ to v, the unique concave,continuous at 0 viscosity solution of $Av = 0$, with $v(0) = 0$.*

Proof. By the Arzela-Ascoli theorem, every subsequence of v^h has a further subsequence that converges, and that limit must be a viscosity solution. Furthermore, that limit must be concave, nondecreasing and continuous at 0 with $v(0) = 0$. Theorem 3.1.8 then follows from Theorem 1.2.10. □

The following is to show that the derivatives $v^h(x)$ converge.

Lemma 3.1.9. [17] *$(v^h)'(x) \to v'(x)$ uniformly on each interval $[a, b]$, $0 < a \leq b < \infty$*

Proof. We take $(v^h)'(ih) = D + V_i^h$. Suppose $x > 0$, $\epsilon > 0$. Choose $\gamma > 0$ such that

$$|x - y| < \gamma \Rightarrow |v'(x) - v'(y)| < \epsilon/2.$$

By the mean value theorem, there exist $\xi_\gamma \in (x - \gamma, x)$ and $\eta_\gamma \in (x, x + \gamma)$ such that

$$v(x) - v(x - \gamma) = v'(\xi_\gamma) \cdot \gamma,$$
$$v(x + \gamma) - v(x) = v'(\eta_\gamma) \cdot \gamma .$$

Let

$$p_\gamma^h = \tfrac{1}{\gamma} \left(v^h(x) - v^h(x - \gamma) \right),$$
$$q_\gamma^h = \tfrac{1}{\gamma} \left(v^h(x + \gamma) - v^h(x) \right).$$

Because v^h is concave, we have $p_\gamma^h \geq v^{h'}(x) \geq q_\gamma^h$. By Theorem 3.1.8, as $h \to 0$

$$p_\gamma^h \to \tfrac{1}{\gamma} \left(v(x) - v(x - \gamma) \right),$$
$$q_\gamma^h \to \tfrac{1}{\gamma} \left(v(x + \gamma) - v(x) \right).$$

Hence, for small h we have,

$$|p_\gamma^h - v'(\xi_\gamma)| < \epsilon/2, \quad |q_\gamma^h - v'(\eta_\gamma)| < \epsilon/2.$$

Thus

$$v'(x) - \epsilon < v'(\eta_\gamma) - \epsilon/2 < q_\gamma^n \le v^{h'}(x) \le p_\gamma^N < v'(\xi_\gamma) + \epsilon/2 < v'(x) + \epsilon.$$

Therefore, for all h sufficiently small, we have $|v^{h'} - v'(x)| < \epsilon$. Thus, for each $x > 0$, $v^{h'}(x) - v'(x)$ as $h \to 0$. Since the functions in question are all nonincreasing, we have $v^{h'}(x) \to v'(x)$ uniformly on each interval $[a, b]$ $(0 < a \le b < \infty)$.

This proves Lemma 3.1.9. □

Note: *This convergence cannot be extended to* $[0, b)$, *because* $v'(0) = \infty$.

Before proving convergence of the controls, it is important to recall the finite difference form of the HJB equation. Let π_i^*, ϕ_i^* give the maximum on the right side of equation (3.1.3). For $x = ih$, $1 \le i \le N - 1$, three cases can occur in the maximization over π, ϕ :

$$-\frac{b - r}{\sigma^2} \frac{D^+V_i}{D^2V_i} \le x.$$

In this case

$$\pi_i^* = -\frac{b - r}{\sigma^2} \frac{D^+V_i}{D^2V_i}, \quad \phi_i^* = 0 - \frac{b - R}{\sigma^2} \frac{D^+V_i}{D^2V_i} - \frac{R - r}{\sigma^2} h \le x < -\frac{(b - r)D^+V_i}{\sigma^2 D^2V_i}.$$

then in the case when $\pi_i^* = x$, $\phi_i^* = 0$,

$$x < -\frac{(b - R)D^+V_i}{\sigma^2 D^2V_i} - \frac{R - r}{\sigma^2} h,$$

when

$$\pi_i^* = \min\left(-\frac{(b - R)D^+V_i}{\sigma^2 D^2V_i} - \frac{R - r}{\sigma^2} h, KNh\right) \phi_i^* = \pi_i^* - x,$$

put

$$G_{N,h}(x, p, m, q) = \max_{\substack{0 \le \pi, \phi < KNh \\ \pi - x \le \phi}} \left(\frac{1}{2}\sigma^2\pi^2 q + (b - r)\pi p - (R - r)\phi m\right) e^{-\beta\Delta t}.$$

The calculations above, yield [17]

$$
G_{N,h} = \begin{cases}
-\frac{1}{2}\frac{(b-r)^2}{\sigma^2}\frac{p^2}{q}, \\[2ex]
\text{if} \quad -\frac{(b-r)}{\sigma^2}\frac{p}{q} \le x, \\[3ex]
\frac{1}{2}\sigma^2 x^2 q + (b-r)xp, \\[2ex]
\text{if} \quad -\frac{(b-R)}{\sigma^2}\frac{p}{q} - \frac{R-r}{\sigma^2}h \le x < -\frac{(b-R)}{\sigma^2}\frac{p}{q} \\[3ex]
\frac{1}{2}\sigma^2 q \left(-\frac{(b-R)}{\sigma^2}\frac{p}{q} - \frac{R-r}{\sigma^2}h\right)^2 \\[1ex]
+(b-r)\left(-\frac{(b-R)}{\sigma^2}\frac{p}{q} - \frac{R-r}{\sigma^2}h\right)p \\[2ex]
-(R-r)\left(-\frac{(b-R)}{\sigma^2}\frac{p}{q} - \frac{R-r}{\sigma^2}h - x\right)m, \\[3ex]
\text{if} \quad x \le -\frac{(b-R)}{\sigma^2}\frac{p}{q} - \frac{R-r}{\sigma^2}h \le KNh, \\[3ex]
\frac{1}{2}\sigma^2 (KNh)^2 q + (b-r)(KNh)p - (R-r)(KNh-x)m, \\[3ex]
\text{if} \quad KNh \le -(-\frac{(b-R))}{\sigma^2}\frac{p}{q} - \frac{(R-r)h}{\sigma^2}h.
\end{cases}
$$

Equation (3.1.3) therefore takes the form

$$
V_i\left(\frac{1-e^{-\beta\Delta t}}{\Delta t}\right) = G_{N,h}(ih, D^+V_i + D^-V_i + D^2V_i) + r(ih)D^+V_i + F_{N,h}(D^-V_i), \quad (3.1.7)
$$

with $F_{N,h}(p)$ as in equation (3.1.2). A tedious calculation shows that the finite difference

equation (3.1.3) can be solved for D^2V_i

$$D^2V_i = H_{N,h}(x, V_i, D^+V_i, D^-V_i),$$

and that, as $h \to 0$,

$$H_{N,h}(x, V_i, D^+V_i, D^-V_i) \to H(x, v, v'),$$

uniformly for $x \in [a, b]$, for each $a > 0$, $b < \infty$, where $v(x)$ solves $v'' = H(x, v, v')$ as in [18]. Since the first and second differences converge to the corresponding derivatives, it follows that the approximate controls π_i^*, ϕ_i^* converge as $h \to 0$ and $ih \to x$ to the optimal controls $\pi^*(x)$, $\phi^*(x)$ for the continuous problem (see [18], Theorem 2.2). The approximate optimal consumption control c_i^* satisfies $c_i^* = (U')^{-1}(e^{-\beta \Delta t} D^- V_i)$. In a similar way, $c_i^* \to c^*(x)$ as $h \to 0$ and $ih \to x$, where $c^*(x) = (U')^{-1}(v'(x))$ as in [17].

3.2 Finite Difference Method

This approach is illustrated by Munk [39] in his PhD dissertation on optimal consumption/portfolio policies and contingent claims pricing and hedging in incomplete markets. We will introduce the approach and just highlight the majors articulations from this work. The set-up provided henceforth is also used in the next approach which will be titled as the finite element approach.

Munk [39] considered a stochastic system on a filtered probability space $(\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P})$, where $\mathfrak{F}$ is a $\sigma - algebra$ on $(\Omega, \mathbb{P})$ and $\mathbb{F} = \{\mathfrak{F}_t | t \in \mathrm{T}\}$ is a non-decreasing and right-continuous filtration. He distinguished between two cases: (i) $\mathrm{T} = [0; T]$ for some $T > 0$ and (ii) $\mathrm{T} = [0; \infty)$. He assumed that (a) $\mathfrak{F}_0$ is the $\sigma - algebra$ generated by the zero sets of the probability measure $\mathbb{P}$ and (b) $\mathfrak{F}_T = \mathfrak{F}$ in the finite horizon case or $\mathfrak{F} = \bigvee_{t \geq 0} \mathfrak{F}_t$ in the infinite horizon case. He assumed that $\mathbb{F}$ is generated by a $d - dimensional$ standard Wiener process w (with independent component processes).

First, he focused on the infinite horizon case $\mathrm{T} = [0; \infty)$, since the notation will be

somewhat messier in the finite horizon case $T = [0; T]$ and assumed that an adapted process $x = \{x(t)|0 \leq t < \infty\}$ exists, such that, at any time $t \geq 0$, the vector $x(t) \in \mathbb{R}^p$ describes the state of the system at that time. In his works, he referred to x (or $x(t)$) as the state variable. The state variable is influenced by the choice of a feedback control $a : \mathbb{R}^p \to \mathbb{R}^q$, with $a(x)$ denoting the control applied at time t when $x(t) = x$, in such a way that x evolves according to the stochastic process

$$dx(t) = f(x(t), a(x(t)))dt + g(x(t), a(x(t)))dw(t), \qquad (3.2.1)$$

where $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ and $g : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{p \times d}$ are continuous functions which satisfy sufficient conditions for equation (3.2.1) to have a solution. Define $S(x, a) = g(x, a)g(x, a)^{\mathrm{T}}$ that is $S : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{p \times p}$. Munk [39] defined a feedback control a is admissible if a is progressively measurable and $a(x(s)) \in \mathbb{A} \subset \mathbb{R}^q$ for all $s \geq 0$. Furthermore, it may be required that the state variable stays within a certain subset of $\mathbb{R}^p$. With such a condition, the set of admissible controls can depend on the initial value x of the state variable. Denote the set of admissible controls given x by $A(x)$. Define

$$W(x, a) = E\left[\int_0^\infty exp\left\{\int_0^t \beta(x(s), a(x(s)))dt\right\} L(x(t), a(x(t)))dt | x(0) = x\right],$$

and

$$V(x) = \sup_{a \in A(x)} W(x, a), \qquad (3.2.2)$$

V is referred to as the value function. L is a continuous function satisfying a polynomial growth condition.

Brennan, Schwartz and Lagnado [8] solved an optimal investment problem by a much particular and specific algorithm, which, applied to the general optimal control problem mentioned above, is as follows.

Consider the HJB equation associated with the control problem,

$$\sup_{a \in \mathbb{A}} \left\{ f(x,a)^T V'(x) + \frac{1}{2} tr(S(x,a)V''(x)) + L(x,a) \Leftrightarrow \beta(x,a)V(x) \right\} = 0.$$

Define a grid on the state space. Let a_0 be any admissible control. Then compute an estimate V_0 of the value function by solving numerically the PDE

$$f(x,a_0)^T V'(x) + \frac{1}{2} tr(S(x,a_0)V''(x)) + L(x,a_0) \Leftrightarrow \beta(x,a_0)V(x) = 0, \qquad (3.2.3)$$

with a finite-difference method. A new control is then computed as

$$a_1(x) = \arg\max_{a \in \mathbb{A}} \left\{ f(x,a)^T V_0'(x) + \frac{1}{2} tr(S(x,a)V_0''(x)) + L(x,a) \Leftrightarrow \beta(x,a)V_0(x) \right\} = 0,$$
$$(3.2.4)$$

for all x in the grid. Then a new estimate V_1 of the value function is found by solving a PDE like equation (3.2.3) replacing a_0 with a_1, etc. Brennan, Schwartz and Lagnado [8] reported that stability of the approach is enhanced, when some upwind difference approximations are used (confer Munk [39] for details).

3.3   Finite Element Method

McGrattan [36] solved a stochastic growth model using finite element method. We look at a description of the method according to McGrattan [36].

Being the most commonly used in modern business cycle and growth literatures, the stochastic growth model has been used to test the performance of alternative numerical methods. McGrattan [36] showed that the method is easy to apply and that, for examples such as the stochastic growth model, it gives accurate solutions within a very short period of time. She also showed how inequality constraints can be handled by redefining the optimization problem with penalty functions.

McGrattan [36] applied a method that is widely used in engineering applications such as structural analysis and aerodynamic design to compute the equilibrium of

a growth model. This method, called the finite element method, is an algorithm for solving functional equations and, for certain problems, is both fast and accurate. Using the Taylor and Uhlig [52] application, she demonstrated that the finite element method works extremely well when applied to a case with an analytical solution. For cases without such a solution, she showed that the method yields decision functions similar to discretized dynamic programming in a fraction of the computing time. Finally, McGrattan [36] showed that the method can also be applied to problems with inequality constraints. Even for a constrained problem, the finite element method performs well.

In their study of alternative methods for solving the stochastic growth model, Taylor and Uhlig [52] focused on the tradeoff between speed and accuracy. In this regard, the finite element method offers several advantages over many of the algorithms that Taylor and Uhlig [52] analyzed. With the finite element method, the first step in solving the functional equation is to subdivide the domain of the state space into nonintersecting subdomains called elements. The domain is subdivided because the method relies on fitting low-order polynomials on subdomains of the state space rather than high-order polynomials on the entire state space. The result is a system of equations that is sparse. Furthermore, as the dimensionality of the problem increases, higher-order functions can be used where needed, with fewer grid points or adaptive grid techniques can be used to better resolve the grid in regions of the state space where nonlinearities occur [52].

In summary, McGrattan's [36] paper described the finite element method by applying it to one example namely the stochastic growth model (the interested reader can refer to McGrattan [36]). She showed that the method is easy to apply and, for examples such as the stochastic growth model, gives accurate solutions within a very short time frame. She also showed how inequality constraints can be handled by redefining the optimization problem with penalty functions.
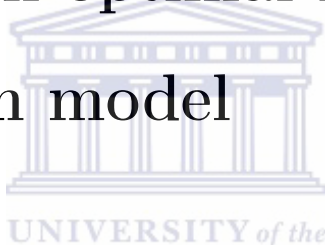
3.4 Infinite series expansion method

Besides the methods presented above, there is also a popular method, namely, Infinite Series Expansion Method, to solve such consumption models. We will explore this method in details to solve an optimal investment consumption model in the next chapter.

3.5 Summary and discussions

We presented different numerical methods to solve an optimization problem. The difference among the methods have been clearly established and it should also be mentioned that each method uses a particular set of conditions before being implemented. Therefore, it is of utmost importance for the investor to look into the conditions before venturing into optimizing any portfolio. In next chapter, we give a detailed discussion of the infinite series expansion method.

Chapter 4

Infinite series expansion method applied to an optimal investment consumption model

In previous chapter we discussed several methods to solve a variety of consumption models. In this chapter we use a specific method, namely the Infinite series expansion method, to solve an optimal investment consumption model.

4.1 Model problem

The model we are solving here originates from the Merton's original problem [38]. When attacking questions on investments, we are always mindful about the investor behavior hence the fundamental concept of utility plays a big role. To begin, we assume that we are dealing with an investor who is risk averse, simply put, an investor that will dismiss an investment if she deems the latter a reasonable target for exploitation, attack or worse. We also move with the assumption of non-satiation, that the investor always prefers more return than less hence the utility function will always be increasing. Since the investor is risk averse, we will expect an either downward or upward concavity depending on the investment, meaning that we will observe a decrease of the marginal

utility of wealth as the wealth itself increases. Presently, the aim would be to maximize
the investor expected utility during his active time given that an infinite time span has
been preferred. We can therefore proceed with this now that all the main parameters
to formulate the problem mathematically have been presented. We define the value
function (refer equation 1.2.8),

$$V(l) = \sup_A E\left\{\int_0^\infty e^{-\beta t} U(c_t)dt\right\}. \tag{4.1.1}$$

The equation above (4.1.1) was originally introduced by Merton and an explicit solution
to the equation exists.

4.2   The Infinite series expansion method

In this section we describe in detail how the optimal consumption problem can be
solved using infinite series expansion. We will utilize the fact that it is possible to
find an infinite series expansion that solves a transformed version of equation (1.2.17).
Once this series expansion has been found, we will be able to return to $W(z)$ and hence
also find the optimal controls using $MATLAB$ as in [3]. We will also be looking at the
behavior of the value function for high and low correlation, income and stock volatility
and finally, we will present the results obtained in [3] and use our own set of values to
generate additional plots that might be helpful in dictating our investor's choices.

4.2.1   Homogeneity transformation: computational solution

We begin by recalling equation (1.2.14) and writing it carefully, we get

$$\frac{\eta^2}{2}(1-\rho^2)zW'' + kzW' - \frac{k_1^2}{2\sigma^2}\frac{(W')^2}{W''} + \delta W' - \log W' - 1 = \beta W. \tag{4.2.1}$$

[To simplify equation (4.2.1), we set a new constant K_i and a function F such that

$$K_1 W + K_2 z W' + K_3 z^2 W'' + K_4 \frac{(W')^2}{W''} + F(W') = 0, \qquad (4.2.2)$$

where

$$K_1 = -\beta,$$

$$K_2 = k = \eta^2 + r - \mu + \delta + \frac{\eta \rho k_1}{\sigma}$$

$$= \eta^2 + r - \mu + \delta - \eta^2 \rho^2 + \frac{\eta \rho}{\sigma}(\alpha - r),$$

$$K_3 = \frac{\eta^2}{2}(1 - \rho^2),$$

$$K_4 = -\frac{k_1^2}{2\sigma^2} = -\frac{(-\eta \rho \sigma + \alpha - r)^2}{2\sigma^2},$$

$$F(x) = \delta x - \log x - 1.$$

Looking at equation (4.2.2), we observe a recurrence of terms of the form $z^k W^{(k)}(z)$ which we will use in finding the solution. Now we need to find where this pattern is disrupted. We can see that

$$K_4 \frac{(W')^2}{W''} = K_4 \frac{(zW')^2}{z^2 W''}, \qquad (4.2.3)$$

but what happens to the term $F(W')$? By observing carefully and applying the *Legendre* transform on the function $F(W')$, the transformation will return a new variable y and a new function $\widetilde{W}(y)$ such that

$$\widetilde{W}(y) = \max_z \left(W(z) - \frac{z}{y} \right). \qquad (4.2.4)$$

By simple mathematics, the maximum on the right hand side gives us $y = \frac{1}{W'(z)}$ and the inverse of this transformation is

$$W(z) = \min_y \left(\widetilde{W}(y) - \frac{z}{y} \right), \qquad (4.2.5)$$

and the maximum on the right hand side gives us $z = y^2 \widetilde{W}'(y)$] [3]. From the above, the following relationships hold

$$y = \frac{1}{W'(z)},$$

$$z = y^2 \widetilde{W}'(y),$$

$$W''(z) = \frac{dW'}{dz} = \frac{d\frac{1}{z}}{dy^2 \widetilde{W}'(y)} = \frac{d\tilde{y}}{d\frac{1}{y^2} \widetilde{W}'(\frac{1}{\tilde{y}})}$$

$$= \frac{1}{-\frac{1}{y^4} W''(\frac{1}{\tilde{y}}) + 2\frac{1}{y^3} \widetilde{W}'(\frac{1}{\tilde{y}})} = \frac{1}{y^4 \widetilde{W''} - 2y^3 \widetilde{W'}},$$

hence the term $F(W')$ takes the form

$$F(W') = F\left(\frac{1}{y}\right) = \frac{\delta}{y} + \log y - 1,$$

which turns equation (4.2.2) into:

$$K_1 \widetilde{W} + (K_1 + K_2 + 2K_4)y\widetilde{W'} - K_3 \frac{(\widetilde{W'})^2}{\widetilde{W''} - \frac{2}{y}\widetilde{W'}} - K_4 y^2 \widetilde{W''} + \frac{\delta}{y} + \log y = 1. \quad (4.2.6)$$

Presently, we can try to find a solution to the ODE. If we assume $\widetilde{W}$ to contain the term $\frac{-1}{K_1} \log y$, by intuition, we can guess that the remaining terms would contain the derivatives of $\log y$ with some constant in front of them. Let us call these constants B_k and see what we get [3].

$$\widetilde{W} = -\frac{1}{K_1} \log y + B_0 + \sum_{n=1}^{\infty} B_n y^{-n}, \quad (4.2.7)$$

$$\widetilde{W'} = -\frac{1}{K_1 y} - \sum_{n=1}^{\infty} n B_n y^{-n-1} = \sum_{n=0}^{\infty} C_n y^{-n-1}, \quad (4.2.8)$$

$$\widetilde{W''} = -\frac{1}{K_1 y^2} + \sum_{n=1}^{\infty} n(n+1) B_n y^{-n-2} = \sum_{n=0}^{\infty} D_n y^{-n-2}. \quad (4.2.9)$$

By comparing this to equation (4.2.6) we can now look at the individual terms y^{-k} for $k = 0, 1, 2...$ to get an expression for B_k. Furthermore,

$$\frac{(\widetilde{W'})^2}{\widetilde{W''} - \frac{2}{y}\widetilde{W'}} = \frac{\sum_{n=0}^{\infty} C_n y^{-n-1}}{\sum_{n=0}^{\infty} D_n y^{-n-2} - 2\sum_{n=0}^{\infty} C_n y^{-n-2}} = y\frac{\sum_{n=0}^{\infty} C_n y^{-n}}{\sum_{n=0}^{\infty} (D_n - 2C_n) y^{-n}}. \quad (4.2.10)$$

At this point we make the assumption that this can be written as an infinite sum $y\sum_{n=0}^{\infty} E_n y^{-n}$. Multiplying both sides by the denominator on the left hand side and dividing by y gives us that

$$\sum_{n=0}^{\infty} C_n y^{-n} = \left(\sum_{n=0}^{\infty} E_n y^{-n}\right)\left(\sum_{n=0}^{\infty} (D_n - 2C_n) y^{-n}\right). \quad (4.2.11)$$

Comparing the individual terms y^{-k} on both sides we get that on the left hand side C_k is the term multiplied by y^{-k} and on the right hand side we will get a sum of terms on the form $E_i(D_j - 2C_j)$ which have the property that $i + j = k$. The following relationships between the constant are as follows [3]

$$C_n = \sum_{n=0}^{\infty} E_{n-i}(D_i - 2C_i), \quad (4.2.12)$$

$$C_0 = -\frac{1}{K_1}, \quad (4.2.13)$$

$$D_n = \frac{1}{K_1}, \quad (4.2.14)$$

$$D_n = -(n+1)C_n = n(n+1)B_n, \ n \neq 0. \quad (4.2.15)$$

The first step is to find E_0 using the fact that C_0 and D_0 are known. [Once E_0 is found, we can use equation (4.2.6)and get B_0 by comparing the constant term]. Then we can use (4.2.12) to express E_1 as a function of B_1. By repeating this process we can get all the constants B_n and E_n. Let us proceed with our theory. First we can use (4.2.12) to express E_n as a function of B_n. It is also assumed that all B_i and E_i are known for

$i < n$ ([3]):

$$C_0 = E_0(D_0 - 2C_0)$$

$$\Rightarrow E_0 = \left(\frac{1}{K_1} - 2\left(-\frac{1}{K_1}\right)\right)^{-1}\left(-\frac{1}{K_1}\right) = -\frac{K_1}{3}\frac{1}{K_1} = -\frac{1}{3},$$

$$C_n = \sum_{i=0}^{n} E_{n-i}(D_i - 2C_i)$$

$$= -\frac{1}{3}(D_n - 2C_n) + \sum_{i=1}^{n-1} E_{n-i}(D_i - 2C_i) + E_n\frac{3}{K_1},$$

$$\Rightarrow E_n = \frac{K_1}{9}n^2 B_n - \frac{K_1}{3}\sum_{i=1}^{n-1} E_{n-i}(i(i+1) + 2i)B_i$$

$$= F_n^1 B_n + F_n^2.$$

For B_0, we get that

$$B_0 = \frac{1}{K_1^2}\left(2K_1 + K_2 + 2K_4 - \frac{1 + K_4}{3}K_3\right). \tag{4.2.16}$$

Now that E_n is expressed as a function of B_n we can insert the results into equation (4.2.6) and solve for B_n.

$$K_1 B_n y^{-n} + (K_1 + K_2 + 2K_4)C_n y^{-n} - K_3 y^{-n}\sum_{i=0}^{n} E_i Cn - i - K_4 D_n y^{-n} = \begin{cases} \delta y^{-1}, & n = 1, \\ 0, & n > 1. \end{cases} \tag{4.2.17}$$

Using that C_n, D_n and E_n can be described as functions of B_n we can solve for B_n and get that (for $n > 1$)
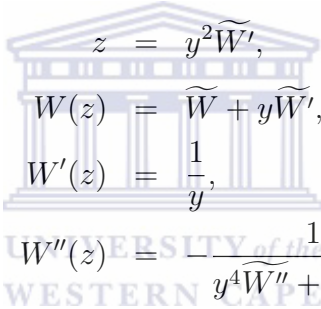
$$\left(K_1 - (K_1 + K_2 + 2K_4)n + K_3\left(\frac{1}{k_1}F_n^1 + \frac{n}{3}\right) - n(n+1)K_4\right)$$

$$= K_3\left(\sum_{i=1}^{n-1} E_i C_{n-i} - \frac{1}{K_1}F_n^2\right), \tag{4.2.18}$$

$$B_n = \frac{K_3 \left(\sum_{i=1}^{n-1} E_i C_{n-i} - \frac{1}{K_1} F_n^2 \right)}{\left(K_1 - (K_1 - K_2 + 2K_4)n + K_3 \left(\frac{1}{K_1} F_n^1 + \frac{n}{3} \right) - n(n+1)K_4 \right)}. \qquad (4.2.19)$$

It is possible to estimate the coefficients numerically now that we have all the required formulae.

4.2.2 Simulating the method

The coefficients B_n, $\widetilde{W}(y)$, $\widetilde{W}'(y)$ and $\widetilde{W}''(y)$ are known therefore it is now possible to recall our previous steps and obtain $W(z), W'(z)$ and $W''(z)$ by remembering that [3]:

$$\begin{aligned}
z &= y^2 \widetilde{W}', \\
W(z) &= \widetilde{W} + y\widetilde{W}', \\
W'(z) &= \frac{1}{y}, \\
W''(z) &= \frac{1}{y^4 \widetilde{W}'' + 2y^3 \widetilde{W}'}.
\end{aligned}$$

Several plots of the value function and optimal controls as a function of z are shown with different values of income volatility δ, correlation ρ, stock volatility σ

Next, we will be presenting the results of the infinite series expansion method and detailed interpretations of our results.

4.3 Numerical results using the infinite series expansion method

In this section we present numerical results for the model under consideration. The values of some parameters used in the simulation are shown in Table 4.3.1

In the plots, we use the ratio $z = l/h$, where l denotes initial wealth and h denotes initial income. The normalization in this plots means that the optimal controls and the money invested in the riskless asset (bond) add up to one (and this is made possible

Table 4.3.1: Parameters for simulation

Correlation	ρ	0.0
Income volatility	η	0.3
Time preference rate	β	0.4
Stock volatility	σ	0.5

since the initial wealth l is greater than zero) . [It should be noted that the wealth process $L(t)$ can go down to zero but that should not be a worry for our investor since positive wealth can be achieved again due to the presence of random income flow][3].

Note that in the following plots $h > 0$. The first parameter we choose to vary, is the correlation ρ, between changes in income and changes in the risky stock market. From figures 4.4.1 and 4.4.2 we observe that the value function follows the same pattern as the correlation that is; the higher the correlation ρ, the higher the value function and the lower the correlation, the lower the value function.

Figure 4.4.3 describes the optimal consumption for different values of ρ and figure 4.4.4 describes the optimal risky investment for different values of ρ. We observe again the same behavior on both figures and that is; the consumption and risky investment should be high for high correlation [3].

The result suggests that the investor should use a large proportion of her wealth on consumption, investment and stock market if the correlation is high. It might sound peculiar at first, but there is a chance that our investor would both make profit on the risky market and also obtain a high income flow (because of the high correlation). Since we assumed an investor that will not accept an investment that is a reasonable target for exploitation or attack, the chance of making profit is indeed positive. Note that the difference in proportion of wealth to be put into risk-free bond, risky stock market and consumption is minimal. The allocation of wealth is estimated to be 25% consumption, 35% risk-free assets and 40% risky investments.
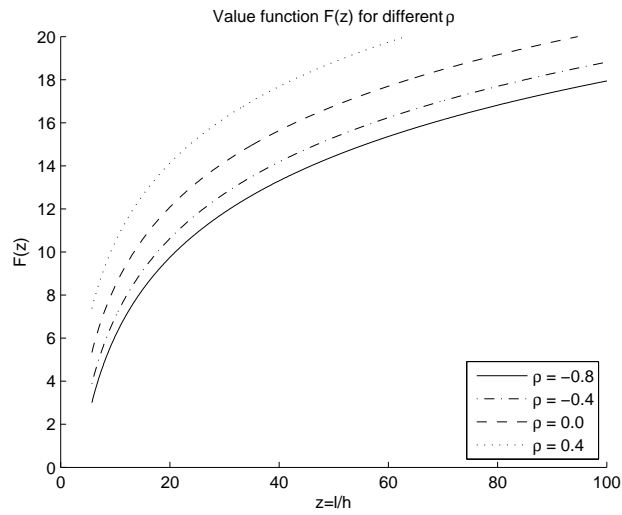
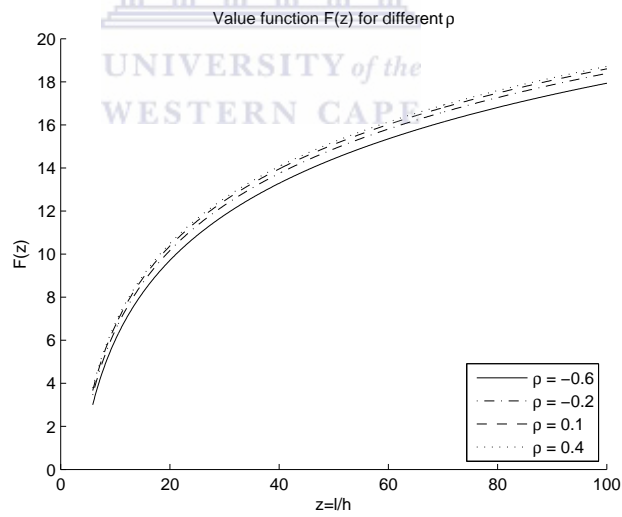Figure 4.3.1: Objective function F(z) for the correlation coefficient ρ = -0.8, -0.4, 0.0, 0.4.

Figure 4.3.2: Objective function F(z) for the correlation coefficient ρ = -0.6, -0.2, 0.1, 0.4.
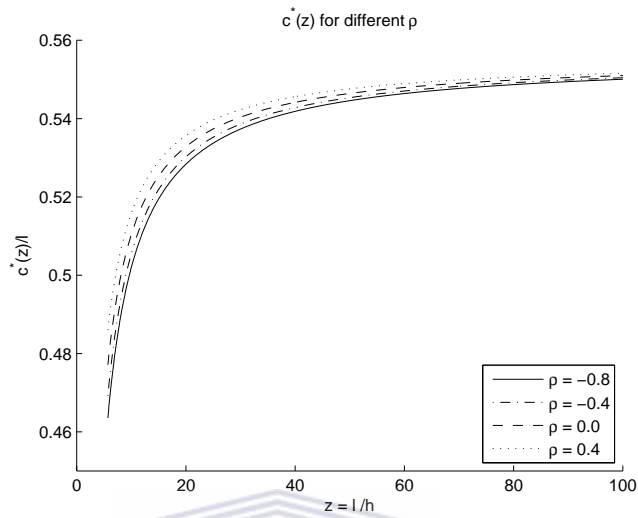
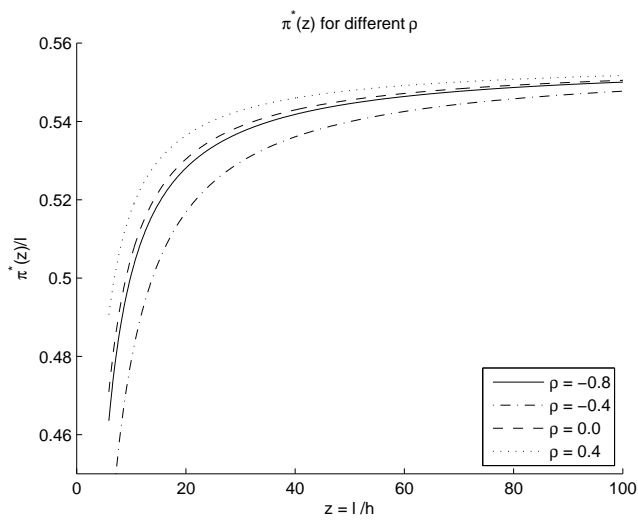Figure 4.3.3: Control process for consumption c^*/l for the correlation coefficient $\rho =$ -0.8, -0.4, 0.0, 0.4.

Figure 4.3.4: Control process for investment π^*/l for the correlation coefficient $\rho =$ -0.8, -0.4, 0.0, 0.4.

The next interesting parameter is the volatility of the stock σ. Figure 4.4.5 and
Figure 4.4.6 illustrate a description of the value function for different values of σ. We
observe that the value function is highest for highest values of stock volatility and
lowest for lowest values of stock volatility. Figures 4.4.7 and 4.4.8 indicate that our
investor should spend a considerable amount of money on the risky assets when there
is small fluctuation in the market (stable market); that is when σ is low and also to
consume a smaller proportion of her wealth. Note that the change in consumption
is small, while the change in investments in risky stock markets is very high. This
means that the distribution between risky and risk-free investments is highly affected
by changes in stock volatility as described in [3].
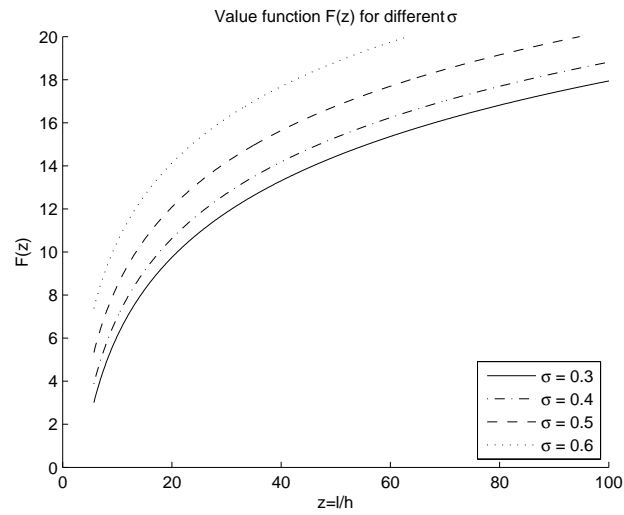
Figure 4.3.5: Objective function F(z) for stock's volatility $\sigma = 0.3, 0.4, 0.5, 0.6$.
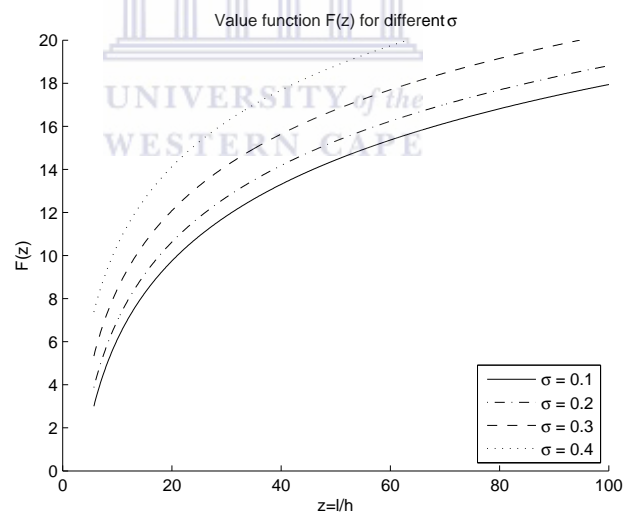
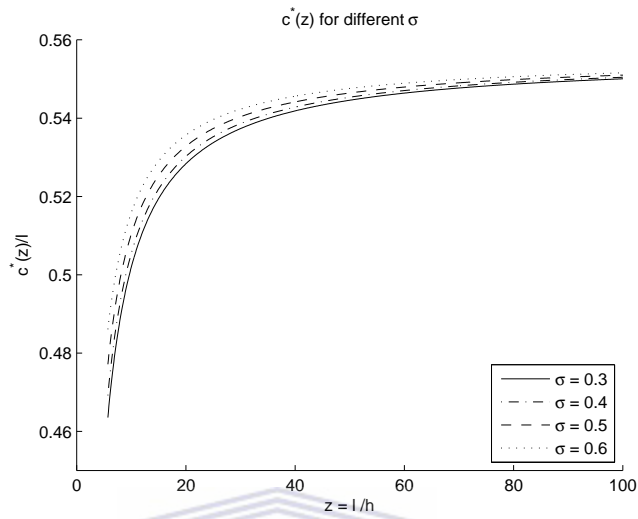Figure 4.3.6: Objective function F(z) for stock's volatility $\sigma = 0.1, 0.2, 0.3, 0.4$.

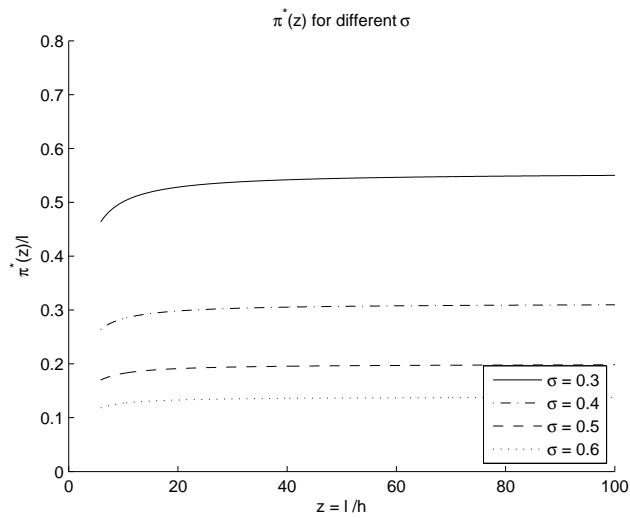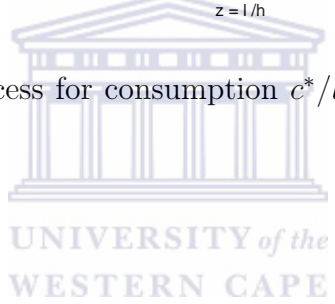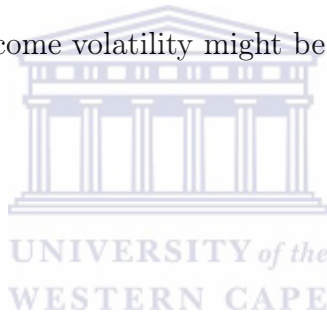Figure 4.3.7: Control process for consumption c^*/l for stock's volatility $\sigma = 0.3$, 0.4, 0.5, 0.6.

Figure 4.3.8: Control process for investment π^*/l for stock's volatility $\sigma = 0.3$, 0.4, 0.5, 0.6.

Finally, another parameter of our interest is the income volatility η. Figure 4.4.9 and figure 4.4.10 declare that the value function is shifted upwards for increasing values of η. In figures 4.4.11 and 4.4.12 we can see that the optimal consumption and optimal risky investment also follow this pattern. In other words the investor should invest more money on the risky stock market and also consume more if the income volatility is high. This can be justified by the fact that our investor needs to make risky investments to avoid bankruptcy and the income generated (profit) will serve for future investments [3]. When studying the result for the optimal consumption, we must always bear in mind what we are trying to optimize. Maximizing wealth is not solely what the investor should focus on, but rather to maximize the utility function as such, a high consumption with high income volatility might be wiser.
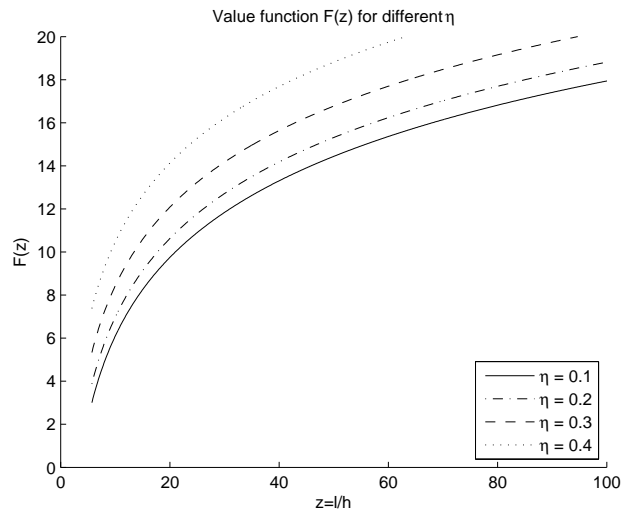
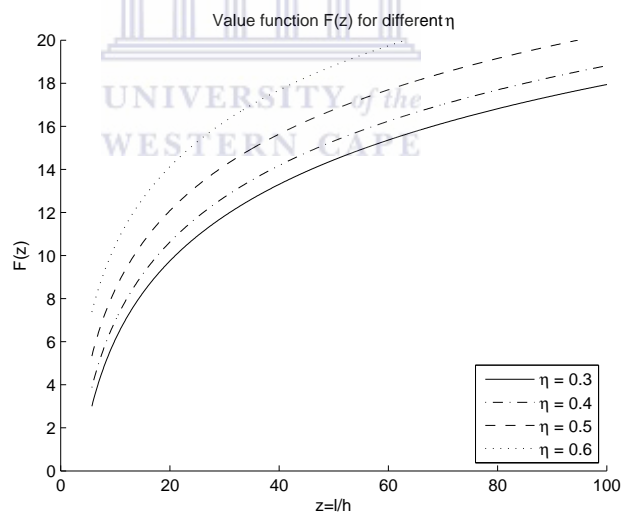Figure 4.3.9: Objective function F(z) for income volatility $\eta = 0.1, 0.2, 0.3, 0.4$.

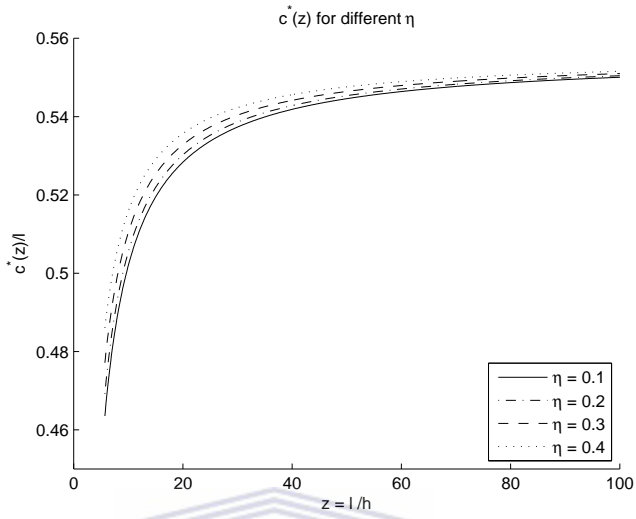Figure 4.3.10:   Objective function F(z) for income volatility $\eta = 0.3, 0.4, 0.5, 0.6$.

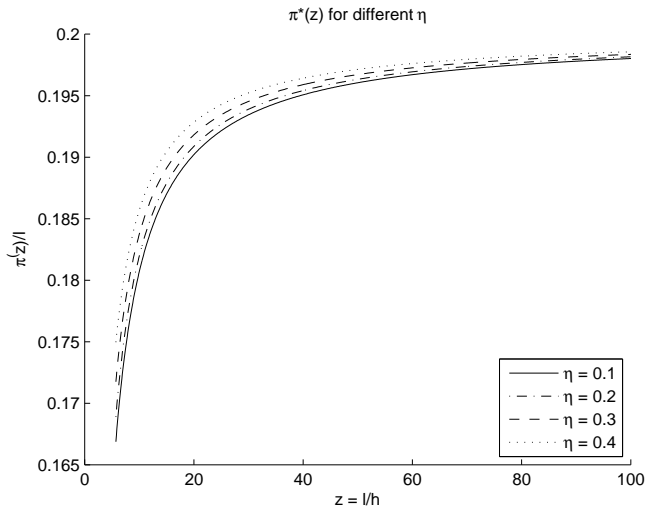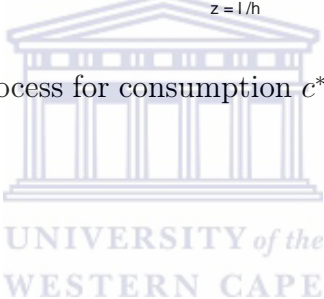Figure 4.3.11: Control process for consumption c^*/l for income volatility $\eta = 0.1$, 0.2, 0.3, 0.4.

Figure 4.3.12: Control process for investment π^*/l for income volatility $\eta = 0.1$, 0.2, 0.3, 0.4.

It is clear from the above sets of numerical simulations that the parameter that has a truly large influence on our optimal investments is the market volatility. It affects the distribution between risky and risk-free investments. The other parameters have interesting effects but they do not affect the total ratios as much.

4.4   Summary and discussions
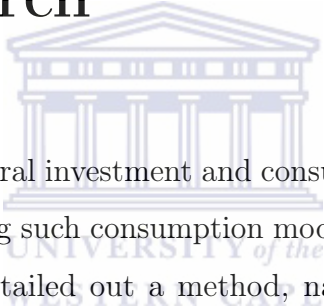
In this chapter, we described the implementation of the method called the infinite series expansion for solving an optimal investment-consumption model. We also presented several numerical results describing the effect of individual parameters in the model.

The next chapter is the conclusion chapter where we give an overall summary of the whole thesis. We highlight challenges that we encountered and mention a few points on how we intend to extend the work contained in this thesis. Some scope for future research is also mentioned.
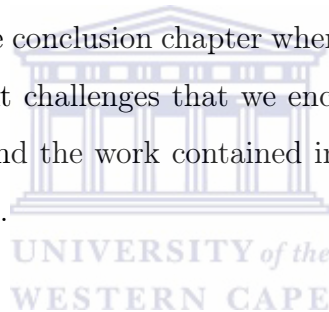
Chapter 5

Concluding remarks and scope for future research

This thesis deals with general investment and consumption problems and discusses numerical methods for solving such consumption models. After discussing some methods from the literature, we detailed out a method, namely, the infinite series expansion method and its application to solve a model problem. Extensive numerical results are presented and along with a thorough discussion on them.

As far as the scope for future research is concerned, we intend to explore the proposed infinite series expansion method by using other types of utility functions than those already considered in this thesis. Further modification on the conditions on utility functions may also lead to a simpler derivation of the HJB equation. This we will explore in future.

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