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Department of Nathematics and Applied Mathematics

# Investigating the difficulties of first year mainstream 

 mathematics students at the University of the Western Cape with "Related Rates" problems.Allen Vernon Taylor

A thesis submitted in fulfilment of the requirements for MSc. (Mathematics Education)

## ACKNOWLEDGEMENTS

There were many who contributed towards the end product of this project. I wish I could mention you all, but limited space prevented me from doing so.

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To all who contributed in any way, no matter how small, I thank you from the bottom of my heart.

Finally, I dedicated this thesis to my late father, Christian Taylor. I will always love you.

## DECLARATION


#### Abstract

I declare that "Investigating the difficulties of first year mainstream mathematics students at the University of the Western Cape with Related Rates problems" is my own work and people who have made any contribution in its compiling have been acknowledged. It has never been submitted for any degree or examination in any other university. All the sources that have been used or quoted have been acknowledged with full reference.




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## ABSTRACT

The aim of the thesis is to research the difficulties that first year mainstream mathematics students at UWC experience when solving Related Rates problems in calculus. In chapter 2, an in-depth study was made of the nature of Related Rates problems by studying a number of examples. The findings of this study are summarized in section 2.12. The study adopted the same model of the solution of all types of Related Rates that was used by Martin (2000) for the solution of geometric Related Rates problems.

In chapter 3 of this thesis, many examples were used to illustrate how the seven step solution procedure of the Standard Solution model is applied.

In the literature review in chapter 4, the underlying concepts which underpin Related Rates problems are identified and specific examples of research on each of these concepts are given. For example, the review of the literature on word problems is done comprehensively and covers extensively the range of issues involved in this topic. Drawing on the work in chapter 2 on the nature of Related Rates problems, it is explained in chapter 5 why this study is underpinned by Constructivism as a theoretical basis.

Chapter 6 of the thesis is devoted to answering the 3 research questions posed in chapter 1. The thesis contains many worked examples of Related Rates problems which can be used by the lecturers assigned to the MAT105 course.

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## CHAPTER 1

## PREAMBLE

### 1.1 INTRODUCTION

In the present climate of globalisation, a country's economic success is dependent on its ability to create, attract and deploy human capital more efficiently than what its competitor nations can do. It is believed that skills and knowledge can help build this competiveness as well as increase employment. It is argued that South Africa does not have a skills crisis at present, because there is not enough economic growth to produce such pressure for skilled artisans. However, South Africa is showing signs of under-skilled artisans as well as specific short-term shortages, which was evident prior to and leading up to the 2010 FIFA Soccer World Cup. Currently, South Africa does not have the capacity to expand economically without foreign scientific and technological expertise (Pratzer, 1994).

South Africa needs to produce qualified professionals such as doctors, engineers and other scientifically orientated professionals and retain them. There is a current emphasis to produce graduates in scare skills areas such as engineering. Engineering programmes are considered cognitively demanding and require at least two years of university mathematics (Collier-Reed \& Smit, 2010). Mathematics forms the core for engineering and science studies at institutions of higher learning. Calculus is often viewed as the gatekeeper for high-level Mathematics, Science and Engineering courses (Moreno \& Muller, 1999).

However, students have problems grasping many of the concepts in calculus courses. The difficulty faced by first-year engineering students in a first year Mathematics course is not unique to South Africa. It is well documented that first year students do not do well in their first year mathematics (Weiss,1997; Wieschenberg, 1994; Budny, LeBold, \& Bjedov, 1997). A calculus course includes differential calculus topics such as Limits, Rates of change, Rules of Differentiation and Related Rates. Research regarding the first three topics is well documented but the topic of Related Rates is not well researched.

The factors influencing poor performance in first year mathematics courses can range from issues related to teaching strategies, content knowledge, motivation, laboratory use, and noncompletion of the syllabus in a year, to social factors such as the role played by parents in their children's education, and general language usage (Mji \& Makgato, 2006). This study will focus mainly on factors relating to content knowledge and teaching strategies.

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### 1.2 PROBLEM STATEMENT

The poor performance of students in the undergraduate mainstream mathematics module (MAT105) at the University of the Western Cape (UWC) has been a concern for a number of years. Students do not perform at an acceptable level to complete the course. Acceptable level in this context, means that students obtain $50 \%$ and above for the course. Below is a table of the through-puts of the Mathematics 105 (MAT105) course for 2010 and 2011. A pass mark is $50 \%$ and above.

## TABLE 1

| Module Code | MODULE DESCRIPTION | ENROLLED | PASS | \% PASS |
| :--- | :--- | :--- | :--- | :--- |
| MAT105 | MATHEMATICS 105 (2010) | 251 | 48 | $19 \%$ |

Source: University of the Western Cape

TABLE 2

| Module Code | MODULE DESCRIPTION | ENROLLED | PASS | \% PASS |
| :--- | :--- | :--- | :--- | :--- |
| MAT105 | MATHEMATICS 105 (2011) | 282 | 60 | $21 \%$ |

Source: University of the Western Cape

This is ironic, since the entrance requirement for the mainstream mathematics (MAT105) module at the UWC is above $50 \%$ in their previous final examination for mathematics.

The National Curriculum Statement for Grades R-12 (NCS) is a policy on curriculum and assessment in the schooling sector. This document was amended as changes were brought in and was called the Revised Curriculum Statement for Grades R - 12 (Department of Education, 2002). The document was then split into two sections to accommodate for grades R-9 and grades $10-12$ separately. Learning Outcome 1 of the Revised Curriculum Statement for Grades $\mathrm{R}-9$ states that "The learner will be able to recognise, describe and represent numbers and their relationships, and to count, estimate, calculate and check with competence and confidence in solving problems"(Department of Education, 2002, p21). This is the outcome that is consistent throughout the primary school. Below are three of the assessment standards that are required for a grade R learner, the grade just before a learner enters grade 1 of their primary school:

- Solves and explains solutions to practical problems that involve equal sharing and grouping with whole numbers of at least 10 and with solutions that include remainders.
- Solves verbally-stated additions and subtraction problems with single-digit numbers and with solutions to at least 10 .
- Explains own solutions to problems.

These three assessment standards are consistent throughout Grades R to Grade 9 except that the numbers that the learner must be able to count are increased as the grades increase. The assessment standards are also expanded to include other learning areas as the grades progress. Learning Outcome 1 of the Revised National Curriculum Statement for Grade10-12 states that "When solving problems, the learner is able to describe, represent and work confidently with numbers and their relationship to estimate, calculate and check answers" (Department of Education, 2002, p12). According to the above, problem solving, including word problems, are taught in grades R through 12. It can be argued that learners had a thorough treatment and assessment of problem solving before they enter university. To test this assumption, students who were registered for the Mathematics 115 (MAM115) course in 2011 were given a revision test on solving word problems during the first week of the university academic year. MAM115 is a service course to all students who study in the field of pharmacy, nursing and education at UWC. The five questions given to the students were based on the work of grades 9 to 12 . This was verified by a panel of three secondary school mathematics teachers. The five questions can be found in addendum A. The results of this test were not part of the initial analysis of this thesis, but it was interesting to note the
outcome. Each question had a total mark of five. The results of this test are summarized in the table below:

## TABLE 3

| SCORE | QUESTION 1 | QUESTION 2 | QUESTION 3 | QUESTION 4 | QUESTION 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $57.8 \%$ | $19.3 \%$ | $90.4 \%$ | $29.5 \%$ | $97.6 \%$ |
| 1 | $15.1 \%$ | $3.0 \%$ | $0.6 \%$ | $11.4 \%$ | $0.0 \%$ |
| 2 | $15.1 \%$ | $11.4 \%$ | $1.2 \%$ | $5.4 \%$ | $1.2 \%$ |
| 3 | $4.8 \%$ | $6.6 \%$ | $0.6 \%$ | $6.0 \%$ | $0.0 \%$ |
| 4 | $0.0 \%$ | $8.4 \%$ | $0.6 \%$ | $6.0 \%$ | $0.0 \%$ |
| 5 | $7.2 \%$ | $50.6 \%$ | $6.6 \%$ | $41.6 \%$ | $1.2 \%$ |

For the first question, the results showed that only $12 \%$ of the students who answered this question obtained a score of more than three. A massive $58 \%$ of students either did not attempt this question or answered this question incorrectly. Only six students attempted question 5, while only 13 students made an attempt to answer question 3. According to the panel who verified the five questions, the fifth question was a popular question that was frequently used in the final examination of the National Certificate exam since 2008. Question 2 and question 4 were the two questions that students answered the best ( $66 \%$ of students obtained a mark of more than 3 out of 5 for question 2 and $54 \%$ of students scored three or more for question 4.

The above results, which were not used in the final analysis of this thesis, gave some indication of the students' deficiencies or lack of preparation when solving word problems. This is in contrast to what is stated in Learning Outcome 1 of the Revised National Curriculum Statement for Grades R-9 and Grades 10-12.

### 1.3 MOTIVATION TO CONDUCT THIS STUDY

The author works in the department of Mathematics and Applied Mathematics at UWC where the poor performance has been identified. The author assisted a colleague to assess a class test involving the topic "Related Rates" during 2011. The results were so poor that it could not be considered for the students' continuous assessment mark. According to the author, despite numerous interventions by the department of Mathematics, no improvements were evident. This prompted the need to investigate the reasons for the students' poor performance in related rates problems.

### 1.4 AIM OF THE STUDY

This study aims to investigate the difficulties that first year mainstream mathematics students at UWC have with related rates problems.

### 1.5 RESEARCH QUESTIONS

By studying worked examples of Related Rates problems in various textbooks (Concepts and Contexts by James Stewart 4th edition, Metric Version ; Calculus With Analytical Geometry by Earl W. Swokowski, 2nd edition, Wadsworth International Student Edition and Calculus With Analytical Geometry by Joe Repka, 1st edition) it is evident that, although not explicitly stated by the authors of these textbooks, a stepwise procedure can be identified. These can range from three to five steps. In her study of geometric related rates problems, Martin (2000) referred to this stepwise procedure as the Standard Solution Model for geometric related rates problems. Several authors (National Assessment of Educational Progress [NAEP], 1988; Cooney et.al., 1975; Hiebert \& Lefevre, 1986) have made a distinction between procedural
knowledge and conceptual understanding. In Martin's Standard Solution Model for geometric related rates problems, the model consists of six steps which can be classified as either relying on students' conceptual understanding or their procedural knowledge. The seventh step is referred to as an auxiliary step. This step can require either conceptual understanding or procedural knowledge depending on the context of the problem. The following table shows these seven steps as well as the conceptual understanding or procedural knowledge it is associated with.

TABLE 4: Martin's Standard Solution Model for geometric related rates problems

| STEP | DESCRIPTION | CLASSIFICATION |
| :---: | :--- | :---: |
| 1 | Sketch the situation and label the sketch with variables or <br> constants | Conceptual |
| 2 | Summarize the problem statement by defining the variables <br> and rates involved in the problem (words to symbols <br> translation) and identifying the given and requested <br> information. | Conceptual |
| 3 | Identifying the relevant equation | Procedural |
| 4 | Implicitly differentiate the equation to transform a statement <br> relating measurements to a statement relating rates | Procedural |
| 5 | Substituting specific values of the variables into the related- <br> rates equation and solve the desired rate | Procedural |
| 6 | Interpret and report results | Conceptual |
| 7 | Solve an auxiliary problem | Either Conceptual or <br> Procedural |

Source: Martin (2000)
This study adopts this Standard Solution model for the solution of all types of related rates problems. The aim is to investigate how effective this model is in improving students' ability to solve related rates problems in general. This will be done by attempting to answer the following research questions:
(i) How do students perform on the conceptual steps no. 1, 2 and 6 of the standard solution module for solving related rates problems?
(ii) How do students perform on the procedural steps no. 3, 4 and 5 of the standard solution module for solving related rates problems?
(iii) How do students perform on the auxiliary step no. 7 of the standard solution module for solving related rates problems?

To clarify the word "perform", the author means that there is an improvement in the students' ability to solve Related Rates problems.

### 1.6 SIGNIFICANCE TO THE FIELD

The findings of this study could shed light on how to improve students' ability to solve Related Rates problems. This study offers a different approach to teaching the topic Related Rates in the first year calculus course which will be elaborated on in Addendum G of this thesis.

### 1.7 DEFINITIONS

For the purpose of this study, the following definitions will apply to terms deemed as important to this study.
(a) Related Rates:

A Related Rates problem refers to the type of calculus problem that requires the determination of the rate of change with respect to time of some variables based on their relationship to other variables whose rate of change are known (Dick \& Patton, 1992, p. 270).
(b) Procedural Knowledge:

Procedural knowledge is characterised by the ability to note, select, and apply the appropriate concrete, numerical, or symbolic procedures required to solve a problem; and to verify and justify the correctness of these procedures (National Assessment of Educational Progress [NAEP], 1988).
(c) Conceptual Understanding Conceptual understanding is characterised by the ability to identify examples and non-examples of a concept; to use, connect, and interpret various conceptual representations; to know, apply, distinguish, and integrate facts, definitions, and principles; and to interpret assumptions and relations in a mathematical setting (National Assessment of Educational Progress [NAEP], 1988).
(d) Poor Performance


Poor performances in this study refer to scores below $50 \%$, which is the pass mark for the module.

### 1.8 DELIMITATIONS

This study is limited to the first year mainstream mathematics students at UWC. Students enrolled in this course achieved a minimum of $50 \%$ in the National Senior Certificate mathematics exams of the previous year or earlier up to 2008. Students who wrote the National Senior Certificate exam (2007 or earlier) had to achieve a minimum of $40 \%$ if mathematics was taken on the higher grade, while a student had to achieve a minimum of $60 \%$ if mathematics was taken on the standard grade.

### 1.9 ETHICAL CONSIDERATIONS

The students who participated in this study agreed on condition that they remain anonymous. To protect the rights, dignity, safety and privacy of research subjects and the integrity of the environment, UWC developed a Research Ethics Policy that aims to govern the ethics of research across the university. This research policy was put in place to prevent harm upon subjects, where the impact of the research may be emotionally damaging to humans. The full document of the Research Policy is available from the university research committee on request.


## CHAPTER 2

## THE NATURE OF RELATED RATES PROBLEMS

### 2.1 INTRODUCTION

In this chapter, an attempt is made to find some of the reasons why students do not perform at an acceptable level when answering Related Rates problems as was mentioned in section 1.3. The reasons appear to be tied up with the nature of these types of problems. It is precisely because of this that textbook authors are able to list a stepwise procedure to solve these problems. This strategy to teach Related Rates is also useful in the assessment process whereby a checklist can be used by the lecturer for the application of the different steps. It is widely used in many universities. However judging from our own experience at UWC, the topic of related rates is very difficult for most students. In studying the nature of these problems, it is hoped that some light will be shed on the reasons for students' low performance in solving such problems and that remedial steps can be taken. In the next section, we discuss the approach of a few selected textbooks.

### 2.2 TEXTBOOKS

The prescribed textbook for a course is the one most important resource in the learning process. Lecturers often provide a list of additional books that students can consult as well as course notes and other reading material. In what follows, the approach to the teaching of Related Rates in different textbooks will be examined.

In 2012, the prescribed textbook for MAT105 was Calculus - Concepts and Contexts by James Stewart (4th edition, Metric Version). Stewart summarises a strategy to solve related rates problems. Here follows his 7 step strategy:

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution .
6. Use the Chain Rule to differentiate both sides of the equation with respect to time.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Stewart then gives forty four exercises after explaining his five worked out examples. The exercises can be found in Calculus - Concepts and Contexts by James Stewart (4th edition, Metric Version). From the forty four exercises, only five are accompanied with a sketch giving a realistic picture of the problem situation or diagram. It is required of students to draw their own sketch, which can lead to an incorrect diagram, depending on how students interpret the problem, and hence an incorrect solution.

The approach used by another textbook that is available to students, Calculus With Analytical Geometry by Earl W. Swokowski (2nd edition, Wadsworth International Student Edition), follows a similar introduction as that of Stewart, but Swokowski introduces a diagram called a schematic diagram, which by definition is a drawing showing all significant components, parts, or tasks, and their interconnections, of a circuit, device, flow, process or project by means of standard symbols. This diagram represents a geometric figure whereby an algebraic
equation can be obtained. No realistic visualization of the situation exists in any of his examples. The worked examples in Swokoswski's book all have schematic diagrams accompanying the solutions. Nothing is mentioned of a strategy or summary to solve Related Rates problems.

Calculus With Analytical Geometry by Joe Repka (1st edition) has more detail when discussing related rates. Repka states that there are no definite rules when solving Related Rates problems, but the general idea is the same. In his examples, Repka's problems are accompanied by the real life picture, followed by a schematic diagram of the problem. The solution then follows the same procedure as the previous two textbooks.

The last textbook that will be discussed is called Calculus by Larson et al. (4th edition). The approach used by Larson et al. is similar to that of Stewart except that the diagram that they use combine the real-life situation with the schematic diagram. They use a four step procedure to solve Related Rates problems that is similar to Stewart's procedure, except that two steps in Stewart's procedure are combined to make one step in the procedure of Larson et al.

### 2.3 CONCEPTUAL UNDERSTANDING AND PROCEDURAL KNOWLEDGE

As mentioned in chapter 1 (p. 6-7), each step in the Standard Solution Model for Related Rates problems are classified as either relying on students' conceptual understanding or their procedural knowledge. Based on previous results of Related Rates assessments, it is evident that students do not possess the required conceptual understanding or procedural knowledge to answer related rates problems confidently and correctly. By definition, conceptual understanding is characterised by the ability to identify examples and non-examples of a
concept; to use, connect, and interpret various conceptual representations; to know, apply, distinguish, and integrate facts, definitions, and principles; and to interpret assumptions and relations in a mathematical setting (National Assessment of Educational Progress [NAEP], 1988). According to Hiebert \& Lefevre (1986), conceptual knowledge is achieved in two ways: by "the construction of relationships between pieces of information" or by the "creation of relationships between existing knowledge and new information that is just entering the system". Hiebert and Lefevre make a secondary distinction between what they call primary level relationships and what they call the reflective level. The primary level refers to pieces of knowledge that are at the same level of abstraction. The reflective level refers to a higher level of abstraction from two pieces of knowledge that are initially conceived as separate pieces of knowledge. If students do not posses conceptual knowledge, students cannot construct relationships between pieces of information given in the problem situation. In Related Rates problems, students need to relate given information and construct a drawing or diagram to visualize the situation. From the diagram, the student has to construct a mathematical relation (equation) which relates the required and given rates. Students lack the ability to set up this equation. Hiebert \& Lefevre (1986) identify procedural knowledge as having a sequential nature, which includes among others:

- knowing the formal language, or the "symbol representation system",
- knowing algorithms and rules for completing tasks and procedures, and
- knowing strategies for solving problems.

Procedural knowledge is characterised by the ability to note, select, and apply the appropriate concrete, numerical, or symbolic procedures required to solve a problem; and to verify and justify the correctness of these procedures (National Assessment of Educational Progress [NAEP], 1988).

Based on previous results, students have little knowledge of algorithms needed to complete procedures and strategies required for problem solving. Furthermore, from the work of Polya (1963), Mason et al, (1982) as well as from the textbook series Discovering Advanced Algebra (2004), it is becoming clear that conceptual knowledge is intricately linked with procedural knowledge and algorithms. In fact, knowledge of procedures is nested in conceptual knowledge (Hiebert \& Lefevre, 1986). They concluded that it is the relationship between conceptual and procedural knowledge that holds the key to improved mathematical understanding. Students should not separate conceptual understanding from procedural knowledge but rather link the two to get a better understanding of Related Rates. In the following section, the way Related Rates problems are stated, is examined because this might have an important bearing on the reasons for students' low performance in solving Related Rates problems.

### 2.4 THE STATEMENT OF RELATED RATES PROBLEMS

The first step in trying to solve a related rates problem is to represent the given information in a realistic picture. A student reading the problem creates a mental picture that is associated with the given information. In the literature, this mental picture is referred to as a concept image (Tall \& Vinner, 1981). It is important that the problem is stated in such a way that students can create such a mental picture that should facilitate a correct solution.

Consider the following example:

1. A kite 50 m above the ground moves horizontally at a speed of $2 \mathrm{~m} / \mathrm{s}$. At what rate is the angle between the string and the horizontal decreasing when 100 m of string has
been let out? (This is an exercise as it appears in Calculus - Concepts and Contexts by James Stewart, 4th edition, Metric Version, p.219).

This statement is problematic. For example, how is the kite controlled? A student might interpret the first line in the problem by drawing the following picture.


Another student might interpret the first line by drawing the two pictures:



In the first picture, the kite is controlled by some mechanism fixed to the ground, while in the second picture, the kite is controlled by a person (either moving or stationary). The next step in solving Related Rates problems is to represent the given information in what some authors like Swokowski (Calculus With Analytical Geometry by Earl W. Swokowski ,2nd edition, Wadsworth International Student Edition) refer to as a schematic diagram. As a notion borrowed from Physics, a schematic diagram is a drawing showing all significant components, parts, or tasks (and their interconnections) of a circuit, device, flow, process, or project by means of standard symbols. In mathematics, a schematic diagram is most often a geometric figure such as a triangle, a circle or cone. In the first picture above, a schematic diagram would be a triangle with vertices $\mathrm{A}, \mathrm{B}$ and C , with A representing the mechanism controlling the kite, C representing the position of the kite at a distance $s$ meters from A, and 50 m above B , a point on the ground, a distance $x$ meters from A. A schematic diagram which represents the first picture is given below:


Let $\Theta$ be the angle between the string and the horizontal. In picture one above, a schematic diagram would be a diagram ADBC , where A represents the position of the person's hand controlling the kite, a height of $y$ metres above the ground from D .

The schematic diagram which represents picture 1 is given below:


It can therefore be seen that that the schematic diagram differs according to the way the problem is interpreted by the student. Of course this will also impact on the solution of the problem.

The problem must therefore be clearly stated to avoid any ambiguities. Hudson (1983) and Vicente et al (2007) have shown in their research that re-phrasing of the problem context has a positive impact on solving word problems by students.

Seifi et al (2012) suggested that rephrasing a word problem context impact positively on solving word problems. They also suggest that textbooks should use appropriate content since unfamiliar contents and language complexities in the problem statement make students unable to recognize the problem. Textbooks should use content that is familiar to the student. Visual representation, both real-life and schematic should accompany Related Rates problems.

### 2.5 REFERENCE POINT

We have seen that the first step in solving Related Rates problem is to draw, if it is not given, a realistic picture (a mental picture) of the problem. From this realistic picture, a schematic diagram of the problem must be produced on which all major components of the problem are depicted as dots or lines and distances are marked with symbols (called variables) which could be known or unknown according to the information provided in the problem. All distances are measured to and/or from a single point on the schematic diagram. This point is referred to as the reference point for the particular problem. A student trying to solve the problem must identify such a point on the schematic diagram. Here are a few examples: (the examples are from Calculus - Concepts and Contexts by James Stewart, 4th edition, Metric Version).

1. A street light is mounted at the top of a 6 -meter-tall pole. A man 2 m tall walks away from the pole with a speed of $1.5 \mathrm{~m} / \mathrm{s}$ along a straight path. How fast is the tip of his shadow moving when he is 10 m from the pole?

A realistic picture of the above problem is given below:


A schematic diagram of this picture is given below with the point of reference indicated:

where B indicate the length of the pole,

2 m indicates the height of the man,
$x$ indicates the distance the man walks from the light pole,
$l$ is the length of the man's shadow on the ground in front of the man and

R indicates the position of the point of reference, the base of the pole.
2. Two carts, A and B, are connected by a rope 12 m long that passes over a pulley P. the point Q is on the floor 4 m directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of $0.5 \mathrm{~m} / \mathrm{s}$. How fast is cart B moving toward Q at the instant when cart A is 3 m from Q .

A realistic picture of the above problem is given below


A schematic diagram of this picture is given below with the point of reference indicated:


Where A indicates the position of cart A, B indicates the position of cart B,
$P$ is the position of the pulley,
$u$ is the length of the rope from A to the P ,
$12-u$ is the length of the rope from P to the B ,
$x$ is the distance from Q to A ,
y is the distance from B to Q and

Q is the point of reference.

It is clear from these two examples that a student's realistic picture of the problem situation, which might not be given, plays an important role in identifying a reference point, which in turn is central to drawing a schematic diagram, an important aid in the mathematical modelling of the problem situation. Without the correct identification of a reference point, the problem difficulty is increased significantly.

### 2.6 VISUALIZATION IN RELATED RATES PROBLEMS

Many Related Rates problems describe the position of one object in relation to another object. For example when two cars approach an intersection from two different directions, two aircraft flying horizontally over an airport; one flying west and the other flying north. Let us consider the following example:

Example 1: At noon, ship A is 150 km east of ship B. Ship A is sailing west at a constant speed of $35 \mathrm{~km} / \mathrm{h}$ and ship B is sailing north at a constant speed of $25 \mathrm{~km} / \mathrm{h}$. How fast is the distance between the ships changing at 16 h 00 ?

When an observer views the two ships say from a harbour or sea shore, the scenario described in the problem may look like the picture below.


This is the position of the two ships at noon.

When ship A sails west and ship B sails north, the observer sees that ship A sails in the direction where ship B was at noon. Ship B sails away from the observer in a straight line. Imagine the sea being a flat horizontal surface and the two ships are moving (sailing) on the surface. Ship A is sailing towards the position of ship B at noon and ship B is sailing away from this position. We can consider the position of ship B at noon as our reference point.

Location of ship A at noon $\qquad$ Location of ship B at noon

Movement of ship A

We can now draw a schematic diagram indicating the movement of the ships.


Indicating all the relevant information and symbols, the schematic diagram looks as follows:

where D is the position of ship A at noon,
A is the position of ship A after $t$ hours,
$C$ is the position of ship $B$ at noon and
B is the position of ship B after $t$ hours.

Note that C is the reference point in this case. If we let the distance of ship A sailed after $t$ hours be $x$, the distance of ship B from the reference point C after $t$ hours be $y$, and the distance between the two ships be $z$ after $t$ hours, the schematic diagram looks as follows:


Note that the distance of DC is 150 km and AC is $150-x$, since $\mathrm{DA}=x$.

From the information above, we can obtain an algebraic relation between $x, y$ and $z$. In triangle $\mathrm{ABC}, z^{2}=(150-x)^{2}+y^{2}$. To obtain the desired rate, we need to differentiate implicitly with respect to time $t$. If we differentiate $z^{2}=(150-x)^{2}+y^{2}$ with respect to time $t$, we obtain $2 z \frac{d z}{d t}=-2(150-x) \frac{d x}{d t}+2 y \frac{d y}{d t}$. The rate at which the distance between the two ships is changing is denoted by $\frac{d z}{d t}$. To solve $\frac{d z}{d t}$, we need to substitute the values of $x, y, z$, $\frac{d x}{d t}$ and $\frac{d y}{d t}$. It should be noted that $\frac{d x}{d t}=35 \mathrm{~km} / \mathrm{h}$ and $\frac{d y}{d t}=25 \mathrm{~km} / \mathrm{h}$. The values of $x$ and $y$ can be calculated after 4 hours using speed $=\frac{\text { dis } \tan c e}{\text { time }}$ or distance $=$ speed times time. For $x$, we have $x=(35 \mathrm{~km} / \mathrm{h})(4 \mathrm{~h})=140 \mathrm{~km}$. Similarly, $y=(25 \mathrm{~km} / \mathrm{h})(4 \mathrm{~h})=100 \mathrm{~km}$. The value of $z$ can be determined by using the theorem of Pythagoras: $z=\sqrt{(150-x)^{2}+y^{2}}$. By substituting $x=140 \mathrm{~km}$ and $y=100 \mathrm{~km}$, we have $z=10 \sqrt{101}$. Substituting these values into the equation $2 z \frac{d z}{d t}=-2(150-x) \frac{d x}{d t}+2 y \frac{d y}{d t}$, we obtain that $\frac{d z}{d t}=21.39 \mathrm{~km} / \mathrm{h}$.

Let us consider another example.

Example 2: An aircraft flying horizontally at an altitude of 2 km and a speed of $800 \mathrm{~km} / \mathrm{h}$ passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 3 km away from the station (This is one of the exercises in Calculus - Concepts and Contexts by James Stewart, 4th edition, Metric Version).

A realistic picture of the information in example 2 is the following:


In order to construct a schematic diagram in example 2, first imagine a point at a height 2 km directly above the radar station. We will call this a reference point for the given problem. The aircraft and the radar station lie in the same vertical plane. A schematic diagram is a right angled triangle in the vertical plane with vertices R (reference point), S (Radar station) and P (aircraft).


If $x$ is the distance the aircraft flies after $t$ hours and $z$ is the distance between the radar station and the aircraft after $t$ hours, the information on the schematic diagram looks like the following:


From the above information, we can find a mathematical relation between the unknown and the constant values. In this case, the mathematical relation is: $z^{2}=x^{2}+2^{2}$. To obtain the desired rate, we need to differentiate implicitly with respect to time $t$. If we differentiate $z^{2}=x^{2}+2^{2}$ with respect to time t , we obtain $2 z \frac{d z}{d t}=2 x \frac{d x}{d t}$. The rate at which the distance from the plane to the station is increasing is denoted by $\frac{d z}{d t}$. To find $\frac{d z}{d t}$, we need to substitute the values of $x, z$ and $\frac{d x}{d t}$ into the equation $2 z \frac{d z}{d t}=2 x \frac{d x}{d t}$. It should be noted that $\frac{d x}{d t}=800 \mathrm{~km} / \mathrm{h}$. We can find the value of $x$ using the equation $x=\sqrt{z^{2}-2^{2}}$, when $z=3$. Therefore $x=\sqrt{3^{2}-2^{2}}=\sqrt{5} \mathrm{~km}$. By substituting $x=\sqrt{5} \mathrm{~km}, z=3$ and $\frac{d x}{d t}=800 \mathrm{~km} / \mathrm{h}$ into the equation $2 z \frac{d z}{d t}=2 x \frac{d x}{d t}$, we obtain that $\frac{d z}{d t}=596.28 \mathrm{~km} / \mathrm{h}$.

Example 3: At a certain instant an aircraft flying due east at $400 \mathrm{~km} / \mathrm{h}$ passes directly over a car travelling due southeast at $100 \mathrm{~km} / \mathrm{h}$ on a straight road, level road. If the aircraft is flying at an altitude of 1 km , how fast is the distance between the aircraft and the car increasing 36 seconds after the aircraft passes directly over the car? (This is an example from the textbook "Single Variable Calculus Revised" by Robert A. Adams, Addison Wesley Publisher 1986)

A realistic picture describing the situation in example 3 is given below:


The schematic diagram that we obtain from the above realistic picture will look as follows:


At a glance, the situation described in example 3 seems similar to that which is described in example 2. However the difference becomes apparent when we try to find an algebraic equation which gives a relationship between the position of the car and the aircraft. We describe how this can be done. The car is in a vertical plane formed by R, C and T. The aircraft is in a different plane formed by $\mathrm{R}, \mathrm{T}$ and P , which is an inclined plane. Because the car and the aircraft are in two different planes, it will be very difficult to describe the movement relative to one another. To make it easier, we make a construction as follows: Draw a vertical line through T to form the rectangle RCTS as in the sketch below. The aircraft is now in the horizontal plane formed by $\mathrm{R}, \mathrm{S}$ and P . It is now easy to find an algebraic relationship describing the movement of the car relative to the aircraft.


Since $\mathrm{CT}=y, \mathrm{RS}=y$ also, since RCTS forms a rectangle. Also if $\mathrm{RC}=1 \mathrm{~km}$, then $\mathrm{ST}=1$. We let $\mathrm{SP}=u$ and $\mathrm{TP}=z$. Note that angle $\mathrm{PRS}=45^{\circ}$, since the car travels due south east. According to the research done by Cankoy and Özder (2011), visual representations in word problems can reduce problem difficulty. To complete the solution of this problem, we start in triangle STP. In $\Delta \mathrm{STP}, z^{2}=1^{2}+u^{2}$. In $\Delta \mathrm{RSP}, u^{2}=x^{2}+y^{2}-2 x y \cos 45^{\circ}$. Substituting this $u^{2}$ into the previous equation, we obtain
$z^{2}=1^{2}+x^{2}+y^{2}-2 x y \cos 45^{\circ}$, which simplifies to $z^{2}=1^{2}+x^{2}+y^{2}-\sqrt{2} x y$. When we differentiate this equation with respect to $\operatorname{time}(t)$, we obtain $2 z \frac{d z}{d t}=2 x \frac{d x}{d t}+2 y \frac{d y}{d t}-\sqrt{2}\left(y \frac{d x}{d t}+x \frac{d y}{d t}\right)$. Making $\frac{d z}{d t}$ the subject of the formula, we obtain $\frac{d z}{d t}=\frac{x \frac{d x}{d t}+y \frac{d y}{d t}-\frac{\sqrt{2}}{2}\left(y \frac{d x}{d t}+x \frac{d y}{d t}\right)}{z}$. Since $\frac{d x}{d t}=400 \mathrm{~km} / \mathrm{h}$ and $\frac{d y}{d t}=100 \mathrm{~km} / \mathrm{h}$ is given, we only need to find the values of $x, y$ and $z$. The problem requires us to find how fast the distance between the aircraft and the car is increasing 36 seconds after the aircraft passes directly over the car. Now 36 seconds converts to 0.01 hours. So if the aircraft flies at a speed of $\frac{d x}{d t}=400 \mathrm{~km} / \mathrm{h}$, the aircraft flies 400 km in 1hour. So in 0.01 hours, the aircraft covers $x=(0.01)(400)=4 \mathrm{~km}$. Also, the car
travels at speed of $\frac{d y}{d t}=100 \mathrm{~km} / \mathrm{h}$, the car travels 100 km in 1 hour. So in 0.01 hours, the car covers $y=(0.01)(100)=1 \mathrm{~km}$. To find z , we know that $z^{2}=1^{2}+x^{2}+y^{2}-\sqrt{2}$; therefore $z=\sqrt{1^{2}+x^{2}+y^{2}-\sqrt{2} x y}$. We know that $x=4 \mathrm{~km}$ and $y=1 \mathrm{~km}$. Substituting these values into $z=\sqrt{1^{2}+x^{2}+y^{2}-\sqrt{2} x y}$, we obtain $z=\sqrt{1^{2}+(4)^{2}+(1)^{2}-\sqrt{2}(4)(1)}=3.513 \mathrm{~km}$. We now substitute $x=4 \mathrm{~km}, y=1 \mathrm{~km}$, $\mathrm{z}=3.513 \mathrm{k} m, \frac{d x}{d t}=400 \mathrm{~km} / \mathrm{h}$ and $\frac{d y}{d t}=100 \mathrm{~km} / \mathrm{h}$ in the equation $\frac{d z}{d t}=\frac{x \frac{d x}{d t}+y \frac{d y}{d t}-\frac{\sqrt{2}}{2}\left(y \frac{d x}{d t}+x \frac{d y}{d t}\right)}{z}$ to solve how fast is the distance between the aircraft and the car increasing 36seconds after the aircraft passes directly over the car. Therefore

$$
\frac{d z}{d t}=\frac{(4)(400)+(1)(100)-\frac{\sqrt{2}}{2}((1)(400)+(4)(100))}{(3.513)}=322.89 \mathrm{~km} / \mathrm{h} . \text { Therefore the distance }
$$

between the aircraft and the car is increasing at a speed of $322.89 \mathrm{~km} / \mathrm{h}$.

Obviously this problem is not easy since it requires the ability to visualize the problem situation. This is sadly lacking in most of our first year students and might easily be one of the most important reasons why students have difficulty with Related Rates problems.

The recommendation of Cankoy and Özder (2011) in their study was that textbooks and other instructional and assessment materials should be enriched by visual representations to reduce the level of cognitive load associated with the problem solving task.

### 2.7 THE CONTEXT OF RELATED RATES PROBLEMS

In South Africa, the three popular national or recreational sports are football, rugby and cricket. Students who were registered for the MAT105 module had diverse backgrounds. Below is a table indicating the demographics of the students in terms of the South African provinces they came from:

TABLE 5

| PROVINCE | EASTERN <br> CAPE | GAUTENG | KWAZULU <br> NATAL | MPUMALANGA | NORTH <br> WEST | NORTHERN <br> PROVINCE | WESTERN <br> CAPE |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| TOTAL | 32 | 18 | 9 | 6 | 2 | 10 | 142 |

Source: University of the Western Cape
Students who were not South African citizens are classified according to their nationality.
Below is a table according different nationalities outside South Africa:

TABLE 6

| COUNTRY | TOTAL |
| :--- | :---: |
| ANGOLA | 2 |
| CAMEROON | 2 |
| CHINA | 1 |
| DEMOCRATIC REPUBLIC OF CONGO | 2 |
| ETHIOPIA | 1 |
| GABON | 1 |
| MOZAMBIQUE | 1 |
| NAMIBIA | 1 |
| NIGERIA | 1 |
| SWAZILAND | 1 |
| TURKEY | 1 |
| UGANDA | 1 |
| ZAMBIA | 3 |
| ZIMBABWE |  |

Source: University of the Western Cape

Let us examine the following example:

1. A baseball diamond is a square with side 90 feet. A batter hits the ball and run towards first base with a speed of $24 \mathrm{ft} / \mathrm{s}$. At what rate is his distance from second base decreasing when he is halfway to first base?

(This example is taken from Calculus - Concepts and Contexts by James Stewart, 4th edition, Metric Version).

In the above example, the terms "baseball diamond", "first base" and "second base" are used. Students from South Africa as well as students from the rest of Africa, more so, students who do the MAT105 course are not familiar with the terms "baseball diamond" and "first and second base". Even with a realistic picture, this problem will be difficult for most students. The next problem might also be difficult for similar reasons.
2. Gravel is being dumped from a conveyor belt at a rate of $3 \mathrm{~m}^{3} / \mathrm{min}$ and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 3 m high?


The familiarity with the concepts in a word problem might reduce problem difficulty and can enhance problem solving. Hembree (1992) in his research concluded that familiar context strongly influences students' problem solving in a positive way. Cordova and Lepper (1996), López and Sullivan (1992) as well as Ku and Sullivan (2002) all concluded in their studies that familiar contexts enhance word problem solving by increasing the meaningfulness of the contexts and thus motivating the students to solve the problem.

Cankoy and Özder (2011) suggested that familiar problem contexts, especially related to students' life, should be considered when the cognitive load associated with the problem solving task is high.

### 2.8 LANGUAGE COMPLEXITY OF RELATED RATES PROBLEM

Some Related Rates problems are difficult due to the complexity of the language used in the statement of the problem. This could include unfamiliar words and phrases which hamper students in visualizing the situation sketched in the problem. For example:

1. A boat is pulled into a dock by a rope attached to the bow of a boat and passing through a pulley on the dock that is 1 meter higher than the bow of the boat. If the rope is pulled in at a rate of $1 \mathrm{~m} / \mathrm{s}$, how fast is the boat approaching the dock when it is 8 meters from the dock?

Difficulty with the language in which the problem is presented has been proposed as an obstacle to students' successful mathematical problem solving. Davidson (1977) found that low verbal ability or lack of familiarity with the language used in problems did hamper students' understanding of word problems.

### 2.9 TRANSLATING VERBAL EXPRESSIONS INTO MATHEMATICAL SYMBOLS IN RELATED RATES PROBLEMS

Consider the following example:
3. A spherical drop of water loses moisture by evaporation at a rate proportional to its surface. What can you say about its radius?

When we examine the sentence "A spherical drop of water loses moisture by evaporation at a rate proportional to its surface", it is expected of a student to obtain an equation relating the water loss, which is a decrease in volume (denoted by $\frac{d V}{d t}$ ) to the surface of the spherical drop, which in this case is the area (denoted by $A$ ). When we examine the National Curriculum statement for Grades R-12 (NCS), the topic proportionality is not properly dealt with in any of the grades R to 12 . The level at which it is treated in these grades are limited to ratios or fractions. In the above example, if a spherical drop of water loses moisture by evaporation at a rate proportional to its surface, the relation in mathematical symbols is: $\frac{d V}{d t} \sim A$. The proportion symbol $(\sim)$ is not discussed in any part of the curriculum of the National Curriculum statement for Grades R-12 (NCS). Therefore students will not be familiar with the notion that if $\frac{d V}{d t} \sim A$, then $\frac{d V}{d t}=-k A$, where $k$ is a positive constant value.

The negative sign indicates that the water is lost through evaporation and hence the volume decreases with time. Therefore it can be assumed that if students do not have the proper treatment of proportionality, then they will not answer the above question correctly. A similar example was also found in another textbook by Larson et. al (1979).
4. A lump of modeling clay is being rolled out so that it maintains the shape of a circular cylinder. If the length is increasing at a rate proportional to itself, show that the radius is decreasing at a rate proportional to itself.

Since the clay is rolled out in the shape of a circular cylinder and the length is increasing at a rate proportional to itself, the equation that relates $V$ its volume, $L$ its length and $r$ its radius, is $V=\pi r^{2} L$. If the shape remains the same, then $V$ remains unchanged. Now if the length is increasing at a rate proportional to itself, then $\frac{d V}{d t}=k L$, where $k$ is any positive constant value. As discussed above, students will find problems dealing with proportionality difficult.

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### 2.10 RECOGNIZING SIMILAR PROBLEMS IN DIFFERENT CONTEXTS

When we examine the solution of Related Rates problems, most of the solutions follow a similar pattern. The question that arises is: Can students recognize similar problems but in a different context? Can students transfer their skills gained by solving a particular problem to a "new" problem? Let us consider the following two examples:

1. A boat is being pulled toward a pier by a rope attached to its bow. A person on the pier is pulling in the rope at a rate of $6 \mathrm{~m} / \mathrm{min}$. If the person's hand is 5 meters higher than the bow of the boat, how fast is the boat moving toward the pier when there are still 13 meters of rope out.

This problem is identical to problem 1 in section 2.8 but students might not recognize this as the problems are phrased differently. A second example is the following problem.
2. Two crates, A and B, are on the floor of a warehouse. The crates are joined by a rope 33 feet long, each crate being hooked at floor level to an end of the rope. The rope is stretched tight and passes over a pulley P that is attached to a rafter 12 feet above a point Q on the floor directly between the two crates. If crate A is 5 feet from Q and is being pulled directly away from Q at a rate of $0.5 \mathrm{ft} / \mathrm{sec}$, how fast is crate B moving toward Q ?

This problem is identical to problem 2 in section 2.5. Again students might fail to recognise this because of the difference in context.

Krutetskii(1976) found that good problem solvers have the ability to see quickly and accurately the mathematical structure of a problem as well as the ability to generalise across a wide range of similar problems. He also concluded in his study that good problem solvers have the ability to remember a problem's formal structure for a long time. Foong (1990) in her study found that successful problem solvers translate the problem statement more correctly and more exactly than unsuccessful problem solvers do.

### 2.11 CURRICULUM TOPICS THAT WERE EXCLUDED FROM THE NATIONAL CURRICULUM STATEMENT FOR GRADES R-12 (NCS)

Another reason for students' difficulty with Related Rates problems is the mathematical background of incoming students. Many topics including Euclidean Geometry and some topics in trigonometry were excluded from the National Curriculum Statement of the senior
phase of the high school. The next example shows that this knowledge is crucial for the solution of certain Related Rates problems.

A man walks along a straight path at a speed of $1,5 \mathrm{~m} / \mathrm{sec}$. A searchlight is located on the ground 6 m from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 8 m from the point on the path closest to the searchlight?

## Solution

Let $x$ be the distance the man walks from the point on the path closest to the searchlight. Then $\frac{d x}{d t}=1,5 \mathrm{~m} / \mathrm{sec}$. Let $\Theta$ be the angle through which the searchlight rotates. Let $s$ be the length of the beam.

$\tan \theta=\frac{x}{6}$
$\therefore \sec ^{2} \theta \frac{d \theta}{d t}=\frac{1}{6} \frac{d x}{d t}$
$\Rightarrow \frac{d \theta}{d t}=\frac{1}{6 \cdot \sec ^{2} \theta} \frac{d x}{d t}$
$\Rightarrow \frac{d \theta}{d t}=\frac{\cos ^{2} \theta}{6} \frac{d x}{d t}$

Also from the diagram, we also have
$s^{2}=x^{2}+6^{2}$

Since $x=8 \mathrm{~m}$, we have

$$
\begin{aligned}
& s^{2}=x^{2}+6^{2} \\
& \Rightarrow s=\sqrt{x^{2}+6^{2}} \\
& \Rightarrow s=\sqrt{8^{2}+6^{2}} \\
& \Rightarrow s=10 .
\end{aligned}
$$

So
$\cos \theta=\frac{6}{10}$
$\Rightarrow \cos ^{2} \theta=\frac{36}{100}$

Therefore, from
$\frac{d \theta}{d t}=\frac{\cos ^{2} \theta}{6} \frac{d x}{d t}$,
we have
$\frac{d \theta}{d t}=\frac{\left(\frac{36}{100}\right)^{2}}{6} 1,5$
$\Rightarrow \frac{d \theta}{d t}=\frac{9}{100}=0,09 \mathrm{rad} / \mathrm{sec}$.
(Step 5)

Therefore the searchlight is rotating at $0,09 \mathrm{rad} / \mathrm{sec}$. recognise that $\tan \theta=\frac{x}{6}$. This does not seem too much of an issue to the student. Finding the derivative of this equation implicitly with respect to time $(t)$, we obtained $\sec ^{2} \theta \frac{d \theta}{d t}=\frac{1}{6} \frac{d x}{d t}$. The ratios $\sec \theta, \operatorname{cosec} \theta$ and $\cot \theta$ are unfamiliar, since these ratios were not part of the curriculum of the National Curriculum Statement for Grades R-12 (NCS) which were implemented in grade 12 from 2008, in grade 11 from 2007 and grade 10 from 2006. These ratios did however form part of the national senior certificate exam prior to 2008. Hence the
students who wrote their final mathematics from 2008 were not taught these ratios as well as the trigonometric identities $\tan ^{2} \theta+1=\sec ^{2} \theta$ and $\cot ^{2} \theta+1=\operatorname{cosec}^{2} \theta$. As we can see, the first identity is also used in the solution of this problem. It must be noted that only 26 students registered for MAT105 in 2012 wrote their final mathematics exam in grade 12 prior to 2008 . Therefore it can be argued that the majority of students from the sample used were not taught the definitions of the ratios $\sec \theta, \operatorname{cosec} \theta$ and $\cot \theta$ as well as the trigonometric identities $\tan ^{2} \theta+1=\sec ^{2} \theta$ and $\cot ^{2} \theta+1=\operatorname{cosec}^{2} \theta$. Therefore the majority of students will not be able to solve this problem as certain topics required in this problem, were not part of their curriculum at school level. Let us look at another example.

A water tank is in the shape of a right circular cone that has a radius of 5 feet and a height of 10 feet. It is positioned so that the cone points straight down. Water is being drained out of the tank at the rate of 2 cubic feet per minute. At what rate is the height of the water in the tank changing when there are $18 \pi$ cubic feet of water in the tank?

Le $V$ be the volume of the water in the tank at any time $t$, where the tank has the shape of a right circular cone. Let $h$ be the height of the water in the tank at any time $t$ and let $r$ be the radius of the water in $r$. Then $\frac{d V}{d t}=-2 f t^{3} / \mathrm{sec}$.

(Step 1)
$\frac{r}{5}=\frac{h}{10}$
$\Rightarrow r=\frac{5 h}{10}=\frac{h}{2}$
$\therefore V=\frac{1}{3} \pi r^{2} h$
$\Rightarrow V=\frac{1}{3} \pi\left(\frac{h}{2}\right)^{2} h$
$\Rightarrow V=\frac{1}{3} \pi \frac{h^{3}}{4}$
$\Rightarrow V=\frac{\pi}{12} h^{3}$
$\therefore \frac{d V}{d t}=\frac{\pi}{4} h^{2} \frac{d h}{d t}$

Also since $r=\frac{h}{2}$, we obtained $\frac{d r}{d t}=\frac{1 d h}{2 d t}$

So from $V=\frac{1}{3} \pi r^{2} h$, we have

$$
\begin{aligned}
& \frac{d V}{d t}=\frac{1}{3} \pi\left(2 r \frac{d r}{d t} h+r^{2} \frac{d h}{d t}\right) \\
& \Rightarrow \frac{d V}{d t}=\frac{1}{3} \pi\left(2\left(\frac{h}{2}\right) \frac{d r}{d t} h+\left(\frac{h}{2}\right)^{2} \frac{d h}{d t}\right) \\
& \Rightarrow \frac{d V}{d t}=\frac{1}{3} \pi\left(h^{2} \frac{d r}{d t}+\frac{h^{2}}{4} \frac{d h}{d t}\right) \\
& \Rightarrow \frac{d V}{d t}=\frac{1}{3} \pi\left(h^{2} \frac{1}{2} \frac{d h}{d t}+\frac{h^{2}}{4} \frac{d h}{d t}\right) \\
& \Rightarrow \frac{d V}{d t}=\frac{1}{3} \pi \frac{3 h^{2}}{4} \frac{d h}{d t} \\
& \Rightarrow \frac{d V}{d t}=\frac{\pi}{4} h^{2} \frac{d h}{d t}
\end{aligned}
$$

Also from $V=\frac{1}{3} \pi r^{2} h$ and $r=\frac{h}{2}$ we obtained $V=\frac{\pi}{12} h^{3}$

But since $V=18 \pi$, we obtain

$$
\begin{align*}
& 18 \pi=\frac{\pi}{12} h^{3} \\
& \Rightarrow h^{3}=216 \\
& \Rightarrow h=\sqrt[3]{216} \\
& \Rightarrow h=6 \tag{Step5}
\end{align*}
$$

So from $\Rightarrow \frac{d V}{d t}=\frac{\pi}{4} h^{2} \frac{d h}{d t}$, we have that
$\frac{d h}{d t}=\frac{4}{\pi h^{2}} \frac{d V}{d t}$
$\Rightarrow \frac{d h}{d t}=\frac{4}{\pi(6)^{2}}(-2)$
$\Rightarrow \frac{d h}{d t}=-\frac{4}{3 \pi}$
$\Rightarrow \frac{d h}{d t} \approx-0,42 f t / \mathrm{sec}$

(Step 5)

Therefore the height of the water in the tank is decreasing at a rate of $0.42 \mathrm{ft} / \mathrm{sec}$.

To obtain $\frac{r}{5}=\frac{h}{10}$ in the first line of the solution, the students needed to understand the concept of similar triangles. The National Curriculum Statement for Grades R-12 (NCS) excluded this topic from the curriculum and hence learners were not taught this section. This topic as well as Euclidean geometry for grades 11 and 12 involving circles were excluded from the curriculum prior to 2008 in grade 12 . Thus the majority of students who wrote their final mathematics exam from 2008 were thus not prepared to deal with topics that were excluded from their curriculum to answer Related Rates problems confidently.

### 2.12 CONCLUSION

In this chapter, we examined some examples of Related Rates problems in order to understand the nature of these problems. Several issues emerged from this exercise that could potentially contribute to the students' weak performance in solving Related Rates problems. Our experiences of working with students in tutorial sessions on Related Rates problems confirm that students might have difficulty with:
(i) The approach in different textbooks.
(ii) The way Related Rates problems are stated.
(iii) Identifying a reference point in a particular problem.
(iv) Visualizing a problem situation.
(v) Understanding the context in which the problem is stated.
(vi) Language complexity of related rates problems.
(vii) Translating verbal expressions into mathematical symbols.
(viii) Recognizing similar problems in different contexts.
(ix) Related Rates problems due to the ill-preparedness for university mathematics as a result of certain topics being excluded from the school curriculum.

## CHAPTER 3

## THE STANDARD SOLUTION MODEL FOR RELATED RATES PROBLEMS

### 3.1 INTRODUCTION

The term "Related Rates problem" refers to the type of word problem that requires the determination of "the rate of change with respect to time of some variables based on their relationship to other variables whose rates of change are known" (Dick \& Patton, 1992, p.270). Students' difficulty with geometric Related Rates problems has been noted by several authors (Balomenos et. al., 1987; White \& Mitchelmore, 1996). Balomenos, Ferrini-Mundy \& Dick (1987) suggested that one way to improve student performance on geometric Related Rates problems would be to have them gain more experience doing the geometry problems that are often embedded in them. In addition, the multi-step, multi-faceted nature of these problems have provided fertile ground to examine students' procedural knowledge and conceptual understanding of several important mathematical concepts that appear throughout the calculus course (Martin, 2000). By studying worked examples of Related Rates problems in various textbooks (Concepts and Contexts by James Stewart 4th edition, Metric Version ; Calculus With Analytical Geometry by Earl W. Swokowski, 2nd edition, Wadsworth International Student Edition and Calculus With Analytical Geometry by Joe Repka, 1st edition) it is evident that, although not explicitly stated by the authors of these textbooks, a stepwise procedure can be identified. These can range from three to five steps. In her study of geometric related rates problems, Martin (2000) referred to this stepwise procedure as the

Standard Solution Model for geometric related rates problems. Several authors (National Assessment of Educational Progress [NAEP], 1988; Cooney et al., 1975; Hiebert \& Lefevre, 1986) have made a distinction between procedural knowledge and conceptual understanding. Procedural knowledge is characterized by the ability to note, select and apply the appropriate concrete, numerical or symbolic procedures required to solve a problem and to verify and justify the correctness of these procedures. Conceptual understanding is characterized by the ability to identify examples and non-examples of a concept; to use, connect, and interpret various conceptual representations; to know, apply, distinguish, and integrate facts, definitions, and principles; and to interpret assumptions and relations in a mathematical setting (NAEP, 1998). White and Mitchelmore (1996) claimed that procedural knowledge and conceptual understanding involve different types of concepts.

Abstract-general concepts are formed by a generalising $\rightarrow$ synthesising $\rightarrow$ abstracting sequence (Dreyfus, 1991) or an interiorisation $\rightarrow$ condensation $\rightarrow$ reification process (Sfard,1991). Such concepts are linked to one another to form conceptual understanding. By contrast, abstract-apart concepts are formed by learning symbolic manipulations without reference to their meaning. Students whose concepts are abstract-apart can only acquire procedural knowledge. In Martin's Standard Solution Model for geometric related rates problems, the model consists of six steps which can be classified as either relying on students' conceptual understanding or their procedural knowledge. The seventh step is referred to as an auxiliary step. This step can require either conceptual understanding or procedural knowledge, depending on the context of the problem. The following table shows these seven steps as well as the conceptual understanding or procedural knowledge it is associated with.

TABLE 7: Martin's Standard Solution Model for geometric related rates problems

| STEP | DESCRIPTION | CLASSIFICATION |
| :---: | :--- | :---: |
| 1 | Sketch the situation and label the sketch with variables or <br> constants | Conceptual |
| 2 | Summarize the problem statement by defining the variables <br> and rates involved in the problem (words to symbols <br> translation) and identifying the given and requested <br> information. | Conceptual |
| 3 | Identifying the relevant equation | Procedural |
| 4 | Implicitly differentiate the equation to transform a statement <br> relating measurements to a statement relating rates | Procedural |
| 5 | Substituting specific values of the variables into the related- <br> rates equation and solve the desired rate | Procedural |
| 6 | Interpret and report results | Conceptual |
| 7 | Solve an auxiliary problem | Either Conceptual or <br> Procedural |

Source: Martin (2000)

### 3.2 EXAMPLES OF RELATED RATES PROBLEMS

In what follows, we show by a few worked examples how Martin's Standard Solution Model can be applied to all Related Rates problems, not just to geometric Related Rates problems, which was the focus of her study. We identify at various stages of the solution the steps in the above table.

Example 1. Two sides of a triangle are $4 m$ and $5 m$ in length and the angle between them is increasing at a rate of $0,06 \mathrm{rad} / \mathrm{s}$. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\frac{\pi}{3}$.

## Solution

We let $\theta$ be the angle between the two sides. We name the triangle $\triangle \mathrm{EBC}$ with $E B=4$ and $B C=5$. We also let $A$ be the area of the triangle at time $t$.

(Step 1)

$$
\begin{align*}
& A=\frac{1}{2} \cdot 4 \cdot 5 \cdot \sin \theta  \tag{Step3}\\
& \Rightarrow A=10 \sin \theta \\
& \frac{d A}{d t}=10 \cos \theta \frac{d \theta}{d t} \tag{Step4}
\end{align*}
$$

But since $\frac{d \theta}{d t}=0,06 \mathrm{rad} / \mathrm{s}$ and $\theta=\frac{\pi}{3}$, we have

$$
\begin{aligned}
\frac{d A}{d t} & =10 \cdot \cos \left(\frac{\pi}{3}\right) \cdot(0,06) \\
\Rightarrow & \frac{d A}{d t}
\end{aligned}=0,3 \mathrm{~m}^{2} / \mathrm{s} .
$$

Example 2: A kite, controlled by a stationary person, 50 m above the ground moves horizontally at a speed of $2 \mathrm{~m} / \mathrm{s}$. At what rate is the angle between the string and the horizontal decreasing when 100 m of string has been let out?

## Solution

Let $x$ be the horizontal movement of the kite, let $y$ be the height of the kite from the ground. Let $s$ be the length of the string that is let out and let $\theta$ be the angle between the string and the horizontal ground. Then $y=50$ and $\frac{d x}{d t}=2 \mathrm{~m} / \mathrm{s}$.


From the diagram, $\sin \theta=\frac{50}{s} \Rightarrow s \cdot \sin \theta=50$

(Step 3)
$\therefore \frac{d s}{d t} \sin \theta+s \cdot \cos \theta \frac{d \theta}{d t}=0$
$\Rightarrow s \cos \theta \frac{d \theta}{d t}=-\frac{d s}{d t} \sin \theta$.
(Step 4)
$\Rightarrow \frac{d \theta}{d t}=-\frac{\frac{d s}{d t} \sin \theta}{s \cos \theta}$

Also $s^{2}=x^{2}+50^{2}$
$\therefore 2 s \frac{d s}{d t}=2 x \frac{d x}{d t}$
$\Rightarrow s \frac{d s}{d t}=x \frac{d x}{d t}$
(Step 7)
$\Rightarrow \frac{d s}{d t}=\frac{x}{s} \cdot \frac{d x}{d t}$

To find $x$ when $s=100$, we use the equation $s^{2}=x^{2}+50^{2}$.
$\therefore x=\sqrt{s^{2}-50^{2}}$
$\therefore x=\sqrt{(100)^{2}-50^{2}}$
$\therefore x=\sqrt{7500}=50 \sqrt{3}$
(Step 7)
$\therefore \frac{d s}{d t}=\frac{50 \sqrt{3}}{100} \cdot 2=\sqrt{3} \mathrm{~m} / \mathrm{s}$
Now $\sin \theta=\frac{50}{100}=\frac{1}{2}$, when $s=100$.
Since $\sin \theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$
(Step 7)
Since $\theta=\frac{\pi}{6}, \cos \frac{\pi}{6}=\frac{\sqrt{3}}{2}$
From $\frac{d \theta}{d t}=-\frac{\frac{d s}{d t} \sin \theta}{s \cos \theta}$ we substitute $\frac{d s}{d t}=\sqrt{3}, \sin \theta=\frac{1}{2}, \cos \theta=\frac{\sqrt{3}}{2}$ and $s=100$, we then have $\frac{d \theta}{d t}=-\frac{\frac{d s}{d t} \sin \theta}{100 \cos \theta} \frac{d \theta}{d t}=-\frac{\sqrt{3} \frac{1}{2}}{100 \frac{\sqrt{3}}{2}}$
$\therefore \frac{d \theta}{d t}=-0.01 \mathrm{rad} / \mathrm{s}$
The angle between the string and the horizontal is decreasing at a rate of $-0.001 \mathrm{rad} / \mathrm{s}$ at the moment the length of the string let out is 100 m .
(Step 6)

Example 3: A plane flying with a constant speed $300 \mathrm{~km} / \mathrm{h}$ passes over a ground radar station at an altitude of 1 km and climbs at an angle of $30^{\circ}$. At what rate is the distance from the plane to the radar station increasing a minute later?

## Solution

We let $x$ be the distance from the plane to a point P immediately above the radar station. Then $\frac{d x}{d t}=300 \mathrm{~km} / \mathrm{h}$. Let $s$ be the distance from the radar station to the plane.

(Step 1)

$$
\begin{align*}
& s^{2}=1^{2}+x^{2}-2(1)(x) \cos \left(90^{\circ}+30^{\circ}\right) \\
& \Rightarrow s^{2}=1+x^{2}-2 x \cos 120^{\circ}  \tag{Step3}\\
& \Rightarrow s^{2}=1+x^{2}-2 x\left(-\frac{1}{2}\right)
\end{align*}
$$

Hence $s^{2}=1+x+x^{2}$

$$
\begin{aligned}
& \therefore 2 s \frac{d s}{d t}=\frac{d x}{d t}+2 x \cdot \frac{d x}{d t} \\
& \Rightarrow 2 s \frac{d s}{d t}=(1+2 x) \frac{d x}{d t} .
\end{aligned}
$$

Therefore $\frac{d s}{d t}=\frac{(1+2 x) \frac{d x}{d t}}{2 s}$

$$
\text { So after } 1 \text { minute }\left(\frac{1}{60} \text { hours }\right), x=300 \cdot \frac{1}{60} \mathrm{~km} \Rightarrow x=5 \mathrm{~km} \text {. }
$$

From $s^{2}=1+x+x^{2}$ we get that

$$
\begin{align*}
& s=\sqrt{1+x+x^{2}} \\
& \Rightarrow s=\sqrt{1+5+5^{2}}  \tag{Step7}\\
& \Rightarrow s=\sqrt{31 .}
\end{align*}
$$

So $\frac{d s}{d t}=\frac{(1+2 x) \frac{d x}{d t}}{2 s}$

$$
\Rightarrow \frac{d s}{d t}=\frac{(1+2(5)) 300}{2(\sqrt{31})}
$$

$$
\Rightarrow \frac{d s}{d t}=\frac{1650}{\sqrt{31}} \approx 296 \mathrm{~km} / \mathrm{h}
$$

Therefore the distance from the plane to the radar station is increasing at a rate of $296 \mathrm{~km} / \mathrm{h}$.
(Step 6)

Example 4: The altitude of a triangle is increasing at a rate of $1 \mathrm{~cm} / \mathrm{min}$ while the area of the triangle is increasing at a rate of $2 \mathrm{~cm}^{2} / \mathrm{min}$. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is $100 \mathrm{~cm}^{2}$.

## Solution

We let $h$ be the altitude of triangle $\triangle \mathrm{EBC}$. We also let $A$ be the area of the triangle and let $x$ be the base of the triangle. $\therefore \frac{d h}{d t}=1 \mathrm{~cm} / \mathrm{min}$ and $\frac{d A}{d t}=2 \mathrm{~cm}^{2} / \mathrm{min}$.

$A=\frac{1}{2} b \cdot h$
$\Rightarrow A=\frac{1}{2} x h$
$\therefore \frac{d A}{d t}=\frac{1}{2}\left(\frac{d x}{d t} \cdot h+x \cdot \frac{d h}{d t}\right)$

Also since
$A=\frac{1}{2} x h$,
we have that
$x=\frac{2 A}{h}$
$\Rightarrow x=\frac{2(100)}{10}$
(Step 7)
$\Rightarrow x=20 \mathrm{~cm}$.
$\therefore 2 \mathrm{~cm}^{2} / \mathrm{min}=\frac{1}{2}\left(\frac{d x}{d t} .(10)+20(1)\right)$
$\therefore 4 \mathrm{~cm}^{2} / \mathrm{min}=10 \frac{d x}{d t}+20$
$\Rightarrow \frac{d x}{d t}=-1,6 \mathrm{~cm} / \mathrm{min}$
So the base is decreasing at a rate of $1,6 \mathrm{~cm} / \mathrm{min}$.

Example 5: A television camera is positioned 1200 m from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is $200 \mathrm{~m} \cdot \mathrm{~s}^{-1}$ when it has risen 900 m .
(a) How fast is the distance from the television camera to the rocket changing at that moment?
(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

## Solution

Let $y$ be the distance the rocket rises. Then $\frac{d y}{d t}=200 \mathrm{~m} \cdot \mathrm{~s}^{-1}$ when $y=900 \mathrm{~m}$.
Let $s$ be the distance from the camera to the rocket. We let $\theta$ be the angle of elevation of the camera.
(Step 2)

(Step 1)
(a) $s^{2}=y^{2}+(1200)^{2}$

$$
\begin{align*}
& \therefore 2 s \frac{d s}{d t}=2 y \frac{d y}{d t} \\
& \Rightarrow \frac{d s}{d t}=\frac{y}{s} \cdot \frac{d y}{d t} \tag{Step4}
\end{align*}
$$

Also since $s^{2}=y^{2}+(1200)^{2}$

$$
\begin{equation*}
s=\sqrt{y^{2}+(1200)^{2}} \tag{Step7}
\end{equation*}
$$

But $\mathrm{y}=900$,

$$
\begin{align*}
& \therefore s=\sqrt{(900)^{2}+(1200)^{2}} \\
& \Rightarrow s=1500 \mathrm{~m} . \tag{Step7}
\end{align*}
$$

Substituting $y=900 \mathrm{~m}$ and $s=1500 \mathrm{~m}$ as well as $\frac{d y}{d t}=200 \mathrm{~m} \cdot \mathrm{~s}^{-1}$ in the expression $\frac{d s}{d t}=\frac{y}{s} \cdot \frac{d y}{d t}$,
we have $\frac{d s}{d t}=\frac{900}{1500} .200$
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(Step 5)

Hence the distance from the television camera to the rocket is increasing at a rate of $120 \mathrm{~m} . \mathrm{s}^{-1}$.
(b) $\tan \theta=\frac{y}{1200}$
$\therefore \sec ^{2} \theta \frac{d \theta}{d t}=\frac{1}{1200} \frac{d y}{d t}$
$\Rightarrow \frac{d \theta}{d t}=\frac{1}{1200 \cdot \sec ^{2} \theta} \frac{d y}{d t}$

Since
$\sec ^{2} \theta=1+\tan ^{2} \theta$ and $y=900$, we have that

$$
\begin{aligned}
& \sec ^{2} \theta=1+\left(\frac{y}{1200}\right)^{2} \\
& \Rightarrow \sec ^{2} \theta=1+\left(\frac{900}{1200}\right)^{2} \\
& \Rightarrow \sec ^{2} \theta=\frac{25}{16}
\end{aligned}
$$

Substituting $\sec ^{2} \theta=\frac{25}{16}$ well as $\frac{d y}{d t}=200 \mathrm{~m} \cdot \mathrm{~s}^{-1}$ in the expression $\frac{d \theta}{d t}=\frac{1}{1200 \cdot \sec ^{2} \theta} \frac{d y}{d t}$
we have
$\frac{d \theta}{d t}=\frac{1}{1200 \cdot\left(\frac{25}{16}\right)} \cdot \frac{200}{1}$
$\Rightarrow \frac{d \theta}{d t}=\frac{8}{75} \approx 0,1067 \mathrm{rad} \cdot \mathrm{s}^{-1}$.
(Step 5)
Therefore the camera's angle of elevation is increasing at a rate of 0,1067 rad.s ${ }^{-1}$. (Step 6)

Example 6: A runner sprints around a circular track of radius 100 m at a constant speed of $7 \mathrm{~m} / \mathrm{s}$. The runner's friend is standing at a distance 200 m from the centre of the track. How fast is the distance between friends changing when the distance between them is 200 m ?

## Solution

Let $x$ be the distance that the runner sprints around the track. Then $\frac{d x}{d t}=7 \mathrm{~m} / \mathrm{s}$.
Also let $s$ be the distance between the runner and the friend.

$\therefore s^{2}=200^{2}+100^{2}-2(200)(100) \cos \theta$
$\Rightarrow s^{2}=50000-40000 \cos \theta$
$\therefore 2 s \frac{d s}{d t}=-40000(-\sin \theta) \cdot \frac{d \theta}{d t}$
(Step 4)
$\therefore \frac{d s}{d t}=\frac{20000 \sin \theta \cdot \frac{d \theta}{d t}}{s}$
Now

$$
\begin{align*}
& x=r \theta  \tag{Step7}\\
& \Rightarrow x=100 \theta .
\end{align*}
$$

$\therefore \frac{d x}{d t}=100 \frac{d \theta}{d t}$
$\Rightarrow 7=100 \cdot \frac{d \theta}{d t}$, since $\frac{d x}{d t}=7 \mathrm{~m} / \mathrm{s}$.
$\therefore \frac{d \theta}{d t}=0,07$
Since $s=200$, we have

$$
\begin{aligned}
& 200^{2}=200^{2}+100^{2}-2(200)(100) \cos \theta \\
& \Rightarrow 10000=40000 \cos \theta \\
& \Rightarrow \cos \theta=\frac{1}{4}
\end{aligned}
$$

since $\sin \theta=\sqrt{1-\cos ^{2} \theta}$,
we get that
$\sin \theta=\sqrt{1-\left(\frac{1}{4}\right)^{2}}$
$\Rightarrow \sin \theta=\sqrt{\frac{15}{16}}$
Using $s^{2}=50000-40000 \cos \theta$,
we obtain
$s^{2}=50000-40000\left(\frac{1}{4}\right)$
$\Rightarrow s^{2}=40000$
$\Rightarrow s=200$

Therefore $\frac{d s}{d t}=\frac{20000 \sin \theta \cdot \frac{d \theta}{d t}}{s}$ becomes
(Step 5)
$\frac{d s}{d t}=\frac{20000\left(\sqrt{\frac{15}{16}}\right) \cdot 0,07}{200}$
$\Rightarrow \frac{d s}{d t}=7 \cdot \sqrt{\frac{15}{16}}$
$\approx 6,78 \mathrm{~m} / \mathrm{s}$.

Example 7: A plane flying horizontally at an altitude of 2 km and a speed of $800 \mathrm{~km} / \mathrm{h}$ passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 3 km away from the station.

## Solution

Let $x$ be the distance travelled by the plane from P , where P is a point directly above the radar station which the plane passes. Let $s$ be the distance from the plane to the radar station. Then $\frac{d x}{d t}=800 \mathrm{~km} / \mathrm{h}$.

(Step 1)
$s^{2}=x^{2}+2^{2}$.
UNIVERSITY of the
$\therefore 2 s \frac{d s}{d t}=2 x \frac{d x}{d t}$
$\Rightarrow \frac{d s}{d t}=\frac{x}{s} \cdot \frac{d x}{d t}$.
Also from $s^{2}=x^{2}+2^{2}$, we obtain

$$
x^{2}=s^{2}-4 \Rightarrow x=\sqrt{s^{2}-4}
$$

Since $\mathrm{s}=3 \mathrm{~km}$, we obtain

$$
\begin{aligned}
& x=\sqrt{(3)^{2}-4} \\
& \Rightarrow x=\sqrt{5}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \frac{d s}{d t}=\frac{x}{s} \cdot \frac{d x}{d t} \\
& \Rightarrow \frac{d s}{d t}=\frac{\sqrt{5}}{3} \cdot(800) \approx 596,3 \mathrm{~km} / \mathrm{h}
\end{aligned}
$$

Thus the distance from the plane to the radar station is increasing at a rate of $596,3 \mathrm{~km} / \mathrm{h}$.
(Step 6)

Example 8: Car A is travelling west at $90 \mathrm{~km} / \mathrm{h}$ and car B is travelling north at $100 \mathrm{~km} / \mathrm{h}$. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 60 m and car B is 80 m from the intersection?

## Solution

Let $x$ be the distance of car A to the intersection at any time $t$ and let $y$ be the distance of car B to the intersection at any time $t$. Then $\frac{d x}{d t}=-90 \mathrm{~km} / \mathrm{h}$, since $x$ is decreasing and $\frac{d y}{d t}=-100 \mathrm{~km} / \mathrm{h}$, since $y$ is decreasing. Let $s$ be the distance between the two cars.
(Step 2)

$s^{2}=x^{2}+y^{2}$.
$\therefore 2 s \frac{d s}{d t}=2 x \frac{d x}{d t}+2 y \frac{d y}{d t}$
$\Rightarrow \frac{d s}{d t}=\frac{x}{s} \frac{d x}{d t}+\frac{y}{s} \frac{d y}{d t}$
$\Rightarrow \frac{d s}{d t}=\frac{x \frac{d x}{d t}+y \frac{d y}{d t}}{s}$.
Also from $s^{2}=x^{2}+y^{2}$ we obtain

$$
\begin{align*}
& s=\sqrt{x^{2}+y^{2}} \\
& \Rightarrow s=\sqrt{(0,06)^{2}+(0,08)^{2}} .  \tag{Step7}\\
& \Rightarrow s=0,1 \mathrm{~km}
\end{align*}
$$

(Note that $60 \mathrm{~m}=0.06 \mathrm{~km}$ and $80 \mathrm{~m}=0.08 \mathrm{~km}$ )
Therefore $\frac{d s}{d t}=\frac{x \frac{d x}{d t}+y \frac{d y}{d t}}{s}$ becomes

$$
\begin{align*}
& \frac{d s}{d t}=\frac{x \frac{d x}{d t}+y \frac{d y}{d t}}{\sqrt{x^{2}+y^{2}}} \\
& \Rightarrow \frac{d s}{d t}=\frac{0,06(-90)+0,08(-100)}{0,1} .  \tag{Step5}\\
& \Rightarrow \frac{d s}{d t}=-134 \mathrm{~km} / \mathrm{h}
\end{align*}
$$

Therefore the cars are approaching each other at $134 \mathrm{~km} / \mathrm{h}$. (the distance is decreasing at a rate of $134 \mathrm{~km} / \mathrm{h}$.)

Example 9: A man walks along a straight path at a speed of $1,5 \mathrm{~m} / \mathrm{sec}$. A searchlight is located on the ground 6 m from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is $8 m$ from the point on the path closest to the searchlight?

## Solution

Let $x$ be the distance the man walks from the point on the path closest to the searchlight. Then $\frac{d x}{d t}=1,5 \mathrm{~m} / \mathrm{sec}$. Let $\theta$ be the angle through which the searchlight rotates. Let $s$ be the length of the beam.
(Step 2)

$\tan \theta=\frac{x}{6}$
$\therefore \sec ^{2} \theta \frac{d \theta}{d t}=\frac{1}{6} \frac{d x}{d t}$
$\Rightarrow \frac{d \theta}{d t}=\frac{1}{6 \cdot \sec ^{2} \theta} \frac{d x}{d t}$
$\Rightarrow \frac{d \theta}{d t}=\frac{\cos ^{2} \theta}{6} \frac{d x}{d t}$

Also
Since $x=8 \mathrm{~m}$,
$s^{2}=x^{2}+6^{2}$
$\Rightarrow s=\sqrt{x^{2}+6^{2}}$
$\Rightarrow s=\sqrt{8^{2}+6^{2}}$
$\Rightarrow s=10$.
So
$\cos \theta=\frac{6}{10}$


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$\Rightarrow \cos ^{2} \theta=\frac{36}{100}$

Therefore from
$\frac{d \theta}{d t}=\frac{\cos ^{2} \theta}{6} \frac{d x}{d t}$, we substitute $\cos ^{2} \theta=\frac{36}{100}$ and $\frac{d x}{d t}=1,5$
$\therefore \frac{d \theta}{d t}=\frac{\left(\frac{36}{100}\right)^{2}}{6} 1,5$
(Step 5)
$\Rightarrow \frac{d \theta}{d t}=\frac{9}{100}=0,09 \mathrm{rad} / \mathrm{sec}$.

Therefore the searchlight is rotating at $0,09 \mathrm{rad} / \mathrm{sec}$.

Example 10: A man starts walking north at $1,2 \mathrm{~m} / \mathrm{s}$ from a point P . Five minutes later a woman starts walking south at $1,6 \mathrm{~m} / \mathrm{s}$ from a point 200 m due east of P. At what rate are the people moving apart 15 minutes after the woman starts walking?

## Solution

Let $x$ be the distance that the man walks from point P and let $y$ be the distance the woman walks from $\mathrm{P}^{\prime}$ (where $\mathrm{P}^{\prime}$ is a point $200 m$ due east of P ). Let $s$ be the distance between the two people 15 minutes after the woman starts walking.
(Step 2)

$s^{2}=200^{2}+(x+y)^{2}$
(Step 3)
$\therefore 2 s \frac{d s}{d t}=2(x+y)\left(\frac{d x}{d t}+\frac{d y}{d t}\right)$.
$\Rightarrow \frac{d s}{d t}=\frac{(x+y)}{s}\left(\frac{d x}{d t}+\frac{d y}{d t}\right)$
(Step 4)

After 20 minutes which equals $(20)(60)=1200$ seconds, $x=(1,2)(1200)=1440 \mathrm{~m}$ and
after 15 minutes which equals $(15)(60)=900$ seconds, $y=(1,6)(900)=1440 \mathrm{~m}$.
(Step 7)
Also from $s^{2}=200^{2}+(x+y)^{2}$
$s=\sqrt{200^{2}+(x+y)^{2}}$
$\Rightarrow s=\sqrt{200^{2}+(1440+1440)^{2}}$.
(Step 7)
$\Rightarrow s=\sqrt{8334400}$
$\Rightarrow s \approx 2886,94 m$
So from $\frac{d s}{d t}=\frac{(x+y)}{s}\left(\frac{d x}{d t}+\frac{d y}{d t}\right)$,
$\frac{d s}{d t}=\frac{(1440+1440)}{2886,94}(1,2+1,6)$
$\Rightarrow \frac{d s}{d t} \approx 2,79 \mathrm{~m} / \mathrm{s}$
(Step 5)

Thus the people are moving apart at a rate of $2,79 \mathrm{~m} / \mathrm{s}, 15$ minutes after the woman started walking.
(Step 6)

Example 11: Gravel is being dumped from a conveyor belt at a rate of $3 \mathrm{~m}^{3} / \mathrm{min}$ and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is $3 m$ high?

## Solution

Let $V$ be the amount of gravel dumped at any time $t$. Let $h$ be the height of the cone at any time $t$. Let $r$ be the radius of the cone.
(Step 2)

$\frac{d V}{d t}=3 m^{3} / \mathrm{min}$

$$
\begin{align*}
& h=2 r=d \\
& \Rightarrow r=\frac{h}{2} \tag{Step2}
\end{align*}
$$

Now
$V=\frac{1}{3} \pi r^{2} h$
$\Rightarrow V=\frac{1}{3} \pi\left(\frac{h}{2}\right)^{2} h$
$\Rightarrow V=\frac{1}{3} \pi\left(\frac{h^{2}}{4}\right) h$
$\Rightarrow V=\frac{1}{12} \pi h^{3}$
$\therefore \frac{d V}{d t}=\frac{\pi}{12} h^{2} \frac{d h}{d t}$
$\Rightarrow \frac{d h}{d t}=\frac{4}{\pi} \cdot \frac{1}{h^{2}} \cdot \frac{d V}{d t}$


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$\Rightarrow \frac{d h}{d t}=\frac{4}{3 \pi} \approx 0,42 \mathrm{~m} / \mathrm{min}$.
Therefore the height of the pile is increasing at a rate of $0,42 \mathrm{~m} / \mathrm{min}$.

## CHAPTER 4

## LITERATURE REVIEW

### 4.1 INTRODUCTION

Little research has been done on the reasons for students' poor performance on Related Rates problems in the first year calculus course at university. Research has mainly focused on topics such as functions (Carlson, 1998), derivatives of functions as limits (Cornu, 1981; Tall \& Vinner,1981; Dubinsky \& Harel, 1992), rates of change, including average and instantaneous rate of change (Orton,1983) and differentiation rules, including the chain rule (Cottrill,1999). These topics form the basis for studying Related Rates. Research also distinguished between procedural knowledge and conceptual understanding (Mahir, 2009) in the solution process of Related Rates problems. Below follows the literature review of articles related to this study.

### 4.2 RATES OF CHANGE

The aim of the study conducted by Bezuidenhout (1998) was to determine first year university students' understanding of the concept of rate of change. The researcher developed his theoretical framework according to the principles of the theory of constructivism, in particular the idea of concept image, since the researcher believes that the understanding of mathematical content is determined by cognitive factors. The researcher used a three phased method, where the first two phases involved administering tests to students studying calculus and the third phase involved interviews with individual students. A total of 107 engineering students participated, 52 students wrote a preliminary test separate from the other 55 students.

The test was conducted during a tutorial session in the second semester of 1994. The second test was compiled from the analysis of the results that were obtained from the first test. A total of 523 first year university students from three South African universities participated in this second test. A random sample was taken from three different groups that participated, which included students doing calculus courses in engineering, physical sciences and service calculus courses for students in the commercial and management sciences. The third phase involved interviews with 15 students who had written both tests. The researcher emphasized that to gain any significant insight into students' concept images, one has to identify misconceptions regarding students' mathematical thinking concerning rate of change. He noted that the method of interviews played a decisive role in detecting students' principles and their reasoning for applying them.

The results from his study suggest that students have some fundamental misconceptions regarding the relationship between the concepts average rate of change, average value of a continuous function and the arithmetic mean. The researcher also stated that a conscious effort should be made to help students relate mathematical concepts to prior experiences and existing knowledge structure.

The study of Herbert and Pierce (2009) was designed to present eight conceptions of rate of change revealed by phenomenographic analysis of interviews with pre-calculus students, with the focus on the appropriateness of the methodology that was employed to reveal those conceptions. The study was done with video recorded interviews with 23 year 10 students at a diversity of Victorian secondary schools. Herbert and Pierce (2009) thought it appropriate to use a qualitative methodology, in this case the methodology of "phenomenography", to establish the categories for conceptions of rate of change held by pre-calculus students, in order to better understand the prior knowledge such students bring to a study of calculus. Herbert and Pierce felt that exploring the conceptions of rate of change held by pre-calculus
students will help explain some difficulties experienced by these students. The results of the study were presented in the form of an outcome space structure and the conceptions which emerged from the phenomenographic analysis are summarised below:


Source: Herbert and Pierce (2009)

Herbert and Pierce (2009) felt that their information may assist future design of calculus material which will take the possible state of students' initial conceptions of rate of change into account.

The study of Hassen and Mitchelmore (2006) was to investigate which of two models of abstraction can help understand how students learn the concept of rate of change. The first model is the empirical abstraction model while the second is the so-called nested Recognition, Building-with and Construction (RBC) model. The sample Hassen and Mitchelmore used were fourteen volunteer year 11 students from five high schools in Sydney, New South Wales, of whom all were exposed to an introduction to calculus prior to their participation in the study. The procedure of the study was designed to allow the identification of the abstraction process either according to the empirical or the so-called nested RBC model that sought to answer the following research questions:
a) How well does each of the two models of abstraction describe the process of learning about rates of change?
b) What can we infer about learning of rate of change concepts?

Each student in the sample was interviewed twice by Hassen. The first interview lasted an hour and the second interview, which was conducted a week later, lasted only half an hour. Both interviews were audio recorded and transcribed.

There were four items used in the first interview, which were titled A, B, C and D. Only one item was used as a follow-up in the second interview, which was titled E. All the items, except item D , were designed to assess the students' understanding of the three aspects of rate of change, which Hauger (1995) identified as macro qualitative, macro quantitative and micro qualitative. Item A was designed to test students' understanding of the concept of speed and
their ability to relate that understanding to the concepts of rate of change. Item B was based on the concept of population growth. The students were given two scenarios and the students' understanding of rate of change was tested in both situations separately before being asked to compare the two different situations. Item C was about a cooling experiment. Item E was about the growth in a student's height over several years. Item D differed from the other items in the sense that it comprised several questions that was intended to summarize what students had learned from working through items A, B and C. One question involved the recognition of anything similar in the three items, while the rest of the questions that were asked were how to find an average rate of change and instantaneous rate of change and to explain the difference between the two concepts.

Students were taught that average rate of change was defined as the ratio of the total change in the dependent variable to the total change in the independent variable over a fixed period. In symbol form, if $x$ represents the independent variable and $y$ represents the dependant variable, then average rate of change $=\frac{\Delta y}{\Delta x}$, where $\Delta x$ represents the total change in the independent variable and $\Delta y$ represents the total change in the dependent variable The significance of the end points of the interval was emphasised to the students. Instantaneous rate of change was taught to students using what is known as the zooming-in approach. The students were shown a series of graphs of the given relationship where the domain and range of each variable were successively reduced. Students were asked to examine the behaviour of the graph around the given point and led to see that the initial non-linear graph appeared to become gradually linear. The students were then asked to predict where the line would be after zooming out back to the original graph if this straight line was extended over a greater domain. The study found that the nested RBC model of abstraction may help educators understand the process of learning the quantitative aspects of rate of change. The use of the
empirical abstraction model is not recommended from the results obtained from the study. Hassen and Mitchelmore (2006) felt that students already gained a fairly global concept of rate of change.

### 4.3 THE CONCEPT OF VARIABLE

The purpose of the study by White and Mitchelmore (1996) was to investigate the performances on some calculus application problems of students who had previously experienced a traditional introductory calculus course. To accomplish this, White and Mitchelmore (2006) used a sample of 40 first year full time university mathematics students which had a satisfactory result in their final high school examination for a mathematics course that contained a large component of calculus. The calculus was taught to the students by White for four hours per week over a six week period as half of a semester course. White followed the approach of Barnes (1992), in which the topic rates of change were investigated using graphs of physical situations. The test items involved four constructed items that dealt with rates of change and optimization. The first two items required students to find a specific rate of change whereas the last two items involved using a derivative to maximize or minimize a given quantity. Each of the four items were constructed in four versions so that the manipulation required to solve each version was the same with the only difference being that each version had successively less translation. The four items can be found in Addendum B. The students were tested before, during and immediately after the course was completed as well as six weeks after the course was completed. The forty students were divided into four parallel groups of ten. Four tests were constructed, each of which included four questions. The tests were administered in a cyclic fashion to each of the four groups over the four data collections. In addition, four students per group were interviewed within three days of each of the four data collections which served to clarify and expand on their written responses. From the responses, White and Mitchelmore (1996) suggested that one major reason why
students find it difficult to apply calculus lies in their underdevelopment of the concept of a variable. Moreover, students treated variables as symbols to be manipulated rather than quantities to be related. This tie in with the results of Martin (2000) with her research of students' difficulty of geometric related rates problems. They further stressed that students have learned to operate with symbols without any regard to their possible contextual meaning, which they termed "abstract-apart knowledge" (Mitchelmore,1994). White and Mitchelmore (1996) concluded that most students have an "abstract-apart" concept of a variable and that they require an "abstract-general" concept of a variable as a prerequisite to a successful study of calculus.

### 4.4 THE CONCEPT OF FUNCTION

The study by Carlson (1998) investigated students' development of the function concept. The sample was students from three different levels of mathematical preparation. The first group comprised of 30 students who have just completed college algebra with a class average of over $95 \%$. The second group consisted of 16 students who have just completed second semester calculus and they achieved a class average of more than $80 \%$. The third group also achieved a class average of more than $80 \%$ and they were graduate students who completed a semester course in either complex analysis or real analysis. The college algebra curriculum involved an early introduction of functions. The calculus curriculum was taught using a traditional text with lectures, while the graduate students were products of the undergraduate and graduate traditional mathematics curriculum. A written exam was administered upon completion of their respective courses. Examinations were scored using a five point rubric for each question. After scoring each examination, group means and standard deviation for each group were computed. Follow-up interviews were conducted on selected students. Final results were obtained by analysing both the quantitative and qualitative results relative to
each group. The findings of Carlson's study range from students having misconceptions about a concept (in this case functions), to concept development that takes longer to evolve. The implication of this study identified that the essential aspects of the function concept needs more attention. These essential aspects that Carlson (1998) tested in her study that needed more attention were the following:

- Characterize "real world" functional relationships using function notation;
- Operate with a particular type of function representation, such as a formula, a table, or a graph;
- Move between different representations of the same function;
- Represent and interpret covariant aspects of the function situation (i.e. recognize and characterize how change in one variable affects change in another);
- Interpret "static" and "dynamic" functional information (i.e. interpret graphs representing position and rate of change);
- Interpret and describe local and global function properties: slope, continuity, and differentiability;
- Construct functions using formulas and other functions;
- Recognize functions, non-functions and function types;
- Conceptualize a function both as a process and as an object;
- Interpret and understand the language of functions; and
- Characterize the relationship between a function and an equation.


### 4.5 CONCEPTUAL UNDERSTANDING \& PROCEDURAL KNOWLEDGE

The purpose of the study by Mahir (2009) was to investigate the conceptual and procedural performances of a group of students on the theory of integration. In line with the definition of Herbert and Leifevre (1986), the main concepts of integral theory are the following: 1) the definite integral of a function as the limit of Riemann sums, 2) the integral-area relation and 3) the fundamental theorem of calculus. Mahir used 62 students that successfully completed
the first year course at the mathematics department of Anadola University in Eskisehir, Turkey. Functions, limits and continuity, differentiation, transcendental functions and sketching graphs of functions were among the topics that these students were exposed to in their first year. They also completed the topics involving techniques and applications of integration, sequences, series and power series. The students wrote an hour long exam of 5 questions. According to Mahir (2009), the first two questions were to test the student's procedural skills, while the next two questions could be solved by either depending on the student's procedural skills or their conceptual understanding. The last question depended chiefly on the students' conceptual understanding to solve the problem. According to the definition that Mahir (2009) uses in his study, procedural skills refer also to procedural knowledge. The first question of the test was correctly solved by $92 \%$ of the students while $74 \%$ solved the second question correctly. Mahir (2009) concluded from these results that the students possess procedural knowledge of integration. He noted that only 5 students attempted the second question by using their conceptual understanding, while 45 students depended on their procedural skills to solve the question. Twelve students did not attempt question 3. Five students who used their conceptual understanding to answer the third question all got it correct, while only $11 \%$ of the 45 students who depended on their procedural skills got it correct. Again Mahir (2009) noted that the number of students who attempted question 4 by using their conceptual understanding were much higher than the number of students who depended on their procedural skills. Regarding the final question, which chiefly depended on the students' conceptual understanding, $40 \%$ of the sample did not attempt the question while a further $36 \%$ answered the question incorrectly. Mahir (2009) concluded from his study that students do not possess satisfactory conceptual understanding to solve integration problems. He also deduced from his study that the students who solved the questions using their conceptual knowledge also possessed procedural skills. Mahir
(2009) believes that deficiencies in student's conceptual understanding should first be determined which will help in future teaching.

The purpose of the study by Engelbrecht et al (2005) was to determine whether there was any relation between students' conceptual understanding and/or procedural skills in mathematics and whether there is any relation between their confidence levels when handling procedural and conceptual problems. Engelbrecht et al (2005) also investigated the relationship between the students' confidence and their actual performances in procedural and conceptual mathematical problems. The sample consisted of 235 first year life science students at the University of Pretoria, South Africa, who all registered in an introductory course in applied calculus in the mathematics department. The assessment instrument consisted of 10 multiple choice items of which $50 \%$ was considered by Engelbrecht et al (2005) to be of a procedural nature and the rest conceptual. The test was thoroughly and independently scrutinised for unbiased view towards the procedural and conceptual division. In addition to the test, the students had to rate their confidence when they answered each question. Each student's Procedural Performance Index (PPI) and their Conceptual Performance Index (CPI) were calculated as a percentage by using a formula constructed by Engelbrecht et al (2005). The students' Procedural Confidence Index (PCI) and their Conceptual Confidence Index (CCI) were also constructed using a similar formula. Engelbrecht et al (2005) firstly compared the performances of each student in the conceptual with their performances in the procedural items, and disregarding the students who indicated that they guessed an answer. The next comparison was between each student's procedural and conceptual confidence indices, while the last two comparisons was between each student's procedural confidence index with their procedural performances and then finally between each student's conceptual confidence index with their conceptual performance index. A scatter plot for the data was constructed in all comparisons. From the results, Engelbrecht et al (2005) concluded that students do not do
better in procedural problems than in their conceptual problems. The students are however more confident about their ability to answer conceptual problems than for procedural problems. Engelbrecht et al (2005) also noted that their teaching methodology followed the more reformed calculus approach, which was different to how their mainstream calculus courses are presented. Engelbrecht et al (2005) recommended that their study be extended to other courses as well as other disciplines at the University of Pretoria.

### 4.6 RELATED RATES

The study conducted by Martin (2000) was to assess the ability of university students registered in an introductory calculus course to solve geometric Related Rates problems. The research questions of the study were guided by the multi-step, multi-faceted nature of the solution of Related Rates problems. After much consultation with examiners, graduate students and several textbooks, Martin (2000) confirmed that the solution to the majority of Related Rates problems consists of the same six steps, with an additional step depending on the context of the problem. In the case of Martin's study (2000), the Related Rates problems that were used come from a geometric context. Martin (2000) referred to the solution of these seven steps as the "Standard Solution model" for geometric related Rates problems, where step 1, step 2 and step 6 were identified as requiring a student to use their conceptual understanding. Step 3, 4 and 5 required students to use their procedural knowledge. Regarding the seventh step, Martin (2000) named this step an auxiliary step, which is additional and may not be used in certain Related Rates problems, and this step Martin (2000) classified as relying either on conceptual understanding or procedural knowledge. Based on these classifications of the different steps, Martin (2000) identified the research questions which were as follows:
a) How do university students, enrolled in an introductory calculus course perform on geometric Related Rates problems?
b) How such students performed on the conceptual steps, the procedural steps and the additional auxiliary step of the "standard solution model" of geometric Related Rates problems?
c) How were performances on the various steps of the "standard solution model" related to each other and to their overall performances?

Martin (2000) designed a non-randomised performance study, using two written instruments. The first instrument consisted of three open-ended geometric Related Rates problems while the second instrument was used to assess students' ability to perform each of the steps of the "standard solution model" independently. The sample for the study were selected from students who were enrolled in two introductory calculus courses at an urban, private university situated in the northeast of the United States of America. Two courses were traditionally taught to the students and the textbook that was used was the textbook by Berkey (1988). There were 120 students from the first course who enrolled at the College of Liberal Arts or the College of Engineering. There were 40 students in the second course, which were designed for the College of Engineering students who performed poorly on the mathematical placement tests administered by the said university. 34 students from the first course and 24 students from the second course wrote both tests and these students were used as the sample of the study.

The first test was administered a month after the course began and was given immediately after the instruction of Related Rates was completed. This first test consisted of 3 problems and was designed to measure the students' ability to solve standard geometric Related Rates problems that were selected from textbooks of Dick and Patton (1992) as well as problems
from Feroe and Steinhorn (1991). The second test, which was fairly long, was given to 52 students within two days after completing the first test. This second test was designed to measure the student's ability to perform the 7 steps which Martin (2000) referred to as the "Standard Solution model" to solve geometric Related Rates problems. The scoring of the first test was done using a rubric that was based on the solutions generated by a panel of experts in the field. Points were awarded for each step of step 3 to step 6 for the first problem, while points were awarded for each of the steps 3 to 7 for the other two problems. Martin (2000) awarded full points for each step even when a mistake was made in a previous step, but the student performed the appropriate procedure. However for the second test, the answers were either correct or incorrect and no partial marks were awarded.

The results showed that the mean score as a whole for the first test was about $43 \%$ with a standard deviation of $22 \%$. In the second test, the mean score of the conceptual steps were lower than that of the procedural steps. The overall performance of the first test achieved an average below $60 \%$, even though the students had recently studied the material. Martin (2000) raised the question whether the degree of thinking which was required to analyse the geometric context was a factor for the poor performance of the students. A second factor which Martin (2000) felt that contributed to the poor performances, was the number of steps that was needed to solve the problem. The students performed poorly regarding the conceptual step 1 and 2. Martin's (2000) results agree with Dreyfus' (1991) argument that students' visualization is rare or disconnected from algebraic representation when students try to solve calculus problems. To complete step 2 of the "Standard Solution module", the students needed to have an abstract-general concept of variable \{See White and Mitchelmore (1996) for definition of abstract-general concept of a variable\}. Based on the low mean score of $22 \%$, Martin (2000) concluded that few students in her study had an abstract-general understanding of variable. The students also had the inability to distinguish variables and
constants and this was one of the prominent error patterns that were noted by White and Mitchelmore (1996). Martin (2000) also highlighted the students' lack of understanding of the mathematical and physical relationship of the problem. They failed to evaluate their solutions to see if it makes sense. Martin's (2000) finding regarding step 6 of the "standard solution model" indicated that students find it easier to decode symbols rather than encoding symbols. The performances on the procedural steps 3,4 and 5 were stronger than that of the conceptual steps, but it still raised concern for Martin (2000), since the average score on the 3 procedural steps were only just above $50 \%$. The conclusion of Martin's (2000) study was that students have a greater difficulty with the conceptual steps of solving Related Rates problems than they have with the procedural steps. Martin (2000) suggested that it is critical for students to be confident, competent users of symbolic representation, as well as being able to make connections among verbal, symbolic and graphical representations. Martin (2000) also suggested that students engage in both processes of encoding as well as decoding symbolic representations, since these two skills represent different levels of conceptual understanding. Martin (2000) found a lower correlation between the conceptual understanding and problem solving performances. Martin's (2000) recommendation was that further research should be undertaken into the matter and ended her study with the assertion that the aim of calculus success need to be identified before designing a curriculum and assessment that match their aim.

The study conducted by Engelke (2006) was a teaching experiment using a computer program which was designed to foster students' exploration of Related Rates problems. Engelke used 3 students from a group of volunteers of her calculus class of 2005, who met outside the regular lectures when Related Rates were taught. The session consisted of 6 teaching sessions and the students were paid for each session, if they attended. All the
teaching sessions of the study were videotaped and transcribed for analysis purposes. The students used a custom computer program to investigate the average rate of change and instantaneous rate of change for some geometric Related Rates problems. The programme allowed the students to a visual representation of the problem at hand, which could be manipulated so that the students can observe what could happen next. A different version of the same problem could be opened via the computer programme to observe what happened to each variable given in the problem and the results could be tabulated. One student could relate two or more variables with respect to time, but that same student could not relate those variables, with respect to time, to each other. The second session dealt with the concerns of students which were identified in the first session. After Engelke explained the chain rule, they could relate rates in other situations, with time being the common variable. Engelke concluded from her data that the possible use of a computer programme to visualize problems situations could assist students' mental model of future problem situations as well as their understanding of the concept of rate.

### 4.7 PROBLEM SOLVING

Based on the literature, some writers believe that solving problems forms the heart of mathematics learning, while other writers consider mathematics as a collection of knowledge which provides the tools for the process of solving mathematical problems. After the 1980's, research concentrated on problem solving as an entity. Garofalo and Lester (1985) claimed that problem solving has come to be viewed as a process involving visualisation, association, abstraction, comprehension, manipulation, masoning, analysis, synthesis and generalisation. Lesh (1981) believes that students should establish relationships and make connections between concepts associated with mathematical content and not separate them. The literature also distinguishes mainly between what they termed "successful" and "unsuccessful"
problem solvers. Dobson (1972) found seven categories which distinguish the differences between "successful" and "unsuccessful" problem solvers. They are:
(1) overall mathematics achievement
(2) verbal and general reasoning ability
(3) spatial ability,
(4) positive attitudes,
(5) resistance to distraction,
(6) level of field independence, and
(7) divergent thinking.

Krutetskii et al (1976) found that students' perception of the important elements of a problem may be the root of the problem between what he termed a "good" and "poor" problem solver. He, as well as Silver (1979), observed that "good" problem solvers tend to recall the structural characteristics of the problem and to forget its details, whereas "poor" problem solvers tend to recall specific details of the problem, with the difference that in the study of Silver, the "good" problem solvers tend to have reasonably accurate recall of contextual details of the problems. Schoenfeld's $(1985,1987)$ research suggested that "good" problem solvers can be distinguished from "poor" problem solvers in at least five important ways. These ways are:
(1) The knowledge of good problem solvers is well connected and composed of rich schemata while that of poor problem solvers is not.
(2) Good problem solvers tend to focus their attention on structural features of problems while poor problem solvers focus on surface features.
(3) Good problem solvers are more aware than poor problem solvers of their strengths and weaknesses as problem solvers.
(4) Good problem solvers are better than poor problem solvers at monitoring and regulating their problem-solving efforts.
(5) Good problem solvers tend to be more concerned than poor problem solvers about obtaining "elegant" solutions to problems.

Foong (1990, 1994) in her research found six differences that distinguishes between "successful" and "unsuccessful" problem solvers. They are:
(1) Successful problem solvers translated the problem statement more correctly and more exactly than did unsuccessful problem solvers.
(2) Unsuccessful problem solvers tended to attend to obvious details, translating statement by statement without having a global representation of the problem situation.
(3) Successful problem solvers planned their solutions in more detail before carrying them out than unsuccessful solvers, who tended to be impulsive in executing a solution without a complete understanding of the problem.
(4) Unsuccessful problem solvers tended towards impulsive solutions and when in difficulty they often returned to the same incorrect method, sometimes repeatedly.
(5) Successful problem solvers used more metacognitive processes which were task directed showing greater awareness of how things were in the solution path and where they should be going in the process.
(6) Negative emotional expressions such as frustration and confusion were found to be more frequent amongst the unsuccessful problem solvers.

Heller and Hungate's (1985) review of several empirical and theoretical analyses found that "poor" problem solvers cannot, or do not, construct problem representations that are helpful in achieving solutions. They also noted that experts solve problems using a process of successive refinements. The strategy used by experts is to perform high-level planning and qualitative analysis before beginning to generate equations. Novices, on the other hand do not have the knowledge required to approach problems in this way, and tend to go directly from the problem text to equations. Experts also have knowledge about when concepts and principles are applicable and useful and procedures for interpreting and applying their factual knowledge, whereas novices lack in much of this knowledge. According to Lester (1994), spatial visualisation, ability to attend to structural features of problems, dispositions such as beliefs and attitudes, and experiential background such as instructional history and familiarity with types of problems, are all causes why students have difficulty in problem solving.

There are also cognitive and affective factors that influence students' difficulties in mathematical problem solving. Mayer (1982) suggested that linguistic and factual knowledge, schema knowledge, algorithmic knowledge and strategic knowledge may be relevant for a psychological basis for understanding mathematical problem solving. Lester (1982) argued that successful problem solving in mathematics consist of at least five components. They are:
a) mathematical knowledge and experience,
b) skill in the use of a variety of generic "tool" skills,
c) the ability to use a variety of heuristics known to be useful in mathematical problem solving,
d) knowledge about one's own cognitions before, during, and after a problem-solving episode and
e) the ability to maintain executive control.

Schoenfeld (1983a, 1983b) distinguished resources, control and belief systems as the three types of knowledge needed in problem solving. Heller and Hungate (1985) summarised the nature of the knowledge required for solving problems in complex subject-matter domains as firstly "knowledge for understanding and representing problems", secondly "strategic knowledge which governs the approach problem solvers take to the task", thirdly " knowledge of basic concepts and principles", and finally "repertoires of familiar patterns and known procedures". Groves and Stacey (1988) claimed that successful problem solving depends on many things, among other, emotional aspects. They also claim that good habits such as not erasing work which may be needed later, as well as awareness and facility with mathematical processes such as generalising, looking at special cases and making conjectures, may influence successful problem solving. Carlson and Bloom (2005) identified four phases of problem solving that were repeated in a cyclic fashion, that is, Orienting, Planning, Executing, and Checking. They also identified a sub-cycle in the planning phase, that of conjecture-imagine-evaluate. Lithner's (2003) proposed a framework to evaluate how students are reasoning about their textbook exercises. Students engage in a reasoning structure that has the following four components, that is: 1) problematic situation, 2) a strategy choice, 3) a strategy implementation, and 4) conclusion. He also identified three reasoning types that play a role in strategy choices and strategy implementation. These are plausible reasoning, reasoning based on established experiences and reasoning based on
identification of similarities. He concluded that students spend more time on reasoning based on established experiences and identification of similarities rather than on plausible reasoning. This caused students to focus on superficial features of the problem and does not lead them to construct meaningful knowledge. In their study of the problem-solving research literature, Kroll and Miller (1993) identified three major cognitive and affective factors that contributed to students' difficulties in problem solving. They are "knowledge", "control" and "beliefs and affects" factors. Under the first cognitive and affective factor, which is knowledge, Kroll and Miller identified five kinds of knowledge.
(a) Algorithmic Knowledge: Computational skills are a necessary but not sufficient component of problem solving. Surveys like the British Assessment of Performance Unit [APU] (Eggleston, 1983) and the National Assessment of Educational Progress [NAEP] (Carpenter et al., 1980) show that the children who are able to solve a problem requiring a particular computation, are much less than the children who can perform the very same computation correctly when not part of a problem.
(b) Linguistic Knowledge: Davidson (1977) found that low verbal ability or lack of familiarity with the language used in problems did hamper children's understanding of word problems. Research by Muth (1984) and Muth and Glynn(1985) concluded that both reading ability and computational ability play important roles in children's successful solution of word problems.
(c) Conceptual Knowledge: Zweng (1979) argued that students had difficulty in deciding which mathematical operation to perform when solving problems. Lester (1985) also found that students' choice of operation was determined by the key words in the problem. Such a situation arises when students are taught to rely on key words to decide which operations to use rather than their knowledge of concepts (Nesher \& Teubal, 1975).
(d) Schematic Knowledge: According to Silver (1981), good problem solvers are more likely to remember information about the structure of the problems they have solved previously, to learn from their mistakes and to relate work they have done on previous problems to the task at hand. Schoenfeld and Herrmann (1982) found that novices attended to surface features of problems, whereas experts categorised problems on the basis of the fundamental principles involved. Span and Overtoom-Corsmit (1986) found that good problem solvers took more time to analyse the situation to produce an appropriate schema and clear representation of the problem.
(e) Strategic Knowledge: According to Polya (1945), students need techniques that will help them develop plans for a solution. Heller and Hungate (1985) found that experts perform high-level planning and qualitative analysis before beginning to generate equations, while novices tend to go directly from the problem text to equations. Span and Overtoom-Corsmit (1986) found that good problem solvers tried to schematise to work systematically with more than one strategy.

The second major cognitive and affective factors identified by Kroll and Miller (1993) were Control factors. Lester (1985) stated that successful problem solving also depends upon knowing when and how to utilise such knowledge and upon having the ability to monitor and evaluate the application of this knowledge, both during and after implementation, even if the student has having sufficient domain-general and domain-specific knowledge. "Metacognition" was first used by Flavell(1976). He used this term to refer to the ability to control one's own thinking processes in problem solving. Silver (1982a) suggested that metacognition should be central in children's mathematical problem solving and is important to success in problem solving. The problem solver must make decisions about which strategy to apply and how long to keep on trying it before stopping and selecting a new strategy
(Silver, 1982b; Schoenfeld, 1983a). Kilpatrick (1985b) also felt that an increase in the awareness of one's emotional influences should give problem solvers greater control over their cognitive processes. In her study, Siemon (1986) found that reflection amongst problem solvers promotes understanding, provides motivation and increases confidence thereby leading to improved problem-solving performance and more efficient learning. In Foong's (1990) study, the successful problem solvers' metacognitive behaviours were task-directed, exhibiting a situational awareness of how things were and where they should be going in the process. In the same study of Foong, she found that the unsuccessful problem solvers' metacognitive behaviours were directionless. Taplin (1994) found that those students who persevere and who don't give up, exhibited more control over their actions during problem solving.

Beliefs and Affective Factors were the third cognitive and affective factors identified by Kroll and Miller (1993). Beliefs and affective factors can either assist or interfere with problem solving. Students' behaviours may be influenced by their feelings of self-esteem, their perceived control over the situation with which they are faced, or their sense of satisfaction in engaging in mathematical tasks (Silver, 1985). Silver (1982a) speculated that affective factors like confidence and willingness to persist may have a substantial effect on the metacognitive processes of problem solvers. Schoenfeld (1983a) suggested that attitudes toward mathematics and confidence about mathematics may be aspects of student belief systems that have an important effect on how students manage their cognitive resources. Research by Lesh (1983) suggested that many students do not believe that mathematics is applicable to solving real-world problems. High school students believe that there was always one correct way to solve any mathematical problem, and that mathematics is mostly memorization (Carpenter et al., 1983). Trimmer (1974) found that confidence, lack of
anxiety, flexibility, lack of rigidity, and an ability to cope with uncertainty were traits associated with successful problem solving.

Hence there are many factors that influence the students' ability to solve mathematical problems successfully. These are not unique and are also applicable to solving related rates problems.

### 4.8 WORD PROBLEM

Real-world word problems that require mathematics for their solutions do not usually come as equations ready to be solved, but rather as verbal or pictorial representations that must be interpreted symbolically, manipulated and then solved. Mathematical word problems mostly deal with relating the real world situation to mathematical concepts. The mathematical word problems are known as instruments which develop the students' ability and talent in solving mathematical problems (De Corte et.al., 1989). However, word problems are defined in terms of a problem classification framework. Verschafel et al. (2000) defined word problems as a verbal description of problem situations wherein one or more questions are raised, the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement.

Students find word problems difficult to solve. Many researchers have confirmed this, for example the work of Gerofsky (1996) as well as that of Craig \& Winter (1991). By researching the existing literature, Gooding (2009) summarized five different categories of difficulties that students may experience while trying to solve mathematical word problems. The five categories of difficulties that students may experience while trying to solve mathematical word problems are:
(i) Reading and understanding the language used within a word problem;
(ii) Recognising and imagining the context in which a word problem is set;
(iii) Forming a number sentence to represent the mathematics involved in the word problem;
(iv) Carrying out the mathematical calculation;
(v) Interpreting the answer in the context of the question.

When students are not able to decode the words used in a word problem, they cannot comprehend a sentence, they cannot understand specific vocabulary and do not have the confidence or the ability to concentrate when reading (Ballew \& Cunningham, 1982; Shuard \& Rothery, 1984; Cummins et al, 1988; Bernardo, 1999). Research done by Caldwell and Goldin (1979) as well as Nunes et al. (1993) concluded that students cannot imagine the context in which a word problem is set or their approach is altered by the context in which the word problem is given. The research done by Carey (1991) and English (1998) concluded that students find it harder to form a number sentence for some word problems structures than others. Category 4, as described by Gooding (2009), also gives more meaning as to why students find it difficult to substitute specific values of variables. This was based on the research by Verschaffel et al. (1999), Nunes et al. (1993) and Anghileri (2001). The size of numbers involved can impact on student's choice of a calculation strategy. Even if the values to be substituted are correct, if it does not seem correct for the student, the student will not complete the calculation. Students do not consider real-life factors and constraints when giving an answer to word problems which can result in giving an answer that is impossible in
the context and therefore incorrect. (Verschaffel et al.,1994; Wyndham \& Säljö, 1997; Cooper \& Dunne ,2000). The existing literature noted the problems that students encounter when attempting to answer word problems, but there are also findings which were very encouraging. The research done by Cankoy and Özder (2011) revealed that visual representations in word problems can reduce problem difficulty.

Gooding's (2009) recommendations in her study include the following:
(i) Encourage students to read the word-problems thoroughly;
(ii) Encourage students to check if an answer is possible in the context of the question.

In the study by Craig (2001), she divided the problems in question into two categories, algorithmic and interpretive. Algorithmic problems are defined as problems that require the student to carry out some calculation, with the aim of obtaining a numerical solution. Interpretive problems, on the other hand, require little or no calculation and require the student to draw a conclusion from some given information using his/her knowledge of mathematics. Galbraith and Haines (2000) made a similar category division in their study, but instead of referring to the one set of problems as algorithmic, they termed it mechanical. The results of Galbraith and Haines (2000) showed clearly that mechanical problems are easier to solve than interpretive problems. Related rates problems can be categorised as a combination of the two. Caldwell and Goldin $(1987,1979)$, as part of their study, categorised their word problems as concrete or abstract, and hypothetical or factual. Concrete and abstract problems are defined in terms of the realism of their context, that is, concrete problems are set in a realistic context and abstract problems have no immediate real world meaning. Hypothetical and factual problems differ in that factual problems simply describe a situation, while hypothetical problems suggest a possible change in the situation. Caldwell and Goldin (1987, 1979) concluded that abstract problems were more difficult than concrete problems.

According to De Corte and Verschaffel (1985) as well as Davies-Dorsey et al. (1991), word problems become easier when they are embedded in a familiar context. The familiar contexts may cause students to pay more attention and it is easier to remember a familiar situation than an unfamiliar one (Stern \& Lehrndorfer, 1992). Smith et al. (1994) in their study measured the readability of problems on a university statistics examination paper according to number of words, number of clauses, and two measures of lexical density. Lexical density is measured as the ratio of lexical words to grammatical words. Unfortunately, they found no correlation between the readability and difficulty level of the problems. Threadgill-Sowder and Sowder (1982) compared the difficulty level of problems presented in verbal format (sentences) versus those presented with detailed diagrams and minimal wording. Their results showed that students found the problems presented almost in diagrammatic form significantly easier than those presented in verbal form only.

Therefore, the existing literature confirm that word problems which include Related Rates problems are not easy to solve, but there are ways and methods to attempt these problems without going to the extremes and give up solving these problems.

So what type of knowledge is required to solve word problems? According to Riley et al (1983) and Riley \& Greeno (1988), solution success of simple arithmetic word problems depends crucially on understanding the semantic relations. By semantic relations, Riley et al (1983) refer to the conceptual knowledge about increases, decreases, combinations and comparisons involving sets of objects. This is not surprising as Martin (2000) also required students to depend on their conceptual understanding when attempting the first 2 steps when solving Related Rates problems using the Standard Solution Model for geometric related rates problem. So if we know what knowledge is required to solve word problems, how do students acquire this knowledge? Riley et al. (1983) argued that the crucial process that drives development of problem-solving skill is the acquisition of knowledge concerning what
they termed part-whole relations. A competent problem solver, including word problem solvers, understands and represents word problems as relations among parts and wholes. They do not see this as two separate entities. A students's logical and mathematical knowledge is presumed to drive understanding of the semantics of the problem text (Riley \& Greeno, 1988). According to them, poor performance on certain word problems reflects a lack of sufficient knowledge concerning part-whole set relations. From existing literature, a student must possess the relevant knowledge in order to solve mathematical problems. Mayer (1982) suggested that some of the types of knowledge that may be relevant for a psychological basis for understanding mathematical problem solving are:
a) linguistic and factual knowledge,
b) schema knowledge,
c) algorithmic knowledge,


### 4.9 CONCLUSION

The studies of Bezuidenhout (1998), Herbert et al.(2009), Mahir (2009) and Engelbrecht et al. (2005) tested the students procedural and conceptual knowledge of a particular calculus topic. It gave insight of how students perceive a certain concept, and relate directly to my study of Related Rates. The studies of White and Mitchelmore (1996) as well as Hassan and Mitchelmore (2006) deal with rates of change which partially lead to Related Rates, but their studies do not give insight as to why students struggle with the topic Related Rates.

The study of Carlson (1998) addresses the issue of the concept of function. Since Related Rates is a rate of a function, it stands to reason that if students do not have a proper
understanding of functions, then they will certainly struggle to understand Related Rates problems. The studies of Martin (2000) as well as Engelke (2006) address the issue of Related Rates. Engelke (2006) uses a computer program to foster students' exploration of Related Rates, but it does not answer the question why students perform badly in answering these types of questions. Martin's (2000) study is restricted to geometric Related Rates questions. In the South African context, geometry questions are badly answered or are given little attention when assessed. Therefore there is a need to study students' ability to answer Related Rates questions in general and not restricted to the geometric context.


## CHAPTER 5

## THEORETICAL FRAMEWORK

### 5.1 INTRODUCTION

From the literature review in chapter 4, the reasons for students' poor performance on Related Rates problems may lie in their weak knowledge and understanding of certain topics associated with related rates. The focus of this study is related rates problems in the first year university calculus course at UWC. Specifically, a certain solution model is investigated as an appropriate teaching strategy to guide students to solve such problems. According to this model, certain thinking processes guide students in arriving at a solution. As a word problem, the first step is to understand the problem. That is, to identify both the known (given) information as well as the required information by reading the problem carefully, perhaps more than once, in order to fully grasp the meaning of words and phrases used in the statement of the problem. This is required to answer questions such as: "Is the context familiar?" or "Is the statement unambiguous?". The second step is to draw a realistic picture (mental image) of how the student interprets the problem. This could be a drawing of a man of a given height walking away from a streetlight at a certain speed, or two aircraft flying over an airport in different directions and at different speeds, or water flowing into a cylindrical tank at a certain speed. Following on this, students produce a schematic diagram in the form of a triangle, a circle or a cone showing the objects as points or lines. A reference point is identified on the schematic diagram from and to which all distances are measured. It is on this schematic diagram that all distances, speed, etc. are displayed and indicated by
symbols (called variables) such as $x, y$ and $z$ for distances and $\alpha, \theta$ etc. for angles of triangles and finally symbols such as $A$ for area, $V$ for volume and $t$ for time, etc. From this schematic diagram, it should become clear which relationships (in the form of equations) exist between the variables. Once this relationship has been identified, the rates of change of all variables with respect to the time are calculated using implicit differentiation. This is followed by a process of substituting all known information and finally the interpretation of the results follows. It is clear from the above discussions that the solution of related rates problems involves three processes. The first part is a process in which conceptual understanding play a major role, while the second process involves mainly procedural knowledge (implicit differentiation, algebraic substitution and manipulation). Finally, a process of interpreting the results involving conceptual understanding provides the solution of the problem. It should now be clear why this study is underpinned by Constructivism as a theoretical basis. In the next section, we therefore first present a brief explanation of the theory of constructivism followed by a discussion of more specific learning theories.

### 5.2 CONSTRUCTIVISM

Constructivism is a theory, based on observation and scientific study, about how people learn. Many definitions of constructivism have been proposed by several authors and researchers such as Henson (2004), Schwandt (2003), Shapiro (2002) and Von Glaserfeld (1996). No single definition has been accepted, but they all adhere to the following characteristics:

- People of all ages construct knowledge, they do not discover it.
- People create knowledge by relating or connecting it to their previous knowledge.
- Knowledge is an autonomous and subjective construction.
- Learning involves active restructuring of how one thinks.
- One's learning and ability to learn is influenced by previous experiences.
- Cognitive growth is stimulated when people are confronted with practical, contextual problems or personal problems that present situations that require a new way of thinking.

Constructivism is thus a philosophy that views knowledge as a subjective process that is shaped and structured by one's experiences (Pelech \& Pieper, 2010). As a person encounter new experiences, then that person connects these new experiences to previous knowledge. These connections restructure the pre-existing knowledge base and also add to the original knowledge base. According to Von Glaserfeld (1989), constructivism is a theory of knowledge with roots in philosophy, psychology and cybernetics.

### 5.2.1 EARLY HISTORY OF CONSTRUCTIVIST WRITERS

The term "constructivism" was only officially used from the 1970's, although evidence exist that the principle of constructivism appeared as far back as the ancient philosophers (Cooney et al. 1993). Although they do not consider themselves true constructivists, their writing contain evidence that people use prior knowledge and senses to construct new knowledge. According to Noddings (1990), one of the four principles that encapsulate the theory of constructivism is that "all knowledge is constructed." One of the ancient writers, Confucius, presented some views of constructivism in his writings, although his writing focuses more on interrelationships between learners and morality (Cooney et al., 1993). Another ancient philosopher, Plato, also used constructivist thought in his writings. He acknowledged that
people construct their own knowledge through their senses (Stevenson \& Haberman, 1998). Aristotle believed that there is an absolute truth, but he allows for human construction of knowledge, a constructivist characteristic. Aristotle hence expressed the constructivist premise that one organizes experiences with the environment (Taylor, 1955).

Medieval writers also contributed towards the constructivist philosophy although they were not explicitly recognised as such. Saint Augustine and Saint Thomas Aquinas, two Catholic theologians, both present experience as a central component of learning. According to Reed and Johnson (2000), St. Augustine not only envisioned the role of the teacher as one who creates the learning environment, but also views the creation of new knowledge as a process of using previous knowledge. St. Augustine voices the constructivist philosophy when he views knowledge not as simply being transmitted but rather as the result of connecting the world to experience. St. Thomas Aquinas acknowledged the role of one's senses in the construction of knowledge, although he was better known for his writing on theology (Baggini \& Strangroom, 2004). Another medieval writer, Vico, presented the subjective construction and context-orientated view of knowledge (Baggini \& Stangroom, 2004). Vico described context as a necessary element of learning. This belief supports Vico as a constructivist. Reed and Johnson (2000) described John Locke as someone who claimed that knowledge comes through the senses. Locke advocated the premise that reflection and the senses are the sources of ideas (Bentley, 1958). Bentley cited that Locke believed that ideas are not innate. One of his categories is made by an act of mind, namely complex ideas. The act of mind was broken down into three actions, namely (i) combining simple ideas, (ii) relating two ideas, and (iii) abstracting from them real existence. Locke's belief that the senses are important to learning, and that the act of the mind involved the combining of ideas, all flow into the constructivist characteristics. Another medieval writer, Kant, believed in
knowledge as it appears in our senses (Bentley, 1958). Kant maintained that people construct their own ways of knowing the physical universe.

Below is a list of the most notable writers who are considered as modern constructivists:

- John Dewey (1859-1952)
- Maria Montessori (1870-1952)
- Wladyslaw Strzeminski (1893-1952)
- Lev Vygotsky (1896-1934)
- Jean Piaget (1896-1980)
- George Kelly (1905-1967)
- Heinz von Foerster (1911 - 2002)
- Jerome Bruner (born 1915)
- Ernst von Glaserfeld (1917-2010)
- Paul Watzlawick (1921 - 2007)
- Edgar Morin (born 1921)
- Humberto Maturana (born 1928)
- Laszlo Garai (born 1935)
- David A. Kolb. (born 1940)

The modern constructivists view knowledge as the result of connecting previous experiences together and the importance of societal and contextual parameters. While some modern writers agree with the concept that knowledge is a process of subjectively constructing from previous knowledge, there are other writers that stipulate that truth is constantly evolving as it reacts to a changing environment. Dewey (1991) believed that knowledge is created by connecting to prior knowledge. Dewey views knowledge as the product of recursive actions,
that is, knowledge grows from what is already known. James (1991) extended the premise of using personal previous knowledge for knowledge construction, to include the role of society. James viewed knowledge as an entity that is created by events. This idea supported constructivists' principle that truth or knowledge is context-based and is created as a result of interacting with the environment. Rorty (1991) claimed that truth is reliant on context and is ever-changing. He maintained that truth or knowledge is an ever-changing entity that adapts to events. Both Rorty and James' notion that man created knowledge as a response to new situations, aligns with the constructivists philosophy of learning through experiences and restructuring prior knowledge. Piaget's theory is based on the premise that students learn by interpreting the results of their interactions with the environment (Fogarty,1999), which aligns with the constructivists' philosophy of constructing knowledge by restructuring previous knowledge. Piaget (1952) viewed the modification of existing knowledge structures as the key element to knowledge creation. William James (1991) went further by stating that the role of society should be included when using personal knowledge for knowledge construction (James, 1991). His view that knowledge is an entity that is created by events supports the constructivists' principle that knowledge is context-related and is created as a result of interacting with the environment. Lev Vygotsky also incorporated the concept of connecting personal knowledge with that of the importance of contextual or societal situations. Vygotsky's (1962) belief that knowledge is a continual personal construction is evidenced in his theory of spontaneous knowledge and scientific knowledge. Spontaneous knowledge is knowledge that students construct from their everyday experience, while scientific knowledge is knowledge that students construct through direct, formal instruction. Vygotsky's view of spontaneous knowledge is representative of the Constructivists philosophy.

### 5.2.2 THE PRINCIPLES OF CONSTRUCTIVISM

According to Noddings (1990), there are four principles that encapsulate the theory of constructivism. They are:
(i) All knowledge is constructed;
(ii) There exist cognitive structures that are activated in the process of construction;
(iii) Cognitive structures are under continual development, and;
(iv) Acknowledgement of constructivism as a cognitive position leads to the adoption of methodological constructivism.

The first principle of constructivism contrasts to a more traditional belief, that knowledge consists of a collection of facts that may be transferred from one person to another by direct instruction (Confrey, 1990). In the constructivist classroom, learning is interactive whereas the learning in a traditionalist classroom is based on repetition. Constructivists believe that knowledge can only be constructed in the mind of the learner and not just passed between individuals.

The second principle describes the nature of the constructions. Constructivists believe that individuals develop mental structures based on their experience, and these structures comprise an individual's knowledge.

The third principle describes cognitive structures as malleable. Knowledge is seen as dynamic, ever changing with our experiences. Traditionalists see knowledge as inert. Silver and Marshall (1990) suggested that learners connect new information to pre-existing knowledge structures rather than adding new information to their already existing store of knowledge.

The fourth principle of constructivism identifies the implication of constructivism. When conducting research, constructivists must use techniques that are best suited to ignite the person's cognitive structures. When teaching, constructivists must use techniques that will challenge and promote modification of students' existing cognitive structures (Noddings, 1990).

### 5.3 TYPES OF CONSTRUCTIVISTS

Not all constructivists are in agreement about the nature of truth which people construct. There is thus a distinction between Radical and Moderate constructivists. Radical constructivists believe that knowledge is constructed in the minds of the learner and not based on a set of objective realities. On the other hand, Moderate constructivists believe that knowledge is constructed in the mind of the learner but contrary to the radical constructivists, they believe there exist objective realities by which the validity of this knowledge can be measured. Von Glaserfeld (1991) cited three principles when he describes radical constructivism: They are:

- Knowledge is built by the cognizing subjects
- The function of cognition is adaptive in the biological sense
- Cognition serves the subject's organization of the experimental world, not the discovery of an objective ontological reality.

Radical constructivists differ from Moderate constructivists in terms of the third principle that Von Glaserfeld cited above. Radical constructivists believe that the content of a person's knowledge can never be judged as true or false. They believe that individuals are protected from constructing false realities because a cognitive image of a false reality would not be viable and would never be incorporated by the cognizing subject in the process of adaptation.

Moderate constructivists reject the Radical constructivists' claim that no external realities exist. Goldin (1990) argues that if one believes that no external realities exist, then one cannot believe that structures exist in mathematics. Moderate constructivists prefer not to think of each individual's constructions as unique. An assessment of the validity of learner constructions is necessitated by the fact that the goal of mathematics education is to help students develop a store of shared knowledge about mathematics.

Noddings (1990) also distinguishes between Strong and Weak constructions. He defined strong acts of construction as those that would be recognized by mathematicians as mathematical. If a student can give an adequate account of why the information, answer or procedure makes sense, the construction would be considered strong. In the same vein, Confrey (1990) defined powerful constructions as those which students believe. Some of the characteristics that Confrey listed as a strong (powerful) construction include the following:

- A structure with a measure of internal consistency
- A convergence among multiple forms and context of representations
- An agreement with experts
- A potential to act as a tool for further constructions
- An ability to be justified and defended.

Strong or powerful constructions are those which can stand up to be the same standard used to evaluate knowledge in the mathematical society.

Noddings (1990) defined weak acts of construction as those that would be evaluated by mathematicians as having limited mathematical value. A student who relies on weak
construction as a tool for solving a problem would have to rely on an external authority for validation of the result, because that student would not be able to justify the chosen technique as appropriate. Similarly, Confrey (1990) defined weak constructions as those which are held up to only one standard. A strong construction can include information to be considered true or false, as long as it can be justified by the student. Similarly, a weak construction can include either true or false information.

Constructivism does not tell us how knowledge is constructed. Piaget (1952) and Vygotsky (1978) describe the processes of how knowledge is constructed. Piaget (1952) claimed that the processes are similar to a biological evolution, with new ideas modifying mental structures so that the cognitive structures will better fit the information being taken in from the environment. Similarly, Vygotsky (1978) describes the construction of knowledge as a process of internalization of ideas and abilities. From interaction with more experienced students, abilities develop and concepts become more refined. The process-object model was one way of conceiving of the maturation process or the evolution of cognitive structures. According to Sfard (1991), this model conceives a subject's understanding of a new mathematical notion as a process. The subject eventually thinks of it as an object as soon as the notion is refined, usually with experience. A more detailed account of Piaget's theory of knowledge construction can be found in Piaget (1947,1952). Similarly, detailed account of Vygotsky's theory of knowledge acquisition can be found in Vygotsky (1978). To summarize, both Piaget and Vygotsky's theories require that subjects be observed in their natural settings in order to obtain and appreciate a full picture of how understanding is evolving in the subjects.

## CHAPTER 6

## RESEARCH METHODOLOGY AND RESULTS

### 6.1 INTRODUCTION

The aim of this study was to investigate the difficulties of first year mainstream mathematics students at the UWC with related rates problems. The mainstream mathematics course is Mathematics 105 (MAT105), hence the subjects of the study were selected from this group. MAT105 is a one year course that introduced students to first year mathematics with a curriculum that includes differential calculus as well as integral calculus. The university also offers what is called an extended curriculum programme (ECP) for mathematics. This is also an extensive calculus course, but is offered over two years. No students that followed the ECP mathematics stream were selected for the purpose of this study.

### 6.2 RESEARCH QUESTIONS

This study is aimed at investigating the effectiveness of the seven steps model as mentioned in chapter 1 (p. 6-7) as a teaching strategy by attempting to answer the following research questions:
(i) How do students perform on the conceptual steps no. 1, 2 and 6 of the standard solution module for solving related rates problems?
(ii) How do students perform on the procedural steps no. 3, 4 and 5 of the standard solution module for solving related rates problems?
(iii) How do students perform on the auxiliary step no. 7 of the standard solution module for solving related rates problems?

### 6.3 RESEARCH DESIGN

Due to the nature of the study, a qualitative approach was used. Qualitative research includes any type of research that produces findings that are not arrived at by means of statistical procedures or other means of quantification. A qualitative approach investigates the quality of relationships, activities, situations or materials. There is a greater emphasis on describing in detail what happens in a specific situation (Fraenkel \& Wallen, 1990). Thus students' responses to the related rates problems will be used to examine how students fare in answering the steps in the Standard Solution Model in a class test on related rates.

### 6.4 PROFILE OF STUDENTS REGISTERED FOR MAT105

There were 238 students registered for the Mathematics105 (MAT105) course of 2012. To protect the right, dignity, safety and privacy of the students, I obtained permission from the university's Faculty Board Research and Ethics Committee and from the UWC Senate Research Committee to conduct this study. I provided reasons for using the MAT105 students as my subjects. Permission was granted after I submitted a research project registration and ethics clearance application form.

Mainstream mathematics students who were registered for the MAT105 course in 2012, obtained The National Senior Certificate for Bachelor's Degree study plus a score of no less than 27 as calculated according to the university's approved points system. This was applicable to students who matriculated from 2008. In addition to this, students also had to adhere to the following specific requirement:

Achieve a level 4 (50-59\%) in English (Home or first additional language);
Achieve a level 3 (40-49\%) in Another Language (Home or first additional language);
Achieve a level 4 (50-59\%) in Mathematics;

Achieve a level 4 (50-59\%) in Physical Science.
Students were also required to write the National Benchmark Test (NBT).

Students who matriculated prior to 2008, had to obtain a full matriculation or age exemption and obtained a pass in Mathematics and either Physical Science or Biology on Higher grade with at least a D symbol (50-59\%) or on Standard grade with at least a C symbol (60-69\%). These students were also required to write the National Benchmark Test (NBT).

Students who were registered for the MAT105 module had diverse backgrounds. Below is a table of the students' background according to their South African provinces of origin:

## TABLE 8

| PROVINCE | EASTERN <br> CAPE | GAUTENG | KWAZULU <br> NATAL | MPUMALANGA | NORTH <br> WEST | NORTHERN <br> PROVINCE | WESTERN <br> CAPE |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| TOTAL | 32 | 18 | 9 | 6 | 2 | 10 | 142 |

Source: University of the Western Cape
Students who were not South African citizens are classified according to their nationality.
Below is a table according different nationalities outside South Africa:

## TABLE 9

| COUNTRY | TOTAL |
| :--- | :---: |
| ANGOLA | 2 |
| CAMEROON | 2 |
| CHINA | 1 |
| DEMOCRATIC REPUBLIC OF CONGO | 2 |
| ETHIOPIA | 1 |
| GABON | 1 |
| MOZAMBIQUE | 1 |
| NAMIBIA | 1 |
| NIGERIA | 1 |
| SWAZILAND | 1 |
| TURKEY | 1 |
| UGANDA | 1 |
| ZAMBIA | 1 |
| ZIMBABWE | 3 |

Source: University of the Western Cape

Below is a table of the students' level of Mathematics competence at National Senior Certificate level from 2008.

## TABLE 10

| CODE | $4(50-59 \%)$ | $5(60-69 \%)$ | $6 \quad(70-79 \%)$ | 7 | $(80-100 \%)$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| TOTAL | 74 | 64 | 35 |  | 22 |

Source: University of the Western Cape
Below is a table of the students' level of Mathematics competence at National Senior Certificate level prior to 2008.

## TABLE 11

| HG | $50-59 \%$ (D) | $60-69 \%$ (C) | $70-79 \%$ (B) | $80-100 \%$ (A) |
| :--- | :---: | :---: | :---: | :---: |
| TOTAL | 4 | 6 | 1 | 2 |
| SG | $50-59 \%$ (D) | $60-69 \%$ (C) | $70-79 \%$ (B) | $80-100 \%$ (A) |
| TOTAL | 1 | 4 | 2 | 4 |

Source: University of the Western Cape

As the above information indicates, the students come from different parts of South Africa as well as other countries on the African continent. Their high school mathematics results suggest that the majority of the students are competent to further their studies in tertiary mathematics. There were two parallel classes for the MAT105 course according to the university timetable. Both groups had 4 lectures of 1hour duration per week, lectured at the same time. My supervisor of this study, with over 36 years of teaching experience at U.W.C, lectured the one group, while a senior colleague lectured the second group. In addition to the lectures, 1 two hour tutorial session per week took place where students work out exercises from the prescribed textbook (Calculus - Concepts and Contexts by James Stewart ,4th edition, Metric Version). The two lecturers held weekly meetings regarding progress in the course. The study took place during the latter part of the third quarter of 2012. One week consisting of 3 lectures and 1 two hour tutorial session were used to teach the topic of Related

Rates. Initially, only two lectures were allocated to lecture Related Rates. However, three lecture slots were allocated to accommodate a proper treatment of Related Rates. Prior to and leading up to the teaching of Related Rates, students were taught in a systematic and comprehensive manner all background topics such as limits, continuity, differentiation rules, implicit differentiation and optimization. The students were thus properly prepared to deal with the concepts needed in Related Rates. Three problems were used from the textbook of Stewart to explain the concepts and understanding of Related Rates. The group used for this study was taught using the Standard Solution Model. It was explained what was required at each step, what needs to be written down and how to obtain the final answer. Specific reference was made to step 7 of the Standard Solution Model. The three problems discussed in class were the following:

1. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of $0.06 \mathrm{rad} / \mathrm{s}$. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\frac{\pi}{3}$.
2. A kite, controlled by a stationary person, 50 m above the ground moves horizontally at a speed of $2 \mathrm{~m} / \mathrm{s}$. At what rate is the angle between the string and the horizontal decreasing when 100 m of string has been let out?
3. Car A is travelling west at $90 \mathrm{~km} / \mathrm{h}$ and car B is travelling north at $100 \mathrm{~km} / \mathrm{h}$. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 60 m and car B is 80 m from the intersection?

Detailed solutions of these problems can be found in Addendum B. After the topic of Related Rates was discussed in the lecture environment, the students were given 8 problems as an exercise in the tutorial session. The tutorial session followed one day after the lectures of

Related Rates were completed. The eight problems with solutions can be found in addendum C. The problems in the tutorial session were done in a collaborative fashion (Barkley et al., 2004) under supervision of the lecturer, a teaching assistant, the researcher as well as tutors. The students were assessed four days after completing the lectures.

### 6.5 SAMPLING

There were two parallel classes for the MAT105 course according to the university timetable as mentioned before. Since I only had access to one group, I only used this group of 127 students for my research. I used this group as my sample because I had access to their tests results and scripts. This method of sampling is called convenience sampling. It is a type of nonprobability sampling which is readily available and convenient (Wilson, 2009). This method of sampling is used to select what is easily accessible to research. The sample drawn by this method does not claim to be representative of the larger population, nor can it claim generalizability. Although this type of sample makes it easier to conduct the research, there is no precise way of generalizing from the sample to any type of population (McMillan \& Schumacher, 2010).

### 6.6 INSTRUMENTATION

Only one written test was used in this study. The test was of thirty minutes duration consisting of two questions on Related Rates. This test was written as part of the students' weekly test and contributed to their Continuous Assessment Mark for MAT105. The test was written four days after completing the topic on Related Rates. The questions for this test were set up in order to answer the research questions of my study. I worked closely with my supervisor. The two questions that were selected for this test were the following:

1. A plane flying with a constant speed $300 \mathrm{~km} / \mathrm{h}$ passes over a ground radar station at an altitude of 1 km and climbs at an angle of $30^{\circ}$. At what rate is the distance from the plane to the radar station increasing a minute later?
2. At noon, ship A is 150 km west of ship B. Ship A is sailing east at $35 \mathrm{~km} / \mathrm{h}$ and ship B is sailing north at $25 \mathrm{~km} / \mathrm{h}$. How fast is the distance between the ships changing at 16h00?

A detailed solution of these two questions can be found in addendum D. In the outlined solution, the different steps of the Standard Solution Model are indicated.

### 6.7 RESEARCH METHOD

### 6.7.1 Introduction

The test referred to in the previous section was assessed by the lecturers according to the memorandum as found in Addendum E. The marks scored by the students were recorded. A score out of 10 was allocated per question for the assessment. The scripts of the students who participated in the research were collected from the lecturer in order to analyse students' performance in the different steps of the Standard Solution Model of related rates problems.

### 6.7.2 Document Analysis

This study will adopt the document analysis method. According to Bowen (2009), this method is a systematic procedure for reviewing or evaluating documents, in this case, the scripts of students.

Below is a list of advantages, as stated by Yin (1994), of document analysis:

- Efficient method: Document analysis is less time-consuming and therefore more efficient
than other research methods. It requires data selection, instead of data collection
- Availability: Many documents are in the public domain, especially since the advent of the Internet, and are obtainable without the authors' permission. This makes document analysis an attractive option for qualitative researchers. As Merriam (1988) argued, locating public records is limited only by one's imagination and industriousness. An important maxim to keep in mind is that if a public event happened, some official record of it most likely exists.
- Cost-effectiveness: Document analysis is less costly than other research methods and is often the method of choice when the collection of new data is not feasible. The data (contained in documents) have already been gathered; what remains is for the content and quality of the documents to be evaluated.
- Lack of obtrusiveness and reactivity: Documents are 'unobtrusive' and 'nonreactive' - that is, they are unaffected by the research process. (Previous studies found in documents are not being considered here.) Therefore, document
analysis counters the concerns related to reflexivity (or the lack of it) inherent in other qualitative research methods. With regard to observation, for instance, an event may proceed differently because it is being observed.

Reflexivity-which requires an awareness of the researcher's contribution to the construction of meanings attached to social interactions and acknowledgment of the possibility UNIVERSITY of the

WESTERN of the investigator's influence on the research-is usually not an issue in using documents for research purposes.

- Stability: As a corollary to being non-reactive, documents are stable. The investigator's presence does not alter what is being studied. Documents, then, are suitable for repeated reviews.
- Exactness: The inclusion of exact names, references, and details of events makes documents advantageous in the research process (Yin, 1994).
- Coverage: Documents provide broad coverage; they cover a long span of time, many events, and many settings (Yin, 1994).


### 6.7.3 Validity and Reliability

Validity and reliability are important in any research. Validity and reliability ensure that the research is considered genuine, that it is believable and convincing. In research, data are considered to the point that they describe or deal at once with the topic under consideration (Mertler \& Charles, 2005).

Reliability is the quality concept in qualitative research which is to be solved in order to state a study as part of proper research (Stenbacka, 2001). McMillan \& Schumacher (2010) argue that validity, on the other hand, is the degree of the agreement between the explanations of the phenomena and the realities of the world. Golafshani (2003) viewed reliability and validity as the idea of trustworthiness, rigor and the quality in qualitative research.

Inter-rater agreement is the degree to which two or more raters using the same rating scale give the same rating to an identical observable situation (Graham, Milanowski \& Miller, 2012). This is to suggest that inter-rater agreement is a measurement of the consistency between the absolute value of the evaluators' ratings (Graham, Milanowski \& Miller, 2012).

To validate the assessment process, twenty scripts were randomly selected from the 127 scripts marked by the researcher. These scripts were then moderated by a senior lecturer in the Mathematics department at UWC. As part of the inter-rater agreement, the assessment process was explained to this lecturer and marked accordingly. If there were discrepancies between the marks of the researcher and that of the moderator, it was discussed and an agreement was reached on the final result.

Since the test scripts were used for this research and it formed part of the students' continuous assessment, all scripts had to be copied and the original scripts were handed back
to the students. The copied scripts were kept in the Mathematics Department's filing system for safe keeping. The students' identity was also hidden in the copied scripts to protect the identity of the participants. As mentioned before, the students agreed to participate in the project on condition of anonymity.

### 6.7.4 Scoring

For the purpose of this study, the scoring was different to the scoring of the lecturers. I was interested if students could identify and answer the different steps of the Standard Solution Model. This scoring will help me in answering my research questions. I examined all the scripts of these students and identified steps 1,2 and 6 in the students' solution. The scoring for steps 1,2 and 6 was as follows: Students obtain a score of 0 if they answered steps 1,2 and 6 incorrectly. If a student answers step 1 and 2 correctly, a score of 1 was allocated. It was felt that steps 1 and 2 should be scored as a single step, since the two steps are dependent on each other. Information of what is given and what should be calculated can only be used if it is extracted from a sketch or diagram or vice versa. A student obtains a score of 2 if they answered steps 1 and 2 as well as step 6 correctly. For the purpose of analysis, the scores of steps 1, 2 and 6 were taken as a total to answer the first research question. The first research question is: How do students perform on the conceptual steps numbers 1,2 and 6 of the standard solution module for solving related rates problems. Thereafter steps 3,4 and 5 in the student's solution were identified. If the student answered steps 3,4 and 5 incorrectly, that student obtained a score of 0 . If the student answered one step correctly and the two remaining steps incorrectly, then that student obtains a score of 1 . If the student answers any two steps correctly and a third step incorrectly, then that student obtains a score of 2 . If all steps are answered correctly, then that student obtains a score of 3 . The scores of steps 3,4 and 5 were taken as a total to answer the second research question. The second research question is: How do students perform on the procedural steps number 3, 4 and 5 of the
standard solution module for solving related rates problems. Finally step 7 in the student's answer was identified. The score of step 7 was used to answer the third research question. The third research question is: How do students perform on the auxiliary step number 7 of the standard solution module for solving related rates problems. A student obtained 0 if he/she answered step 7 incorrectly and 1 mark if it is correct.

In the next section, the results are summarised.

### 6.8 RESULTS

Below is a histogram showing the results of steps 1,2 and 6 of the sample group.

TABLE 12


Close to $45.6 \%$ of students answered steps 1, 2 and 6 incorrectly. More or less $29.1 \%$ could only answer step 1 and 2 or step 6 correctly while only a quarter of students in the sample group could answer all steps correctly.

Below is a histogram showing the results of steps 3, 4 and 5 of the sample group.

TABLE 13


From the above graph, $40.1 \%$ students could not answer any of steps 3,4 and 5 correctly.
Only $25.2 \%$ of student in the sample group could answer all of steps 3,4 and 5 correctly.

Below is the histogram of step 7 of the sample group.

## TABLE 14



The number of students who answered this step correctly is just one percentage point more than the number of students who answered it incorrectly ( $50.4 \%$ to $49.6 \%$ ).

### 6.9 CONCLUSION

From the above results, it is evident that only between 25 and $30 \%$ of students could answer the conceptual understanding $(29.1 \%)$ and procedural knowledge ( $25.2 \%$ ) steps of the Standard Solution Model correctly. From the above results, it can be concluded that students in the MAT105 class do not perform well in both the conceptual understanding (Steps 1, 2 and 6) and the procedural knowledge (Steps 3, 4 and5) of the Standard Solution Model. There was little difference in the number of students who answered step 7 correctly. So students
perform averagely in this step. The reasons why students perform so poorly in both the conceptual understanding (Steps 1, 2 and 6) and the procedural knowledge (Steps 3, 4 and 5) of the Standard Solution Model may vary. The results obtained call into question the use of the Standard Solution Model as an appropriate methodology to teach Related Rates, at least to the students enrolled in the MAT105 course at UWC. It is therefore proposed that the textbook material should be supplemented by additional study material that will address the issues discussed in section 2.12.


## CHAPTER 7

## CONCLUSION AND RECOMMENDATIONS

### 7.1 INTRODUCTION

The aim of the thesis is to research the difficulties that first year mainstream mathematics students at UWC experience when solving Related Rates problems in calculus. In chapter 2, an in-depth study was made of the nature of Related Rates problems by studying a number of examples. The findings of this study are summarized in section 2.12. The study adopted the same model of the solution of all types of Related Rates that was used by Martin (2000) for the solution of geometric Related Rates problems.

In chapter 3 of this thesis, many examples were used to illustrate how the seven step solution procedure of the Standard Solution model is applied.

In the literature review in chapter 4, the underlying concepts which underpin Related Rates problems are identified and specific examples of research on each of these concepts are given. For example, the review of the literature on word problems is done comprehensively and covers extensively the range of issues involved in this topic. Drawing on the work in chapter 2 on the nature of Related Rates problems, it is explained in chapter 5 why this study is underpinned by Constructivism as a theoretical basis.

Chapter 6 of the thesis is devoted to answering the 3 research questions posed in chapter 1 .

The thesis contains many worked examples of Related Rates problems which can be used by the lecturers assigned to the MAT105 course.

### 7.2 RECOMMENDATIONS

7.2.1 A follow-up study will focus on the detail of students' difficulties when solving Related Rates problems. This will be done by analysing the scripts of students, the actual student performances.
7.2.2 The analysis of student performances on the Related Rates questions in the test shows that there is a need to design additional instructional materials. Some work has already been done on this as can be seen in Addendum $G$ of the thesis. This should be continued and improved.
7.2.3 In a follow-up study, the use of the Standard Solution model can be critiqued against the actual student performances.
7.2.4 In a follow-up study, the design of instructional material can be contrasted against existing material in relation to student performances.

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## ADDENDUM A

1. A husband is 7 years older than his wife. Ten years ago, he was twice her age. How old is he?
2. The average weight ( $W$ ) of the antlers of deer is approximately related to the age ( $A$ ) of the deer by the equation: $W=m A+c$. For a particular species, it is found that when $A=30$ months, $W=0.15 \mathrm{~kg}$, whereas when $A=54$ months, $W=0.36 \mathrm{~kg}$. Find $m$ and $c$ and calculate the age at which $W$ reaches 0.5 kg .
3. An individual on a strict diet plans to breakfast on cornflakes, milk and a boiled egg. After allowing for the egg, his diet allows for a further 300 calories for this meal. One ounce of milk contains 20 calories and 1 ounce (about one cupful) of cornflakes (plus sugar) contains 160 calories. What is the relation between the number of ounces of milk and of cornflakes that can be consumed?
4. Sue is 7 years older than Bobby. If the product of their ages is 60 , how old is Bobby?
WESTERN CAPE
5. A box with no top is to be formed from a rectangular sheet of tin by cutting out 4-inch squares from each corner and folded up by the sides. If the width of the box is 3 inches less than the length and the box is to hold 280 cubic inches, find the dimensions of the sheet of tin.

## ADDENDUM B

1. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of $0,06 \mathrm{rad} / \mathrm{s}$. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\frac{\pi}{3}$.

## Solution

We let $\theta$ be the angle between the two sides. We name the triangle $\triangle \mathrm{EBC}$ with $E B=4$ and $B C=5$. We also let $A$ the area of the triangle at time $t$. (Step 2)

(Step 1)
$A=\frac{1}{2} \cdot 4 \cdot 5 \cdot \sin \theta$
(Step 3)
$\Rightarrow A=10 \sin \theta$
$\frac{d A}{d t}=10 \cos \theta \frac{d \theta}{d t} \quad($ Step 4)
But since $\frac{d \theta}{d t}=0,06 \mathrm{rad} / \mathrm{s}$ and $\theta=\frac{\pi}{3}$, we have

$$
\begin{align*}
\frac{d A}{d t} & =10 \cdot \cos \left(\frac{\pi}{3}\right) \cdot(0,06)  \tag{Step5}\\
\Rightarrow \frac{d A}{d t} & =0,3 \mathrm{~m}^{2} / \mathrm{s}
\end{align*}
$$

Thus the area is increasing at a rate of $0,3 \mathrm{~m}^{2} / \mathrm{s}$. (Step 6)
2. A kite, controlled by a stationary person, 50 m above the ground moves horizontally at a speed of $2 \mathrm{~m} / \mathrm{s}$. At what rate is the angle between the string and the horizontal decreasing when 100 m of string has been let out?

## Solution

Let $x$ be the horizontal movement of the kite, let $y$ be the height of the kite from the ground. Let $s$ be the length of the string that is let out and let $\theta$ be the angle between the string and the horizontal ground. Then $y=50$ and $\frac{d x}{d t}=2 \mathrm{~m} / \mathrm{s}$. (Step 2)


From the diagram, $\sin \theta=\frac{50}{s} \Rightarrow s \cdot \sin \theta=50$ RSIT (Step 3)
$\therefore \frac{d s}{d t} \sin \theta+s \cdot \cos \theta \frac{d \theta}{d t}=0$
$\Rightarrow s \cos \theta \frac{d \theta}{d t}=-\frac{d s}{d t} \sin \theta$.
(Step 4)
$\Rightarrow \frac{d \theta}{d t}=-\frac{\frac{d s}{d t} \sin \theta}{s \cos \theta}$

Also $s^{2}=x^{2}+50^{2}$
$\therefore 2 s \frac{d s}{d t}=2 x \frac{d x}{d t}$
$\Rightarrow s \frac{d s}{d t}=x \frac{d x}{d t}$
(Step 7)
$\Rightarrow \frac{d s}{d t}=\frac{x}{s} \cdot \frac{d x}{d t}$
To find $x$ when $s=100$, we use the equation $s^{2}=x^{2}+50^{2}$.
$\therefore x=\sqrt{s^{2}-50^{2}}$
$\therefore x=\sqrt{(100)^{2}-50^{2}}$
$\therefore x=\sqrt{7500}=50 \sqrt{3}$

## (Step 7)

$\therefore \frac{d s}{d t}=\frac{50 \sqrt{3}}{100} \cdot 2=\sqrt{3} \mathrm{~m} / \mathrm{s}$
Now $\sin \theta=\frac{50}{100}=\frac{1}{2}$, when $s=100$.
Since $\sin \theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$
(Step 7)
Since $\theta=\frac{\pi}{6}, \cos \frac{\pi}{6}=\frac{\sqrt{3}}{2}$

From $\frac{d \theta}{d t}=-\frac{\frac{d s}{d t} \sin \theta}{s \cos \theta}$ we substitute $\frac{d s}{d t}=\sqrt{3}, \sin \theta=\frac{1}{2}, \cos \theta=\frac{\sqrt{3}}{2}$ and $s=100$, we then have

$$
\begin{equation*}
\frac{d \theta}{d t}=-\frac{\frac{d s}{d t} \sin \theta}{100 \cos \theta} \frac{d \theta}{d t}=-\frac{\sqrt{3} \frac{1}{2}}{100 \frac{\sqrt{3}}{2}} \tag{Step5}
\end{equation*}
$$

$\therefore \frac{d \theta}{d t}=-0.01 \mathrm{rad} / \mathrm{s}$
The angle between the string and the horizontal is decreasing at a rate of $-0.001 \mathrm{rad} / \mathrm{s}$ at the moment the length of the string let out is 100 m .
(Step 6)
3. A plane flying with a constant speed $300 \mathrm{~km} / \mathrm{h}$ passes over a ground radar station at an altitude of 1 km and climbs at an angle of $30^{\circ}$. At what rate is the distance from the plane to the radar station increasing a minute later?

## Solution

We let $x$ be the distance from the plane to a point P immediately above the radar station. Then $\frac{d x}{d t}=300 \mathrm{~km} / \mathrm{h}$. Let $s$ be the distance from the radar station to the plane.
(Step 2)

$s^{2}=1^{2}+x^{2}-2(1)(x) \cos \left(90^{\circ}+30^{\circ}\right)$
$\Rightarrow s^{2}=1+x^{2}-2 x \cos 120^{\circ}$
(Step 3)
$\Rightarrow s^{2}=1+x^{2}-2 x\left(-\frac{1}{2}\right)$
Hence $s^{2}=1+x+x^{2}$
$\therefore 2 s \frac{d s}{d t}=\frac{d x}{d t}+2 x . \frac{d x}{d t}$
$\Rightarrow 2 s \frac{d s}{d t}=(1+2 x) \frac{d x}{d t}$.
(Step 4)

Therefore $\frac{d s}{d t}=\frac{(1+2 x) \frac{d x}{d t}}{2 s}$

So after 1 minute $\left(\frac{1}{60}\right.$ hours $), x=300 \cdot \frac{1}{60} \mathrm{~km} \Rightarrow x=5 \mathrm{~km}$. (Step 7)
From $s^{2}=1+x+x^{2}$ we get that

$$
\begin{align*}
& s=\sqrt{1+x+x^{2}} \\
& \Rightarrow s=\sqrt{1+5+5^{2}}  \tag{Step7}\\
& \Rightarrow s=\sqrt{31}
\end{align*}
$$

So $\frac{d s}{d t}=\frac{(1+2 x) \frac{d x}{d t}}{2 s}$

$$
\begin{align*}
& \Rightarrow \frac{d s}{d t}=\frac{(1+2(5)) 300}{2(\sqrt{31})} \\
& \Rightarrow \frac{d s}{d t}=\frac{1650}{\sqrt{31}} \approx 296 \mathrm{~km} / \mathrm{h} \tag{Step5}
\end{align*}
$$

Therefore the distance from the plane to the radar station is increasing at a rate of $296 \mathrm{~km} / \mathrm{h}$. (Step 6)

## ADDENDUM C

1. The altitude of a triangle is increasing at a rate of $1 \mathrm{~cm} / \mathrm{min}$ while the area of the triangle is increasing at a rate of $2 \mathrm{~cm}^{2} / \mathrm{min}$. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is $100 \mathrm{~cm}^{2}$.

## Solution

We let $h$ be the altitude of triangle $\triangle \mathrm{EBC}$. We also let $A$ the area of the triangle and let
$x$ be the base of the triangle. $\therefore \frac{d h}{d t}=1 \mathrm{~cm} / \mathrm{min}$ and $\frac{d A}{d t}=2 \mathrm{~cm}^{2} / \mathrm{min}$.
(Step 2)

(Step 1)
$A=\frac{1}{2} b . h$
(Step 3)
$\Rightarrow A=\frac{1}{2} x h$
$\therefore \frac{d A}{d t}=\frac{1}{2}\left(\frac{d x}{d t} \cdot h+x \cdot \frac{d h}{d t}\right)$
(Step 4)

Also since
$A=\frac{1}{2} x h$
we have that
$x=\frac{2 A}{h}$
$\Rightarrow x=\frac{2(100)}{10}$
(Step 7)
$\Rightarrow x=20 \mathrm{~cm}$.
$\therefore 2 \mathrm{~cm}^{2} / \mathrm{min}=\frac{1}{2}\left(\frac{d x}{d t} .(10)+20(1)\right)$
$\therefore 4 \mathrm{~cm}^{2} / \mathrm{min}=10 \frac{d x}{d t}+20$
$\Rightarrow \frac{d x}{d t}=-1,6$

## (Step 5)

So the base is decreasing at a rate of $1,6 \mathrm{~cm} . \mathrm{min}$.
2. A television camera is positioned 1200 m from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is $200 \mathrm{~m}^{-1}$ when it has risen 900 m .
(a) How fast is the distance from the television camera to the rocket changing at that moment?
(b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

## Solution

Let y be the distance the rocket rises. Then $\frac{d y}{d t}=200 \mathrm{~m} \cdot \mathrm{~s}^{-1}$ when $y=900 \mathrm{~m}$.
Let s be the distance from the camera to the rocket. We let $\theta$ be the angle of elevation of the camera. (Step 2)

(a) $s^{2}=y^{2}+(1200)^{2}$
(Step 3)

$$
\begin{aligned}
& \therefore 2 s \frac{d s}{d t}=2 y \frac{d y}{d t} \\
& \Rightarrow \frac{d s}{d t}=\frac{y}{s} \cdot \frac{d y}{d t}
\end{aligned}
$$

(Step 4)
Also since $s^{2}=y^{2}+(1200)^{2}$

$$
\begin{equation*}
s=\sqrt{y^{2}+(1200)^{2}} \tag{Step7}
\end{equation*}
$$

But $\mathrm{y}=900$,

$$
\begin{align*}
& \therefore s=\sqrt{(900)^{2}+(1200)^{2}} \\
& \Rightarrow s=1500 \mathrm{~m} . \tag{Step7}
\end{align*}
$$

Substituting $y=900 \mathrm{~m}$ and $s=1500 \mathrm{~m}$ as well as $\frac{d y}{d t}=200 \mathrm{~m} \cdot \mathrm{~s}^{-1}$ in the expression $\frac{d s}{d t}=\frac{y}{s} \cdot \frac{d y}{d t}$,
we have $\frac{d s}{d t}=\frac{900}{1500} .200$
(Step 5)

$$
=120 \mathrm{~m} \cdot \mathrm{~s}^{-1} .
$$

Hence the distance from the television camera to the rocket is increasing at a rate of $120 \mathrm{~m} . \mathrm{s}^{-1}$. (Step 6)
(b) $\tan \theta=\frac{y}{1200}$
(Step 3)
$\therefore \sec ^{2} \theta \frac{d \theta}{d t}=\frac{1}{1200} \frac{d y}{d t}$
$\Rightarrow \frac{d \theta}{d t}=\frac{1}{1200 \cdot \sec ^{2} \theta} \frac{d y}{d t}$

## (Step 4)

Since
$\sec ^{2} \theta=1+\tan ^{2} \theta$ and $y=900$, we have that
(Step 7)
$\sec ^{2} \theta=1+\left(\frac{y}{1200}\right)^{2}$
$\Rightarrow \sec ^{2} \theta=1+\left(\frac{900}{1200}\right)^{2}$
$\Rightarrow \sec ^{2} \theta=\frac{25}{16}$
Substituting $\sec ^{2} \theta=\frac{25}{16}$ well as $\frac{d y}{d t}=200 m \cdot s^{-1}$ in the expression $\frac{d \theta}{d t}=\frac{1}{1200 \cdot \sec ^{2} \theta} \frac{d y}{d t}$
we have

$$
\begin{aligned}
& \frac{d \theta}{d t}=\frac{1}{1200 \cdot \frac{25}{16} \cdot \frac{200}{1}} \\
& \Rightarrow \frac{d \theta}{d t}=\frac{8}{75} \approx 0,1067 \mathrm{~m} \cdot \mathrm{~s}^{-1}
\end{aligned}
$$

Therefore the camera's angle of elevation is increasing at a rate of $0,1067 \mathrm{~m} \cdot \mathrm{~s}^{-1} \cdot$ (Step 6)
3. A runner sprints around a circular track of radius 100 m at a constant speed of $7 \mathrm{~m} / \mathrm{s}$. The runners friend is standing at a distance 200 m from the centre of the tract. How fast is the distance between friends changing when the distance between them is 200 m ?

Solution
Let $x$ be the distance that the runner sprints around the track. Then $\frac{d x}{d t}=7 \mathrm{~m} / \mathrm{s}$.
Also let $s$ be the distance between the runner and the friend.
(Step 2)

$\therefore s^{2}=200^{2}+100^{2}-2(200)(100) \cos \theta$
$\Rightarrow s^{2}=50000-40000 \cos \theta$
$\therefore 2 s \frac{d s}{d t}=-40000(-\sin \theta) \cdot \frac{d \theta}{d t}$
(Step 4)
$\therefore \frac{d s}{d t}=\frac{20000 \sin \theta \cdot \frac{d \theta}{d t}}{s}$
Now

$$
\begin{align*}
& x=r \theta  \tag{Step7}\\
& \Rightarrow x=100 \theta .
\end{align*}
$$

$\therefore \frac{d x}{d t}=100 \frac{d \theta}{d t}$
$\Rightarrow 7=100 \cdot \frac{d \theta}{d t}$, since $\frac{d x}{d t}=7 \mathrm{~m} / \mathrm{s}$.
$\therefore \frac{d \theta}{d t}=0,07$
Since $s=200$, we have

$$
\begin{align*}
& 200^{2}=200^{2}+100^{2}-2(200)(100) \cos \theta \\
& \Rightarrow 10000=40000 \cos \theta  \tag{Step7}\\
& \Rightarrow \cos \theta=\frac{1}{4}
\end{align*}
$$

since $\sin \theta=\sqrt{1-\cos ^{2} \theta}$,
we get that

$$
\begin{aligned}
& \sin \theta=\sqrt{1-\left(\frac{1}{4}\right)^{2}} \\
& \Rightarrow \sin \theta=\sqrt{\frac{15}{16}}
\end{aligned}
$$

Using $s^{2}=50000-40000 \cos \theta$, we obtain

$$
s^{2}=50000-40000\left(\frac{1}{4}\right)
$$

$$
\Rightarrow s^{2}=40000 \Rightarrow s^{2}=200
$$

Therefore $\frac{d s}{d t}=\frac{20000 \sin \theta \cdot \frac{d \theta}{d t}}{s}$ becomes
$\frac{d s}{d t}=\frac{20000\left(\sqrt{\frac{15}{16}}\right) \cdot 0,07}{200}$
$\Rightarrow \frac{d s}{d t}=7 \cdot \sqrt{\frac{15}{16}}$
$\approx 6,78 \mathrm{~m} / \mathrm{s}$.
Therefore the distance between the two friends is changing at a rate of $6,78 \mathrm{~m} / \mathrm{s}$.
(Step 6)
4. A plane flying horizontally at an altitude of 2 km and a speed of $800 \mathrm{~km} / \mathrm{h}$ passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 3 km away from the station.

## Solution

Let $x$ be the distance travelled by the plane from P , where P is a point directly above the radar station which the plane passes. Let $s$ be the distance from the plane to the radar station. Then $\frac{d x}{d t}=800 \mathrm{~km} / \mathrm{h}$. (Step 2)

$s^{2}=x^{2}+2^{2}$.
(Step 3)
$\therefore 2 s \frac{d s}{d t}=2 x \frac{d x}{d t}$
$\Rightarrow \frac{d s}{d t}=\frac{x}{s} \cdot \frac{d x}{d t}$.
(Step 4)

Also from $s^{2}=x^{2}+2^{2}$, we obtain

$$
x^{2}=s^{2}-4 \Rightarrow x=\sqrt{s^{2}-4} .
$$

Since $\mathrm{s}=3 \mathrm{~km}$, we obtain
(Step 7)
$x=\sqrt{(3)^{2}-4}$
$\Rightarrow x=\sqrt{5}$
Therefore
$\frac{d s}{d t}=\frac{x}{s} \cdot \frac{d x}{d t}$
$\Rightarrow \frac{d s}{d t}=\frac{\sqrt{5}}{3} .(800) \approx 596,3 \mathrm{~km} / \mathrm{h}$

## (Step 5)

Thus the distance from the plane to the radar station is increasing at a rate of $596,3 \mathrm{~km} / \mathrm{h}$. (Step 6)
5. Car A is travelling west at $90 \mathrm{~km} / \mathrm{h}$ and car $B$ is travelling north at $100 \mathrm{~km} / \mathrm{h}$. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 60 m and car B is 80 m from the intersection?

## Solution

Let $x$ be the distance of car A to the intersection at any time $t$ and let $y$ be the distance of car B to the intersection at any time $t$. Then $\frac{d x}{d t}=-90 \mathrm{~km} / \mathrm{h}$ since $x$ is decreasing, and $\frac{d y}{d t}=-100 \mathrm{~km} / \mathrm{h}$ since $y$ is decreasing. Let $s$ be the distance between the two cars.
(Step 2)


$$
\begin{align*}
& s^{2}=x^{2}+y^{2}  \tag{Step3}\\
& \therefore 2 s \frac{d s}{d t}=2 x \frac{d x}{d t}+2 y \frac{d y}{d t} \\
& \Rightarrow \frac{d s}{d t}=\frac{x}{s} \frac{d x}{d t}+\frac{y}{s} \frac{d y}{d t}  \tag{Step4}\\
& \Rightarrow \frac{d s}{d t}=\frac{x \frac{d x}{d t}+y \frac{d y}{d t}}{s}
\end{align*}
$$

Also from $s^{2}=x^{2}+y^{2}$ we obtain

$$
\begin{align*}
& s=\sqrt{x^{2}+y^{2}} \\
& \Rightarrow s=\sqrt{(0,06)^{2}+(0,08)^{2}}  \tag{Step7}\\
& \Rightarrow s=0,1 \mathrm{~km}
\end{align*}
$$

Therefore $\frac{d s}{d t}=\frac{x \frac{d x}{d t}+y \frac{d y}{d t}}{s}$ becomes
$\frac{d s}{d t}=\frac{x \frac{d x}{d t}+y \frac{d y}{d t}}{\sqrt{x^{2}+y^{2}}}$
$\Rightarrow \frac{d s}{d t}=\frac{0,06(-90)+0,08(-100)}{0,1}$.
$\Rightarrow \frac{d s}{d t}=-134 \mathrm{~km} / \mathrm{h}$


Therefore the cars are approaching each other at $134 \mathrm{~km} / \mathrm{h}$.(the distance is decreasing at a rate of $134 \mathrm{~km} / \mathrm{h}$. (Step 6)
6. A man walks along a straight path at a speed of $1,5 \mathrm{~m} / \mathrm{sec}$. A searchlight is located on the ground 6 m from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 8 m from the point on the path closest to the searchlight?

## Solution

Let $x$ be the distance the man walks from the point on the path closest to the searchlight. Then $\frac{d x}{d t}=1,5 \mathrm{~m} / \mathrm{sec}$. Let $\theta$ be the angle through which the searchlight rotates. Let $s$ be the length of the beam. (Step 2)

$\tan \theta=\frac{x}{6}$
(Step 3)
$\therefore \sec ^{2} \theta \frac{d \theta}{d t}=\frac{1}{6} \frac{d x}{d t}$
$\Rightarrow \frac{d \theta}{d t}=\frac{1}{6 \cdot \sec ^{2} \theta} \frac{d x}{d t}$
$\Rightarrow \frac{d \theta}{d t}=\frac{\cos ^{2} \theta}{6} \frac{d x}{d t}$
(Step 4)
Also
Since $x=8 \mathrm{~m}$,

$$
\begin{aligned}
& s^{2}=x^{2}+6^{2} \\
& \Rightarrow s=\sqrt{8^{2}+6^{2}} \\
& \Rightarrow s=10 .
\end{aligned}
$$

(Step 7)

So
$\cos \theta=\frac{6}{10}$
$\Rightarrow \cos ^{2} \theta=\frac{36}{100}$

## (Step 7)

Therefore from
$\frac{d \theta}{d t}=\frac{\cos ^{2} \theta}{6} \frac{d x}{d t}$,
$\frac{d \theta}{d t}=\frac{\left(\frac{36}{100}\right)^{2}}{6} 1,5$
$\Rightarrow \frac{d \theta}{d t}=\frac{9}{100}=0,09 \mathrm{rad} / \mathrm{sec}$.
(Step 5)

Therefore the searchlight is rotating at $0,09 \mathrm{rad} / \mathrm{sec}$.
7. A man starts walking north at $1,2 \mathrm{~m} / \mathrm{s}$ from a point P . Five minutes later a woman starts walking south at $1,6 \mathrm{~m} / \mathrm{s}$ from a point 200 m due east of P . At what rate are the people moving apart 15 minutes after the woman starts walking?

## Solution

Let $x$ be the distance that the man walks from point P and let $y$ be the distance the woman walks from $\mathrm{P}^{\prime}$ (where $\mathrm{P}^{\prime}$ is a point 200 m due east of P ). Let $s$ be the distance between the two people 15 minutes after the woman starts walking. (Step 2)

$s^{2}=200^{2}+(x+y)^{2}$
(Step 3)
$\therefore 2 s \frac{d s}{d t}=2(x+y)\left(\frac{d x}{d t}+\frac{d y}{d t}\right)$.
$\Rightarrow \frac{d s}{d t}=\frac{(x+y)}{s}\left(\frac{d x}{d t}+\frac{d y}{d t}\right)$
(Step 4)

After 20 minutes which equals $(20)(60)=1200$ seconds, $x=(1,2)(1200)=1440 \mathrm{~m}$ and after 15 minutes which equals (15)(60)=900 seconds, $y=(1,6)(900)=1440 \mathrm{~m}$.

Also from $s^{2}=200^{2}+(x+y)^{2}$
$s=\sqrt{200^{2}+(x+y)^{2}}$
$\Rightarrow s=\sqrt{200^{2}+(1440+1440)^{2}}$.
(Step 7)
$\Rightarrow s=\sqrt{8334400}$
$\Rightarrow s \approx 2886,94 m$
So from $\frac{d s}{d t}=\frac{(x+y)}{s}\left(\frac{d x}{d t}+\frac{d y}{d t}\right)$,

$$
\begin{aligned}
& \frac{d s}{d t}=\frac{(1440+1440)}{2886,94}(1,2+1,6) \\
& \Rightarrow \frac{d s}{d t} \approx 2,79 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Thus the people are moving apart at a rate of $2,79 \mathrm{~m} / \mathrm{s}, 15$ minutes after the woman started walking.
(Step 6)
8. Gravel is being dumped from a conveyor belt at a rate of $3 \mathrm{~m}^{3} / \mathrm{min}$ and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 3 m high?

## Solution

Let $V$ be the amount of gravel dumped at any time $t$. Let h be the height of the cone at any time $t$. Let $r$ be the radius of the cone.

$\frac{d V}{d t}=3 m^{3} / \mathrm{min}$
$h=2 r=d$
$\Rightarrow r=\frac{h}{2}$
Now
$V=\frac{1}{3} \pi r^{2} h$
$\Rightarrow V=\frac{1}{3} \pi\left(\frac{h}{2}\right)^{2} h$
$\Rightarrow V=\frac{1}{3} \pi\left(\frac{h^{2}}{4}\right) h$
$\Rightarrow V=\frac{1}{12} \pi h^{3}$
$\therefore \frac{d V}{d t}=\frac{\pi}{12} h^{2} \frac{d h}{d t}$
$\Rightarrow \frac{d h}{d t}=\frac{4}{\pi} \cdot \frac{1}{h^{2}} \cdot \frac{d V}{d t}$
(Step 4)
$\therefore \frac{d h}{d t}=\frac{4}{\pi} \cdot \frac{1}{3^{2}} \cdot 3$
(Step 5)
$\Rightarrow \frac{d h}{d t}=\frac{4}{3 \pi} \approx 0,42 \mathrm{~m} / \mathrm{min}$.
(Step 6)
Therefore the height of the pile is increasing at a rate of $0,42 \mathrm{~m} / \mathrm{min}$.

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## ADDENDUM D

1. A plane flying with a constant speed $300 \mathrm{~km} / \mathrm{h}$ passes over a ground radar station at an altitude of 1 km and climbs at an angle of $30^{\circ}$. At what rate is the distance from the plane to the radar station increasing a minute later?

## Solution

We let $x$ be the distance from the plane to a point P immediately above the radar station. Then $\frac{d x}{d t}=300 \mathrm{~km} / \mathrm{h}$. Let $s$ be the distance from the radar station to the plane. (Step 2)

(Step 1)

Hence $s^{2}=1+x+x^{2}$
$\therefore 2 s \frac{d s}{d t}=\frac{d x}{d t}+2 x \cdot \frac{d x}{d t}$
$\Rightarrow 2 s \frac{d s}{d t}=(1+2 x) \frac{d x}{d t}$.
(Step 4)

Therefore $\frac{d s}{d t}=\frac{(1+2 x) \frac{d x}{d t}}{2 s}$
So after 1 minute ( $\frac{1}{60}$ hours ), $x=300 \cdot \frac{1}{60} \mathrm{~km} \Rightarrow x=5 \mathrm{~km}$. (Step 7)

From $s^{2}=1+x+x^{2}$ we get that

$$
\begin{aligned}
& s=\sqrt{1+x+x^{2}} \\
& \Rightarrow s=\sqrt{1+5+5^{2}} \\
& \Rightarrow s=\sqrt{31} \\
& \text { So } \frac{d s}{d t}= \frac{(1+2 x) \frac{d x}{d t}}{2 s} \\
& \Rightarrow \frac{d s}{d t}=\frac{(1+2(5)) 300}{2(\sqrt{31})} \\
& \Rightarrow \frac{d s}{d t}=\frac{1650}{\sqrt{31}} \approx 296 \mathrm{~km} / \mathrm{h}
\end{aligned}
$$

(Step 7)

## (Step 5)

Therefore the distance from the plane to the radar station is increasing at a rate of $296 \mathrm{~km} / \mathrm{h}$. (Step 6)
2. At noon, ship A is 150 km west of ship B. Ship A is sailing east at $35 \mathrm{~km} / \mathrm{h}$ and ship B is sailing north at $25 \mathrm{~km} / \mathrm{h}$. How fast is the distance between the ships changing at 16h00?

## Solution

Let $x$ be the distance sailed by ship A after noon and let $y$ be the distanced sailed by ship B after noon. Let $s$ be the distance from the ship A to ship B. Then $\frac{d x}{d t}=35 \mathrm{~km} / \mathrm{h}$ and $\frac{d y}{d t}=25 \mathrm{~km} / \mathrm{h}$. (Step 2)

$s^{2}=y^{2}+(150-x)^{2}$
(Step 3)
$\therefore 2 s \frac{d s}{d t}=2 y \frac{d y}{d t}+2(150-x)\left(-\frac{d x}{d t}\right)$
$\Rightarrow \frac{d s}{d t}=\frac{y}{s} \frac{d y}{d t}-\frac{(150-x)}{s} \frac{d x}{d t}$
(Step 4)

After 4 hours, $x=(35)(4)=140 \mathrm{~km}$ and $y=(25)(4)=100 \mathrm{~km}$.
(Step 7)
So from $s^{2}=y^{2}+(150-x)^{2}$ we obtain that

$$
\begin{align*}
& s=\sqrt{y^{2}+\left(150-x^{2}\right)} \\
& \Rightarrow s=\sqrt{(100)^{2}+(150-140)^{2}}  \tag{Step7}\\
& \Rightarrow s=\sqrt{10100 .}
\end{align*}
$$

From $\frac{d s}{d t}=\frac{y}{s} \frac{d y}{d t}-\frac{(150-x)}{s} \frac{d x}{d t}$ we obtain

$$
\begin{align*}
& \frac{d s}{d t}=\frac{100}{\sqrt{10100}}(25)-\frac{(150-140)}{\sqrt{10100}}(35) \\
& \Rightarrow \frac{d s}{d t}=\frac{2500-350}{\sqrt{10100}}  \tag{Step5}\\
& \Rightarrow \frac{d s}{d t}=\frac{2150}{\sqrt{10100}} \approx 21,39 \mathrm{~km} / \mathrm{h}
\end{align*}
$$

The distance between the ships is changing at $21,39 \mathrm{~km} / \mathrm{h}$. (Step 6)


## ADDENDUM E

1. A plane flying with a constant speed $300 \mathrm{~km} / \mathrm{h}$ passes over a ground radar station at an altitude of 1 km and climbs at an angle of $30^{\circ}$. At what rate is the distance from the plane to the radar station increasing a minute later?

## Solution

We let $x$ be the distance from the plane to a point P immediately above the radar station. Then $\frac{d x}{d t}=300 \mathrm{~km} / \mathrm{h}^{\checkmark}$. (Mark is allocated for recognizing $\frac{d x}{d t}=300 \mathrm{~km} / \mathrm{h}$ )
Let $s$ be the distance from the radar station to the plane.


$$
\begin{aligned}
& s^{2}=1^{2}+x^{2}-2(1)(x) \cos \left(90^{\circ}+30^{\circ}\right) \\
& \Rightarrow s^{2}=1+x^{2}-2 x \cos 120^{\circ} \\
& \Rightarrow s^{2}=1+x^{2}-2 x\left(-\frac{1}{2}\right) \\
& \Rightarrow s^{2}=1+x^{2}+x
\end{aligned}
$$

$\checkmark \checkmark$ (1 Mark is allocated for using the cosine-rule and 1 mark is allocated for simplifying and obtaining $s^{2}=1+x+x^{2}$ )

Hence $s^{2}=1+x+x^{2}$
$\therefore 2 s \frac{d s}{d t}=\frac{d x}{d t}+2 x \cdot \frac{d x}{d t}$
$\Rightarrow 2 s \frac{d s}{d t}=(1+2 x) \frac{d x}{d t}$.
Therefore $\frac{d s}{d t}=\frac{(1+2 x) \frac{d x}{d t}}{2 s} \checkmark$ (Mark is allocated for using $s^{2}=1+x+x^{2}$ and deriving implicitly.)
So after 1 minute $\left(\frac{1}{60}\right.$ hours $), x=300 \cdot \frac{1}{60} \mathrm{~km} \Rightarrow x=5 \mathrm{~km} \cdot \checkmark$
(Mark is allocated for finding $x$ )

From $s^{2}=1+x+x^{2}$ we get that

$$
\begin{aligned}
& s=\sqrt{1+x+x^{2}} \\
& \Rightarrow s=\sqrt{1+5+5^{2}} \\
& \Rightarrow s=\sqrt{31 .}
\end{aligned}
$$

$$
\checkmark \text { (mark is allocated for finding s) }
$$

So $\frac{d s}{d t}=\frac{(1+2 x) \frac{d x}{d t}}{2 s}$
$\Rightarrow \frac{d s}{d t}=\frac{(1+2(5)) 300}{2(\sqrt{31})}$
$\Rightarrow \frac{d s}{d t}=\frac{1650}{\sqrt{31}} \approx 296 \mathrm{~km} / \mathrm{h}$
$\checkmark$ (Mark is allocated for the correct substitution)

Therefore the distance from the plane to the radar station is increasing at a rate of $296 \mathrm{~km} / \mathrm{h} . \checkmark$ (Mark is allocated for the interpretation)
2. At noon, ship A is 150 km west of ship B. Ship A is sailing east at $35 \mathrm{~km} / \mathrm{h}$ and ship $B$ is sailing north at $25 \mathrm{~km} / \mathrm{h}$. How fast is the distance between the ships changing at 16h00?

## Solution

Let $x$ be the distance sailed by ship A, after noon and let $y$ be the distanced sailed by ship B after noon. Let $s$ be the distance from the ship A to ship B. Then $\frac{d x}{d t}=35 \mathrm{~km} / \mathrm{h}$ and $\frac{d y}{d t}=25 \mathrm{~km} / \mathrm{h} . \checkmark$ (Mark is allocated for recognizing $\frac{d x}{d t}=35 \mathrm{~km} / \mathrm{h}$ and $\frac{d y}{d t}=25 \mathrm{~km} / \mathrm{h}$ )
$\checkmark \checkmark$ (1 mark for the sketch and 1 Mark for correct information on sketch)


$$
s^{2}=y^{2}+(150-x)^{2} \quad \checkmark \text { (Mark is allocated for using the theorem of Pythagoras.) }
$$

$\therefore 2 s \frac{d s}{d t}=2 y \frac{d y}{d t}+2(150-x)\left(-\frac{d x}{d t}\right)$
$\Rightarrow \frac{d s}{d t}=\frac{y}{s} \frac{d y}{d t}-\frac{(150-x)}{s} \frac{d x}{d t}$
$\checkmark$ (mark is allocated for correct implicit differentiation)

After 4 hours, $x=(35)(4)=140 \mathrm{~km}$ and $y=(25)(4)=100 \mathrm{~km} . \checkmark \checkmark$
(1 Mark is allocated for finding $x$ and 1 mark for finding y.)

So from $s^{2}=y^{2}+(150-x)^{2}$ we obtain that

$$
\begin{aligned}
& s=\sqrt{y^{2}+\left(150-x^{2}\right)} \\
& \Rightarrow s=\sqrt{(100)^{2}+(150} \\
& \Rightarrow s=\sqrt{10100} .
\end{aligned}
$$

$$
\Rightarrow s=\sqrt{(100)^{2}+(150-140)^{2}} \quad \checkmark(\text { mark is allocated for finding s) }
$$

From $\frac{d s}{d t}=\frac{y}{s} \frac{d y}{d t}-\frac{(150-x)}{s} \frac{d x}{d t}$ we obtain
$\frac{d s}{d t}=\frac{100}{\sqrt{10100}}(25)-\frac{(150-140)}{\sqrt{10100}}(35)$
$\Rightarrow \frac{d s}{d t}=\frac{2500-350}{\sqrt{10100}} \quad . \checkmark$ (Mark is allocated for the correct substitution)
$\Rightarrow \frac{d s}{d t}=\frac{2150}{\sqrt{10100}} \approx 21,39 \mathrm{~km} / \mathrm{h}$


The distance between the ships is changing at $21,39 \mathrm{~km} / \mathrm{h} . \quad \checkmark$ (Mark is allocated for the interpretation)

## ADDENDUM F

EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of $100 \mathrm{~cm}^{3} / \mathrm{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm ?

EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1 \mathrm{ft} / \mathrm{s}$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?


FIGURE 2

EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m . If water is being pumped into the tank at a rate of $2 \mathrm{~m}^{3} / \mathrm{min}$, find the rate at which the water level is rising when the water is 3 m deep.

EXAMPLE 4 Car A is traveling west at $50 \mathrm{mi} / \mathrm{h}$ and car B is traveling north at $60 \mathrm{mi} / \mathrm{h}$. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?


FIGURE 4

EXAMPLE 5 A man walks along a straight path at a speed of $4 \mathrm{ft} / \mathrm{s}$. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?


FIGURE 5

## ADDENDUM G

## DESIGNING INSTRUCTIONAL MATERIAL

## G. 1 INTRODUCTION

All the difficulties that students have in solving related rates problems are summarized in section 2.12 . To be successful, any teaching strategy must address these difficulties that the students have. One way this can be achieved is by using scaffolding in the solution process. This is a well-known concept but we briefly discuss its characteristics.

Wood et al (1976) identified six key elements when introducing the term scaffolding. They are:

- recruitment - enlisting the learner's interest and adherence to the requirements of the task;
- reduction in degrees of freedom - simplifying the task so that feedback is regulated to a level that could be used for correction;
- direction maintenance - (verbal prodder and corrector) keeping the learner in pursuit of a particular objective;
- marking critical features - (confirming and checking) accentuating some and interpreting discrepancies;
- frustration control - responding to the learner's emotional state;
- demonstration - or modelling solution to a task.

Tharpe and Gallimore (1988) use the term 'assisted learning' to develop
the classification of adult interactions and identify six interdependent strategies:

- modelling - offering behaviour for imitation;
- contingency management - rewards and punishment arranged to follow on behaviour;
- feeding back - information resulting from experiences;
- instructing - calling for specific action;
- questioning - calling for linguistic response;
- cognitive structuring - providing explanations and belief structures that organise and justify.

When implementing scaffolding, and taking the social dimension in consideration, Wood (1994) proposes two distinct patterns of interaction observed in mathematics lessons. In what Woods termed "the funnel pattern of interactions", students are provided with leading questions in an attempt to guide them to a predetermined solution procedure. This results in students needing only to generate superficial procedures rather than meaningful mathematical strategies. The other pattern called 'focus pattern of interaction' draws students' attention to the critical aspects of a problem with the teacher posing questions to turn the discussion back to the student, leaving responsibility for resolving the situation with the students.

## G. 2 SCAFFOLDING IN SOLVING RELATED RATES PROBLEMS

In order to illustrate how scaffolding is applied to solve related rates problems, six problems were selected from the prescribed textbook for MAT105 (Concepts and Contexts by James Stewart 4th edition, Metric Version). Appropriate questions were formulated to guide or scaffold the students' solutions in a strategic manner to answer the required rates at different levels or steps of the problem. For easy reference, the problems were named the kite problem, the spotlight problem, the lamp post problem, the runner problem, the radar problem and the triangle problem, respectively.
G.2.1 A kite, controlled by a stationary person, flies horizontally at a height of 50 m above the person's hand. The wind speed is $2 \mathrm{~m} / \mathrm{s}$. At what rate is the angle between the string and the horizontal decreasing when 100 m of string has been let out? (Kite problem)

## Solution steps

Step A
(a) Draw a picture of the kite in two different positions indicating clearly the angles between the string and the horizontal as it is blown by the wind.

Answer

(a) What do you notice about the two angles?

Answer:
The angle in position 2 is smaller than the angle in position 1 (that is $\theta_{1}>\theta_{2}$.)
(b) What do you think will happen to the angle between the string and the horizontal as the kite moves further on the horizontal path?
Answer:

The angle gets smaller (the size of the angle decreases).

Draw a picture of the kite at a height of 50 m and making an angle of $\Theta$ with the horizontal.


Represent the height of the kite by a vertical line to the horizontal $l$.


Note: it is assumed that the height makes an angle of $90^{\circ}$ with the horizontal ( $l$ ), but it is not stated explicitly.

## Step B

(a) Draw a schematic diagram in the form of $\triangle \mathrm{ABK}$ indicating the variables $\theta, x, s$ and $y$ where
$\theta$ is the angle between the horizontal and the string, $x$ is the horizontal distance moved by the kite,
$s$ is the length of the string, and $y$ is the height of the kite above the ground.

## Answer:

$x$

(b) Which of the variable(s) in (a) of step B are constant?

Answer:
$y$, since $y=50 m$
(c) For each of the non-constant variables, give a symbolic (mathematical) expression for the rate of change of the variable with respect to time.

Answer:

$$
\frac{d s}{d t}, \frac{d \theta}{d t} \text { and } \frac{d x}{d t} .
$$

(d) Which rates are known from the information given in the problem and which rates are to be found? ( that is, the required rates)

## Answer:

$$
\begin{aligned}
& \frac{d x}{d t} \text { is given. }\left(\frac{d x}{d t}=2 m / s\right) \\
& \frac{d s}{d t} \text { and } \frac{d \theta}{d t} \text { must be found. }
\end{aligned}
$$

(e) (i) What is the horizontal distance the kite flies after 2 seconds?

## Answer:

$$
v=\frac{s}{t} \Rightarrow s=v t
$$

Therefore after 2 seconds, $s=(2)(2)=4 m$.


What is the horizontal distance the kite flies after 3 seconds?
Solution:
$s=(2)(3)=6 m$.
(e)(ii) What is the length of the string let out after 2 seconds?

Answer:
$s=\sqrt{4^{2}+50^{2}}=\sqrt{2516} \approx 50,16 m \quad($ Remember after 2seconds, $s=(2)(2)=4 m)$.
What is the length of the string let out after 3 seconds?
Answer:
$s=\sqrt{6^{2}+50^{2}}=\sqrt{2536} \approx 50,36 m \quad($ Remember after 3seconds, $s=(2)(3)=6 m)$.
(e)(iii) What is the angle between the string and the horizontal after 2 seconds?

Answer:
$\tan \theta=\frac{50}{4}$
$\therefore \theta=1.491 \mathrm{rad}$.
(e)(iii) What is the angle between the string and the horizontal after 3 seconds?

Answer:

$$
\tan \theta=\frac{50}{6}
$$

$\therefore \theta=1.45 \mathrm{rad}$.
(e)(iv) What do you notice about the angle between the string and the horizontal as the kite moves further on its horizontal path?

Answer:
$\theta$ is decreasing.

## Step C

(a)(i) Write down an equation giving the relationship between the variables $s$ and $\theta$.


Answer

$$
\sin \theta=\frac{50}{s} \Rightarrow s \cdot \sin \theta=50
$$

(a)(ii) Write down an equation giving the relationship between the variables $s$ and $x$.

Answer

$$
s^{2}=x^{2}+y^{2} \Rightarrow s^{2}=x^{2}+50^{2}
$$

(a)(iii) Use implicit differentiation to find a relationship between $\frac{d s}{d t}$ and $\frac{d \theta}{d t}$.

## Answer

$$
\frac{d s}{d t} \sin \theta+s \cdot \cos \theta \frac{d \theta}{d t}=0
$$

(a)(iv) Use implicit differentiation to find a relationship between $\frac{d s}{d t}$ and $\frac{d x}{d t}$.

Answer

$$
2 s \frac{d s}{d t}=2 x \frac{d x}{d t} \Rightarrow s \frac{d s}{d t}=x \frac{d x}{d t} \Rightarrow \frac{d s}{d t}=\frac{x}{s} \cdot \frac{d x}{d t}
$$

(a)(v) Use your answer in (a)(iv) to find $\frac{d s}{d t}$ in terms of $x$ when $s=100 m$.

Answer

From (a)(iv), $\frac{d s}{d t}=\frac{x}{s} \cdot \frac{d x}{d t}$. So when $s=100 m$, and given that $\frac{d x}{d t}=2 \mathrm{~m} / s$, we have that $\frac{d s}{d t}=\frac{x}{100} .2 \Rightarrow \frac{d s}{d t}=\frac{x}{50}$.
(a)(vi) Use your answer in (a)(ii) to find $x$ and use this answer and your answer in (a)(v) to find the value of $\frac{d s}{d t}$

## Answer

To find $x$, we use the equation $s^{2}=x^{2}+50^{2}$.

$$
\begin{aligned}
& \therefore x=\sqrt{s^{2}-50^{2}} \\
& \therefore x=\sqrt{(100)^{2}-50^{2}} \\
& \therefore x=\sqrt{7500}=50 \sqrt{3} \\
& \therefore \frac{d s}{d t}=\frac{50 \sqrt{3}}{50}=\sqrt{3} \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Note: the step to find $x$ we call the auxiliary step.
(a)(vii) Use your answer in (a)(iii) to find $\frac{d \theta}{d t}$ in terms of $\sin \theta$ and $\cos \theta$ when $s=100 \mathrm{~m}$.

## Answer

From (a)(iii), $\frac{d s}{d t} \sin \theta+s \cos \theta \frac{d \theta}{d t}=0$

$$
\begin{gathered}
\Rightarrow s \cos \theta \frac{d \theta}{d t}=-\frac{d s}{d t} \sin \theta \\
\Rightarrow \frac{d \theta}{d t}=-\frac{\frac{d s}{d t} \sin \theta}{s \cos \theta}
\end{gathered}
$$

So when $s=100 \mathrm{~m}$, we have that $\frac{d \theta}{d t}=-\frac{\frac{d s}{d t} \sin \theta}{100 \cos \theta}$.
(a)(viii) Use your answer in (a)(ii) to find $\sin \theta$ and hence $\theta$ and use this answer to find the value of $\cos \theta$.

## Answer

From (a)(i), $\sin \theta=\frac{50}{100} \Rightarrow \sin \theta=\frac{1}{2}$ since $s=100 \mathrm{~m}$. Since $\sin \theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$
Since $\theta=\frac{\pi}{6}, \cos \frac{\pi}{6}=\frac{\sqrt{3}}{2}$
(a)(ix) Now use your answers from (a)(viii) and (a)(vi) to find the value of $\frac{d \theta}{d t}$.

Answer
From $\frac{d \theta}{d t}=-\frac{\frac{d s}{d t} \sin \theta}{100 \cos \theta}$ we have $\frac{d \theta}{d t}=-\frac{\sqrt{3}\left(\frac{1}{2}\right)}{100 \frac{\sqrt{3}}{2}}$
$\therefore \frac{d \theta}{d t}=-0.01 \mathrm{rad} / \mathrm{s}$
Note: again the step to find $\sin \theta$ and $\cos \theta$ we call the auxiliary step.
(a)(x) Express your answer in (a)(ix) without the use of symbols (that is, not in symbolic notation)

## Answer

The angle between the string and the horizontal is decreasing at a rate of -0.001 radians $/ \mathrm{s}$ at the moment the length of the string let out is 100 m .
G.2.2 A spotlight on the ground shines on the wall of a building 12 m away. If a man, 2 m tall, walks from the spotlight towards the building at a speed of $1.6 \mathrm{~m} / \mathrm{s}$, how fast is the length of his shadow against the wall decreasing when he is 4 m from the building? (Spotlight problem)

Solution steps:
Step A

(a) Draw a sketch showing the position of the man after he walked 1 m , then 2 m and then 3 m from the spotlight.

(b) Also let $l_{1}, l_{2}$ and $l_{3}$ be the corresponding lengths of the man's shadow against the wall of the building. What do you notice about the length of the man's shadow as he walks further from the spotlight towards the building?

(c) Draw a sketch showing the position of the man after he walked $x$ meters from the spotlight.


## Step B

(a) Draw a schematic diagram of your sketch in step A(c).

Answer


The variables $x$ and $l$ represent the distance the man walks from the spotlight and the length of the man's shadow against the wall of the building, respectively.
(b) Can you identify the two right angled triangles in this diagram?

Answer $\quad \triangle M N S$ and $\triangle P Q S$
(c)(i) What is the meaning of the symbol $\frac{d x}{d t}$ ?

Answer
It represents the rate at which the distance the man walks from the spotlight is changing with time
(c)(ii) From the information given, what can you say about $\frac{d x}{d t}$ ?

Answer

$$
\frac{d x}{d t} \text { is constant }
$$

(c)(iii) What is the numerical value of $\frac{d x}{d t}$ ?

Answer
$1.6 \mathrm{~m} / \mathrm{s}$.
(c)(iv) What is the meaning of the symbol $\frac{d l}{d t}$ ?

Answer
The rate at which the length of the man's shadow against the wall is changing with time.
(c)(v) Is $\frac{d l}{d t}$ positive or negative? Give a reason for your answer.

Answer
UNIVERSITY of the
Negative, the length is decreasing with time
(c)(vi) Is the numerical value of $\frac{d l}{d t}$ known or unknown?

Answer We are required to find this

Step C
(a) Find an equation which gives a relationship between all variables and constants in the schematic diagram?

Answer
$\frac{2}{l}=\frac{x}{12}$ or $l x=24 \quad(0<x<12)$
(b) Use the equation in (a) to answer the question in the problem, that is find the value of $\frac{d l}{d t}$.

From $l x=24$, we have $0=l \frac{d x}{d t}+x \frac{d l}{d t}$.

$$
\therefore \frac{d l}{d t}=-\frac{l}{x} \frac{d x}{d t}
$$

When $x=8$, that is, when he is 4 m from the building, $l=\frac{24}{8}=3 \mathrm{~m}$.

$$
\therefore \frac{d l}{d t}=-\frac{3}{8}(1.6)=-0.6 \mathrm{~m} / \mathrm{s}
$$

(c)(i) Can you interpret your answer in (b) verbally (in words), that is, without the use of symbols.

## Answer

The length of the man's shadow is decreasing at a rate of $0.6 \mathrm{~m} / \mathrm{s}$ when he is 4 m from the building.
(c)(ii) Calculate $\frac{d l}{d t}$ when $x=9, x=10, x=11$ and $x=12$.

Answer When $x=9, \frac{d l}{d t}=-0.470 \mathrm{~m} / \mathrm{s}$

$$
\text { When } x=10, \frac{d l}{d t}=-0.384 \mathrm{~m} / \mathrm{s}
$$

When $x=11, \frac{d l}{d t}=-0.317 \mathrm{~m} / \mathrm{s}$
When $x=12, \frac{d l}{d t}=-0.266 \mathrm{~m} / \mathrm{s}$
(c)(iii) Can you interpret the meaning of the last answer when $x=12$ ?

Answer The instant just before the man is at the wall, the length of the man's shadow is decreasing at a rate of $0.266 \mathrm{~m} / \mathrm{s}$.
G.2.3 A girl, who is 5 feet tall, is approaching a post that holds a lamp 15 feet above the ground. If she is walking at a speed of $4 \mathrm{ft} / \mathrm{s}$, how fast is the end of her shadow moving when she is 17 ft away from the base of the lamp post? (Lamppost problem)

## Solution steps

Step A
(a) Draw a sketch to explain your understanding of the information given in the first sentence of the problem, indicating clearly the position of the girl's shadow.

## Answer



Girls' shadow

## Step B

A schematic diagram of the sketch in Step A looks as follows:

where the line CD represents the girl whose height is $x$ feet, AB represents the lamp post of height $y$ feet, $l$ is the length of the girl's shadow and $m$ is the distance from the girl to the base of the lamp post.
(a)(i) Which of the variables $x, y, l$ and $m$ are constants?

Answer

$$
x \text { and } y
$$

(a)(ii) For those variables that are not constants, write an expression for the rate of change with respect to time for that variable.

Answer

$$
\frac{d l}{d t} \text { and } \frac{d m}{d t}
$$

(a)(iii) Which rates of change are given in the statement of the problem?

Answer

$$
\frac{d m}{d t}=-4 f t / s
$$

(a)(iv) Explain your answer in (a)(iii)

Answer

The rate is decreasing
(a)(v) Can you identify two similar triangles, in the schematic diagram in step B?

Answer $\triangle E C D$ and $\triangle E B A$
(a)(vi) Can you now write down an equation which gives a relationship between the variables.

Answer

$$
\frac{x}{l}=\frac{y}{l+m} \Rightarrow \frac{5}{l}=\frac{15}{l+m}
$$

(a)(vii) Use this equation to write $l$ in terms of $m$.

Answer

$$
\begin{aligned}
& 5 l+5 m=15 l \\
\Rightarrow & 5 m=10 l \\
\Rightarrow & l=\frac{5}{10} m \text { or } l=\frac{1}{2} m
\end{aligned}
$$

(a)(viii) Calculate the length of the girl's shadow when she is $28 \mathrm{ft}, 24 \mathrm{ft}, 20 \mathrm{ft}$ and 18 ft from the lamppost.

Answer

For $m=28: \quad l=\frac{1}{2}(28)=14 f t$.

For $m=24: \quad l=\frac{1}{2}(24)=12 f t$.

For $m=20: \quad l=\frac{1}{2}(20)=10 f t$.

For $m=18: \quad l=\frac{1}{2}(18)=9 f t$.
(a)(xi) Calculate the length of the tip of the girl's shadow when she is $28 \mathrm{ft}, 24 \mathrm{ft}, 20 \mathrm{ft}$ and 18 ft from the lamppost.

## Answer

For $m=28: \quad m+l=28+14=42 f t$.
For $m=24: \quad m+l=24+12=36 f t$.
For $m=20: \quad m+l=20+10=30 f t$.
For $m=18: \quad m+l=18+9=27 f t$.
(a)(x) What do you notice about the length of the girl's shadow as she approaches the lamp post?

Answer
It increases
(a)(xi) What do you notice about the distance from the base of the lamp post to the tip of the girl's shadow?

Answer
It remains the same

## Step C

(a)(i) At what rate is the length of the girl's shadow decreasing when she is 17 ft from the base of the lamp post?

Answer

$$
\begin{aligned}
l & =\frac{1}{2} m \\
\Rightarrow \frac{d l}{d t} & =\frac{1}{2} \frac{d m}{d t} \\
\Rightarrow \frac{d l}{d t} & =\frac{1}{2}(-4) \\
\Rightarrow \frac{d l}{d t} & =-2 f t / s
\end{aligned}
$$

Therefore the length of her shadow is decreasing at a rate of $2 \mathrm{ft} / \mathrm{s}$.
(a)(ii) How fast is the end of her shadow moving when she is 17 ft away from the base of the lamp post?

## Answer

$$
\begin{aligned}
& l+m \\
\Rightarrow & \frac{d}{d t}(l+m) \\
\Rightarrow & \frac{d}{d t} l+\frac{d}{d t} m \\
= & (-2)+(-4) \\
= & -6 f t / s
\end{aligned}
$$

Therefore the end of her shadow is approaching the lamp post at a rate of $6 \mathrm{ft} / \mathrm{s}$, which is faster than the girl is walking.

G.2.4 A runner sprints around a circular track of radius 100 m at a constant speed of $7 \mathrm{~m} / \mathrm{s}$. The runner's friend is standing at a distance 200 m from the centre of the track. How fast is the distance between the friends changing when the distance between them is 200 m ? (Runner problem)

## Solution steps

Step A
(a) Use a sketch to explain your understanding of the first sentence of the problems.

Answer


Suppose P and $\mathrm{P}^{\prime}$ are two points on the track and the distance between them are denoted by
$x$. Let $\theta$ be the angle subtended at the centre by the arc PP'. Note that it is important to
know that both variables $x$ and $\theta$ are functions of $t$.

(a)(i) Write down a formula giving $x$ in terms of $\theta$.

Answer

$$
x=100 \theta
$$

(a)(ii) Find $\frac{d x}{d \theta}$

Answer

$$
\frac{d x}{d \theta}=100
$$

(a)(iii) What is $\frac{d x}{d t}$ in terms of $\theta$ ?

Answer $\quad \frac{d x}{d t}=\frac{d x}{d \theta} \cdot \frac{d \theta}{d t}=100 \frac{d \theta}{d t}$
(a)(iv) Let V be the speed of the runner sprinting around the track. Write V in terms of $x$.

Answer

$$
V=\frac{d x}{d t}
$$

(a)(v) What is the numerical value of V ?

Answer

$$
V=7 \mathrm{~m} / \mathrm{s} .
$$

(a)(vi) What is $\frac{d \theta}{d t}$ ?

Answer


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$$
\Rightarrow \frac{d \theta}{d t}=0.07 \mathrm{rad} / \mathrm{s}
$$

## Step B

(a) Draw a picture of what you read in the second sentence.


For convenience, choose F to be on the horizontal line through O .


## Step C

We now work out the distance between the runner and his friend at different positions as the runner runs around the track.

(a)(i) What is the distance between the runner and his friend when the runner is at position $\mathrm{R}_{1}$ on the circular track?

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## Answer

300m
(a)(ii) What is the distance between the runner and his friend when the runner is at position $\mathrm{R}_{2}$ on the circular track?


Answer let $l_{2}$ represent the distance between the runner and the friend. Then

$$
\begin{aligned}
& \left(l_{2}\right)^{2}=100^{2}+200^{2} \\
& \Rightarrow\left(l_{2}\right)^{2}=50000 \\
& \therefore l_{2}=\sqrt{50000} \\
& \therefore l_{2}=100 \sqrt{5}
\end{aligned}
$$

(a)(iii) What is the distance between the runner and his friend when the runner is at position $\mathrm{R}_{3}$ on the circular track?

$$
\text { Answer } \quad 100 \mathrm{~m}
$$

(a)(iv) What is the distance between the runner and his friend when the runner is at position $\mathrm{R}_{4}$ on the circular track?


Answer let $l_{4}$ represent the distance between the runner and the friend. Then

$$
\begin{aligned}
& l_{4}^{2}=100^{2}+200^{2}-2(100)(200) \cos 135^{\circ} \\
& \Rightarrow\left(l_{4}\right)^{2}=50000+40000\left(\frac{1}{\sqrt{2}}\right) \\
& \Rightarrow l_{4}=279.79 \mathrm{~m}
\end{aligned}
$$

(a)(v) What is the position of the runner on the track when the distance between the runner and his friend is 210 m ?

## Answer

$$
\begin{aligned}
& 210^{2}=100^{2}+200^{2}-2(100)(200) \cos \theta \\
& \Rightarrow 210^{2}=50000-40000 \cos \theta \\
& \Rightarrow \cos \theta=\frac{5900}{40000} \\
& \Rightarrow \cos =0.1475 \\
& \therefore \theta=81.52^{\circ}
\end{aligned}
$$

(a)(vi) Give a symbolic description (in terms of $l$ and $t$ ) of the phrase "how fast is the distance between the friends changing"

Answer
$\frac{d l}{d t}$
(a)(vii) Write down an equation in terms of $l$ and $t$ which will enable you to find $\frac{d l}{d t}$ when $l=200 \mathrm{~m}$ ?

Answer
$l^{2}=100^{2}+200^{2}-2(100)(200) \cos \theta$
$\Rightarrow 2 l \frac{d l}{d t}=2(100)(200) \sin \theta \frac{d \theta}{d t}$
$\Rightarrow \frac{d l}{d t}=\frac{20000 \sin \theta \frac{d \theta}{d t}}{l}$
$\Rightarrow \frac{d l}{d t}=\frac{20000 \sin \theta(0.07)}{l}$
$\Rightarrow \frac{d l}{d t}=\frac{1400 \sin \theta}{l}$
(a)(viii) If $l=200 \mathrm{~m}$ can you find a value for $\sin \theta$ ?

Answer
$200^{2}=100^{2}+200^{2}-2(100)(200) \cos \theta$
$\Rightarrow 40000 \cos \theta=10000$
$\Rightarrow \cos \theta=\frac{10000}{40000}=\frac{1}{4}$
Now $\sin \theta=\sqrt{1-\cos ^{2} \theta}$
Therefore $\sin \theta=\sqrt{1-\left(\frac{1}{2}\right)^{2}}$

$$
\Rightarrow \sin \theta=0.968
$$

(a)(ix) Find $\frac{d l}{d t}$ if $l=200$ and the value of $\sin \theta$ which you calculated above.

Answer

$$
\text { From } \frac{d l}{d t}=\frac{1400 \sin \theta}{l} \text {, we have } \frac{d l}{d t}=\frac{1400(0.968)}{200}=6.776 \mathrm{~m} / \mathrm{s}
$$

After working through the first four problems in which scaffolding was used, it is expected that the students will now have a greater understanding how related rates problems should be solved. It is for this reason that the final two problems have limited scaffolding introduced in order to arrive at a solution.

G.2.5 A plane flying with a constant speed of $300 \mathrm{~km} / \mathrm{h}$, passes over a ground radar station at an altitude of 1 km and climbs at an angle of $30^{\circ}$. At what rate is the distance from the plane to the radar station increasing a minute later? (Radar problem)

## Solution Steps

Step A
At the instant the plane is directly above the radar station, picture (imagine) a vertical line from the radar station to the plane that is perpendicular to the horizontal line representing the path of the plane. At the point of the intersection of these two lines, the plane climbs at an angle of $30^{\circ}$. Let $y$ be the distance between the plane $(\mathrm{P})$ and the radar station (R). A schematic diagram of the situation is as follows.


Given: $\frac{d x}{d t}=300 \mathrm{~km} / \mathrm{h}$
Required: $\frac{d y}{d t}$ when $t=\frac{1}{60}$ hours (1 minute)
Equation relating the variables $x$ and $y$ and the constants:
$y^{2}=1^{2}+x^{2}-2(1)(x) \cos 120^{\circ}$
$y^{2}=1+x^{2}-2 x\left(-\frac{1}{2}\right)$
$y^{2}=1+x+x^{2}$
$\therefore 2 y \frac{d y}{d t}=\frac{d x}{d t}+2 x \frac{d x}{d t}$
$\therefore 2 y \frac{d y}{d t}=\frac{d x}{d t}(1+2 x)$
$\therefore \frac{d y}{d t}=\frac{d x}{d t}\left(\frac{1+2 x}{2 y}\right)$
So when $x=\left(300\left(\frac{1}{60}\right)\right)=5 \mathrm{~km}$ and $y=\sqrt{1+5+5^{2}}=\sqrt{31}$,
$\frac{d y}{d t}=300\left(\frac{1+2(5)}{2(\sqrt{31})}\right)$
$\therefore \frac{d y}{d t}=296.35 \mathrm{~km} / \mathrm{h}$


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G.2.6 Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of $0,06 \mathrm{rad} / \mathrm{s}$. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\frac{\pi}{3}$.(Triangle problem)

## Solution Steps

We let $\theta$ be the angle between the two sides. We name the triangle $\triangle \mathrm{EBC}$ with $E B=4$ and $B C=5$. We also let $A$ the area of the triangle at time $t$.

$A=\frac{1}{2} .4 .5 \cdot \sin \theta$
$\Rightarrow A=10 \sin \theta$
$\frac{d A}{d t}=10 \cos \theta \frac{d \theta}{d t}$
But since $\frac{d \theta}{d t}=0,06 \mathrm{rad} / \mathrm{s}$ and $\theta=\frac{\pi}{3}$, we have

$$
\begin{aligned}
\frac{d A}{d t} & =10 \cdot \cos \left(\frac{\pi}{3}\right) \cdot(0,06) \\
\Rightarrow & \frac{d A}{d t}
\end{aligned}=0,3 m^{2} / \mathrm{s} .
$$

Thus the area is increasing at a rate of $0,3 \mathrm{~m}^{2} / \mathrm{s}$.

## G. 3 DYNAMIC VERSION OF THE PROBLEMS

According to Zimmerman and Cunningham (1991), visualization describes the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated. Computer graphics has expanded the scope of and power of visualization in basically every field, including mathematics. According to a report to the National Science Foundation: Visualization in Scientific Computing (VISC), McCormick et. al. (1987) asserted that visualization transforms the symbolic into the geometric, enabling researchers to observe their simulations and computations. Visualization offers a method of seeing the unseen. From the perspective of mathematical visualization, the constraint that images must be manipulated mentally, without the aid of pencil and paper, seems artificial. In fact, in mathematical visualization, what we are interested in is precisely the student's ability to draw an appropriate diagram (with pencil and paper, or in some cases, with a computer) to represent a mathematical concept or problem and to use the diagram to achieve understanding, and as an aid in problem solving. In mathematics, we are more concerned about visualizing a concept or problem. To visualize a diagram means simply to form a mental image of the diagram, but to visualize a problem means to understand the problem in terms of a diagram or visual image. Mathematical visualization is the process of forming images whether it is mentally, or with pencil and paper, or with the aid of technology and using such images effectively for mathematical discovery and understanding. In related rates problems, we need to visualize the problem to solve the situation. The role of computers in mathematics has shaped our concept of the nature of the mathematics. Computers have a direct role in the visualization of mathematics. The images it generates do not have to be static but can be dynamic or interactive or user controlled. It can be a graphic simulation of a process, such as a plane flying over a radar station. Other technologies can also be considered such as film or software designed to
illustrate its process. To design specific software for mathematics can be very expensive and sometimes not user friendly. In this section we use GeoGebra to design a dynamic version of the problems G.2.1 - G.2.6 which is interactive. Students can experiment by changing the variables in the problem, resulting in a corresponding change in related variables. This is immediately visible to the students; thus giving a visual image which can enhance understanding.

The reader can open the designed examples with the software GeoGebra. This can be downloaded for free by visiting www.geogebra.org. The reader can also use the following link for easier download: http://www.geogebra.org/cms/en/download/. If GeoGebra is loaded on the reader's personal computer (PC), the accompanying CD contains the six problems I designed as well as an explanation of how to use it.


## G. 4 CONCLUSION

In this addendum, we designed instructional material for the teaching of Related Rates. This approach met with very positive responses from colleagues and students. It is especially the dynamic presentation which students enjoyed. More work will be done on developing this aspect of our proposed teaching strategy.

