ANALYSIS OF ERRORS IN DERIVATIVES OF TRIGONOMETRIC FUNCTIONS: A CASE STUDY IN AN EXTENDED CURRICULUM PROGRAMME

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Socio-cultural theory

Social interactions

Zone of proximal development

More knowledgeable others

ABSTRACT

Title: Analysis of errors in derivatives of trigonometric functions: A case study in an extended curriculum programme

The purpose of this study was to explore errors that are displayed by students when learning derivatives of trigonometric functions in an extended curriculum programme. The first aim was to identify errors that are displayed by students in their solutions through the lens of the APOS theory. The second aim was to address students' errors by using the two principles of Vygotsky's socio-cultural theory of learning, namely the zone of proximal development and more knowledgeable others.

The research presented in this thesis is a case study located in the interpretive paradigm of qualitative research. The participants in this study comprised a group of students who registered for mathematics in the ECP at Cape Peninsula University of Technology, Cape Town, South Africa. The study was piloted in 2008 with a group of twenty students who registered for mathematics in the ECP for Chemical Engineering. In 2009 thirty students from the ECP registered for mathematics in Chemical Engineering were selected to participate in the main study.

This study was conducted over a period of four and a half years. Data collection was done through students' written tasks; classroom audio and video recordings and in-depth interviews. Data were analysed through categorising errors from students' written work, and finding common themes and patterns in audio and video recordings and from the in-depth interviews.

The findings of this study revealed that students committed interpretation, arbitrary, procedural, linear extrapolation and conceptual errors. Interpretation errors arise when students fail to interpret the nature of the problem correctly owing to over-generalisation of certain mathematical rules. Arbitrary errors arise when students behave arbitrarily and fail to take account of the constraints laid down in what is given. Procedural errors occur when students fail to carry out manipulations or algorithms although they understand concepts in problem. Linear extrapolation errors happen through an over-generalisation of the property $f(a+b)= f(a)+ f(b)$, which applies only when f is a linear function.

Conceptual errors occur owing to failure to grasp the concepts involved in the problem or failure to appreciate the relationships involved in the problem. The findings were consistent with literature indicated that errors are based on students' prior knowledge, as they over-generalise certain mathematical procedures, algorithms and rules of differentiation in their solutions.

The use of learning activities in the form of written tasks; as well as classroom audio and video recordings assisted the lecturer to identify and address errors that were displayed by students when they learned derivatives of trigonometric functions. The students claimed in their interviews that they benefited from class discussions as they obtained immediate feedback from their fellow students and the lecturer. They also claimed that their performances improved as they continued to practice with the assistance of more knowledgeable students, as well as the lecturer.

This study supports the view from the literature that identification of errors has immense potential to address students' poor understanding of derivatives of trigonometric functions. This thesis recommends further research on errors in various sections of Differential Calculus, which is studied in an extended curriculum programme at Universities of Technology in South Africa.

ACRONYMS

ACE	Activities; Class discussions and Exercises
APOS	Actions; Processes; Objects and Schema
CPUT	Cape Peninsula University of Technology
ECP	Extended Curriculum Programme
FET	Further Education and Training
LCD	Lowest common denominator
MKO	More knowledgeable others
NBTs	National Benchmark Tests
ZPD	Zone of Proximal Development

Declaration

I declare that: *Analysis of errors in derivatives of trigonometric functions: A case study in an extended curriculum programme* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Sibawu Witness Siyepu May 2012

Signed ……………………………………….

DEDICATION

This thesis is dedicated to my late parents Ntombentsha and Tutu 'Donkiri' Siyepu who believed in dedication as a tool of success.

It is also devoted to my late brothers Mncedi and Nkosimbini and my sisters Vuyiswa and Vuyelwa (twins), Nogolide, Vuliwe and Nobuntu who always pray and comfort me during difficult periods.

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CHAPTER ONE

INTRODUCTION

1.1 RATIONALE FOR THE STUDY

Globally there is a high failure rate in mathematics among first year university students (Engelbrecht, Harding and Phiri; 2010; Padayachee, Boshoff, Olivier and Harding; 2011). This is attributed to various reasons such as students who enter universities already at risk. At risk students are students who are not experiencing success in their schooling system. They are usually low academic achievers who show low confidence. Engelbrecht, et al. (2010) asserts that "several lecturers who taught first-year mathematics in 2009 reported on under preparedness of students" (p.4). This is supported by Padayachee et al. (2011) as they explain that "from the experience of teaching first-year mathematics students, they observed that many first year students are under prepared for mathematics (p.1). The identification of at risk students and the development of programmes to prevent their failure are necessary components of educational reform in universities (Margarita, 1987).

To assist the students who are at risk to succeed, the Cape Peninsula University of Technology (CPUT) admits these students in an extended curriculum programme (ECP). ECP is a curriculum that is designed for students who are borderline cases. These students do not meet the minimum academic requirements for admission to the main engineering stream, but show potential based on psychometric testing to succeed in their studies. The minimum requirement for admission in the main engineering stream is that students should at least obtain 50 per cent in Mathematics, Physical Science and English in the matriculation examination, as well as entrance to a university of technology.

The CPUT also established a mathematics support programme which is situated at Fundani centre for higher education and development where I am employed as a mathematics lecturer. One of my job descriptions is to assist students who are at risk to improve their performance in the learning of mathematics.

In an extended curriculum programme, students study the same content of mathematics as other students in the main engineering stream, but instead of completing it within a semester; they have to do it over a period of one year. The way this is done is to add active learning components such as group work, projects, peer work and other related support work such as how to read the subject texts and how to solve problems and represent knowledge in the field.

The current ECP mathematics syllabus is dominated by calculus with the large component of differentiation that consists of derivatives of trigonometric functions. This is a new section that students come across for the first time in their first year level of study in universities. Trigonometry is a section of mathematics that is normally introduced at a later stage of schooling. As a result students tend to reach university with insufficient knowledge of the basics of trigonometry.

Several researchers argue that in spite of many attempts and a variety of approaches that have been adopted to improve students' understanding of derivatives, the problem of poor performance persists with first-year university students (Barnes, 1995; Moru, 2006; Naidoo & Naidoo, 2007; and Tall, 1985, 1992). Similarly there are many studies documented on students' misconceptions and errors in the learning of various topics in mathematics such as differentiation and integration (Kiat, 2005; Luneta & Makonye, 2010; Naidoo & Naidoo, 2007 and Orton, 1983a; 1983b). However, there are no studies reported on exploration of misconceptions and errors in derivatives of trigonometric functions.

There is concern among mathematics educators about the poor performance of first-year students in mathematics, generally, and in differential calculus, in particular. First year mathematics courses at universities of technology consist of basic mathematics and calculus, and much emphasis in the calculus syllabus is placed on differentiation (Naidoo & Naidoo, 2007).

Nationally, poor first year mathematics results throughout South African universities, coupled with evidence from the South African National Benchmark Tests (NBTs), indicate that many students who graduate from the new school system are under-prepared for success in higher education (Wolmarans, Smit, Collier-Reed, & Leather, 2010). They assert that “the mathematics (NBTs) tests set out to measure how much of the new secondary school curriculum has been mastered show that only 7% of the engineering students achieved at the ‘proficient’ level (deemed not to require additional assistance to perform at the degree level” (p. 275). They explain that 73% of engineering students performed in the ‘intermediate’ category, while those at the bottom end of this category are expected to require additional support in the form of augmented programmes such as extra tutorial help. A total of 20% of engineering students were found to have only ‘basic’ mathematical skills and should likely be placed in an extended curriculum programme to have any chance of success (Wolmarans, et al., 2010).

In order to bring a solution to an educational problem one needs to know the roots of the problem, that is, type of problem and its cause. Understanding of the causes may assist researchers to conduct investigations on how to address those identified problems. This study explores students’ errors in derivatives of trigonometric functions in order to develop techniques of addressing the errors that are displayed by the students who participated in this study.

1.2 PURPOSE OF THE STUDY

The aim of this study was to explore errors displayed by the students who were registered for mathematics in an extended curriculum programme in the learning of derivatives of trigonometric functions, and to make some recommendations that hopefully might lead to improvement of learning of this subject among the students.

1.3 RESEARCH QUESTIONS

The study sought to answer the following questions:

1. What kind of errors that are displayed by students in their learning of derivatives of trigonometric functions?
2. In what ways can social interactions address students’ errors and advance them from a basic level to an advanced level of understanding in their learning of derivatives of trigonometric functions?

1.4 SIGNIFICANCE OF THE STUDY

The importance of this study is that it raises awareness in students and lecturers about misconceptions and errors in derivatives of trigonometric functions. This might assist mathematics lecturers to design learning activities that may be employed to address students' misconceptions and errors in the learning process. Once the mathematics lecturer uses class interactions by addressing students' misconceptions and errors, students can become independent and therefore can develop more self-actualisation. Similarly, mathematics lecturers could be more effective in their facilitation of learning among the students.

Several researchers such as Brodie (2010); Hatano (1996); Nesher (1987) and Smith, DiSessa, & Rosehelle (1993) argue that there are benefits in understanding students' misconceptions and errors such as the focus of the feedback in classroom discussion. This study recommends the use of Activities, Class discussion and Exercises (ACE) to design a pedagogical approach that should be used in the teaching and learning of derivatives of trigonometric functions.

For further recommendations this study contributes to the theoretical knowledge, particularly the use of APOS theory to identify students' errors and Vygotsky's Zone of Proximal development to address students' errors in their learning of derivatives of trigonometric functions.

1.5 STRUCTURE OF THE THESIS

The thesis is divided into six chapters.

Chapter one explains rationale and purpose of the study, discusses research questions and gives an outline of the thesis.

In chapter two I provide a review of literature related to errors in learning mathematics. The review discusses the nature of errors in learning mathematics, particularly errors in derivatives of trigonometry functions.

This chapter also deals with students' difficulties when learning the first principles of differentiation, the concept of a derivative and the rules of differentiation. It ends with data analysis techniques used by researchers who conducted research on students' errors when learning differentiation.

Chapter three presents research frameworks which were used to analyse the data collected. It discusses the nature of research frameworks, which covers types of research frameworks, namely theoretical frameworks and conceptual frameworks. It also discusses socio-cultural theory, and APOS theory.

In chapter four I present research methods that were used in this study. These include the research paradigm adopted for this study, the target group, the context of the study, methods of data collection and how the issues of reliability and validity were addressed. It also discusses data analysis, triangulation and research ethics.

Chapter five presents the results of the study and chapter six summarises the results as related to the research questions and research frameworks. Chapter six also discusses the significance and limitations of the study. In conclusion, it discusses avenues for further research and recommendations.

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

This chapter is a literature review of students' errors in mathematics in general and in differentiation, in particular. Students' difficulties when learning mathematics are manifested in errors that they display in their solutions. These errors have been discussed and documented by various researchers such as Brodie (2005; 2006 & 2010); Davis (1984); Drews (2005); Foster (2007); Hatano, (1996); Luneta & Makonye (2010); Nesher (1987); Olivier (1989); Orton (1983a; 1983b); Ryan & Williams (2000); and Smith, et al., (1993).

Luneta and Makonye (2010); Olivier (1989); Ryan and McCrae (2005); Uygur and Ozdas (2005) have written articles on misconceptions and errors on various topics of mathematics such as manipulation of arithmetic numerals; manipulation of algebraic expressions; functions; differentiation; integration; limits; infinity; logarithms and the chain rule. Currently, there is little research, which has been conducted on errors that are displayed by students in derivatives of trigonometric functions.

2.2 THE NATURE OF ERRORS

The nature of errors is based on mistakes displayed by students when they attempt to solve mathematical problems. Students demonstrate different mistakes, which arise for many different reasons. Nesher (1987); Olivier (1989); and Smith, et al., (1993) split these mistakes into three categories namely slips, errors and misconceptions.

Olivier (1989) defines slips, errors and misconceptions as follows:

- Slips are wrong answers owing to processing; they are not systematic, but are carelessly made by both experts and novices; they are easily detected and are suddenly corrected;
- Errors are wrong answers owing to planning; they are systematic in that they are applied regularly in the same circumstances. Errors are the symptoms of the underlying conceptual structures that are the cause of errors; and

- Underlying beliefs and principles in the cognitive structure that are the causes of systematic conceptual errors are known as misconceptions (p.3).

Errors arise from students' prior learning, either in the mathematics classroom or from their interaction with the physical and social world (Smith, et al., 1993). Davis (1984) and Olivier (1989) claim that errors arise from over-generalisation of a concept from one domain to another.

2.2.1 Generalisation over numbers

As indicated above, students develop errors through over-generalisation of a concept from one domain to another. For instance, Olivier (1989) shows an example of students who learn to solve quadratic equations by factoring. He further explains such students tend to make the following error in problems such as $x^2 - 5x + 6 = 12$. They solved this type of a problem without a clear understanding of the zero property. As a result their solution tends to be as follows:

$x^2 - 5x + 6 = 12$

$\Rightarrow (x-3)(x-2) = 12$

$\Rightarrow x - 3 = 12$ or $x - 2 = 12$

$\Rightarrow x = 15$ or $x = 14$

The error in this solution is that these students fail to apply the zero property by transposing 12 first from the right hand side to the left hand side of the equation as follows:

$x^2 - 5x + 6 = 12$

$\Rightarrow x^2 - 5x + 6 - 12 = 12 - 12$

$\Rightarrow x^2 - 5x - 6 = 0$

$\Rightarrow (x-6)(x+1) = 0$

$\Rightarrow x = 6$ or $x = -1$

This error is difficult to eradicate or is at least difficult to eradicate permanently (Olivier, 1989).

He further explains that even with able students who receive excellent instruction emphasising the special role of zero in the zero product principle, this error will continue to crop up in students' work.

Olivier (1989) suggests that over-generalisation of numbers and number properties may be the most important underlying cause of students' errors. Matz (1980) suggests that this error persists due to the two levels of procedures, namely, surface level and deep level procedures. She explains that a surface level procedure arises from low cognitive level engagement where a student acquires knowledge 'by heart' but does not engage with its meaning. She further explains that a deep surface level yields understanding and meaning, at least as the student is able to make sense of the given problem and is also able to interpret it, and the problem will be contextualised by the learning activities in which the student participates.

2.2.2 Generalisation over operations

Olivier (1989) states that students tend to confuse understanding of the commutative property in the four basic operations, namely, addition; subtraction; multiplication and division. Given that in addition $6+4=4+6$, students tend to assume that even in subtraction commutative property holds but it is not true that $6-4=4-6$ because $6-4=2$ whilst $4-6=-2$. He further explains that this arises as teachers introduce integers, since students come from the primary schools with the understanding that they only subtract the smaller number from the bigger number and their answers are always positive. Another contributing factor that may influence students is that commutative property holds under addition and multiplication. As a result, they over-generalise over operations.

The following examples illustrate one of the largest and most frequently occurring class of errors in high school, which Matz (1980) calls linear extrapolation errors.

$\sqrt{(a+b)} = \sqrt{a} + \sqrt{b}$

$(a+b)^2 = a^2 + b^2$

$a(bc) = (ab)(ac)$

$\log(a+b) = \log a + \log b$

$\sin(a+b) = \sin a + \sin b$

These errors arise from the over-generalisation of the property $f(a+b) = f(a) + (b)$, which applies only when f is a linear function, to the form $f(a * \mathrm{b}) = f(a) * f(b)$, where f is any function and * any operation. For example, in a linear function the statement $x(a+b) = ax + bx$ is correct with the focus of applying distributive property law, whereas the same statement does not apply to trigonometric functions, for example, $\cos(a+b) \neq \cos a + \cos b$.

2.2.3 Students' interpretations

In students' experiences of working with multiplication of whole numbers they realise that multiplication always makes bigger (except for 0 and 1, which may be discarded as special cases). Hence, students over-generalise even in cases of decimal numbers and integers, where it is not generally true that multiplication produces a bigger number.

Involving students in the lesson allows students' errors or misconceptions to become visible (Molefe & Brodie, 2010; Nesher, 1987; Swan, 2001). Molefe and Brodie (2010) assert that "getting students to explain themselves create opportunities for discussions of the errors and misconceptions that they produce" (p. 6). This suggests that as students share their mathematical interpretations, they may gain an understanding owing to immediate feedback that they receive in their classroom discussions as they interact with other students and the lecturer to address misconceptions or errors that they produce.

Molefe and Brodie (2010) argue that "errors are signs that students are involved in their learning and their thinking processes are engaged, so further explanations can be encouraged from students to understand why they made those errors" (p. 6). They further argue that in discussing errors, further thinking and reasoning can be provoked and students can develop practices of making meaningful contributions to mathematical discussions (Molefe & Brodie, 2010).

2.2.4 Interference

Olivier (1989) states that students tend to confuse the already existing knowledge with the newly acquired knowledge in their learning of mathematics. He makes an example of students who may say four times four is eight. This might emanate from the fact that these students learnt that four plus four is equal to eight earlier in their learning of addition, as a result they add even in a situation of a multiplication. In this situation the addition schema is constructed first and is well developed. He explains that students tend to replace the newly acquired schema with the one already existing. He further explains that this error of replacing the newly acquired schema with the already existing one also happens in algorithms, and students also tend to interchange algorithms incorrectly. He states that errors that result from the above argument are ascribed to "interference".

Olivier (1989) suggests that errors should be addressed in teaching and learning as they generate further errors if left unattended. He further explains that key errors are those which, if left unresolved have the potential to block or impede further progress.

Errors provide evidence that students are thinking and bring their previous knowledge to bear on new situations (Brodie & Berger, 2010). They further argue that many students' errors are produced by misapplications of standard algorithms. Poor performance of students in mathematics is also ascribed to misconceptions and errors that they bring to higher education from elementary levels (Brodie, 2006; 2010; Kiat, 2005; and Luneta & Makonye, 2010).

2.2.5 Errors in trigonometry

Trigonometry, as a branch of mathematics that deals with the relationships of sides and angles in triangles, forms an important background for the solution of problems to many disciplines (Orhun, 2010). Trigonometry is frequently used in mathematical explanations and definitions of new ideas and concepts. Research studies reveal that many students have not developed clear concepts in trigonometry and that some of them use algebraic notation as informal and inconsistent personal shorthand. For instance in some cases there is use of $f(x) = \sin x$ and sometimes for the same function is written as $y = \sin x$.

Orhun (2010) states that first year university calculus students perform badly in the operations of trigonometric expressions such as addition, subtraction, multiplication and division. This may be due to the fact that there may be no much emphasis in learning of addition, subtraction, multiplication and division of trigonometric functions in the high school curriculum. In addition many students study the derivatives of trigonometric functions for the first time in their first year level in universities. That implies they have not yet developed the schema of addition, subtraction, multiplication and division in derivatives of trigonometric functions. The weaknesses of students in addition, subtraction, multiplication and division of trigonometric functions lead to poor manipulation of trigonometric functions when students are faced with trigonometric problems that require further simplification.

Orhun (2010) suggests that:

> In order for lecturers to account for students' systematic errors from a constructivist perspective, analysing the procedures is not sufficient, since lecturers should analyse students' current schemas and how they interact with each other according to instruction and experience. (p. 182)

Skane and Graeber (1993) claim that some errors displayed by students in the content of algebra, logarithms, exponents and trigonometry are attributed to the distributive law.

They further suggest that traditional instruction is not a sufficient strategy to remediate distributive law errors for some students.

2.3 ROLE PLAYED BY STUDENTS' ERRORS IN INSTRUCTIONAL THEORY

The goal of any mathematical instruction is to assist students to gain an understanding of the concepts and procedures which are relevant to reach a solution of the posed mathematical problem.

Nesher (1987) states that "researchers in the field of mathematics education agree that the process of learning necessarily combines three factors: the student, the lecturer, and subject to be learned" (p.33). She further explains that in order for teaching and learning to be successful, there should be at least two kinds of expertise:

- Subject matter expertise (study material), which can knowledgeably handle the discipline to be learned, and can see the underlying conceptual structure to be learned with its full richness and insights; and
- Expert lecturers whose expertise is in successfully bringing the student to know the given subject matter by various pedagogical techniques that make them experts in teaching (p.33).

Nesher (1987) suggests that "students' expertise is in making errors, that is what they contribute in the learning process" (p.13). She argues that instructional theory should change its perspective from condemning errors to one that seeks and welcomes them. My experience of teaching and learning mathematics tells me that students might gain relational understanding as they recognise their errors and reasons associated with commission of the errors.

Students tend to devise means of addressing their errors by seeking help from other students or from their teachers. Nesher (1987) further suggests that a good instructional programme should predict types of errors and purposefully allow them in the process of learning.

Nesher (1987) suggests that mathematical lessons should bring about the following instructional theoretical aspects, which are outlined below:

- Students should be able to, in the process of learning, test the limitations and constraints of a given piece of knowledge. This can be enhanced by developing learning environments functioning as feedback systems within which students are free to explore their beliefs and obtain specific feedback on their actions.
- In cases where students receive unexpected feedback, if not condemned for it, they will be intrigued and motivated to pursue an inquiry.
- Lecturers cannot fully predict the effect of students' earlier knowledge systems in a new environment. Lecturers should provide opportunities to the students to manifest their errors to develop appropriate instruction that may overcome emerging errors. This study uses learning activities to provide students with opportunities to manifest their errors, while lecturers assist with feedback to address students' errors.
- Errors are usually an outgrowth of an already acquired system of concepts and beliefs that are wrongly applied to an extended domain. They should not be treated as terrible things to be uprooted, since this may confuse students and shake their confidence in their previous knowledge. Instead, the new knowledge should be connected to students' previous conceptual framework and put in the right perspective.
- Errors are found not only behind erroneous performance, but also lurking behind many cases of correct performance. Any instructional theory should shift its focus from erroneous performance to an understanding of the students' whole knowledge system from which they derive their guiding principles.
- Diagnostic items that discriminate between proper concepts and misconceptions are not necessarily ones that lecturers traditionally use in exercises and tests. A special research effort should be made to construct diagnostic items that disclose the specific nature of the misconceptions. (p.38)

The identification of errors might assist lecturers and mathematics teachers to focus on the development of pedagogical techniques that may overcome students' difficulties in their learning of derivatives of trigonometric functions. Brodie and Berger (2010) claim that "the notion 'misconception' empowers lecturers, since it provides them with a way to make sense of pervasive and persistent student errors without blaming students and themselves" (p.170). They argue that this is particularly the case where well-known misconceptions have been identified in certain topics of mathematics (Brodie & Berger, 2010).

Swan (2001) argues that if lecturers become more aware of common errors, they will be in a better position to help students to restructure their knowledge in the direction of more aligned mathematical knowledge. Wood (1988) suggests that to avoid the formation of entrenched errors, there should be open class discussions and interactions that focus on addressing students' errors as they crop up in a classroom situation. He further explains that students might gain better mathematical understanding when they share their interpretations of mathematical problems in classroom discussions.

2.4 STUDENTS' DIFFICULTIES IN CALCULUS

The study of calculus, with its fundamental concepts of limit, derivative and integral, requires an ability to understand algebraic variables as generalised numbers and as functionally related varying quantities (Gray, Loud, & Sokolowski, 2009). Gray et al. (2009) suggest that "calculus instruction should continue to emphasise the differing uses of variables in various contexts and strive to develop students' conceptions of variables as changing and co-varying quantities" (p. 71). Luneta and Makonye (2010) state that "students' performance in calculus is undermined by weak basic algebraic skills of factorisation, handling operations in directed numbers, solving equations, and poor understanding of indices" (p. 167). They further argue that algebraic incompetence has a direct impact on learning calculus. They suggest that lecturers should be aware of the educational backgrounds of the first-year university students in order to design learning activities that may close any gaps that exist between matriculation and the first-year university levels.

Calculus has been a problematic and a difficult subject for many students in many parts of the world (Barnes, 1995). Students often misunderstand the notion of a function and the concept limit which is the cornerstone of several related concepts such as continuity, differentiability, integration and convergence of sequences and series (Tarmizi, 2010).

2.4.1 Students' understanding of first principles of differentiation

The application of first principles of differentiation to learn derivatives of various functions continues to be a significant component to develop the rules of differentiation among students who intend to study advanced mathematics. The derivative can be seen as a concept, which is built from other concepts (Naidoo & Naidoo, 2009). The authors argue that the derivative can be seen as a function, a number if evaluated at a point, a limit of the sequence of secant slopes or rate of change. Each advanced concept in mathematics is based on elementary concepts and cannot be grasped without a solid and specific understanding of the elementary concepts.

Students cannot grasp the concept of a derivative and techniques of differentiation unless they understand the use of the first principles of differentiation to find derivatives of various functions. The understanding of the first principles of differentiation is the building block to the understanding of various rules of differentiation. Ryan (1992) indicates that most students have a limited concept image for the gradient. He recommends that the development of global ideas associated with the gradient of a straight line should be a focus of learning before the idea of gradient of a curve is introduced in beginner calculus. He further elaborates that some students are still indicating fundamental problems with slope as a rate of change and are keying into the x-axis rather than the nature of the slope to state whether it is positive, zero or negative.

The traditional first principles approach has been found to be cognitively demanding for students who demonstrate a 'rush to the rule' for meaning (Ryan, 1992). He recommends that more time should be given to the notion of a tangent to a curve in the first principles approach to differentiation. The situation of a 'rush to the rule' can foster instrumental understanding which may leave students without relational understanding as it is required in order to make sense of mathematics learning.

The notion of instrumental understanding is one of two types of understanding postulated by Skemp (1976), the other being relational understanding. He defines instrumental understanding as knowing the rules of how to apply and carry out a procedure without necessarily understanding the reasoning behind the rules. He describes relational understanding as “knowing what to do and why” (1976, p. 5) and the ability to deduce specific rules or procedures from more general mathematical relationships. He provides an example of teaching the area of a triangle as half the base multiplied by the height. Students find this easy as they know the formula from memory, but they need to understand how the formula is obtained.

Skemp (1982) asserts that communication in mathematics occurs in symbols and words. He argues that it is possible to operate at the level of the symbols by using the syntax of the subject without entering into the meanings of those symbols. He further argues that students should explore the meaning of symbols in order to develop relational understanding. Hiebert and Carpenter (1992) suggest that “once meanings are established for individuals, it is possible to think about creating meanings for rules and procedures that govern actions in symbols” (p.72).

Maharaj (2008) claims that “the teaching implication of identifying errors in the learning process is that before students are required to use and manipulate algebraic and trigonometric functions; the meanings of symbols must be established” (p. 402). The exploration of errors in learning derivatives of trigonometric functions is likely to promote an understanding of special limits and symbols that are involved in learning standard derivatives of trigonometric functions.

Brodie (2010) argues that “errors make sense when understood in relation to the current conceptual system of the student, which is usually a more limited version of a mature conceptual system” (p. 13). The implication of Brodie’s (2010) argument is that lecturers should consider their students’ prior knowledge in order to assess what students know and thus be able to accommodate new knowledge. Correcting students’ errors of current conceptual structures should help them to become more powerful through increasing their understanding in a range of situations (Brodie, 2010).

2.4.2 Students' understanding of the concept 'derivative'

Pillay (2008) and Zandieh (1997a; 1997b; 2000) suggest that the concept of a derivative can be represented in many ways, for instance graphically as the slope of a tangent line to the curve at a point, verbally as the instantaneous rate of change, physically as speed or velocity and symbolically as the limit of the difference quotient. Pillay (2008) claims that students prefer two ways of representation, namely:

- the graphical representation of the derivative with slope as the main focus; and
- to interpret the derivative as a rate of change.

This study focuses on the use of two ways, namely:

- symbolic representation of the derivative; and
- the derivative as a rate of change

The symbolic representation of the derivative is an expression for the average gradient, which is written as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. This study uses this formula to develop understanding of standard derivatives of trigonometric functions. The study also introduced the derivative as a rate of change with students having to show an understanding of using the first principles of differentiation to find standard derivatives. The derivative as the rate of change uses the Leibniz's notation. In Leibniz's notation for differentiation the derivative of the function *f*(*x*) is written $\frac{d(f(x))}{dx}$.

If we have a variable representing a function, for example if we set $y = f(x)$ then we can write the derivative as $\frac{dy}{dx}$. This study also used Lagrange's notation $\frac{d(f(x))}{dx} = f'(x)$.

The findings of Ubuz's (2001) in her research of first year engineering students' understanding of tangency, numerical calculation of gradients and the approximate value of a function at a point through computers reveal that students had the following misconceptions about the derivative:

- the derivative at a point gives the function at a point;
- the tangent equation is the derivative function;

- the derivative at a point is the tangent equation; and
- the derivative at a point is the value of the tangent equation at that point.

The exploration of errors may assist students to address these types of misconceptions as students and lecturers interact in their classroom discussions.

Calculus requires a high level of conceptual understanding, yet many students struggle to make sense of differentiation (Parameswaran, 2007). Some students show an inadequate understanding of the concepts of function and variable (Barnes, 1995). She also claims that there is a lack of awareness that a derivative is a rate of change. Uygur and Ozdas (2005) assert that the derivative is a difficult concept for many students. They further explain that it is worse when the function considered is a composite function. Tall (1993) indicates that "the Leibniz notation $\frac{dy}{dx}$ proves to be almost indispensable in the calculus" (p.19). Yet it causes serious conceptual problems with students whether it represents a fraction or a single individual symbol (Tall, p. 19). He further explains that one difficulty with the notion of the chain rule is the dilemma of whether the du can be cancelled in the equation $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

2.4.3 Students' understanding of rules of differentiation

Studies of students' difficulties with calculus topics may offer insights into understanding misconceptions and errors which are committed by students when doing calculus (Clark, Gordero, Cottrill, Czarnocha, DeVries, John, Tolias and Vidakovic, 1997). Mundy (1984) asserts that there is "a tendency of calculus students to operate at a rote level of procedures and symbol manipulation, which is not supported by an understanding of the concepts involved" (p.171). As a result, students fail to use calculus strategies when dealing with non-routine problems (Clark, et al., 1997). Barnes (1995) suggests that "students should not be taught rules for differentiation until they have developed a good understanding of what a derivative is, and a familiarity with the relationship between a function and its derivative" (p. 4). She further elaborates that students should explore techniques on how to find and investigate derivatives of a variety of different functions.

She emphasises that this may help to avoid what Ryan (1992) has described as 'the rush to the rule', where the meaning is ignored or forgotten, and students operate on a purely mechanical level, pushing symbols around on paper.

The chain rule is a calculus concept that causes difficulties for many students (Wangberg, Engelke, & Karakok, 2010). The derivatives of trigonometric functions become complicated when they need the application of the chain rule. Literature also shows that function composition is particularly problematic for students (Engelke, Ochrtman & Carleson, 2005). Some students are introduced to the chain rule as merely a rule that should be applied without much attempt to reveal the reasons for and justification of the procedure (Orton, 1983b). The chain rule concept, as an example of a schema, requires students' understanding of standard derivatives and the basic rules of differentiation (Clark, et al., 1997). This suggests that the concept of the chain rule should be introduced only when students can operate automatically with other rules of differentiation such as the multiple constant rule, the sum and the difference rule, the power rule, the product rule and the quotient rule.

The literature related to studies in calculus provides evidence that students develop more procedural than conceptual understanding in differentiation with regard to the application of the chain rule (Wangberg, et al., 2010).This suggests that lecturers should design activities that allow students to explore basic concepts of differentiation. Wangberg, et al. (2010) argues that some students fail to recognise that differentiating functions such as $y = \cos \pi x$ require the use of the chain rule.

Jojo, Maharaj and Brijlall (2011) claim that "the complexity of the chain rule deserves exploration because students struggle to understand it and because of its importance in the calculus curriculum" (p. 337). This study explores derivatives of trigonometric functions involving all the rules of differentiation. Uygur and Ozdas (2005) state that many students are able to evaluate derivative of special composite function by memorised rules, but most of them calculate these derivatives without the conscious use of the chain rule.

They further argue that although many students provide the general statement of the chain rule and write down the formula, only few of them can explain the connection between the general statement of the chain rule and memorised rules. They suggest that in teaching the concept of the chain rule, more emphasis should be given to using the Leibniz notation meaningfully, to relate special cases to the general statement of the chain rule and abstract cases in order to avoid such misconceptions. The authors further suggest that another important point when teaching the chain rule is prompting students by relating the composition function notions to various functions, especially abstract problem situations, which embody the chain rule concept.

2.5 ANALYSIS OF STUDENTS' ERRORS

This section reviews the research on analysis of students errors based on the steps of solving problems or sources of difficulties in solving problems. One of the main methods used to analyse students' errors is to classify them into certain categories, which are based on an analysis of students' behaviours (Li, 2006).

Orton (1983a) conducted a study on students' understanding of integration. He used a clinical interview method to investigate students' understanding of elementary calculus. In analysing responses to tasks, Orton (1983a) adopted Donaldson's (1963) three types of error, namely structural, arbitrary and executive. Structural errors are those which arose from some failure to appreciate the relationships involved in the problem or to grasp some principle that was essential to the solution. Arbitrary errors are those in which the subject behaved arbitrarily and failed to take account of the constraints laid down in what was given. Executive errors are those which involved failure to carry out manipulations, though the principles involved may have been understood.

Orton (1983a) claims that the relationship between arbitrary errors and the other two types was difficult to define and arbitrary errors did not appear to be a frequent kind of error with relatively mature students.

Orton (1983b) identifies errors displayed by first year students in differentiation. He followed the same clinical interview methods as in Orton 1983(a).

In summary of the findings, he states that the students showed problems in manipulation of algebraic expressions such as inappropriate cancellation and an inability to factorise correctly. Some students demonstrated poor understanding of the expansion of the quadratic expression such as $3(a+h)^2$. They did not write the middle term $6ah$, which might be a procedural error. Some students omitted negative signs in multiplication of algebraic expressions. The symbols of differentiation and the approach to differentiation were clearly badly understood by the students (Orton, 1983b). In his conclusion, he claims that algebraic difficulties could be obscuring the ideas of calculus. He also argues that some students' difficulties are caused by a poor understanding of the basic concepts of differential calculus.

Orton (1983a) states that although limits are important to a real understanding of integration and differentiation, not much school time is devoted to a consideration of limits before they are suddenly required for calculus. Some students in the study conducted by Orton (1983a) demonstrated poor understanding of simplification and factorisation. He further explains that the greatest problems in differentiation were caused by fractional and negative indices. Some students showed arithmetic errors and were also unable to use brackets correctly.

Kiat (2005) conducted a study of students' difficulties with solving integration problems. He used a six-question test, followed by interviews with selected students to collect data on students' difficulties to solve integration problems. Although it is not mentioned in Kiat's paper but it appears as if he adapted Orton's (1983a) classification of errors.

In Kiat's (2005, p. 42) study students' errors were classified into three categories, namely, conceptual, procedural and technical as shown in Table 2.1 below.

Types of Errors	Description
Conceptual Error	• Failure to grasp the concepts in a problem • Errors from failure to appreciate the relationships in a problem. Example: Area between the curve $y = x(x-4)$ and the x-axis from $x = 0$ to $x = 5$ is: $\int_0^6 x(x-4)dx = \int_0^6 (x^2 - 4x)dx = -8\tfrac{1}{3}\text{units}^2$ Students failed to realise that the part of the curve y= $x(x-4)$ from $x = 0$ to $x = 4$ is below the x-axis, whereas the part from $x = 4$ to $x = 5$ is above the x-axis.
Procedural Error	• Errors from failure to carry out manipulations or algorithms although concepts in a problem are understood. Example: $\int \tan^2 2x dx = \int (\sec^2 2x - 1)dx = \tan 2x - x + c$ Students fail to put a coefficient of $\frac{1}{2}$ in front of tan $2x$.
Technical Error	• Errors owing to lack of mathematical content knowledge in other topics. • Errors owing to carelessness (slips). Example: $\int 2(3+4x)^4 dx = \int (6+8x)^4 dx = \left[\frac{(6+8x)^5}{5\times 8}\right] + c = \frac{(6+8x)^5}{40} + c$ Students wrongly multiplied the constant of 2 into the binomial before integrating.

Table 2.1 Classification of students' errors in Kiat (2005)

According to Kiat's (2005) study, students' responses showed that they confused the appropriate procedures for integration and differentiation. Secondly Kiat (2005) asserts that several students seemed to forget trigonometric identities which are required for manipulation. From the analysis Kiat (2005) concluded that students encountered difficulties in integrating trigonometric functions. This may be analogous to students' understanding of derivatives of trigonometric functions.

Luneta and Makonye (2010) conducted a case study of a grade 12 class in South Africa on student errors in elementary analysis. The purpose of the study was to investigate errors that students displayed in differential calculus, classify errors that students made in response to calculus questions and to explain how students' calculus errors were linked to their misconceptions (Luneta & Makonye, 2010).

Luneta and Makonye (2010) used five different types of errors, but in their interpretation it appears that their types of errors were adapted from Kiat's (2005) three types of errors, namely conceptual errors, procedural errors and technical errors. In their summary of findings they conclude that students' poor understanding of calculus is attributed to knowledge gaps that exist in students' knowledge of algebra (Luneta & Makonye, 2010). They further argue that students' poor understanding of calculus is also owing to language problems and a lack of knowledge in underlying calculus concepts, in which calculus techniques such as rules of differentiation are instrumentally understood (Luneta & Makonye, 2010).

Luneta and Makonye (2010) assert that some students showed poor understanding of functional notations such as $f(x+h)$, and failed to appreciate the mathematical problem. This suggests that these students have conceptual problems with regard to calculus. They further report that some students displayed insufficient knowledge of calculus terminology such as confusing turning points with the axial intercepts (Luneta & Makonye, 2010). This suggests that, when teaching, emphasis should be made to link students' understanding of appropriate concepts and appropriate procedures that are taught in calculus lessons.

Luneta and Makonye (2010) conclude that “errors demonstrated by the students in calculus emanate from prior knowledge as students attempt to construct mathematical meanings” (p. 44). They recommend that further research should be conducted to determine how students’ misconceptions and errors develop and assess how far competency and performance in calculus can be enhanced if lecturers target the misconceptions and errors that have been identified in their students.

Molefe and Brodie (2010) use two theories that inform the notion of teaching and learning in their study. They use Vygotsky’s (1978) socio-cultural theory and Lave and Wenger’s (1991) theory of situated learning. This study uses Vygotsky’s ZPD to address the displayed errors through student-student and student-lecturer interactions. This is done through the use of APOS theory to analyse students’ mental constructions for solutions of differentiation. APOS theory is also used to find out what is happening in the minds of the students. Ubuz (2002) states that the analysis of students’ written and verbal responses revealed significant information regarding the nature and characteristics of students’ understanding of differentiation.

Van Staden (1989) gives examples of inappropriate teaching in elementary calculus. He describes inappropriate teaching as the process through which agents such as teachers and textbooks induce misconceptions in the minds of students. He cited two examples of inappropriate teaching such as, the derivatives of all the trigonometric functions can be obtained from the equation $\frac{d(\sin\theta)}{d\theta} = \cos\theta$ and, the equation $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ is required for the derivation of equation. These are over-generalisation on derivatives of trigonometric functions and special limits of trigonometric functions.

Van Staden only mentioned misconceptions that usually happen in the textbooks but did not investigate in-depth the misconceptions which are committed during the calculation of the derivatives of trigonometric functions.

Dawkins (2006) reports common mathematics errors which also include trigonometric functions. He made examples of trigonometric errors such as linear extrapolation and over-generalisation in multiplication of linear trigonometric functions such as follows:

- $\cos(x+y)=\cos(x)+\cos(y)$
- $\cos(3x)=3\cos(x)$

He also mentioned inappropriate teaching such as stating that:

$$\frac{d(x^n)}{dx}) = nx^{n-1}$$

According to Dawkins (2006) that this is where most instructors leave it, instead of including an important restriction with the above statement. That is in order to use this formula n must be a constant. In other words we cannot use the formula to find the derivative of x^x since the exponent is not a constant. Although Dawkins mentions some common misconceptions that crop up in teaching of trigonometric functions, he does not explore misconceptions and errors in derivatives of trigonometric functions.

Likwambe (2006) claims that classifying errors by using a scheme does not provide insight into the underlying structures of students' concept images at it stops at classification and does not lend itself to developing further insight into why students made those errors. I do not agree with Likwambe's view as this study explores errors displayed by the students in order to assist the lecturer to design classroom interactions that may overcome obstacles and deficiencies, which are demonstrated in students' solutions.

2.6 SUMMARY

This chapter provided some background of what other researchers did to identify students' errors in learning some sections of mathematics. The literature review points out that some errors are linked to prior knowledge of students who tend to over-generalise some procedures, algorithms and rules of differentiation in their solutions.

This chapter discussed the nature of errors, which are displayed by students when learning mathematics. This discussion involved generalisation over numbers, operations, meaning and interference. It also discussed errors that are displayed by students in trigonometry, and the role of errors in instructional theory. The chapter presented some of the causes of difficulties, which are encountered by students in the learning of differentiation. This covered students' understanding of the first principles of differentiation; students' understanding of the concept derivative; and students' understanding of the rules of differentiation.

Lastly, it discussed what other researchers have done about students' misconceptions and errors in various topics of mathematics. I intend to adopt the categories, namely, interpretation, arbitrary, procedural, linear extrapolation and conceptual errors to develop the conceptual framework of my study. The following chapter discusses the research frameworks that serve an interpretive function for this study.

CHAPTER THREE

RESEARCH FRAMEWORKS

3.1 INTRODUCTION

This chapter discusses two theoretical frameworks which underpin this study. The theoretical frameworks of interest are the Vygotskian (1978) socio-cultural theory and Dubinsky's (1991) APOS theory. This chapter discusses socio-cultural theory with its constructs such as the zone of proximal development, semiotic mediation and the centrality of communicative practices. Further it deals with APOS theory which involves the discussion of stages, characteristics, implementation and instructional practices.

3.2 THE NATURE OF RESEARCH FRAMEWORKS

The notion of research framework is central to every field of inquiry (Lester, 2005). A research framework is a basic structure (that is concepts and relationships) of ideas that serve as a basis for a phenomenon that is to be investigated (Lester, 2005). Concepts and interrelationships are used as a basis and justification for all aspects of the research (Lester, 2005).

Lester (2005) discusses four advantages of using a framework to conceptualise and guide one's research as follows:

1. A framework provides a structure for conceptualising and designing research studies. In particular, a research framework helps to determine:
   - The nature of the questions asked;
   - The manner in which questions are formulated;
   - The way the concepts, constructs, and processes of the research are defined.
2. The data without a framework do not make sense of the research study. In order for researchers to interpret, analyse and inform decisions there should be a framework as a guide to validate the evidence brought by the researcher.
3. A good framework allows researchers to exceed common sense.

Lester argues that theory building is a key player in driving practical progress in a research project. Deep understanding that comes from concern for theory building is often essential when dealing with truly important problems.

4. Need for deep understanding, not merely understanding. A research framework helps researchers to develop deep understanding by providing a structure to design research studies, interpret data which result from those studies and draw conclusions. (p.458)

3.2.1 Types of research frameworks

Eisenhart (1991) posits that there are three types of research frameworks, namely theoretical, conceptual and practical frameworks. This thesis is only concerned with the first two.

3.2.2 Theoretical frameworks

Niss (2006) defines a theory as a "system of concepts and claims with certain properties, which are outlined below.

- A theory consists of an organised network of concepts (including ideas, notions, distinctions and terms) and claims about some extensive domain, or a class of domains, of objects, situations and phenomena.
- In theory, the concepts are linked in a connected hierarchy (sometimes of a logical or proto-logical nature) in which a certain set of concepts, taken to be basic, are used as building blocks in the formation of the other concepts in the hierarchy.
- In the theory, the claims are either basic hypotheses, assumptions, or axioms, which are taken as fundamental (that is. not subject to discussion within the boundaries of the theory itself), or statements obtained from fundamental claims by means of formal or material (material means experiential or experimental) derivation (including reasoning).

- In principle, for a system of concepts and claims to be called a theory, the system has to be stable, that is, unchanged over a longer span of time, coherent, that is, the components of the system have to be linked in a clear and non-contradictory way, and consistent in the sense that it is not possible to arrive at contradictory claims by means of the types of derivation permitted in the theory. (p. 2)

The theoretical framework as a structure to guide research investigation is based on researchers' understanding and interpretation of the formal theory to be adopted in which to locate the study.

3.2.3 Conceptual frameworks

Eisenhart (1991) defines a conceptual framework as "a skeletal structure of justification, rather than a skeletal structure of explanation based on formal logic (formal theory) or accumulated experience (practitioner knowledge)" (p.209). She further explains that a conceptual framework is an argument including different points of view and culminating in a series of reasons for adopting some points. She continues stating that a conceptual framework is an argument, which is presented for the concepts chosen for investigation or interpretation, and any anticipated relationships among them that will be appropriate and useful, given the research problem under investigation.

This study integrates socio-cultural and APOS theories to analyse data. The integration of the two theories assisted the researcher to obtain interpretation and understanding of the data.

3.3 SOCIO-CULTURAL THEORY

Socio-cultural theory states that human cognitive developmental processes, or learning processes, are merely products of human society and culture.

Wertsch (1990) asserts that a socio-cultural approach to human development is characterised by three general themes:

(1) a reliance on genetic (developmental) analysis;

(2) the claim that higher order functions in the individual have their origins in social life; and

(3) the claim that an essential key to understanding human social and psychological processes are tools and signs that are used to mediate them. (p.113)

The first theme, namely ‘a reliance on genetics’, emphasises that understanding in the learning situation depends on students’ intellectual capability to assimilate new knowledge. However, intellectual ability cannot work alone; it should be a socio-genetic process with learning coming about through social interactions between students and a more knowledgeable person (Vygotsky, 1978). Tall and Mejia-Ramos (2009) assert that “the student is born with a genetic structure set before birth in the genes, but the generic facilities of perception and action need to be coordinated and refined into coherent perceptions of the world” (p.2). They explain that mathematical procedures are extensions of these tendencies that may be learnt in a basic procedural sense, but are usually better appreciated within a more coherent meaningful framework of connections (Tall & Mejia-Ramos, 2009).

Wertsch (1990) claims that “higher order functions in the individual have their origins in social life” (p.114). This implies that higher mental functions such as abstract reasoning, logical memory, language, voluntary attention, planning and decision-making have their origin in human interaction (Kozulin, 1990; Newman & Holzman, 1993).

Higher mental functions appear gradually during the process of radical transformation of the lower functions, and the transformation is made through the so-called “mediated activity” and “psychological tools” (Kozulin, 1990; Newman & Holzman, 1993). Students bring their previous experiences to bear on new situations that they encounter. Tall and Mejia-Ramos (2009) explain that “technically, *a met-before* is part of the students’ concept image in the form of a mental construct that students use at a given time based on experiences they have met before” (p.1).

Students in their learning process try to link their existing knowledge with the newly acquired knowledge. Socio-cultural approaches are based on the concept that human activities take place in cultural contexts and are mediated by language and other symbol systems, and can be best understood when investigated in their historical development (John-Steiner & Mahn, 1996). In a socio-cultural perspective, human development begins with dependence on caregivers. Caregivers, in the context of this study are lecturers and more knowledgeable students who assist other students to understand difficult concepts and they explain the relevant algorithms in the learning of mathematics.

When it comes to learning activities, inexperienced students depend on others who have more experience. As time goes on, the inexperienced students take on increasing responsibility for their own learning and participation in joint activity (John-Steiner & Mahn, 1996). This refers to the notion of the zone of proximal development (ZPD).

Vygotsky (1978) defines ZPD as "the distance between the actual developmental level, as determined by independent problem solving and the level of potential development, as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 86). To put it differently ZPD, is the difference between what a student can do without help and what he or she can do with help.

Socio-cultural perspective emphasises the use of two concepts, namely the zone of proximal development (ZPD) and the concept of activity to mediate learning among students.

3.3.1 ZPD in learning derivatives of trigonometric functions

In this study the lecturer used the zone of proximal development (ZPD) and structural discussion of learning activities to develop understanding of derivatives of trigonometric functions. Vygotsky argues that student's thinking and problem solving ability fall into three categories. Those that can be performed independently, those that can be performed with assistance and those that cannot be performed even with assistance (Vygotsky, 1978).

The learning activities start from what the students can do alone based on their high school understanding to link the already existing knowledge with knowledge that they can perform with assistance. As students continue to practice they can perform independently in activities that were previously performed with assistance. The shifts the students gain in understanding help them to find a way of attempting the problems that they were unable to solve even with assistance.

Tharp and Gallimore (1990, p. 185) model the zone of proximal development by the following diagram:

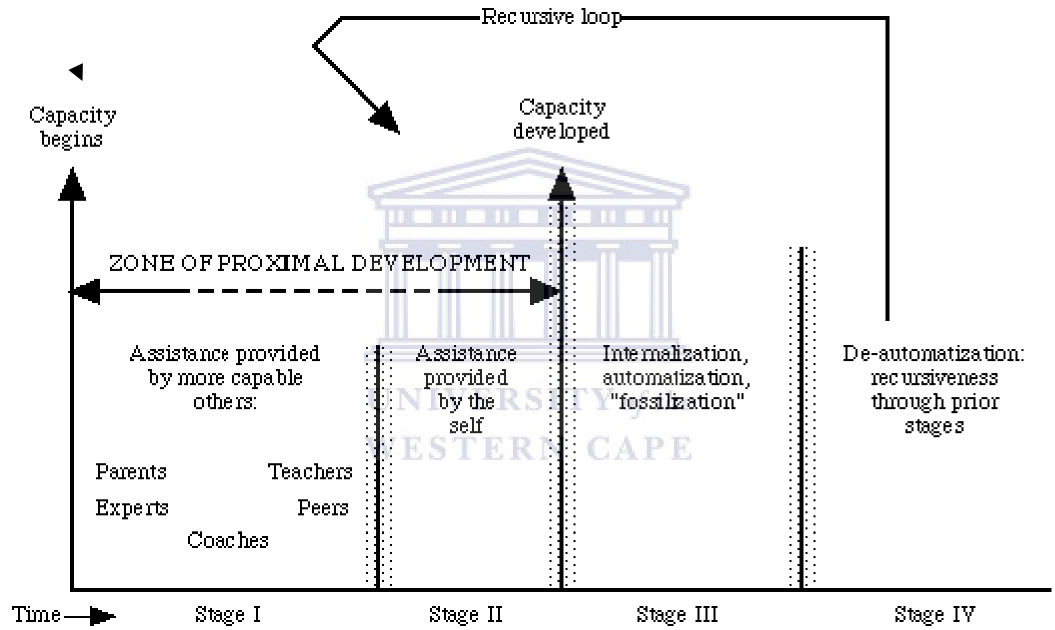

Figure 3.1: Model of four stages in the zone of proximal development (Tharp and Gallimore (1990, p. 185)

Stage 1: The first stage demonstrates how students develop an understanding of language that is appropriate to their study and the basics of the topic under study by relying on others such as instructors to perform the task.

Stage II: In the second stage students use prior knowledge to carry out the task without any guidance. The zone of proximal development occurs between the first and second stages. Students practice alone, which implies that they perform certain activities without assistance. However, they are not at a stage of perfect proficiency and require some assistance sometimes.

Stage III: In the third stage performance is developed, is happening without thinking and knowledge is fixed and it cannot be forgotten. This means that at this stage students reach the stage of independence. In this stage a student does not need help from an adult or even practice more exercises to reinforce the already existing knowledge (Tharp & Gallimore, 1990).

Stage IV: In the fourth stage students are at the de-automatisation of performance that leads to the process of repeating a function, each time applying it to the results of the previous stage through the ZPD. Lifelong learning by any individual is made up of the same regulated, ZPD sequences, from other-assistance to self-assistance recurring over and over again for the development of new capacities (Tharp & Gallimore, 1990).

The interpretation of Vygotsky's socio-cultural approach on cognitive development is that one should understand the two main principles of Vygotsky's work: MKO and ZPD. The MKO refers to someone who has a better understanding or a higher ability level than the student with respect to a particular task, process, or concept (Galloway, 2001).

The ZPD implies that at a certain stage in development, students can solve a certain range of problems only when they interact with people and cooperate with peers (Morris, 2008). Morris (2008) further explains that once student problem solving activities have been internalised; the problems initially solved under guidance and in cooperation with others can be tackled independently.

Vygotsky (1978) highlights that “what is in the ZPD today will be the actual developmental level tomorrow, that is, what a student can do with assistance today, she or he will be able to do it alone tomorrow” (p.87).

Vygotsky believed that when a student is at the ZPD for a particular task, providing the appropriate assistance will give the student advancement to achieve the task (Galloway, 2001). Once the student, with the benefit of assistance, masters the task, the assistance can then be removed and the student will then be able to complete the task on his or her own. Wertsch (1985) states that ZPD “is to deal with two practical problems in the learning situation: the assessment of students’ intellectual abilities and the evaluation of instructional practices” (p. 67). Learning activities challenge students’ thinking within the learning process.

Borchlet (2007) asserts that “learning is determined by the interactions among students’ existing knowledge, established social context, and the problem to be solved” (p. 2). This supports Vygotsky’s (1978) idea that higher order thinking developed first in action and then in thought. Borchlet (2007) argues that “the potential for cognitive development is optimised within ZPD or an area of exploration for which a student is cognitively prepared, but requires assistance through social interaction” (p.2).

The process can be understood in a socio-cultural perspective with reference to Vygotsky’s ZPD, which explains how to advance students’ learning process. This approach is reinforced by Wertsch (1985) who asserts that:

> Any function in the student’s cultural development appears twice, or on two planes. First, it appears on the social plane, and then on the psychological plane. First it appears between people as an inter-psychological category and then within the student as an intra-psychological category. (pp. 60-61)

Wertsch (1990) states that “the fundamental claim is that human activity (on both the inter-psychological and the intra-psychological planes) can be understood only if we take into consideration the “technical tools” and “psychological tools” or “signs’ that mediate the activity” (p. 114). Technical tools refer to physical learning resources such as text books; lecture notes; calculators and classroom written activities.

Psychological tools refer to tools such as language, counting systems, mnemonic techniques, art, writing, diagrams, and maps. Psychological tools are created by society, and are directed towards the control of behaviour (Quek & Alderson, 2002). Psychological tools alter the flow and structure of mental functions, just as physical tools alter the way our work processes evolve (Quek & Alderson, 2002). This study uses learning activities and text books as physical tools to facilitate learning of derivatives of trigonometric functions.

Tools carry with them a historical background. They are instilled with the collected experience and skill that was involved to develop them (Quek & Alderson, 2002). This study uses students' prior knowledge and their experiences of learning differentiation at high school level. For example, in classroom discussions, students explain and justify their understanding and interpretations of the differential calculus problems to other members of the class.

Socio-cultural theory emphasises that the most advantageous learning environment is one where a dynamic interaction between teachers, students and tasks provides an opportunity for students to create their own truth during interaction with others (Atkinson, Derry, Renkl, & Wortham, 2000). Atherton (2005) emphasises that in a socio-cultural classroom, students are active makers of meanings and the role of the lecturer is to guide students to gain meaningful understanding of the learning material.

3.3.2 Semiotic mediation

The process of moving from elementary to higher mental functions is called semiotic mediation and an important mechanism in this transition is the use of tools and symbols (Wertsch, 1991). Semiotic activity is defined as the activity of investigating the relationship between sign and meaning, as well as improving the existing relationship between sign and meaning (van Oers, 1997).

This study focuses on students' methods of making meaning through reading and interpreting learning activities to learn derivatives of trigonometric functions.

3.3.3 Centrality of communicative practices

Vygotsky (1987) places communication at the centre of his theory of language and thought by arguing that "the thought is completed in word" (pp. 249-250). This implies that in designing activities, the lecturer pays attention to arranging interactions where the lecturer and students pause to comment on their problem solving efforts in oral or written reflections (Brown & Cole, 2002).

In this study the students and lecturer use English as the language of teaching and learning. There were situations where students who share the same home language with the lecturer (IsiXhosa) used it during individual attention. Communication is important in developing mathematical understanding (Steele, 2001). Steele (2001) explains that within a socio-cultural perspective students exchange ideas amongst one another and listen actively to one another's views. This creates mutual understanding based on culturally-established mathematical practices. Vygotsky (1994) asserts that language is a cultural tool, and a human instrument of communication.

3.4 APOS THEORY

Dubinsky's (1991) Actions-Processes-Objects-Schemas (APOS) theory was another theoretical framework that this study employed to analyse students' errors in learning of derivatives of trigonometric functions. The aim of applying APOS theory is to reveal the nature of students' understanding rather than to provide a statistical comparison of students' mathematical performances (Weller, Clark, Dubinsky, Loch, McDonald, & Merkovsky, 2003).

The development of APOS theory arose out of an attempt to understand how mathematics can be learned and what an educational programme can do to help with this learning (Dubinsky & McDonald, 2001).

Dubinsky and McDonald (2001) assert that:

> research utilising APOS theory has focused on mathematical concepts such as : functions; various topics in abstract algebra including binary operations, groups, subgroups, cosets, normality and quotient groups; topics in discrete mathematics such as mathematical induction, permutations, symmetries, existential and universal quantifiers; topics in calculus including limits, the chain rule, graphical understanding of the derivative and infinite sequences of numbers; topics in statistics such as mean, standard deviation and the central limit theorem; elementary number theory topics such as place value in base *n* numbers, divisibility, multiples and conversion of numbers from one base to another; and fractions. (p.208)

APOS Theory is an extension of Piaget's theory of reflective abstraction which is applied to the undergraduate mathematics curriculum (Weller, et al., 2003). Thus APOS theory is a constructivist theory of how learning a mathematical concept might take place (Dubinsky & McDonald, 2001). De Vries and Arnon (2004) assert that "according to APOS, the development of every concept begins in the students' mind with action" (p. 55). This theory assists researchers to find out what is going on in the minds of students (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 2004).

The APOS theory has been built from the hypothesis that mathematical knowledge consists of an individual's tendency to deal with perceived mathematical problem situations by constructing mental actions, processes, objects and organising them into schemas to make sense of the situations and solve the problems (Weller, et al., 2003, and Dubinsky & McDonald, 2001). APOS is a description of the mental activities and mental constructions that students might tend to make when formulating their understanding of mathematical concepts (Weller, et al., 2003).

3.4.1 Stages of APOS theory

APOS theory entails four primary stages, namely an action, process, object, and schema stage.

1. An action is "any repeatable mental or physical manipulation that transforms either mentally or physically to obtain an object" (Weyer, 2010, p.10). This is any transformation of object which is perceived by an individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation. For example, in this study the students use first principles to find derivatives of trigonometric functions such as sine and cosine functions and use differentiation rules and techniques to find the derivatives of trigonometric functions. They were expected to be able to write $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ and substitute given values in a correct manner.
2. A process is defined as a "form of understanding of a concept that involves imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under her or his control" (Weyer, 2010, p. 10). That is, an internal construction is made that performs the same action, but now, not necessarily directed by external stimuli. An individual who has a process conception of a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps.
3. A student at a process stage is expected to be able to find the derivative of a given function without using first principles of differentiation and to use the first principle to verify the solution. For example a student might be able to find that $\frac{d(x^3)}{dx} = 3x^2$ by using the power rule and later apply the first principles of differentiation to verify that.
4. Object is a "form of understanding of a concept that sees it as something to which actions and processes may be applied" (Weyer, 2010, p. 10).

At this stage the student sees the procedure as a whole and understands that transformations can be performed on it. For example, in this study the student understands when to apply appropriate rules of differentiation to find the derivative of trigonometric functions.

4. Schema is "a collection of actions, objects, processes and other schemas, together with their relationships that the individual understands" (Weyer, 2010, p.10). At this stage it is expected that students can find derivatives of trigonometric functions that require integration of many rules of differentiation to solve one problem. The curriculum in an extended curriculum programme requires students to be able to apply the rules of differentiation in an integrated manner.

3.4.2 Characteristics of APOS Theory

Dubinsky and McDonald (2001) highlight six characteristics of APOS theory that are used to develop understanding in the learning of mathematics.

Support prediction. The predictive power of APOS theory lies in the assertion that if a student makes certain mental constructions, then he or she will learn a certain mathematical topic. The identification of students' misconceptions might help the lecturer to predict students' errors in similar lessons and support students to develop correct mental constructions.

Possess explanatory power. APOS theory offers explanations of student successes and failures. The researcher's role is to identify and explain students' errors.

This enables the researcher in this study to develop techniques of overcoming students' misconceptions in their learning of the derivatives of trigonometric functions.

Be applicable to a broad range of phenomena. APOS theory has been applied, both by its developers and others, to a large number of undergraduate mathematics topics.

This study explores students' errors in the learning of derivatives of trigonometric functions. There are no studies reported on application of APOS theory to explore students' errors in derivatives of trigonometric functions.

Help organise thinking about learning phenomena. Using APOS theory to develop a genetic decomposition of a mathematical concept is one way of organising one's thinking about how students can learn the concept.

This study uses APOS theory to find out about how students learn the derivatives of trigonometric functions at first-year level in a university of technology.

Serve as a tool for analysing data. This study uses a framework of categories that find common errors in the learning of this subject.

Provide a language for communication about learning. APOS theory provides terms such as action, process, object, schema, interiorisation and encapsulation that are now commonly used in discourse about learning and teaching university mathematics.

3.4.3 How is APOS theory used?

Brijlall and Maharaj (2010) adopted five kinds of construction in reflective abstraction which are outlined below.

- Interiorisation: the ability to apply symbols, language, pictures and mental images to construct internal processes as a way of making sense out of perceived phenomena. Actions in objects are interiorised into a system of operations.
- Coordination: two or more processes are coordinated to form a new process.

Two or more functions can be combined into a problem to necessitate application of differentiation rules such as the product rule, quotient rule, chain rule and the logarithmic differentiation.

For example, to calculate the derivatives of $y = \cos x \sin x$; $y = \cos(2x) + \sin^2 x$ and $y = \frac{\sin(3x)}{5\cos(2x)}$, the students have to apply many rules of differentiation in one problem to find a solution.

- Encapsulation: the ability to conceive a previous process as an object. This is achieved when an individual is aware of totality of the process (Moru, 2006).

For example a student should be able to understand the necessity of applying different rules such as the product rule, quotient rule and the chain rule to solve one problem.

- Generalisation: the ability to apply existing schema to a wider range of contexts. The students move from ability to apply one to ability to apply many rules to find a solution. For example, in a composite function such as $y = \cos^5[\tan(3x)]$, if a student realises that the chain rule should be applied, and then a generalisation on the application of schema is made (Moru, 2006).
- Reversal: the ability to reverse thought processes of previous interiorised processes. For example, reverse process is where the students are able to differentiate a function and integrate it to move back it to its original function.

Brijlall and Maharaj (2010) developed two phases: developmental and research phases where they used APOS theory to design instructional tools and later used students' written work as the data and analysed it to generate further findings. For this study the researcher adopted this investigation cycle to develop theoretical analysis, design and implementation of instruction and to collect and analyse the data.

The investigations cycle through the steps, as shown in Figure 3.2 below.

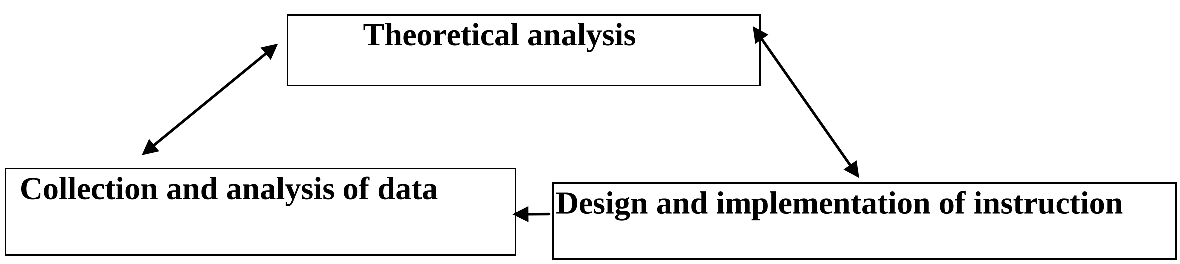

Figure 3.2: Investigation cycle used in APOS theory

Dubinsky and McDonald (2001) assert that "the purpose of theoretical analysis is to propose specific mental constructions (the genetic decomposition) through which a student might learn the concept under consideration" (p. 12).

They further explain that "the role of instructional treatment is to get students to make the proposed mental constructions and use them to construct an understanding of the concept and apply it in both mathematical and non-mathematical situations" (p.12).

The pedagogical strategies for doing this include a few minutes of lecturing and having students doing calculations to implement mathematical ideas, cooperative learning, and a de-emphasis on lecturing in favour of students working in groups to complete mathematical tasks as a data collection process (Dubinsky & McDonald, 2001).

Dubinsky and McDonald (2001) assert that the analysis of data relates to the theoretical analysis in two directions.

1. First, the analysis provides questions to ask from the data. Conversely the data tells something about the effectiveness of the theoretical analysis in terms of mental constructions.
2. The data also tells something about the mathematics that students may or may not have learned. (p.12)

3.4.4 Applying APOS theory on instructional practices

There are several studies which have repeatedly made use of APOS theory. Researchers such as Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas (2004) used APOS theory to design mathematical lessons that were implemented in a mathematics classroom. They suggest the use of cooperative learning groups. Students are organised at the beginning of the semester in small groups of three to five to do all of the course work collaboratively.

In APOS theory, a particular pedagogical approach known as activities, class discussion and exercises (ACE) teaching cycle is used to design instructional strategies (Asiala et al., 2004). In this study the researcher, a lecturer, adapted an ACE teaching cycle.

Activities: In activities students work on the topic to be covered outside classrooms in groups to prepare themselves for classroom discussions. Through these activities students gain experience with mathematical issues that may be developed later in the classroom. The students learn step-by-step instructions on how to perform operations. This is aligned with actions as the first step of learning process in APOS theory.

Class Discussion: Students work in a classroom in teams and towards the end of the lecture they report their calculations to the entire class. The lecturer leads group discussions which are designed to give students an opportunity to reflect on the work that they did in their calculations. The lecturer also intervenes to guide students when necessary to do so. This assists the students and the lecturer to find students' misconceptions through classroom discussion. This is aligned with both processes and objects as the second and third steps of learning processes in APOS theory, that is, an action is repeated and the individual reflects upon it. The students gain understanding as they continue with class discussion.

Exercises: This is aligned with schema where there is a collection of actions, processes, objects, and other schemas, which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept. This study uses exercises such as tutorial tests, sample tests, and formative assessment tests. At the end of each section of differentiation students are assigned tutorials as exercises on what should be done in a classroom to prepare for tests. The students work in pairs to find solutions of various exercises that are given in tutorials. In the last fifteen minutes of the lecture they write a tutorial test. This is a short test of two or three questions. In the following lecture the lecturer reflects on students' performance in tutorial tests. At the end of the topic the students are given a sample test to prepare for a formative assessment test.

Here students work in pairs to write the test and present their calculations in a class, which creates opportunities for other students to question their classmates to gain clarity in the learning of a particular topic to be covered. Lastly, students write a formative assessment test, which comprises five or six questions based on the topic covered.

This study used the three components of the ACE cycle as diagnostic assessment forms to identify students' misconceptions and errors in the learning of derivatives of trigonometric functions. The identification of misconceptions and errors might assist the lecturer to develop remedial activities in the light of evidence of the causes of the original errors and misconceptions (Orton & Frobisher, 1996).

Orton and Frobisher (1996) state that "if the outcomes of the tests are used to diagnose students' misconceptions and errors, then students perceive the tests as helping them overcome problems in the learning of mathematics" (p. 175).

3.4.5 Critiques of APOS theory

There can be no learning theory without weaknesses. APOS, as a theory of learning mathematics concepts at a tertiary level, may not be appropriate to learn mathematics at lower levels where there is an emphasis on concrete, experimental understanding. APOS theory has stages of mathematical development whereby in reality these constructions are not really made in linear sequence (Weyer, 2010). This suggests that stages of APOS theory are not necessarily implemented in a hierarchical classification as discussed earlier. Weyer (2010) proposes that APOS theory is not predictive since it is not diagnostic. This claim reveals that APOS theory may not assist lecturers to find students' understanding of Mathematics and locate where students are in their level of understanding. Hence, APOS theory cannot be used to design baseline assessment. Weyer (2010) claims that "APOS theory is more of a framework for the way people learn mathematics" (p. 15). Weyer (2010) further argues that APOS theory can be time consuming and interpretive in order to determine where students are at their learning process.

3.5 SUMMARY

This chapter discussed the research framework for this study. The discussion involved the nature of research frameworks, as well as socio-cultural and APOS theories. In this study the APOS theory is employed to explain the nature of students' errors when learning derivatives of trigonometric functions in the context of tutorials and formative tests. ACE teaching cycle was used to collect data of the study. Vygotsky's two principles the ZPD and MKO were used to address students' errors and to explain difficult concepts of the derivatives of trigonometric functions. The following chapter examines research methods which were used to collect data for this study.

CHAPTER FOUR

RESEARCH DESIGN AND METHODS

4.1 INTRODUCTION

This chapter discusses the research design and methods employed in data collection and selection of the participants. It also discusses data analysis, reliability, triangulation and validity. It ends with the discussion of research ethics.

4.2 QUALITATIVE RESEARCH

This study is located within the interpretative qualitative research paradigm. Qualitative research is an exploratory approach, which emphasises the use of open-ended questions and probes, which give participants an opportunity to respond in their own words (Devetak, Glazar & Vogrinc, 2010). They further explain that open-ended questions have an ability to evoke responses that are meaningful and important to participants, unanticipated by the researchers which are rich and explanatory in nature (Devetak, et al., 2010).

Denzin and Lincoln (1994) explain that qualitative researchers study phenomena in their natural settings, whilst attempting to make sense of or interpret phenomena in terms of the meanings that people bring to them. They further explain that qualitative research involves the studies used and collection of a variety of empirical materials such as case study, personal experience interview, observations, interactional and visual texts. Terre Blanche and Kelly (1999) state that “interpretive qualitative research relies on firsthand accounts; attempts to describe the situation under consideration in rich detail; and presents its findings in engaging and meaningful language” (p.124).

Jackson (1995) asserts that “qualitative research is based on a small number of participants or an in-depth examination of one group” (p.17).

In this research close attention was paid to a small group of participants in order to provide an in-depth analysis of the study. This research is therefore presented in the form of a case study.

4.3 CASE STUDY

Yin (2009) states that a case study has a twofold technical definition. The first part begins with the scope of a case study:

A case study is an empirical inquiry that

- investigates an existing phenomenon in depth and within its real-life context, especially when the
- boundaries between phenomenon and context are not clearly evident (p. 18).

Yin (2009) argues that "this first part of the technical definition suggests that the case study method is used to understand a real-life phenomenon in depth, but such understanding encompasses important contextual conditions" (p.18).

The second part of a technical definition of a case study is defined as follows:

The case study inquiry copes with the technically distinctive situation in which there will be many more variables of interest than data points, as one result

- relies on multiple sources of evidence, with data needing to converge in a triangulating fashion, and as another result;
- benefits from the prior development of theoretical propositions to guide data collection and analysis (Yin, 2009, p.18).

Yin (2009) claims that "in essence, a twofold definition shows how case study research comprises an all encompassing method covering the logic of design, data collection techniques, and specific approaches to data analysis" (p.18).

This research is a single site case study, which focuses on a group of students that are registered for mathematics in an extended curriculum programme at a university of technology.

4.4 DATA COLLECTION

The researcher collected data for two phases, that is, a pilot and a main study. The purpose of conducting the pilot study was to test the appropriateness of the research instruments, research questions, theoretical framework, methods of data collection and data analysis. In the pilot study the researcher used the differentiation of all sorts of functions such as algebraic, exponential, logarithmic and trigonometric functions.

4.4.1 THE PILOT STUDY

The research participants of the pilot study were the students who were registered for chemical engineering in an extended curriculum programme in the academic year of 2008. They were a group of twenty students chosen from another mathematics lecturer in a class of forty-five students as a group of struggling students in terms of their performance in previous tests. These were the students who obtained an average of fifty percent and below in tests of the first semester. Then in the second semester they were referred to the mathematics support lecturer to work with them to improve their performance. At the end of the year they passed their first semester mathematics and moved to their lecturers to study their second semester mathematics.

This was an unusual case as participants shared a characteristic of interest such as being a borderline case. They shared the following characteristics.

- They were all full time students;
- They were all English second language speakers;
- Three students were foreigners who had equivalent symbols according to the South African Qualifications Authority;
- There were ten female students and ten male students;
- One passed Grade twelve in 2005; and
- Those who were South Africans studied and passed mathematics in high school on a Standard Grade level. The old South African system of education grouped the mathematics syllabus into three categories according to learners' potential.

Those who had high potential were allowed to register on a Higher Grade, which would meet admission requirements for degrees. The Higher Grade syllabus had more advanced problems in terms of content and had a higher cognitive demand compared to the Standard Grade.

The group that demonstrated low ability was allowed to register on a Standard Grade which would meet requirements for diplomas and certificates at universities of technology and technical colleges. The last group, which was Lower Grade, referred to the conversion of a failing symbol from Standard Grade to a passing symbol on a Lower Grade. Those who passed on a Lower Grade were allowed to study mathematics at a Further Education and Training college (FET college) level as a way of improving their content of mathematics, which would hopefully help them to obtain entrance at universities at later stages. In a South African context an FET college is an institution that bridges the gap between secondary schools and universities. It usually offers a wide range of vocational courses and adult education. It is normally offering certificates in certain fields of specialisations which may help a student to obtain an entrance either to a University of Technology or a traditional university.

4.4.2 Research instruments

In this study the lecturer supplied students in the sample group with self-study material to supplement activities obtained in their university study guide. Self-study means activity of learning about something without a teacher or a lecturer to help a learner or a student. Self-study material means learning activities that engages students to a form of study in which one is to a large extent responsible for ones' own learning. The purpose of the self-study material was to allow students to work on differential calculus problems as individuals, in pairs and as groups in between lectures. Self-study activities cover a range of topics as prescribed in the syllabus for differential calculus of an extended curriculum programme. The self-study material could be completed at the students' own pace and convenience, but should be completed prior to the class discussions and tests. The students were encouraged to bring their problems to the class for group discussions, and to obtain assistance from other students and the lecturer.

The instruments were to allow students to write and talk. In talking and writing the students had to explain and justify procedures that they followed in their calculations. The students were given opportunities to do calculations that were required in formative assessments tasks. The lecturer gave the tutorials to the students to give them opportunities to present written solutions and discussion of these solutions in class. At the end of each section students completed formative assessment tasks. Audio and video recordings of the exercise sessions were made to obtain accurate observations. Later, interviews were administered to respondents. The following section gives samples of self-study material that were used to research students' misconceptions and errors in their learning of derivatives of trigonometric functions.

4.4.2.1 Samples of self-study material

The activities were based on the content of differential calculus that is prescribed in an extended curriculum programme.

Initial activities were based on the use of the first principles of differentiation to establish standard derivatives of algebraic expressions; exponential functions; radical functions; logarithmic functions and trigonometric functions.

Prescribed text books by Spiegel (1974); Finney, Thomas and Weir (1994) and Kouba (1998) were consulted to develop the self study material. The self study material was designed to allow students to work on their own, to study the notion and definition of the derivative and the rules of differentiation. The aim here was to develop in the students a basic understanding in the use of the first principles of differentiation of various functions such as algebraic functions, exponential functions, radical functions, logarithmic functions and trigonometric functions. Examples of the activities, which students engaged with during the first stage, are provided in the figures below.

1. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

2. $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$

3. $\lim_{h\to 0}\frac{a^h - 1}{h} = \ln a$

4. $\lim_{h\to \theta}\frac{e^h - 1}{h} = 1$

Figure 4.1: Sample of special limits studied prior to learning the application of the first principles of differentiation in the pilot study

The students' task was to read the above special limits on their own time. They were given tutorial tests to assess their understanding of self-study activities. Once they passed the tutorial tests, they were supplied with other tutorials to reinforce their understanding of these special limits by using the first principles of differentiation to find the derivatives of various functions such as in figure 4.2 below.

In the first principles of differentiation students studied the definition of the derivative as $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$ if this limit exists.

Use the first principles of differentiation to find the derivatives of the following functions (a represents any constants).

1. $y = x$
2. $y = a^x$
3. $y = e^x$
4. $y = \sqrt{x}$
5. $y = \sin x$
6. $y = \cos x$

Figure 4.2: Sample of one of the self-study activities studied in the pilot study

The second set of activities dealt with the introduction of differentiation rules such as the constant multiple rules, power rule, sum and difference rule, product rule, quotient rule and the chain rule. The following figure shows an example of a tutorial task given to the students to practice for classroom discussion purposes.

Find the derivatives of the following functions and simplify where possible

1. $y = 6x^3 - 3x^2 + 7x - 1$
2. $y = (1 - \frac{1}{2}s^2)(3s + 5)$
3. $y = 3\sin x - 4\cos x$
4. $y = x^3 \tan x$
5. $y = \frac{\cos x}{1 + \sin x}$

Figure 4.3: Sample of the self-study activities studied on differentiation in the pilot study

4.4.2.2 Students' written work

On the 27th August 2008 a tutorial test was given to the students. The test consisted of one question. The test required students to differentiate from the first principle.

1. $y = x^2$

Figure 4.4: Sample of the first test given to students in the pilot study

On the 2nd September 2008, the test was given to the students.

Find the derivatives of the following functions:

1. $y = \frac{2x^2 + 4}{x^3 + x}$

2. $y = \ln(3x^3 + 2)$

3. $y = 4\cos 3x$

Figure 4.5: The second test given to the students in the pilot study

On the 4th September 2008 the students in the sample group were given class activities to find the derivatives of functions by using the rules of differentiation. Three exercises were given to students in the sample group that had to be done in class.

1. $y = r^3(3\ln r - 1)$

2. $y = \frac{e^u}{1+u}$

3. $y = \tan x^2$

Figure 4.6: Sample of the third test given to students in the pilot study

On the 18th September 2008 the students were given a formative assessment task. The test was analysed to identify students' misconceptions and errors in their calculations, as well as their progress when learning differential calculus.

1. $y = \frac{4}{x^2}$

2. $y = x^{\ln 3} + (\ln 3)^x + 3\ln x$

3. $y = \frac{\sin x}{1+\cos x}$

4. $y = \cot 4x + \ln(\tan x)$

5. $y = x^3 \operatorname{cosec}(x^2 + 3)$

6. $y = \frac{e^{3x}}{3} + \frac{3}{e^{3x}} + \frac{x}{e^2} + \frac{e^2}{x}$

7. $y = \cos^4(5x^2)$

Figure 4.7: Sample of the fourth test given to students in the pilot study

This was the last written work used to collect data in the pilot study in year 2008. For each test the lecturer marked the scripts, analysed them and gave feedback to the students.

4.4.2.3 Audio and video recordings

On the 12th September 2008 from 8:30 to 10: 30 in the morning, in the pilot study audio and video recordings took place in a lecture room. The students in a sample group gathered in a lecture room to revise the test given as in figure 4.9. The session was audio and video recorded. "The use of audio and video recordings was to capture the lesson as it occurred and, it is then also possible to relive the lesson at the convenience of the researcher" (Mbekwa, 2003).

Audio and video recordings can provide researchers with a more complete sense of who the participants are, and acquaint researchers with the setting in which the people function and types of activities that participants do (Dufon, 2002). Replaying the event also allows researchers more time to study and think about the data before drawing conclusions (Dufon, 2002). The students did calculations on a whiteboard, while the lecturer and other students observed and made additions to correct errors and misconceptions. They also asked questions and assisted in explanation.

The audio and video recordings supported the data collection process through bringing a high level of detail regarding the researcher's and students' interactions (Pelling & Renard, 1999). The lecturer used a tutorial in the audio and video recordings to revise for the upcoming test towards the completion of differentiation.

4.4.2.4 In-depth interviews

On the 25th September 2008 the researcher conducted in-depth interviews for the pilot study, asking questions to assess participants' understanding of differential calculus. The researcher selected six participants in the sample group to participate in the interviews. This comprised three female students and three male students. The aim of the in-depth interview was to find out:

- The work studied by the participants in their high school level to find their prior knowledge with regard to differential calculus.
- Students' understanding and interpretation of differentiation concepts when using self-study activities in the learning of differential calculus.
- The benefits if any of collaboration in their learning process of differentiation.

The results of the pilot study led to some additions and changes in the main study. The researcher added APOS theory to analyse data in the main study as a theory that focuses on cognitive aspects of learning whilst he used socio-cultural theory only to analyse data in the pilot study. The data for the main study focused on the derivatives of trigonometric functions whereas the data collected in the pilot study was done on the differentiation of various functions such as algebraic, exponential, radical, logarithmic and trigonometric functions. The findings showed that the analysis of students' written work should be categorised in both the pilot and the main study. The students who participated in the pilot study proceeded to the second year level of their extended curriculum programme as they passed their first semester course over a period of year 2008. The use of ECP students in this research was found appropriate even in the main study.

4.5 THE MAIN STUDY

As indicated in the first chapter the research participants of the main study are the students who registered for chemical engineering in an extended curriculum programme in the academic year of 2009. They were also a borderline case as they did not meet the minimum requirements of the main engineering stream.

The sample group in the main study shared the same characteristics as the pilot study except for the following:

- The sample group consisted of thirty students who enrolled for chemical engineering in 2009;
- There were sixteen female students and fourteen male students;
- Twenty-two of them passed Grade 12 in 2008;
- Seven of them passed Grade 12 in 2007;
- One passed Grade 12 in 2006;
- Twenty nine students passed Grade 12 mathematics in South African schools;
- One was a foreigner with a qualification equivalent to South African examinations; and
- Their ages ranged from 18 years to 21years, while one student was 25 years.

4.5.1 Research instruments

The main study focused on the derivatives of trigonometric functions. Although the focus was on derivatives of trigonometric functions, there was integration of other functions to build complicated problems of differentiation. The following section deals with students' written work that was used to collect data in the main study in 2009.

4.5.1.1 Samples of self-study material

During the second semester of 2009 (from August to October) the students in the sample group worked with the lecturer using Activities, Class discussions, and Exercises (ACE) teaching cycle to identify misconceptions and errors.

The lecturer intervened in the classroom to deal with students' problems when learning derivatives of trigonometric functions. The lecturer worked with students for four periods per week. The duration of each period was one hour and thirty minutes. The first activity covered the application of the first principles of differentiation to establish standard derivatives of trigonometric functions such as follows:

1. $\frac{d(\sin x)}{dx} = \cos x$;
2. $\frac{d(\cos x)}{dx} = -\sin x$;
3. $\frac{d(\tan x)}{dx} = \sec^2 x$;
4. $\frac{d(\cot x)}{dx} = -\cos ec^2 x$;
5. $\frac{d(\sec x)}{dx} = \sec x \tan x$;
6. $\frac{d(\cos ecx)}{dx} = -\cos ecx \cot x$

Figure 4.8: Sample of the standard derivatives of trigonometric functions studied in the main study

This was done by proving the derivatives of the functions discussed above in a classroom discussion. In the second activity students were required to apply standard derivatives to calculate other derivatives of trigonometric functions using differentiation rules.

The third activity dealt with the application of the logarithmic differentiation rule to find the derivatives of complicated functions. For example to find the derivative of $y = (\tan x)^{\cot x}$ the rule $\frac{d}{dx}(x^n) = nx^{n-1}$ does not apply because of its restrictions that n must be a constant. At the same time the rule $\frac{d}{dx}(a^x) = a^x \ln a$ also does not apply because of its restriction that a must be a constant. As a result in order to find the derivative of $y = (\tan x)^{\cot x}$ we have to multiply throughout by a natural logarithm $(\ln x)$.

The differentiation of complicated functions such as $y = (\tan x)^{\cot x}$ requires the careful use of the following properties of logarithms. Though the following properties are true for a logarithm of any base, only the natural logarithm ($\ln x$) will be used in this set of problems.

Properties of the natural logarithm (base e)

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $\ln y^x = x \ln y$
5. $\ln(xy) = \ln x + \ln y$
6. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

Figure 4.9: Sample of the properties of the natural logarithms studied in the main study

Use logarithmic differentiation to find the derivatives of the following functions:

1. $y = (\sin x)^{x^3}$
2. $y = x^{\ln x} (\sec x)^{3x}$
3. $y = \frac{(\ln x)^x}{2^{3x+1}}$
4. $y = (\tan 2x)^{\cot x}$
5. $y = x^{2x} e^{3x-2} \sqrt{\cos ec 4x}$

Figure 4.10: Sample of self-study activities for logarithmic differentiation studied in class discussions in the main study

The students discussed the activities, shared understanding and completed assessment tasks.

Three kinds of data were collected, namely students' written work; class discussions in the form of audio and video recordings, and in-depth interviews of students concerning mathematical questions, solutions and ways of working with questions in order for students to develop understanding during the learning process of derivatives of trigonometric functions.

4.5.1.2 Students' written work

On the 27 August 2009 the first test was administered to students in the sample group to test their understanding and to identify their errors in calculations.

This was to assist the lecturer to devise means of dealing with students' misconceptions and errors in respect of their solutions. The test comprised one question.

The test required students to differentiate from first principle

$f(x) = \cos x$

Figure 4.11: The first test given to the students in the main study

On the 17th September 2009 the second test was administered to students. The test consisted of 3 main questions with two sub-questions in Question1 and three sub-questions in Question 2 and Question 3.

1.1 $y = 3x^4 - \tan x \cos ecx$ 1.2 $y = 4x^{3/2} \sin x$ 2.1 $\frac{d(\sec x)}{dx} = \sec x \tan x$ 2.2 $\frac{d(\cot x)}{dx} = -\cos ex^2 x$ 2.3 $\frac{d(\cos ecx)}{dx} = -\cos ecx \cot x$ 3.1 $y = \sec^2(x^4)\cot^3(x^4)$ 3.2 $y = \sqrt{\sin(7x) + \ln(5x)}$ 3.3 $y = \tan^3 \sqrt{\cot 7x}$

Figure 4.12: The second test given to the students in the main study

On the 22nd October 2009 the third test was administered to students in the sample group to test their understanding of derivatives of trigonometric functions in application of the quotient rule and logarithmic differentiation. The test comprised three questions.

Find the derivatives of the following functions and leave your answer in the simplest form. 1.1 $y = \frac{\sec\theta}{1 - \cos ec\theta}$ 1.2 $y = x^3 e^{2x+3}\sqrt{\cos x}$ 1.3 $y = \sec^2 x e^{-\tan^2 x}$

Figure 4.13: Sample of the third test given to the students in the main study

For each test the lecturer marked the scripts, analysed them and gave feedback to the students.

4.5.1.3 Audio and video recordings

On the 27th October 2009 from 10:30 to 12:00 audio and video recordings for the main study took place in a lecture room. The students in the sample group gathered in a lecture room to revise the previous test as given in figure 4.13. The revision of the previous test was to give feedback to help students to correct their misconceptions and errors as they worked with other students and the lecturer to differentiate these problems during a class discussion. The students were free to ask questions when they did not understand the concepts and procedures that were applied to reach a correct solution.

4.5.1.4 In-depth interviews

On the 29th October 2009 the researcher conducted in-depth interviews. There were seven female and seven male students. The selection was based on participants' regular attendance and participation in classes.

In-depth interviews are face to face conversations, which explore issues conducted in the study (Boyce & Neale, 2006). The questions that were asked were based on students' written work and possible benefits during the use of self-study activities, Activities, Class discussions and Exercises (ACE) teaching cycle.

Interviews give researchers an opportunity to know people well so that they can understand how the interviewees think and do things (Terre Blanche, Kelly & Durrheim, 2006). Conducting an interview is a more natural form of interacting with people than making them complete a questionnaire, do a test, or perform some experimental task and it fits well with the interpretive approach to research (Terre Blanche et.al, 2006).

This research used in-depth interviews to obtain students' understanding about their use of activities; classroom discussions and exercises to learn the derivatives of trigonometric functions. In-depth interviews are useful when the researcher wants detailed information about a person's thoughts and behaviours or wants to explore new issues in depth (Boyce & Neale, 2006).

The interviews were recorded for analysis. The purpose of the interviews was to find out participants' views about their participation in self-study activities and the ACE teaching cycle in the main study. The questions explored the following areas:

- The work covered in differential calculus at the further education and training phase of schooling to see the links that they bring from high school mathematics to the first year level at university.
- Students' understanding and interpretation of differential calculus concepts when using self-study activities and the ACE teaching cycle when learning derivatives of trigonometric functions.
- The benefits, if any, of using self-study activities and the ACE teaching cycle in the learning of derivatives of trigonometric functions.

4.6 DATA ANALYSIS

Many researchers classify errors to analyse misconceptions and errors displayed by students in their solutions of differentiation problems (Orton 1983a; 1983b; and Kiat, 2005). Upon completion of marking, the researcher carried out item-by-item analysis by examining students' responses for each item. The students' scripts were sorted out and grouped together, putting the scripts with similar errors in one group.

That is the scripts were grouped according to the types of errors, namely, interpretation, arbitrary, procedural, linear extrapolation and conceptual errors.

Thereafter I examined the errors, naming them and trying to find out the causes of errors, referring to the literature as well as the students who committed the errors. I also counted the number of students who committed similar errors and built a frequency table of students' errors in the study.

In the analysis of audio and video recordings, several strategies that were used by students to solve differentiation problems were identified.

As this had been audio and video recorded the CD was replayed and transcribed with errors and misconceptions noted as the students showed their calculations in the video.

This also depicted the lecturer and other students intervening to correct and rectify errors and misconceptions made by other students through open discussion. Patterns and themes emerging out of the discussions in in-depth interviews were noted.

4.7 RELIABILITY

Reliability is concerned with questions of stability and consistency of the study (Singleton & Straits, 2005). Cohen, Manion and Morrison (2000) define reliability as "a synonym for consistency and replicability over time, over instruments and over groups of respondents" (p.117). Hence reliability means obtaining trusted information in that one could obtain the same results if can be in a position to do similar research. This research has used a range of data generation methods such as students' written work, audio and video recordings of class discussions and in-depth interviews, which have been used and tested by other researchers in the field of mathematics education. In this study data collection processes were first piloted and later implemented in the main study.

Cohen et al. (2000) assert that "for research to be reliable it must demonstrate that if it were to be carried out on a similar group of respondents in a similar context (however defined), then similar results would be found" (p. 117). This research was triangulated in terms of the data collection process, investigator process and theoretical process.

With regard to the class discussion and in-depth interviews there were audio and video recordings in the form of a CD, which could be replayed by the researcher, supervisors or anyone who wished to do so in order to assess the suitability of the analysis. Such a process improves the reliability and credibility of the results.

4.8 TRIANGULATION

Triangulation refers to the use of multiple perspectives against which to check one's own position (Kelly, 2006). Cohen and Manion (1986) claim that triangulation "attempts to map out, or explain more fully, the richness and complexity of human behaviour by studying it from more than one standpoint" (p.254). They further explain that in qualitative research triangulation aims to enhance the credibility and validity of the results. In this research three basic types of triangulation were used, as outlined by Kelly (2006):

- Data source triangulation. This entails gathering data through several sampling strategies so that slices of data at different times and social situations, as well as on a variety of people, are gathered. The mixing of data types, known as data triangulation, is often thought to help to validate claims that might arise from an initial pilot study (Olsen, 2004). If the results of a study can be reproduced under similar research methods, then the research instruments are considered to be reliable (Joppe, 2000). This research used two sources of data, namely the pilot study, as well as the main study to find out the extent to which results are consistent over of a certain period of time.
- Investigator triangulation. This refers to the use of several researchers or evaluators, which is useful to draw attention to previously unnoticed researcher effects, which are effects of the researcher on the research context. Supervisors assisted the researcher to pay attention to issues that he did not notice in his data analysis.

This type of triangulation also occurs when researcher presents research papers at national and international conferences, since it helps to receive feedback from reviewers and conference participants.

The researcher also did paper presentations on different phases of the study at university seminars where he received comments from other doctoral students and various specialists in the field of science and mathematics education. This allowed for interpretive validity.

Mbekwa (2003) claims that interpretive validity “refers to the meaning, which research participants attach to all the objects, events and behaviours in the research setting” (p.92).

- Theory triangulation. This refers to the use of multiple perspectives to interpret a single set of data, and this also employs a number of different theories to explain the conclusions of the research. This research integrates socio-cultural and APOS theories to explain and analyse the findings of the study.

To ensure reliability in a qualitative research, examination of trustworthiness is crucial (Joppe, 2000). The use of the three types of triangulation in this study was to establish reliability and validity, which most writers on case study research methods consider to be vital, especially studies that seek explanatory outcomes (Hitchcock & Hughes, 1995).

4.9 VALIDITY

In a broader sense, reliability and validity address issues about the quality of the data and appropriateness of the methods used to carry out a research project. Validity addresses whether the research explains or measures what the researcher intends to measure. Thus validity tries to verify whether research tries to answer the research questions in a suitable manner.

4.9.1 Content validity

This research used the ACE teaching cycle which covered all trigonometric functions in detail in classroom discussions. Data were collected in the form of students' written work, audio and video recordings in the form of class discussions, which showed that the content was a fair representation of the wider issue of the derivatives of trigonometric functions.

Cohen, Manion and Morrison (2000) claim that:

> For content validity, the researcher should ensure that the elements of the main issue to be covered in the research are both fair representation of the wider issue under investigation and that the elements chosen for the research sample are themselves addressed in depth and breadth. (p.109)

4.9.2 Descriptive validity

Mbekwa (2003) states that descriptive validity "refers to the factual accuracy of the researcher's account of what happened in the research setting" (p.92). In qualitative case study research the most important advantage presented by using multiple sources of evidence is the development of bringing together lines of inquiry, a process of triangulation and confirmation (Yin, 2009). The use of multiple sources of evidence in case studies allows an investigator to address a broader range of historical and behavioural issues (Yin, 2009). This research used three types of triangulation such as many sources of data collection, investigator triangulation and theory triangulation to supplement weaknesses that one instrument might have.

4.9.3 Theoretical validity

Mbekwa (2003), states that theoretical validity "refers to the linking of observations and the results of the research to existing theory" (p.93). He further argues that theoretical validity focuses on theory that exists in the community of a particular practice (Mbekwa, 2003). Theoretical validity in this study is addressed in the form of integrating two theoretical frameworks, namely socio-cultural and APOS theories, to analyse errors in the learning of derivatives of trigonometric functions. Vygotsky's ZPD was also used to address students' errors. The combination of two theories brings in theory triangulation, which strengthens the validity of the study.

This research used the literature of other researchers such as Orton (1983a); Orton (1983b); and Kiat (2005) to develop a conceptual framework to classify students' misconceptions and errors in their learning of derivatives of trigonometric functions.

Kanjee (1999) emphasises that "researchers usually develop questions with the assistance of people knowledgeable in the subject area" (p.293). In construct validity, which is theoretical validity (Mbekwa, 2003) emphasis is in the meaning of the responses to one's measuring instrument (Singleton & Straits, 2005). Class discussions and in-depth interviews were audio and video recorded for this research study to allow relevant people such as supervisors to replay them to establish valid understanding of data analysis.

4.10 RESEARCH ETHICS

Central to the case for ethically sound research is the principle that research participants are able to consent freely to their involvement in research (Henn, Weinstein, & Foard, 2006). In this study students were informed about the purpose of research and they agreed to participate. The researcher made it clear that for ethical reasons, the participants' identities would be confidential.

The participants were informed that their participation was voluntary, and that they were free to withdraw at any time they wished.

Christians (2000) states that:

> Proper respect for human freedom generally includes two necessary conditions. Subjects must agree voluntarily to participate. That is without physical or psychological coercion. In addition, their agreement to participate must be based on full and open information. (pp. 138-139)

Pseudonyms have been used to keep students' names anonymous. Confidentiality means that the researcher holds the data in confidence, and keeps it from public consumption (Henn et.al, 2006). All personal data should be secured or concealed and made public only behind a shield of anonymity (Christians, 2000).

The students' written tasks were kept in the researcher's custody where no one else had access to it. The CDs from audio and video recordings and in-depth interviews were used for data analysis only behind a shield of anonymity.

4.11 SUMMARY

This chapter discussed research methods of this study. The discussion involved qualitative research, a case study, research participants, research instruments, data collection and data analysis. It also discussed issues pertaining to qualitative research such as reliability, triangulation, validity and research ethics. The following chapter presents results of the pilot study, as well as the main study.

CHAPTER FIVE

RESULTS OF THE STUDY

5.1 INTRODUCTION

This chapter presents results from the analysis of the data of the pilot study, as well as the main study. The pilot study was conducted to test the suitability of the framework in data analysis. Errors that were displayed by students in all tasks were classified as interpretation, arbitrary, procedural, linear extrapolation and conceptual errors, as discussed below.

- **Interpretation errors:** These errors arise when students fail to interpret the nature of the problem correctly owing to over-generalisation of certain mathematical rules involved in the problem.
- **Arbitrary errors:** These errors arise when students behave arbitrarily and fail to take account of the constraints laid down in what is given.
- **Procedural errors:** In these errors students fail to carry out manipulations or algorithms although they understand concepts in the problem.
- **Linear extrapolation errors:** These errors happen through an over-generalisation of the property $f(a+b) = f(a) + f(b)$, which applies only when f is a linear function.
- **Conceptual errors:** These errors occur owing to failure to grasp the concepts involved in the problem or failure to appreciate the relationships involved in the problem.

5.2 RESULTS FROM THE PILOT STUDY

As indicated in chapter four there were twenty students who participated in the pilot study. This section presents results that were obtained from data analysis of the pilot study. It also covers a discussion of errors displayed by students in the three written tasks in the pilot study.

It further discusses results related to the students' work as gathered from audio and video recordings and students' responses from in-depth interviews. It ends with a frequency table showing students' errors in the three written tasks from the pilot study.

5.2.1 Interpretation errors displayed by students in the three tasks

Three students avoided a constant multiple rule. They used a product rule instead of a constant multiple rule. The constant multiple rule states that the derivative of a constant times a function, is equal to the constant times the derivative of the function. Constant Multiple Rule: If g is a differentiable function and c is a real number;

$f(x) = c \cdot g(x)$ then, $f'(x) = c \cdot g'(x)$.

One student added a coefficient with an index in the application of the power rule. For example, the student wrote that the derivative of $\ln(3x^3 + 2)$ is $\ln 6x^2$. Five students confused the product rule with the logarithmic rule. The following figure shows an example of the kind of the confusion between the product rule and a logarithmic function.

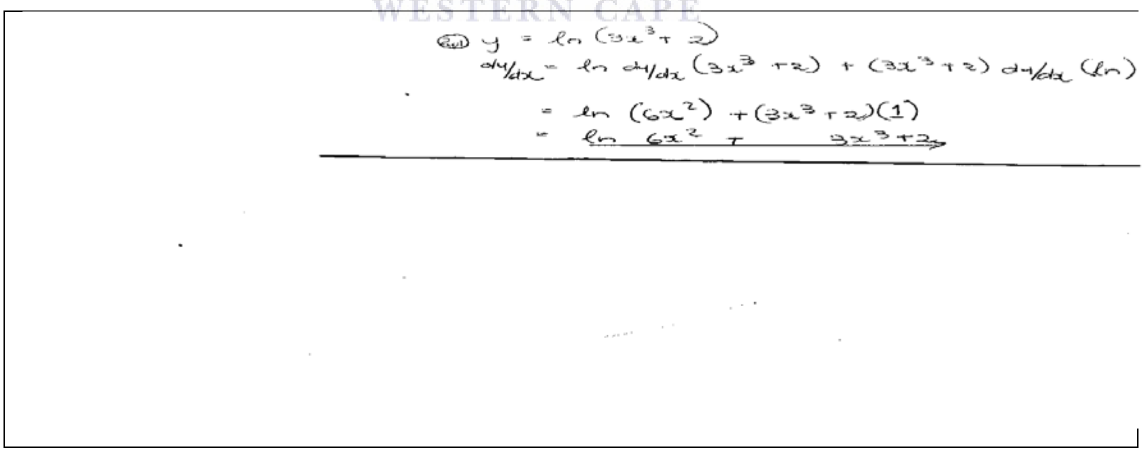

Figure 5.1 Example of an interpretation error

As shown in figure 5.1 the five students did not apply the rule which showed understanding of differentiating the logarithmic function $y = \ln(3x^3 + 2)$ as $\frac{dy}{dx} = \frac{1}{3x^3 + 2} \cdot 9x^2$ to obtain $\frac{dy}{dx} = \frac{9x^2}{3x^3 + 2}$ as the correct derivative.

Instead they applied the product rule which was inappropriate for this problem. They also differentiated $3x^3$ incorrectly as they wrote $6x^2$ as the derivative of $y = 3x^3 + 2$ instead of $9x^2 + 0$.

They also wrote 1 as the derivative of $\ln$. They also applied the second order derivative and differentiated one expression twice.

Two students did not know how to multiply $(3\ln r)$ by $(3r^2)$. One student used a wrong Leibniz's notation in the two functions. For example in differentiation of the two functions $y = r^3(3\ln r - 1)$ and $y = \frac{e^u}{1+u}$ he wrote $\frac{d}{dx}$ instead of $\frac{d}{dr}$ and $\frac{d}{du}$. Four students did not know how to interpret $\tan^2 x$ in order to apply the chain rule. They differentiated $\tan x^2$ as if it was $\tan^2 x$.

Thirteen students differentiated the middle term of $y = x^{\ln 3} + (\ln 3)^x + 3\ln x$ incorrectly. Three of these thirteen students applied the power rule. They wrote that the derivative of $y = (\ln 3)^x$ is $\frac{dy}{dx} = x(\ln 3)^{x-1}$. This is an interpretation error, which originates from over-generalisation of the power rule. Four of these thirteen students wrote that the derivative of $y = (\ln 3)^x$ is $\frac{dy}{dx} = (\ln 3)^x \ln 3$. Two of these thirteen students wrote that the derivative of $y = (\ln 3)^x$ is $\frac{dy}{dx} = (\ln 3)^x$.

This is an interpretation error which originated from an over-generalisation of $\frac{dy}{dx} = a^x \ln a$ as the derivative of $y = a^x$ and the derivative of $y = e^x$ is equal to $\frac{dy}{dx} = e^x$.

They also demonstrated confusion with regard to the properties of logarithms and differentiation of logarithmic functions. For example in the term $x^{\ln 3}$, they applied the property $\ln x^y = y \ln x$. The term $x^{\ln 3}$ requires the application of the power rule as the base is a variable and the index is a constant, and therefore the derivative of $y = x^{\ln 3}$ is $\ln 3x^{\ln 3-1}$. This error is an interpretation error, which originates from an over-generalisation of the logarithmic property $\ln x^y = y \ln x$. Four students confused the sum and the difference rule with the product rule. They also did not distinguish between $\cos ecx^3$ and $x^3 \cos ecx$.

5.2.2 Arbitrary errors displayed by students in the three tasks

One student transcribed the problem incorrectly. One student did not attempt to differentiate the third term of $y = x^{\ln 3} + (\ln 3)^x + 3\ln x$. One student did not write question 2. Another student did not write question 3. Two students did not write question 4. The students were required to differentiate the functions defined by:

2. $y = x^{\ln 3} + (\ln 3)^x + 3\ln x$

3. $y = \dfrac{\sin x}{1 + \cos x}$

4. $y = \cot 4x + \ln(\tan x)$

The above errors show that these students did not know how to differentiate these problems. Another student left a minus sign in the derivative of a cosecant function but showed the minus sign in her answer.

Her solution was as follows:

$$\frac{dy}{dx} = uv' + vu'$$

$$\frac{dy}{dx} = x^3 \cos ec(x^2+3)\cot(x^2+3)2x + \cos ec(x^2+3)3x^2$$

$$\frac{dy}{dx} = -2x^4 \cos ec(x^2+3)\cot(x^2+3) + 3x^2 \cos ec(x^2+3)$$

One student transcribed questions and left them without any attempt to differentiate.

One student omitted one term in her solution and rewrote $y = \cos^4(5x^2)$ as $y = \cos^4 x(5x)$. Another student rewrote $y = \cos^4(5x^2)$ as $y = [\cos^2(5x)]^4$. One student rewrote $y = \cos^4(5x^2)$ as $y = \cos(5x^2)^4$.

5.2.3 Procedural errors displayed by students in the three tasks

One student showed an error in differentiation as he wrote $\frac{dy}{dx} = 3x^2$ instead of $\frac{dy}{dx} = 3x^2 + 1$ as the derivative of $y = x^3 + x$. One student multiplied variables and constants incorrectly, for example, in multiplication of $2x^2 \cdot 3x^2$ she obtained $6x^2$ instead of $6x^4$. Five students failed to manipulate algebraic differentiation. Four students did not write the correct formula of the quotient rule, and they also demonstrated poor application of brackets. One student failed to factorise completely. Six students did not manipulate differentiation of trigonometric functions and logarithmic functions. They did not know correct procedures of differentiation although they understand concepts.

Six students showed a lack of closure as they failed to cancel in their last step in order to leave their answers in the simplest form. Four students demonstrated poor understanding of cancellation with regard to trigonometric identities and a lack of understanding of manipulation of trigonometric functions that are presented in fractional form.

Two students did not know derivatives of trigonometric functions such as sine and cosine functions and they did not manipulate multiplication of trigonometric functions.

5.2.4 Linear extrapolation errors displayed by students in the three tasks

Two students demonstrated linear extrapolation error as they multiplied an algebraic expression by $\ln$ and differentiated the expression by using the sum and difference rule.

Their error shows an over-generalisation of the distributive property as they treated the logarithmic function $\ln x$ as an ordinary variable.

The following figure shows an example of the kind of linear extrapolation error that students displayed in the differentiation of a logarithmic function.

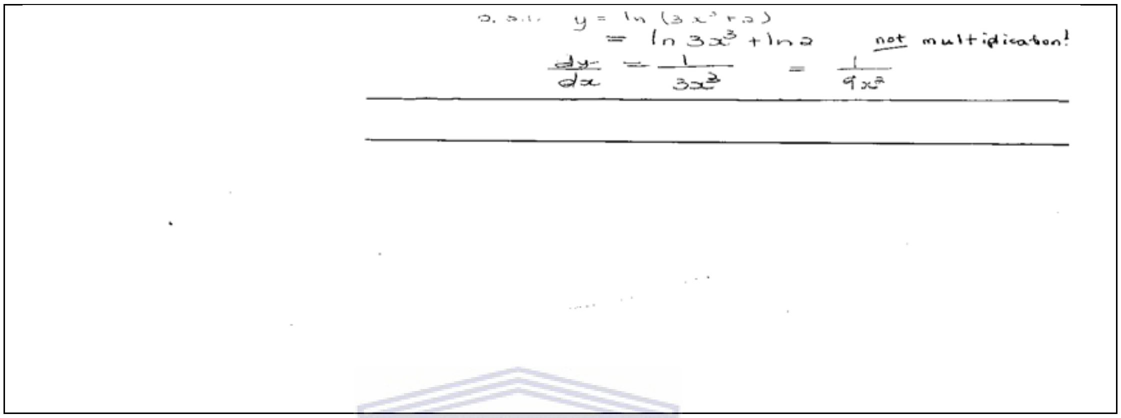

Figure 5.2: Example of a linear extrapolation error displayed by students in differentiation of a logarithmic function

5.2.5 Conceptual errors displayed by students in the three tasks

Three students did not know the constant multiple rule. They applied the product rule instead of the constant multiple rule. One student confused the appropriateness of the product rule and the chain rule. Two students ignored $\ln$, in $y = \ln(3x^2 + 2)$ and they only differentiated $y = 3x^2 + 2$.

Another two students showed poor understanding of Leibniz notation such as $\frac{d}{dx}$. Another two of the twenty students revealed that they did not know that division by zero is not allowed. For example, in differentiation of x^2, using first principles, they failed to simplify $\lim_{h \to 0} \frac{2xh + h^2}{h}$ by taking h in the numerator as a common factor and cancelled. Instead, they applied direct substitution where they divided by zero.

Two of the twenty students did not see the difference in multiplication of two algebraic terms and multiplication of a logarithmic function and an algebraic term.

Three students did not notice that the nature of the middle term of $y = x^{\ln 3} + (\ln 3)^x + 3\ln x$ does not require the application of the power rule. In this case the base is the constant and the index is the variable and, therefore, the power rule is not appropriate. This is a conceptual error, which originated from the over-generalisation of the power rule. Two students wrote that the derivative of $y = (\ln 3)^x$ is $\frac{dy}{dx} = (\ln 3)^x$. This is a conceptual error, which originated from over-generalisation of the derivative of $y = e^x$ is $\frac{dy}{dx} = e^x$. Four students did not understand that e^2 is a constant function. This is a conceptual error that originated from the over-generalisation that a variable represents an unknown.

Table 5.1 below shows errors that were revealed in the three tasks of the pilot study.

Students' errors in the three tasks of the pilot study							
Students' written work in Task 1		No. of interpretation errors	No. of arbitrary errors	No. of procedural errors	No. of linear extrapolation errors	No. of conceptual errors	No. of no errors
Question	1	0	4	7	0	0	8
	2	4	1	6	2	3	6
	3	3	1	8	0	3	8
Students' written work in Task 2							
Question	1	0	0	0	0	2	14
	2	2	0	2	0	2	12
	3	1	1	1	0	2	14
	4	4	0	4	0	0	12
Students' written work in Task 3							
Question	1	0	0	10	0	0	6
	2	8	2	3	0	5	2
	3	2	1	2	0	0	10
	4	0	0	3	**0**	**0**	12
	5	1	4	3	0	0	7
	6	1	5	4	0	4	3
	7	2	2	3	0	0	10
Total number of errors		28	21	56	2	21	124

Table 5.1: Frequency table of students' errors in the three tasks of the pilot study

From the summary above it can be seen that most errors were procedural. This was followed by interpretation errors. The students showed equal number of arbitrary and conceptual errors. The least errors were linear extrapolation.

5.2.6 Results related to students' work as reflected from the audio and video recordings

In audio and video recordings students demonstrated errors that were already identified in their written work. Students demonstrated different approaches in their calculations and also expressed their opinions about certain techniques of differentiation. They also asked questions for clarity and in-depth understanding of differentiation rules. For example one student asked: "How do we see that a problem requires application of the chain rule?" The researcher referred the question to the class. Another student's response was that we should use the chain rule when we cannot apply other rules. No student in the class could provide a satisfactory answer for this question. The researcher as the lecturer referred the students to the self-study learning material which describes the chain rule. The lecturer also explained the chain rule as a composite function rule using the following four examples to illustrate that:

1. $y = x$;

2. $y = (x+2)^3$;

3. $y = (3x^4 + 1)^3$;

4. $y = \cos ec4x$

Another student asked the difference between the product rule and the sum rule. The researcher's response was to find out from the same student the meaning of a sum and a product. The student's response showed that she knew the sum and product as the answer of addition and multiplication, respectively. The researcher explained the two rules, namely the sum and difference rule, and the product rule by means of examples.

5.2.7 Students' responses from in-depth interviews

Six students were interviewed. Their responses showed that five of them studied the first principles of differentiation as part of their high school curriculum.

Four also studied the power rule. Only one among the interviewees studied first principles, the power rule, graphs, and the application of differentiation in high school studies. Five students claimed that they did not work with fractions in their studies of the first principles of differentiation in high school level.

The students' responses indicated that the use of classroom discussions in learning activities had assisted them to develop regular practice to learn differential calculus. They also learnt to practise in groups and learnt from their mistakes and the mistakes of other students.

Students' responses showed that they enter universities with instrumental understanding as a key way of learning mathematics. The fact that they are satisfied with a table of derivatives without showing proof shows that they are not familiar with tracing the origin of differentiation rules.

They highlighted that an advantage of the use of structural discussion of learning activities was to practise examples where students were able to consult the solutions when they were unable to solve them. They explained that their understanding had developed through classroom discussions with other students and interaction with the lecturer.

5.2.8 Outcome of the pilot study

The pilot study indicated that the framework developed for the classification of errors is adequate to analyse students' written work to identify errors that are displayed by students.

The pilot study also assisted the researcher to change the focus of observation to overt behaviour of students rather than looking for errors as these can be observed adequately in students' written activities.

The interviews were also suitable for the study to be taken further to the main study. The following section presents results that were obtained from data analysis of the main study.

5.3 RESULTS FROM THE MAIN STUDY

There were thirty participants who participated in the data collection of the main study. The first assessment task was to use the first principles of differentiation to find the derivative of $\cos x$.

5.3.1 Arbitrary errors displayed by students in Task1

Five students only wrote the formula of the first principles of differentiation to find the derivative of $\cos x$. One student skipped some steps in her solutions. For example, she did not show that $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0}\frac{\sin h}{h} = 1$. She skipped to $\cos x(0) - \sin x(1) = -\sin x$.

5.3.2 Linear extrapolation errors displayed by students in Task1

This section presents linear extrapolation errors displayed by a student in Task 1. One in twenty students multiplied in $\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$ to obtain $\lim_{h\to 0}\frac{\cos x + \cos h - \cos(x)}{h}$.

5.3.3 Conceptual errors displayed by students in Task1

This section presents conceptual errors displayed by students in Task 1. For example, one in twenty students did not manipulate

$\lim_{h\to 0}\frac{-\sin x \sin h}{h} = -\sin x \lim_{h\to 0}\frac{\sin h}{h} = -\sin x \cdot 1 = -\sin x$. She showed a problem in understanding of special limits.

Two students wrote incorrect compound angle formulae. For example, they wrote $\cos(A+B) = \cos A \cos B + \sin A \sin B$.

Table 5.2 below shows errors displayed by students in Task 1.

Students' errors in task 1					
Students' written work in Task1		No, of arbitrary errors	No. of linear extrapolation errors	No. of conceptual errors	No errors
Question	1	6	1	3	20

Table 5.2: Frequency table of students' errors in Task1

5.4 RESULTS FROM ANALYSIS OF STUDENTS' SECOND ASSESSMENT TASK (TASK 2)

5.4.1 Interpretation errors displayed by students in Task 2

One student demonstrated poor understanding of the constant multiple rule in differentiation of $y = 3x^4$. For example, her solution was as follows:

$$\frac{dy}{dx} = 3\frac{dy}{dx}x^4 + x^4\frac{dy}{dx}3$$

$$\frac{dy}{dx} = 3(4x^3) + x^4 \cdot 3$$

$$\frac{dy}{dx} = 12x^3$$

If this method was followed accurately it would come to the correct answer as the derivative of $3 \text{ is } 0$, but this method is laborious and superfluous and hence the more economical rule is the constant multiple rule.

Four students treated $y = 4x^{3/2}$ as a composite function of a trigonometric function. They applied the constant multiple rule and the power rule to obtain $\frac{dy}{dx} = 6x^{1/2}$, and then they differentiated $y = 4x$ again to obtain $\frac{dy}{dx} = 4$. This error originates from an over-generalisation of the differentiation of a composite function of trigonometric functions, which applies only on trigonometric functions and not on algebraic terms such as $4x$.

One student wrote $6x^{1/2}$ as equal to $\sqrt{6x}$ and obtained $y = \sqrt{6x} \cdot \sin x$, as is clear from Figure 5.3 below, then they differentiated $y = \sqrt{6x} \sin x$ as the original problem. The following figure shows an example of the student who demonstrated an interpretation error in differentiation of $y = 4x^{3/2} \sin x$.

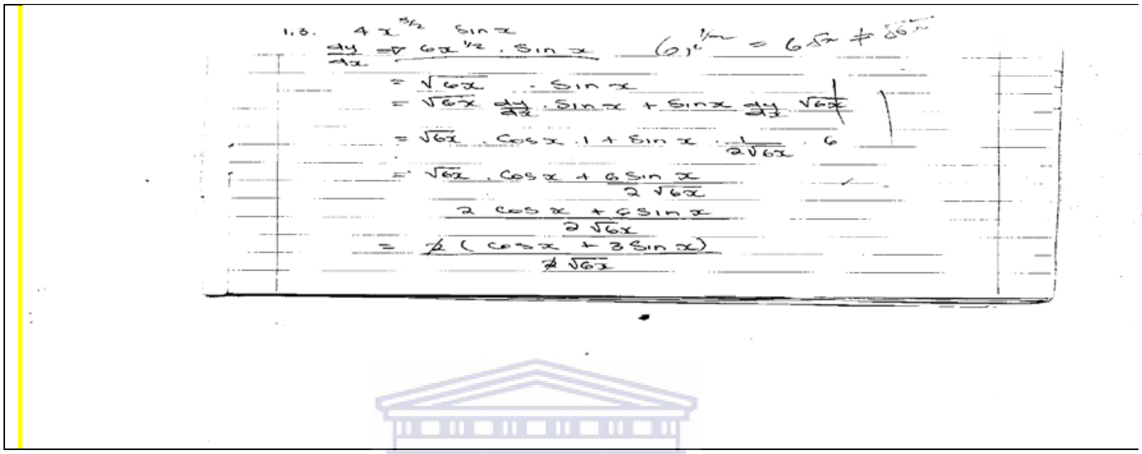

Figure 5.3: Example of the student who demonstrated an interpretation error in differentiation of $y = 4x^{3/2} \sin x$

One student differentiated $y = 4x^{3/2}$ to obtain $\frac{dy}{dx} = 6x^{1/2}$ and differentiated $y = x^{3/2}$ again to obtain $\frac{dy}{dx} = \frac{3}{2} x^{1/2}$. This error originates from an over-generalisation of the differentiation of the composite function of trigonometric functions which does not apply in algebraic terms. She also wrote that the derivative of $y = \sec^2 x^4$ is $\frac{dy}{dx} = \tan(x^4) 4x^3$. This originates from the fact that the derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$ and from an over-generalisation of the symmetric property of equality: if $a = b$ then $b = a$. One students failed to apply the sum and the chain rule to differentiate $y = \sin 7x + \ln 5x$ instead they applied the product rule. Four students differentiated $y = \tan^3 \sqrt{\cot 7x}$ as a product of two functions. As a result, they applied the product rule instead of the chain rule.

5.4.2 Arbitrary errors displayed by students in Task 2

Two students wrote that the derivative of $y = \tan x \cos ecx$ is

$$\frac{dy}{dx} = \sec^2 x \cdot -\cos ecx \cot x$$
$$\frac{dy}{dx} = -\sec^2 x \cos ecx \cot x$$

One of these two students wrote that the derivative of $y = 3x^4$ is $\frac{dy}{dx} = 12x^3 \cdot 3$.

One student transcribed the problem incorrectly. Instead of writing $y = 4x^{3/2} \sin x$, she wrote $\frac{dy}{dx} = 4x^{3/4} \sin x$. Two students differentiated $y = 4x^{3/2}$ first to obtain $\frac{dy}{dx} = 6x^{1/2} \sin x$. Then they differentiated $y = 6x^{1/2} \sin x$ as the original problem.

One student differentiated $y = 4x^{3/2} \sin x$ as $\frac{dy}{dx} 4x^{3/2} \sin x \cdot 6x^{1/2} \cos x$, and wrote $6x^{1/2}$ as equal to $\sqrt{6x}$, and hence obtained $y = \sqrt{6x} \cdot \sin x$. Then she differentiated $y = \sqrt{6x} \sin x$ as the original problem.

Four students showed poor understanding of the differentiation rules of trigonometric functions to prove that the derivative of $y = \sec x$ is $\frac{dy}{dx} = \sec x \tan x$. They failed to apply the quotient rule and as a result they applied the power rule incorrectly. Hence they claimed that $\frac{1}{\sin x} \cdot \frac{-\cos x}{-\sin x} = \sec x \tan x$, which is incorrect. They also showed illogical steps to prove that $\frac{d}{dx}\left[\frac{1}{\sin x}\right] = -\cos ecx \cot x$.

For example, one solution was as follows:

$$\frac{d}{dx}\left[\frac{1}{\sin x}\right]\cdot(\cos x)^{-1}$$

$$=\frac{1}{\sin x}\cdot -1(\cos x)^{-2}$$

$$=\frac{1}{\sin x}\cdot\frac{-}{\cos^2 x}$$

$$=\frac{\sin x}{\cos^2 x}$$

$$=\frac{\sin x}{\cos x}\cdot\frac{-1}{\cos x}$$

$$=-\cos ecx\cot x$$

One student wrote that $\frac{1}{\sin x}$ is equal to $-\cos ec^2 x$. Another one student left out a minus sign in their last step.

Six students did not realise that the problem $y=\sec^2 x^4\cot^3 x^4$ requires the application of the product rule and the chain rule.

Five students did not understand the application of the chain rule to differentiate the sum of a trigonometric function and a natural logarithmic function, which is in the form of a radical expression such as $y=\sqrt{\sin(7x)+\ln(5x)}$ .One student transcribed $y=\tan^3\sqrt{\cot 7x}$ as it is, and left it without any attempt to calculate. One student wrote that the derivative of $y=\tan^3\sqrt{\cot(7x)}$ is $\frac{dy}{dx}=3\tan\cot^{1/2}7x^{1/2}$.

5.4.3 Procedural errors displayed by students in Task 2

One student displayed cancellation errors. She did not write one as the outcome of cancellation. In the last step she wrote $\tan x\cdot\cot x$ as $\frac{\sin x}{\cos x}\cdot\frac{\cos x}{\sin x}$ and cancelled, which was correct, but she did not write $1-\sec^2 x$; she only wrote $-\sec^2 x$ as shown in Figure 5.4.

The following figure shows an example of the student who demonstrated a procedural error in the differentiation of $y = 3x^4 - \tan x \cos ecx$.

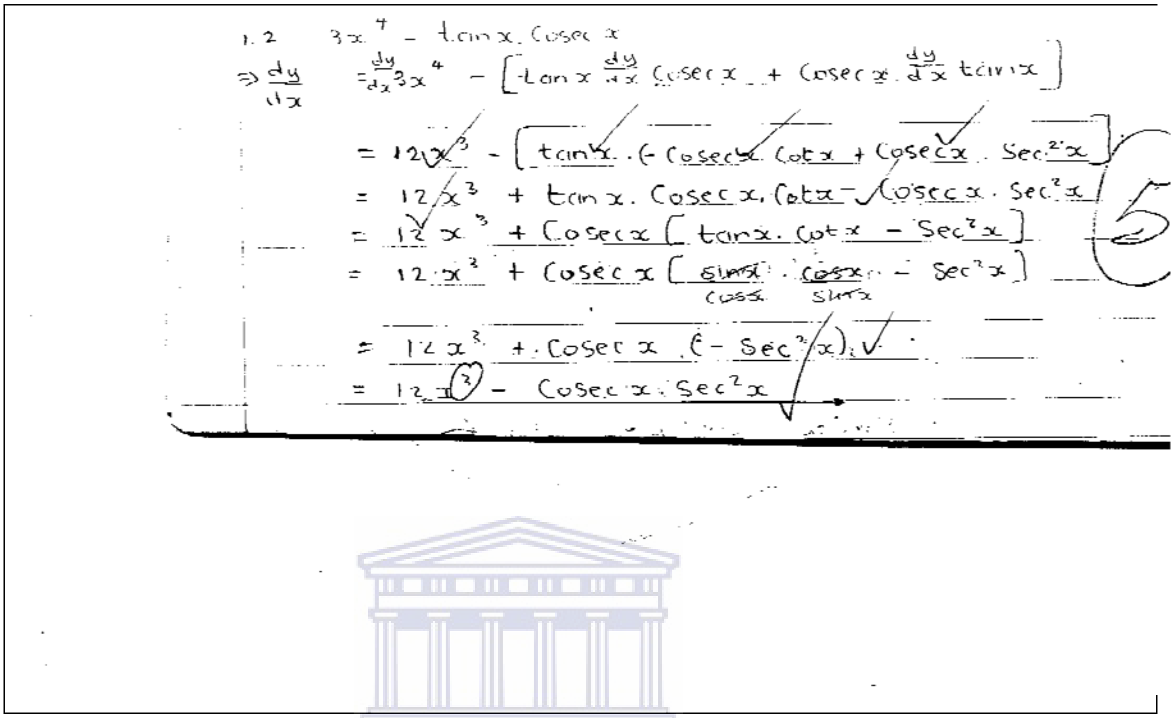

Figure 5.4: Example of the student who demonstrated a procedural error in differentiation of $y = 3x^4 - \tan x \cos ecx$**.**

Nine students did not change a plus sign when multiplied by a minus sign. One student made an error of differentiating $y = 3x^4$ as $\frac{dy}{dx} = 12x$ instead of $\frac{dy}{dx} = 12x^3$. Two students wrote that the derivative of $y = \cos ecx$ is $\frac{dy}{dx} = \cos ecx \cot x$ instead of $\frac{dy}{dx} = -\cos ecx \cot x$. They also did not use brackets to enable them to multiply by a minus sign.

Three students simplified $4x^{3/2} \sin x$ incorrectly. They took out $2\sqrt{x}$ as the highest common factor, which is unnecessary. They tried to simplify it further, but they did not remove the highest common factor.

They removed 2 as a common factor, which was not the highest common factor. Six students indicated that they did not understand the subtraction of fractions. Their solutions indicated that these students applied the power rule to obtain 6, but failed to obtain the correct answer in subtraction of 1 from $\frac{3}{2}$ to obtain $\frac{1}{2}$. One student failed to manipulate the multiplication of $0 \cdot \cos x$ correctly. Instead of obtaining 0 as an answer, she obtained 1. One student in differentiation of $y = \sec^2 x^4 \cot^3 x^4$ failed to add like terms in simplification. Instead, she tried to remove the highest common factor. Three students wrote that the derivative of $y = x^4$ is $\frac{dy}{dx} = 4x$. Eight students failed to apply the power rule to differentiate $y = \cot^3 x^4$ to obtain $\frac{dy}{dx} = (3\cot^2 x^4)(-\cos ec^2 x^4)(4x^3)$.

Sixteen students failed to simplify the differentiation of $y = \sqrt{\sin(7x) + \ln(5x)}$ to reach the correct solution. They did not cancel correctly. They also did not apply the lowest common denominator (LCD) to simplify these fractional trigonometric functions and a fractional algebraic term. Their solutions showed that they multiplied the numerators and left the denominators. They also did not add the fractions correctly. Three students differentiated the radical expression incorrectly. They did not understand that by taking the denominator up as a numerator would leave 2 as the denominator, since the root only affects $\sin(7x) + \ln(5x)$. Twelve students showed poor understanding of multiplication of trigonometric fractions. They also showed a lack of closure in the differentiation of $y = \tan^3 \sqrt{\cot 7x}$ owing to poor simplification of fractional trigonometric functions.

5.4.4 Linear extrapolation errors displayed by students in Task 2

One student split $\sqrt{\sin(7x) + \ln(5x)}$ into $\sqrt{\sin(7x)} + \sqrt{\ln(5x)}$. This error originated from $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, which applies in multiplication of real numbers that are written in a radical form and does not apply in addition and subtraction of real numbers that are written in a radical form.

5.4.5 Conceptual errors displayed by students in Task 2

Two students did not know that the derivatives of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$. One student wrote $\frac{dy}{dx} = -\sec^2 x$ as the derivative of $y = \tan x$ and another one wrote $\frac{dy}{dx} = \sec x \tan x$ as the derivative of $y = \tan x$. One student converted $\sec x$ into $\frac{1}{\sin x}$. He applied the quotient rule and as a result he reached an incorrect solution as $\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$.

One student used incorrect identities as she wrote $\cot x = \frac{\sin x}{\cos x}$ and $\cos ecx = \frac{1}{\cos x}$, which were incorrect as the correct identities are $\cot x = \frac{\cos x}{\sin x}$ and $\cos ecx = \frac{1}{\sin x}$, respectively. Eight students did not apply trigonometric identities in simplification of trigonometric functions, as shown in Figure 5.5.

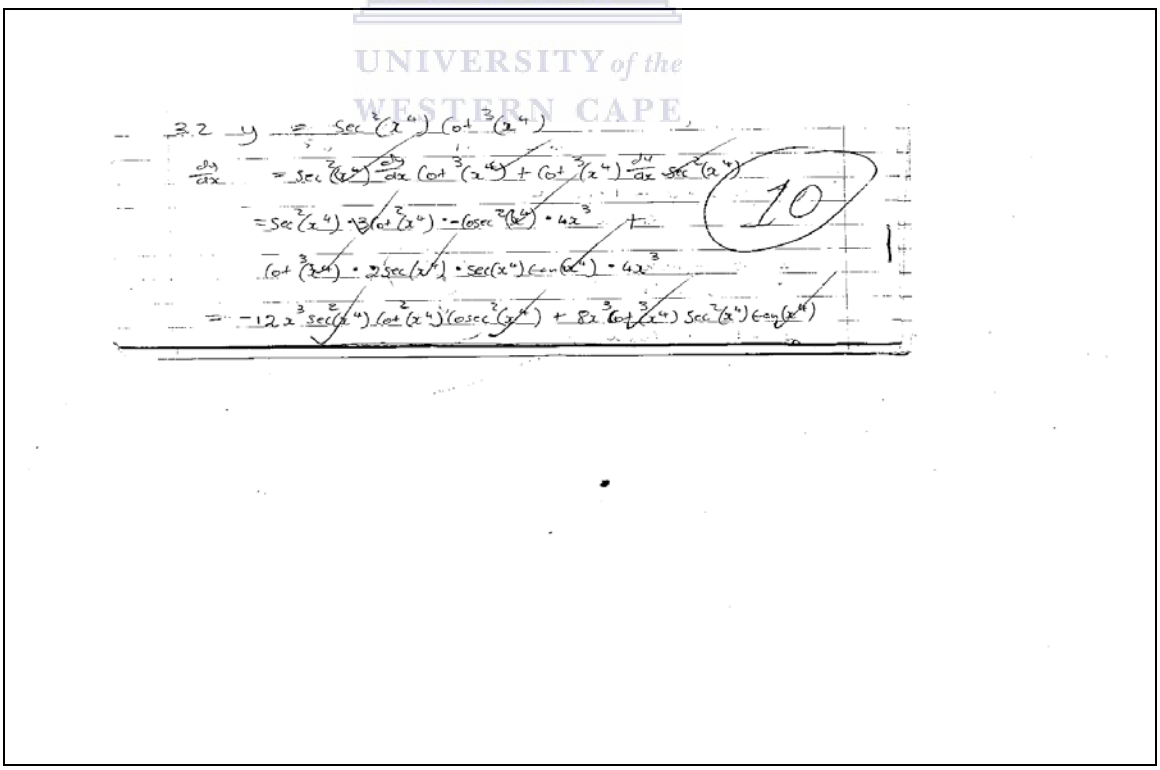

Figure 5.5 Example of a student who did not substitute appropriate trigonometric identities in simplification of trigonometric functions.

They wrote the following as their solutions:

$\frac{dy}{dx} = 2\sec(x^4)\ \sec\tan(x^4)$ as the derivative of $y = \sec^2 x^4$; $\frac{dy}{dx} = 3\cot^2(x^4) - \cos ecx(x^4)$

$\frac{dy}{dx} = 3\cot^2(x^4) - \cos ecx^4 \cot x^4 \cdot \cot x^4 \cdot 4x^3$.

Table 5.3 below shows errors that were displayed by students in Task 2 in the main study

Students' errors in Task 2							
Students' written work in Task 2		No. of interpretation errors	No. of arbitrary errors	No. of procedural errors	No. of linear extrapolation errors	No. of conceptual errors	No errors
Question	1.1	1	2	14	0	6	1
	1.2	6	5	13	0	0	7
Question	2.1	0	1	2	0	1	24
	2.2	0	2	0	0	1	27
	2.3	0	3	1	0	2	25
Question	3.1	1	7	12	0	8	0
	3.2	1	5	16	1	0	0
	3.3	4	2	12	0	0	12
Total number of errors		13	27	68	1	18	96

Table 5.3: Students' errors in Task 2

5.5 RESULTS WITH REGARD TO THE STUDENTS' THIRD FORMATIVE ASSESSMENT TASK (TASK 3)

Errors displayed by students in Task 3 were classified into four categories, namely interpretation, arbitrary, procedural and conceptual errors, as discussed earlier in 5.1.

5.5.1 Interpretation errors displayed by students in Task 3

Fourteen students fused two functions into one function. They treated $x^3 e^{2x+3}$ as the first function and $\sqrt{\cos x}$ as the second function.

One student did not know the derivative of $y = \sec^2 x$, as a result she wrote that the derivative of $y = \sec^2 x$ is $\frac{dy}{dx} = \tan x$.

This error originated from over-generalisation of the symmetric property, which states that for any quantities a and b, if $a = b$, then $b = a$. This is not the case in derivatives.

One student wrote that the derivative of $y = -(\sec^2 x)^2 \sec^2 x$ is $\frac{dy}{dx} = -\tan^2 x$ and also wrote that the derivative of $y = \sec^2 x$ is $\frac{dy}{dx} = (\sec x \tan x)^2$. This originated from the algebraic over-generalisation such as if $a = b$ then $a^2 = b^2$.

5.5.2 Arbitrary errors displayed by students in Task 3

Two students did not show any logic in their differentiation of $y = x^3.e^{2x+3}\sqrt{\cos x}$. They did not apply logarithmic differentiation correctly to obtain $\ln y = \ln x^3 + \ln e^{2x+3} + \ln\sqrt{\cos x}$. Two students only transcribed the problem without any attempt to do calculations.

Two students transcribed the problem incorrectly and also showed illogical steps in their calculations of $y = \sec^2 xe^{-\tan^2 x}$. The first one transcribed the problem as $y = \sec^2 xe^{-\tan x}$ instead of $y = \sec^2 xe^{-\tan^2 x}$. The second one transcribed the problem as $y = \sec^2 xe^{-\tan 2x}$ instead of $y = \sec^2 xe^{-\tan^2 x}$. One student left $(2x+3)\ln e$ without differentiating it.

5.5.3 Procedural errors displayed by students in Task 3

One student showed poor understanding of identities as he wrote $\cot\theta = \tan\theta$. As a result he substituted $\cot\theta$ with $\tan\theta$. One student failed to multiply radical trigonometric functions correctly. He manipulated $\frac{1}{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} \cdot \frac{-\sin x}{1}$ incorrectly. As a result he obtained $\frac{-\sin x}{\sqrt{\cos x}}$ instead of $\frac{-\sin x}{2\cos x}$. One student failed to apply the LCD correctly in $y = x^3 e^{2x+3}\sqrt{\cos x}$. Two students differentiated $y = 2x + 3\ln e$ incorrectly. They did not apply the sum rule.

They also treated $y = 2x + 3\ln e$ as if it required the application of the product rule, treating $2x + 3$ as the first function and $\ln e$ as the second function. They wrote that the derivative of $(2x+3)\ln e$ is $2\ln e$.

5.5.4 Conceptual errors displayed by students in Task 3

Eight students did not know that $\ln e = 1$, hence they applied the product rule to differentiate $\tan^2 x \ln e$. They also did not understand the concept of a natural logarithm ($\ln x$). Hence they differentiated $\frac{1}{2}\ln\cos x$ as $\frac{1}{2}\ln\cos x \cdot -\sin x + \frac{\frac{1}{2} - \sin x}{\cos x}$. One student did not know that $\sqrt{\cos x} = (\cos x)^{\frac{1}{2}} \neq \cos x^{\frac{1}{2}}$, and she differentiated $\ln\sqrt{\cos x}$ incorrectly. For example, her solution was as follows:

$$\frac{d}{dx}(\ln\sqrt{\cos x})$$

$$= \frac{1}{\sqrt{\cos x}} \cdot (-\sin x).$$

Two students showed errors in simplification of trigonometric functions as they did not apply LCD correctly.

One student showed errors in differentiation of trigonometric functions when integrated with logarithmic functions. His solution indicated that he did not distinguish between the power rule and the logarithmic differentiation. He differentiated $\ln x^3$ as $3\ln x^2$ instead of $\frac{3}{x}$. He wrote that the derivative of $y = \ln e^{2x+3}$ is $\frac{dy}{dx} = 2x + 3\ln e \cdot 1$.

They also differentiated $y = \ln\sqrt{\cos x}$ as $\frac{1}{2\sqrt{\cos x}} \cdot \frac{-\sin x}{1}$ instead of $\frac{1}{\sqrt{\cos x}} \frac{1}{2\sqrt{\cos x}} \cdot -\sin x$. One student did not substitute $\frac{-\sin x}{2\cos x}$ with $\frac{-\tan x}{2}$.

One student differentiated the following function as: $y = \ln \sec^2 x = 2\ln \sec x \cdot \frac{1}{\sec x} \cdot \sec x \tan x = 2\ln \sec x \tan x$ instead of writing $\ln \sec^2 x$ as $2\ln \sec x$ first and then differentiate $2\ln \sec x$ as $2 \cdot \frac{1}{\sec x} \cdot \sec x \tan x = 2\tan x$.

One student showed poor understanding of the chain rule as she differentiated $\ln \sec^2 x$ as $\ln \sec^2 x \cdot 2\sec x \cdot \sec x \tan x$. She also differentiated $-\tan 2x$ as $\sec^2 x(-\tan 2x)$ instead of $-\sec^2 x \cdot 2 = -2\sec^2 x$.

One student differentiated $\ln \sec^2 x + \ln e^{-\tan^2 x}$ incorrectly as she wrote that $\frac{y'}{y} = \frac{1}{\sec^2 x} \cdot 2\sec x \cdot \cos ecx \tan x + (-\tan^2 x)\ln e$ is the derivative of $\ln \sec^2 x + \ln e^{-\tan^2 x}$. This solution indicated that this student did not know that the derivative of $\sec x$ is $\sec x \tan x$. One student wrote $\frac{1}{\sin x}$ as the derivative of $\sec x$. One student wrote $\sec x$ as the derivative of $\tan x$, while another one student differentiated $-\tan^2 x$ as $-2\tan x \cdot -\sec^2 x$. The error is to write a minus sign in front of $\sec^2 x$. One student showed poor understanding of the chain rule as they differentiated $-\tan^2 x$ as $-\tan^2 x \sec^2 x$ instead of $-2\tan x \sec^2 x$. One student wrote that the derivative of $\sec^2 x$ is $-\cos ec^2 x$.

One student wrote that the derivative of $-\tan^2 x$ is $-\tan^2 x \cdot 2\tan x \sec^2 x$. One student wrote that the derivative of $\tan x$ is $\cot x$. Two students wrote that the derivative of $-\tan^2 x$ is $\ln \sec x \cdot 0 - \sec^2 x$. Two students did not know how to differentiate a composite function from a trigonometric function such as $-\tan^2 x$. One of these two students wrote $-\sec x$ as the derivative of $-\tan^2 x$. Another one of these two students wrote $-(\sec^2 x)^2 \sec^2 x$ as the derivative of $-\tan^2 x$, and also wrote that the derivative of $\sec^2 x$ is $(\sec x \tan x)^2$. This originated from the algebraic over-generalisation that if $a = b$ then $a^2 = b^2$.

Table 5.4 below shows errors displayed by students in Task 3.

Students' errors in each question of Task 3						
Students' written work in Task 3		No. of interpretation errors	No. of arbitrary errors	No. of procedural errors	No. of conceptual errors	No errors
Question	1	0	0	1	01	0
Question	2	15	6	4	13	4
Question	3	02	3	0	12	6
Total number of errors		17	9	5	26	10

Table 5.4: Students' errors in Task 3

5.6 RESULTS RELATED TO STUDENTS' WORK AS REFLECTED FROM THE AUDIO AND VIDEO RECORDINGS

In the audio and video recordings three students showed their solutions on a whiteboard, whilst representing their groups. The students demonstrated errors that were already identified from students' written work. The lecturer intervened by explaining appropriate procedures, describing concepts that were interpreted incorrectly as the students expressed themselves explaining their understanding of the derivatives of trigonometric functions.

One student argued that her understanding is that logarithmic differentiation is applied only when the base of the function is in the form of a variable, and the index is also a variable. In the case of $y = x^3 e^{2x+3}\sqrt{\cos x}$, all the terms are not in a transcendental form. This student's question indicated that she confused the application of the chain rule with the application of the logarithmic differentiation rule. This problem contains three functions whilst the students were familiar with differentiation of two functions which make it easy for them to apply the product rule.

One student questioned why we do not apply the power rule to differentiate $y = \ln x^3$. His question might be asked for clarity purposes or else it might show that this student did not know the difference between the power rule and logarithmic differentiation. This question might symbolise poor conceptualisation.

The lecturer explained that in the case of a natural logarithmic function we do not apply the power rule. In the explanation the lecturer made examples of x^3 and $\ln x^3$ showing techniques of differentiating these two different functions. He further explained that the first function x^3 requires the application of the power rule with its derivative equal to $3x^2$ and the second function $\ln x^3$ requires application of logarithmic differentiation to obtain its derivative which is equal to $\frac{3}{x}$.This intervention assisted the student working on audio and video recordings to rectify her mistake. The lecturer also intervened by correcting errors as he explained the appropriate procedure involved in cancelling trigonometric functions under addition.

One student raised a question, which reflected a conceptual error. He wanted to know whether it is appropriate to substitute $\sec^2 x$ with $\tan x$.

This question showed that the student did not understand that although the derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$, the derivative of $\sec^2 x$ is not $\tan x$. The lecturer explained the appropriate procedure of obtaining the derivative of $y = \sec^2 x$. One student suggested further simplification of

$y' = [2\tan x - 2\tan^2 x \sec x]\sec^2 xe^{-\tan^2 x}$ to $y' = 2\tan x[1 - \sec^2 x]\sec^2 xe^{-\tan^2 x}$.The lecturer explained that to remove a common factor would be an undesirable closure, as it is the opposite of simplification.

One student requested the use of the product rule to differentiate this problem. The same student attempted the problem by using the product rule in audio and video recorded observations. The student showed that she had poor understanding of the standard derivatives.

5.7 STUDENTS' RESPONSES FROM IN-DEPTH INTERVIEWS

The interviewees' responses regarding what they learned about differentiation in high school revealed that all of them studied limits and the first principles of differentiation.

All the interviewees stated that their learning of limits and the first principles of differentiation focused on the use of real numbers such as 0;1; 2; 3. They claimed that they did not study the application of infinity.

The interviewees explained that their high school learning of differentiation was a basis for what they learnt in an extended curriculum programme. From their responses the interviewer noticed that they entered an extended curriculum programme with different levels of understanding with regard to differentiation. Eight interviewees stated that they only studied the limits and the first principles of differentiation, whereas five interviewees claimed that they also studied differentiation rules such as sum and constant rules.

One interviewee was unable to recall what she did in high school, as she studied high school mathematics in Afrikaans. As a result she could not translate the Afrikaans mathematical concepts into English which is a language of learning and teaching at this University of Technology.

All the interviewees revealed that they did not use the application of the first principles of differentiation to prove standard derivatives of trigonometric functions. They clarified that they only learnt the application of the first principles of differentiation in algebraic functions.

The interviewees' responses regarding the fact that learning material supplementing their study guides contributed to their understanding of differentiation revealed that they gained a deep understanding through the use of the learning material. They claimed that the material had a variety of questions, which kept them practising. The material also had detailed solutions that assisted them to trace their steps when they made certain mistakes in their solutions. They also added that it assisted them to be able to work as individuals when they were alone at home.

One student, Esi (pseudonym), claimed that “the material assists us as we get questions first and try to solve them at home and later the lecturer gives us solutions. In that way we see our mistakes and gain understanding”.

Twelve interviewees stated that collaboration contributed fruitfully to their learning. They claimed that in groups students brought their expertise and as a result they helped one another as they corrected their mistakes during their process of practicing differentiation. They also said that in groups some students asked questions, which others may not have thought of, hence all the group members would benefit. One student claimed that in her case she gained much understanding when she explained a mathematical problem on her own.

She stated that the more students asked questions for clarity, the more she understood through discussion and explanation by other group members.

Two interviewees showed that they prefer to work as individuals as opposed to groups, but that they also consulted others when they were unsure. They claimed that as individuals, they used their pace, as well as solutions from the learning material to deal with their misunderstandings. They also reworked certain problems until they gained understanding.

All the interviewees confirmed that classroom discussion assisted them to understand the derivatives of the trigonometric functions. One student Sama (pseudonym) claimed that “classroom interaction contributes to them as the lecturer send us to the whiteboard that help us to keep on practicing as we do not know when we are going to be chosen to go to the whiteboard”.

They also claimed that class discussions assisted them and that the practical experience of doing calculations on the whiteboard was useful. They also highlighted that when they made mistakes on the whiteboard the lecturer intervened immediately to correct them, which helped them because of the immediate feedback. They claimed that as they also questioned the students doing the calculations on the whiteboard, it assisted them to gain a deep understanding of differentiation.

They also claimed that class discussion assisted them because at times students explained the issue better than the lecturer. They supported this by saying that at times the students brought different approaches to a problem, which were not taught by the lecturer. Hence they gained a deep understanding.

The interviewees' responses regarding the difficult part of their learning differentiation indicated that many of them encountered a problem in understanding of logarithmic differentiation. Three interviewees pointed out the quotient rule and the chain rule as their difficult sections. Four interviewees indicated that they did not know when to apply the logarithmic differentiation or the product rule. One claimed that she did not know logarithmic differentiation because she was absent from class on the day that it was presented.

5.8 SUMMARY

This chapter discussed the results of the analysis of the data that was reported without interpretation. The following final chapter deals with the interpretation and discussion of the results. It also presents the conclusion of the study and recommendations for further research avenues.

CHAPTER SIX

SUMMARY OF RESULTS, RECOMMENDATIONS AND CONCLUSION

6.1 INTRODUCTION

The aim of this study was to explore errors that were displayed in the learning of derivatives of trigonometric functions by students who are registered for mathematics in an extended curriculum programme. The significance of this study was to identify errors and find ways of social interactions among students and lecturers that can help students to gain understanding in their learning of derivatives of trigonometric functions. This chapter provides a summary of the results, limitations, significance, avenues for further research, recommendations and a conclusion.

This chapter also reflects and summarises the results which were presented in Chapter five. The results were presented in five categories or errors, namely interpretation, arbitrary, procedural, linear extrapolation and conceptual errors. This chapter also discusses students' responses from the audio and video recordings and in-depth interviews. The following section relates the results of the main research questions of the study.

6.2 Relating the results to the first research question

The first research question of the study was:

What kind of errors that are displayed by students in their learning of derivatives of trigonometric functions?

6.2.1 Interpretation errors displayed by students in three written tasks.

Few students displayed interpretation errors that were based on poor understanding of concepts. For instance, when differentiating, they treated constants such as π ; e and $\ln 3$ as variables. Students from school level have experience of learning numerals such as 1; 2; 3... As a result they are not familiar with constants such as π ; e and $\ln 3$.

Students were confused about what to do with the index and a coefficient, and instead of multiplying a coefficient with an index in the power rule, they added them. This might originate from the laws of indices "in the case of multiplication if the bases are the same we take one base and add the indices". For example the multiplication law of indices says that symbols $x^m \cdot x^n = x^{m+n}$. This implies that $x^3 \cdot x^2 = x^{3+2} = x^5$. An APOS justification is that these students studied laws of indices as early as Grade eight. In subsequent years problems involving laws of indices reinforced students' understanding of exponents. By repeated manipulation and encapsulation of the laws of indices, students reached an object level of understanding exponents. As late as Grade twelve, the power rule of differentiation was introduced. Less opportunity was provided for high level mental constructs for the power rule to develop for the students beyond an action level.

These students are familiar with laws of indices from their high school syllabus, compared to the power rule in differentiation. Hence, laws of indices are recalled soon in their minds when they deal with differentiation. This suggests that these students were still at stage one of the four stages of the ZPD model. That is, students can only work with assistance of more knowledgeable peers or a lecturer. They need the assistance of more knowledgeable students and the lecturer to gain understanding of the constant multiple rule.

According to the APOS theory, the students require specific teaching; they need to perform mathematical tasks, discuss their results, and listen to the explanations of fellow students and the lecturer. That is, the lecturer should use the ACE teaching cycle to make sense of mathematical meanings in terms of what they are working on.

Students confused the product rule with the logarithmic rule. This originates from poor conceptualisation of a natural logarithm $(\ln x)$. The students' solutions showed that they did not know $\ln x$ as a natural logarithm; they assumed that $\ln x$ is any other variable. As a result, they did not apply properties of a natural logarithm; instead they applied the product rule. For these students there is no difference between $\ln(\sin x + 2)$ and $x(\sin x + 2)$.

This suggests that there should be a thorough explanation of concepts and their properties before the introduction of the rules of differentiation. This also suggests that these students probably did not understand self study activities that were based on the properties of logarithmic functions. These students should be taught the concept of a natural logarithm and its properties, as well as the difference between natural logarithmic functions and algebraic functions.

The errors of students in this study showed that these students were taught at a higher level of understanding than their actual level of understanding. This suggests that lecturers should try to find out students' prior knowledge with regard to the topic that is taught. This also becomes an eye-opener to lecturers as it gives possible errors that may be expected in the learning of derivatives of trigonometric functions. These findings might assist lecturers to develop instructional strategies that will assist students not to get into the trap as discussed above.

Students displayed interpretation errors that are based on poor understanding of differentiation rules such as the power rule; the constant multiple rule and the chain rule. These errors originated from over-generalisation of differentiation of a composite function of trigonometric functions, which apply only to trigonometric functions and not to algebraic functions. This suggests that in the introduction of a composite function and the chain rule, lecturers should explain the difference between the product rule and the chain rule. This should also cover what characterises application of the product rule and the chain rule in the given function.

Students committed interpretation errors that originated from the over-generalisation of the symmetric property of equality that says that if $a = b$ then $b = a$. They also committed errors that originated from the algebraic over-generalisation that if $a = b$, then $a^2 = b^2$. These students knew that the derivative of $y = \sec x$ is $\frac{dy}{dx} = \sec x \tan x$ and they deduced that the derivative of $y = \sec^2 x$ must be $\frac{dy}{dx} = (\sec x \tan x)^2$.

The same applies in the derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$, therefore, they assumed that the derivative of $y = \tan^2 x$ is $\frac{dy}{dx} = (\sec^2 x)^2 \sec^2 x$. They assumed that the process of differentiation is reversible, that is, if the derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$ and, therefore the derivative of $y = \sec^2 x$ is $\frac{dy}{dx} = \tan x$ which is not the case. In the case of differentiation of trigonometric functions the symmetric property of equality does not apply. There is no definite clue on how these students arrived at the process of reversibility as part of their response. A possibility is that these students applied a reversal manipulation that is derived from their schema level, as they recalled that if $y = \tan x$ then $\frac{dy}{dx} = \sec^2 x$. This is possible since they had chosen the $y = \tan x$ function and not anyone of the five trigonometric ratios. This error could have arisen since:

1. These students did not have the relevant mental structure in place that would have allowed them to achieve success at this stage;

2. These students treated the notion "sec" as a function; and

3. They did not have an effective chain rule schema for trigonometric functions.

This might also originate from over-generalisation of trigonometric identities such as $1 + \tan^2 x = \sec^2 x$. Another reason might be poor understanding of trigonometric identities. This suggests that lecturers should highlight in their teaching that differentiation is not reversible. This also suggests that lecturers should also explain the relevance of using trigonometric identities when teaching derivatives of trigonometric functions.

Students displayed interpretation error as they treated $y = x^3 e^{2x+3} \sqrt{\cos x}$ as if it required the application of the product rule instead of the logarithmic differentiation. They treated $x^3 e^{2x+3}$ as the first function and $\sqrt{\cos x}$ as the second function. Their error was to fuse two functions into one.

The nature of this problem tempts students to think of the product rule. They had been working with the product rule for a long time when they are faced with differentiation of two functions joined by a multiplication sign. Now that they came across three functions joined by a multiplication sign the product rule rings first in their minds. As a result, they fused two functions into one function in order to be able to apply the product rule. This suggests that these students were still at an action level of APOS theory with regard to the logarithmic differentiation, as they reacted to the stimuli incorrectly. They knew that they were required to differentiate, but did not know the appropriate differentiation rule.

This error shows a lack of emphasis in teaching what characterises application of logarithmic differentiation in a given function. This suggests that lecturers should explain the nature of functions that require application of logarithmic differentiation. In their explanation they should show the difference between problems that require application of the product rule in contrast to problems that require application of logarithmic differentiation. The lecturer gained an experience in this study as some of the findings and recommendations are also applicable even in his teaching.

6.2.2 Arbitrary errors displayed by students in three written tasks

Students demonstrated arbitrary errors that originate from poor understanding of differentiation rules. They did not know the appropriate procedures to be applied in order to differentiate given functions. These students knew that they had to apply the power rule but they still confused the power rule with the product rule. This indicates that these students were still working between the action level and the process level of APOS theory. The power rule had not been interiorised to shift students from an action level to a process level. They need repeated actions through the ACE teaching cycle in order to reach a schema level of applying the power rule in differentiation of algebraic functions.

Students showed arbitrary errors in differentiation of trigonometric functions. The conclusion here is that these students were still working at an action level of APOS theory.

They reacted to the stimuli as they wrote the derivative of a trigonometric function incorrectly. The mental construction of applying the product rule in differentiation of an integrated problem had not yet been developed. This indicates that they had not yet reached the scheme level of the product rule to differentiate trigonometric functions.

Students only transcribed problems and left them without any attempt of calculation. There is no exact explanation for this reaction. A possibility is that these students did not have a clue on how to start differentiation of this type of problem. This indicates that these students are working at a pre-mature stage of an action level of APOS theory, as they did not react to external stimuli, which was to find the derivative of $y = 4x^{3/2} \sin x$.

Some students were still working at an action level of APOS theory as they understood what was required by the instruction. This indicates that they reacted to external stimuli, but did not know the appropriate procedure to follow in order to reach the correct solution, which suggests that these students had not yet acquired a schema level of the product rule to differentiate problems such as $y = 4x^{3/2} \sin x$.

The students displayed arbitrary errors that originate from poor understanding of concepts. This suggests that these students did not understand that signs of each term change when multiplied by a minus sign. The lecturer should engage students in a variety of activities that allow their fellow students to discuss, explain and make sense of mathematical concepts, which are learned in that particular situation. This study utilised activities involving concepts and derivatives of trigonometric functions. As the students became engaged in the process of knowing that each expression changes its sign when multiplied by a minus sign, they will shift from one level or stage to the second level. According to the Vygotsky's ZPD they will shift from stage 1 where they learn with the assistance of more knowledgeable fellow students and/or the lecturer to a second stage, where they will work independently without any support. This shows clear progress of students as they move from one level to another level. Even in my practice students showed improvement and understanding as we continue working with a variety of activities in derivatives of trigonometric functions.

Some students did not know the relationship between the sine function and the cosecant function, which shows that some arbitrary errors originate from poor conceptualisation of trigonometric concepts. This also suggests that the teaching of derivatives of trigonometric functions should be sequenced starting from concept development and proceeding according to action, process, object and schema levels of the APOS theory. Several students' arbitrary errors revealed that these errors arise from poor understanding of basic concepts of differentiation and poor understanding of appropriate procedures of differentiation.

6.2.3 Procedural errors displayed by students in three written tasks

Few students demonstrated poor cancellation. This error indicated that these students were working at a schema level of the APOS theory, as they demonstrated understanding of differentiation of trigonometric functions. They only made a mistake in the cancellation.

Some students did not change a plus sign when multiplied by a minus sign. They also failed to simplify and showed poor understanding in the use of brackets in manipulation of trigonometric functions. They did not apply brackets to show multiplication signs. A few students demonstrated poor understanding of differentiation rules. They failed to apply the power rule correctly. They also displayed procedural errors that originate from poor understanding of derivatives, as they wrote that the derivative of $y = \cos ecx$ is $\frac{dy}{dx} = \cos ecx \cot x$ instead of $\frac{dy}{dx} = -\cos ecx \cot x$. These students were working at a schema level as they showed careless mistakes of omitting a minus sign from their answers. They also substituted $\cot x$ with $\tan x$. This error showed that these students did not know that $\cot x$ and $\tan x$ are only reciprocals and not identities. These students were still working at an action level of the APOS theory with regard to trigonometric reciprocals.

The students showed poor understanding of closure as they simplified further by removing the highest common factors, which is the opposite of leaving the answer in the simplest form. This indicates that these students had not yet developed the scheme level of APOS theory. Some students showed that they were at an action level of the APOS theory with regard to application of the power rule. Their response indicated that they knew that they had to apply the power rule, but they applied it incorrectly. They also revealed errors when it came to simplification. They did not apply LCD to simplify fractional trigonometric functions and a fractional algebraic term. They multiplied the numerators and left out the denominators. Many students differentiated radical expression incorrectly. For example, they failed to apply the chain rule in differentiation of $y = \sqrt{\cot(7x)}$. The fact that they did not apply the chain rule indicated that these students had not yet developed the mental constructs of the chain rule. It did not occur to them that they had to apply the chain rule. This indicates that they were working at an action level of the APOS theory as they did not react to stimuli to apply the chain rule.

The students did not know that $\ln e$ is equal to one. This research revealed that poor conceptualisation led to application of inappropriate procedures. These students reacted to the stimuli as they differentiated the given function, but owing to poor understanding of the concept $y = \ln e$, they followed incorrect procedures and reached an incorrect answer. They might have been working at an action level with regard to the understanding of $y = \ln e$.

6.2.4 Linear extrapolation errors displayed by students in three written tasks

Students demonstrated an error that originated from over-generalisation of distributive property. These students did not have an understanding of the compound angle formulae of a cosine function. This indicates that they were working at an action level of the APOS theory with regard to the compound angle formulae of a cosine function. The conclusion based on the high school syllabus, the National Curriculum Statement (NCS), was that this student had already passed Grade twelve at a schema level of the APOS theory with regard to the compound angle formulae of a cosine function.

They were expected to enter first-year university level with the understanding that $\cos(x+h) = \cos x \cos h + \sin x \sin h$. The fact that these students did not know the compound angle formulae of a cosine function confirms claims in the literature that several students enter universities under-prepared to study mathematics at first-year university level.

Some students demonstrated a linear extrapolation error with regard to the radical function such as $y = \sqrt{\sin(7x) + \ln(5x)}$. This error originated from $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$. This applies in multiplication of non-negative real numbers that are written in a radical form, and does not apply to addition and subtraction of non-negative real numbers that are written in a radical form. These students were working at an action level of the APOS theory with regard to the differentiation of the radical trigonometric function. Their reaction indicated that they knew that they had to differentiate, but did not know how to differentiate a radical trigonometric function. This suggests that lecturers should revise commutative property of non-negative real numbers that are written in a radical form under the four basic operational signs. This will help students to understand where commutative property holds and where it does not hold.

6.2.5 Conceptual errors displayed by students in three written tasks

Few students showed poor understanding of the special limit $\lim_{h\to 0} \frac{\sin h}{h} = 1$. This indicates that these students were working at a process level of the APOS theory with regard to understanding special limits of trigonometric functions. This might happen owing to the fact that limits of trigonometric functions are only introduced at university level. These students studied limits of algebraic functions only in high school mathematics. As a result they might be working at a schema level of the APOS theory with regard to limits of algebraic functions, but they had not developed mental structures to work at an object level of the APOS theory regarding limits of trigonometric functions.

Some students were working at a process level of the APOS theory with regard to understanding a compound angle of a cosine function.

They knew a compound angle formula of a cosine function, but not in totality as they made the mistake of writing a plus sign instead of a minus sign.

Some students did not know the derivative of trigonometric functions. They might be working at an action level of the APOS theory, as they did not know what should be done to differentiate $y = \tan x$. Few students wrote that the derivative of $y = \tan x$ is $\frac{dy}{dx} = -\sec^2 x$. These students might have been working at an object level of the APOS theory, as they wrote the correct derivative with an error of writing a minus sign in front of $\sec^2 x$. The students did not see a relationship between the derivative of a secant function and a tangent function, while having the correct answer.

Students demonstrated poor understanding of trigonometric reciprocals. They might be working in an action level of the APOS theory, as they did not have mental structures that are in place with regard to the trigonometric identities. These students did not know the derivative of a secant function.

The students did not understand the difference between $\sec x^4 \tan x^4$ and $\sec \tan(x^4)$. These students might have been working at a process level of the APOS theory, as they reacted to the stimuli of finding the derivative correctly, but struggled with the application of the chain rule. This suggests that the schema level of the chain rule was not yet developed with regard to derivatives of trigonometric functions.

Some students did not know the difference between the composite functions and two functions joined by a multiplication sign. They differentiated $y = \cot^3 x^4$ incorrectly. Their errors showed that they did not know that the derivative of $y = \cot x^4$ is $\frac{dy}{dx} = -\cos ec^2 x^4$. These errors show that these students did not have meaningful understanding of trigonometric reciprocals. It may be argued that these students entered an extended curriculum programme under-prepared with regard to the learning of derivatives of trigonometric functions.

One may argue that they did not have a sufficient basis of trigonometry from high school level. Hence, it is difficult for them to grasp derivatives of trigonometric functions. One may argue that even in ECP programme they did not obtain sufficient understanding of derivatives of trigonometric reciprocals. This suggests that the lecturer should put emphasis in derivatives of trigonometric reciprocals. This probably can be done by increasing activities and exercises that involved derivatives of trigonometric function in tutorial tests as well as in formative assessments.

Students did not know that they may substitute trigonometric functions with their trigonometric identities in the process of simplification. One may argue that they did not know trigonometric identity. Some students committed errors that originated from the algebraic over-generalisation that if $a = b$ then $a^2 = b^2$. These students knew that the derivative of $y = \sec x$ is $\frac{dy}{dx} = \sec x \tan x$ and deduced that the derivative of $y = \sec^2 x$ must be $\frac{dy}{dx} = (\sec x \tan x)^2$. The same applies to the derivative of $y = \tan x$ is equal to $\frac{dy}{dx} = \sec^2 x$, therefore, they assumed that the derivative of $y = \tan^2 x$ is equal to $\frac{dy}{dx} = (\sec^2 x)^2 \sec^2 x$. The process of differentiation is not reversible, and this study does not consider this, but other researchers may take this further by trying to find out what makes students think that differentiation may be reversible. Probably, as indicated earlier, students' understanding of trigonometric identities might create a situation where students may think that the process of differentiation is reversible.

At the second level of the APOS theory, an individual who has a process conception of a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps (Weyer, 2010). These students did not try to differentiate trigonometric functions such as $\sec^2 x$. They merely assumed that the derivative is $\tan x$. Perhaps this may be addressed through student-student and student-lecturer interactions in terms of more knowledgeable fellow students or their lecturer.

6.3 Relating the findings to the second research question

The second research question of the study was:

In what ways can social interactions address students' errors and advance them from a basic level to an advanced level of understanding in their learning of derivatives of trigonometric functions?

This question refers to application of the four levels of the APOS theory and the model of the four stages of Vygotsky's zone of proximal development. The APOS theory is linked to the ACE teaching cycle where students work with mathematical tasks in the form of activities in classroom discussions interacting with more knowledgeable fellow students and lecturers to gain an understanding of the work that they do.

The aim of working with mathematical tasks is to gain an understanding of what students learn through the assistance of more knowledgeable fellow students and lecturers who help them to solve mathematical problems that they were unable to solve as individuals independently. For this study students should gain understanding of the first principles of differentiation as their first stage and move to gain understanding of the rules of differentiation and, lastly, be able to integrate all rules of differentiation to find derivatives of complex trigonometric functions.

6.3.1 Students' responses in audio and video recordings

In audio and video recordings students showed their solutions on a whiteboard whilst representing their groups. The students demonstrated errors in their solutions as they showed their approaches of differentiating the two problems by applying logarithmic differentiation and the product rule.

Students demonstrated on the whiteboard while the audience (other students) intervened trying to assist when the one at the whiteboard made a mistake. They also questioned for clarity purposes when they were not certain of what was happening. The lecturer also intervened while students committed some errors in their solutions.

Students showed arbitrary and conceptual errors in differentiation of $y = x^3 e^{2x+3}\sqrt{\cos x}$, as they omitted some concepts in their solutions. This suggests that the students should be referred to the logarithmic rules that were supplied to them as self study activities.

One may suggest more activities and exercises that may bring students' attention to the understanding of logarithmic functions. In a class discussion the students' arguments revealed that they confused the three rules of differentiation, namely the product rule, the chain rule and the logarithmic differentiation rule.

Students showed arbitrary errors as they transcribed the problem incorrectly. The lecturer intervened by showing the students the correct original form of the problem. This student showed understanding of the problem, but revealed conceptual error as they did not know that $-\tan^2 x \ln e = -\tan^2 x$. This originated from the fact that they did not know that $\ln e = 1$. The lecturer intervened by requesting the attention of the entire class. He explained the appropriate procedure of differentiating $\ln e^{-\tan^2 x}$ for the whole class. Students demonstrated an interpretation error, which originated from the poor understanding of cancellation.

6.3.2 Students' responses from in-depth interviews

All the interviewees stated that their learning of limits and the first principles of differentiation in high school only dealt with the use of real numbers. They claimed that they did not study application of infinity. Some interviewees stated that they only studied the limits and the first principles of differentiation whereas others claimed that they also studied differentiation rules such as sum and constant rules.

All the interviewees revealed that they did not use the application of the first principles of differentiation to prove standard derivatives of trigonometric functions. They stated that they only learnt the application of the first principles of differentiation in algebraic functions. The interviewees claimed that the learning material used in class discussions as a supplement to their study guides assisted them as it had a variety of questions, which enabled them to practice.

At the same time the material included detailed solutions which assisted them to check when they had made certain mistakes in their solutions. They also added that it assisted them to be able to work on their own when they were at home.

Regarding the role that collaboration played in their learning of differentiation, twelve interviewees revealed that collaboration was useful. They claimed that all students brought their various proficiencies to the groups; as a result they helped as they corrected their mistakes while practicing their differentiation. They also brought along the idea that in groups some students asked questions that as an individual one never thought of, and hence all group members would benefit. Students claimed that they gained deep understanding when they explained a mathematical problem on their own to other students. They added that when some students asked questions for clarity, they would understand the content even more through discussions and explanations from group members.

They also claimed that class discussions assisted them as it was easier to remember because of the practical experience of doing calculations on the whiteboard. They also highlighted that when they made mistakes on the whiteboard the lecturer would intervene immediately to correct them, which helped them, since they received immediate feedback. They claimed that as they also questioned the student who did calculations on the whiteboard, it helped them to understand differentiation even better.

They also claimed that class discussions assisted them because at times students explained issues clearer than the lecturer. They supported this by saying that at times the students brought different approaches to a problem, which were not taught by the lecturer, and this helped with understanding. This supports Vygotsky's ideas that peer collaboration also helps students to explore other solution strategies through trial and error. They discussed and argued to convince one another with regard to their strategies. Hence they made mathematical meaning among themselves through explaining their thinking and interpretation.

6.4 SIGNIFICANCE OF THE STUDY

The purpose of this study was to explore errors that were displayed by students in their learning of derivatives of trigonometric functions. Differentiation is regarded internationally as the section which presents the most problems to first year mathematics students in universities of technology (Naidoo & Naidoo, 2007). Differentiation is the core concept of an extended curriculum programme and dominates in the current syllabus.

In South Africa high school syllabus in differentiation covers first principles of differentiation, power, sum or difference and a constant multiple rules. It also covers equations of tangents to graphs of functions, sketching graphs of cubic polynomial functions using differentiation to determine stationary points and points of inflection. It also covers solving practical problems concerning optimisation and rates of change, including calculus of motion.

In a university differentiation covers derivatives of algebraic, exponential, logarithmic and trigonometric functions. Students enter universities with only the understanding of differentiation of algebraic functions. This implies that they only come across derivatives of other functions such as exponential, logarithmic and trigonometric for the first time in their first year level at universities.

The current syllabus of differentiation in an extended curriculum is dominated by the derivatives of trigonometric functions. However, students demonstrate poor understanding of differentiation in general and in derivatives of trigonometric functions in particular. Derivatives of trigonometric functions also serve as a foundation for the understanding of integration. Integration is the core in calculus syllabus studied in the second semester in all fields of engineering. Currently there are no studies reported that focused on analysis or exploration of errors and misconceptions in derivatives of trigonometric functions.

Researchers may find the results of this study useful with regard to the identification of errors that were displayed by the students when learning derivatives of trigonometric functions.

The understanding of errors displayed by students in the learning of mathematics may assist mathematics teachers. Mathematics teachers and lecturers may be able to predict possible mistakes that students may do and to develop some techniques of addressing those errors in their teaching of mathematics in general and of derivatives of trigonometric functions in particular.

6.5 LIMITATIONS AND FURTHER RESEARCH OF THE STUDY

Due to the nature of this study only on a group of thirty students in an extended curriculum programme were involved. Research of this nature is also necessary for various levels of learning mathematics. The study focused on students' errors in derivatives of trigonometric functions although algebraic, logarithmic and exponential functions were integrated to build complexity. However, there may be other obstacles that hinder students' progress in their learning of derivatives of trigonometric functions. The full spectrum of differential calculus should be researched as well. A variety of similar case studies will make a remarkable contribution to the understanding of errors in differential calculus across all levels. Future researchers may try other theoretical frameworks to similar studies on other topics of mathematics.

6.6 RECOMMENDATIONS

The results of this study suggest that lecturers should identify students' errors in order to be able to design learning activities that will perhaps enhance students' understanding of derivatives of trigonometric functions. Errors that were displayed by students in this study originated from their prior learning of mathematics. This suggests that errors should be a focus of lectures at all levels in order to assist students to gain meaningful understanding of their learning of derivatives of trigonometric functions.

As indicated in the literature review, few university lecturers have researched errors of students in their learning of mathematics in undergraduate courses. As a result, this study attributes causes of poor performance of students in mathematics to the kind of errors that were displayed by students in their learning of derivatives of trigonometric functions.

As students who participated in this study demonstrated, there were various errors made regarding interpretation, arbitrary, procedural, linear extrapolation and conceptual, which suggests that lecturers should pay special attention to students' errors in order to develop teaching strategies that will address errors. This can be done by encouraging students to self-reflect by trying on their own to identify their errors during class discussions.

The study revealed that students commit several errors due to poor conceptualisation. As a result, poor understanding of concepts leads to poor interpretation, which leads to procedural and linear extrapolation errors. Results of this study also allow the researcher to recommend emphasis on key concepts when teaching derivatives of trigonometric functions. This can be achieved by establishing relational understanding in the learning of derivatives of trigonometric functions.

Results of the study have also revealed that students tend to apply rote learning, and hence memorise rules of differentiation without trying to make sense of what is actually happening. This suggests that lecturers should focus their teaching in development of relational understanding to strengthen students' memorisation. Perhaps the application of the first principles of differentiation to develop standard derivatives may help students to develop insight into their learning of derivatives of trigonometric functions. The results of the study also allow the researcher to recommend the use of a variety of activities to supplement learning activities that are normally practiced by students taken from the prescribed text books and study guides.

Many students demonstrated arbitrary errors, which are based on carelessness such as transfer errors where students failed to transcribe problems as they were given. This may be attributed to a lack of emphasis on the importance of transcribing problems first as they appear.

On the other hand one may link arbitrary errors with anxiety that normally happens when students have fears of tests. Several arbitrary errors originated from conceptual errors as students changed the nature of the problem owing to poor interpretation errors such as $6x^{1/2} = \sqrt{6x}$.

Lecturers should focus their teaching on identification of errors. Once errors are identified, learning activities should be designed in such a way that students are able to address their errors through student-student interactions and student-lecturer interactions whether inside or outside classroom discussions.

The use of Vygotsky's ZPD and the APOS theory to analyse data appears valuable as a framework that can assist lecturers in the teaching of mathematics, in general and derivatives of trigonometric functions, in particular.

The use of the ACE teaching cycle also give students opportunities to understand the importance of student-student interactions and lecturer-student interactions in discussions, debates and arguments inside and outside of the classroom pertaining to the learning of derivatives of trigonometric functions.

The use of both the APOS and Vygotsky's ZPD requires individual attention to obtain students' explanations, discussions and debates on how and why they performed to reach their solutions. This requires extra time such as individual consultations, which may not be available owing to demands of other courses that students registered for in their extended curriculum programme.

6.7 SUMMARY

This chapter has summarised the entire project. The results were discussed, while factors that caused errors among students in an extended curriculum programme were highlighted. Limitations, significance and avenues for further research were also mentioned. Recommendations have been made to suggest possible ways of improving students' performance in derivatives of trigonometric functions.

With the benefit of an improved understanding of errors as forthcoming from this study, reasons uncovered, and with implementation of its recommendations, one may make a substantial contribution towards addressing errors that are displayed by students in their learning of derivatives of trigonometric functions.

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Appendix 1

Interviews for the pilot study and the main study and some students' respondent

In-depth interviews

Explanation of the purpose of the interview: This interview is for my PhD study in mathematics education at UWC. Feel free to express yourself and be open as much as you can. I am going to ask questions about your experience of learning differential calculus during this research project.

First interviewee

Researcher: To what extent did you learn differential calculus in high school?

Beauty: First principle and power rule.

Researcher: How was the level of complexity?

Beauty: We did not do problems involving fractions like $\frac{x}{2x-1}$.

Researcher: Did you learn other differentiation rules like the product, quotient, chain, implicit and the logarithmic in high school?

Beauty: No

Researcher: How do you perceive collaboration on the learning of differential calculus?

Beauty: It helped a lot as we discuss sometimes you get confused but as the lecturer and/or researcher and other students explain then you can see your mistakes, and find the right way of doing it (a mathematical problem).

Researcher: To what extent did you do self-study activities in high school?

Beauty: We had text books and other material like study and master mathematics?

Researcher: Which new strategies that you come across here that you did not learn in high school?

Beauty: The advice of practicing examples prior the lesson presentations in the classroom.

Researcher: Did you find self-study material helpful?

Beauty: Yes

Researcher: How?

Beauty: It helps to read on your own and pay more attention in the classroom once the lecturer or other student try to solve what was a problem in a practice? Then I also ask questions when I do not understand.

Researcher: Find the derivatives of the following e^x; 2^x ; x^2; x^x

Beauty: e^x; 2^x ln x; 2x; apply ln both sides of the equation.

Researcher: find the derivative of the following tan x; cos x; sin x; sec x; cosec x; cot x

Beauty: Sec^2 x; - sin x; cos x I forgot the two sec x and cosec x; for cot x is cosec x cot x

Researcher: Thanks a lot. This is the end.

Second interviewee

Explanation of the purpose of the interview: This interview is for my PhD study in mathematics education at University of Western Cape. Feel free to express yourself and be open as much as you can. I am going to ask questions about your experience of learning differential calculus during this research project.

Researcher: To what extent did you learn differential calculus in high school?

Sweetness: first principle and logarithms but we did not do ln.

Researcher: How was the level of complexity?

Sweetness: Fractions were not involved.

Researcher: Did you learn other differentiation rules like product, quotient, chain, implicit and logarithmic in high school?

Sweetness: No

Researcher: How do you perceive collaboration on the learning of differential calculus?

Sweetness: It helps as we work as students alone I feel free to ask questions for clarity more that I can ask in the classroom.

Researcher: Which section of differential calculus that was easy to understand by only using self-study material?

Sweetness: Product rule and quotient rule, because you write the formula like first function times the derivative of the second function plus the second function times the derivative of the first function and simplify.

Researcher: What about other sections like chain rule; implicit and logarithmic?

Sweetness: They were a bit difficult but through classroom collaboration I grasped and understood the procedures for all differentiation rules.

Researcher: Are you comfortable now in all sections?

Sweetness: Yes; I do not have a problem now.

Researcher: Find the derivatives of the following e^x; 2^x; x^2; x^x

Sweetness: e^x; $2^x \ln x$; $2x$; I have forgotten this one that is x^x

Researcher: Find the derivatives of the following $\tan x$; $\cos x$; $\sin x$; $\sec x$; $\cos ec x$ and $\cot x$

Sweetness: -sin x; cos x; tan $\sec^2$ x; and I do not know the rest?

Researcher: Can you predict what make you do not to know the derivatives of the reciprocals?

Sweetness: They are not common in examples and exercises as the other three trigonometry ratios.

Researcher: Thanks a lot. This is the end.

Third interviewee

Explanation of the purpose of the interview: This interview is for my PhD study in mathematics education at the University of Western Cape. Feel free to express yourself and be open as much as you can. I am going to ask questions about your experience of learning differential calculus during this research project.

Researcher: To what extent did you learn differential calculus in high school?

Mahesh: We did first principle and we applied power rule although we were not expose to the concept power rule.

Researcher: Did you also apply fractions in first principle like $\frac{x}{2x-1}$?

Mahesh: Yes

Researcher: Which grade were you doing?

Mahesh: Standard grade but attending higher grade.

Researcher: Did you learn other differentiation rules like product, quotient, chain, implicit and logarithmic in high school?

Mahesh: No

Researcher: How do you perceive collaboration on the learning of differential calculus?

Mahesh: It is beneficial although at times some students tend to ask something out of order.

Researcher: Like what? Can you make an example?

Mahesh: Questions like why do we waste time by proving something like derivative of cot?

Researcher: Do you think that question was out of order?

Mahesh: Yes because we prove it because we want to know it?

Researcher: Find the derivatives of the following e^x; 2^x ; x^2; x^x

Mahesh: e^x; 2^xln2; 2x; x^x =1

Researcher: Can you justify? Why x^x = 1?

Mahesh: Because the derivative of x is one

Researcher: What if we write as y = x^x

Mahesh: We apply ln?

Researcher: How? Can you explain?

Mahesh: lny = ln x^x

→ln y = x ln x

→1/y.y′ = x.1/x + lnx.1

→1/y.y′ = 1 + ln x

→ y′= (1 + ln x) x^x

Researcher: That is it. Then may you tell me the derivatives of the following trigonometry identities? Cos x; tan x; sin x; sec x; cosec x; cot x?

Mahesh: -sin x; sec^2x; cos x; secx tanx. There was clear facial appearance showing doubt about this one. For cosec Yoo this one I do not know it. For cot x is –cosec x.

Researcher: Which section of differential calculus that was easy to understand by only using self-study material?

Mahesh: Product rule and quotient rule.

Researcher: What are about chain rule, implicit and logarithmic differentiation?

Mahesh: I understood other sections only after explanations and more examples and exercises done in the classroom?

Researcher: Thanks a lot. This is the end.

Fourth interviewee

Explanation of the purpose of the interview: This interview is for my PhD study in mathematics education at the University of Western Cape. Feel free to express yourself and be open as much as you can. I am going to ask questions about your experience of learning differential calculus during this research project.

Researcher: To what extent did you learn differential calculus in high school?

Bethany: We studied limits.

Researcher: What else?

Bethany: Only limits?

Researcher: What about first principle?

Bethany: No we did not study first principle.

Researcher: Did you study derivatives of problems like $x^2 + x + 1$

Bethany: Yes we studied it as $2x + 1$

Researcher: How did you start this?

Bethany: Using power rule and told that the derivative of a constant is zero.

Researcher: Did you use self-study material to learn mathematics in high school?

Bethany: No

Researcher: Which sections did you find easy to learn using self-study material alone?

Bethany: Product and quotient rule.

Researcher: What about other sections of differential calculus?

Bethany: I understood other sections after classroom lessons and collaboration with the lecturer and other students.

Researcher: How do you perceive collaboration on the learning of differential calculus?

Bethany: It is good since we learn from mistakes of us and other students?

Researcher: What do you mean by us and other students?

Bethany: I learn when the lecturer corrects my mistakes and when the lecturer correct mistakes of other students. It also helps us to get more explanation in a classroom?

Researcher: Find the derivatives of the following e^x; 2^x ; x^2; x^x

Bethany: e^x; 2^x ln x; sorry sir 2^x ln 2; 2x ; I cannot remember this one x^x?

Researcher: What if we write y = x^x?

Bethany: Yes I see it now.

Researcher: What is it?

Bethany: We apply ln both sides of the equation.

lny = ln x^x

→ ln y = x ln x

→1/y.y′ = x.1/x + lnx.1

→1/y.y′ = 1 + ln x

→ y′ = (1 + ln x) x^x

Researcher: Find the derivatives of the following Cos x; sin x; tan x; sec x; cosec x and cot x

Bethany: D tan x = sec^2x; D sin x = cos x; D cos x = -sin x; sec x = ….; cosec x = $cosec^2$ x cot x; cot =….;

Researcher: Can you predict what makes you not know the derivatives of the reciprocals?

Bethany: Yes, I do not learn them.

Researcher: Thanks a lot. This is the end.

Fifth interviewee

Explanation of the purpose of the interview: This interview is for my PhD study in mathematics education at the University of Western Cape. Feel free to express yourself and be open as much as you can. I am going to ask questions about your experience of learning differential calculus during this research project.

Researcher: To what extent did you learn differential calculus in high school?

Bhedi: Graphs; first principle; word problems and power rule.

Researcher: Did you learn other differentiation rules like the product, quotient, chain, implicit and the logarithmic in high school?

Bhedi: No

Researcher: Are you familiar with self-study approach on the learning of mathematics?

Bhedi: Yes; we worked with many texts books and other learning material like answer series.

Researcher: Did you work as a group in learning mathematics?

Bhedi: No

Researcher: How do you perceive collaboration on the learning of differential calculus?

Bhedi: It keeps us updated, that is always practicing and alert and it also builds a confidence as we get corrections on our calculations.

Researcher: Which section did you find it difficulty in differential calculus?

Bhedi: Chain rule and logarithmic differentiation.

Researcher: Find the derivatives of the following e^x; 2^x ; x^2; x^x

Bhedi: Got everything right except x^x he thought until the researcher wrote it a y = x^x .

Researcher: Find the derivatives of the following Cos x; sin x; tan x; sec x; cosec x and cot x

Bhedi: Got everything right except the reciprocals in sec x ≈ sec x cot x instead of sec x tan x in cosec x ≈ cosec x cot x instead of –cosec x cot x and cot x ≈ cosec ^{2} x instead of -cosec^2 x .

Researcher: Thanks a lot. This is the end.

Sixth interviewee

Explanation of the purpose of the interview: This interview is for my PhD study in mathematics education at the University of Western Cape. Feel free to express yourself and be open as much as you can. I am going to ask questions about your experience of learning differential calculus during this research project.

Researcher: To what extent did you learn differential calculus in high school?

Palma: Nothing

Researcher: What about first principle?

Palma: No I did not do it

Researcher: What about power rule?

Palma: I did nothing on differentiation.

Researcher: Did you not learn something like the derivative of $2x^2 + 2x + 1$?

Palma: No

Researcher: Did you not learn even the derivative of a constant?

Palma: No

Researcher: How do you perceive collaboration on the learning of differential calculus?

Palma: It was helpful as all questions have answers and the solutions have detailed steps. The extra self-study material, tutorials, sample tests and the university study guide help us to know what to expect in the examinations. The classroom interaction helps us to get clarity on what we do not understand.

Researcher: Find the derivatives of the following e^x; 2^x ; x^2; x^x

Palma: Did everything correct except x^x; did this one correct only after it was written as y = x^x

Researcher: Find the derivatives of the following $\cos x$; $\sin x$; $\tan x$; $\sec x$; $\operatorname{cosec} x$ and $\cot x$

Palma: Got everything right except for $\sec x$

Researcher: Thanks a lot. This is the end.